Optimal reinsurance under dynamic VaR constraint

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Abstract

This paper deals with the optimal reinsurance strategy from an insurer's point of view. Our objective is to find the optimal policy that maximise the insurer's survival probability. To meet the requirement of regulators and provide a tool to risk management, we introduce the dynamic version of Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR) and worst-case CVaR (wcCVaR) constraints in diffusion model and the risk measure limit is proportional to company's surplus in hand. In the dynamic setting, a CVaR/wcCVaR constraint is equivalent to a VaR constraint under a higher confidence level. Applying dynamic programming technique, we obtain closed form expressions of the optimal reinsurance strategies and corresponding survival probabilities under both proportional and excess-of-loss reinsurance. Several numerical examples are provided to illustrate the impact caused by dynamic VaR/CVaR/wcCVaR limit in both types of reinsurance policy.

Keywords: HJB Equation; Dynamic Value-at-Risk (VaR); Conditional Value-at-Risk (CVaR); Worst-case CVaR (wcCVaR); Survival Probability

1 Introduction

The optimal reinsurance problem of minimising ruin probability, or equivalently maximising survival probability for an insurance company through dynamic programming technique has been studied intensively in recent years. As two types of typical reinsurance strategies, proportional and excess-of-loss reinsurance have received great attention from both the academia and practitioners. Some literatures on the proportional reinsurance include Choulli et al. (2003), Højgaard and Taksar (1998a,b, 2004), Schmidli (2001, 2002) and Taksar (2000). Some renowned works on the excess-of-loss reinsurance are Asmussen et al. (2000), Choulli et al. (2001), Irgens and Paulsen (2004) and Meng and Zhang (2010). Recently, several papers deal with the combination of these two types; see, for example, Zhang et al. (2007) and Liang and Guo (2011), both of which draw the conclusion that the optimal combinational reinsurance strategy must be of a pure excess-of-loss type.

With the rapid development of financial markets, the exposure to risk for a commercial institute can be extremely high. Meanwhile, risk management has gained increasing popularity from regulators and practitioners in recent years, with Value-at-Risk (VaR) emerging as a standard tool to measure and control the risk of an insurance company. The VaR is the maximum expected loss over a given horizon period at a given confidence level. Because of its intuitive appeal, VaR is adopted by finance regulators to set up capital requirements on a commercial institution. In spite of its popularity, VaR is extensively criticised for possessing some undesirable theoretical properties. For example, Artzner et al. (1999) show that VaR is not a coherent risk measure since it fails to satisfy the subadditivity property. Accordingly, Conditional Value-at-Risk (CVaR), a convex and coherent risk measure which is also called conditional tail expectation (CTE) and average Value-at-Risk (AVaR), is often advocated as an alternative to VaR. CVaR, which combines aspects of the VaR methodology with more information about the distribution of returns in the tail, is the average loss on the worst given percentage of the possible outcome. To overcome the incomplete information about the loss distribution, worst-case scenario risk measures are investigated recently with worst-case Value-at-Risk (wcVaR) and worst-case Conditional Value-at-Risk (wcCVaR) being the most popular ones. Some related papers include Ghaoui et al. (2003), Zhu and Fukushima (2009), Natarajan et al. (2009) and Čerbáková (2005). Specifically, Čerbáková (2005) points out that wcVaR and wcCVaR are identical for distribution identified by its first two moments and Natarajan et al. (2009) demonstrate that wcCVaR is a coherent risk measure.

Latterly static risk measures have been extended to the dynamic version. Yiu (2004) applies the VaR constraint continuously overtime in portfolio allocation problem. Alexander and Baptista (2004) analyse the portfolio choice implications arising from imposing a dynamic VaR constraint on the mean-variance model, and then compare them with those under a dynamic CVaR constraint. Cuoco et al. (2008)

investigate the optimal dynamic trading strategies with VaR and CVaR limits. Chen et al. (2010) deal with an insurance company's investment-reinsurance problem under dynamic VaR constraint.

However, most existing literatures on the dynamic version of risk measures are based on portfolio selection and the upper boundary of VaR/CVaR is fixed as a constant. In the work of Yiu (2004) and Chen et al. (2010), the dynamic VaR level is fixed as a constant. Specifically, Chen et al. (2010) derive explicit expressions of the minimum ruin probability and optimal investment-proportional reinsurance strategy when the dynamic VaR is bounded by a constant. There is an obvious shortcoming of the constant risk limit assumption since it penalises an insurance company with ample surplus and will not come into effect when the surplus is decreasing. To overcome the disadvantage of constant risk limit, we adopt a more realistic constraint: the risk measure limit level is proportional to insurance company's surplus in hand, which makes the stochastic control problem more complicated than the ones with constant risk measure restriction. In this paper, we introduce three dynamic risk measures for an insurance company, depending on its exposure to claim settlement risks. Subsequently we adopt those risk measures, dynamic VaR, CVaR and wcCVaR when information about loss distribution is partially known, to set up constraints on the reserve of an insurance company.

Here, we investigate the optimal reinsurance problem under dynamic VaR constraint in both proportional and excess-of-loss settings. In the dynamic setting, it is always possible to transform a CVaR/wcCVaR constraint to an equivalent VaR limit. Therefore, the optimisation problem under dynamic CVaR/wcCVaR constraint could be converted to the one under dynamic VaR constraint with a higher confidence level. Our objective is to find the strategy that maximises insurer's survival probability and the corresponding survival probability as a function of initial surplus. Through the standard dynamic programming technique, we obtain the Hamilton-Jacobi-Bellman (HJB) equation that the value function should satisfy. Then under each circumstance, we construct a twice continuously differentiable solution to HJB equation, which coincides with the value function. Meanwhile, the optimal reinsurance strategy is provided by the corresponding maximum point of HJB equation. Results indicate that the dynamic VaR constraint lead to significant qualitative changes of optimal strategy and the maximum survival probability when the surplus is relatively small. For a large initial surplus, the VaR limit does not take that much effect. Besides, concavity of optimal survival probability will no longer hold under dynamic VaR constraint for a sufficient small surplus.

The rest of this paper is organised as follows. In Section 2, we provide a general formulation of the optimal reinsurance problem. Then we investigate the insurance company's maximum survival probability and the corresponding optimal reinsurance strategy in proportional reinsurance settings in Section 3, where dynamic VaR, CVaR and wcCVaR constraints are introduced. Section 4 deals with the optimal excess-of-

loss reinsurance. To illustrate the results, several numerical examples are presented in Section 5. Finally, some additional remarks are provided in Section 6.

2 Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Consider a Cramér-Lundberg model with the surplus process of an insurance company being given by

$$U(t) = u_0 + ct - \sum_{i=1}^{N_t} X_i,$$

where u_0 is the initial surplus, the claim arrival process $\{N_t\}_{t \geq 0}$ is a Poisson process with constant intensity $\lambda > 0$ and the random variables X_i , $i = 1, 2, \dots$, are i.i.d claim sizes independent of N_t . We let T_i denote the i -th claim occurrence time and $F(x)$ denote the claim size distribution with finite first and second moments m_1 , m_2 . The premium rate c is assumed to be calculated via the expected value principle, i.e.,

$$c = (1 + \eta)\lambda m_1,$$

where $\eta > 0$ is the relative safety loading factor.

To manage the risk, an insurance company could choose a reinsurance policy $f(X)$, which will cost $(1 + \theta)\lambda(m_1 - \mathbb{E}[f(X)])t$, where θ ($\theta \geq \eta$) is the loading factor for the reinsurer. Then the insurer's surplus process, under contract f , is given by

$$U^f(t) = u_0 + [(1 + \theta)\lambda \mathbb{E}[f(X)] - (\theta - \eta)\lambda m_1]t - \sum_{i=1}^{N_t} f(X_i).$$

And its diffusion approximation follows

$$dU^f(t) = \lambda [\theta \mathbb{E}[f(X)] - (\theta - \eta)m_1] dt - \sqrt{\lambda \mathbb{E}[f^2(X)]} dW_t.$$

A reinsurance strategy f is an admissible control if $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-decreasing function and satisfies $f(x) \leq x$, $x \geq 0$. We denote the set of admissible controls as Π .

We define the ruin time

$$\tau^f = \inf\{t > 0 : U^f(t) < 0\},$$

where the superscript f emphasises that the surplus process and the ruin time are controlled by an admissible policy f .

The objective is to minimise insurer's ruin probability or equivalently, maximise its survival probability. Thus we define the performance index for f as the survival probability under that policy:

$$J(u, f) = \mathbb{P}(\tau^f = +\infty).$$

Here $u \geq 0$ is the initial surplus. Our objective is to find the value function

$$v(u) = \max_{f \in \Pi} J(u, f)$$

and the optimal policy f^* such that

$$v(u) = J(u, f^*).$$

In the following two sections, we will investigate two types of typical reinsurance strategies: proportional and excess-of-loss reinsurance.

3 Proportional Reinsurance

Under the proportional reinsurance, the insurer could transfer a fraction $1 - q(t)$ of the incoming claims to a reinsurer and $f(X_i) = q(T_i)X_i$, where $q(t)$ is $\mathcal{F}_t$ -measurable and satisfies $0 \leq q(t) \leq 1$ for all t . The diffusion approximation of insurance company's claim process becomes

$$\begin{cases} dC^q(t) = q(t)\lambda m_1 dt + q(t)\sqrt{\lambda m_2} dW_t, \\ C^q(0) = 0, \end{cases} \quad (3.1)$$

where W_t is a standard Brownian motion. The insurer's surplus process satisfies the stochastic differential equation

$$\begin{cases} dU^q(t) = [\theta q(t) - (\theta - \eta)]\lambda m_1 dt - q(t)\sqrt{\lambda m_2} dW_t, \\ U^q(0) = u_0. \end{cases}$$

3.1 Dynamic VaR, CVaR and worst-case CVaR

Yiu (2004) and Cuoco et al. (2008) introduce a dynamic version of Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) constraints for portfolio selection problem. Motivated by their works, we formulate the dynamic VaR and CVaR constraints of an insurance company, we rewrite the claim dynamics (3.1) as

$$\begin{aligned} C^q(t+h) - C^q(t) &= \int_t^{t+h} q(s)\lambda m_1 ds + \int_t^{t+h} q(s)\sqrt{\lambda m_2} dW_s \\ &\doteq \lambda m_1 h q(t) + q(t)\sqrt{\lambda m_2} \int_t^{t+h} dW_s, \end{aligned}$$

where $h > 0$ is small enough and for s in the interval $[t, t+h)$, we approximate $q(s)$ by $q(t)$, i.e., $q(s) \doteq q(t)$, $s \in [t, t+h)$. This is a reasonable approximation since the insurance strategy can only be adjusted at discrete time and the decision is made based on the surplus at time t .

For a given confidence level $1 - \alpha \in (0, 1)$ and a given horizon $h > 0$, the VaR at time t of a proportional reinsurance policy q , denoted by $VaR_t^{\alpha, h}$, is defined as

$$VaR_t^{\alpha, h} \triangleq \inf \{L : \mathbb{P}(C^q(t+h) - C^q(t) \geq L \mid \mathcal{F}_t) < \alpha\}.$$

The dynamic Conditional Value-at-Risk $CVaR_t^{\alpha, h}$ is then given by

$$CVaR_t^{\alpha, h} \triangleq \mathbb{E} \left[C^q(t+h) - C^q(t) \mid C^q(t+h) - C^q(t) \geq VaR_t^{\alpha, h} \right].$$

If we only know the first two moments $\mathbb{E}[C^q(t)] = \lambda m_1 \int_0^t q(s) ds$ and $\mathbb{E}[(C^q(t))^2] = \lambda m_2 \int_0^t q^2(s) ds + (\lambda m_1 \int_0^t q(s) ds)^2$ about the claim process, rather than using (3.1) to approximate it, we define the dynamic worst-case Conditional Value-at-Risk (wcCVaR) risk measure. Worst-case strategies (wcVaR, wcCVaR, etc) are applied when there is incomplete information about the distribution. The definition and properties of wcCVaR are investigated by Zhu and Fukushima (2009), Natarajan et al. (2009) and Čerbáková (2005) and references therein. Here we introduce a dynamic version of wcCVaR. The set $\mathcal{P}_1$ of feasible probability distributions is specified as

$$\begin{aligned} \mathcal{P}_1 = \{ & p(\cdot) : \mathbb{E}_p[C^q(t+h) - C^q(t)] = \lambda m_1 h q(t), \\ & \mathbb{E}_p[(C^q(t+h) - C^q(t))^2] = \lambda m_2 h q(t)^2 + (\lambda m_1 h q(t))^2 \} \end{aligned}$$

and the dynamic worst-case CVaR is defined as

$$wcCVaR_t^{\alpha, h} \triangleq \sup_{p(\cdot) \in \mathcal{P}_1} \inf_{a \in \mathbb{R}} \left\{ a + \frac{1}{\alpha} \mathbb{E}_p[(C^q(t+h) - C^q(t) - a)_+] \right\}, \quad (3.2)$$

where $(x)_+ \triangleq \max\{0, x\}$ and the subscript p indicates that expectation is calculated under distribution $p(\cdot)$.

Proposition 3.1 We have

$$VaR_t^{\alpha, h} = \lambda m_1 h q(t) - \Phi^{-1}(\alpha) \sqrt{\lambda m_2 h} q(t)$$

and

$$CVaR_t^{\alpha, h} = \lambda m_1 h q(t) + \frac{\phi(\Phi^{-1}(\alpha))}{\alpha} \sqrt{\lambda m_2 h} q(t),$$

where $\phi(x)$ and $\Phi(x)$ denote the probability density function and the cumulative distribution function of a standard normal random variable, respectively. $\Phi^{-1}(x)$ is the inverse function of $\Phi(x)$. The dynamic worst-case Conditional Value-at-Risk is

$$wcCVaR_t^{\alpha, h} = \lambda m_1 h q(t) + \sqrt{\frac{1-\alpha}{\alpha}} \sqrt{\lambda m_2 h} q(t).$$

In particular,

$$0 \leq VaR_t^{\alpha, h} \leq CVaR_t^{\alpha, h} \leq wcCVaR_t^{\alpha, h} < U^q(t).$$

Proof. We have

$$\begin{aligned}
& \mathbb{P}(C^q(t+h) - C^q(t) \geq L \mid \mathcal{F}_t) \\
&= \mathbb{P}\left(\lambda m_1 h q(t) + q(t) \sqrt{\lambda m_2} \int_t^{t+h} dW_s \geq L \mid \mathcal{F}_t\right) \\
&= \mathbb{P}\left(\frac{1}{\sqrt{h}} \int_t^{t+h} dW_s \geq \frac{L - \lambda m_1 h q(t)}{q(t) \sqrt{\lambda m_2 h}} \mid \mathcal{F}_t\right) = 1 - \Phi\left(\frac{L - \lambda m_1 h q(t)}{q(t) \sqrt{\lambda m_2 h}}\right),
\end{aligned}$$

where the last equality follows from the fact that $\frac{1}{\sqrt{h}} \int_t^{t+h} dW_s$, conditional on the filtration $\mathcal{F}_t$, is a standard normal random variable. Thus, $\mathbb{P}(C^q(t+h) - C^q(t) \geq L \mid \mathcal{F}_t) < \alpha$ is equivalent to $1 - \Phi\left(\frac{L - \lambda m_1 h q(t)}{q(t) \sqrt{\lambda m_2 h}}\right) < \alpha$, which implies $L \leq \lambda m_1 h q(t) - \Phi^{-1}(\alpha) \sqrt{\lambda m_2 h} q(t)$. Then we have

$$VaR_t^{\alpha, h} = \lambda m_1 h q(t) - \Phi^{-1}(\alpha) \sqrt{\lambda m_2 h} q(t).$$

Therefore, the Conditional Value-at-Risk

$$\begin{aligned}
CVaR_t^{\alpha, h} &= \lambda m_1 h q(t) + \sqrt{\lambda m_2 h} q(t) \mathbb{E}\left[\frac{1}{\sqrt{h}} \int_t^{t+h} dW_s \mid \frac{1}{\sqrt{h}} \int_t^{t+h} dW_s \geq -\Phi^{-1}(\alpha)\right] \\
&= \lambda m_1 h q(t) + \sqrt{\lambda m_2 h} q(t) \frac{1}{\alpha} \int_{-\Phi^{-1}(\alpha)}^{+\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} dx \\
&= \lambda m_1 h q(t) + \frac{\phi(\Phi^{-1}(\alpha))}{\alpha} \sqrt{\lambda m_2 h} q(t).
\end{aligned}$$

Due to the finite expected value and the convexity of objective function in the definition of dynamic wcCVaR (3.2), the supremum and infimum are interchangeable. Considering that

$$\begin{aligned}
\sup_{p(\cdot) \in \mathcal{P}_1} \mathbb{E}_p[(C^q(t+h) - C^q(t) - a)_+] &= \frac{\sup_{p(\cdot) \in \mathcal{P}_1} \mathbb{E}_p[|C^q(t+h) - C^q(t) - a|] + \lambda m_1 h q(t) - a}{2} \\
&= \frac{\sqrt{\lambda m_2 h q(t)^2 + (\lambda m_1 h q(t) - a)^2} + \lambda m_1 h q(t) - a}{2},
\end{aligned}$$

we have

$$\begin{aligned}
wcCVaR_t^{\alpha, h} &= \inf_{a \in \mathbb{R}} \left\{ a + \frac{1}{\alpha} \sup_{p(\cdot) \in \mathcal{P}} \mathbb{E}_p[(C^q(t+h) - C^q(t) - a)_+] \right\} \\
&= \inf_{a \in \mathbb{R}} \left\{ a + \frac{1}{2\alpha} \left[\sqrt{\lambda m_2 h q(t)^2 + (\lambda m_1 h q(t) - a)^2} + \lambda m_1 h q(t) - a \right] \right\} \\
&= \lambda m_1 h q(t) + \sqrt{\frac{1-\alpha}{\alpha}} \sqrt{\lambda m_2 h} q(t),
\end{aligned}$$

where the last equality follows from that the infimum obtains at $a^* = \lambda m_1 h q(t) + \frac{1-2\alpha}{2\sqrt{\alpha(1-\alpha)}}\sqrt{\lambda m_2 h} q(t)$. $\square$

Remark 3.1 Considering that $(\mathbb{E}_p[|C^q(t+h) - C^q(t) - a|])^2 \leq V_p(|C^q(t+h) - C^q(t) - a|) + (\mathbb{E}_p[|C^q(t+h) - C^q(t) - a|])^2 = \mathbb{E}_p[|C^q(t+h) - C^q(t) - a|^2] = (\lambda m_1 h q(t) - a)^2 + \lambda m_2 h q(t)^2$, we obtain $\sup_{p(\cdot) \in \mathcal{P}_1} \mathbb{E}_p[|C^q(t+h) - C^q(t) - a|] = \sqrt{\lambda m_2 h q(t)^2 + (\lambda m_1 h q(t) - a)^2}$

and the supremum is obtained under the distribution $p(\cdot)$ such that $|C^q(t+h) - C^q(t) - a|$ is a constant and $C^q(t+h) - C^q(t)$ follows a binomial distribution.

Remark 3.2 We have $CVaR_t^{\alpha,h} = VaR_t^{\tilde{\alpha},h}$ and $wcCVaR_t^{\alpha,h} = VaR_t^{\hat{\alpha},h}$, where $\tilde{\alpha} = \Phi(-\frac{\phi(\Phi^{-1}(\alpha))}{\alpha})$ and $\hat{\alpha} = \Phi(-\sqrt{\frac{1-\alpha}{\alpha}})$, which implies that in our dynamic setting, it is always possible to transform a Conditional Value-at-Risk or worst-case CVaR constraint into an equivalent VaR constraint, and vice versa. Thus, when changing the confidence level α to $\tilde{\alpha}$ or $\hat{\alpha}$, we obtain the optimal strategy and corresponding value function under dynamic CVaR and wcCVaR constraints, respectively.

3.2 HJB equation

Value-at-Risk is the maximum expected loss over a given horizon period at a given confidence level. It is adopted by regulators to set up capital requirement on an institution. Thus many literatures use VaR as a constraint in their optimisation problem. However, the upper boundary of VaR in previous works is generally fixed as a constant. One of the disadvantages of such assumption is that the constraint penalises an insurance company with ample surplus and is not effective when the surplus is decreasing. In practice, successful companies typically see their VaR limit increased. To capture this fact, we introduce a more realistic constraint: the dynamic VaR level is subject to a fixed proportion of company's surplus in hand, i.e. $VaR_t^{\alpha,h} \leq kU^q(t)$, $0 < k < \infty$.

Under the dynamic VaR constraint

$$VaR_t^{\alpha,h} \leq kU^q(t),$$

from standard dynamic programming technique, we obtain the value function $v(u)$ shall satisfy the following HJB equation

$$\begin{aligned} \max_q \left\{ [\theta q - (\theta - \eta)] \lambda m_1 v'(u) + \frac{1}{2} \lambda m_2 q^2 v''(u) \right\} &= 0, \\ \text{s.t.} \begin{cases} q \in [0, 1], \\ \lambda m_1 h q - \Phi^{-1}(\alpha) \sqrt{\lambda m_2 h} q \leq k u, \\ v(0) = 0, v(+\infty) = 1. \end{cases} \end{aligned} \quad (3.3)$$

Theorem 3.1 (a) If $\theta \geq 2\eta$, the function

$$\delta(u) = \begin{cases} \delta(\frac{A}{k}) - \delta(\frac{A}{k}) \frac{\int_{\frac{A}{k}}^u w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}{\int_0^{\frac{A}{k}} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}, & \text{if } u < \frac{A}{k}, \\ \delta(\frac{A}{k}) + [1 - \delta(\frac{A}{k})][1 - e^{-\frac{2\eta m_1}{m_2}(u - \frac{A}{k})}], & \text{if } u \geq \frac{A}{k}, \end{cases} \quad (3.4)$$

is a smooth ($\mathcal{C}^2$) solution to the HJB equation, where

$$\begin{aligned} A &= \lambda m_1 h - \Phi^{-1}(\alpha) \sqrt{\lambda m_2 h}, \\ \delta\left(\frac{A}{k}\right) &= \frac{\frac{2\eta m_1}{m_2}}{\frac{2\eta m_1}{m_2} + \frac{\left(\frac{k}{A}\right)^{\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A}{k}}}{\int_0^{\frac{A}{k}} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}}. \end{aligned} \quad (3.5)$$

The maximum of the left side of HJB equation is attained at

$$q^*(u) = \begin{cases} \frac{ku}{A}, & \text{if } u < \frac{A}{k}, \\ 1, & \text{if } u \geq \frac{A}{k}. \end{cases} \quad (3.6)$$

(b) If $\eta < \theta < 2\eta$, the function

$$\delta(u) = \begin{cases} \delta(u_1) - \delta(u_1) \frac{\int_{u_1}^u w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}{\int_0^{u_1} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}, & \text{if } u < u_1, \\ \delta(u_1) + [1 - \delta(u_1)][1 - e^{-\frac{\theta^2 m_1}{2(\theta-\eta)m_2}(u - u_1)}], & \text{if } u \geq u_1, \end{cases} \quad (3.7)$$

is a smooth solution to the HJB equation, where

$$\begin{aligned} u_1 &= \frac{2A}{k} \left(1 - \frac{\eta}{\theta}\right), \\ \delta(u_1) &= \frac{\frac{\theta^2 m_1}{2(\theta-\eta)m_2}}{\frac{\theta^2 m_1}{2(\theta-\eta)m_2} + \frac{\left(\frac{\theta}{2(\theta-\eta)} \frac{k}{A}\right)^{\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{\theta m_1}{m_2} \frac{A}{k}}}{\int_0^{u_1} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}}, \end{aligned}$$

and the corresponding maximum point is

$$q^*(u) = \begin{cases} \frac{ku}{A}, & \text{if } u < u_1, \\ 2\left(1 - \frac{\eta}{\theta}\right), & \text{if } u \geq u_1. \end{cases} \quad (3.8)$$

Proof. We solve the HJB equation analytically. First we need to determine the optimal strategy $q^*(u)$. Assume that the maximum of the left hand side of the HJB equation is attained in the interior of the control region, then the maximum point will be

$$q^0(u) = -\frac{\theta m_1}{m_2} \frac{\delta'(u)}{\delta''(u)}. \quad (3.9)$$

Since $\delta(u)$ is an increasing function bounded to be in $[0, 1]$, there must exist a $M \in \mathbb{R}_+$, for $u \geq M$, $\delta(u)$ is concave. Otherwise, $\delta(u)$ cannot be bounded within the interval $[0, 1]$. Then $q^0(u)$ is positive and accordingly, we only need to compare it with the upper boundary of q .

The dynamic VaR constraint implies $q \leq \frac{ku}{A}$, where A is defined by (3.5). Normally, we take $0 < \alpha < \frac{1}{2}$, hence A is always positive.

1. For $u \geq \frac{A}{k}$, we have $\frac{ku}{A} \geq 1$. Then, from $q \leq \frac{ku}{A}$ obtained from the dynamic VaR constraint, and the requirement that the retained proportion of claims $q(t)$ is always within $[0, 1]$, we have $q \in [0, 1]$.

- (a) If $q^0(u) \geq 1$, it is reasonable to let $q^*(u) = 1$, then the HJB equation becomes

$$\eta m_1 \delta'(u) + \frac{1}{2} m_2 \delta''(u) = 0, \quad (3.10)$$

which implies

$$\frac{\delta''(u)}{\delta'(u)} = -\frac{2\eta m_1}{m_2}.$$

Inserting it into (3.9), we obtain

$$q^0(u) = \frac{\theta}{2\eta}.$$

- If $\theta \geq 2\eta$, we have $q^0(u) \geq 1$, consequently $q^* = 1$ and the HJB equation becomes (3.10).
- If $\theta < 2\eta$, we have $q^0(u) < 1$, where conflict exits.

- (b) If $q^0(u) \leq 1$, we have $q^*(u) = q^0(u)$, then the HJB equation becomes

$$\frac{\delta''(u)}{\delta'(u)} = -\frac{\theta^2 m_1}{2(\theta - \eta) m_2}, \quad (3.11)$$

which implies

$$q^0(u) = \frac{2(\theta - \eta)}{\theta} = 2 \left(1 - \frac{\eta}{\theta}\right).$$

- If $\theta \geq 2\eta$, we have $q^0(u) \geq 1$, which conflict with previous assumption.

- If $\theta < 2\eta$, we have $q^0(u) < 1$, consequently $q^* = q^0 = 2(1 - \frac{\eta}{\theta})$ and the HJB equation becomes (3.11).

2. For $0 < u < \frac{A}{k}$, we have $\frac{ku}{A} < 1$, thus $0 \leq q \leq \frac{ku}{A}$.

- (a) When $q^0(u) \geq \frac{ku}{A}$, it is reasonable to let $q^*(u) = \frac{ku}{A}$. Then the HJB equation becomes

$$\left[\frac{k\theta u}{A} - (\theta - \eta) \right] m_1 \delta'(u) + \frac{1}{2} \frac{k^2 u^2}{A^2} m_2 \delta''(u) = 0, \quad (3.12)$$

which implies

$$q^0(u) = \frac{k^2 u^2}{2A^2 \left[\frac{ku}{A} - (1 - \frac{\eta}{\theta}) \right]}.$$

Since $\delta(u)$ is increasing, $\delta'(u)$ is always positive. From HJB equation (3.12), we have $\delta''(u) < 0$ when $u > \frac{A}{k}(1 - \frac{\eta}{\theta})$, accordingly $q^0(u) > 0$.

For $\frac{A}{k}(1 - \frac{\eta}{\theta}) < u < \frac{A}{k}$,

- If $\theta \geq 2\eta$, we have $q^0(u) \geq \frac{ku}{A}$, consequently $q^*(u) = \frac{ku}{A}$ and the HJB equation becomes (3.12).
- If $\theta < 2\eta$, we have $q^0(u) < \frac{ku}{A}$, where conflict exists.

When $u \leq \frac{A}{k}(1 - \frac{\eta}{\theta})$, we have $\delta''(u) \geq 0$ thereof $\delta(u)$ is convex for small u . Through the analysis of the HJB equation (3.3), for $0 < u \leq \frac{A}{k}(1 - \frac{\eta}{\theta})$, the maximum of left hand side of HJB is attained at $q^* = \frac{ku}{A}$ and the HJB equation becomes (3.12).

- (b) When $q^0(u) < \frac{ku}{A}$, we have $q^*(u) = q^0(u)$. From the procedure that is similar to previous analysis we can draw the the following conclusion:

For $0 < u \leq \frac{2A}{k}(1 - \frac{\eta}{\theta})$, $\delta''(u) \geq 0$ hence $\delta(u)$ is convex and $q^*(u) = \frac{ku}{A}$.

For $\frac{2A}{k}(1 - \frac{\eta}{\theta}) < u < \frac{A}{k}$, we have

- If $\theta \geq 2\eta$, $q^0(u) \geq 1 > \frac{ku}{A}$, where conflict exists.
- If $\theta < 2\eta$, $0 < q^0(u) = 2(1 - \frac{\eta}{\theta}) < \frac{ku}{A}$ therefore $q^*(u) = q^0(u) = 2(1 - \frac{\eta}{\theta})$ and HJB is given by (3.11).

From previous analysis, we have the following conclusion:

- If $\theta \geq 2\eta$, the maximum of the HJB equation (3.3) is attained at $q^*(u)$ described by (3.6).
- If $\theta < 2\eta$, the maximum point of the HJB equation (3.3) is the $q^*(u)$ described by (3.8).

In the following, we will solve the HJB equation in each situation.

For $\theta \geq 2\eta$ and $u < \frac{A}{k}$, the corresponding HJB is (3.12), which is equivalent to

$$\frac{\delta''(u)}{\delta'(u)} = -\frac{2\theta m_1}{m_2} \frac{A}{ku} + \frac{2(\theta - \eta)m_1}{m_2} \frac{A^2}{k^2 u^2}.$$

Taking integral from u to $\frac{A}{k}$ we obtain

$$\delta'(u) = \delta' \left(\frac{A}{k} \right) \left(\frac{A}{ku} \right)^{\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta - \eta)m_1}{m_2} \frac{A^2}{k^2} \left(\frac{1}{u} - \frac{k}{A} \right)}.$$

Integrating again leads to

$$\delta(u) = \delta \left(\frac{A}{k} \right) - K \int_u^{\frac{A}{k}} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta - \eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw.$$

Applying the boundary condition $\delta(0) = 0$ we obtain

$$K = \frac{\delta \left(\frac{A}{k} \right)}{\int_0^{\frac{A}{k}} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta - \eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}.$$

For $\theta \geq 2\eta$ and $u \geq \frac{A}{k}$, the corresponding HJB is (3.10) and solving it leads to

$$\delta'(u) = \delta' \left(\frac{A}{k} \right) e^{-\frac{2\eta m_1}{m_2} \left(u - \frac{A}{k} \right)}.$$

Taking integral from $\frac{A}{k}$ to u ,

$$\delta(u) = \delta \left(\frac{A}{k} \right) + \frac{m_2}{2\eta m_1} \delta' \left(\frac{A}{k} \right) \left[1 - e^{-\frac{2\eta m_1}{m_2} \left(u - \frac{A}{k} \right)} \right].$$

Since $\delta(+\infty) = 1$, we have

$$\delta(u) = \delta \left(\frac{A}{k} \right) + \left[1 - \delta \left(\frac{A}{k} \right) \right] \left[1 - e^{-\frac{2\eta m_1}{m_2} \left(u - \frac{A}{k} \right)} \right].$$

Considering that $\delta(u)$ is a smooth solution, it should satisfy $\delta' \left(\frac{A}{k} - \right) = \delta' \left(\frac{A}{k} + \right)$ (meanwhile twice continuously differentiable is satisfied), i.e.

$$\frac{\delta \left(\frac{A}{k} \right)}{\int_0^{\frac{A}{k}} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta - \eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw} \left(\frac{k}{A} \right)^{\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta - \eta)m_1}{m_2} \frac{A^2}{k^2} \frac{k}{A}} = \frac{2\eta m_1}{m_2} \left[1 - \delta \left(\frac{A}{k} \right) \right],$$

which lead to

$$\delta\left(\frac{A}{k}\right) = \frac{\frac{2\eta m_1}{m_2}}{\frac{2\eta m_1}{m_2} + \frac{\left(\frac{k}{A}\right)^{\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A}{k}}}{\int_0^{\frac{A}{k}} w^{-\frac{2\theta m_1}{m_2} \frac{A}{k}} e^{-\frac{2(\theta-\eta)m_1}{m_2} \frac{A^2}{k^2} \frac{1}{w}} dw}}.$$

Thus, if $\theta \geq 2\eta$, the function $\delta(u)$, which is described by (3.4) in Theorem 3.1 is a smooth solution to HJB equation and the corresponding maximum is attained at $q^*(u)$ defined by (3.6).

If $\theta < 2\eta$, the HJB equation becomes (3.12) when $u < u_1 = \frac{2A}{k}(1 - \frac{\eta}{\theta})$ and HJB is given by (3.11) when $u \geq u_1$. Similar to previous calculation, we obtain the function $\delta(u)$ described by (3.7) is a $\mathcal{C}^2$ solution to HJB. The maximum of left hand side of HJB equation is given by (3.8). $\square$

When the value function is twice continuously differentiable, then it is the unique solution of the HJB equation. This verification theorem is supported by various literatures, see, for instance, Fleming and Soner (2006), Yong and Zhou (1999) and references therein. Therefore, we have the following result.

Proposition 3.2 The value function $v(x)$ coincides with the smooth function $\delta(u)$ defined in Theorem 3.1 and the optimal feedback control, which represents the optimal proportional reinsurance strategy, is described by the $q^*(u)$ in Theorem 3.1, where $u = U^{q^*}(t)$ is the corresponding surplus process.

Corollary 3.1 When there is no dynamic VaR, CVaR or wcCVaR constraints, i.e $k = +\infty$ and the model becomes the unconstrained reinsurance problem,

- if $\theta \geq 2\eta$, to maximise its survival probability, the insurer should take the strategy $q^* = 1$, i.e. no reinsurance. The optimal survival probability is

$$\delta(u) = 1 - e^{-\frac{2\eta m_1}{m_2} u},$$

- if $\theta < 2\eta$, insurer's optimal strategy is $q^* = 2(1 - \frac{\eta}{\theta})$ and the corresponding survival probability is

$$\delta(u) = 1 - e^{-\frac{\theta^2 m_1}{2(\theta-\eta)m_2} u}.$$

Remark 3.3 If we do not adopt dynamic VaR/CVaR/wcCVaR to set up capital requirement, we have the results in Corollary 3.1, which coincide with those of Schmidli (2001) or Promislow and Young (2005). Thus, the unconstrained reinsurance problem is a special case of our model.

Without constraint, the optimal survival probability is always a concave function. However, under the dynamic VaR, CVaR or wcCVaR constraints, concavity only holds when initial surplus is large enough. While for small initial surplus, the maximum survival probability becomes a convex function of surplus.

4 Excess-of-loss Reinsurance

When $f(X)$ describes an excess-of-loss reinsurance policy with retention level $b(t)$, i.e., $f(X_i) = X_i \wedge b(T_i)$, where $b(t) \geq 0$ is the control parameter selected at time t by the insurance company. The claim paid by an insurer up to time t is $\sum_{i=1}^{N_t} X_i \wedge b(T_i)$, where $x \wedge y \triangleq \min\{x, y\}$. The diffusion approximation of this claim process is

$$\begin{cases} dC^b(t) = \lambda\mu(b(t))dt + \sqrt{\lambda}\sigma(b(t))dW_t, \\ C^b(0) = 0, \end{cases} \quad (4.1)$$

and the dynamics of insurer's surplus process is governed by

$$\begin{cases} dU^b(t) = [(\eta - \theta)\lambda m_1 + \lambda\theta\mu(b(t))]dt - \sqrt{\lambda}\sigma(b(t))dW_t, \\ U^b(0) = u_0, \end{cases}$$

where

$$\mu(b) \triangleq \mathbb{E}(X_i \wedge b), \quad \sigma^2(b) \triangleq \mathbb{E}[(X_i \wedge b)^2].$$

Integration-by-parts yields

$$\mu(b) = \int_0^b \bar{F}(x)dx, \quad \sigma^2(b) = \int_0^b 2x\bar{F}(x)dx, \quad (4.2)$$

where F is the claim size distribution function and $\bar{F}(x) = \mathbb{P}(X_i > x) = 1 - F(x)$. Define

$$N \triangleq \inf\{x \geq 0 : \bar{F}(x) = 0\}.$$

i.e. N is the support of the claim size distribution. As shown later, our results depend on whether or not N is greater than $\frac{\theta m_2}{2\eta m_1}$. Obviously, both functions $\mu(\cdot)$ and $\sigma^2(\cdot)$ are increasing in $[0, N]$, while in $[N, +\infty)$ they are constants equal to m_1 and m_2 respectively.

4.1 Dynamic risk measures

As in proportional reinsurance setting, we consider the dynamic VaR, CVaR and wcCVaR constraints for excess-of-loss reinsurance. At each instant t , the risk exposure in a given time horizon h is described as

$$\begin{aligned} C^b(t+h) - C^b(t) &= \int_t^{t+h} \lambda\mu(b(s))ds + \int_t^{t+h} \sqrt{\lambda}\sigma(b(s))dW_s \\ &\doteq \lambda h\mu(b(t)) + \sqrt{\lambda}\sigma(b(t)) \int_t^{t+h} dW_s, \end{aligned}$$

where the approximation obtained is under the assumption $b(s) \doteq b(t)$, $s \in [t, t+h)$, i.e. the insurance strategy remain unchanged during $[t, t+h)$.

For a given confidence level $1 - \alpha \in (0, 1)$, the dynamic VaR and CVaR under excess-of-loss circumstances are specified as

$$\begin{aligned} VaR_t^{\alpha,h} &\triangleq \inf \left\{ L : \mathbb{P}(C^b(t+h) - C^b(t) \geq L \mid \mathcal{F}_t) < \alpha \right\}, \\ CVaR_t^{\alpha,h} &\triangleq \mathbb{E} \left[C^b(t+h) - C^b(t) \mid C^b(t+h) - C^b(t) \geq VaR_t^{\alpha,h} \right]. \end{aligned}$$

When the distribution of claim process is only identified by its first two moments, rather than approximated by (4.1), we define the dynamic worst-case CVaR as

$$wcCVaR_t^{\alpha,h} \triangleq \sup_{p(\cdot) \in \mathcal{P}_2} \inf_{a \in \mathbb{R}} \left\{ a + \frac{1}{\alpha} \mathbb{E}_p[(C^b(t+h) - C^b(t) - a)_+] \right\},$$

where

$$\begin{aligned} \mathcal{P}_2 = \{ p(\cdot) : &\mathbb{E}_p[C^b(t+h) - C^b(t)] = \lambda h \mu(b(t)), \\ &\mathbb{E}_p[(C^b(t+h) - C^b(t))^2] = \lambda h \sigma^2(b(t)) + (\lambda h \mu(b(t)))^2 \} \end{aligned}$$

is the set of feasible probability distributions.

Proposition 4.1 We have

$$\begin{aligned} VaR_t^{\alpha,h} &= \lambda h \mu(b(t)) - \Phi^{-1}(\alpha) \sqrt{\lambda h} \sigma(b(t)), \\ CVaR_t^{\alpha,h} &= \lambda h \mu(b(t)) + \frac{\phi(\Phi^{-1}(\alpha))}{\alpha} \sqrt{\lambda h} \sigma(b(t)), \\ wcCVaR_t^{\alpha,h} &= \lambda h \mu(b(t)) + \sqrt{\frac{1-\alpha}{\alpha}} \sqrt{\lambda h} \sigma(b(t)), \end{aligned}$$

and those dynamic risk measures satisfy

$$0 \leq VaR_t^{\alpha,h} \leq CVaR_t^{\alpha,h} \leq wcCVaR_t^{\alpha,h} < U^b(t).$$

Proof. The proof follows similar techniques as that in Proposition 3.1. $\square$

Remark 4.1 Parallel to the proportional reinsurance setting, the dynamic CVaR or wcCVaR constraint can be converted into an equivalent VaR constraint with a different confidence level and vice versa, which can be manifested by $CVaR_t^{\alpha,h} = VaR_t^{\tilde{\alpha},h}$ and $wcCVaR_t^{\alpha,h} = VaR_t^{\hat{\alpha},h}$, where $\tilde{\alpha} = \Phi(-\frac{\phi(\Phi^{-1}(\alpha))}{\alpha})$ and $\hat{\alpha} = \Phi(-\sqrt{\frac{1-\alpha}{\alpha}})$.

4.2 HJB equation

As in the proportional reinsurance model, here we also adopt the dynamic VaR constraint

$$VaR_t^{\alpha,h} \leq k U^b(t), \quad 0 < k < \infty.$$

By the dynamic programming technique, we obtain the following HJB equation satisfied by the value function $v(u)$.

$$\begin{aligned} \max_b \left\{ [(\eta - \theta)m_1 + \theta\mu(b)]v'(u) + \frac{1}{2}\sigma^2(b)v''(u) \right\} &= 0, \\ \text{s.t. } \begin{cases} 0 \leq b \leq N, \\ \lambda h\mu(b) - \Phi^{-1}(\alpha)\sqrt{\lambda h}\sigma(b) \leq ku, \\ v(0) = 0, v(+\infty) = 1. \end{cases} \end{aligned} \quad (4.3)$$

Theorem 4.1 (a) If $\frac{\theta m_2}{2\eta m_1} < N \leq +\infty$, the function

$$\delta(u) = \begin{cases} \delta(u_2) \left[1 - \frac{\int_u^{u_2} e^{\int_v^{u_2} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv}{\int_0^{u_2} e^{\int_v^{u_2} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv} \right], & \text{if } u < u_2, \\ \delta(u_2) + [1 - \delta(u_2)] \left[1 - e^{-\frac{\theta}{g^{-1}(0)}(u-u_2)} \right], & \text{if } u \geq u_2, \end{cases} \quad (4.4)$$

where

$$u_2 = \frac{l(g^{-1}(0))}{k}, \quad (4.5)$$

$$g(b) = (\eta - \theta)m_1 + \theta\mu(b) - \frac{\theta\sigma^2(b)}{2b}, \quad (4.5)$$

$$l(b) = \lambda h\mu(b) - \Phi^{-1}(\alpha)\sqrt{\lambda h}\sigma(b), \quad (4.6)$$

$$\delta(u_2) = \frac{\theta}{\theta + \frac{g^{-1}(0)}{\int_0^{u_2} e^{\int_v^{u_2} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv}}, \quad (4.7)$$

is a smooth solution to the HJB equation. The optimal strategy is given by

$$b^*(u) = \begin{cases} l^{-1}(ku), & \text{if } u < u_2, \\ g^{-1}(0), & \text{if } u \geq u_2. \end{cases} \quad (4.8)$$

(b) If $N \leq \frac{\theta m_2}{2\eta m_1}$, the function

$$\delta(u) = \begin{cases} \delta\left(\frac{l(N)}{k}\right) \left[1 - \frac{\int_u^{l(N)/k} e^{\int_v^{l(N)/k} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv}{\int_0^{l(N)/k} e^{\int_v^{l(N)/k} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv} \right], & \text{if } u < \frac{l(N)}{k}, \\ \delta\left(\frac{l(N)}{k}\right) + \left[1 - \delta\left(\frac{l(N)}{k}\right) \right] \left[1 - e^{-\frac{2\eta m_1}{m_2}\left(u - \frac{l(N)}{k}\right)} \right], & \text{if } u \geq \frac{l(N)}{k}, \end{cases} \quad (4.9)$$

where

$$\delta\left(\frac{l(N)}{k}\right) = \frac{\frac{2\eta m_1}{m_2}}{\frac{2\eta m_1}{m_2} + \frac{1}{\int_0^{l(N)/k} e^{\int_v^{l(N)/k} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv}} \quad (4.10)$$

is a smooth solution to the HJB equation and the corresponding optimal strategy is

$$b^*(u) = \begin{cases} l^{-1}(ku), & \text{if } u < \frac{l(N)}{k}, \\ N, & \text{if } u \geq \frac{l(N)}{k}. \end{cases} \quad (4.11)$$

Proof. Suppose that $\delta(u)$ is a smooth solution to the HJB equation (4.3). Noting that $\mu(b)$ and $\sigma^2(b)$ are defined in (4.2) and taking derivatives, with respect to b , of left hand side of HJB derives

$$\theta \bar{F}(b) \delta'(u) + b \bar{F}(b) \delta''(u).$$

Thus, the maximiser of the left hand side of HJB is equal to $\min(b(u), N)$, where N is the support of claim size distribution and

$$b(u) = \begin{cases} -\frac{\theta \delta'(u)}{\delta''(u)}, & \text{if } \delta''(u) \neq 0, \\ \infty, & \text{if } \delta''(u) = 0. \end{cases}$$

Insert $b(u) = -\frac{\theta \delta'(u)}{\delta''(u)}$ into (4.3), then the HJB equation becomes

$$g(b) \delta'(u) = 0,$$

where $g(b)$ is specified by (4.5).

It is easy to check that $g(b)$ is a continuous and increasing function, $g(0) = (\eta - \theta)m_1 < 0$ and $\lim_{b \rightarrow \infty} g(b) = (\eta - \theta)m_1 + \theta m_1 - 0 = \eta m_1 > 0$. Therefore, $g(b)$ has a unique positive root, we denote it as $\bar{b} = g^{-1}(0)$. The dynamic VaR limit implies $l(b) \leq ku$, with $l(b)$ described by (4.6) being an increasing positive function.

1. The case of $N = \infty$.

If $u \geq u_2$, where $u_2 \triangleq l(g^{-1}(0))/k$, we have $l(\bar{b}) \leq ku$, then the maximum of left hand side of HJB equation is attained at

$$b^* = \bar{b} = g^{-1}(0).$$

Inserting it into $b(u)$ and rearranging we obtain

$$\frac{\delta''(u)}{\delta'(u)} = -\frac{\theta}{g^{-1}(0)}.$$

Taking integral from u_2 to u leads to

$$\delta'(u) = \delta'(u_2) e^{-\frac{\theta}{g^{-1}(0)}(u-u_2)}.$$

Integrating again and using the boundary condition $\delta(\infty) = 1$, we obtain

$$\delta(u) = \delta(u_2) + [1 - \delta(u_2)] \left[1 - e^{-\frac{\theta}{g^{-1}(0)}(u-u_2)} \right], \quad u \geq u_2.$$

If $u < u_2$, it is reasonable to let $b^* = l^{-1}(ku)$, then the HJB equation becomes

$$[(\eta - \theta)m_1 + \theta\mu(l^{-1}(ku))]\delta'(u) + \frac{1}{2}\sigma^2(l^{-1}(ku))\delta''(u) = 0,$$

which can be rewritten as

$$\frac{\delta''(u)}{\delta'(u)} = -\frac{2[(\eta - \theta)m_1 + \theta\mu(l^{-1}(ku))]}{\sigma^2(l^{-1}(ku))}. \quad (4.12)$$

Integrating from u to u_2 leads to

$$\begin{aligned} \delta'(u) &= \delta'(u_2) e^{\int_u^{u_2} \frac{2[(\eta - \theta)m_1 + \theta\mu(l^{-1}(kv))]}{\sigma^2(l^{-1}(kv))} dv}, \\ \delta(u) &= \delta(u_2) - \delta'(u_2) \int_u^{u_2} e^{\int_v^{u_2} \frac{2[(\eta - \theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv. \end{aligned}$$

Noting that $\delta(0) = 0$, we obtain

$$\delta(u) = \delta(u_2) \left[1 - \frac{\int_u^{u_2} e^{\int_v^{u_2} \frac{2[(\eta - \theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv}{\int_0^{u_2} e^{\int_v^{u_2} \frac{2[(\eta - \theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv} \right], \quad 0 \leq u < u_2.$$

By the continuous differentiability of $\delta(u)$, we have

$$\frac{\delta(u_2)}{\int_0^{u_2} e^{\int_v^{u_2} \frac{2[(\eta - \theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv} = [1 - \delta(u_2)] \frac{\theta}{g^{-1}(0)},$$

rearranging this equation we derive the expression of $\delta(u_2)$, which is given by (4.7). Therefore, $\delta(u)$ defined in (4.4) is a smooth solution to HJB and the corresponding optimal strategy is described by (4.8).

2. The case of $\frac{\theta m_2}{2\eta m_1} < N < \infty$.

Noting that $\lim_{b \rightarrow N} g(b) = (\eta - \theta)m_1 + \theta m_1 - \frac{\theta m_2}{2N} > 0$, $g(b) = 0$ admits a unique solution $\bar{b} = g^{-1}(0) \in (0, N]$. Then the optimal retention level and corresponding survival probability is the same as in the case of $N = \infty$.

3. The case of $N \leq \frac{\theta m_2}{2\eta m_1}$.

Noting that $\lim_{b \rightarrow N} g(b) = \eta m_1 - \frac{\theta m_2}{2N} \leq 0$, $g(b)$ will never be positive on $[0, N]$. Then through the analysis of left hand side of the HJB equation, without considering any constraints, the maximum is attained at $b = N$.

If $u \geq l(N)/k$, we have $l(N) \leq ku$ and the maximiser is $b^* = N$. Inserting it into the HJB equation and rearranging, we obtain

$$\frac{\delta''(u)}{\delta'(u)} = -\frac{2\eta m_1}{m_2},$$

integrating from $\frac{l(N)}{k}$ to u gives

$$\delta'(u) = \delta' \left(\frac{l(N)}{k} \right) e^{-\frac{2\eta m_1}{m_2} \left(u - \frac{l(N)}{k} \right)}.$$

Integrating again and using the boundary condition $\delta(\infty) = 1$, we obtain

$$\delta(u) = \delta \left(\frac{l(N)}{k} \right) + \left[1 - \delta \left(\frac{l(N)}{k} \right) \right] \left[1 - e^{-\frac{2\eta m_1}{m_2} \left(u - \frac{l(N)}{k} \right)} \right], \quad u \geq \frac{l(N)}{k}.$$

If $u < l(N)/k$, to meet the dynamic VaR limits we let $b^* = l^{-1}(ku)$, and the HJB becomes (4.12). Following the calculation procedures in the case $N = \infty$ and integrating from u to $l(N)/k$ yields

$$\delta(u) = \delta \left(\frac{l(N)}{k} \right) \left[1 - \frac{\int_u^{l(N)/k} e^{\int_v^{l(N)/k} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv}{\int_0^{l(N)/k} e^{\int_v^{l(N)/k} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv} \right], \quad 0 \leq u < \frac{l(N)}{k}.$$

By the continuous differentiability of $\delta(u)$, $\delta \left(\frac{l(N)}{k} \right)$ satisfies

$$\frac{\delta \left(\frac{l(N)}{k} \right)}{\int_0^{l(N)/k} e^{\int_v^{l(N)/k} \frac{2[(\eta-\theta)m_1 + \theta\mu(l^{-1}(kw))]}{\sigma^2(l^{-1}(kw))} dw} dv} = \left[1 - \delta \left(\frac{l(N)}{k} \right) \right] \frac{2\eta m_1}{m_2}.$$

Simplifying this equation gives the expression of $\delta \left(\frac{l(N)}{k} \right)$, which is given by (4.10). Therefore, $\delta(u)$ defined in (4.9) is a smooth solution to HJB and the optimal strategy is described by (4.11).

□

Remark 4.2 In fact, the concavity of $\delta(u)$ only holds when $u > l(\mu^{-1}(\frac{(\theta-\eta)m_1}{\theta}))/k$. For $u \leq l(\mu^{-1}(\frac{(\theta-\eta)m_1}{\theta}))/k$, we have $\delta''(u) \geq 0$ hence the survival probability is a convex function of u . By analysing the HJB equation, one can find that the maximum is attained at $b^* = l^{-1}(ku)$ and the HJB equation is converted to (4.12).

From the verification theorem in Zhang et al. (2007) or Meng and Zhang (2010), the smooth solution $\delta(u)$, in both cases, coincides with the value function thereof we have the following result.

Proposition 4.2 The value function $v(x)$ coincides with the smooth function $\delta(u)$ defined in Theorem 4.1 and the optimal feedback control, i.e. the optimal excess-of-loss reinsurance strategy, is described by the $b^*(u)$ in Theorem 4.1, where $u = U^{b^*}(t)$ is the corresponding surplus process.

Corollary 4.1 When $k = +\infty$, i.e., our model becomes the unconstrained excess-of-loss reinsurance optimisation problem and our results reduce to

- if $\frac{\theta m_2}{2\eta m_1} < N \leq +\infty$, to maximise its survival probability, the insurer should take the strategy $b^* = g^{-1}(0)$, and the optimal survival probability is

$$\delta(u) = 1 - e^{-\frac{\theta}{g^{-1}(0)}u};$$

- if $N \leq \frac{\theta m_2}{2\eta m_1}$, insurer's optimal strategy is $b^* = N$, i.e. no reinsurance (or transfer the extreme claim to reinsurer at no cost) and the corresponding survival probability is

$$\delta(u) = 1 - e^{-\frac{2\eta m_1}{m_2}u}.$$

Remark 4.3 In our excess-of-loss reinsurance model, if we do not adopt the dynamic version of VaR/CVaR/wcCVaR risk measure to set up capital requirement, we have the results in corollary 4.1, which coincide with the conclusion drawn in Theorem 4 and Theorem 5 in Meng and Zhang (2010) when letting the risk free interest rate go to zero. Therefore, the unconstrained excess-of-loss reinsurance problem is a special case of our model.

5 Numerical Examples

This section provides several numerical calculations to illustrate the results. To determine the expressions of the optimal survival probability, we need compare θ with 2η in proportional setting and compare the support of claim size distribution N with $\frac{\theta m_1}{2\eta m_2}$ in excess-of-loss setting. Noting that there is no parameter satisfying $\theta \leq 2\eta$ and $N \leq \frac{\theta m_1}{2\eta m_2}$, we provide three examples customised to the following three cases.

- $\theta > 2\eta$ and $N = \infty$. We let $\eta = 0.15$, $\theta = 0.4$ and the claim sizes $X_i \sim \exp(1)$ in Example 1.
- $\theta \leq 2\eta$ and $N > \frac{\theta m_1}{2\eta m_2}$. We let $\eta = 0.2$, $\theta = 0.3$ and $X_i \sim U[0, 2]$ in Example 2.
- $\theta > 2\eta$ and $N \leq \frac{\theta m_1}{2\eta m_2}$. We let $\eta = 0.1$, $\theta = 0.4$ and claim sizes follow a truncated exponential distribution with parameter 1 on the interval $[0, 2]$ in Example 3.

In practice, to limit company's risk exposures, regulators will take supervisory activities at regular intervals, as well as commercial institutions' self-monitoring. Normally the supervision is taken every year, each month, fortnightly, weekly or even daily, regarding to the size, business type and complexity of the institution. Thus we could correspondingly choose the time horizon $h = 1$ (yearly basis), $1/4$ (quarterly), $1/12$ (monthly), $1/25$ (fortnightly), $1/50$ (weekly) or $1/250$ (one trading day). In our examples, firstly we assume that the insurance company re-evaluates the risk measures on a fortnightly basis (i.e. $h = 1/25$) and takes the common parameters $\alpha = 0.01$, $k = 1$ (VaR/CVaR/wcCVaR must not exceed the surplus) and $\lambda = 10$ in all the following numerical computations. Then we let η and θ be fixed as specified in each example, and consider different adjustment frequencies of the reinsurance strategy under the dynamic VaR constraint to investigate the effect caused by h .

Example 5.1 Let $\eta = 0.15$, $\theta = 0.4$ and claim sizes follow an exponential distribution with parameter 1. In this case the maximum survival probability $\delta(u)$ under dynamic VaR constraint in the proportional setting is described by (3.4) and Figure 1(a) plots the optimal survival probabilities without constraint and under dynamic VaR, CVaR and worst-case CVaR constraints. In the excess-of-loss setting the analytical expression of optimal survival probability is governed by (4.4) and the graphs of $\delta(u)$ under each limit, together with the case without constraint, are shown in Figure 1(b). Both figures indicate that the maximum survival probability under dynamic risk measures constraints are lower than that without constraints, among which the one under dynamic VaR limit being highest and under wcCVaR being the lowest, providing that the confidential level is the same. It is also notable that the concavity of $\delta(u)$ under VaR/CVaR/wcCVaR constraint no longer holds. Instead, it is convex for small surplus u and becomes concave after the turning point.

Table 1 provides numerical values of the maximum survival probability under proportional and excess-of-loss reinsurance settings, in the case of no constraint and under dynamic VaR/CVaR/wcCVaR constraint. The results show that excess-of-loss reinsurance performs better than proportional types in this parameter setting. As the increase of initial surplus u , the optimal survival probability gradually increases from 0 to 1. Particularly, when u is small, $\delta(u)$ under the dynamic VaR/CVaR/wcCVaR limit increases faster than that with no constraint, which is consistent with the convexity of $\delta(u)$ for small surplus in our constrained problem.

(a) The optimal survival probability for proportional reinsurance under different constraints

(b) The optimal survival probability for excess-of-loss reinsurance under different constraints

(c) The optimal survival probability for proportional reinsurance under dynamic VaR with different h

(d) The optimal survival probability for excess-of-loss reinsurance under dynamic VaR with different h

Figure 1: Graphs of maximum survival probability corresponding to Example 1

Table 1: Maximum Survival Probability in Example 1

u	no constraint		dynamic VaR		dynamic CVaR		dynamic wcCVaR	
	proportional	excess-of-loss	proportional	excess-of-loss	proportional	excess-of-loss	proportional	excess-of-loss
0.5	0.0723	0.0789	0.0062	0.0083	0.0034	0.0048	7.6×10^{-17}	7.4×10^{-12}
1	0.1393	0.1516	0.0498	0.0632	0.0389	0.0514	5.8×10^{-8}	3.5×10^{-7}
2	0.2592	0.2803	0.1738	0.2021	0.1586	0.1888	9.4×10^{-4}	0.0031
4	0.4512	0.4820	0.3878	0.4277	0.3763	0.4186	0.0737	0.1333
6	0.5934	0.6272	0.5465	0.5881	0.5379	0.5816	0.2564	0.3569
8	0.6988	0.7317	0.6640	0.7035	0.6577	0.6988	0.4403	0.5510
12	0.8347	0.8610	0.8156	0.8464	0.8121	0.8440	0.6926	0.7674
16	0.9093	0.9280	0.8988	0.9205	0.8969	0.9192	0.8313	0.8795
20	0.9502	0.9627	0.9445	0.9588	0.9434	0.9581	0.9074	0.9376

For the optimal reinsurance problems under the dynamic VaR constraint, the maximum survival probabilities $\delta(u)$ with different reinsurance strategy adjustment frequencies h , in the proportional and excess-of-loss reinsurance settings, are plotted in Figure 1(c) and Figure 1(d) respectively. We consider the time horizon $h = 1, 1/4, 1/25$ and $1/250$, i.e., yearly, quarterly, fortnightly and daily basis. For both types of reinsurance strategies, the graphs show that $\delta(u)$ will increase with the decreasing of h . This indicates that the more frequently an insurance company adjust its reinsurance strategies, the higher survival probabilities the company will have.

Example 5.2 Let $\eta = 0.2$, $\theta = 0.3$ and claim amounts follow a continuous uniform distribution on the interval $[0, 2]$. Figure 2(a) and Figure 2(b) plots the maximum survival probabilities in the model with no constraint and under dynamic VaR, CVaR and wcCVaR constraints for proportional and excess-of-loss reinsurance, whose analytical expressions are described by (3.7) and (4.4), respectively. The convexity/concavity of $\delta(u)$ under each constraint is similar to Example 1, as well as the magnitude among them. Numerical values of $\delta(u)$ under each constraint are listed in Table 2. Comparing $\delta(u)$ between the two types of reinsurance, we find that the optimal excess-of-loss reinsurance has a higher survival probability than the proportional type when there is no constraint or under dynamic VaR/CVaR constraint. Nevertheless, under the dynamic worst-case CVaR constraint, the optimal proportional reinsurance performs better than excess-of-loss policy in this example.

Under the dynamic VaR constraint, if the insurance company re-evaluates its risks annually, quarterly, fortnightly or daily, correspondingly $h = 1, 1/4, 1/25$ or $1/250$, the optimal survival probabilities $\delta(u)$ for proportional and excess-of-loss reinsurance are described in Figure 2(c) and Figure 2(d) respectively. Both graphs indicate that with the increasing of adjustment frequency of reinsurance strategy (i.e. the decreasing of h), the insurance company will end up with higher survival probabilities.

(a) The optimal survival probability for proportional reinsurance under different constraints

(b) The optimal survival probability for excess-of-loss reinsurance under different constraints

(c) The optimal survival probability for proportional reinsurance under dynamic VaR with different h

(d) The optimal survival probability for excess-of-loss reinsurance under dynamic VaR with different h

Figure 2: Graphs of maximum survival probability corresponding to Example 2

Table 2: Maximum Survival Probability in Example 2

u	no constraint		dynamic VaR		dynamic CVaR		dynamic wcCVaR	
	proportional	excess-of-loss	proportional	excess-of-loss	proportional	excess-of-loss	proportional	excess-of-loss
0.5	0.1553	0.1783	0.0640	0.0779	0.0523	0.0612	1.3×10^{-6}	2.0×10^{-9}
1	0.2864	0.3248	0.2025	0.2377	0.1879	0.2109	0.0034	6.2×10^{-6}
2	0.4908	0.5441	0.4308	0.4855	0.4202	0.4657	0.1200	0.0012
3	0.6367	0.6921	0.5939	0.6526	0.5863	0.6392	0.3322	0.0169
4	0.7408	0.7921	0.7102	0.7654	0.7048	0.7564	0.5183	0.1120
6	0.8680	0.9052	0.8524	0.8930	0.8497	0.8889	0.7546	0.5778
8	0.9328	0.9568	0.9249	0.9512	0.9235	0.9494	0.8750	0.8075
10	0.9658	0.9803	0.9617	0.9778	0.9610	0.9769	0.9364	0.9122
12	0.9826	0.9910	0.9805	0.9899	0.9802	0.9895	0.9676	0.9600

Example 5.3 Let $\eta = 0.1$, $\theta = 0.4$ and claim amounts follow a truncated exponential distribution with probability density function is $f(x) = \frac{e^{-x}}{1-e^{-2}}$, $0 \leq x \leq 2$. In this numerical setting the analytical expressions of optimal survival probability in proportional and excess-of-loss reinsurance are described by (3.4) and (4.9), and the graphs of $\delta(u)$ in both settings are plotted in Figure 3(a) and Figure 3(b), respectively. Numerical results are provided in Table 3, where we can compare the optimal survival probability between two types of reinsurance when there is no constraint and when the dynamic VaR/CVaR/wcCVaR constraint is adopted. Results indicate that when there is no capital requirement, both types of reinsurance work equally well under their optimal strategies. However, after considering the dynamic VaR/CVaR/wcCVaR constraint, proportional reinsurance has a better performance. Figure 3(c) and Figure 3(d) describe the maximum survival probabilities for proportional and excess-of-loss reinsurance under dynamic VaR constraint with different strategy adjustment frequencies h . Like previous two examples, the survival probabilities $\delta(u)$ will increase if the insurance company re-evaluates its risks and adjust its reinsurance strategies more frequently.

Remark 5.1 From the examples, one can notice that when there is no dynamic risk measure constraint, the optimal excess-of-loss reinsurance always has a higher survival probability than the proportional type. This coincides with the conclusion first drawn by Asmussen et al. (2000) that excess-of-loss is better than proportional policy, where their objective is to maximise the total expected present value of dividends. Latterly, such result is extended to the model of minimising insurer's ruin probability, detailed proof is provided in Zhang et al. (2007). However, under the constraint that the dynamic VaR/CVaR/wcCVaR limit is proportional to insurer's surplus, optimal excess-of-loss reinsurance is no longer guaranteed to have the best performance. As evidenced by Example 2, the optimal proportional policy leads to

(a) The optimal survival probability for proportional reinsurance under different constraints

(b) The optimal survival probability for excess-of-loss reinsurance under different constraints

(c) The optimal survival probability for proportional reinsurance under dynamic VaR with different h

(d) The optimal survival probability for excess-of-loss reinsurance under dynamic VaR with different h

Figure 3: Graphs of maximum survival probability corresponding to Example 3

Table 3: Maximum Survival Probability in Example 3

u	no constraint		dynamic VaR		dynamic CVaR		dynamic wcCVaR	
	proportional	excess-of-loss	proportional	excess-of-loss	proportional	excess-of-loss	proportional	excess-of-loss
0.5	0.0878	0.0878	0.0139	0.0068	0.0089	0.0032	2.4×10^{-13}	1.1×10^{-18}
1	0.1678	0.1678	0.0816	0.0411	0.0688	0.0262	3.7×10^{-6}	2.3×10^{-11}
2	0.3075	0.3075	0.2343	0.1814	0.2216	0.1512	0.0074	9.8×10^{-7}
3	0.4237	0.4237	0.3628	0.3188	0.3522	0.2937	0.0690	1.5×10^{-4}
4	0.5204	0.5204	0.4698	0.4331	0.4609	0.4122	0.1852	0.0031
6	0.6679	0.6679	0.6328	0.6074	0.6267	0.5930	0.4280	0.1419
8	0.7700	0.7700	0.7457	0.7281	0.7415	0.7181	0.6039	0.4057
12	0.8897	0.8897	0.8780	0.8696	0.8760	0.8648	0.8100	0.7150
16	0.9471	0.9471	0.9415	0.9375	0.9405	0.9352	0.9089	0.8633

a higher survival probability than excess-of-loss type under the dynamic wcCVaR constraint. The result is also demonstrated by Example 3.

6 Concluding Remarks

This paper investigates the optimal reinsurance problem under dynamic VaR, CVaR and wcCVaR constraints in both proportional and excess-of-loss settings. When the dynamic VaR/CVaR/wcCVaR level is proportional to the surplus of insurance company, closed form expressions for the maximum survival probability and the corresponding reinsurance strategy have been obtained.

In reality, insurance companies will invest their surpluses into financial markets for the pursuit of profit. Under the reinsurance-investment strategy, insurer's risk exposure includes the claims to be settled and possible investment loss. This makes the risk measures more complicated while deserve further studying. To capture more features of the dynamic VaR/CVaR/wcCVaR constraint, the upper boundary can have other forms such as the maximum of a constant and a proportion of the surplus. Comparison between the optimal survival probabilities under different forms of constraints could be interesting. Further studies can also consider other risk measures such as Expectile, which is both coherent and elicitable, for the risk exposure of an insurance company.

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