Beyond Toroidal Multipoles

Shi-Qiang Li and Kenneth B. Crozier

Department of Electrical and Electronic Engineering and School of Physics, University of Melbourne, VIC 3010 Australia
Author e-mail addresses: shiqiang.li@unimelb.edu.au, kenneth.crozier@unimelb.edu.au

Abstract: We propose to separate out the toroidal multipoles in the polar coordinates that is simpler than previous methods done in the Cartesian coordinates. We uncover higher order current configurations other than the toroidal electric multipole.

OCIS codes: (290.4020) Mie theory; (160.3918) Metamaterials.

1. Introduction

Interest in the use of multipole expansions in nanophotonics research has intensified in recent years, due to the useful physical insight they can provide on light-matter interactions in low-loss dielectric nanostructures [1]. They facilitated, for example, the prediction of novel phenomena such as anapoles and toroidal dipoles. These were subsequently demonstrated experimentally (see e.g. [2]).

Although the current configuration of a toroidal dipole is very different to that of an electric dipole, they have the same radiation pattern. It is for this reason that toroidal dipoles, as formulated via a complicated expansion in the Cartesian coordinates, have attracted attention only quite recently, despite the fact that classical multipole expansions in spherical coordinates were discovered more than one century ago.

In this work, we propose an alternative route to separate out the toroidal multipoles in spherical multipole expansions. This method can be used for many different applications such as creating high-Q resonators [3], designing toroidal dielectric metamaterials and cloaking, and creating non-radiating potentials [4].

To prove the generality of this method, we first derive the two dimensional (2D) multipole expansion in polar coordinates, which is simple to perform and understand. In particular, we develop an expansion method that uses the generalized concept of current with the time-harmonic Maxwell Equations, and can be implemented in full-wave frequency-domain solvers. It is applicable to scattering from particles with arbitrary shapes that can be on substrates or in free space. It can be applied to single particles, or to arrays of particles. We separate out the toroidal components through dimensional analysis without resorting to the long-wavelength approximation. We find that light scattering with both the TE and the TM polarizations have toroidal equivalent moments. Further analysis reveals that the term under the TE polarization is the renowned toroidal moment, while the term under the TM polarization is the counterpart of toroidal moments separated from magnetic multipoles. They share the same radiation patterns as magnetic multipole but scale with an additional factor of $(kr)^2$.

2. Results

In the 2D case, the multipole coefficient of light scattering by a particle in polar coordinates can be expressed as,

$$A_m = \frac{\eta k}{4} \int -J_m(kr) \cdot j_z e^{im\psi} ds \quad (1)$$

$$B_m = \frac{i}{4} \int \frac{e^{im\psi}}{r} [(kr)j_\psi \cdot J_{m+1}(kr) + m(ij_r - j_\psi) \cdot J_m(kr)] ds \quad (2)$$

, where $\vec{j} = (j_z, j_\psi, j_r)$ is the scattering current in cylindrical coordinates, η is the wave impedance, k is the wave-vector, $J_m(kr)$ is the m th order Bessel function of the first kind with argument kr . A_m coefficients are solved for the TE wave and B_m are solved for the TM wave. For an infinite cylinder, they reduce to the Mie coefficients [5]. In addition, we can expand the Bessel function as,

$$J_m(kr) = J_{m,0} + J_{m,h} \quad (3)$$

$$J_{m,0} = \frac{1}{m!} \left(\frac{kr}{2}\right)^m \quad (4)$$

$$J_{m,h} = J_m - J_{m,0} \quad (5)$$

, where we have explicitly written down the leading term in the expansion as Eqn (4), while lumping together all the higher order terms in Eqn. (5). By setting m to 1 in Eqn. (2), we can substitute Eqn. (3) to (5) and separate the integrand to two terms based on the order of kr ,

$$B_{1,0} = -\frac{1}{8} \int (j_r + ij_\psi) k e^{i\psi} d\mathbf{s} \quad (6)$$

$$B_{1,h} = \frac{i}{4} \int \frac{e^{i\psi}}{r} [(kr)j_\psi \cdot J_2(kr) + (ij_r - j_\psi) \cdot J_{1,h}(kr)] d\mathbf{s} \quad (7)$$

We can now identify that Eqn. (6) is the pure electric dipole in polar coordinates pointing in the plane. In Eqn (7), the integrand is of order $(kr)^3$ and higher. A careful examination of the terms with $(kr)^3$ gives the toroidal dipole configuration. By the same token, the $J_{m,0}$ term is the pure m th electric multipole and the term containing $J_{m,h}$ and J_{m+1} is the toroidal multipole. Moreover, similar procedure can be performed for magnetic multipole coefficients B_m . Therefore, we conclude from this analysis that toroidal multipole is not the only other current configuration.

We extend our analysis to the 3D case. Due to the space constraint of this abstract, we will not reproduce the multipole expansion results here, which are in a similar form to those of Ref [6]. We expand the spherical Bessel coefficient in terms of kr . The lowest order of kr is the pure electric multipole and the next order is the toroidal multipole. We plot the result of expansion in Fig. 1 and compare with what was calculated by the Cartesian formulation in the literature (Figure 2c of Ref [7]). They are in complete agreement.

Fig. 1. Spherical multipole coefficients for the dipole moment with order 1 and degree 1. The dipole moment with (green dashed line with diamond marker) and without (blue solid line with diamond marker) toroidal moment separation. Toroidal moment is the dashed red curve with diamond marker.

In 3D case, we can again perform separation for magnetic multipole results as well. For the lowest order term, we find that the expression for the magnetic toroidal dipole is,

$$T_m = \frac{1}{10} \int r^2 \nabla \times \vec{j} dV \quad (8)$$

It should be noted that Eqn (8) scales with $(kr)^2$, which is two orders of kr higher than the magnetic dipole moment.

3. Conclusions

In summary, we demonstrated a simple but general method to separate higher order terms from the curvilinear coordinate system, including the classical Mie expansion and the cylindrical expansion. In the extended expression, we found that the zeroth order term comprises pure electric multipoles, while the first order result is the well-known toroidal multipole. In addition, by our method, we also discover the counterparts of toroidal terms in magnetic multipole expansions.

4. References

- [1] A. I. Kuznetsov, A. E. Miroshnichenko, M. L. Brongersma, Y. S. Kivshar, and B. Luk'yanchuk, *Science* **354**, 846 (2016).
- [2] T. Kaelberer, V. Fedotov, N. Papasimakis, D. Tsai, and N. Zheludev, *Science* **330**, 1510 (2010).
- [3] J. S. T. Gongora, A. E. Miroshnichenko, Y. S. Kivshar, and A. Fratalocchi, in *CLEO: QELS Fundamental Science* (Optical Society of America, 2016), p. FTh3A. 7.
- [4] N. Papasimakis, V. Fedotov, V. Savinov, T. Raybould, and N. Zheludev, *Nat. Mater.* **15**, 263 (2016).
- [5] C. F. Bohren and D. R. Huffman, *Absorption and scattering of light by small particles* (John Wiley & Sons, 2008).
- [6] P. Grahm, A. Shevchenko, and M. Kaivola, *New J. Phys.* **14**, 093033 (2012).
- [7] A. E. Miroshnichenko, A. B. Evlyukhin, Y. F. Yu, R. M. Bakker, A. Chipouline, A. I. Kuznetsov, B. Luk'yanchuk, B. N. Chichkov, and Y. S. Kivshar, *Nat. Commun.* **6**, 8069 (2015).