Effect of porosity gradient on the permeability tensor

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Experimental investigations often reveal that there is a correlation between porosity and permeability in rocks. The most commonly used model to predict the porosity-permeability relation is due to Kozeny-Carman or its many modifications. However, these models are problematic because they always involve an empirical constant and, in macroscopically heterogeneous porous media with non-uniform porosity, more than one empirical constant would be required. Moreover, the tensor character of the permeability is not accounted for when the permeability is conceptualized as a plain function of the porosity. To overcome these limitations, we devise an approach by analyzing the drag force in the volume averaging framework of poroelasticity. This allows us to deduce an expression for the inverse permeability tensor. It is the sum of the inverse of the permeability pertaining to a representative volume element and the second spatial derivative of porosity. Therefore, the gradient of porosity changes the permeability depending on the variations of macroscopic porosity variations. This result is thought to be relevant in applications where porosity maps are converted into permeability maps.

KEYWORDS

Porosity-permeability correlations, Analytical solution

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1 | INTRODUCTION

The most fundamental defining characteristics of a porous rock are the porosity and permeability (Sahimi, 2011). The porosity is ratio of pore to bulk volume and quantifies how much fluid can be stored in or extracted from the porous matrix. The permeability is the ability for fluids to flow through the porous matrix. Though porosity and permeability are fundamentally different quantities, they are not independent. There is rich empirical evidence for porosity-permeability relations (Scheidegger, 1974; Schön, 2015). It appears popular to conceptualize these correlations based on the geometrical arrangement of the pores, their inter-connectivity, and the grain size distributions. The perhaps best known is the Kozeny-Carman relation (Schön, 2015). This relation has been adapted and modified in various contexts borrowing ideas from fractal geometry and percolation theory (Costa, 2006).

As for the origin of porosity-permeability relationships, Scheidegger (1974) distinguishes between hydraulic radius theories and drag force theories. The most prominent hydraulic radius theory has been developed by Kozeny and later extended by Carman. Scheidegger critically remarks that these theories come always with some geometrical constants, which cannot be independently measured, and are thus often overrated. He also critiques that a simple correlation between porosity and permeability is physically not plausible, as it is perfectly possible to have two porous media with the same porosity but different permeability. Another concern refers to the spatial heterogeneity of porosity, possibly with large porosity fluctuations. The question is then how this porosity fluctuations affect the correlation between porosity and permeability. Up to now, there is no theoretical analysis which quantifies the effect of porosity variations on the total permeability of porous media. It has to be noted that Darcy's law is valid in porous structures even if the porosity fluctuates (Scheidegger, 1974; Ma, 2015). This fact points to a fundamental flaw in the Kozeny-Carman type theories, which do not capture porosity fluctuations at macro-scale.

Drag force theories essentially consist in calculating the flow resistance in the presence of simple solid obstacles (e.g. spheres). For more complex obstacles Scheidegger sub-classifies the drag force approach as the "Stokes flow approach", which means that the Stokes equation is solved numerically at pore-scale level. While in principle possible, drag force theories and in particular the Stokes flow approach have not been employed to study the permeability in macroscopically heterogeneous porous materials. Drag force approaches based on first principles have been developed by (Bear and Bachmat, 1967; Whitaker, 1986). Bear and Bachmat (1967) consider microscopic (pore-scale) and macroscopic factors of the medium by using statistical averaging procedures. The predicted permeability tensor is proportional to the statistical average of the tortuosity, another difficult-to-determine second rank tensor and the hydraulic conductance of micro-channels (Bear, 1988, page 106). Another theoretical framework to study the permeability from first principles is the volume averaging method (Whitaker, 1986; Bear and Cheng, 2010). It yields to the permeability tensor of heterogeneous porous media. However, the predicted permeability is of rather complex form and the geometrical operators are cumbersome rendering any interpretation difficult or requires heavy computational effort. Indeed, understanding the correlation of permeability with geometrical parameters of porous media remains a challenging problem (Rezaei Niya and Selvadurai, 2017).

Our aim is to derive an analytical relation between the permeability and porosity in macroscopically heterogeneous porous media, i.e. in the presence of porosity gradients. This paper is organized as follows. First, we lay out a few basic concepts that will bring out the general meaning of the permeability (section 2). In section 3 we use the volume averaging theorem to analyze the drag force interaction in a porous medium conceptualized as a granular medium and deduce an analytical expression for the permeability tensor. In the section 4, we numerically illustrate and discuss the effect of porosity gradients leading to a 'heterogeneous' permeability map as opposed to a 'homogeneous' permeability map when there are no porosity-gradients.

2 | CONCEPTS UNDERPINNING THE DRAG FORCE THEORY

2.1 | Permeability, Darcy's law, and concept of RVE

The permeability tensor (κ_{ij}) and fluid shear viscosity (μ^f) are introduced as proportionality constants in Darcy's law connecting the volume of fluid traversing a cross section of a porous medium sample per unit time, the so-called filtration velocity, due to an applied pressure difference across the inlet and outlet. In absence of gravity, the three-dimensional form of Darcy's law is written as the partial differential equation

$$\bar{q}_i = - \frac{\kappa_{ij}}{\mu^f} \partial_j \bar{p}^f. \quad (1)$$

It is a macroscopic law implying that the filtration velocity $\bar{q}_i$ and the fluid pressure $\bar{p}^f$ are macroscopic fields denoted as such with an overbar. The filtration velocity expressed in terms of the fluid velocity is $\bar{q}_i = \eta_0 \bar{v}_i^f$ with the porosity η_0 , or, when expressed relative to the solid velocity, $\bar{q}_i = \eta_0 (\bar{v}_i^f - \bar{v}_i^s)$. The permeability tensor is also a macroscopic quantity which means it requires the existence of a representative volume element (RVE) of the porous medium in the same way the porosity becomes only meaningful at the scale of the RVE.

The spatial derivative operator in Eq. (1) is defined at macroscopic scale, i.e. the scale of the RVE or larger. This becomes clear when the volume averaging method is used to derive Darcy's law (Whitaker, 1986; Bear and Cheng, 2010). In fact, a derivative operator at macro-scale can be only defined in a physically meaningful way if there is a minimum characteristic physical length defined at macroscopic level. This means that fluctuations of physical parameters assume different meaning at different scales. To take into consideration the fluctuation of a physical property at sub-RVE scale, not only we have to use the volume average pertaining to this sub-RVE scale, also we have to use the derivative operators at that scale. Figure 1 schematically shows the RVE for porosity and permeability. It illustrates the domain where averaging theory, or poroelasticity theory become meaningful. One can assume a characteristic volume, i.e. $\Delta U_{0\eta}$ for porosity. It is the minimum volume for which the porosity value is not any longer affected by pore-scale fluctuations. The RVE is not unique for all parameters of a porous medium. This means that if there is a characteristic volume for porosity ($\Delta U_{0\eta}$), then it will not be the same for permeability ($\Delta U_{0\kappa}$). This also means that there is not a *one-to-one correlation* of porosity-permeability even for homogeneous media, unless the fluctuations of these parameters would be directly proportional to each other. In macroscopically heterogeneous media, where the physical properties vary in space, one can expect that any correlation depends on macroscopic fluctuations.

2.2 | Drag force in the volume averaging framework of poroelasticity

Let us consider a porous medium of volume V which is at least of the size of a RVE. In the volume-averaging framework of poroelasticity developed by Sahay et al. (2001) the drag force $\bar{I}_j$ emerges when the equations of motion for the solid and fluid are averaged. In the quasi-static approximation (all acceleration terms neglected) these volume-averaged equations of motion of solid and fluid phase are

$$\partial_k ((1 - \eta_0) \bar{\sigma}_{jk}^s) + \bar{I}_j = 0 \quad (2)$$

$$\partial_k (\eta_0 \bar{\sigma}_{jk}^f) - \bar{I}_j = 0 \quad (3)$$

FIGURE 1 Schematic of the representative volume element (RVE) for porosity (a) and permeability (b). [Adapted after Bear (1988)]

with the macroscopic solid and fluid stress tensors $\bar{\sigma}_{jk}^s$ and $\bar{\sigma}_{jk}^f$. The drag force density is represented as surface integral over the solid-fluid interface A , i.e. the pore boundaries,

$$\bar{I}_j = \frac{1}{V} \int_{A^{sf}} \sigma_{jk}^s \hat{n}_k dA = -\frac{1}{V} \int_{A^{fs}} \sigma_{jk}^f \hat{n}_k dA, \quad (4)$$

where A^{sf} is the surface element with the normal vector pointing into the solid phase (Müller and Sahay, 2019). The integrands are the pore-scale stress tensors of the solid and fluid phase, respectively. That is to say the solid and fluid tractions $\sigma_{jk}^s \hat{n}_k$ and $\sigma_{jk}^f \hat{n}_k$ are acting on the pore-interfacial area contained in the averaging volume. They are equal and opposite on account of Newton's third law. Sahay et al. (2001) further deduce that the drag force is proportional to the relative fluid-solid velocity with the permeability tensor as proportionality coefficient. Thus, its representation in terms of macroscopic fields is set up as (Sahay et al., 2001)

$$\bar{I}_j = -\eta_0^2 \mu^f (\kappa^{-1})_{jk} (\bar{v}_k^s - \bar{v}_k^f) - \bar{p}^f \partial_j \eta_0. \quad (5)$$

This choice guarantees consistency with Darcy's law. Indeed, by substituting Eq. (5) into (3) we recover Darcy's law (Eq. 1). The porosity gradient term on the right-hand side ensures consistency with Darcy's law even in the presence of porosity gradients, i.e. in macroscopically heterogeneous porous media.

3 | GENERALIZED DRAG FORCE APPROACH

3.1 | Decomposition of interfacial area for a granular pack

We now envisage an idealized porous medium made out of a closed pack of N_t grains so that the solid phase is a polydisperse pack. If the fluid is taken out of this porous medium, i.e. in the dry case, it is a random polydisperse pack constituting a granular medium. We further assume that the solid-fluid interfacial area can be decomposed into $A = \sum_{\alpha}^{N_t} A_{\alpha}$, where the A_{α} is the solid-fluid interfacial area of grain α . In view of Eq. (4) the drag force becomes

$$\bar{I}_j = -\frac{1}{V} \sum_{\alpha=1}^{N_t} \int_{A_{\alpha}^{fs}} \sigma_{jk}^f \hat{n}_k dA, \quad (6)$$

where $\int_{A_{\alpha}^{fs}} \sigma_{jk}^f \hat{n}_k dA$ is taken as the fluid force exerted onto a single grain. As the integral is over all interfacial surfaces, in bounded porous media, we have two type of grains. First, those grains which are part of the outer boundary and which are therefore interacting with the outside fluid and boundary condition. Second, the grains in the interior of the porous medium (total number of these interior grains is N_{in}). Therefore, the drag force (6) can be further decomposed. One component referring to a macroscopically homogeneous medium, $\bar{I}_j^h$, and another component that exists due to the presence of macroscopic heterogeneities, $\bar{I}_j^i$. Both components can be represented as a sum over grains. To obtain $\bar{I}_j^h$ it is sufficient to sum up the contributions within one RVE, as all other poroelastic material points yield to the same value. In contrast, to obtain $\bar{I}_j^i$ the summation extends beyond the RVE in order to capture the contribution of the macroscopic heterogeneities. In the following we assume that $\bar{I}_j^h$ is known and concentrate our analysis on $\bar{I}_j^i$. Without loss of generality we represent the drag force as

$$\bar{I}_j = \bar{I}_j^h + \bar{I}_j^i = \bar{I}_j^h - \frac{1}{V} \sum_{\alpha=1}^{N_{in}} \int_{A_{\alpha}^{fs}} \sigma_{jk}^f \hat{n}_k dA, \quad (7)$$

where one has to keep in mind that the summation over the number of grains (α) extends beyond the number of grains contained in the RVE.

A grain is also subjected to an inter-granular contact force, because without it the dry porous medium would be mechanically unstable. This inter-granular contact force must be part of the drag force. Thus, we further assume that the total drag force subjected to a single grain is the sum of the drag force applied from homogeneous and isotropic fluid surrounding the grain and the effective contact force (σ_{jk}^c) from its nearest neighbour grains (see Figure 2). Figures 2(a)-(c) show schematically the decomposition of the drag force into a fluid body force plus the effective interaction from the solid matrix, i.e. grain contacts. Thus, for a single grain, we have

$$\int_{A_{\alpha}^{fs}} \sigma_{jk}^f \hat{n}_k dA \approx \oint_{A_{\alpha}} \sigma_{jk}^f \hat{n}_k dA + \sum_{c=1}^{C_{\alpha}} \int_{A_{\alpha}^c} (\sigma_{jk}^c - \sigma_{jk}^f) \hat{n}_k dA. \quad (8)$$

It is important to note that in this approximation the integral in the first term on the right-hand side becomes a closed surface integral enclosing the grain.

3.2 | Fluid body force at level of single grain

The first integral in the right-hand side of Eq. (8) is interpreted as a fluid body force. This means that each grain is subjected to a body force exerted from the fluid onto the grain. Assuming that the fluid is a Newtonian fluid then the pore-scale fluid stress tensor is $\sigma_{jk}^f = p^f \delta_{jk} + \mu_f \dot{\gamma}_{jk}$. It contains the hydrostatic pressure p^f and viscous shear stress, which is the shear viscosity multiplied with the strain rate tensor $\dot{\gamma}_{jk}$. Using the Gauss divergence theorem we obtain for the fluid body force

FIGURE 2 The decomposition of a porous medium (a) into a single inclusion interaction with fluid (b) and dry granular pack (c).

$$\oint_{A_\alpha} \sigma_{jk}^f \hat{n}_k dA = \int_{V_\alpha} \partial_j p^f dV + \mu_f \int_{V_\alpha} \partial_k \dot{\gamma}_{jk} dV. \quad (9)$$

This drag force is in general a function of the grain shape. We assume that the grain is a sphere with radius a_α . We further assume that this sphere is elastically deformable.

The hydrostatic part of fluid body force is calculated from the pressure field, which only exists in the fluid space, i.e. outside of the deformable sphere. This is mathematically expressed as $p^f(\mathbf{r}) = \bar{p}^f \Theta(\mathbf{r}' - \mathbf{r})$, where $\mathbf{r} \in V_\alpha$, and $\mathbf{r}' = a_\alpha \hat{r}$; and $\Theta(\mathbf{r}' - \mathbf{r})$ is the Heaviside function ensuring that the fluid pressure is defined in the fluid space only. We make use of the equality of derivatives of the Heaviside function $\partial_j \Theta(\mathbf{r}' - \mathbf{r}) = -\partial'_j \Theta(\mathbf{r}' - \mathbf{r})$, in which ∂_j and ∂'_j are, respectively, the derivative operators with respect to coordinates $\mathbf{r}$ and $\mathbf{r}'$. Thus, we find

$$\int_{V_\alpha} \partial_j p^f(\mathbf{r}) d\mathbf{r} = \int_{V_\alpha} -\partial_j \bar{p}^f \Theta(\mathbf{r}' - \mathbf{r}) d\mathbf{r} = \bar{p}^f \partial'_j \int_{V_\alpha} \Theta(\mathbf{r}' - \mathbf{r}) d\mathbf{r} = \bar{p}^f \partial'_j V_\alpha. \quad (10)$$

This means that the isotropic part of the fluid body force is equal to the average of hydrostatic pressure and gradient of grain volume. In the context of soft molecular inclusions, Jackson (1999) shows that the Heaviside function can be replaced by smoother radial distribution functions. Moreover, the variation of solid volume of porous media due to hydrostatic pressure of porous media has been numerically observed (Ahmed et al., 2019).

The second integral in Eq. (9) is the viscous part of the interfacial interaction. It is represented in terms of the strain rate tensor $\dot{\gamma}_{jk} = \frac{1}{2}(\partial_k \partial_j \delta_{kl} + \partial_k \partial_k \delta_{jl}) v_l^f$. Since the pore-scale fluid velocity is taken relative to the solid frame, which can move with its own velocity $v_l^s(\mathbf{r})$, we have

$$\int_{V_\alpha} \partial_k \dot{\gamma}_{jk} dV = \frac{1}{2} \int_{V_\alpha} (\partial_k \partial_j \delta_{kl} + \partial_k \partial_k \delta_{jl}) (v_l^f(\mathbf{r}) - v_l^s(\mathbf{r})) dV. \quad (11)$$

This contribution only exists outside of the deformable sphere. Therefore, as above, we set $v_l^f(\mathbf{r}) - v_l^s(\mathbf{r}) = (v_l^f - v_l^s) \Theta(\mathbf{r}' - \mathbf{r})$ and make use of the Heaviside function property to obtain

$$\int_{V_\alpha} \partial_k \dot{\gamma}_{jk} dV = \frac{1}{2} (\partial'_k \partial'_j \delta_{kl} + \partial'_k \partial'_l \delta_{jl}) \int_{V_\alpha} (v_l^f - v_l^s) \Theta(\mathbf{r} - \mathbf{r}') dV. \quad (12)$$

Since the average of the relative velocity of the fluid to volume is $\frac{1}{V_\alpha} \int_{V_\alpha} (v_l^f - v_l^s) \Theta(\mathbf{r} - \mathbf{r}') dV = (\bar{v}_l^f - \bar{v}_l^s)$ the viscous part of the fluid body force becomes

$$\int_{V_\alpha} \partial_k \dot{\gamma}_{jk} dV = \frac{1}{2} (\bar{v}_l^f - \bar{v}_l^s) (\partial'_k \partial'_j \delta_{kl} + \partial'_k \partial'_l \delta_{jl}) V_\alpha. \quad (13)$$

It is important to mention that these derivative operators (∂_k and ∂'_k) are acting at different physical scales. By shifting from one to another, we are taking into account the variation of the physical quantities at different scales. In other words, the shifting connects the variations of sub-RVE with RVE properties.

3.3 | The fluid body force for the assembly of grains

In view of Eqs. (6), (12) and (15) the interior fluid body force becomes

$$\bar{f}_j^i = -\frac{1}{V} \sum_{\alpha=1}^N \oint_{A_\alpha} \sigma_{jk}^f \hat{n}_k dA = \bar{p}_f \partial_j \left(\sum_{\alpha=1}^N \frac{V_\alpha}{V} \right) + \frac{1}{2} \mu_f (\bar{v}_l^s - \bar{v}_l^f) (\partial_k \partial_j \delta_{kl} + \partial_k \partial_l \delta_{jl}) \left(\sum_{\alpha=1}^N \frac{V_\alpha}{V} \right). \quad (14)$$

Since $1 - 1/V \sum_{\alpha=1}^N V_\alpha$ is the porosity, this expression is re-written as

$$\bar{f}_j^i = -\bar{p}_f \partial_j \eta_0 - \frac{1}{2} \mu_f (\bar{v}_l^s - \bar{v}_l^f) (\partial_k \partial_j \delta_{kl} + \partial_k \partial_l \delta_{jl}) \eta_0. \quad (15)$$

We observe that in macroscopically inhomogeneous porous media, in which the porosity becomes a function of space, $\eta_0(\mathbf{r})$, porosity gradients enter the description of the fluid body force. The variation of porosity is an integral part in poroelasticity (Ahmed et al., 2019) and the first term shows the consistency with poroelasticity theory.

3.4 | Porosity gradient contribution to the permeability tensor

The hydrostatic part of the drag force (15) is proportional to the variation of porosity in a macroscopically heterogeneous porous medium. In a homogeneous porous medium this term vanishes. The existence of this term has been postulated by Sahay et al. (2001). Our analysis reveals that this term enters through the synthesis of the effective hydrostatic pressure force applied to the grain surface for an assembly of grains. The second term in Eq. (15) represents the drag force due to the motion of a Newtonian fluid around the grains. We also observe that this term vanishes in a homogeneous porous medium, as then $\partial_i \partial_j \eta_0 = 0$. This part of the drag force is proportional to the filtration velocity. Darcy's law (Eq. 1) suggests that the filtration velocity is proportional to pressure gradient and the proportionality coefficient contains the permeability tensor. Thus by comparison with Darcy's law we find that this second term of the drag force is a tensor of rank two, which is proportional to the variations of inverse of the permeability tensor, $\delta(\kappa_{jl})^{-1} \propto (\partial_k \partial_j \delta_{kl} + \partial_k \partial_l \delta_{jl}) \eta_0$. Yet, even in the absence of macroscopic heterogeneities, i.e.

in a macroscopically homogeneous porous medium, there is a permeability tensor assigned to a porous medium via Darcy's law. This permeability is solely described by the pore-scale features and is fully determined at the level of a RVE. We label this permeability as κ_{jk}^h . Taken altogether, we infer that the inverse of the permeability tensor of a macroscopically heterogeneous porous medium has the representation

$$(\kappa_{jk})^{-1} = \underbrace{(\kappa_{jk}^h)^{-1}}_{\text{RVE-scale contribution}} + \underbrace{\frac{1}{\eta_0^2}(\partial_j \partial_k + \partial_l \partial_l \delta_{jk})\eta_0}_{\text{macro-scale heterogeneity contribution}}. \quad (16)$$

Eq. (16) can be converted into an equation for the permeability tensor. Noting that $(\kappa_{ij})^{-1} \kappa_{jk} = \delta_{ik}$ we obtain

$$\kappa_{il} \left(\frac{1}{\kappa_{ij}^h} + \frac{1}{\eta_0^2} (\partial_l \partial_j + \partial_k \partial_k \delta_{lj}) \eta_0 \right) = \delta_{ij}. \quad (17)$$

Eq. (17) is partial differential equation for the porosity variable in which the permeability tensor appears as coefficient. This is the main result of this paper. The solution of this equation can be conceived as a porosity-permeability relation. That is to say the permeability tensor components for a macroscopically inhomogeneous porous medium are obtained by solving Eq. (17) and only in the limit of a macroscopically homogeneous porous medium, for which the porosity is constant, one has $\kappa_{jk} = \kappa_{jk}^h$.

4 | NUMERICAL TEST: FROM POROSITY MAP TO HETEROGENEOUS PERMEABILITY MAP

To understand the meaning of the porosity-permeability equation (17) we propose a set of numerical tests in which we evaluate a *heterogeneous permeability map* from a given porosity map. By heterogeneous permeability map we refer to a permeability map that accounts for the macroscopic porosity variations quantified by the second term on the right-hand side of Eq. (16). If these variations are neglected then we will use the notion of *homogeneous permeability map*. In these numerical tests, the resulting permeability map will be subjected to scale dependency questions, i.e. whether the averaging volume is of the size of a RVE or not (see also Figure 1). The porosity-permeability equation (17) is deemed to be valid either at the sub-RVE scale and for macroscopically heterogeneous media. However, the minimum size of a RVE is not known a priori. To avoid any deviation from the main focus of this paper, we devise the numerical test based on a known, analytical porosity-permeability relation for a porous medium realized as randomly distributed spherical inclusions.

4.1 | 2D porosity maps and associated homogeneous permeability maps

We start with a random sphere pack. The spheres have a radius equal to 50μ -meter. We set the target porosity 0.4 and randomly distribute these spheres, thereby obtaining a binary porous medium realisation. Figure 3(a) shows an ensemble of randomly distributed spherical inclusions. We assume that the local volume fraction of a single spherical inclusion, i.e. the local porosity is $\eta_0 = 0.4$. Figure 3(b) shows a 1000×1000 subset of the ensemble. Then we use the moving window average and create different representations of continuous porosity maps at different length scales.

FIGURE 3 (a) random ensemble of spherical inclusions with target porosity 40%. (b) 1000×1000 subset of the ensemble.

Figures 4a, 5a and 6a show the continuous porosity maps resulting from a 200×200 , 1000×1000 and 5000×5000 averaging window, respectively.

Then, we create a homogeneous, scalar permeability map by assigning an effective, scalar permeability value to each effective volume surrounding an inclusion, (i.e. a spherical inclusion in effective medium as shown in Figure 2(b)). To do so, we use the analytical permeability prediction for spherical pack presented by Scheidegger (1974). It reads

$$\kappa = \frac{3}{16} \frac{\eta_0 r^2}{1 - \eta_0} \left(\frac{2}{4} + O(\delta q \rho / (\mu \eta_0)) \right). \quad (18)$$

The porosity and radius of grains are, respectively, η_0 and r . The second term $O(\delta q \rho / (\mu \eta_0))$ accounts for non-Darcian flow and will be neglected in our study. The homogeneous permeability map for different length scales has been also created by moving window averaging. For the homogeneous permeability map, we use the harmonic averaging. That means, in each window, we use harmonic averaging of the permeability. These homogeneous permeability maps are shown in Figures 4b, 5b and 6b, where the permeability value is given in unit m^2 . We note that another option for calculating the homogeneous permeability map would be to use the KC relationship or any other porosity-permeability correlation. We use this analytical prediction in order to avoid any systematic error.

4.2 | Heterogeneous permeability maps and the tensorial character of the permeability

Having constructed continuous porosity maps (Figures 4a, 5a and 6a) and the corresponding homogeneous permeability maps (Figures 4b, 5b and 6b) we now use Eq. (17) to construct heterogeneous permeability maps. This heterogeneous permeability map takes into account the porosity variations. It is also important to note that Eq. (17) allows us to analyze the tensorial character of the permeability. This is not possible with any porosity-permeability relation of the KC-type which always yields a scalar permeability value only. Figures 7-9 show the heterogeneous permeability maps for the three sizes of the moving window average (these are 200×200 , 1000×1000 and 5000×5000). In these Figures, from left to right, the permeability tensor elements are shown, i.e. (a) κ_{xx} , (b) κ_{yy} and (c) κ_{xy} . The

FIGURE 4 Upscaled porosity (a) and permeability (b) obtained by moving window averaging with a window size of 200×200 .

FIGURE 5 Same as Figure 4 except for the window size 1000×1000 .

FIGURE 6 Same as Figures 5 and 4 except for window size (5000×5000).

FIGURE 7 Heterogeneous permeability map. The permeability tensor components are respectively, κ_{xx} , κ_{yy} and κ_{xy} . The size of the averaging window is 200×200 .

permeability values are in unit of m^2 .

Comparing these heterogeneous permeability maps with the homogeneous maps we make the following observations. For the κ_{xy} component we note that the spatial variations of porosity do not appear to matter. This is expected since there are no obvious spatial alignments in the porosity maps which would render a heterogeneous κ_{xy} distribution. We further observe that the κ_{xx} and κ_{yy} components are virtually equal. This is expected for precisely the same reason: The original porosity maps are isotropic and hence no preferred direction for flow should be generated and hence $\kappa_{xx}(\mathbf{r}) = \kappa_{yy}(\mathbf{r})$. Yet, the permeability values are different if compared with their respective homogeneous permeability map. This difference is due to the presence of porosity fluctuations. A direct comparison of the homogeneous and heterogeneous permeabilities for a 1D cross section taken at $y = 5000m$ is shown in Figure 10. This means that macroscopic porosity fluctuations, depending on their spatial arrangement, either strengthen or weaken the ability to flow. These local fluid pathway alterations could not be detected when a KC-type porosity-permeability relation is used to generate a permeability map from a given porosity map.

We note that in these numerical examples the porosity is a smoothly varying field. Correspondingly the porosity

FIGURE 8 Same as Figure 7 except for the size of the averaging window, which is 1000×1000 .

FIGURE 9 Same as Figure 7 except for the size of the averaging window, which is 5000×5000 .

FIGURE 10 1D cross section showing the fluctuations of the heterogeneous permeability versus the homogeneous permeability at $y = 5000$ for different x coordinate for the averaging window (a) 1000×1000 and (b) 5000×5000 .

gradients are moderate and the incurring correction to the permeability is relatively small. However, in real geological structures porosity gradients can become strong. Fractured porous rock masses would be a prime example (Olson et al., 2009; Gale et al., 2014). In such circumstances we anticipate that the heterogeneous permeability can be substantially different from the homogeneous permeability.

5 | CONCLUSION

We have developed a drag force approach to determine the permeability tensor in macroscopically heterogeneous porous media. The notion of macroscopic heterogeneity implies that the porosity is varying in space. The construction principle is based on the empirical fact that Darcy's law remains its validity in macroscopically heterogeneous porous media. By comparison with the area integral representing the drag force in the volume averaging theory of poroelasticity and making a single solid inclusion approximation we derive a partial differential equation for the porosity variable in which the permeability tensor appears as a coefficient. We therefore conclude that in macroscopically heterogeneous porous media, the porosity-permeability relation is not simply a complex function of the porosity as previously thought. Instead, our result suggests that the permeability depends on the gradient of the porosity and how this gradient varies in space. We conjecture that the deduced partial differential equation for the permeability tensor provides a new avenue to convert porosity maps into maps for the permeability tensor components.

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Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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references

- Ahmed, S., Müller, T. M., Madadi, M. and Calo, V. (2019) Drained pore modulus and biot coefficient from pore-scale digital rock simulations. *International Journal of Rock Mechanics and Mining Sciences*, **114**, 62 – 70.
- Bear, J. (1988) *Dynamics of Fluids in Porous Media*. Dover Civil and Mechanical Engineering Series. Dover.
- Bear, J. and Bachmat, Y. (1967) A generalized theory on hydrodynamic dispersion in porous media. *Int. Union Geod. Geophys. Publ.*, **72**.
- Bear, J. and Cheng, A. (2010) *Modeling Groundwater Flow and Contaminant Transport*. Theory and Applications of Transport in Porous Media. Springer Netherlands.
- Costa, A. (2006) Permeability-porosity relationship: a reexamination of the kozeny-carman equation based on a fractal pore-space geometry assumption. *Geophysical Research Letters*, **33** (L02318), 1 – 5. Publisher: American Geophysical Union.
- Gale, J., Laubach, S., Olson, J., Eichhubl, P. and Fall, A. (2014) Natural fractures in shale: A review and new observations. *AAPG Bulletin*, **98**, 2165–2216.
- Jackson, J. D. (1999) *Classical electrodynamics*. New York, NY: Wiley, 3rd ed. edn.
- Ma, J. (2015) Review of permeability evolution model for fractured porous media. *Journal of Rock Mechanics and Geotechnical Engineering*, **7**, 351 – 357.
- Müller, T. M. and Sahay, P. N. (2019) Elastic potential energy in linear poroelasticity. *GEOPHYSICS*, **84**, W1–W20.
- Olson, J., Laubach, S. and Lander, R. (2009) Natural fracture characterization in tight gas sandstones: Integrating mechanics and diagenesis. *AAPG Bulletin*, **93**, 1535–1549.
- Rezaei Niya, S. M. and Selvadurai, A. P. S. (2017) The estimation of permeability of a porous medium with a generalized pore structure by geometry identification. *Physics of Fluids*, **29**, 037101.
- Sahay, P. N., Spanos, T. J. T. and De La Cruz, V. (2001) Seismic wave propagation in inhomogeneous and anisotropic porous media. *Geophysical Journal International*, **145**, 209–222.
- Sahimi, M. (2011) *Flow and transport in porous media and fractured rock : from classical methods to modern approaches*. Wiley-VCH, 2nd edn.
- Scheidegger, A. (1974) *The physics of flow through porous media*. University of Toronto Press.
- Schön, J. (2015) *Physical Properties of Rocks: Fundamentals and Principles of Petrophysics*. Developments in Petroleum Science. Elsevier Science.
- Whitaker, S. (1986) Flow in porous media i: A theoretical derivation of darcy's law. *Transport in Porous Media*, **1**, 3–25.

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