Addendum to: Multilevel dual approach for pricing American style derivatives

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Abstract In this note, we show how the dual approach in its particular form presented in [1] can be fitted into the framework of the recent work [2].

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In the recent paper [2], a class of methodologies for developing upper bounds for Bermudan derivatives via Monte Carlo simulation is studied. This class, in particular, is designed to study methods that involve sub-simulations. Unfortunately, one of the most popular upper bound methodologies, that of Andersen and Broadie [1], does not lie within this class, since while the process used is an approximation to a martingale, the approximation itself is not a martingale with respect to some enlarged filtration as in Example 3.1 in [2].

We recall the dual approach, originally proposed by Rogers [4] and Haugh and Kogan [3], in the setup of [2]. Let

$$(Z_i : i = 0, 1, \dots, T), \quad T \in \mathbb{N},$$

be a discrete-time, nonnegative stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and adapted to the filtration $\mathbb{F} := (\mathcal{F}_i : 0 \leq i \leq T)$. It is assumed that

$$\mathbb{E}[Z_i] < \infty \quad \text{for } 0 \leq i \leq T.$$

The problem is to find the optimal time to stop in order to maximize the value of (Z_i) . Let $\mathcal{T}_i$ denote the set of stopping times taking values in $\{i, i+1, \dots, n\}$. A well-known

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fact is that the value of the optimal stopping problem is given by the Snell envelope

$$Y_i^* = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}_i[Z_\tau], \quad 0 \leq i \leq T,$$

at time $i = 0$, i.e., by Y_0^* . In the following, τ denotes a stopping time, $\mathbb{E}_i := \mathbb{E}_{\mathcal{F}_i}$ denotes the conditional expectation with respect to the σ -algebra $\mathcal{F}_i$, and $\sup$ ($\inf$) is to be understood as *essential supremum* (*essential infimum*).

The dual approach is based on the following observation: for any martingale (M_j) with $M_0 = 0$, we have

$$Y_0^* = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0[Z_\tau] = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}_0[Z_\tau - M_\tau] \leq \mathbb{E}_0 \left[\max_{0 \leq j \leq T} (Z_j - M_j) \right], \quad (0.1)$$

so that the right-hand side provides an upper bound for Y_0^* . In [3] and [4], it is shown that (0.1) holds with equality (and the equality is almost sure) for the martingale part of the Doob decomposition of Y^* , that is,

$$Y_j^* = Y_0^* + M_j^* - A_j^*,$$

where

$$M_j^* = \sum_{i=1}^j (Y_i^* - \mathbb{E}_{i-1}[Y_i^*]), \quad A_j^* = \sum_{i=1}^j (Y_{i-1}^* - \mathbb{E}_{i-1}[Y_i^*]). \quad (0.2)$$

In practice, there is a variety of ways to implement the dual method of [4] and [3]. A straightforward way is to approximate the Doob martingale of the Snell envelope by using sub-simulations to estimate continuation values, for example due to a given approximate value function or due to a given suboptimal exercise strategy. These approaches naturally lead to an upper-biased estimate and one objective in [2] was to treat them in a unified way. To this end, Belomestny, Schoenmakers and Dickmann [2] choose a setup where the *approximated martingales are martingales* themselves with respect to some enlarged filtration, thus allowing direct application of the results of [4] and ensuring upper-biased estimates. Although various sub-simulation-based algorithms fall into this setup, the particular Andersen and Broadie [1] algorithm (based on suboptimal stopping families) does not.

Let us recap in more detail. First, a strategy is fixed. Since we have to consider values from forward starting points, this is a vector (τ_j) of stopping times with $\tau_j \geq j$ and

$$\tau_j > j \implies \tau_{j+1} = \tau_j.$$

Given these, we define a value process

$$Y_j = \mathbb{E}_j[Z_{\tau_j}].$$

[1] define a martingale analogously to (0.2) by

$$\begin{aligned} M_j^{AB} &= \sum_{i=1}^j (Y_i - \mathbb{E}_{i-1}[Y_i]) = \mathbb{E}_j[Y_{j+1}] - Y_0 + \sum_{i=0}^j (Y_i - \mathbb{E}_i[Y_{i+1}]) \\ &= \mathbb{E}_j[Y_{j+1}] - Y_0 + \sum_{i=0}^j (Z_i - \mathbb{E}_i[Y_{i+1}]) 1_{\{\tau_i=i\}}. \end{aligned}$$

The expectations in this martingale are not immediate and have to be estimated via Monte Carlo simulation. The term Y_0 may be computed with high accuracy based on non-nested Monte Carlo simulations, and so M_j^{AB} may be approximated by

$$M_j^{AB,k} = \frac{1}{k} \sum_{\ell=1}^k \xi_{j+1}^{(\ell)} - Y_0 + \sum_{i=0}^j \left(Z_i - \frac{1}{k} \sum_{\ell=1}^k \xi_{i+1}^{(\ell)} \right) 1_{\{\tau_i=i\}}, \quad (0.3)$$

where, just as in [2], the conditionally on $\mathcal{F}_T$ independent random variables $\xi_j^{(\ell)}$ are characterized by

$$\mathbb{E}_T[\xi_j^{(\ell)}] = \mathbb{E}_{j-1}[\xi_j^{(\ell)}] = \mathbb{E}_{j-1}[Y_j] = \mathbb{E}_{j-1}[Z_{\tau_j}], \quad j = 1, \dots, T,$$

and in particular it holds that

$$\mathbb{E}_j[M_j^{AB,k}] = M_j^{AB} \quad (0.4)$$

is a martingale with respect to the filtration $\mathbb{F}$.

The problem is that $(M_j^{AB,k})$ is not a martingale with respect to the canonically enlarged filtration $\mathbb{F}' := (\mathcal{F}'_i : 0 \leq i \leq T)$ with

$$\mathcal{F}'_j := \mathcal{F}_j \vee \sigma(\xi_p^{(\ell)}, p = 1, \dots, j, \ell = 1, \dots, k),$$

unlike the martingale (3.2) of Example 3.1 in [2], since (0.3) is even not adapted to $\mathbb{F}'$. In fact, the adaptedness is destroyed by the presence of the indicator in (0.3), that is, by the fact that in general $1_{\{\tau_i=i\}} \neq 1$ with positive probability.

The lack of the martingale property for the “true” algorithm from [1] is not taken into account in [2]; but it turns out that the full martingale condition is not necessary to obtain an upper bound. Consider equation (0.1). There we assumed that (M_j) is a martingale with respect to $\mathbb{F}$. We now show that the weaker condition that $(\mathbb{E}_j[M_j])$ is a martingale with respect to the filtration $\mathbb{F}$ is sufficient (cf. (0.4)). Indeed, if (M_j) , with $M_0 = 0$, is adapted to some extended filtration such that $(\tilde{M}_j) := (\mathbb{E}_j[M_j])$ is a martingale, we have

$$\begin{aligned} Y_0^* &:= \sup_{\tau} \mathbb{E}[Z_{\tau}] = \sup_{\tau} \mathbb{E}[Z_{\tau} - \tilde{M}_{\tau}] = \sup_{\tau} \mathbb{E} \left[\sum_{j=0}^T 1_{\{\tau=j\}} (Z_j - \mathbb{E}_j[M_j]) \right] \\ &= \sup_{\tau} \mathbb{E} \left[\sum_{j=0}^T \mathbb{E}_j[1_{\{\tau=j\}} (Z_j - M_j)] \right] = \sup_{\tau} \mathbb{E}[Z_{\tau} - M_{\tau}] \leq \mathbb{E} \left[\max_{j=0, \dots, T} (Z_j - M_j) \right]. \end{aligned}$$

Under this extension of the framework, we can set $M_j = M_j^{AB,k}$, and then the algorithm from [1] is encompassed by the proofs of [2] and all the results go over. In particular, it then follows that the rate of convergence of the bias caused by sub-simulations is as in that paper and a multi-level methodology can be implemented.

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