Maximizing the Net Present Value of a Steiner tree

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Abstract The theory of Steiner trees has been extensively applied in physical network design problems to locate a Steiner point that minimizes the total length of a tree. However, maximizing the total generated cash flows of a tree has not been investigated. Such a tree has costs associated with its edges and values associated with nodes. In order to reach the nodes in the tree, the edges need to be constructed. The edges are constructed in a particular order and the costs of constructing the edges and the values at the nodes are discounted over time. These discounted costs and values generate cash flows. In this paper, we study the problem of optimally locating a single Steiner point so as to maximize the sum of all the discounted cash flows, known as the Net Present Value (NPV).

An application of this problem occurs in underground mining where, we want to optimally locate a junction point in the underground access network to maximize the NPV. We propose an efficient iterative algorithm to optimally locate a single degree-3 Steiner point. We show this algorithm converges quickly and the Steiner point is unique subject to realistic design parameters.

Keywords Network optimization · Optimal mine design · Net Present Value · Steiner points.

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1 Introduction

The Steiner tree problem deals with constructing a minimum cost tree to interconnect a given set of locations known as terminals. In practical applications the associated costs may vary with time, making this optimization problem particularly hard to solve. Such trees may have construction and operational costs associated with their edges and values associated with their terminals. Further, the costs and values which generate cash flows in the trees may be discounted over time. Therefore time has a crucial impact on the value of such trees. The discount rate is the key variable of this process and is used to discount the cash flows. The sum of all the cash flows is known as the Net Present Value (NPV).

This paper is motivated by a problem in optimal mine design which is to locate a junction point in the access network of an underground mine to maximize the NPV. If the location of the junction point is known, the process of constructing the mine access can be scheduled. To date, the scheduling and access design of an underground mine have only been considered as two separate optimization problems. A weakness of this approach is that the costs of access construction fail to be reflected in the NPV calculation. In this paper we formulate scheduling and access design as a single optimization problem.

In the literature, the problem of underground access design is generally treated as a tree network problem where, the locations of the ore resource draw points are given and the junction points of the network are to be obtained for a given objective function. To find a minimum length network connecting a set of given terminals or points, Steiner points or junctions can be added to reduce the network length. This is the classical Steiner problem [1] and in the Euclidean minimal Steiner tree, the angles at the Steiner point are exactly $2\pi/3$. However, the problem we need to analyse is different from this, as the Steiner points will be located to maximize the NPV. The optimization of underground mine access has been studied by Brazil *et al* [2,3] to minimize the development and haulage costs over the life of a mine. They modelled the problem as a variation of the Steiner problem with a constraint on the gradient in order to ensure navigability by trucks. The *gradient constraint* defines a safe climbing limit for trucks, typically in the range 1:9 to 1:7. However, the discounted cost was not taken into account in their model [2,3]. When scheduling is considered, time value of money has a crucial effect on locating the junction points in the access network for maximum NPV.

Optimization of NPV for underground mines was introduced by Lane [4] in 1988. He formulated the cash flows for the complete underground mining process. However, his theory can only be applied to underground mines with a given access geometry. In the past decade there has been some development towards maximizing the NPV in underground mines. Mixed integer programming is used in the optimization processes in [5–7]. In each paper, the optimal mine design has its own unique features influencing maximization of the NPV. The NPV is simply a measure of the time based value of the ore for given specific mine design parameters and underlying assumptions applied to a known or assumed ore body size and geometry [8]. However, different techniques have been applied for different underground mines and it is always assumed that the underground access is given. From the literature, it appears that no-one has tried to develop a generic theory which can be used for any underground mine. The recognition of the need for reliable

optimization techniques and better algorithms for the mining industry is the main motivation for investigating this fundamental optimization problem.

We consider a tree where construction costs are associated with the edges and values are associated with the nodes and the edges are constructed in a given order. Such a tree is defined as a *discounted Steiner tree* T which is the maximum NPV tree spanning a given set of nodes N in space. The nodes in T but not in N are referred to as *discounted Steiner points*. The discounted Steiner points are added to the tree in order to maximize the NPV. We propose a *discounted Steiner point algorithm* to locate the discounted Steiner point when T has three terminals. Such a network will be called a *simple discounted Steiner tree*. We define NPV by taking the values of the ore bodies, construction costs, development rate and discount rate into account. The location of a discounted Steiner point depends in particular on the time discount factor. It also depends on the order of reaching and extracting the ore resources and on the value of the mined material.

The proposed algorithm will find the optimal location of the discounted Steiner point. Once this point is known, the access to the mine can be determined by including the link between the discounted Steiner point and each of the given terminals. In general, these links should represent minimum length navigable paths between the end points. However, in order to make the model as simple and general as possible we relax the navigability conditions in this paper, and assume the links are straight line segments. A consequence of this, is that the theory and the algorithm outlined in this paper can be directly applied for underground mines where, the terminals points lie in a near-horizontal plane.

This paper consists of seven sections. Section 2 formulates and explains the optimization problem. Section 3 discusses degenerate cases of the location of the discounted Steiner point. Section 4 introduces the discounted Steiner point algorithm to solve the problem. The uniqueness of the discounted Steiner point and convergence of the algorithm are discussed in Section 5. Section 6 investigates the performance of the discounted Steiner point algorithm. Finally, Section 7 contains the conclusions and future research work.

2 Problem formulation

In this section, we model underground mine access as a discounted Steiner tree. This is a physical design problem and the parameters: locations of ore deposits and their values, development rate D m p.a., cost rate C \$/m and the discount rate d %p.a. are assumed to be given. In real situations, there will be additional constraints such as a gradient constraint which defines a safe climbing limit for trucks. In this paper we study the unconstrained problem without considering the more complex constrained problem.

Let $p_0 = (x_0, y_0, z_0)$, $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ be three given terminal points as shown in Fig. 1. The point p_0 is the surface portal or breakout point from existing infrastructure. The points p_1 and p_2 represent draw points for resources (ore deposits) with values $\$V_1$ and $\$V_2$, respectively. The aim is to locate the discounted Steiner point s with coordinates (x, y, z) to maximize the NPV.

The line segments p_0s , sp_1 and sp_2 are the edges in the tree. In fact the edges represent the decline links of the underground mine. The order of construction of the edges is given. First the p_0s edge, then the sp_1 edge and finally the sp_2

Fig. 1 A schematic representation of a simple underground mine

edge is constructed. The sp_1 edge is constructed first as it is assumed $\$V_1 > \V_2 . Theorem 1 provides a formula for the discounted cost for constructing an edge of length l metres. We define $r = 1 + d$, $V_c = CD/\ln r$, $k = V_1 \ln r / (V_2 \ln r + CD)$ and use them throughout this paper.

Theorem 1 *The discounted cost of constructing an edge of length l is expressed as,*

$$\text{Construction cost of the edge} = V_c(1 - r^{-l/D})$$

Proof The time taken to construct a portion of the edge of length x is x/D . Therefore, the discount factor is $(1 + d)^{-x/D}$. The discounted cost of developing the edge can be written,

$$\begin{aligned} \text{Construction cost of the edge} &= \int_0^l C(1 + d)^{-x/D} dx = \frac{CD}{\ln r}(1 - r^{-l/D}) \\ &= V_c(1 - r^{-l/D}) \end{aligned}$$

□

The Net Present Value is defined by the cash flows generated from the resource extraction and construction costs of the access network. Since the construction is a cost, cash flows generated from access construction have a negative value. Finally, the NPV is the sum of all the cash flows. We do not consider other fixed costs in this model. The lengths l_0 , l_1 and l_2 are construction lengths from p_0 , p_1 , and p_2 to s , respectively which are given by Euclidean distances.

Lemma 1 *In a simple discounted Steiner tree, the cash flows generated from access construction are given by NPV_{con} where,*

$$NPV_{con} = V_c(r^{-(l_0+l_1+l_2)/D} - 1) \quad (1)$$

Proof In order to construct the entire tree, the edges p_0s , sp_1 and sp_2 need to be constructed. Therefore, the discounted costs for constructing the entire tree are,

$$\begin{aligned} \text{Construction costs for the entire tree} &= \int_0^{l_0+l_1+l_2} C(1 + d)^{-x/D} dx \\ &= \frac{CD}{\ln r}(1 - r^{-(l_0+l_1+l_2)/D}) \end{aligned}$$

The cash flows generated from the access construction are the negative of the above and are given by NPV_{con} where,

$$NPV_{con} = V_c(r^{-(l_0+l_1+l_2)/D} - 1)$$

□

Lemma 2 *In a simple discounted Steiner tree, the cash flows generated from resource extraction are given by NPV_{ext} where,*

$$NPV_{ext} = V_1r^{-(l_0+l_1)/D} + V_2r^{-(l_0+l_1+l_2)/D} \quad (2)$$

Proof The resource (ore) at point p_1 is extracted before p_2 since the sp_1 edge is constructed before the sp_2 edge. Therefore, the total times taken to reach p_1 and p_2 are $t_0 + t_1$ and $t_0 + t_1 + t_2$, respectively where, $t_0 = l_0/D$, $t_1 = l_1/D$, $t_2 = l_2/D$. The NPV generated from the ore extraction is expressed as NPV_{ext} where,

$$NPV_{ext} = V_1r^{-(t_0+t_1)} + V_2r^{-(t_0+t_1+t_2)} \quad (3)$$

Substituting the expressions for t_0 , t_1 and t_2 into (3) gives the result. □

Theorem 2 *In a simple discounted Steiner tree, the NPV is given by,*

$$NPV = V_1r^{-(l_0+l_1)/D} + (V_2 + V_c)r^{-(l_0+l_1+l_2)/D} - V_c \quad (4)$$

Proof We consider only the cash flows generated from resource extraction and access construction because these directly depend on the location of the discounted Steiner point. Therefore, the total NPV is expressed as,

$$NPV = NPV_{ext} + NPV_{con} \quad (5)$$

Substituting the expressions from (1) and (2) into (5) gives the result. □

The lengths l_0 , l_1 and l_2 are functions of the discounted Steiner point coordinates x , y and z . Equation (4) is the objective function for the problem of optimally locating the discounted Steiner point. Therefore, the problem we will analyse can be expressed as the following optimization problem;

$$\text{maximize } V_1r^{-(l_0+l_1)/D} + (V_2 + V_c)r^{-(l_0+l_1+l_2)/D} - V_c$$

$$\text{such that } l_0, l_1, l_2 \geq 0$$

where V_1 , V_2 , V_c , D and r are constants. The decision variables are l_0 , l_1 and l_2 and they are functions of the coordinates of the discounted Steiner point, x , y and z . There will be additional constraints on l_0 , l_1 and l_2 that are imposed by the geometry of the network. Later, we will obtain such a constraint which is given by (13).

A maximum always exists since the NPV is bounded above by $V_1 + V_2$. It is obvious that, if the discounted Steiner point is not on the plane defined by the three vertices, then the NPV can be increased by moving the discounted Steiner point towards and perpendicular to the plane, since this decreases l_0 , l_1 and l_2 . Therefore, the discounted Steiner point always lies in the interior or on the boundary of the triangle given by the three vertices p_0 , p_1 and p_2 .

The objective function is differentiable on the problem domain with respect to x , y and z except at p_0 , p_1 and p_2 since these points are degenerate Steiner points. We will discuss these cases further in Section 3.

The problem is a non-convex problem, therefore methods that assume convexity cannot be used. The optimal point which gives the maximum NPV is found by differentiating the objective function and setting this equal to zero.

$$\begin{aligned} \frac{\partial NPV}{\partial x} &= \frac{-V_1 \ln r}{D} r^{-(l_0+l_1)/D} \left(\frac{\partial l_0}{\partial x} + \frac{\partial l_1}{\partial x} \right) \\ &\quad - \left(\frac{V_2 \ln r}{D} + C \right) r^{-(l_0+l_1+l_2)/D} \left(\frac{\partial l_0}{\partial x} + \frac{\partial l_1}{\partial x} + \frac{\partial l_2}{\partial x} \right) \\ &= 0 \end{aligned}$$

Therefore,

$$\frac{V_1 \ln r}{D} \left(\frac{\partial l_0}{\partial x} + \frac{\partial l_1}{\partial x} \right) + \left(\frac{V_2 \ln r}{D} + C \right) r^{-l_2/D} \left(\frac{\partial l_0}{\partial x} + \frac{\partial l_1}{\partial x} + \frac{\partial l_2}{\partial x} \right) = 0$$

Similarly, differentiating the objective function with respect to y and z and setting these equal to 0, we obtain,

$$(A + B)\nabla(l_0 + l_1) + B\nabla l_2 = 0 \quad (6)$$

where $A = V_1 \ln r / D$ and $B = (V_2 \ln r / D + C) r^{-l_2/D}$.

Next, we develop an important property of the angles at the discounted Steiner point in a simple discounted Steiner tree.

Theorem 3 *If the discounted Steiner point does not coincide with p_0 , p_1 or p_2 , then the paths from the discounted Steiner point to p_0 and p_1 make equal angles with the path from the discounted Steiner point to p_2 and that angle θ is expressed as,*

$$\cos \theta = -\frac{1}{2(kr^{l_2/D} + 1)} \quad (7)$$

Proof Let $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ be the unit vectors which are directed from the fixed points p_0 , p_1 , and p_2 , respectively towards the discounted Steiner point. Let θ be the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$ and θ' be the angle between $\mathbf{u}_0$ and $\mathbf{u}_2$ as shown in Fig. 2. The unit vectors are expressed as the gradients $\mathbf{u}_0 = \nabla l_0$, $\mathbf{u}_1 = \nabla l_1$ and $\mathbf{u}_2 = \nabla l_2$.

Substituting the values of ∇l_0 , ∇l_1 and ∇l_2 into (6),

$$(A + B)(\mathbf{u}_0 + \mathbf{u}_1) + B\mathbf{u}_2 = 0 \quad (8)$$

By taking the dot product of (8) with $(\mathbf{u}_0 - \mathbf{u}_1)$,

$$(A + B)(\mathbf{u}_0 + \mathbf{u}_1) \cdot (\mathbf{u}_0 - \mathbf{u}_1) + B\mathbf{u}_2 \cdot (\mathbf{u}_0 - \mathbf{u}_1) = 0$$

Since $(\mathbf{u}_0 + \mathbf{u}_1) \cdot (\mathbf{u}_0 - \mathbf{u}_1) = |\mathbf{u}_0|^2 - |\mathbf{u}_1|^2 = 1 - 1 = 0$, the expression above reduces to,

$$\begin{aligned} \mathbf{u}_2 \cdot \mathbf{u}_0 &= \mathbf{u}_2 \cdot \mathbf{u}_1 \\ |\mathbf{u}_2| |\mathbf{u}_0| \cos \theta &= |\mathbf{u}_2| |\mathbf{u}_1| \cos \theta' \\ \cos \theta &= \cos \theta' \\ \theta &= \theta' \end{aligned}$$

Fig. 2 The vector representation of the problem

By taking the dot product of (8) with $\mathbf{u}_2$,

$$(A + B)(\mathbf{u}_0 + \mathbf{u}_1) \cdot \mathbf{u}_2 + B\mathbf{u}_2 \cdot \mathbf{u}_2 = 0 \quad (9)$$

Since $\mathbf{u}_0 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_2 = \cos \theta$ and $\mathbf{u}_2 \cdot \mathbf{u}_2 = 1$, by substituting these values into (9),

$$\begin{aligned} 2(A + B) \cos \theta + B &= 0 \\ \cos \theta &= -\frac{1}{2(A/B + 1)} = -\frac{1}{2(kr^{l_2/D} + 1)} \end{aligned}$$

□

Lemmas 3 and 4 provide the upper and lower bounds of the angle θ which are defined for zero and infinite discount rates, respectively.

Lemma 3 *If the operating discount rate is zero, then the discounted Steiner point coincides with the classical Steiner point.*

Proof If $d = 0$, then $r = 1$ and $k = 0$. Substituting the values of r and k into (7),

$$\begin{aligned} \cos \theta &= -\frac{1}{2(kr^{l_2/D} + 1)} = -1/2 \\ \theta &= 2\pi/3 \end{aligned}$$

which corresponds to the classical Steiner position and this angle is the upper bound for the angle θ . □

Lemma 4 *For the limiting case of an infinite discount rate the discounted Steiner point is at the intersection of the line connecting p_0 and p_1 and the line perpendicular to it and passing through p_2 .*

Proof If $d \rightarrow \infty$, then $r \rightarrow \infty$. By substituting the limiting value of r into (7),

$$\begin{aligned} \lim_{r \rightarrow \infty} \cos \theta &= \lim_{r \rightarrow \infty} -\frac{1}{2(kr^{l_2/D} + 1)} = 0 \\ \lim_{r \rightarrow \infty} \theta &= \pi/2 \end{aligned}$$

Hence, $\pi/2$ is the lower bound of angle θ . □

Therefore, from Lemmas 3 and 4, for any discount rate between 0 and ∞ , the range of an optimal θ is $\pi/2 < \theta < 2\pi/3$.

Corollary 1 *If the operating discount rate is zero, then the problem of maximizing the NPV is reduced to minimizing the total length $l_0 + l_1 + l_2$.*

Proof By expanding the objective function in (4),

$$NPV = V_1 r^{-(l_0+l_1)/D} + V_2 r^{-(l_0+l_1+l_2)/D} + \frac{CD}{\ln r} (r^{-(l_0+l_1+l_2)/D} - 1) \quad (10)$$

If $d = 0$, then $r = 1$ so the first two terms reduce to $V_1 + V_2$ and the third term is an indeterminate form as r approaches 1. Therefore, by using l'Hôpital's rule [9] for the third term,

$$\lim_{r \rightarrow 1} \frac{CD}{\ln r} (r^{-(l_0+l_1+l_2)/D} - 1) = -C(l_0 + l_1 + l_2)$$

By substituting the values above into (10), the new NPV function becomes $NPV_{(d=0)}$, where

$$NPV_{(d=0)} = V_1 + V_2 - C(l_0 + l_1 + l_2)$$

which implies,

$$\max [V_1 + V_2 - C(l_0 + l_1 + l_2)] \equiv \min [C(l_0 + l_1 + l_2) - V_1 - V_2]$$

Therefore, the problem is reduced to a length minimization problem when the operating discount rate is zero and the solution is indeed the classical Steiner point. $\square$

So far, we have developed one equation in terms of θ and l_2 , Equation (7), using the objective function. Now, we relate θ and l_2 geometrically and generate a second equation, Equation (13), in terms of θ and l_2 .

Fig. 3 The geometric representation of the problem

Let $\gamma_0 = \angle sp_0p_2$, $\gamma_1 = \angle sp_1p_2$, $\nu = \angle p_0p_2p_1$ and d_0, d_1, d_2 be the given Euclidean distances as shown in Fig. 3. By applying the cosine rule to triangle $p_0p_1p_2$,

$$\nu = \arccos \left(\frac{d_0^2 + d_1^2 - d_2^2}{2d_0d_1} \right) \quad (11)$$

By applying the sine rule to triangles p_0p_2s and p_1p_2s ,

$$\sin \gamma_0 = \frac{l_2 \sin \theta}{d_0} \quad \sin \gamma_1 = \frac{l_2 \sin \theta}{d_1}$$

Summing the angles in the quadrilateral $p_0p_2p_1s$ yields:

$$\gamma_0 + \nu + \gamma_1 + 2\theta = 2\pi \quad (12)$$

By substituting the values for γ_1 and γ_0 into (12), l_2 can be obtained, where

$$l_2 = \frac{d_0 d_1 |\sin(2\theta + \nu)|}{\sin \theta \sqrt{d_0^2 + d_1^2 + 2 \cos(2\theta + \nu) d_0 d_1}} \quad (13)$$

since $\sin(2\theta + \nu) < 0$.

3 Degenerate cases of the discounted Steiner point

Next, we consider the degenerate cases where the discounted Steiner point coincides with one of the vertices p_0 , p_1 or p_2 . Equation (7) is used to calculate some special angles for which the discounted Steiner point is located at p_0 , p_1 and p_2 . These angles are called *critical angles*. For each vertex the critical angle is the minimum angle for which the discounted Steiner point coincides with the vertex. The critical angle determines whether the discounted Steiner point lies at p_0 , p_1 or p_2 .

Fig. 4 The discounted Steiner point degenerate cases

Let λ , μ and ν be the angles $\angle p_1p_0p_2$, $\angle p_0p_1p_2$ and $\angle p_1p_2p_0$, respectively. By applying the cosine rule for triangle $p_0p_1p_2$ in Fig. 4,

$$\lambda = \arccos \left(\frac{d_0^2 + d_2^2 - d_1^2}{2d_0d_2} \right) \quad (14)$$

$$\mu = \arccos \left(\frac{d_1^2 + d_2^2 - d_0^2}{2d_1d_2} \right) \quad (15)$$

By substituting the values of d_0 , d_1 and d_2 into (11), (14) and (15) the angles ν , λ , and μ can be found respectively.

Let φ , ω and ψ be the critical angles for which the discounted Steiner point lies at p_0 , p_1 and p_2 , respectively. The discounted Steiner point coincides with a vertex if the angle at the vertex is greater than or equal to the critical angle.

Lemma 5 *If $\nu \geq \psi$, then the discounted Steiner point coincides with p_2 and*

(I) *the critical angle ψ is given by,*

$$\psi = 2 \arccos \left(\frac{1}{2(k+1)} \right) \quad (16)$$

(II) *the maximum NPV is given by $NPV_{(s=p_2)}$, where*

$$NPV_{(s=p_2)} = (V_1 + V_2 + V_c)r^{-(d_0+d_1)/D} - V_c \quad (17)$$

Proof The conditions that need to be satisfied so that the discounted Steiner point is located at p_2 are $l_2 = 0$ and $2\pi - 2\theta = \psi$. By substituting these values into (7),

$$\begin{aligned} \cos \left(\pi - \frac{\psi}{2} \right) &= -\frac{1}{2(k+1)} \\ \psi &= 2 \arccos \left(\frac{1}{2(k+1)} \right) \end{aligned}$$

Therefore, if $\nu \geq \psi$ the discounted Steiner point coincides with p_2 . When the discounted Steiner point is at p_2 , $l_0 = d_0$, $l_1 = d_1$ and $l_2 = 0$. Substituting these values into (4) gives the resultant NPV. $\square$

Equation (16) can be verified for a special case, where if $d = 0$, then $r = 1$ and $k = 0$ so ψ should be $2\pi/3$ as for the classical Steiner tree. By substituting the value of k into (16),

$$\psi = 2 \arccos \frac{1}{2} = \frac{2\pi}{3}$$

By substituting $k = V_1 \ln r / (V_2 \ln r + CD)$ into (16),

$$\psi = 2 \arccos \left(\frac{V_2 \ln r + CD}{2(V_1 \ln r + V_2 \ln r + CD)} \right)$$

Note that $V_1 \ln r > 0$. Therefore, $\psi \geq 2\pi/3$ in general.

Lemma 6 *If $\mu \geq \omega$, then the discounted Steiner point coincides with p_1 and*

(I) *the critical angle ω is given by,*

$$\omega = \arccos \left(\frac{-1}{2(kr^{d_1/D} + 1)} \right) \quad (18)$$

(II) *the maximum NPV is given by $NPV_{(s=p_1)}$, where*

$$NPV_{(s=p_1)} = V_1 r^{-d_2/D} + (V_2 + V_c)r^{-(d_1+d_2)/D} - V_c \quad (19)$$

Proof The conditions that need to be satisfied so that the discounted Steiner point is located at p_1 are $l_2 = d_1$ and $\theta = \omega$. By substituting these values into (7),

$$\begin{aligned} \cos \omega &= \frac{-1}{2(kr^{d_1/D} + 1)} \\ \omega &= \arccos \left(\frac{-1}{2(kr^{d_1/D} + 1)} \right) \end{aligned}$$

Hence, if $\mu \geq \omega$ the discounted Steiner point lies at point p_1 . When the discounted Steiner point is at p_1 , $l_0 = d_2$, $l_1 = 0$ and $l_2 = d_1$. Substituting these values into (4) gives the resultant NPV. $\square$

Equation (18) can be verified as before for the classical Steiner tree case. If $d = 0$, then $r = 1$ and $k = 0$. By substituting the value of k into (18),

$$\omega = \arccos \frac{-1}{2} = \frac{2\pi}{3}$$

Similar to the result in Lemma 5, $\omega \leq 2\pi/3$ in general.

Lemma 7 *If $\lambda \geq \varphi$, then the discounted Steiner point coincides with p_0 and*

(I) the critical angle φ is given by,

$$\varphi = \arccos \left(\frac{-1}{2(kr^{d_0/D} + 1)} \right) \quad (20)$$

(II) the maximum NPV is given by $NPV_{(s=p_0)}$, where

$$NPV_{(s=p_0)} = V_1 r^{-d_2/D} + (V_2 + V_c) r^{-(d_0+d_2)/D} - V_c \quad (21)$$

Proof The conditions that need to be satisfied so that the discounted Steiner point is located at p_0 are $l_2 = d_0$ and $\theta = \varphi$. By substituting these values into (7),

$$\begin{aligned} \cos \varphi &= \frac{-1}{2(kr^{d_0/D} + 1)} \\ \varphi &= \arccos \left(\frac{-1}{2(kr^{d_0/D} + 1)} \right) \end{aligned}$$

Therefore, if $\lambda \geq \varphi$ the discounted Steiner point lies at point p_0 . Similarly, when the discounted Steiner point is at p_0 , $l_0 = 0$, $l_1 = d_2$ and $l_2 = d_0$. Substituting these values into (4) the resultant NPV can be obtained. $\square$

Equation (20) can be verified for a special case as before, where if $d = 0$, then $r = 1$ and $k = 0$. By substituting the value of k into (20),

$$\varphi = \arccos \frac{-1}{2} = \frac{2\pi}{3}$$

Similar to Lemma 5, $\varphi \leq 2\pi/3$ in general.

For a particular given example, first ψ , ω and φ should be calculated and compared with λ , μ and ν . If one of the inequalities $\nu \geq \psi$, $\mu \geq \omega$ or $\lambda \geq \varphi$ is satisfied, then the optimal location of the discounted Steiner point will be at one of the vertices.

4 Discounted Steiner Point Algorithm (DSPA)

In Section 2, (7) and (13) were developed in terms of θ and l_2 . In this section, an iterative process is presented to find the optimal values for θ and l_2 . If these two parameters are known, the discounted Steiner point coordinates can be calculated by solving three quadratic simultaneous equations.

The discounted Steiner point algorithm is a fixed point iteration algorithm where we want to solve for θ . It seeks the solution by an iterative process such that $\theta_{n+1} = f(\theta_n)$ starting from some initial value θ_0 .

Algorithm 1: Discounted Steiner Point Algorithm

Input: $V_1, V_2, C, D, d, \epsilon$ and locations of p_0, p_1 and p_2
Output: The optimal location of the discounted Steiner point and NPV

- 1 Calculate d_0, d_1, d_2 and then $\nu, \mu, \lambda, \psi, \omega$ and φ using (11), (14), (15), (16), (18) and (20), respectively.
- 2 **if** $\nu \geq \psi$ **then**
- 3 | the discounted Steiner point coincides with p_2 and NPV is given by (17).
- 4 **else if** $\mu \geq \omega$ **then**
- 5 | the discounted Steiner point coincides with p_1 and NPV is given by (19).
- 6 **else if** $\lambda \geq \varphi$ **then**
- 7 | the discounted Steiner point coincides with p_0 and NPV is given by (21).
- 8 **else**
- 9 Initialization: $\theta(0) = 2\pi/3$
- 10 $i = 0$
- 11 **repeat**
- 12 | Compute l_2

$$l_2(i+1) = \frac{d_0 d_1 |\sin(2\theta(i) + \nu)|}{\sin \theta(i) \sqrt{d_0^2 + d_1^2 + 2 \cos(2\theta(i) + \nu) d_0 d_1}}$$
- 13 | Update θ

$$\theta(i+1) = \arccos \left(-\frac{1}{2(kr^{l_2(i+1)/D} + 1)} \right)$$
- 14 | $i = i + 1$
- 15 **until** $|\theta(i) - \theta(i-1)| < \epsilon$
- 16 Calculate l_0 and l_1 using (22) and (23), respectively.
- 17 The optimal NPV is given by (24).
- 18 The optimal location of the discounted Steiner point is calculated using (25), (26) and (27).

Let $\hat{\theta}$ and $\hat{l}_2$ be the optimal values obtained from the iterative process. Then $\hat{l}_0$ and $\hat{l}_1$ are calculated by applying the sine rule for triangles p_0sp_2 and p_1sp_2 in Fig. 3.

$$\hat{l}_0 = \frac{d_0 \sin(\hat{\theta} + \gamma_0)}{\sin \hat{\theta}} \quad (22)$$

$$\hat{l}_1 = \frac{d_1 \sin(\hat{\theta} + \gamma_1)}{\sin \hat{\theta}} \quad (23)$$

The maximum NPV is expressed as NPV^* , where

$$NPV^* = V_1 r^{-(\hat{l}_0 + \hat{l}_1)/D} + (V_2 + V_c) r^{-(\hat{l}_0 + \hat{l}_1 + \hat{l}_2)/D} - V_c \quad (24)$$

Since $\hat{l}_0, \hat{l}_1$ and $\hat{l}_2$ are known, x, y, z can be calculated by solving three quadratic simultaneous equations (25), (26) and (27).

$$\hat{l}_0^2 = (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 \quad (25)$$

$$\hat{l}_1^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad (26)$$

$$\hat{l}_2^2 = (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 \quad (27)$$

The iterative process converges to the optimal solution very quickly and convergence is independent of the initial value of θ . The uniqueness and convergence will be established in the next section.

5 Uniqueness of the discounted Steiner point and the convergence of the algorithm

We investigate the problem of determining conditions for the discounted Steiner point to be unique and for the algorithm to converge. The DSPA is a fixed point iteration of the function,

$$f(\theta) = \arccos \left(-\frac{1}{2(kr^{l_2/D} + 1)} \right)$$

where l_2 is a function of θ and is given by (13).

Previously we have shown that $\pi/2 < \theta < 2\pi/3$ but it also needs to be established for the θ values at each iteration. Let $f(\theta) = \arccos(-1/x)$ where x is a function of θ and is given by $x = 2(kr^{l_2/D} + 1)$. Since $x > 2$, it follows that $-1/2 < -1/x < 0$ and so $\pi/2 < f(\theta) < 2\pi/3$. Therefore, the n^{th} iterate, θ_n , satisfies $\pi/2 < \theta_n < 2\pi/3$ because $\theta_n = f(\theta_{n-1})$.

It is sufficient to find a condition ensuring that $|f'(\theta)| < 1$ which establishes the uniqueness of the discounted Steiner point and the convergence of the algorithm. Since $dx/dl_2 = 2kr^{l_2/D} \ln r/D = (x-2) \ln r/D$,

$$f'(\theta) = -\frac{1/x^2}{\sqrt{1-1/x^2}} \frac{dx}{dl_2} \frac{dl_2}{d\theta} = -\frac{1}{x\sqrt{x^2-1}} \frac{dx}{dl_2} \frac{dl_2}{d\theta} = -\frac{\ln r}{D} \frac{x-2}{x\sqrt{x^2-1}} \frac{dl_2}{d\theta} \quad (28)$$

Next, we will find a bound for $dl_2/d\theta$. For convenience, let $v = (d_0^2 + d_1^2)/d_0d_1 + 2\cos(2\theta + \nu)$. Then,

$$l_2 = -\frac{\sqrt{d_0d_1} \sin(2\theta + \nu)}{v^{1/2} \sin \theta} \quad (29)$$

and,

$$\frac{dv}{d\theta} = -4\sin(2\theta + \nu) \quad (30)$$

Differentiating (29) with respect to θ yields,

$$\frac{dl_2}{d\theta} = -\sqrt{d_0d_1}w \quad (31)$$

where

$$w = \frac{2v^{1/2} \sin \theta \cos(2\theta + \nu) - \sin(2\theta + \nu)[v^{1/2} \cos \theta + (v^{-1/2}/2)(dv/d\theta) \sin \theta]}{v \sin^2 \theta}$$

If we can find an upper bound, M , for w , then $w \leq M$ and so $dl_2/d\theta \geq -\sqrt{d_0 d_1} M$.

$$\begin{aligned} w &= \frac{2v \sin \theta \cos(2\theta + \nu) - \sin(2\theta + \nu)[v \cos \theta - 2 \sin(2\theta + \nu) \sin \theta]}{v^{3/2} \sin^2 \theta} \\ &= \frac{2 \sin \theta \cos(2\theta + \nu) - \sin(2\theta + \nu) \cos \theta}{v^{1/2} \sin^2 \theta} + \frac{2 \sin^2(2\theta + \nu) \sin \theta}{v^{3/2} \sin^2 \theta} \\ &= \frac{2 \sin \theta [\cos^2(\theta + \nu/2) - \sin^2(\theta + \nu/2)] - 2 \sin(\theta + \nu/2) \cos(\theta + \nu/2) \cos \theta}{v^{1/2} \sin^2 \theta} \\ &\quad + \frac{8 \sin^2(\theta + \nu/2) \cos^2(\theta + \nu/2) \sin \theta}{v^{3/2} \sin^2 \theta} \end{aligned}$$

Now,

$$v = (d_0 - d_1)^2 / d_0 d_1 + 2 + 2 \cos(2\theta + \nu) \geq 2[1 + \cos(2\theta + \nu)] = 4 \cos^2(\theta + \nu/2)$$

Therefore, using the fact that $\sin \theta > 0$ because $\pi/2 < \theta < 2\pi/3$,

$$\begin{aligned} w &\leq \frac{2 \sin \theta [\cos^2(\theta + \nu/2) - \sin^2(\theta + \nu/2)] - 2 \sin(\theta + \nu/2) \cos(\theta + \nu/2) \cos \theta}{v^{1/2} \sin^2 \theta} \\ &\quad + \frac{8 \sin^2(\theta + \nu/2) \cos^2(\theta + \nu/2) \sin \theta}{4v^{1/2} \cos^2(\theta + \nu/2) \sin^2 \theta} \\ &= \frac{2 \sin \theta \cos^2(\theta + \nu/2) - 2 \sin \theta \sin^2(\theta + \nu/2) - 2 \sin(\theta + \nu/2) \cos(\theta + \nu/2) \cos \theta}{v^{1/2} \sin^2 \theta} \\ &\quad + \frac{2 \sin^2(\theta + \nu/2) \sin \theta}{v^{1/2} \sin^2 \theta} \\ &= \frac{2 \sin \theta \cos^2(\theta + \nu/2) - 2 \sin(\theta + \nu/2) \cos(\theta + \nu/2) \cos \theta}{v^{1/2} \sin^2 \theta} \\ &= \frac{2 \cos(\theta + \nu/2) [\sin \theta \cos(\theta + \nu/2) - \cos \theta \sin(\theta + \nu/2)]}{v^{1/2} \sin^2 \theta} \\ &= \frac{-2 \cos(\theta + \nu/2) \sin(\nu/2)}{v^{1/2} \sin^2 \theta} \end{aligned}$$

Since $\theta > \pi/2$, $\nu > 0$, and 2θ and ν are two of the angles in the quadrilateral $p_0 s p_1 p_2$, it follows that $\pi < 2\theta + \nu < 2\pi$. Thus, $\pi/2 < \theta + \nu/2 < \pi$ and so $\cos(\theta + \nu/2) < 0$. Since $v \geq 4 \cos^2(\theta + \nu/2)$, it follows that $v^{1/2} \geq -2 \cos(\theta + \nu/2)$. Hence,

$$w \leq \frac{-2 \cos(\theta + \nu/2) \sin(\nu/2)}{-2 \cos(\theta + \nu/2) \sin^2 \theta} = \frac{\sin(\nu/2)}{\sin^2 \theta} \leq \frac{4}{3}$$

where the last inequality follows from the fact that $\pi/2 < \theta < 2\pi/3$ and so $\sin \theta > \sqrt{3}/2$. Therefore from (31),

$$\frac{dl_2}{d\theta} \geq -\sqrt{d_0 d_1} M \quad (32)$$

where $M = 4/3$ is an upper bound for w . Provided that $dl_2/d\theta < 0$ (which is proved below) it follows from (28) that,

$$|f'(\theta)| \leq \frac{x-2}{x\sqrt{x^2-1}} \times \frac{4\sqrt{d_0 d_1} \ln r}{3D}$$

The maximum value of $(x - 2)/(x\sqrt{x^2 - 1})$ with $x > 2$ occurs at the largest solution to the equation $x^3 - 4x^2 + 2 = 0$. Denoting this maximum value by β , we have $\beta \approx 0.129$. Therefore,

$$|f'(\theta)| \leq \frac{4\beta\sqrt{d_0d_1} \ln r}{3D}$$

In order to ensure that $|f'(\theta)| < 1$, we must have,

$$\sqrt{d_0d_1} < \left(\frac{3}{4\beta}\right) D / \ln r \quad (33)$$

The numeric factor $3/4\beta$ is approximately equal to 5.80. The left hand side of (33) is the geometric mean of the distances from the surface portal to the second resource point and the first resource point to the second resource point; it can be interpreted as a length scale for the geometry of the problem. The expression $D / \ln r$ is the length of decline that would be constructed during the characteristic time for the discounting. At a typical construction rate of 2000m p.a. and with a discount rate of 10% p.a., the right hand side of (33) is more than 120km. For non-uniqueness, inequality (33) would have to be violated meaning either d_0 or d_1 would have to be greater than 120km, which is unrealistic for a real underground mine. Therefore the condition (33) is easily satisfied in practice.

Fig. 5 Proof that $dl_2/d\theta < 0$.

In order to complete the proof, we need to show that $dl_2/d\theta < 0$. Let C_0 be the circle through the given points p_0 and p_2 and the Steiner point s , and let C_1 be the circle through p_1 , p_2 and s , as shown in Fig. 5. Since $\theta > \pi/2$, the angles of the arcs p_0sp_2 and p_1sp_2 are both less than π . Hence, the arc of C_0 from s to p_2 and not passing through p_0 and the arc of C_1 from s to p_2 and not passing through p_1 are minor arcs of their respective circles.

If s is now moved while the circles remain fixed, then the angle p_0sp_2 is greater than, equal to, or less than θ according to whether s is inside, on or outside C_0 , and similarly for the angle p_1sp_2 and the circle C_1 . Hence, if s is moved to a location where the two angles remain equal to each other and are greater than θ , then s must be in the lens-shaped region where the interiors of C_0 and C_1 intersect. For s within this region, the length of the line segment sp_2 is less than l_2 . Therefore, $dl_2/d\theta < 0$.

6 Numerical trials

Numerical trials have been carried out to analyse the performance of the discounted Steiner point algorithm. The algorithm is applied to data values that are typical for an underground mine and then the improvement of the NPV is discussed.

The developed algorithm was applied using the two data sets below for a range of discount rates and then the optimal locations of the discounted Steiner point were obtained. The discount rates 5, 10 and 20 % p.a. are the most realistic in underground mine operations.

Case 1:

$$V_1 = \$60\text{M}, V_2 = \$40\text{M}$$

$$C = \$6000/\text{m}, D = 1560 \text{ m p.a.}$$

$$d = 0, 5, 10, 20, 50, 100, \infty \% \text{ p.a.}$$

$$p_0 = (200, 0, 1000), p_1 = (0, 0, 0) \text{ and } p_2 = (1000, 0, 0)$$

Case 2:

$$V_1 = \$100\text{M}, V_2 = \$1\text{M}$$

$$C = \$4000/\text{m}, D = 2000 \text{ m p.a.}$$

$$d = 0, 5, 10, 20, 50, 100, \infty \% \text{ p.a.}$$

$$p_0 = (0, 0, 1000), p_1 = (600, 0, 0) \text{ and } p_2 = (1250, 0, 750)$$

Fig. 6 The optimal locations of the discounted Steiner point for Case 1

Table 1 Variation of the NPV for a range of discount rates for Case 1

Discount rate / % p.a.	Location of the DSP	θ / deg	l_2 / m	NPV / \$M	Improvement / \$
0	$s_0 = (319, 0, 259)$	120	728	88.660	0
5	$s_1 = (247, 0, 267)$	113	798	84.618	49000
10	$s_2 = (215, 0, 267)$	110	828	80.997	159000
20	$s_3 = (184, 0, 265)$	106	857	74.700	393000
50	$s_4 = (150, 0, 259)$	103	888	60.875	958000
100	$s_5 = (128, 0, 253)$	100	907	46.971	1516000
∞	$s_6 = (38, 0, 192)$	90	980	0	0

Fig. 7 The optimal locations of the discounted Steiner point for Case 2**Table 2** Variation of the NPV for a range of discount rates for Case 2

Discount rate / % p.a.	Location of the DSP	θ / deg	l_2 / m	NPV / \$M	Improvement / \$
0	$s_0 = (662, 0, 500)$	120	638	93.120	0
5	$s_1 = (546, 0, 418)$	108	777	90.221	159000
10	$s_2 = (508, 0, 381)$	102	827	87.674	435000
20	$s_3 = (479, 0, 344)$	98	870	83.183	995000
50	$s_4 = (457, 0, 307)$	94	908	72.778	2269000
100	$s_5 = (449, 0, 289)$	92	923	61.269	3449000
∞	$s_6 = (441, 0, 264)$	90	943	0	0

The results are shown in Fig. 6 and Table 1 for Case 1 and in Fig. 7 and Table 2 for Case 2. In the tables, DSP refers to the Discounted Steiner Point. Fig. 6 and Fig. 7 illustrate the optimal trees for the discount rates that were considered. In Tables 1 and 2, for zero and infinite discount rates the optimal values of θ are $2\pi/3$ and $\pi/2$, respectively, which verifies Lemmas 3 and 4 for this set of data.

We have compared the NPV by placing the discounted Steiner point at the classical Steiner point for all the discount rates as shown in the last column (Improvement) of the Tables 1 and 2. For positive discount rates the discounted Steiner point gives better NPV, and the improvement in the NPV increases with increasing discount rate.

7 Conclusion

This paper proposes an efficient algorithm to locate the discounted Steiner point for a simple discounted Steiner tree. We prove the uniqueness of the discounted Steiner point and the convergence of the discounted Steiner point algorithm subject to realistic values of the input parameters. We show in a maximum NPV tree the paths from the discounted Steiner point to the first terminal and the second terminal make equal angles with the path from the discounted Steiner point to the third terminal (surface portal). For zero discount rate, the problem is reduced to a length minimization problem and the solution is indeed the classical Steiner point. When the DSPA is applied to typical underground mine data, it yields higher NPV compared with the placement of the Steiner point at the classical position. This algorithm can be applied in a real mine situation when the resource points and the surface portal, or breakout point from existing infrastructure, lie in a plane that is nearly horizontal and hence the gradient constraint would be satisfied.

In future, a new algorithm will be developed to locate a single discounted Steiner point with more realistic constraints such as the gradient constraint, which is the most important physical constraint on the access network, and the optimal configurations will be studied.

The discounted Steiner point algorithm will also be extended to locate multiple discounted Steiner points in a network. The idea is to use the discounted Steiner point algorithm iteratively. Initial numerical trials show that the new algorithm works satisfactorily and converges for two discounted Steiner points. A similar iterative process will be investigated to locate multiple Steiner points.

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