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The Firefighter problem: Saving Sets of Vertices on Cubic Graphs

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Abstract

In the context of the Firefighter problem, a deterministic model of the spread of a fire or virus on a graph, SFIRE is the decision problem that asks if a specified set of vertices can be prevented from burning. We show SFIRE remains NP-complete even when restricted to graphs with maximum degree 3 even when the fire starts at a vertex of degree 2.

Keywords: Graph Theory, Algorithms, Complexity, Firefighter

A *rooted graph* is an ordered pair (G, r) , where G is a graph and $r \in V(G)$. The *Firefighter problem on (G, r)* is the following discrete-time process, which was introduced by Hartnell in 1995 [18]. At $t = 0$ the vertex r is *burning*. At each subsequent time step a vertex that is not burning is chosen to be *defended* and the fire spreads from all burning vertices to all undefended neighbours. Once a vertex is defended or burning it remains so for the remainder of the process. The process continues until the neighbourhood of every burning vertex contains only burning vertices and defended vertices.

Different objectives can be considered within the context of the Firefighter problem. For example, one may examine the expected number of burning vertices if the fire breaks out at a random vertex of a fixed graph [8, 20, 26, 27]; or whether or not the fire can be contained on an infinite grid [12, 16, 25]. Many of these problems have also been considered for the variant of the Firefighter problem in which more than one vertex can be defended in each time step [4]. Much of the recent work in the area has focused on classifying the computational complexity of decision problems defined within the context of the Firefighter problem [3, 10, 14, 17].

The parameter $MVS(G, r)$ is the maximum number of vertices that can be prevented from burning if the fire begins at $r \in V(G)$. The problem of deciding if $MVS(G, r) \geq k$ is NP-complete, even when restricted to trees with maximum degree 3 [15]. In [10] the authors give a dichotomy theorem for this decision problem with respect to path width and maximum degree of the input graph. The problem is NP-complete, even when restricted to trees with path width at most 3. The problem is fixed parameter tractable when parametrized by path width and maximum degree. In [3] the authors show that the problem is W[1]-hard when parametrized by the number of saved vertices, protected vertices and burned vertices. The

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problem can be solved in polynomial time for interval graphs, split graphs, permutation graphs, P -free graphs (for fixed k), and graphs with maximum degree 3 where the fire starts at a vertex of degree 2 [15, 17].

Given that the problem of deciding if $MVS(G, r) \leq k$ is NP-complete researchers have also focused on approximation algorithms for this parameter. In [19] the authors give a $\frac{1}{2}$ -approximation algorithm for trees for $MVS(T, r)$, which was later improved to a $(1 - \frac{1}{e})$ -approximation algorithm [9]. In [1] and [2] the authors give approximation algorithms for other graph families and other variants of the problem.

In this paper we study the problem of preventing a specified set of vertices from burning. This problem is NP-complete when restricted to trees with maximum degree $b + 2$, where b vertices can be defended in each round and the specified set to be prevented from burning consists of the leaves of the tree. The problem can be solved in polynomial time under the additional restriction that the fire starts at a vertex of degree at most $b + 1$ [4, 22].

All graphs under consideration are assumed to be simple and connected. A graph is called *properly subcubic* if it is a proper subgraph of a connected cubic graph. Recall that for a connected graph G and vertices $u, v \in V(G)$, the *distance from u to v* , denoted $d_G(u, v)$, is the number of edges in a shortest path from u to v . As the graphs under consideration are connected, this parameter is defined for every pair $u, v \in V(G)$. For all other graph theoretic definitions and notation we refer the reader to [7].

A *strategy* is a sequence $D = (d_1, d_2, \dots, d_\ell)$ of vertices to be defended; the vertex d_i is to be defended at time $t = i$. We assume d_i is not burning at time i . Denote by D_i the subsequence $(d_1, d_2, \dots, d_i)$. Denote by H_{D_i} the set of vertices that are burning at time i if d_j is defended at time j for all $0 \leq j \leq i$. Let $H_D = H_{D_\ell}$ be the set of burning vertices when the process terminates.

We call D *minimal* if there exists no strategy D^2 such that $H_{D^2} \subset H_D$. We call D *adjacent* if for all $1 \leq i \leq \ell$, we have that d_i is adjacent to a burning vertex after time $i - 1$. Otherwise, a strategy is called *non-adjacent*. For a fixed set $S \subseteq V(G) \setminus \{r\}$, D is called *successful* for S if $S \cap H_D = \emptyset$.

As the process proceeds on (G, r) with strategy D , we can consider each time step i as the start of a new process (G, H_{D_i}) proceeding with strategy $(d_{i+1}, d_{i+2}, \dots, d_\ell)$. Removing $d_1, d_2, \dots, d_i$ from G at this stage does not impact the evolution of the process; nor does identifying the vertices of H_{D_i} into a single vertex, r_i . Let G_i^D be the graph resulting from these two

operations after components not containing r_i are removed. Such components may be removed as for each vertex v in such a component there is no path from r to v . And so v cannot burn, nor should it be defended. Note that if $G \in \mathcal{F}$ for some family of graphs $\mathcal{F}$, then it need not be that $G_i^D \in \mathcal{F}$.

Observation 1. For a fixed strategy $D = (d_1, d_2, \dots, d_\ell)$, the evolution of the process starting time $t = i$ is identical to the evolution of the process on (G_i^D, r_i) using the strategy $(d_i, d_{i+1}, \dots, d_\ell)$ in the sense that, in each case, the collection of new burning vertices at each subsequent time step is the same.

Consider the following three decision problems defined for the Firefighter problem.

FIREFIGHTER

INSTANCE: An rooted graph (G, r) and an integer $k \in \mathbb{N}$.

QUESTION: Is $MVS(G, r) \leq k$? That is, if the fire breaks out at $r \in V(G)$, is there a strategy such that no more than k vertices burn?

SFIRE

INSTANCE: An rooted graph (G, r) and a subset $S \subseteq V(G) \setminus \{r\}$.

QUESTION: If the fire breaks out at r , is there a strategy under which no vertices in S burn?

WFIRE

INSTANCE: An rooted graph (G, r) , a weight function $w: V(G) \rightarrow \mathbb{N}$ and an integer k .

QUESTION: If the fire breaks out at r , is there a strategy so that the total weight of the burned vertices does not exceed k ?

Theorem 2. [15] FIREFIGHTER is NP-complete, even when restricted to graphs with maximum degree 3. The problem is Polynomial for properly subcubic graphs provided the fire starts at a vertex of degree at most 2.

Theorem 3. [22] SFIRE is NP-complete, even when restricted to trees with maximum degree 3. The problem is Polynomial for binary trees provided the fire starts at a vertex of degree at most 2.

The decision problems SFIRE and FIREFIGHTER can be unified under a single model, WFIRE, whereby vertices are assigned weights and the objective is to prevent a specified total weight from burning. An instance of FIREFIGHTER can be expressed as an instance of WFIRE by letting all weights be 1. An instance of SFIRE can be expressed as an instance of WFIRE by letting vertices in S have weight 1, all other vertices have weight 0 and by letting

$k = |S|$. The decision problem, WFIRE, is NP-complete even when restricted to binary trees [13, 14]. Further, the optimization version of the problem is NP-hard under the same restriction [11].

We find properly subcubic graphs to be of particular interest for the Firefighter problem. Suppose the fire starts at a vertex of degree 2. By defending a vertex adjacent to the fire, there is only one newly burning vertex at the end of $t = 1$. That is, r_1 has degree at most 2. Continuing to defend, in each round, a vertex adjacent to a burning vertex ensures that at the end of each round there is at most one new burning vertex. By applying such a strategy, the spread of the fire can be contained to a path. In this way, the fire can be *guided* to a vertex of degree at most 2, or to a vertex that has a neighbour that has already been defended. This idea forms the basis of the polynomial-time algorithm given in [15] for FIREFIGHTER for when the fire breaks out at a vertex of degree at most 2 on a properly subcubic graph.

In this article we examine the restriction of SFIRE to instances where the fire breaks out at a vertex of degree 2 on a properly subcubic graph. We show, in contrast to the result for FIREFIGHTER, SFIRE remains NP-complete under this restriction.

We begin with the following lemmas about strategies in the Firefighter problem. We show that for a fixed set of vertices to be defended, the order in which they are defended does not matter, so long as no vertex that is supposed to be defended burns before it can be defended.

Lemma 4. (The Ordering Lemma) Let $D = (d_1, d_2, \dots, d_t)$ be a strategy for the Firefighter problem on (G, r) so that there exists i such that $d_{G_{t-1}}(d_{i+1}, r_{i-1}) \leq d_{G_{t-1}}(d_i, r_{i-1})$. If D' is strategy formed by changing the order of d_i and d_{i+1} in D , then $H_D = H_{D'}$.

Proof. By Observation 1, we may assume $d_G(d_2, r) \leq d_G(d_1, r)$. That is, d_2 is not further from r than d_1 in G . Since d_2 is not burning at the end of $t = 1$, it must be that neither d_1 nor d_2 is adjacent to r . Therefore the set of burning vertices under D after $t = 2$ is exactly the set of burning vertices under D' . Since the set of defended vertices in both cases is also the same after $t = 2$, it follows that $H_D = H_{D'}$.

Using the Ordering Lemma, we may, when convenient, consider the sequence of defended vertices to be a set. In each round we defend the vertex of this set that minimizes the distance to a burning vertex.

The following lemma generalizes an observation by MacGillivray and Wang [23] about the Firefighter process on trees: to save the maximum vertices in a tree, each defended vertex must be adjacent to a burning vertex.

Lemma 5. (The Tree Lemma) Let D be a strategy for the Firefighter problem on (G, r) so that for some $1 \leq i \leq \ell$

- d_i is not adjacent to r_{i-1} ; and
- there is a unique path P from r_{i-1} to d_i in G_{i-1}^D

If D^2 is the strategy formed by replacing d_i with the neighbour of r_{i-1} in P , then $H_{D^2} \subset H_D$.

Proof. By Observation 1, we may assume d_1 is not adjacent to r , and there is a unique path $P = rx_1x_2 \dots x_s d_1$ from r to d_1 in G .

Consider the strategy D^2 formed by replacing d_1 with x_1 in D . Since there is a unique path from r to d_1 in G no vertex of P burns when using strategy D^2 . Therefore there is a vertex contained in H_D that is not contained in H_{D^2} . Since $G_{i-1}^{D^2} = G_{i-1}^D$ we have that $G_{i-1}^{D^2}$ is a proper subgraph of G_i^D . Therefore every vertex that burns by using strategy D^2 must also burn by using D . Therefore $H_{D^2} \subset H_D$.

To study the complexity of SFIRE restricted to properly subcubic graphs, we introduce a reduction to instances of SFIRE. Let (G, r) be a rooted graph and $S \subseteq V(G) \setminus \{r\}$. Let D be a strategy such that no vertex of S burns. Edges between vertices in S and vertices not in the same component as r in $G - S$ do not influence whether all vertices in S can be prevented from burning. We *reduce* (G, r) in two steps:

- (1) Delete every vertex $v \in V(G)$ which has the property that every path from r to v necessarily has an internal vertex contained in S .
- (2) Delete every edge whose ends are both contained in S .

Some of the deleted vertices may belong to S , so we also say that S has been *reduced*. We call a (G, r) with $S \subseteq V(G) \setminus \{r\}$ *reduced* if S is an independent set and for every $v \in V(G) \setminus S$ there is a path from r to v containing no vertex of S . Consider the example given in Figure 1. The rooted graph on the right is the result of reducing the rooted graph on the left.

Theorem 6. Let (G, r) be a rooted graph and let $S \subseteq V(G) \setminus \{r\}$. Let (G^R, r) and $S^R \subseteq V(G^R) \setminus \{r\}$ be the result of reducing (G, r) and $S \subseteq V(G) \setminus \{r\}$. There is a strategy for (G, r) so that no vertex of S burns if and only if there is a strategy for (G^R, r) so that no vertex of S^R burns.

Proof. Let D be a minimal strategy for (G, r) so that no vertex of S burns. Since G^R is a subgraph of G and $S^R \subseteq S$, D is a strategy for (G^R, r) so that no vertex of S^R burns.

Let D^2 be a strategy for (G^R, r) so that no vertex of S^R burns. Consider using D^2 for the Firefighter problem on (G, r) . To derive a contradiction, assume that some vertex of S burns

under D^2 . Let $t = i$ be the earliest time such that a vertex $s \in S$ is burning. At time i there is a path P from r to s that contains only burning vertices. By choice of i , s is the only vertex of S on this path. Therefore no vertex or edge contained in P is deleted by the reduction of (G, r) . That is, P is a subgraph of G^R . Therefore s burns at time i for the Firefighter problem on (G^R, r) with strategy D^2 . This is a contradiction.

Once we restrict our attention to reduced instances, we find those instances for which there exists an adjacent strategy can be fully characterized.

Proposition 7. Let (G, r) and set $S \subseteq V(G) \setminus \{r\}$ be reduced, where G is properly subcubic and $\deg(r) \leq 2$. There is an adjacent strategy D such that no vertex of S burns if and only if one of the following conditions hold.

- (1) there exists a path $P = rx_2x_3 \dots x_\ell$ so that x_ℓ has degree at most 2 and $P \cap S = \emptyset$
- (2) $G - S$ contains a cycle;
- (3) $G - S$ is a tree and there is a vertex $s \in S$ such that s has a pair of neighbours, u and v such that u is an ancestor of v in $G - S$.

Proof. Let (G, r) with set S be reduced where G is properly subcubic and $\deg(r) \leq 2$. Assume there is an adjacent strategy D such that no vertex of S burns. Since D is an adjacent strategy, $\deg(r) \leq 2$, and G has maximum degree 3, then at each time step at most one new vertex burns. If $\deg(r) < 2$, then (1) holds directly with $r = x_\ell$. As such we assume $\deg(r) = 2$. Let $r = b_0$ and let $b_0, b_1, b_2, \dots, b_t$ be the sequence of burned vertices, with b_i burning at time $t = i$. If b_t has degree 1 or 2 in G , then (1) holds with $P = rb_1b_2 \dots b_t$. Otherwise, let $D = (d_1, d_2, \dots, d_{t+1})$. When b_t burns, it has only one unburned neighbour, d_{t+1} . As such, its other neighbour must be d_i for some $1 \leq i \leq t$. If $d_i \notin S$, then $d_ib_{i-1}b_i \dots b_t$ is a cycle in $G - S$ and (2) holds. Otherwise if $d_i \in S$ and $G - S$ does not contain a cycle, then (3) holds with $d_i = s$, $u = b_{i-1}$ and $v = b_t$.

Assume (1) holds. Let $P = x_1x_2x_3 \dots x_\ell$ be a shortest such path, where $x_1 = r$ and x_ℓ has degree 1 or 2. Let $Z = \{z_1, z_2, \dots, z_{\ell-1}\}$ so that z_i is adjacent to x_i but $z_i \notin P$ for all $1 \leq i \leq \ell - 1$. That is, z_i is the vertex adjacent to x_i so that z_i is not contained in P . If x_ℓ has degree 2, let z be the neighbour of x_ℓ not contained in P . Let $D = (z_1, z_2, \dots, z_{\ell-1})$, or $D = (z_1, z_2, \dots, z_\ell)$ if x_ℓ has degree 2. Observe that by using D only vertices of P burn. Therefore if there exists a path $P = rx_2x_3 \dots x_\ell$ so that x_ℓ has degree at most 2 and $P \cap S = \emptyset$, then there is an adjacent strategy under which no vertex of S burns.

Assume (2) holds, but (1) does not. It then follows that each vertex of $G - S$ other than r has degree exactly 3, as otherwise a path from r to such a vertex satisfy (1). Choose $C = v_1v_2 \dots v_kv_1$ to be a chordless cycle in $G - S$ so as to minimize the distance between r and a vertex nearest to r that is contained in C . Without loss of generality, let v_1 be a nearest vertex to r on C . Let

$P = x_1 x_2 \dots x_\ell$ be a shortest path in $G - S$ from $r = x_1$ to $x = v_1$. Note that if r is contained in a cycle, then P consists of a single vertex, namely $r = x_1 = x = v_1$. Let $Z = \{z_1, z_2, \dots, z_{\ell-1}\}$ so that z_i is adjacent to x_i but $z_i \notin P$ for all $1 \leq i \leq \ell - 1$. That is, z_i is the vertex adjacent to x_i that is not contained in P .

Let $W = \{w_2, w_3, \dots, w_{k-1}\}$ so that w_j is adjacent to v_j but $w_k \notin C$ for all $1 \leq j \leq k - 1$. That is, w_i is the vertex adjacent to v_i that is not contained in C . By choice of C and P , we have $Z \cap W = \emptyset$. Let $D = (z_1, z_2, \dots, z_{\ell-1}, v_k, w_2, w_3, \dots, w_{k-1})$. The strategy D defends, in order, all vertices of Z , v_k , and then all vertices of W . Observe that by using D , only vertices of $P \cup C \setminus \{v_k\}$ burn. Therefore if $G - S$ contains a cycle then there is an adjacent strategy such that no vertex of S burns.

Assume (3) holds, but (1) and (2) do not. Let $s \in S$ be the nearest vertex to r so that s has a pair of neighbours, u and v where u is an ancestor of v in $G - S$. Since $G - S$ is a tree, there exists a unique path Q from r to s that contains no other vertex of S . Let P be the subpath of Q that begins at r and ends at u . Let C be the cycle in $G - (S \setminus \{s\})$ containing u, v, s . The proof proceeds as in (2) with $s = v_k, u = v_1$ and $v = v_{k-1}$.

We look now at rooted graphs (G, r) and sets $S \subseteq V(G) \setminus \{r\}$ for which the only strategies that prevent all vertices in S from burning are those that, at some time step, defend a vertex not adjacent to a burning vertex. Consider the example given in Figure 2a. There are 3 vertices of S and they are each at distance at least 3 from r . A strategy that defends, at each step, a vertex of S will prevent the vertices in S from burning. In general if

$$|\{s : d_G(r, s) \leq i, \quad s \in S\}| \leq i$$

for all $i > 0$, then the greedy strategy of defending at time i the undefended vertex in S closest to r_{i-1} will prevent all vertices in S from burning. There exist examples, however, for which a non-adjacent strategy exists, but the strategy defends, at some time steps, vertices not contained in S . Figure 2b gives such an example. To ensure that no vertex of S burns the fire is first guided to the unique vertex not in S at distance 3 from r by defending vertices adjacent to a burning vertex. Following this, the vertices of S at distance 5 from r are defended one by one. This second example typifies those for which a non-adjacent minimal strategy must be used – every defended vertex is either adjacent to a burning vertex or is contained in S .

Proposition 8. Let (G, r) with $S \subseteq V(G) \setminus \{r\}$ be reduced. If every strategy that ensures no vertex of S burns is a non-adjacent strategy, then for every minimal such strategy every defended vertex is either adjacent to a burning vertex or is contained in S .

Proof. Let (G, r) and $S \subseteq V(G) \setminus \{r\}$ be reduced so that every strategy that ensures no vertex of S burns is a non-adjacent strategy. By Proposition 7, we have that $G - S$ is a tree. Let D be a minimal strategy so that there exists d_i that is neither contained in S nor adjacent to r_{i-1} in G_{i-1}^D . Since S is reduced, and $G - S$ is a tree, then G is a tree and so is G_{i-1}^D . Therefore there is a unique path P from d_i to r_{i-1} in G_{i-1}^D . By the Tree Lemma, replacing d_i with the neighbour of r_{i-1} in P yields a strategy D' such that $H_{D'} \subset H_D$. This contradicts that D is a minimal strategy.

Recall that a graph is properly subcubic when it has maximum degree three, but contains a vertex of degree at most two. We apply Proposition 8 in our study of the complexity of SFIRE restricted to properly subcubic graphs. To complete our reduction we require the following decision problem.

3FLFIRE

INSTANCE: A rooted tree (T, r) with $\Delta(T) \leq 3$.

QUESTION: If the fire breaks out at r , is there a strategy under which no leaf of T burns?

Theorem 9. [22] 3FLFIRE is NP-complete.

Let J_k be the properly subcubic graph shown in Figure 3. It consists of k copies of a full and complete binary tree of height 2, with x_i, y_i, z_i and q_i ($1 \leq i \leq k$) attached as shown. Three additional vertices are added: u, v and w . Vertices u and w are adjacent to p_k and w is adjacent to q_k . This completes the construction of J_k .

Let $X = \{x_1, x_2, \dots, x_k\}$, $Y = \{y_1, y_2, \dots, y_k\}$, and $Z = \{z_1, z_2, \dots, z_k\}$. The following lemmas describe a pair of important properties of (J_k, p_0) .

Lemma 10. There are three adjacent strategies for (J_k, p_0) with set $S = X \cup Y \cup Z$ so that no vertex of S burns. In each of these strategies, no vertex of Z is defended.

Proof. Let D be an adjacent strategy so that no vertex in S burns. The result follows from Proposition 7. It is easily checked that neither (2) nor (3) hold. Therefore the only adjacent strategies are to guide the fire to a vertex of degree at most 2 in $G - S$. The vertices u, v and w are the only such vertices. It is easily verified that the adjacent strategy that guides the fire to u (respectively v and w) contains no vertex of Z .

Lemma 11. If D is a minimal non-adjacent strategy for (J_k, p_0) so that no vertex of $S = X \cup Y \cup Z$ burns, then $p_k, q_k \in H_D$.

Proof. Let D be a minimal non-adjacent strategy for (J_k, p_0) so that no vertex of $S = X \cup Y \cup Z$ burns. Let j be the earliest time such that d_j is not adjacent to r_{j-1} in G_{j-1}^D . By Proposition 8, we may assume $d_j \in X \cup Y \cup Z$. We consider three cases.

Case i. $j \equiv 0 \pmod{3}$: After time $j - 1$ there is a single vertex burning at distance $j - 1$ from p_0 . This vertex is a parent of either p_i or q_i in the tree $J_k - S$ for some $1 \leq i \leq k$. Each of these vertices has a neighbour in S . By choice of j we have that y_i and z_i are undefended at time j . Since d_j is not adjacent to a burning vertex we have $d_j \neq y_i$ and $d_j \neq z_i$. Therefore one of y_i or z_i burns at time j . This contradicts that no vertex of S burns.

Case ii. $j \equiv 2 \pmod{3}$: After time $j - 1$ there is a single vertex burning at distance $j - 1$ from p_0 . This vertex is a child of p_i in the tree $J_k - S$ for some $1 \leq i \leq k$. Since $d_j \in X \cup Y \cup Z$, we have $d_j \in \{x_i, y_i, z_i\}$. However, defending one of these vertices at $t = j$ implies that at least one of them will burn at $t = j + 2$. This contradicts that no vertex of S burns.

Case iii. $j \equiv 1 \pmod{3}$: After time $j - 1$ there is a single vertex burning at distance $j - 1$ from p_0 . This vertex is either p_i or q_i for some $1 \leq i \leq k - 1$. If q_i is this burning vertex, then by choice of j , q_i has two undefended neighbours in S . Therefore at $t = j + 1$ a vertex of S burns, a contradiction. Therefore p_i is this burning vertex. Since no vertex in S burns, $\{d_j, d_{j+1}, d_{j+2}\} = \{x_i, y_i, z_i\}$. At $t = j + 3$, both p_{i+1} and q_{i+1} are burning. If $i + 1 = k$, our claim is proven. Otherwise $i + 1 < k$. Since q_{i+1} adjacent to x_{i+2} we have $d_{j+3} = x_{i+2}$. After $t = j + 3$ both of the children of p_{i+1} are burning. Since no vertex of S burns, we have $\{d_{j+4}, d_{j+5}\} = \{y_{i+2}, z_{i+2}\}$. If $i + 4 = k$, our claim is proven. Otherwise we repeat this process $\left(k - \frac{i+4}{3}\right)$ times and conclude that q_k and p_k burn. That is, $q_k, p_k \in H_D$.

Theorem 12. SFIRE remains NP-complete when the fire breaks out at a vertex of degree 2 on a properly subcubic graph.

Proof. The problem is known to be in NP [22]. The transformation is from 3FLFIRE.

Let (T^2, a^3) be a rooted tree with $\deg(T^2) \leq 3$. Let L^2 be the set of leaves of T^2 , and $S' = \{s'_1, s'_2, \dots, s'_k\}$ be the set of degree two vertices of T^2 . Note that $k = |S'|$.

We construct a rooted graph (H, r) and a set S so that:

- H is properly subcubic;
- r has degree at most 2; and
- there is a strategy to prevent each vertex of S from burning if and only if there is a strategy to prevent each vertex of L' from burning.

Construct (H, r) from J_k and T^2 as follows:

- Add the edge $z_i s_i'$ for all $1 \leq i \leq k$;
- identify the neighbours of a' in T' with u, v and w , respectively;
- remove a' ; and
- let $r = p_0$.

We note that H is a properly subcubic graph. Further observe that since $k < V(T^2)$, the construction can be carried out in polynomial time. We claim that all leaves of T^2 can be prevented from burning if and only if all vertices in $S = L^2 \cup X \cup Y \cup Z$ can be prevented from burning.

Suppose no vertex in L^2 burns using some strategy D^2 . We use D^2 to construct a strategy that ensures no vertex of S burns. Consider the strategy that begins $(x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_{k+1}, y_{k+1}, z_{k+1})$. After defending these vertices both q_k and p_k are burning. At this point combining the burning vertices into a single vertex and removing all defended vertices yields the rooted graph (T^2, a^2) with a^2 burning. Proceeding with strategy D^2 from this point ensures that no vertex in S burns.

Suppose all vertices in S can be prevented from burning using strategy $D = (d_1, d_2, \dots, d_t)$. By Lemmas 10 and 11, we have that by the end of time $3k$ either no vertex of Z has been defended and p_k is burning, or some subset of the vertices of Z has been defended and both p_k and q_k are burning. In the first case, p_k is the root of a complete binary tree in H_{3k}^D . Therefore some vertex of L^2 must burn, a contradiction. And so we conclude that some subset of the vertices of Z has been defended and both p_{k+1} and q_{k+1} are burning. Note that T^2 is a subgraph of H_{3k}^D . Thus the strategy $D^2 = (d_{3k+1}, d_{3k+2}, \dots, d_{3k+t})$, where t is the height of T^2 , is a strategy that prevents all of the leaves of T^2 from burning. This completes the proof.

As mentioned above, much of the recent research in this area has been in the study algorithms that approximate $MVS(G, r)$. That *FIREFIGHTER* and *SFIRE* can be unified under a weighted model (with associated decision problem *WFIRE*) suggests that the ongoing research in to approximation algorithms for $MVS(G, r)$ may be adapted to provided bounded error approximation algorithms for an optimization version of the *SFIRE* problem.

Recent work in the broader study of discrete-time graph contagions has focused on a model of graph burning for which the objective is to burn the entirety of the graph in the minimum number of time steps. Under this model, in each step rather than deciding which vertex to defend, the strategist much decide which vertex to set aflame. This model was first introduced in [6] and has subsequently been studied by researchers looking to bound the so-called *burning number* in deterministic and probabilistic frameworks [5, 21, 24]. Though this model

has the opposite objective to the process studied herein, these two models share the same underlying mechanic. And so objectives for the Firefighter process provide natural areas of study for this model of graph burning.

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Figure 1 An example of reduction. The rooted graph on the right results from reducing the rooted graph on the left.

Figure 2 Instances of SFIRE that require non-adjacent strategies

Figure 3 J_k

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