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21. SNAPSHOTS OF CREATIVITY IN UNDERGRADUATE MATHEMATICS EDUCATION

Creativity in undergraduate learning is not solely the preserve of a student. We believe that to enable students to be creative they need to be given problems with creative potential and experience creative instructional engagements. We first establish our view of what creativity entails and discuss what can be the relevance for mathematics education from the work of the 'masters'. We then illustrate creative engagements in undergraduate mathematics for students and teachers by presenting examples of problems that have creative solutions, problems that promote creativity and creative pedagogical approaches.

INTRODUCTION

Mathematics can be a mundane learning of techniques – certainly many techniques need to be learned. But mathematics also gives the chance for students to reach heights of creativity and we cannot always predict who these students will be. We believe that all students should be given the opportunity to be creative by solving interesting mathematical problems and experiencing creative instructional engagements.

In presenting our view on creativity we focus on three situations in which it can apply: an idea or solution; a question, problem or task; and an instructional engagement. We start with several examples of problems and their solutions that we consider 'creative'. With the help of these examples we establish what we consider as demonstrated creativity in a solution or a proof.

Furthermore, building on the existing theoretical approaches, as well as on the experiences of the 'masters', we present a selection of mathematical problems that promote creativity in students. We then exemplify and discuss creative approaches in teaching undergraduate mathematics. We believe that the 'snapshots' presented here will stimulate the recollection of similar examples from the readers' previous experience.

WHAT 'COUNTS' AS CREATIVE?

The nature of creativity is discussed intensively in this volume, relying on a variety of theoretical perspectives. However, before any exposure to a theoretical analysis, there is a shared recognition of creativity among mathematicians and teachers of

mathematics, when it is observed in approaches to mathematical problems. In this section we attempt to clarify our perspective by outlining what contributes to this 'shared recognition'. Later, in sections *LEARNING FROM THEORIES* and *LEARNING FROM MASTERS*, we draw connections between our perspective and prevalent theoretical approaches in psychology and mathematics education.

First, we would like to situate creativity in undergraduate mathematics in a similar way to that in which 'problem solving' is situated in mathematics education. That is, we want to situate it with respect to the individual's prior knowledge and experience. A student is considered to be engaged in a problem solving activity if s/he does not have a previously learnt algorithm for approaching a problem. As such, what is a 'problem' for one learner may not be a problem for another.

Similarly, whether an idea, a solution or a proof is considered creative depends on the context. It is 'creative' if it employs methods or arguments not previously considered or expected in the given context by the given person. For example, Gauss' solution of pairing the numbers in calculating the sum of the first 100 numbers, fact or fiction, was definitely creative when it was suggested by a 9 year old.¹ It is quite possible that another student may come up with that same creative idea, having never before heard about young Gauss or being exposed to the method. Research of Ching (1997) demonstrates exactly this, presenting work of young students who essentially reinvented the infamous Gaussean solution in designing a method for finding the sum of the sequence of numbers from 1 to 100. Consequently, we consider an event to be creative even if it has been experienced by many people many times before the particular experience of a given student. (Of course once the Gaussean method is introduced in class, it becomes a standard approach for dealing with arithmetic sequences).

Another aspect of creativity that we want to emphasise is its novelty or unexpectedness. If every student in a class comes up with the same idea, then it is highly probable that the idea is not a creative one. Elegance is also a word that we might want to associate with creativity. If the potentially creative idea is longer or more convoluted than the standard solution method, then what has been gained by it? Clearly the creative must have some advantage over the standard approach.

Further, for a student to be creative, the instructor has to have provided a problem where creativity can be shown. If a question asks what is the derivative of x^6 , there is very little that a student can do than answer $6x^5$. So it is important that the chosen tasks provide an opportunity for some students to produce creative ideas.

Finally we note that creativity is an important part of research mathematics. We believe with Hardy (1940/2002) that mathematics at this level is essentially a creative activity, rather than an explanatory or expository one in that researchers are trying to produce new ideas. Often in early courses in mathematics, though, a greater emphasis is placed on teaching new content than teaching the way that new mathematics is produced by research mathematicians. However, we feel that it is important, and makes the subject more interesting to students, if mathematics is presented as being more than content. By using all the methods at their disposal to solve problems, students gain a better understanding of how research is carried out

and they also develop techniques to solve any problem (mathematical or otherwise) that they may meet in later life.

In what follows we apply the adjective ‘creative’ to (1) an idea or solution, (2) a question, problem or task and (3) an instructional engagement. We use the word for any of these that is unusual or novel in some way or possibly elegant or neat. We note that instructors have to value creativity. This can be done in their approach to the subject in class (see CREATIVITY IN TEACHING UNDERGRADUATE MATHEMATICS). This leads to the presentation of creative problems (see MATHEMATICAL PROBLEMS THAT PROMOTE CREATIVITY) and hopefully creative solutions (see the next section).

THREE PROBLEMS WITH CREATIVE SOLUTIONS

In this section we look at three problems and present solutions of them that we think of as being creative. We encourage the reader to try the problems first simply because it is hard to appreciate a creative idea unless you have been thinking along another route first. Once you have a solution you can try to find a ‘better’, more creative one.

From Graph Theory

Problem 1: How many regular graphs of degree three are there on six vertices and what do they look like?

A regular graph of degree three is one in which there are three edges incident with every vertex. In some ways this is a trivial problem because all you have to do is to produce a list using a method of exhaustion. This can be done reasonably easily because there are only two such graphs. The difficulty, implied in the question, is to justify that no more than two such graphs exist. It is not good enough for a student to say “I can’t find any more.” Many students get no further than showing the actual two graphs, see Figure 1. Many seem not to understand that they need to justify that there are no more graphs.

Figure 1. Regular graphs of degree 3 on 6 vertices.

A creative method here is to note that if G is regular of degree 3, then its complement, H , is regular of degree 2. It is easy to show that the regular graphs of

degree 2 are cycles and unions of cycles. So on 6 vertices H can only be the cycle on 6 vertices or the two cycles on 3 vertices. The two graphs we require can quickly be recovered by complementation from these two graphs. We have also justified that there are only two such graphs.

From Number Theory

Problem 2: Prove that $n^5 - n$ is divisible by 3 for any n .

This looks as though it would fall to a standard proof using Mathematical Induction and it does. However, pursuing this route we might well need an induction within an induction. This becomes a bit messy.

On the other hand, there are two creative solutions to this problem that we want to show. The first one relies on 'factorisation'. We see that $n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = n(n + 1)(n - 1)(n^2 + 1)$. Since $n - 1$, n , and $n + 1$ are consecutive numbers, one of them has to be divisible by 3. The result follows.

The second creative solution relies on modular arithmetic, where for all n , $n \equiv 0, 1$, or $2 \pmod{3}$. Further, in that natural order, $n^5 \equiv 0, 1$, or $2 \pmod{3}$. Then in all three cases $n^5 - n \equiv 0 \pmod{3}$. This assures the required result. (Note that a student who has never met the concept of congruence can still produce this result but without the standard notation or definition).

From Geometry

Problem 3: Show that three medians in a triangle (a) intersect at one point and (b) create segment ratio 2:1 at the point of intersection.

Though this famous theorem can be proved using standard methods of Euclidean geometry, a creative proof starts by assigning Affine coordinates to the vertices of a triangle, as shown in Figure 2. The assignment of coordinates for A, B and C is arbitrary, but the numbers are chosen for simplicity of calculation.

Then, considering pairs of points, the equations of lines are calculated as follows:

$$\begin{aligned} \text{AL: } y &= x \\ \text{BM: } y &= -\frac{1}{2}x + 3 \\ \text{CK: } y &= -2x + 6 \end{aligned}$$

Further, the intersection point for each pair of these lines is calculated to be (2,2). This completes the proof of (a). In (b) we need to prove that $AO:OL = BO:OM = CO:OK = 2:1$. (A number indicating the length cannot be assigned to segments in Affine coordinates, since the unit is chosen differently on each axis. However, comparing the coordinates on parallel lines it is possible to determine the

Figure 2. Medians in a triangle.

ratio of lengths). Without loss of generality, let us consider BO and OM . Comparing the coordinates of $B(6,0)$ to coordinates of $O(2,2)$ there is a difference of 4 in the X-coordinate and a difference of 2 in the Y-coordinate. Comparing the coordinates of O to those of $M(0,3)$, there is a difference of 2 in the X-coordinate and the difference of 1 on the Y-coordinate. Since those segments are on the same line, the ratio of lengths is established as 2:1. The same applies to other segments as well and completes the proof of (b).

Creativity in Solutions

Through experience we have come to consider the solutions presented above as creative not because they were once produced by one student. On the contrary, they have been produced many times over a period of years. However, these are the solutions that have been produced less often than other solutions. In the majority of cases, without help, students have produced either no solution, incomplete solutions or the 'standard' ones. So relative rarity is a feature of the three solutions presented above.

In the Graph Theory problem the creativity in the solution is in shifting perspective to an isomorphic problem by considering the complement of the graph. Also, we consider the presented solution to the Geometry creative because it relies on unexpected use of tools from a different domain, in this case Affine coordinates. In the Number Theory problem the tools of modular arithmetic serve as means to this end.

Further, the creative solutions in these problems are more elegant than the standard ones. In the Graph Theory problem, when making a systematic list to

produce all regular degree 3 graphs on 6 vertices it can be hard to justify that all graphs have been considered. However, this easily follows from the complement method. In the Geometry problem the elegance is achieved in the simplicity of calculations. It is further appreciated by the fact that the presented proof, unlike the standard one, does not require any auxiliary constructions. In the Number Theory problem the routine induction is not straightforward. This contributes to the perception of elegance in both creative methods. Somehow this elegance appears to make the ideas easy to remember and the method is then able to be applied on later occasions. (We illustrate this idea in *What if not?*).

In considering these three problems we added another dimension to the ideas of novelty, rarity and elegance mentioned in WHAT COUNTS AS CREATIVITY. This dimension relates to the use of tools in the solution. We suggest that an important identifier of creativity in mathematics can also be in the use of 'unexpected' tools. Such tools employ a change of perspective that may be achieved by transformation, auxiliary construction, reformulation, consideration of an isomorphic problem, or application of methods that fall outside of the domain in which the problem appears to be situated.

We note that the ability to use a variety of tools is strongly connected to the prior knowledge of an individual. That is to say that extensive background knowledge in different mathematical areas extends an individual's capability to produce a creative solution.

Further, we suggest that the use of out-of-domain tools in a problem solving situation is a feature that identifies mathematical creativity at a professional level. In fact, significant progress in mathematics as a discipline is made by introducing tools from other, seemingly unrelated, areas of the discipline. Consider the classical example the ancient mystery of doubling the cube or trisecting an angle. The problems are situated in geometry but they were solved by using extensions of fields, that is, introducing tools from Abstract Algebra. Or the well known Fermat's Last Theorem. This problem is apparently situated in elementary Number Theory but the solution involved elliptic curves and ideas from ring theory.

LEARNING FROM THEORIES

In what follows we consider attributes of creativity as described by scholars of creativity.

Torrance: Approach to Creativity in Education and Psychology

The most frequent approach of creativity in education follows Torrance (1974), whose test of creativity attends to fluency, flexibility, originality and elaboration. A short description of each of these categories is provided below.

- Fluency:* The total number of interpretable, meaningful, and relevant ideas generated in response to the stimulus.
- Flexibility:* The number of different categories of relevant responses.
- Originality:* The statistical rarity of the responses among the test subjects.

Elaboration: The amount of detail in the responses.

For example, finding unusual uses for common objects, such as bricks or envelopes, is commonly considered as an indicator of an individual's creativity. However, consider the task of finding various uses for the Pythagorean theorem, and a student's suggestion of exhibiting the theorem as graffiti on a wall. This may be considered 'creative' in some contexts, but not in ours.

In an attempt to apply Torrance's approach to what we intuitively considered as 'creative' in the solutions and proofs presented above, we conclude that the issue of originality appears to be most applicable. In particular, we consider the use of tools from a different domain and shifting perspective to an isomorphic problem as indicators of originality.

Though we cannot consider 'statistical rarity' of any particular solution, we can build on a collective experience and consider relative rarity compared to a standard, expected and readily available methods.

However, the question comes to mind, as to in what mathematical tasks fluency, flexibility and elaboration can be exhibited as an indicator of creativity? Fluency in relation to mathematics is usually seen as a skill or proficiency in applying a known procedure. Further, in order to consider fluency and flexibility we need to consider a situation in which a relatively large number of responses to a given task are offered. Can we develop such experiences/situations in undergraduate mathematics education? We return to this point in the discussion of teacher's creativity in the CREATIVITY IN TEACHING UNDERGRADUATE MATHEMATICS section

Approaches to creativity specific to Mathematics Education

Silver (1997) describes the 'genius' view of creativity, where creative experience is considered as 'rare mental feats', 'exceptional thought process' and 'occasional bursts of insight' (p. 75). He suggests that this view has limited applicability in education and contrasts it with a more contemporary view, where creativity is related to personal inclination, deep personal knowledge, long periods of work and rich instruction. (The connection between 'occasional bursts of insight' and 'long periods of work' is clear in the work of Hadamard, explored in the next section). Silver views problem-posing and problem-solving activities as playing a central role in developing creativity in students. This view is generally supported in mathematics education research.

Shiraman (2005) distinguishes between mathematical creativity at the professional level and at a student level. At a professional level he considers the indicators of creativity to be "original work that significantly extends the body of knowledge" and "ability to open avenues of new questions for other mathematicians". In a similar fashion, he defines creativity of a student as "the process that results in unusual (novel) and/or insightful solutions to a given problem" and "formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle requiring imagination" (p. 24). The strong analogy in this distinction echoes the view of Pólya (1945), that the difference in problem solving between the work of a student and a mathematician is that of a degree of sophistication, rather than in the nature of activity.

Shiraman's definition also strongly draws on both problem solving and problem posing. Accepting this general definition, we illustrated in Problems 1, 2 and 3 above what features of solutions are viewed as insightful and we added further detail on what the 'new angle' may entail.

LEARNING FROM MASTERS

Though Silver (1997) suggested that 'genius' perspective on creativity may not be helpful for the purpose of education, in this section we describe how 'masters' view creativity and what from their experience may be relevant for undergraduate mathematics education.

Hadamard and Edison

As Silver mentioned the connection to genius, we turn here to the renowned view of genius attributed to Thomas Edison, "Genius is one percent inspiration and ninety-nine percent perspiration". In fact this is a paraphrase of Edison's saying: "None of my inventions came by accident. I see a worthwhile need to be met and I make trial after trial until it comes. What it boils down to is one per cent inspiration and ninety-nine per cent perspiration."

Further, the illumination, which is in the centre of creative activity, is preceded by preparation and incubation (referring to the 4-step creative process described by Hadamard (1945): preparation, incubation, illumination and verification). This view connects the idea of sudden insight, also known as the 'AHA!' moment, to the extensive work that is an essential condition for developing insight. A related saying attributed to Edison is that "A genius is often a talented person who has simply done all of his homework."

What do We Learn from Hadamard and Edison?

Considering the work of Hadamard and Edison the inevitable conclusion from the perspective of mathematics education calls for preparation and perspiration. Both relate to the work of a student as well as the teacher. For students to do 'all their homework' may be beneficial not only for developing creativity. However, this will strongly depend on the nature of the homework and a thoughtful combination between skill development and problem solving may lead to a desirable result. This is the 'perspiration' of the teacher.

Moreover, an important element, often ignored in mathematical instruction at any level, is incubation. Our courses work on strict deadlines for submission of assignments, not really allowing for extensive prolonged thinking on any given problem. Though deadlines cannot be totally avoided, they can be treated with some flexibility and instructional planning should allow for longer 'thinking time' on at least some problems. While for some students such scheduling may promote procrastination, for others the period of incubation embedded in the task may turn to be productive and promote creative activity.

Leonardo da Vinci and Einstein

Curiosity is often identified by historians as one of the main features of da Vinci's genius. What is also mentioned is his commitment to test results and his willingness to embrace ambiguity and uncertainty. Further, Einstein's view of curiosity is evident in the following quote: "The important thing is to not stop questioning. Curiosity has its own reason for existing." Imagination or 'thought experiment' is Einstein's answer to uncertainty.

Another important feature in Leonardo da Vinci's creativity is in seeing how everything connects to everything else. According to Gelb (2000), "That's systems thinking, [...] you have to see patterns, relationships, and processes and how they all fit together." A similar feature of connectivity appears in Einstein's work. For example, according to G. Holton (1995), Einstein insisted that the separate laws of physics could be brought together into one set that is true everywhere in the universe. This not only illustrates his search for overarching and generalised explanation, but also describes his aesthetic preference for the simple and universal.

What do We Learn from da Vinci and Einstein

Ambiguity and uncertainty are usually not associated with mathematical experience at the undergraduate level. However, dealing with ambiguity may be a valuable component in developing creativity. This may be achieved when tackling open ended problems and seeking patterns for generalization. Further, dealing with ambiguity is one of the means to develop students' curiosity.

We further learn that risk taking, questioning, and problem posing are unavoidable components of creativity. This emphasises the need for instructors to choose problems that allow for the development of patterns and relationships, problems that can be approached with a variety of tools and problems that lead to possible generalizations. We exemplify several of such problems in the next section.

MATHEMATICAL PROBLEMS THAT PROMOTE CREATIVITY

As we mentioned above, not all mathematical tasks are equally suited for creative engagement. However, in what follows we present several problems that we believe encourage creative pursuits.

Squares and Rectangles

Problem 4: How many squares can be found on a chessboard?

While the immediate answer is 64, this is not the answer being sought. The idea is to consider all the squares that can be drawn on a not necessarily coloured 8×8 grid, where a square has its vertices as intersections of the grid lines and its edges parallel to the sides of the grid. The general solution, often preceded by consideration of smaller grids, leads to counting squares of different sizes, as shown in Table 1:

Table 1. Counting squares.

Size of a square	8×8	7×7	6×6	5×5	4×4	3×3	2×2	1×1
Number of squares	1	4	9	16	25	36	49	64

This further leads to the realization that the total number of squares on a grid (204 in case of the 8×8 grid) is given by the sum of consecutive square numbers. This observation can be easily generalized to determine the number of grid squares for the $n \times n$ board.

This problem leads itself to a variety of possible variations and additional generalizations. For example, keeping the vertices of the squares at the intersection of the grid lines, the limitation that the sides of the squares are parallel to the sides of the board can be omitted. How many squares are there on the 8×8 board in this case? On the $n \times n$ board?

Further, extending the idea of a 'square' that is found on a board, we can ask, why square? A variation on this problem asks how many rectangles can be found on a chessboard. A standard solution may list all the possible sizes of rectangles. However, an alternative view suggests that this number is given by $\binom{9}{2} \times \binom{9}{2}$, where

a rectangle is determined by a choice of 2 vertical grid-lines out of 9, followed by a choice of 2 horizontal lines, out of 9.

The creativity in this solution is in unexpected application of combinatorial structure. Making connections between different domains is the genius of da Vinci, exemplified with a simple mathematical problem.

Further generalizations include: How many rectangles are there on the $m \times n$ board? How many rectangles are there if the requirements that the sides be parallel to the sides of the board is omitted? How many squares are there on the $m \times n$ board?

Functions with Absolute Value

Problem 5: Are there any values of b for which the system of equations $y = |x|$ and $y = 6 - |x - b|$ have an infinite number of solutions? Find all of these values and the corresponding solutions if such value(s) exist or show that no finite set of solutions exists.

This requires imagination if not creativity. First the modulus function is something that students in a first year Calculus course find hard to come to grips with; it really worries them. Second, because the functions are made up of lines they expect the lines to meet in discrete places. Some students will give this as their reason why no infinite set of solutions exists. Third, a neat solution requires shifting perspective to look at the graphs of these functions. So students might need to play with graphs and with values of b to see that, by changing the values of b , the second graph can be made to slide 'on top of' the first graph. It requires a further effort to see that

there are TWO values of b here. Students usually stop and relax once they have found one – surely there can't be more? Fourth, students are generally not very good at geometry and don't have a feel for it. Hence they try algebra at every opportunity. The algebra here is a mess.

Algebraic solution (an outline): The function $y = |x|$ is defined as x for $x \geq 0$ etc. and $y = 6 - |x - b|$ as $6 - (x - b)$ for $x \geq b$ etc. So we have to take the pieces of the functions where they agree. If $b \geq 0$, we have to look at the three cases $x \geq b$, $0 \leq x < b$ and $x < 0$. To find a solution, for the first of these we get $x = 6 - (x - b)$ so $x = 3 + b/2$. Given b students assume that it is unique. And the same thing happens for the other two restrictions on x . Hence the solution is unique for all b . A similar proof follows for $b < 0$.

What students miss is that $x = 3 + b/2$ AND $x \geq b$. So $3 + b/2 \geq b$ or $b \leq 6$ for this solution. This has ramifications that need to be followed up but it is somewhat tedious to do this.

Geometric solution: We suggest starting with an experiment by choosing an arbitrary b , for example $b = 3$ (Figure 3).

Figure 3. Graphs of $f(x) = 6 - |x - 3|$ and $g(x) = |x|$.

The graph can be moved for any b by translation. And at some stage one graph appears on the top of the other so that there are an infinite number of points in common. This happens twice for $b = +6$ and -6 , as shown in Figure 4.

Figure 4. Graphs of $f(x) = 6 - |x - 6|$, $h(x) = 6 - |x + 6|$ and $g(x) = |x|$.

The creative steps here are first taking a geometric perspective; then being willing to experiment with values of b . Both changing perspective and experimenting are valuable problem solving tools that will be useful for the students to try in other situations.

Parabolas and Tangents

Problem 6: Consider the graphs $y = 3x^2 + x + 1$ and $y = -x^2 + 9x - 3$. Show that there is a point on these two graphs for which they and the graph $y = x^2 - 3x + 7$ have a common tangent. Find another parabola that has the same tangent in common with the two parabolas above or prove that no such parabola exists. (This problem follows an idea of Harradine (2007)).

Students have to find the point of intersection of the two curves and the tangent at that point. Then they can check that the third graph has the relevant properties.

There are two ways to go. Recognise the pattern: what do 3, 1, 1; 1, -3, 7; and -1, 9, -3 have in common? The pattern can be guessed and then it needs to be checked to make sure that all graphs in that family ($a, 7 - 10a, 9a - 2$) have the same tangent and pass through the same point. It can also be shown that this is the unique infinite family that satisfies these constraints.

Alternatively, we could start with $y = ax^2 + bx + c$, find the conditions for the graph to have the two key properties and write out the general form of the infinite family.

The problem here is that students don't like infinite families. It is likely that they don't see parabolas as Objects (see Dubinsky & McDonald, 2001, for example on APOS theory) and can't conceive the possibility and so can't bring the problem to a solution. (This may also be the problem that students have with the modulus function in the last example).

It is also interesting to follow this up in two ways. First by animating the infinite set. Graph all of the infinite family by letting the coefficient 'a' move from a suitably large negative number to a suitably large positive number in small steps. (This can be done on a CAS calculator). And second by asking if they can find an infinite family through any given point with any common tangent. In practice this is just a matter of translating the original functions and changing the gradient at the intersection point.

This is an example of a creative problem in calculus. It is an application that is relatively routine but is hopefully more interesting than asking students to simply find the equation of the tangent to a given curve at a given point. It forces the students to think creatively about curves that may or may not exist as well as conceiving an infinite family. Students are encouraged to make guesses and have to justify these guesses. They also have to be sure that there is no counterexample. This also needs them to use their imaginations.

What we have not discussed here nor in any of the other problems is the incubation period that is needed by most students to consider the problem, experiment with approaches, and see which of the approaches found are useful.

A Variation on Trigonometric Functions

Problem 7: Sine and cosine are often defined via coordinates on the unit circle. Define squine and cosquine via coordinates on the unit square (oriented as you wish). What properties do these functions have?

Use the square in Figure 5 to define the functions squine and cosquine by $x = \text{cosquine}(\theta)$ and $y = \text{squine}(\theta)$. What properties do sine and squine share? What relations are there between squine and cosquine? Explore.

This is really a revision of sine and cosine so that students start to understand the basics of the trigonometric functions and their properties as well as how those properties arise. The question is an open one and gives students room to explore and be creative as well as to practice their trigonometric knowledge.

Figure 5. The basic diagram for the squine and cosquine functions.

Among other things, in the process of examining this question, students will see that the squine function is periodic and be able to define it in terms of trigonometric functions. They might be encouraged to invent other shapes than the square of Figure 5. The diamond with unit side is also of interest.

Couples at Dinner Party

Problem 8: Five couples went to a dinner party. At the beginning of the party some people shook hands. It turns out that everybody I spoke to later on said that they had shaken hands a different number of times. Now no one shook hands with the same person more than once and no one shook hands with their partner. How many times did my wife shake hands? Extend.

This problem can be found in several places on the web so we omit a solution here. This can be thought of in terms of Graph Theory. Its advantage is that it gets students thinking away from the standard questions usually set to illustrate the theory. There is a pretty clear generalisation.

What is Creative in These Problems?

The problems presented above have several features in common. First, they engage learners in ‘real’ problem solving. That is to say, they do not have well defined and predetermined algorithms for approach and that is where creativity of a problem-solver has room to flourish. Further, they can be approached via a variety of methods, giving the students an opportunity to seek different methods and chose a more elegant one. They present a variety of opportunities for exploration by asking questions, seeking patterns, and inviting extension, variation or generalization.

CREATIVITY IN TEACHING UNDERGRADUATE MATHEMATICS

We turn our attention now to what creative teachers of mathematics create. In fact, we have already partially considered this question by considering the features of mathematical problems or tasks that promote students' creativity: creative teachers employ and develop such tasks. While problem posing and problem solving appear 'obvious' means for developing creativity, there are several reported attempts of more specific rerouting of these activities.

Snapshots from Literature

Research of Watson and Mason (2005) offers 'learner-generated examples' as a powerful strategy for developing and extending students' knowledge. Consider for example the following task: "Write down two numbers that have a difference of 2" (p. 34). While it may appear obvious, Watson and Mason offer extensions: "Write down another pair. And another. And another." And "Write down two numbers that differ by 2, but for which the fact that the difference is 2 is as obscure as possible." This challenge of creating 'another', and of making the difference 'obscure' may serve as the means for developing creativity, as well as for demonstrating creativity. It is also a good device for revising known material.

This instructional strategy of learner-generated examples can be applied at any level. We find the following among the examples Watson and Mason offer for the undergraduate level: "Find an oscillating function on a finite open interval that does not attain its least upper bound on the interval." (p. 82). (See Koichu, *If not, what yes*, for a similar idea). Watson and Mason leave it up for the reader to determine the meaning of "oscillating" and consider a variety of meanings.

Zazkis and Leikin (2007) used Watson and Mason's ideas related to generating of examples and asked a variety of individuals to give an example of an irrational number, and another, and another. While the examples of some participants were limited to $\sqrt{2}$ and π , others generated a resourceful variety that included cubic roots and arithmetic operations with roots of primes. Zazkis & Leikin concluded that learner-generated examples serve not only as a useful pedagogical strategy, but also they provide a valuable lens for a researcher to examine participants' understanding of mathematics. (They also provide a tool to examine students' understanding.)

The relation of example generation tasks and creativity brings us back to the criteria of fluency and flexibility suggested by Torrance. However, in considering these attributes, that is, the number of generated examples and the variety of categories in which the examples fall, it may be impossible to distinguish between a person's creativity and his/her prior knowledge and experience. This reinforces the consideration of extensive knowledge as a contributing factor to creativity.

The idea of 'and another' can be applied in different ways. For example, Leikin (2007) asked her undergraduate students to produce multiple solutions/proofs for geometry problems. She claimed that it is more beneficial for a learner to solve one problem in five different ways than to solve five different problems. Through this

twist in a problem solving engagement her students develop valuable connections and push their boundaries of creativity.

Zaslavsky (1995) advocated the use of open-ended problems. She exemplified the ways in which 'standard' tasks can be modified to encourage open-ended investigation. Such modifications allow for flexibility of solutions and various levels of difficulty and generalisation, that in turn encourage creativity.

Shriki (2008) reported on a study that explicitly engaged her students in creating 'new' mathematical concepts. Following a structured discussion with her students, pre-service teachers, Shriki identified their belief that 'new' ideas and theorems are the privilege reserved exclusively for research mathematicians. She assigned the task of inventing a new concept and then exploring its properties. Following the initial resistance, the students reported excitement and enhanced motivation.

What all the above examples have in common is teacher-created opportunities for students to be creative. Some of these opportunities emerge spontaneously, but most of them are purposefully selected. In what follows we elaborate on several opportunities that encourage students' creativity in more detail. The first two are examples from the experience of the authors. The last three we adopt from the literature.

A Creative Start for a Course

D. Holton (2002) starts his first course in Graph Theory by the following question.

A graph is a thing that has points joined by edges. How many graphs are there on one vertex, two vertices, three vertices, ...?

This aims to produce a creative interactive lesson/lecture (or two) and allows students to discover for themselves the basic ideas of graph theory. What usually comes out of this is (i) loops, multiple, directed edges, digraphs, labelled and unlabelled graphs, and 'simple' graphs; (ii) a nice formula for the number of labelled graphs on n vertices and the fact that the corresponding formula for unlabelled graphs is hard to find; (iii) degree of a vertex, sum of degrees equals twice the number of edges; (iii) self-complementary graphs and on what numbers of vertices they exist; (iv) connectivity ideas; (v) isomorphism ideas; and (vi) 'regularity'.

This approach is taken because it enables students to begin to explore the territory and is better in the long run than just giving them a set of definitions. Starting any lecture course just with definitions, an example, and then going straight on to theorems is a problem. In the early years of university especially, students need to get some examples and understanding of the field behind them before they can take an intelligent interest in theorems. They need to have Action and Process knowledge of a topic before they can fully comprehend it at an Object level as they are required to do when being shown theorems (see APOS theory in, for example, Dubinsky and McDonald, 2001).

Moving with a Flow

Zazkis' students, prospective elementary school teachers, in one of the class sections calculated the ratio of the circumference to diameter in different circles. They exhibited their excitement and appreciation with the resulting approximation of π . The instructor attempted to have her students see the constant in more general terms – as a constant ratio of corresponding parts in similar objects. In a rather naïve question, some students wondered why the relationship is featured for circles only. After all, all the squares are similar, and so are equilateral triangles and other regular polygons. The instructor suggested that students should not only define and calculate the desired ratios, but also name them. The chosen name was 'cake' (in analogy with a 'pie'), that was defined as the ratio between the perimeter of a polygon and a segment connecting the 'centre' to the 'corner'. Then, triangular cakes were calculated, followed by square, pentagonal and hexagonal ones. (Some calculations were done using algebra and trigonometry, others were carried out empirically).

It was further noted that when the number of sides in a polygon increased, the value of 'cakes' approached the value of π . Indeed, a similar creative method was developed by Archimedes long before the students thought of it. However, this should not degrade the students' creativity. As we mentioned at the start, we consider an approach to be creative if it presents novelty or originality of an individual (or in this case a group of individuals), even if the derived facts or methods were known to the humanity before. However, the name for the invention was totally original. The instructor's role in this event is crucial: it manifests itself in attending to the 'flow' of students thinking, rather than setting the boundaries of formal mathematical curriculum.

What if Not?

'What if not' is a powerful strategy developed by Brown and Walter (1983), that has its place in undergraduate mathematics. Changing conventional parameters is a worthwhile engagement that invites creativity and enhances understanding of mathematical concepts.

'What if not' invites a variety of ways of extending and varying problems. An illustrative example of implementing this strategy is presented in Problem 4, *Squares and rectangles*. In this section we reconsider the Geometry Problem about the properties of medians in a triangle. But what if not a median? Let us define a tridian as a segment that connects a vertex of a triangle to a point that marks $1/3$ of the opposite side. Then the ratio of segments at the points of intersection of tridians is 3:3:1 (see Figure 6). Students can be invited to discover this relationship for themselves.

The proof using standard Euclidian geometry is rather complicated. However, the – once considered creative – method of assigning Affine coordinates to the vertices of the triangle appears useful.

Figure 6. Tridians in a triangle.

Then, why stop with tridians? In a similar way, a point can mark $1/4$, $1/5$ or $1/n$ of a side and ratios of segments at the points of intersection be considered. We leave the pleasure of discovering these ratios to the creative reader.

Creative Attention: If Not, What Yes

Koichu (2008) describes an activity that he engages in with classes of pre-service and in-service mathematics teachers that is a development of the classical “what if not” strategy (Brown and Walter, 1993). The question raised by Koichu in his classes is as follows.

Recently I looked at many examples of even functions and noticed an interesting property: *Every even function has a local strict extremum at $x = 0$.* To me, the statement looks just right, but it does not appear in any algebra or calculus textbook I've looked at. Is there somebody familiar with this property of even functions? What do you think about it?

The statement in *italics* is then examined by each class. This involves students working individually, in groups, and as a whole class to try to see if it is a true statement. Apparently the classes tend to go through the following stages.

First they consider well-known even functions and see that they satisfy the italic statement. However, since the statement does not appear in textbooks, there is some hesitancy to accept the statement. After a short while some counterexamples are suggested such as $y = c$ a constant, and $y = \frac{1}{x^2}$. This leads to a reformulation of

the statement that excludes constant functions and functions that are not defined for all real numbers. Then counterexamples are found for this revised formulation.

By a series of several reformulations and counterexamples, the classes usually end up with the final formulation:

Every even function that is monotonous and not constant in a right-side neighbourhood of $x = 0$ has a local strict extremum at $x = 0$.

Although the students are somewhat disappointed by the final formulation, which is not too hard to prove, they value the experience of working on an open question.

This is an example of a lesson that involves more than just regurgitating knowledge presented by the instructor. The original question, which could have been simply treated with a counterexample, has been developed into a creative activity in defining, verifying, searching for counterexamples and finding a proof. This kind of activity may be not too hard to produce for a given course after a little thinking or after listening to students' questions.

Technology Supporting Creativity?

Technology can also be used creatively to promote creative thinking. Here we cite some work of Hillel (2001) in the area of linear algebra where he considers three activities.

His first activity is labelled 'Surprises' in that it enables students to discover something that is unexpected, namely that the fifth power of a matrix can be the zero matrix and that such a matrix can have few, if any, zero entries. Given technological assistance, students can handle large matrices very early in their experience of linear algebra and come to see that complicated looking matrices have interesting properties. This early contact with a range of examples gives "students quite a different insight about behaviour of matrices" (Hillel, 2001, p. 373). So one of the aims of this activity is along the lines of the material in *A Creative start for a course*.

Hillel's second activity is about clarification — clarification of theory. This is done by taking a large homogeneous system of equations and performing elementary row operations on its coefficient matrix. All of the matrices along the way from the coefficient matrix to the row reduced form are kept and students are led to see that all of the relevant intermediate systems of equations have the same solutions. This activity is continued further but the technology is used to help students see further into the theory that is presented than they would by simply calculating solutions of linear equations.

The third activity is an investigation into what Hillel calls the 'life span' of a vector under a matrix. The life span of a vector v under a matrix A is the smallest k for which $A^k v = 0$. The value of k for different given A and v is investigated by the students as well as for A and v of their choosing. This enables links to be made with such concepts as eigenvalue, nullspace, nilpotent matrices and minimal polynomials.

The work here would be extremely difficult, not to mention tedious, using only hand calculations. However, technology can be used in a positive way here to

enable students to do mathematics creatively and not just be confined to routine problems.

CONCLUSION

The task of mathematics education is not to measure creativity (we leave this to psychologists), but to develop and enhance it. In this chapter we presented several examples of how this may be achieved through instructional engagements. These engagements present just a few examples rather than the 'state of the art' in undergraduate mathematics education.

Though our examples focus on undergraduate mathematics education, our conclusion is not limited to this level. We believe that a creative pedagogical approach – in task design and in supporting diversity and flexibility – invites creative responses from students. As such, echoing the ideas of Bell and Hadamard, we emphasize 'preparation' on the part of the instructor to invoke 'perspiration' on the part of a student. Hopefully, at times this will result in 'illumination'.

What do creative teachers create? They create open problems that allow experimentation by individuals, groups or the whole class. They also encourage the use of a wide range of mathematical skills. In short: they create opportunities for students to develop their own creativity.

NOTES

- ¹ See, for example, <http://www.nzmaths.co.nz/HelpCentre/Seminars/Gauss.aspx>.

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