STOCHASTIC LIFE ANNUITIES

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Abstract

This paper gives analytic approximations for the distribution of a stochastic life annuity. It is assumed that returns follow a geometric Brownian motion. The distribution of the stochastic annuity may be used to answer questions such as "What is the probability that an amount F is sufficient to fund a pension with annual amount y to a pensioner aged x?" The main idea is to approximate the future lifetime distribution with a combination of exponentials, and then apply a known formula (due to Marc Yor) related to the integral of geometric Brownian motion. The approximations are very accurate in the cases studied.

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1. Introduction

Defined contribution pension arrangements, whether in the form of employer-sponsored pension plans or individual savings, have enjoyed great popularity lately. The main drawback of these arrangements is the uncertainty of future lifetime and rates of return. As a consequence, retirees face two difficult decisions: how much to draw form their savings every month, and how to invest what is left. A mathematical difficulty is that the distribution of a life annuity including random duration of life and random rates of return is not known in closed form. The main goal of this paper is to present an apparently new way to approximate the distribution of such "stochastic life annuities."

This problem has already appeared in the actuarial literature. Pollard (1971) was apparently the first to consider the problem of finding the distribution of a life annuity, when both mortality and the discount rates are random. Several authors have studied the same problem since, including Boyle (1976), Waters (1978), Wilkie (1978), Panjer & Bellhouse (1980), Bellhouse & Panjer (1981), Westcott (1981), Dufresne (1989, 1990, 1992), Beekman and Fuelling (1990), Frees (1991), De Schepper *et al.* (1992abc, 1994), Vanneste *et al.* (1994, 1997), Parker (1994a, 1994b), and Hoedemakers *et al.* (2005). The majority of the contributions focused on calculating the moments of the annuity, the goal being to reconstruct or approximate the distribution from its moments; a few actuarial papers dealt with the distribution of annuities with random rates of interest, or with their "comonotonic" approximation. This paper proceeds differently. The distribution of a stochastic life annuity with an exponential lifetime distribution is already known explicitly, in the case where rates of discount are independent over time (more details below). It will then be seen that the same holds if the lifetime has a density given by a combinations of exponentials. Next, because combinations of exponentials are dense in the set of distributions on $\mathbb{R}_+$, any lifetime distribution may be approximated to any degree of precision by a combination of exponentials. Consequently, the distribution of a stochastic life annuity, with a given (arbitrary) lifetime distribution, may be approximated by the distributions

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of stochastic life annuities with lifetimes distributed as combinations of exponentials; moreover, the error of this approximation tends to zero as the approximating lifetime distributions converge to the true lifetime distribution. Each approximating stochastic life annuity distribution is known exactly, and will be identified. Numerical illustrations are given.

Some of the papers cited above assumed that rates of discount form an autoregressive process, probably to reflect the well-known serial dependence of interest rates. In this paper, the discounted values of annuities are not based on an interest rate model, but are instead the amount of money required to fund the annuity, given a particular investment policy. Let U_t be the time-t value of one dollar invested at time 0. This dollar may have been invested in stocks, bonds, and so on. The amount required at time 0 to fund a \$ C cash-flow at time t is thus \$ C/U_t. The amount initially required to fund several cash flows $\{C_j\}$ at times $\{t_j\}$ is then

$$S \;=\; \sum_j \frac{C_j}{U_{t_j}}.$$

This amount is of course random, and is not a "price" in any sense of the word. (No-arbitrage pricing (as described, for instance, in Harrison & Pliska (1981)) does not lead to a unique price in cases where cash flows depend on mortality, as the market is then incomplete.) Notwithstanding this, there is an interest in knowing the probability distribution of S, for instance when comparing investment strategies, or, possibly, when informing insurance contractholders of the likely amounts of their benefit payments, if there is a "variable" component in their contracts.

It will be assumed that the rates of return on the investments chosen are serially independent. This is the usual assumption for stocks, adopted by Black & Scholes (1973), though their "log-normal" model goes back to Osborne (1959); the first appearance of Brownian motion was also in the context of a financial model (Bachelier, 1900), and also featured returns which are independent over time, though it is the asset values themselves (and not their logarithm) which were normally distributed. Bond yields, however, are highly correlated over time. Nevertheless, it can be argued that a managed portfolio of bonds would often have returns which are less correlated over time.

Let $\{W_t; t \geq 0\}$ be a standard Brownian motion, and define

$$U_t \;=\; e^{mt+\sigma W_t}.$$

The process $\{U_t; t \geq 0\}$ is called "geometric Brownian motion." As indicated above, U_t will represent the time-t value of one dollar invested at time 0. The parameters (m, σ) depend on the available securities and on the investment strategy followed. The variable S defined above is then a sum of lognormals, a topic studied from the point of view of Asian options by several authors (see Dufresne (2004) for references).

There is another source of randomness in a stochastic life annuity. The variable T will represent the future lifetime of an insured or pensioner. We will always assume that investment results and survival do not affect one another, or, expressed mathematically, that T and the Brownian motion W are stochastically independent. The amount required at time 0 to fund an annuity of 1 per annum payable continuously while the pensioner is alive is

$$D_T \;=\; \int_0^T U_s^{-1}\, ds \;=\; \int_0^T e^{-ms-\sigma W_s}\, ds. \tag{1.1}$$

This formula corresponds to a level annuity, but note that obvious modifications allow for (1) payments that increase exponentially (say, with inflation), by including an exponential factor e^{yt} in the integral; and (2) guarantee periods, by replacing T with $\max(T, n)$. It does not seem possible to apply our results to discretely paid annuities, though the difference between the discounted values of annuities paid monthly and annuities paid continuously is not very great. Observe that the distribution of T does not have to be continuous (see Sections 3 and 4).

Section 2 recalls some known results about the integral of Brownian motion which are used in the sequel. Section 3 deals with the approximation of distributions on $\mathbb{R}_+$ by combinations of exponentials, and Section 4 gives a proof that the distribution of a stochastic life annuity can be approximated to any degree of precision by a stochastic annuity for a duration T that has a combination of exponentials as density. Section 5 looks at transforms and moments of stochastic life annuities, and shows that the distribution of a stochastic life annuity is in general NOT determined by its moments (however, it may be determined by its reciprocal moments).

Section 6 lists the assumptions made in the numerical examples in the subsequent sections. Section 7 identifies the Jacobi approximations used for the distribution of the future lifetime. Numerical distributions of stochastic life annuities are given in Section 8. Section 9 shows how, in certain cases, it is numerically feasible to use a more direct route to find the distribution of a stochastic life annuity, and one such case is used as a check on the method described in Section 8. Section 10 is about deferred annuities, and Section 11 shows how the results presented may be applied to stochastic annuities-certain.

Notation and vocabulary. The cdf of a probability distribution is its cumulative distribution function, the ccdf is the complement of the cdf (one minus the cdf), and the pdf is its probability density function. An *atom* is a point $x \in \mathbb{R}$ where a measure has positive mass, or, equivalently, where its distribution function has a discontinuity. A sequence of random variables $\{X_n\}$ is said to *converge in distribution* to a random variable X if $\mathsf{P}(X_n \leq x)$ converges to $\mathsf{P}(X \leq x)$ for all x where the distribution function of X is continuous; convergence in distribution is also called *weak convergence* (see Billingsley, 1986). Finally, use will be made of the Pochhammer symbol

$$(a)_0 \;=\; 1, \quad (a)_n \;=\; a(a+1)\cdots(a+n-1), \quad n \;=\; 1,2,\ldots$$

2. Fundamental results on the integral of Brownian motion

This section collects some known results about the integral of Brownian motion. More details can be found in the references; in particular, Yor (2001) contains several papers on these topics, while Dufresne (2005) gives a summary of several of the results, with detailed references.

The expression in equation (1.1) depends on T, m and σ. The scaling property of Brownian motion makes it possible to transform equation (1.1) in such a way that any one of these parameters has a preset value. The connection with Bessel processes (Yor, 1992) makes the choice $\sigma = 2$ more convenient in many developments. We thus adopt Yor's notation

$$A_t^{(\mu)} \;=\; \int_0^t e^{2(\mu s + W_s)}\,ds, \quad t \geq 0, \tag{2.1}$$

where W is a standard Brownian motion. From the scaling property of Brownian motion, the conversion rule between formulas (1.1) and (2.1) is:

$$D_T = \int_0^T e^{-ms-\sigma W_s}\, ds$$

has the same distribution as

$$\frac{4}{\sigma^2} A_t^{(\mu)}, \quad \text{where } t = \frac{\sigma^2 T}{4},\ \mu = -\frac{2m}{\sigma^2}. \tag{2.2}$$

The process $A_t^{(\mu)}$ has been studied in several contexts, notably in relation to Asian options (for instance, Geman & Yor (1993)).

Moments of $A_t^{(\mu)}$. The distribution of $A_t^{(\mu)}$ has all moments (positive and negative) finite, that is,

$$\mathsf{E}(A_t^{(\mu)})^r < \infty \qquad \forall r \in \mathbb{R}.$$

Ramakrishnan (1954) showed that, for $n = 1, 2 \ldots$,

$$\mathsf{E}(A_t^{(\mu)})^n = \sum_{k=0}^{n} b_{n,k} e^{a_k t}, \quad a_k = 2k\mu + 2k^2, \quad b_{n,k} = n! \Big/ \prod_{\substack{j=0 \\ j\neq k}}^{n} (a_j - a_k). \tag{2.3}$$

Proofs of this result may also be found in Dufresne (1989) and in Yor (1992). The conversion rule (2.2) may be used in an obvious way to find corresponding formulas for

$$\mathsf{E}\Big(\int_0^T e^{-ms-\sigma W_s}\, ds\Big)^n, \qquad n = 1, 2 \ldots$$

(see Section 5). Formulas for $\mathsf{E}[(A_t^{(\mu)})^{-r}]$, $r > 0$, can be found in Dufresne (2000).

The distribution of $A_t^{(\mu)}$ ***sampled at an independent exponential time***. The result below first appeared in Yor (1992). Let $S_\lambda \sim$ **Exponential**(λ), that is, let S_λ be a random variable with density

$$f_{S_\lambda}(x) = \lambda e^{-\lambda x} \mathbf{1}_{\{x>0\}}.$$

Suppose, moreover, that S_λ is independent of the Brownian motion W in definition (2.1). A summary of the proof of the following theorem is given in the Appendix.

Theorem 2.1 (Yor). *The distribution of* $A_{S_\lambda}^{(\mu)}$ *has a density given by equation (2.4), and is the same as that of* $B_{1,\alpha}/2G_\beta$*, where* $B_{1,\alpha}$ *and* G_β *are independent,* $B_{1,\alpha} \sim$ **Beta**$(1,\alpha)$*,* $G_\beta \sim$ **Gamma**$(\beta, 1)$*, with*

$$\alpha = \frac{\gamma+\mu}{2}, \qquad \beta = \frac{\gamma-\mu}{2}, \qquad \gamma = \sqrt{2\lambda+\mu^2}.$$

This is equivalent to

$$f_\lambda(u) = \lambda (2u)^{(\mu-\gamma)/2-1} \frac{\Gamma\big(\frac{\mu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\tfrac{\gamma-\mu}{2}+1, \gamma+1; -\tfrac{1}{2u}\big) \mathbf{1}_{\{u>0\}}, \tag{2.4}$$

where ${}_1F_1$ is the confluent hypergeometric function:

$${}_1F_1(a,b;z) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}.$$

The distribution of $A_S^{(\mu)}$ is also known for three other distributions for S, see Yor (2001, pp.104-105).

Laplace transform of the distribution of $A_t^{(\mu)}$. It has been proved that the distribution of $A_t^{(\mu)}$ is not determined by its moments (Hörfelt, 2005); this makes sense intuitively, as the same situation prevails for the lognormal distribution. Moreover, $\mathsf{E}\exp(sA_t^{(\mu)}) = \infty$ for all $s > 0$, as for the lognormal distribution. By contrast, and a little surprisingly, $\mathsf{E}\exp(s/(2A_t^{(\mu)})) < \infty$ if $s < 1$, which implies that the distribution of $1/A_t^{(\mu)}$ is determined by its moments (Dufresne, 2001). The same reference shows formulas for the moments of $1/A_t^{(\mu)}$. A formula is given for the Laplace transform of $A_t^{(\mu)}$ in Yor (2001, p.106).

The distribution of $A_t^{(\mu)}$. Several expressions have been found for the density of the integral of geometric Brownian motion at a fixed time, but none of those expressions is simple. We give only two examples, the reader is referred to Dufresne (2005) for more details. Other expressions for the density are given in De Schepper *et al.* (1992ab) (see the comments in Deelstra & Delbaen (1992)).

The following expression for the density of $A_t^{(\mu)}$ is due to Yor (1992):

$$g(t,x) \;=\; \frac{e^{-\mu^2 t/2}}{x}\int_{-\infty}^{\infty} e^{\mu u-\frac{1}{2x}(1+e^{2u})}\theta_{e^u/x}(t)\,du,$$

where

$$\theta_r(t) \;=\; \frac{re^{\frac{\pi^2}{2t}}}{\sqrt{2\pi^3 t}}\int_0^{\infty} \exp(-y^2/2t)\exp(-r\cosh y)\sinh(y)\sin\left(\tfrac{\pi y}{t}\right)dy.$$

Another expression for the distribution of $A_t^{(\mu)}$ is given in Dufresne (2001). If

$$q(y,t) \;=\; \frac{e^{\frac{\pi^2}{8t}-\frac{y^2}{2t}}}{\pi\sqrt{2t}}\cosh y,$$

then the density of $1/(2A_t^{(\mu)})$ is

$$2^{-\mu}x^{-\frac{\mu+1}{2}}e^{-\mu^2 t/2}\int_{-\infty}^{\infty} e^{-x\cosh^2 y}q(y,t)\cos\left(\tfrac{\pi}{2}\left(\tfrac{y}{t}-\mu\right)\right)H_\mu(\sqrt{x}\sinh y)\,dy. \qquad (2.5)$$

Here H_μ is the Hermite function (Lebedev, 1972, Chapter 10). The above expression reduces to a single integral when $\mu = 0, 1, 2, \ldots$, but is otherwise a double integral.

The distribution of $A_\infty^{(\mu)}$. For any $m, \sigma > 0$ (Dufresne, 1990),

$$\frac{2}{\sigma^2}\left(\int_0^\infty e^{-mt-\sigma W_t}\,dt\right)^{-1} \sim \mathbf{Gamma}\big(\tfrac{2m}{\sigma^2}, 1\big) \qquad \forall\, m, \sigma > 0. \tag{2.6}$$

If $m \le 0$, then

$$\lim_{t\to\infty}\int_0^t e^{-ms-\sigma W_s}\,ds \;=\; \infty \quad \text{a.s.}$$

3. Approximating distributions on $\mathbb{R}_+$ by combinations of exponentials

By "combination of exponentials" we mean a function of the form

$$f(t) \;=\; \sum_{j=1}^n a_j\lambda_j e^{-\lambda_j t}\mathbf{1}_{\{t>0\}}, \tag{3.1}$$

where $\{a_j\}$, $\{\lambda_j\}$ are constants. This function is a probability density function if

$$(1)\quad \sum_{j=1}^n a_j = 1; \quad (2)\quad \lambda_j > 0 \quad \forall\, j; \quad (3)\quad f(x) \;\ge\; 0 \quad \forall\, x \ge 0.$$

Conditions (1) and (2) imply that the function $f(\cdot)$ integrates to one over $\mathbb{R}_+$, but they do not imply (3); for example, consider

$$f(x) \;=\; e^{-x} - 16e^{-2x} + 24e^{-3x} \;=\; e^{-x}[24(e^{-x} - \tfrac{1}{3})^2 - \tfrac{5}{3}].$$

If $a_j > 0$ for all j, then (3.1) is called a "mixture of exponentials." A proof of the following result is given in the Appendix.

Theorem 3.1. *(a) Suppose T is a non-negative random variable. Then there exists a sequence of random variables $\{T_n\}$ each with a pdf given by a combination of exponentials and such that T_n converges in distribution to T.*

(b) If the distribution of T has no atom, then

$$\lim_{n\to\infty}\sup_{0\le t<\infty}|F_T(t) - F_{T_n}(t)| \;=\; 0.$$

A method will now be presented for approximating probability distributions by combinations of exponentials. More details can be found in Dufresne (2006) (two other methods are presented in that paper).

For $\alpha, \beta > -1$, the *shifted Jacobi polynomials* are defined as

$$R_n^{(\alpha,\beta)}(x) \;=\; \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1, \alpha+1; 1-x\right) \;=\; \sum_{j=0}^n \rho_{nj}x^j,$$

where ${}_2F_1$ is the Gauss hypergeometric function and

$$\rho_{nj} \;=\; \frac{(-1)^n(\beta+1)_n(-n)_j(n+\lambda)_j}{(\beta+1)_j n! j!}.$$

The shifted Jacobi polynomials are orthogonal on $[0,1]$, for the weight function

$$w^{(\alpha,\beta)}(x) \;=\; (1-x)^\alpha x^\beta.$$

Based on the properties of the Jacobi polynomials, it is possible to prove that for a wide class of functions $\phi(\cdot)$ defined on $(0,1)$ (including all continuous and bounded functions),

$$\phi(x) \;=\; \sum_{n=0}^{\infty} c_n R_n^{(\alpha,\beta)}(x), \qquad c_n \;=\; \frac{1}{h_n}\int_0^1 \phi(x)(1-x)^\alpha x^\beta R_n^{(\alpha,\beta)}(x)\,dx$$

$$h_n \;=\; \int_0^1 (1-x)^\alpha x^\beta [R_n^{(\alpha,\beta)}(x)]^2\,dx \;=\; \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\lambda)n!\Gamma(n+\lambda)}.$$

This can be applied to probability distributions on $\mathbb{R}_+$ in the following way. Let

$$\overline{F}(t) \;=\; 1-F(t) \;=\; \mathsf{P}(T>t)$$

and define, for some $r>0$,

$$g(x) \;=\; \overline{F}\Big(-\frac{1}{r}\log(x)\Big), \quad 0<x\le 1, \quad g(0) \;=\; 0.$$

(If T represents the time-until-death of a life currently aged x, then $\overline{F}(t) = {}_tp_x$.) This maps the interval $[0,\infty)$ onto $(0,1]$, $t=0$ corresponding to $x=1$, and $t\to\infty$ corresponding to $x\to 0+$; since $\overline{F}(\infty)=0$, we set $g(0)=0$. If α, β, p and $\{b_k\}$ are found such that

$$g(x) \;=\; x^p \sum_{k=0}^{\infty} b_k R_k^{(\alpha,\beta)}(x), \quad 0<x\le 1,$$

then

$$\overline{F}(t) \;=\; e^{-prt}\sum_{k=0}^{\infty} b_k \sum_j \rho_{kj} e^{-jrt}.$$

If $p>0$, a combination of exponentials is obtained when this series is truncated. The constants $\{b_k\}$ can be found by

$$\begin{aligned} b_k \;&=\; \frac{1}{h_k}\int_0^1 x^{-p} g(x) R_k^{(\alpha,\beta)}(x)(1-x)^\alpha x^\beta\,dx \\ &=\; \frac{r}{h_k}\int_0^\infty e^{-(\beta-p+1)rt}(1-e^{-rt})^\alpha R_k^{(\alpha,\beta)}(e^{-rt})\overline{F}(t)\,dt. \end{aligned}$$

This is a combination of terms of the form

$$\int_0^\infty e^{-(\beta-p+j+1)rt}(1-e^{-rt})^\alpha \overline{F}(t)\,dt, \quad j=0,1,\ldots,k. \tag{3.2}$$

If $\alpha = 0, 1, 2\ldots$, then this integral is a combination of values of the Laplace transform of $\overline{F}(\cdot)$; the latter may be expressed in terms of the Laplace transform of the distribution of T:

$$\int_0^\infty e^{-st}\overline{F}(t)\,dt \;=\; -\frac{1}{s}\int_0^\infty \overline{F}(t)\,de^{-st} \;=\; \frac{1}{s}[1-\mathsf{E}e^{-sT}], \quad s>0. \tag{3.3}$$

The following results are consequences of the classical theory of orthogonal polynomials (see Dufresne (2006) for proofs).

Theorem 3.2. *Suppose* $\alpha, \beta > -1$*, that* $\overline{F}(\cdot)$ *is continuous on* $[0,\infty)$*, and that the function*

$$e^{prt}\overline{F}(t)$$

has a finite limit as t *tends to infinity for some* $p \in \mathbb{R}$ *(this is always true when* $p \geq 0$*). Then*

$$\overline{F}(t) \;=\; e^{-prt}\sum_{k=0}^{\infty} b_k R_k^{(\alpha,\beta)}(e^{-rt})$$

for every t *in* $(0,\infty)$*, and the convergence is uniform over every interval* $[a,b]$*, for* $0<a<b<\infty$.

Not all distributions satisfy the condition in Theorem 3.2 for some $p>0$. The next result does not need this assumption.

Theorem 3.3. *Suppose* $\alpha, \beta > -1$ *and that for some* $p \in \mathbb{R}$ *and* $r>0$

$$\int_0^\infty e^{-(\beta+1-2p)rt}(1-e^{-rt})^\alpha \overline{F}(t)^2\,dt \;<\; \infty$$

(this is always true if $p < (\beta+1)/2$*). Then*

$$\lim_{N\to\infty}\int_0^\infty \left[\overline{F}(t) - e^{-prt}\sum_{k=0}^{N} b_k R_k^{(\alpha,\beta)}(e^{-rt})\right]^2 e^{-(\beta+1-2p)rt}(1-e^{-rt})^\alpha\,dt \;=\; 0.$$

Theorem 3.2 gives convergence at every point in $(0,\infty)$, while Theorem 3.3 gives mean square convergence.

The truncated series obtained using this method are not true distribution functions. The function may be smaller than 0 or greater than 1 in places, or it might decrease over some intervals. This is good enough for the purposes of this paper, but the method can be modified in such a way that a true distribution function results. See Sections 3 and 4 of Dufresne (2006).

4. The distribution of a stochastic life annuity

Given that T is independent of the Brownian motion W, the distribution of $A_T^{(\mu)}$ can be found by conditioning on T:

$$\mathsf{P}(A_T^{(\mu)} \le x) \;=\; \int \mathsf{P}(A_t^{(\mu)} \le x)\, dF_T(t). \tag{4.1}$$

However, in light of the known expressions for the density of $A_t^{(\mu)}$ (see Section 2), it appears that formula (4.1) may not be easy to use in numerical applications. In fact, the computation of the density of $A_t^{(\mu)}$ for fixed t is itself a problem, see Dufresne (2004), Ishiyama (2005). The only exceptions are when $\mu = 0, 1, 2, \ldots$, in which case the pdf of $A_t^{(\mu)}$ may be expressed as a single integral (see equation (2.5)); this is illustrated in Section 9 below. The approach suggested in this paper is to approximate the distribution of T by a combination of exponentials, and then apply Theorem 2.1.

The following result says that the error made in approximating the distribution of $A_{T_1}^{(\mu)}$ by $A_{T_2}^{(\mu)}$ is never larger than the error made in approximating T_1 by T_2, which is a very good thing numerically.

Theorem 4.1. *Let $T_1, T_2 \ge 0$ be random variables with distribution functions $F_{T_1}(\cdot), F_{T_2}(\cdot)$, respectively, and that are independent of W. Then*

$$|\mathsf{P}(A_{T_1}^{(\mu)} \le x) - \mathsf{P}(A_{T_2}^{(\mu)} \le x)| \;\le\; \sup_{0\le t<\infty} |F_{T_1}(t) - F_{T_2}(t)| \quad \forall\, x. \tag{4.2}$$

If $\mu < 0$, then

$$|\mathsf{P}(A_{T_1}^{(\mu)} \le x) - \mathsf{P}(A_{T_2}^{(\mu)} \le x)| \;\le\; C(x) \sup_{0\le t<\infty} |F_{T_1}(t) - F_{T_2}(t)|,$$

where

$$C(x) \;=\; \frac{1}{\Gamma(-\mu)} \int_0^{\frac{1}{2x}} y^{-\mu-1} e^{-y}\, dy \;<\; 1.$$

Proof. Integration by parts in formula (4.1), we get, for $x > 0$,

$$\begin{aligned}\mathsf{P}(A_T^{(\mu)} \le x) \;&=\; -\int \mathsf{P}(A_t^{(\mu)} \le x)\, d[1 - F_T(t)] \\ &=\; 1 - \int [1 - F_T(t)]\, d_t \mathsf{P}(A_t^{(\mu)} > x).\end{aligned}$$

The function $t \mapsto \mathsf{P}(A_t^{(\mu)} > x)$ is non-decreasing. Hence

$$\begin{aligned}|\mathsf{P}(A_{T_1}^{(\mu)} \le x) - \mathsf{P}(A_{T_2}^{(\mu)} \le x)| &\le \int |F_{T_1}(t) - F_{T_2}(t)|\, d_t\mathsf{P}(A_t^{(\mu)} > x) \\ &\le \Big(\sup_{0\le t<\infty} |F_{T_1}(t) - F_{T_2}(t)|\Big)\mathsf{P}(\lim_{t\to\infty} A_t^{(\mu)} > x).\end{aligned}$$

This proves inequality (4.2). If $\mu < 0$, we may then apply result (2.6), which says that, if $\Gamma_{-\mu} \sim$ **Gamma**$(-\mu, 1)$,

$$\mathsf{P}(A_\infty^{(\mu)} > x) \;=\; \mathsf{P}(\Gamma_{-\mu} < \tfrac{1}{2x}) \;=\; C(x). \qquad \square$$

Remark. This theorem does not rest on the particular definition of the process $\{A_t^{(\mu)}\}$, as the same proof holds more generally: *if T_1, T_2 are independent of a non-decreasing process $\{X_t\}$ with $X_t = 0$ for $t < 0$, then, for all x,*

$$|\mathsf{P}(X_{T_1} \le x) - \mathsf{P}(X_{T_2} \le x)| \;\le\; \mathsf{P}(\lim_{t\to\infty} X_t > x) \sup_{0\le t<\infty} |F_{T_1}(t) - F_{T_2}(t)|.$$

A question arises, however, in cases where the cdf of $A_T^{(\mu)}$ is approximated by a function which is not a true distribution function. Specifically, the approximation for $F_T(\cdot)$, call it $G(\cdot)$, may be smaller than 0 or greater than 1 in places, or may decrease in places. The approximation for $\mathsf{P}(A_{T_1}^{(\mu)} \le x)$ is then

$$H(x) \;=\; \int \mathsf{P}(A_t^{(\mu)} \le x)\, dG(t).$$

Do the upper bounds in Theorem 4.1 still hold? A review of the proof of Theorem 4.1 shows that the result becomes:

Theorem 4.2. *Suppose $G(\cdot)$ has bounded variation, with $G(t) = 0$ for $t < 0$, and suppose also that*

$$\lim_{t\to\infty} G(t) \;=\; G(\infty)$$

exists. Then

$$\begin{aligned}|\mathsf{P}(A_T^{(\mu)} \le x) - H(x)| &\le |1 - G(\infty)|\mathsf{P}(\lim_{t\to\infty} A_t \le x) + \sup_{0\le t<\infty} |F_T(t) - G(t)|\mathsf{P}(\lim_{t\to\infty} A_t > x) \\ &\le \sup_{0\le t<\infty} |F_T(t) - G(t)| \quad \forall\, x.\end{aligned}$$

An immediate consequence of Theorems 3.1 and 4.1 is that if $T, \{T_n\}$ are independent of W, if $F_T(\cdot)$ is continuous, and if T_n converges in distribution to T, then $A_{T_n}^{(\mu)}$ converges in distribution to $A_T^{(\mu)}$. A more general result is easily obtained.

Theorem 4.3. *If $T, \{T_n\}$ are independent of W, and if T_n converges in distribution to T, then $(e^{W_{T_n}^{(\mu)}}, A_{T_n}^{(\mu)})$ converges in distribution to $(e^{W_T^{(\mu)}}, A_T^{(\mu)})$.*

Proof. Write the joint characteristic function of the pair $(e^{W^{(\mu)}_{T_n}}, A^{(\mu)}_{T_n})$ as

$$\int \mathsf{E}\, e^{is_1 e^{W^{(\mu)}_t} + is_2 A^{(\mu)}_t}\, dF_{T_n}(t), \quad s_1, s_2 \in \mathbb{R}.$$

The function $f(t) = \mathsf{E}\,\exp(is_1 e^{W^{(\mu)}_t} + is_2 A^{(\mu)}_t)$ is continuous and bounded, and thus the weak convergence of $\{T_n\}$ to T implies that $\mathsf{E}\, f(T_n)$ converges to $\mathsf{E}\, f(T)$ (this is a classical result, see for instance Billingsley (1986), p.344, Theorem 25.8). $\square$

Corollary 4.4. *Suppose $\{T_n\}$ converges in distribution to T. Then*

$$\lim_{n\to\infty} \mathsf{P}(A^{(\mu)}_{T_n} \le x) \;=\; \mathsf{P}(A^{(\mu)}_T \le x) \quad \forall x.$$

Proof. Since the distribution of $A^{(\mu)}_T$ is continuous, this is a direct consequence of Theorem 4.3. $\square$

Let $g(t,x)$ be the the density function of $A^{(\mu)}_t$. If $S_\lambda \sim$ **Exponential**(λ) is independent of W, then the density of $A^{(\mu)}_{S_\lambda}$ is (see formula (2.4))

$$f_\lambda(x) \;=\; \int_0^\infty \lambda e^{-\lambda t} g(t,x)\, dt \;=\; \lambda(2x)^{(\mu-\gamma)/2-1} \frac{\Gamma\big(\frac{\mu+\gamma}{2}\big)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\tfrac{\gamma-\mu}{2}+1, \gamma+1; -\tfrac{1}{2x}\big)\mathbf{1}_{\{x>0\}}.$$

Recall that $\gamma = \sqrt{2\lambda+\mu^2}$ in this expression. Now, suppose that T has an arbitrary distribution on $\mathbb{R}_+$, that it is independent of W, and that $F_T(\cdot)$ is approximated by $G(\cdot)$, a combination of exponentials, with

$$\frac{dG(t)}{dt} \;=\; \sum_{j=1}^n a_j \lambda_j e^{-\lambda_j t}, \qquad t > 0. \tag{4.3}$$

The density of $A^{(\mu)}_t$ is then approximated by

$$\sum_{j=1}^n a_j f_{\lambda_j}(x). \tag{4.4}$$

It is possible to find the ccdf of the corresponding approximation for $A^{(\mu)}_T$ in closed form. Theorem 2.1 implies that, if $x > 0$,

$$\begin{aligned}
\mathsf{P}(A^{(\mu)}_{S_\lambda} > x) &= \mathsf{P}\bigg(\frac{B_{1,\alpha}}{G_\beta} > 2x\bigg) \\
&= -\int_0^1 \mathsf{P}\bigg(\frac{u}{G_\beta} > 2x\bigg)\, d_u(1-u)^\alpha \\
&= -\int_0^1 \int_0^{u/2x} \frac{v^{\beta-1}}{\Gamma(\beta)} e^{-v}\, dv\, d_u(1-u)^\alpha \\
&= \frac{1}{(2x)^\beta \Gamma(\beta)} \int_0^1 u^{\beta-1}(1-u)^\alpha e^{-\frac{u}{2x}}\, du \\
&= (2x)^{-\beta} \frac{\Gamma(\alpha+1)}{\Gamma(\gamma+1)}\, {}_1F_1\big(\beta, \gamma+1; -\tfrac{1}{2x}\big).
\end{aligned}$$

We have thus proved the following result.

Theorem 4.5. *If G is as in equation (4.3), then*

$$\int \mathsf{P}(A_t^{(\mu)} > x)\, dG(t) \;=\; \sum_{j=1}^{n} a_j (2x)^{\frac{\mu-\gamma_j}{2}} \frac{\Gamma(\frac{\mu+\gamma_j}{2}+1)}{\Gamma(\gamma_j+1)}\, {}_1F_1\big(\tfrac{\gamma_j-\mu}{2}, \gamma_j+1; -\tfrac{1}{2x}\big), \qquad (4.5)$$

where $\gamma_j = \sqrt{2\lambda_j + \mu^2}$.

5. Moments and Laplace transform of stochastic life annuities

Using results (2.2)-(2.3) and the independence of T and W, we obtain formulas for the moments of stochastic life annuities. Recall that

$$\mu \;=\; -\frac{2m}{\sigma^2}.$$

Theorem 5.1. *If $n \in \mathbb{N}_+$ and $\mathsf{E}\exp[(-km + k^2\sigma^2/2)T] < \infty$ for $k = 1, \ldots, n$, then*

$$\mathsf{E}\, D_T^n \;=\; \left(\frac{4}{\sigma^2}\right)^n \sum_{k=0}^{n} b_{n,k}\, \mathsf{E}\, e^{(-km+k^2\sigma^2/2)T}.$$

Note that $\mathsf{E}\, e^{-\delta T}$ is the actuarial value of an insurance of one unit payable at the moment of death (usually written $\overline{A}_x$), with an instantaneous rate of interest equal to δ. In practical applications it would be finite for any δ (observe that $-km + k^2\sigma^2/2$ may be positive or negative depending on k, m and σ).

The Laplace transform of $A_T^{(\mu)}$ may also be obtained from that of $A_t^{(\mu)}$ by conditioning on T,

$$\mathsf{E}\, e^{-sA_T^{(\mu)}} \;=\; \int \mathsf{E}\, e^{-sA_t^{(\mu)}}\, dF_T(t);$$

this is finite for all $s > 0$.

Theorem 5.2. *(a) If $\mathsf{P}(T > 0) > 0$, then*

$$\mathsf{E} e^{sA_T^{(\mu)}} \;=\; \infty \quad \forall\, s > 0.$$

(b) If the distribution of T has one or more atoms in $(0, \infty)$, then the distribution of $A_T^{(\mu)}$ is not determined by its moments.

(c) If $T \geq \epsilon$ for some $\epsilon > 0$, then

$$\mathsf{E} e^{s/(2A_T^{(\mu)})} < \infty \quad \forall\, s < 1,$$

and the distribution of $1/A_T^{(\mu)}$ *is determined by its moments.*

Proof. (a) Condition on T, and use the result for fixed t (Section 2).

(b) Hörfelt (2005) has shown that the distribution of $A_t^{(\mu)}$ is not determined by its moments, for any fixed $t > 0$. Suppose t_0 is an atom of the distribution of T. Then there are two distinct distributions with the same moments as $A_{t_0}^{(\mu)}$. We can then produce two distinct distributions with the same moments as $A_T^{(\mu)}$, by changing the distribution of $A_T^{(\mu)}$ on the set $\{T = t_0\}$.

(c) The result is obvious if $s \leq 0$, so let $0 < s < 1$. It is known that $\mathsf{E}e^{s/(2A_t^{(\mu)})} < \infty$ for every $t > 0$ (Dufresne, 2001). Since $A_t^{(\mu)}$ is non-decreasing in $t, T \geq \epsilon > 0$ implies

$$\mathsf{E}e^{s/(2A_T^{(\mu)})} \ \leq \ \mathsf{E}e^{s/(2A_\epsilon^{(\mu)})} \ < \ \infty.$$

It is well known that a distribution which has a finite Laplace transform in a neighborhood of the origin is determined by its moments. □

Part (b) above shows that, at least when the distribution of T has one or more atoms, there is no justification in approximating the distribution of the stochastic life annuity based on its moments only, as was suggested by several authors. It is plausible that the same holds for arbitrary distributions for T.

6. Model assumptions

The next sections will use the following mortality assumptions. The lifetimes of the annuitants obey Makeham's law, with

$$\mu_x \ = \ A + Bc^x.$$

The parameters used will be those chosen by Bowers *et al*. (1997, p.78):

$$A = .0007, \quad B = 5 \times 10^{-5}, \quad c = 10^{.04}.$$

For the joint and last survivor example both annuitants have the same future lifetime distribution.

On the financial side, one dollar invested at time 0 grows to

$$e^{mt+\sigma W_t}$$

at time t, where W is standard Brownian motion. This is the same assumption as for the risky asset in the usual Black-Scholes. The benchmark scenario will be: $m = .06, \sigma = .20$.

7. Jacobi approximations of lifetime distributions

The technique for finding an approximation for the future lifetime distribution is described in Section 3. The parameters of the 20-term Jacobi expansion of

$$\overline{F}(t) \ = \ \mathsf{P}(T_{65} > t)$$

[Table 1 about here]

are given in Table 1. The other parameters required (see Section 3) are

$$\alpha \;=\; 0, \quad \beta \;=\; 0, \quad p \;=\; .2, \quad r \;=\; .08.$$

The same procedure was applied to the duration of the last-to-die status $T_{\overline{65:65}}$, and the parameters are also shown in Table 1. The other parameters are

$$\alpha \;=\; 0, \quad \beta \;=\; 0, \quad p \;=\; .1, \quad r \;=\; .055.$$

The precision of an approximation $\hat{F}$ of F was chosen as the sup norm of the difference:

$$\|\hat{F} - F\| \;=\; \sup_{t\geq 0} |\hat{F}(t) - F(t)|.$$

In the case of one life, the estimated precisions are

$$\|F - F_3\| \;=\; 0.082, \quad \|F - F_5\| \;=\; 0.043, \quad \|F - F_{10}\| \;=\; 0.0065$$
$$\|F - F_{20}\| \;=\; 0.00024, \quad \|F - F_{40}\| \;=\; 5.0 \times 10^{-6}.$$

In the case of two lives,

$$\|F - F_3\| \;=\; 0.15, \quad \|F - F_5\| \;=\; 0.082,$$
$$\|F - F_{10}\| \;=\; 0.012 \quad \|F - F_{20}\| \;=\; 0.00030.$$

[Figure 1 about here]

Figure 1 shows some of the low-order approximations of the ccdf of $T_{\overline{65:65}}$; the higher order approximations cannot be distinguished visually from the exact ccdf. Table 2 compares the exact values of $\overline{a}_{65}$ and $\overline{a}_{\overline{65:65}}$ with the values obtained using the 20-term approximations of the distributions of T_{65} and $T_{\overline{65:65}}$.

[Table 2 about here]

Some practical considerations

The numerical examples in this paper uses a Makeham survival function, but in practice the life table will most likely be given by a table of ℓ_x or q_x values at integer ages x. If this is the case, then the first step is to turn this table into a survival function $\overline{F}(t) = \mathsf{P}(T > t)$ for a continuous parameter $t \geq 0$, in order to be able to compute the integrals in expression (3.2). Theorem 4.1 and Corollary 4.4 hold for any variable $T \geq 0$, but, since a combination of exponentials is continuous, the maximum error

$$\sup_{0\leq t<\infty} |\hat{F}(t) - F(t)|$$

does not tend to 0 if T has one or more atoms. This would make the error bound in Theorem 4.2 useless for estimating the error on the distribution of the stochastic annuity. We will thus consider continuous functions $\overline{F}(t)$.

Any interpolation technique is acceptable here, provided the resulting function $\overline{F}(t)$ is a true ccdf, but linear interpolation is particularly convenient, as will now be shown, at least when the parameter α is an integer. Given a retirement age w (an integer), let

$$\overline{F}(t) \;=\; (1-u)_k p_w + u \cdot {}_{k+1}p_w, \quad u \;=\; t-k \in (0,1), \quad k = 0, 1 \ldots.$$

This is equivalent to the uniform distribution of deaths assumption. When α is an integer, then expression (3.2) may be expanded as a combination of integrals of the form

$$\int_0^\infty e^{-\gamma t}\overline{F}(t)\,dt.$$

Under the assumption that $\overline{F}(t)$ is linear between integers, this is

$$\sum_{k=0}^\infty \int_0^1 e^{-\gamma(k+u)}[(1-u)\overline{F}(k) + u\overline{F}(k+1)]\,du$$

$$= \sum_{k=0}^\infty e^{-\gamma k}{}_k p_w \int_0^1 [e^{-\gamma u} + e^{\gamma u}](1-u)\,du - \int_0^1 (1-u)e^{\gamma u}\,du.$$

The second integral in the last expression equals

$$\frac{1}{\gamma^2}(e^\gamma - 1 - \gamma),$$

while the first integral is

$$2\int_0^1 \cosh(\gamma u)(1-u)\,du \;=\; \frac{2}{\gamma}\int_0^1 \sinh(\gamma u)\,du \;=\; \frac{2}{\gamma^2}[\cosh(\gamma) - 1].$$

Hence,

$$\int_0^\infty e^{-\gamma t}\overline{F}(t)\,dt \;=\; \frac{2}{\gamma^2}[\cosh(\gamma)-1]\ddot{a}_w - \frac{1}{\gamma^2}(e^\gamma - 1 - \gamma),$$

where the annuity is computed at rate $i = e^\gamma - 1$. This agrees with the limit as m tends to infinity of formula (5.4.14) in Bowers *et al*. (1997, p.152).

8. Distributions of stochastic life annuities: numerical illustrations

In this section, we present two numerical examples which involve the distribution of stochastic life annuities. In all cases (except when $\sigma = 0$) the distribution of D_T is approximated by formula (4.5), using the 20-term approximations for the distribution of T shown in Section 7.

Example 8.1. Distibution function of D_T for $\boldsymbol{\sigma} = 0$ and $\boldsymbol{\sigma} = .2$. Figure 2 compares the ccdf's of D_T (for a single-life annuity) in the cases where $\sigma = 0$ and $\sigma = .2$. A simple check on the accuracy of the approximation is to calculate the first and second moments of D_T. When $\sigma = 0$, the exact first moment is 9.2709, while with the approximate distribution function is also

9.2709 to five decimal places; the exact and approximate standard deviations are both 3.5725 to five decimal places. When $\sigma = .20$, the exact and approximate first moments are both 10.823; the exact and approximate standard deviations are 7.6716 and 7.6667, respectively. The very small errors in moments are not surprising, since the moments of D_T are combinations of annuity values, and the Jacobi approximation is based on the Laplace transform of $\overline{F}$, which is itself an annuity value when $\alpha = 0$ (see equations (3.2)-(3.3)).

The standard deviation is significantly higher when $\sigma = .20$, which could have been guessed from the thicker tail of the distribution of D_T. For instance, the probability that \$ 15 is sufficient to fund a \$ 1 per annum annuity is .9993 when $\sigma = 0$, while it is only .7981 when $\sigma = .2$; the same computation for an intial amount of \$ 12 yields probabilities of .7339 and .6739, respectively; recall that the expected value of D_T here is 9.2709 when $\sigma = 0$, and 10.823 when $\sigma = .20$.

[Figures 2 and 3 about here]

Example 8.2. In this example we look at the shortfall probability

$$\mathsf{P}(D_T > (1+q)K)$$

for fixed q and K but when σ varies. Figure 3 shows the shortfall probability in the case of single life, if

$$K \;=\; \bar{a}_{65}^{.06} \;=\; 9.2709$$

and $q = 0, .5$. It can be seen that the probability of not having enough funds is more or less constant when $q = 0$, but that it increases significantly as a function of σ when $q = .5$; this means that a larger loading is required for a given shortfall probability when returns have greater volatility.

Figure 4 applies the same reasoning to the joint-and-last-survivor annuity. Here

$$K \;=\; \bar{a}_{\overline{65:65}}^{.06} \;=\; 10.823,$$

and the same conclusions are reached as in the case of a single-life annuity.

[Figure 4 about here]

9. Check against exact distribution when $\mu = 0$

Expression (2.5) for the density of $1/(2A_t^{(0)})$ simplifies to a single integral if $\mu = 0, 1, 2, \ldots$ (this is because the Hermite function H_μ reduces to the usual Hermite polynomial in those cases). The resulting pdf is not really "simple," but it can be used more easily in numerical calculations than when the pdf is a double integral. We will use the case $\mu = 0$ to perform a check on the results of the preceding section. The pdf of $A_t^{(0)}$ is

$$g(t,u) \;=\; \frac{e^{\frac{\pi^2}{8t}}}{\pi u^{\frac{3}{2}}\sqrt{t}} \int_0^\infty e^{-\frac{1}{2u}\cosh^2 y - \frac{y^2}{2t}} \cosh(y)\cos\Big(\frac{\pi y}{2t}\Big)\, dy. \tag{9.1}$$

To obtain the pdf of $A_T^{(0)}$, we will now integrate the pdf of $A_t^{(0)}$ times the pdf of T:

$$f_{A_T^{(0)}}(x) \;=\; \int_0^\infty g(t,x) f_T(t)\, dt. \tag{9.2}$$

A numerical problem arises when t is small, however, because of the oscillatory nature of the function inside the integral (9.1). In order to circumvent this difficulty, the lognormal approximation will be used for t small. It is known (Dufresne, 2004) that the distribution of the appropriately scaled variable $A_t^{(\mu)}$ tends to a lognormal as $t \to 0+$.

Figure 5 compares the pdf's of $A_{.05}^{(0)}$, as computed from equation (9.1), with its lognormal approximation (the latter is obtained by matching the first and second moments). The difference is not very great, and becomes smaller when $t < .05$. It was decided that, in computing integral (9.2), the integral density (9.1) would be replaced by its lognormal approximation for $t < .05$; this should not cause a lot of difference in the numerical values of integral (9.2).

In order to find out whether the last claim is correct, the distribution of $A_T^{(0)}$ was computed using this approximation when T is exponential, and then compared with the known exact distribution (Theorem 2.1). Figure 6 shows both curves (they cannot be distinguished visually). The largest difference between the two pdfs is approximately .046, and is attained around $x = .04$. The absolute difference between the two pdf's quickly decreases as x increases, and is less than 10^{-9} for $.2 \le x < 25$. The error introduced in replacing the exact pdf with the lognormal approximation for $x < .05$ thus appears very small.

Finally, a check was performed assuming T has the Makeham distribution of Section 7 (for a single life). The pdf of $A_T^{(0)}$ was calculated using formula (9.2), and then integrated numerically (using the trapeze rule) to give the ccdf of $A_T^{(0)}$. This was compared with the Jacobi approximation described in Section 8. The maximum absolute difference was estimated (see third column below). The second column gives the precision of the Jacobi approximation of the distribution of T, from Section 7. Theorem 4.2 says that, in theory, the maximum error in approximating the distribution function of $A_T^{(0)}$ is never bigger than the error made in approximating the distribution of T. The table below is in agreement with the theorem, and appears to confirm that the integral formula (9.2) gave more precise results than the 20-term Jacobi approximation (of course at the cost of a greater programming and computing effort).

No. of terms	Precision Makeham ccdf	Max. Difference (Integral-Jacobi)
3	.082	.078
5	.043	.034
10	.0065	.0026
20	.00024	.00024

10. Deferred annuities

Suppose the annuity is due to start being paid in n years, when the annuitant, if alive, is age w. If T is the future lifetime at the current age, $w - n$, then the amount of money required to fund this deferred annuity is

$$D = \frac{\mathbf{1}_{\{T>n\}}}{U_n} \int_0^{\max(T-n,0)} \frac{U_n}{U_{n+s}}\, ds, \quad U_t = e^{mt+\sigma W_t}.$$

Let T' be the random variable representing the future lifetime of the annuitant at age w, given survival up to that age, and let

$$Y \;=\; e^{-mn-\sigma W_n}, \quad D' \;=\; \int_0^{T'} e^{-ms-\sigma W'_s}\,ds,$$

where W' is another Brownian motion, independent of W.

The independent increments property of Brownian motion means that, for $x \geq 0$,

$$\mathsf{P}(D > x) \;=\; \mathsf{P}(T > n)\mathsf{P}(Y \cdot D' > x),$$

Hence, conditionally on survival to age w, the distribution of D is as the product convolution of a lognormal and the distribution of a stochastic life annuity, the latter calculated as in previous sections.

11. Other techniques for approximating the density of $A_t^{(\mu)}$ (t fixed)

The problem of approximating the distribution of $A_t^{(\mu)}$ for fixed t may also be solved by finding a combination of exponentials which approximates the degenerate distribution at t. In particular, it is known (Dufresne, 2006) that if $X_a \sim$ **LogBeta**(a, b, c) (see Appendix for definition) with $c > 0$ fixed, $b = \kappa a$ for some constant $\kappa > 0$, then

$$X_a \xrightarrow{\mathrm{d}} x_0 \;=\; \frac{1}{c}\log\left(1 + \frac{c}{\kappa}\right) \quad \text{as } a \to \infty.$$

Rather than to use a combination of exponentials, it is possible to use a gamma approximation for a degenerate distribution. Another way to look at this is to go back to the derivation of the distribution of $A_{S_\lambda}^{(\mu)}$ in Section 2. The proof of the following theorem is in the Appendix.

Theorem 11.1. *The density of $A_t^{(\mu)}$ is*

$$g(t,x)$$
$$= \lim_{n\to\infty} \frac{(-1)^{n-1}}{(n-1)!}\left(\frac{n}{t}\right)^n \left[\frac{\partial^{n-1}}{\partial\lambda^{n-1}}\left((2u)^{(\mu-\gamma)/2-1}\frac{\Gamma\left(\frac{\mu+\gamma}{2}\right)}{\Gamma(\gamma+1)}\,{}_1F_1\left(\tfrac{\gamma-\mu}{2}+1, \gamma+1; -\tfrac{1}{2x}\right)\right)\right]_{\lambda=\frac{n}{t}}.$$

Conclusion

This paper has proposed an analytic technique for computing the distribution of stochastic life annuities, when the returns are lognormal. The numerical examples presented indicate that the method can be very accurate.

Some possible applications of the results are:

(1) Given particular values for m and σ (which represent the type of investment strategy chosen), find the probability that a pension will be paid in full by the initial amount invested.

(2) Find the effect on the distribution of the discounted annuity of introducing an n-year or a joint-life guarantee.

(3) Among the set of pairs (m, σ) available in the market, find the one which maximise the probability of a certain amount being sufficient to fund a pension.

APPENDIX

Proofs Theorems 2.1 and 11.1

The idea of the proof of Theorem 2.1 is to find a function $h_\lambda(\cdot,\cdot)$ such that for any non-negative functions $f(\cdot)$ and $g(\cdot)$,

$$\mathsf{E}[f(e^{W^{(\mu)}_{S_\lambda}})g(A^{(\mu)}_{S_\lambda})] \;=\; \int_0^\infty\!\!\int_0^\infty f(r)g(u)h_\lambda(r,u)\,dr\,du. \qquad (A.1)$$

(Here $W_t^{(\mu)} = \mu t + W_t$.) Then $h_\lambda(\cdot,\cdot)$ is the joint density function of $(e^{W^{(\mu)}_{S_\lambda}}, A^{(\mu)}_{S_\lambda})$.

By the Cameron-Martin theorem,

$$\mathsf{E}[f(e^{W_t^{(\mu)}})g(A_t^{(\mu)})] = e^{-\mu^2 t/2}\mathsf{E}[e^{\mu W_t}f(e^{W_t})g(A_t)],$$

and so

$$\mathsf{E}[f(e^{W^{(\mu)}_{S_\lambda}})g(A^{(\mu)}_{S_\lambda})] \;=\; \mathsf{E}\int_0^\infty \lambda e^{-\lambda t-\mu^2 t/2}e^{\mu W_t}f(e^{W_t})g(A_t)\,dt.$$

Yor (1992) used a stochastic time change and the properties of Bessel processes to show that, if $\gamma = \sqrt{2\lambda+\mu^2}$,

$$h_\lambda(r,u) \;=\; \frac{\lambda}{u}r^{\mu-1}e^{-(1+r^2)/2u}I_\gamma(\tfrac{r}{u})\mathbf{1}_{\{r,u>0\}}, \qquad (A.2)$$

where $I_\cdot(\cdot)$ is the modified Bessel function of the first kind:

$$I_p(z) \;=\; \sum_{n=0}^\infty \frac{(z/2)^{p+2n}}{n!\Gamma(n+p+1)}.$$

The density of $A^{(\mu)}_{S_\lambda}$ may be found by integrating out r in formula (A.2); this yields formula (2.4).

Now, turn to Theorem 11.1. Suppose $S_{n,\lambda} \sim \mathbf{Gamma}(n,\lambda)$; then $S_{n,\lambda}$ converges in distribution to the degenerate distribution at t if $n\to\infty$ and $n/\lambda \to t$. Replacing S_λ with $S_{n,\lambda}$ in equation (A.1), we get

$$\begin{aligned}\mathsf{E}[f(e^{W^{(\mu)}_{S_{n,\lambda}}})g(A^{(\mu)}_{S_{n,\lambda}})] \;&=\; \mathsf{E}\int_0^\infty \frac{\lambda^n t^{n-1}}{\Gamma(n)}e^{-\lambda t-\mu^2 t/2}e^{\mu W_t}f(e^{W_t})g(A_t)\,dt\\ &=\; \frac{(-1)^{n-1}\lambda^n}{(n-1)!}\frac{\partial^{n-1}}{\partial\lambda^{n-1}}\mathsf{E}\int_0^\infty e^{-\lambda t-\mu^2 t/2}e^{\mu W_t}f(e^{W_t})g(A_t)\,dt.\end{aligned}$$

Then the joint density function of $(e^{W^{(\mu)}_{S_{n,\lambda}}}, A^{(\mu)}_{S_{n,\lambda}})$ is

$$h_{n,\lambda}(r,u) \;=\; \frac{1}{u}r^{\mu-1}e^{-(1+r^2)/2u}\frac{(-1)^{n-1}\lambda^n}{(n-1)!}\frac{\partial^{n-1}}{\partial\lambda^{n-1}}I_\gamma(\tfrac{r}{u})\mathbf{1}_{\{r,u>0\}}$$

and the density of $A^{(\mu)}_{S_{n,\lambda}}$ is

$$f_{n,\lambda}(u) \;=\; \frac{(-1)^{n-1}\lambda^n}{(n-1)!}\frac{\partial^{n-1}}{\partial\lambda^{n-1}}\left[(2u)^{(\mu-\gamma)/2-1}\frac{\Gamma\left(\frac{\mu+\gamma}{2}\right)}{\Gamma(\gamma+1)}\,{}_1F_1\left(\tfrac{\gamma-\mu}{2}+1,\gamma+1;-\tfrac{1}{2u}\right)\right].$$

Letting $n \to \infty$ we get Theorem 11.1. This result may be seen as an application of a classical inversion theorem for Laplace transforms (see Feller, 1971, p.233).

Proof of Theorem 3.1

In the proof we use the logbeta distribution, which we now define.

Definition. *For parameters $a, b, c > 0$, the **logbeta distribution** has density*

$$f_{a,b,c}(x) \;=\; \frac{c\,\Gamma(\frac{b}{c}+a)}{\Gamma(\frac{b}{c})\Gamma(a)}e^{-bx}(1-e^{-cx})^{a-1}\mathbf{1}_{\{x>0\}}.$$

This law is denoted **LogBeta**(a,b,c).

The name chosen comes from the fact that

$$X \sim \mathbf{LogBeta}(a,b,c) \quad\Leftrightarrow\quad e^{-cX} \sim \mathbf{Beta}(b/c,a).$$

It can be seen that when $a = 1, 2, \ldots$, the pdf of the logbeta distribution is a combination of exponentials.

(a) A distribution function $F(\cdot)$ can be approximated to any degree of precision (at all points t) by some discrete distribution. For instance, one may define a new distribution by first giving it the same atoms as $F(\cdot)$, and then, between those atoms, making it a step function close to $F(\cdot)$. The problem then reduces to approximating discrete distributions that have a finite number of possible values; the latter are convex combinations of degenerate distributions (that is, that take only one value). Part (a) will then be proved if we find how to approximate a degenerate distribution at a point $t_0 \geq 0$ by combinations of exponentials. It is known (Dufresne, 2006) that if $X \sim$ **LogBeta**(a,b,c), $c > 0$ is fixed, $b = \kappa a$ for a fixed constant $\kappa > 0$, then

$$X \xrightarrow{\mathrm{d}} x_0 \;=\; \frac{1}{c}\log\left(1+\frac{c}{\kappa}\right) \quad \text{as } a \to \infty.$$

Rescaling as necessary, and letting $a \in \mathbb{N}$ tend to infinity, we see that there is indeed a sequence of combinations of exponentials which converges in distribution to any constant in $\mathbb{R}_+$.

(b) Suppose $F_T(\cdot)$ is continuous, and let $\{T_n\}$ be a sequence of random variables converging to T in distribution. Let $\epsilon > 0$, and find points $t_1 < \cdots < t_m$ such that

$$F_T(t_1) \ < \ \epsilon, \quad F_T(t_{j+1}) - F_T(t_j) \ < \ \epsilon \quad \forall\, j = 1, \ldots, m-1, \quad F_T(t_m) \ > \ 1 - \epsilon.$$

Then there exists n_0 such that

$$|F_{T_n}(t_j) - F_T(t_j)| \ < \ \epsilon, \quad j = 1, \ldots, m, \quad \forall\, n \geq n_0.$$

This implies

$$|F_{T_n}(t) - F_T(t)| \ \leq \ 2\epsilon \quad \forall t \in \mathbb{R}. \qquad \square$$

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Figure 1. Section 7, Last−to−die Lifetime Distribution

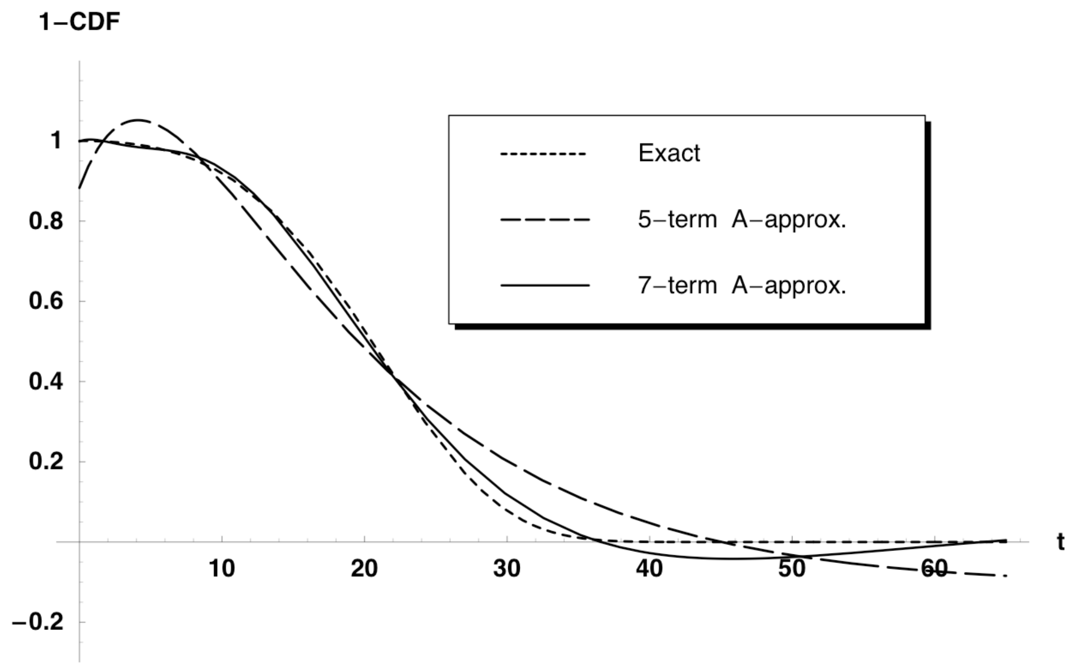

Figure 2. Example 8.1, Single−Life Annuity

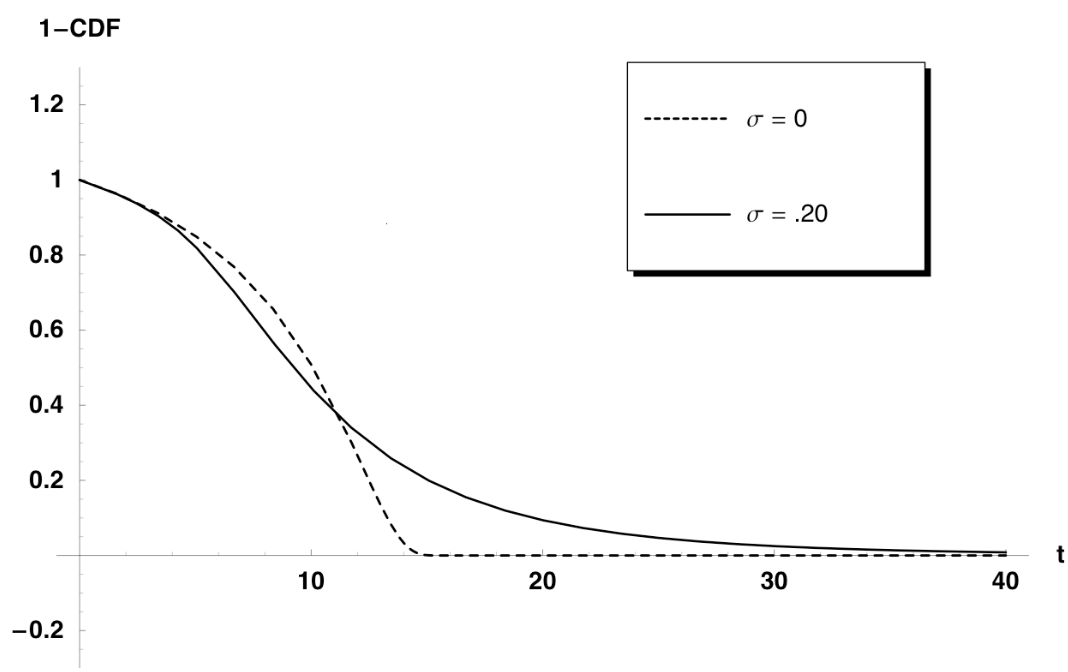

Figure 3. Example 8.2, Single–Life Annuity

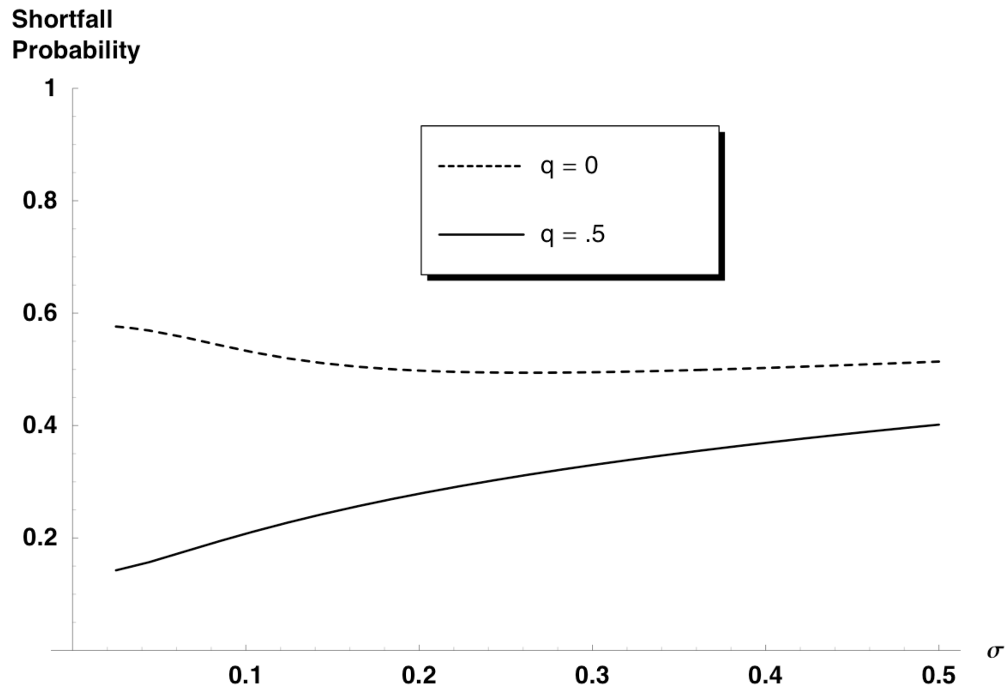

Figure 4. Example 8.2, Joint–Life Annuity

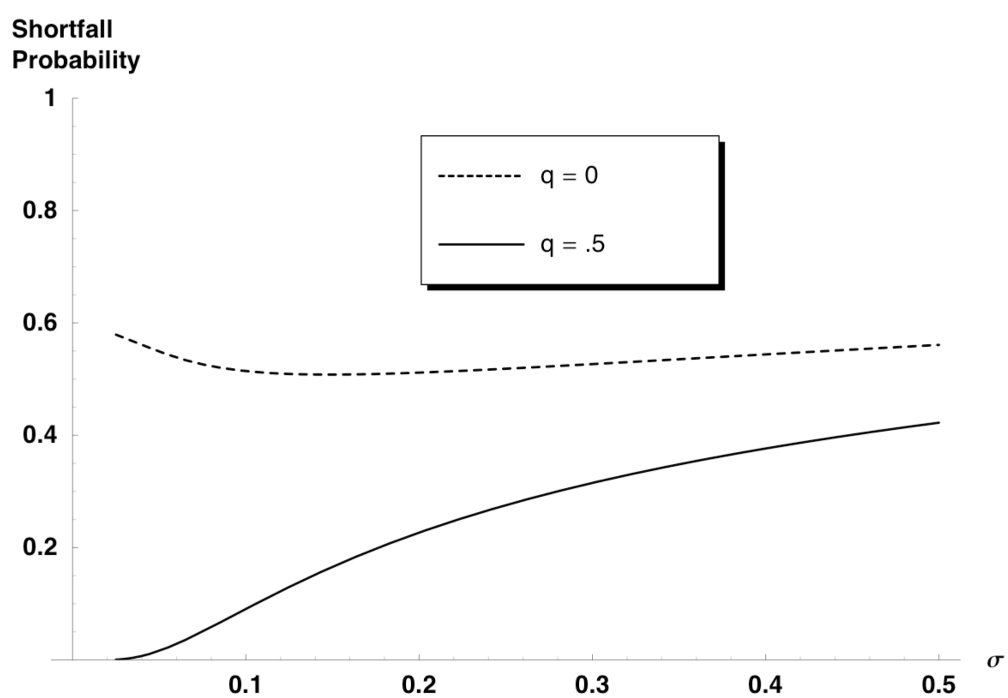

Figure 5. Lognormal vs exact PDFs

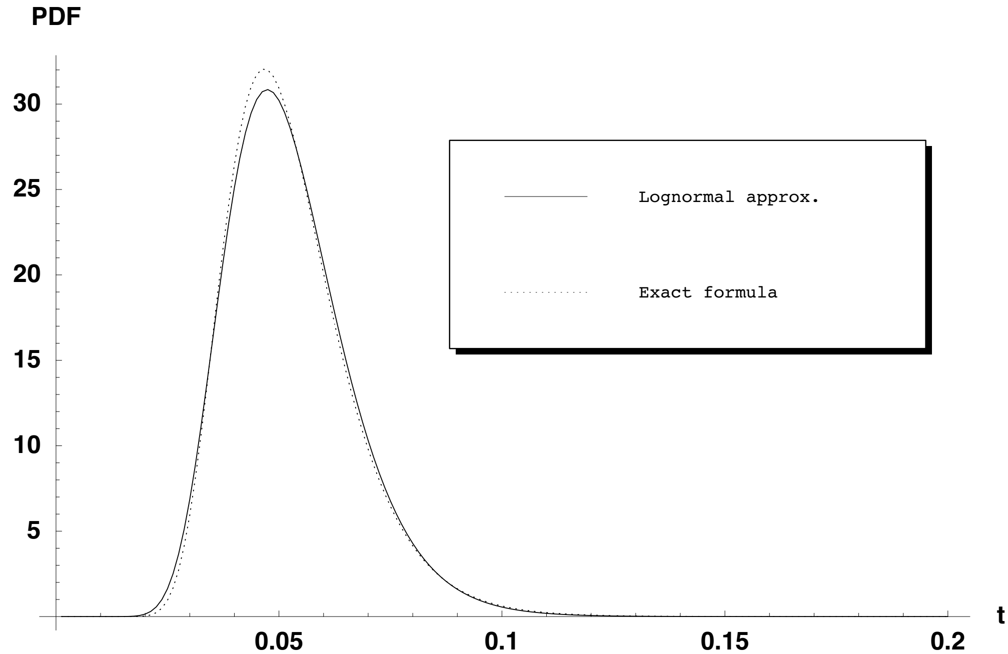

Figure 6. PDF of A_T when T is exponential