Magnetic Fields and Chaos in Coupled Atom-Cavity Systems

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This thesis explores the manifestation of many-body quantum physics in coupled atom-cavity systems, through the lens of the Jaynes-Cummings-Hubbard model. This model captures the behaviour of a strongly interacting photon system, with the entailing complex many-body quantum states. In particular, this work explores two distinct, but related, topics in many-body quantum mechanics.

The first part of the work is concerned with a number of phenomena arising from the introduction of (synthetic) magnetic fields to the Jaynes-Cummings-Hubbard model. In this context, the Mott-superfluid phase transition is suppressed by strong magnetic fields, and vortices are observed in the superfluids regime.

Also studied is the regime of high synthetic magnetic field to particle density, where the fractional quantum Hall effect manifests. Laughlin-like states are indicated by a number of metrics. Pfaffian like states are also show to exist when a three-body interaction is induced by the inclusion of a three level atom in the photonic cavities.

In the latter part of the thesis, the relationship between classical and quantum chaos is explored through the introduction of a periodically perturbing force to a pair of coupled cavities. Demonstrated in this system are such chaotic behaviours such as localization, partially chaotic phase space, dynamic localization and dynamic tunneling.
• This thesis comprises only my original work towards the PhD.

• Due acknowledgement has been made in the text to all other material used.

• The thesis is less than 100,000 words in length, exclusive of tables, bibliographies and appendices.

Andrew L.C. Hayward
The work in this thesis is based on a number of published articles, original work and collaborations. The contributions for each chapter are detailed below. All calculations and figures are the author’s own work.

- Chapter 1 is an introduction to the work in this thesis and a review of the literature.
- Chapter 2 reviews the current state of cavity QED as it pertains to quantum emulation, and introduces the relevant formalism and physics for the work in this thesis.
- Chapter 3 covers the introduction of synthetic magnetic fields into a JCH system.
- Chapter 4 studies the Mott-superfluid phase transition in the JHC, and the effect of magnetic fields on this transition. Furthermore, it looks into the behaviour of vortices formed in the superfluid regime.
- Chapter 5 studies the manifestation of fractional quantum Hall effect physics in the JCH model with a magnetic field.
- Chapter 6 shows how the substitution of a three level atom into the Jaynes Cummings model can lead to three-body interactions. This leads to the presence of the exotic Pfaffian state in the JHC system when the appropriate magnetic field is present.
- Chapter 7 introduces the background physics required for the study in chapter 8.
- Chapter 8 studies the manifestation of chaos a pair of coupled JC cavities. The chaotic behaviour is present in both the quantum model, and in a semi-classical model.
- Chapter 9 Summarises the work in this thesis, and the outlook for future work.


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Symbols and abbreviations used in this thesis.

\[ [\hat{A}, \hat{B}] \] Commutator of quantum mechanical operators \( \hat{A} \) and \( \hat{B} \). I.e. \( \hat{A}\hat{B} - \hat{B}\hat{A} \)

\( \dot{a} \) Time derivative: \( \frac{da}{dt} \)

\( \mathbf{x}, \mathbf{p}, \mathbf{L} \) bold symbols denote spacial vectors

\( \theta, \varphi, \phi, \varphi \) Symbols used for angles and phases

\( U(t) \) Propagator from time \( t = 0 \) to time \( t \)

\( U(T, t_0) \) Propagator from time \( t_0 \) to time \( T \)

\( z \) a complex number \( z = x + iy \)

BEC Bose-Einstein Condensate

BH Bose-Hubbard

EIT electronically induced transparency

EM Electro-magnetic

HC Hard core

JC Jaynes-Cummings

JCH Jaynes-Cummings-Hubbard

QED Quantum electrodynamics

RWA Rotating wave approximation

TLS Two level system
A problem faced by physicists in trying to understand complex states of matter is the computational intractability of large interacting quantum systems [3]. Where in a classical system the computational complexity increases linearly with the number of particles in the system, a quantum mechanical system grows exponentially. This places severe limitations on our ability to both comprehend, and engineer quantum systems. Notable examples where this computational roadblock hinders progress are in chemistry [4], biology [5,6], materials science, and solid state physics.

If the suspicions of computer scientists are correct, then there is a fundamental difference between the kind of computation done by classical and quantum systems. The implication being that there is an intrinsic limitation to using classical computing to simulate quantum systems.

In 1982 Feynman [7] conjectured that to effectively understand quantum systems, we would have to harness the complexity of quantum mechanics itself. That is, only by using a medium that was physically quantum mechanically itself would one faithfully simulate quantum mechanics.

Since then, efforts have been made along several fronts to realise such a quantum simulator. These efforts fall into two distinct categories. The first, one might call quantum computers, or digital quantum simulators, seek to recreate the success of
CHAPTER 1. INTRODUCTION

modern computers through the creation and manipulation of quantum bits (qubits). These efforts have strong theoretical grounding, in that a quantum computing is universal and physically realisable. Continual progress is being made towards the construction of larger quantum computers.

The second kind of quantum simulator, which one might call analog quantum simulation, or sometimes quantum emulation. Here, one creates a correspondence between two quantum systems. One of these systems is an object that we wish to understand. The other a system that we have control over, and can measure and manipulate at will. By creating states in the second system, evolving and observing results one can infer properties of the first system via the correspondence\(^1\).

It is analog quantum computing that is the jumping off point for this thesis. In particular, it is shown how a number of phenomena from traditional condensed matter can be mapped onto a system of coupled photonic cavities. The superior capacity for measurement and control in photonic systems opens the potential for understanding a range of effects that have hitherto been inaccessible to investigation.

This thesis studies the manifestation of a number of quantum phenomena that arise in bespoke atom-cavity systems. The physics of a single atom in a cavity have been investigated since the earliest days of quantum theory [9–12], and studied intensely in the during the quantum atom-optics explosion of the 1980’s [13–18]. The developments in this field were recognized in the 2012 Nobel Prize\(^2\).

Although a venerable field, cavity quantum electrodynamics (QED) is still an area of intense research owing to continual improvements in fabrication techniques, and to the current interest in quantum computing and emulation. Indeed, atom-cavity systems offer perhaps the most pristine environments in which to explore fundamental quantum phenomena that are only now coming to light with developments in quantum information science [20–23]. These developments speak not only to the requirements in

\(^1\)Buluta and Nori [8] provide a comprehensive list of the kind of correspondences that have been investigated.

\(^2\)The 2012 Nobel prize in Physics was awarded to Wineland and Haroche ‘for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems’ [19]
building a functioning quantum computer [24–27], but also to the properties of solid state materials [28–30], phase transitions [31], lattice gauge theory [32], and to the general underlying structure in complex quantum systems [33].

The Jaynes-Cummings (JC) model is the canonical model for atom-light interactions, and the starting point for all the models studied in the thesis. The JC interaction induces photon-photon interactions, lifting the behaviour of the light from simple linear dynamics\(^3\) to the rich realm of strongly correlated many-body quantum systems.

The major work in this thesis concerns the many-body physics of JC systems coupled together. This behavior is described by the Jaynes-Cummings-Hubbard (JCH) model [31, 36–39], introduced in section 2.3. Quantum systems with highly correlated states exhibit an exponential growth of Hilbert space with the number of particles, making the study of arbitrary states of even modest systems computationally intractable. This problem has motivated efforts in the field of quantum emulation [8]. A quantum emulator is designed to replicate the physics of some target system. Such emulators require scalable and convenient state preparation and measurement, and control over single and many-body interactions. Proposals for emulation platforms include ultra cold-atoms, superconductors, and superfluids [40–42]. There has been significant progress towards realising this goal [43–47].

Decoherence is the fundamental difficulty encountered in attempting manipulation of quantum systems; quantum states which interact strongly with their environment. Electromagnetic waves, which are more weakly interacting, can remain coherent for significantly longer periods. Indeed, the presence of polarization in the cosmic microwave background [48] implies that photons can stay coherent for at least billions of years. While this makes photons a great storage and transmission medium for quantum information, this very independence from the environment can make the task of extracting or manipulating the information challenging.

A photonic cavity primarily acts as a resonator. The quality of the resonator is

---

\(^3\)Even without interactions, many-photon states can in fact be extremely complicated. Indeed, the general task of describing the distribution of many photons falls into the complexity class \#P [34], and linear optical systems with adaptive measurements can be used for universal quantum computation [35].
defined primarily through a single factor, \( Q \). The quality factor, \( Q = \frac{\omega_0}{\gamma} \), quantifies the photonic confinement. Here, \( \omega_0 \) is the resonant frequency of the cavity, and \( \gamma \) is the dissipation rate. Longer storage times allow finer control and manipulation of quantum states stored within the cavity.

The design of photonic cavities with large \( Q \)'s is progressively improving. For example, photonic bandgap materials [49] allow for the construction cavities modes that lie in the band gap. Table 1.1 lists some common photonic cavities used in recent experiments, along with their characteristic properties.

Introducing a atom into a cavity enhances the atom-light interaction; the Purcell effect [56]. The atom-light interaction leads to a non-linearity in the energy levels of the system. This non-linearity leads to effects such as photon blockade [43, 57], and electronically induced transparency (EIT) [58]. When the atom light coupling is strong, such that the Rabi frequency is greater than the dissipation rate, \( \gamma \), the system can be described by the Jaynes-Cummings Hamiltonian [12].

Interactions with other systems, such as atoms, admit a quantum mechanical description when the electromagnetic fields are sufficiently small, and hence the broad field is given the name Cavity quantum electrodynamics, or Cavity QED. High quality cavities are the basic building blocks from which the systems considered in this work are constructed.

Developments in the fabrication of electromagnetic cavities with very high confinement have drastically improved the situation. By confining light to a small volume for long times, strong interactions between photons can be orchestrated via the presence of

<table>
<thead>
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<th>Cavity</th>
<th>( \omega_0 ) (Hz)</th>
<th>( \gamma ) (Hz)</th>
<th>( Q ) (unitless)</th>
</tr>
</thead>
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<tr>
<td>Fabry-Perot [50]</td>
<td>(1.9 \times 10^{14})</td>
<td>(1.9 \times 10^8)</td>
<td>(1.0 \times 10^6)</td>
</tr>
<tr>
<td>Microsphere [51]</td>
<td>(3.5 \times 10^{14})</td>
<td>(2.3 \times 10^8)</td>
<td>(1.5 \times 10^6)</td>
</tr>
<tr>
<td>Microtoroid [52]</td>
<td>(1.9 \times 10^{14})</td>
<td>(1.9 \times 10^6)</td>
<td>(1.0 \times 10^8)</td>
</tr>
<tr>
<td>Photonic band gap [53]</td>
<td>(3.5 \times 10^{14})</td>
<td>(8.8 \times 10^9)</td>
<td>(4.0 \times 10^4)</td>
</tr>
<tr>
<td>Coplanar waveguide resonator [54]</td>
<td>(5.4 \times 10^9)</td>
<td>(5.4 \times 10^1)</td>
<td>(1.0 \times 10^8)</td>
</tr>
<tr>
<td>LC resonator [55]</td>
<td>(8.1 \times 10^9)</td>
<td>(8.1 \times 10^6)</td>
<td>(1.0 \times 10^3)</td>
</tr>
</tbody>
</table>

Table 1.1: Characteristics of some cavities from recent experiments.
electronic systems that interact non-linearly with light. This has lead to the proposals of a wealth of applications for electromagnetic cavities for use in a wide variety of purposes in quantum computing, simulation and exploring the fundamentals of quantum mechanics.

Although originally developed for describing real atoms in photonic cavities, the Jaynes Cummings Hamiltonian is quite general, and has been employed in a variety of environments. That is, any two level system (TLS) weakly, and non-diagonally\(^4\), coupled to a Bosonic field mode. Desirable properties for quantum applications are a strong atom-light coupling, large cavity Q, long coherence times, \(T_2\), and tunablity of the system. In practice, there is usually a trade off between these quantities for any particular system.

Atomic systems described by cavity QED can be broadly split into two categories\(^6\): natural atoms, including neutral atoms and ions, and artificial atoms, such as superconducting circuits, quantum dots, or spins in solids.

Many atomic species have been found that posses accurately quantified energy structures with extremely long lifetimes. Typically, neutral atoms can have \(T_1\) on the order of seconds\(^6\) and \(T_2 \approx 40\) ms. Such systems are the playground of quantum atom-optics, and most quantum phenomena have been demonstrated in this context. Charge neutrality also means that the atom only interacts weakly with the environment when extraneous electromagnetic fields are precluded from the atom’s location. Thus, in the context of an intra-cavity atom, the atom-cavity system is extremely well defined by the model presented in Eq. (2.15), and consequently the JC Hamiltonian in the appropriate circumstance. Manipulation of atomic levels and positions though lasers and magnetic fields allows experimentalists to engineer precisely specified quantum environments. Neutral atoms typically have energy spacing in the optical regime (\(\approx 10^{14}\) Hz). Neutral atoms can be coupled to a wide range of photonic cavities, such as micro-toroids\(^6\) and photonic crystals\(^7\).

Superconducting qubits exploit the macroscopic quantum effects present in superconducting materials, with the non-linearities arising from Josephson junctions to create

\(^4\)A diagonally coupled TLS, such as is found in opto-mechanical systems\(^5\), is not described by the JC Hamiltonian, although such terms do appear in atom-cavity systems when considering dynamics beyond the RWA (see Eq. (2.30)).
CHAPTER 1. INTRODUCTION

Figure 1.1: A transmon qubit coupled to a transmission line. Fink et al. *Nature* **454**, 315-318 (17 July 2008) [1].

effective two-level systems that mimic those found in atoms. These superconducting qubits come in several forms; Flux [66], and charge [67, 68] qubits utilize respective quantized flux and charge. In the superconducting circuits that form these qubits, charge and flux are conjugate variables. Other qubits, such as the transmon [69, 70], lie somewhere between these two.

These qubits are of particular importance to the field of cavity QED since their macroscopic size allows precise control over the system. Furthermore, this technology is becoming increasingly robust, with an exponential growth [60] in the coherence times as researchers continually find new ways to isolate these circuits from their environment. This growth has been spurred by improvements in fabrication [71, 72], and optimizing geometries [73, 74].

Systems of coupled JC cavities have been suggested for a diverse range of optical applications such as an optical analog for the Josephson junction [75] and Q-switching [76]. Networks of JC systems have also been predicted to exhibit phase transitions [31, 77]. The JCH has been studied analytically using strong-coupling theory [78–80], slave-Boson [81], one-electron\(^5\) approximation [82], and mean field theory [31, 83]. Numerical approaches

\(^5\)Really one-polariton.
include Monte Carlo [84], exact diagonalization [85], and density matrix renormalization group [86].

Improvements in the realization of photonic cavities in the lab have made possible exploration of Jaynes-Cummings systems [43, 87, 88] in the strong coupling regime. Experimental realizations of a JCH system have been proposed for a number of frameworks. Cavities in photonic crystals are making significant progress [45]. Photonic crystals, particularly in photonic band-gap structures, can offer very high Q values, and small mode volumes. However, there is a fundamental limitation to the achievable volumes that limit the strengths of the atom-cavity coupling. Nevertheless proposals for cold atoms coupled to photonic crystals [89] have shown potential to achieve strong interactions.

Circuit QED offers the most promising framework for realizing JCH phenomena, owing mostly to the high atom-cavity interactions achievable, and long lifetimes. Systems of coupled qubits have been achieved [90]. A current implementation of interest in circuit QED, where a superconducting optical resonator is capacitively coupled to a Cooper-pair box (Fig. 1.1). This is equivalent to a single cavity mode of the EM field coupling to a two level atom. The advantage of circuit QED is that coherence times and atom-field coupling much greater than that can be achieved with visible and near infra-red systems. This makes circuit QED a potential medium for quantum computing, and already has been used to implement an 2 qubit Shor’s algorithm [91].

Recently, Toyoda [47] has implemented a two cavity JCH system with trapped ions in the scheme proposed by Ivanov [92] and was able to detect signatures of the Superfluid-Mott transition in the cavity excitation number variances.

The work in this thesis falls into two categories. The first section deals with the effects of synthetic magnetic fields in coupled atom-cavity systems.

The problem of studying magnetic phenomena in condensed matter systems that do not couple to the magnetic field has been a focus in the condensed community for a long time [93–95]. From the study of gauge symmetry one can see that the effect of a magnetic field is to break time-reversal symmetry in fairly generic way. This opens up a path to introducing synthetic magnetic fields into systems, such as BEC’s, by explicitly
CHAPTER 1. INTRODUCTION

breaking the time-reversal symmetry [96]. This was first attempted through rotation of the BEC, where, once one moves to a non-rotating frame, the rotation appears in the Hamiltonian describing the system as a perpendicular magnetic field\(^6\). This approach was fruitful, yielding effects such as vortices [97, 98] and vortex lattices [99, 100] and vortex liquids [101].

Introducing synthetic magnetic fields though rotation or stirring is fundamentally limited in the realizable strengths of field, due to the centrifugal force. This limitation precludes the study of some phenomena which are of great interest. In particular, achieving the physics of the fractional quantum Hall effect is difficult, since this requires very high fields.

To overcome this limitation, a number of schemes were proposed that broke time-reversal symmetry though various mechanisms, such as though periodically driven potentials [103] and multi-component BEC’s [104–107].

The Sagnac effect [108,109] describes the effect of rotation on light. This is equivalent to the effect of rotation or stirring on a BEC. However, the magnitude of the effective magnetic field is far too tiny to observe any quantum Hall effect like phenomena. Therefore, researchers have taken alternative routes to achieving these effects in photonic systems.

The introduction of synthetic magnetic fields to a coupled cavity array [28,110–112] allows for the investigation of some fundamental phenomena in the context of correlated states of light.

Chapter 3 looks at the introduction of magnetic fields into two dimensional JCH lattices. Inducing magnetic fields requires the breaking of time reversal symmetry in some manner. This chapter looks in detail at how photon assisted tunneling (PAT) can be adapted to the JCH model. In PAT, one introduces an explicitly time dependent variation of the cavity resonant frequency. Explicit comparisons with an effective Hamiltonian show that this method can be used to induce the desired (synthetic) magnetic fields.

Chapter 4 studies the effect of a magnetic field in the JCH system in the context of the superfluid-Mott transition. It has been shown that in the case of Bose-Hubbard systems

\(^6\)See Fetter [102] for a comprehensive review of the physics of rotating BEC’s
that the Mott-superfluid phase transition is modified by the presence of a magnetic field \[113–115\]. This analysis is adapted to the JCH model a similar suppression of the phase transition is found.

In the superfluid regime of the phase diagram we look at behaviour of vortices, which form when superfluid is stirred. Using a mean field model for the states of light we find the formation of Abrikosov vortices when the JC induced non-linearity is sufficiently large.

Moving from the mean field picture to the lower density regime, in chapter 5 we look at the emergence of fraction quantum Hall like states in the JCH model. By studying the wavefunction and energy spectrum of the low lying states we demonstrate evidence for Laughlin-like states. This is confirmed through the computation of topological invariants that distinguish fraction quantum Hall states.

In Chapter 6 we extend the JCH model to include a 3-level intra-cavity atom. Appropriately tuned, this system can be used to induce an effective 3-body interaction for the polaritons in the system. With the three-body interaction, we show that Pfaffian like states \[116\] can be found in the groundstates of this modified JCH lattice. These states possess interesting characteristics, such as non-abelian excitations \[117\]. Such excitations might be used for quantum computing \[118\].

The later part of the thesis concerns the use of JCH systems to study the onset of quantum chaos. There is a rich history of using atom-light cavity systems to investigate chaotic phenomena \[57,119–124\]. Chapter 7 provides some background on chaos, quantum chaos, and the tension between the two.

Chapter 8 introduces a system of coupled atom cavities that exhibits chaotic behaviour. Here, we consider two Jaynes-Cummings cavities that are periodically coupled together via a Hubbard like coupling. Each cavity by it’s own is totally well behaved. However, the coupling disturbs the regular Rabi oscillations of the individual cavities and leads to cavity if sufficiently strong.

The investigation of this model begins by considering a semi-classical version of the Jaynes-Cummings model. By considering a system with classical dynamics, one can find
CHAPTER 1. INTRODUCTION

chaotic phenomena. In the context of quantum chaos we find the presence of dynamically localized states, which have exponentially suppressed dispersion. We also observe chaos induced tunneling between these localized states, a process forbidden in classical chaos.
Review of Cavity Quantum Electrodynamics

2.1 Electromagnetic Cavities

An optical resonator confines electromagnetic modes to a spatial region. The characteristic properties of an optical resonator are the cavity quality, $Q_\omega$, which quantifies the lifetime of the cavity mode, and the mode volume, $V_\omega$, which quantifies the spacial extent of the mode.

The canonical example is the Fabry-Pérot cavity, schematically shown in Fig. 2.1. A Fabry-Pérot cavity consists of a pair of opposing flat mirrors, which trap the electromagnetic field. This localization constrains the electromagnetic-field to certain standing wave modes in the cavity, with a discrete eigenmode spectrum. For the Fabry-Pérot cavity, these modes are evenly separated, with the frequency of the $n$th mode $\omega_n = \frac{2\pi(n+1)c}{2L}$, where $L$ is the length of the cavity and $c$ is the speed of light.

Each cavity mode has two characteristic time scales: the round-trip time, $t_\omega = \frac{2\omega_0 L}{c}$, and the cavity confinement time $t_c = 1/\gamma_\omega$, where $\gamma_\omega$ is the dissipation rate of the cavity. An important parameter for classifying cavities is the cavity quality factor,

$$Q_\omega = \frac{t_c}{t_\omega} = \frac{\omega_0}{2\gamma_\omega}.$$  \hspace{1cm} (2.1)

$Q_\omega$ can be interpreted as the average number of round trips that light makes in cavity
CHAPTER 2. REVIEW OF CAVITY QUANTUM ELECTRODYNAMICS

Figure 2.1: Schematic of a Fabry-Pérot cavity, with the first two modes shown, with frequencies $\omega_1$ and $\omega_2$. $\gamma_\omega$ is the dissipation rate of the cavity, i.e. the inverse of the confinement time.

before it is lost to the environment. The second defining characteristic of a cavity mode is the cavity mode volume, $V_\omega$, which characterizes the spacial extent of the cavity mode:

$$V_\omega = \frac{\int d^3r \ [n(r)E(r)]^2}{\max [n(r)E(r)]^2},$$

where $n(r)$ is the refractive index, and $E(r)$ is the magnitude of the electric-field. If the ambition is the confinement and manipulation of light, then in general, the higher the $Q_\omega$ and the lower the mode volume, the better.

Quantization of the Electromagnetic Field

Cavity QED describes the behaviour of quantized modes of the EM field in a cavity. Therefore, one must move from the classical description of EM resonators to a quantum mechanical theory. The standard route for deriving the quantum theory for the EM field in a cavity follows the canonical quantization procedure, and is covered in detail in most quantum optics texts (e.g. [125]). What follows is a brief sketch that glosses over the minutiae, but is sufficient for the purposes of this thesis\(^1\).

\(^1\)For a rigorous study of EM quantization in a medium see [126].
2.1. ELECTROMAGNETIC CA VITIES

to express the EM field in a cavity as a superposition of single cavity modes\(^2\). In the non-homogeneous Coulomb gauge \((\nabla \cdot (\epsilon(r)A(r)) = 0)\), the scalar potential vanishes for source free equations. The Maxwell equations in a source free, time independent dielectric medium obey a second order differential equation for vector potential:

\[
[\nabla^2 - \epsilon(r)\omega^2] A(r) = 0, \tag{2.3}
\]

where we have assumed that the relative permeability is 1 \([127]\)\(^3\). As this defines a Hermitian generalized eigenvalue problem, the vector potential can be expanded as a linear superposition of the eigenmodes:

\[
A(r) = \sum_k \frac{1}{\sqrt{N_k}} [A_k u_k(r)e^{-i\omega_k t} + A_k^* u_k^*(r)e^{i\omega_k t}], \tag{2.4}
\]

and the electric field is:

\[
E(r) = \frac{d}{dt} A(r) = i \sum_k \frac{\omega_k}{\sqrt{N_k}} [A_k u_k(r)e^{-i\omega_k t} - A_k^* u_k^*(r)e^{i\omega_k t}], \tag{2.5}
\]

where \(A_k\) are the mode coefficients, \(u_k(r)\) the field amplitudes, and \(N_k\) the mode normalization. The normalization is defined through the inner product:

\[
\int_V d\nu \epsilon(r) u_k^*(r) u_k^*(r) = \delta_{kk} \omega_k. \tag{2.6}
\]

By virtue of the Hermicity of the defining eigenvalue problem, the field modes are orthogonal. Furthermore, time-reversal symmetry implies that \(u_k^*(r) = u_{-k}(r)\).

The canonical quantization procedure consists of replacing the mode coefficients with operators, making the substitution \(A_k(A_k^*) \rightarrow \hat{a}_k (\hat{a}_k^\dagger)\). The operators \(\hat{a}_k\) and \(\hat{a}_k^\dagger\) act to remove or add a single photon to mode \(k\) respectively. These operators act on the

\(^2\)This is under the assumption of a completely discreet spectrum, true if the cavity and medium is finite in extent. In principal there will be a continuum component to the field, but this can be ignored for our purposes, assuming some low frequency cutoff.

\(^3\)Mostly true for naturally occurring materials at optical frequencies, but can be high in some metals such as iron \([O(10^3)]\).
quantum state, as represented in Fock space:

\[
\hat{a}_k |n_1, \ldots, n_k, n_{k+1}, \ldots \rangle = \sqrt{n_k} |n_1, \ldots, n_k - 1, n_{k+1}, \ldots \rangle
\]

\[
\hat{a}^\dagger_k |n_1, \ldots, n_k, n_{k+1}, \ldots \rangle = \sqrt{n_k + 1} |n_1, \ldots, n_k + 1, n_{k+1}, \ldots \rangle,
\]

and the creation ($\hat{a}^\dagger_k$), and annihilation ($\hat{a}_k$) operators obey the usual commutation relations:

\[
[\hat{a}_k, \hat{a}^\dagger_l] = \delta_{kl}.
\]

The classical expression for the EM field Hamiltonian is given by:

\[
H = \frac{1}{2} \int_V dV \mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B}.
\]

Substituting the quantized expressions for the fields into Eq. (2.9) results in the quantum Hamiltonian:

\[
\hat{H} = \sum_k \hbar \omega_k \left( \hat{a}^\dagger_k \hat{a}_k + \frac{1}{2} \right).
\]

The mode normalization is chosen such that the energy of a single photon in mode $k$ is $\hbar \omega_k$. There is also the notorious and physically irrelevant constant factor of $\omega_k^2$ in each mode that can be trivially removed via a gauge transformation. Since the operator $\hat{a}^\dagger_k \hat{a}_k = \hat{n}_k$, i.e. the number of photons is mode $k$, the Hamiltonian describing an electromagnetic field in a linear medium is seen to be:

\[
\hat{H}_{\text{EM}} = \sum_k \hbar \hat{n}_k,
\]

where the factor of 1/2 has been discarded. Throughout the remainder of the thesis, notation is simplified further by setting $\hbar = 1$, so that energies are in unit’s of angular frequency.

The annihilation and creation operators for each cavity mode can be expanded in
2.1. ELECTROMAGNETIC CAVITIES

terms of the real space operators:

\[ \hat{a}_k = \sum_{\sigma} \int d^3x \frac{\hat{a}_{x,\sigma}}{\sqrt{V}} u_k(x) \cdot \epsilon_{\sigma}, \]  

(2.12)

where \( \sigma \) is a sum over polarizations \( \epsilon_{\sigma} \) in 3 dimensional space. This expansion follows directly from the definition of the modes \( u_k(x) \) that solve Maxwell’s equations in media, and the linearity of electromagnetism.

Cavity Quality

An isolated, ideal cavity will localize a mode of light perfectly inside the cavity, leading to an infinite life time for the photon. For real cavities, this is not possible: there will always be some loss out of the cavity. For any non-ideal cavity, each mode \( k \) of the cavity will have some support outside of the cavity, and there will be overlap with the cavity and the continuum of modes of the surrounding medium. This can be transformed, using the theory of quasi-mode optics [128], to a description in terms of perfect cavity modes, and non-overlapping continuum modes, with a coupling between them:

\[ H = \sum_{\text{quasi-modes}} \omega_k \hat{\alpha}_k^\dagger \hat{\alpha}_k + \int d\omega(p) \hat{b}_k^\dagger \hat{b}(p) + \sum_k \int d\omega(p) \hat{\alpha}_k^\dagger \hat{b}(p) + \gamma(p) \hat{\alpha}_k^\dagger \hat{b}(p). \]  

(2.13)

Thus a photon in mode \( k \) will tunnel into the external modes \( p \) over time.

If these external modes are large in extent, then the probability of the photon returning to the cavity is low. The loss can be then be treated as a Markovian process, and the external modes can be traced out. This leads to a total dissipation factor for the cavity mode:

\[ \gamma_k \approx \int d\omega \left( \frac{\Delta_k(p)}{2} \right)^2 - \frac{|\Delta_k(p)|^2}{2}. \]  

(2.14)

By discarding the external modes, it is possible to talk of a decay rate, \( \gamma_k \) of a mode, which will account for all the loss of photons to the environment.

The Q-factor, for a mode \( k \), is the ratio of the mode loss rate to the mode frequency:

\[ Q_k = \frac{\omega_k}{\gamma_k}, \]

where \( \omega_k \) is the frequency of the mode.

---

\(^4\)The condition on longitudinal polarizations, \( \epsilon(k) \cdot k = 0 \), is implied in the definitions of \( u_k(x) \).


\[ Q_k = \frac{2k}{\epsilon_k} \] A high Q-factor is desirable, since it implies there is more time for photonic manipulation in a cavity before escape.

### 2.2 Atom-Light Interaction

For a charged particle in an electric field, the interaction is found by replacing the kinetic energy of each particle, \( \frac{p_i^2}{2m_i} \), by \( \frac{1}{2m_i} (p_i + qA(\mathbf{r}_i))^2 \). From this, the interaction Hamiltonian is:

\[
\hat{H}_{\text{int}} = \hat{H}_{\text{dipole}} + \hat{H}_{\text{A}} = \sum_i \frac{q_i}{2m_i} (A(\mathbf{r}_i) \cdot \hat{p}_i + \hat{p}_i \cdot A(\mathbf{r}_i)) + \frac{q_i^2}{m_i} |A(\mathbf{r}_i)|^2.
\]

The dipole approximation begins by assuming that \( A(\mathbf{r}) \) is constant over the support of the atomic wavefunction\(^5\), a reasonable assumption given that atomic wavefunctions are on the order of angstroms\((O(10^{-10}))\), and optical wavelengths in media are typically hundreds of nanometers. In this case, the atom-photon interaction is separable. Using the commutation relation \[ [\hat{r}, \hat{H}_{\text{atom}}] = i\hbar \hat{p}/m \] one can then write the dipole interaction as:

\[
\hat{H}_{\text{dipole}} = \sum_j \sum_{mn} \hat{A}(0) \cdot (|m\rangle \langle m| H_{\text{atom}}^{\text{f}} - \hat{r}_j H^{\text{atom}}_{\text{f}} |n\rangle \langle n|)
\]

\[ \approx i\hbar \sum_{mn} A(0)(\hat{a}^\dagger + \hat{a}) |m\rangle \langle n| \epsilon_{mn} \cdot D_{mn}, \]

where the \(|m\rangle\)'s are atomic eigenstates and \( D_{mn} = \sum_i \langle m| r_i |m\rangle \) are the dipole elements. This form shows that the EM field induces a coupling between atomic eigenstates that is proportional to the field strength, dipole element, \( D_{mn} \), and the atomic energy gap \( \epsilon_{nm} = \epsilon_n - \epsilon_m \)\(^7\).

---

\(^5\)The dipole approximation effectively ignores the motional degrees of freedom of the atom inside the cavity. This is not always a valid approximation, and motion in the cavity can have a deleterious effect on atom-cavity coherence [129]. However, recent experiments have shown how to control these effects to a degree [130].

\(^6\)This ignores relativistic contributions to the kinetic energy of the electrons.

\(^7\)This is different from another common expression for the atom-photon interaction: \( H'_{\text{dipole}} = E(0)(\hat{a}^\dagger + \hat{a}) \hat{D} \) (see e.g. [125]). Indeed, a significantly different path is followed, involving a gauge transformation of the vector potential, to extract this form. This difference highlights the fact that only for transitions where \( \epsilon_n - \epsilon_m \approx \omega \) is the dipole approximation a good one. Luckily, this is exactly the regime of interest to this thesis, where effects of atom-photon interactions are the strongest.
The $A^2$ term in Eq. (2.15) is proportional to the electric field strength, and so can have important effects for high fields in a cavity. It is also proportional to the number of electrons in the cavity, and can play an important role in the Dicke super-radiance phase transition (see e.g. [131]). In the case of our single cavity with small photon number, the $A^2$ term adds a small shift to the cavity frequency, and some non-photon number conserving terms which are removed in the rotating wave approximation\(^8\).

The atom-field interaction is given by:

$$
\sum_{k} \sum_{mn} \frac{\epsilon_{mn}}{\sqrt{V_k \omega_k \epsilon_0}} \langle m|\hat{d}|n \rangle (a^\dagger_k + a_k).
$$

For the case of a single cavity mode, and two level atom, the interaction Hamiltonian is expressed in terms of the Pauli matrices:

$$
H_I = g (a^\dagger + a) \sigma_x,
$$

where all the coefficients have been collected in $g$, and with

$$
g = \frac{\epsilon_{eq}}{\sqrt{2\hbar \omega_0}} (D_{eg})_x.
$$

The dipole interaction describes the dominant dynamics of atoms situated in a cavity mode. This is the starting point for Cavity quantum electrodynamics.

There are three ways in which one can change the size of $g$: Higher frequencies, smaller mode volume, and larger atomic dipoles. For the purpose of introducing strong atom-light interactions, critical factor is $\frac{g}{\gamma} = \frac{g}{\omega Q}$. This is the ratio of the Rabi frequency to the cavity dissipation rate. For $\frac{g}{\gamma} \ll 1$, photons do not remain in the cavity long enough for coherent quantum effects to manifest.

For a large electric field $E$, and $n$ intra-cavity atoms, the semi-classical evolution is given by the Optical-Bloch equations [125], which will appear in chapter 8. When operating in the quantum regime, the dynamics are described by the Jaynes-Cummings

---

\(^8\)See section 2.2.2.
CHAPTER 2. REVIEW OF CAVITY QUANTUM ELECTRODYNAMICS

model which is described in the following section.

2.2.1 The Jaynes-Cummings Hamiltonian

The Jaynes-Cummings (JC) Hamiltonian is the canonical model for atom-light interactions, describing a single confined Bosonic mode interacting with a two level system (TLS).

The cavity mode is described in second quantised form with the Bosonic creation and annihilation operators.

\[ H_{\text{cavity}} = \omega a^\dagger a \]
\[ [a, a^\dagger] = 1. \]  

The two level approximation for the atom can be made when the atomic energies are significantly enharmonic. In this case, the system is equivalent the form of a spin 1/2 system.

\[ H_{\text{atom}} = \epsilon \sigma^+ \sigma^- \]
\[ [\sigma^-, \sigma^+] = \sigma^z. \]

The dipole interaction, up to a gauge transformation, is then given by

\[ \mathbf{E} \cdot \sigma = (a^\dagger + a)\sigma^x. \]

The, the Jaynes-Cummings Hamiltonian is given by:

\[ H^{\text{JC}} = \omega a^\dagger a + \frac{\epsilon}{2} (\sigma_z + 1) + \frac{\beta}{2} \sigma_x E \\
= \omega a^\dagger a + \epsilon \sigma^\dagger \sigma + \beta (\sigma^\dagger + \sigma)(a^\dagger + a), \]

where the three terms correspond to the resonator mode, the atomic mode, and the atom-cavity interaction respectively.

The interaction Hamiltonian, and the full Hamiltonian, have a parity symmetry \( P = \sigma^z e^{i\pi a^\dagger a/2} \) with eigenvalues \( p = \pm 1 \).
2.2. ATOM-LIGHT INTERACTION

2.2.2 Rotating Wave Approximation

The dipole interaction term in Eq. (2.23) includes terms, $\beta a\sigma^\dagger$ and $\beta a\sigma$ which behave quite differently to the terms $\beta a^\dagger\sigma$ and $\beta a^\dagger$. The later act by annihilating(creating) a photonic mode while simultaneously exciting(de-exciting) the TLS, while the former, referred to as the counter-rotating terms, act to excite(de-excite) the atom and the photon field together.

The action of the $a\sigma^\dagger$ like terms conserves the total number of excitations, $\ell$, in the cavity, i.e. the excitation number operator

$$L = a^\dagger a + \sigma^+\sigma^-$$  \hspace{1cm} (2.24)

commutes with these terms. This is not the case for the other two terms, which act to shift $\ell$ by ±2. This conserves the odd/even parity of the state, but not the total excitation number.

Considering the JC Hamiltonian of Eq. (2.23), the sub-manifold spanned by constant $\ell$ has energy $\ell\omega \pm \frac{\Delta}{2}$, where $\Delta = \omega - \epsilon$. The quantity $\Delta$, referred to as detuning, denotes the difference in energy between the atomic and photonic excitation. For the remainder of this thesis, it will be assumed that $\Delta \ll \omega$. That is, the photonic and atomic energies are relatively close to resonance. The energy within each $\ell$ sub-manifold is small compared to the gap between each sub-manifold.

Under a further assumption, that $\beta \ll \omega$, the coupling between these sub-manifolds induced by the counter-rotating terms will be very small. The effect can be seen by transforming to a rotating frame, via the unitary operator:

$$U^{RW} = e^{-i\omega t(a^\dagger a + \sigma^+\sigma^-/2)}.$$  \hspace{1cm} (2.25)

Under a time-dependent unitary transformation, the Hamiltonian of a system transforms as:

$$H'(t) = U(t)H(t)U^\dagger(t) - iU^\dagger(t)\partial_t U(t).$$  \hspace{1cm} (2.26)
CHAPTER 2. REVIEW OF CAVITY QUANTUM ELECTRODYNAMICS

In this new frame, the $H_{JC}$ becomes:

$$H^{\text{rot}}(t) = \frac{\Delta}{2} \sigma^z + \beta (a^\dagger \sigma + a \sigma^\dagger) + \beta (a^\dagger \sigma^z e^{2i\omega t} + a \sigma^z e^{-2i\omega t}).$$  \hfill (2.27)

If the magnitude of the fast terms is much less than the frequency of oscillation, then the fast terms are slaved to the slow terms when the interaction strength is small\(^9\). When the fast oscillations are much slower, the two can be separated.

The Magnus expansion, detailed further in section 3.4, allows one to define an effective time-independent Hamiltonian. Evolution of the system over one period is given by:

$$U(t) = \exp \left[i \Omega(t) \right]$$  \hfill (2.28)

where $\Omega$ is an hermitian operator. As the Hamiltonian is periodic, with period $T = 2\pi/\omega$, an effective time-independent Hamiltonian governing the evolution of the system is the time average of $\Omega$ over $T$, i.e.

$$H^{\text{eff}} = \frac{\Omega(T)}{T}. \hfill (2.29)$$

If the magnitude of $\Omega$ is small, then it may be approximated by the Magnus expansion. This will indeed be the case when $\beta T = \beta / \pi \omega \ll 1$:

$$\frac{\Omega(T)}{T} = \frac{\Delta}{2} \sigma^z + \beta (a^\dagger \sigma + a \sigma^\dagger)$$

$$- \frac{\beta}{\omega} \left( \beta (a^\dagger a^\dagger + aa - a^\dagger a - \frac{1}{2}) \sigma^z + \frac{\beta}{2} (\sigma^z a^\dagger + a \sigma^z) \right) + \mathcal{O} \left( \left(\frac{\beta}{\omega}\right)^2 \right) \hfill (2.30)$$

When $\frac{\beta}{\omega} \ll 1$, the second line in Eq. (2.30) can be discarded\(^{10}\), an approximation known as the rotating wave approximation (RWA). The remaining terms then constitute the

---

\(^{9}\)This is true of any linear system, which includes all quantum systems described by a Schrödinger equation. Not so for non-linear systems, where the microscopic behaviour of the fast terms can feedback to the slow driving motion.

\(^{10}\)Practically the entire corpus of quantum atom-optics research begins with the RWA. However, current experiments in circuit QED are approaching the limit of this approximation, opening a whole field of relatively unexplored physics [132–136].
2.2. ATOM-LIGHT INTERACTION

canonical Jaynes - Cummings Hamiltonian:

\[ H^{JC} = \frac{\Delta}{2} \sigma^{\dagger} \sigma + \beta (\sigma^{\dagger} a + \sigma a^{\dagger}) , \quad (2.31) \]

arguably one of the most important models in quantum optics\(^{11}\). For the remainder of this thesis, this is the form of the Jaynes-Cummings Hamiltonian that is used.

The discarded terms, if not sufficiently small, will manifest in two separate ways. Firstly, they introduce a coupling between different excitation numbers. However, the stratification of energies of each excitation manifold (in the non-rotating frame) implies that:

\[ \| P_{\ell \perp} U^{\text{exact}}(t) P_{\ell \parallel} \| < C_{\ell} \beta / \omega \quad \forall t \quad (2.32) \]

where \( P_{\ell \perp} / P_{\ell \parallel} \) is the projection out of/into the \( \ell \) excitation manifold, and \( C_{\ell} \) is a different constant of order \( O(\ell) \) for each \( \ell \). That is, fluctuations of a state initially in an excitation eigenstate into other eigenstates will be bounded for all time. However, the discarded terms also induce an energy shift, and modify the coupling, within the excitation sub-manifold, on the order of \( \beta^{2} / \omega \). This implies a modified Rabi frequency will manifest on time scales of order \( t \sim \omega / \beta^{2} \).

Once the inter-excitation terms have been discarded one may move back to a non-rotating frame by applying the inverse of the transform in Eq. (2.25). \( U^{\text{RW}} \) commutes with the JC Hamiltonian now, so the effect is simply the return of the mode energy term: \( \omega a^{\dagger} a \).

The popularity of the JC Hamiltonian in Eq. (2.31) is due to the fact that, firstly, it is a very good approximation for most atom-cavity systems that have been considered, and secondly, it is exactly solvable, which greatly expands the field of techniques which can be applied to JC systems. For a comprehensive review that discusses the validity of the RWA, and the two level approximation, in various regimes, see [138].

\(^{11}\)See [137] for a comprehensive study of the JC model, and it’s applications and extensions.
CHAPTER 2. REVIEW OF CAVITY QUANTUM ELECTRODYNAMICS

Solution to the JC Hamiltonian with the RWA

It is worth noting that the cavity frequency does not appear in the Jaynes-Cummings Hamiltonian. This is representative of the fact that the number of excitations, \( \ell \), in each cavity is a good quantum number for the JC Hamiltonian, i.e.

\[
[H_{JC}, L] = 0. \tag{2.33}
\]

where \( L \) is the excitation number operator given in Eq. (2.24).

For \( \ell \) an excitation quantum number, if \( \ell > 0 \) then \( H_{JC} \) is restricted to a two dimensional subspace:

\[
H_{JC}^{\ell} = \begin{bmatrix}
0 & \sqrt{\ell}\beta \\
\sqrt{\ell}\beta & \Delta
\end{bmatrix}. \tag{2.34}
\]

The normalized eigenvectors, and eigenvalues, of the two dimensional matrix are easily found:

\[
\psi_{\ell, \pm} = \begin{bmatrix}
a \\
b
\end{bmatrix}, \quad \chi(\ell, \Delta) = \frac{1}{2} \sqrt{4\beta^2 \ell + \Delta^2}. \tag{2.35}
\]

The solutions can be cast in a slightly more informative manner:

\[
|+\ell\rangle = \sin \theta_{JC}^{\ell} |g, \ell\rangle + \cos \theta_{JC}^{\ell} |e, \ell - 1\rangle
\]

\[
\mid -\ell\rangle = \cos \theta_{JC}^{\ell} |g, \ell\rangle - \sin \theta_{JC}^{\ell} |e, \ell - 1\rangle, \tag{2.36}
\]

where

\[
\tan \theta_{JC}^{\ell} = \frac{2\beta \sqrt{\ell}}{(\Delta + 2\chi(\ell))},
\]

\[
H_{JC}^{| \pm \ell \rangle} = (\pm \chi(\ell) - \Delta/2) | \pm \ell \rangle. \tag{2.37}
\]

Here, the angle \( \theta_{JC}^{\ell} \) is the degree of mixing between atomic and photonic components of the eigenstates with excitation number \( \ell \). For example, at \( \theta_{JC}^{\ell} = \pi/4 \), the eigenstates are in equal superpositions of the two basis states, and at \( \theta_{JC}^{\ell} = 0, \pi/2 \) the two modes are entirely decoupled.

Figure 2.2 shows the energies for given excitation numbers (\( \ell \)) as a function of detuning.

\[^{12}\text{For } \ell = 0, H_{JC}^{\ell} \text{ is just the trivial 1 dimensional Hamiltonian.}\]
2.2. ATOM-LIGHT INTERACTION

As there are only two levels, there is necessarily an avoided crossing. The minimum separation of energies is $2\beta \sqrt{\ell}$, which occurs at $\Delta = 0$ with $\theta_{\ell}^{JC} = \pi/4$.

2.2.3 Decoherence, Dissipation & Pumping

In general, a quantum system will interact with its environment. Sometimes this is desired or deliberate, as in the case of pumped/driven systems. However, uncontrolled interactions induce system-environment correlations that manifest as decoherence in the system. This has the effect of washing out quantum effects, and the environment constitutes the quantum degrees of freedom that are not included in the system under consideration, and are either ignored or not monitored. Interactions with these degrees of freedom lead to system-environment correlations. When considering the system by itself, these correlations lead to decoherence.

In general, the behaviour of the isolated system depends on the specifics of the coupling to the environment, and the dynamics within the environment. However, in most cases, the interaction takes a general form determined by only a few parameters.
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Figure 2.3: a) A two level atom in a Fabry-Pérot Cavity. Atom oscillates coherently between the ground and excited states with the Rabi frequency $\beta$. b) Simulated cavity with dephasing and dissipative effects. (Blue) – $\langle \sigma^+ \sigma^- \rangle$, (purple) – $\langle L \rangle$. Note the suppression of Rabi oscillations by the dephasing, and the slow loss of mean excitation number.

In the context of an atom-cavity system there are three non-unitary processes that affect the quantum dynamics. Dissipation, whereby photons leak out of the cavity, atomic relaxation to the ground state, and atomic dephasing, where the different atomic levels become incoherent. The atomic relaxation and dephasing are are usually given as the characteristic times $T_1$ and $T_2$ respectively. The dephasing rate is driven by random fluctuations in the atomic energy, while relaxation arises from an off-diagonal coupling to environmental modes. Typically, $T_1$ is much longer than $T_2$, but can be increased by shielding the atom from errant electromagnetic fields in the environment.

Under the assumption of weak coupling, and a large environment, the full dynamics of the cavity can be described in by the Lindbald equation for the atom-cavity density matrix:

$$\dot{\rho} = -i \left[ H^{JC}, \rho \right] - \left[ \delta \omega \hat{a} \hat{a}^\dagger + P \hat{a} + P^* \hat{a}^\dagger, \rho \right] + \sum_i \gamma_i \left[ \hat{G}_i \rho \hat{G}_i^\dagger - \frac{1}{2} \left\{ \rho, \hat{G}_i^\dagger \hat{G}_i \right\} \right], \quad (2.38)$$

where $\delta \omega$ is the pump detuning $(\omega - \omega^{pump})$, and $P$ is the pump strength. The $\hat{G}_i$ are the decoherence channels, with strength $\gamma_i$. For the atom-cavity systems, these are the cavity dissipation $\gamma_\omega$, relaxation, $\gamma_{\sigma^-} = 2\pi/T_1$, and dephasing, $\gamma_{\sigma_z} = 2\pi/T_2$. The channels are respectively $\hat{a}$, $\hat{\sigma}^-$, and $\hat{\sigma}_z$. 

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A schematic of a realistic cavity is illustrated in Fig. 2.3. Dissipation and relaxation leads a slow leaking of excitations out of the cavity, while the dephasing leads to a suppression of Rabi oscillations, as demonstrated in Fig. 2.3b). The study of these environmental interactions in the context of coupled cavity systems is in its infancy. The general problem of driven-dissipative cavity arrays is one of a non-linear, non-equilibrium system, and in general difficult to approach. Nevertheless, recent investigations have uncovered a wealth of non-trivial phenomena exhibited by these systems. Some of these phenomena have direct correspondences with equilibrium behaviour [139], while others are unique to driven-dissipative systems [140].

In this thesis, the effects interactions with the environment are studied in the context of the chaotic dynamics of two coupled JC cavities in chapter 8. However, it is the purely quantum behaviour that is of interest, so the effects of the environment are ignored for most part.

2.2.4 Photon Blockade

Photon blockade [57,141] is a striking phenomena that exists as a consequence of strongly atom-photon interactions as a photonic analogue of Coulomb blockade (found in e.g. quantum dots [142]), and has been demonstrated experimentally [43,143].

For a single mode atom-less cavity, there is a single stationary state that all initial conditions for the density matrix converge to in the long time limit. The expectation of the photon number is given by:

$$\langle \hat{n} \rangle = \frac{|P|^2}{\delta \omega_1^2 + \gamma_1^2},$$

where $P$ is the pump strength, $\delta \omega_1$ is the pump detuning, and $\gamma_1$ is the dissipation rate for the cavity mode.

One can see that the presence of an energy gap between the cavity and pump frequency ($\delta \omega_1$) suppresses the entry of photons into the cavity mode.

The presence of a strongly coupled intra-cavity atom induces an energy shift cost for
additional photons, as seen in Fig. 2.2. Thus the induced non-linearity acts to restrict the occupation of higher photonic occupations. That is, higher phonton numbers have an increased effective $\delta \omega_1$, which, with reference to Eq. (2.39), lowers the expected occupation for these states.

This is demonstrated in Fig. 2.4, which compares the ratio of finding two or one photons in the cavity with increasing atom-cavity coupling $\beta$ (and therefore larger effective $\delta \omega_1$). Furthermore, the effect of atomic dephasing $\gamma_1$ suppresses the coherence between photonic and atomic modes, reducing the blockade effect.

### 2.2.5 Multi-level Atoms

There are no two level systems; any physical atom has an infinite number of discrete states which can be coupled into via the atom-light interaction. The two level approximation is often very good, but there will always be some contribution from the other levels. Indeed, many important quantum-optical devices make use of the more complicated dynamics that are possible with multi-level atoms, such as stimulated Raman adiabatic passage (STIRAP) [144] or electromagnetic induced transparency (EIT) [58]. Out of the three levels, only one may be coupled to both other levels, since the dipole interaction is non-zero only between states of different parity. Therefore there are three different
configurations for three level systems that are can be considered, shown schematically in Fig. 2.5.

In chapter 6 I study a Ξ-configuration atom scheme that can replace the two level system in the photonic cavity. This configuration can be tuned such that the intra-cavity atom induces an effective three-body interaction.

Multiple Cavity Modes

A third extension to the Jaynes-Cummings model is to include multiple photon modes. Indeed for realistic cavities, the polarization degrees of freedom will often manifest in a doubly degenerate cavity spectrum. Furthermore, if it is not the lowest cavity mode that is being considered, spacial symmetries will contribute multiple modes of similar energies to the spectrum. In general, these modes will all couple via the dipole interaction, and cannot be ignored.

As in the case of multiple atomic levels, the rotating wave approximation ensures that the excitation number is still a good quantum number. For the Jaynes-Cummings model with multiple modes, this means that the Hamiltonian is still solvable within some excitation number sub-manifold.

The extra Bosonic degrees of freedom afforded by multi-mode cavities can be used in a variety of contexts, such as to contrive phenomena found in multi-species Bose
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gasses [145,146] and induce abelian [28], and non-abelian [147], artificial magnetic fields.

2.3 Coupled Atom-Cavity Systems

In 2006/7 several authors proposed a lattice model based on JC systems that would mimic certain behaviours seen in solids [31, 36, 37]. These models, now known as Jaynes-Cummings-Hubbard (JCH) models, consist of a lattice of coupled photonic cavities, each of which interacts with a TLS. This interaction, which introduces a non-linear photon-photon interaction, can be described by the JC Hamiltonian, and the inter-cavity tunneling is reminiscent of that found in the Hubbard model, which gives the JCH its name.

2.3.1 Tight Binding Model

The model central to this thesis will involved arrays of coupled cavities. Consider a dielectric material supporting several photonic cavities with similar mode frequencies. If the cavities are sufficiently separated then one can take the separate cavity modes as a basis, and the overlap of modes with different cavities can be treated as a perturbation whereby photons tunnel from one cavity to another. The Hamiltonian for such a system is given by:

\[ \hat{H}^{\text{TB}} = \sum_{i} \sum_{k} \omega'_{iik} \hat{a}^\dagger_{iik} \hat{a}_{iik} - \sum_{ij} \sum_{kk'} \kappa_{ikjk'} \hat{a}^\dagger_{ik} \hat{a}_{jk'}, \quad (2.40) \]

where \( \omega'_{i} \) are the perturbed mode frequencies, and \( \kappa_{ikjk'} \) is the tunneling rate between modes \( i \) and \( j \). A fully justified derivation of this form requires a subtle treatment of the defining electromagnetic equations, since the set of isolated cavity modes is not orthonormal under the inner product defined by the permittivity of the full cavity array. Nevertheless, Eq. (2.40) is fully general for linear optics, the only question being on the accurate derivation of the coupling constants and modified mode frequencies.
2.3. COUPLED ATOM-CAVITY SYSTEMS

Figure 2.6: A linear array of coupled Jaynes-Cummings cavities with uniform hopping rate $\kappa$

For our purposes, we consider only the lowest mode of each cavity, assuming that the higher modes are significantly greater than the inter cavity coupling rates. Furthermore, the spacial structure implies that $\kappa_{ij}$ should decrease as the distance between cavity $i$ and $j$ increases. In the case of high-Q cavities, the inter-cavity coupling decreases exponentially with distance, so that only the nearest neighbours need be considered. Thus the cavity array system can be modeled well by the Hamiltonian

$$\hat{H}^{\text{Hubbard}} = \sum_{i, j}^{\text{cavities}} \kappa_{ij} \hat{a}_i^\dagger \hat{a}_j,$$

where $<i,j>$ indicates that the sum is to be taken over nearest neighbours only. Hermicity of the Hamiltonian requires that the $\kappa_{ij} = \kappa_{ji}^*$. 

2.3.2 Jaynes-Cummings Hubbard Model

The tight binding for photons in a cavity array provides the starting point for condensed matter analogies in photonic systems. As this model is completely linear, the range of phenomena that arises in such a system is somewhat limited. As shown in section 2.2, the introduction of an atomic system to the cavity induces non-linear interactions. This opens up a plethora of non-linear behaviours to manifest in the photonic system. The JCH model describes a cavity array, as described by a tight-binding model, augmented
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by two level systems in each cavity interacting via a Jaynes-Cummings like term:

\[ \hat{H}^{JCH} = \sum_x \hat{H}^{JC}(x) + \hat{K} \]
\[ \hat{K} = -\sum_{x,x'} \kappa(x,x') \hat{a}_x \hat{a}_{x'}^\dagger. \]  

(2.42)

Figure 2.6, provides a cartoon picture of the JCH model in 1 dimension.

The excitation number conservation of an individual JC cavity is broken with the introduction of the cavity-cavity tunneling, \( \hat{K} \). However, the total excitation operator,

\[ \hat{L} = \sum_x \hat{L}_x, \]  

(2.43)

commutes with \( \hat{H}^{JCH} \). The JCH model then describes a system with a well defined number of excitations, or particles. In this case, these particles are referred to as polaritons. Polaritons exists as superpositions of photonic and atomic excitations, and, via the non-linearity inherent in the Jaynes-Cummings cavities, experience polariton-polariton interactions.

2.3.3 Single Excitation in a JCH Lattice

Bloch’s Theorem

Consider a single particle in a periodic potential, i.e. \( V(r) = V(r + a_i) \), where \( a_i \) are the lattice vectors. The Hamiltonian describing this particle is invariant under translations \( x \to x + R \). We make the Bloch ansatz [148], that the wavefunction is:

\[ \psi_k(x + R) = e^{ik \cdot R} \psi(x) \]

(2.44)

If periodic boundary conditions are assumed, then there is a constraint on the values that \( k \) can take:

\[ k_i L_i = 2\pi n, \]

(2.45)

where \( L_i \) are the dimensions of the system, and \( n \) is an integer.
Substituting the Bloch ansatz into the Hamiltonian in Eq. (2.40) we can see that $\psi_k$ are the eigenvalues:

$$\hat{H}^{TB}\psi_k = 2 \sum_i \kappa_i \cos (k_i) \psi_k,$$

where the $\psi_k$ are the momentum eigenstates for free particles.

Bloch’s theorem allows the JCH model to be expressed in terms of momentum states.

Writing

$$\hat{a}_x = \sum_k \hat{a}_k \exp [ik \cdot cx],$$

$$\hat{\sigma}_x = \sum_k \hat{\sigma}_k \exp [ik \cdot cx],$$

the JCH Hamiltonian (in 2 dimensions) in Eq. (2.42) becomes:

$$\hat{H}^{JCH} = \sum_k 2\kappa (\cos(k_x) + \cos(k_y)) \hat{a}_k^\dagger \hat{a}_k + \Delta \hat{\sigma}_k^+ \hat{\sigma}_k^- + \beta \sum_{k'} \left( \hat{a}_k^\dagger \hat{\sigma}_{k'}^- + \text{h.c.} \right).$$

In this form the momentum conservation is manifest. The solutions are simply symmetric and anti-symmetric superpositions of atomic and photonic momentum eigenstates, in the same form as the single site JC system:

$$|k, +\rangle = \sum_x e^{-ik \cdot x} (\sin \theta_k |x, 1, g\rangle + \cos \theta_k |x, 0, e\rangle)$$

$$|k, -\rangle = \sum_x e^{-ik \cdot x} (\cos \theta_k |x, 1, g\rangle + \sin \theta_k |x, 0, e\rangle)$$

$$\tan \theta_k = \frac{2\beta}{\Delta + \sqrt{4\beta^2 + (\Delta - 2\kappa \cos k_x + \cos k_y)^2}},$$

and

$$E_{k, \pm} = \frac{\Delta}{2} - \kappa [\cos k_x + \cos k_y] \pm \frac{1}{2} \sqrt{4 \beta^2 + (\Delta - 2\kappa \cos k_x + \cos k_y)^2}.$$

A number of observations about these solutions are in order. Firstly, the photon-atom mixing (denoted by the JC mixing angle, $\theta^{JC}$ for the single cavity) depends on $k$. Indeed, the transformation $\Delta \rightarrow \Delta - 2\kappa [\cos k_x + \cos k_y]$ maps this problem back to the original single excitation JC system. Thus the kinetic energy term leads to an effective detuning in the atom/photon subspace.
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When the detuning is larger than both the tunneling strength, and the atom-photon coupling, the atomic and photonic modes separate out, leading to independent propagation of photonic and atomic excitations. This is elaborated in section 2.4.4, and plays an important role in understanding the results of chapter 5.

In the case where the tunneling strength is much greater than the atom-cavity coupling \( \kappa \gg \beta \) then the effective atom-photon coupling will be momentum dependent to some degree. That is, for high momentums, the JC term will be somewhat transparent to the photons. This manifests directly in the photon-photon scattering cross sections (as discussed in the following section).

2.4 Regimes of the JCH Model

2.4.1 Continuum Limit

The single particle solutions for the JCH model (and other lattice models) are a function of the wave-vector \( k \). When \( k \cdot a \ll 1 \) the lattice structure becomes invisible, and it is expected that the system will reduce to some continuous space model. Expanding Eq. (2.48) in terms of \( k \cdot a \) results in:

\[
\hat{H}_{\text{JCH}} \approx \sum_k \kappa (-1 + (k \cdot a)^2) \hat{a}_k^\dagger \hat{a}_k + \sum_x \Delta \sigma^z(x) + \beta(x) a(x)^\dagger \sigma + a(x) \sigma^+).
\]

The mapping for the JCH model then results in a two mode continuum model with \( \kappa = 2/m^* \), where \( m^* \) is the effective mass for the photon mode.

Alternatively, when the \( \Delta \approx 0 \), expanding the single particle energy to second order in the momentum yields \( E_{\pm} \approx \pm \beta + (\kappa \pm 2\kappa^2/\beta)k^2 \) for the upper/lower polariton branches.

2.4.2 Bose-Hubbard Model

The JCH model, at its core, describes interacting Bosonic particles confined to a discreet lattice. The same physics is found in cold atoms confined by an optical lattice. Since the successful creation of a Bose-Einstein condensate (BEC) [149] these systems have
2.4. REGIMES OF THE JCH MODEL

been studied intensely, both theoretically and experimentally. Here, an optical lattice is formed by interfering lasers, which create a periodic confining potential for the cold atoms \[150,151\]. The primary theoretical framework for describing such a configuration is the Bose-Hubbard (BH) model \[152–154\]. Like the JCH, a single Bosonic mode of each lattice site can tunnel to nearby sites via a Hubbard hopping term. However, in the case of cold atoms, the interaction is a direct two body scattering cold atoms:

\[
\hat{H}_{BH} = \sum_x \frac{U(x)}{2} \hat{b}^\dagger_x \hat{b}^\dagger_x \hat{b}_x + V(x) \hat{b}^\dagger_x \hat{b}_x + \sum_{x \neq x'} \kappa(x, x') \hat{b}^\dagger_{x'} \hat{b}_x. \tag{2.52}
\]

2.4.3 Large Detuning in a JC Cavity

It was shown previously that the polariton angle \(\theta_{JC}\) is very small for atoms with large detuning. That is, the atomic and photonic degrees of freedom are largely independent. Adiabatic elimination can be used to remove either the photonic or atomic degrees of freedom from the problem completely. The formalism for computing this can be found in appendix A.

For the single atom cavity, the Hamiltonian is decomposed into the required form to apply the adiabatic elimination, with \(\hat{H}_0 = \omega \hat{a}^\dagger \hat{a} + \epsilon \hat{\sigma}^+ \hat{\sigma}^-\), \(V_X = \beta (\hat{a}^\dagger \hat{\sigma}^- + \text{h.c})\) and \(V_D = 0\). The only contributions are from the 2nd and 4th order.

\[
\begin{align*}
W^{(2)} &= \frac{\beta^2}{\Delta} (2\hat{a}^\dagger \hat{a} \hat{\sigma}^+ \hat{\sigma}^- + \hat{\sigma}^+ \hat{\sigma}^- - \hat{a}^\dagger \hat{a}) \\
W^{(4)} &= \frac{\beta^4}{\Delta^2} (\hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} - 2\hat{a}^\dagger \hat{a} \hat{\sigma}^+ \hat{\sigma}^- - \hat{\sigma}^+ \hat{\sigma}^-)
\end{align*} \tag{2.53}
\]

The effective Hamiltonian has no off-diagonal terms, allowing the elimination of either the atomic or photonic degree of freedom by applying initial conditions.

This example is somewhat trivial, as the JC model can be solved exactly. However, it makes salient the non-linearity in the cavity energy spectrum resulting from the presence of the inter-cavity atom. Indeed, assuming no initial atomic population, the photonic
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Hamiltonian can be recast as

\[ \hat{H}^{\text{photon}} = \left( \omega - \frac{\beta^2}{\Delta} - \frac{\beta^4}{\Delta^3} \right) \hat{a}^\dagger \hat{a} + \frac{\beta^4}{\Delta^3} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}. \]  

This is in the canonical form of a two-body contact interaction, found, for example, in the Bose-Hubbard model, among others. Note that the sign of the detuning determines whether the interaction is repulsive or attractive. As this work concerns the behaviour of cavity arrays, the direct link with the physics found in the extensively studied Bose-Hubbard model is valuable.

2.4.4 Adiabatic Elimination of Photonic States in the JCH Model

The JCH model shares many properties with other lattice models. As was just shown, the presence of an inter-cavity atom can reproduce the atom-atom interaction found in the Bose-Hubbard model. However, in the decoupled regime this interaction is necessarily small. To access the important regime of \( U \gg \kappa \), the intra-cavity hopping must be smaller still.

It is now shown that in the large detuning limit, the atomic states have an effective site to site coupling, mediated by the photonic field. Assuming the photonic field is initially in a state with some photonic occupation \( |\psi_{\text{ph}}\rangle = \prod_x |n_x\rangle \), application of perturbation theory to the JCH Hamiltonian in Eq. (2.42) yields the atom-atom matrix elements. The presence of the photon-photon hopping results in a third order correction for the atomic Hamiltonian.

\[ W^{(3)}_\sigma = -\frac{\beta^2}{\Delta^2} \sum_{xx'} (\hat{n}_x + 1)(\hat{n}_{x'} + 1)(\kappa_{xx'} \hat{\sigma}_x^\dagger \hat{\sigma}_{x'} + \text{h.c.}), \]

where \( \hat{n}_i = |\psi| \hat{a}_i^\dagger \hat{a}_i |\psi\rangle \)\(^{13}\).

\(^{13}\) It should be noted that the condition on the adiabatic elimination requires that the energy subsets of the full Hamiltonian must be well separated. In the case of large \( \kappa \), the energies of the photonic field will be shifted by \( O(\kappa) \), which may make the adiabatic approximation invalid.
The full atomic Hamiltonian is then:

\[ \hat{H}_{JCH}^s = \sum_x \hat{\sigma}_x^+ \hat{\sigma}_x^- - \sum_{xx'} (\kappa'_{xx'} \hat{\sigma}_x^+ \hat{\sigma}_{x'}^- + \text{h.c.}), \]

where \( \kappa'_{xx'} \) is the effective coupling between sites, given by:

\[ \kappa'_{xx'} = -\frac{\beta^2}{\Delta^2} \sum_{<xx'>} (n_x + 1)(n_{x'} + 1) \kappa_{xx'}. \]

The resulting system is exactly that of the hard-core Bose-Hubbard model, a repulsive Bose-Hubbard model in the limit where the on-site repulsion tends to infinity.

### 2.5 summary

The Jaynes-Cummings-Hubbard model is a theoretical framework for describing arrays of coupled photonic cavities with strongly coupled resonant intra-cavity atoms. The intra-cavity atoms, described by the Jaynes-Cummings model, induce non-linearities to the photonic field, leading to non-trivial many-body behaviour. While the atom-photon system displays complicated motion, when the atom is moved away from resonance, the atomic and photonic degrees of freedom decouple. This reduces the system to an effective Bose-Hubbard model, one that has an infinite on-site potential (Hardcore Bose-Hubbard) for the atomic case, and a weak potential for the photonic field.
3.1 Magnetic Fields in Quantum Mechanics

Classically, the electro-magnetic (EM) field admits two physically equivalent descriptions, firstly in terms of EM field tensor $F$, or as the 4-vector potential $A$, from which the EM fields can be derived:

$$ F = dA. $$

The classical perspective was that the potential was a mathematically convenient formulation of the ‘real’ fields. This perspective was somewhat motivated by the presence of physically irrelevant gauge freedom in the potential field. That is, the transformation $A \to A + d\chi$, leaves $F$ unchanged, for any function $\chi(x)$.

This perspective changed with the discovery of quantum field theory, where the potential field becomes primary. Here, the gauge freedom actually becomes a defining feature of the theory, prescribing the form of the interaction between charged particles and the EM field. However, the primacy of the potential field as the physically relevant quantity can already be seen in ordinary, non-relativistic, quantum mechanics.

In non-relativistic quantum mechanics, the modern perspective is of the vector...
potential, $A$, as a gauge connection on the $\mathbb{R}^3$ manifold that the electron inhabits. Thus in the presence of the magnetic field the covariant derivative becomes:

$$\nabla \rightarrow \nabla - iqA,$$  \hfill (3.2)

where $q$ is the charge on the electron. Here, the magnetic field is purely external and is not a dynamic quantity of the system, as is the case in QED.

For external magnetic fields, gauge transformations remain physically irrelevant, but implies a simultaneous transformation of the wavefunction:

$$A' = A + \nabla \chi(x),$$

$$\psi' = \psi e^{-i\chi(x)}.$$  \hfill (3.3)

Furthermore, given the desired gauge transformation from $A$ to $A'$, one can find $\chi$ by integrating over a path: $\chi(x) = \int_{x_0}^{x} (A' - A) \cdot dr = \int_{x_0}^{x} \nabla \chi \cdot dr$.

### 3.2 Uniform Magnetic Fields in 2 Dimensions

A major part of this thesis is concerned with particles moving in a 2D plane, subject to a constant, perpendicular, magnetic field. Indeed, the physics of even a single particle in such a configuration are alone sufficiently interesting. For the purposes of this work, the single particle physics forms the basis for understanding the repercussions of more complicated situations; in particular, the behaviour of many interacting particles, and the introduction of a lattice structure. This model is of huge importance, owing to the unique structure of the eigenstates in this system. These states are the famous Landau levels, which consist of a harmonic ladder of massively degenerate states. I will now cover the necessary mathematics of Landau level physics required for the rest of this thesis.

Consider a particle of charge $q$, confined to the $x - y$ plane, moving in the magnetic field $B = B\hat{z}$. In the absence of interactions, the Hamiltonian takes a simple form:

$$H_B = \frac{\Pi^2}{2m},$$  \hfill (3.4)
where $\Pi = p - qA(x)$ is the canonical momentum.

The vector potential defining a constant magnetic field must obey $\nabla \times A(x) = B\hat{z}$, but otherwise has significant freedom in it’s form, since $A$ is invariant under gauge transformations. There are two gauges commonly used; The Landau gauge, $A^L = -By\hat{x}$, and the symmetric gauge, $A^S = B(x\hat{y} - y\hat{x})/2$. Both forms are useful, depending on the application, but solutions in one gauge can be easily transformed into the other via a gauge transformation, since

$$A^S = A^L - \nabla(Bxy/2). \tag{3.5}$$

The Landau gauge becomes useful when considering particles moving on a torus, and the symmetry of the symmetric gauge is advantageous when considering the open plane. Both these geometries appear in the thesis, so I will briefly discuss both.

In the Landau gauge the magnetic Hamiltonian takes on a simple form:

$$H_B = \frac{1}{2m} \left[ (\hat{p}_x + y/\ell_B^2)^2 - \nabla^2_{y} \right], \tag{3.6}$$

where $\ell_B = 1/\sqrt{qB}$ is the magnetic length. Since $H_B$ is independent of $x$, one can write $\psi(x, y) = \psi(x)e^{-ikx}$, yielding:

$$H_B\psi(x, y) = \frac{1}{2m} \left[ (-\nabla^2_{y} + (k - y/\ell_B^2)^2) \right] \psi(x, y). \tag{3.7}$$

Solutions to this Hamiltonian are harmonic oscillator states, centred around $\ell_B^2k$ and with the spring constant $K = \sqrt{\ell_B^2/m}$. The eigenstates are then

$$\psi_{nk} \propto e^{-y^2/2\ell_B^2 - ikx} H_n(y - k\ell_B^2), \tag{3.8}$$

where $H_n(y)$ is the $n^{th}$ Hermite polynomial. The energy of these states is

$$\hat{H}\psi_{nk} = \omega_c(n + 1/2)\psi_{nk}, \tag{3.9}$$
where $\omega_c = \frac{qB}{m}$ is the cyclotron frequency. As $E_{nk}$ is independent of $k$, there exists a large number of states with the same energy.

The set of states in the $n$th energy level are collectively known as the $n$th Landau level. The number of states in a level is exactly the same as the number of flux quanta piercing the surface. That is, for a surface of dimensions $L_x$ by $L_y$, the total flux is given by $qBL_xL_y$. In these units, a quantum of flux, $\Phi_0$, is $2\pi$. Thus there are $qBL_xL_y/2\pi\ell_B$ flux quanta penetrating the system. One can see that $\ell_B$ completely determines the length scale for the system. It is therefore convenient to move to coordinates where $\ell_B = 1$ for the remainder of this chapter.

This state counting becomes clear on the cylindrical geometry. Here, there is a condition on $\psi$: $\psi(x,y) = \psi(x + L_x, y)e^{i\phi_x}$, where $L_x$ is the circumference of the cylinder. This imposes a condition on $k$:

$$kL_x - \phi_x = 2\pi l \quad l \in \mathbb{Z}. \quad (3.10)$$

That is, $k$ takes on a discrete set of values. The states $\psi_{nk}$ are then equally spaced along the $\hat{y}$ direction at $y_k = 2\pi l/L_x + \phi_x/L_x$, which are separated by $\Delta y = 2\pi/L_x$. If the length of the cylinder is $L_y$, then there are $L_xL_y/2\pi$ states in total. Also note that as the phase $\phi_x$ around the cylinder is increased from 0 to $2\pi$, the centres of the Landau states shift forward by $\Delta x$ along the $y$ axis.

### 3.2.1 Lowest Landau Level

Since the levels $\psi_{nk}$ all have the same energy for some $n$, if follows that any linear combination $\psi_n(a) = \sum_k a_k \psi_{nk}$ is also an eigenstate. This has some very profound consequences, in particular for the lowest Landau level.

A particularly useful formulation of the system is to move to the complex plane and express the coordinates as $z = x + iy$. It turns out that any wavefunction of the form

$$\psi(z) = f(z)e^{-y^2/2} \quad (3.11)$$
3.2. UNIFORM MAGNETIC FIELDS IN 2 DIMENSIONS

is an eigenstate of $\hat{H}_B$ that also lies in the lowest Landau level (LLL) when $f(z)$ is a holomorphic function [155]. Such functions have the property that they are completely determined (up to a constant) by the locations of the zeros of the function.

This is more clearly seen in the symmetric gauge. Here, eigenstates are labeled by two integers, the Landau level, $n$, and the angular momentum $m$. In the LLL, these eigenstates are:

$$\chi_{0,m} \propto z^m e^{-z\bar{z}/4}, m \geq 1,$$

where $\bar{z}$ is the complex conjugate of $z$. Since any analytic function $g(z)$ can be written as $g(z) \sum_m c_m z^m$, $\psi = g(z)e^{-z\bar{z}/4}$ will clearly lie in the LLL.

The states $\chi_{0,m}$ form rings around $z = 0$, with their maximum amplitude at $\sqrt{2m}$. These states are quite different from the states found in the Landau gauge. Since the two gauges are related by a physically irrelevant gauge transformation, it is possible to transform between the two. However, the details of this transformation are not particularly relevant for this thesis.

Landau Levels on a Torus

The previous discussion of Landau levels forms the basis for understanding the physics of the Hall effect in 2DEG. However, these states exist only on the infinite plane. For practical computations, some different geometry must instead be used, so to make the problem finite. These geometries fall into two classes, open or closed, which are used depending on which aspects of the QHE are being investigated. Open geometries include the open disk, or plane, or the cylinder.

Open surfaces, such as the disk, or rectangle, allow one to study finite size, and edge effects. Closed geometries, like the torus, or sphere, remove the presence of edge effects, leaving just the physics of the bulk. The cylinder is another common geometry, which, being open in only one dimension, lies somewhere between the open plane and the torus.

In chapter 5, the Quantum Hall effect in the JCH lattice is studied. Both the toroidal and spherical geometries are used to study quantum Hall effect physics, each with it’s
own advantages and disadvantages. For the study of systems confined to a lattice, the toroidal geometry is most suitable (mapping a lattice to a sphere introduces unnecessary complications owing to its non-zero curvature). It will therefore be helpful to consider the corresponding structure of Landau levels on the toroidal geometry. Consider the torus with dimensions $\mathbf{L} = \{L_x, L_y\}$, and skewness $L_\Delta$, as shown in Fig. 3.1. The wavefunction on a torus is subject to periodic boundary conditions (PBC), since the torus identifies points $\mathbf{r}$, and $\mathbf{r} + mL_x\hat{x} + n(L_y\hat{y} + L_\Delta\hat{x})$, for integers $m$ and $n$. Comparing the wavefunction at two points requires performing a gauge transformation, since the vector potential does not itself respect the periodicity of the torus, i.e.

$$A(\mathbf{r} + \mathbf{L}) = A(\mathbf{r}) - \nabla \int_{\mathbf{r}}^{\mathbf{r}+\mathbf{L}} A(l)dl. \quad (3.13)$$

In the Landau gauge, $A = -By\hat{x}$, this puts two conditions on the wavefunction:

$$\psi(z + L_x) = e^{i\phi_x}\psi(z)$$

$$\psi(z + \tau L_x) = e^{-i2\pi z/L_x + i\phi_y}\psi(z) \quad (3.14)$$

where the $\phi_{x,y}$ are ‘twist’ angles. The twist angles are required to access the full Hilbert
3.3. MAGNETIC FIELDS FOR A PARTICLE ON A LATTICE

space, since the restriction to the torus for a single value of $\phi_{x,y}$ picks out a restricted subspace of possible states [156].

From the Gauss-Bonnet theorem [157], these boundary conditions, coupled with the requirement that $\psi$ is single-valued, imply that the number of flux through the torus, $N_\phi$, is exactly quantised:

$$2\pi N_\phi = L_x \times L_y.$$  \hfill (3.15)

The solutions to the PBC’s can be found by expanding the conditions in terms of the planar magnetic solutions [156]. These are most conveniently expressed as:

$$\psi_n(z) = e^{-\frac{1}{2}y^2} \frac{1}{\sqrt{N_\phi L_x \pi}} \theta \left[ \begin{array}{c} -N_s \\ 0 \end{array} \right] \left( \frac{\pi}{L_x} (z + x_s)|N_s \tau \right),$$  \hfill (3.16)

where $\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|\tau)$ is the generalized elliptic theta function [158], the details of which can be found in the Appendix B. These functions simultaneously satisfy the imposed boundary conditions, and the magnetic Hamiltonian in Eq. (3.4).

3.3 Magnetic Fields for a Particle on a Lattice

A particle on a lattice, with periodic potential, has a block diagonal decomposition into orthogonal, separated bands, each with an absolutely continuous spectrum. This results naturally from the periodicity of the underlying potential, which is parametrised by the lattice vectors $a_n$. When these bands are well separated it is common to restrict the dynamics to one of these bands (which is the approach taken in the JCH model). In a similar fashion, the band structure for a particle in 2 dimensions in a constant magnetic field is divided into gapped, orthogonal bands. This band structure is also a consequence of the periodicity of the magnetic lattice, which has lattice length $\ell_B$. In this case, it is often valid to restrict dynamics to the lowest Landau level.

The combination of a periodic system with a magnetic field results in an interesting interplay between these two symmetries, since the two Hamiltonians will not commute.
in general. This interaction is encapsulated in the parameter $\alpha = \ell_B^2/(2\pi a^2)$, the ratio of magnetic to lattice areas, and the amount of flux that passes though each lattice plaquette.

In the situation where the Landau level gap is much greater than the Bloch gap, the lattice potential acts as a perturbation within each Landau level $V(-i\nabla_y/\ell_B^2, y)$.

In this thesis, it is the opposite limit that is studied, i.e. one where the Landau level separation is small, and the tight binding model is still valid. In this case, the effect of the magnetic field enters into the tight binding Hamiltonian via the Peierls substitution [159].

The periodic structure of the lattice disrupts and mixes the Landau level structures, but in an interesting and structured way. In the JCH model, the atom-cavity coupling introduces an additional energy scale that modifies this structure further. These modifications are interesting in themselves, and also play a role in the later study of the quantum Hall effect on the JCH lattice.

3.3.1 Peierls Substitution

We consider now the effect of magnetic fields to particles confined to a lattice. Specifically, we restrict ourselves to tight-binding models, in which the kinetic part of the Hamiltonian is of the of Eq. (2.40):

$$\hat{H}_K = -\sum_{x,x'} \kappa_{x',x} \hat{a}^\dagger_{x'} \hat{a}_x,$$

where the sum is over all the lattice sites. This term is the lattice analogue of the $p^2$ term in the continuum. In the continuum, the effect of a geometric curvature is a modification of the momentum $p \rightarrow p - qA$. In the case of a lattice, the appropriate replacement, known as the Peierls substitution [159], is:

$$\kappa_{x',x} \rightarrow \kappa_{x',x} e^{i\phi_{xx'}}$$

$$\phi \equiv \int_x^{x'} A(r) \cdot dr$$

That is, the particle must accumulate a geometric phase as it is travels between sites. By considering a single particle on a lattice, one can see that in the continuum, the Peierls
substitution reduces to the usual minimal substitution.

The lattice Hamiltonian is gauge dependent. However, the phase accumulated around
a closed path is not dependent on the gauge. If one has a square lattice, with \( \hat{R} \) and
\( \hat{U} \) operators that translate particles right and up respectively, then these translation
operators have the property that:

\[
U^\dagger R^\dagger UR = \sum_x |x\rangle \langle x| e^{i2\pi \alpha(x)},
\]

\[
2\pi \alpha(x) = \phi_{x,x+a_1} + \phi_{x+a_1,x+a_2} + \phi_{x+a_1+a_2,x+a_2} + \phi_{x+a_2,x+} = 2\pi \Phi_x / \Phi_0
\]

where \( a_{1,2} \) are the lattice vectors.

For a particle in a 2D lattice, \( \alpha(x) \) completely specifies the effect of the magnetic
field, even in the case of non-uniform fields.

### 3.3.2 The Hofstadter Butterfly

In [160], Hofstadter studied a single electron in a two dimensional square lattice tight-
binding model under the application of a constant, perpendicular, magnetic field. In the
Landau gauge, the Schrödinger equation for the lattice wavefunction can be separated
into \( x \) and \( y \) components: \( \psi_{x,y} = \psi_x e^{i\nu y} \), where \( \nu \) is the wave number. Then the equation
for \( \psi_x \) is:

\[
E\psi_x = \kappa(\psi_{x-1} + \psi_{x+1}) - 2 \cos (2\pi \alpha x - \nu)\psi_x,
\]

where \( \kappa \) is the complex valued hopping matrix given in Eq. (3.18). Note the similarity to
the continuum Landau problem, where the wavefunction could be written as a product
state. Here the momentum is replaced by the discrete lattice momentum, and the
harmonic potential is replaced by the cosine. As one goes to the continuum limit, with
very small lattice constant, the Hofstadter equation will reduce to Eq. (3.7).

For the constant field, the flux per plaquette, \( \alpha \), is a constant. Note that, since
\( \alpha = q Ba^2 / \Phi_0 = a^2 / \ell_B^2 \); the two length scales in the system are in competition, which is
quantified by \( \alpha \). This competition leads to a fractal band structure, shown in Fig. 3.2.
The Schrödinger equation for the Hofstadter model takes the form:

$$E\psi(m,n) = t_m \left( e^{-iA_x(m,n)}\psi(m+1,n) + e^{iA_x(m,n)}\psi(m-1,n) \right) + t_n \left( e^{-iA_y(m,n)}\psi(m,n+1) + e^{iA_y(m,n)}\psi(m,n-1) \right).$$

(3.21)

In the Landau gauge Eq. (3.21) becomes:

$$E\psi(m,n) = t_m \left( e^{i\tau\alpha n}\psi(m+1,n) + e^{-i\tau\alpha n}\psi(m-1,n) \right) + t_n \left( \psi(m,n+1) + \psi(m,n-1) \right).$$

(3.22)

This Schrödinger equation has no dependence on $m$. Bloch’s theorem can be applied via the ansatz $\psi(m+p,n) = \psi(m,n)e^{ip\nu}$. This results in the Harper equation:

$$\epsilon\psi_n = H_B(\alpha)\psi_n = \psi_{n+1} + \psi_{n-1} + 2\cos(\nu + \tau\alpha n)\psi_n$$

(3.25)

where $\epsilon = \frac{E}{t}$.

There is in general no closed form expression for the energy levels of the Hofstadter problem. The spectrum as a function of $\alpha$ is plotted in Fig. 3.2. This complicated and intricate figure has many non-trivial properties, and is the focus of much attention [160–164], owing to the fractal nature of it’s band structure. However, in this thesis, there are a few aspects of the spectrum that are of particular relevance. The fractal nature of the spectrum is intimately related to the irrationality of $\alpha$, in the Diophantine sense, when we write $\alpha = p/q$, for $p,q \in \mathbb{N}$ and $\gcd(p,q) = 1$. The first property to note is that the spectrum of $H_B(\alpha,\nu)$ has $q$ bands over the Brillouin zone $0 \leq \nu \leq 2\pi/q$.

In the limit $\alpha = 1/q \to 0$, the energy spectrum of the lower bands converges to the continuum Landau level spectrum, $\omega_c(q + 1/2)$. In these units, the cyclotron energy $\omega_c = \kappa\alpha/2$. Even for relatively large values of $\alpha$ this approximation works quite well. The corresponding eigenstates also converge to those of the continuum Landau states.

When $\alpha \ll 1$, but is not of the form $1/q$, but rather $p/q$, then the lower Landau levels
3.4. EFFECTIVE MAGNETIC FIELDS

A large array of coupled photonic cavities bears many resemblances to other systems in condensed matter and solid state physics. This allows the possibility of the simulation and investigation of aspects of the physics which may be difficult to access in other
systems, so it is natural to ask how a JCH model might act in the presence of a magnetic field.

Of course, photons do not couple to magnetic fields, and so a different route to inducing a geometric phases must be found if one wishes to explore these ideas in experiment. There have been several proposals within the literature for achieving this, most extensively in neutral atom BECs \[104, 165–168\]. Most of these mechanisms for time-reversal symmetry (TRS) breaking do not translate into atom-cavity systems, as they rely on mechanisms unavailable to quantum-optics, and so novel solutions must be sought.

In cavity QED systems several mechanisms have been proposed for inducing synthetic magnetic fields. Trapped ions \[169\], photonic crystals \[28, 107, 170\], circuit QED \[110\]. In this section I explore the nature of synthetic magnetic fields, and show how photon assisted tunneling \[103\] can be adapted to the JCH model to induce such fields.

**Time Reversal Symmetry Breaking**

A classical system has time-reversal symmetry if for any solution \(x(t), p(t)\) to the equations of motion there is a corresponding reversed motion solution: \(x'(t), p'(t)\) such that \(x, p\) can be related to \(x', p'\) by \(t \to -t\) and \(p \to -p\).

External magnetic fields, as discussed in this chapter, break this time reversal symmetry. For example, an electron in a magnetic field undergoes cyclotron motion with only one chirality. Time reversal switches the chirality, and so the system has no orbits invariant under this operation.

The laws of physics were thought to be fundamentally time-reversal symmetric until the discovery of parity violation, and charge-parity violation in interaction mediated by the weak force. It is now thought that the universe is symmetric under charge-parity-time reversal.

TRS is not so important in the study of classical mechanics as discreet symmetries are not useful for solving the equations of motion. However, in quantum mechanics, the presence, or lack, of TRS has important implications for the energy spectrum and
3.4. EFFECTIVE MAGNETIC FIELDS

eigenstate structure [171,172].

In quantum mechanics, the time-reversal operator, \( \hat{\Theta} \) is an anti-unitary operator related to the complex conjugation operator, \( \hat{K} \);

\[
\hat{\Theta} = \hat{U} \hat{K},
\]

(3.26)

where, \( \hat{U} \) is some unitary operator. Let \( \hat{H} = \frac{(\hat{p} - q\hat{A})^2}{2m} + V(\hat{x}) \). Then under the time reversal, the Hamiltonian transforms as:

\[
\hat{\Theta} \hat{H} \hat{\Theta} \to (-\hat{p} - q\hat{A})^2 2m + V(\hat{x}) + \frac{1}{2m} (\hat{p} \cdot \hat{A} + \hat{A} \cdot \hat{p} + \hat{A}^2) - \hat{U} \nabla \hat{U}^\dagger.
\]

(3.27)

One can see that, in the case that the vector potential is removable by a gauge transformation by choosing \( \hat{U} = \exp \left[ -i \int A(x) \cdot dx \right] \), then \( \hat{H} \) will be time reversal symmetric. A system that breaks TRS possesses a non-trivial vector potential. The implication is that a magnetic field directly implies TRS breaking. The converse is not necessarily true. The time reversal symmetry of a system can be induced by explicitly introducing a time dependent term into the potential energy. This is schematically outlined in Fig. 3.4.

Periodic Hamiltonians

An alternative means for TRS breaking is to explicitly introduce a time dependence into the system which lacks chiral symmetry. Floquet theory, which describes periodic time-dependent systems, will be covered in detail in Chapter 8. However, the idea will be briefly introduced here for convenience of reading.

Consider the periodic time dependent Hamiltonian \( H_0(t) \) with \( H_0(t) = H_0(t + nT), \quad n \in \mathbb{Z} \). Then the evolution of this system from time \( t = nT \) to \( t = (n + 1)T \) is given by the unitary

\[
U_f \equiv U_0(T) = \hat{T} \exp \left[ i \int_0^T dt H(t) \right],
\]

(3.28)
Figure 3.4: (a) Schematic of a square JCH lattice with a constant effective magnetic field. Photons moving around a plaquette acquire a phase $\Delta \phi$. (b) A single mode photonic cavity with frequency $\omega$ coupled to a two level atom with strength $\beta$. (c) Scheme for breaking TRS in photonic cavities: a potential $V = [V^{DC} + V^{AC} \sin(\omega rf t + \Delta \phi)] x$ ($x$ and $y$ in units of the lattice spacing) is applied to the cavities (indicated by green arrows) by dynamically tuning $\omega$. The phase offset, $\Delta \phi$, along $y$ results in the synthetic magnetic field seen in (a).
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where \( \hat{T} \) is the time ordering operator.

The Floquet Hamiltonian, \( H_f \), is defined as the time independent Hamiltonian that produces \( U_f \) over \( T \), i.e.

\[
U_f = \exp \left[ i H_f T \right].
\]  

(3.29)

\( H_f \) is not uniquely determined from this definition. However, if the period \( T \) is much shorter than other characteristic time scales in \( H_0 \), then all possible \( H_f \)'s will reduce to the same effective Hamiltonian after adiabatic elimination of fast dynamics.

This process is similar to that used in chapter 2 to produce the rotating wave approximation, whereby the fast dynamics are eliminated, and one is left with an effective Hamiltonian. The general study of these effective Hamiltonians falls under the broad banner of Floquet theory. Floquet theory has found a large number of application in driven quantum systems [173], and the interpretation of the induced effective Hamiltonians is an active area of research [174–176].

The Magnus expansion [177], is tool for approximating the propagator\(^1\) for a time dependent Hamiltonian\(^2\):

\[
U(t) = \exp[\Omega(t)] \\
\Omega(t) = \sum_{n=1}^{\infty} \Omega^{(n)}.
\]  

(3.30)

Here, the propagator is the exponentiation of an operator \( \Omega \), which is given by the sum of the Magnus series \( \Omega^{(n)} \). For the evolution given by the first two of which are:

\[
\Omega^{(1)} = -i \int_{0}^{t} d\tau_1 H(\tau_1) \\
\Omega^{(2)} = \frac{1}{2} \int_{0}^{t} d\tau_1 \int_{0}^{\tau_1} d\tau_2 [H(\tau_1), H(\tau_2)].
\]  

(3.31)

In general, the expressions for the terms in this series are rather complicated, much more so than the terms shown suggest. See e.g. [178] for techniques to find the higher terms.

\(^1\)Without loss of generality, in this section the evolution is presumed to begin from \( t = 0 \).

\(^2\)The utility of the Magnus expansion ranges far beyond quantum mechanics, see [178] for a comprehensive review.
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provide a solution that is accurate for all time, under certain convergence criteria. We can also use the expansion to recover an effective time independent Hamiltonian from a periodic system.

For a system with period $T$, we have $U(T) = \exp \Omega(T)$. If $\Omega$ satisfies the convergence criteria, (i.e. not to large in magnitude), then, if we are interested only in dynamics much longer than $T$, the propagator is approximately $U(t) \approx \exp \left[\Omega(T) \frac{t}{T}\right]$. We make the identification $\frac{\Omega}{T} \equiv iH$.

**Photon Assisted Tunneling**

Kolovsky [103] has shown how TRS breaking can be achieved in an optical lattice through photon assisted tunneling (PAT). In PAT, two off resonant cavities can be strongly coupled by the application of an oscillating potential. Extending this process to an array of cavities allows for the control of the effective tunneling matrix elements.

Consider a BEC in a 1D optical lattice which can be described with a Bose-Hubbard model of localized modes, coupled via the usual tight-binding term. In photon assisted tunneling, an electric field with is applied along the lattice axis with both a DC and an AC component. The setup is described by the Hamiltonian:

$$H = \sum_m \omega a_m^\dagger a_m + \kappa (a_{m+1}^\dagger a_m + a_m^\dagger a_{m-1}) + \left[V^{DC} + V^{AC} \cos (\omega_{rf} t + \phi)\right] a_m^\dagger a_m. \quad (3.32)$$

The DC field provides a constant detuning between cavities, which, on it’s own, suppresses inter-site hopping. The effect of the oscillating term can be seen by moving to a rotating frame, via Eq. (2.26), with the unitary transformation:

$$U = e^{-i \sum_m [\omega t + V^{DC} m t + \frac{V^{AC}}{\omega_m} \sin (\omega_{rf} t + \phi)] a_m^\dagger a_m}. \quad (3.33)$$

Using $[a_i^\dagger a_i, a_j^{(\dagger)}] = (-)a_j^{(\dagger)}$, and the Baker-Campbell-Hausdorff relations, we have:

$$Ua_i^\dagger a_j U^\dagger = \exp [i(\Theta(i) - \Theta(j))] a_i^\dagger a_j. \quad (3.34)$$
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The transformed Hamiltonian, $H'$ is then

$$H' = \sum_m \kappa (a_i^\dagger a_{m+1} \Theta(t) + a_i^\dagger a_{m-1} \Theta^*(t)), \tag{3.35}$$

$$\Theta(t) = \exp [iV_{DC} - iV_{AC} \frac{\omega_{rf}}{\omega_r} \sin (\omega_{rf} t + \phi)].$$

If the period of $\Theta(t)$, $T$, is much less than $1/\kappa$ then $\langle a_i^\dagger a_{m\pm1} \rangle$ is approximately constant over $T$, and the inter-site coupling can be given as an effective time-independent coupling $\kappa_{\text{eff}}$, which is found by averaging $\kappa \Theta(t)$ over $T$:

$$\kappa_{\text{eff}} = \frac{\kappa}{T} \int_{-T/2}^{T/2} dt \Theta(t). \tag{3.36}$$

Approximating $\frac{V_{DC}}{\omega_{rf}} \approx \frac{q}{p}$, with $p$ and $q$ coprime, gives $T = q\tau \omega_{rf}$. Transforming $t \rightarrow (t - \phi)/q$:

$$\kappa_{\text{eff}} = \frac{\kappa_{m,m+1} e^{-ip\phi/q}}{q\tau} \int_{-q\tau/2}^{q\tau/2} dt \exp \left[ \frac{p}{q} t - iV_{AC} \sin (t)/(\omega_{rf}) \right] \tag{3.37}$$

which, for $q = 1$ is

$$= \kappa J_p (V_{AC}/\omega_{rf}) e^{-ip\phi}, \tag{3.38}$$

where $J_p$ is a Bessel function of the first kind.

The Hamiltonian $H$ is invariant under time reversal. The phase picked up for the 1D array is inconsequential, as it can be eliminated simply by the translation $t \rightarrow t - \phi/\omega_{rf}$. However, if the phase $\phi$ offset is not constant throughout the system, this is no longer true. An effective magnetic field can be produced in a 2D lattice by utilizing PAT along one dimension, with the offset phase changing along the other, i.e.

$$H = \sum_{mn} \omega_{mn} a_m^\dagger a_m + \left[ V_{DC} + V_{AC} \cos (\omega_{rf} t + n\phi) \right] a_m^\dagger a_m m$$

$$+ \kappa_x (a_m^\dagger a_{m+1,n} + a_m^\dagger a_{m-1,n}) + \kappa_y (a_m^\dagger a_{m,n+1} + a_m^\dagger a_{m,n-1}).$$

Clearly this approximation can be made arbitrarily accurate.
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The effective coupling in the $y$ direction is given by:

$$
\kappa_{\text{eff}}^y = \kappa_y T \int_{T/2}^{T/2} dt e^{i V^AC_m \sin(\omega^r t) - \sin(\omega^r t + \phi)} \\
= \kappa_y J_0(2mV^AC \sin[\phi/2]/\omega^r).
$$

The effective coupling in the $y$ direction has a dependence on phase offset and the magnitude of the oscillating potential, which varies along the $x-$axis. Nevertheless, a judicious choice of inter-cavity couplings leads to the desired Hamiltonian in Eq. (3.25), with the phase per plaquette $\tau\alpha = p\phi/q$.

We choose now to have $\omega^r \approx \Delta$. For small values of $x$, $J_m(x) \approx (\frac{x}{2})^m / m!$. For sufficiently small $V^AC/\omega^r$:

$$
\kappa_{\text{eff}}^x \approx \kappa V^AC / 2\omega^r
$$

and

$$
\kappa_{\text{eff}}^{ym} \approx \kappa \left[1 - \left(mV^AC \sin(\phi/2)/\omega^r\right)^2\right].
$$

3.4.1 Validity of PAT in the Presence of On-site Interactions

There is a trade off between the various choices of parameters here. The accuracy of the approximation relies on $\kappa \ll \omega^r$. However, increasing $\omega^r$ will reduce the effective tunneling, slowing the dynamics (or equivalently reducing energy gaps), unless accompanied by a proportional increase in the driving strength, $V^AC$. Arbitrary manipulation of these parameters in a physical system may not be possible. It is therefore important to have an idea of the range of parameters for which the PAT effective magnetic field scheme will achieve a faithful simulation of the JCH system with magnetic field. The description of PAT developed in the previous section was in the context of a single photon, in an infinite, atomless cavity array. In this section, a more thorough analysis is developed for the JCH model, which puts better bounds on the situations pin which PAT may be used as a mechanism for producing gauge fields.

I therefore explicitly introduce the JC inter-atom coupling to the driven lattice system.
to study the validity of the PAT scheme in this setting.

\[ H(t) = \sum_x \left[ V^{\text{DC}} + V^{\text{AC}} \cos (\omega t + 2\pi y\alpha) \right] (a_x \dagger a_x + \frac{1}{2}\sigma_{z,x}) x + H_{JCH} \]  

The time dependent potential is applied to the atomic energy simultaneously so that the atom-cavity detuning remains constant in time. Without the modulated atomic energy the JC coupling would still remain, but at a reduced magnitude, as per Eq. (3.38).

A realistic JCH lattice with the PAT effective field will deviate from the ideal case in two ways. Firstly, the behaviour of the system will deviate from the infinite lattices due to the presence of edges. The edges lead to coupling between Landau levels.

Secondly, the JC interaction does not commute with the inter-cavity coupling, and therefore will affect the effective dynamics. For the effective Hamiltonian approximation to be valid, we must have \( \beta/\omega \ll 1 \), similar to the condition on \( \kappa \). That is, there must be sufficiently small transfer between atomic and cavity populations during each period of the oscillation driving field to ignore the 2nd order Magnus term, \( \Omega^{(2)} \) in Eq. (3.30).

The accuracy of the approximation can be tested by direct comparison of the time dependent unitary evolution to the evolution of the effective Hamiltonian. Let \( U_{\text{PAT}}(t) \) be the exact unitary evolution under the driving fields, and \( U_{\text{eff}}(t) \) be the unitary evolution of a system with the effective Hamiltonian. Then the difference in the unitary evolution can be captured by the minimum fidelity. That is,

\[ F_{\text{min}}(t) = \min_\psi | \langle \psi | U_{\text{eff}}(t) U_{\text{PAT}}^\dagger(t) | \psi \rangle | \]  

If the unitary matrices are sufficiently similar, then the fidelity can be bounded by:

\[ F_{\text{min}} \geq \cos \max_i \arg v_i \geq 1 - (\max_i \arg v_i)^2 / 2 \]  

where the \( v_i \) are eigenvalues of \( U_{\text{eff}}(t) U_{\text{PAT}}^\dagger(t) \).

We can therefore bound the error during evolution as \( \delta \leq (\max_i \arg v_i)^2 \). That is, the
deviation of any expectation value from the ideal magnetic field case will be proportional to $\delta$. Essentially, $\delta$ quantifies the maximum deviation of the wavefunction overlap from unity under the evolution of two different time evolutions. This puts a lower bound on the similarity of two unitary maps.

I simulate the PAT magnetic field scheme on a $10 \times 10$ site lattice for a selection of parameters. The $\hat{x}$ and $\hat{y}$ tunneling strengths are modified such that the effective inter-site coupling is $\kappa_{\text{eff}} = 1$, and the inter-cavity coupling, $\beta = 1$.

Figure 3.5 shows how the PAT generated time evolution compares with evolution under a magnetic field. Here, $\delta$ scales with the various system parameters, which shows how closely the PAT scheme approximates an explicit (synthetic) magnetic field. Smaller values for $\delta$ indicate that the system evolves similarly under both Hamiltonians.

From inspection of the Magnus transform used to derive the effective Hamiltonian, it is expected that the error should scale as

$$\delta(t) \propto \left[ \left( \frac{V_{\text{AC}}}{\omega_{\text{rf}}} \right)^2 + \left( \frac{\beta}{\omega_{\text{rf}}} \right)^2 \right] \left( \omega_{\text{rf}} t \right)^2.$$  \hfill (3.44)

This scaling is confirmed numerically, and is evident in Fig. 3.5. It can be seen from Fig. 3.5 that the error per cycle is proportional to $\left( V_{\text{AC}} / \omega_{\text{rf}} \right)^2$, as expected. However, since the effective coupling is $\approx \kappa V_{\text{AC}} / \omega_{\text{rf}}$, the number of cycles required must increase, for a given $\kappa$, to allow sufficient time for the same evolution to occur. Therefore, we require that $V_{\text{AC}} / \omega_{\text{rf}} \ll 0$ for the simulation to succeed.

Hence, in principal, the magnetic lattice may be simulated through the PAT scheme arbitrary accurately by increasing the driving frequency. In practice this is not always possible, and the precise choice of system parameters will depend on the particulars of the application.

3.4.2 Photon Assisted Tunneling in Circuit QED

Within the circuit QED framework, the present state of the art should be sufficient to realise these effective Hamiltonians. The on-site driving energy supposed in Eq. (3.32)
Figure 3.5: Scaling of $\delta$ as a function of different parameters. a) $\delta(t)$ for a JCH lattice with $\omega^{\text{rf}} = 2\pi \times 20\kappa$ and $V^{\text{AC}}/\omega^{\text{rf}} = 0.1$. $\delta$ after one period for: b) $V^{\text{AC}}/\omega^{\text{rf}}$. c) $2\pi/\omega$. d) $\beta$. 
can be directly achieved in a transmon qubit by moderating the applied flux $\Phi$.

In section 2.4, it was shown that a large, constant detuning of atoms in a JCH lattice decouples the photonic and atomic modes. When the atomic energy is much less then the photonic energy, the system can be described as a hardcore boson model, where the effective atom-atom coupling is mediated by the photonic coupling. In circuit QED, it is possible to construct this hardcore model simply by capacitively coupling the qubits directly.

Photons, in general, and in the circuit QED architecture, do not respond directly to electric fields. However, a gradient in the cavity frequency has the identical effect. Recently, cavities with tunable resonances have been fabricated [179, 180]. This is achieved by the inclusion of an intra-cavity Josephson junction, which changes the cavity boundary conditions, and can be tuned via a magnetic field. Transmission line resonator experiments [179] have shown $\omega_{rf}$ can be driven at $\mathcal{O}(10^3)$ times the cavity dissipation frequency. Such fast driving is sufficient to achieve the required rates to realise photon tunneling as described in this chapter. A laboratory demonstration of the proposed scheme is therefore imminently realizable with present technology.

3.4.3 Floquet States

The photon assisted tunneling mechanism hitherto described is a time-dependent, and therefore non-equilibrium system. While the evolution of a state is well described by the JCH with the effective tunnelling $\kappa_{x,y}^{\text{eff}}$ in Eq. (3.40), it is not sensible to speak of a groundstate in the usual thermodynamic sense. Indeed, the periodic driving continually adds energy to the system. Much of this thesis is concerned with the behaviour of a JCH lattice as considered from the perspective of thermodynamic, condensed matter theory. In what sense then can the system in question be viewed as a many-body system in thermal equilibrium?

The problem of dissipation in periodically driven systems has been considered by several authors [181]. The broad conclusion is that, under some assumptions, the time averaged energies, eigenstates of the effective Hamiltonian, will stand in for the energies of
a time independent system. These results require that the inter-state coupling induced by
the dissipative terms be much smaller than the quasi-energy differences. In general, this
will not be satisfied. However, the fast-driving condition already required for the analysis,
$E/\omega \ll 2\pi$, guarantees this, since the lattice energy is bounded. For our purposes we
presume that the dissipative terms can be made arbitrarily small.

3.5 Single Excitation in the JCH model

The Hofstadter problem finds a natural counterpart in the JCH lattice with effective
magnetic field. For a single excitation in a JCH system, Bloch’s theorem can be applied
to simplify the problem in the same manner as the Hofstadter case, except that now
the wavefunction $\psi$ has two components: $\psi^{a}$ and $\psi^{s}$, corresponding to the photonic and
atomic parts respectively. The energy equation is:

$$E\psi_{n} = \begin{pmatrix} -2\kappa \cos (\nu + 2\pi \alpha n) & \beta \\ \beta & \Delta \end{pmatrix} \psi_{n} - \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}(\psi_{n-1} + \psi_{n+1}).$$  \hspace{1cm} \text{(3.45)}

The photonic part has the same solutions as the Hofstadter model, but the JC term causes
mixing between atomic and photonic parts. The problem can be solved by assuming a
separable wavefunction:

$$\psi_{n} = \psi_{n}^{H}\psi_{JC},$$  \hspace{1cm} \text{(3.46)}

where $\psi^{H}$ is a function over the lattice sites, and $\psi^{JC}$ is a state of a JC cavity. By
assuming a wavefunction of the form of Eq. (3.46), with $\psi^{H}$ a solution to the Harper
equation with normalized energy $\epsilon$, the problem can be reduced to a single site:

$$E\psi_{JC} = \begin{pmatrix} \kappa \epsilon & \beta \\ \beta & \Delta \end{pmatrix} \psi_{JC}. $$  \hspace{1cm} \text{(3.47)}

This matrix equation has the two solutions:

$$E_{\pm}(\epsilon) = \frac{1}{2} \left( \pm \sqrt{(\Delta - \kappa \epsilon)^2 + 4\beta^2} + \Delta + \kappa \epsilon \right),$$  \hspace{1cm} \text{(3.48)}
I.e. the solutions are polaritons with a kinetic component corresponding to the harper energy.

Figure 3.6 shows the energy spectrum of the JCH lattice with magnetic field over $\alpha$. The butterfly structure of the single particle is duplicated, each copy corresponding to one of the two polariton species. Note first that for the $\Delta = 0$ case, the lowest energy states are almost completely projected into the photonic modes. When the detuning is large, the two modes are either completely atomic or photonic, such as in Fig. 3.6 c,d).

3.6 Summary

The introduction of magnetic fields to two dimensional quantum systems can lead to a whole range of interesting physics. Here, it has discussed how synthetic magnetic fields can be achieved in JCH systems. In particular, the photon assisted tunneling scheme was adapted for the JCH system. Here, one finds that the synthetic field can be introduced with high fidelity by appropriately modulating the atomic exciting energy along with the cavity frequency. The JCH model with a magnetic has an interesting spectrum that maps onto the Hofstadter butterfly.
3.6. SUMMARY

Figure 3.6: Normalized energy spectrum as a function of alpha for the JCH for a) $\Delta = 0$ b) $\Delta = 2$. c) and d) show the spectrum for $\Delta = -7$, where d) is showing only the atomic states.
In this chapter I study superfluid properties of the JCH model with an effective magnetic field. Using a mean-field theory, derived from the Gutzwiller Ansatz, the super-fluid-Mott phase transition is found to be moderated by the presence of the magnetic field. Inside the super-fluid phase, the non-linear interactions favor the formation of a triangular vortex lattice. The equations for the superfluid near the phase transition are found to map to the Gross-Pitaevskii equation in certain limits.

4.1 Mott-Superfluid Phase Transition

A superfluid is a state of matter in which the particles exhibit a viscousless and irrotational flow [182]. These properties require that quantum fluctuations exist across the full volume of the fluid, such that the current is independent of the linear extent of the system. The degree of superfluidity can be quantified by the existence of off-diagonal long range order (ODLRO) in the reduced density matrix. For a Bosonic superfluid, this occurs for the first order reduced density matrix $\rho_{xy}^1 = \text{Tr} a_x^\dagger a_y \rho$, where $\rho$ is the density matrix of
the state. A state can be said to have ODLRO when

$$\lim_{|x-x'| \to \infty} \rho^1_{xx'} \neq 0.$$  \hfill (4.1)

A coherent collection of photons, being themselves bosons, qualifies as a superfluid.

There are several mechanisms by which superfluidity in a system can be destroyed, with temperature and disorder being the usual culprits [182]. The presence of either of these in sufficient amounts will lead to a breakdown of the long-range fluctuations, and a transition to a different state of matter. The Mott-superfluid quantum phase transition occurs without the presence of any randomness at all, and so is of a different character to these other transitions.

The Mott phase is a quantum phase without long-range fluctuations that can exists in interacting lattice models. In a system with a periodic potential, interactions can lead to the loss of superfluidity. In this case, inter-particle on-site interactions induce destructive interference that, once the interaction strength exceeds some critical value, leads to a complete loss of superfluidity. In the momentum representation, it can be thought of as product of many particles with random phases induced by particle-particle interactions. Thus, from this perspective, the single particle density matrix with $N_p$ particles:

$$\rho^1_{xy} \approx \frac{1}{N_p} \sum_n \exp(i\phi_n |x-y|),$$  \hfill (4.2)

which, for random $\phi_n$’s, will drop to 0 in the thermodynamic limit ($N_p \to \infty$). This phase is gapped when there is exactly integer filling of the cavity sites $\langle L \rangle \in \mathbb{N}$.

The Jaynes-Cummings-Hubbard model is similar in structure to the Bose-Hubbard one, with the Jaynes-Cummings interaction acting as an on-site particle-particle interaction. Naturally, the two models share some similar properties. The JCH model exhibits a phase transition between a Mott phase, and a super-fluid phase which is seen in the Bose-Hubbard case. That the JCH model possesses a Mott-superfluid transition was briefly discussed in chapter 1. The particulars of the Mott-superfluid phase transition
4.1. MOTT-SUPERFLUID PHASE TRANSITION

derive from competition between the kinetic and interaction energy in the JCH model. In
general, the groundstate will lie somewhere between the pure superfluid and pure Mott
states.

In the limit of no atom-cavity coupling there will be no effective interactions between
photons in the system. The groundstate of an $N_p$ photon system will be a basic superfluid:

$$|\Psi_{sf}\rangle = (\hat{a}_{k=0}^\dagger)^{N_p} |0\rangle.$$  \[4.3\]

Here, the behaviour of the atomic degrees of freedom are inconsequential, since they are
completely decoupled from the photons.

The presence of a non-zero atom-photon interaction mixes the atomic and photonic
states, inducing scattering between polaritons. At some critical strength, the system
undergoes a phase transition to the insulating Mott phase.

In contrast to the non-interacting limit, we can consider the limit of very large
interaction strength : $\beta/\kappa \rightarrow \infty$. In this case the groundstate is simply a product state
of decoupled JC cavities,

$$|\Psi_{\text{Mott}}\rangle = \prod_i^N P_{x_i}^\dagger |0\rangle,$$  \[4.4\]

where, with a slight abuse of notation$^1$, $P_{x\pm}^\dagger$ is the creation operator for a polariton
at site $x$, i.e. each cavity has some integer number of excitations in the lower branch
polariton state.

The behaviour of the system in-between these two limits is the focus of this chapter.
While this is a fairly well studied problem [31,83], the work here explores new ground
by considering the effect that external (artificial) magnetic fields have on the JCH
Mott-superfluid phase transition.

$^1$The polaritons are non-linear excitations, and so there is no lowering/raising operator in the normal
sense. However, a consistent, but cumbersome, approach can be defined [83].
CHAPTER 4. SUPERFLUID-MOTT TRANSITION AND VORTICES IN THE JCH MODEL

4.1.1 Gutzwiller Ansatz, and Mean Field Theory

Somewhere between the well defined limits for the interacting and non-interacting limits of the JCH model lies a quantum phase transition. To answer questions regarding the location and properties of this transition requires knowledge of the system’s groundstate. For large system sizes, the exact quantum mechanical problem of the JCH model is too computationally difficult to approach directly, either analytically or numerically. Various approximations have been used to study the superfluid Mott insulator phase transition in the JCH model: Mean field approximations \[31,83\], Monte-Carlo simulations \[84\], and density matrix renormalization group \[183,184\].

The Gutzwiller ansatz \[154\] is an approximation that reduces the complexity of the many-body problem, including strongly interacting Bosons \[185,186\]. It assumes that the groundstate can be approximated by a projection onto a product of individual cavity states:

\[
|\Psi_{GA}\rangle = P_{N_\ell} \prod_x |\psi_x\rangle,
\]

where the projection \( P_{N_\ell} \) is onto the subspace of total excitation number \( N_\ell \). The Gutzwiller ansatz allows one to find a groundstate via the variational technique; varying the single site wavefunctions so as to minimize the system’s energy.

The Gutzwiller ansatz may be used to find an approximate groundstate for some Hamiltonian, \( H \), by minimizing the total energy over free parameters of \( |\Psi_{GA}\rangle \), i.e.

\[
\frac{\delta E}{\delta |\Psi_{GA}\rangle} = 0, \quad \text{where} \quad E \left[ |\Psi_{GA}\rangle \right] = \frac{\langle \Psi_{GA} | H | \Psi_{GA} \rangle}{\langle \Psi_{GA} | \Psi_{GA} \rangle}.
\]

If \( |\Psi_{GA}\rangle \) minimizes Eq. (4.6) then the product state structure implies that there exists a Hamiltonian on each site \( h_i \) for which \( |\psi_i\rangle \) is the groundstate. Thus there is a Hamiltonian \( H_{\text{eff}} = \sum h_i \) for which \( |\Psi_{GA}\rangle \) is an exact ground state.

To find this effective Hamiltonian one evaluates the JCH energy with the Gutzwiller state. The JCH Hamiltonian has an on-site Jaynes-Cummings term which is trivial to compute. It also has a kinetic energy term which involves operators acting on separate
cavities. Evaluating this term requires a bit more work.

In the thermodynamic limit, the local product form of $|\Psi^{GA}\rangle$ justifies the assumption of separability of non-local products of operators that commute with the total excitation number, i.e.

$$\frac{\langle \Psi^{GA}|A_x B_{x'}|\Psi^{GA}\rangle}{\langle \Psi^{GA}|\Psi^{GA}\rangle} = \langle \psi_x|A_x|\psi_x\rangle \langle \psi_{x'}|B_{x'}|\psi_{x'}\rangle \quad \forall x \neq x'$$ \hspace{1cm} (4.7)

if $[A_x B_{x'}, N] = 0$. This separability is true under assumptions on the character of variances in excitation number of each site’s wavefunction. To see this, one can write

$$\frac{\langle \Psi^{GA}|A_x B_{x'}|\Psi^{GA}\rangle}{\langle \Psi^{GA}|\Psi^{GA}\rangle} = \sum_{i=n}^{N_{\ell}} \langle \psi_x| P_k A_x B_{x'} P_k |\psi_x\rangle \times \left( \prod_{i \neq x, x'} \langle \psi_i| \right) P_{N_{\ell}-n} \left( \prod_{j \neq x, x'} \langle \psi_j| \right).$$ \hspace{1cm} (4.8)

By the central limit theorem, the quantity in the second line should be approximately unity over the support of $\langle \psi_x| P_k A_x B_{x'} P_k |\psi_x\rangle$ when the variance in the excitation number satisfy the Lyapunov condition [187] and $N_{\ell} \approx \sum_{i=1}^{n} \langle \psi_i|n_i|\psi_i\rangle$. This all simply justifies the intuition that, for a large system, the global conservation of excitation number should have little consequences for local observables.

Using Eq. (4.8), the inter-site tunneling can be factorized as:

$$\langle \Psi^{GA}|a_x^{\dagger} a_{x'}|\Psi^{GA}\rangle = \langle \psi_x|a_x^{\dagger}|\psi_x\rangle \langle \psi_{x'}|a_{x'}|\psi_{x'}\rangle.$$ \hspace{1cm} (4.9)

The total energy of the Gutzwiller state is then:

$$E(\Psi^{GA}) = \sum_x \langle \psi_x|H^{JC}_x|\psi_x\rangle + \sum_{xx'} \kappa_{xx'} \left( \langle \psi_x|a_x|\psi_x\rangle \langle \psi_{x'}|a_{x'}^{\dagger}|\psi_{x'}\rangle + c.c. \right).$$ \hspace{1cm} (4.10)

From this expression of the energy, it is possible to write down a Hamiltonian which has
the Gutzwiller state as a groundstate with the same energy as Eq. (4.10).

\[ H_{\text{eff}} = \sum_x H_{\text{eff}}^x, \]

which is a sum of local, commuting Hamiltonians that reflect the product state nature of the trial wavefunction. The local effective Hamiltonians are

\[ H_{\text{eff}}^x = H_{JC}^x - \sum_{x'} \left[ \kappa_{xx'} \psi_{x'}^{*} \left( a_x - \frac{1}{2} \psi_x \right) + \text{h.c.} \right], \]

where we have introduced the notation \( \psi_x = \langle \psi_x | a_x | \psi_x \rangle \).

The \( \psi_x \) turn out to be a very important quantity for the system. They qualify as a local superfluid order parameter. Furthermore, each \( H_{\text{eff}}^x \) depends non-linearly on the state of neighboring sites through the \( \psi_x \)'s at each site. Indeed, this effective Hamiltonian is the same as arrived at via mean-field theory \[182\], where each site experiences only the averaged effects from the rest of the system. However, the approach taken here makes explicit the connection between the underlying product state assumption, and the effective low energy dynamics.

The local order parameters, \( \psi = \{ \psi_x \} \) determine the nature of the groundstate. From Eq. (4.8) it follows that the off diagonal elements of the single particle density matrix are determined by \( \psi \): \( \rho_{xx'} = \psi_x \psi_{x'}^{*} \). Since these off-diagonal elements factorize exactly under the Gutzwiller ansatz, the system can be said to be superfluid whenever there is system wide connected non-zero \( \psi_i \)'s. The Mott insulating phase is characterised by \( \psi_i = 0 \) across the whole system \[188\].

The procedure for finding this groundstate, and studying the solutions is now detailed. However, it is worth noting that the implications drawn from this analysis are only valid in the regime where inter-cavity entanglement is small, where the Gutzwiller wavefunction, \( |\Psi^{\text{GA}}\rangle \), is a good groundstate. \( |\Psi^{\text{GA}}\rangle \) becomes increasing accurate as the lattice dimensions increases, but for smaller dimensions will deviate significantly for the ‘true’ groundstate. For example, in the 1 dimensional case there is no true Mott-superfluid
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phase transition [189], but the transition is predicted under the Gutzwiller ansatz. In the opposite limit, of a lattice with infinite dimensions, correlations between particles are divided between an infinite number of lattice sites, and the approximation becomes exact [190].

For the study of (synthetic) magnetic fields in the JCH the system is restricted to two dimensions. In this case it has been shown, for non-magnetic systems, that while the correlations discarded under the Gutzwiller ansatz can affect the details of the system’s groundstate, the broad properties are consistent with results from more rigorous methods, such as Monte-Carlo simulations [84], and density matrix renormalization group [183,184].

The method presented can be applied to any locally interacting system which has limited long-range correlations. This set of systems includes the JCH model, along with related models, such as other Hubbard models [191], and Ising models [182].

4.1.2 Chemical Potential and Open System Effects

The effective Hamiltonian arrived at though the Gutzwiller procedure removes the restriction of the system to a specific number of excitations, since, except in the \( \psi = 0 \) case, the Hamiltonian does not conserve the particle number. However, it is possible to restrict the particle density implicitly though the introduction of a chemical potential:

\[
H(\mu) = H_{\text{JCH}} - \mu L. \tag{4.13}
\]

That is, the system is now studied in the grand-canonical ensemble, and the system’s groundstate at the desired density is found by varying the chemical potential. In the case of the effective Hamiltonian in Eq. (4.12), this is necessary, since this Hamiltonian does not preserve total excitation number.

There has been some confusion in the literature [31,192] as to the physical significance, or meaning, of the chemical potential in photonic systems. From a mathematical perspective, the chemical potential introduced in this manner acts simply as a Lagrange multiplier. The choice of \( \mu \) defines an excitation density for the groundstate.
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The chemical potential can also appear as a real physical quantity. Consider the JCH weakly coupled to a bath of photons, all with frequency $\omega_b$ such that $\omega - \omega_b = \mu$. Assuming a mechanism for thermalization, such a coupling of the bath to atomic or photonic modes, the pump will act to drive the system to some groundstate, determined by $\mu = \omega_{pump}$. Weakly blue band pumping will act to create a positive chemical potential for photons in the system. Although a JCH system under such pumping, when paired with photon losses, will exhibit non-equilibrium dynamics, when these effects are small, the system will be well described by Eq. (4.13) in the groundstate \[193\].

For real world cavity lattices, the photons in the system will not be conserved, due to dissipation. In contrast to cold atom systems, where some set number of particles can be loaded into the lattice, a photonic lattice is driven by a laser, and will be in a steady state where losses and driving equilibrate \[139\]. Still, for an array of high quality cavities, one may consider a regime of weak pumping, and weak losses, such that the mean-field approximation made here captures the broad physics of the system \[140\].

The mean field Hamiltonian in Eq. (4.12) is a sum of local commuting Hamiltonians. The chemical potential can be similarly decomposed, such that the full Hamiltonian is:

$$H^{\text{eff}} = \sum_x H_x(\psi) - \mu L_x.$$  \hspace{1cm} (4.14)

This describes a system in the grand canonical ensemble, which can exchange particles with the environment. The advantage is that Hilbert space is a product of local spaces, which is not the case for the canonical ensemble. Depending on the particular approach, such as the one taken in this chapter, this can significantly simplify things, although not always (as is the case in chapter 5, where it is more parsimonious to restrict the Hilbert space to a specific excitation number). This approach is particularly useful for considering infinite systems. In this case, it is more useful to think of the chemical potential as a parameter.
4.2 Mott-Superfluid Phase Transition: Zero Synthetic Magnetic Field ($\alpha = 0$)

The simplest situation in which to study the Mott-superfluid transition is in the homogeneous lattice. That is, one in which we can write $\kappa_{xx'} = \kappa$ across the entire lattice, where $\kappa$ is the constant tunneling rate. The Hamiltonian describing this system is invariant under translations $T_i$ along lattice vectors $a_i$:

$$H = T_iHT_i^\dagger.$$  \hspace{1cm} (4.15)

In general, the groundstate of a translation invariant groundstate will not itself be invariant (e.g. in systems with inter-cavity interactions [194]). However, for the case considered here, the mean field groundstate is in fact invariant. This fact allows one to reduce the problem to that of a single site. That is, since

$$|\psi_x\rangle = |\psi_{x'}\rangle \forall x, x',$$  \hspace{1cm} (4.16)

there is just a single $\psi = \langle a_i \rangle$ that characterises the entire groundstate, i.e. $\psi = \psi 1$, where $1$ is a vector of 1’s.

Under this assumption we can reduce the kinetic term in Eq. (4.12) by making the substitution

$$\sum_{x'} \kappa_{xx'} \psi_{x'}^* = z \kappa \psi,$$  \hspace{1cm} (4.17)

where $z$ is the coordination number, the number of neighbors of each site. For example, in the two dimensional square lattice (the case considered in this chapter), $z = 4$. This factor comes from the fact that each adjacent site adds the same factor of $\kappa \psi$ to the mean field Hamiltonian. In general this will not always be the case, and is relevant only for the homogeneous lattice. The position of $z$ in the mean-field Hamiltonian also shows how the mean-field approach is unable to discriminate with respect to the dimensionality of the lattice, prohibiting the discovery of any dimensional dependent effects (re. the


absence of a phase transition in 1 dimension).

The mean-field Hamiltonian for the homogeneous system is therefore:

\[ H_{\text{MF}}(\psi) = H_{\text{JC}} - zK\psi(a^\dagger + a - \psi) - \mu L, \quad (4.18) \]

where one can assume \( \psi \in \mathbb{R} \) without loss of generality\(^2\). The groundstate of the system is then the state that minimizes the energy of \( H_{\text{MF}} \) w.r.t. the single parameter \( \psi \).

### 4.2.1 The Self-Consistent Groundstate Algorithm

There is no closed form expression for the solutions of the Hamiltonian in Eq. (4.18), so approximate or numerical techniques must be applied. Gradient decent of the energy functional is sufficiently efficient at finding minima the ground state of \( H_{\text{MF}} \) in most cases, but is susceptible to frustration in some circumstances (although not in the present case of a homogeneous lattice).

An alternate approach is to converge to the groundstate by self-consistent iteration. By finding the groundstate of \( H_{\text{MF}} \) for some choice of \( \psi \), and using the resultant state to update \( \psi \) by computing \( \psi_i = \langle \hat{a}_i \rangle \). This method guarantees the self-consistency of \( \psi \), and will also be monotonic in the total energy [195].

The general procedure is fairly straight forward:

1. Choose some non-zero initial value for \( \psi \).
2. Find the groundstate \( |\psi\rangle \) of \( H_{\text{eff}}(\psi) \) [Eq. (4.18)].
3. Evaluate \( \psi' = \langle \psi|a|\psi \rangle \).
4. (a) if \( \psi - \psi' < \delta \), then the solution has been found.
   (b) otherwise, let \( \psi = \eta \psi + (1 - \eta)\psi' \) and return to step 2.

Here, \( \eta \) is a dampening factor that can help with convergence. It is not useful in the single site case (and can be set to 0), but has been found to be helpful in the magnetic case, as well as in the work in [194].

\(^2\)Due to the absence of time reversal symmetry breaking and gauge symmetry
4.2. MOTT-SUPERFLUID PHASE TRANSITION: ZERO SYNTHETIC MAGNETIC FIELD ($\alpha = 0$)

Figure 4.1: Phase diagram for the JCH model at $\Delta = 0$. Note the well separated lobes (in black) for low $\kappa$, and the strange behaviour near $\mu = 0$ resulting from the truncated Hilbert space. The red line along the phase boundary is the analytically determined value for the critical $\kappa$ where the transition occurs.

4.2.2 Numerical Computation of the Phase Diagram

The phase of the JCH is dependent upon several parameters. These are (for the present model) $\beta$, $\Delta$, $\kappa$, and $\mu$. $\beta$ can be factored out of the model without affecting the structure, since it just defines an energy scale. The phase diagram as a function of $\Delta$ is not particularly interesting in this context. In the limits of $|\Delta/\beta| \gg 1$, the JCH model can be mapped to the Bose-Hubbard model (section 2.4). Since $\Delta$ more or less governs the strength of the interactions in the system, it has the effect of deforming the phase boundary, but not changing it in any essential way. For this reason, I restrict the analysis of the JCH phase transition to the case of $\Delta = 0$, without any real loss of generality.\footnote{A more thorough investigation of the role of $\Delta$ in the JCH Mott-superfluid phase transition can be found in [83].}

A numerical computation of the phase diagram is shown in Fig. 4.1. One can readily observe the well-known result [31, 83] that the parameter space is separated into two distinct phases. For low hopping strength $\bar{\kappa} = \kappa/\beta$, one finds lobes of vanishing superfluid order parameter, i.e. the Mott-insulating phase. Each lobe corresponds to a state with an
integer number of strongly localised excitations per site. For example the first (second)
Mott lobe, with $\mu = (\mu - \omega)/\beta < -1$ ($-1 < \mu = (\mu - \omega)/\beta < -0.4$) corresponds to the
states $|-,0\rangle$ ($|-,1\rangle$). For low chemical potential, there are no excitations in the system.
Raising the chemical potential, the lobe is successively filled with one, two, and more
excitations per site. There is an accumulation point for Mott lobes at $\mu = 0$. The absence
of this in the phase diagram is due to a lack of numerical precision, since higher and
higher excitation numbers are required.

At sufficiently large $\kappa$, the system undergoes a phase transition into a phase of
finite superfluid order parameter. The excitations are homogeneously distributed and
delocalised. As $\kappa$ increases past the transition the order parameter increases, as does the
number of excitations.

### 4.2.3 Analytic Theory at the Phase Boundary

Near the Mott-superfluid phase boundary the superfluid parameter $\psi$ is very small across
the entire system. We seek to minimize the energy functional in Eq. (4.6) over the
Gutzwiller wavefunction. However, it is also possible to consider the groundstate as an
independent function of $\psi$. This is possible, since [182]

$$
\psi = \langle a \rangle \quad \text{if} \quad \frac{\delta E}{\delta \psi} = 0. \quad (4.19)
$$

Therefore, one can treat effect of the mean-field perturbatively in $\psi$.

Consider the JCH system at some value of $\kappa$ and $\mu$. For $\psi = 0$ the groundstate of
the cavity will be a lower branch polariton eigenstate of $H^{\text{JCH}}$, with an integer number
of excitations $|\ell, -\rangle$. Which eigenstate corresponds to the groundstate is determined by
finding the value for $\ell$ which minimizes $E^{\text{JCH}}_{\ell,-} - \ell \mu$. Eq. (2.36) gives an analytic form of
$E^{\text{JCH}}_{\ell,-}$, so that for $\Delta = 0$ one needs the $\ell$ that minimizes $-\sqrt{\ell} - \frac{\mu}{\beta} \ell$. Or, equivalently, the
$\ell^{\text{th}}$ Mott lobe lies between $\frac{\mu}{\beta} = -\sqrt{\ell}$ and $\frac{\mu}{\beta} = -\sqrt{\ell + 1} + \sqrt{\ell}$.

Taking $z\kappa(a + a^\dagger)\psi$ to be the perturbation we can find an expression for the energy
4.2. MOTT-SUPERFLUID PHASE TRANSITION: ZERO SYNTHETIC MAGNETIC FIELD ($\alpha = 0$)

![Graph showing properties of the superfluid near the phase transition.](image)

in terms of $\psi$:

$$E(\psi) = E_{JC}^{\ell} - R_\ell \kappa^2 \psi^2 + S_\ell \kappa^4 \psi^4 - \kappa \psi^2,$$

where $R_\ell$ and $S_\ell$ are positive functions of $\mu$, $\beta$ and $\delta$. These are scalars which come from the 2nd and 4th order perturbative corrections to the JC energy. The precise form for these parameters are derived later, but are shown in Fig. 4.2. For now, we focus on the solution to these equations, making no assumptions about the specific values of $S_\ell$ and $R_\ell$.

The location of the transition occurs when some non-zero values for the $\psi_i$ give an energy lower than the groundstate energy in the Mott, i.e.

$$-\kappa^2 R_\ell \psi^2 + \kappa \psi^2 < 0,$$

the solution of which is given by:

$$\kappa_\ell = (z R_\ell)^{-1}.$$

The analytically determined phase transition is shown as a red line in Fig. 4.1.
Perturbative Corrections to the Mean-Field Hamiltonian

I now compute the constants that determine the near-boundary behaviour of the photons using perturbation theory. The corrections only depend on $\psi$. Without loss of generality we can assume that $\psi$ is real and positive, since the phase can be removed by a simple gauge transformation.

The unperturbed Hamiltonian is simply the single site mean field Jaynes-Cummings Hamiltonian, $H^{\text{JCH}}$. For the $\ell$th Mott lobe the zero order state is $|\psi^{(0)}_{\ell}\rangle = |\ell, -\rangle$ with energy $E_{\ell,-}$. The first order corrections to the single site wave functions are:

$$|\psi^{(1)}_{\ell}\rangle = z\kappa \psi \sum_{m=\ell\pm 1} \sum_{p=+, -} |m, p\rangle \frac{\langle m, p|a^\dagger + a|\ell, -\rangle}{E_{\ell,-} - E_{mp}}.$$  \hspace{1cm} \text{(4.23)}

In terms of the JC mixing angle, the terms in Eq. (4.23) are given by

$$\langle \ell + 1, - | a^\dagger | \ell, -\rangle = \cos \theta_{\ell} \cos \theta_{\ell+1},$$
$$\langle \ell + 1, + | a^\dagger | \ell, -\rangle = \cos \theta_{\ell} \sin \theta_{\ell+1},$$
$$\langle \ell - 1, - | a | \ell, -\rangle = \cos \theta_{\ell} \cos \theta_{\ell-1},$$
$$\langle \ell - 1, + | a | \ell, -\rangle = \cos \theta_{\ell} \sin \theta_{\ell-1},$$  \hspace{1cm} \text{(4.24)}

where $|0, -\rangle = |g, 0\rangle$, $\cos \theta_0 = 1$, and $\sin \theta_\ell = \cos \theta_\ell = 0$ for $\ell < 0$.

From the perturbed states one can compute the order parameter,

$$\langle a_{\ell}(\psi) \rangle = \psi \sum_{\alpha=\pm} \frac{|\langle \ell + 1, \alpha | a^\dagger | \ell, -\rangle|^2}{E_{\ell+1, \alpha} - E_{\ell,-} - \mu} + \frac{|\langle \ell - 1, \alpha | a | \ell, -\rangle|^2}{E_{\ell-1, \alpha} - E_{\ell,-} - \mu} \equiv R_{\ell} \psi$$  \hspace{1cm} \text{(4.25)}

$R_{\ell}(\mu)$ is plotted in Fig. 4.2a). One can see the source of Mott lobe structure, since the critical $\kappa$ is inversely proportional to $R_{\ell}$. Notably, the value of $R_{\ell}$ diverges when the energy gap between two adjacent Mott lobes closes, implying a critical $\kappa = 0$.

The fourth order corrections to the energy are similarly computed, and plotted in Fig. 4.2b). The explicit expressions for the corrections are unwieldy, and offer little insight into the behaviour, but can be found in the Appendix C.
4.2.4 Superfluid Behaviour Past the Phase Boundary

Since the general form of the energy functional near the phase transition does not depend on the specifics of the JCH model, one expects to see the same universal behaviour as is seen in other Mott-superfluid transitions, such as the Bose-Hubbard model. Taking the derivative of Eq. (4.20) with respect to $\psi$ yields an equation for $\psi$ inside the superfluid region:

$$0 = 2R_{\ell}\kappa^2 - 4S_{\ell}\kappa^4 + \kappa\psi,$$

which, when solved, yields:

$$\psi = \sqrt{\frac{\kappa R_{\ell} - 1}{2S_{\ell}\kappa^3\epsilon^3}} = \sqrt{\frac{R_{\ell}^2\kappa'}{2S_{\ell}(\kappa' + 1)^3}},$$

where $\kappa' = (\kappa - \kappa_c)/\kappa_c$, with $\kappa_c$ being the critical $\kappa$ for the phase transition for that value of $\mu$. As the system crosses the phase transition $\psi \propto \sqrt{\kappa'}$, which confirms the universal scaling behaviour of the model [83].

The JCH model has one differing characteristic from the Bose Hubbard model. The effective non-linearity induced by the intra cavity atoms decreases as the number of photons in the cavity increases. That is, the energy functional is asymptotically linear in $\psi$. This is in contrast to the BH model, for which the energy is convex. The consequence is that the kinetic energy of the superfluid will always dominated over the photon-photon interaction, precluding the possibility for Thomas-Fermi like states as seen in BEC's [196].

4.3 Mott-Superfluid Transition in a Synthetic Magnetic Field

$(\alpha \neq 0)$

The Mott-superfluid transition in the JCH lattice is now considered in the presence of a synthetic magnetic field, applied in the manner discussed in chapter 3. This situation has been previously investigated in the context of the Bose-Hubbard model [113–115, 195]
CHAPTER 4. SUPERFLUID-MOTT TRANSITION AND VORTICES IN THE JCH MODEL

The magnetic field case differs in several key ways from the non-magnetic case. Firstly, as in the case of a single particle, the field introduces a "magnetic lattice" which breaks the translational symmetry of the underlying lattice. Hence, with a magnetic field, $\psi_i$’s cannot be assumed identical across the system. To capture this, a large lattice must be simulated, in contrast to the single site reduction that was possible for the homogeneous case.

The magnetic field also induces vorticity into the superfluid component. This additional structure is worthy of closer investigation; the interaction between Mott physics, and vortex dynamics leads to several interesting phenomena. The emergence and behaviour of vortices in the JCH model is covered in section 4.4.

The governing equation for the system is the mean-field Hamiltonian Eq. (4.11). This contains an on-site Jaynes-Cummings term, and a kinetic term. The character of the kinetic term is determined by the tunneling elements $\kappa_{xx'}$. Since it is of interest to study the transition in terms of the strength of the inter-site tunneling, it is convenient to divide the elements into a tunneling rate, $\kappa$, that is constant across the system, and the adjacency matrix elements $A_{xx'}(\alpha)$, which are of unit magnitude, but in general complex, and a function of the magnetic density $\alpha$. Thus one can write the kinetic term in matrix notation Eq. (4.11) as:

$$K = -\kappa \psi^\dagger A(\alpha) (a - \frac{1}{2} \psi).$$  \hspace{1cm} (4.28)

where $a$ is the vector of photonic raising operators. The energy functional can then be succinctly written as

$$E(\psi) = E^{JC} + K(\alpha) - \mu L.$$  \hspace{1cm} (4.29)

The effect of magnetic fields on the JCH system (under the Gutzwiller ansatz) can then be understood through the solutions to this functional.

4.3.1 Numerical Computation of the Mott-Superfluid Transition

Since the order parameters $\psi$ are not uniform a larger lattice must be used to compute the groundstate of the mean-field Hamiltonian. To capture this, a large lattice must
4.3. MOTT-SUPERFLUID TRANSITION IN A SYNTHETIC MAGNETIC FIELD

be simulated so that the magnetic and cavity lattices are consummate. Under this condition, the groundstate will be periodic, so that a finite lattice with periodic boundary conditions can be used. Specifically, for the magnetic parameter $\alpha = \frac{p}{q}$, there must be $q = V = L_x L_y$. Hence, for large denominators, the number of lattice sites may be large. However, the algorithm is reasonable efficient, and on a modest computer will converge within a short time (seconds) for large lattice areas (order 1000).

With periodic boundary conditions, the tunnelling parameters in the mean-field Hamiltonian are the same as those in the single particle case studied in section 3.5. Convergence of the algorithm, even with the self-consistent method, can suffer from convergence issues, particularly in the superfluid region near the phase transition although the general properties of the superfluid will be found. This can be ameliorated by a good choice of initial state. This initial guess is the known solution to the continuum case, where $\alpha$ goes to zero. This solution is simply the triangular Abrikosov lattice [94]. This works well for small $\alpha$. However, at larger values of $\alpha$ convergence can sometimes be a problem.

The details of the Abrikosov lattice are elaborated on in section 4.4. However, Fig. 4.3 demonstrates the effect of frustration by showing the different groundstates found by the self-consistent method for different initial guesses for $\psi$. An Abrikosov lattice as the initial state, and a randomly chosen initial state, results in significantly different converged groundstates.

These convergence issues do not seriously impede the study of the phase transition since, although the true groundstate has not been found, the general properties of the state will be similar between the converged and frustrated states.

To study the general properties of the phase transition under a magnetic field one is interested in the maximum value that $\psi$ takes on the lattice. In Fig. 4.4, $\text{max}(\psi)$ is plotted against the magnetic density $\alpha$ and the tunnelling strength $\kappa$ at $\bar{\mu} = -0.78$, which corresponds to the tip of the first Mott lobe. Here, one can see that the magnetic field acts to suppress the superfluid phase. This suppression comes from the interaction between
CHAPTER 4. SUPERFLUID-MOTT TRANSITION AND VORTICES IN THE JCH MODEL

Figure 4.3: Order parameters $\psi$ of the JCH groundstate after convergence of the iterative self-consistent method for a $64 \times 64$ lattice with $\alpha = 1/64$. a) Using an appropriate Abrikosov vortex lattice as the initial state. b) Using a random guess as the initial state.

The spatial lattice and magnetic lattice. Indeed, the structure of the transition, as a function of $\alpha$ is directly related to the Hofstadter butterfly [160]. The precise relationship between the butterfly and phase transition structures is elucidated in section 4.3.2.

The addition of a magnetic field does not change the structure of the Mott-superfluid phases transition diagram in any qualitative way. This is shown in Fig. 4.5. Here, I compare the superfluid density at non-zero $\alpha$ to the $\alpha = 0$ case (at some $\mu$ and $\kappa$) via

$$\Delta\psi(\alpha) = \frac{\max[\psi(\alpha = 0)] - \max[\psi(\alpha)]}{\max[\psi(\alpha = 0)] + \max[\psi(\alpha)] + \epsilon},$$

(4.30)

where $\epsilon$ is a factor used to avoid divide by zero errors, but which plays no physical role. One sees that the transition to superfluid is suppressed for $\bar{\kappa}$ near the transition. Deep inside the superfluid regime, there is a relatively small difference in the magnitude of $\max(\psi)$. The major divergence occurs near the transition boundary.
4.3. MOTT-SUPERFLUID TRANSITION IN A SYNTHETIC MAGNETIC FIELD

\((\alpha \neq 0)\)

Figure 4.4: The maximum superfluid order parameter \((\text{max}[\psi])\) as a function of the synthetic magnetic field \(\alpha\) and the interactive hopping \(\kappa\), with \(\bar{\tau} = -0.78\) for a \(20 \times 20\) lattice. The solid black curve corresponds to the evaluation of the transition between the Mott insulator state and the superfluid state as determined by Eq. (4.35). We have introduced the following dimensionless parameterisation: \(\pi = \kappa/\beta\), \(\bar{\tau} = (\mu - \omega)/\beta\) and \(\Delta = (\omega - \epsilon)/\beta = 0\).
Figure 4.5: (a) The superfluid order parameter ($\psi$) as a function of the chemical potential $\mu$ and the interactive hopping $\kappa$, with a lattice of 4 × 5 sites in the absence of synthetic magnetic field ($\alpha = 0$). (b,c) $\Delta \psi$ as a function of the chemical potential $\mu$ and the interactive hopping $\kappa$, for $|\alpha| = 2/5$ and $3/5$. (b) and $|\alpha| = 1/10$ and $9/10$. (c), with $\epsilon = 0.001$. For (a-c) we have introduced the following dimensionless parameterisation: $\overline{\kappa} = \kappa/\beta$, $\overline{\mu} = (\mu - \omega)/\beta$ and $\overline{\Delta} = \Delta/\beta = 0$. 
4.3. MOTT-SUPERFLUID TRANSITION IN A SYNTHETIC MAGNETIC FIELD \((\alpha \neq 0)\)

4.3.2 Analytic Determination of the Transition.

The analysis for finding the location of the critical \(\kappa\) for the non-magnetic case in section 4.2.3 can be extended to the magnetic case in a fairly straightforward manner. Instead of treating just a single order parameter, the full lattice must be considered. As in the case homogeneous case, for some Mott lobe \(\ell\), the energy functional for the system can be expanded in terms of the order parameters, the coefficients of which can be found through perturbation theory. In terms of the matrix formulation in Eq. (4.28) this expansion is:

\[
E_0 - \kappa^2 r_\ell \psi^\dagger A^2 \psi + \kappa \psi^\dagger A \psi + \kappa^4 s_\ell \sum_i |(A \psi)_i|^4.
\]

The \(r_\ell\) and \(s_\ell\) are functions of the chemical potential, and are same as in the homogeneous case. This can be seen by expanding the mean-field Hamiltonian:

\[
H_i^{\text{eff}} = H_i^{\text{JC}} - \mu L_i - t \sum_j \left[ \left( a_i^\dagger - \frac{1}{2} \psi_i^* \right) \Psi_i + \text{h.c.} \right],
\]

where \(\Psi_x = \sum_{x'} A_{xx'} \psi_{x'}\). Using the Mott phase groundstate for some \(\ell\) as a zeroth order wavefunction, the perturbative corrections to the energy for each site have exactly the same structure as in the homogeneous case.

The groundstate of the system is again determined by minimizing the energy in Eq. (4.31). The structure of the groundstate inside the superfluid region is studied in section 4.4. For now, I focus on the location of the phase boundary.

The location of the transition occurs when some non-zero values for \(\psi\) give an energy lower than the groundstate energy in the Mott phase, i.e.

\[
-\kappa^2 r_\ell \psi^\dagger A^2 \psi + \kappa \psi^\dagger A \psi < 0
\]

Assuming for now that the system is confined to some finite lattice, with periodic boundary conditions, one can write the order parameter vectors as \(\psi = \psi \tilde{\psi}\), where \(\tilde{\psi}\) is normalized to 1, and \(\psi\) is a real positive scale factor for the order parameter. The
equation for the phase boundary can be reduced to:

\[-\kappa r \bar{\psi}^\dagger A^2 \bar{\psi} + \bar{\psi}^\dagger A \bar{\psi} < 0\]  

(4.34)

Since the expectation value of \( \bar{\psi}^\dagger A \bar{\psi} \) is bounded above by the maximum eigenvalue of \( A \), the groundstate energy is minimized when \( \psi \) is a maximum eigenstate of \( A \), with energy \( \epsilon(\alpha) \). Therefore we have the condition for the superfluid transition:

\[ t_n = [r_n \epsilon(\alpha)]^{-1}. \]  

(4.35)

This energy is dependent on \( \alpha \), of which \( A \) is a function. The eigenvalue spectrum of \( A \), which is the Hofstadter butterfly, was shown in Fig. 3.2. Comparing the outline of the Mott-superfluid transition in Fig. 4.4 to the outline of the Hofstadter butterfly one can see immediately the correspondence.

As \( \alpha \to 0 \), the value of \( \epsilon(\alpha) \) approaches 4. This is the equal to the coordination number, \( z = 4 \), of the square lattice. Hence in this limit, the magnetic field has negligible effect on the phase transition.

In Fig. 4.4 we note that only some of the very fine structure in \( \epsilon(\alpha) \) is visible, both in the location of the boundary, and within the superfluid region. This structure is perhaps over represented in the figure since, in finding the groundstate wavefunction, we limited the system size to a \( 20 \times 20 \) lattice. As expected [160], small fluctuations in the system lead to a blurring of the fractal structure, and so the very fine structure seen in Fig. 4.4 would be challenging to resolve in any experiment.

This analysis of the JCH system is undertaken in the language of a particle-conserving condensed matter system. How does the existence or absence of long-range correlations in the groundstate of this model translate to a realistic photonic system; one driven by a laser (or some external voltage in the case of circuit QED) with dissipation and decoherence? Essentially, spacial correlations within the system will be exponentially vanishing past a critical point in the atomic interaction (Mott phase). The groundstate of the JCH system is simply one of many channels open to photonic states within the
optical system. So for some configuration, one should consider the chemical potential as a means by which to look at the spectrum of states that will be occupied.

From this perspective our results regarding the response of a Jaynes-Cummings-Hubbard lattice is readily interpretable: frustration induced by the magnetic field amplifies the effect of interactions acting to suppress long range correlations in the photonic fluid. This is a readily observable phenomena in isolation. However, since various open-system effects, will have similar effects, disentangling these may prove difficult. The dynamics of the open JCH system are far richer [139] and more complicated than the situation we have considered. The effects that we identify in this work, in regards to the response of the Mott-superfluid phase transition to magnetic fields, translates readily into the open systems context. As with the case non-magnetic case [140], one should expect to find phenomena dependent on the various open system effects; phenomena not identified here. However, investigation into this we leave for future work.

4.4 Vortices

The eigenstates of $A$ also have the property of non-uniformity on the lattice. That is, in the superfluid regime the order parameter has spacial structure. This structure is described in the continuum limit by the vortex dynamics, with appropriate modifications to include the nature of the JCH system.

In dilute Bose gas superfluids the introduction of synthetic magnetic fields [104,165–168], or the application of rotation to break time reversal symmetry, has lead to the observation of single vortices [106], vortex lattices [98,197,198], and the prediction of the emergence of fractional quantum Hall states [101,165,199–203]. The application of a synthetic magnetic field in a JCH lattice is expected to produce similar effects in the superfluid state. Considerable effort has already been applied to the study of of fractional quantum Hall states [204] in the ‘high’ synthetic magnetic field regime, where the number of flux quanta though the lattice is larger than the number of excitations. However, less attention has been paid to the emergence of a vortex and vortex lattice states in the ‘low’ synthetic magnetic field regime. Below we demonstrate that the JCH system does admit
vortex and vortex lattice solutions in the superfluid regime, upon the introduction of a synthetic magnetic field.

4.4.1 Single Vortex in the JCH

In the context of the meanfield description of JCH model, the local order parameter shares all the characteristics of a superfluid. Figures 4.6(a,b) show the magnitude of the mean field order parameter (a) and its phase (b) for a periodic lattice with a single flux quantum penetrating $64 \times 64$ sites. Minimizing the meanfield energy functional[Eq. (4.28)] results in a single vortex structure where the superfluid density rises monotonically from
4.4. VORTICES

the centre of the vortex core and the superfluid phase rotates by $2\pi$. Figure 4.7 shows a cross section of a single vortex for increasing values of the chemical potential. As the chemical potential increases at a set $\kappa$ and $\Delta$, the superfluid density increases. This leads to a stronger repulsive force within the superfluid, and, as in the case of conventional superfluids, a decreasing core size for the vortex. Note that, since the Jaynes Cummings interaction increases slowly with excitation (square root), the effect of higher $|\psi|$ is diminishing as $\mu$ rises. Also note that as the superfluid fraction increases, the atomic population will be smaller, as shown in Fig. 4.7.

4.4.2 Multiple Vortices

At higher synthetic magnetic fields one expects multiple vortices to be admitted into the superfluid. In the absence of an underlying lattice structure the vortices will arrange themselves into a triangular Abrikosov vortex lattice. This configuration arises, in the presence of a local repulsive non-linearity, to minimize the self-interaction of the superfluid at a given density of vortices. Figures 4.6(c,d) show that energy minimization of the meanfield JCH, in the presence of a synthetic magnetic field, also leads to the formation of an Abrikosov vortex lattice state.

One can attack this problem analytically in the continuum limit, by using the torus
CHAPTER 4. SUPERFLUID-MOTT TRANSITION AND VORTICES IN THE JCH MODEL

solutions presented in chapter 3. When $\alpha \ll 1$, the low energy solutions to the Harper equation converge to the solutions to the Landau problem with periodic boundary conditions. In this case, the bandgap is $\Delta_\alpha \approx 2\pi \alpha$, and the bandwidth is exponentially small, and

$$E/t = f_\epsilon_0 (1 - tae_0) + \sum_x t^3 f^2 b e_0^4 |\bar{\psi}|^4,$$

where we have $\psi = f\bar{\psi}$. The minimum energy for this state will be one that minimizes the quartic term.

By computing the energy over different possible symmetries one finds that the triangular lattice gives the smallest energy. Worth noting is that the energy difference between the triangular and square lattices is very small: approximately 1.3%. The implication is that a small amount of disorder will dominate the vortex location. This small energy difference is the source of the convergence problems for numerically finding the mean-field groundstate, seen in Fig. 4.3. This can also be seen in cases where $\alpha$ is not so small. Here, the intra-band lattice effects dominate, and determine the position of the vortex centres.

4.5 Summary

We have shown that the introduction of a synthetic magnetic field to the JCH model modifies the boundary between the Mott-Insulator and superfluid regimes. This modification arises from a competition between the magnetic lattice and the spatial lattice. Additionally, we predict that in the superfluid regime the introduction of a synthetic magnetic field leads to the formation of vortices which, due to the local non-linear atom-photon interaction, forms a triangular lattice in the groundstate.
5

Hall Effect Physics in the JCH Model

5.1 The Hall Effects

The Hall effect concerns the effect of a magnetic field applied perpendicular to a potential difference across a conductor. The classical Hall effect\(^1\), was discovered by Hall in 1879 [206]. In 1980, just over a century after Hall’s discovery, Klaus von Klitzing [207] measured the Hall conductance for a Metal-Oxide-Semiconductor Field Effect Transistor (MOSFET) at a temperature of 1.5\(K\) and a magnetic field of 18\(T\). In this Nobel prize winning experiment, von Klitzing showed that the Hall conductivity is precisely quantized in this regime. The quantization of the Hall Current had been previously predicted [208], but it’s experimental discovery precipitated a torrent of theoretical work. This work resulted in a new understanding of the Hall Effect that drew from topics from from topology [209,210], and percolation and disorder [211,212].

Just two years later a second upheaval followed, with the completely unexpected discovery of systems with fractionally quantized Hall conductances [213]. These fractional systems seemed to require a significantly different physics again, and so lead to the partition of the two quantum Hall effects, fractional (FQHE) and integer (IQHE). The FQHE is fundamentally a many-body phenomenon, and is therefore a much richer theory.

\(^1\)The idea of the Hall experiment arose from a passage in Maxwell’s text on Electro-Magnetism, where it was claimed that the magnetic field acted on the conductor of charge and not the current itself. Hall disagreed, and developed his experiment as a means of distinguishing the two cases [205].
CHAPTER 5. HALL EFFECT PHYSICS IN THE JCH MODEL

Figure 5.1: Basic schematic of the Hall effect experiment: Application of a current, $I$, to the two-dimensional electron gas of a conductor in a perpendicular magnetic field induces chiral edge currents(in green).

The strongly correlated states that describe the FQHE have forced physicists to reassess their understanding.

Thirty years after their discovery, the Integer [207] and Fractional [213] Quantum Hall Effects are still the focus of intense theoretical and experimental attention [214,215]. The FQHE relies on the presence of particle-particle interactions to form highly correlated states. These states can exhibit anyonic, and sometimes non-abelian, excitations which are explicitly non-local. As such, the investigation of large systems suffers strongly from the exponential explosion in Hilbert space. While there exist exact solutions for some FQHE systems, such as the Laughlin ansatz [216], these have yet to be observed directly in experiment. For this reason, emulation of the FQHE, particularly emulating the strong magnetic fields required, has become a major topic of interest in the scientific community [103–105,217].

This chapter explores how the fractional quantum Hall effect manifests in coupled atom-cavity systems described by the Jaynes-Cummings-Hubbard model. Specifically, I show the existence of Laughlin-like FQHE states in the JCH model in the presence of an artificial magnetic field. These states constitute new, strongly correlated states of light.
5.1. THE HALL EFFECTS

5.1.1 Classical Hall Effect

In the classical Hall effect, a current, \( j \), is applied to a conductor across one dimension, conventionally \( x \), while a constant magnetic field, \( B = B \hat{z} \), is applied perpendicularly (see Fig. 5.1).

\[
    j = \sigma E, \quad E = \rho j.
\]

By definition, \( \sigma \) is the conductivity tensor and \( \rho = \sigma^{-1} \) is the resistivity tensor. The Hall conductivity is defined as the off diagonal conductivity \( \sigma_H = \sigma_{xy} \). When the magnetic field is applied across the current, electrons are subject to a Lorentz force \( F \):

\[
    F = \frac{1}{nc} j \times B \Rightarrow \sigma_H = \frac{nec}{B},
\]

where \( n \) and \( e \) are the electron carrier density and electron charge respectively, and \( c \) is the speed of light. Note that the resulting off-diagonal conductivity is thus inversely proportionate to \( B \), with the constant of proportionality \( R_H = \frac{1}{nce} \) dependent on the particular conductor in question. Thus the classical Hall Effect is useful for measuring the charge carrier concentration of a conductor, which cannot be determined from the linear resistivity alone, and for measuring magnetic field strength.

5.1.2 Integer Quantum Hall Effect

In the 80s, measurements of the Quantum Hall Effect in MOSFET’s revealed that the Hall conductivity is strictly quantized [207], with

\[
    \sigma_H = \frac{ne^2}{h}, \quad n \in \mathbb{N}
\]

The constant \( \frac{e^2}{h} \) is independent of the specifics of the conductor being studied. That is, \( \sigma_H \) is a quantity that derives from some global property of states in the Hall system, and is not dependent on local details of the system. The integer factor \( n \), is given by the
CHAPTER 5. HALL EFFECT PHYSICS IN THE JCH MODEL

Figure 5.2: From Willett et al. [2]. The Hall resistance measured as a function of magnetic field. Drops in longitudinal resistance are marked by the filling factor. The principal series of $\nu = \frac{n}{2n+1}$ fractional states is clearly visible.

The filling factor, $\nu$:

$$n = \nu = \frac{\Phi_0}{B},$$  \hspace{1cm} (5.3)

the ratio of electric to magnetic density. An integer value of $\nu$ corresponds to $n$ completely filled Landau levels.

At the same time, the longitudinal conductivity, $\sigma_{xx}$ vanishes. These signatures mark a total departure from the behaviour of the classical Hall effect. The theoretical developments in the understanding of the role of topology in quantum systems spawned by this discovery are a major touchstone in the work presented in this chapter.

5.1.3 Fractional Quantum Hall Effect

In contrast to the IQHE, the discovery of Hall currents with non-integer $N$ came as a complete surprise. Figure 5.2, from [2], shows a relatively clean Hall experiment where the different fractional Hall states are manifest.
Laughlin provided an explanation for the presence of these levels by considering a strongly correlated many-body wave-function:

$$\Psi^L = \prod_{i<j} (z_i - z_j)^q \prod_i \exp \left( -\frac{|z_i|^2}{2} \right).$$  (5.4)

The Laughlin wavefunction for systems with filling factor $\nu = \frac{1}{q}$, with $q$ odd. Further, the excitations in these systems have fractional charge $\nu$ and exhibit anyonic statistics. These theoretical predictions have since been observed experimentally [218,219].

In the FQHE, electrons are confined to a partially filled Landau level. Since these levels are completely degenerate (at least in the ideal case), the repulsive Coulomb force between electrons drives strong correlations between the occupied states within the Landau level.

Subsequent discoveries of Hall plateaus at other filling factors [2,220] increased the theoretical interest in the FQHE, and Laughlin’s wavefunction was extended to more exotic states [116,221,222]. Such states can exhibit non-abelian statistics, some of which have been proposed for use as qubits with which to do quantum computing. The theory of such states is by no means complete, and many of the predictions lack experimental confirmation.

5.2 FQHE in the Continuum

This chapter is concerned with the manifestation of fractional quantum Hall physics in atom-cavity lattices. However, it is worthwhile to first consider the simpler case of particles in the more common case of a homogeneous 2-dimensional medium. The homogeneous case is both much simpler to approach analytically, and has been studied in much greater detail than the case of particles in a lattice potential (let alone in the couple atom-cavity case).
CHAPTER 5. HALL EFFECT PHYSICS IN THE JCH MODEL

5.2.1 Laughlin Ansatz

The Laughlin wavefunction \(^2\) was conceived in an attempt to provide some insight into the unexpected appearance of Hall plateaus at non-integer filling factors, the origin of which at the time was a mystery \(^3\). Laughlin strove to write down a wave function that: 1) was fully in the lowest landau level. 2) had 0 interaction energy for the contact potential. 3) was a minimum angular momentum state, and 4) obeyed the statistics required of fermions.

There are a number of exact analytical solutions for quantum Hall states in the ideal situation of particles in a constant magnetic field with a delta function interaction. The first of these to be discovered was the Laughlin wavefunction. This gives a functional form for the groundstate of an electron (or Bosonic) gas in a strong magnetic field with delta-function particle-particle interactions, and a filling factor \(1/n\), where \(n\) is odd for Fermionic systems, and even for Bosonic.

In chapter 3, it was shown that the wavefunction of a single particle in a constant magnetic field, confined to the LLL could be given as a product of a Gaussian factor \(\exp\left(\frac{-|z_i|^2}{4}\right)\) multiplied by any holomorphic function. In this chapter I consider multi-particle wavefunctions of the form:

\[
\Psi(z_1, z_2, \ldots, z_{N_p}) = F(z_1, z_2, \ldots, z_{N_p}) \times \exp\left[\sum_i -\frac{|z_i|^2}{4}\right].
\]  \(5.5\)

For \(\Psi\) to be a good candidate wavefunction for a FQHE state \(F(z)\) is required to be holomorphic in the \(z_i\)'s, and to respect the appropriate exchange symmetry, i.e.

\[
F(z) = (-1)^s F(z : z_i \leftrightarrow z_j)
\]  \(5.6\)

where, for the case of photons (electrons), \(s = 0 (-1)\). The wavefunction that Laughlin

\(^2\)For which eponymous inventor was appropriately awarded a 1/3 part of the 1998 Nobel prize.
wrote down which satisfies these criteria is:

$$\Psi^L_q(z_1, z_2, \ldots, z_N) \propto \left(\prod_{i>j}^N (z_i - z_j)^q\right) \left(\prod_{k=1}^N e^{-|z|^2/4}\right), \quad (5.7)$$

where $N$ is the number of particles in the system and $q = 1/\nu$. The wavefunction is composed of single particle states which lie in the LLL, as it is composed of functions of the form $f(z)\psi^{LLL}$. Particles are repulsed by the Jastrow product term, which has angular momentum $q$ and goes to 0 as any two particles approach one another. This means that, for a delta function interaction, Eq. (5.8) has no interaction energy, and the total energy is just given by $E = N\omega_c$.

Laughlin Wavefunction on a Torus

When working on a toroidal geometry, the wavefunction must satisfy the periodic boundary conditions, as in the case of single particle wavefunctions in section 3.2.1. The wavefunction for a single particle on a torus has periodic boundary conditions that constrain the location of the zeros. Taking these solutions as basis functions, we want to write a wavefunction for multiple particles with the same properties as the planar Laughlin wavefunction. That is, with the correct behaviour as two particles approach, and correct symmetry properties. From an analysis of the symmetry of the torus [223], it follows that any state in the LLL on a torus can be written as a product of a centre of mass term, $F_{CM}$, and a relative motion term, $f_{rel}$.

$$\Psi(z) \propto F_{CM}(Z)f_{rel}(z). \quad (5.8)$$

The centre of mass term is determined completely from the symmetry of the torus, and is given by:

$$F_{CM}(Z) = 0 \left[ \frac{k}{q} + \frac{(N_\phi - 2)}{2q} + \frac{\phi_1}{2\pi q} - \frac{(N_\phi - q)}{q^2 - \phi_2/2\pi} \right] \left( \frac{qZ}{L_x} \right) \left( \frac{\eta}{q^\tau} \right). \quad (5.9)$$
CHAPTER 5. HALL EFFECT PHYSICS IN THE JCH MODEL

The wavefunction is a product of two functions [158]. \( \Psi^{CM} \) depends only on the centre of mass of the particles, and \( \Psi^{rel} \) only on their relative motion \( (z_i - z_j) \), where

\[
\Psi^{rel} = \prod_{i<j}^{N_p} \theta_1 \left( \frac{z_i - z_j}{L_x} | \tau \right)^q . \tag{5.10}
\]

and \( k \) is an integer that runs from 0 to \( q - 1 \). This defines \( q \) degenerate groundstates. For the case of \( \nu = 1/2 \), this implies that the groundstate degeneracy is 2.

The quantization of the average transverse conductivity in the QHE is independent of the specific implementation and experimental conditions. This remarkable result is a consequence of the energy gap in energy spectrum, and a relationship between the transverse conductance, \( \sigma_H \), and the topology of the groundstate. The groundstate of a FQHE state is topologically distinct from the vacuum, and the excited states. This itself implies an energy gap. In the last section, it was shown that the groundstate of a system with large \( \alpha \) deviates significantly from the Laughlin ansatz. Topological invariants provide an alternative means of classifying the groundstate.

5.3 Fractional Quantum Hall Effect in the JCH Lattice

The extension of the JCH model to include (synthetic) magnetic fields opens up the possibility of creating FQHE like physics in atom-cavity systems. This creates a great opportunity for studying aspects of the FQHE in a totally new context, one which has the attendant advantages of the JCH platform over experiments with the traditional electronic system. To this end, I investigate the manifestation of Laughlin like states as they arise in the JCH model.

As shown in chapter 2, the JCH model has limits within it’s parameter space that map to continuum, point interaction models. Since the Laughlin wavefunction is an exact groundstate for such a system, with the appropriate magnetic field, one can say with certainty that FQHE physics will exist within the JCH model. However, the existence of such physics in the regime where the Jaynes-Cummings interaction is important is not so
obvious. I therefore explore the JCH model with magnetic field in the regime where the atom-cavity interaction is most pertinent.

The JCH model approaches the hardcore Bose-Hubbard model in the limit of a system wide large detuning, where only the atomic states are occupied in the groundstate. In the opposite limit, where the atomic energy is much greater then the cavity’s, the model is a Bose-Hubbard model, with the effective on-site interaction decreasing as $\Delta$ is increased. Tuning the system between these two extremes requires passing through a resonance, where the atomic and photonic states mix.

From the previous study of the JCH model in a magnetic field in section 3.5 it was shown that the atom-cavity interference is greatest at $\Delta \approx -4\kappa$. That is, in the lowest Landau level, the JCH single particle wavefunction is an equal superposition of atomic and photonic states.

The competition between the JC interaction and the kinetic energy will be most strongly felt when the strengths of these interactions are of the same order. The studies of the JCH model in this section are therefore restricted to $\kappa = \beta$. Here, the different effects in the JCH lattice are most manifest. Extrapolation to the situation where $\kappa \neq \beta$ is fairly obvious.

The simplest Laughlin-like state for a Bosonic system is the $\nu = 1/2$ state. States at higher filling factors are somewhat unstable for a $\delta$-function potential, although they can be made stable by the inclusion of a longer range potential [224]. This chapter therefore restricts it’s self to the $\nu = 1/2$ state.

There are a number of possible metrics that can be used to identify the existence of FQHE physics in the JCH model. Firstly, the groundstate manifold should be degenerate with a dimension of two for $\nu = 1/2$. The energy spectrum of the low lying states, and the wavefunction overlap with the Laughlin wavefunction also quantify the nature of the groundstate. A Chern number of $\mathcal{C} = 1$ for the groundstate manifold is a strong indication of a Laughlin-like state.
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<table>
<thead>
<tr>
<th>Model</th>
<th>Dimension</th>
<th>Example ( N_p = 4, N_s = 36 )</th>
</tr>
</thead>
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<tr>
<td>Hardcore</td>
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</tr>
<tr>
<td>Bose-Hubbard</td>
<td>[^{N_p + N_s - 1 \choose N_p}]</td>
<td>82,251</td>
</tr>
<tr>
<td>JCH</td>
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<td>1,121,472</td>
</tr>
</tbody>
</table>

Table 5.1: Vector space dimensions of different lattice models.

5.3.1 Computing the Ground States

To identify FQHE physics requires constructing and numerically computing the low lying states of the JCH Hamiltonian with magnetic field.

The Laughlin state has a filling factor \( \nu = \frac{1}{2} \). For an \( N \times M \) lattice with a total of \( N_\ell \) excitations in the system this implies \( \alpha = \frac{2N_\ell}{NM} \). We compute on a toroid to remove edge effects and to compute the Chern number.

To calculate the groundstate for a JCH system I construct the JCH Hamiltonian explicitly, and use numerical diagonalization to compute the low lying states. Code for these computations was written in MATLAB [225], and is available online [226].

Table 5.1 shows formula for the vector space dimension of the hardcore Boson, Bose-Hubbard, and, JCH lattice models. The full JCH model is significantly larger, since it has many more ways to distribute some set number of particles. This restricts the size of system that it is possible to simulate. While 4 excitations in the lattice is computationally tractable size, even going to 5 excitations becomes rather unfeasible. Nevertheless, even with a restricted system size the presence of FQHE like physics is observable.

In Fig. 5.3 the results of the groundstate computation as a function of the atomic detuning is presented for a number of lattice configurations for \( \nu = 1/2 \). Shown in this figure for each configuration is the energy gap between the groundstate manifold and the next highest state, and the overlap of the groundstate with the Laughlin wavefunction. The specifics of these measures are discussed in more detail below, but for the present I
make a few observations.

There is a large variation in the behaviour of the energy, and wavefunction overlap, for the different lattice configurations. This is even true of states with similar $\alpha$. For many configurations there appears to be a region in $\Delta$ where the wavefunction overlap begins to fall, and the energy gap closes. This marks a transition from a FQHE state, to an uncorrelated state, where the polariton-polariton interaction is in completion with the complicated band structure of the Hofstadter butterfly.

The overlap with the trial wavefunction is very large for some configurations, which is a very good indication of FQHE physics. However, this is not an unambiguous metric, since some states with low overlap can still possess the correct topology, and, for smaller systems, the overlap can still be surprisingly large, even when the energy structure is not correct.

These energies and overlaps are computed for just one value of the twist angles. These quantities can vary significantly over the full range of periodic boundary conditions, and so the specific values of the energy gap presented in Fig. 5.3 do not represent the true bandgap. Such a computation is presented in Fig. 5.4, but is too computationally intensive to determine for all points as a function of $\Delta$.

### 5.3.2 Energy Gap

The Laughlin state at $\nu = 1/2$ on a torus has a groundstate with two degenerate groundstates. Fig. 5.4 shows the energy above the lowest state for a $6 \times 6$ lattice with 3 particles at $\Delta = 0$. One sees that, over the twist angles the groundstate manifold has dimensions 2, and is separated from the higher states. Note that the minimum gap does not necessarily appear at $\theta = 0$.

#### Estimation of the Energy Gap

In Eq. (3.45), the kinetic energy of a particle in the lattice is given by $\kappa_{\text{eff}} = \kappa \cos^2 \theta^{JC}(\Delta)$. 
Figure 5.3: Energy gap above the degenerate groundstate and Laughlin wavefunction overlap for different lattice configurations as a function of detuning. a) $\ell = 2$ b) $\ell = 3$ c) $\ell = 4$. Lattice sizes are indicated by colour, as specified in the legend.
5.3. FRACTIONAL QUANTUM HALL EFFECT IN THE JCH LATTICE

Figure 5.4: Energy of low lying states in a $6 \times 6$ JCH lattice with 3 particles, and $\Delta/\beta = 0$. The lowest state here intersects with the groundstate, but is gapped from the state above, implying a groundstate manifold of dimension 2.

Figure 5.5: Energy gap for a $6 \times 6$ JCH lattice with 3 particles, and $\alpha = 1/12$. Blue dots are energies from the exact diagonalization, the red dashed line is from the gap estimate described in the text, where it is assumed that $\Delta E^{HC} = \alpha/2$. 
The energy scale of the hardcore model is determined completely by $\alpha$ and $\kappa$. We therefore expect the energy gap in the JCH model $\delta E^{JCH} \rightarrow \kappa_{\text{eff}} E^{HC}$. It makes sense therefore to use the Hardcore model as a reference. As the atomic energy approaches 0, the atomic-photonic mixing increases, and the effective on-site repulsion in reduced.

It is possible to put a rough estimate on the energy of the lowest eigenstate by assuming a state of constant density. The interaction energy of such a state is then approximately:

$$E_{\text{int}} \approx \frac{N_p(N_p - 1)}{N_s} U_{\text{eff}},$$

where $U_{\text{eff}}$ is the effective 2-body interaction of the JC cavity:

$$U_{\text{eff}} = E^{JCH}_2(\Delta - \kappa f(\alpha)) - 2E^{JCH}_1(\Delta - \kappa f(\alpha)).$$

The energy gap is then $E^{\text{JCH}}_{\text{gap}} \approx \min(E^{\text{HC}}_{\text{gap}}, \kappa_{\text{eff}}(\Delta, \alpha), E_{\text{int}})$. For $\alpha \ll 1$, we can assume the lower levels of the Hofstadter butterfly are separated by the cyclotron energy, $\alpha/2$.

That is, at some point, there is a crossing between the first excited state of the hardcore model, and the lowest state of the non-interacting model. The situation is shown in Fig. 5.5. This approximation provides a good asymptotic estimate of the gap for $|\Delta/\beta| >> 1$.

Note that, in this model, the gap persists up to arbitrarily high $\Delta$ and there is no transition to a lower state. Inspection of Fig. 5.3 reveals that, for some of the systems under investigation, this is not the case. For cases where $\alpha \neq 1/q$, the FQHE state, in the hardcore limit, must project significantly into the higher bands of the Hofstadter butterfly. This raises the groundstate energy above that of the lowest possible energy state. When $\alpha \neq 1/q$, this energy can be significantly higher. This implies that, at some point, the trial groundstate of free particles in the lowest band will be energetically favorable. This is the source of the topological phase transition seen in our results. The transition is not seen in the $\alpha = 1/4$ and $1/6$ cases, but occurs in the others. Note that there is a transition in the $\alpha = 1/9$ case that this model fails to explain.
5.3. FRACTIONAL QUANTUM HALL EFFECT IN THE JCH LATTICE

5.3.3 JCH Laughlin Wavefunction

One source of evidence for the presence of Laughlin like states is to find the wavefunction overlap between the numerically determined groundstate, and an appropriate trial wavefunction. In the Bose-Hubbard model, one can take a suitable normalised continuum Laughlin wavefunction (on the appropriate topology). In the case of the JCH model, the presence of the atomic degrees of freedom mean that this wavefunction cannot be used directly. Instead, one can define an equivalent function for the JCH continuum limit simply by replacing $\psi^{\text{LLL}}$ with the appropriate JCH single particle functions, as defined in Eq. (3.46):

$$\Psi^{\text{JCH}} = \left[ \prod_{N \geq i > j \geq 1} (z_i - z_j)^{1/\nu} \right] \prod_{k=1}^{N} \psi^{\text{JCH}-\text{LLL}}(z_k).$$

These wavefunctions are coherent superpositions of atomic and photonic components, but, as one can see from the structure of Eq. (5.13), there are no extra correlations between the atomic degrees of freedom past the spatial correlations inherent in the Laughlin wavefunction.

A product state of single polaritons is in general not well defined at points where multiple excitations coincide at a site. However, as the Laughlin wavefunction vanishes when particles coincide, this issue is avoided, and Eq. (5.13) is valid. $\Psi^{\text{JCH}}_L$ is an exact groundstate for filling factor $\nu$ of $H^{\text{JCH}}$ in the $\alpha \to 0$ limit.

The Laughlin wavefunction can be a very good approximation to states on a JCH, or BH, lattice, where the strong particle-particle interactions are on-site only. In some situations this is not true, even if the topological properties of the groundstate correspond to a Laughlin-like state. This can happen when $\alpha$ becomes large, and there is significant mixing of Landau levels for the single particle solutions. This breaks the assumption of the LLL in the Laughlin ansatz, and one does not expect a high overlap.

Another possibility is that the on-site interaction is not a delta-function, but some finite value. This allows the possibility of multiple occupations per site, which is precluded in the Laughlin wavefunction. Indeed, both these situations arise in the JCH model.
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under certain parameter regimes.

To compare the JCH Laughlin wavefunction to the groundstate of the JCH lattice one computes the groundstate via exact diagonalization, and construct the trial wavefunction in the computational basis. The overlaps computed between the trial and computed wavefunctions are shown in Fig. 5.3.

5.3.4 Chern Numbers and Topological Phases

Degeneracy of the groundstate manifold, and large overlap with the Laughlin wavefunction are good indications that the groundstate of the JCH lattice is indeed a photonic analog of the FQHE. However, there are cases where the overlap with the trial wavefunction is low, but where one expects to find a fractional ground state, for example, in the case of 3 particles in $4 \times 4$ lattice. This deviation is expected for large $\alpha$, where the single particle states deviate significantly from the continuum LLL’s. Hafezi et al. [224] have studied this situation in the Bose-Hubbard model.

In these cases, the Chern number can provide an unambiguous indication of FQHE physics, since it is topologically distinct from other quantum Hall states, and corresponds directly to the quantised fractional conductance. This also means that the Chern number gives an unambiguous indication of a quantum phase transition, since it changes discretely. The results in Fig. 5.3 suggest the presence of such a phase transition. In Fig. 5.6, the computation of the Chern number for several systems is used to determine the location of a phase transition as $\Delta$ is increased from the hardcore Boson limit ($\Delta/\beta \to -\infty$) to the non-interacting limit ($\Delta/\beta \to \infty$). These results are also presented in table 5.2. We carry this out for only a few systems, since the computation can be somewhat lengthy. The details of the Chern number computation are given in section 5.3.4.

Chern Number

The Chern number is a topological quantity coming from the field of differential geometry. Following [156], let $\mathcal{M}$ be an oriented closed 2-manifold. There exists a triangulation of $\mathcal{M}$ with vertices labeled $\alpha_i$. The curvature is given by the closed 2-form $\Omega = dA^{\alpha\beta} = F^{\alpha\beta}$
5.3. FRACTIONAL QUANTUM HALL EFFECT IN THE JCH LATTICE

in each chart $U_\alpha$.

$$C_1 = \frac{1}{2\pi i} \int T \text{Tr} d_\beta A^{\alpha\beta} = \frac{1}{2\pi i} \int T \text{Tr} F^{\alpha\beta}. \quad (5.14)$$

The Chern number classifies the homotopy class of the fiber bundle, which is a topological invariant.

The connection between this topological invariant, and the physics of the quantum Hall effect is as follows. Consider a set of $k$ degenerate wavefunctions, $|\Psi^\alpha(\theta)\rangle$, defined on a torus with generalized periodic boundary conditions. Then there is a 1-form defined as

$$A^{\alpha\beta} = \left\langle \Psi^\alpha | \partial_\theta_y \Psi^\beta \right\rangle d\theta_x - \left\langle \Psi^\alpha | \partial_\theta_x \Psi^\beta \right\rangle d\theta_y. \quad (5.15)$$

and therefore

$$C = \frac{1}{2\pi i} \int T d\theta_x \wedge d\theta_y \left\langle \partial_\theta_x \Psi^\alpha | \partial_\theta_y \Psi^\beta \right\rangle - \left\langle \partial_\theta_y \Psi^\alpha | \partial_\theta_x \Psi^\beta \right\rangle. \quad (5.16)$$

The degenerate groundstates, $\Psi^k(\theta)$, define a principal fiber bundle over the $T^2$ manifold. That is, the Chern number is insensitive to small perturbations relative to the energy gap. Only if the gap closes can a transition to topologically different states occur.

The conductivity tensor measures the response of the system to an external force, namely:

$$\mathbf{j} = \sigma \mathbf{E}. \quad (5.17)$$

Linear response theory quantifies the time dependent effect on an observable by the application of some force. This is encapsulated, to a first order perturbation in the strength of the applied force, by the Kubo relation [227]:

$$\sigma_{\mu\nu} = \int_0^{1/kT} \int_0^\infty d\lambda dt \left\langle \dot{J}_\nu(-i\lambda) \dot{J}_\mu(t) \right\rangle. \quad (5.18)$$
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The Hall conductivity at zero temperature is then given by:

\[ \sigma_{xy} = \frac{\hbar}{iB} \sum_{\alpha, \beta \neq \alpha} \frac{v^y_{\alpha\beta} v^x_{\beta\alpha} - v^y_{\alpha\beta} v^x_{\alpha\beta}}{(E^\alpha - E^\beta)^2}. \] (5.19)

where \( v^\mu \) is the current operator. The current, \( v^\mu \), in this case is the sum over the particle density flow. The current density operator, \( j^\mu \) is found from the continuity equation, \( \partial_t \rho + \nabla \cdot j = 0 \):

\[ \partial_t \rho = i[H, n_i] = a_i^\dagger \sum_{j \in \text{n.n.}} (\kappa_{ij} a_j) + \text{h.c.} \] (5.20)

The sum for \( \alpha \) is over energy levels occupied by the system, which on our case is presumed to be the groundstate, and, assuming a gap, the sum over \( \beta \) includes all other levels.

The relationship between the First Chern numbers calculated in the previous section, and the Hall conductance can be appreciated with the following observation:

\[ \frac{\partial H}{\partial \alpha_\mu} = v^\mu. \] (5.21)

Using \( \langle \alpha | \frac{\partial H}{\partial \sigma_{\mu}} | \beta \rangle = (E^\alpha - E^\beta) \langle \frac{\partial \alpha}{\partial \sigma_{\mu}} | \beta \rangle \), and \( \sum_{\beta \neq \alpha} | \beta \rangle \langle \beta | = 1 - \sum_{\alpha} | \alpha \rangle \langle \alpha | \), the Hall conductance becomes:

\[ \sigma_{xy} = \frac{1}{iB} \left( \left\langle \frac{\partial \alpha}{\partial \phi_y} \bigg| \frac{\partial \alpha}{\partial \phi_x} \bigg\rangle - \left\langle \frac{\partial \alpha}{\partial \phi_x} \bigg| \frac{\partial \alpha}{\partial \phi_y} \bigg\rangle \right. \right). \] (5.22)

The generalized boundary conditions, parameterized by \( \phi_{x,y} \), which were discussed in chapter 3, induce translations on the system. The factors \( \phi_\mu \) can be considered to be voltage drops [228].

Explicit computation of the curvature, \( \text{Tr} F^{\alpha\beta} \), of the continuum Laughlin states for arbitrary \( N_P \) has been carried out by Varnhagen [229]. In this lengthy, but straightforward computation, the suppression of fluctuations by increasing particle number becomes clear:

\[ \text{Tr} F^{\alpha\beta} = \frac{1}{\sqrt{2qN_p\pi^2}} \theta_3 \left( \frac{2\tau \phi_x}{qN_p\pi} \bigg| \frac{\tau}{qN_p} \right) \exp \left[ -\frac{2i\tau \pi \phi_x^2}{qN_p} \right] \] (5.23)
In essence, the centre of mass function becomes increasingly smooth w.r.t the twist angles $\phi_{x,y}$ as the phase of the generalized boundary conditions becomes distributed among an increasing number of particles. This can be seen by inspection of Eq. (5.23).

The quantity in Eq. (5.23) is independent of $\phi_y$, and exponentially approaches a constant value of $1/2\pi$ as $N_p \to \infty$. Therefore one can see that, in the limit of many particles, the Hall conductivity is directly proportional to the Chern number.

**Computing the Chern number**

The degenerate groundstates, $\Psi_0(\theta)$, define a principal fiber bundle over the $T^2$ manifold. The Chern number classifies the homotopy class of the fiber bundle, which is a topological
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<table>
<thead>
<tr>
<th>$L_x \times L_y$</th>
<th>$N_p$</th>
<th>Dim($H$)</th>
<th>$\alpha$</th>
<th>Laughlin overlap$^1$</th>
<th>Transition($\Delta_c$)</th>
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<td>0.99</td>
</tr>
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</table>

1 These overlaps are taken for the hard-core boson limit.
2 For these lattice configurations, there is no identifiable phase transition.

Table 5.2: Results for systems of size $L_x \times L_y$ sites with $N_p$ excitations. All systems have $C_1 = 1/2$ below the transition strength $\Delta = \Delta_c$. Also shown is the Hilbert space dimensions, Dim($H$), and the Laughlin overlap.

invariant. That is, the Chern number is insensitive to small perturbations relative to the energy gap. Only if the gap closes can the transition to topologically different states occur. Explicit computation of Eq. (5.16)) is computational intensive. We instead use the method first proposed in [230], and used in [224], which allows for efficient computation of the Chern number in the presence of degeneracies. In this method, a phase is defined for the ground state at each $\theta_{x,y}$ with respect to two reference states. The Chern number is given as the signed sum of the vortices which occur at the zeros of the overlap with one of the reference states.

We wish to compute the Chern Number $C_1$, as defined in Eq. (5.16). To directly compute this value by integrating over the entire torus is possible, but computationally expensive. Since $F^{\alpha\beta}$ may be significantly oscillatory, convergence to the correct value is hard to guarantee, and accurate determination of $C_1$ may be very computationally intensive. Evaluation of each point on the torus requires finding the eigenvalues of the Hamiltonian.

Instead, it is possible to determine $C_1$ through the analysis of the of the vector bundle

$$\mathcal{A}^{\alpha\beta}(\theta)_{\mu} = i \left( \psi^\alpha | \partial_\mu \Psi^\beta \right), \quad \mu \in x, y. \quad (5.24)$$
Since $\mathcal{A}$ is on a closed base manifold, the integral $\int_{\gamma} \mathcal{A} \cdot ds$ around any closed loop $\gamma$ must be zero. Stokes theorem would then imply that $\int_{\Gamma} d\mathcal{A} = 0$. The non-zero Chern number derives from singularities in $A_{\alpha}^{\beta}$. Any vector bundle with a global chart is topologically isomorphic to the trivial bundle. Transition functions that map between different local charts completely characterise the topology of the vector bundle. That is, there is no function which is analytic over the whole torus such that $\mathcal{A}^{\alpha\beta}(\theta) = f(\theta') \mathcal{A}^{\alpha\beta}(\theta')$, since there will exist singularities in $\mathcal{A}$.

Let $\mathcal{A}'$ be the vector field in a region of the torus where $\mathcal{A}$ is not well defined. Then there is a gauge transformation

$$\mathcal{A} = \mathcal{A}' - \nabla \chi \Psi$$ \hfill (5.25)

that maps between the two vector fields. Since $\mathcal{A}$ is not defined over the entire torus, there can be a non-zero value for the winding number. If the loop integral is taken over a region where these two fields overlap, then

$$\int_{\gamma} \mathcal{A}' \cdot ds = \int_{\gamma} \mathcal{A} \cdot ds + \int_{\gamma} \nabla \chi \cdot ds$$ \hfill (5.26)

$$= 0 + \int_{\gamma} \nabla \chi \cdot ds.$$

Defining the gauge transformation through the transition function

$$\Psi(\theta) \to e^{i f(\theta)} \text{ and } \mathcal{A}(\theta) \to \mathcal{A}(\theta) - \nabla f(\theta),$$ \hfill (5.27)

the Chern number is just the total vorticity in $\exp[i f(\theta)]$.

It is possible to find $\exp[i f(\theta)]$ by noting that a choice of gauge on the torus can be defined through

$$\mathcal{A}_1 = \nabla \Lambda_1, \quad \Lambda_1 = \langle \Phi_1 | \Psi \rangle,$$ \hfill (5.28)

where $|\Phi\rangle = |\Psi(\theta_1)\rangle$ is the groundstate at $\theta = \theta_1$. The Chern number is then determined by the transition function at the points around where $\Lambda_1$ vanishes. We then have $\exp i f = \Lambda_1 \Lambda_2^*$, where $\Lambda_2$ is defined similarly at the point $\theta_2$, which is chosen such that
CHAPTER 5. HALL EFFECT PHYSICS IN THE JCH MODEL

\( \Lambda_2 \neq 0 \) at the zero of \( \Lambda_1 \).

In the case of the Laughlin state, the groundstate is twice degenerate, and hence this analysis must be extended. If \( |\Psi^\alpha\rangle \) defines the ground state manifold, then the proper extension is

\[
\Lambda_{\alpha\beta}^1 = \langle \Phi_1^\alpha | \Psi^\beta \rangle.
\]

Defining the function \( \Omega = \det \Lambda_1 \Lambda_2^* \), the Chern number is then equal to the sum of the vorticity of \( \Omega \) around the zeros of \( \Lambda_1 = \det \Lambda_1^{\alpha\beta} \).

This approach to computing the Chern number is significantly more efficient than direct integration, since the groundstate only needs to be computed at a set number of points, such that the vorticity of the functions can be discerned. In practise, this can be determined from a fairly low resolution.

The situation is demonstrated in Fig. 5.7. Here, \( \Omega \) and \( \Lambda_1 \) are computed for the JCH system with 3 particles on a \( 6 \times 6 \) lattice. The phase of \( \Omega \) is plotted as vectors in the complex plane over \( T_\theta \), along with the amplitude of \( \Lambda_1 \) (represented by color). See that there is a single vortex with winding number \( w = 1 \) at the point where \( \Lambda_1 \) vanishes. This indicates that \( C_1 = 1 \) for the system.

Also note that there is another vortex in \( \Omega \). This exists as \( \Omega \) must have total vorticity 0. This vortex appears at the point where \( \Lambda_2 \) vanishes.

Topological Phase Transitions

From inspection of Fig. 5.6 it appears that for some lattice configurations there is a qualitative change in the nature of the groundstate as the detuning increases. Computation of the Chern number shows that at these points, the groundstate manifold changes from a Laughlin-like state indicated by \( C = 1 \) to a topologically distinct state with different Chern number. Fig. 5.7 shows \( \Lambda \) on two different sides of this transition in the \( 4 \times 4 \) lattice with 3 excitations. On the Laughlin state side of this transition the Chern number is indeed \( C = 1 \). On the other side of the transition the groundstate has a degeneracy of 1, and \( C = 9 \). The Chern number for the lowest energy single particle in this lattice
5.3. FRACTIONAL QUANTUM HALL EFFECT IN THE JCH LATTICE

Figure 5.7: Phase of $\det[\Omega(\theta)]$ for 3 particles in a $4 \times 4$ JCH lattice. a) $\Delta = -10$. The total vorticity in the $\Lambda_1 = 0$ region sums to 1. b) $\Delta = -8$. Here, $C = -9$. Lattice effects mean that the groundstate is no longer in the same topological class as the Laughlin wavefunction. The black contours denote the region where $\Lambda_1 = 0$.

is $C = 3$, which indicates that the system has changed from a strongly correlated one, to an uncorrelated product state, since the total Chern number for a product state is simply the sum of the Chern numbers of the constituent states. This situation arises due to the energy spectrum of this particular lattice configuration. There is a single gapped lowest energy state, which becomes preferred for all excitation to occupy when the polariton-polariton interaction is sufficiently weak.

A similar situation arises in other lattice configurations, as indicated in table 5.2. However, for some lattice configurations, such as in the $6 \times 6$ lattice with 3 excitations, the single excitation groundstate manifold is triply degenerate. In this case, the Laughlin like state remains preferred even for very weak interactions, and no topological phase transition takes place.

It is perhaps pertinent at this time to make a distinction between two separate, but closely related concepts. Namely, a Quantum Hall state and a Quantum Hall phase. A phase is defined by its phase transitions, which, from basic considerations, can only occur in the limit of an infinite number of particles.

With the discovery of the QHE came a new type of phase transition that defied
CHAPTER 5. HALL EFFECT PHYSICS IN THE JCH MODEL

the Landau symmetry breaking paradigm, topological phases that are defined through completely non-local properties. Nevertheless, the requirement for a discontinuity in the value of some operator is still required to mark the transition between phases.

This is all in marked contrast to the topological states that one can still define on a finite number of particles. The systems considered in this chapter are confined to small lattices with small numbers of particles. The Chern number that quantifies the topology of these small states, and in theory remains stable as the size of the system increases. Indeed, as the number of particles is increased (for a set value of $\alpha = p/q$), the fluctuations in energy as a functions of the periodic boundary conditions is exponentially suppressed. In the same way, system specific properties, and inhomogeneities are suppressed, which leads to the ‘exact’ quantization of the Hall conductance.

These phase transitions depend on the energy spectrum for the particular lattice configuration. The question of how this will scale to larger lattices is non-trivial, but computational limitations preclude the numerical exploration of this matter. However, since the single particle groundstate manifold for some $\alpha$ does not change as the lattice size increases, one expects that these transitions to persist.

5.4 Summary

Strong evidence for the existence of fractional quantum Hall states in JCH systems has been demonstrated. A Laughlin wavefunction like ansatz for the JCH was introduced by using the single particle solutions to a JCH system in a magnetic field studied in chapter 3. High overlap with exact groundstates of small systems indicate FQHE. Furthermore, by computing the Chern number for theses groundstates, the groundstates of the system were seen to possess the appropriate topological qualities.

The Chern number was also able to indicate that there is a transition from the strongly correlated FQHE to an uncorrelated state as the atomic excitation energy is increased. The exact nature of these transitions is unclear, due to the small size of the systems it was possible to simulate.
6

Pfaffian States in JCH Lattices

6.1 The Moore-Read Pfaffian State

There exists now a plethora of Hall plateaus at favourable filling factors discovered by experimentalists. The vast majority of these belong to the composite Boson series of Haldane \[221\]:

\[
\nu = \frac{n}{(2p - 1)n \pm 1} \quad \text{and} \quad 1 - \frac{n}{2pn \pm 1}.
\]  

(6.1)

For electrons, these states all possess odd denominators. The discovery by Willett \[2\] of an even-denominator quantum Hall state at \(\nu = 5/2\) came as a surprise to many \[231\]. Since then, several other states at even filling factor have been found \[232\]. There exist a plethora of proposed physical mechanisms for the origin of these states \[116,233,234\].

While the jury is still not in, there is significant evidence that the \(\nu = 5/2\) state in the electronic FQHE is described by the Pfaffian state of Moore and Read \[116\]. Further more, several numerical studies in spherical \[235–237\], and toroidal \[101,237,238\] geometries have found large overlaps with the Bosonic Pfaffian at \(\nu = 1\).

The Pfaffian state shares the structure of most proposed FQHE states, that is single particle Gaussian terms, \(\exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right]\), multiplied by a Jastrow Factor, an analytic function \(F(\tilde{z})\). The Pfaffian state is built upon a pairing hypothesis \[233\], that assumes that, like in a BCS superconductor, electrons partition into pairs via some attractive
CHAPTER 6. PFAFFIAN STATES IN JCH LATTICES

interaction. This partitioning is constrained by the symmetries of the particles in the system, and is succinctly expressed in terms of the matrix Pfaffian:

$$\Psi^{\text{Pf}} \propto \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j)^{(1/\nu)} \exp \left( -\frac{1}{4} \sum_i |z_i|^2 \right). \tag{6.2}$$

The Pfaffian is of the matrix with $ij$th elements $1/(z_i = z_j)$ and

$$\text{Pf} (M_{ij}) = \left( \frac{2^{N/2}(N/2)!}{(N/2)!} \right)^{-1} \sum_{\sigma \in S_N} \text{sgn} \sigma \prod_{k=1}^{N/2} M_{\sigma(2k-1)\sigma(2k)}, \tag{6.3}$$

where $S_N$ is the set of permutations of $N$ elements. The Pfaffian factor acts to cancel a factor of $(z_i - z_j)$ for each pair of particles in the wavefunction. Essentially, the particles described by this theory are paired. For $\nu = 1$ the paired nature of the groundstate is seen by the equivalent expression of $\Psi^{\text{Pf}}$:

$$\Psi^{\text{Pf}} \propto \sum_{\sigma \in S_N} \left[ \prod_{i<j=1}^{N/2} (z_{\sigma_i} - z_{\sigma_j})^2 \right] \left[ \prod_{i<j=N/2+1}^{N} (z_{\sigma_i} - z_{\sigma_j})^2 \right]. \tag{6.4}$$

That is, one can think of it as a product of two $\nu = 1/2$ Laughlin states, appropriately symmetrized. On the torus, the two copies of each $\nu = 1/2$ Laughlin states each have a degeneracy of 2, which leads to three linearly independent groundstates for the Pfaffian state on the torus.

The addition of particles or vortices into the Pfaffian state create quasi-particles. There are a number of ways that these excitations can be assigned to the different internal Laughlin states \[239\], which leads to a total state space for $2n$ quasi-particles of $2^{n-1}$ dimensions. These states are completely degenerate as the separation between quasi-particles increases to infinity. This degeneracy is topological as no local perturbations can lift the degeneracy, or cause transitions between states in the degenerate subspace. The non-abelian statistics of the Pfaffian quasi-particles manifest as rotations within this subspace as particles are braided around one another \[240\]. The braiding of these particles in the $\nu = 1$ boson Pfaffian state have been numerically simulated in Bose-Hubbard
6.2 Pfaffian-Like States in the JCH Model

Inspired by results in cold atom simulations [101, 235–237], one might expect to find evidence of a Pfaffian groundstate in the JCH model at $\nu = 1$.

A comprehensive search over many lattice configurations, and interaction strengths was conducted within the JCH model. However, the tell-tail signature of a triple degeneracy groundstate proved elusive. Instead, the simulations revealed, for some lattice configurations, a single separated groundstate in the strongly interacting limit, with a transition to a gapless phase as the effective 2-body interaction decreased. Other configurations possessed a gapless groundstate extending all the way to the hardcore limit. While these results jar with the BEC findings, other lattice boson simulations have failed similarly [244].

Although the evidence from BECs suggests that a Pfaffian groundstate should be preferred, there are other possible states at $\nu = 1$ filling for Bosons which are in competition with the Pfaffian. For example, Read [245] proposes a groundstate in which, approximately, a single vortex is attached to each Boson. This assignment exactly cancels out the external magnetic field, which reduces the problem to that of a Fermi-Liquid. Alternatively, it is conjectured [246] that a striped phase with charge density order may exist at $\nu = 1$, with some numerical simulations [247] finding evidence for this.

The Pfaffian state requires a large amount of Landau level degeneracy for the pairing of bosons to occur, which may be significantly violated when the particles are confined to a lattice, suppressing a correlated groundstate. Such a groundstate would explain the presence of gapless states for some of the simulations at $\nu = 1$, although it is not clear how to verify this conjecture at this time. It would require simulations of very large systems to separate out the impact of various factors such as finite size, and lattice potentials and limited particle number.
CHAPTER 6. PFAFFIAN STATES IN JCH LATTICES

6.3 3-Body Interactions

While there was no indication of Pfaffian like physics in the JCH model at filling factor $\nu = 1$, realizing quantum states of light with the non-trivial properties of the Moore-Read Pfaffian state remains a worthwhile pursuit.

For $\nu = 1$, the Pfaffian groundstate is the highest density groundstate of the 3-body delta potential Hamiltonian:

$$H_{3\text{-body}} = \sum_i N_i^2 + \sum_{ijk} \delta(z_i - z_j)\delta(z_j - z_k).$$

where the sum is over unique combinations of particles. The Pfaffian state lies in the nullspace of $H_{3\text{-body}}$, which is clear from inspection of Eq. (6.4); for each pair of particles, there is a number of terms in Eq. (6.4) for which each particle is in a different partition. However, there is no term for which 3 or more particles coincide that does not vanish.

In this chapter, I show that by inducing an effective 3-body interaction, it is possible to realize a Pfaffian like state in coupled atom-cavity systems.

6.3.1 3-Level JCH Model

3-body interactions (without corresponding 2-body ones) are unnatural, and do not arise in many physical systems. A number of schemes for creating effective 3-body interactions have been proposed in the context of BECs in recent times [242, 244, 248, 249]. In this section, I show that an effective 3-body interaction can be induced in atom-cavity lattice by replacing the 2-level atoms in the JCH model with appropriately tuned 3-level atoms. Furthermore, it is demonstrated via simulation that an atom-cavity lattice consisting of these 3-level atoms can possess a Pfaffian like state as its groundstate.

Consider the 3 level system atom in the $\Xi$ configuration, as shown in Fig. 6.1. This configuration consists of two evenly spaced excited states, with the atom cavity system
6.3. 3-BODY INTERACTIONS

Figure 6.1: Three-level atom configuration for inducing a 3-body interaction in the JCH lattice. Engineering the excited states to have identical excitation energies the two-excitation anharmonicity is removed when $\beta_2 = \sqrt{2}\beta_1$.

described by the Hamiltonian:

$$H^{3L} = \omega a^\dagger a + \epsilon_1 \langle e_1 | e_1 \rangle + (\beta_1 \sigma_1^+ a + h.c.) + \epsilon_1 \langle e_2 | e_2 \rangle + (\beta_2 \sigma_2^+ a + h.c.).$$  \hspace{1cm} (6.6)

Here, $\sigma_{1(2)}^+$ raises the atomic level from $g \leftrightarrow e_1$ ($e_1 \leftrightarrow e_2$), and levels 1(2) have energies $\epsilon_{1(2)}$. Choosing an atom with energy levels:

$$\epsilon_1 = \omega - \Delta, \quad \epsilon_2 = 2\omega - 2\Delta,$$  \hspace{1cm} (6.7)

and transition strengths:

$$g \leftrightarrow e_2 \equiv \beta_1, \quad e_1 \leftrightarrow e_2 \equiv \beta_2 = \sqrt{2}\beta_1,$$  \hspace{1cm} (6.8)

leads to an effective 3-body interaction. This can be seen by considering the formulation of the JC system as a two-mode Bosonic system with inter-mode tunneling. If one imposes a hardcore boson condition on one of the modes, then the system corresponds exactly to the JC cavity with a two-level atom. In the single excitation subspace, the hardcore
CHAPTER 6. PFAFFIAN STATES IN JCH LATTICES

condition is automatically satisfied, and the system is simply a free boson model.

If, instead of the hardcore condition, one imposes a 3-particle hardcore condition ($U_3 \to \infty$), then a similar situation arises, except that for both the single and double excitation subspaces, there are no interactions. Mapping this model back to the atomic model, for 1 and 2 particles, the equivalent atom-cavity model corresponds exactly to the one presented previously, where the factor of $\sqrt{2}$ arises from the indistinguishably of the bosons.

The 3-body non-linearity is demonstrated in Fig. 6.2 a). Here, the energy cost per particle for the lower polariton branch is plotted as a function of the detuning, $\Delta$. For 1 and 2 excitations, the energy per particle is the same. However, for $\ell = 3$ and above, there is an increased cost for adding additional particles. Fig. 6.2b) shows that, at $\beta_2/\beta_1 = \sqrt{2}$, the 2 and 3 level atom-cavity systems share several properties. The non-linearity is unbounded as the detuning is lowered, and disappears as the detuning is increased. Also, the non-linearity does not grow quadratically with excitation number, as is the case for a pure 3-body interaction, similar to the 2-level atom case.

Figure 6.2b) demonstrates how the two body interaction is affected by the strength of $\beta_2$. As $\beta_2$ is tuned away from $\beta_2/\beta_1 = \sqrt{2}$ the effective two-body interaction becomes +ve or -ve. However, at low detunings ($\Delta/\beta_1 = -5$ in Fig. 6.2b) the relative strength of this effective two body interaction is much smaller than the three body one.
This method for generating 3-body interactions opens up unique possibilities for investigating the physics of topological quantum states that has proved elusive in traditional environments. Furthermore, this same technique can be extended to higher order interactions. There is a hierarchy of states that generalize the Pfaffian \([250]\) state, which are expected to be groundstates of these higher order interactions. The JCH model with this modification is, to our knowledge, the only system in which such higher order interactions might be achieved, outside of a fully functional quantum computer.

In practice, engineering the three level system as described lies well within the capabilities of current cavity QED fabrication techniques. Engineering a system like this in circuit QED has been discussed in \([251]\). For cavity atoms, most \(\Xi\) configurations tend to be unstable, with fast relaxation rates that would preclude large scale coherence in the system. This instability can be mitigated by instead using an \(M\) like configuration (as in \([252]\)), where classical driving can be used to create an effective three level JCH system.

### 6.3.2 Pfaffian Like States in the 3-Level JCH Model

To identify the presence of Pfaffian like physics in this modified JCH model I follow a similar strategy to the previous chapter, by looking at the band structure, trial wavefunction overlaps, and the Chern numbers of numerically computed groundstates.

As with the Laughlin wavefunction, the Pfaffian wavefunction must be adapted to the JCH lattice, with periodic boundary conditions, if comparisons are to be made to the numerical determined groundstates. The Pfaffian state on a torus takes a similar form to the Laughlin form \([158]\):

\[
\Psi^\text{Pf}_\alpha = \text{Pf} \left[ \frac{\theta_\alpha((z_i - z_j)/L_x|\tau)}{\theta_1((z_i - z_j)/L_x|\tau)} \right] \Psi^\text{JCH}_{L,q} \quad \alpha \in \{2, 3, 4\},
\]

where \(\Psi^\text{JCH}_{L,q}\) is the JCH Laughlin wavefunction on a torus used in chapter 5, with \(q = 1\) for the \(\nu = 1\) filling factor considered here. \(\alpha\) denotes the three degenerate states.

Computation of the groundstates is restricted by the increased dimension of the
CHAPTER 6. PFAFFIAN STATES IN JCH LATTICES

3-level JCH system, further above the 2-level case. The Pfaffian state is defined only for an even number of particles, with $N_\ell \geq 4$. Thus the computations in this chapter are restricted to the $N_\ell$ case, since 6 excitations is beyond the computational resources available.

Figure 6.3 shows the Pfaffian band structure for the JCH model with 3-level atoms. For 4 particles there is a large modulation of the energy as a function of the twist angles. However, a quasi-gap exists, and computation of the Chern number ($C = 3$), coupled with the 3-fold degeneracy provides strong evidence for Pfaffian physics.

In section 5.3.2, the gap above the Laughlin groundstate manifold was estimated as a function of $\Delta$. One expects this estimate to translate directly to the Pfaffian state. However, this does not seem to apply in this case. As discussed above, it is possible that the first excited state of this system does not correspond to an excitation lying in a higher Landau level. Rather, it is some combination of quasi-particles and quasi-holes. These lie in the LLL, with 0 excitation energy in the continuum limit, which would explain the rapid decrees in the gap as the $\alpha$ decreases. It may be possible to verify this hypothesis through more extensive computations, and analysis of the degeneracy of excitations. However, I leave the resolution of this problem for future work.

With strong evidence for the Pfaffian state, I proceed to investigate the 3-level JCH model in more detail, by exploring the properties of the groundstate over a range of lattice sizes and system parameters. These investigations are presented in Fig. 6.4.

In Fig. 6.4a) the groundstate changes as a function of the detuning, $\Delta$. As in the case of the Laughlin case [85], increasing the atomic detuning, which alters the effective interaction strength (Fig. 6.2b), can induce a transition from a Pfaffian state to an uncorrelated one. This transition is accompanied by a closing of the bandgap, and, for most cases, a drop off of the overlap with the trial wavefunction.

In the case of Laughlin states on a lattice, the gap has been shown [224] to scale proportionally to the flux density per lattice plaquette. This does not seem to apply in the case of the Pfaffian state. It is possible that the first excited state of this system
6.3. 3-BODY INTERACTIONS

Figure 6.3: 3-level JCH model in a $4 \times 4$ lattice with 4 particles. a) Band structure: First 5 levels above the groundstate (orange, blue, green, red, purple) and b) phase of det $\Omega$ for the 3 dimensional groundstate manifold. The circle corresponds to the region where det $\Lambda = 0$. Since the total vorticity of this region is 3, the sum of the Chern numbers for these three states is 3, indicative of Pfaffian physics.

is not an excitation lying in a higher Landau level. Rather, it is some combination of quasi-particles and quasi-holes. These excitations lie in the LLL, with no energy gap in the continuum limit, which would explain the rapid decrees in the gap as the flux density per lattice plaquette decreases.

The Pfaffian states at $\nu = 1$ have a straightforward interpretation as the symmetrised product of two Laughlin states at $\nu = 1/2$ [158]. Assigning each particle to one of two Laughlin states. The wavefunction will vanish as two particles in the same Laughlin state approach each other, but not if those two particles are in different states. However, if any three particles coincide, then by construction the wavefunction will be zero at this point. The three degenerate states in the torus setting correspond to the singlet and two doublet states one can construct from the doubly degenerate Laughlin states.

This re-expression of the Pfaffian wavefunction also allows one to translate findings from investigations into the Laughlin state in the JCH into the current work. For example, the detuning for which the Pfaffian state undergoes a transition (Fig. 6.4b,d) is the same for the equivalent single Laughlin state as per Fig. 5.6. Furthermore, the overlap with the trial Pfaffian wavefunction is very well approximated by the overlap with a single
Figure 6.4: Bandgap and Pfaffian overlap as a function of a) $\Delta/\beta_1$ (with $\beta_2/\beta_1 = \sqrt{2}$) and b) $\beta_2/\beta_1$ (at $\Delta/\beta_1 = -5$) for 4 particles on $4 \times 4$ (blue) $4 \times 5$ (orange) $5 \times 5$ (green) $6 \times 4$ (red) $6 \times 5$ (purple) and $6 \times 6$ (brown) lattice. The dashed line b) indicates $\beta_2/\beta_1 = \sqrt{2}$. 
6.4. SUMMARY

Laughlin function, to the power of two.

In Fig. 6.4b) the groundstate changes as a function of $\beta_2$. The groundstate experience a transition from the Pfaffian state away from $\beta_2/\beta_1 = \sqrt{2}$. For $\beta_2/\beta_1 > \sqrt{2}$, where the effective two body interaction becomes attractive, there is a very sharp transition to a collapsed state. On the other side, $\beta_2/\beta_1 < \sqrt{2}$, the gap and Pfaffian overlap remain fairly stable, although there is a transition in the $4 \times 5$ lattice configuration.

6.4 Summary

While there was no evidence for a Moore-Read Pfaffian like state in the standard JCH model, it was shown how such states might be induced. A method by which three-body interactions can be induced in Jaynes-Cummings-Hubbard systems was developed by the introduction of a three-level atom into the photonic cavity. In the presence of synthetic magnetic fields, such interactions, strongly correlated states of light, with Pfaffian-like topological properties, will exist. This opens up exciting possibilities for the exploration of exotic quantum states within the cavity QED framework, including states with non-abelian quasi-particles pertaining to topological quantum computing.
7

Review of Classical and Quantum Chaos

7.1 Chaotic Systems

In 1890 Poincaré’s Sur La Problème des trios corps et les équations de la dynamique [253] was published in Acta Mathematica as the winning entry in the competition celebrating the 60th birthday of Oscar II\(^1\). Proving that some periodic orbits in the restricted 3-body problem were unstable\(^2\), it is generally considered to be the first assault on the determinism which had triumphed since the publication of the Principia in 1686. Progress was made throughout the 20th century towards the understanding of unstable mechanical systems (particularly the celestial kind). However, the full implications were not fully realized until the advent of computers, which allowed for extensive numerical experimentation [257]. These developments were grouped together under the umbrella of chaos theory [258].

Consider a system with \(N\) degrees of freedom. The state of the system is given by a point in its phase space, and its location in phase space is sufficient to determine everything (by definition).

Unless a system is trivially boring, then it will change its location in phase space

---

\(^1\)The full story of the competition is a compelling piece of mathematical history in itself. For a comprehensive, including detailed analysis of the significance of Poincaré’s work, see [254].

\(^2\)It a common misconception that this makes the general 3-body problem insoluble; In fact, there is a convergent expansion (in \(t^{(1/3)}\)) for not only the 3-body problem [255], but for the \(n\)-body [256].
over time. This is the *dynamics* of the system. The nature of a systems path through phase space is the subject of chaos theory.

For a classical system, the path through phase space that a system will trace is completely determined. That is, given a system at some point $\mathbf{x}$ in phase space, combined with it’s governing equations, will totally determine it’s future behaviour exactly.

Trajectories through phase space are called *orbits*. Orbits can be either periodic, where the system returns to it’s initial state eventually, (and by implication an infinite number of times), and non-periodic orbits.

We understand a dynamic system by the behaviour of it’s orbits in phase space. Orbits in this context refer to the 1 dimensional lines which are solutions to the equations of motion for the system in question. This includes periodic orbits, which are closed, and therefore of finite extent, and open trajectories, which are infinite.

For a system to be chaotic, it must satisfy three criteria:

1. Sensitivity to initial conditions – The famous butterfly effect. All regions of phase space have hyperbolic instability. That is, any two trajectories will eventually diverge exponentially, no matter how close in phase space they are initially. This is sometimes quantified by the system having a positive Lyapunov exponent [259].

2. Topological Mixing – any open region of phase space will eventually overlap with any other region. The set orbits passing through some volume in phase space will pass through all other regions, ‘mixing’ the phase space.

3. Dense Periodic Orbits – All points in phase space are approached by periodic orbits.

For a system to be chaotic, all of these must be satisfied. However, for certain types of dynamics, such as in systems where the dynamics is derived from a governing Hamiltonian, just the second criterion will imply the other two.
7.2 Hamiltonian Chaos

The previous definition of chaos is quite general, and serves to describe behaviour across a wide range of disciplines [260–264]. The focus of this work is on the holonomic dynamics found in classical and quantum mechanics.

For a classical system in \( N \) dimensions, its phase space has \( N \) spatial dimensions and \( N \) momentum dimensions, giving a total of \( 2N \) degrees of freedom.

The equations of motion governing these systems can be derived from a Hamiltonian, which is a function of the conjugate variables \( x_i \) and \( p_i \), and time, \( t \).

\[
H(x_1, p_1, \ldots, x_N, p_N, t) \quad \dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \tag{7.1}
\]

A Hamiltonian system is said to be integrable if there are \( N \) functionally independent constants of motion, also called isolating integrals of motion, which define an orbit in phase space. When a system is integrable, then every orbit is stable, and there is no chaotic motion. These constants arise from continuous symmetries of the Hamiltonian. This can be shown by transforming the system to action-angle variables [265].

If one can find a canonical transformation for the momenta and position variables,

\[
p_i \to J_i, \\
x_i \to \theta, \tag{7.2}
\]

such that the transformed Hamiltonian, \( \mathcal{H}(\mathcal{J}) \), is a function of only the momenta, then the dynamics become trivial. Hamilton equations give the equations of motion:

\[
\dot{\theta} = \frac{\partial \mathcal{H}}{\partial J_i} = \omega_i(\mathcal{J}) \\
\dot{\mathcal{J}} = \frac{\partial \mathcal{H}}{\partial \theta} = 0. \tag{7.3}
\]

This leads to trivial motion for the angular variables: \( \theta = \omega_i t + \omega_0 \).

Thus orbits in an integrable system trace out \( N \)-tori in coordinate space. If the \( \omega \) are relatively irrational, then the orbits will completely cover the surface of the torus. In the
case where the $\omega$ are commensurate, the orbits will be periodic.

A system governed by a Hamiltonian automatically has one constant of motion: the energy. This means that there can be no chaotic behaviour for $N = 1$ dimensional systems, since the system is automatically integrable. The two dimensional phase space is partitioned by non-intersecting orbits, making the dynamics inherently stable. For $N > 1$, this is not necessarily the case, and chaotic dynamics can emerge.

Time dependent Hamiltonians do not in general conserve energy, and hence do not have $E$ as an constant of integration. Time dependent systems are often used to study the transition to chaos in 1 dimensional systems.

An $N$-dimensional time dependent Hamiltonian can be re-framed as an $N + 1$-dimensional Hamiltonian:

$$H(x_1, p_1, \ldots, x_N, p_N, t) \rightarrow H(x_1, p_1, \ldots, x_N, p_N, x_{N+1}) + p_{N+1}$$

Hamiltonians with a periodic time dependence can be useful for studying the transition to chaos. Since the time dimension is periodic, it makes constructing a Poincarré surface trivial.

7.2.1 KAM theory

The breakdown of perturbation theory for Hamiltonian systems is due to the presence of resonances in the perturbative expansion. Consider an integrable system described by the Hamiltonian $H(J)$, with a perturbing interaction $\lambda V(\theta)$. Corrections to the action-angle variables will include terms of the form:

$$\sum_n V_n \cos \left( \frac{n \cdot \theta}{n \cdot \omega} \right)$$

where $V_n$ are the Fourier coefficients of the perturbing potential. If the frequencies of the system are commensurate, such that $n \cdot \omega = 0$, then the denominator of Eq. (7.5) will vanish, making the perturbation infinite. Even for quasi-periodic orbits, where $\omega$ are incommensurate, the denominator can become arbitrarily small for certain $n$, and so
perturbation can become arbitrarily large. Poincaré’s contribution was to show that this behaviour was generic [253]. However, many systems (e.g. the solar system) do not present obviously chaotic tendencies. KAM theory [172] seeks to explain this apparent contradiction.

Kolmogorov [266] showed how to construct a perturbation theory which was rapidly convergent for non-resonant tori. This was later made rigorous by Moser [267] and Arnol’d [268]. In the KAM picture, tori that are far from resonance are stable for small perturbations.

Orbits that are resonant are separated from each other by the stable KAM tori. Thus trajectories in phase space are confined to regions, partitioned by the KAM tori.

As the strength of a perturbation is increased further, the KAM tori break up into chains of isolated islands. These chains only partially block the flow of orbits, and so the phase space becomes increasingly mixed, with some regions becoming chaotic, and some remaining stable. At sufficiently high strength, all KAM tori can be destroyed, making the entire phase space chaotic. This transition to chaos is illustrated in Fig. 7.1 for the kicked rotor.

### 7.2.2 Kicked Rotor

The kicked rotor is toy model that is illustrate of this. Consider a rotor with angular mass $I$, angular momentum $L$, that is periodically ‘kicked’, by a $\theta$ dependent potential:

$$H_R = \frac{L^2}{2I} + K \cos (\theta) \delta_T(t),$$

where $K$ is the kick parameter which sets the strength of the perturbation and

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

During the period between kicks, the angular momentum is conserved, and $\theta$ evolves according to $\theta = \theta_0 + Jt$. During the kick, the momentum is shifted by a discrete amount,
so that, over the course of a full period, the system makes a transition:

\[
\begin{align*}
J_{n+1} &= J_n + K \sin (\theta_n) \\
\theta_{n+1} &= \theta_n + J_{n+1} T.
\end{align*}
\] (7.8)

Both $J$ and $\theta$ are invariant up to factors of $2\pi$, and so the phase space of the system need only be considered in the $[0, 2\pi] \times [0, 2\pi]$ region. Also, without loss of generality we can rescale so that $T = 1$.

Kicked systems make good models to study this transition to chaos [269], since the perturbation is easily controlled, and the dynamics can usually be easily solved, since evolution from kick to kick is trivially written down. The periodicity of the potential makes defining a Poincarré map simple, without deducing the intersection of a trajectory with some chosen plane.

When the kick parameter is 0, the rotor has both periodic and non-periodic orbits. For $J_0 = \frac{p}{2\pi q}$ the system will return to it’s initial state, $x_0$, after $M$ kicks. If $J/2\pi$ is irrational, then the system never repeats. $\omega = p/q$ defines the systems winding number. The smaller $q$, the stronger the resonance. Regions in phase space near these resonances are the first places where the KAM tori begin to break down and the system becomes chaotic. Where $\omega$ is more irrational, the system remains stable for larger values of $K$.

These phenomena can be observed in Fig. 7.1. The Poincarré map for a set of initial points is plotted for different values of $K$. For small $K$, the system is perturbed, but warped. As $K$ is increased, sections of the phase space form KAM tori. Once $K$ is sufficiently high ($K > 9.72$), sections of phase space are completely chaotic, but some islands of stability remain.
Figure 7.1: Transition to chaos in the kicked rotor. Trajectory for 1000 kicks for a) $K = 0.5$ b) $K = 0.971635$ c) $K = 3.0$
CHAPTER 7. REVIEW OF CLASSICAL AND QUANTUM CHAOS

7.3 Quantum Chaos

The development of chaos theory lead to a fairly drastic change in perspective for physics. Fairly quickly it was seen that the purview of the new theory extended into any field where feedback mechanisms played an important role.

The question now stands, “How do these ideas translate to the quantum world?” There is not yet an unambiguous response. Certainly, elements of chaos theory are present in a quantum system. But not all elements, and many aspects appear with significant differences. Contra-wise, there are phenomena in quantum chaos which have no correspondence in a classical setting. But a correspondence there must necessarily be, and the clarifying of this constitutes a large and active part of research on classical chaos.

In the Bohr-Sommerfield quantization scheme, the allowable states are ones where the constants of integration are integer multiples of $\hbar$, Plank’s constant. This has been successful for classically integrable systems. However, as noted by Einstein [270], if the model is non-integrable, there is no clear path to quantization.

A quantum system with $N$ degrees of freedom can be called integrable if there exists $N$ independent operators $I_i$ such that

$$[I_i, I_j] = 0 \quad \forall i, j.$$  \hspace{1cm} (7.9)

That is, there is a complete set of ‘good’ quantum numbers, and eigenstates of the systems Hamiltonian can be written are mutual eigenstates of all the $I_i$.

Chaos, as defined for classical systems, does not translate directly to the quantum setting. First of all, the Heisenberg uncertainty principal precludes a well defined phase space.

Secondly, there can be no exponential dependence on initial conditions, since the evolution of a quantum state is strictly linear. That is, the overlap of two different
7.3. QUANTUM CHAOS

quantum states remains constant throughout the evolution in time, i.e.

$$\langle \psi_1(t)|\psi_2(t) \rangle = \langle \psi_1(t = 0)|U^\dagger(t)U(t)|\psi_2(t = 0) \rangle = \langle \psi_1(t = 0)|\psi_2(t = 0) \rangle.$$  \hspace{1cm} (7.10)

Despite this, there are still ways in which chaotic behaviour can manifest in quantum systems.

The transition to chaos in a quantum system can be seen in the structure of its eigenstates and energy spectrum. For example, random matrix theory [171,271,272] is the study of the spectral properties of random matrices, which have been shown to share similar characteristics of the quantum versions of classically chaotic systems.

One way to observe a transition to chaos is through the participation number,

$$P(\psi) = \left( d \sum_i \left| \langle \psi | \psi^i_0 \rangle \right|^4 \right)^{-1},$$  \hspace{1cm} (7.11)

which is normalized by the total dimension \(d\) of the space. \(P\) is \(1/d\) when \(\left| \langle \psi | \psi^i_0 \rangle \right| = 1\) for some \(i\) and 1 when \(\psi\) projects evenly onto the \(|\psi^i_0\rangle\). One can consider this to be an indication of quantum ergodicity [273].

Naturally, this is a basis dependent measure, but comparing to the unperturbed states can present evidence towards chaotic behaviour.

7.3.1 Quantum Kicked Rotor

The kicked rotor has in section 7.2.2 has a quantum mechanical counterpart. Here, the Schrödinger equation governing the system is:

$$i\hbar \frac{\partial \psi(\theta, t)}{\partial t} = \frac{\hbar^2}{2I} \frac{\partial^2}{\partial \theta^2} + K \cos(\theta) \delta_T(t) |\psi(\theta, t).$$  \hspace{1cm} (7.12)
This can be expanded in the angular momentum eigenbasis, $|n\rangle$, where $L |n\rangle = \hbar n |n\rangle$. The Hamiltonian then becomes:

$$\frac{\hbar^2 n^2}{2I} + \frac{K}{2} \delta_T(t) \sum_n (|n + 1\rangle \langle n| + |n - 1\rangle \langle n|).$$  \hspace{1cm} (7.13)

As with the classical kicked rotor, one can define a map from the state at time $t$ to the time $t + T$ that defines the dynamics. For a quantum system this takes the form of a unitary operator, which is known as a Floquet map, i.e.

$$|\psi_{n+1}\rangle = U |\psi_n\rangle.$$  \hspace{1cm} (7.14)

For the case of the quantum kicked rotor the Floquet map is given by:

$$U_{mn} = (-i)^{m-n} J_{m-n} \left( \frac{K}{\hbar} \right) \exp \left( -\frac{i\hbar m^2 T}{2I} \right),$$  \hspace{1cm} (7.15)

where $J_m$ is the $m$th Bessel function.

For quantum systems with time-periodic potentials, the eigenstates, $|\phi_k\rangle$, and eigenphases, $\phi_k$, of the Floquet map $U$ completely define the evolution, i.e.

$$|\psi_N\rangle = \sum_k \exp [i\phi_k N] |\phi_k\rangle \langle \phi_k|\psi_0\rangle.$$  \hspace{1cm} (7.16)

In Fig. 7.2 the average participation number for a state at $n = 0$ with the Floquet eigenstates of the kicked rotor as a basis for a range of parameters. Identifiable here are a number of chaotic signatures. For resonant values of $T$, the participation number is large, implying that the Floquet eigenstates are spread out in momentum space. However, away from the resonances, the participation number drops. This is called dynamically localization [274]. Essentially, the ‘random’ phases seen by the momentum states destructively interfere, suppressing diffusion. These localized states can be considered to be the quantum equivalent of KAM tori [275], although the correspondence is far from straightforward [276].

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7.4. SUMMARY

Figure 7.2: Participation Number of the quantum kicked rotor. The localization length increases as the kicking parameter $K$ is increased, but remains small for off resonant values of $T$.

7.4 Summary

Background theory for chaos has been introduced, where the basic properties of both classical and quantum chaos were elaborated upon. The kicked rotor offers a clean and straightforward model in which these phenomena manifest. In the next chapter, this theory is used in the context of a pair of coupled JC cavities, where the relationship between classical and quantum chaos is explored.
8.1 Coupled Cavities

The original proposals for observing quantum phase transitions in JCH systems [31,36,37] called for large numbers of identical systems. Constructing large arrays of cavities which are sufficiently coherent and identical poses a significant challenge. Exploiting long coherence times can allow some analogous effects to be studied by trading large-scale phenomena for small-scale, long time phenomena. For example, there is an isomorphism between the periodically kicked rotor and the Anderson tight binding model [277]. The Anderson model predicts localization for particles in a disordered lattice, and for dimension greater than three exhibits a second order phase transition between metallic and super fluid phases. This has been recently demonstrated in the time-domain as a kicked system with cold atoms [278].

Periodic systems, such as delta kicked rotors and tops, are widely used to study the link between classical and quantum chaos [269]. Several interesting correspondences between the two regimes have been identified such as dynamic localization with regions of stability [279] and Lyapunov exponents with entanglement generation [280]. There are many open questions about the nature of quantum systems with semi-classical dynamics that exhibit chaotic behavior, particularly in time varying systems [281].

This chapter explores the properties of the kicked coupled cavities system. Here,
two JC cavities are periodically coupled via the usual Hubbard-like hopping term. The starting point is an analysis using Floquet theory, which reveals a rich cast of effects associated with chaotic behaviour. These include a transition to chaos, dynamic tunneling and localization. The quantum behaviour is then compared to a semi-classical version, which exhibits true chaos. The two descriptions converge in the correspondence limit, which prompts a discussion on the relation between quantum and classical chaos.

### 8.1.1 Model for the Kicked Coupled Cavity System

The kicked coupled JCH system, illustrated in Fig. 8.1, can be modeled by a time dependent Hamiltonian, \( H_{\text{kicked}}(t) \), composed of a constant term, \( H_0 \), and a time dependent kicking term \( H_K(t) \).

\[
H_0 = H_{1}^{\text{JC}} + H_{2}^{\text{JC}} \\
H_K(t) = \kappa \delta_T(t)(a_1 a_2 + a_1^\dagger a_2)
\]  

As in the case of the other kicked systems [Eq. (7.6)], modelling the kick profile by a \( \delta \)-function significantly simplifies the analysis. When it comes to a real-world implementation, the interaction must necessarily be on for some finite time. Simulations show that a finite time kick does not significantly alter the behaviour of the system, so that the broad conclusions drawn from working with the kicked approximation can be trusted.

This is shown in Fig. 8.2. The effect of a finite time kick is quantified using the bound on the deviation of fidelity, \( \delta \), as defined in Eq. (3.43). Here, one can see that for small \( \tau \), the deviation of the evolution between using a \( \delta \)-kick and a finite one is quite minimal. Of course, if the system is chaotic, then this small change in the kick will cause any...
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Figure 8.2: Effect of a finite kick time, $\tau$. Comparing the Floquet matrix for the finite kick and the $\delta$-kick, as measured by $\delta$, for $\beta T = 1.2$.

trajectory to radically change. However, one does not expect it to affect significantly any non-chaotic trajectories, due to the nature of their stability to perturbation.

In this system, the two JCH cavities undergo ordinary evolution between kicks. Since the two cavities are decoupled, the evolution of the system during this time is trivial, since the exact solutions to the JC cavity are known [Eq. (2.35)], and will in general consist of coherent Rabi oscillations. The kick couples the two cavities together transferring photons between cavities. This mixes the two systems, leading to complex dynamics.

The three dimensionless parameters, $\kappa \tau$, $T/\beta$ and $\Delta/\beta$, are sufficient to specify the dynamics of $H$. For simplicity only the quasi-resonant case, $\Delta \sim 0$, is considered, where the key features of the system are most easily elucidated. This makes $\sin \theta_l = \cos \theta_l = \frac{1}{\sqrt{2}}$.

Symmetries of the Coupled Cavity System

The inter-cavity coupling term breaks the individual excitation conservation of each JC system, but commutes with the total $\ell_T = \ell_1 + \ell_2$, thus we can consider cases of total excitation number individually. For a single excitation, $\ell = 1$, the excitation oscillates between cavities trivially, with frequency $\kappa \tau / T$. For all $\ell > 1$ rich behavior with signatures of quantum chaos can be seen. However, here the $\ell = 2$ case is focused
CHAPTER 8. CHAOS IN COUPLED CAVITIES

on in the quantum case, and the semi-classical equivalent. Although the dimension of Hilbert space is just 8 for \( \ell = 2 \), many of the features of quantum chaos are already present, and it is this case which will be most accessible experimentally.

When the detuning of both cavities is equal then Hamiltonian is invariant under exchange of the cavities. This gives another quantum number \( p = \pm 1 \), where

\[
P | \psi_p \rangle = \sum \delta_{m_1,n_2} \delta_{m_2,n_1} \delta_{s_1,r_2} \delta_{s_2,r_1} | m_1, r_1, m_2, r_2 \rangle \langle n_1, s_1, n_2, s_2 | \psi_p \rangle = p | \psi_p \rangle \tag{8.2}
\]

States with \( \ell_T \) excitations form a \( 4\ell_T \) dimensional subspace, which is further separated into the two parity states \( p = \pm 1 \) of dimension \( 2\ell_T + p \).

8.2 Semi-Classical Coupled Jaynes-Cummings Model

The Jaynes-Cummings model, as defined in Chapter 2, is explicitly quantum mechanical, deriving from the interaction between light and atoms, as opposed being a quantized version of some classical system. However, the correspondence between classical and quantum chaos is of great theoretical interest. Of course, ultimately, all classical systems must be derivable from quantum mechanical systems, the latter being more fundamental. In the case of the Jaynes-Cummings model, there is a classical approximation which is valid when the quantum fluctuations become negligible.

8.2.1 Semi-Classical Equations of Motion

To find the classical description for the Jaynes-Cummings model, begin with the Heisenberg equations of motion for a set of observables:

\[
\begin{align*}
\dot{\sigma}^- &= i[H^{JC}, \sigma^-] = i\Delta \sigma_- + i\beta \sigma_z a \\
\dot{\sigma}_z &= i[H^{JC}, \sigma_z] = -2i\beta (\sigma^- a^\dagger + \sigma^+ a) \\
\dot{a} &= i[H^{JC}, a] = -i\beta \sigma^-
\end{align*}
\tag{8.3}
\]
8.2. SEMI-CLASSICAL COUPLED JAYNES-CUMMINGS MODEL

Under the assumption that quantum fluctuations are negligible one can make the mean
field approximation [282]:

\[ \langle AB \rangle = \langle A \rangle \langle B \rangle. \quad (8.4) \]

Then the expectation value of these operators can then be treated as classical variables.
I.e.

\[
\begin{align*}
E &= \langle a \rangle \\
S_z &= \langle \sigma_z \rangle \\
S &= \langle \sigma^- \rangle
\end{align*}
\]

This results in a set of equations governing the dynamics of a classical system:

\[
\begin{align*}
\langle \dot{a} \rangle &= \dot{E} = -i\beta S, \\
\langle \dot{\sigma}^- \rangle &= \dot{S} = i\Delta S + i\beta E S_z, \\
\langle \dot{\sigma}_z \rangle &= \dot{S}_z = 2\beta i(SE^* - S^*E),
\end{align*}
\]

where \( E \), the E-field, and \( S, S_z \), vectors on the Bloch sphere, are now classical quantities.

This model has been studied extensively [283]. The atomic motion is given by the
well known Optical Bloch Equations [125]. However, the electric field in this case depends
upon the atomic state and so the dynamics are considerably more complicated [283].

For no detuning the uncoupled equations of motion are equivalent to that of a
pendulum with the momentum, \( E \), and the height of the bob. This motion has two
constants of motion,

\[
\begin{align*}
N &= |E|^2 + \frac{1}{2}(S_z + 1) \\
S^2 + 4S^*S &= 1.
\end{align*}
\] (8.7)

one can therefore reduce the dynamics of a JC cavity to two degrees of freedom, \( E \), and
\( \theta \), with \( S = i \sin (\theta)/2 \) and \( S_z = \cos (\theta) \). While this has an analytical solution in terms of
elliptical functions [130], in practice it is easier to numerically integrate.

For the two cavities we have two sets of variables, \( E_{1,2} \) and \( \theta_{1,2} \). The kick which
couples the two cavities is governed by Heisenberg equations of motion for the electric
field variables during the kick duration:

\[
\begin{align*}
\dot{E}_1 &= -i\kappa E_2 \\
\dot{E}_2 &= -i\kappa E_1
\end{align*}
\]  

These equations are simply solved. Integrating over the \(\delta\)-function yields a map that exchanges the field between the two cavities:

\[
\begin{pmatrix}
E_1 \\
E_2
\end{pmatrix}_{n+1} = \begin{pmatrix}
\cos \kappa \tau & \sin \kappa \tau \\
-\sin \kappa \tau & \cos \kappa \tau
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2
\end{pmatrix}_n.
\]  

The kick map is simply a rotation matrix acting on the two electric field components. This implies the conservation of the \(E_1^2 + E_2^2\) during the kick, and therefore the conservation of \(N_1 + N_2\) for the full dynamics.

Equation (8.9) also shows that the kick transition is a linear map, and therefore cannot by itself induce chaotic dynamics. Only when combined with the non-linearity inherent in the Jaynes-Cummings dynamics will the system be chaotic.

### 8.2.2 Transition to Chaos

The kicked hopping leads to non-integrable dynamics, so that the only constant of motion is now \(N_1 + N_2 = N\). In general this results in a chaotic phase space, however, for some values of \(\kappa\) and \(T\) there will be regions in which the motion is semi-regular. These regions are described by KAM (Kolmogorov-Arnold-Moser) theory [172]. In an unperturbed system the path in the \(d\) dimensional phase space in action-angle variables lies on the surface of a \(d\)-torus. If the periods in each dimension are sufficiently incommensurate then the system is confined near a deformed torus for small perturbations. The system becomes increasingly chaotic as the perturbation is turned up, leading to destruction of some tori. The phase space is then a chaotic sea with islands of stability which are topologically separate, from the chaos as well as each other. Eventually the perturbation destroys all these regions and the dynamics are fully chaotic.
8.2. SEMI-CLASSICAL COUPLED JAYNES-CUMMINGS MODEL

Figure 8.3: Classical phase space strobos of the dimensionless E-field in each cavity over 200 kicks from several initial points. a) $\kappa \tau = 1.3$, $\beta T = 0.1$. In the small $T$ limit the total total energy in the electric field, $E_1^2 + E_2^2$ is stable, leading to non overlapping rings. b) $\kappa \tau = 0.4$, $\beta T = 1.7$. Phase space is mostly chaotic except for the four regions where the energy in the system is confined mostly to one cavity. As there are 4 degrees of freedom, and only a single constant of motion, plots of the electric field in each cavity do not convey the entire dynamics.

The kicked cavity system exhibits a transition to chaos in a similar fashion to the kicked rotor. The system is defined by the three parameters, $\tau$, $T$ and $N_0$.

For $\tau = 0$ or $T = 0$, the system is exactly solvable, and therefor not chaotic. As either of these parameters are increased, the system becomes increasingly chaotic. In general, a larger value of $N_0$ leads to a more unstable system. These results will now be shown in more detail.

The centers of stability that survive the longest are usually found around short periodic orbits. In this kicked system, however, there are in general no single-period orbits, making the motion difficult to determine the precise point at which the phase space becomes fully chaotic. However, numerical simulations for the $N = 2$ case indicate that for small $\kappa \tau$ the most persistent KAM tori are around $N_{1,2} = \sqrt{2} \sin (\kappa \tau)^2$, $N_{2,1} = \sqrt{2} \cos (\kappa \tau)^2$ (Fig. 8.3b). That is, in these four regions of phase space the energy in the system remains localized to a single cavity. As $\kappa \tau$ is increased these regions become leaky (cantori) and eventually disappear, after which the phase space is fully chaotic.

The value of $\kappa \tau$ at which the system becomes chaotic is dependent on $T$. The period for a small electric field in a cavity is $2\pi$; when $\beta T$ is resonant with this the KAM tori are destroyed with much smaller $\kappa \tau$. Unlike other kicked systems, this system is still
regular for some $\kappa \tau$ at the resonances due to the non-linear nature of the perturbation that each cavity sees. The range of parameters in which this mode occurs is shown in Fig. 8.4a) where the destabilizing effect of the resonances can be seen around $\beta T = 2n\pi$.

Alternatively, one can also consider the limit in which $\kappa \tau$ is larger than the kick period, $\beta T$. In this limit the electric field decouples from the atomic degrees of freedom and the energy in the electric field oscillates between the two cavities Fig. 8.3b) and one sees separate regions which conserve the total energy of the field. For small kick period, $T \ll \beta$ there is a center of stability around $S_{z1} = S_{z2} = 0$, dynamically confining the atoms to their ground states.

### 8.2.3 3 Dimensional Reduction

The phase space of the two cavities is 4 dimensional, with $E_{1,2}$ and $\theta_{1,2}$. However, the conserved quantity $N$ reduces this to 3. One can reduce the two electric field variables to an angular variable, $\theta_E = \arctan(E_1, E_2)$. This reduction to 3 dimensions allows for
easier visualization of the system at varies points in parameter space.

The two sets of parameters in Fig. 8.3 are reproduced in Fig. 8.5, shows the trajectories for a few initial conditions. One can clearly see the different dynamics at work. In particular, the lower dimensionality of some trajectories is easy to make out, indicative of regular dynamics.
Figure 8.5: Poincaré map at $t = T$ for a) $\kappa \tau = 1.3, \beta T = 0.1$ b) $\kappa \tau = 0.4, \beta T = 1.7$. Note that while most of the points are spread throughout the entire phase space, the points in blue are confined to a single cavity and are experiencing regular Rabi oscillations.
8.3 Quantum Dynamics

Having studied the semi-classical dynamics of the kicked coupled cavity system, it is expected that, by the correspondence principal, the quantum dynamics exhibit some qualitatively similar behavior to the classical case, however, there are also effects which arise which are specifically quantum in nature.

8.3.1 Floquet Operator

In general, the inclusion of a time dependent operator breaks energy conservation. The Hamiltonian is symmetric under transformations \( t \rightarrow t + nT \), which admits a new discrete symmetry. This allows the dynamics to be studied in a time-independent manner by considering the Floquet operator. The Floquet operator is the propagator, \( U_f = U(T) \), which maps the system from time \( t = nT \) to \( t = (n + 1)T \).

The behaviour of the system for all time can then be expressed simply in terms of the eigenvectors \( |\phi_i\rangle \) and eigenphases \( \phi_i \) of \( U_f \). That is,

\[
\psi(t = nT) = \sum_i e^{i\phi_i n} |\phi_i\rangle \langle \phi_i| \psi(t = 0)\rangle. \tag{8.10}
\]

All information about the system’s dynamics is contained therein, and the presence of chaotic behaviour will manifest itself in various ways within the structure of the eigenvectors and eigenvalues.

The Floquet operator for the coupled cavity system can be factorised into the perturbed and unperturbed evolution, due to the delta-function kick:

\[
U_f = e^{\int_T^t iH \, dt} = e^{iH_0 T} e^{iK} = e^{(H_{1C} + H_{2C})T} e^{e\tau(a_1^\dagger a_2 + a_2^\dagger a_1)}. \tag{8.11}
\]

8.3.2 Transition to Quantum Chaos

The quantum equivalent of KAM tori can be understood as dynamic localization [275]: States which are initially in the localized regions have exponentially suppressed diffusion.
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into chaotic areas of phase space.

If some state $|\psi\rangle$ is well represented by a small number of basis states, $|\psi_0\rangle$, one may consider $|\psi\rangle$ to be localized to some degree. This is quantified by the participation number in Eq. (7.11).

While $P$ is dependent on the choice of basis (i.e. one can always choose some basis with $\psi$ as a base), comparing the eigenstates of the unperturbed Hamiltonian to the perturbed best represents the degree of mixing [284]. Taking the $\kappa = 0$ eigenstates as the basis, and increasing $\kappa$, leads to Floquet states with increasing $P$.

Figure 8.4b) shows the average participation number of the Floquet states over a range of $\kappa \tau$ and $\beta T$ for a system with two excitations. Let $|\psi_2^2\rangle$ denote the subspace of states with two excitations in the one cavity, and likewise the states with one excitation in each cavity as $|\psi_1^1\rangle$. The regions where $P$ is small corresponds to states with both excitations in the same cavity being dynamically separated from states with excitations in both cavities, i.e. an approximate symmetry of $U_f$.

The suppression is destroyed by resonances which occur at $T = \frac{1}{2\pi} = \frac{n\sqrt{2}}{2}, n(1 + \frac{\sqrt{2}}{2}), n(1 - \frac{\sqrt{2}}{2})$, which are solutions to

$$\sqrt{2}T = mT, m \in \mathbb{I}.$$ 

At these values the phase accrued after each period is 0, and so there is no destructive interference. This implies that there is indeed dynamical localization suppressing dispersion in the system. For example, when $T = \frac{n\sqrt{2}}{2}$, the states in $|\psi^2\rangle$ pick up no relative phase to states with $E = 0$. This removes the interference suppressing transmission into these states, and destroys the localization.

In Fig. 8.4b), for the atomic limit, shows that the dependence of localization on the parameters correspond qualitatively to the semi-classical case (Fig. 8.4a), though with important differences. The frequency at which the classical cavities oscillate depends continuously on the energy in the cavity, and in general is different from the Rabi frequency of the quantum case; these two only coincide in the limit $l \to \infty$. Thus, the
Figure 8.6: Average participation number and long term averages of observables (mean expectation of an ensemble of random states) over increasing $\kappa \tau$, with $\beta T = 1.2$. Circles (Dashed): $(\sigma_{z1}^\dagger + 1)/2$, Diamonds (Dots): $a_1^\dagger a_1$, Squares (Dot-Dashed): $\sigma_{z1}\sigma_{z2}$, Triangles: Average participation number (Dot-Dashed).

locations of resonances are different in the two regimes.

Note also that in contrast to the classical case, the resonance removes the localization for arbitrarily small $\kappa \tau$. Resonances in the classical case are not sharp, due to the energy dependent frequencies.

For time independent systems, chaos can be studied via the statistics of energy levels, however, in periodic systems, the eigenphases of the unitary operator are not observable. However, the observables of a chaotic system are ergodic. That is, the mean of some observable $\hat{O}$ over an ensemble of random states is identical to the mean of $\hat{O}$ over a sufficiently long time, which is experimentally accessible. For a chaotic system, the unitary map $U_f$ has no symmetries, and so one expects the average state to be no different from a random one chosen with the appropriate measure. Figure 8.6 shows the long-time mean of some experientially observable quantities, and the mean over random states.

**Perturbation Theory**

When the kick is small, Eq. (8.11) can be approached analytically via perturbation theory. Consider the kicked Floquet operator:

$$U_f = \exp [i\lambda V] \exp [iH_0T],$$

(8.12)
where $\lambda$ is the perturbative parameter, and $H_0$ is the unkicked Hamiltonian with solutions

$$\exp[iH_0T]|n\rangle = \exp[i\phi_n]|n\rangle.$$  \hfill (8.13)

Here, $|n\rangle$ are eigenstates of $H_0$ with energy $E_n$, and $\phi_n = E_nT$ the eigenphases.

Following the usual procedure [285], one can expand the exact solutions in powers of $\lambda$:

$$\phi_n = \phi_n^{(0)} + \lambda \phi_n^{(1)} + O(\lambda^2)$$

$$|n\rangle = |n\rangle^{(0)} + \lambda |n\rangle^{(1)} + O(\lambda^2).$$  \hfill (8.14)

Substituting this expansion into Eq. (8.13), and collecting the first order terms yields the first order corrections to the eigenphase and eigenstates of the Floquet operator:

$$\phi^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle$$

$$|n^{(1)}\rangle = i \sum_{m \neq n} \frac{\langle m^{(0)}|V|m^{(0)}\rangle}{1 - \exp[i(\phi_m^{(0)} - \phi_n^{(0)})]} |m^{(0)}\rangle.$$  \hfill (8.15)

This result looks quite similar to the ordinary time independent perturbation theory, with the exception of the denominator in the corrections to the eigenstates. Here, it is the separation of the eigenphases, modulus $2\pi$, that affects the magnitude of the perturbation. Thus, a resonance can occur even if the energies of $H_0$ are different, but the unkicked Hamiltonian has degenerate eigenphases, as in the case in Fig. 8.4.

Equation (8.15) also shows that, in the case of a small kicking strength, and well separated eigenphases, the Floquet eigenstates will not derivate far from

$$V = a_1^\dagger a_2 + a_2^\dagger a_1 \text{ and } \lambda = \kappa \tau$$  \hfill (8.16)

From this one can see how a resonance leads to the mixing of eigenstates, and a higher participation number.

### 8.3.3 Tunneling

In the case of degenerate states, the situation is more complicated.
Classically, islands of stability are topologically separated, forbidding transitions between them. Quantum dynamics admit such flow of probability in phase space by a mechanism called dynamic tunneling and has been observed experimentally in a variety of systems [286]. Although this mechanism is distinct from the usual tunneling, as there is no potential barrier to overcome, the system nevertheless moves across classically forbidden regions in phase space.

In the $\kappa = 0$ limit there is a two fold degeneracy for all Floquet states due to the $H_1^{JC}, H_2^{JC}$ symmetry. A state with both excitations initially in one cavity is in a superposition of two Floquet states, $|\pm f^2\rangle$, which both have equal projections onto both cavities, and are both in the $|\psi^2_i\rangle$ subspace:

$$|\psi\rangle = |\psi^2_1\rangle = \frac{1}{\sqrt{2}} (|+f^2\rangle + |-f^2\rangle).$$  \hspace{1cm} (8.17)

The perturbation breaks the degeneracy, leading to an approximate separation in the eigenphases, $\phi$. Each kick, the two Floquet states composing $|\psi\rangle$ are separated by a phase-angle of $\phi$. After $\frac{\pi}{2\kappa}$ kicks the phase separation is $\pi$, and $|\psi\rangle$ has evolved to the state $\frac{1}{\sqrt{2}}(|+f^2\rangle - |-f^2\rangle) = |\psi^2_2\rangle$, i.e. completely in the other cavity. Figures 8.7a) and 8.7b) show the transmission between the two separated localized states for $\kappa\tau = 0.1$ and $\kappa\tau = 0.2$ respectively. The two excitations in the system oscillate between cavities, though are strongly localized to the $\psi^2_i$ subspace. As $\kappa\tau$ increases so does $\phi$, and the localization to the $|\psi^2_i\rangle$ subspace decreases.
Chapter 8. Chaos in Coupled Cavities

Figure 8.7: Evolution of the system initially in $|\psi^2\rangle$ over 1000 kicks with $\beta T = 1.2$ and a) $\kappa \tau = 0.1$ b) $\kappa \tau = 0.2$. Blue line is the expectation of excitations in cavity 1. Purple line is the expectation of finding both excitations together in the one cavity.

8.4 Experimental Implementation

Figure 8.8 shows a possible experimental implementation in a circuit QED system, compatible with the current state of the art and thus allowing an experimental investigation of quantum chaos effects in a fast developing field. Superconducting strip-line cavities coupled to transmon qubits provide a atom-cavity interaction well into the strong coupling regime [87], and the architecture provides a simple means for producing the kicked coupling ($\kappa$) through an intermediate qubit [287].

While the effects discussed apply to any implementation of JC systems, circuit QED presents itself as one of the most viable platforms due to the large coupling coefficients and long coherence time, relative to other cavity QED systems.

Current experiments in cavity QED, where a transmon is coupled to a resonating microwave cavity, have characteristics which could allow a successful realization of this kicked system. A cavity QED setup with $\omega/2\pi = 6.92$GHz, $\beta/2\pi = 347$MHz and coherence time of order $1\mu$s has been achieved recently [43, 87].

The localization transition occurs around $\kappa \tau \approx 0.1$ and so for the delta-function kick approximation to be valid the pulse time should satisfy $\tau \ll 1/\beta$. For the coupling strengths cited above, this requires a pulse time of $\tau \approx 10^{-10}$s and, therefore, $\kappa$ order
8.4. EXPERIMENTAL IMPLEMENTATION

Figure 8.8: Schematic of a possible superconducting stripline cavity implementation of the kicked system. Transmon qubits are centered on each cavity at a and b with an atom-photon coupling $\beta$, and with the inter-cavity coupling, $\kappa$, controlled by an applied voltage at c.

1GHz. Between pulses $\kappa$ must be of the same order as the decoherence rate (i.e. $\sim 1$MHz) such that the dispersion due to the constant inter-cavity coupling is small over the time of the experiment. Thus a sequence of $\sim 100$ kicks could be applied within the coherence time. This is long enough to observe dynamic tunneling and localization/delocalization by including the decoherence and dephasing explicitly in the simulation.

The tunable hopping term could be achieved using an intermediate qubit coupling such as in [287,288]. In such scheme’s the effective coupling is of order

$$\kappa_{\text{eff}} \sim \beta_{13}^3 \beta_{23}^3 / \Delta^3$$

where $\beta_{13}$, $\beta_{23}$ and $\Delta$ are the coupling strengths of each resonator to the intermediate qubit and it’s detuning respectively and $\Delta \gg \beta$. This requires the coupling to the intermediate qubit to be significantly greater than the other couplings. The detuning can be controlled in situ, allowing the coupling to be switched on and off.

Spectroscopic measurements can be used to determine the final state [1]. Although there will be significant interaction with the environment, the only final states of interest are those that still have two excitations. One can therefore largely remove the effects of atomic relaxation and photon dissipation with a post-selection scheme, given a temperature smaller than the characteristic energies of the system. De-phasing terms will still be relevant, however, these are generally ignorable over the time frames.
8.5 Summary/Conclusions

The phenomena discussed have been observed in other systems, such as dynamic tunneling and localization in cold atoms \cite{286,289}. Circuit QED allows direct control over many system parameters and direct measurement of the state of the system. This can be used, for example, to study the effect of noise by controlling the detuning parameter in situ.

As circuit QED is proving to be an important field, with a wide range of possible applications, understanding chaotic behavior in these systems will be crucial. An experimental realization of the system seems quite possible, although it is not without challenges, specifically in achieving a sufficiently large inter-cavity coupling. It would allow the study of the rich behavior that can be expected in coupled Jaynes-Cummings systems, and open up new regimes for investigating quantum chaos.

I have presented a simple model which exhibits a transition from localization to ergodicity and dynamic tunneling. Importantly, this behavior persists even for small Hilbert space dimension, which, although interesting behavior can be seen for any number of excitations above two, the lowest case most clearly conveys the aspects emphasized. Furthermore, the two excitation case will most likely be the easiest to implement experimentally. Constantly improving control in circuit QED systems means that it will be possible to study the higher dimensional cases. This could potentially allow a novel means for probing the transition between classical and quantum chaos.
Conclusion and Future Work

The major theme of the work in this thesis had been the adaptation of many-body physics to coupled atom cavity systems.

The first few chapters focused on introducing magnetic fields into coupled atom cavity systems, and how this manifests in the JCH model.

The introduction of magnetic fields to two dimensional quantum systems can lead to a whole range of interesting physics. Here, it has discussed how synthetic magnetic fields can be achieved in JCH systems. In particular, the photon assisted tunneling scheme was adapted for the JCH system. Here, one finds that the synthetic field can be introduced with high fidelity by appropriately modulating the atomic exciting energy along with the cavity frequency. The JCH model with a magnetic has an interesting spectrum that maps onto the Hofstadter butterfly.

The introduction of a synthetic magnetic field to the JCH model modifies the boundary between the Mott-Insulator and superfluid regimes. This modification arises from a competition between the magnetic lattice and the spatial lattice. Additionally, we predict that in the superfluid regime the introduction of a synthetic magnetic field leads to the formation of vortices which, due to the local non-linear atom-photon interaction, forms a triangular lattice in the groundstate. This work opens up possible avenues for future research into the role of the JCH atom-cavity interaction in frustrating the formation
of triangular Abrikosov vortex lattices, and, since this is an inherently two dimensional system, the possibility of observing a BKT transition.

In the mean field limit it was shown how the Mott-Superfluid phase transition in the JCH model is affected by the presence of a synthetic magnetic field. Using the Gutzwiller ansatz numeric and analytic techniques show how the magnetic field suppresses the phase transition, according to the frustration induced by the magnetic length. The presence of vortices in the superfluid regime was seen though similar techniques.

While this confirms the presence of these well known magnetic phenomena, the connection with open system photonic lattices requires further investigation to allow these phenomena to be confirmed experimentally.

Strong evidence for the existence of fractional quantum Hall states in JCH systems has been demonstrated. A Laughlin wavefunction like ansatz for the JCH was introduced by using the single particle solutions to a JCH system in a magnetic field studied in chapter 3. High overlap with exact groundstates of small systems indicate FQHE. Furthermore, by computing the Chern number for theses groundstates, the groundstates of the system were seen to possess the appropriate topological qualities.

The Chern number was also able to indicate that there is a transition from the strongly correlated FQHE to a uncorrelated state as the atomic excitation energy is increased. The exact nature of these transitions is unclear, due to the small size of the systems it was possible to simulate.

While there was no evidence for a Moore-Read Pfaffian like state in the standard JCH model, It was shown how such states might be induced. A method by which three-body interactions can be induced in Jaynes-Cummings-Hubbard systems was developed by the introduction of a three-level atom into the photonic cavity. In the presence of synthetic magnetic fields, such interactions, strongly correlated states of light, with Pfaffian-like topological properties, will exist.

The existence of non-abelian states in coupled-atom cavities opens up a number of exciting possibilities for future work. These states possess gapped excitations with degenerate manifolds. The non-abelian properties of these excitations mean that braiding
operations can act on this degenerate subspace, forming the basis for quantum computation. Extending the analysis of the three-level JCH system to include the behaviour of these excitations would be an important contribution to the field. This would include developing methods by which to observe such properties in a driven-dissipative context.

The final two chapters looked at how phenomena drawn from chaos theory manifest in the JCH. Specifically, introducing a periodically kicked coupling between two cavities resulted in a wide range of chaotic behaviour.

The phenomena discussed have been observed in other systems, such as dynamic tunneling and localization in cold atoms [286, 289]. Circuit QED allows direct control over many system parameters and direct measurement of the state of the system. This can be used, for example, to study the effect of noise by controlling the detuning parameter in situ.

As circuit QED is proving to be an important field, with a wide range of possible applications, understanding chaotic behavior in these systems will be crucial. An experimental realization of the system seems quite possible, although it is not without challenges, specifically in achieving a sufficiently large inter-cavity coupling. It would allow the study of the rich behavior that can be expected in coupled Jaynes-Cummings systems, and open up new regimes for investigating quantum chaos.

I have presented a simple model which exhibits a transition from localization to ergodicity and dynamic tunneling. Importantly, this behavior persists even for small Hilbert space dimension, which, although interesting behavior can be seen for any number of excitations above two, the lowest case most clearly conveys the aspects emphasized. Furthermore, the two excitation case will most likely be the easiest to implement experimentally. Constantly improving control in circuit QED systems means that it will be possible to study the higher dimensional cases. This could potentially allow a novel means for probing the transition between classical and quantum chaos.

The JCH model is proving to be a flexible framework for studying many-body quantum physics. This thesis has shown a number of important potential instantiations of this.


BIBLIOGRAPHY


A Adiabatic Elimination

When a quantum system is composed of two or more subsystems which have energies which are relatively well separated and are weakly coupled, then in general, the basis states of each system will not mix together. However, the presence of the other states leads to interactions within each subsystem via virtual exchanges. The aim of adiabatic elimination is to find an effective description for one of these subsystems, without reference to the rest of the system. This is a very common situation in quantum atom-optics, where the radiation field and atomic energies can be quite different. A rigorous technique for adiabatic elimination thesis of well separated states, which is used in this thesis, is presented here. This method allows for the proper treatment of degeneracies. More details can be found in Shavitt [290].

Consider a system described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{V}$ is a perturbation on $\hat{H}_0$. Let $\mathcal{E}_0$ be the spectrum of $\hat{H}_0$. Suppose that $\mathcal{E}_0$ can be separated into disjoint sets $\mathcal{E}_\alpha$ such that:

$$|E_\alpha - E_\beta| \ll V \quad \forall E_{\alpha,\beta} \in \mathcal{E}_{\alpha,\beta}, \quad \alpha \neq \beta,$$  \hspace{1cm} (A.1)$$

where $V$ is the strength of the perturbation. Let $P_\alpha$ be the projection operator onto
the subspace spanned by the eigenvectors of $H_0$ corresponding to the energies in $E_\alpha$, ie.

$$S(P_\alpha H_0 P_\alpha) = E_\alpha$$  \hspace{1cm} (A.2)

$$P_\alpha H_0 P_\beta = 0.$$  \hspace{1cm} (A.3)

We can decompose any operator $\hat{A}$ into two parts, $A_X$, and $A_D$, which act between and within different subspaces. In cases of interest, $V$ has some interaction between subspaces, ie. $V_X \neq 0$. The method here involves finding a unitary transformation

$$H' = U H U^\dagger.$$  \hspace{1cm} (A.4)

which removes the cross terms from the interaction, resulting in an effective Hamiltonian, $H'$, which is diagonal with respect to the subspaces. The technique presented here determines a convergent perturbative expansion of $U$ and the resulting effective Hamiltonian.

$$W^{(1)}_\alpha = P_\alpha V D P_\alpha$$  \hspace{1cm} (A.5)

$$W^{(2)}_\alpha = \frac{1}{2} [V_X, G^{(1)}]$$  \hspace{1cm} (A.6)

$$W^{(3)}_\alpha = \frac{1}{2} [V_X, G^{(2)}]$$

where

$$G^{(1)}_\alpha = \sum_{\alpha} \sum_k |k\rangle \langle k|_{(E_\alpha - E_k)} (\alpha| - h.c.$$  \hspace{1cm} (A.7)

$$G^{(2)}_\alpha = \sum_{\alpha} \sum_{kk'} |k\rangle \langle k|_{(E_\alpha - E_k)(E_{\alpha'} - E_{k'})} (\alpha| + \sum_{\alpha\alpha'} \sum_k |k\rangle \langle k|_{(E_\alpha - E_k)(E_{\alpha'} - E_{k'})} (\alpha| - h.c.$$  \hspace{1cm} (A.8)

**B Jacobi Theta Functions**

The Jacobi theta functions are analytic functions of $z$ which satisfy the generalized periodic boundary conditions of the toroidal geometry [158]:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) \equiv \sum_{n=-\infty}^{\infty} \exp \left[ i \pi \tau (n+a)^2 + i 2 \pi (n+a)(z+b) \right].$$  \hspace{1cm} (A.9)
C. 2ND AND 4TH ORDER CORRECTIONS FOR THE MOTT-SUPERFLUID TRANSITION

\( \vartheta \) is doubly periodic:

\[
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+1|\tau) = e^{i2\pi a} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau)
\]

\[
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z+\tau|\tau) = e^{i2\pi(z+b)}e^{-i\pi\tau} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau).
\]

There are 4 commonly used theta functions, \( \vartheta_{1,2,3,4} \):

\[
\vartheta_1 \equiv \vartheta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \quad \vartheta_2 \equiv \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}
\]

\[
\vartheta_3 \equiv \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vartheta_4 \equiv \vartheta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.
\]

and, from the definition of \( \vartheta \):

\[
\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z|\tau) = \vartheta_3 (z+\tau a + b|\tau) \exp \left[ i\pi a^2 + 2\pi ia(z + b) \right].
\]

C 2nd and 4th order corrections for the mott-superfluid transition

In chapter 4, the mean field energy above the \( \ell \)th Mott lobe is approximated by:

\[
E_{\ell}^{\text{MF}}(\psi) \approx E_{\ell}^{\text{JC}}(\ell) - R_{\ell}\kappa^2 \psi^2 + S_{\ell}\kappa^4 \psi^4 - \kappa \psi^2.
\]
For the \( n \)th state, if \( \langle n | V | n \rangle = 0 \), the 2nd and 4th order corrections to the \( n \)th energy, \( E_n \), are given by:

\[
E^{(2)}_n = \lambda^2 \sum_{m \neq n} \frac{|\langle n | V | m \rangle|^2}{(E_n - E_m)},
\]

\[
E^{(4)}_n = \sum_{m \neq n} \sum_{l \neq n} \sum_{m \neq n} \lambda^4 \frac{|\langle n | V | k \rangle| \langle k | V | l \rangle| \langle l | V | m \rangle| \langle m | V | n \rangle}{(E_n - E_k)(E_n - E_l)(E_n - E_m)} - E^{(2)}_n \sum_{m} \frac{|\langle n | V | m \rangle|^2}{(E_n - E_m)^2}.
\]

The Jaynes-Cummings energies are:

\[
E_{\ell \pm} = \frac{1}{2} \left[ \Delta \pm \sqrt{4\ell \beta^2 + \Delta^2} \right].
\]

And with polariton states, expressed in the photonic/atomic basis:

\[
|+, \ell\rangle = \sin \theta^{JC}_\ell |g, \ell\rangle + \cos \theta^{JC}_\ell |e, \ell - 1\rangle,
\]

\[
|-, \ell\rangle = \cos \theta^{JC}_\ell |g, \ell\rangle - \sin \theta^{JC}_\ell |e, \ell - 1\rangle,
\]

\[
\theta^{JC}_\ell = \frac{1}{2} \arctan \left( \frac{2\beta \sqrt{\ell}}{\Delta} \right), \quad \theta^{JC}_0 = \frac{\pi}{2}.
\]

Abbreviating \( \sin \theta^{JC}_\ell (\cos \theta^{JC}_\ell) \) as \( S_\ell (C_\ell) \), one can write the matrix elements of the perturbation interaction as:

\[
V_{(\ell,+)(\ell-1,+)} = \sqrt{\ell} C_\ell S_{\ell-1} + \sqrt{\ell - 1} S_\ell C_{\ell-1},
\]

\[
V_{(\ell,+)(\ell-1,-)} = \sqrt{\ell} S_\ell C_{\ell-1} - \sqrt{\ell - 1} T_\ell S_{\ell-1},
\]

\[
V_{(\ell,-)(\ell-1,+)} = \sqrt{\ell} C_\ell S_{\ell-1} - \sqrt{\ell - 1} T_\ell C_{\ell-1},
\]

\[
V_{(\ell,-)(\ell-1,-)} = \sqrt{\ell} T_\ell C_{\ell-1} + \sqrt{\ell - 1} S_\ell S_{\ell-1}.
\]

The other matrix elements can be related via \( V_{(\ell,\alpha)(\ell+1,\beta)} = V_{(\ell+1,\beta)(\ell,\alpha)} \). From this one can compute the corrected energies:

\[
R_\ell = \sum_{\alpha = \pm} \sum_{\beta = \pm 1} \frac{V_{(\ell,-)(\ell+1,\alpha)}^2}{E_{\ell,-} - E_{\ell+1,\alpha} + \beta \mu},
\]

and

\[
S_\ell = A_\ell + B_\ell - C_\ell.
\]
where

\[ A_\ell = \sum_{a=\pm1} \sum_{\beta=\pm} \sum_{\gamma=\pm} \sum_{\delta=\pm} A^{\alpha\beta\gamma\delta}_\ell, \]

\[ B_\ell = \sum_{a=\pm1} \sum_{\beta=\pm} \sum_{\gamma=\pm1} \sum_{\delta=\pm} B^{\alpha\beta\gamma\delta}_\ell, \]

\[ C_\ell = R_\ell \sum_{a=\pm1} \sum_{\beta=\pm} C^{\alpha\beta}_\ell, \]

and

\[ A^{\alpha\beta\gamma\delta}_\ell = \frac{V_{(\ell,-)(\ell+\alpha,\beta)} V_{(\ell+\alpha,\beta)(\ell+2\alpha,\gamma)} V_{(\ell+2\alpha,\gamma)(\ell+\alpha,\delta)} V_{(\ell+\alpha,\delta)(\ell,-)}}{(E_{\ell,-} - E_{\ell+\alpha,\beta} - \alpha \bar{\mu})(E_{\ell,-} - E_{\ell+2\alpha,\gamma} - 2\alpha \bar{\mu})(E_{\ell,-} - E_{\ell+\alpha,\delta} - \alpha \bar{\mu})}. \]

\[ B^{\alpha\beta\gamma\delta}_\ell = \frac{V_{(\ell,-)(\ell+\alpha,\beta)} V_{(\ell+\alpha,\beta)(\ell,+)} V_{(\ell,+)(\ell+\gamma,\delta)} V_{(\ell+\gamma,\delta)(\ell,-)}}{(E_{\ell,-} - E_{\ell+\alpha,\beta} - \alpha \bar{\mu})(E_{\ell,-} - E_{\ell,+})(E_{\ell,-} - E_{\ell+\gamma,\delta} - \gamma \bar{\mu})}. \]

\[ C^{\alpha\beta}_\ell = \frac{V^2_{(\ell,-)(\ell+\alpha,\beta)}}{(E_{\ell,-} - E_{\ell+\alpha,\beta} - \alpha \bar{\mu})^2}. \]