A Few Families of Cayley Graphs and Their Efficiency as Communication Networks

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Cayley graphs are highly attractive structures for communication networks because of their many desirable properties, including vertex-transitivity and efficient routing algorithms. The families of circulants and cube-connected graphs are among the most popular Cayley graphs for efficient communication networks. The diameter, forwarding and optical indices, bisection width and Wiener index of a network are among the most important parameters to measure the efficiency of the network.

Circulant graphs and, in particular, circulant graphs with small degrees are interesting models for communication networks. However, our knowledge of many of their parameters, including arc-forwarding index, edge-forwarding index, directed and undirected optical indices, are very limited, except for very few special cases. We study the family of circulant graphs of degree 4 and obtain lower and upper bounds for their forwarding and optical indices. We give approximation algorithms for the corresponding problems of the forwarding indices and optical indices with a small constant performance ratio.

The family of recursive cubes of rings has received a lot of attention for communication networks, but many aspects of them have remained unknown. We study this family of graphs by redefining each of them as a Cayley graph on the semidirect product of an elementary abelian group by a cyclic group in order to facilitate the study of them by using algebraic tools. We give an algorithm for computing shortest paths and obtain the exact value of their diameters. We obtain sharp bounds on the Wiener index, vertex-forwarding index, edge-forwarding index and bisection width of recursive cubes of rings. The cube-connected cycles and cube-of-rings are special recursive cubes of rings, and our results apply to these well-known networks.

We introduce cube-connected circulants as a new family of cube-connected Cayley graphs and study their efficiency for communication networks. We give an algorithm for computing shortest path routing and the exact value of the di-
ameter of a cube-connected circulant. We observe that while recursive cubes of rings are special cube-connected circulants, these two families of cube-connected graphs have significantly different routing behaviours in general. Hence we develop results for ‘proportional’ graphs which will be useful in obtaining bounds for the edge-forwarding index of cube-connected circulants. We give sharp lower and upper bounds for the Wiener index, vertex-forwarding and edge-forwarding indices of cube-connected circulants. We study the embedding of cube-connected circulants into hypercubes and the embedding of hypercubes into cube-connected circulants. We show that cube-connected circulants outperform a few well-known network topologies in many aspects.
Gratefully dedicated to

My parents and

My wife
DECLARATION

This is to certify that:

(i) the thesis comprises only my original work towards the PhD except where indicated in
    the preface;

(ii) due acknowledgement has been made in the text to all other material used; and

(iii) the thesis is fewer than 100,000 words in length.

Hamid Mokhtar
Preface

The results in Chapter 4 of this thesis form the paper ‘Forwarding and optical indices of 4-regular circulant networks’ which is published in Journal of Discrete Algorithms [12]. Heng-Soon Gan and Sanming Zhou are coauthors of this paper. A preprint of the results in Chapter 5 is submitted for publication as a paper with title ‘Recursive cubes of rings as models for interconnection networks’, and Sanming Zhou is the coauthor of this paper. Sanming Zhou provided considerable assistance with revision of these two papers. A comment of Professor Graham Brightwell led to Lemma 4.2.7 and subsequent improvement of a lower bound. All other results in this thesis are my own original work. A preprint of the results in Chapter 6 is ready and will be submitted for publication as a paper in near future.

I would like to acknowledge the University of Melbourne for scholarships MIFRS and MIRS.
Acknowledgments

I would like to express my deep gratitude to Dr. Sanming Zhou, who has been a supportive and insightful supervisor, and set high standards for my research. It would be impossible to understand the beauty of science and significantly enhance my research skills without such a supervisor. Also I would like to acknowledge my advisory committee members, Dr. Andrew Wirth and Dr. Heng-Soon Gan, for their helpful feedback, and anonymous examiners for their careful examinations of my thesis. I am grateful to the University of Melbourne and its School of Mathematics and Statistics for this extraordinary academic opportunity.

This long journey became smoother and enjoyable with many friends and colleagues with whom I shared words, experiences and smiles. I am grateful to Nojan Madinehi, Dr. Abasalt Bodaghi, Dr. Michael Payne, Dr. Charl Ras, and especially Dr. Tohid Erfani who has been an invaluable friend and an inspirational colleague. I am very grateful to Dr. Aidani who has been an inspirational teacher and opened a new window of vision to my being in the world.

My family has been a source of confidence and love for me through my life, and their constant motivation, inspiration and support have been an invaluable treasure without which this thesis would not get into reality. My deepest gratitude is to my parents for their unconditional love and support, my father, Hassan, for his lifelong inspiration, dedication and love, and my mother, Zahra, for her unique and unforgettable sacrifice, love and motivation. I am especially grateful to my sister, Mahdiyeh, for her exceptional care and affectionate, my younger brother, Majid, for his great inspiration and support, my brothers-in-law for their valuable support, my parents-in-law for their special care, and my sister-in-law, Nasrin, for all her extraordinary support and kindness.

My very special and inmost gratitude is to Zahra, my incredible friend and amazing wife. Her companion added a new dimension of being into my life and her extreme patience and dedication provided me a unique opportunity to pursue my ambitions. Surely this thesis could not get to this point without her limitless understanding, sacrifice, support and love, and it is not possible for me to express my gratitude for having her in my life.
If we knew what it was we were doing, it would not be called research, would it?

Albert Einstein
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Chapter 1

Introduction

This chapter provides a brief overview of the problems of interconnection network design and routing in communication networks as the main research topics of this thesis. This overview is followed by a statement of goals and contributions of this thesis to the study of these problems. Finally, we present an overall structure of the thesis with a short description of each chapter.

1.1 Overview

An inevitable element of our modern society is the interconnection networks which have an enormous impact on the quality of communications between users and data transmissions. The dependency of communications, technology and science on efficient interconnection networks has brought about a growing demand in networks with large capacity and high performance. Important classes of interconnection networks are parallel computers (supercomputers), VLSI, Internet services, multimedia services, virtual private networks and social networks.

An interconnection network is a collection of different components, or nodes, which are required to be interconnected. The nodes can be processors, memories, switches, terminals, people and end-users containing data. Any node interconnects with other nodes via links, which can be either physical wires, optical links or radio transmission.

The complexity of interconnection network design has been increased as a result of the growing demand to solve arising hard problems in reasonable time using parallel computers and the huge growth of the number of network users which rely on interconnection networks. For instance, the strength of powerful supercomputers is
due to deploying many processors in parallel with efficient underlying interconnection networks rather than relying on fast processor units. In other words, the quality of a system of interconnected users mostly relies on the performance and invulnerability of its underlying network in practice. The construction of interconnection networks for parallel computers is by methodological design and well planned in advance, whereas the topologies of some other communication networks, for instance the Internet, evolve and expand without any general plan. Therefore, a fundamental study on high-performance interconnection networks and their design is essential and unavoidable. This study has become an important field of research in computer science and optimisation since the 1970s. It was more emphasised in the 1980s due to Synchronous Optical Networking (SONET) and recently for designing structures for all-optical networks and new supercomputers with millions of processor units.

A powerful and analytical tool in studying interconnection networks is graph theory because the topology of any interconnection network is basically a graph. In fact, every interconnection network can be theoretically modelled by a finite graph such that each node is represented by a vertex and each link is represented by an edge (see Section 2.3). This modelling has provided a theoretical framework for formulating and dealing with problems arising from network design.

The research problems in this thesis are the design of efficient interconnection networks for a large number of nodes and efficient communication schemes for them. The efficiency of a network can be measured variously according to the requirements and limitations. A few important criteria to evaluate the efficiency of a network are the existence of an efficient interconnecting scheme, efficient wavelength assignment, the minimal congestion on edges and nodes under optimal interconnecting schemes, the minimum delay in data transmission, implementation of algorithms in the network and the costs of its construction and expansion. The related parameters for these criteria in graph theory are the forwarding and optical indices, diameter, bisection width and embedding parameters of graphs. Moreover, the design of an efficient routing, which determines a path for every pair of vertices, has a crucial role in understanding the network performance in the busiest situation (see Section 2.3).

Cayley graphs, which are defined using group theory, were first proposed and studied for interconnection networks in 1989 [5]. These graphs are theoretically and practically attractive for communication networks because they acquire 'nice' features by nature, and there are plenty of theoretical results in group theory to support related studies. This is our primary motivation to research on Cayley graphs for interconnection net-
works. Having said that, it is difficult to find the exact values or good estimates for some of the invariants of a Cayley graph in general. The circulants and cube-connected graphs are two families of Cayley graphs which have received considerable attention in the literature. Our knowledge about some basic communication-related invariants of circulant graphs such as arc-forwarding, edge-forwarding and optical indices is quite limited, even for circulant graphs of degree 4. One aim of this thesis is to study these invariants with a focus on the family of 4-regular circulant graphs. Another goal of this thesis is obtaining a better understanding of the family of recursive cubes of rings, as a family of cube-connected graphs, for interconnection networks by studying some of their parameters, including the Wiener index, vertex-forwarding and edge-forwarding indices, diameter and bisection width. We also propose the family of cube-connected circulant graphs as efficient models for interconnection networks and study their parameters. By comparing with a few well-known graphs, we will show that the proposed cube-connected circulant graph outperforms existing graphs in some invariants.

1.2 Contributions and thesis outline

In the following, the major contributions of this dissertation to the research field of interconnection network design are presented. These are followed by an overview of the structure of this thesis and a summary of content for each chapter.

Contributions

1. Obtaining lower and upper bounds for the arc-forwarding and edge-forwarding indices for the family of 4-regular circulant graphs;

2. Giving an approximation algorithm for the wavelength assignment of the family of 4-regular circulant graphs;

3. Developing a shortest path routing for the recursive cube of rings;

4. Giving sharp lower and upper bounds for the Wiener index, edge-bisection width, vertex-forwarding and edge-forwarding indices of the recursive cube of rings;

5. Proposing the cube-connected circulants as efficient models for interconnection networks;

6. Developing a shortest path routing for the cube-connected circulants;
7. Giving sharp lower and upper bounds for the Wiener index, edge-bisection width, vertex-forwarding and edge-forwarding indices of the cube-connected circulants;

8. Developing results for the edge-forwarding index of proportional graphs;

9. Giving results on the embedding problems for the cube-connected circulants.

Results 1 and 2 are given in Chapter 4. Results 3 and 4 are covered in Chapter 5. Results 5, 6, 7 and 8 are covered in Chapter 6.

Overview of chapters

Chapter 2: Preliminaries. Basic definitions and standard notations which will be used throughout this thesis will be introduced in this chapter.

Chapter 3: Literature review. A related history of interconnection network modelling and related routing problems will be presented in this chapter. Then we will give a literature review on the usage of Cayley graphs for designing interconnection networks and related routing problems. After that, a more detailed survey on the circulant graphs and related routing problems will be given. Finally we will review a subclass of the cube-connected graphs and existing results about our problems on such graphs. In the context of literature review we will propose our research questions in this chapter.

Chapter 4: 4-regular circulant graphs. In this chapter, we will focus on the family of 4-regular circulant graphs with connection set \( \{ \pm 1, \pm s \} \), \( 1 < s < n/2 \). We will give a routing scheme for this family of graphs and obtain lower and upper bounds for the edge-forwarding and arc-forwarding indices of these graphs. An approximation algorithm for wavelength assignment with performance ratio a small constant for the directed and undirected optical indices of this family of graphs will be given. This chapter is based on our paper [49].

Chapter 5: Recursive cube of rings. We will study recursive cubes of rings as models for interconnection networks. We will first redefine a recursive cube of rings as a Cayley graph on the semidirect product of an elementary abelian group by a cyclic group to facilitate the study of them by using algebraic tools. We will give an algorithm for computing shortest paths and the distance between any two vertices in recursive cubes of rings, and obtain the exact value of their diameters. We obtain sharp bounds on the Wiener index, vertex-forwarding index, edge-forwarding index and bisection width of recursive cubes of rings. The cube-connected cycles and cube-of-rings are
special recursive cubes of rings, and hence all results obtained in the paper apply to these well-known networks. This chapter is based on our submitted paper [81].

Chapter 6: Cube-connected circulants. We will propose cube-connected circulants as a new family of cube-connected Cayley graphs as models for interconnection networks. We will give an algorithm for computing shortest paths and obtain the distance between any two vertices, and the exact value of the diameter of any cube-connected circulant. We will also give sharp lower and upper bounds for their Wiener index, vertex-forwarding and edge-forwarding indices for this family of graphs. We will present our results on the embedding of cube-connected circulants into hypercube graphs. We will also compare a few invariants of this new family of graphs with other well-known graphs and show they outperform many existing network topologies in some aspects. While we will observe that recursive cubes of rings are special cube-connected circulants, we will show they have significantly different behaviours in routing. In fact, in contrast to recursive cubes of rings, cube-connected circulants are not ‘orbit-proportional graphs’ in general. Hence we will develop a different method which will be useful in obtaining the edge-forwarding index of a cube-connected circulant. Moreover, cube-connected circulants have more complicated shortest path algorithms since their factor graphs are multiplicative circulants and hypercubes. This chapter is based on a preprint of our paper [82].

Chapter 7: Conclusion and future works. We will give a few concluding remarks and propose a few open problems for future research in this area.
Chapter 2

Preliminaries

The purpose of this chapter is to provide necessary concepts, mathematical definitions and notations as an essential foundation for the literature review and results in the following chapters. Section 2.1 offers terminology and definitions from graph theory. Cayley graphs and concepts from group theory are covered in Section 2.2. A mathematical modelling of interconnection networks, related parameters and problems are given in Section 2.3.

2.1 Graphs

In this thesis, the reference for graph theory terminology and notation is [36]. A graph $X$ is a pair of disjoint sets $(V, E)$ such that $E \subseteq [V]^2$, where $[V]^2$ is the set of all 2-element subsets of $V$. Any element of $V$ is called a vertex and any element of $E$ is called an edge of $X$. A graph whose vertex set is empty is called the empty graph. We denote the vertex set of $X$ by $V(X)$ and its edge set by $E(X)$. Two distinct vertices $x$ and $y$ are adjacent in $X$ if $\{x, y\}$ is in $E(X)$; then $x$ and $y$ are end-vertices of $\{x, y\}$ and they are incident with the edge $\{x, y\}$.

The order of a graph $X$ is the number of its vertices, that is, $|V(X)|$, and its size is the number of its edges, that is, $|E(X)|$. The number of edges incident to a vertex $x$ is called the degree (or valency) of $x$ and is denoted by $\deg(x)$. The maximum and minimum degrees of vertices in $X$ are denoted by $\Delta(X)$ and $\delta(X)$, respectively. The graph $X$ is called $r$-regular if $\deg(x) = r$ for every $x \in V(X)$.

For $V' \subseteq V(X)$, the subgraph of $X$ induced by $V'$ is the graph $(V', E')$, where $\{x, y\} \in E'$ if and only if $\{x, y\} \in E(X)$ and $x, y \in V'$. 

7
A **multigraph** is a graph such that distinct edges may have the same end-vertices (called **parallel edges** and an edge may have the same end-vertices (called a **loop**). A graph is **simple** if it does not contain any parallel edge and loop. In this thesis, the term ‘graph’ refers to a (undirected) simple graph unless otherwise stated.

Given vertices \(x\) and \(y\) in a graph \(X\), a **path** from \(x\) to \(y\), or for short an \(xy\)-path, is a sequence \(v_0, e_1, v_1, e_2, \ldots, e_t, v_t\) such that \(v_0, v_1, v_2, \ldots, v_t\) are pairwise distinct vertices in \(X\), the edge \(e_i = \{v_{i-1}, v_i\}\) for \(1 \leq i \leq t\), belongs to \(E(X)\), \(v_0 = x\) and \(v_t = y\). The vertices \(x\) and \(y\) in an \(xy\)-path are **end-vertices** or **terminals** of the path, and a vertex in a path is an **internal vertex** of the path if it is not an end-vertex. Alternatively, a path in a graph can be represented by its vertex sequence or edge sequence. The length of a path is the number of edges in the path, and an \(xy\)-path with the minimum length is called a **shortest \(xy\)-path**. The distance of two vertices \(x\) and \(y\) is the length of a shortest \(xy\)-path and is denoted by \(\text{dist}(x, y)\). For every vertex \(x\) we set \(\text{dist}(x, x) = 0\). If there is no \(xy\)-path in the graph, we set \(\text{dist}(x, y) = 1\).

Furthermore, the **diameter** of \(X\), denoted by \(\text{diam}(X)\), is the maximum of distances between any two vertices in \(X\), that is,

\[
\text{diam}(X) = \max_{x, y \in V(X)} \text{dist}(x, y). \tag{2.1}
\]

For a positive integer \(t\), any two vertices at distance \(t\) are said to be \(t\)-**neighbours** of each other. The set of \(t\)-neighbours of a vertex \(x\) is denoted by \(N_t(x)\).

A graph is **connected** if there exists a path between every pair of vertices in the graph. A graph which is not connected is called **disconnected**. Two paths are **vertex-disjoint** (or **disjoint** for short) if they do not have any common internal vertex.

Two graphs \(X\) and \(X'\) are **isomorphic** if there exists a bijection \(f : V(X) \rightarrow V(X')\) such that the adjacency and non-adjacency of vertices are preserved by \(f\). In other words, \(\{f(x), f(y)\} \in E(X')\) if and only if \(\{x, y\} \in E(X)\). Such a bijection is called an **isomorphism**. In particular, if \(X = X'\), then \(f\) is called an **automorphism** of \(X\). A graph \(X\) is **vertex-transitive** if for any two vertices \(x, y \in V(X)\), there exists an automorphism \(f\) of \(X\) such that \(y = f(x)\). As a result, every vertex-transitive graph is regular.

A **directed graph** (or **digraph** for short) is a pair of disjoint sets \(D = (V, A)\), where \(V\) is the vertex set, \(A\) is the arc set and \(A \subseteq V \times V\). The **tail vertex** of an arc \(a = (x, y)\) is \(x\) and its **head vertex** is \(y\). A path \(v_0, e_1, v_1, e_2, \ldots, e_t, v_t\) in a (directed or undirected) graph is called an **oriented path** if edge \(e_i\) is directed from \(v_{i-1}\) to \(v_i\) for \(1 \leq i \leq t\). Then, \(v_0\) is the **source** or **origin**, and \(v_t\) is the **destination** of this oriented path.
2.2 Algebraic graph theory

In this thesis, the reference for algebraic graph theory is [22]. A non-empty set $G$ together with a binary operation $\cdot$ forms a group if (i) $g \cdot h \in G$ for every $g, h \in G$, (ii) there is an identity (or unit element) $1_G \in G$ such that $g \cdot 1_G = 1_G \cdot g = g$ for every $g \in G$, (iii) there exists $g^{-1} \in G$ such that $g^{-1} \cdot g = g \cdot g^{-1} = 1_G$ for every $g \in G$ ($g^{-1}$ is the inverse of $g$ in $G$) and (iv) $g \cdot (h \cdot f) = (g \cdot h) \cdot f$ for every $g, h, f \in G$. We only deal with finite groups here. A subset $S \subseteq G$ is a generating set for group $G$ if every element in $G$ can be written as a product of some elements of $S$, and is a minimal generating set if $G$ cannot be generated by $S \setminus \{s\}$ for any $s \in S$.

A group $G$ acts on a set $V$ if each pair $(g, x) \in G \times V$ corresponds to an element $g(x) \in V$ such that $1_G(x) = x$ and $g(h(x)) = (gh)(x)$ for any $g, h \in G$ and $x \in V$. $G$ is said to act transitively on $V$ if $G$ acts on $V$ and for every $x, y \in V$, there exists $g \in G$ such that $g(x) = y$. It is known that the set of all automorphisms of a graph $X$, denoted by $\text{Aut}(X)$, forms a group. If $\text{Aut}(X)$ acts transitively on $V(X)$, then $X$ is vertex-transitive [23].

Given a graph $X$ and a subgroup $H$ of $\text{Aut}(X)$, the $H$-orbit on $E(X)$ containing a given edge $\{x, y\} \in E(X)$ is $\{\{f(x), f(y)\} : f \in H\}$. An $H$-orbit on $E(X)$ is also called an edge-orbit of $X$ with respect to $H$.

Cayley graphs

Let $G$ be a group and $S$ be a subset of $G$ such that $S$ is unit free and closed under taking inverse, that is, $1_G \notin S$ and $S = S^{-1}$ (which means $s^{-1} \in S$ whenever $s \in S$). The Cayley graph on $G$ with respect to the connection set $S$, denoted by $\text{Cay}(G, S)$, is the graph with vertex set $G$ such that $x, y \in G$ are adjacent if and only if $x^{-1}y \in S$. The Cayley graph $\text{Cay}(G, S)$ is connected if and only if $S$ is a generating subset of $G$. In the case that $S$ is not closed under taking inverses, $\text{Cay}(G, S)$ is a Cayley digraph on $G$ with respect to the connection set $S$ such that $(x, y)$ is an arc in the digraph if and only if $x^{-1}y \in S$. In this thesis we only consider Cayley graphs on finite groups. It is well-known that $G$ acts on itself by left-regular multiplication as a group of automorphisms of $\text{Cay}(G, S)$. In other words, every $g \in G$ gives rise to an automorphism $\hat{g} : G \to G$, $u \mapsto g^{-1}u$, of $\text{Cay}(G, S)$, and the group of these permutations $\hat{g}$ forms a vertex-transitive subgroup of $\text{Aut}(\text{Cay}(G, S))$ which is isomorphic to $G$. Therefore, we have the following theorem:

**Theorem 2.2.1.** ([22, Theorem 3.1.2]) Every Cayley graph is vertex-transitive.
The converse of this theorem is not correct. For example, the famous Peterson graph is the smallest vertex-transitive graph which is not Cayley.

**The circulant graphs**

The family of circulant graphs is an important class of Cayley graphs. For a positive integer $n \geq 3$, let $\mathbb{Z}_n$ denote the additive group of integers modulo $n$. For any $S \subseteq \mathbb{Z}_n$ such that $0 \notin S$ and $S = -S$, the **circulant graph** $C_n(S)$ of order $n$ with respect to the connection set $S$ is the Cayley graph $\text{Cay}(\mathbb{Z}_n, S)$. In other words, $C_n(S)$ is the graph on the vertex set $\mathbb{Z}_n$ such that vertices $i$ and $j$ are adjacent if and only if $j - i \in S$. We use $C_n(s_1, s_2, \ldots, s_k)$ to denote the circulant graph on $n$ vertices with respect to the connection set $S = \{\pm s_1, \pm s_2, \ldots, \pm s_k\}$. See Figure 2.1 for an example.

Since circulants belong to the family of Cayley graphs, any undirected circulant graph $C_n(S)$ is vertex-transitive and $|S|$-regular. It is well-known that the circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is connected if and only if $\gcd(n, s_1, s_2, \ldots, s_k) = 1$. The circulant graphs are also called ‘multi-loop graphs’ in the community of computer science.

**Semidirect product graphs**

Let $K$ and $H$ be two groups such that $H$ acts on $K$ as a group. This is to say that, for any $k \in K$, $h \in H$, there corresponds an element of $K$ denoted by $\varphi_h(k)$ such that $\varphi_{1_H}(k) = k$, $\varphi_{h_2}(\varphi_{h_1}(k)) = \varphi_{h_2h_1}(k)$ and $\varphi_h(k_1k_2) = \varphi_h(k_1)\varphi_h(k_2)$ for any $k, k_1, k_2 \in K$ and $h, h_1, h_2 \in H$. (In other words, $\varphi : h \mapsto \varphi_h$ defines a homomorphism.
from $H$ to $\text{Aut}(K)$.) The *semidirect product* of $K$ by $H$ with respect to this action, denoted by $[K \rtimes H]$, is the group defined on $K \times H = \{(k, h) : k \in K, h \in H\}$ with operation given by

$$(k_1, h_1)(k_2, h_2) = (k_1\varphi h_1(k_2), h_1 h_2).$$

(2.2)

(A few equivalent definitions of the semidirect product exist in the literature. We use the one in [6, pp. 22–23] for convenience of our presentation.) The semidirect product is a strong tool to construct new Cayley graphs which benefit from advantages of two known Cayley graph.

The *Cartesian product* of $X$ and $X'$ is the graph with vertex set $V(X) \times V(X')$ such that $(u, u')$ is adjacent to $(v, v')$ if and only if either $uv \in E(X)$ and $u' = v'$, or $u'v' \in E(X')$ and $u = v$.

### 2.3 Interconnection network design

Interconnection network design is a methodological creation of a structure for a collection of nodes which are required to be interconnected with a specific performance. Any communication network can be theoretically modelled by a finite graph $X = (V, E)$ such that the order of $X$ is equal to the number of nodes, each node is uniquely represented by a vertex in $X$ and the corresponding vertices of any two nodes are adjacent in $X$ if and only if the nodes are connected directly (see Figure 2.2 for an illustration). Since there is a clear correspondence between a communication network and its graph model, ‘network’ and ‘graph’, ‘node’ and ‘vertex’, and ‘link’ and ‘edge’ are used interchangeably with each other henceforth.

Among various settings for interconnecting multi-processor systems, a common network setting is the one in which each processor has a local memory and processors execute codes independently. In this thesis we only consider full-duplex models [17] for which links can transfer data in both directions simultaneously, and each edge in the graph model represents two links in opposite directions.

The design of interconnection networks can be very complicated because of diverse requirements including low cost, high performance, and reliability. Many of important challenges for the interconnection network design can be modelled by measurable factors and investigated as optimisation problems. Among these challenges, the construction costs, network throughput, delay, resilience and fault-tolerance have received significant attention for research in this area. There is no known network for which all factors of networks are optimal simultaneously. In fact, there are some factors which are
incompatible with each other, such as 'high connectivity' and 'small number of ports for nodes' in the network. Therefore, choosing a network structure is a trade-off between properties and desired characteristics of the network. In the following, we formally state some problems and concepts related to the network design.

Routing problems

In a given network $X$, a connection request is an ordered pair of vertices $(x, y)$ if the source vertex $x$ has some data to send to the destination vertex $y$. A connection request set is a set of connection requests for each of which an (oriented) path must be established. A routing $R$ for a connection request set $I$ is a set of paths which consists of exactly one path for every request in $I$, that is,

$$R = \{P_{x,y} : (x, y) \in I\}, \quad (2.3)$$

where $P_{x,y}$ is an $xy$-path and $|R| = |I|$. Note that $P_{y,x}$ is not necessarily $P_{x,y}$ with reverse orientation. Among different types of connection request sets, an important one is the all-to-all connection request set, denote by $I_A$, and it is the set of all ordered pairs of vertices in the graph, that is,

$$I_A = \{(x, y) : (x, y) \in V(X) \times V(X)\}. \quad (2.4)$$

Hence an all-to-all routing contains $|V(X)|(|V(X)| - 1)$ oriented paths. The all-to-all connection request set is also called 'complete exchange'. In this thesis, we only deal
with the all-to-all routing, or ‘routing’ for brevity, since it is important in understanding the network performance in the busiest situation and therefore the total network throughput. R is a symmetric routing if for every \((x,y) \in I_A\), \(P_{y,x} \in R\) is the same as \(P_{x,y} \in R\) with reverse orientation. Also a routing \(R\) is consistent if \(P_{x,y} \in R\) and \(z \in P_{x,y}\) implies that \(P_{x,z}\) and \(P_{z,y}\). Furthermore, a routing \(R\) is a minimal routing or shortest path routing if every path in \(R\) is a shortest path.

The forwarding indices

Any routing scheme imposes a running cost and restrictions on the network throughput which can be measured by the number of usages of vertices and edges. For a given graph \(X\) and a routing \(R\) for \(X\), the load of \(R\) on a vertex \(v \in V(X)\) is the number of paths in \(R\) with \(v\) as an internal vertex, and the maximum load of \(R\) on vertices of \(X\) is denoted by \(\xi(X, R)\). The vertex-forwarding index of \(X\), denoted by \(\xi(X)\), is obtained by

\[
\xi(X) = \min_R \xi(X, R),
\]

where the minimum is taken over all routings \(R\) for \(X\). The minimal vertex-forwarding index of \(X\), denoted by \(\xi_m(X)\), is defined similarly with the minimum in \(2.5\) over all shortest path routings. Clearly,

\[
\xi(X) \leq \xi_m(X). \tag{2.6}
\]

**Problem 2.3.1** \((\ref{30})\). The vertex-forwarding index problem.

Instance: A graph \(X\) and a positive integer \(k\).

Question: \(\xi(X) \leq k\)?

The load of a routing \(R\) on an edge \(e \in E(X)\) is the number of paths in \(R\) passing through \(e\) in either directions and the maximum load of \(R\) on the edges of \(X\) is denoted by \(\pi(X, R)\). The edge-forwarding index of \(X\), denoted by \(\pi(X)\), is defined as

\[
\pi(X) = \min_R \pi(X, R), \tag{2.7}
\]

where the minimum is taken over all routings \(R\) for \(X\). If the minimum is taken over all shortest path routings, we obtain the minimal edge-forwarding index of \(X\), denoted by \(\pi_m(X)\). Thus,

\[
\pi(X) \leq \pi_m(X). \tag{2.8}
\]

**Problem 2.3.2** \((\ref{64})\). The edge-forwarding index problem.

Instance: A graph \(X = (V, E)\) and a positive integer \(k\).

Question: \(\pi(X) \leq k\)?
Note that the equality in (2.6) or (2.8) hold for trees, while the inequalities hold strictly for wheel graphs [13].

Similarly, the arc-forwarding index is defined for directed or full-duplex undirected graphs such that the load of routings on arcs with respect to their directions is measured. In this view, the load of a routing \( R \) on an arc \( a \) is the number of paths which pass through \( a \) in the same direction as \( a \). For a routing \( R \) in \( X \), \( \overrightarrow{\pi}(X, R) \) denotes the maximum load of \( R \) on arcs of \( X \) and the \emph{arc-forwarding index} of \( X \), denoted by \( \overrightarrow{\pi}(X) \), is defined as

\[
\overrightarrow{\pi}(X) = \min_{R} \overrightarrow{\pi}(X, R),
\]

where the minimum is taken over all routings for \( X \). If the minimum in the above equation is taken over all shortest path routings, we obtain the \emph{minimal arc-forwarding index}, denoted by \( \overrightarrow{\pi}_m(X) \). Definitely \( \pi(X) \leq \pi_m(X) \). It follows from the definitions that

\[
\frac{\pi(X)}{2} \leq \overrightarrow{\pi}(X) \leq \pi(X). \tag{2.9}
\]

Intuitively, the forwarding indices of a network reveal how the network structure is capable of carrying data transmission. In practical terms, the edge-forwarding and arc-forwarding indices measure the minimum heaviest load on edges and arcs of a given network, respectively, with respect to the all-to-all communication.

**Vertex-transitivity (Symmetry)**

A favourable property for a communication network is the vertex-transitivity since the same hardware can be installed for every node in the network. For instance a similar number of ports and the same routing scheme can be applied for every node.
Connectivity and fault-tolerance

An important consideration in a network design is the reliability of the network in an event of hardware failures. The fault-tolerance of a network is the maximum number of nodes in the network whose impairment does not block interconnection of other nodes. A similar concept in graph theory is vertex connectivity. A vertex-cut of a connected non-complete graph is a subset of vertices whose removal disconnects the graph. For a positive integer $k$, a graph is $k$-connected if every subset of vertices $U$ with $|U| < k$ is not a vertex-cut for the graph. The vertex connectivity of $X$, $\kappa(X)$, is the maximum integer $k$ such that $X$ is $k$-connected. We can assume $\kappa(X) = 0$ if $X$ is the trivial graph or a disconnected graph. Menger’s Theorem is a foundation in graph connectivity:

**Theorem 2.3.3** (Menger 1927). A graph is $k$-connected if and only if there are $k$ pairwise internally vertex-disjoint paths between any two distinct vertices in the graph.

For any (non-complete) graph $X$ with connectivity $\kappa$, there are $\kappa$ pairwise vertex-disjoint paths between any pair of distinct vertices $(x, y)$. So there is at least one $xy$-path even if $\kappa - 1$ vertices are deleted from the graph and hence the graph is $(\kappa - 1)$-fault-tolerant. If $X$ is the complete graph $K_n$, the connectivity and fault tolerance are equal. A $r$-regular graph is optimal fault-tolerant if the connectivity of the graph is $r$.

Analogously, the edge-connectivity corresponds to the tolerance against faulty links of networks. The edge-connectivity of a vertex-transitive graph is equal to the degree of its vertices [113, Lemma 3.3.3], that is to say, any vertex-transitive graph is optimally edge-connected.

Degree, diameter and mean distance

The degrees of vertices are directly related to the construction cost of networks since the number of links attached to a node is determined by its degree. In most cases, there is a constraint on the maximum degree of vertices in network construction. Also it is desired that the degree of vertices remains fixed during network expansion since it cuts the costs of altering all existing hardware instruments.

Another important factor of interconnection network efficiency is the transmission delay of messages in the network. The latency of data transmission between any two nodes and the cost of interconnection between them are proportional to the number of links of the used path for carrying the data. The diameter and average distance are meaningful factors in measuring performance of networks for transmission delay.
provided that the store-forward time of messages in all vertices of a network are identical.

In general, specifying the diameter of a graph, as defined in (2.1), can be obtained in polynomial time by the breadth-first search algorithm [82]. The mean distance in a graph is the average distance between pairs of vertices. It is an indicator of the average data transmission delay in general. The mean distance of $X$ is defined as

$$\text{dist}(X) = \frac{1}{|V(X)|||V(X)| - 1|} \sum_{(x,y) \in V(X) \times V(X)} \text{dist}(x,y). \quad (2.10)$$

### Wiener index and total distance

The **Wiener index** of $X$ [110, Chapter 11] is defined as

$$W(X) = \sum_{\{x,y\} \in |V(X)|^2} \text{dist}(x,y). \quad (2.11)$$

Obviously $W(X) = |V(X)|||V(X)| - 1|\text{dist}(X)/2$. The Wiener index is one of the most popular topological indices in combinatorial chemistry [33, 108]. However, obtaining formulas for the Wiener index of families of graphs is difficult in general. The Wiener index is also used in obtaining an estimation of the vertex-forwarding and edge-forwarding indices of a network, or computation of these parameters for some classes of graphs (see [102, 105, 116] and Theorems 3.1.1 and 3.1.2).

If $X$ is a vertex-transitive graph, then the **total distance** $\text{td}(X)$ of $X$ is the sum of the distances from any fixed vertex to all other vertices in $X$. In other words, for any $y \in V(X)$,

$$\text{td}(X) = \sum_{x \in V(X)} \text{dist}(y,x). \quad (2.12)$$

One can verify that, for a vertex-transitive graph $X$, the average distance of $X$ is equal to $\text{td}(X)/||V(X)| - 1|$ and the Wiener index of $X$ is given by $\text{td}(X) = 2W(X)/|V(X)|$.

### Bisection width

The bisection width scales the ‘bottleneck’ of the network by finding the minimum number of edges which connect two (almost) equal size parts of the network [102]. In a graph $X$ and a subset $U$ of $V(X)$, let $\delta(U, \overline{U})$ be the subset of $E(X)$ consisting of those edges with one end-vertex in $U$ and the other in $\overline{U} := V(X) \setminus U$. A **bisection** of $X$ is a partition $\{U, \overline{U}\}$ of $V(X)$ such that $|U|$ and $|\overline{U}|$ differ by at most one. The **bisection width** of $X$, denoted by $\text{bw}(X)$, is the minimum of $|\delta(U, \overline{U})|$ over all bisections $\{U, \overline{U}\}$.
of $X$. It is known [81] that the decision problem for $bw(X)$ is NP-complete for general graphs $X$.

Let $R$ be a routing in $X$ such that $\pi(X) = \pi(X, R)$ and $\{U, \overline{U}\}$ a partition of $V(X)$. Then the total load on the edges of $\delta(U, \overline{U})$ under $R$ is at most $\pi(X)|\delta(U, \overline{U})|$. On the other hand, there are exactly $2|U||\overline{U}|$ paths in $R$ with one end-vertex in $U$ and the other in $\overline{U}$. Therefore,

$$\pi(X)\delta(U, \overline{U}) \geq 2|U||\overline{U}|.$$  \hspace{1cm} (2.13)

In particular, for a bisection $\{U, \overline{U}\}$ of $X$ with $\delta(U, \overline{U}) = bw(X)$, this yields [52]

$$bw(X) \geq \frac{2[V(X)]/2}{\pi(X)} = \frac{[V(X)]^2/2}{\pi(X)}.$$  \hspace{1cm} (2.14)

In general, it is desirable that the bisection width of a network to be at least the same order of the number of vertices. This way, the maximum congestion of routings on edges is proportional to the number of vertices.

**Embedding**

The embedding problem is a significant topic in interconnection network design, which deals with the efficient implementation of algorithms in parallel computing systems. In fact, any parallel algorithm can be represented by a graph, and it is executable in a parallel system if its graph is isomorphic to a subgraph of the parallel system [61, 78]. Also, a host network can simulate a guest network in order to execute an existing parallel algorithm on the guest network.

An *embedding* of a guest graph $X$ into a connected host graph $Y$ is an injective mapping $\phi$ from $V(X)$ to $V(Y)$ such that each edge $\{x, y\}$ in the guest graph $X$ is associated with a $\phi(x)\phi(y)$-path in the host graph $Y$. The length of $\phi(x)\phi(y)$-path is the dilation of $\{x, y\}$ under $\phi$ and the maximum dilation of edges is the dilation of $\phi$.

In other words, the *dilation* of $\phi$ is

$$dil(\phi)(X, Y) := \max \{\text{length of the } \phi(x)\phi(y)\text{-path} : \{x, y\} \in E(X)\}.$$  \hspace{1cm} (2.15)

An embedding with minimum dilation is an embedding with optimal dilation. Define

$$dil(X, Y) = \min_\phi \text{dil}(\phi)(X, Y),$$

where minimum is taken over all embeddings of $X$ into $Y$. When the orders of the guest and the host graphs are equal, any embedding is an one-to-one mapping.

The cost of an embedding can also be measured by congestion and expansion. The *congestion* of an edge $e \in E(Y)$ is the number of edges of $X$ whose corresponding
paths use \( e \). A large edge congestion may cause a delay in a communication process by putting the data in queue for transmission. The \textit{expansion} of an embedding is the order of \( Y \) over the order of \( X \), that is \(|V(Y)|/|V(X)|\). Minimising the expansion is important since a smaller host graph is more favourable in costs. In many research works on embedding problems, it is assumed that the guest and host graphs have the same order.

**Optical communication networks**

\( \text{Optical networking} \) is a new generation of communication networks such that data transmission occurs via light through optical fibres. In these networks, special devices are used at nodes which convert traversing data to light or vice versa. A stream of light can transmit data with 10 Gbit/s rate through an optical fibre. Using \textit{Wavelength Division Multiplexing} (WDM) technology, an optical fibre can host several light streams simultaneously, which increases the capacity of the optical fibre. With recent technology, a fibre can transmit data with 1.6 Tbit/s rate by up to 160 light streams. An important consideration in using WDM technology is to avoid interference of data streams on each other in each optical fibre. In other words, distinct data streams which use the same optical fibre must be assigned distinct wavelengths. An assumption in this thesis is that each path must use the same wavelength from the source to the destination so as to be consistent with the optical networks without wavelength converters.

A \textit{valid wavelength assignment} (or \textit{valid colouring}) for a routing in a given optical network is the allocation of one wavelength to each path in the routing such that any pair of paths which have an edge in common (regardless of the orientation of the paths) are allocated distinct wavelengths (see Figure 2.4 for an illustration). A valid colouring of \( R \) is also called an \textit{edge-conflict-free colouring} of \( R \).

For a graph \( X \) and a routing \( R \) for \( X \), define \( w(X, R) \) to be the minimum number of wavelengths required for a valid wavelength assignment of \( R \). The \textit{optical index} of \( X \), \( w(X) \), is defined as

\[
w(X) := \min_R w(X, R),
\]

where the minimum is taken over all routings \( R \) for \( X \).

Analogous to the (undirected) optical index, the directed optical index of graphs is defined for a full-duplex model graph. A valid wavelength assignment in this case is similarly defined as above with the requirement that any two oriented paths using a same arc in its direction must receive distinct wavelengths. Such a colouring is also called an \textit{arc-conflict-free colouring}. Let \( \overrightarrow{w}(X, R) \) be the minimum number of
wavelengths required for a valid wavelength assignment of $R$. The \textit{directed optical index} of $X$, $\overrightarrow{w}(X)$, is defined as

$$\overrightarrow{w}(X) := \min_{R} \overrightarrow{w}(X, R).$$  \hspace{1cm} (2.17)

Since the number of wavelengths needed is no less than the number of paths on a most loaded edge (or arc in the directed version), we have (see e.g. [14])

$$w(X) \geq \pi(X), \quad \overrightarrow{w}(X) \geq \overrightarrow{\pi}(X).$$  \hspace{1cm} (2.18)

In general, equality in (2.18) is not necessarily true (see e.g. [73,114]). The \textit{routing and wavelength assignment problem} is the problem of computing $w(X)$, and its \textit{oriented version} is the one of finding $\overrightarrow{w}(X)$.

**Problem 2.3.4.** Optical index problem.

\textit{Instance:} A graph $X = (V,E)$ and an integer $k$.

\textit{Question:} $w(X) \leq k$?

\section*{Notable graphs for communication networks}

There are a few families of graphs which have been well known as interconnection networks in the literature. In this section we briefly mention those graphs that we will use in the following chapters.

A \textit{cycle}, or \textit{ring}, on $n$ vertices is the Cayley graph $\text{Cay}(\mathbb{Z}_n, \{-1,+1\})$. A wheel is the graph on $n+1$ vertices $\{0,1,\ldots,n\}$ and edges $\{0,i\}$ and $\{i,i+1 \mod n\}$ for $1 \leq i \leq n$. The \textit{hypercube} of dimension $d$, denoted by $Q_d$, can be defined as the Cayley graph on $\mathbb{Z}_2^d$ with bitwise additive operation, with respect to the connection set $\{e_1,e_2,\ldots,e_d\}$, where $e_i \in \mathbb{Z}_2^d$ is the row vector whose $i$th entry is 1 and all other entries 0, $1 \leq i \leq d$. 
The cube-connected cycle $CC_n$ can be defined as the Cayley graph on $Z_2^2 \times Z_n$ such that $(a, x)$ is adjacent to $(b, y)$ if and only if either $a = b$ and $x \equiv y \pm 1 \mod n$, or $b = a + e_{1+x}$ and $x = y$.

The star graph $ST_n$ is the Cayley graph defined on the symmetric group $S_n$ of $n$ letters with respect to the connection set $\{T_{i,i} : 2 \leq i \leq n\}$, where the transposition $T_{i,j}$ swaps letters $i$ and $j$. In other words, the vertices of $ST_n$ are the permutations of $\{1, 2, \ldots, n\}$. The $n$th bubble sort graph is the Cayley graph on $S_n$ with respect to the connection set $\{T_{i,i+1} : 1 \leq i \leq n-1\}$. On the same group, the Cayley graph with respect to the connection set $\{i(i-1)\ldots321(i+1)\ldots n : 1 < i \leq n\}$ is called a pancake graph. The complete transposition graph $CT_n$ is the Cayley graph on $S_n$ with respect to the connection set $\{T_{i,j} : 1 \leq i < j \leq n\}$.

The multiplicative circulant $C(r,m)$ is the Cayley graph on $Z_r^m$ with respect to the connection set $\{1, -1, r, -r, \ldots, r^{m-1}, -r^{m-1}\}$. A recursive circulant $G(cr^n, r)$ is a generalisation of a multiplicative circulant: $G(cr^n, r)$ is a circulant graph defined on $Z_{cr^n}$ with respect to the connection set

$$\{1, -1, r, -r, \ldots, r^{[\log_r (cr^n)]-1}, -r^{[\log_r (cr^n)]-1}\}.$$

The Knödel graph, denoted by $W_{\Delta,n}$, is a circulant graph on $n$ vertices with maximum degree $\Delta$, where $n$ is even and $1 \leq \Delta \leq \lceil \log_2 n \rceil$. The vertex set of $W_{\Delta,n}$ is $\{(x,y)|x = 1, 2 \text{ and } 0 \leq y \leq n/2 - 1\}$ and there is an edge between vertices $(1, y)$ and $(2, y + 2^k - 1 \mod (n/2))$, for each $k = 0, 1, \ldots, \Delta - 1$ and $0 \leq y \leq n/2 - 1$. The Knödel graph is a generalisation of the pancake graph.
Chapter 3

Literature review

In this chapter a brief survey of research works in network design and routing problems is presented to provide a context for results of this thesis. An overview of network modelling and routing problems for interconnection networks is presented in Section 3.1. Studies on Cayley graphs for interconnection networks and related routing problems are surveyed in Section 3.2. In Section 3.3, we review circulant graphs and their special families for network design, and in Section 3.4 we review cube-connected graphs for interconnection networks. We will propose our research problems whenever relevant.

3.1 Interconnection network design

Important applications of interconnection network design include the design of parallel computer topologies, Very-Large-Scale-Integration (VLSI), distributed memory systems and all-optical networks with high performance and low cost \[15, 110\]. Many network structures have been proposed and studied \[32, 76\] for different purposes. One of the earliest supercomputers was ILLIAC IV with 256 processors, which was built in 1964 and, as of today, Sunway TaihuLight is the most powerful supercomputer with around \(10.6 \times 10^6\) processors and speed of \(9.30 \times 10^{16}\) flops per second \[1\]. This constantly growing and unavoidable need for more efficient network structures justifies many existing research works and various proposed graphs in the literature \[31, 72, 75, 89, 90, 101, 103, 104\]. So theoretical approaches are essential to study effective practice in communication networking to meet demands.

Notable supercomputer topologies, which are used in practice, are the hypercube
of dimension 12, torus graphs of dimensions 2, 3, 5 and 6 and some circulant graphs \[2, 3, 5, 6\]. The current network topologies may not have satisfactory performance for a higher number of end-users. Hence due to growth in the number of end-users, there have been many research works on designing topologies to keep up with demands.

In this chapter, we review theoretical methods for designing and studying large graphs and their properties. A reference to other problems, applications and aspects of the communication networks is \[91\], Chapter 11.

**Routing in communication networks**

There are different routing problems about interconnections in networks. Among them, the vertex-forwarding, arc-forwarding and edge-forwarding index problems are important in the study of network capacity as they measure the load on network components in the busiest situations. The vertex-forwarding index problem, Problem \[2.3.1\], was first introduced by Chung et al. in 1987 \[30\] to address the maximum number of required message forwarding tasks for each processor in a network. A solution to this problem determines the minimum load on the most congested vertex of a given graph. By an observation on the load of a vertex on average and in the worst situation, the following theorem can be proved.

**Theorem 3.1.1** \([30]\). For a connected graph \(X = (V, E)\), we have

\[
\frac{1}{|V|} \sum_{x \in V} \sum_{y \in V, x \neq y} (\text{dist}(x, y) - 1) \leq \xi(X) \leq \xi_m(X) \leq (n - 1)(n - 2).
\]

Moreover, \(\xi(X)\) and \(\xi_m(X)\) are equal to the lower bound above if and only if there exists a shortest path routing which loads vertices uniformly.

Heydemann et al. analogously defined the edge-forwarding index problem, Problem \[2.3.2\] to obtain the minimum load on the most congested edge when the connection request set is all-to-all in a given graph \[64\]. This parameter is an indicator for the required bandwidth for links of the network in the busiest situation. The lower and upper bounds for \(\pi(X)\) given in the following theorem are obtained by the load on the edges in the average case and in the worst case scenario, respectively.

**Theorem 3.1.2** \([64]\). For a connected graph \(X = (V, E)\), we have

\[
\frac{1}{|E|} \sum_{(x, y) \in V \times V} \text{dist}(x, y) \leq \pi(X) \leq \pi_m(X) \leq \left\lfloor \frac{n^2}{2} \right\rfloor.
\]

Furthermore, \(\pi(X)\) and \(\pi_m(X)\) are equal to the lower bound above if and only if there exists a shortest path routing which loads edges uniformly.
Saad showed that Problem 2.3.1, the vertex-forwarding index problem, is NP-complete in general [92]. Heydemann et al. proved that the minimal vertex-forwarding index problem for graphs with diameters at least 4 is NP-complete, even if routings are restricted to shortest paths [66]. The complexity of this problem remains the same even for graphs with diameters 2; however, Problem 2.3.1 for graphs of diameter 2 is polynomially solvable if restricted to shortest path consistent symmetric routings [66] or even restricted to shortest path routings [92]. Not surprisingly, Problem 2.3.2, the edge-forwarding index problem, is NP-complete for \( k = 3 \) in general [73]. This problem remains hard even for shortest path consistent symmetric routings and graphs with diameters at least 3 [66].

The forwarding indices of a few simple structure graphs, like paths, trees, wheels, complete bipartite graphs, cycles, and hypercubes have been obtained [64]. For instance, \( \xi(C_n) = \xi_m(C_n) = \lfloor (n - 2)^2/4 \rfloor \), \( \xi(Q_d) = \xi_m(Q_d) = 2^{d-1}(d - 2) + 1 \) [30], \( \pi(C_n) = \pi_m(C_n) = \lfloor n^2/4 \rfloor \) and \( \pi(Q_d) = \pi_m(Q_d) = 2^d \), where \( n \geq 3 \) and \( d \geq 1 \). There have been different approaches to the forwarding index problems and results have been obtained for special families of graphs. This is an indication that obtaining results for these parameters is challenging in general.

**Proposition 3.1.3** ([64]). For any graph \( X \) of order \( n \), with maximum degree \( \Delta \),

(i) \( 2\xi(X) + 2(n - 1) \leq \Delta \pi(X) \);

(ii) \( \pi(X) \leq \xi(X) + 2(n - 1) \);

Apart from the above proposition, there is no known relation between vertex-forwarding and edge-forwarding indices of a graph in general.

A graph is called vertex-optimal [30] if its vertex-forwarding index is equal to the lower bound in Theorem 3.1.1 and edge-optimal [11] if its edge-forwarding index is equal to the lower bound in Theorem 3.1.2. An interesting topic here is the characterisation of graphs which are vertex-optimal or edge-optimal (see Section 6.2). Note that the vertex- and edge-forwarding indices for a graph are not closely related. For instance, a graph can be vertex-optimal while not edge-optimal.

The edge-optimality (vertex-optimality, respectively) of a graph indicates that there exists a shortest path routing for the graph whose load on all edges (vertices, respectively) is uniform. However, an edge-optimal (vertex-optimal, respectively) graph may not be an efficient topology in general. For instance, while the cycles and hypercubes are vertex-optimal and edge-optimal, \( \xi(C_n) \) and \( \pi(C_n) \) are of order of the upper bounds in Theorems 3.1.2 and 3.1.1, that is, \( O(n^2) \), but \( \xi(Q_d) = O(n \log n) \) and \( \pi(Q_d) = O(n) \),

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where $n = |V(Q_d)|$. So a (vertex-) edge-optimal graph among graphs with a similar set of properties might not be an efficient interconnection network. For instance, $C_n$ has much larger forwarding indices than those of $Q_d$, where $2^d = n$. Therefore, it is desirable to understand the minimum possible values for forwarding indices of graphs with a given set of properties. Chung et al. asked an interesting question: for given integers $n \geq 2$ and $\Delta \geq 1$, find a connected graph $X^*$ among all graphs of $n$ vertices and maximum degree at most $\Delta$ with minimum vertex-forwarding index $[30]$. In other words, find graph $X^*$ such that $\xi(X^*) = \xi_{\Delta,n}$, where

$$
\xi_{\Delta,n} = \min_{|V(X)|=n, \Delta(X)\leq \Delta} \xi(X).
$$

This question was formerly known as the ‘vertex-forwarding problem’.

Similarly, the same problem for the edge-forwarding index is proposed by Heydemann et al. [33]: for given integers $n \geq 2$ and $\Delta \geq 1$, find a connected graph $X^*$ such that $\pi(X^*) = \pi_{\Delta,n}$, where

$$
\pi_{\Delta,n} = \min_{|V(X)|=n, \Delta(X)\leq \Delta} \pi(X).
$$

Intuitively, finding a graph and a routing such that the vertex-forwarding or edge-forwarding index is equal $\xi_{\Delta,n}$ or $\pi_{\Delta,n}$, respectively, is equivalent to finding a graph with $n$ vertices of degree at most $\Delta$ with maximum throughput regarding vertex or link capability, respectively.

Chung et al., Heydemann et al. and Bouabdallah and Sotteau in [28, 30, 65] obtained $\xi_{\Delta,n}$ and $\pi_{\Delta,n}$ for very large or small $\Delta$. It is shown that $\xi_{2,n} = \xi(C_n) = \xi_m(C_n) = [(n-1)^2/4]$ [30] and $\pi_{2,n} = \pi(C_n) = \pi_m(C_n) = [n^2/4]$; moreover, $\pi_{n-1,n} = \pi(K_n) = \pi_m(K_n) = 2$, $\pi_{n-2,n} = 4$ for $n = 4, 5, 7$ and $\pi_{n-2,n} = 3$ for $n \geq 6, n \neq 7$ [28]. For any positive integer $k$ and graphs of order $n$ we have $\xi_{n-2k-1,n} = 2k$ if $n \geq 3k+2$ [65] and $\pi_{n-2k-1,n} = 3$ if $n > 10k$, $\pi_{n-2k-1,n} = 4$ if $4k < n \leq 10k$, and $\pi_{n-2k-1,n} = 6$ if $4k < n < 14k/3+1$ [28].

The exact values of $\xi_{\Delta,n}$ and $\pi_{\Delta,n}$ are unknown in general. However, it is shown that as $n \to \infty$ we have $\xi_{\Delta,n} \geq (1+o(1))n \log_{\Delta-1} n$ for $\Delta \geq 3$ [30, Theorem 6], and $\xi_{\Delta,n} \leq (3+O(1/\log \Delta))n \log_{\Delta-1} n$ for $\Delta \geq 6$ [30, Theorem 7]. Using Proposition 3.1.3, lower and upper bounds of the same order can be obtained for $\pi_{\Delta,n}$ [13]. Hence, for $\Delta \geq 6$ and as $n \to \infty$, we have

$$
\xi_{\Delta,n} = \Theta(n \log_{\Delta-1} n), \quad \pi_{\Delta,n} = \Theta(n \log_{\Delta-1} n).
$$

One might expect $k$-vertex or $k$-edge connected graphs to have smaller vertex-forwarding and edge-forwarding indices in general. Comparing to the bounds for the
forwarding indices in Theorems 3.1.1 and 3.1.2, there are slightly stronger upper bounds for the forwarding indices of $k$-vertex ($k$-edge) connected graphs. For any 2-vertex connected graph $X$ on $n$ vertices, $\xi(X) \leq (n - 2)(n - 3)/2$ [63] and if $n \geq 7$, then $\xi_m(X) \leq n^2 - 7n + 12$ [63]. If $X$ is $k$-vertex connected with $n$ vertices, $\xi(X) \leq (n - 1)[(n - k - 1)/k]$ and $\pi(X) \leq n[(n - k - 1)/k]$ [63]; furthermore, if $n \geq 8k - 10$ and $k \geq 3$, we have $\xi_m(X) \leq n^2 - (2k + 1)n + 2k$ [63]. Also it is shown in [34] that $\pi_m(X) \leq \lfloor n^2/2 \rfloor - n - 2(k - 1)^2$ if $n \geq \max\{3k + 3, (k + 1)^2/2\}$ and $k \geq 3$. For any 2-edge connected graph $X$ on $n$ vertices, $\pi_m(X) \leq \lfloor (n - 1)^2/2 \rfloor$ [63] and $\pi(X) \leq \lfloor n^2/4 \rfloor$ [63]. Mansouddakis and Tuza gave almost similar bounds for the forwarding indices of $k$-vertex connected digraphs in [72]. Comparing the bounds for the forwarding indices in Theorems 3.1.1 and 3.1.2 and those for $k$-vertex connected or $k$-edge connected (undirected) graphs, we observe that the connectivity does not influence the forwarding indices of a graph substantially.

**Optical index**

The optical index of a network is an important invariant for optical networking and there have been many papers and methods to deal with the optical index problem, Problem 2.3.4, in the literature. It is easy to observe that the complexity of computing the optical index of a graph is at least that of the edge-forwarding index of the graph. In fact, Kosowski showed that deciding whether $w(X) \leq 3$ for a given graph $X$ is NP-complete in general [73]. There are a few other versions of the routing and wavelength assignment problem [12, 51], but in this thesis, we only study this problem for the all-to-all connection request set on full-duplex graphs.

In general, equalities in (2.18) do not hold, that is, the optical (directed optical, respectively) index may be strictly larger than the edge-forwarding (arc-forwarding, respectively) index of a graph. In fact, Kosowski constructed a family of planar graphs $X$ each of whose arc-forwarding index is strictly less than its directed optical index, and the ratio of $\overrightarrow{w}(X)$ to $\overrightarrow{\pi}(X)$ is at least $20/19$ when $V(X)$ is sufficiently large [73]. However, equalities in (2.18) hold for a few classes of graphs. It is known that the directed optical index is equal to the arc-forwarding index for stars and trees [51], rings, hypercubes [13], weighted trees of rings [12], a few families of recursive circulant graphs and compound graphs [7] (see Section 3.3), Cartesian products of $k$ rings of the same order (i.e. the $k$-dimensional torus), paths and complete graphs [11, 43].

Related problems to wavelength assignment problem are the ‘graph colouring problem’ [30] and generalisation of ‘path intersection problem’ in graph theory [51]. Given
a routing $R$ for graph $X$, the conflict graph $X_R$ is the graph such that there is a one-to-one correspondence between its vertices and the paths of $R$, and any two vertices in $X_R$ are adjacent if and only if their corresponding paths in $R$ have an edge in common. Therefore, any graph colouring of $X_R$ with minimum number of colours determines a solution for a wavelength assignment of $R$ in $X$ with $w(X, R)$ colours. Similarly, the directed optical index of $X$ can be modelled by the conflict graph, with the only difference being that vertices of $X_R$ are adjacent if and only if their corresponding paths share an arc in the same direction.

Obtaining the optical index has remained open for many families of graphs even for problems with connection request sets other than the all-to-all instance. It is shown that for a graph $X = (V, E)$ and any connection request set $I$, we have $\overbar{w}(X, I) \leq 2\sqrt{|E|} \overbar{w}(X, I)$. In general, obtaining a tight bound for the optical index is quite difficult unless it is restricted to a special family of graphs. For a given routing $R$ (for an arbitrary connection request set) in the tree of rings, there exists a polynomial algorithm for the wavelength assignment of $R$ achieving approximation ratio of 2.75 asymptotically, and an algorithm of approximation ratio 2 for the same problem when restricted to a subfamily of the trees of rings.

3.2 Cayley graphs and network design

Cayley graphs are widely used in interconnection network design because of their vertex-transitivity, efficient implementation of distributed algorithms, hierarchical structure and methodological analysis of their properties. There are signs indicating that the number of non-Cayley vertex-transitive graphs is much smaller than that of Cayley graphs; hence, algebraic graph theory can be used to study vertex-transitive graphs in general. A huge interest in using algebraic graph theory to design and study communication networks in the literature is a sign of a strong bond between these two research fields. In fact, several families of Cayley graphs have been studied from the viewpoint of routing algorithms, diameters, and forwarding indices. On the other hand, existing results in group theory provide powerful tools for developing results in related studies. As an example, Babai et al. proved that there exists a connected Cayley graph of order $n$ and degree at most 7 on any non-Abelian finite group whose diameter is $O(\log n)$.

In 1989 Akers and Krishnamurthy first used group-theoretical methods to represent symmetric networks and construct new graphs on finite groups for interconnection
networks in [3]. They also proposed two families of graphs as models, namely the star and pancake graphs as alternatives for hypercubes. Hypercubes are famous due to their favourable features including, vertex-transitivity, high connectivity, logarithmic diameter and simple routing algorithms. However, the family of hypercubes suffers from large vertex degree which is a major obstacle for its practicality. In comparison with a hypercube with almost similar order, a star graph $ST_n$ has smaller diameter, degree and connectivity, which are $\lfloor 3(n - 1)/2 \rfloor$, $n - 1$ and $n - 1$, respectively. However, the degrees of both graphs are large and grow with the order of the network, and the orders of these graphs increase very fast. In other words, there might not exist a star or hypercube graph whose order is close to the number of nodes of the network we want to design (for example, the difference of orders of $ST_{n+1}$ and $ST_n$ is $(n + 1)! - n!$, which is huge for large $n$). The bubble sort and pancake graphs of dimension $n$ are Cayley graphs defined on the permutation group $S_n$ with $n!$ vertices and degree $(n - 1)$, and they have similar connectivities as that of the star graph $ST_n$ [3,13,77].

The contribution of Akers and Krishnamurthy to network design is mostly due to building up a conversation for adopting group-theory models in this area. This work has inspired researchers to propose new construction methods for new graphs using group theory. Annexstein et al. studied ‘group action graphs’, which are associated with Cayley graphs but are not vertex-transitive in general [8,12]. The shuffle-exchange and de Bruijn graphs are examples of group action graphs. In 1997, Heydemann surveyed related results on Cayley graphs for interconnection networks and problems in this area with an emphasis on routing problems [62]. Lakshmivarahan et al. studied Cayley graphs on permutation groups and gave results on symmetric properties of various Cayley graphs and introduced a vertex- and edge-transitive graph, namely the complete-transposition graph $CT_n$, with $n!$ vertices, degree $n(n - 1)/2$ and diameter $n - 1$ [71].

In addition to simple construction, vertex-transitivity and small diameter of Cayley graphs, they acquire attractive features for designing large graphs. In fact, the Cartesian product of any two Cayley graphs is Cayley, and it inherits some characteristics of the factor graphs [12]. Similarly, the weak and strong products of Cayley graphs are Cayley [12]. Moreover, the semidirect product of two groups is a strong tool for defining Cayley graphs. An example is the generalised cube-connected cycles [27]. In some cases Cayley graphs are optimally connected. It is shown in [55] and [111, Theorem 2.2.17] that the connectivity of $\text{Cay}(G, S)$ is $|S|$ if $S$ is a minimal generating set of $G$. 

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Routing in Cayley graphs

The majority of results on the forwarding index problems have been produced on vertex-transitive and Cayley graphs \[110\]. A reason is that these problems are very challenging and hard in general. The following theorem indicates that the class of Cayley graphs are attractive for routing problems.

**Theorem 3.2.1** (\[64\]). For any Cayley graph \(X = (V, E)\) and any vertex \(u \in V\),

\[
\xi(X) = \xi_m(X) = \sum_{u \in V} \text{dist}(u, v) - (|V| - 1).
\]

Consequently, any Cayley graph is vertex-optimal by Theorem 3.1.1. This theorem cannot be generalised for general vertex-transitive graphs \[97\]. An interesting question here is to characterise the family of edge-optimal graphs. Solé introduced the family of orbital-regular graphs and proved that they are edge-optimal \[98\]. In other words, there exists a shortest path routing for any orbital-regular graph which loads edges of the graph uniformly. Fang et al. in \[45\] showed that all orbital-regular graphs (except cycles and stars) are Frobenius graphs, which are certain Cayley graphs on the kernels of finite Frobenius groups. The cycles \(C_n\) and stars \(K_{1,n}\) have simple structures and obtaining a shortest path routing for them is straightforward \((\pi(C_n) = \pi_m(C_n) = \lfloor n^2/4 \rfloor \[64\] and \(\pi(K_{1,n}) = \pi_m(K_{1,n}) = 2n \[45\]). The Frobenius graphs are classified into two subclasses, namely first-kind and second-kind Frobenius graphs. Zhou in \[116\] gave a shortest path routing algorithm for first-kind Frobenius graphs and Fang and Zhou in \[46\] gave a shortest path routing algorithm for a class of second-kind Frobenius graphs. The edge-forwarding index of Frobenius circulants is also investigated in the literature (see Section 3.3). It is conjectured in \[64\] that any distance-transitive graph is edge-optimal and its forwarding indices are achievable by a shortest path routing.

Except for orbital-regular graphs, a Cayley graph may not admit a shortest path routing which load edges uniformly in general. Guayacq showed in \[52\] that if a routing \(R\) for a Cayley graph \(\text{Cay}(G, S)\) loads vertices uniformly, then \(R\) loads edges of label \(s\) (that is, \(\{u, us\}\) for \(u \in G\)) uniformly, for each \(s \in S\). Using this result, he obtained the edge-forwarding indices of the star and complete-transposition graphs and showed that these parameters are equal to their lower bounds in Theorem 3.1.2 within 1 \[52\].

Shahrokhi et al. formulated the edge-forwarding index problem as a variation of the integer multicommodity flow problem \[97\]. They called a graph ‘\(H\)-orbit proportional’ if there exists a shortest path routing for the graph such that it loads each edge-orbit with
respect to $H$ uniformly, and the loads of the routing on edge-orbits are independent (see Section 5.7), where $H$ is a subgroup of the automorphism group of the graph. They showed that the edge-forwarding index of ‘orbit proportional’ graphs, including cube-connected cycles and butterfly graphs, can be obtained by some shortest path routing.

Within the class of Cayley graphs, circulants and cube-connected graphs comprise large subclasses for efficient communication networks. In the rest of this chapter, we will review the routing problems for these subclasses of graphs.

### 3.3 Circulant graphs

The family of circulant graphs has been studied for local computer networks, optical networks such as SONET networks, parallel computing systems (with practical usage for ILLIAC IV [10] and Cray T3D [3]), and also in research areas other than communication network design, including designing binary codes [72] and neural networks [57]. Inheriting nice features as Cayley graphs, they also acquire recursive construction and optimal fault-tolerance in some cases. It is known that any connected circulant graph of degree $k$ has vertex-connectivity $k$ if it does not contain $K_4$ [20], and edge-connectivity $k$ [71]. Therefore, the family of circulant graphs are potentially good models for communication networks.

Although circulant graphs are simple from a group-theoretic point of view, there have been many hard questions proposed for them in the fields of number theory, algorithm design and graph theory, and some of them remain open [80]. Therefore, the related problems have been narrowed down for specific families of the circulant graphs, either with a restriction on the degree or on the connection set. Among them, the problems of determining the connection set for the minimum diameter or mean distance, designing routing algorithms, determining the diameter, the vertex-forwarding, edge-forwarding and optical indices for circulants are related to communication network design.

Wong and Coppersmith carried out one of the early research works on circulant graphs to propose an ‘optimal’ underlying graph for a distributed memory network in 1974 [107]. The diameter of a 2$k$-regular circulant graph $C_n(\pm s_1, \ldots, \pm s_k)$ is at least $\sqrt{k}n/2 - (k + 1)/2$ and for given $n \geq 3$ and $k \geq 2$, a circulant graph is optimal if its diameter achieves this lower bound. Using a similar method, Hwang obtained a lower
bound for the mean distance of $C_n(\pm s_1, \ldots, \pm s_k)$ in [71, Theorem 4.6], that is,
\[ \frac{k}{2(k+1)!n} \left( \sqrt{k!n} - 3k - 1 \right)^{k+1}. \] (3.4)
Boesch et al. gave a stronger lower bound for the diameter of circulant graphs of degree 4, which is $d(n) := \lceil (\sqrt{2n^2 - 1} - 1)/2 \rceil$, and showed that this lower bound is achievable for $C_n(\pm s_1, \pm s_2)$ with $s_1 = d(n)$ and $s_2 = d(n) + 1$ [21]. Monakhova also obtained similar results [30]. A variation of the problem of finding optimal circulants of degree 4 is determining $s$ such that $C_n(\pm 1, \pm s)$ is optimal. The subclass of circulants $C_n(1, s)$ with $n = 2d^2 + 2d$ have diameters strictly larger than $d(n)$ [11], while there are infinitely many instances for which the diameter is equal to the lower bound $d(n)$ [18].

Analogously, Yebra et al. in 1985 [10] used periodic plane tessellations to represent and characterise circulant graphs with the maximum order for given degree and diameter. For circulants of degree 4, they associated an integer in the 2-dimensional lattice to each vertex such that the associated integers of adjacent vertices are neighbours in the lattice. The optimal double-loop graph with diameter $d$ is $C_{n_d}(a, b)$, where $n_d = 2d^2 + 2d + 1$ and $(a, b)$ is any pair of integers satisfying $(d+1)a - db \equiv 0 \mod n_d$ and $da + (d+1)b \equiv 0 \mod n_d$ (for instance, $(d, d+1)$), where the equation system is obtained from the plane tessellation method to maximise $n_d$ when the diameter is $d$. Similarly, they showed that $C_{n_d}(3d+1, 1, -3d-2)$ is an optimal triple-loop graph, where $n_d = 3d^2 + 3d + 1$ [10]. Thomson and Zhou studied Frobenius circulant graphs which include the family of circulants $C_{n_d}(d, d+1)$ and $C_{n_d}(3d+1, 1, -3d-2)$ (up to isomorphism), using group-theoretic methods [10]. Similarly, finding optimal circulant digraphs has also been investigated in the literature [13, 10, 11]. The problem of characterising optimal circulant graphs of degrees more than 6 is still open.

The family of (undirected) multiplicative circulant graphs $C_n(1, r, r^2, \ldots, r^{m-1})$, where $r, m$ are positive integers and $n = r^m$, was first introduced by Wong and Copper-smith [10] as models for interconnection networks. These circulant graphs are optimal and acquire simple routing algorithms and recursive construction [10]. The diameter and average distance of $C_n(1, r, r^2, \ldots, r^{m-1})$ is $m \lfloor r/2 \rfloor$ and $m(r^2 - 1)/(4r)$, respectively, when $r$ is odd [10]; and $mr/2 - \lfloor m/2 \rfloor$ and $m(r-1)/4 + t$, respectively, for some $|t| \leq m/4$ and even $r \geq 4$ [10]. In the same era, Park and Chwa proposed the recursive circulant graph $G(\ell r^m, r)$ [51] for integers $m \geq 1$ and $1 \leq c < r$, which is a generalisation of multiplicative circulant graphs (see Section 4.3). This family also enjoy recursive construction, logarithmic diameter and simple routing algorithm.

The family of Knödel graphs is a subclass of circulant graphs which are bipartite and are non-isomorphic to recursive circulants and hypercubes in general [12]. The
Knödel graph $W_{\Delta,n}$, as defined in Chapter 2, can be defined on the semidirect product group $\mathbb{Z}_2 \rtimes \mathbb{Z}_{n/2}$ equipped with operation $(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 + (-1)^{x_1}y_2)$ and the connection set $\{(1, 2^k - 1) : 0 \leq k \leq \Delta - 1\}$ [17]. In the case $n = 2^k$, for some integer $k \geq 1$, the diameter of $W_{k,2^k}$ is $[k/2] + 1$.

Thomson and Zhou studied the families of Frobenius circulant graphs of degree four [103], and six [104] as subfamilies of Frobenius circulants. The family of Frobenius circulants is a subclass of Frobenius graphs which are orbital-regular and hence edge-optimal [46, 103]. The Frobenius circulants of degree four are $C_n(1,s)$ for some $s$ satisfying $s^2 + 1 \equiv 0 \mod n$ (up to isomorphism), for instance $C_n(1,2d+1)$, where $n = 2d^2 + 2d + 1$. The Frobenius circulant graphs of degree 6 are $C_n(a,b,1)$ such that $n \equiv 1 \mod 6$, $b \equiv a - 1 \mod n$ and $a^2 - a + 1 \equiv 0 \mod n$ [105]. This family includes the triple-loop graphs $C_n(3d+1,1,-(3d+2))$ where $n = 3d^2 + 3d + 1$ [111], which is used in practice for a computing system known as HARTS [40]. Furthermore, both families of graphs are optimal and have the recursive construction property [105, 117].

**Routing in Circulant graphs**

While the shortest path problem in some families of circulant graphs is solvable in polynomial time, this problem can be complex in general. As the input size of this problem is $O(\log n)$ (that is $s_1, s_2, n$), an algorithm is polynomial if and only if the number of operations is $O(\log n)$. Cai et al. modelled this problem by a shortest vector problem in the integer lattice with $l_1$ norm. They reduced this problem to the general case of the shortest vector to prove that the shortest path problem for circulant graphs when the connection set is not fixed is NP-complete [24]. The idea of an integer lattice with $l_1$ norm is rooted from the well-known Minokowski theorem [24, 56]. Moreover, the problem remains NP-complete even for an approximation of factor $2 - \epsilon$, for any constant $\epsilon > 0$ [24].

The complexity of the shortest path problem for $C_n(\pm s_1, \ldots, \pm s_k)$ for a fixed $k \geq 3$ has been controversial. Zerovnik et al. [115] and Dobravec et al. [38] conjectured the shortest path problem (with fixed $k \geq 3$) is NP-complete. However, Gómez et al. asserted that the shortest path problem on circulant graphs with fixed $k$ is solvable in polynomial time [55]. They modelled the problem by integer linear programming. Since an integer linear model with fixed number of constraints and binary encodings of size $O(\log n)$ are solvable in polynomial time, they claimed that the shortest path problem for circulant graphs with fixed $k$ is also solvable in polynomial time. Using the closest vector problem in the integer lattice, they developed a polynomial time algorithm for
the problem in the case $k = 2$. However, they divulged that they could not derive a similar algorithm for $k \geq 3$ \[14,15\]. It is unknown whether an efficient algorithm for this problem with $k \geq 3$ can be obtained or not.

The best known algorithm for the shortest path problem for circulants of degree four has polynomial running time \[26,44,115\]. Žervonik et al. developed a polynomial time algorithm for obtaining the diameter of the circulant graph $C_n(\pm s_1, \pm s_2)$ \[115\], which can be easily generalised to obtain shortest paths.

Similarly, Cheng et al. developed an $O(\log(n))$ algorithm in 1988 \[28\] to obtain the diameter of the weighted circulant graph of degree 4. Later, Cheng et al. obtained a closed-form formula for the diameter of $C_n(s_1, s_2)$ in 2005 \[27\], restricted to the case $s_1 = 1$, which excludes those circulants such that $\gcd(n, s_1) > 1$ or $\gcd(n, s_2) > 1$.

Circulants of degree 4 or more have been investigated for different versions of the shortest path problem. Dobravec et al. designed a dynamic shortest path algorithm for circulants with $k \geq 2$ \[38\]. This idea was also used for directed circulant graphs by Guan \[59\]. Later, Dobravec et al. generalised their algorithm for a shortest path in the directed circulant graph $C_n(\pm s_1, s_2)$ \[37\].

Results on the forwarding index problems for circulants have been obtained only for some special families, including the recursive circulants, a class of Knödel graphs and two classes of Frobenius circulant graphs.

The total distance (or equivalently, average distances) of the recursive circulants and the multiplicative circulants is calculated in \[88,100\], which in turn give the vertex-forwarding indices of these families of graphs \[53\]. Moreover, Gauyacq et al. studied the edge-forwarding index of the recursive circulants and showed that \[53\]

$$cs^{m-1}s_2^2 - \frac{cs^{m-1}s_2^2}{4} \leq \pi(G(cs^{m-1}, s)) \leq \max \left\{ s\pi(G(cs^{m-1}, s)), cs^{m-1}s_2^2 - \frac{cs^{m-1}s_2^2}{4} \right\}. \tag{3.5}$$

There is no routing with a uniform load on edges for $G(2^m, 4)$ with $m \geq 3$, and so recursive circulant graphs are not edge-optimal in general \[53\].

Our knowledge about the Knödel graphs and their characteristics is quite limited in general. Most of the known results on the Knödel graph $W_{k,n}$ are for the particular case $n = 2^k$ for some positive $k$. The diameter of $W_{k,2^k}$ is $\lfloor k/2 \rfloor + 1$, its edge-forwarding index is $2^k$ and its edge bisection width is $2^{k-1}$ \[47\]. Only recently an approximation algorithm has been developed for shortest paths in general Knödel graphs \[58\]. Except for a formula for the diameter of Knödel graphs (within 2) \[58\], many other invariants remain unknown in general.

Any Frobenius circulant graph is Cayley and orbital-regular \[45\], and so admits a shortest path routing that has uniform loads on the vertices and edges. Thomson and
Zhou [104, 105] gave shortest path routings for Frobenius circulant graphs of degree 4 and 6, and gave formulas for their vertex-forwarding and edge-forwarding indices by following the general framework developed in [116].

In the literature, different routing problems have been studied for special families of circulant graphs. However, except for these special cases, most of the routing problems for the family of circulant graphs are still open in general. Most important questions for circulants are complex in general. More specifically, our knowledge on arc-forwarding, edge-forwarding and optical indices of them are very limited for circulant graphs in general, even when restricted to circulants of degree 4. On the other hand, circulant graphs are attractive for communication models. For instance, any connected circulant graph of degree 4, $C_n(\pm s_1, \pm s_2)$, is optimally fault-tolerant and can be decomposed into two Hamiltonian cycles [116, 32]. Hence, it is desirable to study the family of circulant graphs of degree 4. Therefore, the following problem is noteworthy in this research area.

**Problem 3.3.1.** Let $C_n(1, s)$ be a connected circulant graph of degree 4, where $n \geq 5$ and $s \in \mathbb{Z}_n \setminus \{0\}$. Determine $\pi(C_n(1, s))$ and $\overrightarrow{\pi}(C_n(1, s))$.

Note that $C_n(s_1, s_2)$ is connected if and only if $\gcd(n, s_1, s_2) = 1$. Moreover, for every pair $(s_1, s_2)$, where either $s_1$ or $s_2$ is coprime with $n$, that is, $\gcd(s_1, n) = 1$ or $\gcd(s_2, n) = 1$, $C_n(s_1, s_2)$ is isomorphic to $C_n(1, s)$, where $s \equiv s_1^{-1}s_2 \mod n$. Hence, Problem 3.3.1 excludes circulant graphs $C_n(s_1, s_2)$ with $\gcd(n, s_1) > 1$ and $\gcd(n, s_2) > 1$.

Problem 3.3.1 is difficult in general due to the complexity of the forwarding index problems and also the generality of this family (in fact two graphs in this family may extremely differ in some properties, e.g. the diameter). We deal with this problem in Chapter 4 by giving upper and lower bounds for these invariants. In general, a uniform-loading routing may not exist for a circulant graph, likewise Cayley graphs in general. Note that existing results on circulant graphs of degree 4 (except for special cases) for the gossiping and the bisection width problems are only approximate answers [80, 85], which hints at the difficulty of routing problems on this family of graphs.

**Optical index of circulants**

The optical index of circulant graphs has not been investigated in the literature except for very special cases. Amar et al. proved that for integers $2 \leq p_1 \leq p_2 \leq \ldots \leq p_n$,

$$w(K_{p_1}[K_{p_2}[\ldots[K_{p_n}]]]) = p_1p_2\ldots p_n,$$

(3.6)
where $K_p$ is the complete graph of order $p$ and $G_1[G_2]$ is the compound graph \[\mathbb{I}\]. The compound graph $G_1[G_2]$ is obtained by replacing each vertex of $G_1$ by a copy of $G_2$ and adding a perfect matching between two copies of $G_2$ if and only if their corresponding vertices in $G_1$ are adjacent. Using this result, they obtained the optical index of recursive circulants for some special cases: $w(G(2^{m}, 2)) = 2^{m-2}$, $w(G(3^{m}, 3)) = 3^{m-1}$, $w(G(2^{m}, 4)) = 2^{m-1}$ and $w(G(4^{m}, 4)) = 2^{2m-1} \[\mathbb{I}\]$. Fertin and Raspaud obtained the optical index of the Knödel graph $W_{k;n}$ for the special case $n = 2^k$, that is, $w(W_{k,2^k}) = 2^{k-1}$, by showing that $W_{k,2^k}$ is a compound graph obtained from $K_2$ and using $\langle \mathbb{I7} \rangle$. Narayanan et al. gave an approximation algorithm to compute the directed optical index of the optimal circulant graph of degree 4, that is $C_n(1, 2d+1)$, where $n = 2d^2 + 2d + 1$, and showed that $\overrightarrow{w}(C_n(1, 2d+1)) \leq 1.006 \overrightarrow{\mathcal{F}}(C_n(1, 2d+1)) \[\mathbb{I7}\]$. Apart from these results, the problem of determining the (directed and undirected) optical index of other circulant graphs is still open.

**Problem 3.3.2.** Let $C_n(1,s)$ be a connected circulant graph of degree 4, where $n \geq 5$ and $s \in \mathbb{Z}_n \setminus \{0\}$. Determine $w(C_n(1,s))$ and $\overrightarrow{w}(C_n(1,s))$.

This problem is difficult because of the complexity of the optical index problem and the complexity arising from the nature of this graph family in general. We deal with this problem in Chapter \[\mathbb{I}\] by giving a 4.85-approximation for these invariants.

### 3.4 Cube-connected graphs

Graphs obtained by some product of two or more graphs compose a huge class of graphs in the design of large communication networks. In product graphs, some known graphs become ‘factor’ graphs and some of their features are inherited by the product graph. Vertex-transitivity, being Cayley and short diameter are some possible inherited features. For instance, the torus graph is a Cayley graph, obtained by the Cartesian product of $k$ cycles on $n$ vertices. A well-known product in algebraic graph theory is the semidirect product. By **cube-connected graphs**, we refer to the class of graphs defined on a semidirect product group $\mathbb{Z}_n^k \rtimes \varphi H$ for some group $H$ and a homomorphism $\varphi : H \rightarrow \text{Aut}(\mathbb{Z}_n^k)$. A general definition of cube-connected graphs is obtained by substituting $\mathbb{Z}_n^k$ with $\mathbb{Z}_k^n$ for some $k \geq 2$.

The $n$-dimensional hypercube can be thought as a Cayley graph on $\mathbb{Z}_2^n \rtimes \varphi H$, where $H$ is the trivial group. The crossed cube graph, obtained by a modification to the adjacency relation of the hypercubes \[\mathbb{I2}\], has many of the nice features of hypercubes and its diameter is $[(n + 1)/2]$. However, the vertex degrees of both graphs are large.
Preparata and Vuillemin proposed the cube-connected cycles as a more efficient alternative to the shuffle exchange graph and described the implementation of a few parallel algorithms on this structure \cite{15}. The cube-connected cycle of dimension \( n \), \( CC_n \), is a cube-connected graph defined on \( \mathbb{Z}_2^n \times \mathbb{Z}_n \), where \( \varphi_x(a_1a_2\ldots a_n) = a_{1+x}a_{2+x}\ldots a_{n+x} \) with subscripts taken modulo \( n \) for any \( x \in \mathbb{Z}_n \), with respect to the connection set \( \{(0_n, 1), (0_n, n-1), (e_1, 0)\} \) \cite{20}. The cube-connected cycle \( CC_n \) is a Cayley graph with \( n2^n \) vertices of degree 3, and its diameter is \( 2n \) if \( n = 2, 3 \), and \( \lfloor 5n/2 \rfloor - 2 \) if \( n \geq 4 \) \cite{15}. The butterfly graph \( B(n) \) is a Cayley graph defined on the same group as that of \( CC_n \), but with respect to the connection set \( \{(0_n, -1), (0_n, 1), (e_{n-1}, -1), (e_1, 1)\} \). This graph is 4-regular with diameter \( \lfloor 3n/2 \rfloor \). These two families of graph have logarithmic order diameter and, in contrast to the hypercube, their vertex degrees are fixed.

The \( k \)-ary \( n \)-cube \( Q^k_n \), defined on \( \mathbb{Z}_k^n \) for some integers \( k \geq 2 \) and \( n \geq 1 \), belongs to the class of cube-connected graphs and it can be thought as a generalisation of 2- and 3-dimensional torus as well as binary hypercube \cite{20}. The \( k \)-ary \( n \)-cube has \( k^n \) vertices of degree \( 2n \) if \( k \geq 3 \), and \( n \) if \( k = 2 \), and its diameter is \( n(k/2) \). A variation of the butterfly graph is the \( k \)-ary wrapped-around butterfly \( B_k(n) \) with vertex set \( \mathbb{Z}_k^n \times \mathbb{Z}_n \), \( nk^n \) vertices of degree \( 2k \) and diameter \( \lfloor 3n/2 \rfloor \) \cite{15, 19}. The cyclic-cubes, defined independently, is isomorphic to the \( k \)-ary wrapped-around butterfly \cite{19}. By contracting each cycle of cyclic-cubes to a vertex and adding edges between vertices whose reduced cycles are connected by at least an edge, we obtain a \( k \)-ary cube graph.

A variation of \( Q^k_n \) is the torus-connected cycles, \( TCC(n, k) \), proposed in \cite{22}. It is basically a \( k \)-ary \( n \)-cube whose vertices are replaced with a cycle of order \( 2n \). So it has \( 2nk^n \) vertices of degree 3 and its diameter is \( nk \), but as a major drawback, there is no evidence that it is a vertex-transitive graph.

A class of cube-connected graphs are obtained by wreath product of permutation groups. A family of such graphs \( WG^m_n \) are defined on the wreath product of \( \mathbb{Z}_m \) by \( S_n \). Note that the wreath product of a group \( K \) by a subgroup \( H \) of the permutation group \( S_n \) is the semidirect product group \( K^n \rtimes H \) where \( (k_1, k_2, \ldots, k_n)^h = (k_{1^h}, k_{2^h}, \ldots, k_{n^h}) \) for \( (k_1, k_2, \ldots, k_n) \in K^n, h \in H \). \( WG^m_n \) is a \((2m - 1)\)-regular Cayley graph with order \( m^n n! \) and diameter \( 9n^2/8 - n/4 + 2 \) \cite{15}. According to the value of \( m \), either the degree of \( WG^m_n \) is larger or the diameter of \( WG^m_n \) is larger than \( O(n(\log m + \log n)) \) which is the logarithmic order of the graph order.

Carlsson et al. defined generalised cube-connected cycles on a semidirect product group as a generalisation of the cube-connected cycle \cite{23}. They defined the generalised cube-connected cycles \( GCC_n \) on the vertex set \( \mathbb{Z}_r \times \mathbb{Z}_2^n \) with respect to the same
connection set as that of the cube-connected cycles $CC_n$. This research work inspired the design of other Cayley graphs, including the cube-of-rings and the recursive cubes of rings. The cube-of-rings is a Cayley graph on the same group as $GCC_n$, but with a different connection set such that $n = kr$ for some $k \geq 1$ [31]. Later in 2000, Sun et al. defined the recursive cubes of rings on $\mathbb{Z}_2^n \times \mathbb{Z}_r$ by specifying an adjacency relation function [110]. This graph contains $r2^n$ vertices of degree $d + 2$. In 2013, Xie et al. redefined recursive cubes of rings by proposing a new adjacency function and posing a new condition on their parameters, and showed that recursive cubes of rings under the condition is vertex-transitive [110] (see Chapter 5 for a detailed story of this graph).

In Chapter 6, we define the recursive cube of rings on a semidirect product group. Any recursive cube of rings $Q_n(d,r)$ is realised by three different parameters, namely $n, d$ and $r$ with $n \geq d \geq 1$ and $r \geq 3$, and its vertex is $\mathbb{Z}_2^n \times \mathbb{Z}_r$. An advantage of the recursive cube of rings is that three different parameters give freedom in realizing a graph with fairly close to demanded characteristics. Moreover, this graph contains the cube-connected cycles and the cube-of-rings as special cases. Hence this family of graphs is attractive for further studies.

**Problem 3.4.1.** What is the diameter of a recursive cube of rings? What are the bisection width, forwarding indices and Wiener index of a recursive cube of rings?

Cycles are a factor for the most of the generalisation of the known graphs. For instance, the $k$-ary $n$-cube $Q_n^k$ has $2^n$ cycles of length $k$, and the cube-connected cycles has $2^n$ cycles of length $n$. However, cycles themselves are not efficient for practical usage in general due to their weak fault-tolerance, their large diameters and loads on vertices and edges. On the other hand, a few families of circulant graphs are known to have nice features. Therefore, a generalisation of a cube-connected graph with a circulant factor graph can introduce efficient graphs for interconnection networks. Therefore, we propose a new family of graphs, named ‘cube-connected circulants’, and study their properties for communication networks in Chapter 6.

**Routing in cube-connected graphs**

Despite diverse families of cube-connected graphs, the known results for their properties are limited to the diameter and (shortest) path algorithms, except for a few special families, including the hypercubes, cube-connected cycles and butterfly networks. The hypercubes are well-studied in the literature, especially for communication networks. For instance, it is known that $\overrightarrow{d}(Q_d) = 2^{d-1}$ [13]. In general,
for $k$-ary $n$-cube of $Q_k^n$, we have $\xi(Q_k^n) = \xi_m(Q_k^n) = nk^{n-1}[k^2/4] - k^n + 1$ and $\pi(Q_k^n) = \pi_m(Q_k^n) = k^{n-1}[k^2/4] \[^{30},^{64}\].

Hou and Xu gave an upper bound for the vertex-forwarding index of $k$-ary wrapped butterfly, that is $\xi(B_k(n)) < 5n^22^{-2}k^n - 2n(k^{[n/2]} + k^{[n/2 - 1]}) + (3n + 1)$ with an approximation ratio less than 2 when $k = 2 \[^{67}\]$. Yan et al. \[^{112}\] proved that $\xi(CC_n) = 7n^22^{n-2}(1 - o(1))$ for $n \geq 2$, and Shahrokhi and Székely \[^{96}\] obtained an asymptotic value for the edge-forwarding index of the cube-connected cycles and butterfly graphs, that is $\pi(CC_n) = 5n^22^{n-2}(1 - o(1))$ and $\pi(B_2(n)) = 5n^22^{n-3}(1 + o(1))$.

In Chapter \[^{5}\] we give a shortest path routing and a formula for the diameter of recursive cubes of rings. We also obtain sharp bounds for the bisection width, forwarding indices and the Weiner index of this family of cube-connected graphs. In Chapter \[^{1}\] we give sharp bounds for the Wiener index, bisection width, vertex-forwarding index and edge-forwarding index of cube-connected circulants, and study some embedding problems for cube-connected circulants.
Chapter 4

4-regular circulant graphs

In this chapter, we consider a family of 4-regular circulant graphs. We first investigate the arc-forwarding and edge-forwarding problems for these graphs. Using a two dimensional lattice model, we obtain lower bounds for the arc-forwarding and edge-forwarding indices of such graphs. We construct an all-to-all routing for them and give an upper bound for the arc-forwarding and edge-forwarding indices. We give an approximation algorithm for the optical indices of this family of circulant graphs. The results in this chapter answer Problems 6.3.1 and 6.3.2 with approximation ratio a small constant.

4.1 Introduction

We focus on 4-regular circulant graphs on \( n \) vertices with respect to the connection set \( \{ \pm 1, \pm s \} \), denoted by \( C_n(1, s) \). Without loss of generality, we assume \( 1 < s < n/2 \). As a subclass of Cayley graphs, \( C_n(1, s) \) is vertex-transitive. In fact, for any \( i, j \in \mathbb{Z}_n \), \( x \mapsto x + (j - i), x \in \mathbb{Z}_n \), is a permutation of \( \mathbb{Z}_n \) that maps \( i \) to \( j \) and preserves the adjacency and non-adjacency relation of \( C_n(1, s) \). Note that when \( n \) and \( a \) are coprime, \( C_n(a, b) \) is isomorphic to \( C_n(1, s) \), where \( s \equiv a^{-1}b \mod n \). Therefore, the results in this chapter do not cover circulant graphs \( C_n(a, b) \) with \( \min\{\gcd(n, a), \gcd(n, b)\} \geq 2 \).

4.1.1 Main results

In what follows we assume that \( n \) and \( s \) are integers with \( n \geq 5 \) and \( 1 < s < n/2 \), and \( q \) and \( r \) are integers defined by

\[
q := \lfloor n/s \rfloor, \quad n := qs + r.
\]
Observe that $0 \leq r < s$ and $q \geq 2$ as $s < n/2$. Denote by $x \equiv y \pmod{n}$ ($x \equiv y \pmod{n}$, respectively) the integer $x+y \pmod{n}$ ($x-y \pmod{n}$, respectively) between 0 and $n-1$, where $x,y \in \mathbb{Z}_n$.

Denote

$$\delta(x) := \begin{cases} 1, & \text{if } x \text{ is an odd integer} \\ 0, & \text{if } x \text{ is an even integer}. \end{cases}$$

The first main result in this chapter is as follows.

**Theorem 4.1.1.** The following hold:

(a) if $2 \leq s \leq \sqrt{n} - 1$, then

$$\frac{n^2 - \epsilon(n)}{8(s+1)} \leq \frac{\pi(C_n(1,s))}{2} \leq \frac{\overline{\pi}(C_n(1,s))}{8s} \leq \frac{(n-r)(n+r+2) + s^2}{8s};$$

(b) if $s = \sqrt{n}$, then

$$\frac{\sqrt{n}(n-1)}{8} \leq \frac{\pi(C_n(1,s))}{2} \leq \frac{\overline{\pi}(C_n(1,s))}{8} \leq \frac{\sqrt{n}(n-\epsilon(s))}{8};$$

(c) if $\sqrt{n} + 1 \leq s < n/2$, then

$$\max \left\{ \frac{n^2 - \epsilon(n)}{8(s+1)}, \frac{(n-1)(\sqrt{2n}-7)^2}{24n} \right\} \leq \frac{\pi(C_n(1,s))}{2} \leq \frac{\overline{\pi}(C_n(1,s))}{8s} \leq \frac{s^2(n+r+2) - \epsilon(s)(n-r)}{8s}. $$

Let $\text{ratio}$ denote the ratio of the upper bound to the lower bound in the same equation above. In (4.1), $\text{ratio} = \frac{n+1}{s} \left( (1 - \frac{c}{n}) (1 + \frac{r+2}{n}) + \frac{s^2}{n^2 - \epsilon(n)} \right)$, which is asymptotically $(s+1)/s (\leq 3/2)$ as $n \to \infty$. In (4.2), $\text{ratio} = \frac{n - \epsilon(n)}{n-1}$, which tends to 1 as $n \to \infty$. By using the lower bound $(n^2 - \epsilon(n))/8(s+1)$ and the upper bound $s(n+r+2)/8$ in (4.3), we have $\text{ratio} \leq (s^2 + s)(n+r+2)/(n^2 - \epsilon(n))$. If $s = c\sqrt{n}$, for $c \in (1, \sqrt{3}/2)$, then $\text{ratio} \leq c^2 + o(1)$, which is at most $3/2$ asymptotically. If $\sqrt{3n}/2 < s < n/2$, then the lower bound in (4.3) is equal to $(n-1)(\sqrt{2n}-7)^2/24n$ and $\text{ratio} \leq 3sn(n+1)/(n-1)(\sqrt{2n}-7)^2$. In the latter case, the upper bound increases with $s$ and can be $O(\sqrt{n})$ in the worst case scenario when $s \approx cn$ for some constant $c < 1/2$.

We will prove the lower bounds in (4.1)-(4.3) in the next section. The upper bounds will be proved in Section 4.3; see Lemma 4.3.3 which gives more information and better upper bounds in some cases. To establish the upper bounds, we will give a specific routing (see Construction 4.3.1), which can be viewed as an approximation algorithm for computing $\overline{\pi}(C_n(1,s))$ and $\pi(C_n(1,s))$. From the discussion above, for $2 \leq s \leq \sqrt{3n}/2$, the performance ratio of this algorithm is at most $3/2$ asymptotically. So we obtain the following corollary of Theorem 4.1.1.
Corollary 4.1.2. There exists a 1.5-factor approximation algorithm to solve the edge-forwarding and arc-forwarding problems for 4-regular circulant graphs $C_n(1, s)$ with $n$ sufficiently large and $2 \leq s \leq \sqrt{3n}/2$.

In the worst case when $s \approx cn$ is large, where $c < 1/2$ is a constant, the ratio obtained from (4.3) (and that of the approximation algorithm from Construction 4.3.1) is $O(\sqrt{n})$. It seems that this large ratio is due to the fact that the lower bound in (4.3) is unsatisfactory when $s$ is large. Our next result shows that sometimes we can significantly improve this lower bound for large $s$. This enhanced lower bound together with the upper bound in (4.3) implies that for $s \approx cn$ our algorithm can achieve a constant performance ratio $1/c$ in some cases.

Theorem 4.1.3. If $r \leq q$ or $r + q \geq s + 1$, then

$$\pi(C_n(1, s)) \geq \frac{1}{2} \left( \frac{(s+1)^2}{2} \right).$$

(4.4)

We prove Theorem 4.1.3 by computing the sum of the distances between all pairs of nodes in $C_n(1, s)$, which is done by investigating an equivalent problem [115] for the integer lattice $\mathbb{Z}^2$.

The second main result in this chapter is the following theorem on the optical indices of $C_n(1, s)$. Denote

$$\kappa(a) := a + \frac{\epsilon(s) + \epsilon(q)}{2}.$$  

(4.5)

Theorem 4.1.4. The following hold:

(a) if $2 \leq s \leq \sqrt{n - r + (\kappa(-2))^2 + \kappa(-2)}$, then

$$\frac{n^2 - \epsilon(n)}{8(s+1)} \leq \overrightarrow{w}(C_n(1, s)) \leq \frac{s + 2}{24} \left( 6q^2 + 3q(s+4) + s(4s+10) + \epsilon(q)(2q+3s+3) \right);$$  

(4.6)

(b) if $\sqrt{n - r + (\kappa(-1))^2 + \kappa(-1)} \leq s \leq \sqrt{n - r + (\kappa(0))^2 + \kappa(0)}$, then

$$\frac{n^2 - \epsilon(n)}{8(s+1)} \leq \overrightarrow{w}(C_n(1, s)) \leq \frac{q(q+2)(5q+2)}{24} + \frac{s(s+2)(2s+5)}{6} + \epsilon(q)\frac{5q^2 + 13q + 7}{8};$$  

(4.7)

(c) if $\sqrt{n - r + (\kappa(1))^2 + \kappa(1)} \leq s < n/2$, then

$$\max \left\{ \frac{n^2 - \epsilon(n)}{8(s+1)} + \frac{(n-1)(\sqrt{2n} - 7)^3}{24n}, \frac{(q+10)}{24} + \frac{s(s+2)(q+1)}{2} + \epsilon(q)\frac{(q+5)^2 + 4(s+1)^2}{8} \right\} \leq \overrightarrow{w}(C_n(1, s))$$  

(4.8)
Moreover, if \( \overline{w}(C_n(1,s)) \) is replaced by \( w(C_n(1,s))/2 \) in (4.8)-(4.9), then the same lower and upper bounds are valid.

Let \( \text{ratio} \) denote the ratio of the upper bound to the lower bound in the same equation above. In (4.8), \( \text{ratio} \leq 2(1 + \frac{1}{s})(1 + \frac{2}{s}) \left( 1 + \frac{s^2}{2n^2} + \frac{2s}{3n^3} \right) \frac{n^2}{n^2 - c(n)} + O \left( \frac{1}{\sqrt{n}} \right) \),

which is \( 2(1 + \frac{1}{s})(1 + \frac{2}{s}) \) asymptotically when \( s^2 = o(n) \). When \( s^2 = \Omega(n) \), in (4.9) we have \( \text{ratio} \leq 13/3 \) asymptotically since \( s \leq \sqrt{n - r + (\kappa(-2))^2} + \kappa(-2) \). In (4.3), \( \text{ratio} \leq 13/3 \) asymptotically as \( s \approx q \approx \sqrt{n} \) in case (b). Let

\[
\delta(n) := 3n(n^2 - \epsilon(n))/(n - 1)(\sqrt{2n} - 7)^3 - 1. \tag{4.9}
\]

Then \( \delta(n) + 1 > 3\sqrt{n}/2\sqrt{2} > 1 \) if \( n \geq 25 \), \( \delta(n) \approx 3\sqrt{n}/2\sqrt{2} \) as \( n \to \infty \), and \( \delta(n) + 1 \leq 91\sqrt{n}/80 \) for sufficiently large \( n \). If \( s \leq \delta(n) \), then the lower bound in (4.8) is equal to \( (n^2 - \epsilon(n))/8(s + 1) \) and \( \text{ratio} \leq \frac{q(q+2)(n+10)(s+1)}{3(n^2 - \epsilon(n))} + \frac{4(s+1)(s+2)(n-r+s)}{n^2 - \epsilon(n)} + O \left( \frac{1}{\sqrt{n}} \right) \).

This upper bound increases with \( s \) and approaches a constant between 13/3 and 4.85, depending on the value of \( s \), as \( n \to \infty \). The upper bounds in (4.8)-(4.9) will be proved by giving a specific colouring (see Construction 4.3.1) of the routing obtained from Construction 4.3.1, and this can be viewed as an approximation algorithm for the routing and wavelength assignment problem and its oriented version for \( C_n(1,s) \). Thus the discussion above yields the following corollary of Theorem 4.1.4.

**Corollary 4.1.5.** There is a 4.85-factor approximation algorithm to solve the routing and wavelength assignment problem and its oriented version for 4-regular circulant graphs \( C_n(1,s) \) with \( n \) sufficiently large and \( 3 \leq s \leq 3\sqrt{n}/2\sqrt{2} - 1 \).

If \( \delta(n) < s < n/2 \), then the lower bound in (4.8) is equal to \( (n - 1)(\sqrt{2n} - 7)^3/24n \) and the ratio from (4.8) (and so the approximation ratio of the algorithm from Construction 4.4.1) is at most \( \frac{n}{n-1} \left( \frac{q(q+2)(n+10)}{(\sqrt{2n} - 7)^3} + \frac{s(s+2)(n+10)}{2(\sqrt{2n} - 7)^3} \right) + O \left( \frac{1}{\sqrt{n}} \right) \). This upper bound increases with \( s \) and is \( O(\sqrt{n}) \) in the worst case scenario when \( s \approx cn \) for some constant \( c < 1/2 \). However, in the case when \( r \leq q \) or \( r + q \geq s + 1 \), our stronger lower bound on \( w(C_n(1,s)) \) obtained from (4.4) and (4.18) implies that the approximation ratio is at most \( 4(1/c + \epsilon(q)) \) asymptotically.

The rest of the chapter is organized as follows. We will prove the lower bounds in the theorems above in the next section. In Section 4.3.1 we will establish the upper bounds in Theorem 4.1.4 by devising a specific routing (Construction 4.3.1). In Section 4.4.1 we will prove the upper bounds in Theorem 4.1.4 by giving a specific colouring (Construction 4.4.1) for the routing obtained from Construction 4.3.1. Note that the lower bounds in (4.8)-(4.9) are obtained from (4.18) and the lower bounds in Theorem 4.1.4.
4.2 Lower bounds

4.2.1 Two lower bounds

Lemma 4.2.1.

\[ \pi(C_n(1, s)) \geq \frac{[n/2][n/2]}{s + 1} = \frac{n^2 - \epsilon(n)}{4(s + 1)} \]

Proof. We apply (4.10) to \(C_n(1, s)\). Choose \(U = \{0, \ldots, [n/2] - 1 \} \subset \mathbb{Z}_n\) so that \(|U| = [n/2]\) and \(|\overline{U}| = [n/2]|\). Consider the neighbours \(i + s, i - s\) of \(i \in U\). We have: \([n/2] \leq i + s \leq n - 1\) if and only if \([n/2] - s \leq i \leq [n/2] - 1\), and \(i - s (\equiv n - (s - i) \mod n)\) lies in \( \overline{U}\) if and only if \(0 \leq i \leq s - 1\). Thus \(\delta(U)\) consists of edges \(\{i, i + s\}\) \((|n/2] - s \leq i \leq [n/2] - 1\), \(\{i, i - s\}\) \((0 \leq i \leq s - 1\), \(\{0, n - 1\}\) and \(\{[n/2] - 1, [n/2]\}\). Hence \(|\delta(U)| = 2s + 2\). This together with (4.10) yields \(\pi(C_n(1, s)) \geq [n/2]/[n/2]/(s + 1). \)

As observed in [64], we have

\[ \pi(G) \geq \frac{\sum_{(x,y) \in V(G) \times V(G)} \text{dist}(x, y)}{|E(G)|} = \frac{n(n - 1) \overline{\text{dist}}(G)}{|E(G)|}, \tag{4.10} \]

where \(\text{dist}(x, y)\) is the distance between \(x\) and \(y\) in \(G\) and \(\overline{\text{dist}}(G)\) is the mean distance among all unordered pairs of vertices of \(G\). It was proved in [71, Theorem 4.6] that, if \(G\) is a circulant network of order \(n\) and degree 4, then

\[ \overline{\text{dist}}(G) \geq \frac{(\sqrt{2n} - 7)^3}{6n}. \]

Applying this and (4.10) to \(C_n(1, s)\), we obtain:

Lemma 4.2.2.

\[ \pi(C_n(1, s)) \geq \frac{(n - 1)(\sqrt{2n} - 7)^3}{12n}. \]

It can be verified that the lower bound in Lemma 4.2.1 is no less than that in Lemma 4.2.2 if and only if \(s \leq 3n(n^2 - \epsilon(n))/(n - 1)(\sqrt{2n} - 7)^3 - 1\). Thus the lower bounds in (4.1)-(4.3) follow from (4.10) and Lemmas 4.2.1 and 4.2.2 immediately.

4.2.2 Proof of Theorem 4.1.3

To prove Theorem 4.1.3, we need two results from [115]. Let \(\mathbb{Z}^2\) be the 2-dimensional \(\mathbb{Z}\)-module lattice. Define \(l : \mathbb{Z}^2 \to \mathbb{Z}_n\) by

\(l(x) := x_1 \oplus x_2 s, \ x = (x_1, x_2) \in \mathbb{Z}^2, \)

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We may view $l$ as a labelling that labels each point $(x_1, x_2)$ of $\mathbb{Z}^2$ by the node $x_1 \oplus x_2 s$ of $C_n(1, s)$. We observe that when two points $(x_1, x_2)$ and $(y_1, y_2)$ are neighbors in the lattice $\mathbb{Z}^2$ (that is, either $x_1 = y_1$ and $|x_2 - y_2| = 1$, or $|x_1 - y_1| = 1$ and $x_2 = y_2$), the corresponding labels $l(x_1, x_2)$ and $l(y_1, y_2)$ are adjacent nodes of $C_n(1, s)$.

Denote by $\| \cdot \|$ the $L_1$-norm in $\mathbb{Z}^2$ defined by

$$\| x \| := |x_1| + |x_2|, \ x = (x_1, x_2) \in \mathbb{Z}^2.$$ 

The length of a path $x_0, x_1, \ldots, x_k$ in $\mathbb{Z}^2$, connecting $x_0$ and $x_k$ is defined as $k$, and the distance between two points of $\mathbb{Z}^2$ is defined to be the length of a shortest path in $\mathbb{Z}^2$ connecting them, where $x_{i-1}$ and $x_i$ are neighbors in the lattice. Thus the distance between $x$ and $y$ in $\mathbb{Z}^2$ is equal to $\| x - y \|$.

Each path $x_0, x_1, \ldots, x_k$ in $\mathbb{Z}^2$ gives rise to the oriented path $l(x_0), l(x_1), \ldots, l(x_k)$ in $C_n(1, s)$. Note that even if the former is a shortest path in $\mathbb{Z}^2$, the latter is not necessarily a shortest path in $C_n(1, s)$.

Denote by $X$ the set of points of $\mathbb{Z}^2$ with label $0$. That is,

$$X := \{(x_1, x_2) \in \mathbb{Z}^2 : l(x_1, x_2) = 0\}.$$ 

Note that $X$ relies on $s$ implicitly. A basis for $X$ is a set of two independent vectors $\{a, b\}$ in $X$ such that any vector in $X$ is a linear combination of them with integer coefficients. The parallelogram generated by a basis $\{a, b\}$ is defined as

$$[a, b] := \{x \in \mathbb{Z}^2 : x = \alpha a + \beta b, 0 \leq \alpha, \beta \leq 1\},$$ 

where $\alpha$ and $\beta$ are real. Note that its corner points, $0, a, b$ and $a + b$, are in $X$. Similarly, the half-open parallelogram generated by $\{a, b\}$ is defined as

$$[a, b) := \{x \in \mathbb{Z}^2 : x = \alpha a + \beta b, 0 \leq \alpha, \beta < 1\}.$$ 

In what follows $\text{dist}(i, j)$ denotes the distance between $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_n$ in $C_n(1, s)$. It is observed in [113] that, for every $v \in \mathbb{Z}^2$, we have

$$\text{dist}(0, l(v)) := \| v - X \| := \min \{ \| v - x \| : x \in X \}. \quad (4.11)$$

**Lemma 4.2.3.** ([174, Proposition 2]) Let $\{a, b\}$ be a basis for $X$. Then $[a, b)$ has exactly $n$ points and each label in $\{0, 1, \ldots, n-1\}$ appears exactly once as $l(x)$ for some $x \in [a, b)$.

Thus the labelling $l$ induces a bijection between the points in $[a, b)$ and the nodes in $\mathbb{Z}_n$. This together with (4.11) implies that for any $i \in \mathbb{Z}_n$, $\text{dist}(0, i) = \| v_i - X \|,$
where \( v_i \) is the unique point in \([a, b]\) with \( l(v_i) = i \). The next lemma says that, if \( \{a, b\} \) is a packed basis, then \( \|v_i - X\| \) is attained at a corner point of \([a, b]\), where a basis \( \{a, b\} \) is packed if it satisfies

\[
\max\{\|a\|, \|b\|\} \leq \min\{\|a - b\|, \|a + b\|\}. \tag{4.12}
\]

**Lemma 4.2.4.** \((\text{H13, Lemma 2})\) Let \( \{a, b\} \) be a packed basis for \( X \). Then, for any \( v \in [a, b] \), we have

\[
\|v - X\| = \min\{\|v - x\| : x = 0, a, b, a + b\}.
\]

It is not immediate that a basis exists for \( X \). In the following we give a packed basis for \( X \) in all cases except when \( r \geq q \) and \( r + q \leq s \).

**Lemma 4.2.5.** The following hold:

(a) If \( r \leq q \) and \( 2r \leq s + 1 \), then \( \{(s, -1), (r, q)\} \) is a packed basis for \( X \);

(b) If \( r \leq q \) and \( 2r \geq s + 1 \), then \( \{(s, -1), (r - s, q + 1)\} \) is a packed basis for \( X \);

(c) If \( r \geq q \) and \( r + q \geq s + 1 \), then \( \{(s, -1), (r - s, q + 1)\} \) is a packed basis for \( X \).

**Proof.** It can be shown (see [27, pp. 6]) that any point in \( X \) is of the form \( i(s, -1) + j(r, q) \) for some integers \( i \) and \( j \). (In fact, for any \((x_1, x_2) \in X\), we have \( x_1 + x_2 s = kn \) for some integer \( k \), and so \((x_1, x_2) = (kq - x_2)(s, -1) + k(r, q)\).) It follows that the pair \( \{a, b\} \) in each case is a basis for \( X \). It remains to verify that \( \{a, b\} \) satisfies (H12).

Recall that \( q \geq 2 \) as \( s < n/2 \).

(a) Let \( a = (s, -1) \) and \( b = (r, q) \). Since \( r \leq q \) and \( 2r \leq s + 1 \) and \( q \geq 2 \), we have \( \|a\| = s + 1 \), \( \|b\| = r + q \), \( \|a - b\| = s - r + q + 1 \), \( \|a + b\| = s + r + q - 1 \), and \( \{a, b\} \) satisfies (H12).

(b) Let \( a = (s, -1) \) and \( b = (r - s, q + 1) \). Since \( r \leq q \) and \( 2r \geq s + 1 \), we have \( \|a\| = s + 1 \), \( \|b\| = s - r + q + 1 \), \( \|a - b\| = 2s - r + q + 2 \) and \( \|a + b\| = r + q \). Hence \( \|a\| \leq \|b\| \leq \|a + b\| \leq \|a - b\| \) and \( \{a, b\} \) satisfies (H12).

(c) Let \( a \) and \( b \) be as in (b). Since \( r \geq q \) and \( r + q \geq s + 1 \), the norms of \( a, b, a - b, a + b \) are the same as in (b). Hence \( \|b\| \leq \|a\| \leq \|a + b\| \leq \|a - b\| \) and \( \{a, b\} \) satisfies (H12). \(\square\)

**Lemma 4.2.6.** If \( r \leq q \) or \( r + q \geq s + 1 \), then

\[
\sum_{i \in \mathbb{Z}_n} \text{dist}(0, i) \geq \left[ \frac{(s + 1)^2}{2} \right]. \tag{4.13}
\]

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Proof. Since \( r \leq q \) or \( r + q \geq s + 1 \), one of the three cases in Lemma \ref{lem:dist} occurs, and in each case we have a packed basis \( \{a, b\} \) for \( X \) as given in Lemma \ref{lem:dist}. By Lemma \ref{lem:dist}, for any \( i \in \mathbb{Z}_n \), there exists at least one \( v \in [a, b] \) such that \( l(v) = i \). Moreover, by Lemma \ref{lem:dist}, \( \text{dist}(0, i) = \min\{\|v - x\| : x = 0, a, b, a + b\} \). We now compute the sum of these distances \( \text{dist}(0, i) \) for \( i \) in a certain subset of \( \mathbb{Z}_n \).

Case 1: \( r \leq q \) and \( 2r \leq s + 1 \). In this case we have \( a = (s, -1), b = (r, q) \) by Lemma \ref{lem:dist}. Set \( v_i = (i, 0), w_i = (a + b) - (i, 0) \), \( \alpha_i = iq/n, \beta_i = i/n, \alpha_i' = 1 - iq/n \) and \( \beta_i' = 1 - i/n \) for \( 1 \leq i \leq s \). Then \( \alpha_i, \beta_i, \alpha_i', \beta_i' \in [0, 1] \), \( v_i = \alpha_i a + \beta_i b, w_i = \alpha_i' a + \beta_i' b \) and \( v_i, w_i \in [a, b] \) for each \( i \). Since \( l(v_i) = i, l(w_i) = n - i \) and \( s < n/2, l(v_1), \ldots, l(v_s), l(w_1), \ldots, l(w_s) \) are pairwise distinct. It can be verified that

\[
\|v_i - 0\| = \|w_i - (a + b)\| = i, \quad \|v_i - a\| = \|w_i - b\| = s - i + 1,
\]

\[
\|v_i - (a + b)\| = \|w_i - 0\| = r - i + q - 1, \quad \|v_i - b\| = \|w_i - a\| = r - i + q.
\]

Assume \( 1 \leq i \leq \lfloor s/2 \rfloor \). Since \( r \leq q \), we have \( |r - i| + q \geq i \) (as \( i - r + q \geq i \) if \( i > r \) and \( r - i + q \geq r \geq i \) if \( i < r \)) and \( s + r - i + q - 1 \geq s + 1 - i \geq i \). Therefore,

\[
\|v_i - X\| = \|v_i - 0\| \quad \text{and} \quad \|w_i - X\| = \|w_i - (a + b)\|.
\]

For \( \lfloor s/2 \rfloor < i \leq s \), it can be verified that \( \|v_i - X\| = \|v_i - a\| \) and \( \|w_i - X\| = \|w_i - b\| \) (see Figure \ref{fig:dist} for an illustration). Therefore,

\[
\sum_{i \in \mathbb{Z}_n} \text{dist}(0, i) \geq \sum_{i=1}^{s} (\|v_i - X\| + \|w_i - X\|)
\]

\[
= \sum_{i=1}^{\lfloor s/2 \rfloor} 2i + \sum_{\lfloor s/2 \rfloor + 1}^{s} 2(s - i + 1)
\]

\[
\geq (s + 1)^{2}/2.
\]

Case 2: \( r \leq q \) and \( 2r \geq s + 1 \), or \( r \geq q \) and \( r + q \geq s + 1 \). Then \( r + q \geq s + 1, 2r \geq s + 1 \), and \( a = (s, -1), b = (r - q, q + 1) \) by Lemma \ref{lem:dist}. Set \( v_i = (i, 0), w_i = (a + b) - (i, 0) \), \( \alpha_i = iq + 1/n, \beta_i = i/n, \alpha_i' = 1 - iq + 1/n, \beta_i' = 1 - (i/n) \) for \( 1 \leq i \leq s - 1 \) and \( u = (1, 1) \). Then \( \alpha_i, \beta_i, \alpha_i', \beta_i' \in [0, 1] \), \( v_i = \alpha_i a + \beta_i b, w_i = \alpha_i' a + \beta_i' b \) and \( v_i, w_i \in [a, b] \) for each \( i \). Moreover, \( u = ((s - r + q + 1)/n)a + ((s + 1)/n)b \in [a, b] \). Since \( l(v_i) = i, l(w_i) = n - i, l(u) = s + 1 \) and \( s < n/2, l(v_1), \ldots, l(v_s), l(w_1), \ldots, l(w_s), l(u) \) are pairwise distinct. It can be verified that

\[
\|v_i - 0\| = \|w_i - (a + b)\| = i, \quad \|v_i - a\| = \|w_i - b\| = s - i + 1,
\]

\[
\|v_i - (a + b)\| = \|w_i - 0\| = |r - i| + q, \quad \|v_i - b\| = \|w_i - a\| = s - r + i + q + 1.
\]

Assume \( 1 \leq i \leq \lfloor s/2 \rfloor \). Since \( r + q \geq s + 1 \), we have \( r - i + q \geq s - i + 1 \geq i \) and \( s - r + i + q + 1 \geq i \). Hence \( \|v_i - X\| = \|v_i - 0\| \) and \( \|w_i - X\| = \|w_i - (a + b)\| \). If

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Figure 4.1: \( \text{dist}(0, i) \) is either \( \|v_i - 0\| \) or \( \|v_i - a\| \); \( \text{dist}(0, n - i) \) is either \( \|w_i - (a + b)\| \) or \( \|w_i - b\| \);

\[|s/2| < i \leq s - 1, \text{ then } \|v_i - X\| = \|v_i - a\| \text{ and } \|w_i - X\| = \|w_i - b\|. \]

Note that \( \|u - X\| = 2 \). Therefore,

\[
\sum_{i \in \mathbb{Z}_n} \text{dist}(0, i) \geq \sum_{i=1}^{s-1}(\|v_i - X\| + \|w_i - X\|) + \|u - X\|
= \sum_{i=1}^{[s/2]} 2i + \sum_{[s/2]+1}^{s-1} 2(s - i + 1) + 2
\geq \lfloor (s + 1)^2/2 \rfloor.
\]

Proof of Theorem 4.1.3. Suppose \( r \leq q \) or \( r + q \geq s + 1 \). Since \( C_n(1, s) \) is vertex-transitive, we have \( \sum_{(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n} \text{dist}(i, j) = n \sum_{i \in \mathbb{Z}_n} \text{dist}(0, j) \). This together with (4.10) and (4.13) implies (4.13).

As shown in the proof of Lemma 4.2.6, we obtained (4.13) by computing the total distance from the vertex 0 to \( 2s \) or \( 2s - 1 \) other vertices. This implies that the difference between the two sides of (4.13) is small if and only if \( s \) is close to \( n/2 \).

The sum of the distances can be precisely computed when \( s = \sqrt{n} \), leading to the following better lower bound than Theorem 4.1.3 in this special case.
Lemma 4.2.7. If $s = q = \sqrt{n}$, then
\[
\pi(C_n(1, s)) \geq \frac{\sqrt{n}(n-1)}{4}.
\]

Proof. By Lemma 4.2.5, $\{((\sqrt{n}, -1), (0, \sqrt{n}))\}$ is a packed basis for $X$ in this case. One can see that the following is a closest corner point to $(i, j)$:

(a) $(0, 0)$, if $0 \leq i \leq \lfloor \sqrt{n}/2 \rfloor$ and $0 \leq j \leq \lfloor (\sqrt{n} - 1)/2 \rfloor$;
(b) $(\sqrt{n}, -1)$, if $\lfloor \sqrt{n}/2 \rfloor + 1 \leq i \leq \sqrt{n} - 1$ and $0 \leq j \leq \lfloor \sqrt{n}/2 \rfloor - 1$;
(c) $(0, \sqrt{n})$, if $0 \leq i \leq \lfloor (\sqrt{n} - 1)/2 \rfloor$ and $\lfloor (\sqrt{n} + 1)/2 \rfloor \leq j \leq \sqrt{n} - 1$ and
(d) $(\sqrt{n}, \sqrt{n} - 1)$, if $\lfloor (\sqrt{n} + 1)/2 \rfloor \leq i \leq \sqrt{n} - 1$ and $\lfloor \sqrt{n}/2 \rfloor \leq j \leq \sqrt{n} - 1$.

Since every point $(i, j)$ in $X$ appears in exactly one of these four cases, we have
\[
\sum_{k \in \mathbb{Z}_n} \text{dist}(0, k) = \sum_{i=0}^{\lfloor \sqrt{n}/2 \rfloor} \sum_{j=0}^{\lfloor (\sqrt{n} - 1)/2 \rfloor} (i + j) + \sum_{i=\lfloor \sqrt{n}/2 \rfloor + 1}^{\lfloor \sqrt{n}/2 \rfloor - 1} \sum_{j=0}^{\lfloor \sqrt{n}/2 \rfloor - 1} (\sqrt{n} - i + j + 1) + \sum_{i=0}^{\lfloor (\sqrt{n} - 1)/2 \rfloor} \sum_{j=\lfloor (\sqrt{n} + 1)/2 \rfloor}^{\lfloor (\sqrt{n} + 1)/2 \rfloor - 1} (i + \sqrt{n} - j) + \sum_{i=\lfloor (\sqrt{n} + 1)/2 \rfloor}^{\lfloor (\sqrt{n} + 1)/2 \rfloor - 1} \sum_{j=\lfloor \sqrt{n}/2 \rfloor}^{\lfloor \sqrt{n}/2 \rfloor} (\sqrt{n} - j + \sqrt{n} - 1 - j) = \frac{\sqrt{n}(n-1)}{2}.
\]
The result then follows from (4.10) and vertex-transitivity of $C_n(1, s)$. \qed

4.3 A routing scheme, and proof of Theorem 4.1.1

In this section we give a specific routing scheme for $C_n(1, s)$ which yields the required upper bounds on the forwarding indices of $C_n(1, s)$. The same routing will be used in the next section to give upper bounds on the optical indices of $C_n(1, s)$. We will use the words ‘link’ and ‘arc’ interchangeably and we call a link of $C_n(1, s)$ of the form $(x, x \oplus 1)$ ($(x, x \ominus 1)$, respectively) a clockwise (anticlockwise, respectively) ring link, and a link of the form $(x, x \oplus s)$ ($(x, x \ominus s)$, respectively) a clockwise (anticlockwise, respectively) skip link. We define a routing as follows.

Construction 4.3.1. Define
\[
\mathcal{R} := \{P_{x,y} : x, y \in \mathbb{Z}_n, x \neq y\},
\] (4.14)
where $P_{x,y}$ is the path in $C_n(1, s)$ from $x$ to $y$ specified as follows.
1. For $d = 1, \ldots, [n/2]$, say, $d = is + j$ for some $i, j$ with $0 \leq i \leq [q/2]$ and $0 \leq j \leq s - 1$,

(a) if $j \leq [s/2]$, then define $P_{0,d} : 0, s, 2s, \ldots, is, is + 1, is + 2, \ldots, is + j$;

(b) if $j > [s/2]$, then define $P_{0,d} : 0, s, 2s, \ldots, (i + 1)s, (i + 1)s - 1, (i + 1)s - 2, \ldots, (i + 1)s - (s - j)$.

2. For $d = [n/2] + 1, \ldots, n - 1$, letting $P_{0,n - d} : v_1, v_2, \ldots, v_k$ be the path from 0 to $n - d$ constructed in Step 1, define $P_{0,d} : v_1, n - v_2, \ldots, n - v_k$.

3. For $1 \leq x, y \leq n - 1$ with $x \neq y$, letting $v_1, v_2, \ldots, v_k$ denote the path $P_{0,y \ominus x}$ from 0 to $y \ominus x$ constructed in Step 1 or 2, define $P_{x,y} : x \oplus v_1, x \oplus v_2, \ldots, x \oplus v_k$.

The routing $R$ constructed above is symmetric in the sense that, for any $x, y, k \in \mathbb{Z}_n$ with $x \neq y$, $P_{x \oplus k,y \ominus k}$ is the path obtained by adding $k$ to each node of $P_{x,y}$. We say that $P_{x \oplus k,y \ominus k}$ is obtained from translation of $P_{x,y}$ by $k$. This feature is crucial in the following computation of $\overline{f}(C_n(1, s))$.

Denote

$$\Delta := \begin{cases} \frac{s}{4} + \frac{1}{2} \left\lfloor \frac{s}{2} \right\rfloor \left( s - \left\lfloor \frac{r + 2}{2} \right\rfloor \right), & \text{if } s \text{ is even} \\ \frac{1}{2} \left\lfloor \frac{r + 1}{2} \right\rfloor \left( s - \left\lfloor \frac{r + 1}{2} \right\rfloor \right), & \text{if } s \text{ is odd} \end{cases}$$

One can verify that the four numbers involved in the next lemma are integers.

**Lemma 4.3.2.**

(a) If $q$ is even, then

$$\overline{f}(C_n(1, s), R) = \max \left\{ \frac{q}{4} \left\lfloor \frac{s^2}{2} \right\rfloor + \frac{1}{2} \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r + 2}{2} \right\rfloor, \frac{q^2s}{8} + \frac{q}{2} \left( \left\lfloor \frac{r}{2} \right\rfloor + \frac{\epsilon(s)}{2} \right) \right\}.$$  

(b) If $q$ is odd, then

$$\overline{f}(C_n(1, s), R) = \max \left\{ \frac{q}{4} \left\lfloor \frac{s^2}{2} \right\rfloor + \Delta, \frac{(q^2 - 1)s}{8} + \frac{q + 1}{2} \left\lfloor \frac{r + \epsilon(s)}{2} \right\rfloor \right\}.$$  

**Proof.** Observe that if a path $P_{x,y}$ in $R$ passes through a clockwise ring link $(v, v \oplus 1)$, then the path $P_{x \oplus (v' - v), y \ominus (v' - v)}$ (which is also in $R$) passes through the clockwise ring link $(v', v' \oplus 1)$. Hence the loads on all clockwise ring links under $R$ are equal. Similarly, all anticlockwise ring links have the same load, all clockwise skip links have the same load, and all anticlockwise skip links have the same load. So the load on each clockwise ring link (skip link, respectively) is the total number of clockwise ring links (skip links, respectively) used by paths of $R$ divided by $n$. The same can be said for anticlockwise ring or skip links.
On the other hand, for any fixed \( x \in \mathbb{Z}_n \), the paths \( P_{x,y} \), \( x \neq y = 0, \ldots, n-1 \), use the same number of ring (skip, respectively) links as the paths \( P_{0,d} \), \( d = 1, \ldots, n-1 \), because \( P_{x,y} \) is the translation of \( P_{0,y \oplus x} \) by \( x \) and therefore there is a bijection between these two sets of paths. Since there are \( n \) translations for any path \( P_{0,d} \), \( d = 1, \ldots, n-1 \), we conclude that the load on any ring (skip, respectively) link in each direction is the total number of used ring (skip, respectively) links in that direction by paths \( P_{0,d} \), \( d = 1, \ldots, n-1 \).

Claim 1: The maximum load on ring links under \( \mathcal{R} \) is equal to the number of clockwise ring links used when \( q \) is even, and anticlockwise ring links used when \( q \) is odd, by paths \( P_{0,d} \), \( d = 1, \ldots, n-1 \).

Claim 2: The maximum load under \( \mathcal{R} \) on skip links is equal to the number of clockwise skip links used by paths \( P_{0,d} \), \( d = 1, \ldots, n-1 \).

Proof of Claims 1 and 2. For any path \( P_{0,d} \) in \( \mathcal{R} \), where \( d < n/2 \), the path \( P_{0,n-d} \) is in \( \mathcal{R} \) and is distinct from \( P_{0,d} \). Moreover, if \((u,v)\) is a link in one of these two paths, then \((n \ominus u,n \ominus v)\) is a link in the other (with opposite direction). If \( n \) is odd, then \( d \neq n-d \) for all \( d \in \mathbb{Z}_n \), and so the number of clockwise ring links (skip links, respectively) used is equal to the number of anticlockwise ring links (skip links, respectively) used by the paths \( P_{0,d} \).

Assume \( n \) is even. If \( d = n/2 \), then \( P_{0,d} \) and \( P_{0,n-d} \) are identical, and this path does not use any anticlockwise skip link. Hence \( P_{0,n/2} \) uses fewer anticlockwise skip links than clockwise skip links. So the maximum load on skip links is equal to the number of clockwise skip links used by the paths \( P_{0,d} \).

If \( q \) is even, then \( n/2 = qs/2 + r/2 \); in this case \( P_{0,n/2} \) uses \( r/2 \) clockwise ring links, and so it uses fewer anticlockwise ring links than clockwise ring links. Thus the maximum load on ring links is equal to the number of clockwise ring links used by the paths \( P_{0,d} \).

If \( q \) is odd, then \( n/2 = (q - 1)s/2 + (s + r)/2 \); in this case \( P_{0,n/2} \) uses \( (s - r)/2 \) anticlockwise ring links, and so it uses fewer clockwise ring links. Hence the maximum load on ring links is equal to the number of anticlockwise ring link used by the paths \( P_{0,d} \). This completes the proof of Claims 1 and 2.

We now count the number of links in \( C_n(1,s) \) used by paths \( P_{0,d} \), \( d = 1, \ldots, n-1 \), for even \( q \) and odd \( q \) separately.

Ring links: We first count the number of clockwise (anticlockwise, respectively) ring links used by paths \( P_{0,d} \) when \( q \) is even (odd, respectively), where \( P_{0,d} \) is defined in
Construction [3.3.1] in Step 1 when \( d = is + j \leq \lfloor n/2 \rfloor \) and in Step 2 when \( d = n - (is + j) > \lfloor n/2 \rfloor \), where \( 0 \leq i \leq \lfloor q/2 \rfloor \) and \( 0 \leq j \leq s - 1 \).

**Case 1**: \( q \) is even. If \( d \leq \lfloor n/2 \rfloor \), then \( is + j \leq q\lfloor s/2 \rfloor + \lfloor r/2 \rfloor \); if \( d > \lfloor n/2 \rfloor \), then \( d = n - (is + j) \) and so \( is + j < q\lfloor s/2 \rfloor + \lfloor r/2 \rfloor \). When \( d \leq \lfloor n/2 \rfloor \), by Step [2], the path \( P_{0,d} \) uses some clockwise ring link if and only if either \( 0 \leq i \leq q/2 - 1 \) and \( 0 \leq j \leq \lfloor s/2 \rfloor \), or \( i = q/2 \) and \( 0 \leq j \leq \lfloor r/2 \rfloor \). When \( d > \lfloor n/2 \rfloor \), by Steps [2](b) and [3], \( P_{0,d} \) uses some clockwise ring link if and only if \( 1 \leq i \leq q/2 \) and \( \lfloor s/2 \rfloor + 1 \leq j \leq s - 1 \). Moreover, \( P_{0,d} \) uses \( j \) (\( s-j \), respectively) clockwise ring links if \( d \leq \lfloor n/2 \rfloor \) (\( d > \lfloor n/2 \rfloor \), respectively). Therefore, the total number of clockwise ring links used by the paths \( P_{0,d}, d = 1, \ldots, n - 1 \), is equal to

\[
\sum_{i=0}^{q/2-1} \sum_{j=0}^{\lfloor s/2 \rfloor} j + \sum_{i=q/2}^{q/2} \sum_{j=0}^{\lfloor r/2 \rfloor} j + \sum_{i=1}^{q/2} \sum_{j=\lfloor s/2 \rfloor+1}^{s-1} (s-j) = \frac{q}{4} \left( \frac{s^2}{2} \right) + \frac{1}{2} \left( \frac{r}{2} \right) \left( r + \frac{2}{2} \right). \tag{4.15}
\]

**Case 2**: \( q \) is odd. If \( d \leq \lfloor n/2 \rfloor \), then \( is + j \leq (q-1)s/2 + \lfloor (s+r)/2 \rfloor \); if \( d > \lfloor n/2 \rfloor \), then \( d = n - (is + j) \) and so \( is + j < (q-1)s/2 + \lfloor (s+r)/2 \rfloor \). When \( d \leq \lfloor n/2 \rfloor \), by Step [3](b), \( P_{0,d} \) uses some anticlockwise ring link if and only if either \( 1 \leq i \leq (q-1)/2 \) and \( \lfloor s/2 \rfloor + 1 \leq j \leq s - 1 \), or \( i = (q+1)/2 \) and \( \lfloor s/2 \rfloor + 1 \leq j \leq \lfloor (s+r)/2 \rfloor \). When \( d > \lfloor n/2 \rfloor \), by Steps [3](a) and [4], \( P_{0,d} \) uses some anticlockwise ring link if and only if \( 0 \leq i \leq (q-1)/2 \) and \( 1 \leq j \leq \lfloor s/2 \rfloor \). The path \( P_{0,d} \) uses \( s-j \) (\( j \), respectively) anticlockwise ring links if \( d \leq \lfloor n/2 \rfloor \) (\( d > \lfloor n/2 \rfloor \), respectively). Therefore, the total number of anticlockwise ring links used by the paths \( P_{0,d}, d = 1, \ldots, n - 1 \), is given by

\[
\sum_{i=1}^{(q-1)/2} \sum_{j=\lfloor s/2 \rfloor+1}^{s-1} (s-j) + \sum_{i=(q+1)/2}^{(q+1)/2} \sum_{j=\lfloor s/2 \rfloor+1}^{\lfloor (s+r)/2 \rfloor} (s-j) + \sum_{i=0}^{(q-1)/2} \sum_{j=1}^{\lfloor s/2 \rfloor} j = \frac{q}{4} \left( \frac{s^2}{2} \right) + \Delta. \tag{4.16}
\]

**Skip links**: Now we evaluate the load on clockwise skip links. Note that \( P_{0,d} \) uses clockwise skip links if \( d \leq \lfloor n/2 \rfloor \).

**Case 3**: \( q \) is even. In this case \( P_{0,d} \) uses exactly \( k \) clockwise skip links if and only if either (i) \( i = k, 0 \leq j \leq \lfloor s/2 \rfloor \), if \( 1 \leq k \leq q/2 - 1 \); (ii) \( i = k, 0 \leq j \leq \lfloor r/2 \rfloor \), if \( k = q/2 \); or (iii) \( i = k - 1, \lfloor s/2 \rfloor + 1 \leq j \leq s - 1 \), if \( 1 \leq k \leq q/2 \). So the total number of clockwise skip links used by the paths \( P_{0,d}, d = 1, \ldots, n - 1 \), is equal to

\[
\sum_{k=1}^{q/2-1} k \left( \left\lfloor \frac{s}{2} \right\rfloor + 1 \right) + \sum_{k=q/2}^{q/2} k \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) + \sum_{k=1}^{q/2} k \left( s - \left\lfloor \frac{s}{2} \right\rfloor - 1 \right) = \frac{q^2s}{8} + \frac{q}{2} \left( \left\lfloor \frac{r}{2} \right\rfloor + \frac{r}{2} + \frac{s}{2} \right). \tag{4.17}
\]

**Case 4**: \( q \) is odd. By Step [2], \( P_{0,d} \) uses exactly \( k \) clockwise skip links if and only if either (i) \( i = k, 0 \leq j \leq \lfloor s/2 \rfloor \), if \( 1 \leq k \leq (q-1)/2 \); (ii) \( i = k - 1, \lfloor s/2 \rfloor + 1 \leq j \leq s - 1 \),
if } 1 \leq k \leq (q - 1)/2; \text{ or (iii) } i = k - 1, \lfloor s/2 \rfloor + 1 \leq j \leq \lfloor (s + r)/2 \rfloor, \text{ if } k = (q + 1)/2.

Thus the total number of clockwise skip links used by the paths } P_{0,d}, \text{ } d = 1, \ldots, n - 1, \text{ is equal to

\[
\sum_{k=1}^{(q-1)/2} k \left\lfloor \frac{s + 2}{2} \right\rfloor + \sum_{k=1}^{(q-1)/2} k \left\lceil \frac{s - 2}{2} \right\rceil + \sum_{k=(q+1)/2}^{(q+1)/2} k \left\lfloor \frac{r + \epsilon(s)}{2} \right\rfloor = \frac{(q^2 - 1)s}{8} + \frac{q + 1}{2} \left\lfloor \frac{r + \epsilon(s)}{2} \right\rfloor.
\]

(4.18)

Using (4.15)-(4.18), Claims 1 and 2 imply the required results immediately. \hfill \square

By comparing the two terms in each case of Lemma 4.3.2, we can identify the maximum term for different ranges of } s, \text{ which is presented in the following lemma.

Lemma 4.3.3. The following hold:

(a) if } q \text{ is even and } 2 \leq s \leq \sqrt{n} - 1, \text{ then

\[
\overline{\lambda}(C_n(1, s), R) \leq \frac{q(n + r + 2\epsilon(s))}{8};
\]

(4.19)

(b) if } q \text{ is odd and } 2 \leq s \leq \sqrt{n} - 1, \text{ then

\[
\overline{\lambda}(C_n(1, s), R) \leq \frac{q(n + r + 2\epsilon(s)) + s}{8};
\]

(4.20)

(c) if } q = s = \sqrt{n}, \text{ then the loads on all links of } C_n(1, s) \text{ are equal and

\[
\overline{\lambda}(C_n(1, s), R) = \frac{\sqrt{n}(n - \epsilon(s))}{8};
\]

(4.21)

(d) if } q \text{ is even and } s \geq \sqrt{n} + 1, \text{ then

\[
\overline{\lambda}(C_n(1, s), R) \leq \frac{sn + r - \epsilon(s)q}{8};
\]

(4.22)

(e) if } q \text{ is odd and } s \geq \sqrt{n}, \text{ then

\[
\overline{\lambda}(C_n(1, s), R) \leq \frac{s(n + r + 2) - \epsilon(s)q}{8}.
\]

(4.23)

Moreover, when } \sqrt{n} \text{ is not an integer, if } s = \lfloor \sqrt{n} \rfloor \text{ and } q \text{ is odd or } s = \lfloor \sqrt{n} + 1 \rfloor \text{ and } q \text{ is even, then the greater term in each case in Lemma 4.3.2 relies on } r. \text{ However, the two terms are almost equal for sufficiently large } n. \text{ When } \sqrt{n} \text{ is an odd integer and } q = s = \sqrt{n}, \text{ the conditions in cases (c) and (e) are satisfied simultaneously. Although the right hand sides of (4.21) and (4.23) are equal in this special case, the result in case (c) is slightly stronger as we have equality in (4.21).}
Proof of Theorem 4.1.4. As mentioned in Section 4.2.1, the lower bounds in (4.11)-(4.13) follow from (4.23) and Lemmas 4.2.1, 4.2.2 and 4.2.7 immediately. The upper bounds in (4.11)-(4.13) follow from Lemma 4.3.3 and the fact that \( \bar{w}(C_n(1, s)) \leq \bar{w}(C_n(1, s), R) \). In fact, if \( 2 \leq s \leq \sqrt{n} - 1 \), then (4.14) or (4.21) applies. By (4.15) and (4.21), we obtain \( \bar{w}(C_n(1, s)) \leq (q(n + r + 2\epsilon(s))/8) + (s/8) = ((n - r)(n + r + 2) + s^2)/8s \). If \( s = \sqrt{n} \), then \( q = [n/s] = \sqrt{n} \) and by (4.21) we obtain \( \bar{w}(C_n(1, s)) \leq (\sqrt{n}(n - \epsilon(s))/8 \). If \( \sqrt{n} + 1 \leq s < n/2 \), then by (4.22) and (4.23) we have \( \bar{w}(C_n(1, s)) \leq (s(n + r + 2) - \epsilon(s)q)/8 = (s^2(n + r + 2) - \epsilon(s)(n - r))/8s \). \( \square \)

4.4 Proof of Theorem 4.1.4

The lower bounds in (4.11)-(4.13) follow from (4.23) and Theorem 4.1.4 immediately. Let \( R \) be the routing defined in (4.23). Since \( \bar{w}(C_n(1, s)) \leq \bar{w}(C_n(1, s), R) \) and \( w(C_n(1, s)) \leq w(C_n(1, s), R) \), it suffices to prove the upper bounds for \( \bar{w}(C_n(1, s), R) \) and \( w(C_n(1, s), R)/2 \). In the following we give an arc-conflict-free colouring of \( R \) and compute the number of colours used. This number gives the required upper bound for \( \bar{w}(C_n(1, s), R) \).

By Construction 4.3.1, the unique path in \( R \) connecting \( x \) to \( y = x \oplus (is + j) \) is \( P_{ij} \), which we denote by \( P_{ij}^x \) in the rest of this proof, where \( |i| \leq \lceil q/2 \rceil \), \( |j| \leq \lceil s/2 \rceil \) and \( |is + j| \leq n/2 \). In other words, \( P_{ij}^x \) connects \( x \) to \( y \) by \( i \) successive skip links followed by \( j \) successive ring links, where negative \( i \) and \( j \) stand for anticlockwise skip and ring links, respectively. Given \( i \) and \( j \), set

\[
\alpha := |j|/\gcd(s, |j|), \quad \beta := s/\gcd(s, |j|),
\]

where \( \gcd(s, |j|) \) is the greatest common divisor of \( s \) and \( |j| \). Then \( \alpha \) is the smallest positive integer such that \( |j| \) divides \( \alpha s \). Note that \( \alpha s = \beta |j| \).

Construction 4.4.1. We define a colouring \( f : R \to \mathbb{Z}^3 \) by using elements of \( \mathbb{Z}^3 \) as colours.

1. If \( j > 0 \) and \( |i| \leq j \), let \( x_{\alpha} = \lfloor x/\alpha s \rfloor \).

   (a) If \( x \leq n/2 \), define \( f(P_{ij}^x) = (\epsilon(x_{\alpha})j + (x + \lfloor x_{\alpha}/2 \rfloor \mod j), i, j) \);

   (b) if \( x > n/2 \), define \( f(P_{ij}^x) = ((2 + \epsilon(x_{\alpha}))j + (x + \lfloor x_{\alpha}/2 \rfloor \mod j), i, j) \).

2. If \( |i| > j \geq 0 \), let \( x_0 = x \mod s \).

   (a) If \( x < \lfloor q/2 \rfloor s \) and \( x_0 \leq s - j \), define \( f(P_{ij}^x) = (x_0 + \lfloor x/s \rfloor \mod |i|, i, j) \);
(b) if \( x < (\lfloor q/2 \rfloor - 1)s \) and \( x_0 > s - j \), define \( f(P^x_{ij}) = (i + (x_0 + \lfloor x/s \rfloor \mod |i|), i, j) \);

(c) if \( (\lfloor q/2 \rfloor - 1)s \leq x < \lfloor q/2 \rfloor s \) and \( x_0 > s - j \), define \( f(P^x_{ij}) = (2i + x_0 + j - s, i, j) \);

(d) if \( \lfloor q/2 \rfloor s \leq x \leq n - j \) and \( x_0 \leq s - j \), define \( f(P^x_{ij}) = (i + (x_0 + \lfloor x/s \rfloor + s - q - r - 1 \mod |i|), i, j) \);

(e) if \( \lfloor q/2 \rfloor s \leq x \leq n - j \) and \( x_0 > s - j \), define \( f(P^x_{ij}) = (x_0 + \lfloor x/s \rfloor - q - r \mod |i|, i, j) \);

(f) if \( x > n - j \), define \( f(P^x_{ij}) = (2i + x + j - n, i, j) \).

3. If \( j < 0 \), define \( f(P^x_{ij}) = f(P^x_{(-i)(-j)}) \).

By the definition above, \( f(P^x_{ij}) \neq f(P^y_{kl}) \) if \( k \neq i \) or \( l \neq j \) when \( jl \geq 0 \), and if \( k \neq -i \) or \( l \neq -j \) when \( j < 0 \). Since \( P^x_{ij} \) and \( P^y_{ij} \) do not have any common link in the same direction, to prove that \( f \) is arc-conflict-free, it suffices to verify that \( P^x_{ij} \) and \( P^y_{ij} \) do not have any common link if \( f(P^x_{ij}) = f(P^y_{ij}) \), or equivalently \( f(P^x_{ij}) \neq f(P^y_{ij}) \) if \( P^x_{ij} \) and \( P^y_{ij} \) have a common link.

Fix \( i \) and \( j \). Assume \( x < y \). Note that \( P^x_{ij} \) and \( P^y_{ij} \) share a skip link if and only if \( y \ominus x = hs \) for some \( h \) with \( 0 < |h| < |i| \), and they share a ring link if and only if \( y \ominus x < |j| \).

**Case 1:** \( j > 0 \) and \( |i| \leq j \). Assume \( f(P^x_{ij}) = f(P^y_{ij}) \). Then by Step 1, \( x_\alpha \) and \( y_\alpha \) have the same parity and either \( x, y \leq n/2 \) or \( x, y > n/2 \). So if \( P^x_{ij} \) and \( P^y_{ij} \) share a ring link, then \( y - x < j \), and if they share a skip link, then \( y - x = hs \) for some \( h \) with \( 0 < h < |i| \). In the following we show that neither of these can happen, and therefore \( P^x_{ij} \) and \( P^y_{ij} \) cannot have any common link.

**Subcase 1.1:** \( x_\alpha = y_\alpha \). We have \( x + \lfloor x_\alpha/2 \rfloor = y + \lfloor y_\alpha/2 \rfloor \mod j \) as \( f(P^x_{ij}) = f(P^y_{ij}) \). So \( y = x + \gamma j \) as \( x_\alpha = y_\alpha \), where \( \gamma \geq 1 \). Hence \( P^x_{ij} \) and \( P^y_{ij} \) do not share any ring link as \( y - x \geq j \). If they share a skip link, then \( y = x + hs \), \( 0 < h < |i| \), which together with \( y = x + \gamma j \) gives \( hs = \gamma j \). Since \( x_\alpha = y_\alpha \), we have \( y - x < \alpha s \) and so \( h < \alpha \). The latter inequality together with \( hs = \gamma j \) contradicts the choice of \( \alpha \). Hence \( P^x_{ij} \) and \( P^y_{ij} \) do not share any skip link.

**Subcase 1.2:** \( x_\alpha \neq y_\alpha \). We have \( x = x_\alpha \alpha s + l_x s + x_0 \) and \( y = y_\alpha \alpha s + l_y s + y_0 \), where \( 0 \leq l_x, l_y \leq \alpha - 1 \), \( x_0 = x \mod s \) and \( y_0 = y \mod s \). Since \( x_\alpha \) and \( y_\alpha \) have the same parity, we have \( y_\alpha - x_\alpha \geq 2 \) and so \( y - x = (y_\alpha - x_\alpha) \alpha s + (l_y - l_x) s + y_0 - x_0 > \alpha s = \beta j \geq j \) as \( |l_y - l_x| \leq \alpha - 1 \) and \( |y_0 - x_0| < s \). So \( P^x_{ij} \) and \( P^y_{ij} \) do not share any ring link.
Suppose by way of contradiction that $P_{ij}^x$ and $P_{ij}^y$ have a common skip link. Then $y = x + hs$ with $h = \alpha + t$, for a positive integer $t$, as $y_\alpha - x_\alpha \geq 2$. Also $y_\alpha \alpha s + l_\gamma s = x_\alpha \alpha s + l_\gamma s + hs$ as $y_0 = x_0$. Since $f(P_{ij}^x) = f(P_{ij}^y)$, we have $x + |x_\alpha / 2| \equiv y + |y_\alpha / 2| \equiv x + hs + |y_\alpha / 2| \mod j$. Hence $(y_\alpha - x_\alpha) / 2 \equiv -hs \equiv -(\alpha s + ts) \equiv -tc \mod j$ as $\alpha s = \beta j$, where $c = s \mod j$. If $c = 0$, then $y_\alpha - x_\alpha \geq 2j$ as $y_\alpha > x_\alpha$ and $(y_\alpha - x_\alpha)\alpha s \geq 2j s$. If $c \neq 0$, then $(y_\alpha - x_\alpha) / 2 = kj - tc = \gcd(c, j)(ka_1 - ta_2)$, where $\gcd(a_1, a_2) = 1$ and $k$ is an integer. So $y_\alpha - x_\alpha \geq 2 \gcd(c, j)$ as $y_\alpha > x_\alpha$. Since $\alpha = j / \gcd(s, j)$ and $\gcd(s \mod j, j) = \gcd(s, j)$, we have $\alpha = j / \gcd(c, j)$, $c \neq 0$. So $(y_\alpha - x_\alpha)\alpha s \geq 2 \gcd(c, j)\alpha s = 2js$. Since $(y_\alpha - x_\alpha)\alpha s = hs + (l_\gamma - l_\gamma) s \geq 2js$, and so $hs \geq 2js + (l_\gamma - l_\gamma) s \geq 2js + (1 - \alpha) s = s + (2s - \beta) j \geq (j + 1) s > |i| s$ as $\beta \leq s$ and $j \geq |i|$, contradicting the fact $h < |i|$. Hence $P_{ij}^x$ and $P_{ij}^y$ have no common skip link.

Case 2: $|i| > j \geq 0$. We show that $P_{ij}^x$ and $P_{ij}^y$ are assigned distinct colours if they share a link. We first assume that they share a ring link, so that either $y - x \leq j - 1$ or $x - y + n \leq j - 1$. Denote by $f(P_{ij}^x)_1$ the first coordinate of $f(P_{ij}^x)$.

Subcase 2.1: $y - x \leq j - 1$. Since $j \leq |s| / 2$, either $|y / s| = |x / s| + 1$ or $|y / s| = |x / s|$. If $|y / s| = |x / s| + 1$, then $y_0 \leq j$ and $x_0 > s - j$ as $y - x \leq j - 1$, which implies that $y$ and $x$ respectively satisfy either (a) and (b), or (d) and (c), or (d) and (e), or (f) and (e) in Step 2, and so $f(P_{ij}^x)_1$ and $f(P_{ij}^y)_1$ differ by at least $i$. Now assume $|y / s| = |x / s|$. If $x_0 \leq s - j$ and $y_0 > s - j$, then $f(P_{ij}^x)_1$ and $f(P_{ij}^y)_1$ differ by at least $i$; otherwise $x_0 + |x / s| \neq y_0 + |y / s| \mod i$ as $y_0 - x_0 \leq j - 1 < |i|$. So $f(P_{ij}^x) \neq f(P_{ij}^y)$.

Subcase 2.2: $x - y + n \leq j - 1$. In this case we have $0 \leq x < j$ and $n - j < y < n$, and so $f(P_{ij}^x)_1$ and $f(P_{ij}^y)_1$ differ by at least $2i$ by (a) and (f) in Step 2.

Now we assume that $P_{ij}^x$ and $P_{ij}^y$ share a skip link, so that either $y = x + hs$ or $x = y + hs - n$, where $0 < h < |i|$. Subcase 2.3: $y = x + hs$. So $y_0 = x_0$ and $|y / s| - |x / s| = h$. If both $x$ and $y$ satisfy the same condition in Step 2 (namely, one of (a)-(f)), then $f(P_{ij}^x) \neq f(P_{ij}^y)$ since $x_0 + |x / s| \neq x_0 + h + |x / s| \equiv y_0 + |y / s| \mod |i|$. If $|q / 2 - 1| s \leq x < |q / 2| s$ and $y > n - j$, then $x_0 = y_0 > s - j (j < |s| / 2)$ and $y \geq (q - 1) s$, and so $\lfloor y / s\rfloor - \lfloor x / s\rfloor \geq |q / 2| > |i| > h$, which contradicts $y = x + hs$. For other ranges of $x$ and $y$, we have $x_0 = y_0$ and so $f(P_{ij}^x) \neq f(P_{ij}^y)$.

Subcase 2.4: $x = y + hs - n$. In this case we have $x < hs$ and $y > n - hs = (q - h) s + r$. Since $h < |i| \leq |q / 2|$, we then have $0 \leq x < [(q - 2) / 2] s$ and $[(q + 2) / 2] s \leq y < n$. Also we have $y_0 = x_0 + r \mod s$ and $|y / s| = \lfloor (x + r) / s\rfloor + q - h$. 55
We have the following: (i) when \( r = 0 \) (which implies \( y_0 = x_0 \)) or \( n - j < y \), \( f(P_{ij}^r) \) and \( f(P_{ij}^y) \) differ by at least \( i \) by Step 2; (ii) when \( x_0 \leq s - j \) and \( y_0 > s - j \), \( f(P_{ij}^r) \equiv x_0 + \lfloor x/s \rfloor \mod |i| \) and \( f(P_{ij}^y) = (x_0 + r) + (\lfloor x/s \rfloor + q - h) - q - r \equiv x_0 + \lfloor x/s \rfloor - h \mod |i| \), and so \( f(P_{ij}^r) \neq f(P_{ij}^y) \); (iii) when \( x_0 > s - j \) and \( y_0 \leq s - j \), we have \( y_0 = x_0 + r - s \) and \( \lfloor y/s \rfloor = \lfloor x/s \rfloor + 1 + q - h \); hence \( f(P_{ij}^r) \equiv i + (x_0 + \lfloor x/s \rfloor) \mod |i| \) and \( f(P_{ij}^y) \equiv i + ((x_0 + r - s) + (\lfloor x/s \rfloor + 1 + q - h) + s - q - r - 1) \mod |i| \equiv i + (x_0 + \lfloor x/s \rfloor - h) \mod |i| \), implying \( f(P_{ij}^r) \neq f(P_{ij}^y) \); (iv) when \( x_0 > s - j \) and \( y_0 > s - j \), or \( x_0 \leq s - j \) and \( y_0 \leq s - j \), \( f(P_{ij}^r) \) and \( f(P_{ij}^y) \) differ by at least \( i \) by Step 2.

In summary, whenever \( P_{ij}^r \) and \( P_{ij}^y \) share a link, they are assigned different colours.

**Case 3:** \( j < 0 \). If \( f(P_{ij}^r) = f(P_{ij}^y) \), then \( f(P_{(i-j)(-i)}^r) = f(P_{(i-j)(-i)}^y) \) by Construction \( \text{xiii} \). Thus, by what we proved in Cases 1 and 2, \( P_{(i-j)(-i)}^r \) and \( P_{(i-j)(-i)}^y \) do not have any common link. Therefore, \( P_{ij}^r \) and \( P_{ij}^y \) have no common link.

So far we have proved that the colouring \( f : \mathcal{R} \to \mathbb{Z}^3 \) is arc-conflict-free.

The number of colours used by \( f \) is \( |f(\mathcal{R})| \). We now estimate this number and thus obtain the required upper bounds for \( \overrightarrow{w}(C_n(1, s), \mathcal{R}) \) by using \( \overrightarrow{w}(C_n(1, s), \mathcal{R}) \leq |f(\mathcal{R})| \).

For fixed \( i \) and \( j \), \( f \) uses \( 4j \) colours if \( j \geq |i|, 2|j| + j \) colours if \( 0 \leq j < |i| \), and no new colours if \( j < 0 \). Note that \( |i| \leq \lfloor q/2 \rfloor \) and \( |j| \leq \lfloor s/2 \rfloor \) as mentioned earlier. Thus, if \( j \geq |i| \) and \( j > \lfloor q/2 \rfloor \), then \( -\lfloor q/2 \rfloor \leq i \leq \lfloor q/2 \rfloor \); and if \( 0 \leq j \leq |i| - 1 \) and \( |i| - 1 > \lfloor s/2 \rfloor \), then \( 0 \leq j \leq \lfloor s/2 \rfloor \). Therefore, by setting \( \gamma_1 := \min\{\lfloor s/2 \rfloor, \lfloor q/2 \rfloor \} \) and \( \gamma_2 := \min\{\lfloor s/2 \rfloor + 1, \lfloor q/2 \rfloor \} \) and noting \( \sum_{i=-k}^{k} 2|i| = \sum_{i=1}^{k} 4i \), we obtain

\[
|f(\mathcal{R})| = \sum_{j=1}^{\gamma_1} \sum_{i=-j}^{j} 4j + \sum_{j=1+\gamma_1}^{\gamma_2} \sum_{i=-\lfloor q/2 \rfloor}^{\lfloor q/2 \rfloor} 4j + \sum_{i=1}^{\gamma_2} \sum_{j=0}^{\lfloor q/2 \rfloor} (4i + 2j) + \sum_{i=1+\gamma_2}^{\gamma_1} \sum_{j=0}^{\lfloor s/2 \rfloor} (4i + 2j).
\] (4.24)

If \( \lfloor s/2 \rfloor \leq \lfloor q/2 \rfloor - 2 \), then (4.24) is equal to

\[
\frac{1}{6} \left[ \frac{s + 2}{2} \right] \left( 3q^2 + 6q + (3q + 10) \right) \left( \frac{s}{2} \right) + 8 \left( \frac{s + 2}{2} \right) + \frac{\epsilon(q) \lfloor s + 2 \rfloor}{6} \left( \frac{s}{2} \right) + \frac{\epsilon(q) \lfloor s/2 \rfloor}{2} + \frac{3}{2}.
\] (4.25)

If \( \lfloor q/2 \rfloor - 1 \leq \lfloor s/2 \rfloor \leq \lfloor q/2 \rfloor \), then (4.24) is equal to

\[
\frac{5q^3 + 12q^2 + 4q}{24} + \frac{1}{3} \frac{s + 2}{2} \left( 4 \left( \frac{s}{2} \right) + 5 \right) + \epsilon(q) \frac{5q^2 + 13q + 7}{8}.
\] (4.26)

If \( \lfloor s/2 \rfloor \geq \lfloor q/2 \rfloor + 1 \), then (4.24) is equal to

\[
\frac{q^3 + 12q^2 + 20q}{24} + (2q+2) \left( \frac{s + 2}{2} \right) + \epsilon(q) \left( \frac{q^2 + 9q + 11}{8} + 2 \left( \frac{s + 2}{2} \right) \right).
\] (4.27)

Recall from (1.7) that \( \kappa(a) = a + (\epsilon(s) + \epsilon(q))/2 \). Thus, for \( s = \sqrt{n - r + (\kappa(a))^2 + \kappa(a)} \), we have \( \lfloor s/2 \rfloor = \lfloor q/2 \rfloor + a \) as \( q = (n - r)/s \). Therefore, by applying \( \lfloor s/2 \rfloor \leq s/2 \)
in (1.25)-(1.27), we obtain upper bounds for $|f(R)|$ which yields the upper bounds in (1.11)-(1.13).

To prove that the upper bounds in (1.10)-(1.12) also apply to $w(C_n(1, s), R)/2$, we modify the definition of $f$ as follows. Define $f$ in the same as in Construction 4.4.1 except that in Step 3 we redefine $f(P_{ij}^z) = -f(P_{(i-j)}^z)$ for $j < 0$. Obviously, $f(P_{ij}^z) \neq f(P_{kl}^y)$ if $k \neq i$ or $l \neq j$. Moreover, when $P_{ij}^z$ and $P_{ij}^y$ share an edge, they share an arc and so are assigned distinct colours by the discussion above. Therefore, this modified colouring $f$ is edge-conflcit-free. Since it uses twice as many colours as in the directed version, the upper bounds in (1.10)-(1.12) are also upper bounds for $w(C_n(1, s), R)/2$. \□
Chapter 5

Recursive cubes of rings

In this chapter, we define recursive cubes of rings on the semidirect products of elementary abelian groups by cyclic groups. This construction facilitates the study of these graphs by using algebraic tools. We give an algorithm for computing shortest paths and obtain the Wiener index, vertex-forwarding and edge-forwarding indices of recursive cubes of rings. We also give bounds for their bisection width and the exact value of their diameter which improve existing results on these parameters. The well-known cube-connected cycles and the cube-of-rings are special recursive cubes of rings, and hence all results obtained in the chapter apply to these networks.

5.1 Introduction

Since the class of Cayley graphs is huge, it is not a surprise that not every Cayley graph has all desired network properties. For instance, the degrees of hypercubes and recursive circulants increase with their orders, and the diameters of low degree circulants are larger than the logarithm of their orders. In order to overcome shortcomings of existing graphs, Cayley graphs with better performance are in demand. Inspired by the work in [31], an interesting family of graphs, called recursive cubes of rings, were proposed as interconnection networks in [101]. A recursive cube of rings is not necessarily a Cayley graph, as shown in [53,109] by counterexamples to [101, Property 4]. Nevertheless, under a natural condition this graph is indeed a Cayley graph as we will see later. In [29] the vertex-disjoint paths problem for recursive cubes of rings was solved by using Hamiltonian circuit Latin squares, and in [101] the recursive construction of them was given. The diameter problem for recursive cubes of rings has attracted
considerable attention: An upper bound was given in [101, Property 5] but shown to be incorrect in [109, Example 6]; and another upper bound was given in [101, Theorem 13] but it was unknown whether it gives the exact value of the diameter. A result in [69] on the diameter of a recursive cube of rings was also shown to be incorrect in [101].

5.1.1 Main results

A comprehensive study of recursive cubes of rings is conducted in this chapter. As mentioned above, a recursive cube of rings as defined in [101, 109] is not necessarily a Cayley graph. We will give a necessary and sufficient condition for this graph to be a Cayley graph (see Theorem 5.2.7). We will see that, under this condition (given in (5.1)), a recursive cube of rings as in [109] can be equivalently defined as a Cayley graph on the semidirect product of an elementary abelian group by a cyclic group (see Definition 5.2.1). We believe that this definition is more convenient for studying various network properties of recursive cubes of rings. For example, from our definition it follows immediately that the cube-connected cycles [90] and cube-of-rings [31] are special recursive cubes of rings.

The above-mentioned condition (see (5.1)) will be assumed from Section 5.3 onwards. In Section 5.3, we give a method for finding a shortest path between any two vertices and a formula for the distance between them in a recursive cube of rings (see Theorems 5.3.2 and 5.3.3). In Section 5.4, we give an exact formula for the diameter of any recursive cube of rings (see Theorem 5.4.1). This result shows that the upper bound for the diameter given in [109] is not tight in general, though it is sharp in a special case. In Section 5.5, we give nearly matching lower and upper bounds on the Wiener index of a recursive cube of rings, expressed in terms of the total distance from a fixed vertex to all other vertices (see Theorems 5.5.2 and 5.5.3). These results will be used in Sections 5.6 and 5.7 to obtain the vertex-forwarding index (see Theorem 5.6.1) and nearly matching lower and upper bounds for the edge-forwarding index (Theorem 5.7.4) of a recursive cube of rings. Another tool for obtaining the latter is the theory [96] of integral uniform flows in orbital-proportional graphs. In Section 5.8, we give nearly matching lower and upper bounds for the bisection width of a recursive cube of rings, which improve the existing upper bounds in [69, 101, 109].

Since the cube-connected cycles [90] and cube-of-rings [31] are special recursive cubes of rings, all results obtained in this chapter are valid for these well known networks. In particular, we recover a couple of existing results for them in a few cases, and obtain new results for them in the rest of the cases. All results here are also valid
for the network $RCR-II(d, r, n - d)$ with $dr \equiv 0 \mod n$ (see the discussion in Section 5.2.2).

Our study here shows that recursive cubes of rings enjoy fixed degree, logarithmic diameter and relatively small forwarding indices in some cases, and flexible choice of order and other invariants when their defining parameters vary. Therefore, they are promising topologies for interconnection networks.

Throughout this chapter we assume that $n$, $d$ and $r$ are positive integers with $n \geq 2$, and $n \geq d$, and $\log a$ is meant $\log_2 a$. From Section 5.3 onwards we assume that $r \geq 3$.

5.2 Definitions and properties

In this section we give our definition of a recursive cube of rings. This network is essentially the network $RCR-II$ defined in [109], which in turn is a modified version of the original recursive cube of rings introduced in [101]. However, unlike [101] and [109], we impose a condition (see (5.1) below) to ensure that the network is a Cayley graph and so has the desired symmetry. Without this condition a recursive cube of rings does not behave nicely – it may not even be regular – as shown in [69, 109]. The treatment in this chapter is different from that in papers [101] and [109]: We define a recursive cube of rings (under condition (5.1)) as a Cayley graph on the semidirect product of an elementary abelian 2-group by a cyclic group. This definition makes the adjacency relation easier to understand and also facilitates subsequent studies of such networks as we will see later.

5.2.1 Recursive cubes of rings

Denote by $e_i$ the row vector of $\mathbb{F}_2^n$ (the $n$-dimensional vector space over the 2-element field $\mathbb{F}_2 = \{0, 1\}$) with the $i$th coordinate 1 and all other coordinates 0, and denote its transpose by $e_i^\top$, $1 \leq i \leq n$. An important convention for our discussion is that the subscripts of these vectors are taken modulo $n$, so that $e_0$ is $e_n$, $e_{n+1}$ is $e_1$, $e_{-2}$ is $e_{n-2}$, and so on. Define

$$M = [e_1^\top, \ldots, e_n^\top, e_1^\top]$$

and treat it as an element of the multiplicative group $GL(n, 2)$ of invertible $n \times n$ matrices over $\mathbb{F}_2$. Then $M^n = I_n$ is the identity element of $GL(n, 2)$ and

$$e_i M^j = e_{i+j}$$

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for any integers $i$ and $j$. It can be verified that, under the condition
\[ dr \equiv 0 \mod n, \quad (5.1) \]
the mapping
\[ \varphi : \mathbb{Z}_r \rightarrow Aut(\mathbb{Z}_n^2) = GL(n,2), x \mapsto \varphi_x \]
defined by
\[ \varphi_x(a) = aM^{dx}, \ a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_n^2, x \in \mathbb{Z}_r \quad (5.2) \]
is a homomorphism from $\mathbb{Z}_r$ to $Aut(\mathbb{Z}_n^2)$. In other words, the rule (5.2) defines an action as a group of the cyclic group $\mathbb{Z}_r$ on the elementary abelian 2-group $\mathbb{Z}_n^2$. (Since $M^n = I_n$, the exponent $dx$ of $M$ can be thought as taken modulo $n$.) In fact, for any integers $x, y$ with $x \equiv y \mod r$, by (5.1) and the fact $M^n = I_n$ we have $M^{dx} = M^{dy}$ and so $a^x$, where $a^x = \varphi_x(a)$ as defined in (5.2), does not rely on the choice of the representative $x \in \mathbb{Z}_r$. Moreover, for $a, b \in \mathbb{Z}_n^2$ and $x, y \in \mathbb{Z}_r$, we have $\varphi_0(a) = a$, $\varphi_y(\varphi_x(a)) = \varphi_y(aM^{dx}) = (aM^{dx})M^{dy} = aM^{dx+y} = \varphi_{x+y}(a)$ and $\varphi_x(a + b) = (a + b)M^{dx} = aM^{dx} + bM^{dx} = \varphi_x(a) + \varphi_x(b)$. Since the operations of $\mathbb{Z}_n^2$ and $\mathbb{Z}_r$ are additions, it follows that indeed (5.2) defines an action of $\mathbb{Z}_r$ on $\mathbb{Z}_n^2$ as a group.

Define
\[ G := \mathbb{Z}_n^2 \rtimes_{\varphi} \mathbb{Z}_r \]
to be the semidirect product of $\mathbb{Z}_n^2$ by $\mathbb{Z}_r$ with respect to the action (5.2). In view of (5.2), the operation of $G$ is given by
\[ (a, x)(b, y) = (a + bM^{dx}, x + y), \]
where the second coordinate $x + y$ is taken modulo $r$. It can be verified that the identity element of $G$ is $(0_n, 0)$ and the inverse of $(a, x)$ in $G$ is $(-aM^{-dx}, r - x)$, where $0_n = (0, 0, \ldots, 0)$ is the identity element of $\mathbb{Z}_n^2$.

**Definition 5.2.1.** Let $r \geq 1$, $n \geq 2$ and $d \geq 1$ be integers such that $n \geq d$ and $dr \equiv 0 \mod n$. Define $Q_n(d, r)$ to be the Cayley graph $\text{Cay}(G, S)$ on $G = \mathbb{Z}_n^2 \rtimes_{\varphi} \mathbb{Z}_r$ with respect to the connection set
\[ S := \{(0_n, 1), (0_n, r - 1), (e_1, 0), (e_2, 0), \ldots, (e_d, 0)\}. \quad (5.3) \]
In other words, $Q_n(d, r)$ has vertex set $G$ such that for any $(a, x) \in G$ the neighbours of $(a, x)$ are:

(a) $(a, x)(e_i, 0) = (a + e_{i+dx}, x), 1 \leq i \leq d;$
(b) \((a, x)(0_n, 1) = (a, x + 1)\) and \((a, x)(0_n, r - 1) = (a, x - 1)\).

We call \(Q_{n}(d, r)\) a recursive cube of rings.

The edge joining \((a, x)\) and \((a + e_i + \delta_x, x)\) is called a cube edge of \(Q_{n}(d, r)\) with direction \(e_i\), and \((a + e_i + d_y, x)\) is called a cube neighbour of \((a, x)\).

The edges joining \((a, x)\) and \((a, x + 1)\), \((a, x - 1)\) are two ring edges of \(Q_{n}(d, r)\), and these two vertices are the ring neighbours of \((a, x)\).

The cycle \((a, 0), (a, 1), \ldots, (a, r - 1), (a, 0)\) of \(Q_{n}(d, r)\) with length \(r\) is called the \(a\)-ring of \(Q_{n}(d, r)\).

Since \(dr\) is a multiple of \(n\) by our assumption, whenever \(x \equiv y \mod r\) we have \((a + e_i + d_x, x) = (a + e_i + d_y, y)\), and so \(Q_{n}(d, r)\) is well-defined as an undirected graph. We may think of \(Q_{n}(d, r)\) as obtained from the \(n\)-dimensional cube \(Q_n\) (with vertex set \(\mathbb{Z}_n^2\)) by replacing each vertex \(a\) by the corresponding \(a\)-ring and then adding cube edges by using rule (a) in Definition 5.2.1. See Figure 5.1 for an illustration.

The next lemma shows that recursive cubes of rings are common generalizations of three well-known families of interconnection networks, namely, hypercubes, cube-connected cycles \(CC_n\) and cube-of-rings \(COR(d, r)\) \cite{31}. \(CC_n\) can be defined as the Cayley graph on \(\mathbb{Z}_2^n \times \mathbb{Z}_n\) such that \((a, x)\) is adjacent to \((b, y)\) if and only if either \(a = b\) and \(x \equiv y \pm 1 \mod n\), or \(b = a + e_{i + x}\) and \(x \equiv y \mod n\) (see e.g. \cite{35}). \(COR(d, r)\) can be defined \cite{31, Lemma 2} as the Cayley graph on the semidirect product of \(\mathbb{Z}_2^d\) by \(\mathbb{Z}_r\) with operation given by \((a, x)(b, y) = (aM^{dy} + b, x + y)\), with respect to the connection set \(\{(0_{dr}, 1), (0_{dr}, r - 1), (e_1, 0), (e_2, 0), \ldots, (e_d, 0)\}\).

Lemma 5.2.2. The following hold:

\[ Q_n \cong Q_n(n, 1), \quad CC_n \cong Q_n(1, n), \quad COR(d, r) \cong Q_{dr}(d, r). \]
In other words, hypercubes, cube-connected cycles and cubes-of-rings are special recursive cubes of rings.

Proof. When \( r = 1 \), we have \( \mathbb{Z}_2^n \times_\varphi \mathbb{Z}_1 \cong \mathbb{Z}_2^n \) and \( Q_n \cong Q_n(n, 1) \) by the definitions of the two graphs. By the discussions above, \( CC_n \) is the Cayley graph on \( \mathbb{Z}_2^n \times_\varphi \mathbb{Z}_n \) with respect to the connection set \( \{(0_n, 1), (0_n, n - 1), (e_1, 0)\} \); hence, \( CC_n \) is isomorphic to \( Q_n(1, n) \). Similarly, \( COR(d, r) \) is isomorphic to \( Q_{dr}(d, r) \). In fact, the permutation of \( \mathbb{Z}_2^n \times \mathbb{Z}_r \) defined by \( (a, x) \mapsto (a, x)^{-1} = (-aM^{-dx}, r - x) \) is an isomorphism from \( COR(d, r) \) to \( Q_{dr}(d, r) \).

Since hypercubes have been well studied, we will not consider them anymore. Also, we will not consider the less interesting case where \( r = 2 \), for which the neighbours \((a, x + 1)\) and \((a, x - 1)\) of \((a, x)\) are identical and the ring edges \(\{(a, x), (a, x + 1)\}\) and \(\{(a, x), (a, x - 1)\}\) are parallel edges. We assume \( r \geq 3 \) in the rest of this chapter.

The following observation follows from the definition of \( Q_n(d, r) \) immediately.

**Lemma 5.2.3.** Suppose \( r \geq 3 \). Then \( Q_n(d, r) \) is a connected \((d + 2)\)-regular graph with \(2^nr\) vertices and \(2^{n-1}r(d + 2)\) edges.

Proof. Only the connectedness requires justification. Since \( dr \equiv 0 \mod n \), we may assume \( dr = tn \) for some integer \( t \). Since \( i + dx \) runs over all integers from 1 to \( tn \) when \( i \) is running from 1 to \( d \) and \( x \) from 0 to \( r - 1 \), the set \( S \) given in (5.4) is a generating set of \( \mathbb{Z}_2^n \times_\varphi \mathbb{Z}_r \). Hence \( Q_n(d, r) \) is connected.

It is worth mentioning that in general \( Q_n(d, r) \) may not be edge-transitive as \( CC_n \) is not edge-transitive [12].

Denote
\[
D(x) := \{i + dx \mod n : 1 \leq i \leq d\}, \ x \in \mathbb{Z}_r. \tag{5.4}
\]

**Lemma 5.2.4.** For any fixed \( a \in \mathbb{Z}_2^n \) and \( j \) with \( 1 \leq j \leq n \), there are exactly \( dr/n \) distinct cube edges of \( Q_n(d, r) \) with direction \( e_j \) that are incident to some vertices of the \( a \)-ring, namely the edges joining \((a, x_1)\) and \((a + e_j, x_1)\), where \( x_1 = \lfloor (j + ln - 1)/d \rfloor \), \( 0 \leq l < dr/n \).

Proof. The cube neighbours of \((a, x)\) are precisely those \((a + e_j, x)\) such that \( j \in D(x) \). By (5.4) we have \( dr = tn \) for some positive integer \( t \). Since \( 1 \leq j \leq n \) and \( \{i + dx : 1 \leq i \leq d, 0 \leq x < r\} = \bigcup_{l=0}^{r-1} \{ln + 1, \ldots, (l + 1)n\} \) is the set of integers from 1 to \( tn \), there are exactly \( t \) distinct pairs \((i, x)\) such that \( j = i + dx \mod n \), namely \((i_l, x_l)\) defined by \( x_l = \lfloor (j + ln - 1)/d \rfloor \) and \( i_l = j + ln - dx_l, 0 \leq l < dr/n \). From this and (5.3) the result follows. 

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In the special case when \( r = n \), by Lemma 5.2.4, there are exactly \( d \) cube edges in each direction incident to any given \( a \)-ring in \( Q_n(d, n) \). Thus \( Q_n(d, n) \) can be thought of as a generalization of cube-connected cycles; we call it the \( d \)-ply cube-connected cycles of dimension \( n \).

### 5.2.2 A larger family of networks

We now justify that, under condition (6.1), \( Q_n(d, r) \) is isomorphic to a recursive cube of rings in the sense of [109], and vice versa. In [101], a recursive cube of rings was defined to have vertex set \( \mathbb{Z}_2^d \times \mathbb{Z}_r \) such that \((a, x)\) is adjacent to \((b, y)\) if and only if either \( a = b \) and \( x \equiv y \pm 1 \mod r \), or \( x \equiv y \mod r \) and \( b = a + e_j \), where \( j = n - x(n - d) - i \) if \( n \geq i + x(n - d) \) and \( j = i - dx \mod n \) otherwise for some \( 1 \leq i \leq d \). It was claimed in [101] that this is a Cayley graph. However, as shown in [59, 109], in general this graph may not even be regular and so not even be vertex-transitive without condition (6.1). A modified definition of a recursive cube of rings was given in [110]. We now restate this definition using a different language. \( Q_n^- (d, r) \) below is defined to be the graph \( RCR-II(d, r, n - d) \) in [110].

**Definition 5.2.5.** Let \( n \geq 2, d \geq 1 \) and \( r \geq 1 \) be integers. Define \( Q_n^- (d, r) \) to be the graph with vertex set \( \mathbb{Z}_2^d \times \mathbb{Z}_r \) such that \((a, x)\) and \((b, y)\) are adjacent if and only if either \( a = b \) and \( x \equiv y \pm 1 \mod r \), or \( x \equiv y \mod r \) and \( b = a + e_i \) for some \( i \) with \( 1 \leq i \leq d \).

We call this graph a *general recursive cube of rings* and the edge between \((a, x)\) and \((a + e_i, x)\) a ‘cube edge’ with ‘direction’ \( e_i \). It is known that \( Q_n^- (d, r) \) is connected if and only if \( dr \geq n \) [109, Theorem 3]. Note that (5.1) is not required in the definition of \( Q_n^- (d, r) \). In [110, Theorem 9] it was shown that, if (5.1) is satisfied, then \( Q_n^- (d, r) \) is vertex-transitive. The next lemma asserts that under this condition \( Q_n^- (d, r) \) is isomorphic to \( Q_n(d, r) \) and hence is actually a Cayley graph.

**Lemma 5.2.6.** If \( dr \equiv 0 \mod n \), then \( Q_n^- (d, r) \cong Q_n(d, r) \).

**Proof.** Since \( dr \equiv 0 \mod n \), similar to (5.1) the rule \( \theta_x(a) = a M^{-dx} \), \( a \in \mathbb{Z}_2^n \), \( x \in \mathbb{Z}_r \), defines an action of \( \mathbb{Z}_r \) on \( \mathbb{Z}_2^n \). The operation of the corresponding semidirect product of \( \mathbb{Z}_2^n \) by \( \mathbb{Z}_r \) is given by \((a, x)(b, y) = (a + b M^{-dx}, x + y)\). It can be verified that the Cayley graph on this semidirect product with respect to the same connection set \( S \) as in (5.1) is exactly \( Q_n^- (d, r) \). Moreover, the permutation of the set \( \mathbb{Z}_2^n \times \mathbb{Z}_r \) defined by \((a, x) \mapsto (a, r - x)\) is an isomorphism from \( Q_n^- (d, r) \) to \( Q_n(d, r) \). \( \square \)
The next result shows that, if \( r \geq 3 \) and \( n \geq 2d \), then condition (5.1) is necessary and sufficient for \( Q_d^{-}(d, r) \) to be a Cayley graph. Therefore, all results in the rest of this chapter are about \( RCR-I\!I(d, r, n - d) \) with \( dr \equiv 0 \mod n \).

**Theorem 5.2.7.** Let \( r \geq 3 \) and \( n \geq 2d \). Then \( Q_d^{-}(d, r) \) is a connected Cayley graph if and only if \( dr \equiv 0 \mod n \).

**Proof.** In a graph two vertices are \( t \)-neighbours of each other if the distance between them is equal to \( t \). Obviously, in a vertex-transitive graph any two vertices have the same number of \( t \)-neighbours for any integer \( t \geq 1 \).

If \( dr \equiv 0 \mod n \), then \( Q_d^{-}(d, r) \cong Q_n(d, r) \) by Lemma 5.2.6. So \( Q_d^{-}(d, r) \) is a connected Cayley graph by Lemmas 5.2.8 and 5.2.10.

In the rest of the proof we assume that \( Q_d^{-}(d, r) \) is connected but \( dr \not\equiv 0 \mod n \). We will prove that \( Q_d^{-}(d, r) \) is not vertex-transitive and hence not a Cayley graph. We achieve this by showing that the numbers of \( t \)-neighbours of \((0_n, r - 1)\) and \((0_n, r')\) in \( Q_d^{-}(d, r) \) are distinct for some \( t \) to be defined later, where

\[
r' = \lfloor (r - 1)/2 \rfloor.
\]

Note that \( r' \geq 1 \) as \( r \geq 3 \). Set

\[
D_x^- := D(-x) = \{ i - dx \mod n : 1 \leq i \leq d \}, \ x \in \mathbb{Z}_r,
\]

with the understanding that the elements of \( F_x^- \) are between 1 and \( n \). Obviously, all these sets have size \( d \). Note that \((a, x)\) and \((a + e_j, x)\) are adjacent in \( Q_d^{-}(d, r) \) if and only if \( j \in D_x^- \). We partition the set of \( t \)-neighbours \((a_t, x_t)\) \((0 \leq x_t \leq r - 1)\) of \((0_n, r - 1)\) into \( N_{t-2}, N_{t-1} \) and \( N_t \), according to whether \( \min\{r - x_t - 1, x_t + 1\} \) is at most \( t - 2 \), exactly \( t - 1 \), and exactly \( t \), respectively. Similarly, we partition the set of \( t \)-neighbours \((b_t, y_t)\) \((0 \leq y_t \leq r - 1)\) of \((0_n, r')\) into \( N_{t-2}', N_{t-1}', N_t' \) according to whether \( \min\{r' - y_t - 1, y_t + 1\} \) is at most \( t - 2 \), exactly \( t - 1 \), and exactly \( t \), respectively. We will prove that \( |N_{t-2}| \leq |N_{t-2}'|, |N_{t-1}| < |N_{t-1}'| \) and \( |N_t| = |N_t'| \), and therefore \((0_n, r - 1)\) and \((0_n, r')\) have different numbers of \( t \)-neighbours as required.

Since \( dr \not\equiv 0 \mod n \), we can write \( dr = an - q \), where \( a \) and \( q \) are integers with \( 1 \leq q \leq n - 1 \) and \( a \geq 2 \). So

\[
D_{r-1}^- = q + D_{r-1}^- := \{ q + d + i \mod n : 1 \leq i \leq d \}.
\]

Note that the sets \( D_i^- \) and \( D_{i+1}^- \) have consecutive elements (modulo \( n \)) and \( |D_{r-1}^- \cap D_t^-| = |D_r^- \cap D_{r+i}^-| \), for every \( 0 \leq i < r' \).

**Claim 1:** There exists \( k \) with \( 0 \leq k \leq r' - 1 \) such that \( |D_r^- \cup D_{r+k+1}^-| \neq |D_{r-1}^- \cup D_k^-| \).
Proof. First, we show that $D_{r-1}^{-1} \cap D_k^{-} \neq \emptyset$ for some $0 \leq k < r'$. In fact, if $q > n - 2d$, then $D_{r-1}^{-1} \cap D_0^{-} \neq \emptyset$ since $n - d < q + d < n + d$ implies $q + d + i \mod n \in D_0^{-}$ for some $1 \leq i \leq d$. Assume $1 \leq q \leq n - 2d$. Then $dr = an - q \geq an - (n - 2d) = (a-1)n + 2d \geq (a-1)(2d+1) + 2d = 2ad + (a-1)$, that is, $r \geq 2a + (a-1)/d$. Since $a \geq 2$, it follows that $r \geq 5$ and $r' \geq 2$. Let $\epsilon = r - 1 - 2r'$. Then $dr' = (dr - (\epsilon+1)d)/2$, and $\epsilon = 0, 1$ depending on whether $r$ is odd or even. Note that $\cup_{x=0}^{r'-1} D_x^{-} = \{1, \ldots, d\} \cup \{n, n-1, \ldots, 1-d(r'-1) \mod n\}$. Thus, if $D_{r-1}^{-1} \cap (\cup_{x=0}^{r'-1} D_x^{-}) = \emptyset$, then the elements of $D_{r-1}^{-1}$ are between $d$ and $1 - d(r'-1) \mod n$, that is, $d < q + d + i < n + 1 - d(r'-1) = (2-a)n/2 + d + 1 + (q + (1+\epsilon)d)/2$ for $1 \leq i \leq d$. In particular, when $i = d$, this inequality yields $(q + d)/2 < (2-a)n/2 + ed/2 + 1$. However, this is impossible, because when $r$ is even and $a = 2$, we have $q = an - dr$ is even and so $q \geq 2$, and in the remaining case we have $\epsilon = 0$ or $a > 2$. This contradiction shows that there exists some $0 \leq k \leq r' - 1$ such that $D_{r-1}^{-1} \cap D_k^{-} \neq \emptyset$.

Now we show that for this particular $k$ we have $|D_{r-1}^{-1} \cap D_k^{-} | \neq |D_{r-1}^{-1} \cap D_k' |$ or $|D_{r-1}^{-1} \cap D_k^{-1} | \neq |D_{r-1}^{-1} \cap D_k'^{-1} |$. If $k = 0$, then $|D_{r-1}^{-1} \cap D_0^{-} | \neq |D_{r-1}^{-1} \cap D_0'^{-} |$ since $D_{r-1}^{-1} \cap D_0'^{-} = \emptyset$. Assume that $1 \leq k \leq r' - 1$ and $|D_{r-1}^{-1} \cap D_k'^{-} | = |D_{r-1}^{-1} \cap D_k^{-} | = s$ for some $s$ with $0 < s < d$ (note that $s < d$ since $D_{r-1}^{-1} \neq D_{r-1}^{-}$). Since $D_k'^{-} \cup D_k'^{-1} = \{1 - dk, \ldots, 2d - dk \mod n\}$ and $2d \leq n$, we have $|D_k'^{-} \cup D_k'^{-1} | = 2d$. So one of $D_{r-1}^{-1}$ and $D_{r-1}^{-1}$ has $d - s$ elements in $D_{r-1}^{-1}$, and the other has $d - s$ elements in $D_{r-1}^{-1}$. Since $D_{r-1}^{-1} = g + D_{r-1}^{-}$ for $1 \leq q < n$, it follows that $|D_{r-1}^{-1} \cap D_k'^{-1} | \neq |D_{r-1}^{-1} \cap D_k'^{-} |$. Since $|D_x|^ = d$ and $|D_x'^{-} \cup D_{x'}^{-} | = |D_x'^{-} \cup D_{x'}^{-}\cup D_{x'}^{-} |$ for any $x, x', j \in \mathbb{Z}_r$, we have $|D_r'^{-} \cup D_{r+k+1}'| = |D_{r-1}^{-1} \cup D_k'^{-} | \neq |D_{r-1}^{-1} \cup D_k'^{-} |$. \hfill \Box

Let $k$ be the smallest integer satisfying the conditions in Claim 1 and set $t = k + 2$. Then $2 \leq t \leq r' + 1$ by Claim 1.

Claim 2: We have $d(k+1) < n$ and $|D_{r-1}^{-1} \cup (\cup_{i=0}^{k+1} D_{r-1}^{-}) | < |\cup_{i=0}^{k+1} D_{r-1}^{-} |$.

Proof. Suppose $d(k+1) \geq n$. Let $j$ be the largest integer such that $d(j+1) < n$. Since $1 \leq j < k$, by Claim 1 and the minimality of $k$, $|D_{r-1}^{-1} \cup D_{r-1}^{-1} | = |D_{r'}^{-} \cup D_{r'+1}^{-} | = |D_{r-1}^{-1} \cup D_{r-1}^{-1} |$ for every $1 \leq i \leq j + 1$. In particular, $D_{r-1}^{-1} \cup D_{j}^{-} = D_{r-1}^{-} \cup D_{j}^{-}$. Clearly, $D_{r-1}^{-1} \cap D_{r-1}^{-1} = \emptyset$ for $1 \leq i \leq j$. Denote $s = |D_{r-1}^{-1} \cap D_{r-1}^{-}$. Then $0 < s < d$, $D_{r-1}^{-1} \cap D_{r-1}^{-} = \emptyset$ for $1 \leq i \leq j$, and $|D_{r-1}^{-1} \cap D_{r-1}^{-} | = s$. Note that $\cup_{i=1}^{d} D_{r-1}^{-} = \{l+1, \ldots, l+jd \mod n\}$ and $D_{j}^{-} = \{l-d+1, \ldots, l \mod n\}$ for some integer $l$. Since $D_{r-1}^{-1} \cap D_{j}^{-} = D_{r-1}^{-1} \cap D_{j}^{-} = \{l - d + 1, \ldots, l - d + s \mod n\}$, we have $|D_{r-1}^{-1} \cup D_{r-1}^{-} | = |\cup_{i=0}^{k+1} D_{r-1}^{-} | = (k+2) - n \leq d(k+2) - n$. Therefore, $d(k+1) < n$.

Consider the case $n < d(k+2)$ first. In this case, $(\cup_{i=1}^{k} D_{r-1}^{-} \cup D_{r-1}^{-1}) \cap (D_{r-1}^{-} \cup D_k'^{-} ) = \emptyset$, and $(\cup_{i=1}^{k} D_{r-1}^{-} \cup D_{r-1}^{-1}) \cap D_{r-1}^{-1} = \emptyset$ by Claim 1 and the choice of $k$. So $|D_{r-1}^{-1} \cup D_k'^{-} | = d(k+2) - n \leq d(k+2) - n$. 67
We prove this by showing the existence of an injective mapping from \(N_{t-2}\) to \(N'_{t-2}\). In fact, for any \((a_t, x_t) \in N_{t-2}\), we choose a shortest path \(P(a_t, x_t) : (0_n, r - 1), (a_1, x_1), \ldots, (a_t, x_t)\) such that \((a_{j+1}, x_{j+1}) = (a_j, x_j + 1)\) or \((a_{j+1}, x_{j+1}) = (a_j + e_{i_j}, x_j)\) and \(i_j \in D_{x_j}, 0 \leq j < t\). (Hence the directions of all cube edges on \(P(a_t, x_t)\) are in \(\cup_{i=1}^{t-1} D_{x_i}\).) We have \(a_t \neq 0_n\), for otherwise the distance between \((a_t, x_t)\) and \((0_n, r - 1)\) would be \(t - 2\), a contradiction. We construct a path from \((0_n, r')\) to \((b_t, y_t)\) as follows. First, set \(y_i := x_i - r + 1 + r' \mod r\) with \(0 \leq y_i < r\), for \(1 \leq i \leq t\). Since \(\max\{|x_i - x_j|, r - |x_i - x_j|\} \leq t - 2\), we have \(\max\{|y_i - y_j|\} \leq t - 2\), where the maximum is taken over all \((i, j)\) such that \((a_i, x_i), (a_j, x_j) \in P(a_t, x_t)\). This together with \(d(t-1) < n\) (Claim 2) implies that \(D_{y_1}, D_{y_2}, \ldots, D_{y_t}\) are mutually disjoint, and so \(|\cup_{i=0}^{t-1} D_{y_i}| = |D_{y_i}| = td\). Hence we can choose an injective mapping \(g : \cup_{i=0}^{t-1} D_{x_i} \to \cup_{i=0}^{t-1} D_{y_i}\) such that \(g(p) \in D_{y_i}\) for \(p \in D_{x_i}\). Set \(a_0 = b_0 = 0_n\). For \(0 \leq i < t\), set \(b_{i+1} = b_i\) if \(a_{i+1} = a_i\), and \(b_{i+1} = b_i + e_j\) if \(a_{i+1} = a_i + e_j\). In this way we construct a path \(P'(a_t, x_t) : (0_n, r'), (b_1, y_1), \ldots, (b_t, y_t)\). We now prove that \(P'(a_t, x_t)\) is a shortest path. That is, any path \(P : (0_n, r'), (b'_1, y'_1), \ldots, (b'_t, y'_t) = (b_t, y_t)\) in \(Q_n(d, r)\) has length at least \(t\). In fact, since \(b_t \neq 0_n\), \(P\) contains at least one cube edge and there exist \(i, j\) and \(l\) such that the \(i\)th coordinate of \(b_l\) is 1, and \(i \in D_{y_i} \cap D_{y_j}\), but \(y'_i \neq y'_j\). Since the number of ring edges on \(P\) is at least \(|y'_i - y'_j|\), it suffices to prove that \(|y'_i - y'_j| + |y_i - y_j| \geq t - 1\).

In fact, since \(d(t-1) < n\) and \(D_{r+i} \cap D_{r+j} = \emptyset\) for every pair \((i, j)\) with \(|j' - i'| \leq t - 2\), we have \(|y'_i - y_j| \geq t - 1\). Since \(b_t \neq 0_n\), we have

\[
|y_j - r'| + |y_i - y_j| \leq t - 1, \quad \text{for } 0 \leq j \leq t.
\]  

(5.6)

If \(r' \leq y_t \leq y_j\) and \(y'_i \leq y_j - (t - 1)\), then \(r' + y_t \geq 2y_j - (t - 1)\) by (5.6) and so \(|y'_i - r'| + |y_i - y'_i| = r' - y_i + y_i - y'_i \geq 2y_j - (t - 1) - 2y'_i \geq t - 1\). If \(y_t \leq r' \leq y_t\) and \(y'_i \geq y_j + (t - 1)\), then \(-r' - y_t \geq -2y_j - (t - 1)\) by (5.6) and so \(|y'_i - r'| + |y_t - y'_i| = y'_i - r' + y'_i - y_t \geq 2r' - 2y_j - (t - 1) \geq 2(t - 1) - (t - 1) = t - 1\). Similarly, \(|y'_i - r'| + |y_t - y'_i| \geq t - 1\) in all other cases.
So far we have proved that \( P'(a_t, x_t) \) is a shortest path in \( N_{t-2}' \). Since \( g \) is an injection, different \((a_t, x_t) \in N_{t-2}\) gives rise to different \((b_t, y_t) \in N_{t-2}'\).

Claim 4: \(|N_{t-1}| < |N_{t-1}'|\).

**Proof.** We claim that, for \( k \in D_{r-1}^{-} \cup (\cup_{i=1}^{t-1} D_{r-1}^{-}) \) and \( l \in \cup_{j=1}^{t} D_{r-j}^{-}, (e_k, t-2) \) and \((e_l, r - t + 1)\) are in \( N_{t-1} \). In fact, for every \( 0 \leq x \leq t - 1 \), the paths \((0_n, r-1), (0_n, 0), \ldots , (0_n, r-1+x), (e_k, r-1+x), (e_k, r+x), \ldots , (e_k, t-2) \) and \((0_n, r-1), (0_n, r-2), \ldots , (0_n, r-x-1), (e_l, r-x-1), (e_l, r-x-2), \ldots , (e_l, r-t)\) are shortest paths. Similarly, for \( k' \in \cup_{j=0}^{t-1} D_{r+j}^{-}, l' \in \cup_{j=0}^{t-1} D_{r-j}^{-}, (e_{k'}, r'+t-1) \) and \((e_{l'}, r'-t+1)\) are in \( N_{t-1}' \), because for every \( 0 \leq y \leq t - 1 \) the paths \((0_n, r'), (0_n, r'+1), \ldots , (0_n, r+y), (e_{k'}, r'+y), (e_{k'}, r'+y+1), \ldots , (e_{k'}, r'+t-1) \) and \((0_n, r'), (0_n, r'-1), \ldots , (0_n, r'-y), (e_{l'}, r'-y), (e_{l'}, r'-y-1), \ldots , (e_{l'}, r'-t+1)\) are shortest paths. Since \(|\cup_{j=1}^{t} D_{r-j}^{-}| = |\cup_{j=0}^{t-1} D_{r-j}^{-}|\), the number of \( t \)-neighbours \((e_t, r-t)\) of \((0_n, r-1)\) is equal to the number of \( t \)-neighbours \((e_{t'}, r'-t+1)\) of \((0_n, r')\). However, by Claim 2, the number \(|D_{r-1}^{-} \cup (\cup_{i=1}^{t-1} D_{r-1}^{-})|\) of \( t \)-neighbours \((e_k, t-2)\) of \((0_n, r-1)\) is strictly less than the number \(|\cup_{j=0}^{t-1} D_{r+j}^{-}\)| of \( t \)-neighbours \((e_{k'}, r'+t-1)\) of \((0_n, r')\).

Claim 5: \(|N_t| = |N_t'|\).

**Proof.** Clearly, \( N_t \) and \( N_t' \) consist of the vertices on the \( 0_n \)-ring which are at distance \( t \) from \((0_n, r-1)\) and \((0_n, r')\), respectively. The claim follows immediately.

Combining Claims 3-5, the number of \( t \)-neighbours of \((0_n, r-1)\) is strictly less than that of \((0_n, r')\). Therefore, \( Q_n^{-}(d, r) \) is not vertex-transitive and so is not a Cayley graph.

5.3 Shortest paths in \( Q_n(d, r) \)

Since \( Q_n(d, r) \) is a Cayley graph, for any \((a, x), (b, y) \in G\), if

\[
P_{(a, x)} : (0_n, 0), (a_1, x_1), \ldots , (a_t, x_t) = (a, x)
\]

is a path from \((0_n, 0)\) to \((a, x)\), then

\[
(b, y)P_{(a, x)} : (b, y), (b, y)(a_1, x_1), \ldots , (b, y)(a_t, x_t)
\]

is a path from \((b, y)\) to \((b, y)(a, x)\). Moreover, the former is a shortest path if and only if the latter is a shortest path. Therefore, to find a shortest path between any two vertices, it suffices to find a shortest path from \((0_n, 0)\) to any \((a, x) \in G\). This is what we are going to do in this section.
Segments of a path from \((0_n, 0)\) to \((a, x)\)

Figure 5.2: Segments of a path from \((0_n, 0)\) to \((a, x)\)

Suppose that \(P\) is a path in \(Q_n(d, r)\) from \((0_n, 0)\) to \((a, x)\) with \(s\) cube edges. Removing these \(s\) cube edges from \(P\) results in \(s + 1\) subpaths, each of which is a path in a ring and is called a *segment*. Such a segment may contain only one vertex, and this happens if and only if this vertex is incident to two cube edges or it is \((0_n, 0)\) or \((a, x)\) and incident to a cube edge on \(P\). The first segment must be on the \(0_n\)-ring, say from \((0_n, 0)\) to \((0_n, x_1)\) for some \(x_1 \in \mathbb{Z}_r\). If the cube edge on \(P\) incident to \((0_n, x_1)\) is in direction \(e_{i_1}\), then the second segment must be on the \(e_{i_1}\)-ring from \((e_{i_1}, x_1)\) to, say, \((e_{i_1}, x_2)\). In general, for \(1 \leq t \leq s + 1\), we may assume that the \(t\)th segment is on the \((e_{i_1} + \cdots + e_{i_{t-1}})\)-ring connecting \((e_{i_1} + \cdots + e_{i_{t-1}}, x_{t-1})\) and \((e_{i_1} + \cdots + e_{i_{t-1}}, x_t)\) for some \(i_1, \ldots, i_{t-1} \in \{1, \ldots, n\}\) and \(x_{t-1}, x_t \in \mathbb{Z}_r\), where \(e_{i_0}\) is interpreted as \(0_n\) and \(x_0 = 0\). This implies that, for \(1 \leq t \leq s\), the \(t\)th cube edge on \(P\) is in direction \(e_{i_t}\) and it connects \((e_{i_1} + \cdots + e_{i_{t-1}}, x_{t-1})\) and \((e_{i_1} + \cdots + e_{i_{t-1}} + e_{i_t}, x_t)\) (see Figure 5.2).

By the definition of \(Q_n(d, r)\), we have \(i_t \in D(x_t)\) for \(1 \leq t \leq s\). So every path \(P\) in \(Q_n(d, r)\) from \((0_n, 0)\) to \((a, x)\) determines two tuples, namely, \((x_0, x_1, \ldots, x_s, x_{s+1})\) and \((i_1, \ldots, i_s)\), where \(x_0 = 0, x_{s+1} = x\) and \(e_{i_1} + \cdots + e_{i_s} = a\). Conversely, any two tuples

\[ \hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1}), \hat{i} = (i_1, \ldots, i_s), \]  

(5.7)

such that \(i_t \in D(x_t)\) for each \(t\), \(x_0 = 0, x_{s+1} = x\) and \(e_{i_1} + \cdots + e_{i_s} = a\), give rise to \(2^{s+1}\) paths in \(Q_n(d, r)\) from \((0_n, 0)\) to \((a, x)\) with \(s\) cube edges and \(s + 1\) segments, because the \(t\)th segment can be one of the two paths from \((e_{i_1} + \cdots + e_{i_{t-1}}, x_{t-1})\) to \((e_{i_1} + \cdots + e_{i_{t-1}}, x_t)\) on the \((e_{i_1} + \cdots + e_{i_{t-1}})\)-ring. If we choose the shorter of these two paths for every \(t\), then we get a path from \((0_n, 0)\) to \((a, x)\) with shortest length among all these \(2^{s+1}\) paths, and this shortest length is \(s + l(\hat{x})\) (which is independent of \(\hat{i}\)), where we define

\[ l(\hat{x}) := \sum_{t=1}^{s+1} \min\{|x_t - x_{t-1}|, r - |x_t - x_{t-1}|\}. \]  

(5.8)

If a path contains two cube edges with the same direction \(e_i\), then it has a subpath
of the form \((b, y_0), (b + e_i, y_0), \ldots, (b + e_i, y_1), (b + e_i + e_{j_1}, y_1), \ldots, (b + e_i + e_{j_1} + \cdots + e_{j_k}, y_k)\), and by replacing this subpath with \((b, y_0), \ldots, (b, y_1), (b + e_{j_1}, y_1), \ldots, (b + e_{j_k} + \cdots + e_{j_k}, y_k)\) we obtain a shorter path with the same end-vertices. Therefore, any shortest path from \((0_n, 0)\) to \((a, x)\) contains exactly one cube edge in direction \(e_i\) if \(a_i = 1\) and no cube edge in direction \(e_i\) if \(a_i = 0\). Thus the number of cube edges in any shortest path from \((0_n, 0)\) to \((a, x)\) is equal to \(\|a\|\), where

\[
\|a\| := \sum_{i=1}^{n} a_i
\]  

(5.9)

is the Hamming weight of \(a\).

Define an \((a, x)\)-sequence to be a tuple \(\hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1})\) with \(x_t \in \mathbb{Z}_r\) for each \(t\) such that \(x_0 = 0\), \(x_{s+1} = x\), \(s = \|a\|\), and for every \(i\) with \(a_i = 1\) there is a unique \(t\) with \(i \in D(x_t)\). Denote

\[
\|a\| := \min_{\hat{x}} \ell(\hat{x}),
\]  

(5.10)

with the minimum running over all \((a, x)\)-sequences \(\hat{x}\). An \((a, x)\)-sequence achieving the minimum in (5.10) is said to be optimal. Denote by \(\text{dist}((0_n, 0), (a, x))\) the distance between \((0_n, 0)\) and \((a, x)\) in \(Q_n(d, r)\). The discussion above implies the following results.

**Lemma 5.3.1.**

(a) Any \((a, x)\)-sequence \(\hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1})\) and any \(\hat{i} = (i_1, \ldots, i_s)\) such that \(a_{i_t} = 1\) and \(i_t \in D(x_t)\) for \(1 \leq t \leq s\) and \(s = \|a\|\), give rise to \(2^{s+1}\) paths in \(Q_n(d, r)\) from \((0_n, 0)\) to \((a, x)\).

(b) The minimum length among the paths obtained from \(\hat{x}\) and \(\hat{i}\) is equal to \(\|a\| + \ell(\hat{x})\).

(c) \(\text{dist}((0_n, 0), (a, x)) = \|a\| + \ell(\hat{x})\).

In the rest of this section, we give a method for finding optimal \((a, x)\)-sequences (or equivalently shortest paths from \((0_n, 0)\) to \((a, x)\)). We need to handle the cases \(dr = n\) and \(dr \geq 2n\) separately because for each \(i\), by Lemma 5.3.1, \(x \in \mathbb{Z}_r\) with \(i \in D(x)\) is unique in the former but not in the latter. Algorithm 5.3 gives a shortest path in \(Q_n(d, r)\) from \((0_n, 0)\) to a given vertex \((a, x)\) provided that we have an optimal \((a, x)\)-sequence. Algorithms 5.3 and 6.3 give optimal \((a, x)\)-sequences for cases \(dr = n\) and \(dr \geq 2n\), respectively.
Algorithm 5.1 A shortest path from \((0_n, 0)\) to \((a, x)\)

1: Input: \((a, x)\) and an optimal \((a, x)\)-sequence \(\hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1})\), where \(s = \|a\|\);
2: Output: a shortest path from \((0_n, 0)\) to \((a, x)\);
3: Choose \(i = (i_1, i_2, \ldots, i_s)\) such that \(i_t \in D(x_t)\) for \(1 \leq t \leq s\), and \(e_{i_1} + \cdots + e_{i_s} = a\);
4: for \(t = 1\) to \(s + 1\) do
5: \hspace{1cm} Let \(P_t\) be the shorter path on \((e_{i_1} + \cdots + e_{i_t-1}, x_{t-1})\) and \((e_{i_t} + \cdots + e_{i_{t+1}}, x_t)\);
6: \hspace{1cm} The path \(P_1, e_{i_1}, P_2, e_{i_2}, \ldots, P_s, e_{i_s}, P_{s+1}\) is a shortest path from \((0_n, 0)\) to \((a, x)\).

5.3.1 Case \(dr = n\)

In this case, by Lemma 5.2.3, for every \(1 \leq i \leq n\) there is a unique \(y \in \mathbb{Z}_r\) such that \(i \in D(y)\). Hence any \((a, x)\)-sequence can be obtained from any other \((a, x)\)-sequence by permuting entries (while fixing the first and last entries). So a sequence \((y_0, y_1, \ldots, y_{s+1})\) satisfying \(y_{t-1} \leq y_t, 1 \leq t \leq s + 1\), is obtained by reordering the entries of any \((a, x)\)-sequence. This sequence is uniquely determined by \((a, x)\), with \(y_0 = 0\) and \(x = y_t\) for some \(0 \leq t^* \leq s + 1\). If \(y_t < y_s + 1\), then \((y_0, y_1, \ldots, y_{s+1})\) is not an \((a, x)\)-sequence as \(y_{s+1} \neq x\). Denote
\[
\hat{y}_{a,x} := (y_0, y_1, \ldots, y_{s+1}, y_{s+2}), \quad y_{s+2} = r. \quad (5.11)
\]

Define
\[
L_1(a, x) := \begin{cases} 
\max\{y_t - y_{t-1} : 1 \leq t \leq t^*\}, & x \neq 0, \\
0, & x = 0
\end{cases},
\]
\[
L_2(a, x) := \max\{y_t - y_{t-1} : t^* + 1 \leq t \leq s + 2\}.
\]

Since \(y_{s+2} > y_{s+1}\), \(L_2(a, x) \geq 1\), and if \(x \neq 0\), then \(L_1(a, x) \geq 1\). Choose \(t\) with \(1 \leq t \leq t^*\) such that \(y_t - y_{t-1} = L_1(a, x)\) when \(x \neq 0\), and \(t = 0\) when \(x = 0\). Now let
\[
\hat{x}^1 = (y_0, y_1, \ldots, y_{t-1}, y_s, \ldots, y_{t+1}, y_t, x), \quad (5.12)
\]
with assumption that \(y_{t-1} = 0\) when \(t = 0\). Choose \(t\) with \(t^* + 1 \leq t \leq s + 2\) such that \(y_t - y_{t-1} = L_2(a, x)\), and let
\[
\hat{x}^2 = (y_0, y_s, \ldots, y_t, y_1, y_2, \ldots, y_{t-1}, x). \quad (5.13)
\]
Case dr = n: $(a, x)$-sequences $\hat{x}^1$ and $\hat{x}^2$

(See Figure 5.3 for an illustration.) It is clear that $\hat{x}^1$ and $\hat{x}^2$ are $(a, x)$-sequences. There exists $\hat{r}^1 = (i_1, i_2, \ldots, i_{t-1}, i_t, i_{t+1}, \ldots, i_r)$ which together with $\hat{x}^1$ satisfies (5.14). A path from $(0_n, 0)$ and $(a, x)$ can be obtained from $\hat{x}^1$ and $\hat{r}^1$ as described above, whose length is $\|a\| + l(\hat{x}^1)$ by Lemma 5.3.1. Similarly, a path from $(0_n, 0)$ and $(a, x)$ can be obtained from $\hat{x}^2$ and $\hat{r}^2 = (i_s, i_{s+1}, \ldots, i_t, i_{t+1}, \ldots, i_r)$, whose length is $\|a\| + l(\hat{x}^2)$. We now show that either $\hat{x}^1$ or $\hat{x}^2$ is an optimal $(a, x)$-sequence and so $l(a, x) = \min\{l(\hat{x}^1), l(\hat{x}^2)\}$.

Theorem 5.3.2. Suppose $dr = n$. Then

$$l(a, x) = \min\{r + x - 2L_1(a, x), r + (r - x) - 2L_2(a, x)\} \quad (5.14)$$

and so

$$\text{dist}((0_n, 0), (a, x)) = \|a\| + \min\{r + x - 2L_1(a, x), r + (r - x) - 2L_2(a, x)\}. \quad (5.15)$$

Moreover, if $l(\hat{x}^1) \leq l(\hat{x}^2)$ (respectively, $l(\hat{x}^2) \leq l(\hat{x}^1)$), then $\hat{x}^1$ (respectively, $\hat{x}^2$) is an optimal $(a, x)$-sequence.

Proof. Let $\check{x} = (x_0, x_1, \ldots, x_{s+1})$ be an arbitrary $(a, x)$-sequence, where $x_0 = 0$, $x_{s+1} = x$ and $s = \|a\|$. From the discussion above, the sequence $(y_0, y_1, \ldots, y_{s+1})$ is obtained by reordering the entries of $\check{x}$ such that $y_{t-1} \leq y_t$, for $1 \leq t \leq s + 1$. Let $C_r$ be the cycle with vertex set $\{0, 1, 2, \ldots, r - 1\}$ and edges joining 0 and 1, 1 and 2, $\ldots$, $r - 1$ and 0. Any path $P$ from $(0_n, 0)$ to $(a, x)$ by using $\hat{x}$ with minimum length gives rise to a walk $W$ from 0 to $x$ on $C_r$, obtained by treating each segment of $P$ as a path on $C_r$.

The length of $W$ is equal to $l(\hat{x})$.

**Case 1:** $W$ contains all edges of $C_r$. In this case we have $l(\hat{x}) \geq \min\{r + 2r - x\}$.

**Case 2:** At least one edge of $C_r$ is not contained in $W$. In this case there is exactly one $t$ with $1 \leq t \leq s + 2$ such that the path $y_{t-1}, y_{t-1} + 1, \ldots, y_t - 1, y_t$ is not in $W$. 

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Conversely, for any $1 \leq t \leq s + 2$, there is a walk $W$ as above which does not use the path $y_{t-1}, y_{t-1} + 1, \ldots, y_t, y_t$. If $x \geq y_t$, then $l(\hat{x}) \geq 2y_{t-1} + (r - y_t) + (x - y_t) = r + x - 2(y_t - y_{t-1})$; if $x \leq y_{t-1}$, then $l(\hat{x}) \geq 2(r - y_t) + y_{t-1} + (y_t - x) = 2r - x - 2(y_t - y_{t-1})$.

By the definition of $L_1(a, x)$ and $L_2(a, x)$, the smallest lower bound for $l(\hat{x})$ obtained in Case 2 is

$$l(\hat{x}) \geq \min\{r + x - 2L_1(a, x), 2r - x - 2L_2(a, x)\}. \quad (5.16)$$

Since $L_1(a, x) \geq 0$ and $L_2(a, x) \geq 1$, we have $\min\{r + x, 2r - x\} \geq \min\{r + x - 2L_1(a, x), 2r - x - 2L_2(a, x)\}$. In addition, for $\hat{x}^1$ and $\hat{x}^2$ defined in (5.12) and (5.13), respectively, we have $l(\hat{x}^1) \leq r + x - 2L_1(a, x)$ and $l(\hat{x}^2) \leq 2r - x - 2L_2(a, x)$. Therefore, by (5.12), $l(a, x) = \min\{l(\hat{x}^1), l(\hat{x}^2)\} = \min\{r + x - 2L_1(a, x), 2r - x - 2L_2(a, x)\}$, which together with Lemma 5.3.1 implies (5.13). Moreover, $\hat{x}^j$ with $l(\hat{x}^j) = \min\{l(\hat{x}^1), l(\hat{x}^2)\}$, $j \in \{1, 2\}$, is an optimal $(a, x)$-sequence.

We remark that $l(\hat{x}^1) \leq r + x - 2L_1(a, x)$ and equality holds if $\hat{x}^1$ is an optimal $(a, x)$-sequence. Similarly, $l(\hat{x}^2) \leq 2r - x - 2L_2(a, x)$ and equality holds if $\hat{x}^2$ is an optimal $(a, x)$-sequence. Algorithm 5.2 is a procedure to obtain an optimal $(a, x)$-sequence for $Q_n(d, r)$ when $dr = n$.

**Remark.** In the special case when $d = 1$ and $r = n$ (that is, when $Q_n(d, r) = CC_n$), Theorem 5.3.2 gives rise to [99, Lemma 1].

**Algorithm 5.2** A procedure to obtain optimal $(a, x)$-sequence when $dr = n$

1: Input: $(a, x)$;
2: Output: an optimal $(a, x)$-sequence $\hat{x}$;
3: Let $\hat{y}_{a,x} = (y_0, y_1, \ldots, y_{s+2})$ be as given in (5.11), and $t^*$ such that $x = y_{t^*}$, where $s = \|a\|$;
4: If $x = 0$, let $p = 1$, and if $x = 0$, let $p$ be a positive integer such that $1 \leq p \leq t^*$ and $y_p - y_{p-1} = L_1(a, x)$;
5: Let $q$ be a positive integer such that $t^* + 1 \leq q \leq s + 2$ and $y_q - y_{q-1} = L_2(a, x)$;
6: if $2r - x - 2L_2(a, x) \geq x + r - 2L_1(a, x)$ then
7: $\hat{x} := (y_0, y_1, \ldots, y_{p-1}, y_s, \ldots, y_q, x)$;
8: else
9: $\hat{x} := (y_0, y_s, \ldots, y_q, y_1, \ldots, y_{q-1}, x)$.
5.3.2 Case \( dr \geq 2n \)

Given \((a, x) \in G\), let \( \hat{i} = (i_1, i_2, \ldots, i_s)\) be such that \( a_{i_t} = 1 \) for \( 1 \leq t \leq s \) and \( i_1 < i_2 < \cdots < i_s\), where \( s = \|a\| \). Since \( dr/n \geq 2 \), by Lemma 5.3.2 applied to \( j = i_1, i_2, \ldots, i_s\), there exist two \((a, x)\)-sequences

\[
\hat{y} = (y_0, y_1, \ldots, y_s, y_{s+1}), \quad \hat{z} = (z_0, z_1, z_2, \ldots, z_s, z_{s+1})
\]

(5.17)

such that \( y_t = \lfloor (i_t - 1)/d \rfloor \), \( z_t = \lfloor (i_t + dr - n - 1)/d \rfloor \) and \( i_t \in D(y_t) \cap D(z_t) \) for \( 1 \leq t \leq s \). It is clear that \( 0 \leq y_{t-1} \leq y_t \leq \lfloor (n - 1)/d \rfloor = \lfloor n/d \rfloor - 1 \) and \( r - \lfloor n/d \rfloor \leq z_{t-1} \leq z_t \leq r - 1 \) for \( 2 \leq t \leq s \). Denote \( k = \lfloor n/d \rfloor d - n \). Then \( 0 \leq k \leq d - 1 \). Since \( (i_t - d)/d \leq y_t \leq (i_t - 1)/d \), we have \( i_t = k_t + dy_t \) for some \( 1 \leq k_t \leq d \) and every \( 1 \leq t \leq s \). Therefore, \( z_t = \lfloor (k_t + dy_t + dr - n - 1)/d \rfloor \) and so

\[
z_t = y_t + r - \lfloor n/d \rfloor + q_t, \quad 1 \leq t \leq s,
\]

(5.18)

where \( q_t = \lfloor (k_t + k - 1)/d \rfloor = 0 \) or \( 1 \). Note that if \( n \equiv 0 \mod d \), then \( q_t = 0 \) for every \( t \). In the following we show how to obtain an optimal \((a, x)\)-sequence from \( \hat{y} \) and \( \hat{z} \).

In the case when \( x \leq \lfloor r/2 \rfloor \), if \( x < y_s \), then let \( h, 1 \leq h \leq s \), be such that \( y_{h-1} \leq x < y_h \). Define

\[
L_1(a, x) := \begin{cases} 
\max\{y_h - x + q_h, y_j - y_{j-1} + q_j, \lfloor n/d \rfloor - y_s : h < j \leq s\}, & x < y_s, \\
\lfloor n/d \rfloor - x, & y_s \leq x \leq \lfloor r/2 \rfloor.
\end{cases}
\]

Similarly, if \( x > \lfloor r/2 \rfloor \) and \( x > z_1 \), then let \( 1 \leq l \leq s \) be such that \( z_l < x \leq z_{l+1} \). Define

\[
L_2(a, x) := \begin{cases} 
\max\{x - z_l + q_l, z_{j+1} - z_j + q_j, z_1 - r + \lfloor n/d \rfloor : 0 \leq j \leq l\}, & x > z_l, \\
\lfloor n/d \rfloor - (r - x), & z_1 \leq x > \lfloor r/2 \rfloor.
\end{cases}
\]

If \( x < y_s \), then \( L_1(a, x) \geq y_h - x + q_h \geq 1 \); if \( x > z_1 \), then \( L_2(a, x) \geq z_l - x + q_l \geq 1 \).

**Theorem 5.3.3.** Suppose \( dr \geq 2n \). Then the following hold.

(a) If \( 0 \leq x \leq \lfloor r/2 \rfloor \), then \( l(a, x) = 2\lfloor n/d \rfloor - x - 2L_1(a, x) \) and so

\[
\text{dist}(0_n, 0_n, (a, x)) = \|a\| + 2\lfloor n/d \rfloor - x - 2L_1(a, x).
\]

(b) If \( \lfloor r/2 \rfloor < x \leq r - 1 \), then \( l(a, x) = 2\lfloor n/d \rfloor - (r - x) - 2L_2(a, x) \) and so

\[
\text{dist}(0_n, 0_n, (a, x)) = \|a\| + 2\lfloor n/d \rfloor - (r - x) - 2L_2(a, x).
\]
Proof. (a) Let $\hat{t}, \hat{y}$ and $\hat{z}$ be as defined in (5.14). If $x \geq y_s$, then $\hat{y}$ is an optimal $(a, x)$-sequence since $l(a, x) \geq \min\{x, r - x\}$ and $l(\hat{y}) = x$ as $x \leq |r/2|$ and $0 = y_0 \leq y_1 \leq \cdots \leq y_s \leq y_{s+1} = x$. Hence the length of the path obtained from $\hat{y}$ and $\hat{t}$ is $\|a\| + x$ and the result follows if $x \geq y_s$. If $x < y_s$, then let

$$\hat{x}^t = (0, z_s, z_{s-1}, \ldots, z_t, y_1, y_2, \ldots, y_{t-1}, x), \ 1 \leq t \leq s + 1,$$

with the understanding that $\hat{x}^1 = (0, z_s, z_{s-1}, \ldots, z_1, x)$ and $\hat{x}^{s+1} = \hat{y}$. Since $i_j \in D(y_j) \cap D(z_j)$ for $1 \leq j \leq s$, each $\hat{x}^t$ is an $(a, x)$-sequence corresponding to $\hat{t} = (i_s, i_{s-1}, \ldots, i_t, i_1, \ldots, i_{t-1})$, $1 \leq t \leq s + 1$. (See Figure 5.4 for an illustration.) Thus, for $2 \leq t \leq s$, we have

$$l(\hat{x}^t) = \min\{z_s, r - z_s\} + \sum_{j=t+1}^{s} \min\{z_j - z_{j-1}, r - (z_j - z_{j-1})\} + \min\{|z_t - y_1|, r - |z_t - y_1|\}$$

$$+ \sum_{j=2}^{t-1} \min\{y_j - y_{j-1}, r - (y_j - y_{j-1})\} + \min\{|x - y_{t-1}|, r - |x - y_{t-1}|\}.$$ 

Since $r > 2[n/d] - 2$ (as $dr \geq 2n$), we have $z_t - y_1 \geq z_t - y_t \geq |r/2|$ by (5.18). Moreover, since $r > 2[n/d] - 2$, $r - [n/d] \leq z_j \leq r - 1$ and $0 \leq y_j \leq [n/d] - 1$, we have $z_j - z_{j-1} \leq |r/2|$ and $y_j - y_{j-1} \leq |r/2|$, for $1 \leq j \leq s$. Hence, for $2 \leq t \leq s$, the computation above together with (5.18) gives

$$l(\hat{x}^t) = (r - z_t) + \sum_{j=t+1}^{s} (z_j - z_{j-1}) + r - (z_t - y_1) + \sum_{j=2}^{t-1} (y_j - y_{j-1}) + |x - y_{t-1}|$$

$$= 2(r - z_t) + y_{t-1} + |x - y_{t-1}|$$

$$= 2([n/d] - y_t - q_t) + y_{t-1} + |x - y_{t-1}|, \ 2 \leq t \leq s.$$ 

In addition, $l(\hat{x}^1) = (r - z_1) + \min\{z_1 - x, r - (z_1 - x)\}$ and $l(\hat{x}^{s+1}) = y_s + \min\{|y_s - x|, r - |y_s - x|\} = 2y_s - x$ as $0 \leq x < y_s \leq |r/2|$. As above, let $h$ be such that $1 \leq h \leq s$ and $y_{h-1} \leq x < y_h$. Then

(i) $l(\hat{x}^{s+1}) = 2[n/d] - x - 2([n/d] - y_s);$

(ii) $l(\hat{x}^t) = 2[n/d] - x - 2(y_t - y_{t-1} + q_t), \ h < t \leq s;$

(iii) $l(\hat{x}^h) = 2\lfloor n/d \rfloor + x - 2(y_h + q_h) = 2\lfloor n/d \rfloor - x - 2(y_h - x + q_h);$ 

(iv) $l(\hat{x}^t) = 2\lfloor n/d \rfloor + x - 2(y_t + q_t) \geq l(\hat{x}^h), \ 2 \leq t < h; \text{ and}$

(v) $l(\hat{x}^1) = \min\{r - x, 2(r - z_1) + x\} \geq \min\{2\lfloor n/d \rfloor - x - 2, 2\lfloor n/d \rfloor - y_1 - q_1\} + x\} \geq \min\{2\lfloor n/d \rfloor - x - 2(y_h - x + q_h), 2\lfloor n/d \rfloor - x - 2(y_1 - x + q_1)\}$ as $r > 2\lfloor n/d \rfloor - 2$.

Note that $y_1 - x + q_1 \leq y_h - x + q_h$ since either $h = 1$, or $h > 1$ and $y_1 < y_h$, or $h > 1$ and $q_1 \leq q_h$ if $y_1 = y_h$ as $i_1 < i_h$. Thus $l(\hat{x}^1) \geq l(\hat{x}^h)$ in each case.
Figure 5.4: Case $dr \geq 2n$: $(a, x)$-sequences

Therefore,
\[
\min_{1 \leq t \leq s+1} l(\hat{x}^t) = \min_{1 \leq t \leq s+1} l(\hat{x}^t) = 2[n/d] - x - 2L_1(a, x). \tag{5.20}
\]

Now it remains to show that $l(\hat{w}) \geq 2[n/d] - x - 2L_1(a, x)$ for any $(a, x)$-sequence $\hat{w} = (w_0, w_1, \ldots, w_{s+1}) \neq \hat{x}^t$, $1 \leq t \leq s+1$. Let $\hat{k} = (k_1, k_2, \ldots, k_s)$ be a permutation of $\hat{t}$ such that $k_j \in D(w_j)$, $1 \leq j \leq s$. By Lemma 5.24, $w_j = [(k_j + lj - 1)/d]$ for $1 \leq j \leq s$ and some $0 \leq l_j \leq dr/n - 1$.

**Case 1:** There exists $1 \leq j \leq s$ such that $w_j$ is contained in neither $\hat{y}$ nor $\hat{z}$. Then $1 \leq l_j \leq dr/n - 2$ and so $[n/d] \leq w_j \leq r - [n/d]$. Since $x < y_s < [n/d]$, it follows that $l(\hat{w}) \geq \min\{w_j, r - w_j\} + \min\{x - w_j, r - x - w_j\} \geq 2y_s - x = l(\hat{x}^{s+1})$.

**Case 2:** $w_j$ is contained in either $\hat{y}$ or $\hat{z}$ for every $j$ with $1 \leq j \leq s$. If $w_j$ is contained in $\hat{y}$ (respectively, $\hat{z}$) for all $j$ with $1 \leq j \leq s$, then $\hat{w}$ is obtained from $\hat{x}^{s+1}$ (respectively, $\hat{x}^1$) by permuting its entries and so $l(\hat{w}) \geq l(\hat{x}^{s+1})$ (respectively, $l(\hat{w}) \geq l(\hat{x}^1)$). Otherwise, let $2 \leq t \leq s$ be the smallest integer such that $z_t$ is an entry of $\hat{w}$ and so $\hat{w}$ must contain $y_{t-1}$. Then either $\hat{w} = (0, \ldots, z_t, \ldots, y_{t-1}, \ldots, x)$ or $\hat{w} = (0, \ldots, y_{t-1}, \ldots, z_t, \ldots, x)$. Therefore, $l(\hat{w}) \geq \min\{(r - z_t) + (r - (z_t - y_{t-1})) + |x - y_{t-1}|, y_{t-1} + r - (z_t - y_{t-1}) + \min\{z_t - x, r + x - z_t\}\} = \min\{2(r - z_t) + y_{t-1} + |x - y_{t-1}|, 2y_{t-1} + r - x, 2y_{t-1} + 2(r - z_t) + x\}$ as $z_t - y_{t-1} \geq \lfloor r/2 \rfloor$. Hence $l(\hat{w}) \geq \min\{l(\hat{x}^t), l(\hat{x}^1), l(\hat{x}^t)\}$.

In both cases above, there exists some $1 \leq t \leq s+1$ such that $l(\hat{w}) \geq l(\hat{x}^t)$. So, by (5.20), $l(a, x) = l(\hat{x}^t)$ for some $h \leq t \leq s+1$, and any $\hat{x}^t$ achieving the minimum in (5.20) is an optimal $(a, x)$-sequence. Therefore, the result follows from Lemma 5.3.1.

(b) The proof is similar to that in case (a) and so is omitted.

**Remark.** From the proof of Theorem 5.3.4, for any $(a, x) \in G$ with $x \leq \lfloor r/2 \rfloor$, $\hat{x}^t$ given in (6.14) is an optimal $(a, x)$-sequence whenever $l(\hat{x}^t) = 2[n/d] - x - 2L_1(a, x)$, $h \leq t \leq s+1$. Thus, $\hat{x}^t$ and its corresponding $\hat{t}$ give rise to a shortest path from $(0_n, 0)$
to \((a, x)\) by Lemma 5.3.1. Similarly, for any \((a, x) \in G\) with \(x > |r/2|\), let \(\hat{y}\) and \(\hat{z}\) be as defined in (5.17) and \(1 \leq l \leq s\) be such that \(z_l < x \leq z_{l+1}\). Let

\[
\hat{x}^t = (0, y_1, \ldots, y_t, z_{s-1}, \ldots, z_{l+1}, x), \quad 1 \leq t \leq s + 1,
\]

(5.21)

where \(\hat{x}^1\) and \(\hat{x}^{s+1}\) are respectively interpreted as \(\hat{z}\) and \(\hat{y}\). Then \(\hat{x}^t\) is an optimal \((a, x)\)-sequence whenever \(t(\hat{x}^t) = 2[n/d] - (r - x) - 2L_2(a, x), 0 \leq t \leq l\). By Lemma 5.3.1, \(\hat{x}^1\) and \(\hat{y} = (i_1, i_2, \ldots, i_{l-1}, i_s, \ldots, i_t)\) give rise to a shortest path from \((0, a)\) to \((a, x)\). Algorithm 5.3 is a procedure to obtain an optimal \((a, x)\)-sequence for \(Q_n(d, r)\) when \(dr \geq 2n\).

**Algorithm 5.3** A procedure to obtain optimal \((a, x)\)-sequence when \(dr \geq 2n\)

1: Input: \((a, x)\);
2: Output: an optimal \((a, x)\)-sequence \(\hat{x}\);
3: Let \(s = \|a\|\) and \(\tilde{t} = (i_1, i_2, \ldots, i_s)\) such that \(a_{i_t} = 1\) for \(1 \leq t \leq s\) and \(i_1 < i_2 < \cdots < i_s\);
4: Let \(\tilde{y} = (y_0, y_1, \ldots, y_s, y_{s+1})\) and \(\tilde{z} = (z_0, z_1, z_2, \ldots, z_s, z_{s+1})\) such that \(y_t = \lfloor(i_t - 1)/d\rfloor\) and \(z_t = \lfloor(i_t + dr - n - 1)/d\rfloor\) for \(1 \leq t \leq s\), \(y_0 = z_0 = 0\) and \(y_{s+1} = z_{s+1} = x\);
5: if \(y_h \leq x \leq |r/2|\) then
6: \(\tilde{y}\) is an optimal \((a, x)\)-sequence;
7: if \(y_{h-1} \leq x < y_h\) for some \(1 \leq h \leq s\) then
8: \begin{itemize}
9: \(t^* = h\) if \(y_h - x + q_h = L_1(a, x)\), or \(t^* = j\) if \(y_j - y_{j-1} + q_j = L_1(a, x)\)
10: for some \(h + 1 \leq j \leq s\), or \(t^* = s + 1\) if \(\lfloor n/d\rfloor - y_s = L_1(a, x)\);
11: \end{itemize}
12: \(\hat{x} = (y_0, z_s, z_{s-1}, \ldots, z_{i^*}, y_1, \ldots, y_{i^*-1}, x)\) is an optimal \((a, x)\)-sequence;
13: if \(z_l \geq x > |r/2|\) then
14: \(\hat{z}\) is an optimal \((a, x)\)-sequence;
15: if \(z_{l-1} \leq x \leq z_l\) for some \(1 \leq l \leq s\) then
16: \begin{itemize}
17: \(t^* = 0\) if \(z_1 - r + \lfloor n/d\rfloor = L_2(a, x)\), or \(t^* = j\) if \(z_{j+1} - z_j + q_j = L_2(a, x)\)
18: for some \(1 \leq j \leq l - 1\), or \(t^* = l\) if \(x - z_l + q_l = L_2(a, x)\);
19: \end{itemize}
20: \(\hat{x} = (y_0, y_1, \ldots, y_{i^*}, z_s, z_{s-1}, \ldots, z_{i^*-1}, x)\) is an optimal \((a, x)\)-sequence;
5.4 Diameter of $Q_n(d, r)$

In [101, Theorem 5] it was claimed that the diameter of $Q_n^-(d, r)$ (see Section 5.2.2) is equal to $n + [(r - 3)/2]$, and in [101, Theorem 3] it was claimed that $\text{diam}(Q_n^-(d, r)) \leq n + [r/2] + 1$. As noticed in [101], these results are incorrect. In [101, Theorem 13] it was proved that $\text{diam}(Q_n^-(d, r))$ is bounded from above by $n + [3r/2] - 1$ if $r \leq 3$ and $n + [3r/2] - 2$ if $r \geq 4$. But still the precise value of $\text{diam}(Q_n^-(d, r))$ was unknown.

We give the exact value of $\text{diam}(Q_n(d, r))$ in the following theorem. Our result shows in particular that the bound $\text{diam}(Q_n^-(d, r)) \leq n + [3r/2] - 2$ ($r \geq 4$) is tight when $dr = n$ but not in general (by Lemma 5.2.6, $Q_n^-(d, r) \cong Q_n(d, r)$ when $dr = n$).

**Theorem 5.4.1.** If $dr = n$, then

$$\text{diam}(Q_n(d, r)) = \left\{ \begin{array}{ll} n + r, & \text{if } r = 3, \\ n + [3r/2] - 2, & \text{if } r \geq 4. \end{array} \right.$$ 

If $dr \geq 2n$, then

$$\text{diam}(Q_n(d, r)) = n + \max\{[r/2], 2[n/d] - 2\}.$$ 

**Proof.** Since $Q_n(d, r)$ is vertex-transitive, $\text{diam}(Q_n(d, r)) = \max \text{dist}((0_n, 0), (a, x))$, where maximum is taken over all $(a, x) \in G$.

Suppose $dr = n$ first. By (5.15),

$$(0_n, 0), (a, 0)) = \|a\| + \min\{r, 2r - 2L_2(a, 0)\} \leq n + r.$$ 

We claim that this upper bound is achieved by $(a, 0) = (1_n, 0)$. In fact, for $1 \leq i \leq n+2$, we have $y_i - y_{i-1} = 0$ or $1$ in $\hat{y}_{1_n,0}$ (given in (5.14)). Hence $L_2(1_n, 0) = 1$. So

$$\min\{r, 2r - 2L_2(1_n, 0)\} = r$$ 

and $\text{dist}((0_n, 0), (1_n, 0)) = n + r$.

By (5.14), for any $(a, x) \in G$ with $x \neq 0$, since $L_1(a, x), L_2(a, x) \geq 1$, we have

$$\text{dist}((0_n, 0), (a, x)) \leq \|a\| + \min\{r + x - 2, 2r - x - 2\} \leq n + 3[r/2] - 2.$$ 

This upper bound is achieved by $(a, x) = (1_n, [r/2])$. In fact, for $1 \leq i \leq n + 2$ we have $y_i - y_{i-1} = 0$ or $1$ in $\hat{y}_{1_n,[r/2]}$ and hence $L_1(1_n, [r/2]) = L_2(1_n, [r/2]) = 1$. Note that the maximum of $\min\{r + x - 2, 2r - x - 2\}$ is $3[r/2] - 2$, which is attained when $x = [r/2]$. Thus, $\text{diam}(Q_n(d, r)) = \max\{n + r, n + [3r/2] - 2\}$ if $dr = n$, as claimed.

Now suppose $dr \geq 2n$. For $(a, x) \in G$ with $0 \leq x \leq [r/2]$, $\text{dist}((0_n, 0), (a, x)) = \|a\| + 2[n/d] - x - 2L_1(a, x)$ by Theorem 5.3.3. If $L_1(a, x) = [n/d] - x$, then $2[n/d] - x -
Thus, 
\[
\text{dist}((0_n, 0), (a, x)) \leq n + \max\{|r/2|, 2[n/d] - 2\}.
\]

Note that for any \((1_n, x)\), in \(\hat{g} = (0, y_1, \ldots, y_n, x)\) as given in (5.17), we have \(y_0 = 0\), \(y_n = \lfloor n/d \rfloor - 1\), and either \(y_t = y_{t-1} + 1\) or \(y_t = y_{t-1} + 1, 1 \leq t \leq n\). In particular, for \((a, x) = (1_n, \lfloor r/2 \rfloor)\), we have \(y_0 < \lfloor n/d \rfloor \leq x\) and so \(L_1(1_n, \lfloor r/2 \rfloor) = \lfloor n/d \rfloor - x\). This implies \(\text{dist}((0_n, 0), (1_n, \lfloor r/2 \rfloor)) = n + \lfloor r/2 \rfloor\). On the other hand, for \((a, x) = (1_n, 0)\), we have \(0 \leq y_t - y_{t-1} + q_t \leq 2\), where \(q_t = \lfloor (k+k_t-1)/d \rfloor = 0\) or 1 and \(0 \leq k, k_t - 1 \leq d - 1\) as defined in the beginning of Section 5.3.2. For any \(t\) with \(y_t = y_{t-1} + 1\), \(i_{t-1} \in D(y_{t-1})\) and \(i_t \in D(y_t)\), we have \(dy_t + k_t = i_t = i_{t-1} + 1 = dy_{t-1} + k_{t-1} + 1\), that is, \(d + k_t = 1 + k_{t-1}\). Since \(1 \leq k_{t-1}, k_t \leq d\), we have \(k_{t-1} = d\) and \(k_t = 1\). Therefore, \(q_t = 0, y_t - y_{t-1} + q_t = 1\) and so \(L_1(1_n, 0) = \max_{1 \leq t \leq n}(y_t - y_{t-1} + q_t) = 1\). Hence 
\[
\text{dist}((0_n, 0), (1_n, 0)) = n + 2\lfloor n/d \rfloor - 2.
\]

Similar to the case \(0 \leq x \leq \lfloor r/2 \rfloor\), for any \((a, x) \in G\) with \(\lfloor r/2 \rfloor < x \leq r - 1\), 
\[
\text{dist}((0_n, 0), (a, x)) \leq n + \max\{\lfloor r/2 \rfloor - 1, 2\lfloor n/d \rfloor - 3\} \leq n + \max\{\lfloor r/2 \rfloor, 2\lfloor n/d \rfloor - 2\}.
\]

Therefore, \(\text{diam}(Q_n(d, r)) = n + \max\{\lfloor r/2 \rfloor, 2\lfloor n/d \rfloor - 2\}\) if \(dr \geq 2n\), and \(\text{diam}(Q_n(d, r))\) is attained by \((0_n, 0), (1_n, \lfloor r/2 \rfloor)\) or \((0_n, 0), (1_n, 0)\). 

It would be ideal if the diameter of a network is of logarithmic order of its number of vertices. In view of Theorem 5.3.1, \(Q_n(d, r)\) has this property when \(r = O(n)\).

Applying Theorem 5.3.1 to the \(d\)-ply cube-connected cycles \(Q_n(d, n)\), the cube-of-rings \(\text{COR}(d, r)\) and the cube-connected cycles \(\text{CC}_n\) (see Lemma 5.2.2), we obtain the following corollary. In particular, we recover the formulas for \(\text{diam}(\text{COR}(d, r))\) and \(\text{diam}(\text{CC}_n)\) as special cases of Theorem 5.3.1. It was claimed in [111], Theorem 4] that the diameter of \(\text{COR}(d, r)\) is \(d(r + 1) + \lfloor r/2 \rfloor - 2\) when \(r \geq 4\). Unfortunately, its proof contains a computation error and this formula is incorrect except when \(d = r\).

**Corollary 5.4.2.** (a) \(\text{diam}(Q_n(d, n)) = n + \max\{\lfloor n/2 \rfloor, 2\lfloor n/d \rfloor - 2\}\) for \(n \geq d \geq 2\);

(b) \(\text{diam}(\text{COR}(d, r)) = (d + 1)r\) if \(r = 3\), and \(\text{diam}(\text{COR}(d, r)) = (d + 1)r + \lfloor r/2 \rfloor - 2\) if \(r \geq 4\);

(c) \(\text{diam}(\text{CC}_n) = 2n\) if \(n = 3\), and \(\text{diam}(\text{CC}_n) = \lfloor 5n/2 \rfloor - 2\) if \(n \geq 4\) (see [113]).
5.5 Total distance

In this section we give bounds for \( \text{td}(Q_n(d, r)) = \sum_{(a, x) \in G} \text{dist}((0_n, 0), (a, x)) \). Since

\[
\sum_{(a, x) \in G} \|a\| = r \sum_{a \in \mathbb{Z}_2^n} \|a\| = r \sum_{i=0}^n \binom{n}{i} i = 2^{n-1}nr,
\]

by Lemma 5.3.1,

\[
\text{td}(Q_n(d, r)) = \sum_{(a, x) \in G} (\|a\| + l(a, x)) = 2^{n-1}nr + \sum_{(a, x) \in G} l(a, x). \tag{5.22}
\]

It remains to estimate \( \sum_{(a, x) \in G} l(a, x) \), and for this purpose we will use the notions of integer partitions and \( k \)-compositions of integers.

5.5.1 Case \( dr = n \)

Since \( dr = n \), by (5.11), \( \sum_{(a, x) \in G} l(a, x) = \sum_{(a, x) \in G} \min\{r + x - 2L_1(a, x), 2r - x - 2L_2(a, x)\} \). In order to give a good estimate of this sum, we will give a lower bound for the number of vertices \((a, x)\) such that \( L_1(a, x) \leq g \) (or \( L_2(a, x) \leq g \)) for a certain \( g \geq 1 \).

Let \( 2 \leq z \leq r \) be an integer. For any \( c = (c_{d+1}, c_{d+2}, \ldots, c_{ds}) \in \mathbb{Z}_2^{d(z-1)} \), let

\[
\hat{w}_c = (w_0, w_1, \ldots, w_s, w_{s+1})
\]

be such that \( w_0 = 0, w_{s+1} = z, 1 \leq w_i \leq w_{i+1} \leq z - 1 \) for \( 1 \leq i \leq s - 1 \), and \( c_i \in D(w_i) \) for \( 1 \leq i \leq s \), where \( s = \|c\| \). Since \( dr = n \), \( \hat{w}_c \) is well defined and unique. Define

\[
V(z) := \left\{ c \in \mathbb{Z}_2^{d(z-1)} : \max_{0 \leq i \leq s} (w_{i+1} - w_i) \leq \lceil \log^2 z \rceil \right\}.
\]

Lemma 5.5.1. For \( 2 \leq z \leq r - 1 \), we have

\[
|V(z)| \geq 2^{d(z-1)} \left( 1 - 2z^{-1-d\log z} \right).
\]

Proof. Given non-negative integers \( q \) and \( p \geq 1 \) with \( p + q \leq z \), let \( L_{q,p} \) be the subset of \( \mathbb{Z}_2^{d(z-1)} \) such that for any \( c \in L_{q,p} \) with \( \hat{w}_c = (w_0, w_1, \ldots, w_s, w_{s+1}) \), we have \( w_j = q \), \( w_{j+1} = p + q \) for some \( 1 \leq j \leq s - 2 \), \( w_0 = 0 \), \( w_{s+1} = z \) and \( w_i \leq w_{i+1} \) for \( 0 \leq i \leq s \), where \( s = \|c\| \). Hence, if \( c \in L_{q,p} \), then \( c_i = 0 \) for \( i \in \cup_{j=q+1}^{p+q-1} D(j) \) (that is, \( d(q + 1) < i \leq d(q + p) \), \( c_i = 1 \) for at least one \( i \in D(q) \) and at least one \( i \in D(p + q) \). Hence \( |L_{q,p}| = 2^{d(z-p-1)}(2^d - 1) \) if \( q = 0 \) or \( q = z - p \) and \( |L_{q,p}| = 2^{d(q-1)}(2^d - 1)^22^{d(z-q-p-1)} = (2^d - 1)^22^{d(z-p-2)} \) if \( 1 \leq q \leq z - p - 1 \). Thus, for any \( p \) and \( q \),

\[
|L_{q,p}| \leq 2^{d(z-p)}. \tag{5.23}
\]
For any $1 \leq g < z$, let $V_g := \{c \in \mathbb{Z}_2^d(z-1) : \max_{0 \leq i \leq s} (w_{i+1} - w_i) \leq g\}$ and denote $p^* = \max_{0 \leq i \leq s} (w_{i+1} - w_i)$. (Note that $p^*$ relies on $(c, z)$.) For any $c \in \mathbb{Z}_2^d(z-1)$ with $p^* > g$, we have $c \in L_{q, p^*}$ for some $q \geq 0$. Therefore, by (5.23),

$$|V_g| = 2^d(z-1) - \left| \bigcup_{p > g} 0 \leq g \leq z - p \bigcup L_{q, p}\right|$$

$$\geq 2^d(z-1) - \sum_{p > g} (z - p + 1)|L_{0, p}|$$

$$\geq 2^d(z-1) - \sum_{p = g+1}^z (z - p + 1)2^d(z-p)$$

$$= 2^d(z-1) - 2^d \sum_{i=1}^{z-g} i2^{di}.$$ 

Since $\sum_{i=1}^k it^i = t(1 - (k + 1)t^k + kt^{k+1})/(t - 1)^2$, we have $\sum_{i=1}^k it^i \leq 2kt^k$ for $t = 2$ and $\sum_{i=1}^k it^i \leq kt^{k+2}/(t - 1)^2 \leq 2kt^k$ for $t = 2^d$ and $d \geq 2$. Hence $2^{-d} \sum_{i=1}^{z-g} i2^{di} \leq (z - g)2^{d(z-g-1)+1}$ and therefore $|V_g| \geq 2^d(z-1) - (z - g)2^{d(z-g-1)+1}$. Hence, for $g = \lfloor \log^2 z \rfloor$, we have $V_g = V(z)$, $g \geq \log^2 z$ and so $|V(z)| \geq 2^d(z-1)(1 - (z - g)2^{1-dg}) \geq 2^d(z-1)(1 - 2z1-d\log z)$.

\textbf{Theorem 5.5.2.} Suppose $dr = n$. If $r \geq 2^9$, then

$$2^{n-2}r^2(2d + 5) \left(1 - \frac{20 \log^2 r}{2n + 5r}\right) \leq \text{td}(Q_n(d, r)) \leq 2^{n-2}r^2(2d + 5) \left(1 - \frac{8(r - 1)}{2nr + 5r^2}\right);$$

and if $3 \leq r < 2^9$, then

$$2^{n-2}(2nr + r^2) \leq \text{td}(Q_n(d, r)) \leq 2^{n-2}(2nr + 5r^2 - 8r + 8).$$

\textbf{Proof.} Note that for any $(a, x)$ with $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_2^n$, the sequence in (5.11) is exactly the same as that for $(a', r - x)$ but with reverse order for all entries except the first and last ones, where $a' = (a_n, \ldots, a_2, a_1)$. Therefore, $L_1(a, x) = L_2(a', r - x)$ and $L_2(a, x) = L_1(a', r - x)$. Consequently, we have $l(a, x) = l(a', r - x)$ and so

$$\sum_{a \in \mathbb{Z}_2^n} l(a, x) = \sum_{a \in \mathbb{Z}_2^n} l(a', r - x).$$

By (5.11), if $x = 0$, then $l(a, x) \leq r$; and if $x \neq 0$, then $l(a, x) \leq r + x - 2$ since $L_1(a, x) \geq 1$. Setting $\delta = 0$ if $r$ is odd and $\delta = 1$ if $r$ is even, then, by (5.20), we have

$$\sum_{(a, x) \in G} l(a, x) \leq \sum_{a \in \mathbb{Z}_2^n} \left(2 \sum_{x=1}^{\lfloor (r-1)/2 \rfloor} l(a, x) + \delta l(a, \lfloor r/2 \rfloor) + l(a, 0)\right)$$

$$\leq 2^{n+1} \sum_{x=1}^{\lfloor r/2 \rfloor} (r + x - 2) + \delta 2^n (\lfloor 3r/2 \rfloor - 2) + 2^nr$$

$$\leq 2^{n-2}(5r^2 - 8r + 8).$$

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which together with (5.24) gives the upper bounds in (5.24) and (5.25) after straightforward manipulations.

It remains to prove the lower bounds in (5.24) and (5.25). Observe that (5.14) and (5.26) together yield

$$\sum_{(a,x) \in G} l(a, x) \geq 2 \sum_{x=1}^{[(r-1)/2]} \sum_{a \in \mathbb{Z}_2^n} \min\{r + x - 2L_1(a, x), 2r - x - 2L_2(a, x)\}.$$  (5.28)

**Case 1:** Assume first that \( r \geq 2^9 \) and denote \( h = \lfloor \log^2 r \rfloor \). Then \( 2h \leq \lfloor r/2 \rfloor \). For any \( (a, x) \) with \( h \leq x \leq \lfloor r/2 \rfloor - h \), since \( L_1(a, x) \geq 1 \) and \( L_2(a, x) \geq 1 \), if

\[
L_1(a, x) \leq \lfloor \log^2 x \rfloor \quad \text{and} \quad L_2(a, x) \leq \lfloor \log^2(r - x) \rfloor, \tag{5.28}
\]

then \( 2L_2(a, x) - 2L_1(a, x) \leq 2L_2(a, x) - 2 \leq 2h \leq r - 2x \), that is, \( 2r - x - 2L_2(a, x) \geq r + x - 2L_1(a, x) \).

Denote by \( N_x \) the number of elements \( (a, x) \in G \) such that \( 2r - x - 2L_2(a, x) \geq r + x - 2L_1(a, x) \). If \( b \in V(x) \) and \( b' \in V(r - x) \), then \( (a, x) = ((a_1, a_2, \ldots, a_d, b, a_{dx+1}, \ldots, a_{dx+d}, b')) \) satisfies (5.28) for arbitrary \( (a_1, a_2, \ldots, a_d), (a_{dx+1}, \ldots, a_{dx+d}) \in \mathbb{Z}_2^d \). Conversely, for any \( (a, x) \) satisfying (5.28), we have \( (a_{d+1}, a_{d+2}, \ldots, a_{dx}) \in V(x) \) and \( (a_{d(x+1)+1}, \ldots, a_{dx-1}, a_{dr}) \in V(r - x) \). Thus, by Lemma (5.21), for \( h \leq x \leq \lfloor r/2 \rfloor - h \), there are at least \( 2^{2d}2^d x^{-1}(1 - 2x^{1-d\log x} + 2^{d} (r-x)^{-1}(1 - 2(r-x)^{-1-d\log(r-x)}) \) elements in \( \mathbb{Z}_2^n \) satisfying (5.28), that is, \( N_x \geq 2^n(1 - 2x^{1-d\log x})/((r-x)^{-1-d\log(r-x)}) \). Since \( h \leq x \leq \lfloor r/2 \rfloor - h \), we have \( x^{1-d\log x} \leq 1/(2r^0 r) \) and \( (r-x)^{-1-d\log(r-x)} \leq 1/(2r^0 r) \), and therefore \( N_x \geq 2^n(1 - 1/(2^9 r)^2) \). This together with \( r + x - 2L_1(a, x) \geq (r + x)(1 - 2\lfloor \log^2 x \rfloor/(r + x)) \) implies that for \( r \geq 2^9 \),

\[
\sum_{(a,x) \in G} l(a, x) \geq 2 \sum_{x=1}^{[(r-1)/2]} \sum_{a \in \mathbb{Z}_2^n} N_x(r + x - 2L_1(a, x)) \geq 2 \sum_{x=1}^{[(r-1)/2]} 2^n(1 - 1/(2^9 r)^2) (r + x)(1 - 2\lfloor \log^2 h \rfloor/(r + h)) \geq 2^{n+1}(1 - 2\log^2 h)/r \sum_{x=1}^{[(r-1)/2]} (r + x) = 2^{n-2}(1 - 2\log^2 h)/r (5r^2 - (20r - 8)h - 10r) \geq 5.2^{n-2}r^2(1 - (4\log^2 r)/r). \tag{5.29}
\]

Combining this with (5.22), we obtain the lower bound in (5.24).

**Case 2:** Now assume \( 3 \leq r < 2^9 \). Note that \( l(a, 0) \geq 2 \) except for the \( 2^d \) vertices \( (a, 0) \) with \( a = (a_1, \ldots, a_d, 0, \ldots, 0) \) for which \( L_2(a, 0) = r \). Hence \( \sum_{a \in \mathbb{Z}_2^n} l(a, 0) \geq 2(2^n -
2^d \geq 2^n \) by \( d \leq n \) and (6.14). If \( 1 \leq x \leq r/2 \), then \( l(a, x) \geq x \) since \( L_1(a, x) \leq x \) and \( L_2(a, x) \leq r - x \). Therefore,

\[
\sum_{(a, x) \in \mathcal{C}} l(a, x) \geq 2 \sum_{a \in 2^d_2} \sum_{x=1}^{\lfloor r/2 \rfloor} x + \sum_{a \in 2^d_2} l(a, 0) \geq 2^{n+1} \sum_{x=1}^{\lfloor r/2 \rfloor} x + 2^n \geq 2^{n-2} r^2. \tag{5.30}
\]

This together with (5.22) implies the lower bound in (5.20).

### 5.5.2 Case \( dr \geq 2n \)

Similar to Section 6.5.1, in order to estimate \( \text{td}(Q_n(d, r)) \) we will give a lower bound for the number of vertices \((a, x)\) with \( L_1(a, x) \leq g \) for a certain \( g \geq 1 \). For this purpose we will consider integer sequences \( 0 = x_0 < x_1 < \cdots < x_k = x_{k+1} = m \) such that \( \max_{1 \leq i \leq k+1} (x_i - x_{i-1}) \leq g \). We call a solution \((r_1, r_2, \ldots, r_k)\) to the equation:

\[
r_1 + r_2 + \cdots + r_k = m
\]

with all \( r_i \)'s positive integers a \textit{k-composition} of \( m \). It is known that the number of \( k \)-compositions of \( m \) is \( \binom{m-1}{k-1} \). Any \( k \)-composition \((r_1, r_2, \ldots, r_k)\) of \( m \) gives rise to a sequence \( 0 = x_0 < x_1 < \cdots < x_k = x_{k+1} = m \), where \( x_i = \sum_{j=1}^{i} r_j \), \( 1 \leq i \leq k \). Clearly, \( \max_{1 \leq i \leq k+1} (x_i - x_{i-1}) = \max_{1 \leq i \leq k} r_i \). Also any \((k+1)\)-composition \((r_1, r_2, \ldots, r_k, r_{k+1})\) of \( m \) gives rise to a sequence \( 0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = m \), where \( x_i = \sum_{j=1}^{i} r_j \), \( 1 \leq i \leq k \), which satisfies \( \max_{1 \leq i \leq k+1} (x_i - x_{i-1}) = \max_{1 \leq i \leq k+1} r_i \). Hence, for any fixed \( k \) with \( \lceil m/g \rceil \leq k \leq m \), the number of sequences \( 0 = x_0 < x_1 < \cdots < x_k = x_{k+1} = m \) (respectively, \( 0 = x_0 < x_1 < \cdots < x_k < x_{k+1} = m \)) with \( \max_{1 \leq i \leq k+1} (x_i - x_{i-1}) \leq g \) is equal to the number of \( k \)-compositions \((r_1, r_2, \ldots, r_k)\) (respectively, \((k+1)\)-compositions \((r_1, r_2, \ldots, r_{k+1})\)) of \( m \) with \( 1 \leq r_i \leq g \) for each \( i \).

Given integers \( a < b \) and \( k \) with \( (b - a)/\lceil \log(b - a) \rceil \leq k \leq b - a \), we defined an \([a, b]_k\)-sequence to be an integer sequence \( 0 = x_0 < x_1 < \cdots < x_k = x_{k+1} = m \) such that \( \max_{1 \leq i \leq k+1} (x_i - x_{i-1}) \leq \lceil \log(b - a) \rceil \). The next lemma is a key step towards an asymptotic formula for \( \text{td}(Q_n(d, r)) \) to be given in Theorem 5.5.3.

**Lemma 5.5.3.** Given an integer \( m \geq 9 \), for \( k \) with \( m/\lceil \log m \rceil \leq k \leq m \) let \( b_k \) be the number of \([0, m]_k\)-sequences. Then, for \( g = \lceil \log m \rceil \) and any real number \( z \geq 2 \),

\[
\left( 1 - \frac{2}{(z+1)^{\lceil \log m \rceil}} \right) (z+1)^m \leq \sum_{k=\lceil m/g \rceil}^{m} b_k z^k \leq (z+1)^m.
\]

**Proof.** From the discussion in Section 5.5.2, \( b_k \) is the number of all \( k \)-compositions \((r_1, r_2, \ldots, r_k)\) of \( m \) with \( 1 \leq r_i \leq g \) for each \( i \), plus the number of all \((k+1)\)-compositions \((r_1, r_2, \ldots, r_{k+1})\) with solutions \( 1 \leq r_i \leq g \) for each \( i \).
Given non-negative integers \(i, q, p\), denote by \(C_{q,p}(i)\) the family of \(k\)-compositions \((r_1, r_2, \ldots, r_k)\) of \(m\) such that \(\sum_{j<i} r_j = q\), \(r_i = p\) and \(\sum_{j>i} r_j = m - q - p\).

If \(i = 1\) (\(i = k\), respectively), then \(q = 0\) (\(q = m - p\), respectively) and

\[
|C_{q,p}(i)| = \binom{m-p-1}{k-2}.
\]  

(5.31)

For \(2 \leq i \leq k - 1\), we have \(i - 1 \leq q \leq m - p - k + i\) and

\[
|C_{q,p}(i)| = \binom{q-1}{i-2} \binom{m-p-q-1}{k-i-1}.
\]  

(5.32)

Therefore, the set of \(k\)-compositions \((r_1, r_2, \ldots, r_k)\) with \(\max_{1 \leq i \leq k} r_i \geq g + 1\) is

\[
\left( \bigcup_{p > g} C_{0,p}(1) \right) \cup \left( \bigcup_{p > g} \bigcup_{i = 2}^{k-1} \bigcup_{q = i-1}^{m-p-k+i} C_{q,p}(i) \right) \cup \left( \bigcup_{p > g} C_{m-p,p}(k) \right).
\]

By (5.31) and (5.32), the size of this set is at most

\[
2 \sum_{p > g} \binom{m-p-1}{k-2} + \sum_{p > g} \sum_{i = 2}^{k-1} \sum_{q = i-1}^{m-p-k+i} \binom{q-1}{i-2} \binom{m-p-q-1}{k-i-1}.
\]  

(5.33)

To simplify the expression above, we use the following known identity:

\[
\sum_{l=0}^{\alpha} \binom{l}{j} \binom{\alpha - l}{\beta - j} = \binom{\alpha + 1}{\beta + 1},
\]  

(5.34)

where \(\alpha, \beta\) and \(j\) are integers with \(0 \leq j \leq \beta \leq \alpha\). Applying (5.34) to the case where \(\alpha = m - p - 2\), \(\beta = k - 3\) and \(l = q - 1\), one can verify that (5.33) is less than or equal to

\[
2 \sum_{p > g} \binom{m-p-1}{k-2} + \sum_{p > g} \sum_{i = 2}^{k-1} \binom{m-p-1}{k-2} = k \sum_{p > g+1} \binom{m-p-1}{k-2} = k \sum_{l=k-2}^{m-g-2} \binom{l}{k-2}.
\]

Again by (5.34) applied to \(\alpha = m - g - 2\) and \(j = \beta = k - 2\), the right-hand side of the above equation is equal to \(k \binom{m-g-1}{k-1}\). So the number of \(k\)-compositions of \(m\) with \(1 \leq r_i \leq g\) for \(1 \leq i \leq k\) is at least \(\binom{m-1}{k-1} - k \binom{m-g-1}{k-1}\). Similarly, the number of \((k+1)\)-compositions of \(m\) with \(1 \leq r_i \leq g\) for \(1 \leq i \leq k+1\) is at least \(\binom{m-1}{k} - (k+1) \binom{m-g}{k}\).

Thus, since \(\binom{m-1}{k-1} + \binom{m-1}{k} = \binom{m}{k}\), we have \(b_k \geq \binom{m}{k} - (k+1) \binom{m-g}{k}\). On the other hand, the number of \(k\)-compositions of \(m\) plus the number of \((k+1)\)-compositions of \(m\) is an upper bound for \(b_k\), that is, \(b_k \leq \binom{m-1}{k-1} + \binom{m-1}{k} = \binom{m}{k}\). Therefore, we have

\[
\sum_{k=0}^{[m/g]-1} \binom{m}{k} z^k - \sum_{k=0}^{m} \binom{m}{k} z^k - \sum_{k=[m/g]}^{(m-1)} \binom{m-g}{k} z^k \leq \sum_{k=[m/g]}^{m} b_k z^k \leq \sum_{k=0}^{m} \binom{m}{k} z^k = (z + 1)^m.
\]  

(5.35)

(5.36)
Note that
\[ \sum_{k=\lfloor m/g \rfloor}^m (k+1) \binom{m-g}{k} z^k \leq (m-g+1) \sum_{k=\lfloor m/g \rfloor}^{m-g} \binom{m-g}{k} z^k \]
\[ < (m-g+1)(z+1)^{m-g}. \quad \text{(5.37)} \]

Using \( \sum_{k=0}^{t} \binom{m}{k} \leq \binom{m}{\lfloor t \rfloor} \frac{m-t+1}{m-2t+1} \) and \( \binom{m}{t} \leq \binom{n}{t} \frac{m}{m/t} \), we obtain
\[ \sum_{k=0}^{\lfloor m/g \rfloor-1} z^k \binom{m}{k} \leq z^{\frac{m}{g}} \sum_{k=0}^{\lfloor m/g \rfloor-1} \binom{m}{k} \]
\[ \leq z^{\frac{m}{g}} \frac{m-\lfloor m/g \rfloor}{m-2\lfloor m/g \rfloor -1}. \quad \text{(5.39)} \]

This together with (5.37) and the lower bound in (5.34) yields
\[ \sum_{k=\lfloor m/g \rfloor}^m b_k z^k \geq (z+1)^m \left( 1 - \frac{z}{z+1} \right)^{\frac{m}{g}} \frac{(m-\lfloor m/g \rfloor)^{1/m}}{((z+1)^{g-1})^{\frac{m}{g}} (m-2\lfloor m/g \rfloor -1)^{1/m}} \leq g^{1/m} \frac{(z+1)^{g/m}}{(z+1)^g} \text{ for } m \geq 9. \]

One can verify that \( \left( \frac{z}{z+1} \right)^{\frac{1}{g}} \left( \frac{(eg)^{\frac{1}{g}}}{((z+1)^{g-1})^{\frac{m}{g}}} \right)^{\frac{m-\lfloor m/g \rfloor}{m-2\lfloor m/g \rfloor -1}} \leq 2 \frac{(z+1)^{g/m}}{(z+1)^g} \) for \( m \geq 9. \)

Using this, the expression above follows that
\[ \sum_{k=\lfloor m/g \rfloor}^m b_k z^k \geq (z+1)^m \left( 1 - \frac{2}{(z+1)^g} \right). \]

This together with the upper bound in (5.30) completes the proof. \( \Box \)

Set
\[
\ell_{n,d,r} := \begin{cases} 
(12[n/d]^{3/2} \log(2[n/d]))/(2nr + r^2 + 8[n/d]^2), & \text{if } [n/d] \geq 100, \\
8[n/d]^2/(2nr + r^2 + 8[n/d]^2), & \text{if } [n/d] < 100.
\end{cases} \quad \text{(5.40)}
\]

Note that \( 0 < \alpha_{n,d,r} < 1 \) and \( \alpha_{n,d,r} \) can be arbitrarily small for sufficiently large \( 2^n r \).

**Theorem 5.5.4.** Suppose \( dr \geq 2^n \). Then
\[
2^{n-1} \left( nr + \lfloor r^2/2 \rfloor + 4 \lfloor n/d \rfloor^2 \right) (1 - \alpha_{n,d,r}) \leq t d(Q_n(d,r)) \leq 2^n \left( nr + \lfloor r^2/2 \rfloor + 4 \lfloor n/d \rfloor^2 \right). \quad \text{(5.41)}
\]

Proof. Set \( q := [n/d] \) in this proof. For any \((a, x) \in G\), we have \( L_1(a, x) = L_2(a', r-x) \) in view of (5.11) and so \( l(a, x) = l(a', r-x) \), where \( a = (a_1, a_2, \ldots, a_n) \) and \( a' = (a_n, \ldots, a_2, a_1) \). So \( \sum_{a \in \mathbb{Z}_q^n} l(a, x) = \sum_{a \in \mathbb{Z}_q^n} l(a, r-x) \) for any \( 1 \leq x \leq \lfloor (r-1)/2 \rfloor \).

Setting \( \delta = 0 \) if \( r \) is odd and \( \delta = 1 \) if \( r \) is even, we then have
\[
\sum_{(a,x) \in G} l(a, x) = 2 \sum_{a \in \mathbb{Z}_q^n} \sum_{x=0}^{\lfloor (r-1)/2 \rfloor} l(a, x) - \sum_{a \in \mathbb{Z}_q^n} l(a, 0) + \delta \sum_{a \in \mathbb{Z}_q^n} l(a, \lfloor r/2 \rfloor). \quad \text{(5.42)}
\]
Note that \( \sum_{a \in \mathbb{Z}_2^n} l(a, 0) \leq \sum_{a \in \mathbb{Z}_2^n} (2q - 2) < 2^{n+1}q \) and when \( r \) is an even integer, \( \sum_{a \in \mathbb{Z}_2^n} l(a, \lfloor r/2 \rfloor) = \sum_{a \in \mathbb{Z}_2^n} (r/2) = 2^{n-1}r. \)

Denote

\[ V := \{ (a, x) \in G : y_s \leq x \leq \lfloor (r - 1)/2 \rfloor, \text{ where } \hat{y} \text{ is as in (5.11)} \}, \]

where \( s = \|a\| \). So \( (a, x) \in V \) if and only if either \( x \geq q - 1 \), or \( 0 \leq x \leq q - 2 \) and \( a_i = 0 \) for \( d(x + 1) < i \leq n \). Thus, for any \( (a, x) \in V \), we have \( L_1(a, x) = q - x \) and so \( l(a, x) = x \) by Theorem 5.3.3. Therefore,

\[ \sum_{a \in \mathbb{Z}_2^n} l(a, x) = A_1 + A_2 + A_3, \tag{5.43} \]

where

\[ A_1 = \sum_{x=0}^{q-2} \sum_{a \in V \text{ for } x} (2q - x - 2L_1(a, x)), \]

\[ A_2 = \sum_{x=0}^{q-2} \sum_{a \in V} x = \sum_{x=0}^{q-2} 2^{d(x+1)}x = \frac{2^d}{(2^d - 1)^2} \left((2^d - 1)(q - 2) - 1 \right) + \frac{2^d}{(2^d - 1)^2}, \tag{5.44} \]

\[ A_3 = \sum_{a \in \mathbb{Z}_2^n} \sum_{x=q-1}^{[\lfloor (r-1)/2 \rfloor]} x = 2^{n-1} \lfloor (r - 1)/2 \rfloor \lfloor (r + 1)/2 \rfloor - 2^{n-1} (q^2 - 3q + 2). \tag{5.45} \]

In (5.44) we used the fact that, for a fixed \( x \) with \( 0 \leq x \leq q - 2 \), the number of elements \( (a, x) \in V \) is equal to \( 2^{d(x+1)} \). Since \( 2^{d(q-2)}(2^d - 1) \leq 2^n \), we have

\[ 2^{n-d+1}q \leq A_2 \leq 2^n(q - 2) + 4. \tag{5.46} \]

Since \( L_1(a, x) \geq 1 \), we have

\[ A_1 \leq \sum_{x=0}^{q-2} \sum_{a \in \mathbb{Z}_2^n} (2q - x - 2) \leq 2^n \sum_{x=0}^{q-2} (2q - x - 2) = 2^{n-1}(3q^2 - 5q + 2) < 3 \cdot 2^{n-1}q^2. \tag{5.47} \]

Combining this with (5.42), (5.43), (5.44) and (5.46), we obtain

\[ \sum_{(a, x) \in G} l(a, x) \leq 2^{n-1} \left( \lfloor r^2/2 \rfloor + 4q^2 \right), \tag{5.48} \]

which together with (5.22) gives the upper bound in (5.11) for all possible \( q \).

Now we give a lower bound for \( A_1 \) and thus a lower bound for \( td(Q_n(d, r)) \).

**Case 1:** \( q \geq 100 \). For a fixed \( x \) with \( 0 \leq x \leq q - \sqrt{q} \), denote \( g = \lfloor \log(q-x-1) \rfloor \). (Note that \( q-x-1 \geq 9 \) and \( g > 3 \) as \( q \geq 100 \).) Denote by \( W_{x,k} \) the set of \( [x, q-1]_k \)-sequences. Denote by \( N_{x,k} \) the number of vertices \((a, x)\) with \( \hat{y} = (y_0, y_1, \ldots, y_t, y_{t+1}, \ldots, y_{t+k}, x) \).
such that \( y_t \leq x \) and the sequence \( x < y_{t+1} < \cdots < y_{t+k} \leq q - 1 \) belongs to \( W_{x,k} \). Note that for any such \((a, x)\), we have \((a, x) \notin V\) and \( L_1(a, x) \leq g + 1 \). For a fixed sequence \( y_{t+1}, \ldots, y_{t+k} \), the number of vertices \((a, x)\) with \( y \) as above is \( 2^{d(x+1)}(2^d - 1)^k \) since \((a_1 + dy_1, \ldots, a_k + dy_k) \neq (0, 0, \ldots, 0)\) for \( t + 1 \leq j \leq t + k \). Thus, for any \( x \) with \( 0 \leq x \leq q - \sqrt{q} \) and \( k \) with \((q - x - 1)/g \leq k \leq q - x - 1 \), we have

\[
N_{x,k} = 2^{d(x+1)}(2^d - 1)^k|W_{x,k}|.
\]

On the other hand, \( 2q - x - 2L_1(a, x) \geq 2q - x - 2(g + 1) = (2q - x)(1 - (g + 1)/(q - x/2)) \geq (2q - x)(1 - (\log q)/\sqrt{q}) \). Denote \( l_1 = \lfloor (q - x - 1)/g \rfloor \) and \( l_2 = q - x - 1 \).

Applying Lemma 5.5.3 to \( m = q - x - 1 \), we have

\[
A_1 \geq \sum_{x=0}^{q-\sqrt{q}} \sum_{k=1}^{l_2} N_{x,k}(2q - x) \left(1 - \frac{\log q}{\sqrt{q}}\right) \\
= \sum_{x=0}^{q-\sqrt{q}} 2^{d(x+1)}(2q - x) \left(1 - \frac{\log q}{\sqrt{q}}\right) \sum_{k=1}^{l_2} (2^d - 1)^k|W_{x,k}| \\
\geq 2^{d(y)} \left(1 - \frac{\log q}{\sqrt{q}}\right) \left(1 - \frac{2}{2^2/2}\right) \sum_{x=0}^{q-\sqrt{q}} (2q - x) \\
\geq 3 \cdot 2^{n-1} q^2 \left(1 - \frac{\log q}{\sqrt{q}}\right) \left(1 - \frac{2}{q^2}\right) \left(1 - \frac{\sqrt{q} - 1}{q} - \frac{q - 2\sqrt{q}}{2q^2}\right) \\
\geq 3 \cdot 2^{n-1} q^2 \left(1 - \frac{\log(2q)}{\sqrt{q}}\right).
\]

From this and (5.42), (5.43), (5.44) and (5.45), we obtain

\[
\sum_{(a,x) \in G} l(a, x) \geq 2^{n-1} \left( [r^2/2] + 4q^2 \right) \left(1 - \frac{12q^{3/2} \log(2q)}{r^2 + 8q^2}\right).
\]  

(5.49)

Plugging this into (5.22) yields the lower bound in (5.11) for \( q \geq 100 \).

**Case 2:** \( 1 \leq q < 100 \). Suppose \( q \geq 2 \) first. For any \((a, x) \notin V\) with \( 0 \leq x \leq q - 2 \), we have \( L_1(a, x) \leq q - x - 1 \) since \( y_s > x \). Moreover, for any fixed \( x \) with \( 0 \leq x \leq q - 2 \), there are \( 2^n - 2^{d(x+1)} \) elements \((a, x)\) in \( G \setminus V \). This together with the fact that \( \sum_{i=0}^{k} iz^i \leq 2kz^k \) for \( z = 2^d \) implies that

\[
A_1 \geq \sum_{x=0}^{q-2} \sum_{(a,x) \notin V} (x + 2) = \sum_{x=0}^{q-2} (2^n - 2^{d(x+1)})(x + 2) \geq 2^{n-1}(q^2 - q).
\]

This together with (5.42), (5.43), (5.44) and (5.45) yields

\[
\sum_{(a,x) \in G} l(a, x) \geq 2^{n-1}[r^2/2].
\]

On the other hand, if \( q = 1 \), then \( V = G \) and it can be verified that \( \sum_{(a,x) \in G} l(a, x) = 2^{n-1}[r^2/2] \). Hence, for \( 1 \leq q < 100 \), we have

\[
\sum_{(a,x) \in G} l(a, x) \geq 2^{n-1} \left( [r^2/2] + 4q^2 \right) \left(1 - \frac{8q^2}{r^2 + 8q^2}\right).
\]  

(5.50)

Combining this with (5.22), we obtain the lower bound in (5.11) for \( 1 \leq q < 100 \).  \( \square \)
Remark. (a) When $2^n r$ is large, $\alpha_{n,d,r}$ is small and so (5.41) gives

$$\text{td}(Q_n(d, r)) \approx 2^{n-1} \left(nr + \lfloor r^2/2 \rfloor + 4\lceil n/d \rceil^2\right).$$

(b) Define

$$\beta_{n,d,r} := \begin{cases} 
(12\lceil n/d \rceil^{3/2} \log(2\lceil n/d \rceil))/(r^2 + 8\lceil n/d \rceil^2), & \text{if } \lceil n/d \rceil \geq 100, \\
8\lceil n/d \rceil^2/(r^2 + 8\lceil n/d \rceil^2), & \text{if } \lceil n/d \rceil < 100. 
\end{cases} \tag{5.51}$$

Since $r \geq 2n/d$, $\beta_{n,d,r}$ can be arbitrarily small for sufficiently large $r$. In the next section we will use the following bounds obtained from the proof of Theorem 5.5.4:

$$2^{n-1} \left(\lfloor r^2/2 \rfloor + 4\lceil n/d \rceil^2\right) (1 - \beta_{n,d,r}) \leq \sum_{(a,x) \in G} l(a, x) \leq 2^{n-1} \left(\lfloor r^2/2 \rfloor + 4\lceil n/d \rceil^2\right). \tag{5.52}$$

The lower bound here is sharp when $n = d$ (see Case 2 in the proof of Theorem 5.5.4), while the upper bound is nearly tight for sufficiently large $2^n r$.

### 5.6 Vertex-forwarding index

In this section, $\alpha_{n,d,r}$ is as given in (5.40). By Theorems 3.2.1 and 3.1.1, any Cayley graph $X$ admits a shortest path routing that loads all vertices uniformly. It follows that $\xi(X) = \xi_m(X) = \sum_{v \in V(X)} \text{dist}(u, v) - (|V(X)| - 1)$, where $u$ is any fixed vertex of $X$.

**Theorem 5.6.1.** We have $\xi(Q_n(d, r)) = \xi_m(Q_n(d, r))$ and the following hold:

(a) if $dr = n$ and $r \geq 2^9$, then

$$2^{n-2}r^2(2d+5) \left(1 - \frac{4\log_2 r + 4}{2n + 5r^2}\right) \leq \xi(Q_n(d, r)) \leq 2^{n-2}r^2(2d+5) \left(1 - \frac{12r - 8}{2nr + 5r^2}\right);$$

and if $dr = n$ and $3 \leq r < 2^9$, then

$$2^{n-2}(2nr + r^2 - 4r) \leq \xi(Q_n(d, r)) \leq 2^{n-2}(2nr + 5r^2 - 12r + 8);$$

(b) if $dr \geq 2n$, then

$$2^{n-1}(nr + \lfloor r^2/2 \rfloor + 4\lceil n/d \rceil^2) (1 - \alpha_{n,d,r}) \leq \xi(Q_n(d, r)) \leq 2^{n-1}(nr + \lfloor r^2/2 \rfloor + 4\lceil n/d \rceil^2).$$

**Proof.** Since $\xi(Q_n(d, r)) = \xi_m(Q_n(d, r)) = \text{td}(Q_n(d, r)) - 2^nr + 1$, the results follow from Theorems 5.5.2 and 5.5.4. □
Corollary 5.6.2. (a) \( \xi(Q_n(d, n)) = \xi_m(Q_n(d, n)) = 2^{n-1}(n^2 + \lceil n^2/2 \rceil + 4\lceil n/d \rceil^2) \) \((1 - O((\log n) / (\sqrt{n/d} (3d^2 + 8))))\) for \( d \geq 2 \);

(b) if \( r \geq 2^9 \), then \( \xi(COR(d, r)) = \xi_m(COR(d, r)) \) and
\[
2^{dr-2} r^2 (2d + 5) \left(1 - \frac{4 \log^2 r}{(2d + 5)r} \right) \leq \xi(COR(d, r)) \leq 2^{dr-2} r^2 (2d + 5) \left(1 - \frac{12r - 8}{(2d + 5)r^2} \right);
\]
and if \( 3 \leq r < 2^9 \), then \( \xi(COR(d, r)) = \xi_m(COR(d, r)) \) and
\[
2^{dr-2} (2dr^2 + 2r^2 - 12r) \leq \xi(COR(d, r)) \leq 2^{dr-2} (2dr^2 + 5r^2 - 12r + 8);
\]

(c) If \( 3 \leq n < 2^9 \), then \( \xi(CC_n) = \xi_m(CC_n) \) and
\[
2^{n-2} (3n^2 - 4n) \leq \xi(CC_n) \leq 2^{n-2} (7n^2 - 12n + 8);
\]
and if \( n \geq 7 \), then \( \xi(CC_n) = \xi_m(CC_n) \) and \((\mathbb{C} l)\)
\[
7 \cdot 2^{n-2} n^2 (1 - (4 \log^2 n)/(7n^2)) \leq \xi(CC_n) \leq 7 \cdot 2^{n-2} n^2 (1 - (12n - 8)/(7n^2)).
\]

5.7 Edge-forwarding index

In this section, \( \alpha_{n, d, r} \) is as given in \((\mathbb{L} 10)\). We will use the theory of orbit proportional Cayley graphs \((\mathbb{D} 9)\) in our study the edge-forwarding index problem for \( Q_n(d, r) \).

Given a graph \( X = (V, E) \) and a subgroup \( H \) of \( Aut(X) \), the \( H \)-orbit on \( E(X) \) containing a given \( e \in E(X) \) is \( \{g(e) : g \in H\} \), and the stabiliser of \( u \in V(X) \) in \( H \) is \( H_u = \{g \in H : g(u) = u\} \). Define
\[
H_{u, v} = (H_u)_v = \{g \in H : g(u) = u, g(v) = v\}
\]
for distinct vertices \( u, v \in V(X) \).

Let \( \bar{R} = \cup_{(u, v) \in V \times V} \bar{R}_{uv} \) be the set of all paths in \( X \), where \( \bar{R}_{uv} \) is the set of all \( uv \)-paths in \( X \). A uniform flow \((\mathbb{K} 7)\) in \( X \) is a function \( f : \bar{R} \to [0, 1] \) such that \( \sum_{P \in \bar{R}_{uv}} f(P) = 1 \) for any distinct vertices \( u, v \in V \). A path \( P \) in \( X \) is active (under \( f \))(\((\mathbb{K} 7)\)) if \( f(P) > 0 \). The flow \( f \) is called integral if \( f(P) \in \{0, 1\} \) for any \( P \in \bar{R} \). An integral uniform flow is essentially the same as an all-to-all routing. Given a subgroup \( H \leq Aut(X) \), a uniform flow \( f \) is called \( H \)-invariant if \( f(P) = f(g(P)) \) for all \( g \in H \) and \( P \in \bar{R} \), where \( g(P) \) is the image of \( P \) under \( g \).
Lemma 5.7.1. ([13], Theorem 1) Let $X = (V, E)$ be a graph and $H$ a subgroup of $Aut(X)$. Then there exists an $H$-invariant uniform flow $f^*$ in $X$ such that any active path under $f^*$ is a shortest path and the number of active paths is at most $|H_{uv}|$. 

The $H$-invariant uniform flow in Lemma 5.7.1 is integral if $|H_{uv}| = 1$ for every pair $u, v$ of distinct vertices. Denote by $E_1, E_2, \ldots, E_k$ the $H$-orbits on $E(X)$. Of course $\{E_1, E_2, \ldots, E_k\}$ is a partition of $E(X)$. We say that $X$ is $H$-orbit proportional if for any shortest $uv$-path $P$ and any $uv$-path $P'$ in $X$,

$$|E(P) \cap E_i| \leq |E(P') \cap E_i|, \quad i = 1, 2, \ldots, k.$$ (5.53)

In particular, for $1 \leq i \leq k$, $|E(P) \cap E_i| = \min |E(P') \cap E_i|$ with the minimum running over all $P' \in \tilde{R}_{uv}$. Moreover, for $1 \leq i \leq k$, we have $|E(P) \cap E_i| = |E(P') \cap E_i|$ if both $P$ and $P'$ are shortest $uv$-paths. Not all Cayley graphs are orbit proportional.

It was proved in [13], Theorem 4] that if $X$ is $H$-orbit proportional and $f^*$ is an $H$-invariant uniform flow in $X$ such that any active path is a shortest path (the existence of $f^*$ is guaranteed by Lemma 5.7.1), then $\pi(X) = \max_{e \in E(X)} \sum_{P \in P(f^*)} f^*(P)$. On the other hand, for $e, e' \in E_i$, we have $\sum_{P \in P} f^*(P) = \sum_{P, e' \in P} f^*(P)$ since $f^*$ is $H$-invariant. Therefore, what was proved in [12], Lemma 5] is the following result:

$$\pi(X) = \pi_m(X) = \max_{e \in E(X)} \sum_{P \in P} f^*(P) = \max_{1 \leq i \leq k} \frac{\sum_{(u,v)} |E(P_{uv}) \cap E_i|}{|E_i|},$$ (5.54)

where $P_{uv}$ is any shortest $uv$-path in $X$.

Now let us return to the edge-forwarding index problem for $Q_n(d, r)$. Since $Q_n(d, r)$ is vertex-transitive, by [13], we have

$$\pi(Q_n(d, r)) \geq |G| \frac{td(Q_n(d, r))}{|E|}.$$ (5.55)

Define

$$E_0 := \{(a, x), (a, x + 1) : (a, x) \in G\},$$

$$E_i := \{(a, x), (a + e_i + dx, x) : (a, x) \in G\}, \quad 1 \leq i \leq d.$$ 

Then $|E_0| = 2^r r$ and $|E_i| = 2^{n-1} r$ for $1 \leq i \leq d$. It can be verified that $\{E_0, E_1, \ldots, E_d\}$ is a partition of the edge set of $Q_n(d, r)$.

$G$ can be viewed as a subgroup of $Aut(Q_n(d, r))$ since $Q_n(d, r)$ is a Cayley graph on $G$ (see Section 2.2). So we can talk about $G$-orbits on $E(Q_n(d, r))$.

Lemma 5.7.2. $E_0, E_1, \ldots, E_d$ are the $G$-orbits on $E(Q_n(d, r))$. 

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Proof. Since \((0_n, 0)\) is the identity element of \(G\), by Definition 5.7.1, \(E_0\) is the \(G\)-orbit on \(E(Q_n(d, r))\) containing \(\{(0_n, 0), (0_n, 1)\}\) and \(E_i\) is the \(G\)-orbit on \(E(Q_n(d, r))\) containing \(\{(0_n, 0), (e_i, 0)\}\), \(1 \leq i \leq d\). Since \(\{E_0, E_1, \ldots, E_d\}\) is a partition of \(E(Q_n(d, r))\), these are all \(G\)-orbits on \(E(Q_n(d, r))\).

Lemma 5.7.3. \(Q_n(d, r)\) is \(G\)-orbit proportional if and only if \(n \equiv 0 \mod d\).

Proof. Suppose that \(n \equiv 0 \mod d\). Let \((a, x) \in G\) and let \(P\) and \(P'\) be paths from \((0_n, 0)\) to \((a, x)\) in \(Q_n(d, r)\). Suppose that \(P\) is a shortest path so that \(|P| = \|a\| + l(a, x)\) by Lemma 5.7.1. For \(1 \leq j \leq n\), if \(a_j = 1\) (respectively, \(a_j = 0\)), then \(P'\) contains an odd (respectively, even) number of cube edges \(\{(b, y), (b + e_j, y)\}\) in direction \(e_j\), and \(P\) contains exactly one (respectively, zero) such edges by Lemma 5.7.1. On the other hand, we claim that any two cube edges \(\{(b, y), (b + e_j, y)\}\) and \(\{(b', y'), (b' + e_j, y')\}\) in the same direction \(e_j\) are in the same \(G\)-orbit \(E_k\) for some \(k\). In fact, by Definition 5.7.1, we have \(j \in D(y) \cap D(y')\) and so \(k + dy \equiv k' + dy' \mod n\) for some \(1 \leq k, k' \leq d\). Since \(n \equiv 0 \mod d\) by our assumption, \(k = k'\) and hence \((b + e_j, y) = (b, y)(e_k, 0)\) and \((b' + e_j, y') = (b', y')(e_k, 0)\). In other words, both \(\{(b, y), (b + e_j, y)\}\) and \(\{(b', y'), (b' + e_j, y')\}\) are in the \(G\)-orbit \(E_k\). This together with what we proved above implies that \(|E(P) \cap E_i| \leq |E(P') \cap E_i|\) for \(1 \leq i \leq d\). Moreover, \(|E(P) \cap E_0| \leq |E(P') \cap E_0|\), for otherwise there exists an \((a, x)\)-sequence \(\hat{x}\) obtained from segments of \(P'\) such that \(l(\hat{x}) < l(a, x)\), which is a contradiction. Therefore, (5.56) is satisfied and so \(Q_n(d, r)\) is \(G\)-orbit proportional.

Suppose that \(n \not\equiv 0 \mod d\). Then \(dr \geq 2n\) by (5.1). By Lemma 5.7.2, there exist \(1 \leq j \leq n\), \(1 \leq k, k' \leq d\) and \(y, y' \in \mathbb{Z}_r\) such that \(y \neq y'\), \(k \neq k'\) and \(j = dy + k = dy' + k' \mod n\). Let \(y, 0 \leq y < r\), be such that \(j \in D(y)\) and \(\min\{y, r - y\} \leq \min\{y', r - y'\}\) for any \(0 \leq y' < r\) with \(j \in D(y')\). The path \((0_n, 0), \ldots, (0_n, y), (e_j, y), \ldots, (e_j, 0)\) is a shortest path with exactly one edge in \(E_k\), and the path \((0_n, 0), \ldots, (0_n, y'), (e_j, y'), \ldots, (e_j, 0)\) has exactly one edge in \(E_{k'}\). Since the first path has an edge in \(E_k\) while the second path does not contain any edge in \(E_{k'}\), (5.56) is not satisfied by these paths and \(E_k\). Hence \(Q_n(d, r)\) is not \(G\)-orbit proportional.

In view of the discussion at the beginning of Section 5.8, any set \(\{P(a, x) : (a, x) \in G\}\) of shortest paths in \(Q_n(d, r)\) starting from \((0_n, 0)\) gives rise to a shortest path routing in \(Q_n(d, r)\) defined by

\[
\{(b, y)P(a, x) : (a, x), (b, y) \in G\}.
\]
Theorem 5.7.4.

(a) Suppose \( dr = n \). We have \( \pi(Q_n(d, r)) = \pi_m(Q_n(d, r)) \) and the following hold:

(i) if \( 3 \leq r \leq 6 \), then

\[
\pi(Q_n(d, r)) = 2^n r^2;
\]

(ii) if \( 7 \leq r < 2^9 \), then

\[
2^n r^2 \leq \pi(Q_n(d, r)) \leq 2^n - 2(5r^2 - 8r + 8);
\]

(iii) if \( r \geq 2^9 \), then

\[
2^n - 2r^2 \max \{4, 5(1 - (4 \log^2 r/r))\} \leq \pi(Q_n(d, r)) \leq 2^n - 2(5r^2 - 8r + 8).
\]

(b) Suppose \( dr \geq 2n \) and \( n \equiv 0 \mod d \). Then \( \pi(Q_n(d, r)) = \pi_m(Q_n(d, r)) \) and

\[
2^{n-1} \max \left\{ \frac{4nr}{d} \left( \frac{r^2}{2} + 4n^2/d^2 \right) (1 - \beta_{n,d,r}) \right\}
\leq \pi(Q_n(d, r)) \leq 2^{n-1} \left( \frac{r^2}{2} + 4n^2/d^2 \right).
\]

(c) If \( dr \geq 2n \), then

\[
\frac{2^n}{d+2} (nr + \frac{r^2}{2} + 4\lfloor n/d \rfloor^2)(1 - \alpha_{n,d,r}) \leq \pi(Q_n(d, r)) \leq 2^{n-1} \max \{ 4nr/d + 2r, (\frac{r^2}{2} + 4\lfloor n/d \rfloor^2) \}.
\]

Proof. Let \( P_{(a,x)} \) be a shortest path in \( Q_n(d, r) \) from \( (0_n, 0) \) to \( (a, x) \). Since \( G \) is regular on \( G \) in its left-regular multiplication, we have \( G_u = \{(0_n, 0)\} \) and so \( G_{uv} = \{(0_n, 0)\} \) for any two distinct vertices \( u \) and \( v \) of \( Q_n(d, r) \). Thus, by Lemma 5.7.3, there is exactly one \( uv \)-path \( P_{uv} \) such that \( f^*(P_{uv}) = 1 \) and \( P_{uv} \) is a shortest path.

If \( n \equiv 0 \mod d \), then by Lemma 5.7.3, \( \pi(Q_n(d, r)) \) is given by (5.59). Since

\[
\sum_{(u,v) \in G \times G} |E(P_{uv}) \cap E_i| = 2^n r \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_i|, 0 \leq i \leq d,
\]

we have

\[
\pi(Q_n(d, r)) = \pi_m(Q_n(d, r)) = \max_{0 \leq i \leq d} \frac{2^n r \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_i|}{|E_i|}
\]

\[
= \max \left\{ \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_0|, \max_{1 \leq i \leq d} 2 \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_i| \right\}
\]

\[
= \max \left\{ \sum_{(a,x) \in G} l(a, x), \max_{1 \leq i \leq d} 2r \sum_{a \in \mathbb{Z}_2} |E(P_{(a,0)}) \cap E_i| \right\}.
\]

(5.59)
(a) Suppose $dr = n$. For $1 \leq i \leq d$ and $0 \leq y \leq r - 1$, $P_{(a,x)}$ contains exactly one edge in $E_i$ if and only if $a_{dy+i} = 1$. Hence $\sum_{a \in \mathbb{Z}_2^n} |E(P_{(a,0)}) \cap E_i| = 2^{n-r} \sum_{y=0}^r y(y + 1) = 2^{n-1}r$. This together with (5.27), (5.24), (5.22), and (5.21) yields the result.

(b) Suppose $dr \geq 2n$ and $n \equiv 0 \mod d$. For $1 \leq i \leq d$, $P_{(a,x)}$ has exactly one edge in $E_i$ if and only if $a_{dy+i} = 1$ for some $0 \leq y \leq r - 1$. Since by Lemma 5.2.4 the number of distinct values of $dy+i$ is $n/d$, we have $\sum_{a \in \mathbb{Z}_2^n} |E(P_{(a,0)}) \cap E_i| = 2^{n-n/d} \sum_{y=0}^{n/d} y(y+n/d) = 2^{n-1}n/d$. Therefore, by (5.25) and (5.23) we get (5.57) immediately.

(c) Suppose $n \not\equiv 0 \mod d$. Denote by $R$ the routing (5.59) based on our chosen shortest paths $P_{(a,x)}$. Let $0 \leq x < r$ and $i \in D(x)$ be fixed. For any pair of vertices $((b,y),(c,z))$, the path $P \in R$ from $(b,y)$ to $(c,z)$ passes through $e = \{(a,x), (a + e_i,x)\}$ for some $a \in \mathbb{Z}_2^n$ provided that its corresponding optimal sequence (obtained from (5.14)) contains $x$. By the construction of the optimal sequence for $P$, we have $r - [n/d] + 1 \leq x - y \leq [n/d] - 1$ and $b_i \neq c_i$. Therefore, there are at most $2^{n-1}(2[n/d] - 1)r \leq 2^{n-1}nr/d + 2^{n-1}r$ paths in $R$ containing $e$. On the other hand, for any path $P : (b,y), \ldots, (a,x), (a + e_i,x), \ldots, (c,z)$ in $R$ that passes through $e$ and any $a' \in \mathbb{Z}_2^n$, the path $gP : g(b,y), \ldots, (a',x), (a' + e_i,x), \ldots, g(c,z)$ in $R$, where $g = (a' - a,0)$, passes through the edge $e' = \{(a',x), (a' + e_i,x)\}$. Therefore, the paths in $R$ uniformly load the edges of $\{(a,x), (a + e_i,x)\} : a \in \mathbb{Z}_2^n$. Thus, the load on each cube edge $\{(a,x), (a + e_i,x)\}$ under $R$ is at most $2^{n+1}nr/d + 2^{n}r$.

On the other hand, the load on ring edges of $Q_n(d,r)$ under $R$ is uniform since $R$ is $G$-invariant and the set of ring edges forms a $G$-edge orbit. Similar to (b), the load of $R$ on ring edges is given by (5.52). This together with the upper bound for the load on the cube edges under $R$ gives the upper bound in (5.58) for $\pi(Q_n(d,r))$ and $\pi_m(Q_n(d,r))$. The lower bound in (5.58) follows from (5.55) and Theorem 5.7.4.

**Remark.** (a) By Theorem 5.7.4, when $r$ is large, we have

$$\pi(Q_n(d,r)) \approx 5 \cdot 2^{n-2}r^2, \quad \text{if } dr = n$$

$$\pi(Q_n(d,r)) \approx 2^{n-1}\left([r^2/2] + 4n^2/d^2\right), \quad \text{if } dr \geq 2n \text{ and } n \equiv 0 \mod d.$$
(ii) $1.82(5d + 10)/(4d + 10) \leq 1.95$ if $dr = n$ and $r \geq 2^9$, (iii) $1/(1 - \alpha_{n,d,r})$ if $n \equiv 0 \mod d$ and $dr = kn$ for some integer $k \geq 3$, and (iv) $\max\{2(d + 2)(2n + d)/(2n + r), 1 + d(k^2 + 8)/(2dk + k^2 + 8)(1 - \alpha_{n,d,r})\} < 6$ if $dr = kn$ for some integer $k \geq 2$. In the first three cases $Q_n(d, r)$ has relatively small edge-forwarding index.

By Lemma 5.2.2, we obtain the following corollary of Theorem 5.7.4. In particular, we recover the formula $\pi(CC_n) = 5n^22^{n-2}(1 - o(1))$ ([10], Theorem 3). In fact, we give more accurate lower and upper bounds for $\pi(CC_n)$ and $\pi_m(CC_n)$. We observe that the $d$-ply cube-connected cycles $Q_{kd}(d, kd)$ with $k, d \geq 2$ have smaller edge-forwarding indices than the usual cube-connected cycles.

**Corollary 5.7.5.**

(a) $2^{n-2}n^2 \max\{4, 5(1 - 4(\log^2 n)/n)\} \leq \pi(CC_n) = \pi_m(CC_n) \leq 5.2^{n-2}n^2(1 - (8n - 8)/(5n^2))$ if $n \geq 2^9$; $2^n2 \leq \pi(CC_n) = \pi_m(CC_n) \leq 5.2^{n-2}n^2(1 - (8n - 8)/(5n^2))$ if $7 \leq n < 2^9$; and $\pi(CC_n) = \pi_m(CC_n) = 2^n2$ if $3 \leq n \leq 6$.

(b) $2^{dr-2}r^2 \max\{4, 5(1 - 4(\log^2 r)/r)\} \leq \pi(COR(d, r)) = \pi_m(COR(d, r)) \leq 2^{dr-2}(5r^2 - 8(r - 1))$ if $r \geq 2^9$; $2^{dr-2} \leq \pi(COR(d, r)) = \pi_m(COR(d, r)) \leq 2^{dr-2}(5r^2 - 8(r - 1))$ if $7 \leq r < 2^9$; and $\pi(COR(d, r)) = \pi_m(COR(d, r)) = 2^{dr-2}$ if $3 \leq r \leq 6$.

(c) $2^{kd-1} \max\{2k^2d, ([k^2d^2/2] + 4k^2)(1 - \beta_{kd,d,kd})\} \leq \pi(Q_{kd}(d, kd)) = \pi_m(Q_{kd}(d, kd)) \leq 2^{kd-1}([k^2d^2/2] + 4k^2)$.

### 5.8 Bisection width

It was claimed in [11] Corollary 1] and [8], Theorem 2] (with a difference for case $r = 1$) that the bisection width of $Q_n^-(d, r)$ is equal to $2^{n-1}dr/n$ and shown in [10], Theorem 12] that $\bw(Q_n^-(d, r)) \leq 2^{n-1}dr/n$. However, it has been unknown whether the upper bound on $\bw(Q_n^-(d, r))$ in [10] is sharp or not. Using Theorem 5.7.4, we give sharp bounds on $\bw(Q_n(d, r))$ in the following theorem. Our result shows in particular that in some cases $\bw(Q_n^-(d, r)) < 2^{n-1}dr/n$ and so the related results in [8, 11] are incorrect and the upper bound on $\bw(Q_n^-(d, r))$ in [10] is not sharp in general.

**Theorem 5.8.1.**

(a) If $dr = n$, then

(i) $\bw(Q_n(d, r)) = 2^{n-1}$ when $3 \leq r \leq 6$;

(ii) $2^{n-1}/(5 - (8(r - 1)/(r^2))) \leq \bw(Q_n(d, r)) \leq 2^{n-1}$ when $r \geq 7$.

(b) If $dr \geq 2n$, then
(i) \(2^{n+1}r^2/(r^2+8n^2/d^2) \leq \text{bw}(Q_n(d, r)) \leq 2^n \min\{dr/2n, 2\} \) when either \(r\) is even or \(n = kd\) for some \(k \geq 2\);

(ii) \(2^{n+1}r^2/(r^2+8) \leq \text{bw}(Q_n(d, r)) \leq 2^{n-1} \min\{r, 5\} \) when \(r\) is odd and \(n = d\);

(iii) \(2^n \min\{dr/(4n+2d), 2r^2/(r^2+(8|n/d|^2))\} \leq \text{bw}(Q_n(d, r)) \leq 2^n \min\{dr/2n, 2\} \) when \(n \neq 0 \mod d\).

Proof. The lower bound in each case is obtained from (2.34) and the corresponding upper bound for \(\pi(Q_n(d, r))\) in Theorem 5.7. Hence we have

\[
\text{bw}(Q_n(d, r)) \geq \begin{cases} 
2^{n-1}, & \text{if } dr = n \text{ and } 3 \leq r \leq 6, \\
2^{n+1}/(5 - (8(r-1)/r^2)), & \text{if } dr = n \text{ and } r \geq 7, \\
2^{n+1}r^2/(r^2+8n^2/d^2), & \text{if } dr \geq 2n \text{ and } n \equiv 0 \mod d, \\
2^n \min\{dr/(4n+2d), 2r^2/(r^2+8|n/d|^2)\}, & \text{if } dr \geq 2n.
\end{cases}
\]

(5.60)

It remains to prove the upper bounds for \(\text{bw}(Q_n(d, r))\). Let \(U\) be the set of vertices \((a, x)\) of \(Q_n(d, r)\) with \(a = (a_1, a_2, \ldots, a_{n-1}, 1)\). Then \(\overline{U}\) is the set of vertices \((a, x)\) such that \(a = (a_1, a_2, \ldots, a_{n-1}, 0)\) and \(\{U, \overline{U}\}\) is a bisection of \(Q_n(d, r)\). There is an edge between \((a, x) \in U\) and \((a + e_n, x) \in \overline{U}\) if and only if \(n \in D(x)\). Thus,

\[
\delta(U, \overline{U}) = \{(a, x), (a + e_n, x) : n \in D(x), 0 \leq x \leq r - 1\}.
\]

By Lemma 5.2.2, there are \(dr/n\) distinct elements \(x\) such that \(n \in D(x)\). Hence \(\mid \delta(U, \overline{U}) \mid = 2^{n-1}dr/n\) and so

\[
\text{bw}(Q_n(d, r)) \leq 2^{n-1}dr/n.
\]

(5.61)

We now use other bisections to obtain better upper bounds in some cases.

Case 1: \(r\) even. Let \(U = \{(a, x) : a \in \mathbb{Z}_2^n, 0 \leq x \leq r/2 - 1\}\). Then \(\{U, \overline{U}\}\) is a bisection of \(Q_n(d, r)\) and

\[
\delta(U, \overline{U}) = \{(a, x), (a, x + 1) : a \in \mathbb{Z}_2^n, x = r/2 - 1, r - 1\}.
\]

Hence \(\text{bw}(Q_n(d, r)) \leq \mid \delta(U, \overline{U}) \mid = 2^{n+1}\).

Case 2: \(r\) odd. Let \(U = \{(a, x) : a \in \mathbb{Z}_2^n, 1 \leq x \leq (r-1)/2\} \cup \{(a, 0) : a \in \mathbb{Z}_2^n, a_n = 0\}\). Then \(\overline{U} = \{(a, x) : a \in \mathbb{Z}_2^n, (r+1)/2 \leq x \leq r - 1\} \cup \{(a, 0) : a \in \mathbb{Z}_2^n, a_n = 1\}\) and \(\{U, \overline{U}\}\) is a bisection of \(Q_n(d, r)\). Two vertices \(((a_1, \ldots, a_{n-1}, 1), 0)\) and \(((b_1, \ldots, b_{n-1}, 0), 0)\) are adjacent if and only if \(n \in D(0)\). Hence, if \(n > d\), then
these two vertices are not adjacent since \( n \notin D(0) \). Thus, if \( n > d \), then

\[
\delta(U, \overline{U}) = \{(a, (r - 1)/2), (a, (r + 1)/2) : a \in \mathbb{Z}_2^n\} \cup \\
\{(a, 0), (a, 1) : a \in \mathbb{Z}_2^n, a_n = 1\} \cup \{(a, 0), (a, r - 1) : a \in \mathbb{Z}_2^n, a_n = 0\}.
\]

Therefore, \(|\delta(U, \overline{U})| = 2^{n+1}\). If \( n = d \), then \(((a_1, \ldots, a_{n-1}, 0), 0)\) and \(((b_1, \ldots, b_{n-1}, 0), 0)\) are adjacent and so \(bw(Q_n(d, r)) \leq |\delta(U, \overline{U})| = 2^{n+1} + 2^{n-1}\).

Combining (5.61) and the upper bounds for \(bw(Q_n(d, r))\) in Cases 1 and 2, we obtain

\[
bw(Q_n(d, r)) \leq \begin{cases} 
\min\{2^{n-1}dr/n, 2^{n+1}\}, & \text{if } r \text{ even or } n > d, \\
\min\{2^{n-1}dr/n, 5 \cdot 2^{n-1}\}, & \text{if } r \text{ odd and } n = d.
\end{cases}
\]

This together with (5.61) completes the proof. \(\square\)
Chapter 6

Cube-connected circulants

In this chapter, we introduce cube-connected circulants as efficient models for communication networks. We give an algorithm for computing a shortest path between any pair of vertices in a cube-connected circulant. We give formulas for the diameter of a cube-connected circulant and the distance between any pair of vertices in such a graph. Then we provide sharp lower and upper bounds for the Wiener index, vertex-forwarding and edge-forwarding indices of any cube-connected circulant. While recursive cubes of rings are special cases of cube-connected circulants, obtaining the edge-forwarding index of a cube-connected circulant is more challenging than that of recursive cubes of rings in general.

6.1 Introduction

There are many graph models suggested for interconnection networks. However, there is no graph model which possesses all good properties in general. Hence new graphs are introduced to overcome the deficiency of their predecessor network models [74, 101, 118]. For instance, the cube-connected cycles, the multiplicative circulants and the recursive circulants were proposed as alternative network models for the hypercubes and shuffle-exchange graphs [88, 90, 100]. Recursive cubes of rings were introduced as more general and alternative models for the cube-connected cycles [101, 109] and they have many desirable properties such as logarithmic diameter, vertex-transitivity and fixed degree [90]. In this chapter we introduce a new and efficient family of Cayley graphs, namely cube-connected circulants, as models for interconnection networks. As we will see, cube-connected circulants outperform many existing graphs, including re-
cursive cubes of rings, in several invariants. The factor graphs for recursive cubes of rings are hypercubes and rings. It is known that rings are not efficient models for interconnection networks in general since they have low connectivity and large edge-forwarding index, vertex-forwarding index and diameter. Hence one reason for shortcomings of recursive cubes of rings is ring graphs as factors of this family of graphs. So overcoming such shortcomings is our primary motivation to introduce the family of cube-connected graphs such that its factor graphs are more efficient. Since the multiplicative circulants outperform a few well-known graphs, e.g. hypercubes, in many parameters, they are appropriate candidates for being factor graphs of some cube-connected graphs. Therefore, we construct cube-connected circulants with hypercubes and multiplicative circulant graphs as factor graphs.

By adopting the algebraic tools used in Chapter 5, we define 'cube-connected circulants' on the semidirect products of elementary abelian groups by cyclic groups. We will show that this family of graphs can be constructed recursively in Section 6.2. In Section 6.3 we develop a shortest path algorithm for cube-connected circulants by retrieving techniques from Section 5.3. The diameter of cube-connected circulants will be given in Section 6.3 and the total distance and Wiener index of any cube-connected circulant will be given in Section 6.4. We will use results in Section 6.5 to obtain the vertex-forwarding and edge-forwarding indices of cube-connected circulants in Sections 6.6 and 6.7. We study the bisection width of this family of graphs in Section 6.8 and embedding of cube-connected circulants into hypercubes and embedding of hypercubes into cube-connected circulants in Section 6.9. We also compare cube-connected circulants with well-known graph models with respect to a few invariants. A cube-connected circulant can be thought of as a generalisation of a recursive cube of rings, but, in addition to all advantages of recursive cubes of rings, it acquires significantly smaller diameter and edge-forwarding index when its parameters are set appropriately. Despite similarities between these two families of graphs, they are notably different in routing. For instance, cube-connected circulants are not orbit-proportional, whereas recursive cubes of rings are (see Section 6.7). Hence we will develop results in Section 6.7 to obtain the edge-forwarding index of this family of graphs. Our method can also be used for other graphs which satisfy certain conditions. Furthermore, we show that cube-connected circulants have almost optimal vertex-forwarding and edge-forwarding indices in view of (5.3). Finally, we remark that cube-connected circulants outperform many existing graphs in several invariants.
Cube-connected circulants

A multiplicative circulant $C(r,m)$ is a circulant graph on the vertex set $\mathbb{Z}_{r^m}$ with connection set $\{\pm r^j : 0 \leq j < m\}$ (see Section 2.3). It is known that multiplicative circulants have simple routing algorithms, recursive construction and short diameter (in fact, they are optimal circulants (see Section 3.3)) \cite{89,100,107}. In the following we use multiplicative circulants as a factor graph to construct a cube-connected graph.

Throughout this chapter we assume $n$, $d$, $m$ and $r$ are positive integers such that $n \geq d$ and $r \geq 3$. Furthermore, we assume

$$dr \equiv 0 \mod n. \quad (6.1)$$

Similar to Section 5.2.1, let $e_i$ denote the row vector of $\mathbb{F}_2^n$ with its $i$th coordinate 1 and all other coordinates 0, denote its transpose by $e_i^\top$, $1 \leq i \leq n$, and take the subscripts of these vectors modulo $n$. Recall that

$$M = [e_2^\top, \ldots, e_n^\top, e_1^\top]$$

is an element of $GL(n,2)$. Hence $M^n = I_n$ is the identity element of $GL(n,2)$ and $e_i M^j = e_{i+j}$ for any integers $i$ and $j$. The mapping $\varphi : \mathbb{Z}_r \rightarrow Aut(\mathbb{Z}_2^n) = GL(n,2)$ with $x \mapsto \varphi_x$ defined by

$$\varphi(x)(a) = a M^x, \quad a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_2^n, \quad x \in \mathbb{Z}_r$$

is a homomorphism from $\mathbb{Z}_r$ to $Aut(\mathbb{Z}_2^n)$. In other words, the rule (6.2) defines an action as a group of the cyclic group $\mathbb{Z}_r$ on the elementary abelian 2-group $\mathbb{Z}_2^n$.

Now define

$$G := \mathbb{Z}_2^n \rtimes_{\varphi} \mathbb{Z}_r,$$  \quad (6.3)

to be the semidirect product of $\mathbb{Z}_2^n$ by $\mathbb{Z}_r$ with respect to the action (6.2). In view of (6.2), the operation of $G$ is given by

$$(a, x)(b, y) = (a + b M^x, x + y),$$

where the second coordinate $x + y$ is taken modulo $r^m$. It can be verified that the identity element of $G$ is $(0_n, 0)$ and the inverse of $(a, x)$ in $G$ is $(-a M^{-x}, r^m - x)$, where $0_n = (0, 0, \ldots, 0)$ is the identity element of $\mathbb{Z}_2^n$.

**Definition 6.2.1.** Let $n$, $d$, $m$ and $r$ be positive integers such that $n \geq 2$, $n \geq d$, $r \geq 3$ and condition (6.1) is satisfied. Define $CQ_n(d,r,m)$ to be the Cayley graph $Cay(G,S)$. 110
on \( G = \mathbb{Z}_2^d \times_{\varphi} \mathbb{Z}_{r^m} \) with respect to the connection set

\[
S = \{ (0_n, 1), (0_n, r), \ldots, (0_n, r^{m-1}), (0_n, -1), \ldots, (0_n, -r^{m-1}), (e_1, 0), \ldots, (e_d, 0) \}.
\]

(6.4)

In other words, the vertex set of \( CQ_n(d, r, m) \) is \( G \) and a vertex \( (a, x) \) in \( CQ_n(d, r, m) \) is adjacent to vertices

(i) \( (a, x)(e_i, 0) = (a + e_{i+d}x, x) \), \( 1 \leq i \leq d \);

(ii) \( (a, x)(0_n, r^j) = (a, x + r^j) \) and \( (a, x)(0_n, r^j) = (a, x - r^j) \), \( 0 \leq j \leq m - 1 \).

We call \( CQ_n(d, r, m) \) a cube-connected circulant.

The edges between \( (a, x) \) and \( (a, x + r^j) \) are circulant edges of \( CQ_n(d, r, m) \), and \( (a, x + r^j) \) and \( (a, x - r^j) \) are circulant neighbours of \( (a, x) \).

The edges between \( (a, x) \) and \( (a + e_i dx, x) \) are cube edges of \( CQ_n(d, r, m) \) in direction \( i + dx \), and \( (a + e_i dx, x) \) is a cube neighbour of \( (a, x) \).

The multiplicative circulant graph on vertices \( (a, 0), (a, 1), \ldots, (a, r^m - 1), (a, 0) \) of \( CQ_n(d, r, m) \) is called the \( a \)-circulant of \( CQ_n(d, r, m) \).

A cube-connected circulant \( CQ_n(d, r, m) \) can be thought of as a hypercube of dimension \( n \) whose vertices are replaced with a multiplicative circulant graph \( C(r, m) \) and their cube edges are modified according to Definition 6.2.1. See Figure 6.1 for an example of \( CQ_n(d, r, m) \).

From Definition 6.2.1 we observe that \( (a, x) \) is adjacent to \( (a + e_j, x) \) for any \( x \in \mathbb{Z}_{r^m} \), if and only if \( j \in D(x) \), where

\[
D(x) := \{ i + dx \pmod{n} : 1 \leq i \leq d \}.
\]

(6.5)

From the definition of \( CQ_n(d, r, m) \) and \( Q_n(d, r) \), any recursive cube of rings is a special cube-connected circulant. From Definition 6.2.1 and Lemma 6.2.2 the following lemma is obtained.

**Lemma 6.2.2.** \( Q_n(d, r) \cong CQ_n(d, r, 1) \) for \( n \geq d \geq 1, r \geq 3 \) and \( CC_n \cong CQ_n(1, n, 1) \) for \( n \geq 3 \). Moreover, \( Q_n(d, r^m) \) is a subgraph of \( CQ_n(d, r, m) \).

**Lemma 6.2.3.** \( CQ_n(d, r, m) \) is a connected graph with \( r^m 2^n \) vertices of degree \( (d + 2m) \) and \( (d + 2m) r^m 2^{n-1} \) edges.

**Proof.** \( CQ_n(d, r, m) \) is connected since it contains \( Q_n(d, r^m) \) as a connected subgraph by Lemmas 6.2.1 and 6.2.2. Since \( CQ_n(d, r, m) \) is a Cayley graph, the number of vertices of \( CQ_n(d, r, m) \) is \( |G| \) and its degree is \( |S| \).

\[ \Box \]
In the following we show that cube-connected circulants enjoy a recursive construction.

**Theorem 6.2.4.** \( \text{CQ}_n(d, r, m) \) can be constructed from \( r \) copies of \( \text{CQ}_n(d, r, m - 1) \).

**Proof.** For \( 0 \leq k \leq r - 1 \), denote by \( G_k \) the subset of \( G \) consisting of \((a, kr^{m-1} + x)\) with \( a \in \mathbb{Z}_2^n \) and \( 0 \leq x < r^{m-1} \). Obviously, \( \{G_0, \ldots, G_{r-1}\} \) is a partition of the vertex set of \( \text{CQ}_n(d, r, m) \). Denote by \( \text{CQ}_k \) the subgraph of \( \text{CQ}_n(d, r, m) \) induced by \( G_k \).

First we show that \( \text{CQ}_k \) is isomorphic to \( \text{CQ}_n(d, r, m - 1) \), \( 0 \leq k \leq r - 1 \). By Definition 6.2.1, \( \text{CQ}_n(d, r, m - 1) \) is a Cayley graph on the vertex set \( G' = \{(a, x) : a \in \mathbb{Z}_2^n, 0 \leq x \leq r^{m-1} - 1\} \) with respect to the connection set

\[ \{(0_n, 1), (0_n, r), \ldots, (0_n, r^{m-2}), (0_n, -1), \ldots, (0_n, -r^{m-2}), (e_1, 0), \ldots, (e_d, 0)\}. \]

For a fixed \( k \), \( 0 \leq k \leq r - 1 \), the mapping \( h_k : G' \mapsto G_k \) with \( h_k(a, x) = (a, kr^{m-1} + x) \) defines an isomorphism from \( \text{CQ}_n(d, r, m - 1) \) to \( \text{CQ}_k \). In fact, for any \( i \) with \( 1 \leq i \leq d \), \((a, x)\) and \((a + e_i dx, x)\) are adjacent in \( \text{CQ}_k \) if and only if \( h_k(a, x) \) and \( h_k(a + e_i dx, x) = (a + e_i dx, kr^{m-1} + x) \) are adjacent in \( \text{CQ}_n(d, r, m) \) since \( D(x + kr^{m-1}) = D(x) \) by (6.1). On the other hand, for \( 0 \leq j \leq m - 2 \) and \( \delta \in \{-1, 1\}, (a, x) \) and \((a, x + \delta r^j)\) are adjacent in \( \text{CQ}_n(d, r, m) \) if and only if \( h_k(a, x) \) and \( h_k(a, x + \delta r^j) = (a, kr^{m-1} + x + \delta r^j) \) are adjacent in \( \text{CQ}_k \). Hence \( \text{CQ}_k \cong \text{CQ}_n(d, r, m - 1), 0 \leq k \leq r - 1 \).

By using isomorphism \( h_k \) and labelling the vertices of the \( k \)th copy of \( \text{CQ}_n(d, r, m - 1) \) with vertices of \( \text{CQ}_k \) for \( 0 \leq k \leq r - 1 \), we get a one-to-one mapping from the vertex...
set of $CQ_n(d, r, m)$ to the vertex set of the union of these $r$ copies of $CQ_n(d, r, m - 1)$. Clearly the graph obtained by the union of these $r$ copies of $CQ_n(d, r, m - 1)$ has vertex set $G$, cube edges with end-vertices $(a, x)$ and $(a + e_j, x)$ for $a \in \mathbb{Z}_2^n$, $0 \leq x \leq r^m - 1$ and $j \in D(x)$, and circulant edges with end-vertices $(a, x)$ and $(a, x + r^j)$ for $a \in \mathbb{Z}_2^n$, $0 \leq x \leq r^m - 1$ and $0 \leq j \leq r - 2$. Therefore, this union of $r$ copies of $CQ_n(d, r, m - 1)$ together with circulant edges with end-vertices $(a, x)$ and $(a, r^{m-1} + x)$ for $a \in \mathbb{Z}_2^n$ and $0 \leq x \leq r^m - 1$, is isomorphic to $CQ_n(d, r, m)$.

It is worth mentioning that (6.1) is essential for the vertex-transitivity and connectivity of a cube-connected circulant $CQ_n(d, r, m)$. In fact, recursive cubes of rings are special cube-connected circulants and, by Theorem 5.2.7, a variation of them with $dr \not\equiv 0 \mod n$ is not vertex-transitive.

**Lemma 6.2.5.** For any fixed $a \in \mathbb{Z}_2^n$ and $j$ with $1 \leq j \leq n$, there are exactly $dr^m/n$ distinct cube edges of $CQ_n(d, r, m)$ with direction $e_j$ that are incident to some vertices of the $a$-circulant, namely the edges joining $(a, x_1)$ and $(a + e_j, x_1)$, where $x_1 = [(j + ln - 1)/d]$, $0 \leq l < dr^m/n$.

**Proof.** The proof is similar to that of Lemma 6.2.4 and so is omitted.

### 6.3 Shortest path in $CQ_n(d, r, m)$

In this section, we give an algorithm for a shortest path between any two vertices in $CQ_n(d, r, m)$. This will give a formula for the distance of any pair of vertices in $CQ_n(d, r, m)$. Since there are similarities between recursive cube of rings and cube-connected circulants, we will generalise some techniques and terminologies from Section 5.3. Since $CQ_n(d, r, m)$ for $m = 1$ is isomorphic to $Q_n(d, r)$ and the shortest path for this case is known (see Section 5.3), we assume $m \geq 2$ in this section.

Suppose $P$ is a path in $CQ_n(d, r, m)$ from $(0_n, 0)$ to a given $(a, x) \in G$ with $s$ cube edges. Removing these $s$ cube edges from $P$ results in $s + 1$ subpaths; each is a path in a multiplicative circulant and is called a segment. Note that such a segment may contain only one vertex, and this happens if and only if this vertex is incident to two cube edges or it is $(0_n, 0)$ or $(a, x)$ and incident to one cube edge on $P$. The first segment must be on the $0_n$-circulant, say from $(0_n, 0)$ to $(0_n, x_1)$ for some $x_1 \in \mathbb{Z}_n^m$. If the cube edge on $P$ incident to $(0_n, x_1)$ is in direction $e_{i_1}$, then the second segment must be on the $e_{i_1}$-circulant from $(e_{i_1}, x_1)$ to, say, $(e_{i_1}, x_2)$. In general, for $1 \leq t \leq s + 1$, we may assume that the $t$th segment is on the $(e_{i_1} + \cdots + e_{i_{t-1}})$-circulant connecting
Any $d$-tuple in $CQ$ paths in $\mathbb{Z}^n$, where $e_i$ is interpreted as $0$ or $1$. This implies that, for $1 \leq t \leq s$, the $t$th cube edge on $P$ is in direction $e_i$ and it connects $(e_i + \cdots + e_{i-1}, x_t)$ and $(e_i + \cdots + e_{i}, x_t)$. By the definition of $CQ_n(d,r,m)$, we have $i_t \in D(x_t)$ for $1 \leq t \leq s$. So every path $P$ in $CQ_n(d,r,m)$ from $(0_n,0)$ to $(a,x)$ determines two tuples, namely, $(x_0, x_1, \ldots, x_s, x_{s+1})$ and $(i_1, \ldots, i_s)$, where $x_0 = 0$, $x_{s+1} = x$ and $e_i + \cdots + e_{i} = a$. Conversely, any tuples

$$\hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1}), \hat{i} = (i_1, \ldots, i_s),$$

such that $i_t \in D(x_t)$ for each $t$, $x_0 = 0$, $x_{s+1} = x$ and $e_i + \cdots + e_i = a$, give rise to paths in $CQ_n(d,r,m)$ from $(0_n,0)$ to $(a,x)$ with $s$ cube edges and $s+1$ segments, where the $t$th segment can be any paths from $(e_i + \cdots + e_{i-1}, x_t)$ to $(e_i + \cdots + e_{i}, x_t)$ in the $(e_i + \cdots + e_{i})$-circulant. If we choose a shortest path among these paths for every $t$, then we get a path from $(0_n,0)$ to $(a,x)$ with shortest length among all these paths and the length of this path is $s + \ell_e(\hat{x})$ (which is independent of $\hat{i}$), where

$$\ell_e(x) = \sum_{t=0}^{s+1} d_m(x_{t-1}, x_t)$$

and $d_m(x,y)$ is the distance of $x$ and $y$ in the multiplicative circulant graph $C(r,m)$ such that $x, y$ are taken modulo $r^m$.

One can verify that any shortest path from $(0_n,0)$ to $(a,x)$ contains exactly one cube edge in direction $e_i$ if $a_i = 1$ and no cube edge in direction $e_i$ if $a_i = 0$. Thus, the number of cube edges in any shortest path from $(0_n,0)$ to $(a,x)$ is equal to $\|a\|$, as given in (6.6).

Define an $(a,x)$-sequence to be a tuple $x = (x_0, x_1, \ldots, x_s, x_{s+1})$ with $x_t \in \mathbb{Z}^n$, $0 \leq t \leq s+1$, such that $x_0 = 0$, $x_{s+1} = x$, $s = \|a\|$, and for every $i$ with $a_i = 1$ there is a unique $t$ with $i \in D(x_t)$. Denote

$$\ell_e(a,x) := \min_{\hat{x}} \ell_e(\hat{x}),$$

where the minimum is running over all $(a,x)$-sequences $\hat{x}$. An $(a,x)$-sequence achieving the minimum in (6.7) is said to be optimal. From the discussion above, we obtain the following lemma in which $\text{dist}((0_n,0),(a,x))$ denotes the distance between $(0_n,0)$ and $(a,x)$ in $CQ_n(d,r,m)$.

Lemma 6.3.1.

(a) Any $(a,x)$-sequence $\hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1})$ and $\hat{i} = (i_1, \ldots, i_s)$ such that $a_i = 1$ and $i_t \in D(x_t)$, for $1 \leq t \leq s$ and $s = \|a\|$, gives rise to paths from $(0_n,0)$ to $(a,x)$.
Figure 6.2: $P^*$ is obtained from $P$ by permuting all edges with label $(0_n, \delta r^j)$, $\delta \in \{-1, 1\}$, $1 \leq j < m$, to the last segment

(b) the minimum length among paths obtained from $\hat{x}$ and $\hat{i}$ is equal to $\|a\| + \ell_c(\hat{x})$;

c) $\text{dist}((0_n, 0), (a, x)) = \|a\| + \ell_c(a, x)$.

In the rest of this section, we find an optimal $(a, x)$-sequence and obtain a formula for $\ell_c(a, x)$ for a given vertex $(a, x)$ in $CQ_n(d, r, m)$. Using $\ell_c(a, x)$ and Lemma 6.3.1, we obtain formulas for the distance between $(0_n, 0)$ and $(a, x)$, which give the distance between any pair of vertices in $CQ_n(d, r, m)$ as this graph is vertex-transitive. In the following we describe a relation between optimal sequences in $Q_n(d, r)$ as given in Chapter 5 and those of $CQ_n(d, r, m)$, and then we use results in Chapter 5.

Any path in a Cayley graph can be uniquely represented by its edge labels (namely, jumps) from the connection set of the graph and one of its end-vertices. Hence by representing a given path $P$ from $(0_n, 0)$ to $(a, x)$ in $CQ_n(d, r, m)$ with its jumps from $S$ as given in (5.4), the $t$th segment of $P$ is represented by a sequence of labels from $\{(0_n, \pm 1), (0_n, \pm r), \ldots, (0_n, \pm r^{m-1})\}$, $1 \leq t \leq s + 1$, and any cube edge of $P$ is represented by a label $(e_i, 0)$ for some $1 \leq i \leq d$. Denote by $\hat{x} = (x_0, x_1, \ldots, x_s, x_{s+1})$ and $\hat{i}$ the corresponding tuples of $P$. In the following we obtain two paths $P'$ and $P^*$ from $P$. Intuitively, $P'$ will be obtained by omitting any jump $(0_n, r^k)$ and $(0_n, -r^k)$ with $1 \leq k \leq m - 1$ from segments of $P$, and $P^*$ will be obtained by permuting all jumps $(0_n, r^k)$ and $(0_n, -r^k)$ with $1 \leq k \leq m - 1$, in $P$ to the last segment while preserving their relative order (see Figure 6.2 for an illustration).

Construct a new path $P'$ whose $t$th segment is obtained from all jumps $(0_n, 1)$ and $(0_n, -1)$ of the $t$th segment of $P$ while their relative orders from $P$ are preserved, $1 \leq t \leq s + 1$. Let $\hat{w} = (w_0, w_1, \ldots, w_s, w_{s+1})$ be the corresponding tuple to the
segments of }\mathcal{P}'\text{ such that the }t\text{th segment of }\mathcal{P}'\text{ is a path from }\langle a_t, w_{t-1} \rangle \text{ to }\langle a_t, w_t \rangle\text{ for some }a_t \in \mathbb{Z}_n^d.\text{ The jumps of segments of }\mathcal{P}\text{ which are not included in the segments of }\mathcal{P}'\text{ are }\langle 0_n, r^k \rangle \text{ or }\langle 0_n, -r^k \rangle\text{ for some }1 \leq k \leq m - 1.\text{ It follows that }x_t \equiv w_t \mod r\text{ and so }D(x_t) = D(w_t), 0 \leq t \leq s + 1.\text{ Hence }\hat{w} \text{ is an }\langle a, w \rangle\text{-sequence, where }w = w_{s+1}.\text{ By Lemma }6.3.1, \hat{w} \text{ together with }\hat{i}\text{ determines a path, i.e. }\mathcal{P}', \text{ from }\langle 0_n, 0 \rangle \text{ to }\langle a, w \rangle.\text{ Moreover, since any segment of }\mathcal{P}'\text{ is only composed of jumps }\langle 0_n, 1 \rangle \text{ or }\langle 0_n, -1 \rangle,\text{ the length of the }t\text{th segment is at least }\min\{ |w_t - w_{t-1}|, r^m - |w_t - w_{t-1}| \}.\text{ Therefore, the total length of segments of }\mathcal{P}'\text{ is at least}\begin{align*}
(l(\hat{w})) := \sum_{t=1}^{s+1} \min\{ |w_t - w_{t-1}|, r^m - |w_t - w_{t-1}| \},
\end{align*}\text{ and the total length of segments of }\mathcal{P}'\text{ is }\tilde{l}(\hat{w})\text{ if }\mathcal{P}'\text{ is a shortest path from }\langle 0_n, 0 \rangle \text{ to }\langle a, w \rangle.\text{ Now we construct }\mathcal{P}^*\text{ by setting its }t\text{th segment to be the same as the }t\text{th segment of }\mathcal{P}'\text{ for }1 \leq t \leq s,\text{ and its }\langle s + 1 \rangle\text{st segment to be the }\langle s + 1 \rangle\text{st segment of }\mathcal{P}'\text{ followed by those jumps of the }s + 1\text{ segments of }\mathcal{P}\text{ which are not included in any segment of }\mathcal{P}',\text{ namely all jumps }\langle 0_n, r^k \rangle \text{ or }\langle 0_n, -r^k \rangle\text{ in }\mathcal{P}\text{ with }1 \leq k \leq m - 1.\text{ Then }\hat{x}^* = (w_0, w_1, \ldots, w_s, x)\text{ is the }\langle a, x \rangle\text{-sequence corresponding to }\mathcal{P}^*\text{ and its }\langle s + 1 \rangle\text{st segment is a path from }\langle a, w_s \rangle \text{ to }\langle a, x \rangle\text{ in the }a\text{-circulant of the form }e_1, e_2, \ldots, e_h, e_{h+1}, \ldots, e_t,\text{ where }e_i = \langle 0_n, 1 \rangle \text{ or }\langle 0_n, -1 \rangle\text{ for }1 \leq i \leq h, e_i = \langle 0_n, r^{k_i} \rangle \text{ or }\langle 0_n, -r^{k_i} \rangle\text{ for }h + 1 \leq i \leq t\text{ and some }1 \leq k_i \leq m - 1,\text{ and }e_1, e_2, \ldots, e_h \text{ is a path from }\langle a, w_s \rangle \text{ to }\langle a, w \rangle.\text{ Thus, }\mathcal{P}\text{ and }\mathcal{P}^*\text{ have the same length and it is equal to the length of }\mathcal{P}'\text{ plus the number of jumps in the segments of }\mathcal{P}\text{ which are }\langle 0_n, \delta r^k \rangle\text{ for some }1 \leq k < m \text{ and }\delta \in \{-1, 1\},\text{ that is, the length of }e_{h+1}, e_{h+2}, \ldots, e_t.\text{ The discussion above gives the following lemma.}

**Lemma 6.3.2.** Suppose }\mathcal{P}\text{ is a path from }\langle 0_n, 0 \rangle \text{ to }\langle a, x \rangle\text{ in }\mathbb{C}Q_n(d, r, m).

\begin{enumerate}[(a)]
\item A path }\mathcal{P}'\text{ can be obtained from }\mathcal{P}\text{ whose edges are labelled by either }\langle 0_n, 1 \rangle, \langle 0_n, -1 \rangle \text{ or }\langle e_i, 0 \rangle\text{ for some }1 \leq i \leq d,\text{ and the number of edges of }\mathcal{P}\text{ with label }\langle 0_n, 1 \rangle \text{ or }\langle 0_n, -1 \rangle\text{ is equal to that of }\mathcal{P}'.\text{ Moreover, }\mathcal{P}'\text{ is a path from }\langle 0_n, 0 \rangle \text{ to }\langle a, w \rangle\text{ for some }w \text{ with }w \equiv x \mod r.
\item If }\mathcal{P}\text{ is a shortest path, then }\mathcal{P}'\text{ is a shortest path from }\langle 0_n, 0 \rangle \text{ to }\langle a, w \rangle\text{ and so the number of edges of }\mathcal{P}\text{ labelled with either }\langle 0_n, 1 \rangle \text{ or }\langle 0_n, -1 \rangle\text{ is }\tilde{l}(\hat{w}),\text{ where }\hat{w}\text{ is an optimal }\langle a, w \rangle\text{-sequence corresponding to }\mathcal{P}'.\text{ Moreover, the number of edges of }\mathcal{P}\text{ labelled with }\langle 0_n, r^k \rangle\text{ and }\langle 0_n, -r^k \rangle\text{ with some }1 \leq k < m,\text{ is }d_m(w, x).\text{ Therefore, by Lemma }6.3.1,\text{ the length of }\mathcal{P}\text{ is equal to}\begin{align*}
dist(\langle 0_n, 0 \rangle, \langle a, x \rangle) = \|a\| + \tilde{l}(\hat{w}) + d_m(w, x).
\end{align*}
\end{enumerate}
For a path $P$, let $\hat{w}$ be the $(a, w)$-sequence as given in Lemma 6.3.2, and let $w = kr + y$ for some $0 \leq k < r^{m-1}$ and $0 \leq y < r$. Now we use $(a, y)$-sequences in recursive cubes of rings $Q_n(d, r)$ and results from Chapter 6, where $(a, y) \in \mathbb{Z}_r^n \times \mathbb{Z}_r$, and obtain a relation between $\hat{w}$ and some $(a, y)$-sequence $\hat{y}$ in $Q_n(d, r)$. It can be verified that

$$
\tilde{l}(\hat{w}) \geq \min \{ kr + y, r^m - kr - y \}.
$$

(6.10)

An $(a, y)$-sequence $\hat{y}$ in $Q_n(d, r)$ can be obtained from $\hat{w}$ by taking its entries modulo $r$. In other words, $\hat{y} = (y_0, y_1, \ldots, y_s, y)$ is an $(a, y)$-sequence in $Q_n(d, r)$, where $y_t \equiv w_t \mod r$ and $0 \leq y_t < r$, $0 \leq t \leq s+1$. Since $w_t - w_{t-1} = y_{t-1} + jr$ for some integer $0 \leq j < r^{m-1}$, we have $\min\{|w_t - w_{t-1}|, r^m - |w_t - w_{t-1}|\} \geq \min\{|y_{t-1}, r - |y_{t-1}|\}$. Thus

$$
\tilde{l}(\hat{w}) \geq l(\hat{y}) = \sum_{t=1}^{s+1} \min\{|y_t - y_{t-1}|, r - |y_t - y_{t-1}|\},
$$

(6.11)

where $l(\hat{y})$ is the sum of the lengths of corresponding segments of $\hat{y}$ in $Q_n(d, r)$ (see (6.10)).

**Construction 6.3.3.** Any $(a, y)$-sequence $\hat{y} = (y_0, y_1, \ldots, y_{s+1})$ in $Q_n(d, r)$ gives rise to an $(a, x)$-sequence $\hat{x'} = (x'_{0}, x'_{1}, \ldots, x'_{s+1})$ in $\mathcal{Q}_n(d, r, m)$, where $y \equiv x' \mod r$.

That is, $x'_0 = y_0$, $x'_t = x'_{t-1} + (y_t - y_{t-1})$ if $|y_t - y_{t-1}| \leq r - |y_t - y_{t-1}|$, and $x'_t = x'_{t-1} + r^m - (r - (y_t - y_{t-1}))$ if $|y_t - y_{t-1}| > r - |y_t - y_{t-1}|$ for $1 \leq t \leq s+1$ (see Figure 6.3). By this construction, $x' = x'_{s+1}$ and

$$
\tilde{l}(\hat{x'}) = l(\hat{y}).
$$

(6.12)

**Theorem 6.3.4.** Suppose $dr = n$. Given $(a, x) \in G$, let $x = ir + y$, for some integers $0 \leq i < r^{m-1}$ and $0 \leq y < r$. The following hold:

(a) if $y \neq 0$, then

$$
\ell_c(a, x) = \min\{ r + y - 2L_1(a, y) + d_m(y - r, x), 2r - y - 2L_2(a, y) + d_m(y, x) \}
$$

and

$$
\text{dist}(0_n, 0, (a, x)) = ||a|| + \min\{ r + y - 2L_1(a, y) + d_m(y - r, x), 2r - y - 2L_2(a, y) + d_m(y, x) \};
$$

(b) if $y = 0$, then

$$
\ell_c(a, x) = \min\{ 2r - 2L_2(a, 0) + d_m(0, x), r + d_m(-r, x) \}
$$

and

$$
\text{dist}(0_n, 0, (a, x)) = ||a|| + \min\{ 2r - 2L_2(a, 0) + d_m(0, x), r + d_m(-r, x) \}.\]
Figure 6.3: Obtaining an optimal \((a, x)\)-sequence in \(CQ_n(d, r, m)\) from \((a, y)\)-sequences \(\hat{y}\) in \(Q_n(d, r)\), where segments are projected into a circulant and a cycle.

Moreover, an optimal \((a, x)\)-sequence in \(CQ_n(d, r, m)\) can be obtained from \(\hat{y}^1\) or \(\hat{y}^2\), where \(\hat{y}^1\) and \(\hat{y}^2\) are the two \((a, y)\)-sequences in \(Q_n(d, r)\), given in (6.12) and (6.13).

Proof. Note that we have \(l(\hat{y}^1) \leq r + y - 2L_1(a, y)\), \(l(\hat{y}^2) \leq 2r - y - 2L_2(a, y)\) and \(\min\{l(\hat{y}^1), l(\hat{y}^2)\} = \min\{r + y - 2L_1(a, y), 2r - y - 2L_2(a, y)\}\) (see Section 6.1).

Using Construction 6.12, \(\hat{w}^1\) and \(\hat{w}^2\) can be obtained from \(\hat{y}^1\) and \(\hat{y}^2\), respectively, such that

\[ \hat{l}(\hat{w}^i) = l(\hat{y}^i), \quad i = 1, 2, \]

by (6.12), and \(\hat{w}^1\) is an \((a, y - r)\)-sequence and \(\hat{w}^2\) is an \((a, y)\)-sequence in \(CQ_n(d, r, m)\).

By Lemma 6.3.1, \(\hat{w}^1\) gives rise to a path \(P_{y - r}\) in \(CQ_n(d, r, m)\) form \((0_n, 0)\) to \((a, y - r)\) with length \(\|a\| + l(\hat{y}^1)\), and \(\hat{w}^2\) gives rise to a path \(P_y\) form \((0_n, 0)\) to \((a, y)\) with length \(\|a\| + l(\hat{y}^2)\).

Suppose \(P\) is a shortest path from \((0_n, 0)\) to \((a, x)\) in \(CQ_n(d, r, m)\). By Lemma 6.3.2, a shortest path from \((0_n, 0)\) to \((a, w)\) for some \(w \equiv x \mod r\), can be obtained such that \(\hat{w}\) is its corresponding \((a, w)\)-sequence and the length of \(P\) is \(\|a\| + \hat{l}(\hat{w}) + d_m(w, x)\) by (1.4). Hence \(w = kr + y\) for some \(0 \leq k < r^{m-1}\). In the following we show that either \(0 \leq w \leq r\) or \(r^m - r \leq w < r^m\).

If \(r < w \leq r^m/2\), then \(k \geq 1\). So, by (6.10) and since \(d_m(w, y) \leq k\), we have

\[ l(\hat{y}^2) + d_m(y, w) \leq r + y - 2 + d_m(y, w) \leq r + y - 2 + k < r + y + (k - 1)r \leq \hat{l}(\hat{w}). \]

This implies that \(\hat{l}(\hat{w}) + d_m(y, w) < \hat{l}(\hat{w})\). Similarly, if \(r^m/2 < w < r^m - r\), then \(k \leq r^{m-1}-2\).

So, by (6.10) and since \(d_m(y - r, w) \leq r^{m-1} - k - 1\), we have \(l(\hat{y}^1) + d_m(y - r, w) \leq 2r - y + r^{m-1} - k - 3 < \hat{l}(\hat{w})\). This implies that \(\hat{l}(\hat{w}) + d_m(y - r, w) < \hat{l}(\hat{w})\). Therefore, \(\hat{w}\) is not an optimal \((a, w)\)-sequence in both cases. Hence we must have \(0 \leq w \leq r\) or
\[ r^m - r \leq w < r^m, \] and so either \( \tilde{w}^1 \) or \( \tilde{w}^2 \) is an optimal \((a, w)\)-sequence for some \( w \equiv x \) mod \( r \). Thus, if \( y \neq 0 \), then by (6.13) and (6.11) we have

\[
\text{dist}((0_n, 0), (a, x)) = \|a\| + \min\{\tilde{l}(\tilde{w}^1) + d_m(y - r, x), \tilde{l}(\tilde{w}^2) + d_m(y, x)\},
\]

which implies \( \ell_c(a, x) = \min\{l(\tilde{y}^1) + d_m(y - r, x), l(\tilde{y}^2) + d_m(y, x)\} \). Thus we have the results when \( y \neq 0 \).

If \( y = 0 \), then either \( w = 0, r \) or \(-r\). Similarly,

\[
\text{dist}((0_n, 0), (a, x)) = \|a\| + \min\{\tilde{l}(\tilde{w}^1) + d_m(-r, x), \tilde{l}(\tilde{w}^2) + d_m(-r, x), \tilde{l}(\tilde{w}^2) + d_m(r, x)\},
\]

and \( \ell_c(a, x) = l(\tilde{y}^1) + d_m(-r, x), l(\tilde{y}^2) + d_m(-r, x), l(\tilde{y}^2) + d_m(r, x) \). \( \square \)

**Theorem 6.3.5.** Suppose \( dr \geq 2n \) and \( x = ir + y \), for some integers \( 0 \leq i < r^{m-1} \) and \( 0 \leq y < r \). The following hold:

(a) if \( dr \geq 2n \) and \( 0 \leq y \leq [(r - 1)/2] \), then

\[
\ell_c(a, x) = 2\lceil n/d \rceil - y - 2L_1(a, y) + d_m(y, x)
\]

and

\[
\text{dist}((0_n, 0), (a, x)) = \|a\| + 2\lceil n/d \rceil - y - 2L_1(a, y) + d_m(y, x);
\]

(b) if \( dr \geq 2n \) and \( y \geq [(r + 1)/2] \), then

\[
\ell_c(a, x) = 2\lceil n/d \rceil - (r - y) - 2L_2(a, y) + d_m(y - r, x)
\]

and

\[
\text{dist}((0_n, 0), (a, x)) = \|a\| + 2\lceil n/d \rceil - (r - y) - 2L_2(a, y) + d_m(y - r, x);
\]

(c) if \( r \) is even and \( y = r/2 \), then

\[
\ell_c(a, x) = \min\{d_m(-r/2, x), d_m(r/2, x)\} + r/2
\]

and

\[
\text{dist}((0_n, 0), (a, x)) = \|a\| + \min\{d_m(-r/2, x), d_m(r/2, x)\} + r/2.
\]

Moreover, an optimal \((a, x)\)-sequence in \( CQ_n(d, r, m) \) can be obtained from an optimal \((a, y)\)-sequence \( \tilde{y} \) in \( Q_n(d, r) \) such that \( \ell_c(\tilde{x}) = l(\tilde{y}) + d_m(y, x) = l(a, y) + d_m(y, x) \).

**Proof.** (a) Let \( \tilde{y}^1 \) be an \((a, y)\)-sequence in \( Q_n(d, r) \) as in (6.13), and denote by \( \tilde{w}^1 \) the \((a, y)\)-sequence in \( CQ_n(d, r, m) \) obtained from \( \tilde{y}^1 \) using Construction 6.3.3 such that \( \tilde{l}(\tilde{w}^1) = l(\tilde{y}^1) = 2\lceil n/d \rceil - y - 2L_1(a, y) \leq r - 1 \) by (6.12).

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For a given shortest path $P$ from $(0_n, 0)$ to $(a, x)$ in $CQ_n(d, r, m)$, a shortest path from $(0_n, 0)$ to $(a, w)$ for some $w \equiv x \mod r$, with corresponding $(a, w)$-sequence $\hat{w}$ can be obtained by Lemma 6.3.2, and the length of $P$ is $\|a\| + \hat{l}(\hat{w}) + d_m(w, x)$ by (6.9).

Suppose $w = kr + y$ for some $0 \leq k < r^{m-1}$. If $r < w < r^{m-1} - r$, then, by (6.10),
\[ \hat{l}(\hat{w}) \geq \min\{kr + y, r^m - kr - y\} > r + k - 1 \geq \hat{l}(\hat{w}^1) + d_m(y, w). \]
Hence, $\hat{w}$ cannot be an optimal $(a, w)$-sequence. Thus either $w = y$ or $w = y - r$ when $y \neq 0$, and either $w = 0, r$ or $w = -r$ when $y = 0$. Therefore, if $y \neq 0$, then, by Lemma 6.3.2 and (6.12), we have
\[ \text{dist}(0_n, a, x) = \|a\| + \hat{l}(\hat{w}^1) + d_m(y, x). \]
Similarly, if $y = 0$, then $\hat{l}(\hat{w}) \geq \hat{l}(\hat{w}^1) + d_m(y, w)$ and hence $\text{dist}(0_n, a, x) = \|a\| + \hat{l}(\hat{w}^1) + d_m(y, x)$.

(b) The proof is similar to Case (a) and hence is omitted.

(c) In this case, both $\hat{y}^1$ and $\hat{y}^2$ are optimal in $Q_n(d, r)$ and hence the $(a, y)$-sequences $\hat{w}^1$ and $\hat{w}^2$ can be optimal. Thus, $\text{dist}(0_n, a, x) = \|a\| + \min\{\hat{l}(\hat{w}^1) + d_m(y, x), \hat{l}(\hat{w}^2) + d_m(r - y, x)\}$. Since $l(\hat{y}^1) = l(\hat{y}^2) = r/2$ in this case, we have the result. \hfill $\Box$

**Algorithm 6.1** A shortest path from $(a, y)$ to $(a, x)$ in a-circulant

1. Let $x - y := c_0 + c_1 r + c_2 r^2 + \ldots + c_{m-1} r^{m-1}$ for $0 \leq c_i < r$, $0 \leq i < m$, and $w := y$;
2. for $i = 0$ to $m - 1$ do
3. if $|c_i| \leq \lfloor r/2 \rfloor$ then
4. begin
5. Send message from $(a, w)$ to $(a, w + c_i r^i)$ using $c_i$ jumps $(0_n, r^i)$;
6. Add $c_i r^i$ to $w$;
7. end
8. else
9. begin
10. Send message from $(a, w)$ to $(a, w + c_i r^i)$ using $r - c_i$ jumps $(0_n, -r^i)$;
11. Add $c_i r^i$ to $w$;
12. end

Using Lemma 6.3.2 and Construction 6.3.3, we develop an algorithm to compute a shortest path between vertices $(0_n, 0)$ and $(a, x)$ in $CQ_n(d, r, m)$. The last segment of our shortest path contains a shortest path between two given vertices in the $a$-circulant. We modify the shortest path algorithm in $C(r, m)$ given in [100] to obtain Algorithm 111.
for a shortest path in the $a$-circulant. Using Algorithms 6.1, 6.2 and 5.3, we develop Algorithm 6.2 for a shortest path in $CQ_n(d, r, m)$.

For a given vertex $(a, x)$ in $CQ_n(d, r, m)$, Algorithm 6.2 gives a shortest path from $(0_n, 0)$ to $(a, x)$. One can verify that the length of the shortest path obtained from Algorithm 6.2 is equal to $\text{dist}((0_n, 0), (a, x))$. Since $CQ_n(d, r, m)$ is a vertex-transitive graph, this algorithm can be used to obtain a shortest path between any two vertices in the graph. That is, for any two vertices $(b, y)$ and $(c, z)$ in $CQ_n(d, r, m)$,

$$(b, y), (b, y)(a_1, x_1), \ldots, (b, y)(a_l, x_l) = (c, z)$$

is a shortest path from $(b, y)$ to $(c, z)$ if $(0_n, 0), (a_1, x_1), \ldots, (a_l, x_l) = (a, x)$ is a shortest path from $(0_n, 0)$ to $(a, x)$ and $(a, x) = (b, y)^{-1}(c, z)$. As a result,

$$\text{dist}((b, y), (c, z)) = \text{dist}((0_n, 0), (a, x)),$$

which implies that the distance of any two vertices in $CQ_n(d, r, m)$ can be obtained by Theorems 6.3.4 and 6.3.5.

### 6.4 Diameter of $CQ_n(d, r, m)$

In this section, we give a formula for the diameter of $CQ_n(d, r, m)$. The diameter of the multiplicative circulants $C(r, m)$ is given by

$$\text{diam}(C(r, m)) = \left\lceil \frac{m(r-1)}{2} \right\rceil. \quad (6.13)$$

The parities of $m$ and $r$ are important in determining formulas for the diameter of $CQ_n(d, r, m)$. It is shown in [SS] that when $r$ is even, $\text{diam}(C(r, m)) = d_m(0, x) = d_m(0, x+1)$ for some $x \in V(C(r, m))$ if and only if $m$ is even. It can be verified by induction that there exists $x \in V(C(r, m))$ such that $\text{diam}(C(r, m)) = d_m(0, x) = d_m(0, x+1)$ when $r$ is odd. As in [SS] corresponding to $C(r, m)$ define $\alpha_m = 1$ if $m$ is odd and $r$ is even, and $\alpha_m = 0$ in all other cases. By the discussion above, $\text{diam}(C(r, m)) = d_m(0, x) = d_m(0, x+1)$ for some $x \in V(C(r, m))$ if and only if $\alpha_m = 0$.

**Lemma 6.4.1 ([SS]).** Suppose $m \geq 1$. The following hold:

(a) $d_m(x, y) = d_{m+1}(rx, ry)$ for any $x, y \in \mathbb{Z}_{r^m};$

(b) If $r$ is odd, then

$$d_m(0, kr) < d_m(0, kr+1) < \cdots < d_m(0, kr+\lfloor r/2 \rfloor)$$
Algorithm 6.2 A shortest path from \((0_n, 0)\) to \((a, x)\) in \(CQ_n(d, r, m)\)

1: If \(dr = n\), then let \(y \equiv x \mod r\), \(0 \leq y < r\) and let \(\hat{y}^1\) and \(\hat{y}^2\) be two \((a, y)\)-sequences in \(Q_n(d, r)\) obtained from Algorithm 5.2. If \(l(\hat{y}^1) + d_m(y - r, x) \leq l(\hat{y}^2) + d_m(y, x)\), set \(\hat{y} = \hat{y}^1\); otherwise, set \(\hat{y} = \hat{y}^2\);

2: If \(dr \geq 2n\), then set \(\hat{y} = \hat{y}^1\) when \(0 \leq y \leq \lfloor r/2 \rfloor\), and set \(\hat{y} = \hat{y}^2\) when \(y > \lfloor r/2 \rfloor\), where \(\hat{y}^1\) and \(\hat{y}^2\) are obtained from Algorithm 5.3;

3: Set \(w_0 = y_0\), \(b = 0_n\) and \(P = (0_n, 0)\);

4: for \(t = 1\) to \(\|a\| + 1\) do

5: \hspace{1em} if \(|y_t - y_{t-1}| \leq r - |y_t - y_{t-1}|\) then

6: \hspace{2em} Set \(w_t = w_{t-1} + (y_t - y_{t-1})\);

7: \hspace{1em} else

8: \hspace{2em} Set \(w_t = w_{t-1} + r^m - (r - (y_t - y_{t-1}))\);

9: for \(t = 1\) to \(s\) do

10: \hspace{1em} begin

11: \hspace{2em} Let \(t\)th segment of \(P\) be a shortest path from \((b, w_{t-1})\) to \((b, w_t)\);

12: \hspace{2em} Add the cube edge \(\{(b, w_t), (b + e_i, w_t)\}\) to \(P\);

13: \hspace{2em} set \(b = b + e_i\);

14: \hspace{1em} end

15: Let \((s + 1)\)th segment of \(P\) be a shortest path from \((a, w_s)\) to \((a, w_{s+1})\);

16: Concatenate the shortest path from \((a, w_{s+1})\) to \((a, x)\), obtained from Algorithm 6.1, to \(P\).

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and
\[ d_m(0, (k + 1)r) < d_m(0, (k + 1)r - 1) < \cdots < d_m(0, (k + 1)r - [r/2]); \]

(c) If \( r \) is even, then
\[ d_m(0, kr) < d_m(0, kr + 1) < \cdots < d_m(0, kr + r/2 - 1) \leq d_m(0, kr + r/2) \]

and
\[ d_m(0, (k + 1)r) < d_m(0, (k + 1)r - 1) < \cdots < d_m(0, kr + r/2 + 1) \leq d_m(0, kr + r/2). \]

**Theorem 6.4.2.** The diameter of \( CQ_n(d, r, m) \) is given by
\[
\text{diam}(CQ_n(d, r, m)) = \begin{cases} 
n + m + 2, & \text{if } dr = n \text{ and } r = 3, \\
n + \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{(m-1)(r-1)}{2} \right\rceil - \alpha_{m-1} - 2, & \text{if } dr = n \text{ and } r \geq 4, \\
n + \max \left\{ \left\lceil \frac{(m-1)(r-1)}{2} \right\rceil + 2 \left\lfloor \frac{m}{4} \right\rfloor - 2 \left\lceil \frac{m(r-1)}{2} \right\rceil \right\}, & \text{if } dr \geq 2n.
\end{cases}
\]

Proof. As \( CQ_n(d, r, m) \) is vertex-transitive, \( \text{diam}(CQ_n(d, r, m)) = \max \text{dist}((0, 0), (a, x)) \), where the maximum is taken over all \((a, x) \in G\).

Suppose \( dr = n \) and first assume \( x \equiv 0 \mod r \). Since \( L_2(a, 0) \geq 1 \), by Theorem 6.4.1 we have
\[ \text{dist}((0, 0), (a, x)) \leq n + \min\{2r - 2 + d_m(0, x), r + d_m(-r, x), r + d_m(r, x)\}. \quad (6.14) \]

The right-hand side above is equal to \( n + r + \min\{d_m(-r, x), d_m(r, x)\} \) since \( 2r - 2 + d_m(0, x) \geq r + d_m(r, x) \). By Lemma 6.4.1, there does not exist any \( w \) such that \( \text{diam}(C(r, m)) = d_m(0, w - 1) = d_m(0, w + 1) \), for otherwise \( d_m(0, w) > \min\{d_m(0, w + 1), d_m(0, w - 1)\} = \text{diam}(C(r, m)) \), which is a contradiction. So, for \( x \) with \( x \equiv 0 \mod r \),
\[ \text{dist}((0, 0), (a, x)) \leq n + r + \text{diam}(C(r, m - 1)) - 1. \quad (6.15) \]

This upper bound is attainable for \((a, x) = (1, wr)\) with \( w \) such that \( \text{diam}(C(r, m - 1)) = d_{m-1}(0, w) \) as \( L_2(1, 0) = 1 \).

Now suppose \( x \not\equiv 0 \mod r \), \( x = ir + y \) and \( 0 < y < r \). Let \( z_i = d_m(0, (i + 1)r) - d_m(0, ir) \in \{-1, 0, 1\} \). Since \( L_1(a, x) \geq 1, L_2(a, x) \geq 1, d_m(y, x) = d_m(0, ir) \) and \( d_m(y - r, x) = d_m(0, (i + 1)r) \), the upper bound in \( (6.14) \) is at most
\[ \max \left\{ n + d_m(0, ir) + \min\{2r - y - 2, z_i + r + y - 2\} \right\}, \quad (6.16) \]
where the maximum is taken over all \( 0 < y < r \) and \( 0 \leq i < r^{m-1} \). Note that \( \min\{2r - y - 2, z_i + r + y - 2\} = r + [(r + z_i)/2] - 2 \). So \((6.16)\) is equal to \( \max_i \{ n + \)
\[ d_m(0, ir) + r + \lfloor (r + z_t)/2 \rfloor - 2 \leq n + r + \text{diam}(C(r, m - 1)) + \lfloor (3r + z_t)/2 \rfloor - 2 \]

for some \( t \) such that \( d_m(0, tr) = \text{diam}(C(r, m - 1)) \) and so \( z_t \in \{-1, 0\} \). Moreover, \( z_t = 0 \) if \( \alpha_{m-1} = 0 \); and \( z_t = -1 \) if \( \alpha_{m-1} = 1 \). Therefore, using (6.10),

\[
\text{dist}((0_n, 0), (a, x)) \leq n + \text{diam}(C(r, m - 1)) + \lfloor 3r/2 \rfloor - 2 - \alpha_{m-1}.
\] (6.17)

This upper bound is attainable for \((a, x) = (1_n, ir + \lfloor r/2 \rfloor)\), for \( i \) such that \( \text{diam}(C(r, m - 1)) = d_{m-1}(0, i) \). By comparing (6.13) and (6.17), we have the results when \( dr = n \).

Now assume \( dr \geq 2n \) and \( x = ir + y \), for some \( 0 \leq y \leq \lfloor (r - 1)/2 \rfloor \). By Theorem 6.3.4,

\[
\text{dist}((0_n, 0), (a, x)) \leq n + 2\lfloor n/d \rfloor - y - 2L_1(a, y) + d_m(y, x).
\] (6.18)

If \( L_1(a, y) = \lfloor n/d \rfloor - y \), then the right-hand side of (6.18) is at most

\[
n + y + d_m(y, x) = n + d_m(0, x) \leq n + \text{diam}(C(r, m))
\] (6.19)

since \( y + d_m(y, x) = d_m(0, x) \) by Lemma 6.4.1 and \( 0 \leq y \leq \lfloor (r - 1)/2 \rfloor \). Note that for \( y \) with \( d_m(0, y) = \text{diam}(C(r, m)) \), we have \( y = ir + \lfloor r/2 \rfloor \) by Lemma 6.4.1. Hence, the upper bound \( n + \text{diam}(C(r, m)) \) is attainable by \((a, x) = (1_n, y)\) as \( L_1(a, y) = \lfloor n/d \rfloor - \lfloor r/2 \rfloor \).

Now assume \( L_1(a, y) \neq \lfloor n/d \rfloor - y \). Since \( L_1(a, y) \geq 1 \), the right-hand side of (6.18) is at most

\[
n + 2\lfloor n/d \rfloor - 2 + d_m(0, ir) \leq n + 2\lfloor n/d \rfloor - 2 + \text{diam}(C(r, m - 1)).
\] (6.20)

In this case, this upper bound is attainable by \((1_n, ir)\), for \( i \) such that \( \text{diam}(C(r, m - 1)) = d_{m-1}(0, i) \) as \( L_1(1_n, 0) = 1 \). One can verify that for case \( dr \geq 2n \) and \( x = ir + y \) with \( \lfloor r/2 \rfloor < y < r \) we have the same results. Hence (6.20) together with (6.19) implies \( \text{diam}(CQ_n(d, r, m)) = n + \max\{\text{diam}(C(r, m - 1)) + 2\lfloor n/d \rfloor - 2, \text{diam}(C(r, m))\} \). We have the result by (6.13).

\section{Total distance \(CQ_n(d, r, m)\)}

In this section we obtain the total distance of cube-connected circulants and present them in Theorems 6.5.2 and 6.5.3. The problem of computing the total distance of cube-connected circulants will be broken down into a few smaller problems such that known results on the total distances of multiplicative circulants and recursive cubes of rings from Section 6.4.2 can be used.
In the following discussion we present some results for multiplicative circulants. Since \( d_m(0, i) \) and \( d_m(0, i + 1) \) differ by at most 1 for any \( 0 \leq i < r^m \), the vertex set of \( C(r, m) \) can be partitioned into three subsets \( V_m, V_m^+ \) and \( V_m^- \) as follows:

(i) \( i \in V_m \) if and only if \( d_m(0, i + 1) = d_m(0, i) \);
(ii) \( i \in V_m^+ \) if and only if \( d_m(0, i + 1) = d_m(0, i) + 1 \);
(iii) \( i \in V_m^- \) if and only if \( d_m(0, i + 1) = d_m(0, i) - 1 \).

**Lemma 6.5.1.** For \( m \geq 1 \), the following hold:

(a) if \( r \) is odd, then \( |V_m^+| = |V_m^-| = (r^m - 1)/2 \) and \( |V_m| = 1 \);

(b) if \( r \) is even, then \( |V_m^+| = |V_m^-| = r^m/2 - r^{m-1}/2 + r^{m-2}/2 + \ldots + (-1)^{m-1}r/2 \), \( |V_1| = 0 \), and for \( m \geq 2 \), \( |V_m| = r^{m-1} - r^{m-2} + \ldots + (-1)^mr \).

**Proof.** (a) For \( m = 1 \), the graph is a cycle and the claimed is obvious. Suppose \( m \geq 2 \). By Lemma 6.4.1, \( ir + j - 1 \in V_m^+ \) and \( (i + 1)r - j \in V_m^- \) for any \( 0 \leq i < r^{m-1} \) and \( 1 \leq j \leq (r - 1)/2 \). Furthermore, \( ir + (r - 1)/2 \in V_m \) if and only if \( i \in V_m^- \); \( ir + (r - 1)/2 \in V_m^+ \) if and only if \( i \in V_m^+ \); \( ir + (r - 1)/2 \in V_m^- \) if and only if \( i \in V_m^- \). Therefore, \( |V_m^-| = |V_m^+| = r^{m-1}(r - 1)/2 + |V_{m-1}^+| \) and \( |V_m| = 1 \).

(b) It is clear that \( |V_1| = 0 \) and \( |V_i^+| = |V_i^-| = r/2 \). Assume \( m \geq 2 \). By Lemma 6.4.1, \( ir + j - 1 \in V_m^+ \) and \( (i + 1)r - j \in V_m^- \) for any \( 0 \leq i < r^{m-1} \) and \( 1 \leq j \leq r/2 - 1 \). Furthermore, \( ir + r/2 - 1 \in V_m^+ \) and \( ir + r/2 \in V_m \) if and only if \( i \in V_m^- \); \( ir + r/2 - 1 \in V_m^+ \) and \( ir + r/2 \in V_m^- \) if and only if \( i \in V_m^- \). Therefore, \( |V_m| = |V_{m-1}^+| + |V_{m-1}^-| \), \( |V_m| = r^{m-1}(r/2 - 1) + |V_{m-1}^+| + |V_{m-1}^-| \), and \( |V_m| = r^{m-1}(r/2 - 1) + |V_{m-1}^+| + |V_{m-1}^-| \). Clearly \( |V_m| = |V_m^+| \) as \( |V_1^+| = |V_1^-| \). Using this recursive formula, the result can be verified easily.

The total distance of recursive circulants is obtained in [KS]. Since multiplicative circulants are special recursive circulants, the total distance of multiplicative circulants is already known. However, we obtain an equivalent but simpler formula for \( \text{td}(C(r, m)) \) when \( r \) is even in the following theorem. Without loss of generality we may assume \( \text{td}(C(r, 0)) = 0 \) for any integer \( r \geq 3 \).

**Theorem 6.5.2.**

(a) If \( r \) is odd, then \( \text{td}(C(r, m)) = mr^{m-1}(r^2 - 1)/4 \);

(b) if \( r \) is even, then \( \text{td}(C(r, m)) = mr^{m+1}/4 + \sum_{i=1}^{m-1}(-1)^{i}(m - i)r^{m-i}/2 \).

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Proof. (a) When $r$ is odd, the result is known in [SS, Theorem 7].

(b) Since $d_m(0, kr + j) = d_{m-1}(0, k) + |j|$ for $-r/2 + 1 \leq j \leq r/2 - 1$ by Lemma 6.4.1, we have $\sum_{j=-r/2+1}^{r/2-1} d_m(0, kr + j) = (r-1)d_{m-1}(0, k) + (r^2 - 2r)/4$. On the other hand, $d_m(0, kr + r/2) = d_{m-1}(0, k) + r/2$ if $k \in V_{m-1} \cup V_{m-1}^+$, and $d_m(0, kr + r/2) = d_{m-1}(0, k) + r/2 - 1$ if $k \in V_{m-1}^-$. Therefore,

$$td(C(r, m)) = \sum_{i=0}^{m-1} d_m(0, i) = \sum_{j=-r/2+1}^{r/2-1} d_m(0, kr + j) + \sum_{k=0}^{r-1} d_m(0, kr + r/2)$$

$$= \sum_{k=0}^{r-1} ((r-1)d_{m-1}(0, k) + (r^2 - 2r)/4) + \sum_{k=0}^{r-1} (d_{m-1}(0, k) + r/2 - |V_{m-1}^-|)$$

$$= r^{m+1}/4 + rtd(C(r, m - 1)) - |V_{m-1}^-|.$$ 

Using this recursive formula, we have

$$td(C(r, m)) = m r^{m+1}/4 - r^{m-2}|V_1^-| - r^{m-3}|V_2^-| - \ldots - r|V_{m-2}^-| - |V_{m-1}^-|$$

$$= m r^{m+1}/4 + \sum_{i=1}^{m-1} (-1)^i(m - i)r^{m-1}/2.$$

Now we are ready to deal with the total distance of cube-connected circulants. This parameter will be presented for cases $dr = n$ and $dr \geq 2n$ separately in the following theorems. Note that

$$\sum_{a,x \in G} \|a\| = r^m \sum_{a \in \mathbb{Z}_2^n} \|a\| = r^m \sum_{i=0}^{n} \binom{n}{i} = n2^{n-1}r^m.$$

Therefore, by Lemma 6.3.1, we have

$$td(CQ_n(d, r, m)) = \sum_{a,x \in G} (\|a\| + \ell_c(a, x)) = n2^{n-1}r^m + \sum_{a,x \in G} \ell_c(a, x). \quad (6.21)$$

**Theorem 6.5.3.** Suppose $dr = n$. If $r \geq 2^9$, then

$$2^{n-2}r^{m+1}(2d + 5) \left(1 - \frac{4 + 20r + 20r \log^2 r}{2nr + 5r^2} \right) + r2^mtd(C(r, m - 1)) \leq$$

$$td(CQ_n(d, r, m)) \leq 2^{n-2}r^{m+1}(2d + 5) \left(1 - \frac{8(r - 1)}{2nr + 5r^2} \right) + r2^mtd(C(r, m - 1));$$

and if $3 \leq r < 2^9$, then

$$2^{n-2}r^{m-1}(2nr + r^2 - 2r - 4) + r2^mtd(C(r, m - 1)) \leq td(CQ_n(d, r, m)) \leq$$

$$2^{n-2}r^{m-1}(2nr + 5r^2 - 8r + 8) + r2^mtd(C(r, m - 1)).$$
Suppose $m = 1$. Since $CQ_n(d, r, 1)$ is isomorphic to $Q_n(d, r)$, $\text{td}(CQ_n(d, r, m))$ is equal to $\text{td}(Q_n(d, r))$. Hence

\[
\text{td}(CQ_n(d, r, m)) = nr2^{n-1} + \sum_{a \in \mathbb{Z}_2^n, 0 \leq y < r} l(a, y),
\]  

(6.22)

as $\ell_c(a, y) = l(a, y)$ for $a \in \mathbb{Z}_2^n$ and $0 \leq y < r$, and, by Theorem 6.3.2,

\[
l(a, y) = \min\{2r - y - 2L_2(a, y), r + y - 2L_1(a, y)\}.
\]  

(6.23)

Now suppose $m \geq 2$. For any $0 \leq i < r^{m-1}$, let $z_i = d_{m-1}(0, i + 1) - d_{m-1}(0, i)$. Then, by Theorem 6.3.3(a), for any $0 < y < r$, we have

\[
\ell_c(a, ir + y) = d_{m-1}(0, i) + \min\{2r - y - 2L_2(a, y), z_i + r + y - 2L_1(a, y)\},
\]  

(6.24)

and, by Theorem 6.3.4(b), we have

\[
\ell_c(a, ir) = d_{m-1}(i) + \min\{2r - 2L_2(a, 0), r + z_i, r - z_{i-1}\}.
\]  

(6.25)

By definition, $z_i = 1$ if $i \in V_{m-1}^+; z_i = 0$ if $i \in V_{m-1};$ and $z_i = -1$ if $i \in V_{m-1}^-$. Using the partition $\{V_{m-1}, V_{m-1}^+, V_{m-1}^-\}$ of the vertex set of $C(r, m-1)$, we now obtain a partition for the vertex set of $CQ_n(d, r, m)$. For $0 \leq i < r^{m-1}$, let

\[
U_i^\leq := \{(a, ir + y) | a \in \mathbb{Z}_2^n, 0 \leq y < r, 2r - y - 2L_2(a, y) \leq r + y - 2L_1(a, y)\}.
\]

Similarly, we define $U_i^\geq, U_i^=, U_i^>$ and $U_i^<_i$ to be the set of vertices $(a, x)$ such that $2r - y - 2L_2(a, y)$ is greater than or equal to, equal to, strictly greater and strictly less than $r + y - 2L_1(a, y)$, respectively. Obviously $\{U_i^\leq : i \in V_{m-1} \cup V_{m-1}^+\} \cup \{U_i^> : i \in V_{m-1} \cup V_{m-1}^+\} \cup \{U_i^= : i \in V_{m-1}^-\} \cup \{U_i^<_i : i \in V_{m-1}^-\}$ is a partition for the vertex set of $CQ_n(d, r, m)$. In the following we obtain formulas for $\ell_c(a, ir + y)$ for every $(a, ir + y) \in G$ by considering the cases $0 < y < r$ and $y = 0$ separately.

**Case 1:** Suppose $0 < y < r$. For any $(a, ir + y) \in U_i^\leq$ with $i \in V_{m-1}^+$, we have $2r - y - 2L_2(a, y) < z_i + r + y - 2L_1(a, y)$ as $z_i = 1$ for each $i \in V_{m-1}^+$. Hence, by (6.23) and (6.24), $\ell_c(a, ir + y) = d_{m-1}(0, i) + l(a, y)$. The same formula can be obtained for any $(a, ir + y) \in U_i^\leq$ with $i \in V_{m-1}^-$, and also for any $(a, ir + y) \in U_i^\leq$ with $i \in V_{m-1}^-$. Thus, for $(a, x) \in \{(a, x) \in U_i^\leq : i \in V_{m-1} \cup V_{m-1}^+\} \cup \{(a, x) \in U_i^> : i \in V_{m-1}^-\}$, we have

\[
\ell_c(a, ir + y) = d_{m-1}(0, i) + l(a, y).
\]

On the other hand, for any $(a, ir + y) \in U_i^\geq$ with $i \in V_{m-1}^+$, we have $2r - y - 2L_2(a, y) \geq z_i + r + y - 2L_1(a, y)$ and so $\ell_c(a, ir + y) = d_{m-1}(0, i + 1) + l(a, y)$. The same formula is
valid for any \((a, ir+y) \in U^>_i\) with \(i \in V_{m-1}\), and also for any \((a, x) \in U^>_i\) with \(i \in V^>_m\). That is, for any \((a, x) \in \{(a, x) \in U^>_i : i \in V_{m-1} \cup V^+_m\} \cup \{(a, x) \in U^>_i : i \in V^>_m\}\), we have
\[\ell_c(a, ir+y) = d_{m-1}(0, i+1) + l(a, y).\]

**Case 2:** Suppose \(y = 0\). That is, \(x = ir\) for some \(0 \leq i < r^{m-1}\). Recall that in this case \(L_1(a, 0) = 0\). Since by (6.22), \(z_i\) and \(z_{i-1}\) are involved in the expression of \(\ell_c(a, x)\), we need to consider more cases than in Case 1. Assume \(i \in V_{m-1} \cup V^+_m\) and \((a, ir) \in U^>_i\). So \(z_i \geq 0\) and \(r \leq 2L_2(a, 0)\). If \(z_{i-1} \leq 0\) or \(r < 2L_2(a, 0)\), then, by (6.23) and (6.22),
\[\ell_c(a, x) = d_{m-1}(0, i) + 2r - 2L_2(a, 0) = d_{m-1}(0, i) + l(a, 0);\]
if \(z_{i-1} = 1\) and \(r = 2L_2(a, 0)\), then \(\ell_c(a, x) = d_{m-1}(0, i) + l(a, 0) - 1\). For any \(i \in V_{m-1} \cup V^+_m\) and \((a, x) \in U^>_i\), if \(z_{i-1} = -1\), then \(\ell_c(a, x) = d_{m-1}(0, i+1) + l(a, 0);\) if \(z_{i-1} \geq 0\), then \(\ell_c(a, x) = d_{m-1}(0, i+1) + \min\{r, r-z_i-z_{i-1}\} = d_{m-1}(0, i+1) + l(a, 0) - (z_i + z_{i-1})\) since \(2r - 2L_2(a, 0) > r\) and \(z_i + z_{i-1} \geq 0\). For any \(i \in V_{m-1}\) and \((a, x) \in U^>_i\), we have \(\ell_c(a, x) = d_{m-1}(0, i) + l(a, 0)\) and for any \(i \in V^>_m\) and \((a, x) \in U^>_i\), we have \(\ell_c(a, x) = d_{m-1}(0, i+1) + l(a, 0)\).

Therefore, using formulas from Cases 1 and 2,
\[
\sum_{(a, x) \in G} \ell_c(a, x) =
\sum_{i \in V_{m-1} \cup V^+_m} \left( \sum_{(a, ir+y) \in U^>_i} (d_{m-1}(0, i) + l(a, y)) + \sum_{(a, ir+y) \in U^>_i} (d_{m-1}(0, i + 1) + l(a, y)) \right)
+ \sum_{i \in V^>_m} \left( \sum_{(a, ir+y) \in U^>_i} (d_{m-1}(0, i) + l(a, y)) + \sum_{(a, ir+y) \in U^>_i} (d_{m-1}(0, i + 1) + l(a, y)) \right) - \beta,
\]

where
\[\beta := \sum_{i \in V_{m-1} \cup V^+_m} \sum_{z_{i-1}=1}^{r=2L_2(a, 0)} \sum_{z_{i-1}=0}^{r > 2L_2(a, 0)} (z_i + z_{i-1}).\]

Setting \(\beta_1\) to be the size of \(\{(a, 0) : r = 2L_2(a, 0)\}\) and \(\beta_2\) to be the size of \(\{(a, 0) : r > 2L_2(a, 0)\}\), we have
\[\beta \leq |V_{m-1} \cup V^+_m| \beta_1 + |V_{m-1}| \beta_2 + 2|V^+_m| \beta_2.\]

By Lemma (6.24), \(|V_{m-1} \cup V^+_m| \leq r^m / (2(r + 1)) \leq r^{m-1} / 2, |V_{m-1}| + 2|V^+_m| = |V_{m-1}| + |V^+_m| + |V^>_m| = r^{m-1}\) and since \(\beta_1 + \beta_2 \leq 2^n\), we have
\[\beta \leq 2^n r^{m-1}.\] (6.26)
For any $0 \leq i < r^{m-1}$, we have
\[
L^* := \sum_{(a, ir+y) \in U_i^<} l(a, y) + \sum_{(a, ir+y) \in U_i^\geq} l(a, y)
= \sum_{(a, ir+y) \in U_i^<} l(a, y) + \sum_{(a, ir+y) \in U_i^\geq} l(a, y)
= \sum_{a \in \mathbb{Z}_2^n, 0 \leq y < r} l(a, y),
\]
where tight bounds for $L^*$ were given in (6.24), (6.24) and (6.34). Therefore,
\[
\sum_{(a, x) \in G} \ell_c(a, x) = |V_{m-1} \cup V_{m-1}^+ \cup V_{m-1}^-|L^* + \sum_{i \in \mathbb{Z}_{m-1}} \left( |U_i^<| + |U_i^\geq| \right) d_{m-1}(0, i)
+ \sum_{i \in \mathbb{Z}_{m-1}^+} \left( |U_i^<| d_{m-1}(0, i) + |U_i^\geq| (d_{m-1}(0, i) + 1) \right)
+ \sum_{i \in \mathbb{Z}_{m-1}^-} \left( |U_i^<| d_{m-1}(0, i) + |U_i^\geq| (d_{m-1}(0, i) - 1) \right) - \beta.
\]
Since $|U_i^<| + |U_i^\geq| = |U_i^<| + |U_i^\geq| = r2^n$ for every $0 \leq i < r^{m-1}$ and $|V_{m-1} \cup V_{m-1}^+ \cup V_{m-1}^-| = r^{m-1}$, the equation above is
\[
\sum_{(a, x) \in G} \ell_c(a, x) = r^{m-1}L^* + r2^n \sum_{i \in \mathbb{Z}_{m-1}} d_{m-1}(0, i) + \sum_{i \in \mathbb{Z}_{m-1}^+} |U_i^\geq| - \sum_{i \in \mathbb{Z}_{m-1}^-} |U_i^\geq| - \beta.
\]
By Lemma 6.3.4, $|V_{m-1}^-| = |V_{m-1}^+| \leq r^{m-1}/2$. So, $-\sum_{i \in \mathbb{Z}_{m-1}^+} |U_i^\geq| + \sum_{i \in \mathbb{Z}_{m-1}^-} |U_i^\geq| = \sum_{i \in \mathbb{Z}_{m-1}^-} |U_i^\geq| \leq r^{m-1}2^{n-1}$. Therefore, by (6.28) and (6.29), we have
\[
r^{m-1}L^* + r2^n \text{td}(C(r, m - 1)) - r^{m-1}2^n - r^{m-1}2^{n-1}
\leq \sum_{(a, x) \in G} \ell_c(a, x) \leq r^{m-1}L^* + r2^n \text{td}(C(r, m - 1)).
\]
So, by (6.28) and (6.29), we have
\[
r^{m-1}(nr2^{n-1} + L^*) + r2^n \text{td}(C(r, m - 1)) - r^{m-1}2^n - r^{m-1}2^{n-1}
\leq \text{td}(CQ_n(d, r)) \leq r^{m-1}(nr2^{n-1} + L^*) + r2^n \text{td}(C(r, m - 1)).
\]
Note that $nr2^{n-1} + L^* = \text{td}(Q_n(d, r))$. Hence the equation above together with Theorem 6.5.2 implies the required result.

In the following let
\[
\alpha'_{n,d,r} := \alpha_{n,d,r} + \frac{2}{2nr + r^2 + 8[nd]^2},
\]
where $\alpha_{n,d,r}$ is defined in (6.30).
Theorem 6.5.4. Suppose \( dr \geq 2n \). Then
\[
2^{n-1}r^{m-1} \left( nr + \lceil r^2/2 \rceil + 4\lceil n/d \rceil^2 \right) (1 - \alpha'_{d,r}) + r2^n \text{td}(C(r, m - 1)) \leq \text{td}(CQ_n(d, r, m)) \leq 2^{n-1}r^{m-1} \left( nr + \lceil r^2/2 \rceil + 4\lceil n/d \rceil^2 \right) + r2^n \text{td}(C(r, m - 1)).
\]

Proof. When \( m = 1 \), \( CQ_n(d, r, m) \) is isomorphic to \( Q_n(d, r) \) and hence
\[
\text{td}(QC_n(d, r, 1)) = rn2^{n-1} + \sum_{a \in \mathbb{Z}_2^n, 0 \leq y < r} l(a, y) = \text{td}(Q_n(d, r)), \quad (6.30)
\]
where, by Theorem 6.3.3,
\[
l(a, y) = 2\lceil n/d \rceil - y - 2L_1(a, y), \quad (6.31)
\]
if \( 0 \leq y \leq \lfloor r/2 \rfloor \), and
\[
l(a, y) = 2\lceil n/d \rceil - (r - y) - 2L_2(a, y) \quad (6.32)
\]
if \( \lceil (r + 1)/2 \rceil < y < r \).

Suppose \( m \geq 2 \). By Theorem 6.5.3 and using (6.31) and (6.32), for any \( (a, ir + y) \in G \) with \( 0 \leq i < r^{m-1} \) and \( 0 \leq y \leq \lfloor (r - 1)/2 \rfloor \), we have
\[
\ell_c(a, ir + y) = l(a, y) + d_{m-1}(0, i);
\]
for any \( (a, ir + y) \in G \) with \( 0 \leq i < r^{m-1} \) and \( y \geq \lceil (r + 1)/2 \rceil \), we have
\[
\ell_c(a, ir + y) = l(a, y) + d_{m-1}(0, i + 1).
\]
Furthermore, if \( y = r/2 \) when \( r \) is even, \( \ell_c(a, ir + y) = \min\{d_m(-r/2, x), d_m(r/2, x)\} + r/2 \), that is \( \ell_c(a, ir + y) = l(a, r/2) + d_{m-1}(0, i) - 1 \) if \( i \in V_{m-1}^- \), and \( \ell_c(a, ir + y) = l(a, r/2) + d_{m-1}(0, i) \) if \( i \in V_{m-1} \cup V_{m-1}^+ \).

\[
\sum_{(a,x) \in G} \ell_c(a, x) = \sum_{a,y \leq \lfloor r/2 \rfloor} \sum_{i=0}^{r^{m-1}-1} (d_{m-1}(i) + l(a, y)) - \sum_{i \in V_{m-1}, a \in \mathbb{Z}_2^n} \beta_r \\
+ \sum_{a,y > \lceil (r+1)/2 \rceil} \sum_{i=0}^{r^{m-1}-1} (d_{m-1}(i + 1) + l(a, y)) \\
= r^{m-1} \sum_{a,0 \leq y < r} l(a, y) + \sum_{(a,y)} \sum_{i=1}^{r^{m-1}-1} d_{m-1}(i) - \sum_{i \in V_{m-1}, a \in \mathbb{Z}_2^n} \beta_r,
\]
where \( \beta_r = 1 \) and \( \sum_{i \in V_{m-1}, a \in \mathbb{Z}_2^n} \beta_r < 2^{n-1}r^{m-1} \) if \( r \) is even and \( \beta_r = 0 \) and \( \sum_{a,i} \beta_r = 0 \) if \( r \) is odd by Lemma 6.5.1. Therefore, by (6.21) and (6.30), for some \( 0 \leq c \leq 1 \),
\[
\text{td}(CQ_n(d, r, m)) = r^{m-1} \left( nr2^{n-1} + \sum_{a,0 \leq y < r} l(a, y) \right) + r2^n \text{td}(C(r, m - 1)) - c2^{n-1}r^{m-1}.
\]

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Therefore, the equation above together with the tight bounds for \((6.3.1)\) obtained in \((6.3.4)\) yields the required results.

**Remark.** Set \(\epsilon = 2^{n-1} \sum_{i=1}^{m-2} (-1)^i (m - i - 1)r^{m-i}\).

(a) Suppose \(dr = n\). If \(r \geq 2^9\) and \(2^n r\) is large, then Theorem \(6.5.3\) yields

\[
\text{td}(CQ_n(d, r, m)) \approx \begin{cases} 
2^n r^{m+1} (2nr + (m + 4)r^2 - (m - 1)), & \text{if } r \text{ is odd,} \\
2^n r^{m+1} (2nr + (m + 4)r^2) + \epsilon, & \text{if } r \text{ is even.}
\end{cases}
\]

(b) Suppose \(dr \geq 2n\). When \(2^n r\) is large, \(\alpha'_{n,d,r}\) is small and so Theorem \(6.5.4\) yields

\[
\text{td}(CQ_n(d, r, m)) \approx \begin{cases} 
2^n r^{m+1} (2nr + m(r^2 - 1) + 8\lceil n/d \rceil^2), & \text{if } r \text{ is odd,} \\
2^n r^{m+1} (2nr + mr^2 + 8\lceil n/d \rceil^2) + \epsilon, & \text{if } r \text{ is even.}
\end{cases}
\]

### 6.6 Vertex-forwarding index

The vertex-forwarding index of a cube-connected circulant can be obtained by its total distance as it belongs to the family of Cayley graphs.

**Theorem 6.6.1.**

(a) Suppose \(dr = n\). Then \(\xi(CQ_n(d, r, m)) = \xi_m(CQ_n(d, r, m))\) and the following hold:

(i) if \(r \geq 2^9\), then

\[
2^n r^{m+1} (2d + 5) \left(1 - \frac{4 + 6r + 20r \log_2 r}{2nr + 5r^2} \right) + r^2 n \text{td}(C(r, m - 1)) \leq \xi(CQ_n(d, r, m)) \leq 2^n r^{m+1} (2d + 5) \left(1 - \frac{12r - 8}{2nr + 5r^2} \right) + r^2 n \text{td}(C(r, m - 1));
\]

(ii) if \(3 \leq r < 2^9\), then

\[
2^n r^{m+1} (2nr + r^2 - 6r - 4) + r^2 n \text{td}(C(r, m - 1)) \leq \xi(CQ_n(d, r, m)) \leq 2^n r^{m+1} (2nr + 5r^2 - 12r + 8) + r^2 n \text{td}(C(r, m - 1)) + 1.
\]

(b) Suppose \(dr \geq 2n\). Then \(\xi(CQ_n(d, r, m)) = \xi_m(CQ_n(d, r, m))\) and

\[
2^n r^{m+1} ((n - 2)r + \lceil r^2/2 \rceil + 4 \lceil n/d \rceil^2) (1 - \alpha_{n,d,r}) + r^2 n \text{td}(C(r, m - 1)) \leq \xi(CQ_n(d, r, m)) 2^n r^{m+1} ((n - 2)r + \lceil r^2/2 \rceil + 4 \lceil n/d \rceil^2) + r^2 n \text{td}(C(r, m - 1)) + 1.
\]

**Proof.** By Theorem 6.2.1, \(\xi(CQ_n(d, r, m)) = \xi_m(CQ_n(d, r, m)) = \text{td}(CQ_n(d, r, m)) - 2^n r^{m+1}\) as \(CQ_n(d, r, m)\) is a Cayley graph. Therefore, the results follow from Theorems 6.5.3 and 6.5.4. \(\square\)
6.7 Edge-forwarding index

In this section we obtain tight bounds for the edge-forwarding index of cube-connected circulants. While there are similarities between recursive cubes of rings and cube-connected circulants, these two families of graphs have notably different characteristics and behaviours in routings unless they are isomorphic. In fact, any recursive cube of rings is an orbit-proportional graph and so its edge-forwarding index can be obtained by a shortest path routing (see Section 5.7). However, a cube-connected circulant is not orbit-proportional in general. There is no known method which gives the exact value or even a good estimate of these parameters for such graphs. Therefore, obtaining the edge-forwarding index of $CQ_n(d,r,m)$ is a challenging task.

Recall from Section 5.7 that an integral uniform flow for $X$ is a function $f : R \rightarrow \{0,1\}$ such that there is exactly one active path with respect to $f$ for every pair of vertices $(u,v)$ in $X$, where $P$ is active with respect to $f$ if $f(P) > 0$ and $R$ is the set of all paths in $X$. For given $H \leq Aut(X)$, $f$ is $H$-invariant if and only if $f(P) = f(gP)$ for any $P \in R$ and any $g \in H$. We say a uniform flow $f$ is minimal if any active path with respect to $f$ is a shortest path. The uniform flow $f$ is optimal if $(X,f) = (X)$, where $(X,f) = \max_{e \in E(X)} \sum_{P : e \in P} f(P)$.

Let $E = \{E_1,E_2,\ldots,E_t\}$ be a partition of $E(X)$. We call $X$ an $E$-proportional graph if for any pair of vertices $(u,v)$ and any two $uv$-paths $P$ and $P'$ in $X$ such that $P$ is a shortest path, we have

$$|E(P) \cap E_i| = |E(P') \cap E_i|, \quad i = 1,2,\ldots,t. \tag{6.33}$$

In the case when $E_1,E_2,\ldots,E_t$ are $H$-edge orbits of $X$, then $X$ is called ‘$H$-orbit proportional’ \cite{H}, where $H \leq Aut(X)$.

Recursive cubes of rings and cube-connected cycles are orbit-proportional and the results in \cite{H} are useful for obtaining their edge-forwarding indices (see Section 5.7 and \cite{H}). In contrast to these two families of graphs, multiplicative circulants and cube-connected circulants are not orbit proportional. In the following we develop results to obtain the edge-forwarding index for $E$-proportional graphs. This will be used in Theorem 6.7.1 to obtain the edge-forwarding index of cube-connected circulants and multiplicative circulants.

**Theorem 6.7.1.** Suppose $E = \{E_1,E_2,\ldots,E_t\}$ is a partition of $E(X)$ and there exists an integral uniform flow $f^*$ in $X$ such that $f^*$ loads edges of $E_i$ uniformly for every $1 \leq i \leq t$. The following hold:
(a) 
\[ \pi(X, f^*) = \max_{1 \leq i \leq t} \frac{\sum_{e \in E_i} \sum_{P : \delta \in P} f^*(P)}{|E_i|}. \]  
(6.34)

(b) If \( X \) is \( E \)-proportional and \( f^* \) is minimal, then \( f^* \) is optimal and
\[ \pi(X) = \pi_m(X) = \pi(X, f^*) = \max_{1 \leq i \leq t} \frac{\sum_{(u,v) \in V^2(X)} |E(P_{u,v}) \cap E_i|}{|E_i|}, \]
where \( P_{u,v} \) is a shortest path from \( u \) to \( v \) in \( X \).

Proof. (a) Since \( f^* \) loads edges of \( E_i \) uniformly for each \( i \), the result follows from the definition of the edge-forwarding index.

(b) Let \( \delta : E(X) \to \mathbb{R}^+ \) be a non-negative non-zero function. For every \( P \in R \), define \( \delta(P) := \sum_{e \in P} \delta(e) \) and for every pair of distinct vertices \( (u,v) \) in \( X \), define \( \delta(u,v) := \min \{ \delta(P) : P \in R_{u,v} \} \). Moreover, let
\[ \pi(X, \delta) := \frac{\sum_{(u,v) \in V^2(X)} \delta(u,v)}{\sum_{e \in E(X)} \delta(e)}. \]

For any integral uniform flow \( f \) and a pair of vertices \( (u,v) \) in \( X \), there is exactly one \( uv \)-path \( P_{u,v}^* \) such that \( f(P_{u,v}^*) = 1 \). Clearly, \( \delta(u,v) = \delta(P_{u,v}^*) = \sum_{P \in R_{u,v}} f(P) \sum_{e \in P} \delta(e) \), where \( R_{u,v} \) is the set of paths from \( u \) to \( v \). Thus
\[ \sum_{(u,v) \in V^2(X)} \delta(u,v) \leq \sum_{(u,v) \in V^2(X)} \sum_{P \in R_{u,v}} f(P) \sum_{e \in P} \delta(e) \]
\[ = \sum_{(u,v) \in V^2(X)} \sum_{e \in E(X)} \sum_{P \in R_{u,v} : e \in P} f(P) \delta(e) \]
\[ = \sum_{e \in E(X)} \delta(e) \sum_{(u,v) \in V^2(X)} \sum_{P \in R_{u,v} : e \in P} f(P) \]
\[ = \sum_{e \in E(X)} \delta(e) \sum_{P \in E : e \in P} f(P) \]
\[ \leq \pi(X, f) \sum_{e \in E(X)} \delta(e). \]

Hence \( \pi(X, \delta) \leq \pi(X, f) \) for any integral uniform flow \( f \). Therefore, if there exists a function \( \delta \) and an integral uniform flow \( f \) such that \( \pi(X, \delta) = \pi(X, f) \), then \( f \) is optimal uniform flow. In particular, if \( f \) is a minimal uniform flow, then \( \pi(X) = \pi_m(X) = \pi(X, f) \).

Define \( \delta_i(e) = 1 \) if \( e \in E_i \), and \( \delta_i(e) = 0 \) otherwise, \( 1 \leq i \leq t \). Since \( X \) is \( E \)-proportional, for every pair of distinct vertices \( (u,v) \) and any shortest path \( P \) between \( u \) and \( v \) we have \( \delta_i(u,v) = \delta_i(P) \) by (6.33). In particular, \( \delta_i(u,v) = \delta_i(P_{u,v}) = |E(P_{u,v})| \cap
Let $E_i$, where $P_{u,v} \in R_{u,v}$ and $f^*(P_{u,v}) = 1$. Thus $|E(P_{u,v}) \cap E_i| = \sum_{\mathbb{E} \subseteq E_i} \sum_{P \in R_{u,v}, v \in P} f^*(P)$.

Therefore, as $\sum_{e \in E} \delta_i(e) = |E_i|$, we have

$$\frac{\sum_{(u,v) \in V^2(X)} \delta_i(u,v)}{\sum_{e \in E} \delta_i(e)} = \frac{\sum_{(u,v) \in V^2(X)} |E(P_{u,v}) \cap E_i|}{|E_i|} = \frac{\sum_{e \in E} \sum_{P \in R_{u,v}} f^*(P)}{|E_i|}.$$ 

Therefore, by (6.33), we have $max_{1 \leq i \leq t} \pi(X, \delta_i) = \pi(X, f^*)$ and so $f^*$ is an optimal minimal uniform flow.

Theorem 6.7.1 can be used to obtain the edge-forwarding index of a large class of graphs, including edge-transitive graphs and orbit-proportional graphs. In a special case, if $X$ is a Cayley graph, there exists an $H$-invariant uniform flow in $X$ which loads each $H$-edge orbit of $X$ uniformly for some $H \leq Aut(X)$ with $|H_{u,v}| = 1$ [16, Theorem 1].

A multiplicative circulant graph $C(r, m)$ is not orbit-proportional for $m \geq 2$. The edge-forwarding index of $C(r, m)$ with $r$ odd is known to be $\pi(C(r, m)) = r^{m-1}[r^2/2]$ [58, Theorem 1]. Using Theorem 6.7.1, we recover this result:

**Corollary 6.7.2.** If $r$ is an odd positive integer, then

$$\pi(C(r, m)) = \pi_m(C(r, m)) = \frac{r^{m-1}(r^2 - 1)}{4}.$$ 

**Proof.** Let $R = \{uP_v : u, v \in V(C(r, m))\}$ be a routing for $C(r, m)$, where $P_v$ is path from 0 to $v$, obtained from the shortest path algorithm in $C(r, m)$ given in [101] (which is equivalent to Algorithm 6.1 except that it is designed for $C(r, m)$), and $uP_v$ is the translation of $P_v$ under $u$. One can verify by induction that $R$ loads all edges of $C(r, m)$ uniformly. Thus $C(r, m)$ is $\{E(C(r, m))\}$-proportional. Note that $|E(P_v) \cap E(C(r, m))| = dist(0, v)$ and so $\sum_{(u,v) \in V^2} |E(P_{u,v}) \cap E(C(r, m))| = r^m \sum_{v \in V} dist(0, v)$. Hence, by Theorems 6.7.1 and 6.5.2, we have

$$\pi(C(r, m)) = \pi_m(C(r, m)) = \frac{|V(C(r, m))|}{|E(C(r, m))|} = \frac{r^{m-1}(r^2 - 1)}{4}.$$ 

Unfortunately, there does not exist any partition of the edge set of cube-connected circulants which satisfy condition (6.33). However, there is a partition of the edge set of cube-connected circulants such that (6.33) is ‘partially’ satisfied. In the following we prove results on such conditions which will be used to obtain $\pi(CQ_u(d, r, m))$.

**Lemma 6.7.3.** Given a graph $X$, let $E = \{E_1, E_2, \ldots, E_t\}$ be a partition of $E(X)$. Suppose there exist an integer $t'$, $1 \leq t' \leq t$, and some integers $\beta_{u,v}^i \geq 0$ for $i$, $1 \leq i \leq t'$.  

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and \((u, v) \in V(X) \times V(X)\) such that for any \(P, P' \in R_{u,v}\) with \(P\) a shortest path, we have
\[
|E(P) \cap E_i| \leq |E(P') \cap E_i| + \beta_{u,v}^i, \quad i = 1, 2, \ldots, t'.
\]
Then the following holds:
\[
\pi(X) \geq \max_{1 \leq i \leq t'} \sum_{(u,v) \in V(X) \times V(X)} \frac{|E(P_{u,v}) \cap E_i| - \beta_{u,v}^i}{|E_i|}.
\]

**Proof.** Let \(f\) be an arbitrary integral uniform flow in \(X\). By (6.35), for any \(i, 1 \leq i \leq t'\), \((u,v) \in V(X) \times V(X)\) and a shortest path \(P_{u,v}\) from \(u\) to \(v\), we have
\[
\sum_{e \in E_i} \sum_{P \in R_{u,v} \in P} f(P) \geq |E(P_{u,v}) \cap E_i| - \beta_{u,v}^i.
\]
Hence for any \(i, 1 \leq i \leq t'\), we have
\[
\pi(X, f) |E_i| \geq \sum_{e \in E_i} \sum_{(u,v) \in V(X) \times V(X)} \sum_{P \in R_{u,v} \in P} f(P) = \sum_{(u,v) \in V(X) \times V(X)} \sum_{e \in E_i} P \in R_{u,v} \in P \sum_{P \in R_{u,v} \in P} f(P) \geq \sum_{(u,v) \in V(X) \times V(X)} (|E(P_{u,v}) \cap E_i| - \beta_{u,v}^i).
\]
By taking minimum over all integral uniform flows on the left-hand side, we obtain the result.

As mentioned above, it is known that \(\pi(C(r,m)) = r^{m-1} \frac{r^2 / 2}{4}\), Theorem 1], and we obtained a similar result when \(r\) is odd in Corollary 6.7.1. Using Lemma 6.7.3, we recover a similar result when \(r\) is even in the following corollary.

**Corollary 6.7.4.** If \(r\) is an even positive integer, then
\[
\pi(C(r,m)) = \pi_m(C(r,m)) = \frac{r^{m+1}}{4}.
\]

**Proof.** Let \(F\) denote the set of edges of \(C(r,m)\) with label 1, \(-1\) and \(\hat{F} = E(C(r,m)) \setminus F\). For integers \(u, v, k\) and \(k'\) with \(0 \leq u, v < r\) and \(0 \leq k, k' < r^{m-1}\), one can verify that any path from \(kr + u\) to \(k'r + v\) in \(C(r,m)\) has at least \(\min\{|v-u|, r-|v-u|\}\) edges with labels either 1 or \(-1\), and when it is a shortest path, the number of its edges with labels either 1 or \(-1\) is exactly \(\min\{|v-u|, r-|v-u|\}\). In other words, for any two paths \(P, P'\) from \(kr + u\) to \(k'r + v\) such that \(P\) is a shortest path, we have
\[
|E(P) \cap F| = \min\{|v-u|, r-|v-u|\} \leq |E(P') \cap F|.
\]

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Hence, by Lemma 6.7.3,

\[
\pi(C(r, m)) \geq \sum_{(kr+u,k'r+v) \in (\mathbb{Z}/m)^2} \min\{|v-u|, r-|v-u|\} / |F|
\]

\[= r^{2m-1} \sum_{0 \leq u < r} \min\{u, r-u\} / r^m \quad (6.36)
\]

\[= r^{m+1}/4.
\]

On the other hand, an all-to-all routing \( R \) can be obtained from the shortest path algorithm in \( C(r, m) \) given in [40] (which is similar to Algorithm 6.2) such that \( \pi(C(r, m), R) \leq r^{m+1}/4 \) [43, Proposition 5]. This together with (6.36) completes the proof. □

**Remark.** Note that the lower bound in (6.36) for \( \pi(C(r, m)) \) is strictly larger than the lower bound \( |V(C(r, m))| \) (6.36) therefore, by Theorem 6.5.2 and Corollaries 6.7.2 and 6.7.4, \( C(r, m) \) is not edge-optimal when \( r \) is even, but is edge-optimal when \( r \) is odd.

Define

\[
E_i := \{(a, x), (a + e_{i+dx}, x)\} : (a, x) \in G, \ 1 \leq i \leq d;
\]

\[
F_j := \{(a, x), (a, x + r^j)\} : (a, x) \in G, \ 0 \leq j \leq m - 1.
\]

Note that \( E_1, E_2, \ldots, E_d, F_0, F_1, \ldots, F_{m-1} \) are \( G \)-edge orbits of \( CQ_n(d, r, m) \), where \( G \leq Aut(CQ_n(d, r, m)) \). Hence

\[
\mathcal{E} = \{E_1, E_2, \ldots, E_d, F_0, F_1, \ldots, F_{m-1}\}
\]

is a partition of the edge set of \( CQ_n(d, r, m) \). Note that \( |E_i| = r^{m}2^{n-1} \) for \( 1 \leq i \leq d \) and \( |F_j| = r^{m}2^n \) for \( 0 \leq j < m \).

**Lemma 6.7.5.** For \((a, x) \in G\), let \( P \) and \( \bar{P} \) be two paths from \((0, 0)\) to \((a, x)\) such that \( P \) is a shortest path. Assume \( x = kr + y \) for some integers \( k \) and \( y \) with \( 0 \leq k < r^{m-1} \) and \( 0 \leq y < r \). The followings hold:

(a) \( |E(P) \cap E_i| \leq |E(\bar{P}) \cap E_i| \) for \( 1 \leq i \leq d \) if and only if \( n \equiv 0 \) mod \( d \);

(b) if \( dr = n \) and \( |r - 2y - 2L_2(a, y) + 2L_4(a, y)| \leq 1 \), then \( |E(P) \cap F_0| \leq |E(\bar{P}) \cap F_0| + 1 \);

(c) if \( dr = n \) and \( |r - 2y - 2L_2(a, y) + 2L_4(a, y)| \geq 2 \), then \( |E(P) \cap F_0| \leq |E(\bar{P}) \cap F_0| \);

(d) if \( dr \geq 2n \), then \( |E(P) \cap F_0| \leq |E(\bar{P}) \cap F_0| \).
Proof. (a) The proof is similar to that of Lemma 5.7.3.

(b)-(c) Denote by \( \hat{y}^1 \) and \( \hat{y}^2 \) the two \((a, y)\)-sequences in \( Q_n(d, r) \) given in (5.12) and (5.13). So \( l(\hat{y}^1) \leq r + y - 2L_1(a, y), \ l(\hat{y}^2) \leq 2r - y - 2L_2(a, y) \) and \( \min\{l(\hat{y}^1), l(\hat{y}^2)\} = \min\{r + y - 2L_1(a, y), 2r - y - 2L_2(a, y)\} \) by Theorem 6.3.2.

By Lemma 6.3.4, a path \( \hat{P}' \) from \((0_n, 0)\) to \((a, w)\) can be obtained from \( \hat{P} \) for some \( w \equiv x \mod r \) such that the number of edges of \( \hat{P}' \) with labels either \((0_n, 1)\) or \((0_n, -1)\) is equal to that of \( \hat{P} \). Denote by \( \hat{w}' \) the corresponding \((a, w)\)-sequence of \( \hat{P}' \). Therefore, by Lemma 6.3.2 and (6.11), we have

\[
|E(\hat{P} \cap F_0| = \hat{l}(\hat{w}') \geq \min\{l(\hat{y}^1), l(\hat{y}^2)\}.
\]

Now denote by \( \hat{x} \) the corresponding \((a, x)\)-sequence of \( P \). By Theorem 10.1.4,

\[
\ell_c(\hat{x}) = \min\{l(\hat{y}^1) + d_m(y-r, x), l(\hat{y}^2) + d_m(y, x)\},
\]

and by Lemma 6.3.2 the number of edges of \( P \) with labels from \( F_0 \) is either \( l(\hat{y}^1) \) or \( l(\hat{y}^2) \).

If \( |r - 2y - 2L_2(a, y) + 2L_1(a, y)| \leq 1 \), then \( |l(\hat{y}^1) - l(\hat{y}^2)| \leq 1 \) and the number of edges of \( P \) with labels from \( F_0 \) is at most \( 1 + \min\{l(\hat{y}^1), l(\hat{y}^2)\} \). This together with (6.18) yields \( 1 + |E(\hat{P} \cap F_0| = 1 + \hat{l}(\hat{w}') \geq |E(P \cap F_0| \) and completes the proof of (b).

If \( |r - 2y - 2L_2(a, y) + 2L_1(a, y)| \geq 2 \), then \( |l(\hat{y}^1) - l(\hat{y}^2)| \geq 2 \). Since \( |d_m(y, x) - d_m(y-r, x)| = |d_{m-1}(0, k) - d_{m-1}(0, k+1)| \leq 1 \), we have \( \ell_c(\hat{x}) = \min\{l(\hat{y}^1), l(\hat{y}^2)\} + d_m(w, x) \) and the number of edges in \( F_0 \) is \( \min\{l(\hat{y}^1), l(\hat{y}^2)\} \) by Theorem 6.3.2, where \( w = y \) or \( w = y - r \). Hence \( |E(P) \cap F_0 = \min\{l(\hat{y}^1), l(\hat{y}^2)\} \leq |E(\hat{P}) \cap F_0| \) by (6.38). This yields part (c).

(d) Using Lemma 6.3.4, let \( \hat{P}' \) be the path obtained from \( P \) such that the number of edges of \( \hat{P}' \) with labels either \((0_n, 1)\) or \((0_n, -1)\) is equal to that of \( \hat{P} \). If \( 0 \leq y \leq [(r - 1)/2] \), then for the optimal \((a, y)\)-sequence \( \hat{y}^1 \) in \( Q_n(d, r) \) obtained from (5.11), we have \( |E(P) \cap F_0| = |E(P') \cap F_0| \geq l(\hat{y}^1) \) by (6.11). Similarly, if \([ (r + 1)/2 \leq y < r \), then for the optimal \((a, y)\)-sequence \( \hat{y}^2 \) in \( Q_n(d, r) \) obtained from (6.21), we have \( |E(\hat{P}) \cap F_0| \geq l(\hat{y}^2) \). One can check that when \( r \) is even and \( y \neq r/2 \), we have \( l(\hat{y}^1) = l(\hat{y}^2) \) and similar to the above \( |E(\hat{P}) \cap F_0| \geq l(\hat{y}^i) \) for \( i = 1, 2 \).

By Algorithm 6.4, we can construct an all-to-all routing for \( CQ_n(d, r, m) \) which loads edges of each block of \( E \) uniformly. However, this routing may not be a shortest path routing.

Lemma 6.7.6. The routing \( R^* \) obtained from Algorithm 6.3 is a \( G \)-invariant all-to-all routing that loads edges with the same label uniformly. Furthermore,
Algorithm 6.3 A routing for $CQ_n(d, r, m)$ with uniform loading on blocks of $E$

1: for all $(a, x) \in G$ do
2:   Begin
3:     Let $x = kr + y$ for some integers $0 \leq k < r^{m-1}$ and $0 \leq y < r$;
4:     if $dr = n$ then
5:       Denote by $\hat{y}$ an optimal $(a, y)$-sequence obtained from Procedure 6.2
6:     else
7:       Denote by $\hat{y}$ an optimal $(a, y)$-sequence obtained from Procedure 6.3
8:       Using Construction 6.3.2, obtain $\hat{w} = (w_0, w_1, \ldots, w_{s+1})$ from $\hat{y}$;
9:       Denote by $P^*_a$ the path $P_1$ concatenated with $P_2$, where $P_1$ is a path from 
0:       $(0_n, 0)$ to $(a, w_{s+1})$ with $(a, w_{s+1})$-sequence $\hat{w}$, and $P_2$ is a path from $(a, w_{s+1})$ to 
1:       $(a, x)$ obtained using Algorithm 6.1
10:   End
11: Set $R^* := \emptyset$;
12: for all pairs of vertices $((b, y), (c, z))$ do
13:   Add $(b, y)P^*_a$ to $R^*$, where $(a, x) = (b, y)^{-1}(c, z)$.

(a) if $dr = n$, then $\pi(CQ_n(d, r, m), R^*) \leq r^{m-1}2n^2 \max\{4r^2, 5r^2 - 8r + 8\}$;
(b) if $dr \geq 2n$ and $n \equiv 0 \mod d$, then $\pi(CQ_n(d, r, m), R^*) \leq r^{m-1}2n^2 \left(\left\lfloor r^2/2\right\rfloor + 4(n/d)^2\right)$.

Proof. By construction of $R^*$ in Algorithm 6.3, it is obvious that $R^*$ is $G$-invariant. 
Note that the set of edges in $CQ_n(d, r, m)$ with the same label is a $G$-edge orbit of 
$CQ_n(d, r, m)$. Hence the load of $R^*$ on edges of any $G$-edge orbit of $CQ_n(d, r, m)$ is uniform, where edge-orbits of $CQ_n(d, r, m)$ are given in (5.34). Therefore, $\pi(CQ_n(d, r, m), R^*)$ can be obtained using (5.34).

Note that for any path $P_{u,v} \in R^*$ from $u$ to $v$, we have $P_{u,v} = uP_{(a,x)}$, where 
$u^{-1}v = (a, x)$. So $\sum_{(u,v) \in G} |E(P_{u,v}) \cap A| = 2^{n-rm} \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap A|$ for any 
$A \in E$. Hence, by (5.34), we have 

$$\pi(CQ_n(d, r, m), R^*) = \max \left\{ \max_{1 \leq i \leq d} 2 \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_i|, \max_{0 \leq j < m} \sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_j| \right\}.$$

The number of edges of $P_{(a,x)} \in R^*$ in $E_i$ is the number of unique values $dt+i \mod n$ between 1 and $n$ such that $a_0 = 1$, $0 \leq t < r^m$, which is at most $n/d$. 
Hence $\sum_{a \in Z^2} |E(P_{(a,0)}) \cap E_i| = 2^{n-n/d} \sum_{t=0}^{r^m/n} t^{(n/d)} = 2^{n-1}n/d$, and so 

$$\sum_{(a,x) \in G} |E(P_{(a,x)}) \cap E_i| = 2^{n-1}r^m n/d, \ 1 \leq i \leq d. \quad (6.39)$$

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On the other hand, for fixed $a \in \mathbb{Z}_2^r$, $k$ and $y$ with $0 \leq k < r^{m-1}$ and $0 \leq y < r$, $|P_{(a,kr+y)} \cap F_j|$ is equal to the number of edges of $P_{0,kr}$ with label $(0_n, r^j)$ by construction of $R^*$, $1 \leq j < m$. Since $P_{0,kr}$ in $R^*$ is obtained by a shortest path routing in $C(r, m - 1)$, we have $\sum_{0 \leq k < r^{m-1}} |P_{(a,kr+y)} \cap F_j| \leq r^{m-2}|r^2/4|$ by Corollaries 6.7.2 and 6.7.4 for any $a \in \mathbb{Z}_2^r$, $0 \leq y < r$ and $1 \leq j < m$. Therefore,

$$\sum_{(a,x) \in G} |E(P_{(a,x)}) \cap F_j| \leq 2^r r^{m-1}|r^2/4|, \quad 1 \leq j < m. \quad (6.40)$$

Clearly, $|P_{(a,kr+y)} \cap F_0| = l(a, y)$ for any $0 \leq k < r^{m-1}$. Hence

$$\sum_{(a,x) \in G} |E(P_{(a,x)}) \cap F_0| = r^{m-1} \sum_{(a,y), 0 \leq y < r} l(a, y).$$

(a) When $dr = n$, $\sum_{(a,y), 0 \leq y < r} l(a, y)$ can be specified using (5.31), (5.34) and (5.32). Therefore, when $dr = n$, by (6.39) and (6.40),

$$\pi(CQ_n(d, r, m), R^*) \leq \max \left\{ r^{m+1} n, 2^2 r^{m-1}|r^2/4|, r^{m-1} 2^{m-2}(5r^2 - 8r + 8) \right\}$$

$$\leq r^{m-1} 2^{m-2} \max \left\{ 4r^2, 4|r^2/4|, 5r^2 - 8r + 8 \right\}.$$

(b) When $dr \geq 2n$, we have $\sum_{(a,y), 0 \leq y < r} l(a, y)$ using (6.42), and so, by (6.39) and (6.40),

$$\pi(CQ_n(d, r, m), R^*) \leq r^{m-1} 2^{m-1} \max \{2nr/d, 2|r^2/4|, |r^2/2| + 4(n/d)^2\}.$$ 

This completes the proof. \qed

**Theorem 6.7.7.**

(a) Suppose $dr = n$. If $r \geq 2^9$, then

$$r^{m+1} 2^{m-2} \max \left\{ 4, 5-\frac{20\log^2 r}{r} - \frac{4}{r^m} \right\} \leq \pi(CQ_n(d, r, m)) \leq r^{m+1} 2^{m-2} \left( 5 - \frac{8r - 8}{r^2} \right);$$

if $7 \leq r < 2^9$, then

$$r^{m+1} 2^n \leq \pi(CQ_n(d, r, m)) \leq r^{m-1} 2^{m-2} \max \{ 4r^2, 5r^2 - 8r + 8 \};$$

and if $3 \leq r \leq 6$, then

$$\pi(CQ_n(d, r, m)) = r^{m+1} 2^n.$$

(b) If $n \equiv 0 \mod d$ and $dr \geq 2n$, then

$$r^{m-1} 2^{n-1} \max \left\{ \frac{2nr}{d}, \left( \frac{r^2}{2} \right)^2 + 4n^2 \frac{d^2}{r} \right\} (1 - \beta_{n,d,r})$$

$$\leq \pi(CQ_n(d, r, m)) \leq r^{m-1} 2^{n-1} \left( \frac{r^2}{2} + 4n^2 \frac{d^2}{r} \right).$$

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Proof. For \((a, x) \in G\), let \(P(a, x)\) and \( \bar{P}(a, x)\) denote two paths from \((0, 0)\) to \((a, x)\) in \(CQ_n(d, r, m)\) such that \(P(a, x)\) is a shortest path. By Lemma \(\ref{lemma:shortest-path-condition}\), \(|E(P(a, x)) \cap E_i| \leq |E(\bar{P}(a, x)) \cap E_i|\) for \(1 \leq i \leq d\), and \(|E(P(a, x)) \cap F_0| \leq |E(\bar{P}(a, x)) \cap F_0| + \beta(a, x)\), where \(\beta(a, x) = 1\) if \(dr = n\) and \(|r - y - L_2(a, y) + L_1(a, y)| \leq 1\), and \(\beta(a, x) = 0\) otherwise. On the other hand, \(\sum_{u, v} |E(P_{u, v}) \cap A| = 2^m r^m \sum_{u \in G} |E(P_u) \cap A|\) for any \(A \subseteq E\), where \(P_u\) and \(P_a\) are shortest paths. Note that \(|E(P(a, x)) \cap F_0| \geq l(a, y)\) by \(\ref{equation:shortest-path-condition}1\), where \(x = kr + y\) for some integers \(k\) and \(y\) with \(0 \leq k < r^{m-1}\) and \(0 \leq y < r\). Thus, by Lemmas \(\ref{lemma:shortest-path-condition}\) and \(\ref{lemma:shortest-path-condition}\), we have

\[
\pi(CQ_n(d, r, m)) \geq \max \left\{ \frac{\max_{1 \leq i \leq d} \sum_{(u, v) \in E_i} |E(P_{u, v}) \cap E_i|}{|E_i|}, \frac{\sum_{(u, v) \in E_i} (|E(P_{u, v}) \cap F_0| - \beta(u, v))}{|F_0|} \right\} 
= \max \left\{ \frac{\sum_{(a, x) \in G} |E(P_{a, x}) \cap E_i|}{|E_i|}, \frac{\sum_{(a, x) \in G} (|E(P_{a, x}) \cap F_0| - \beta(a, x))}{|F_0|} \right\} 
\geq \max \left\{ \sum_{(a, x) \in G} |E(P_{a, x}) \cap E_i|, \sum_{a \in G} l(a, y), r^{m-1} \sum_{a \in G} \beta(a, x) \right\} .
\]

(6.41)

(a) Suppose \(dr = n\). Since \(\sum_{(a, x) \in G} \beta(a, x) \leq r 2^n\), if \(r \geq 2^n\), then, by \(\ref{equation:shortest-path-condition}1\), \(\ref{equation:shortest-path-condition}3\) and \(\ref{equation:shortest-path-condition}4\),

\[
\pi(CQ_n(d, r, m)) \geq r^{m+1} 2^{n-2} \max\{4, 5(1 - 4(\log^2 r)/r) - 4/r^m\} .
\]

and if \(3 \leq r < 2^n\), then, by \(\ref{equation:shortest-path-condition}1\), \(\ref{equation:shortest-path-condition}3\) and \(\ref{equation:shortest-path-condition}4\),

\[
\pi(CQ_n(d, r, m)) \geq r^{m+1} 2^{n-2} \max\{4, 1 - 4/r^m\} .
\]

Since \(\pi(CQ_n(d, r, m)) \leq \pi(CQ_n(d, r, m), R)\) for any routing \(R\), Lemma \(\ref{lemma:shortest-path-condition}\) together with \(\ref{equation:shortest-path-condition}2\) and \(\ref{equation:shortest-path-condition}3\) implies the results when \(dr = n\).

(b) When \(dr \geq 2n\) and \(n \equiv 0 \mod d\), by \(\ref{equation:shortest-path-condition}1\), \(\ref{equation:shortest-path-condition}3\) and \(\ref{equation:shortest-path-condition}4\), we have

\[
\pi(CQ_n(d, r, m)) \geq r^{m-1} 2^{n-1} \max\{2nr/d, (r^2/2) + 4(n/d)^2\} (1 - \beta_{n, d, r}) ,
\]

which together with Lemma \(\ref{lemma:shortest-path-condition}\) implies the result when \(dr \geq 2n\). \(\square\)

### 6.8 Bisection width

**Lemma 6.8.1.**

(a) \(2^{r_m-1} \leq \text{bw}(C(r, m)) \leq 2(r^m - 1)/(r - 1)\);
(b) if \( r \) is even, then \( \text{bw}(C(r,m)) = 2r^{m-1} \).

**Proof.** (a) It is known that \( 2(|s_1| + |s_2| + \cdots + |s_k|) \) is an upper bound for the bisection width of a circulant graph with respect to the connection set \( \{\pm s_1, \pm s_2, \ldots, \pm s_k\} \) [6.1, Lemma 1]. So, the bisection width of \( C(r,m) \) is at most \( 2(1 + r + \cdots + r^{m-1}) \). Using (6.7.7) and Corollaries 6.7.2 and 6.7.3, we obtain the lower bound \( 2r^{m-1} \) for \( \text{bw}(C(r,m)) \).

(b) Multiplicative circulants are special cases of recursive circulants, and the bisection width of the latter is given in [6.3, Theorem 2], which gives the formula. \( \square \)

**Theorem 6.8.2.** (a) If \( dr = n \), then

\[
\text{(i)} \quad 2^{n-1}r^{m-1} \min \left\{ 1, \frac{4r^2}{5r^2-8r+8} \right\} \leq \text{bw}(CQ_n(d,r,m)) \leq r^{m-1}2^{n-1} \quad \text{when} \quad 3 \leq r < 2^9.
\]

\[
\text{(ii)} \quad \frac{2^{n+1}r^{m+1}}{5r^2-8r+8} \leq \text{bw}(CQ_n(d,r,m)) \leq r^{m-1}2^{n-1} \quad \text{when} \quad r \geq 2^9.
\]

(b) If \( dr \geq 2n \) and \( n \equiv 0 \mod d \), then

\[
\frac{2^{n+1}r^{m+1}}{r^2 + 8n^2/d^2} \leq \text{bw}(CQ_n(d,r,m)) \leq dr^{m-1}/n.
\]

Furthermore, if \( r \) is even, then \( \text{bw}(CQ_n(d,r,m)) \leq 2^{n-1}r^{m-1} \min\{dr/n,4\} \).

**Proof.** Using (6.3) and the upper bounds for \( \pi(CQ_n(d,r,m)) \) in Theorem 6.7.4, we obtain lower bounds for \( \text{bw}(CQ_n(d,r,m)) \). Hence, if \( dr = n \), then

\[
\text{bw}(CQ_n(d,r,m)) \geq \begin{cases} 
2^{n-1}r^{m-1}, & \text{if } 3 \leq r \leq 6, \\
2^{n-1}r^{m-1} \min \left\{ 1, \frac{4r^2}{5r^2-8r+8} \right\}, & \text{if } 7 \leq r < 2^9, \\
\frac{2^{n+1}r^{m+1}}{5r^2-8r+8}, & \text{if } r \geq 2^9,
\end{cases}
\]

(6.44)

and if \( dr \geq 2n \) and \( n \equiv 0 \mod d \), then

\[
\text{bw}(CQ_n(d,r,m)) \geq \frac{2^{n+1}r^{m+1}}{r^2 + 8n^2/d^2}.
\]

(6.45)

In the following we give bisections for \( CQ_n(d,r,m) \). Let \( U = \{(a_1 \ldots a_n, x) : 0 \leq x < r^m, a_i \in \mathbb{Z}_2, 2 \leq i \leq n\} \) and \( \overline{U} = \{(0a_1 \ldots a_n, x) : 0 \leq x < r^m, a_i \in \mathbb{Z}_2, 2 \leq i \leq n\} \). Then \( \{U, \overline{U}\} \) is a partition for \( G \) so that \( U \) and \( \overline{U} \) have the same size, namely \( 2^{n-1}r^m \).

Denote by \( V_1 \) the set of all \( x \), \( 0 \leq x < r^m \), such that \( 1 \in D(x) \). By Lemma 6.2.5, \( |V_1| = dr^m/n \). Therefore,

\[
\delta(U, \overline{U}) = \{((1a_1 \ldots a_n, x), (0a_1 \ldots a_n, x)) : a_i \in \mathbb{Z}_2, 2 \leq i \leq n, x \in V_1\}.
\]

Thus, the bisection size of \( \{U, \overline{U}\} \) is \( |\delta(U, \overline{U})| = 2^{n-1}r^md/n \), which implies the claimed upper bound for the bisection width of \( CQ_n(d,r,m) \).
Here we give another bisection for $CQ_n(d, r, m)$ with $r$ even which gives a smaller upper bound for its bisection width in some cases. When $r$ is even, by Lemma 6.8.1, there exists a bisection $\{W, \overline{W}\}$ for multiplicative circulants such that $|\delta(W, \overline{W})| = 2r^{m-1}$ and $|W| = |\overline{W}| = r^m/2$. Now let $U = \cup_{a \in \mathbb{Z}_2^a} W_a$ and $\overline{U} = \cup_{a \in \mathbb{Z}_2^a} \overline{W}_a$, where $(a, x) \in W_a$ (respectively $(a, x) \in \overline{W}_a$) if and only if $x \in W$ (respectively $x \in \overline{W}$), for any $a \in \mathbb{Z}_2^a$. It can be verified that $\{U, \overline{U}\}$ is a partition of $G$ and $|U| = |\overline{U}| = r^m2^{n-1}$. Since for any $(a, x) \in U$ and $(a', x') \in \overline{U}$ we have $x \neq x'$, there is no cube edge with one end-vertex in $U$ and the other in $\overline{U}$ and so $|\delta(U, \overline{U})| = 2^{n+1}r^{m-1}$. By combining this upper bound and the one obtained earlier, we have $bw(Q_n(d, r)) \leq 2^{n-1}r^{m-1}\min\{dr/n, 4\}$ when $r$ is even. \hfill \Box

Remark. According to Table 6.1, recursive cubes of rings and cube-connected circulants outperform existing well-known network models in important characteristics. Many well-known graphs have either a large degree or a large diameter. For instance, the degree of hypercube with $N$ vertices is $\log N$, and the diameter of circulants of degree 4 or 6 with $N$ vertices is $O(\sqrt{N})$. An advantage of recursive cubes of rings and cube-connected circulants with $N$ vertices is that their degrees can be chosen to be small while their diameters are $O(\log N)$. In view of (5.3), while $DL_d$ and $TL_d$ are edge-optimal, their forwarding indices are much larger than $\pi_{4, N} = \Theta(N \log_3 N)$ and $\pi_{6, N} = \Theta(N \log_3 N)$, respectively, where $N$ is the number of vertices. Although cube-connected circulants and recursive cubes of rings are not edge-optimal in general, their forwarding indices are close to the optimal order of forwarding indices for graphs of given order and degree. That is, $\pi(Q_n(d, r)) = |V(Q_n(d, r))| 5r/4$ and $\xi(Q_n(d, r)) = |V(Q_n(d, r))| (5r/4 + n/2)$, where $\pi_{d+2, r^{2n}} = \xi_{d+2, r^{2n}} = O(r^{2n}(\log_{d+1}(r) + n\log_{d+1}2))$; and $\pi(CQ_n(d, r, m)) = |V(CQ_n(d, r, m))| 5r/4$ and $\pi(CQ_n(d, r, m)) = |V(CQ_n(d, r, m))| ((m+4)r+n/2$ where $\pi_{2m+d, 2r^m} = \xi_{2m+d, 2r^m} = \Theta(2^m r^m (m \log_{2m+d-1}2 + n \log_{2m+d-1}2))$. As another desirable property, the bisection widths of cube-connected circulants are almost of the same order of its vertex size. In addition, there are four different parameters in the definition of a cube-connected circulant whereas for most of well-known network models there are at most two such parameters. These four parameters grant a freedom of choice in specifying characteristics of a network model, including the number of vertices, degree, diameter and forwarding indices. Note that a cube-connected circulant with appropriate parameters can have much smaller diameter and forwarding indices than a given recursive cube of ring. Finally, vertex-transitivity is an important factor of recursive cubes of rings and cube-connected circulants, and so these families of graphs outplay the shuffle-exchange and de Bruijn graphs in this regards [12, 17].

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| Network $X$ | $|V(X)|$ | $\text{deg}(X)$ | $\text{diam}(X)$ | $\xi(X)$ | $\pi(X)$ | $\text{bw}(X)$ |
|---|---|---|---|---|---|---|
| $C_n$ | $n$ | $2$ | $\left\lceil \frac{n}{2} \right\rceil$ | $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ | $\left\lfloor \frac{n^2}{4} \right\rfloor$ | $2$ |
| $CC_n$ | $2^n n$ | $3$ | $\left\lceil \frac{5n}{2} - 2 \right\rceil$ | $7n^2 2^{n-2}(1 - o(1))$ | $5n^2 2^{n-2}(1 - o(1))$ | $2^n - 1$ |
| $B(n)$ | $2^n n$ | $4$ | $\left\lceil \frac{3n}{2} \right\rceil$ | $\leq 5n^2 2^{n-2} - n(2^n + 2^{n/2+1} - 3)$ | $5n^2 2^{n-3}(1 + o(1))$ | $2^n$ |
| $DL_d$ | $2d^2 + 2d + 1$ | $4$ | $d$ | $4(d^3 - d)/3$ | $d(d + 1)(2d + 1)/3$ | $\leq 4d + 2$ |
| $TL_d$ | $3d^3 + 3d + 1$ | $6$ | $d$ | $2(d^3 - d)$ | $d(d + 1)(2d + 1)/3$ | $\leq 12d + 4$ |
| $C(r, m)$ | $r^m$ | $2m$ | $\left\lceil \frac{(r-1)m}{2} \right\rceil$ | $r^m \left( \frac{mr}{4} - 1 \right) - \epsilon_1$ | $r^{m-1} \left\lceil \frac{r^2}{4} \right\rceil$ | $\leq \frac{2(r^{m-1})}{r-1}$ |
| $Q_n$ | $2^n$ | $n$ | $n$ | $2^{n-1}(n - 2) + 1$ | $2^{n-1}$ | $2^{n-1}$ |
| $Q_n(d, r)$ | $2^n r$ | $d + 2$ | $n + \left\lceil \frac{3r - 4}{2} \right\rceil - 2$ | $\approx 2^{n-2}r^2(2d + 5)$ | $5 \cdot 2^{n-\frac{3}{2}}$ (if $r \geq 2^9$) | $2^{n-1}$ |
| (if $dr = n$) | | | | | | |
| $Q_n(d, r)$ | $2^n r$ | $d + 2$ | $n + \left\lceil \frac{3r - 4}{2} \right\rceil$ | $\approx 2^{n-1} \left( nr + \left\lceil \frac{r^2}{4} \right\rceil + 4n^2 \frac{r}{d} \right)$ | $\approx 2^{n-1} \left( \left\lceil \frac{r^2}{4} \right\rceil + 4n^2 \frac{r}{d} \right)$ | $\approx 2^n \min \{ \frac{dr}{m}, 2 \}$ |
| (if $dr \geq 2n$, $n \equiv 0 \mod d$) | | | | | | |
| $CQ_n(d, r, m)$ | $2^n r^m$ | $d + 2m$ | $n + \left\lceil \frac{3r - 4 - 2nm}{2} \right\rceil + \left\lceil \frac{(r-1)(m-1)}{2} \right\rceil$ | $\approx 2^{n-2}r^m \left( 2nr + (m + 4)r^2 - (m - 1) \right)$ | $\approx 5 \cdot 2^{n-2}r^{m+1}$ | $\approx 2^{n-1}r^{m-1}$ |
| (if $dr = n$) | | | | | | |
| $CQ_n(d, r, m)$ | $2^n r^m$ | $d + 2m$ | $n + \max \left\{ \left\lceil \frac{(r-1)(m-1)-4}{2} \right\rceil + 2 \left\lceil \frac{nr}{d} \right\rceil \right\}$ | $\approx 2^{n-2}r^{m-1} \left( 2nr + m(r^2 - 1) + 2n^2 \frac{r}{d} \right)$ | $\approx 2^{n-1}r^{m-1} \left( \left\lceil \frac{r^2}{4} \right\rceil + 4n^2 \frac{r}{d} \right)$ | $\approx 2^{n-1}r^{m-1} \frac{d}{n}$ |
| (if $dr \geq 2n$, $n \equiv 0 \mod d$) | | | | | | |

Table 6.1: Comparison of $Q_n(d, r)$ and $CQ_n(d, r, m)$ with a few known network topologies
6.9 Embedding

In this section we study embeddings of cube-connected circulants into hypercubes and embeddings of hypercubes into cube-connected circulants.

Theorem 6.9.1. Suppose $k, m$ are positive integers and $k$ divides $m$. Then $CQ_n(d, 2^k, m/k)$ can be embedded into $Q_{n+m}$ with dilation 2 and congestion 4.

Proof. It is known that the multiplicative circulant $C(2^k, m/k)$ can be embedded into the hypercube $Q_m$ with dilation 2 and congestion 4 [SS, Corollary 3]. Assume $\phi : \mathbb{Z}_{2^m} \rightarrow \mathbb{Z}_{2^m}$ is the embedding of $C(2^k, m/k)$ into $Q_m$ such that $dil_\phi(C(2^k, m/k), Q_m) = 2$. We define $\varphi : \mathbb{Z}_2^n \times \mathbb{Z}_{2^m} \rightarrow \mathbb{Z}_2^n \times \mathbb{Z}_{2^m}$ as follows:

$$\varphi(a, x) = (a, \phi(x)) \quad \text{for any } (a, x) \in \mathbb{Z}_2^n \times \mathbb{Z}_{2^m}. \quad (6.46)$$

Clearly, $\varphi$ is an one-to-one mapping. Any two adjacent vertices in $CQ_n(d, 2^k, m/k)$ are joined by a cube edge or a circulant jump. If $(a, x)$ and $(b, y)$ are joined by a cube edge in direction $e_i$ for some $1 \leq i \leq n$, then $b = a + e_i, i \in D(x)$ and $x = y$. In this case, the length of $\varphi((a, x), (b, y))$ is 1 since $(a, \phi(x))$ is joined to $(a + e_i, \phi(x))$ in $Q_{n+m}$. On the other hand, if $(a, x)$ and $(b, y)$ are joined by a circulant jump, then there is a path from $(a, \phi(x))$ and $(b, \phi(y))$ in $Q_{n+m}$ with length at most 2 as $dil_\phi(C(2^k, m/k), Q_m) = 2$.

Similarly, one can see that the congestion of $\varphi$ on edges $\{a, a + e_i\}$ for $1 \leq i \leq n$ in $Q_{n+m}$ is 1. And the congestion of $\varphi$ on edges $\{a, a + e_i\}$ for $n + 1 \leq i \leq n + m$ in $Q_{n+m}$ is equal to the congestion of $\phi$ on links of $Q_m$. This completes the proof. \qed

Theorem 6.9.2. $Q_{n+m}$ can be embedded into $CQ_n(d, 2^k, m/k)$ with dilation

(a) $\max\{2^k + 1, \sum_{i=0}^{k-1} \left(\frac{i}{i/2}\right)\}$ if $dr = n$; and

(b) $\max\{2\lceil n/d \rceil - 1, \sum_{i=0}^{k-1} \left(\frac{i}{i/2}\right)\}$ if $dr \geq 2n$.

Proof. There exists an embedding $\nu : \mathbb{Z}_{2^m}^n \rightarrow \mathbb{Z}_{2^m}$ of $Q_m$ into $C(2^k, m/k)$ with dilation $\sum_{i=0}^{k-1} \left(\frac{i}{i/2}\right)$ [SS, Theorem 13]. For $a = (a_1a_2\ldots a_n)$ and $b = (b_1b_2\ldots b_m)$, let $ab$ denote $(a_1a_2\ldots a_nb_1b_2\ldots b_m)$. We define $\psi : V(Q_{n+m}) \rightarrow V(CQ_n(d, r, m))$ as follows:

$$\psi(ab) = (\nu(b)) \quad \text{for any } a \in \mathbb{Z}_2^n, b \in \mathbb{Z}_{2^m}.$$

Any edge in $Q_{n+m}$ is of the form $\{ab, ab + e_i\}$ for some $1 \leq i \leq n + m$. If $n + 1 \leq i \leq n + m$, then $\psi(ab) = (\nu(b))$ and $\psi(ab + e_i) = (\nu(b + e_{i-n}))$. Hence, the dilation of $\{ab, ab + e_i\}$ is the length of $\nu(b)$ to $\nu(b + e_{i-n})$.

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Now assume \( \{ab, ab + e_i\} \in E(Q_{n+m}) \) for some \( 1 \leq i \leq n \). If \( i \in D(\nu(b)) \), then \((a, \nu(b))\) and \((a + e_i, \nu(b))\) are adjacent in \( CQ_n(d, r, m) \) and so the dilation of such edge is 1 under \( \psi \). If \( i \not\in D(\nu(b)) \), then assume \( x \in \mathbb{Z}_{2^k} \) is a closest vertex to \( \nu(b) \) in \( C(2^k, m/k) \) such that \( i \in D(x) \). By Lemma 6.2.5, \( |\nu(b) - x| \leq \lfloor r/2 \rfloor \) if \( dr = n \) and \( |\nu(b) - x| \leq \lceil n/d \rceil - 1 \) if \( dr \geq 2n \). So the dilation of \( \{ab, ab + e_i\} \) under \( \varphi \) is at most \( 2|x - \nu(b)| + 1 \). This completes the proof. \( \Box \)
Chapter 7

Concluding remarks and further research topics

In Section 7.1, we give an overview of major achievements in this thesis. We discuss possible future research directions in Section 7.2.

7.1 Remarks

Our knowledge on the routing of circulants, even for 4-regular circulants, is very limited, except for a few special families of circulants. Our results in Chapter 4 are contributions to the study of circulants for communication networks which sheds light on the edge-forwarding and arc-forwarding indices of general 4-regular circulant graphs. Our results show the behaviour of such circulant networks and their throughput capabilities. By Theorem 4.1.1, the lower and upper bounds for these two parameters are exact for $C_n(1, \sqrt{n})$, where $\sqrt{n}$ is an integer, and $\pi(C_n(1, \sqrt{n}))/2 = \pi'(C_n(1, \sqrt{n})) \approx n^{3/2}/8$. The approximation ratio of the obtained bounds for these parameters are of a constant factor in most cases. It can be implied from Theorem 4.1.1 that a 4-regular circulant graph $C_n(1, s)$ has almost maximum network throughput if $s$ is of order of $\sqrt{n}$. Also our 4.85-approximation algorithm for the optical indices of this family of graphs in Chapter 4 is a contribution towards implementation of such networks in optical networking (see Theorem 4.1.4). To our knowledge, our results on the optical index of circulant graphs, apart from very special classes of them, are the first results on this parameter in the literature.

Although recursive cubes of rings have received a lot of attention, our knowledge
about this family of graphs was very limited and a few known results were shown to be incorrect or weak in general. Our study on these graphs in Chapter 5 provided a solid ground for understanding and research on this family of graphs. A noteworthy contribution of this chapter is the sharp bounds for the total distance of $Q_n(d;r)$ in Theorems 5.5.2 and 5.5.4 and the methods used to obtain these bounds. We used these results to give the vertex-forwarding index, edge-forwarding index and bisection width of recursive cubes of rings. Algorithm 5.1 gives detailed procedure of routing in this family of graphs. Since cube-connected cycles and cube-of-rings are subfamilies of recursive cubes of rings, our results apply to these subfamilies. Our results not only provide a comprehensive picture of this family of graphs for interconnection networks, but also give us an insight by which we introduced cube-connected circulants that outperform recursive cubes of rings for some invariants.

The introduction of cube-connected circulants in Chapter 6 is another contribution to efficient communication network design. In view of the properties of cube-connected circulants, including simple routing, logarithmic diameter, high data throughput, this family of graphs are promising network structures, especially for large interconnection networks. This family of graphs admits recursive constructions, and as four different parameters are involved, they enjoy flexible design with desirable properties including degree, diameter, forwarding indices and bisection width. It can be observed from Table 6.1 that in comparison with some well-known interconnection networks, the cube-connected circulant is an efficient model for many invariants. One can observe that $CQ_n(d, r, m)$ and $Q_n(d, r)$ have almost similar edge-forwarding indices while $CQ_n(d, r, m)$ can have much larger number of vertices than that of $Q_n(d, r)$.

Interestingly, cube-connected circulants have nearly the minimum possible values for forwarding indices among all graphs with the same maximum degree and order. In view of (6.9), for graphs with $2^n r^m$ vertices and degree at most $2m + d$, we have $\xi_{2m+d,r^m2^n} \geq r^m2^n(n + m \log(r))(1 + o(1))/\log(2m + d - 1)$. From Theorem 6.6.1, $\xi(CQ_n(d, r, m))$ is close to the lower bound for $CQ_n(d, r, m)$ if $dr = n$. Hence $\xi(CQ_n(d, r, m))$ is very close to the lower bound for $\xi_{2m+d,r^m2^n}$, especially when $m \geq r/\log r$. From Theorem 6.7.7, we have $\pi_2m+d,r^m2^n \geq r^m2^{n+1}(n + m \log(r))/(2m + d)$ log(2m + d - 1)). From Theorem 7.4.1, we have $\pi(CQ_n(d, r, m)) \leq \frac{5m^{m+1}2^{n-2}}{3}$ if $n \equiv 0 \mod d$ and $dr \geq 2n$. Hence $\pi(CQ_n(d, r, m))$ is very close to the lower bound for $\pi_{2m+d,r^m2^n}$. Therefore, cube-connected circulants are efficient in terms of the load on edges and vertices for given maximum degree and order, that is, they
are efficient models for maximum network throughput.

A significant contribution of Chapter 6 is Lemma 6.7.3 which is used to obtain a strong lower bound for the edge-forwarding index of $CQ_n(d, r, m)$. This result can be used for obtaining lower bounds for the edge-forwarding indices of a large class of graphs, including the edge-transitive, orbit-proportional and $E$-proportional graphs, for a specific partition $E$ of the edge set of a given graph.

7.2 Further research topics

In Theorem 4.1.1, ratios of the upper bounds of the forwarding indices of $C_n(1, s)$ to their corresponding lower bounds are constant for $s$ with $2 \leq s \leq \sqrt{3n/2}$, and they are $O(\sqrt{n})$ for $s$ with $\sqrt{3n/2} < s < n/2$. As the forwarding indices of $C_n(1, s)$ are $O(n^2)$ for $2 \leq s \leq \sqrt{n}$, our feeling is that they should be $O(n^2)$ for $\sqrt{3n/2} < s < n/2$ as well. This suggests that there is space for improving the lower bounds for these parameters for the case $\sqrt{3n/2} < s < n/2$. Since obtaining a ‘packed basis’ for the case $\sqrt{3n/2} < s < n/2$ is difficult, 2-dimensional lattice method might not be an effective method for improving the lower bounds in this case.

Moreover, an important and challenging problem on the family of circulants of degree 4 is to obtain their average distances. In [27] a formula for the diameter of $C_n(1, s)$ is obtained. With a formula or a good estimate of the average distance of $C_n(1, s)$, stronger lower bounds for the forwarding indices of such graphs will be obtained, and hence a better understanding of routing and forwarding indices of this family of graphs can be derived in general. Therefore, we believe that the problem of determining the average distance of $C_n(1, s)$ is an interesting research topic.

By Theorem 4.1.1, a lower bound for the edge-forwarding index of circulant graph $C_n(1, s)$ is approximately $\sqrt{n^3/(3\sqrt{2})}$, where $s \notin \{1, n-1\}$. We showed that the edge-forwarding index of $C_n(1, \sqrt{n})$ is almost $\sqrt{n^3}/4$, which is very close to this lower bound, where $\sqrt{n}$ is an integer. On the other hand, it is known [103] that the edge-forwarding index of Frobenius circulants of degree 4 with diameter $d$ and $n_d = 2d^2 + 2d + 1$ vertices is $d(d+1)(2d+1)/3 \approx \sqrt{n_d^3}/(3\sqrt{2})$. It is known that the diameter of $C_n(s_1, s_2)$ is equal to its lower bound if $s_1 = D$ and $s_2 = D + 1$, where $D = [(\sqrt{2n-1}-1)/2]$ (e.g. [71, Theorem 5.2]). Hence the family of circulants $C_n(D, D + 1)$ is a good candidate for efficient networks. Moreover, their diameters increase by at most 1 if a vertex becomes faulty [71, Theorem 5.5]. An interesting research topic is to obtain $\pi(C_n(D, D + 1))$, $\overline{\pi}(C_n(D, D + 1))$, $w(C_n(D, D + 1))$ and $\overline{w}(C_n(D, D + 1))$. In view of the discussion
above, we conjecture that $C_n(D, D + 1)$ has forwarding index close to the lower bound $\sqrt{n^3}/(3\sqrt{2})$.

**Conjecture 7.2.1.** Let $D = \lfloor (\sqrt{2n} - 1)/2 \rfloor$. Then $\pi(C_n(D, D+1))/2 = \pi(C_n(D, D+1)) \approx \sqrt{n^3}/(6\sqrt{2})$.

In the same direction, one may investigate the forwarding and optical indices for circulants $C_n(1, \sqrt{cn})$ for positive integers $c$ such that $\sqrt{cn}$ in an integer.

Our results in Chapter 4 are about connected circulants $C_n(1, s)$. These circulants belong to a larger family of circulants $C_n(a, b)$ where gcd$(n, a, b) = 1$. It is shown in [10] that any circulant $C_n(a, b)$ can be decomposed into two Hamiltonian cycles. The same problems in Chapter 4 for this family of circulants is a potential research topic in future.

Our results in Chapter 4 only deal with the forwarding and optical indices of 4-regular circulants. However, the family of circulant graphs is large and interesting in many fields of research. It is conjectured that the shortest path problem for $k$-regular circulants, $k \geq 3$, is NP-hard [115]. The problems of designing heuristic algorithms for shortest path and obtaining the forwarding indices for such graphs can add insight to understanding this family of graphs.

Knödel graphs are an interesting family of circulants. For a very special family of Knödel graphs, an efficient shortest path algorithm is known and their forwarding indices is obtained (see Section 3.3). Recently, a shortest path algorithm for Knödel graphs is designed in the general case [58]. We would like to explore the Wiener index and forwarding index problems in this general case.

So far we have discussed undirected circulant graphs, but the family of semi-directed 2-circulant $\text{Cay}(\mathbb{Z}_n, \{s_1, s_2\})$, where $s_1 \neq 0$ and $s_2 \notin \{0, s_1, -s_1\}$, is an interesting family of circulants (e.g. see [37]). The arc-forwarding index and directed optical index of semi-directed 2-circulants are unknown. We believe that these problems are interesting and challenging.

We obtained sharp bounds for $\text{td}(Q_n(d, r))$ in Theorems 5.5.2 and 5.5.4 in Chapter 5, which led to the results on the vertex-forwarding index, edge-forwarding index and bisection width of recursive cubes of rings. These results are asymptotic due to the complexity of computing or estimating the summation of $L_i(a, x)$ over all vertices $(a, x)$ of $Q_n(d, r)$, $i = 1, 2$. However, as remarked in Chapter 5, the results give good approximations of the vertex-forwarding index, edge-forwarding index and bisection width of $Q_n(d, r)$ when $r$ is large enough and $n \equiv 0 \mod d$. For the case $n \not\equiv 0 \mod d$, the ratio of the upper bound of $\pi(Q_n(d, r))$ in our results to its corresponding lower bound is at least 2. One might investigate improving this ratio, but we did not
find it worthwhile since $Q_n(d, r)$ is not orbit-proportional in this case, and hence there is no guarantee that cube edges of the graph are loaded uniformly under an optimal routing.

There is a close relation between the edge-forwarding index and the isoperimetric number (or equivalently, expanding factor) of a graph (see e.g. [32]). An interested reader may use our results on the edge-forwarding index of studied families of graphs in this thesis, i.e. $C_n(1, s)$, recursive cubes of rings and cube-connected circulants, to study their isoperimetric numbers.

A primary motivation for introducing recursive cube of rings in [101] was studying a more general graph than cube-connected cycles. On the other hand, the butterfly graph is a sibling of cube-connected cycles since both are Cayley graphs defined on the same group but different connection sets. It may be interesting to study Cayley graphs on $G$, as given in (6.3) with condition $n \geq 2d$, with respect to the connection set

$$\{(0_n, 1), (0_{n-1}, 1), (e_1, 1), (e_2, 1), \ldots, (e_d, 1), (e_{n-1}, -1), (e_{n-2}, -1), \ldots, (e_{n-d}, -1)\}.$$

This family of graphs contains the family of butterfly graphs. We believe that our methods in Chapters 5 and 6 can be used in the study of this larger family of Cayley graphs.

Since cube-connected circulants are potentially good candidates for interconnection networks, it is desirable to have further studies of them. For instance, it is unknown whether a cube-connected circulant is Hamiltonian or not. However, it is known that cube-connected cycles contain cycles of almost every length [68]. Since cube-connected cycles are special cases of cube-connected circulants, we conjecture that cube-connected circulants are Hamiltonian and they contain cycles of almost every length.

Recall that recursive circulants are generalisations of multiplicative circulants. One may be interested in extending the definitions and results about cube-connected circulants in Chapter 6 to the more general case of cube-connected circulants, where the factor graph of $CQ_n(d, r, m)$ is the recursive circulant $G(cr^m, r)$ instead of the multiplicative circulant $C(r, m)$. This study may benefit from existing results about cube-connected circulants and recursive circulants in the literature.
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Glossary of Symbols

$\alpha_{n,d,r}$  -  
$\alpha_m$ is 1 if $m$ odd and $r$ even, 0 otherwise.

$Aut(X)$ automorphism group of graph $X$.

$\beta_{n,d,r}$  -  
$bw(X)$ bisection width of $X$.

$Cay(G,S)$ Cayley graph on group $G$ with respect to connection set $S$.

$CC_n$ cube-connected cycle of dimension $n$.

$C_n(S)$ circulant graph of order $n$ with respect to connection set $S$.

$COR(d,r)$ cube-of-rings.

$\ell(\hat{w})$ length of $\hat{w}$ in ring.

$CQ_n(d,r,m)$ cube-connected circulant.

$C(r,m)$ multiplicative circulant.

$diam(X)$ diameter of graph $X$.

$dil(X,Y)$ dilation of embedding $X$ into $Y$.

$dist(X)$ mean distance of $X$.

$D(x)$ set of directions of cube edges incident to $(a,x)$.

$\epsilon(x)$ is 1 if $x$ is odd, 0 if $x$ is even.

$E(X)$ edge set of $X$.

$l(a,x) = \min_{\hat{x}} l(\hat{x})$ in $Q_n(d,r)$.

$l_c(a,x) = \min_{\hat{x}} l_c(\hat{x})$ in $CQ_n(d,r,m)$.

$L_1(a,x)$.

$L_2(a,x)$.

$l(x) = x_1 \oplus x_2 s, x = (x_1, x_2)$.

$\oplus$ addition taken modulo $n$. 

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\( \ominus \) subtraction taken modulo \( n \).

\( \pi(X) \) edge-forwarding index of \( X \).

\( \pi_{\Delta,n} \) minimum edge-forwarding index of graphs with \( n \) vertices and degree at most \( \Delta \).

\( \pi_m(X) \) minimal edge-forwarding index of \( X \).

\( \pi(X,R) \) maximum load of \( R \) on edges of \( X \).

\( P_{x,y} \) a path from \( x \) to \( y \).

\( Q_n(d,r) \) recursive cube of rings.

\( K \rtimes_{\varphi} H \) semidirect product of \( K \) by \( H \) with respect to action \( \varphi \).

\( \text{td}(X) \) total distance of \( X \).

\( [V]^2 \) set of all 2-element subsets of \( V \).

\( V(X) \) vertex set of \( X \).

\( W(X) \) Wiener index of \( X \).

\( w(X) \) optical index of \( X \).

\( \xi(X) \) vertex-forwarding index of \( X \).

\( \xi_{\Delta,n} \) minimum vertex-forwarding index of graphs with \( n \) vertices and degree at most \( \Delta \).

\( \xi_m(X) \) minimal vertex-forwarding index of \( X \).

\( \xi(X,R) \) maximum load of \( R \) on vertices of graph \( X \).

\( \mathbb{Z}_2^n \) elementary abelian 2-group under bitwise binary addition.

\( \mathbb{Z}_r \) additive group of integers modulo \( r \).
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