Abstract

In this thesis we primarily focus on developing tractable dynamic market models, together with applications of these models. We also address several open questions concerning the statistical properties of static market models.

The original research contained in this thesis begins when we define a new distribution, called a general non-central hypergeometric distribution, which models biased sampling without replacement. We then consider a classic mechanism design market model with unit traders in which buyer valuations and seller costs are independent and, on each side of the market, identically distributed. We show that the equilibrium quantity traded has a general non-central hypergeometric distribution under a broad class of mechanisms, where the sampling bias encodes information about the distribution of buyer values relative to seller costs, strategic behaviour of market participants and rent extraction by the market intermediary. We then extend this work by developing a general approach to approximating outcomes in large markets. In particular, we show that the joint distribution of the equilibrium quantity traded and welfare is asymptotically normal, compute the parameters of the approximating normal distribution and bound the approximation rate.

We then turn our attention to dynamic market models in which traders with persistent types arrive over time and focus on optimally assessing the tradeoff between the benefits of increasing market thickness and the cost of delay. We start by considering a two-period extension of the classic model of Myerson and Satterthwaite [117] and derive the class of Bayesian optimal α-mechanisms. Investigating the properties of these mechanisms, we compare them to benchmark static mechanisms and find that dynamics tends to increase the convexity of the region of trade in period one. Motivated by the computational complexity of these optimal mechanisms, we consider approximate implementation in the form of a price-posting mechanism, before turning our attention to models with simpler type spaces.

Finally, we consider an infinite-horizon market model in which agents have binary type spaces. Assuming that a buyer-seller pair arrives in each period and agents are privately informed about their types, we introduce notions of dynamic efficiency and optimality and construct the class of Bayesian optimal α-mechanisms. We demonstrate that, provided the discount factor is sufficiently large, a profit-maximising two-sided platform creates higher welfare compared to a less centralised, welfare-targeting market maker. Further, the main benefits from dynamic mechanisms are reaped by clearing markets at fixed, optimally chosen frequencies. We consider a variety of economic applications including in-house production by the designer, taxation policy and asymptotically optimal prior-free mechanisms. With minor qualifications (including a dynamic generalisation of Myerson [116] regularity), our analysis and main results apply to more general arrival processes and type spaces.
Declaration

This is to certify that:

i. the thesis comprises only my original work towards the PhD except where indicated in the Preface,

ii. due acknowledgment has been made in the text to all other material used,

iii. the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Signed

Ellen Victoria Muir

Ellen Victoria Muir
Preface

This thesis contains original research in Chapters 3 through 7. Chapter 3 is based on the following papers.


Chapter 4 is based on the following paper, which has received a revise and resubmit from *Stochastic Models*.


Chapter 5 is based on the following working paper.


Chapter 6 is based on the following working paper, which is has been submitted for publication.


Finally, Chapter 7 is based on the following working papers, the second of which was submitted as an entry to the INFORMS George Nicholson Student Paper Competition.


All co-authorship has taken place in accordance with the Graduate Research Training Policy of the University of Melbourne.
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\textbf{Notation}

\begin{itemize}
  \item $\land$ \quad Minimum
  \item $\subset$ \quad Subset (not necessarily strict)
  \item $\| \cdot \|_\infty$ \quad Supremum norm
  \item $\sim$ \quad Set of preferences
  \item $\succsim_i$ \quad Preference relation of agent $i$
  \item $\succ_i$ \quad Strict preferences derived from $\succsim_i$
  \item $A$ \quad Set of actions available under Markov decision process
  \item $A_x$ \quad Set of actions available in state $x$ under Markov decision process
  \item $A_t$ \quad Period $t$ action for Markov decision process
  \item $a$ \quad Action available under Markov decision process
  \item $a_E$ \quad Number of efficient pairs cleared under action (binary type space)
  \item $a_I$ \quad Number of identical suboptimal pairs cleared under action (binary type space)
  \item $\mathcal{A}$ \quad Directed subset of $\mathbb{N}^2$ endowed with natural pre-order
  \item $\alpha$ \quad Weight placed on revenue under $\alpha$-mechanism
  \item $\mathcal{B}$ \quad Message space of general mechanism
  \item $\mathcal{B}_i$ \quad Message space of agent $i$
  \item $(\mathcal{B}, Q_\mathcal{B}, M_\mathcal{B})$ \quad General mechanism
  \item $\mathcal{B}$ \quad Brownian bridge process on $[0, 1]$
  \item $\beta$ \quad Beta distribution
\end{itemize}
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<thead>
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<th>Description</th>
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<td>Beta function</td>
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<tr>
<td>Bn</td>
<td>Binomial distribution</td>
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<tr>
<td>$B_t$</td>
<td>Buyer arriving in period $t$</td>
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<td>Cov</td>
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<tr>
<td>$\mathcal{C}$</td>
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<td>$\mathcal{C}_j$</td>
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<td>$c$</td>
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<td>$\delta$</td>
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<td>$F_i$</td>
<td>Marginal distribution of buyer $i$ value</td>
</tr>
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</table>
\( f \) Joint density of buyer valuations
\( f_i \) Marginal density of buyer \( i \) value
\( F \) Marginal distribution of buyer \( i \) value (i.i.d. case)
\( f \) Marginal density of buyer \( i \) value (i.i.d. case)
\( G_j \) Marginal distribution of seller \( j \) cost
\( g \) Joint density of seller costs
\( g_j \) Marginal density of seller \( j \) cost
\( G \) Marginal distribution of seller \( j \) cost (i.i.d. case)
\( g \) Marginal density of seller \( j \) cost (i.i.d. case)
\( \Gamma_{\alpha,j} \) Weighted virtual cost of seller \( j \)
\( \Gamma_{\alpha} \) Weighted virtual cost of seller \( j \) (i.i.d. case)
\( \Gamma_j \) Virtual cost of seller \( j \)
\( \Gamma \) Virtual cost of seller \( j \) (i.i.d. case)
\( G_m \) Geometric distribution
\( \mathcal{H}_t \) Set of period \( t \) histories
\( h_t \) Complete period \( t \) history of agent types
\( \hat{h}_t \) Complete period \( t \) history of agent reports
\( H_{\text{g}} \) Standard hypergeometric distribution
\( H_{\text{gF}} \) Fisher’s non–central hypergeometric distribution
\( H_{\text{gG}} \) General non–central hypergeometric distribution
\( H_{\text{gW}} \) Wallenius’ non–central hypergeometric distribution
\( I \) Indicator function
\( I \) Identity matrix
\( \mathcal{I} \) Set of agents
\( K \) Equilibrium quantity traded
\( K^* \) Ex–post efficient quantity traded
\( K_\alpha \)  
Quantity traded under \( \alpha \)-mechanism

\( \lambda \)  
Ratio of number of sellers to buyers

\( m \)  
Number of sellers

\( M \)  
Set of sellers

\( M_\alpha^* \)  
\( \alpha \)-Walrasian set of sellers

\( M^* \)  
Walrasian set of sellers

\( M \)  
Payment rule for direct mechanism

\( M_i \)  
Payment rule for agent \( i \)

\( m_i \)  
Interim expected payment of agent \( i \)

\( M_t \)  
Period \( t \) payment rule

\( M_i^{B_t} \)  
Period \( i \) payment rule of period \( t \) buyer

\( m^{B_t}_i \)  
Expected discounted payment of period \( t \) buyer

\( M_i^{S_t} \)  
Period \( i \) payment rule for period \( t \) seller

\( m^{S_t}_i \)  
Expected discounted payment for period \( t \) seller

\( M_B \)  
Payment rule for general mechanism

\( M^V \)  
Payment rule under VCG mechanism

\( M^A \)  
Payment rule under AGV mechanism

\( M^\alpha \)  
\( \alpha \)-Mechanism payment rule

\( \mathcal{M} \)  
Market with independent and, on each side, identically distributed types

\( \mu \)  
Belief system

\( \mathbb{N} \)  
Set of natural numbers \( \{1, 2, \ldots \} \)

\( n \)  
Number of agents or buyers

\( \langle n, m, F, G \rangle \)  
Trade setup with independent and, and on each side, identically distributed types

\( \mathcal{N} \)  
Set of buyers

\( \mathcal{N}_\alpha^* \)  
\( \alpha \)-Walrasian set of buyers
$N^*$  Walrasian set of buyers
$\langle N, M, \succ \rangle$  Marriage problem
N  Normal distribution
NB  Bivariate normal distribution
NB  Negative binomial distribution
$O$  Type space of all agents
$O_i$  Type space of agent $i$
$O_{it}$  Type space of agent $i$ in period $t$
$O_{it}^{t-1}$  Space of period $t$ type histories of agent $i$
$O$  Conditional observation probability function of partially observable Markov decision process
$P$  Probability
$\mathcal{P}$  Space of probability distributions over given set
$P$  Transition matrix of Markov chain
$p_B$  Price posted for buyers
$p_S$  Price posted for sellers
$p_{it}$  Payment made by agent $i$ in period $t$
$P$  Transition probability function of Markov decision process
$P_a(x, \cdot)$  Transition probabilities of Markov decision process, conditional on previous action and state
$p$  Probability that an efficient type arrives (binary type case)
$\Phi_{\alpha, i}$  Weighted virtual value of buyer $i$
$\Phi_{\alpha}$  Weighted virtual value of buyer $i$ (i.i.d. case)
$\Phi_i$  Virtual value of buyer $i$
$\Phi$  Virtual value of buyer $i$ (i.i.d. case)
$\pi$  Policy of Markov decision process
\[ \pi^* \] Optimal policy of Markov decision process

\[ \pi_\tau \] Threshold policy with threshold \( \tau \)

\[ \Psi \] Set of all allocations

\[ \Psi_{it} \] Set of period \( t \) allocations for agent \( i \)

\[ \Psi_{i}^{t-1} \] Space of period \( t \) allocation histories for agent \( i \)

\[ \psi \] Allocation

\[ \psi_i \] Allocation of agent \( i \)

\[ \psi_{-i} \] Allocation of agents, excluding agent \( i \)

\[ \psi_{it} \] Allocation of agent \( i \) in period \( t \)

\[ \psi_{i}^{t-1} \] Period \( t \) history of agent \( i \) allocations

\[ Q \] Space of allocation rules

\[ Q_i \] Allocation rule for direct mechanism

\[ Q_{it} \] Allocation rule for agent \( i \)

\[ q_i \] Interim expected allocation of agent \( i \) under direct mechanism

\[ Q_{t} \] Allocation rule for period \( t \)

\[ Q_{i}^{B_t} \] Period \( i \) allocation rule for period \( t \) buyer

\[ q_{B_t} \] Discounted probability of trade for period \( t \) buyer

\[ Q_{i}^{S_t} \] Period \( i \) allocation rule for period \( t \) seller

\[ q_{S_t} \] Discounted probability of trade for period \( t \) seller

\[ Q_B \] Allocation rule for general mechanism

\[ Q^* \] Ex-post efficient allocation

\[ Q^x \] \( \alpha \)-Mechanism allocation rule

\[ \langle Q, M \rangle \] Direct mechanism

\[ \langle Q^*, M^V \rangle \] VCG mechanism

\[ \langle Q^*, M^A \rangle \] AGV mechanism

\[ \langle Q^\alpha, M^\alpha \rangle \] Bayesian optimal \( \alpha \)-mechanism
**NOTATION**

- **Q**: Matching rule
- **R**: Profit under Bayesian optimal direct mechanism
- **R**: Lower bound of profit under constrained efficient mechanism
- **\( \mathbb{R}_n \)**: Uniform empirical process defined on \([0, 1]\)
- **r**: Reward function of Markov decision process
- **\( r_a(x) \)**: Reward earned when under Markov decision process action \( a \) in state \( x \)
- **\( \mathcal{R}_{(n,m)} \)**: Realisation of a market
- **s**: Agent strategy profile
- **s_i**: Strategy profile of agent \( i \)
- **s_{it}**: Strategy of agent \( i \) in period \( t \)
- **s^***: Equilibrium agent strategy profile
- **S**: Signal space of partially observable Markov decision process
- **S_l**: Seller arriving in period \( t \)
- **T**: Threshold policy mapping
- **\( \tau \)**: Threshold associated with threshold policy
- **\( \tau^* \)**: Threshold associated with optimal threshold policy
- **\( \bar{\tau} \)**: Optimal threshold of bounding Markov decision process
- **\( \theta \)**: Profile of agent types
- **\( \theta_i \)**: Type of agent \( i \)
- **\( \underline{\theta}_i \)**: Lower bound of support of agent \( i \) type
- **\( \overline{\theta}_i \)**: Upper bound of support agent \( i \) type
- **\( \theta_{it} \)**: Type of agent \( i \) in period \( t \)
- **\( \theta^{t-1}_i \)**: Period \( t \) history of types of agent \( i \)
- **\( \theta^{t-1}_{-i} \)**: Period \( t \) history of types, excluding agent \( i \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_{-i} )</td>
<td>Profile of agent types, excluding agent ( i )</td>
</tr>
<tr>
<td>( \theta_t )</td>
<td>Types of agents arriving in period ( t ) (persistent type case)</td>
</tr>
<tr>
<td>( \hat{\theta} )</td>
<td>Profile of agent reports</td>
</tr>
<tr>
<td>( \hat{\theta}^* )</td>
<td>Equilibrium profile of agent reports</td>
</tr>
<tr>
<td>( U )</td>
<td>Uniform distribution</td>
</tr>
<tr>
<td>( u_i )</td>
<td>Quasilinear payoff function of agent ( i )</td>
</tr>
<tr>
<td>( \hat{u}_i )</td>
<td>Value of agent ( i ) (interdependent or common values case)</td>
</tr>
<tr>
<td>( U_i )</td>
<td>Interim expected payoff of agent ( i )</td>
</tr>
<tr>
<td>( \mathbb{U}_n )</td>
<td>Uniform empirical quantile function on ([0,1] )</td>
</tr>
<tr>
<td>( \text{Var} )</td>
<td>Variance</td>
</tr>
<tr>
<td>( \mathcal{V} )</td>
<td>Type space of all buyers</td>
</tr>
<tr>
<td>( \mathcal{V}_i )</td>
<td>Type space of buyer ( i )</td>
</tr>
<tr>
<td>( v )</td>
<td>Profile of buyer values</td>
</tr>
<tr>
<td>( v_i )</td>
<td>Type of buyer ( i )</td>
</tr>
<tr>
<td>( v_{(i)} )</td>
<td>The ( i )th largest buyer value</td>
</tr>
<tr>
<td>( v_{i} )</td>
<td>Lower bound of support of buyer ( i ) value</td>
</tr>
<tr>
<td>( v_i )</td>
<td>Lowest bound of support of buyer ( i ) value (i.i.d. case)</td>
</tr>
<tr>
<td>( \pi_i )</td>
<td>Upper bound of support of buyer ( i ) value</td>
</tr>
<tr>
<td>( \pi_i )</td>
<td>Upper bound of support of buyer ( i ) value (i.i.d. case)</td>
</tr>
<tr>
<td>( v_t )</td>
<td>Valuation of period ( t ) buyer (persistent type case)</td>
</tr>
<tr>
<td>( \hat{v} )</td>
<td>Profile of buyer reports</td>
</tr>
<tr>
<td>( \mathbb{V}_n )</td>
<td>Empirical quantile function for sample of buyer valuations</td>
</tr>
<tr>
<td>( V_\pi )</td>
<td>Value function of Markov decision process under policy ( \pi )</td>
</tr>
<tr>
<td>( V_\pi^D )</td>
<td>Value function under discriminatory market clearing with threshold ( \tau )</td>
</tr>
<tr>
<td>( V_\pi^{D\tau} )</td>
<td>Value function under discriminatory market clearing with threshold ( \tau^* )</td>
</tr>
</tbody>
</table>


\( V^U_\tau \) Value function under uniform market clearing with threshold \( \tau \)

\( V^{U*}_{\tau*} \) Value function under uniform market clearing with threshold \( \tau^* \)

\( W \) Welfare under Bayesian optimal direct mechanism

\( W^* \) Welfare under ex–post efficient allocation

\( W^-_{-i} \) Welfare under ex–post efficient allocation, excluding agent \( i \)

\( W^{D,\alpha} \) Welfare under discriminatory market clearing Bayesian optimal \( \alpha \)–mechanism

\( W^{U,\alpha} \) Welfare under uniform market clearing Bayesian optimal \( \alpha \)–mechanism

\( W^{F,\alpha} \) Welfare under fixed frequency market clearing Bayesian optimal \( \alpha \)–mechanism

\( W^{0,\alpha} \) Welfare under decentralised market Bayesian optimal \( \alpha \)–mechanism

\( \mathbb{W} \) Brownian motion defined on \([0, 1]\)

\( \{X_1, \ldots, X_m\} \) Sample of independent and identically distributed random variables drawn from distribution \( F \)

\( X_{(i)} \) The \( i \)th smallest random variable in \( \{X_1, \ldots, X_m\} \)

\( X_n \) Empirical quantile function defined on \([0, 1]\)

\( \mathcal{X} \) State space of Markov decision process

\( X_t \) State of Markov decision process in period \( t \)

\( x \) State of Markov decision process

\( x_E \) Number of efficient pairs present (binary type space)

\( x_I \) Number of identical suboptimal pairs present (binary type space)

\( \langle \mathcal{X}, \mathcal{A}, P, r, \delta \rangle \) Markov decision process

\( \langle \mathcal{X}, \mathcal{A}, P, r, S, O, \delta \rangle \) Partially observable Markov decision process
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${Y_1, \ldots, Y_n}$</td>
<td>Sample of independent and identically distributed random variables drawn from distribution $G$</td>
</tr>
<tr>
<td>$Y_{[j]}$</td>
<td>The $j$th largest random variable in ${Y_1, \ldots, Y_n}$</td>
</tr>
<tr>
<td>$Y_t$</td>
<td>State of period $t$ order book Markov chain</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

We introduce and motivate the class of problems considered in this thesis and survey the relevant economics literature.

1.1 Outline

Standard economic models of markets assume static centralised trading. However, real–world markets differ with respect to the degree of centralisation and the frequency and flexibility with which they clear. In this thesis, we are primarily concerned with the development of tractable dynamic models of markets together with applications of these models. We also address several open questions concerning the statistical properties of static market models.

This thesis is structured as follows. The remainder of Chapter 1 provides an introduction to the mechanism design concepts and methodologies that are the basis of subsequent chapters. We also survey the related economics literature concerning intermediation in two–sided markets, static mechanism design, dynamic mechanism design and dynamic matching. Finally, we describe the main contributions of the thesis in the context of this literature. Mathematical preliminaries are covered in Chapter 2. In particular, this chapter provides background on order statistics, empirical quantile processes, Brownian bridge processes, Markov decision processes and dynamic programming. Our results concerning the statistical properties of static market models are presented in Chapters 3 and 4, while dynamic market making is discussed in Chapters 5, 6 and 7. The contents of Chapters 3 to 7 are outlined in more detail in Section 1.4. Concluding remarks and suggestions for future research can be found in Chapter 8.

1.2 Mechanism Design Background

Mechanism design is a field of economics that provides an approach to modelling and solving resource allocation problems involving self–interested strategic agents with private allocation–relevant information. Practical applications of mechanism design range from the allocation of government resources (in-
including land, mining rights, radio spectrum licenses and university places) to kidney exchange programs and advertisement placement in Internet search engines (see Loertscher, Marx, and Wilkening [91] for a recent survey). We begin by reviewing the key game-theoretic concepts, methodologies and insights of mechanism design that underpin this thesis. In particular, we introduce and formally define the class of mechanisms studied throughout this thesis: Bayesian optimal $\alpha$-mechanisms. We then review the relevant literature in Section 1.3, discussing the contributions of each paper in the context of the results presented in this section.

1.2.1 Agents, Types and Payoff Functions

We consider an economy in which an allocation decision involving several agents must be made. Specifically, we consider a canonical mechanism design model with $n$ agents, indexed by $i \in I := \{1, \ldots, n\}$. The preferences of agents depend on their types and we denote the type of agent $i$ by $\theta_i \in O_i$, where $O_i \subset [\theta_i, \bar{\theta}_i]$ is a closed and bounded subset of $\mathbb{R}$. Let $\theta := (\theta_1, \ldots, \theta_n) \in \mathcal{O} := \prod_{i=1}^n O_i$ denote the vector of type profiles and denote by $\theta_{-i}$ the vector of type profiles excluding that of agent $i$. The finite set of feasible allocations is $\Psi$, so that for a given allocation $\psi \in \Psi$, the allocation of agent $i$ is $\psi_i$. Furthermore, we let $\psi_{-i}$ denote the allocation of agents other than $i$.

We assume each agent $i$ reports a type $\hat{\theta}_i \in O_i$ to a direct mechanism. A direct mechanism $\langle Q, M \rangle$ consists of an allocation rule $Q : \mathcal{O} \rightarrow \Psi$ and a payment rule $M : \mathcal{O} \rightarrow \mathbb{R}^n$. The allocation rule then determines, as a function of agent reports $\hat{\theta}$, the allocation $Q_i(\hat{\theta})$ of each agent. Similarly, the payment rule determines the payment $M_i(\hat{\theta})$ made by each agent.

Agent $i$’s payoff function $u_i : \mathcal{O} \times \Psi \times \mathbb{R} \rightarrow \mathbb{R}$ depends on the type vector $\theta$, the allocation $\psi$ and her monetary transfer. We assume agents have quasilinear preferences, so that if $i$ is of type $\theta_i$ and reports $\hat{\theta}_i$ to a direct mechanism $\langle Q, M \rangle$, her payoff is given by

$$ u_i(\theta, Q(\hat{\theta}), M_i(\hat{\theta})) = \tilde{u}_i(\theta, Q(\hat{\theta})) - M_i(\hat{\theta}). $$

This general specification allows for allocative externalities, as the payoff of agent $i$ may depend directly on the allocations $\psi_{-i}$ of other agents. From this point forward, we will assume agent preferences do not exhibit externalities and restrict ourselves to considering payoff functions $u_i : \mathcal{O} \times \Psi_i \times \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$ u_i(\theta, Q_i(\hat{\theta}), M_i(\hat{\theta})) = \tilde{u}_i(\theta)Q_i(\hat{\theta}) - M_i(\hat{\theta}). $$

Notice that this still allows the value $\tilde{u}_i$ of agent $i$ to exhibit informational externalities and depend on the types $\theta_{-i}$ of other agents. In the mechanism design literature, this case is referred to as interdependent values. The extreme case where, for all $i, j \in I$, we have

$$ \tilde{u}_i(\theta) = \tilde{u}_j(\theta) $$
is known as the *common values* model. For the remainder of this thesis, we will assume that agents have *private values*. That is, we will assume that agent preferences do not exhibit externalities and the quasilinear payoff function $u_i : O_i \times \Psi_i \times \mathbb{R} \to \mathbb{R}$ of agent $i$ is of the form

$$u_i(\theta_i, Q_i(\hat{\theta}), M_i(\hat{\theta})) = \tilde{u}_i(\theta_i)Q_i(\hat{\theta}) - M_i(\hat{\theta}).$$

Notice that without loss of generality, we can restrict attention to the case

$$u_i(\theta_i, Q_i(\hat{\theta}), M_i(\hat{\theta})) = \theta_iQ_i(\hat{\theta}) - M_i(\hat{\theta}).$$

### 1.2.2 Bayesian Nash Equilibria and the Revelation Principle

Suppose temporarily that agent types are common knowledge among agents and a mechanism designer. Given a direct mechanism $(Q, M)$ we can regard the agents as participating in a game of strategic iteration, in which the payoff of agent $i$ is affected by the reports $\hat{\theta}_{-i}$ of other agents. In such games, agents are uncertain about the strategies of other agents and the solution concept used in non–cooperative game theory is the *Nash equilibrium*. A strategy profile $\hat{\theta}^*$ of agent reports constitutes a Nash equilibrium if no unilateral deviation by any player improves the payoff of that player. Formally, $\hat{\theta}^*$ is a Nash equilibrium if, for all $i \in I$,

$$\hat{\theta}_i^* \in \arg\max_{\hat{\theta}_i \in O_i} u_i(\theta_i, Q_i(\hat{\theta}_i, \hat{\theta}_{-i}^*), M_i(\hat{\theta}_i, \hat{\theta}_{-i}^*)).$$

We now assume that the type of each agent is *private information*. Specifically, we assume that the vector of agent types is a random variable $\Theta$ drawn from some measurable distribution function $F$ with support $O$. Notice that this allows for *dependence* between agent types. We assume that agent $i$ learns the realisation $\theta_i$ of her type $\Theta_i$ prior to reporting to the mechanism. While the distribution $F$ is common knowledge among agents and the designer, the realisation of agent $i$’s type is private information (known only by agent $i$).

When agent types are private information, this introduces an additional layer of uncertainty when an agent reports to the mechanism. Agents now participate in a *Bayesian game* in which they are uncertain about the preferences of other agents. The corresponding solution concept from non–cooperative game theory is the *Bayesian Nash equilibrium*. Under a Bayesian Nash equilibrium with rational *risk–neutral* agents, each agent selects a strategy profile that maximizes their expected payoff, given their beliefs about the types of other agents and given the strategy profiles of other agents. Formally, the strategy of agent $i$ is a function $s_i : O_i \to O_i$ such that $i$ reports $\hat{\theta}_i = s_i(\theta_i)$ when her type if $\theta_i$. The strategy profile $s^*(\theta) := (s_1^*(\theta_1), \ldots, s_n^*(\theta_n))$ is a Bayesian Nash equilibrium if, for all $i \in I$ and $\theta_i \in O_i$,

$$s_i^*(\theta_i) \in \arg\max_{\hat{\theta}_i \in O_i} \mathbb{E}_{\Theta_{-i}} \left[ u_i(\theta_i, Q_i(\hat{\theta}_i, s_{-i}^*(\Theta_{-i})), M_i(\hat{\theta}_i, s_{-i}^*(\Theta_{-i})) \mid \Theta_i = \theta_i) \right].$$
An equilibrium in which agents report their true types is known as a \textit{truthful equilibrium}. We can see from the previous equation that truth–telling is a Bayesian Nash equilibrium of the direct mechanism $\langle Q \circ s^*, M \circ s^* \rangle$. Thus, the outcome of any direct mechanism can be implemented as a truthful equilibrium. We can extend this result by considering a \textit{general mechanism} $\langle B, Q_B, M_B \rangle$ under which agent $i$ reports a message from the set $B_i$. As before, the allocation rule $Q_B : B \rightarrow \Psi$ and payment rule $M_B : B \rightarrow \mathbb{R}^n$ map agent reports to mechanism outcomes. The strategy $s_i : O_i \rightarrow B_i$ is now a mapping such that $s_i(\theta_i)$ is the message agent $i$ reports to the mechanism when her type is $\theta_i$. If the strategy profile $s^*(\theta)$ is a Bayesian Nash equilibrium of the mechanism, truth–telling is a Bayesian Nash equilibrium of the mechanism $\langle Q \circ s^*, M \circ s^* \rangle$. Thus, the equilibrium outcome of any mechanism can be implemented as a truthful equilibrium of a direct mechanism and we may consider direct mechanisms without loss of generality. This result is known as the \textit{revelation principle}.

\subsection*{1.2.3 Notions of Incentive Compatibility and Individual Rationality}

In light of the revelation principle, it is natural to introduce notions of incentive compatibility and individual rationality. Incentive compatibility constraints ensure that agents are appropriately incentivised to reveal their private information to the mechanism. If these constraints hold for a direct mechanism, truth–telling is a Bayesian Nash equilibrium of that mechanism. Individual rationality constraints ensure agents have an incentive to participate in a direct mechanism. Before introducing the notions of incentive compatibility and individual rationality that pertain to our setup, some further notation is useful. Given a direct mechanism $\langle Q, M \rangle$, let

$$q_i(\hat{\theta}_i) = E_{\Theta_{-i}} \left[ Q_i(\hat{\theta}_i, \Theta_{-i}) \mid \Theta_i = \theta_i \right]$$

denote agent $i$’s interim expected allocation when $i$ reports $\hat{\theta}_i$ and all other agents report truthfully. Similarly, let

$$m_i(\hat{\theta}_i) = E_{\Theta_{-i}} \left[ M_i(\hat{\theta}_i, \Theta_{-i}) \mid \Theta_i = \theta_i \right]$$

denote agent $i$’s interim expected payment when $i$ reports $\hat{\theta}_i$ and all other agents report truthfully. A direct mechanism $\langle Q, M \rangle$ is said to be \textit{Bayesian incentive compatible} if, for all $i \in I$ and $\theta_i \in O_i$,

$$\theta_i \in \arg \max_{\hat{\theta}_i \in O_i} u_i(\theta_i, q_i(\hat{\theta}_i), m_i(\hat{\theta}_i))$$

\footnote{Notice that direct mechanisms are a special case of these more general mechanisms, with $B = O$.}
However, there is a stronger notion of incentive compatibility. A Bayesian Nash equilibrium $s^*(\theta)$ is said to be implementable in dominant strategies if, for all $i \in I$ and $\theta \in O$,

$$s^*(\theta_i) \in \arg \max_{\hat{\theta}_i \in O_i} u_i(\theta_i, Q_i(\hat{\theta}_i, \theta_{-i})) , M_i(\hat{\theta}_i, \theta_{-i}) .$$

Notice that a Bayesian Nash equilibrium implementable in dominant strategies does not depend on the beliefs of agents. A direct mechanism $\langle Q, M \rangle$ is said to be dominant strategy incentive compatible if, for all $i \in I$ and $\theta \in O$,

$$\theta_i \in \arg \max_{\hat{\theta}_i \in O_i} u_i(\theta_i, Q_i(\hat{\theta}_i, \theta_{-i})) , M_i(\hat{\theta}_i, \theta_{-i}) .$$

Dominant strategy incentive compatible mechanisms are generally regarded as more “robust” than Bayesian incentive compatible mechanisms because the equilibrium behaviour of agents does not depend on their beliefs.

Suppose agents can elect not to participate in the mechanism, thereby guaranteeing themselves a payoff of zero. Interim individual rationality then requires that for all $i \in I$ and $\theta_i \in O_i$,

$$\max_{\hat{\theta}_i \in O_i} u_i(\theta_i, q_i(\hat{\theta}_i), m_i(\hat{\theta}_i)) \geq 0 .$$

Ex–post individual rationality is a stronger notion of individual rationality. It requires that, for all $i \in I$ and $\theta \in O$,

$$\max_{\hat{\theta}_i \in O_i} u_i(\theta_i, Q_i(\hat{\theta}_i, \theta_{-i})) , M_i(\hat{\theta}_i, \theta_{-i})) \geq 0 .$$

### 1.2.4 The Revenue Equivalence Theorem

We now restrict attention to the case in which types are independently distributed. That is, we now consider the independent private values model. For the remainder of this section we also suppose that for all $i$, $O_i = [\theta_i, \bar{\theta}_i]$ and that $F_i$ is absolutely continuous. Bayesian incentive compatibility implies that the interim expected payoff function $U_i : O_i \rightarrow \mathbb{R}$ for agent $i$ is given by

$$U_i(\theta_i) = \max_{\hat{\theta}_i \in O_i} u_i(\theta_i, q_i(\hat{\theta}_i), m_i(\hat{\theta}_i)) = \max_{\hat{\theta}_i \in O_i} \{\theta_i q_i(\hat{\theta}_i) - m_i(\hat{\theta}_i)\} .$$

Since $U_i$ is the maximum of a family of affine functions, this immediately implies that $U_i$ is convex. Thus, $U_i(\theta_i) = q_i(\theta_i)$ almost everywhere and $q_i$ is a non–decreasing function (see, for example, Royden and Fitzpatrick [130]). We have

$$U_i(\theta_i) = U_i(\bar{\theta}_i) + \int_{\theta_i}^{\bar{\theta}_i} q_i(\vartheta) d\vartheta .$$
which implies that
\[ m_i(\theta_i) = -U_i(\theta_i) + q_i(\theta_i)\theta_i - \int_{\theta_i}^{\theta_i} q_i(\theta_i) d\theta_i, \]  
(1.1)
where \( U_i(\theta_i) \) is a constant. Thus, for any Bayesian incentive compatible mechanism \( (Q, M) \) the payment rule \( M \) is fixed, up to a constant, by the allocation rule \( Q \). This is known as the revenue equivalence theorem. Conversely, suppose an allocation rule \( Q \) is such that, for every \( i \in \mathcal{I}, q_i \) is non-decreasing. Then we can construct a Bayesian incentive compatible mechanism using a payment rule given by (1.1).

1.2.5 The VCG and AGV Mechanisms

Given a profile \( \theta \in \mathcal{O} \) of agent types and an allocation rule \( Q \), social welfare is given by
\[ W(\theta, Q) = \sum_{i=1}^{n} \theta_i Q_i(\theta). \]
Let \( Q \) denote the set of all feasible allocation rules \( Q : \mathcal{O} \to \Psi \). The ex–post efficient allocation rule \( Q^* \) maximises social welfare. That is, for all \( \theta \in \mathcal{O}, \)
\[ Q^*(\theta) \in \arg \max_{Q \in Q} \sum_{i=1}^{n} \theta_i Q_i(\theta). \]
Furthermore, let
\[ W^*(\theta) := \sum_{i=1}^{n} \theta_i Q_i^*(\theta) \quad \text{and} \quad W_{-i}^*(\theta) := \sum_{j \neq i} \theta_j Q_j^*(\theta). \]

We now introduce the celebrated VCG mechanism \( (Q^*, M_V^*) \), named for Vickrey [140], Clarke [31] and Groves [69], which implements the ex–post efficient allocation \( Q^* \) in dominant strategies. Transfers under the VCG mechanism are given by the payment rule \( M_V^* : \mathcal{O} \to \mathbb{R}^n \) with
\[ M_i^V(\theta) = W^*(\theta_i, \theta_{-i}) - W_{-i}^*(\theta). \]
Under the VCG mechanism, the transfer paid by each agent \( i \) is equal to difference between social welfare when \( i \) reports \( \theta_i \) and the welfare of all other agents when \( i \) reports \( \theta_i \). Essentially, \( i \) pays for the externality she imposes on the other agents. Suppose agent \( i \)'s type is \( \theta_i \). If all other agents report \( \theta_{-i} \) and \( i \) reports \( \hat{\theta}_i \), then \( i \)'s payoff is given by
\[ \theta_i Q_i^*(\hat{\theta}_i, \theta_{-i}) - M_i^V(\hat{\theta}_i, \theta_{-i}) = \sum_{j=1}^{n} \theta_j Q_j^*(\hat{\theta}_i, \theta_{-i}) - W^*(\theta_i, \theta_{-i}). \]  
(1.2)
By definition, the first term on the right-hand side of (1.2) is maximised at \( \hat{\theta}_i = \theta_i \) and the second term is independent of \( \hat{\theta}_i \). The VCG mechanism ensures it is a dominant strategy for agents to report truthfully by setting transfers that make each agent a residual claimant of the ex–post social surplus. Furthermore, since \( W^*(\hat{\theta}_i, \theta_{-i}) - W^*(\hat{\theta}_i, \theta_{-i}) = 0 \), the VCG mechanism is ex–post individually rational. Finally, the interim expected payoff \( U^V_i \) of agent \( i \) is given by

\[
U^V_i(\theta_i) = E_{\Theta_{-i}}[W^*(\theta_i, \Theta_{-i}) - W^*(\theta_i, \Theta_{-i})].
\]

(1.3)

Recall that Bayesian incentive compatibility implies that \( U^V_i(\theta_i) \) is convex and increasing and notice that \( U^V_i(\theta_i) = 0 \). It immediately follows from the revenue equivalence theorem that, among all ex–post efficient, Bayesian incentive compatible and interim individually rational mechanisms, the VCG mechanism maximises the expected payment made by each agent.

A direct mechanism \( (Q, M) \) is said to be interim budget balanced if, for all \( \theta \in \mathcal{O} \),

\[
\sum_{i=1}^{n} m_i(\theta_i) = 0
\]

and ex–post budget balanced if, for all \( \theta \in \mathcal{O} \),

\[
\sum_{i=1}^{n} M_i(\theta) = 0.
\]

The VCG mechanism is not necessarily interim budget balanced. However, the AGV mechanism \( (Q^A, M^A) \), independently suggested by Arrow [5] and d’Aspremont and Gérard-Varet [46, 47], provides an example of an ex–post efficient and budget balanced mechanism. Under the AGV mechanism, the payment rule \( M^A : \mathcal{O} \to \mathbb{R}^n \) is given by

\[
M^A_i(\theta) = \frac{1}{n-1} \sum_{j \neq i} E_{\Theta_{-j}}[W^*_j(\theta_j, \Theta_{-j})] - E_{\Theta_{-i}}[W^*_i(\theta_i, \Theta_{-i})].
\]

(1.4)

By inspection, the AGV mechanism is ex–post budget balanced. Next, suppose all other agents report truthfully and agent \( i \) is of type \( \theta_i \) but reports \( \hat{\theta}_i \). Agent \( i \)'s interim expected payoff is then

\[
E_{\Theta_{-i}}[\theta_i Q^*_i(\hat{\theta}_i, \Theta_{-i}) - M^A_i(\hat{\theta}_i, \Theta_{-i})] = E_{\Theta_{-i}} \left[ \sum_{j=1}^{n} \theta_j Q^*_j(\hat{\theta}_i, \Theta_{-i}) \right] - E_{\Theta_{-i}} \left[ \frac{1}{n-1} \sum_{j \neq i} E_{\Theta_{-j}}[W^*_j(\theta_j, \Theta_{-j})] \right].
\]
By the definition of $Q^*$, the first term on the right-hand-side of the above equation is maximised by setting $\hat{\theta}_i = \theta_i$. The second term is independent of $\hat{\theta}_i$. Thus, the AGV mechanism is Bayesian incentive compatible. However, the AGV mechanism is not implementable in dominant strategies and is not necessarily interim individually rational.

The “incentive payment” in (1.4) ensures the agent $i$ is a residual claimant of the interim expected social surplus. This term does not depend on the strategies of other agents and it incentivises $i$ to report truthfully. The “balance payment” in (1.4) ensures the AGV mechanism is ex-post budget balanced by requiring that each agent pays her share of the incentive payments of all other agents.

Note that more generally, both the VCG and AGV mechanisms can be used to construct transfers that implement any allocation rule maximising an objective function of the form

$$W(\theta, Q) = \sum_{i=1}^{n} \hat{W}_i(\theta_i)Q_i(\theta).$$

### 1.2.6 Two-Sided Bayesian Optimal Mechanisms

A prior-free direct mechanism $(Q, M)$ is such that the allocation rule $Q$ and payment rule $M$ do not depend on the distributions from which agents draw their types. Under a Bayesian direct mechanism, these functions may depend on agent type distributions. For example, the VCG mechanism is a prior–free mechanism and the AGV mechanism is a Bayesian mechanism. We now discuss Bayesian optimal mechanisms, which maximise some expected object, where this expectation is taken over the distribution of agent types, subject to incentive compatibility and individual rationality constraints.\(^3\)

We now consider the following problem in which agents with private types trade units of a homogeneous indivisible good. Suppose there are a group of $n$ buyers, indexed by $i \in \mathcal{N} := \{1, \ldots, n\}$, and a group of $m$ sellers, indexed by $j \in \mathcal{M} := \{n+1, \ldots, n+m\}$. Buyers demand at most one unit of the good and have valuations $v_i \in [\underline{v}_i, \overline{v}_i] := \mathcal{V}_i$, independently drawn from absolutely continuous distributions function $F_i$. Similarly, sellers have the capacity to produce at most one unit and have costs $c_j \in [\underline{c}_j, \overline{c}_j] := \mathcal{C}_j$, independently drawn from absolutely continuous distribution functions $G_j$. We additionally assume that buyer valuations and seller costs are independent of one another and that for all $i \in \mathcal{N}$ and $j \in \mathcal{M}$, $f_i(v_i) > 0$ for all $v_i \in [\underline{v}_i, \overline{v}_i]$ and $q_j(c_j) > 0$

\(^2\)Although the AGV mechanism is not implementable in dominant strategies, the assumption of independent types can be relaxed. See, for example, Riordan [126] and Crémé and Riordan [39].

\(^3\)Notice that these definitions have nothing to do with the strategies of agents. For example, the second-price auction with an optimally chosen reserve price is a Bayesian mechanism that endows agents with dominant strategies. A first-price auction is a prior–free mechanism under which agent strategies are Bayesian.
for all $c_j \in [\underline{c}_j, \overline{c}_j]$. Finally, we let the functions $f$ and $g$ denote the joint densities of buyer valuations and seller costs, respectively and set $V := \prod_{i=1}^{n} V_i$ and $C := \prod_{j=1}^{m} C_j$.

Once again, we assume agents are risk–neutral and have quasilinear preferences. Suppose buyer $i$ has valuation $v_i$, receives a unit of the good with probability $q_i$ and pays an expected transfer of $m_i$. The expected payoff of buyer $i$ is then given by

$$u_i(v_i, q_i, m_i) := v_i q_i - m_i.$$  

Similarly, if seller $j$ with cost $c_j$ produces the good with probability $q_j$ and receives a transfer of $m_j$, his expected payoff is given by

$$u_j(c_j, q_j, m_j) := m_j - c_j q_j.$$  

We assume that buyer and sellers trade via an intermediary and that this intermediary is the mechanism designer. Suppose buyers report types $\hat{v}$ and sellers report types $\hat{c}$ to the intermediary. The allocation rule $Q : V \times C \to [0, 1]^{n+m}$ is now such that $Q_i(\hat{v}, \hat{c})$ specifies the probability that buyer $i$ receives a unit and $Q_j(\hat{v}, \hat{c})$ specifies the probability that seller $j$ produces a unit. Similarly, the transfer rule $M : V \times C \to \mathbb{R}^{n+m}$ is such that $M_i(\hat{v}, \hat{c})$ specifies the payment made by buyer $i$ and $M_j(\hat{v}, \hat{c})$ specifies the payment received by seller $j$. A direct mechanism is feasible if, for all $\hat{v} \in V$ and $\hat{c} \in C$,

$$\sum_{i \in N} Q_i(\hat{v}, \hat{c}) \leq \sum_{j \in M} Q_j(\hat{v}, \hat{c}). \quad (1.5)$$

Let $W$ denote expected social welfare and $R$ denote expected profit of the intermediary. We now derive the class of $\alpha$–mechanisms which are the class of Bayesian optimal mechanisms that maximise the convex combination $(1 - \alpha)W + \alpha R$, given any $\alpha \in [0, 1]$. By (1.1), under any Bayesian incentive compatible, direct mechanism we have

$$m_i(v_i) = -U_i(\overline{v}_i) + v_i q_i(v_i) - \int_{\overline{v}_i}^{v_i} q_i(t) \, dt \quad (1.6)$$

$$m_j(c_j) = U_j(\overline{c}_j) + c_j q_j(c_j) + \int_{c_j}^{\overline{c}_j} q_j(t) \, dt, \quad (1.7)$$

where (1.7) accounts for the fact that trade involves sellers incurring a cost of production and receiving payments. Interim individual rationality requires $U_i(\overline{v}_i) \geq 0$ and $U_j(\overline{c}_j) \geq 0$. Under any Bayesian optimal mechanism these constraints must bind since the intermediary places a weight of $\alpha$ on maximising profit. Thus, the intermediary maximises payments made by buyers and minimises payments made to sellers and we set $U_i(\overline{v}_i) = 0$ and $U_j(\overline{c}_j) = 0$ in (1.6) and (1.7).
The profit of the intermediary under any Bayesian optimal mechanism is given by

\[ R = \sum_{i \in \mathcal{N}} \mathbb{E}_{V_i} [m_i(V_i)] - \sum_{j \in \mathcal{M}} \mathbb{E}_{C_j} [m_j(C_j)]. \]

Using (1.6) and (1.7) with \( U_i(\nu_i) = 0 \) and \( U_j(\tau_j) = 0 \), changing the order of integration and using the fact that types are independent we obtain

\[ R = \int_{\mathcal{Y} \times \mathcal{C}} \left( \sum_{i \in \mathcal{N}} \Phi_i(v_i)Q_i(v, c) - \sum_{j \in \mathcal{M}} \Gamma_j(c_j)Q_j(v, c) \right) f(v)g(c) \, dv \, dc. \]

Here, the functions

\[ \Phi_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \quad \text{and} \quad \Gamma_j(c_j) := c_j + \frac{G_j(c_j)}{g_j(c_j)} \]

are known as the virtual type functions and can be interpreted in terms of marginal revenue and marginal cost functions.\(^4\)

Since social welfare \( W \) is given by

\[ W = \int_{\mathcal{Y} \times \mathcal{C}} \left( \sum_{i \in \mathcal{N}} v_iQ_i(v, c) - \sum_{j \in \mathcal{M}} c_jQ_j(v, c) \right) f(v)g(c) \, dv \, dc, \]

the objective function \((1 - \alpha)W + \alpha R\) can be written

\[ \int_{\mathcal{Y} \times \mathcal{C}} \left( \sum_{i \in \mathcal{N}} \Phi_{\alpha,i}(v_i)Q_i(v, c) - \sum_{j \in \mathcal{M}} \Gamma_{\alpha,j}(c_j)Q_j(v, c) \right) f(v)g(c) \, dv \, dc. \quad (1.8) \]

Here, the functions

\[ \Phi_{\alpha,i}(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \quad \text{and} \quad \Gamma_{\alpha,j}(c_j) := c_j + \frac{G_j(c_j)}{g_j(c_j)} \]

are known as the weighted virtual type functions. Notice that the weighted virtual types \( \Phi_{\alpha,i}(v_i) \) and \( \Gamma_{\alpha,j}(c_j) \) are given by the convex combinations \( \Phi_{\alpha,i}(v_i) = (1 - \alpha)v_i + \alpha\Phi_i(v_i) \) and \( \Gamma_{\alpha,j}(c_j) = (1 - \alpha)c_j + \alpha\Gamma_j(c_j) \). Furthermore, when \( \alpha = 1 \) we obtain the virtual type functions as a special case. We assume that Myerson’s [116] regularity conditions are satisfied. That is, we assume that the functions \( \Gamma_i \) and \( \Phi_i \) are strictly increasing on their supports \([\underline{\nu}_i, \bar{\nu}_i]\)

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\(^4\)To see this, suppose we have a continuum of buyers of mass 1 with valuations drawn from the distribution \( F_i \). At a market price of \( p \), the quantity demanded is given by \( q(p) = 1 - F_i(p) \). The inverse demand curve is then given by \( p(q) = F_i^{-1}(1 - q) \) and revenue is \( p(q)q = qF_i^{-1}(1 - q) \). Differentiating the expression for revenue with respect to \( q \) and using \( p = F_i^{-1}(1 - q) \) we obtain \( M R(p) = \Phi_i(p) \). The argument for sellers is analogous.
and $[c_j, \tau_j]$, respectively. Notice that this implies that $\Phi_{\alpha,i}$ and $\Gamma_{\alpha,i}$ are also strictly increasing on their respective supports.

We must now determine the allocation rule $Q^\alpha$ that maximises (1.8), subject to the incentive compatibility and individual rationality constraints. Fix a realisation $(v, c)$ of agent types. If the sets of trading buyers and sellers are given by $N_\alpha \subset N$ and $M_\alpha \subset M$, the designer’s payoff is

$$\sum_{i \in N_\alpha} \Phi_{\alpha,i}(v_i) - \sum_{j \in M_\alpha} \Gamma_{\alpha,j}(c_j).$$

(1.9)

Denote by $N^\alpha_*$ and $M^\alpha_*$ the $\alpha$–Walrasian sets, which maximise (1.9) subject to the feasibility constraint $|N^\alpha_*| \leq |M^\alpha_*|$. In the event that the sets $N^\alpha_*$ and $M^\alpha_*$ are not unique, this is resolved using a tie–breaking rule. We then set $Q^\alpha_i(v, c) = Q^\alpha_j(v, c) = 1$ for all buyers and sellers who belong the $\alpha$–Walrasian sets and $Q^\alpha_i(v, c) = Q^\alpha_j(v, c) = 0$ otherwise. This allocation rule is known as the $\alpha$–allocation rule. The $\alpha$–allocation rule $Q^\alpha$ maximises (1.8) since it maximises the integral pointwise. Myerson’s [116] regularity conditions guarantee that the interim expected allocation $q^\alpha_i$ of buyer $i$ is non–decreasing for all $i \in N$ and the interim expected allocation $q^\alpha_j$ of seller $j$ is non–increasing for all $j \in M$, so that $Q^\alpha$ can be implemented by an incentive compatible mechanism.\(^5\) Thus, the Bayesian optimal $\alpha$–mechanism is given by $(Q^\alpha, M^\alpha)$, where $M^\alpha$ is a payment rule such that (1.6) and (1.7) hold with $U_i(v_i) = 0$ and $U_j(\tau_j) = 0$.

We obtain the welfare–maximising and profit–maximising mechanisms as a special case when $\alpha = 0$ and $\alpha = 1$, respectively. Under welfare–maximisation, the $\alpha$–Walrasian sets become the usual Walrasian sets $N^*$ and $M^*$ and the allocation rule is the ex–post efficient allocation rule $Q^*$.

We now consider constrained efficient mechanisms, which maximise social welfare subject to some profit constraint $R \geq R$. In this context, the welfare–maximising mechanism is sometimes referred to as the first–best mechanism. The mechanism that sets $R = 0$ is sometimes referred to as the second–best mechanism. The Lagrangian associated with this problem is $\max_{Q \in Q} (W + \lambda(R)(R - R))$, where $\lambda(R)$ is the Lagrange multiplier. Let $\lambda^*(R)$ denote the value of the Lagrange multiplier that solves this optimisation problem and set $\alpha = \lambda^*(R)/(1 + \lambda^*(R))$ so that $\alpha \in [0, 1]$. Then an equivalent optimisation problem is $\max_{Q \in Q} \left( (1 - \alpha)W + \alpha R \right)$. Thus, the class of constrained optimal mechanisms is equivalent to the class of $\alpha$–mechanisms.

\(^5\)To see this, let a realisation $(v, c)$ be given and suppose that $v_i$ increases. Since $\Phi_{\alpha,i}$ is strictly increasing, it follows directly from the definition of $Q^\alpha$ that this leads to a weak increase in $Q^\alpha_i(v, c)$. Since this holds for every realisation of types, it follows immediately that $q^\alpha_i(v_i)$ is non-decreasing. The argument for sellers is analogous.
1.2.7 The Impossibility of Ex–post Efficient Trade

We consider our setup from Section 1.2.6 but now assume that buyer values are such that \( v_i \in [v, \bar{v}] \) for all \( i \in \mathcal{M} \) and seller costs are such that \( c_j \in [c, \bar{c}] \) for all \( j \in \mathcal{N} \). We now show that provided \( v \leq c \) and \( \bar{v} \leq \bar{c} \), there does not exist a mechanism that is ex–post efficient, Bayesian incentive compatible, interim individually rational and interim budget balanced. We start by letting \( K^* \) denote the ex–post efficient quantity traded. Given realisations \( v \) of buyer valuations and \( c \) of seller costs, let \( v[k] \) denote the \( k \)th highest buyer value and \( c(k) \) denote the \( k \)th smallest seller cost. Under ex–post efficiency, the associated realisation \( k \) of the quantity traded satisfies

\[
\begin{align*}
v[k] &\geq c(k) & \text{and} & \quad v[k+1] < c(k+1),
\end{align*}
\]

where we set \( v[0] = \bar{v} \), \( v[n+1] = \underline{v} \), \( c(0) = \underline{c} \) and \( c(m+1) = \bar{c} \) for convenience.\(^6\)

Under the VCG mechanism described in Section 1.2.5, the mechanism sets a price of \( p_B = \max\{c(k), v[k+1]\} \) for buyers and \( p_S = \min\{c(k+1), v[k]\} \) for sellers. Thus, since buyer valuations and seller costs are almost surely distinct, the VCG mechanism almost surely runs an ex–post deficit (see Table 1.1).

Recall that among all ex–post efficient, Bayesian incentive compatible and interim individually rational mechanisms, the VCG mechanism maximises the expected payment made by each agent. Thus, if the VCG mechanism runs an expected deficit, all ex–post efficient, Bayesian incentive compatible and interim individually rational mechanisms must run an expected deficit. This result regarding the impossibility of ex–post efficient trade is known as the Myerson–Satterthwaite impossibility theorem and was first proven by Myerson and Satterthwaite [117] in the context of bilateral trade.

We now discuss the importance of the assumptions \( v \leq c \) and \( \bar{v} \leq \bar{c} \) for the impossibility of ex–post efficient trade. For example, suppose that \( \bar{v} > \bar{c} \). This implies that the interim probability of trade for a worst–off seller of type \( \bar{c} \) is positive. The mechanism designer can then fully extract the surplus from trades involving sellers of this type and, provided the market contains a sufficiently large number of buyers, the mechanism will not run an expected deficit.

\(^6\)The Walrasian sets \( \mathcal{N}^* \) and \( \mathcal{M}^* \) described in Section 1.2.6 are then given by the buyers with valuations \( \{v[1], \ldots, v[k]\} \) and sellers with costs \( \{c(1), \ldots, c(k)\} \).
deficit. Makowski and Mezzetti [100] illustrate a possibility result of this type when \( m = 1 \) and Williams [142] provides a general characterisation of how the impossibility of ex–post efficient trade depends on the supports \([c, \bar{c}]\) and \([\underline{v}, \bar{v}]\).

### 1.2.8 Dynamic Mechanism Design

Up to this point, we have focused on static mechanism design problems in which a single allocation decision must be made. However, many real–world allocation problems are dynamic in nature. Agents can arrive and depart over time, groups of agents with private information can interact repeatedly and private information can evolve and depend on previously observed signals and allocations. The work contained in Chapters 5, 6 and 7 of this thesis are closely related to the emerging literature concerning dynamic mechanism design. In order to provide a brief overview of this literature in Section 1.3.2 we now describe a general dynamic mechanism design setup (based on that of Pavan, Segal, and Toikka [121]) and describe the equilibrium concepts and notions of incentive compatibility that pertain to dynamic allocation problems.

Suppose that time is discrete and indexed by \( t \in \mathbb{N} \) and that there are \( n \) agents indexed by \( i \in \mathcal{I} \). In each period \( t \) agent \( i \) observes a signal \( \theta_{it} \in \mathcal{O}_{it} \subset \mathbb{R} \) and sends a message to a mechanism which determines the allocation \( \psi_{it} \in \Psi_{it} \) and payment \( p_{it} \) of agent \( i \). The set of feasible allocation sequences is given by \( \Psi \subset \prod_{t=1}^{\infty} \prod_{i=1}^{n} \Psi_{it} \). Let \( \theta_{t}^{-1} \in \mathcal{O}_{t}^{t-1} := \prod_{s=1}^{t-1} \mathcal{O}_{is} \) and \( \psi_{t}^{t-1} \in \Psi_{t}^{t-1} := \prod_{s=1}^{t-1} \Psi_{is} \) denote the period \( t \) history of the signals and allocations, respectively, of agent \( i \). We assume that in each period \( t \) agent \( i \) observes only \( \theta_{it} \) and \( \psi_{it} \). A direct mechanism \( \langle Q, M \rangle \) is now such that \( Q_{t} \) and \( M_{t} \) specify the period \( t \) allocation and payment rules, respectively.

We now introduce the concept of a perfect Bayesian equilibrium, which is a natural refinement of Bayesian Nash equilibrium that applies to dynamic games. Let \( \mu \) denote a belief system that specifies agent \( i \)'s beliefs regarding unobserved past signals and agent reports \( (\theta_{i}^{-1}, \bar{\theta}_{i}^{-1}) \) at each information set \( (\theta_{it}, \bar{\theta}_{it}, \psi_{i}^{t-1}) \). Given a belief system \( \mu \), the strategy \( s_{it} \) of agent \( i \) is now a set of mappings \( s_{it} : \mathcal{O}_{it} \times \mathcal{O}_{i}^{t-1} \times \Psi_{i}^{t-1} \rightarrow \mathcal{O}_{i} \) such that \( s_{it}(\theta_{it}, \bar{\theta}_{i}^{t-1}, \psi_{i}^{t-1}) \) specifies the report of agent \( i \) in period \( t \). A strategy profile \( s \) is said to be sequentially rational if, given the belief system \( \mu \), no agent \( i \) can improve their expected discounted payoff in any period by unilaterally deviating from \( s_{i} \). A belief system \( \mu \) is consistent if beliefs are consistent with the equilibrium strategies \( s^{*} \) and updated using Bayes’ rule wherever possible. A perfect Bayesian equilibrium is then a belief system and strategy profile \( \langle s^{*}, \mu \rangle \) such that \( s^{*} \) is sequentially rational and \( \mu \) is consistent.

\footnote{An information set is a set that, at a given point in a game, contains all possible moves that have taken place in the game so far. In this example, agent \( i \) observes \( (\theta_{it}, \bar{\theta}_{i}^{t-1}, \psi_{i}^{t-1}) \) in period \( t \) and updates her beliefs regarding \( (\theta_{i}^{-1}, \bar{\theta}_{i}^{-1}) \) accordingly. The belief system \( \mu \) specifies how all agents form their posteriors.}
We are now equipped to introduce several notions of incentive compatibility that apply to dynamic mechanisms. A perfect Bayesian Nash equilibrium \( \langle s^*, \mu \rangle \) is on–path truthful if agents report truthfully on the equilibrium path.\(^8\) A direct mechanism is perfect Bayesian incentive compatible (see Pavan, Segal, and Toikka [121]) if it has an on–path truthful perfect Bayesian equilibrium.\(^9\) A Bayesian Nash equilibrium \( \langle s^*, \mu \rangle \) is strongly truthful if agents report truthfully both on and off the equilibrium path.\(^10\) A direct mechanism is strongly perfect Bayesian equilibrium–implementable (see Pavan, Segal, and Toikka [121]) if it has a strongly truthful perfect Bayesian equilibrium. Furthermore, a direct mechanism is periodic ex–post incentive compatible (see Athey and Segal [6] and Bergemann and Välimäki [17]) if it has a strongly truthful perfect Bayesian equilibrium that is robust to agents observing the current and historical signals \( \theta_{-i} \) of other agents.

1.2.9 Example Dynamic Mechanism Design Problem

In environments in which the private information of agents evolve dynamically, constructing incentive compatible transfers is challenging. In static environments, agents can only deviate by misreporting a single piece of information. However, in a dynamic environment, agents can misreport multiple pieces of information. This requires dealing with multi–period deviations from truth–telling, as well as deviations that are contingent on any information previously observed by agents. To illustrate the difficulty of constructing incentive compatible transfers in dynamic environments, we consider a simple two–period example of Athey and Segal [6] involving a single buyer and seller.

Consider a two–period model involving a buyer \( B \) and seller \( S \) with time indexed by \( t = 1, 2 \). The timing is as follows. In period 1 the seller learns his type \( \theta_S \in [1, 2] \) and reports a type \( \hat{\theta}_S \) to the designer. The buyer then purchases \( \psi_1 \) units from the seller. In period 2, the buyer learns her type \( \theta_B \in [0, 1] \) and reports a type \( \hat{\theta}_B \) to the designer before purchasing \( \psi_2 \) units from the seller. We assume the buyer’s total value is \( \psi_1 + \theta_B \psi_2 \) and the seller’s cost is \( \frac{1}{2} \theta_S \psi_1^2 \) in each period. The efficient mechanism implements \( Q_1(\theta_S) = 1/\theta_S \) and \( Q_2(\theta_S, \theta_B) = \theta_B/\theta_S \). Thus, under the efficient mechanism, period 1 trade reveals the seller’s type to the buyer.

Suppose we naively use the buyer’s updated beliefs to construct transfers that generalise the static AGV mechanism described in Section 1.2.5. Computing the expected externality the buyer imposes on the seller, the buyer’s

\(^8\)That is, for all \( i \in I \) and \( t \in \mathbb{N} \), \( s_i(\theta_{it}, \theta_i^{t-1}, \psi_i^{t-1}) = \theta_{it} \).

\(^9\)Interim incentive compatibility (see Bergemann and Välimäki [17]) is a stronger version of perfect Bayesian incentive compatibility, under which on–path truth–telling is robust to agents observing the complete allocation history \( \psi_i^{t-1} \).

\(^10\)That is, for all \( i \in I, t \in \mathbb{N} \) and \( \theta_i^t, \theta_i^{t-1}, s_i(\theta_{it}, \theta_i^{t-1}, \psi_i^{t-1}) = \theta_{it} \).
1.2. MECHANISM DESIGN BACKGROUND

incentive payment is given by

$$\tilde{m}_B(\hat{\theta}_B) = -\left(\frac{1}{2}\theta_S Q^2_S(\theta_S) + \frac{1}{2}\theta_S Q^2_S(\theta_S, \hat{\theta}_B)\right).$$

The seller’s incentive payment is equal to the expected value of the buyer and is given by

$$\tilde{m}_S(\hat{\theta}_S) = E_{\Theta_B} \left[ Q_1(\hat{\theta}_S) + \Theta_B Q_2(\hat{\theta}_S, \Theta_B) \right].$$

The transfers are then given by

$$m_B(\hat{\theta}_B, \hat{\theta}_S) = \tilde{m}_B(\hat{\theta}_B) - \tilde{m}_S(\hat{\theta}_S) \quad \text{and} \quad m_S(\hat{\theta}_B, \hat{\theta}_S) = \tilde{m}_S(\hat{\theta}_S) - \tilde{m}_B(\hat{\theta}_B).$$

However, these transfers are not incentive compatible because they induce the seller to over-report his type in period 1 in order to decrease $\tilde{m}_B(\hat{\theta}_S, \hat{\theta}_B)$. In general, we cannot directly use updated beliefs to construct incentive compatible transfers as this may undermine incentives for truthful reporting in previous periods. We need to make agents’ incentive payment depend on updated beliefs in a manner that does not destroy incentives for reporting truthfully in previous periods. Suppose instead the buyer is charged the change in the seller’s expected cost,

$$\tilde{m}_B(\hat{\theta}_S, \hat{\theta}_B) = -\left(\frac{1}{2}\theta_S \left( Q_2(\hat{\theta}_S, \hat{\theta}_B) \right)^2 - E_{\Theta_B} \left[ \frac{1}{2}\theta_S \left( Q_2(\hat{\theta}_S, \Theta_B) \right)^2 \right] \right)$$

$$= -\left( \hat{\theta}_B^2 - E_{\Theta_B} \left[ \Theta_B^2 \right] \right) / 2\theta_S.$$

This incentive payment induces the buyer to internalise the effect of her report on the seller. Furthermore, it cannot be manipulated by the seller because by the law of iterated expectations, its expected value is always zero,

$$E_{\Theta_B} \left[ \tilde{m}_B(\theta_S, \Theta_B) \right] = -\left( E_{\Theta_B} \left[ \Theta_B^2 \right] - E_{\Theta_B} \left[ E_{\Theta_B} \left[ \Theta_B^2 \right] \right] \right) / 2\theta_S = 0.$$

Thus, we have successfully constructed period ex-post Bayesian incentive compatible transfers.

1.2.10 Two-sided Matching

In Chapter 6 we will consider a dynamic market making problem that is closely related to the burgeoning literature on dynamic matching. Matching theory provides a useful approach to solving problems involving markets in which participants care about the identity of agents they trade with. Possible applications include the allocation of graduate doctors to hospitals, the allocation of students to university places, marriage markets, kidney exchanges and adoption markets. In this section, we provide a brief overview of the matching theory terminology used in later chapters.
Consider a setup in which we have women, indexed by $i \in \mathcal{N}$, on one side of the market and men, indexed by $j \in \mathcal{M}$, on the other. Let $\succsim^i$ denote the preference relation of woman $i$ over $\mathcal{M} \cup \{i\}$ and $\succ^i$ denote the strict preferences derived from $\succsim^i$. We write $j \succ^i j'$ if woman $i$ prefers to be matched with man $j$ over $j'$, $j \succ^i i$ if woman $i$ prefers to be matched with man $j$ over remaining single and $i \succ^i j$ if woman $i$ prefers to main single over being matched with man $j$. Similarly, we let $\succsim^j$ denote the preference relation of man $j$ over $\mathcal{N} \cup \{j\}$ and $\succsim^j := \{\succsim^l\}_{l \in \mathcal{M} \cup \mathcal{N}}$ denote the set of preferences. A *marriage problem* (Gale and Shapley [57]) is a triplet $\langle \mathcal{N}, \mathcal{M}, \succsim \rangle$.

The allocation rule $Q : \mathcal{N} \cup \mathcal{M} \to \mathcal{N} \cup \mathcal{M}$ is known now as a *matching rule*. If woman $i$ and man $j$ are matched, the matching rule sets $Q(i) = j$ and $Q(j) = i$. If any agent $l \in \mathcal{N} \cup \mathcal{M}$ is unmatched, we have $Q(l) = l$. Put differently, a matching rule $Q$ is such that for all $i \in \mathcal{M}$ and $j \in \mathcal{N}$,

$$Q(i) \notin \mathcal{M} \Rightarrow Q(i) = i, \quad Q(j) \notin \mathcal{N} \Rightarrow Q(j) = j \quad \text{and} \quad Q(i) = j \Leftrightarrow Q(j) = i.$$  

A matching $Q$ is *Pareto efficient* if there is no matching $Q'$ such that, for all $l \in \mathcal{N} \cup \mathcal{M}$, $Q'(l) \succsim_{l} Q(l)$ and for some $l \in \mathcal{N} \cup \mathcal{M}$, $Q'(l) \succ_{l} Q(l)$.\footnote{That is, it is not possibility to strictly improve the match of any agent, without making another agent strictly worse off.} It is *blocked by agent* $l \in \mathcal{N} \cup \mathcal{M}$ if $l \succ_{l} Q(l)$.\footnote{That is, agent $l$ strictly prefers remaining unmatched over the match assigned by $Q$.} A matching is then *individually rational* if it is not blocked by any agent. A matching is *blocked by a pair* $(i, j) \in \mathcal{N} \times \mathcal{M}$ if these agents strictly prefer matching over the matches assigned by $Q$. That is,

$$j \succ^i Q(i) \quad \text{and} \quad i \succ^j Q(j).$$

A matching rule is *stable* if it is individually rational and not blocked by any pair. The concept of stability was first introduced by Gale and Shapley [57] in a classic paper. It immediately follows from these definitions that if a matching is stable, it is also Pareto optimal.

### 1.3 Literature Review

First and foremost, this thesis relates to the literature concerning mechanism design, market making and intermediation. We begin this literature review by providing a brief overview of the related static mechanism design literature and explain the contributions of these papers in relation to the concepts and methodologies presented in Section 1.2. We then discuss the relevant dynamic mechanism design and dynamic matching literature in more detail. Finally, we conclude by outlining the emerging industrial organisation literature concerning two-sided platforms, as we will drawn on an important theme from this literature in later chapters.
1.3. LITERATURE REVIEW

1.3.1 Static Mechanism Design

There are a number of extensive surveys, including Krishna [83] and Borgers [21], that provide detailed overviews of the mechanism design literature. We now give a cursory summary of some of the most important papers from this literature, with a particular focus on papers whose contributions are closely related to the exposition presented in Section 1.2 and the results presented in subsequent chapters of this thesis.

The discussion in Section 1.2 focused on the independent private values model. This model has been studied extensively in the literature on mechanism design and auction theory, see, for example, Myerson [116], Milgrom and Weber [113], Chatterjee and Samuelson [29], Myerson and Satterthwaite [117], Gresik and Satterthwaite [68], Williams [142], Baliga and Vohra [10], Muir [114] and Loertscher and Marx [90]. Interdependent values are studied, for example, by Myerson [116], Milgrom and Weber [113], Maskin [102], Dasgupta and Maskin [45], Jehiel and Moldovanu [77], Bergemann and Välimäki [16], Perry and Reny [122] and Ausubel [8]. The work of Mezzetti [108, 109] forms the basis of the dynamic AGV mechanism from Section 1.2.9 and shows that ex-post efficiency and full surplus extraction is possible with interdependent values. The fact that the revenue equivalence theorem does not hold for the correlated values case was first noted by Myerson [116], who provided an illustrative example. These ideas were later formalised by Crémer and McLean [38, 37] and McAfee and Reny [105], who proved that full surplus extraction is possible when agent types are correlated random variables. However, the independent private values model is robust to the introduction of a small amount of dependence among valuations and costs when prices are bounded in magnitude (see Kosmopoulos and Williams [82]). Furthermore, results concerning full surplus extraction with correlated types have been shown to hinge crucially on common knowledge of beliefs (see Neeman [118]). For this reason, models with correlated types are considered to be “fragile”.

The original version of the revelation principle was developed in the context of dominant strategy incentive compatible mechanisms (see Gibbard [63] and Green and Laffont [67]) and was later extended to Bayesian incentive compatible mechanisms (see, among others, Holmstrom [74], Dasgupta, Hammond, and Maskin [44] and Myerson [115]). The version of the revelation principle presented in Section 1.2.2 was developed by Myerson [116] in the context of auction theory. The revenue equivalence theorem presented in Section 1.2.4 is due to Vickrey [140], Myerson [116], Riley and Samuelson [125] and Harris and Raviv [71]. The profit–maximisation mechanism presented in Section 1.2.6 was originally developed by Myerson [116] in the context of optimal auctions. This work was extended to bilateral trade and α–mechanisms by Myerson and Satterthwaite [117] and to multilateral trade by Baliga and Vohra [10]. The interpretation of virtual valuations as marginal revenues is due to Bulow and Roberts [26].
The impossibility of ex–post efficient trade was originally proven by Myerson and Satterthwaite [117] in the context of bilateral trade and was generalised to two–sided setups by Gresik and Satterthwaite [68]. The two–sided version of the VCG mechanism presented in Section 1.2.7 can be found in Loertscher and Marx [89]. The argument showing the impossibility of ex–post efficient trade presented in this section, based on the observation that the VCG mechanism maximises the expected payment made by each agent, is due to Makowski and Mezzetti [100, 99], Williams [142] and Krishna and Perry [84]. Furthermore, α–mechanisms have been studied by Myerson and Satterthwaite [117], Gresik and Satterthwaite [68], Tatur [137] and Loertscher, Marx, and Wilkening [91] for private goods and Neeman [118] and Hellwig [72] in the context of public goods. The equivalence of α–mechanisms and constrained efficient mechanisms was first observed by Bulow and Roberts [26] for the bilateral trade setup of Myerson and Satterthwaite [117].

Many mechanism design papers concerning intermediation in two–sided markets consider the mechanism designer as a broker which facilitates (and possibly exploits) trade by receiving funds (if the mechanism generates positive revenue) or providing funds (if the mechanism runs a deficit). Starting with the work of Myerson and Satterthwaite [117], this problem has been considered in a variety of settings. Baliga and Vohra [10] consider a profit maximising broker intermediating a two–sided market in which the distribution of traders’ private signals is not known by the traders or the broker. Segal and Whinston [135] extend the results of Myerson and Satterthwaite to settings with more general property rights, allowing for private information, joint ownership of an asset (as in Cramton, Gibbons, and Klemperer [36]) and liability rules. Loertscher and Marx [90] consider club goods, which requires accounting for network externalities.\footnote{In literature on two–sided platforms, network externalities refer to the property that the expected gains for platform users is increasing in the number of users on the other side of the market. In the case of club goods there exists an additional externality which must be considered as each buyer provides additional revenue which increases the probability that the good is produced.}

1.3.2 Dynamic Mechanism Design

The vast literature concerning mechanism design has to date largely focused on static setups. However, there has been a recent upsurge of interest in dynamic mechanism design (refer to Bergemann and Said [15] and Bergemann and Pavan [14] for recent surveys). One of the earliest contributions was made by Courty and Li [33] who consider optimal advanced ticket sales by a monopolist to a group of buyers with uncertain valuations in a two–period model. Broadly speaking, the dynamic mechanism design literature can be divided into two strands; one involving setups with a static population of agents whose types evolve over time and a second which considers dynamic populations of agents...
1.3. LITERATURE REVIEW

with persistent types. Recent contributions to the first strand of literature include Bergemann and Välimäki [17], Athey and Segal [6], Pavan, Segal, and Toikka [121] and Skrzypacz and Toikka [136], while recent contributions to the second include Parkes and Singh [120], Gershkov and Moldovanu [60] and Board and Skrzypacz [20].

Bergemann and Välimäki [17] consider an infinite–horizon dynamic allocation problem involving a central planner and a group of agents who receive private and independent signals over time (which may depend on previously observed signals and allocations). For example, once could consider the problem of efficiently renting units of an indivisible resource to a group of agents whose signals evolve dynamically due to learning. Assuming that agents are substitutes\footnote{Put differently, assuming the presence of any given agent imposes a negative externality on all other agents.}, they propose the \textit{dynamic pivot mechanism}, an infinite-horizon generalisation of the static VCG mechanism, which has non-negative transfers and yields a weak budget surplus for the central planner. The dynamic pivot mechanism satisfies periodic ex–post incentive compatibility and individual rationality constraints. It also satisfies efficient exit conditions (agents neither make nor receive transfers once they are not pivotal for the mechanism).

To solve a related problem in which agents have private types, Athey and Segal [6] propose the \textit{team mechanism}. Under the team mechanism, transfers may be negative and the mechanism is not necessarily periodic ex–post budget balanced. When types are independent, the team mechanism may be modified to create the \textit{balanced mechanism}, a dynamic generalisation of the AGV mechanism which is illustrated in Section 1.2.9. In general, the Myerson–Satterthwaite impossibility theorem applies and the balanced team mechanism does not satisfy periodic ex–post individual rationality constraints (agents may wish to exit the mechanism after some histories). However, Athey and Segal show that under additional assumptions (agents are sufficiently patient and types evolve under the balance team mechanism according to an ergodic finite Markov chain), this mechanism can be made period ex–post individually rational and self–enforcing\footnote{Meaning no central planner is needed and in each period agents make a public report before taking publicly observed action and making publicly observed payments.}. Skrzypacz and Toikka [136] build on earlier work of Athey and Segal [7] and study efficient contracting in dynamic environments in the context of a repeated bilateral trade problem in which the private information of the buyer and seller evolve according to a pair of Markov processes. In the model of Skrzypacz and Toikka, agents’ private information is initially multi–dimensional as they allow for private parameters to appear in the agents’ Markov processes. This may correspond to, for example, the seller having superior information about their long–run average cost.

Pavan, Segal, and Toikka [121] provide necessary conditions (and some sufficient conditions) for perfect Bayesian incentive compatibility in settings...
with dynamic private information. This is achieved by deriving a dynamic envelope formula that captures the effect of a change in the current type of an agent on that agent’s discounted expected payoff, accounting for the impact this has on the distribution of future types. Their analysis also yields a dynamic version of Myerson’s [116] revenue equivalence theorem. That is, given a dynamic mechanism satisfying perfect Bayesian incentive compatibility constraints, the allocation rule determines, up to a constant, the net expected present value of an agent’s payments for any realised sequence of types. Thus, efficient mechanisms may differ only by these constants and the spread of payments over future periods. The dynamic pivot mechanism of Bergemann and Välimäki [17] fixes these constants to satisfy period ex–post individual rationality and efficient exit conditions. The balanced team mechanism of Athey and Segal [6] achieves the same allocation but spreads the ex–post payments in a manner which ensures their mechanism has a balanced budget.

The focus of the aforementioned papers is constructing incentive compatible transfers, leaving optimal allocation rules implicitly defined as solutions to dynamic programming problems. Thus, none of these papers are able to draw meaningful comparisons between outcomes under welfare–maximisation and revenue maximisation in dynamic environments. A notable exception is Fershtman and Pavan [52], who exploit the results of Bergemann and Välimäki and Pavan, Segal, and Toikka to compare outcomes under welfare and profit maximisation in a two–sided many–to–many matching economy in which agent valuations may evolve exogenously (for example, as a result of shocks) or endogenously (for example, as a result of learning through experimentation).

In setups involving a dynamic population of agents with persistent types, the characterisation of incentive compatibility is a straightforward generalisation of the static case. However, the current allocation affects the sets of possible allocations in future periods. The mechanism designer chooses when to run a static mechanism, which gives rise to interesting dynamics. Pavan, Segal, and Toikka call this the mechanism designer’s optimal timing problem. Parkes and Singh [120] consider sequential allocation problems in a very general setup in which agents with independent persistent types randomly arrive and depart. Their online VCG mechanism generalises the static VCG mechanism and implements the efficient outcome. However, the allocation rule for their mechanism is implicitly defined as the optimal policy of a Markov decision process. Gershkov and Moldovanu [60] explicitly compute the efficient allocation in a special case of Parkes and Singh’s setup in which agents are perfectly impatient.

Another subset of this literature considers revenue maximisation. Pai and Vohra [119] consider a general setup in which a monopolist seller has multiple units of an indivisible good to sell to a group of buyers within a finite number of periods. Buyers have persistent types and randomly arrive and depart. Pai and Vohra are able to determine the dynamic analogue of Myerson’s [116] optimal mechanism. In related work, Board and Skrzypacz [20] determine
the profit–maximising mechanism in a single–sided setup in which buyers are patient, form rational expectations and are able to strategically time their purchases.

1.3.3 Dynamic Matching

This thesis is also closely related to the burgeoning dynamic matching literature which, motivated by applications such as kidney exchanges, adoption markets and the allocation of public housing, consider one–to–one dynamic matching markets with permanent matches. Optimally addressing the tradeoff between market thickness and the cost of delay is the fundamental economic insight for efficiency in such an environment. To the best of our knowledge, this tradeoff was first studied by Ünver [138] in the context of dynamic kidney exchange. Ünver shows that if a central planner minimises total discounted waiting costs, it may be optimal to withhold bilateral matches in the hope of subsequently executing multilateral matches.

In more recent work, Akbarpour, Li, and Gharan [2] build on the work of Ünver and Anderson et al. [3] to determine the efficient mechanism in a dynamic bilateral matching model without transfers in which exchange possibilities have a network structure. Baccara, Lee, and Yariv [9] consider a model in which agents have simpler preferences but face an explicit per period waiting cost. They are able to determine the efficient mechanism by optimally addressing the tradeoff between the benefits of increasing market thickness and the cost of delay. In a setup with one–sided matching in which the match–specific surplus function induces an element of interdependent valuations, Herbst and Schickner [73] also assess the tradeoff between the quality of matches and the cost of waiting.

In contrast to the aforementioned matching papers, Doval [50] proposes an extension of static matching stability which applies to these dynamic matching environments and investigates the existence of stable, rather than optimal, matchings. Doval finds that dynamically stable matchings may not exist in two–sided economies (agents on both sides of the matching problem are discounted expected utility maximisers) and suggests that unravelling may be prevented if matching is facilitated by a clearing house which restricts agents from waiting for a better match.

There is another strand of dynamic matching literature motivated by applications such as labour markets, college admissions and marriage markets which study environments in which matches are not permanent as agents have limited commitment power and may make interim revisions. See Kadam and Kotowski [78] and references therein.

17 Doval builds on previous work concerning stability in dynamic matching markets (see, for example, Kurino [85] and Du and Livne [51]) and coalition formation (see, for example, Pycia [124]).

18 In contrast, the form of unravelling previously emphasised in matching literature is associated with agents matching prior to participating in a centralised platform which implements a static matching (see, for example, McKinney, Niederle, and Roth [106]).
1.3.4 Two-Sided Platforms in Industrial Organisation

There is a vast literature on intermediation in two-sided markets, often motivated by the study of financial markets; see, for example, Mendelson [107] and Budish, Cramton, and Shim [25]. Furthermore, there is a large number of mathematics and statistics papers concerning limit order books (see Kelly and Yudovina [79] and references therein). In contrast to the mechanism design literature concerning two-sided markets, this literature does not consider agents’ incentives in two-sided intermediated markets. It also leaves aside the possibility that the market maker’s objective may entail generating profit, which, together with increasing market thickness in the guise of “getting both sides on board”, looms large in the literature on two-sided markets in industrial organisation such as Caillaud and Jullien [28, 27], Rochet and Tirole [127, 129], Armstrong [4] and Gomes [66].

With the notable exception of Gomes, these papers take a reduced-form approach as to how the intermediary generates surplus. Typically, the setups analysed in this literature are one-shot. For example, Rochet and Tirole [127] consider a monopolist two-sided platform and analyse the welfare implications of a cooperatively determined interchange fee in the context of payment card associations. Caillaud and Jullien [28, 27] study competing intermediaries who provide online platforms that match two sides of a market, such as eBay and Amazon. Rochet and Tirole [129] obtain a pricing model for two-sided competing platforms which encompasses the models of Rochet and Tirole [128] (who consider a pay per usage fee structure) and Armstrong [4] (who considers a pay per membership fee structure). Finally, Gomes [66] studies optimal auction design for an online profit-maximising platform that provides users with links to advertisers. Here, the platform faces a tradeoff between extracting rents from advertisers and maintaining a high number of users by providing links to relevant advertisers.

1.4 Summary of Main Contributions

As alluded to in Section 1.1, the primary contribution of this thesis is the development of tractable dynamic models of two-sided markets together with applications of these models. The secondary contribution of this thesis is that it addresses several open questions concerning the statistical properties of static market models.

The original research contained in this thesis begins with this secondary contribution in Chapter 3. Here, we consider static market models with unit traders in which buyer valuations and seller costs are independent and, on each side of the market, identically distributed. Although this family of models has been studied extensively in the literature on Bayesian mechanism design, little is known about the statistical properties of mechanism outcomes in such models. In particular, one open question is: what is the distribution of the
equilibrium quantity traded? To address this question Chapter 3 defines a new probability distribution, called a \textit{general non–central hypergeometric distribution}, which models biased sampling without replacement. The sampling bias is determined by a weighting function and the hypergeometric distributions previously defined in the statistics literature correspond to specific classes of weighting functions. We then consider a market in which \( n \) buyers with unit demand independently draw their valuations from some distribution \( F \) and \( m \) sellers with unit capacity independently draw their costs from some distribution \( G \). We show that the efficient quantity traded has a general non–central hypergeometric distribution, which in this case depends on \( n \), \( m \) and the weighting function \( G \circ F^{-1} \). In particular, when \( F = G \) (buyer valuations and seller costs are drawn from the same distribution), the quantity traded has a standard hypergeometric distribution. We extend these results by showing that the equilibrium quantity traded under a variety of market mechanisms, including \( \alpha \)-mechanisms, \( k \)-double auctions and first– and second–price auctions, has a general non–central hypergeometric distribution. In addition to encoding information about the distribution of buyer valuations relative to seller costs, in these extensions the weighting function can also depend on the degree of rent extraction by the market maker and the strategic behaviour of market participants.

Given that limit economies play an important role in neoclassical economics, another open question of interest concerns the asymptotic behaviour of the quantity traded and welfare in large markets. In Chapter 4 we study our previous model of a market involving trade between \( n \) buyers and \( m \) sellers and show that any such market can be represented in terms of two independent empirical quantile processes, one associated with demand and the other with supply. In markets with a large number of buyers and sellers, these can be approximated by appropriately weighted Brownian bridge processes, allowing any market statistic to be asymptotically expressed as a functional of a Gaussian process. Using an appropriate generalisation of the delta method (see, for example, Cramér [35]) we show that the joint distribution of welfare and the quantity traded is asymptotically normal. Moreover, we provide an upper bound on the approximation rate. To the best of our knowledge, this chapter is the first manuscript to compute higher order distributional approximations (of the central limit theorem–type) to mechanism outcomes. One advantage of this approach is that it enables the direct comparison of different mechanisms using the parameters of the approximating normal distributions. Furthermore, our approach immediately generalises to any mechanism which can be appropriately represented in terms of transformed empirical quantile functions, which covers a large class of mechanisms studied in the Bayesian mechanism design literature.

The second part of this thesis contains our primary contribution concerning dynamic models of two–sided markets. Characterising efficiency in a dynamic environment in which traders arrive stochastically is challenging, in part be-
cause the frequency at which the market clears is a design variable. In such setups the market maker faces a non–trivial tradeoff between the benefits of market thickness stemming from the accumulation of traders and the associated cost of delay. This tradeoff gives rise to the fundamental economic insight for efficiency in a dynamic setting. A natural starting point is a two–period extension of the classic bilateral trade model of Myerson and Satterthwaite [117] that we analyse in Chapter 5. We assume that in each period one buyer and one seller arrive and draw their types from the absolutely continuous distribution functions \( F \) and \( G \) respectively. We assume all agents discount their period two payoffs by a common factor \( \delta \) and use backward induction to determine the class of Bayesian optimal \( \alpha \)–mechanisms. These optimal dynamic mechanisms are complex insofar as the optimal allocation and payment rules are non–trivial functions of agent reports in period one. This motivates us to construct a price–posting mechanism that approximates the outcome of the optimal mechanism and outperforms two benchmark static mechanisms, provided the discount factor is sufficiently large.

We solve this two–period dynamic model with continuous type distributions using backward induction. This approach requires computing payoffs for every possible realisation of agent types and becomes intractable if more periods are added to the model. However, the inclusion of additional periods is necessary to understand which results are driven by end–of–the–world effects. To obtain a tractable infinite–horizon dynamic market model we consider agents with simple type distributions. In Chapter 6 we analyse an infinite–horizon model in which buyers and sellers with binary types and a common discount factor \( \delta \) arrive in pairs. We begin by assuming the market maker is unconstrained and has the ability to dictate which agents trade in each period. We determine the efficient market–clearing policy by showing that the market maker’s optimisation problem reduces to determining the optimal policy of a Markov decision process with a denumerable state space. We then prove that a particular type of policy, a threshold policy, is optimal. This allows the state space to be partitioned so that the optimal policy can be computed using dynamic programming techniques.

In the infinite–horizon model with binary types, the optimal policy is difficult to derive but remarkably simple to describe and implement. It provides the tractable baseline model that we use to explore a variety of questions of economic interest. Among other things, a dynamic notion of efficiency emerges from our analysis. Unlike its static counterpart of ex–post efficiency, this notion of efficiency depends on the distributions from which agent types are drawn. Matching low (high) value buyers with low (high) cost sellers, a feature that would indicate inefficiency in a static setting, is actually a requirement of dynamic efficiency. Furthermore, for a fixed discount factor, a reduction in these apparently suboptimal matches indicates greater rent extraction by the market maker. Profit–maximising market makers have an incentive to create inefficiently thick markets because, relative to welfare, less
1.4. SUMMARY OF MAIN CONTRIBUTIONS

profit can be extracted from suboptimal matches.

We compare the unconstrained optimal market–clearing policy, which we call discriminatory market clearing, with various alternatives, including uniform and fixed frequency market clearing. Under uniform market clearing all agents must exit the market whenever it is cleared, regardless of whether they trade. Fixed frequency market clearing restricts uniform market clearing to occur at fixed (but optimally chosen) intervals. Of these policies, discriminatory market clearing is the most sophisticated and results in a market with the highest degree of centralisation. Markets that clear at a fixed frequency are the least centralised. To compare outcomes under each of the market–clearing policies we exploit properties of the Poisson equation, which characterises certain constrained dynamic optimisation problems. We show that the main gains from centralisation relative to a decentralised market are already achieved with fixed frequency market clearing, which is the least sophisticated form of centralisation, if the discount factor is sufficiently large. Furthermore, provided that agents are sufficiently patient, we find that a market with a profit–maximising market maker generates higher welfare compared to a market with a lower degree of centralisation in which the market maker (possibly) subsidises welfare–maximising trade. Thus, it is possible for a market to simultaneously extract revenue and benefit traders. This resonates with a key theme in the burgeoning literature on two–sided platforms that it is imperative for the platform to “bring both sides on board”.

We investigate three applications of our dynamic market model. First, we consider an extension in which the market maker also has the ability to produce. We find that suboptimal matches create incentives for socially wasteful, profitable vertical integration by the market maker. Second, we compare ad valorem taxes to specific taxes and show that, although these taxation schemes are equivalent in static markets, ad valorem taxes outperform specific taxes in dynamic markets. Specific taxes distort the relative value of efficient and suboptimal matches, which induces the market maker to create an excessively thick market and reduces the welfare of traders. Finally, we show that it is possible to construct asymptotically optimal, detail–free, incentive compatible and individually rational mechanisms in environments in which the market maker does not know the distributions from which agents draw their types.

In Chapter 7 we allow for several generalisations of the baseline model including the arrival of unpaired agents, independent continuous–time arrival processes and discrete types spaces which obey a dynamic generalisation of Myerson’s [116] regularity condition. Each of these extensions increases the size of the state space of the underlying Markov decision process so that the optimal market–clearing policies are not simple threshold policies. However, in each case it suffices to consider a finite partition of the state space, which allows the optimal policy to be computed by modifying standard algorithms (policy and value iteration). This is achieved by exploiting coupling arguments to construct an appropriate related Markov decision process. The threshold
policy of this new Markov decision process is then used to partition the state space of the original Markov decision process.

We believe that the tractability and flexibility of the model developed in Chapter 6 provides a promising basis for future research and details a future research agenda is outlined in Chapter 8.
Chapter 2

Mathematical Preliminaries

We introduce the main mathematical results required for the subsequent chapters.

2.1 Probability Distributions

In this section we introduce and fix notation for the probability distributions used throughout this thesis. In particular, we introduce the existing general non-central hypergeometric distributions from the statistics literature.

We start by defining the hypergeometric distributions discussed in Chapter 3. Let \( N \in \mathbb{N}, n \in \{0, 1, \ldots, N\} \) and \( D \in \{0, 1, \ldots, N\} \) be given. If, for \( x \in \max\{0, n + D - N\}, \ldots, \min\{n, D\} \), \( X \) has a probability mass function given by

\[
p_X(x; n, N, D) = \binom{D}{x} \binom{N-D}{n-x} \binom{N}{n},
\]

then we say \( X \) has a standard hypergeometric distribution and write \( X \overset{d}{=} \text{Hg}(n, D, N) \). Further, let \( \omega_1, \omega_2 \in (0, \infty) \) be given, with \( \omega = \omega_1/\omega_2 \). Suppose for \( x \in \max\{0, n + D - N\}, \ldots, \min\{n, D\} \), \( X \) has a probability mass function given by

\[
p_X(x; n, N, D, \omega) = \frac{\binom{D}{x} \binom{N-D}{n-x} \omega^x}{P_0},
\]

where

\[
P_0 = \sum_{y=\max\{0, n+D-N\}}^{\min\{n, D\}} \binom{D}{y} \binom{N-D}{n-y} \omega^y.
\]

Then we write \( X \overset{d}{=} \text{HgF}(n, D, N, \omega) \) and say \( X \) has Fisher’s non-central hypergeometric distribution. Finally, suppose that for \( x \in \max\{0, n + D - N\}, \ldots, \min\{n, D\} \), \( X \) has a probability mass function given by

\[
p_X(x; n, N, D, \omega) = \binom{D}{x} \binom{N-D}{n-x} \int_0^1 (1 - t^{1/d})^{n-x}(1 - t^\omega)^x dt, \quad (2.1)
\]
where \( d = x - D + N - n + \omega(D - x) \). Then \( X \) has Wallenius’ non-central hypergeometric distribution and we write \( X \overset{d}{=} \text{HgW}(n, D, N, \omega) \).

We now introduce the remaining discrete distributions used in this thesis. Let \( n \in \mathbb{N} \) and \( p \in (0, 1) \) be given. If, for \( x \in \{0, 1, \ldots, n\} \), \( X \) has a probability mass function given by

\[
p_X(x; n, D, N) = \binom{n}{x} p^x (1 - p)^{n-x},
\]

then we say that \( X \) has a binomial distribution and write \( X \overset{d}{=} \text{Bn}(n, p) \).

If, for \( x \in \mathbb{N} \), \( X \) has a probability mass function given by

\[
f_X(x; p) = (1 - p)^{x-1} p,
\]

then we say that \( X \) has a geometric distribution and we write \( X \overset{d}{=} \text{Gm}(p) \).

Finally, let \( r \in \mathbb{N} \) be given. Suppose for \( x \in \mathbb{N} \), \( X \) has a probability mass function given by

\[
f_X(x; r, p) = \binom{x-1}{r-1} (1 - p)^r p^{x-r}.
\]

Then we say that \( X \) has a negative binomial distribution and we write \( X \overset{d}{=} \text{NB}(r, p) \).

We now define the continuous distributions used throughout this thesis. Let a closed interval \([a, b]\) \( \subset \mathbb{R} \) be given. Suppose for \( x \in [a, b] \), \( X \) has a probability density function given by

\[
f_X(x; a, b) = \frac{1}{a - b}.
\]

Then we say that \( X \) is uniformly distributed on \([a, b]\) and write \( X \overset{d}{=} \text{U}[a, b] \).

Now suppose that we have any \( a, b \in \mathbb{R} \) and define the Beta function, which is given by

\[
\text{Beta}(a, b) = \int_0^1 t^{a-1} (1 - t)^{b-1} dt.
\]

Suppose for \( x \in [0, 1] \), \( X \) has a probability density function given by

\[
f_X(x; a, b) = \frac{x^{a-1}(1 - x)^{b-1}}{\text{Beta}(a, b)}.
\]

Then we say that \( X \) has a Beta distribution and write \( X \overset{d}{=} \text{B}(a, b) \).

Let \( \mu \in \mathbb{R} \) and \( \sigma^2 \in \mathbb{R}_{>0} \) be given. Suppose for \( x \in \mathbb{R} \), \( X \) has a probability density function given by

\[
f_X(x; \mu, \sigma^2) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.
\]
Then we say that $X$ has a normal distribution and we write $X \overset{d}{=} \mathcal{N}(\mu,\sigma^2)$.

Next, let $\mu \in \mathbb{R}^2$ be given and suppose $\Sigma \in \mathbb{R}^{2 \times 2}$ is positive definite. Suppose for $x \in \mu + \text{span}(\Sigma) \subseteq \mathbb{R}^2$, $X$ has a probability density function given by

$$f_X(x;\mu,\Sigma) = \frac{1}{\sqrt{2\pi\det(\Sigma)}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}.$$  

Then we say that $X$ has a bivariate normal distribution and we write $X \overset{d}{=} \mathcal{N}(\mu,\Sigma)$.

Finally, let $\lambda \in \mathbb{R}_{>0}$ be given. Suppose that for $x \in \mathbb{R}_{>0}$, $X$ has a probability density function given by

$$f_X(x;\lambda) = \lambda e^{-\lambda x}.$$  

Then we say that $X$ has an exponential distribution and we write $X \overset{d}{=} \text{Exp}(\lambda)$.

### 2.2 Order Statistics

In Chapter 3 we require several well-known order statistics results (see, for example, David [48]). Suppose $\{X_1,\ldots,X_m\}$ is a sample of independent and identically distributed random variables drawn from the absolutely continuous distribution function $G$. We let $X_{(i)}$ denote the $i$th smallest random variable contained in the sample. That is, $X_{(i)}$ denotes the $i$th order statistic of the sample. For $i = 1,\ldots,m$, the density function of the order statistic $X_{(i)}$ is given by

$$g_{X_{(i)}}(y) = \frac{m!}{(i-1)!(m-i)!} F^{i-1}(x)[1 - F(x)]^{m-i} f(x), \quad (2.2)$$

where $x \in \mathbb{R}$. For $i = 1,\ldots,n-1$, the joint density function of $(X_{(i)},X_{(i+1)})$ is given by

$$g_{(X_{(i)},X_{(i+1)})}(x,x') = \frac{n!}{(i-1)!(n-i-1)!} G^{i-1}(x)[1 - F(x')]^{n-i-1} \times g(x)g(x'), \quad (2.3)$$

where $x, x' \in \mathbb{R}$ and $x < x'$.

Similarly, we let $\{Y_1,\ldots,Y_n\}$ denote a sample of independent and identically distributed random variables drawn from the absolutely continuous distribution function $F$. We let $Y_{[j]}$ denote the $j$th largest random variable contained in the sample. By symmetry, for $j = 1,\ldots,n$, the density of the order statistic $Y_{[j]}$ is given by

$$f_{Y_{[j]}}(y) = \frac{n!}{(j-1)!(n-j)!} F^{n-j}(y)[1 - F(y)]^{j-1} f(y), \quad (2.4)$$
where \( y \in \mathbb{R} \). Furthermore, for \( j = 1, \ldots, m - 1 \), the joint density function of \((Y[j], Y[j+1])\) is given by

\[
f_{(Y[j], Y[j+1])}(y, y') = \frac{m!}{(j-1)!(m-j-1)!} F^{m-j-1}(y') [1 - F(y)]^{j-1} \times f(y)f(y'), \tag{2.5}
\]

where \( y, y' \in \mathbb{R} \) and \( y' < y \).

### 2.3 Empirical Quantile Processes and Brownian Bridges

In Chapter 4, we will exploit some well-known results from large sample theory to approximate welfare and the quantity traded in large markets. We now provide some background on Brownian bridge process, empirical quantile processes and the functional central limit theorem that applies to uniform empirical processes.

A Brownian bridge is a Gaussian process \( \{B(t)\}_{t \in [0, 1]} \) such that

1. \( \mathbb{E}[B(t)] = 0 \) for all \( t \in [0, 1] \),
2. the function \( t \rightarrow B(t) \) is continuous with probability one, and
3. \( \text{Cov}(B(s), B(t)) = \min\{s, t\} - st \) for all \( s, t \in [0, 1] \) with \( s \neq t \).

A Brownian bridge defined on the interval \([0, 1]\) is essentially a standard one-dimensional Brownian motion \( \mathbb{W}(t) \), conditional on having \( \mathbb{W}(1) = 0 \) (see, for example, Cox and Miller [34], Freedman [56] and Billingsley [18]). That is, for \( t \in [0, 1] \), \( B(t) = \mathbb{W}(t) - t\mathbb{W}(1) \) defines a Brownian bridge process.

For a sample \( \{X_1, \ldots, X_n\} \) with order statistics \( X^{(1)} \leq \cdots \leq X^{(n)} \), the respective empirical quantile function \( X_n(t), t \in [0, 1] \), is defined by

\[ X_n(0) := X^{(1)}, \quad X_n(t) := X^{(i)}, \quad (i-1)n^{-1} < t \leq in^{-1}, \quad i = 1, \ldots, n. \]

For the empirical quantile function \( U_n(t) \) constructed from a sample of \( U[0, 1] \)-distributed independent and identically distributed \( U_1, \ldots, U_n \),

\[ R_n(t) := \sqrt{n} (U_n(t) - t), \quad t \in [0, 1], \]

denotes the respective uniform quantile process. Recall that both \( U_n \) and \( R_n \) are random elements of the Skorokhod space \( \mathcal{D}[0, 1] \) of càdlàg functions on \([0, 1]\).

By the well-known Donsker’s theorem (see, for example, Section 14 of Billingsley [18]), as \( n \to \infty \), the distribution of \( R_n \) in \( \mathcal{D}[0, 1] \) converges weakly to that of the Brownian bridge process \( \mathbb{B} \). A sharp bound on the rate of convergence is given in Csörgő and Révész [41]. A corollary of that result
states that there exists a probability space carrying a sequence of processes $\mathbb{R}_n^* \overset{d}{=} \mathbb{R}_n$, $n = 1, 2, \ldots$, and a Brownian bridge process $B$ such that, as $n \to \infty$,

$$\|\mathbb{R}_n^* - B\|_{\infty} = O\left( n^{-1/2} \ln n \right) \text{ a.s.,}$$

where $\|\cdot\|_{\infty}$ denotes the supremum norm, $\|h\|_{\infty} := \sup_{t \in [0,1]} |h(t)|$, $h \in \mathbb{D}[0,1]$.

Therefore, as $n \to \infty$,

$$U_n(t) = t + n^{-1/2} \mathbb{R}_n(t) \overset{d}{=} t + n^{-1/2} \mathbb{R}_n^*(t)$$

for $t \in [0,1]$, where $\|\theta_n\|_{\infty} = O(1)$ a.s.

### 2.4 Markov Decision Processes

Markov decision processes provide a useful modelling tool for situations in which decisions are made over time and outcomes are random and depend on the actions taken by the decision–maker. These processes assume that decisions are made in a Markovian environment, where the transitions of the underlying Markov chain depend on the actions of the decision–maker. For an overview of the classes of Markov decision processes and their solution concepts, see Puterman [123]. In this section we fix notation for the Markov decision processes constructed in Chapter 6.

We denote a Markov decision process by $\langle X, A, P, r, \delta \rangle$. Here, $X$ is the state space of the Markov decision process and $X_t \in X$ denotes the state of the Markov decision process in period $t$. The set of actions available to the decision–maker is $A$ and $A_x$ denotes the set of actions available in state $x \in X$. We let $A_t$ denote the action of the decision–maker in period $t$. The function $P$ encodes the transition probabilities of the underlying Markov chain. We denote by

$$P_a(x, \cdot) := \mathbb{P}(X_{t+1} = x' \mid X_t = x, A_t = a)$$

the probability that the process transitions from state $x \in X$ in period $t$ to state $x' \in X$ in period $t + 1$, when the decision–maker took the action $a \in A_x$ in period $t$. Similarly the function $r$ encodes the immediate rewards earned by the decision–maker. Here, $r_a(x)$ is the immediate reward the decision–maker earns when the action $a$ is implemented in state $x$. Finally, $\delta \in [0,1]$ is the discount factor. We assume that both the state space $X$ and the action space $A$ are finite.

A policy $\pi : X \to A$ of a Markov decision process is such that $\pi(x) \in A_x$ specifies the action taken by the decision–maker in state $x$. The optimal policy
\( \pi^* \) maximises the total expected discounted reward received by the decision-maker. It can be determined using dynamic programming techniques.

Let a policy \( \pi \) be given. We denote the total expected discounted reward earned by the decision-maker, starting from state \( x \), under the policy \( \pi \) by \( V_\pi(x) \). We can characterise \( V_\pi(x) \) recursively,

\[
V_\pi(x) = r_\pi(x) + \delta \sum_{x' \in \mathcal{X}} P_\pi(x, x') V_\pi(x').
\] (2.7)

The Bellman equation (see Bellman [12]) characterises the optimal policy and is given by

\[
V_{\pi^*}(x) = \max_{a \in A_x} \left\{ r_a(x) + \delta \sum_{x' \in \mathcal{X}} P_a(x, x') V_{\pi^*}(x') \right\}.
\]

In this thesis, we will exploit two well-known methods for optimally solving Markov decision processes. The first is the value iteration method, which is due to Bellman [12]. The value iteration algorithm starts with \( i = 0 \) and an initial guess \( V_0 \) for the value function. The algorithm then iteratively computes

\[
V_{i+1}(x) = \max_{a \in A_x} \left\{ r_a(x) + \delta \sum_{x' \in \mathcal{X}} P_a(x, x') V_i(x') \right\},
\]

until \( \| V_{i+1} - V_i \|_\infty < \varepsilon \), where \( \varepsilon \) is some pre-specified tolerance. Bellman [12] showed that, theoretically, the above recursion converges to the Bellman equation regardless of the initial guess \( V_0 \). The value iteration method has two disadvantages. First, the algorithm does not have a definite stopping condition and second, the algorithm does not return the optimal policy (only the value function associated with the optimal policy).

The policy iteration method is due to Howard [76]. The policy iteration algorithm starts with \( i = 0 \) and an initial guess \( \pi_0 \) for the optimal policy. At step \( i \), the algorithm computes \( V_{\pi_{i-1}} \) iteratively using

\[
V_{j+1}(x) = r_{\pi_{i-1}}(x) + \delta \sum_{x' \in \mathcal{X}} P_{\pi_{i-1}}(x, x') V_j(x').
\]

Alternatively, the algorithm can compute \( V_{\pi_{i-1}} \) exactly by solving the linear system defined in 2.7. The algorithm then computes an updated policy \( \pi_i \) using

\[
\pi_{i+1}(x) = \arg \max_{a \in A} \left\{ r_a(x) + \delta \sum_{x' \in \mathcal{X}} P_a(x, x') V_{\pi_{i-1}}(x') \right\}.
\]

The algorithm terminates once the policy \( \pi_i \) is invariant and Howard [76] showed that this occurs in finitely many steps. The advantage policy iteration
is that the stopping condition is definite and the algorithm returns both $\pi^*$ and $V_{\pi^*}$.

In Chapter 6 we will also consider partially observable Markov decision processes $\langle X, A, P, r, S, O, \delta \rangle$. Under one of these processes, the decision-maker observes a state dependent signal $s \in S$ in each period, rather than the true state of the process. The function $O(x, s)$ then specifies the probability that the true state of the process is $x$ given that the signal $s$ was observed. Under a partially observable Markov decision process policy $\pi : S \rightarrow A$ is such that $\pi(s) \in A$ specifies the action taken by the decision-maker upon observing the signal $s$. 
Chapter 3

A General Non–central Hypergeometric Distribution and the Equilibrium Quantity Traded

We introduce a general non–central hypergeometric distribution and show that the equilibrium quantity traded has this distribution under a large class of market mechanisms.

3.1 Introduction

In this chapter we construct a general non–central hypergeometric distribution, which models biased sampling without replacement. The distribution is constructed from the combined order statistics of two samples; the first of independent and identically distributed random variables with absolutely continuous distribution $F$ and the second of independent and identically distributed random variables with absolutely continuous distribution $G$. We show the distribution depends on $F$ and $G$ only through $G \circ F^{-1}$ (G composed with the quantile function of $F$) and that the standard hypergeometric distribution and Wallenius’ non–central hypergeometric distribution arise as special cases. We provide an application for our general non–central hypergeometric distribution by showing it can be used to model the distribution of the equilibrium quantity traded under a variety of market mechanisms, including the Bayesian optimal mechanisms described in Section 1.2.6.

The standard hypergeometric distribution models the number of marked objects obtained when an unbiased sample is collected from a finite population without replacement. Fisher’s and Wallenius’ non–central hypergeometric distributions (see definitions in Section 2.1) are widely used generalisations of this standard distribution. Fisher’s non–central hypergeometric distribution was first described by Fisher [53] in the context of contingency tables. Wallenius [141] conceived a non–central hypergeometric distribution in his PhD thesis.
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(later published as a technical report by Stanford University) for use in competitive models of Darwinian evolution. These non–central hypergeometric distributions may be considered in the context of an urn problem with simple biased sampling. Specifically, sampling is undertaken such that if an urn contains balls of two different weights, the probability of drawing a given ball is proportional to its weight.

The difference between the Fisher and Wallenius distributions is subtle and has led to nomenclature confusion, which was resolved by Fog [55]. Fisher’s distribution models a situation in which balls are sampled independently and Wallenius’ distribution models sampling without replacement. However, when sampling is unbiased – that is, all balls are of equal weight – both distributions give rise to the standard hypergeometric distribution. This is discussed in Section 3.2 in more detail.

In this chapter we show it is possible to consider the standard hypergeometric distribution in terms of samples taken from two independent sets of order statistics drawn from the same distribution. It is natural to extend this to samples of sets of order statistics drawn from different distributions, \( F \) and \( G \). This allows us to construct a general non–central hypergeometric distribution which models cases in which marked objects are more or less likely to be sampled. Let \( F^{(-1)} \) denote the quantile function of \( F \) (which in this case is simply the inverse of \( F \)) and denote by \( F^{(\omega)}(x) = (F(x))^\omega \). We will show our non–central hypergeometric distribution depends on the functions \( F \) and \( G \) only through \( G \circ F^{(-1)} \). In some robust special cases, we obtain a distribution which depends on \( F \) and \( G \) only through a small number of parameters. In particular, when \( F = G^{(\omega)} \), it depends only on \( \omega \). The function \( G \circ F^{(-1)} \) can be thought of as a weighting function which determines the sampling bias.

The non–central hypergeometric distribution defined in this chapter arises naturally in many applications. In particular, we consider a market in which \( n \) buyers with unit demand independently draw their valuations from some distribution \( F \) and \( m \) sellers with unit capacity independently draw their costs from some distribution \( G \). We show that the ex–post efficient quantity traded has a general non–central hypergeometric distribution, which in this case depends on \( n \), \( m \) and the weighting function \( G \circ F^{-1} \). In particular, when \( F = G \) (buyer valuations and seller costs are drawn from the same distribution), the quantity traded has a standard hypergeometric distribution. We also consider multi–unit traders and present an application for our result regarding Wallenius’ non–central hypergeometric distribution. We extend these results by showing that the equilibrium quantity traded under a variety of market mechanisms, including \( \alpha \)–mechanisms, \( k \)–double auctions and first– and second–price auctions, has a general non–central hypergeometric distribution. In addition to encoding information about the distribution of buyer valuations relative to seller costs, in these extensions the weighting function can also depend on the degree of rent extraction by the market maker and the
strategic behaviour of market participants.

The remainder of the chapter is organised as follows. In Section 3.2 we discuss the three well–known existing hypergeometric distributions. Section 3.3 provides a construction of our general non–central hypergeometric distribution using the combined order statistics of two samples, one comprised of independent and identically distributed random variables with an absolutely continuous distribution F. For the other, the absolutely continuous distribution function is G. In Section 3.4, our non–central hypergeometric distribution is explicitly computed in terms of the functions F and G. Section 3.5 shows the standard hypergeometric distribution and Wallenius’ non–central hypergeometric distribution arise as special cases. In Section 3.6 we show the general non–central hypergeometric distribution depends only on F and G through G ◦ F(−1). Finally, in Section 3.7 we shows that the general non–central hypergeometric distribution arises naturally in the study of economic markets. Concluding remarks are contained in Section 3.8.

3.2 Sampling Without Replacement

The hypergeometric distributions introduced in 2.1 may described in terms of urn problems involving sampling without replacement. Suppose an urn contains N balls, D of which are marked. If n successive draws are performed without replacement, the number of marked balls contained within the sample has a standard hypergeometric distribution.

Fisher’s distribution (see 2.1) may be described in the context of an urn problem in the following manner. Suppose an urn contains N balls and D of these balls are marked. A sample of A1 marked balls is collected by including each ball with probability \( \frac{\omega_1}{\omega_1 + \omega_2} \). A sample of A2 unmarked balls is then collected by including each ball with probability \( \frac{\omega_2}{\omega_1 + \omega_2} \). With this method of sampling, the event that one ball is included in the sample is independent of the inclusion of other balls. Fisher’s non–central hypergeometric distribution is given by the number of marked balls \( Y_1 \), conditional on \( A_1 + A_2 = n \). Thus, Fisher’s distribution is the conditional distribution of two independent binomial random variables, given their sum. If \( \omega_1 = \omega_2 \), marked and unmarked balls are equally likely to be included in the sample. In this case, Fisher’s non–central hypergeometric distribution is equivalent to the standard hypergeometric distribution.

Wallenius’ non–central hypergeometric distribution (see 2.1) may also be described in terms of sampling without replacement. Consider an urn which contains D marked balls of weight \( \omega_1 \) and \( N - D \) unmarked balls of weight \( \omega_2 \). Assume n successive draws are performed such that the probability of selecting a given ball is its proportion of the total weight of all remaining balls. Then the number of marked balls within the sample of size n has Wallenius’ non–central hypergeometric distribution. Wallenius [141] showed
that the probability mass function of the number of marked balls selected satisfies the combinatorial recursion

\[
p_X(x; n, N, D, \omega) = \frac{\omega(D - x + 1)}{\omega(D - x + 1) + N - D - n + x} p_X(x - 1; n - 1, N, D, \omega) \\
+ \frac{N - D - n + x + 1}{\omega(D - x) + N - D - n + x + 1} p_X(x, n - 1, N, D, \omega)
\]

and the solution to this recursion is given by (2.1). If the sampling is performed without bias, then marked and unmarked balls have an equal weight. Thus, the standard hypergeometric distribution is obtained as a special case of Wallenius’ non–central hypergeometric distribution when \( \omega_1 = \omega_2 \).

We have obtained the standard non–central hypergeometric distribution as a special case of both Fisher’s and Wallenius’ distributions. However, Wallenius’ non–central hypergeometric distribution is the natural generalisation of the standard hypergeometric distribution, as it models sampling without replacement. When balls are sampled without replacement from an urn, it is necessarily the case that successive draws are dependent. Since Fisher’s non–central hypergeometric distribution is constructed such that individual draws are independent, no method of sampling without replacement will give rise to this distribution in general, with the notable exception being the case in which \( \omega_1 = \omega_2 \).

The remainder of this chapter will focus on the construction of a general non–central hypergeometric distribution, which models biased sampling without replacement. We do not expect Fisher’s non–central hypergeometric distribution to arise naturally as a special case, as this distribution does not model sampling without replacement, in general.

### 3.3 Order Statistics and Sampling Without Replacement

In this section, we show it is possible to consider sampling without replacement in terms of the order statistics of two samples of random variables. For ease of exposition, we consider absolutely continuous probability distribution functions only. However, the results presented in this chapter can be generalised to Riemann–Stieltjes integrable functions.

Take \( N \in \mathbb{N} \), \( n \in \{0, 1, \ldots, N\} \) and \( D \in \{0, 1, \ldots, N\} \) and let \( F \) and \( G \) be absolutely continuous probability distribution functions on \( \mathbb{R} \). Let \( \mathcal{Y} = \{Y_1, \ldots, Y_{N-D}\} \) be a sample of independent and identically distributed random variables with distribution \( F \). Similarly, let \( \mathcal{X} = \{X_1, \ldots, X_D\} \) be a sample of independent and identically distributed random variables with distribution \( G \). For ease of exposition, we assume the random variables in the set \( \mathcal{Z} = \mathcal{X} \cup \mathcal{Y} \) are distinct (an event which occurs almost surely since \( F \) and \( G \) are absolutely continuous).
3.3. ORDER STATISTICS & SAMPLING WITHOUT REPLACEMENT

Order the random variables in the set $Z$ from smallest to largest. We consider the first $n$ random variables from the ordered list (the $n$ smallest random variables from the set $Z$) and count the number of random variables from the set $X$ within this sample. Denote this random variable by $L$. Here, $L$ lies within the set $\{\max\{0, n + D - N\}, \ldots, \min\{n, D\}\}$. Formally, let $Z_1 = X_1, \ldots, Z_D = X_D, Z_{D+1} = Y_1, \ldots, Z_N = Y_{N-D}$. Using indicator functions, $L$ may be written as

$$L = \sum_{i=1}^{n} \sum_{j=1}^{D} I(Z_{(i)} = X_j),$$

(3.1)

where $Z_{(i)}$ is the $i$th order statistic, that is, the $i$th lowest random variable in the set $Z$. Refer to David [48] for an introduction to order statistics.

By construction, $L$ models biased sampling without replacement. Thus, $L$ has a non-central hypergeometric distribution. However, the nature of the biased sampling depends non-trivially on the distribution functions $F$ and $G$. We thus have defined a general non-central hypergeometric distribution, dependent on $n$, $D$, $N$, $F$ and $G$.

We next show it is possible to define $L$ in terms of the order statistics of the sets $X$ and $Y$. Let $Y_{[j]}$ denote the $j$th largest random variable from the set $Y$.

We start by proving the following lemma. Let $l = \max\{0, n + D - N\}$ and $\bar{l} = \min\{n, D\}$ so the support of $L$ becomes $\{l, \ldots, \bar{l}\}$ for the remainder of the chapter.

Lemma 3.3.1. Take $l \in \{l + 1, \ldots, \bar{l}\}$. Then $X_{(l)} < Y_{[N-D-n+l]}$ if and only if there are at least $l$ random variables from $X$ among the $n$ smallest random variables from $Z$. That is, $X_{(l)} < Y_{[N-D-n+l]}$ if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{D} I(Z_{(i)} = X_j) \geq l.$$  

(3.2)

Proof. Suppose (3.2) holds and there are at least $l$ random variables from $X$ among the $n$ smallest random variables from $Z$. Then there are at least $N - D - n + l$ random variables from $Y$ among the $N - n$ largest random variables from $Z$. Thus, $X_{(l)} < Y_{[N-D-n+l]}$. Conversely, suppose $X_{(l)} < Y_{[N-D-n+l]}$. Then exactly $D - l$ random variables from $X$ are greater than $X_{(l)}$ and at least $N - D - n + l$ random variables from $Y$ are greater than $X_{(l)}$. Therefore, at least $N - n$ random variables from $Z$ are greater than $X_{(l)}$ and $X_{(l)}$ is contained in the lowest $n$ random variables from $Z$. Thus, the lowest

\[1\] If $F$ and $G$ are Riemann–Stieltjes integrable the random variables in the set $Z$ are not necessarily distinct and a strict ordering of the set is required to proceed. One possibility is to resolve ties randomly and uniformly. Once the set $Z$ has been strictly ordered, all results presented in the chapter generalise.
n random variables from $\mathcal{Z}$ contains at least $l$ random variables from $\mathcal{X}$ and (3.2) holds.

Using Lemma 3.3.1 we prove a theorem expressing $L$ in terms of order statistics.

**Theorem 3.3.2.**
1. $L = \bar{l}$ if and only if $Y_{[N-D-n+\bar{l}+1]} < X_{\bar{l}+1}$
2. For $\underline{l} < l < \bar{l}$, $L = l$ if and only if $X_{(l)} < Y_{[N-D-n+l]}$ and $X_{(l+1)} > Y_{[N-D-n+l+1]}$
3. $L = \underline{l}$ if and only if $X_{(\underline{l})} < Y_{[N-D-n+\underline{l}]}$

**Proof.** Suppose $\underline{l} < l < \bar{l}$. By Lemma 3.3.1

\[
\sum_{i=1}^{n} \sum_{j=1}^{D} \mathbb{I}(Z_{(i)} = X_{j}) \geq l \tag{3.3}
\]

if and only if $X_{(l)} < Y_{[N-D-n+l]}$ and

\[
\sum_{i=1}^{n} \sum_{j=1}^{D} \mathbb{I}(Z_{(i)} = X_{j}) < l + 1 \tag{3.4}
\]

if and only if $Y_{[N-D-n+l+1]} < X_{(l+1)}$ (since $Y_{[N-D-n+l+1]} \neq X_{(l+1)}$ by assumption). Combining (3.3) and (3.4) shows

\[
\sum_{i=1}^{n} \sum_{j=1}^{D} \mathbb{I}(Z_{(i)} = X_{j}) = l \tag{3.5}
\]

if and only if $X_{(l)} < Y_{[N-D-n+l]}$ and $Y_{[N-D-n+l+1]} < X_{(l+1)}$. So by (3.5) and the definition of $L$, $L = \bar{l}$ if and only if $X_{(\bar{l})} < Y_{[N-D-n+\bar{l}]}$ and $X_{(\bar{l}+1)} > Y_{[N-D-n+\bar{l}+1]}$.

The $L = \underline{l}$ and $L = \bar{l}$ cases are similar. \qed

### 3.4 The Probability Mass Function of $L$

As seen in Section 3.3, $L$ has a non-central hypergeometric distribution which depends non-trivially on the functions $F$ and $G$. This gives rise to a general non-central hypergeometric distribution. Using Theorem 3.3.2, the probability mass function of $L$ can be computed.

**Proposition 3.4.1.** When $\underline{l} = 0$ we have

\[
P(L = \underline{l}) = \frac{(N-D)!}{(N-D-n)!(n-1)!} \int_{-\infty}^{\infty} [1 - G(s)]^D F^{n-1}(s) \times [1 - F(s)]^{N-D-n} f(s) ds, \tag{3.6}
\]
and when \( l = n + D - N \),
\[
\mathbb{P}(L = l) = \frac{D!}{(n + D - N)!(N - n - 1)!} \int_{-\infty}^{\infty} G_{n+D-N}(s) \times [1 - G(s)]^{N-n-1} g(s) F^{N-D}(s) \, ds.
\]  
(3.7)

For the case in which \( l < l < \bar{l} \),
\[
\mathbb{P}(L = l) = \frac{D!(N - D)!}{l!(n - l)!(D - l - 1)!(N - D - n + l)!} \left( \int_{-\infty}^{\infty} G_{l}(s) \times [1 - G(s)]^{D-l-1} g(s) F^{n-l}(s) [1 - F(s)]^{N-D-n+l} \, ds + \frac{N - D - n + l}{D - l} \int_{-\infty}^{\infty} G_{l}(s) [1 - G(s)]^{D-l} F^{n-l}(s) \times [1 - F(s)]^{N-D-n+l-1} f(s) \, ds \right).
\]  
(3.8)

Finally, if \( \bar{l} = D \),
\[
\mathbb{P}(L = \bar{l}) = \frac{(N - D)!}{(n - 1)!(n - D)!} \int_{-\infty}^{\infty} G_{n-1}(s) F^{n-D}(s) \times [1 - F(s)]^{N-n-1} f(s) \, ds,
\]  
(3.9)

and if \( \bar{l} = n \),
\[
\mathbb{P}(L = \bar{l}) = \frac{D!}{(n - 1)!(D - n)!} \int_{-\infty}^{\infty} G_{n-1}(s) [1 - G(s)]^{D-n} g(s) \times [1 - F(s)]^{N-D} \, ds.
\]  
(3.10)

**Proof.** We start by considering the case \( l = 0 \). Using Theorem 3.3.2, (2.2) and (2.4) we have
\[
\mathbb{P}(L = \bar{l}) = \mathbb{P}(X_{(1)} \geq Y[N-D-n+1])
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_{X_{(1)}}(x) f_{Y[N-D-n+1]}(y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D [1 - G(x)]^{D-1} g(x) f_{Y[N-D-n+1]}(y) \, dx \, dy
\]
\[
= \int_{-\infty}^{\infty} [1 - G(y)]^{D} f_{Y[N-D-n+1]}(y) \, dy
\]
\[
= \frac{(N - D)!}{(N - D - n)!(n - 1)!} \int_{-\infty}^{\infty} [1 - G(s)]^{D} F^{n-1}(s) \times [1 - F(s)]^{N-D-n} f(s) \, ds.
\]
Similarly, when \( l = n + D - N \),

\[
\mathbb{P}(L = l) = \mathbb{P}(X_{n+D-N+1} > Y_1)
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{x} g_{X_{n+D-N+1}}(x) f_{Y_1}(y) dy dx
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{x} g_{X_{n+D-N+1}}(x)(N-D)F^{N-D-1}(y)f(y) dy dx
\]
\[
= \int_{-\infty}^{\infty} g_{X_{n+D-N+1}}(x)F^{N-D}(x) dx
\]
\[
= \frac{D!}{(n + D - N)!(n - n - 1)!} \int_{-\infty}^{\infty} G^{n+D-N}(s)(1 - G(s))^{N-n-1}
\]
\[
\times g(s)F^{n-D}(s) ds.
\]

We next consider the case \( l = D \). Using Theorem 3.3.2, (2.2) and (2.4) we obtain

\[
\mathbb{P}(L = l) = \mathbb{P}(Y_{N-n} > X_{D})
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{y} g_{X_{D}}(x) f_{Y_{N-n}}(y) dx dy
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{y} D G^{D-1}(x)g(x)f_{Y_{N-n}}(y) dx dy
\]
\[
= \int_{-\infty}^{\infty} G^{D}(y)f_{Y_{N-n}}(y) dy
\]
\[
= \frac{(N - D)!}{(N - n - 1)!(n - D)!} \int_{-\infty}^{\infty} G^{D}(s)F^{n-D}(s)(1-F(s))^{N-n-1}
\]
\[
\times f(s) ds.
\]

Similarly, when \( l = n \),

\[
\mathbb{P}(L = l) = \mathbb{P}(Y_{N-D} > X_{n})
\]
\[
= \int_{-\infty}^{\infty} \int_{x}^{\infty} g_{X_{n}}(x) f_{Y_{N-D}}(y) dy dx
\]
\[
= \int_{-\infty}^{\infty} \int_{x}^{\infty} g_{X_{n}}(x)(N-D)[1 - F(y)]^{N-D-1}g(y) dy dx
\]
\[
= \int_{-\infty}^{\infty} g_{X_{n}}(x)[1 - F(x)]^{N-D} dx
\]
\[
= \frac{D!}{(n - 1)!(D - n)!} \int_{-\infty}^{\infty} G^{n-1}(s)[1 - G(s)]^{D-n}g(s)
\]
\[
\times [1 - F(s)]^{N-D} ds.
\]
Finally, take $l < l < L$ and by Theorem 3.3.2 we have

$$
P(L = l) = P(Y_{N-D-n+l} > X_{l}, X_{l+1} > Y_{N-D-n+l+1})
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{\min\{y, x'\}} g(x_l, x_{l+1})(x, x')
\times f(Y_{N-D-n+l}, Y_{N-D-n+l+1})(y, y') \, dx \, dy \, dx'

= \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{\max\{y, x'\}} g(x_l, x_{l+1})(x, x')
\times f(Y_{N-D-n+l}, Y_{N-D-n+l+1})(y, y') \, dx \, dy \, dx'

+ \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{y} g(x_l, x_{l+1})(x, x')
\times f(Y_{N-D-n+l}, Y_{N-D-n+l+1})(y, y') \, dx \, dy \, dx'

+ \int_{-\infty}^{\infty} \int_{y}^{\infty} \int_{-\infty}^{y} g(x_l, x_{l+1})(x, x')
\times f(Y_{N-D-n+l}, Y_{N-D-n+l+1})(y, y') \, dx \, dy \, dx' \, dy.

(3.11)

To simplify notation, let

$$
I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{x'} g(x_l, x_{l+1})(x, x')
\times f(Y_{N-D-n+l}, Y_{N-D-n+l+1})(y, y') \, dx \, dy \, dx'

$$

and

$$
I_2 = \int_{-\infty}^{\infty} \int_{y}^{\infty} \int_{-\infty}^{y} g(x_l, x_{l+1})(x, x')
\times f(Y_{N-D-n+l}, Y_{N-D-n+l+1})(y, y') \, dx \, dy \, dx' \, dy,

$$

so (3.11) becomes

$$
P(L = l) = I_1 + I_2.

(3.12)
Recomputing $I_1$ using (2.3) and (2.5) gives

\[
I_1 = \frac{D!}{(l-1)!(D-l-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{y} \int_{-\infty}^{y'} G^{l-1}(x)[1 - G(x')]^{D-l-1}
\times g(x)g(x')f(y)[Y(N-D-n+l)Y(N-D-n+l+1)](y, y') \, dx \, dy \, dy' \, dx'
\]

\[
= \frac{D!}{l!(D-l)!} \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{y} \int_{-\infty}^{y'} G^{l}(x)[1 - G(x')]^{D-l-1}
\times g(x')f(y)[Y(N-D-n+l)Y(N-D-n+l+1)](y, y') \, dy \, dy' \, dx'
\]

\[
= \frac{D!(N-D)!}{l!(n-l)!(N-D-n+l)!} \int_{-\infty}^{\infty} \int_{-\infty}^{x'} \int_{-\infty}^{y} \int_{-\infty}^{y'} [1 - G(x')]^{D-l-1}
\times G^{l}(x')g(x')F^{n-l-1}(y')[1 - F(y)]^{N-D-n+l} f(y) \, dy \, dy' \, dx'
\]

\[
= \frac{D!(N-D)!}{l!(n-l)!(D-l)!(N-D-n+l-1)!} \int_{-\infty}^{\infty} G^{l}(x)[1 - G(x')]^{D-l-1}
\times g(x')F^{n-l}(x)[1 - F(x')]^{N-D-n+l} \, dx.
\] (3.13)

Recomputing $I_2$ using (2.3) and (2.5) gives

\[
I_2 = \frac{D!}{(l-1)!(D-l-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y} \int_{-\infty}^{y} \int_{-\infty}^{y} G^{l-1}(x)[1 - G(x')]^{D-l-1}
\times g(x)g(x')f(y)[Y(N-D-n+l)Y(N-D-n+l+1)](y, y') \, dx \, dy' \, dx' \, dy
\]

\[
= \frac{D!}{l!(D-l)!} \int_{-\infty}^{\infty} G^{l}(y)[1 - G(x')]^{D-l-1} g(x')
\times f(y)[Y(N-D-n+l)Y(N-D-n+l+1)](y, y') \, dy' \, dx' \, dy
\]

\[
= \frac{D!(N-D)!}{l!(n-l)!(N-D-n+l-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y} \int_{-\infty}^{y} \int_{-\infty}^{y} [1 - G(x')]^{D-l-1}
\times G^{l}(y)g(x')F^{n-l-1}(y')[1 - F(y)]^{N-D-n+l-1} f(y) \, dy' \, dy' \, dx'
\]

\[
= \frac{D!(N-D)!}{l!(n-l)!(D-l)!(N-D-n+l-1)!} \int_{-\infty}^{\infty} G^{l}(y)
\times [1 - G(x')]^{D-l} g(x')F^{n-l}(y)[1 - F(y)]^{N-D-n+l-1} f(y) \, dx' \, dy
\]

\[
= \frac{D!(N-D)!}{l!(n-l)!(D-l)!(N-D-n+l-1)!} \int_{-\infty}^{\infty} G^{l}(y)[1 - G(y)]^{D-l}
\times F^{n-l}(y)[1 - F(y)]^{N-D-n+l-1} f(y) \, dy.
\] (3.14)
Combining (3.12), (3.13), (3.14) we have
\[
\mathbb{P}(L = l) = \frac{D!(N-D)!}{l!(n-l)!(D-l-1)!(N-D-n+l)!} \left( \int_{-\infty}^{\infty} G^l(s) \times [1 - G(s)]^{D-l-1} g(s) F^{n-l}(s) [1 - F(s)]^{N-D-n+l} ds \right)
\]
\[
+ \frac{N - D - n + l}{D - l} \int_{-\infty}^{\infty} G^l(s) [1 - G(s)]^{D-l} F^{n-l}(s) \times [1 - F(s)]^{N-D-n+l-1} f(s) ds.
\]

3.5 Robustness with Respect to \( F \) and \( G \)

3.5.1 The Standard Hypergeometric Distribution

When \( F = G \), \( L \) has a standard hypergeometric distribution. To determine \( L \) we order the random variables from the set \( Z \), sample the \( n \) smallest random variables and count the number of random variables from the set \( X \) in the sample. When \( F = G \) this is equivalent to sampling \( n \) balls from an urn of \( N \) balls containing \( D \) marked balls without bias. This combinatorial argument leads us to the following theorem.

**Theorem 3.5.1.** When \( F = G \), \( L \) has a standard hypergeometric distribution. In particular,

\[ L \overset{d}{=} \text{Hg}(n, D, N). \quad (3.15) \]

This theorem may alternatively be proved by substituting \( F = G \) into the probability mass function derived in Section 3.4.

**Proof.** To prove Theorem 3.5.1, we start with the probability mass function of \( L \) and impose the requirement that \( F = G \). The resulting expression is then rewritten in terms the Beta function. Finally, everything is written in terms of factorials using the relationship between the Beta and Gamma function (refer to Abramowitz and Stegun [1]) and simplified. Take \( l < l \ll l \) and compute the integral terms in (3.8). We write

\[
\mathbb{P}(L = l) = I_1 + I_2, \quad (3.16)
\]

where \( I_1 \) and \( I_2 \) are defined in (3.13) and (3.14). Using \( F = G \), \( I_1 \) becomes

\[
I_1 = \frac{D!(N-D)!}{l!(n-l)!(D-l-1)!(N-D-n+l)!} \int_{-\infty}^{\infty} G^n(s) \times [1 - G(s)]^{N-n-1} g(s) ds. \quad (3.17)
\]
Making the change of variables \( t = G(s) \) gives
\[
I_1 = \frac{D!(N - D)!}{l!(n - l)!(D - l - 1)!(N - D - n + l)!} \int_0^1 t^n(1 - t)^{N-n-1} \, dt
\]
\[
= \frac{D!(N - D)!}{l!(n - l)!(D - l - 1)!(N - D - n + l)!} \text{Beta}(n + 1, N - n)
\]
\[
= \frac{D!(N - D)!}{l!(n - l)!(D - l - 1)!(N - D - n + l)!} \frac{n!(N - n - 1)!}{N!}
\]
\[
= \frac{D!n!(N - D)!(N - n - 1)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!}.
\]
(3.18)

Using \( F = G \), \( I_2 \) becomes
\[
I_2 = \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!} \int_{-\infty}^{\infty} F^n(s) \, ds
\]
\[
\times [1 - F(s)]^{N-n-1} f(s) \, ds.
\]
Making the change of variables \( t = F(s) \) we obtain
\[
I_2 = \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!} \int_0^1 t^n(1 - t)^{N-n-1} \, dt
\]
\[
= \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!} \text{Beta}(n + 1, N - n)
\]
\[
= \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!} \frac{n!(N - n - 1)!}{N!}
\]
\[
= \frac{D!n!(N - D)!(N - n - 1)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!}.
\]
(3.20)

Finally, substituting (3.18) and (3.20) into (3.16) gives
\[
\mathbb{P}(L = l) = \frac{D!n!(N - D)!(N - n - 1)!(D - l) + (N - D - n + l)]}{l!(n - l)!(D - l)!(N - D - n + l)!} \frac{D!n!(N - D)!(N - n)!}{l!(n - l)!(D - l)!(N - D - n + l)!}
\]
\[
= \frac{D!n!(N - D)!(N - n - 1)!(D - l) + (N - D - n + l)}{(N - D) - l} \frac{(N - D)}{N}
\]
\[
= \frac{(D)\binom{N-D}{n-l}}{(N)_l}.
\]
(3.21)

The \( L = \ell \) and \( L = \bar{\ell} \) cases are similar. Thus, \( L \) has the standard hypergeometric distribution when \( F = G \) and
\[
L \overset{d}{=} \text{Hg}(n, D, N).
\]
(3.22)
3.5. ROBUSTNESS WITH RESPECT TO F AND G

3.5.2 Wallenius’ Noncentral Hypergeometric Distribution

We also obtain Wallenius’ non-central hypergeometric distribution as a special case.

**Theorem 3.5.2.** When \( F = G^\omega \), for some \( \omega \in (0, \infty) \), \( L \) has Wallenius’ non-central hypergeometric distribution. In particular,

\[
L \overset{d}{=} \text{HgW}(n, D, N, \omega).
\]  

(3.23)

**Proof.** Suppose \( R \overset{d}{=} \text{HgW}(n, D, N, \omega) \). From (20) in Fog [55] the probability mass function of \( R \) may be written

\[
P(R = r) = \binom{D}{r} \binom{N-D}{n-r} d \int_0^1 (1 - t^{1/\omega})^r (1 - t)^{n-r} t^{d-1} dt,
\]  

(3.24)

where

\[
d = N - D - n + r + \omega(n - r)
\]  

(3.25)

and \( r \) lies in the range \( \{\max\{0, n + D - N\}, \ldots, \min\{n, D\}\} \).

In the context of an urn problem, the number of marked balls included in a sample also characterises the number of marked balls not included. The probability of sampling \( r \) marked balls of weight \( \omega \) in \( n \) draws is equal to the probability of sampling \( D - r \) marked balls of weight \( 1/\omega \) in \( N - n \) draws. Thus, the density function for \( R \) may be rewritten

\[
P(R = r) = \binom{D}{r} \binom{N-D}{n-r} d \int_0^1 (1 - t^{1/\omega})^{D-r} (1 - t)^{N-D-n+r} t^{d-1} dt
\]  

(3.26)

where

\[
d = n - r + r/\omega
\]  

(3.27)

and \( r \) lies in the range \( \max\{0, n + D - N\}, \ldots, \min\{n, D\} \).

We now prove that when \( F = G^\omega \), \( L \) has the same distribution as the random variable \( R \). Take \( l < l' \) and compute \( P(L = l) \). We write

\[
P(L = l) = I_1 + I_2,
\]  

(3.28)

where \( I_1 \) and \( I_2 \) are defined in (3.13) and (3.14).

Making the change of variables \( t = F(s) = G^\omega(s) \), (3.13) becomes

\[
I_1 = \frac{D!(N-D)!}{l!(n-l)!(D-l-1)!(N-D-n+l)!} \frac{1}{\omega} \int_0^1 t^{l/\omega} (1 - t^{1/\omega})^{D-l-1} 
\times t^{1/\omega-1} t^{n-l-1} (1 - t)^{N-D-n+l} dt
\]  

\[
= \binom{D}{l} \binom{N-D}{n-l} \frac{D-l}{\omega} \int_0^1 (1 - t^{1/\omega})^{D-l-1} (1 - t)^{N-D-n+l} 
\times t^{n-l+(l+1)/\omega-1} dt.
\]  

(3.29)
Similarly, making the change of variables \( t = G(s) = F^{\omega}(s) \) in (3.14) gives

\[
I_2 = \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!} \int_0^1 t^{l/\omega} (1 - t^{1/\omega})^{D - l} t^{n - l} \, dt
\]

\[
= \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l - 1)!} \int_0^1 (1 - t^{1/\omega})^{D - l} \times (1 - t)^{N - D - n + l - 1} t^{n - l + l/\omega} \, dt
\]

\[
= \frac{-D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l)!} \times [ (1 - t^{1/\omega})^{D - l} (1 - t)^{N - D - l + l/\omega}]_0^1
\]

\[
+ \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l)!} \left( n - l + \frac{l}{\omega} \right) \int_0^1 (1 - t^{1/\omega})^{D - l} (1 - t)^{N - D - n + l + l/\omega - 1} \, dt
\]

\[
- \frac{D!(N - D)!}{l!(n - l)!(D - l)!(N - D - n + l)!} \frac{D - l}{\omega} \int_0^1 (1 - t^{1/\omega})^{D - l - 1} (1 - t)^{N - D - n + l + l + (l + 1)/\omega - 1} \, dt
\]

\[
= \binom{D}{l} \binom{N - D}{n - l} \left( n - l + \frac{l}{\omega} \right)
\]

\[
\times \int_0^1 (1 - t^{1/\omega})^{D - l} (1 - t)^{N - D - n + l + l/\omega - 1} \, dt
\]

\[
- \binom{D}{l} \binom{N - D}{n - l} \frac{D - l}{\omega}
\]

\[
\times \int_0^1 (1 - t^{1/\omega})^{D - l - 1} (1 - t)^{N - D - n + l + l + (l + 1)/\omega - 1} \, dt. \quad (3.30)
\]

Substituting (3.29) and (3.30) into (3.28) gives

\[
P(L = l) = \binom{D}{l} \binom{N - D}{n - l} \left( n - l + \frac{l}{\omega} \right) \int_0^1 (1 - t^{1/\omega})^{D - l} (1 - t)^{N - D - n + l}
\]

\[
\times t^{n - l + l/\omega - 1} \, dt.
\]

When \( r = l \) in (3.27), \( d = n - l + l/\omega \), and so

\[
P(L = l) = \binom{D}{l} \binom{N - D}{n - l} d \int_0^1 (1 - t^{1/\omega})^{D - l} \times (1 - t)^{N - D - n + l + d - 1} \, dt, \quad (3.31)
\]
which is identical to (3.26) when \( r = l \) as required. The \( L = l \) and \( L = \bar{l} \) cases are similar.

Notice when \( F = G^\omega \), the resulting distribution is dependent on \( F \) and \( G \) only through a single parameter \( \omega \). This is a generalisation of the case \( F = G \) with \( \omega = 1 \).

As was the case with Theorem 3.5.1, the result presented in Theorem 3.5.2 can be explained intuitively in the context of an urn problem. Recall \( L \) is the number of random variables from the set \( X \) contained in the \( n \) smallest random variables from the set \( Z \). When \( F = G^\omega \), we have

\[
\mathbb{P}(X_j < Y_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} \omega g(x)G(y)^{\omega-1}g(y)\,dx\,dy
= \int_{-\infty}^{\infty} \omega G(y)^{\omega}g(y)\,dy
= \frac{\omega}{\omega + 1}
\]

and

\[
\mathbb{P}(Y_i < X_j) = 1 - \frac{\omega}{\omega + 1} = \frac{1}{\omega + 1}. \tag{3.32}
\]

Thus, this is equivalent to a Wallenius scheme in which \( n \) marked balls are sampled from an urn containing \( D \) marked balls of weight \( \omega \) and \( N - D \) unmarked balls of weight 1.

### 3.6 A General Noncentral Hypergeometric Distribution

From the results presented in Theorem 3.5.1 and Theorem 3.5.2, one might anticipate our general non-central hypergeometric distribution does not depend directly on both \( F \) and \( G \). We show in the following theorem that, in fact, it depends only on \( G \circ F^{(-1)} \).

**Theorem 3.6.1.** The distribution of \( L \) is parameterised by \( n \), \( D \), \( N \) and \( G \circ F^{(-1)} \).

**Proof.** We show for general \( F \) and \( G \), the resulting non-central hypergeometric distribution is parameterised by \( n \), \( D \), \( N \) and \( G \circ F^{(-1)} \). For \( \frac{1}{2} < t < 1 \), we have

\[
\mathbb{P}(L = t) = I_1 + I_2, \tag{3.33}
\]
where \( I_1 \) and \( I_2 \) are defined in (3.13) and (3.14). Starting from (3.13) and making the change of variables \( t = G(s) \) gives

\[
I_1 = k_1 \int_{-\infty}^{\infty} g_l(s)[1 - G(s)]^{D-l-1} [F(G^{-1}(G(s)))]^{n-l} \times [1 - F(G^{-1}(G(s)))]^{N-D-n+l} ds \\
= k_1 \int_{0}^{1} t^l (1 - t)^{D-l-1} [F(G^{-1}(t))]^{n-l} [1 - F(G^{-1}(t))]^{N-D-n+l} dt \\
= k_1 \int_{0}^{1} t^l (1 - t)^{D-l-1} [(G \circ F^{-1})^{-1}(t)]^{n-l} \times [1 - (G \circ F^{-1})^{-1}(t)]^{N-D-n+l} dt,
\]

where

\[
k_1 = \frac{D!(N-D)!}{l!(n-l)!(D-l-1)!(N-D-n+l)!}.
\]

Similarly, making the change of variables \( t = F(s) \) in (3.14) we have

\[
I_2 = k_2 \int_{-\infty}^{\infty} [F(F^{-1}(F(s)))]^l [1 - G(F^{-1}(F(s)))]^{D-l} F^{n-l}(s) \times [1 - F(s)]^{N-D-n+l-1} f(s) ds \\
= k_2 \int_{0}^{1} [G(F^{-1}(t))]^l [1 - G(F^{-1}(t))]^{D-l} t^{n-l}(1 - t)^{N-D-n+l-1} dt \\
= k_2 \int_{0}^{1} [(G \circ F^{-1})(t)]^l [1 - (G \circ F^{-1})(t)]^{D-l} t^{n-l} \times (1 - t)^{N-D-n+l-1} dt,
\]

where

\[
k_2 = \frac{D!(N-D)!}{l!(n-l)!(D-l-1)!(N-D-n+l)!}.
\]

From (3.34) and (3.36) it can be seen that in this case the distribution of \( L \) depends only on \( n, D, N \) and \( G \circ F^{-1} \) as required. The \( L = l \) and \( L = \overline{l} \) cases are similar.

Theorem 3.6.1 tells us any choices of \( F \) and \( G \) that give rise to the same function \( G \circ F^{-1} \) will produce the same general non–central hypergeometric distribution. That is, \( G \circ F^{-1} \) is essentially a weighting function that encodes the sampling bias of the distribution. This leads us to the following definition.

**Definition 3.6.2.** Suppose a random variable \( X \) has the same distribution as the random variable \( L \). Then we say that \( X \) has a general non–central hypergeometric distribution and write \( X \overset{d}{=} \text{HgG}(n, D, N, G \circ F^{-1}) \), where HgG stands for general hypergeometric.
3.7. THE EQUILIBRIUM QUANTITY TRADED

We now provide an explicit example in which our non–central hypergeometric distribution has a distribution other than the standard hypergeometric distribution and Wallenius’ non–central hypergeometric distribution.

**Example 3.6.3.** Consider \( X \overset{d}{=} \text{HgG}(n, D, N, G \circ F^{-1}) \), where \( F(x) = x \) for \( x \in [0, 1] \) and \( G(x) = 1 - e^{-\lambda x} \) with \( \lambda > 0 \) for \( x \in [0, \infty) \). Using the results from the proof of Theorem 3.6.1, for \( l < l < \overline{l} \) we have

\[
P(X = l) = I_1 + I_2,
\]

where

\[
I_1 = k \int_0^1 t^l (1 - t)^{D-l-1} \log^{n-l}(1-t)^{-1/\lambda} \times [1 - \log(1-t)^{-1/\lambda}]^{N-D-n+l} \, dt
\]

and

\[
I_2 = \frac{k(N - D - n + l)}{(D-l)} \int_0^1 (1 - e^{-\lambda t}) e^{-\lambda(D-l)} t^{n-l}(1-t)^{N-D-n+l-1} \, dt,
\]

with

\[
k = \frac{D!(N - D)!}{l!(n-l)!(D-l-1)!(N - D - n + l)!}.
\]

It is clear \( X \) does not have a standard hypergeometric distribution. Applying the binomial theorem to (3.31) in the proof of Theorem 3.5.2, shows \( P(X = l) \) may be written as a sum of Beta functions if \( X \) has Wallenius’ non–central hypergeometric distribution. If we apply the Binomial theorem to (3.40), it may be written as a sum of Kummer’s confluent hypergeometric functions (refer to Abramowitz and Stegun [1]), while (3.39) cannot be written nicely as a sum of well-known functions. Thus, \( X \) does not have Wallenius’ non–central hypergeometric distribution either.

Aside from complete enumeration of all probabilities, there are no exact formulas for the mean and variance of Wallenius’ non–central hypergeometric distribution (see Fog [55]). The general non–central hypergeometric distribution shares this property. However, given a set of parameter values, the mean and variance may be numerically computed using the probability mass function given in Section 3.4. Refer to Table 3.1 for some example calculations.

### 3.7 The Equilibrium Quantity Traded

In this section we show that our general non–central hypergeometric distribution arises naturally in the study of markets. In particular, the equilibrium quantity traded has a general non–central hypergeometric distribution under a variety of market mechanisms.
3.7.1 The \( \langle n, m, F, G \rangle \) Setup with Unit Traders

We consider a canonical economic model for a market in which a homogeneous, indivisible good is traded among \( n \) buyers and \( m \) sellers. Each buyer demands at most one unit of the good and sellers have the capacity to produce at most one unit of the good. Assume buyer valuations \( V_1, \ldots, V_n \) are independent and identically distributed absolutely continuous random variables drawn from the distribution \( F \) with density \( f \). Assume seller costs \( C_1, \ldots, C_m \) are independent and identically distributed absolutely continuous random variables drawn from the distribution \( G \) with density \( g \). Furthermore, assume buyer valuations and sellers costs are independent of one another.

This unitary \( \langle n, m, F, G \rangle \) trade setup is a special case of the independent private values model discussed in Section 1.2.6. It has been studied extensively in the two-sided mechanism design literature, for example, by Gresik and Satterthwaite [68], Williams [142], Baliga and Vohra [10] and Muir [114], Loertscher and Marx [90]. The assumption of independently distributed types is standard but not without loss of generality; see Myerson [116], Crémer and McLean [38, 37] and McAfee and Reny [105], as well as other references in Section 1.3.1.

In our setup, we order sellers from lowest cost to highest cost. The lowest cost sellers are more efficient and should trade more often in the market than higher cost sellers. We let \( C_{(j)} \) denote the \( j \)th most efficient seller. Buyers are ordered from highest valuation to lowest valuation. Higher valuation buyers are more efficient and should trade more often than lowest valuation buyers. We let \( V_{[i]} \) denote the \( i \)th most efficient buyer.
3.7. THE EQUILIBRIUM QUANTITY TRADED

3.7.2 Ex–post Efficiency

Recall that in the context of a market, social welfare is given by the sum of trading buyer valuations less the sum of trading seller costs and can be thought of as the total gain from trade experienced by market participants. Recall also that market is said to be ex–post efficient if the welfare of market participants is always maximised. The quantity traded in an ex–post efficient market is known as the \textit{ex–post efficient quantity}. For the \(\langle n,m,F,G \rangle\) trade setup, we now provide a mathematical definition of the ex–post efficient quantity.

\textbf{Definition 3.7.1.} The \textit{ex–post efficient quantity}, \(K^*\), is such that

\[ K^* = \arg \max_{k=0,1,\ldots,n \wedge m} \sum_{i=1}^{k} (V_i - C_{(i)}). \tag{3.42} \]

By virtue of the ordering of buyers and sellers, there is a characterisation of the ex–post efficient quantity which is more tractable. We have \(K^* = 0\) if and only if \(V_1 < C_{(1)}\), and \(K^* = \min\{n,m\}\) if and only if \(V_{n \wedge m} \geq C_{(n \wedge m)}\). In all other cases, \(K^*\) is the unique quantity such that

\[ V_{K^*} \geq C_{(K^*)} \quad \text{and} \quad V_{K^*+1} < C_{(K^*+1)}. \tag{3.43} \]

We now show the ex–post efficient quantity \(K^*\) is distributed according to our general non–central hypergeometric distribution.

\textbf{Lemma 3.7.2.} For any \(i \in \{1, \ldots, n \wedge m\}\), \(V_i > C_{(i)}\) if and only if the \(n\) lowest agent types contain at least \(i\) seller costs.

\textit{Proof.} The result directly follows from applying Lemma 3.7.2 with appropriate parameter values. \(\square\)

\textbf{Lemma 3.7.3.} \(K^*\) is given by the number of seller costs contained in the \(n\) lowest types.

\textit{Proof.} If \(\min\{n,m\}\) of the \(n\) lowest types are seller costs, clearly \(K^* = \min\{n,m\}\). Similarly, if none of the \(n\) lowest types are seller costs, \(K^* = 0\). Otherwise, \(K^*\) is the unique quantity satisfying (3.43). By Lemma 3.7.2, \(C_{(K^*)} \leq V_{K^*}\) implies at least \(K^*\) of the \(n\) lowest types are seller costs. Furthermore, \(C_{(K^*+1)} \leq V_{K^*+1}\) implies at least \(K^* + 1\) of the \(n\) lowest types are seller costs. Thus, having \(C_{(K^*)} \leq V_{K^*}\) and \(V_{K^*+1} < C_{(K^*+1)}\) requires that precisely \(K^*\) of the \(n\) lowest types are seller costs. \(\square\)

This leads us to the following theorem.

\textbf{Theorem 3.7.4.} \(K^*\) has the general non–central hypergeometric distribution. Specifically,

\[ K^* \overset{d}{=} \text{HgG}(n, m, n + m, G \circ F^{(-1)}). \tag{3.44} \]
The result follows directly from Lemma 3.7.3 and Definition 3.6.2.

Theorems 3.5.1 and 3.7.4 imply the ex–post efficient quantity has the standard hypergeometric distribution when \( F = G \).

**Corollary 3.7.5.** When \( F = G \),

\[
K^* \overset{d}= Hg(n, m, n + m).
\]  

(3.45)

Corollary 3.7.5 states that, when \( F = G \), the distribution of \( K^* \) does not depend on the distribution from which types are drawn. Using the mechanism design literature terminology introduced in Section 1.2, one would say that \( K^* \) has a prior–free distribution in this case.

### 3.7.3 Multi–Unit Traders

A natural extension of the unitary \( \langle n, m \rangle \) setup is to consider buyers with multi–unit demand and sellers with multi–unit capacities. Specifically, we now assume each buyer demands \( \nu \) units of the homogeneous good and each seller has the capacity to produce \( \upsilon \) units. Since the good is indivisible, \( \nu \) and \( \upsilon \) must be positive integers. Assume buyer valuations \( V_1^i, \ldots, V_\nu^i \) for each unit of the good are independent and identically distributed with distribution \( F \). Similarly, assume seller costs \( C_1^j, \ldots, C_\upsilon^j \) for each unit of the good are independent and identically distributed with distribution \( G \). If buyers are allowed to trade up to \( \nu \) units and sellers are allowed to trade up to \( \upsilon \) units, this multi–unit setup can simply be considered as a unitary \( \langle \nu n, \upsilon m \rangle \) setup. This multi–unit setup is studied, for example, by Loertscher and Mezzetti [92].

By Theorem 3.7.4,

\[
K^* \overset{d}= HgG(\nu n, \upsilon m, \nu n + \upsilon m, G \circ F^{(-1)})
\]  

(3.46)

Assume next that sellers have unit capacities, which are independent and identically distributed with distribution \( G \). Buyers have multi–unit demands as just described but are restricted to buy one unit at most (for example, because of government imposed rationing). In this case, a buyer’s valuation for one unit of the good will be given by

\[
V_i = \max\{V_1^i, \ldots, V_\nu^i\}.
\]  

(3.47)

Thus, we may consider this situation as an \( \langle n, m \rangle \) setup in which the distribution of buyer valuations is given by \( F^\nu \). Finally, if \( F = G \), we obtain a prior–free distribution for the ex–post efficient quantity.

**Corollary 3.7.6.** Suppose the distribution of buyer valuations is given by \( F^\nu \) and \( G = F \), then

\[
K^* \overset{d}= HgW(n, m, n + m, \nu).
\]  

(3.48)

Proof. The result follows directly from Theorems 3.5.2 and 3.7.4.

\[\square\]
3.7.4 The Equilibrium Quantity Traded

We now return to our original setup and assume there exists an intermediary who accepts bids from buyers and sellers and facilitates trade by selecting the market mechanism. We assume the intermediary selects a mechanism such that the quantity traded \( K \) is such that

\[
K = 0 \text{ if } B(V_{[1]}) < S(C_{(1)}),
\]

\[
K = \min\{n, m\} \text{ if } B(V_{[n \wedge m]}) \geq S(C_{(n \wedge m)}),
\]

where \( B \) and \( S \) are non-decreasing functions. Otherwise, \( K \) is such that

\[
B(V_{[K]}) \geq S(C_{(K)}) \quad \text{and} \quad B(V_{[K+1]}) < S(C_{(K+1)}).
\]

(3.49)

In other words, we are interested in mechanisms that induce the ex-post efficient quantity traded with respect to non-decreasing transformations of the true demand and supply schedules. The functions \( B \) and \( S \) depend on the objective of the intermediary and the equilibrium bidding strategies of buyers and sellers. We immediately have the following proposition, which can be proven in the same manner as Theorem 3.7.4.

**Proposition 3.7.7.** When the quantity traded in the economy \( \langle n, m, F, G \rangle \) is characterised by (3.49), we have

\[
K = H_{G}(n, m, n + m, (G \circ S^{-1}) \circ (F \circ B^{-1})^{-1}).
\]

We next show that this setup subsumes the cases of Bayesian optimal mechanisms, \( k \)-double auctions and first-price and second-price auction.

3.7.5 Bayesian Optimal Mechanisms

Recall that many mechanism design papers (see, for example, Myerson and Satterthwaite [117] and references in Section 1.3.1) consider the class of Bayesian optimal mechanisms known as \( \alpha \)-mechanisms, which maximise a convex combination of market welfare, \( W \), and profit \( R \). More precisely, these mechanisms maximise the objective \((1 - \alpha)W + \alpha R\), where \( \alpha \in [0, 1] \) is a parameter. Under such a mechanism, we call the quantity traded \( K_{\alpha} \) the \( \alpha \)-quantity traded.

Suppose we have a design problem that is regular in the sense of Myerson [116]. Then by the analysis presented in Section 1.2.6, the quantity traded under an \( \alpha \)-mechanism is characterised by (3.49) with

\[
B(v) = \Phi_{\alpha}(v) = v - \alpha \frac{1 - F(v)}{f(v)} \quad \text{and} \quad S(c) = \Gamma_{\alpha}(c) = c + \alpha \frac{G(c)}{g(c)},
\]

where the function \( \Phi_{\alpha} \) and \( \Gamma_{\alpha} \) are the weighted virtual type functions. It immediately follows from Proposition 3.7.7 that \( K_{\alpha} \) has a general non-central hypergeometric distribution. Furthermore, it immediately follows that the quantity traded under any constrained efficient mechanism has a general non-central hypergeometric distribution, since the equivalence between the class
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of constrained efficient mechanisms and the class of \( \alpha \)-mechanisms was also established in Section 1.2.6. In the special case \( \alpha = 1 \), we obtain the profit-maximising mechanism and we have

\[
B(v) = \Phi(v) = v - \frac{1 - F(v)}{f(v)} \quad \text{and} \quad S(c) = \Gamma(c) = c + \frac{G(c)}{g(c)},
\]

where \( \Phi \) and \( \Gamma \) are the virtual type functions.

3.7.6 \( k \)-Double Auctions

Under a \( k \)-double auctions (see, for example, Chatterjee and Samuelson [29] and Satterthwaite and Williams [132]), the equilibrium quantity traded also has a general non–central hypergeometric distribution. In this case, the functions \( B \) and \( S \) are the respective symmetric equilibrium bidding functions of buyers and sellers. The auctioneer sets a market-clearing price \( p(\hat{v}, \hat{c}) \), which is a function of the respective bids \( \hat{v} \) and \( \hat{c} \) submitted by buyers and seller. The market clearing price lies between the \( m \)th and the \((m + 1)\)th highest bid.

3.7.7 First–Price and Second–Price Auctions

We now assume there are \( J \geq 1 \) units for sell, each unit \( j \) costing \( c_j \) to the seller, with \( c_j \geq c_{j+1} \) for all \( j \). The buyer’s side of the market is unchanged. This is a standard auction theory setup (see, for example, Segal [134]). If the units are sold using a second–price auction format, the quantity traded still has a general non–central hypergeometric distribution. However, the joint distribution of the the seller costs is degenerate. Assuming sellers do not set reserve prices we have

\[
\forall j \in \{1, \ldots, J - 1\}, \quad \mathbb{P}(K = j) = \frac{n!}{j!(n-j)!} F^{n-j} (c_{j+1}) (1 - F(c_j))^j,
\]

(3.50)

with \( \mathbb{P}(K = 0) = F^n(c_1) \) and \( \mathbb{P}(K = J) = (1 - F(c_J))^J \). In the standard multi–unit homogeneous good setup, where \( c_j = c \) for all \( j \), the distribution of the quantity traded simplifies to a binomial distribution.

The quantity traded under the second–price mechanism with optimal reserve prices also has a non–central hypergeometric distribution. The distribution of \( K \) is characterised by (3.50) with \( c_j \) replaced by the optimal reserve price \( \Phi^{-1}(c_j) \) for all \( j \), where \( \Phi \) denotes the virtual valuation function.

In the case of a first–price auction, the symmetric equilibrium bidding function is given by

\[
B(v) = v - \frac{\int_v^w F^{n-1}(t) \, dt}{F^{n-1}(v)}
\]

and the distribution of \( K \) is characterised by (3.50) with \( c_j \) replaced by \( B^{-1}(c_j) \) for all \( j \).
3.8 Conclusion

In this chapter we constructed a general non-central hypergeometric distribution from the combined order statistics of two independent and identically distributed samples of random variables with absolutely continuous distribution functions $F$ and $G$. Our non-central hypergeometric distributions model biased sampling without replacement.

It was shown that the standard hypergeometric distribution arises as a special case when $F = G$. We obtain Wallenius’ non-central hypergeometric distribution with weight parameter $\omega$ when $F = G^\omega$. These results are robust in the sense that we obtain distributions independent of $F$ and $G$. This motivated us to prove any choices of $F$ and $G$ which give rise to the same function $G \circ F^{(-1)}$ produce the same non-central hypergeometric distribution. Thus, our non-central hypergeometric distribution depends on $F$ and $G$ only through $G \circ F^{(-1)}$.

Finally, it was shown the general non-central hypergeometric distribution models the distribution of the equilibrium quantity traded in economic markets. Specifically, in the unitary $\langle n, m, F, G \rangle$ setup from Bayesian mechanism design, the ex-post quantity traded is distributed according to our non-central hypergeometric distribution. Furthermore, the equilibrium quantity traded under $\alpha$-mechanisms, $k$-double auctions and first- and second-price auctions has a general non-central hypergeometric distribution. An extension of this model in which buyers have multi-unit demand provided an application of our result regarding Wallenius’ non-central hypergeometric distribution.

Future work could involve developing efficient numerical techniques for computing the moments of the general non-central hypergeometric distribution. Furthermore, there is scope to investigate asymptotic properties of the distribution. Our non-central hypergeometric distribution is closely related to the two-sample problem in statistics, which considers comparing the order statistics of two distributions. There exists a body of literature deriving asymptotic results for this problem (see Kochar [80]). Moreover, there is also scope to apply our general non-central hypergeometric distribution to additional economic applications as well as some of the biological applications for Wallenius’ and Fisher’s non-central hypergeometric distributions. For some of these biological applications, it may be useful to create a multivariate version of our non-central hypergeometric distribution, as Chesson [30] did for Wallenius’ non-central hypergeometric distribution.
Chapter 4

Approximating the Equilibrium Quantity Traded and Welfare in Large Markets

*We investigate the asymptotic behaviour of the equilibrium quantity and welfare in large markets*

4.1 Introduction

The problem of how to achieve efficient market outcomes has been of fundamental importance in economics, dating back to the pioneering work of Léon Walras. A significant and sometimes insurmountable barrier to designing efficient markets is the information required to set prices that equate market demand and supply. In many practical applications this information is privately held by strategic buyers and sellers who will not freely reveal this information, given the impact this will have on market prices. For a market intermediary to elicit this information and implement the ex–post efficient outcome, agents with private information must be appropriately compensated. Thus, in many cases it is not possible to design a market which is ex–post efficient, correctly elicits private information from agents (is Bayesian incentive compatible and interim individually rational) and does not require an intermediary to subsidise trade (does not run an interim budget deficit). This result, known as the impossibility of ex–post efficient trade, was formally established in Section 1.2.7 and was first emphasised by Myerson and Satterthwaite [117].

In light of the Myerson–Satterthwaite impossibility theorem, a prominent strand of mechanism design literature focuses on designing markets that are ex–post efficient, Bayesian incentive compatible, interim individually rational and interim deficit-free asymptotically, as the number of buyers and sellers grows large (see, for example, McAfee [104] and Rustichini, Satterthwaite, and Williams [131]). This requires approximating mechanism outcomes (such as the equilibrium quantity traded and welfare) in large markets.
In this chapter we consider the canonical mechanism design market model introduced in Section 1.2, known as the independent private values model. We devise a general methodology for approximating mechanism outcomes in large markets under the independent private values assumption. To do this, we restrict attention to approximating the equilibrium quantity traded and welfare under the ex–post efficient mechanism. We show that, as both the number of buyers and number of sellers tend to infinity, the joint distribution of the equilibrium quantity traded and welfare is asymptotically normal, and give an upper bound for the approximation rate. This is accomplished by constructing, on a common probability space, an empirical quantile process representing market demand and an independent empirical quantile process representing supply together with two independent and appropriately weighted Brownian bridges approximating the above-mentioned quantile processes. The distribution of interest can then be approximated by that of a functional of a Gaussian process.

Several papers analyse the performance of mechanisms in large markets with independent private values, including Gresik and Satterthwaite [68], McAfee [104], Satterthwaite and Williams [133] and Rustichini, Satterthwaite, and Williams [131]. However, this literature has previously focused on computing the rate at which mechanism outcomes converge to efficiency. To the best of our knowledge, this manuscript is the first to compute higher order distributional approximations (of the central limit theorem-type) to mechanism outcomes. One advantage of this approach is that it enables the direct comparison of different mechanisms using the parameters of the approximating normal distributions. Furthermore, our approach immediately generalises to any mechanism which can be appropriately represented in terms of transformed empirical quantile functions, which covers a large class of mechanisms studied in the Bayesian mechanism design literature, as is discussed in detail in Section 4.3. The common probability space method also allows us to compute the covariance of mechanism outcomes, which is problematic if a rate of convergence approach is adopted and is important in some settings (see, for example, p. 447 of McAfee [104]). Finally, unlike papers (such as Gresik and Satterthwaite [68]) which consider sequences of markets with a fixed ratio of buyers and sellers, we formulate more general convergence results that apply to nets of markets in which this ratio is not necessarily fixed.

The remainder of this chapter is structured as follows. In the next section we introduce the problem setup and state our main results. In Section 4.3, we discuss extensions and applications of these results. Proofs are included in Section 4.4.
4.2 Setup and main results

We again consider the unitary $⟨n, m, F, G⟩$ setup presented in Chapter 3. We consider a market in which units of a homogeneous, indivisible good are traded among $n$ buyers and $m$ sellers. Each buyer is interested in purchasing one unit of the good, and each seller has the capacity to produce and sell one unit of the good. Buyers are willing to purchase the good at a price not exceeding their respective private reservation valuations $V_1, \ldots, V_n$ that are assumed to be independent and identically distributed random variables with a common distribution function $F$ with support $[v, \overline{v}]$. Sellers are willing to produce at a price that is not less than their respective private production costs $C_1, \ldots, C_m$ which are also assumed to be independent and identically distributed random variables, with a distribution function $G$ with support $[c, \overline{c}]$. We assume that buyers valuations and sellers costs are independent of each other and call the $(n+m)$-tuple $R_{(n,m)} := (V_1, \ldots, V_n; C_1, \ldots, C_m)$ a realisation of the market $\mathcal{M} := ⟨n, m, F, G⟩$.

This independent private values model has been studied extensively in the literature on mechanism design and auction theory, see, for example, Myerson [116], Milgrom and Weber [113], Chatterjee and Samuelson [29], Myerson and Satterthwaite [117], Gresik and Satterthwaite [68], Williams [142], Baliga and Vohra [10], Muir [114] and Loertscher and Marx [90]. The results of our analysis immediately generalise to settings in which sellers produce multiple units (and similarly, buyers demand multiple units), provided the cost of production for each unit is independently drawn from the distribution $G$. We may also relax the assumption of identical distributions. For example, we can suppose a fixed proportion of sellers draw their costs from some distribution $G_1$ and the rest draw their costs from some distribution $G_2$, provided we use the appropriate mixture distribution in our asymptotic analysis.

Within a market, the welfare generated by trade is defined as the sum of trading buyer valuations less the sum of trading seller costs:

$$\sum_{i \in \mathcal{N}^T} V_i - \sum_{j \in \mathcal{M}^T} C_j,$$

where $\mathcal{N}^T$ and $\mathcal{M}^T$ are, respectively, the subsets of buyers and sellers who trade in the market. Feasibility requires $|\mathcal{N}^T| \leq |\mathcal{M}^T|$. Recall that a market is said to be ex–post efficient if market welfare is always maximised for given buyer valuations and seller costs. The quantity traded in an ex–post efficient market is called the ex–post efficient quantity. To compute it, we form a demand curve by ordering buyer valuations $V_i$ from highest to lowest: $V_{[1]} \geq \cdots \geq V_{[n]}$, and we form a supply curve by ordering seller costs $C_j$ from lowest
Figure 4.1: The ex–post efficient quantity $K$ is given by the abscissa of the intersection of the plots of the buyer and seller order statistics, and the respective value of welfare $W$ is equal to the area of the shaded region.

to highest: $C_{(1)} \leq \cdots \leq C_{(m)}$. The ex–post efficient quantity is then given by

$$K := \arg \max_{0 \leq k \leq n \wedge m} \sum_{i=1}^{k} (V_{[i]} - C_{(i)}) \equiv \left\{ \{k \in \{1, \ldots, n \wedge m\} : V_{[k]} - C_{(k)} \geq 0 \} \right\}.$$  

(4.1)

In an ex–post efficient market, the value of welfare is given by

$$W := \sum_{i=1}^{K} (V_{[i]} - C_{(i)}).$$  

(4.2)

Figure 4.1 provides an illustration for the quantities $K$ and $W$. Note that it is not necessary to specify a pricing scheme in order to define the ex–post efficient quantity and welfare. However, the ex–post efficient outcome can be achieved if the market intermediary sets a price of $\max\{C_{(K)}, V_{[K+1]}\}$ for buyers and $\min\{C_{(K+1)}, V_{[K]}\}$ for sellers, where we set $V_{[n+1]} = \varnothing$ and $C_{(m+1)} = \varnothing$ for convenience (see Loertscher and Marx [89]). Furthermore, the “matching” of buyers and sellers that appears in (4.1) and (4.2) does not necessarily indicate that these agents trade directly with one another. Indeed, because of the homogeneous goods assumption, it does not matter which buyer trades with which seller. The “matching” in (4.1) and (4.2) is an algorithm for computing the ex–post efficient quantity traded and welfare.

Before stating the main results of this chapter, we must introduce some additional model assumptions. First of all, in our large market setup we will consider a family of markets of increasing size with the same distribu-
4.2. SETUP AND MAIN RESULTS

tion functions \( F \) and \( G \) that satisfy the following standard mechanism design assumption:

(A1) The distribution functions \( F \) and \( G \) are absolutely continuous, with respective densities \( f(x) \) and \( g(x) \) bounded and bounded away from zero on their respective supports \([u, v]\) and \([c, \overline{c}]\), such that \([u, v] \cap [c, \overline{c}] \neq \emptyset\) (in other words, \( v < c \) and \( c < v \)).

Given the distribution functions \( F \) and \( G \), it is natural and convenient to consider a net (also known as a Moore–Smith sequence) of markets

\[ \mathcal{M}_a := \langle a := (n, m), F, G \rangle, \quad a \in \mathcal{A}, \]

indexed by the directed set \( \mathcal{A} \) about which we will make the following assumption:

(A2) We assume that

\[ \mathcal{A} := \{ a = (n, m) \in \mathbb{N}^2 : \lambda_a := mn^{-1} \in I \}, \]

where \( I := [1 - F(\overline{\tau}) + \epsilon, 1/(G(v) + \epsilon)] \) for a fixed \( \epsilon > 0 \).

Here \( \epsilon \) is assumed to be small enough so that \( I \neq \emptyset \); note that \( F(\overline{\tau}) > 0 \) and \( G(v) < 1 \) by virtue of the assumption \( v < \overline{c} \), see (A1). Note also that \( \inf I > 0 \).

The set \( \mathcal{A} \) is endowed with the natural pre–order: for \( a = (n, m) \) and \( a' = (n', m') \), one has \( a \leq a' \) iff \( n \leq n' \) and \( m \leq m' \). We will be interested in the limiting distributions of the ex–post efficient quantities \( K_a \) and welfares \( W_a \) for the respective markets from the net \( \{ \mathcal{M}_a \}_{a \in \mathcal{A}} \).

The assumption \( \lambda_a \in I \) from (A2) excludes trivial cases. Indeed, with probability tending to one, for large \( a \) the number of \( v_i \)'s exceeding \( \overline{\tau} \) will be equal to \( (1 - F(\overline{\tau}) + o(1))m \). So if \( \lambda_a < 1 - F(\overline{\tau}) - \epsilon \) for a fixed \( \epsilon > 0 \), it would mean that the total number of sellers \( m = \lambda_a n \) would be less than the number of buyers with valuations higher than the maximum possible production cost \( \overline{\tau} \), meaning that all sellers trade and there is rationing on the demand side of the market. Likewise, the situation when \( \lambda_a > 1/G(v) \) corresponds to a market with excess supply (all buyers trade and there is rationing on the supply side of the market).

To state the main results, we need some further notation. We will frequently deal with scaled functions of the form \( h(\lambda^{-1}_a t) \). For convenience, for any \( h : [0, 1] \to \mathbb{R} \) we define

\[ \widehat{h}(s) := h(s \wedge 1), \quad s \geq 0. \]

Note that the function \( \widehat{h}(\lambda^{-1}_a t), t \in [0, 1], \) is well-defined for all \( \lambda_a > 0 \).

Using notation \( h(\cdot)^{-1} \) for the inverse of function \( h \) (to avoid confusion with the reciprocal \( h^{-1} \)), introduce the functions

\[ E_a(t) := F(\cdot)^{-1}(1 - t) - G(\cdot)^{-1}(\lambda^{-1}_a t), \quad t \in [0, 1], \quad (4.3) \]
Figure 4.2: The scaled empirical quantile functions, which are demand and supply schedules, may be approximated by the true scaled quantile functions. The errors associated with these approximations are of order $1/\sqrt{n}$ and are given by appropriately scaled Brownian bridges.

and put

$$\mathcal{H}(h) := \sup\{t \in (0,1) : h(t) \geq 0\}$$

(4.4)

(which is well-defined for any $h : [0,1] \to \mathbb{R}$ with $h(0) > 0$) and

$$t_a := \mathcal{H}(E_a) \in (0, \lambda_a \wedge 1),$$

(4.5)

where the right-hand relation holds due to the assumption on $I$ from (A2).

Due to the a.s. convergence of empirical quantile functions to the theoretical ones, for large markets the function $E_a(t)$ approximates the difference between the step-functions whose plots are depicted, respectively, by the dashed and solid lines in Figure 4.2 (see (4.13) below). Simulations illustrating this approximation are shown in Figure 4.3. So $t_a n$ will be the “first order approximation” to $K_a$, while the integral of $E_a$ over $(0, t_a)$ will specify a deterministic approximation to $W_a$. The “second order approximation” to both $K_a$ and $W_a$ will be obtained in this chapter using the second order approximation to empirical quantile functions provided by the sum of the theoretical quantile function and a Brownian bridge process.

Now observe that, provided that $f, g$ are continuous inside their respective supports, the function $E_a(t)$ is continuously differentiable for $t \in (0, 1) \setminus \{\lambda_a\}$, and

$$E_a'(t) = -\frac{1}{f(F^{-1}(1-t))} - \frac{1}{\lambda_a g(G^{-1}(\lambda_a^{-1} t))}, \quad t \in (0, \lambda_a \wedge 1).$$

(4.6)
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Figure 4.3: Three simulations illustrating the approximation of the scaled empirical quantile functions by the theoretical quantile functions when $F(x) = x$ and $G(x) = x^2$ for $x \in [0, 1]$. The code used to produce this figure can be found in Appendix A.1.1.

We will need one more technical assumption on the distribution functions $F$ and $G$:

(A3) The densities $f$ and $g$ are differentiable on $(a, b)$ and $(c, d)$, respectively. Moreover, the functions

$$
\frac{d}{dt} \frac{1}{f(F^{-1}(t))} = -\frac{f'(F^{-1}(t))}{f^3(F^{-1}(t))} \quad \text{and} \quad \frac{d}{dt} \frac{1}{g(G^{-1}(t))} = -\frac{g'(G^{-1}(t))}{g^3(G^{-1}(t))}
$$

are bounded on $(0, 1)$.

Finally, let

$$
\sigma_a^2 := \frac{1}{(E_a(t_a))^2} \left[ \frac{t_a(1 - t_a)}{f^2(F^{-1}(1 - t_a))} + \frac{t_a(1 - \lambda_{-1}^{-1} t_a)}{\lambda_{-1}^2 g^2(G^{-1}(\lambda_{-1}^{-1} t_a))} \right], \quad (4.7)
$$

and

$$
\varsigma_a^2 := 2 \int_0^{t_a} S_a(t) \, dt, \quad (4.8)
$$

where the function $S_a(t)$, $t \in (0, t_a)$, is given by

$$
\frac{1 - t}{f(F^{-1}(1 - t))} \int_{F^{-1}(1 - t)}^{\pi} (1 - F(x)) \, dx + \frac{1 - \lambda_{-1}^{-1} t}{g(G^{-1}(\lambda_{-1}^{-1} t))} \int_{\lambda_{-1}^{-1} t}^{G^{-1}(\lambda_{-1}^{-1} t)} G(x) \, dx.
$$

Our main result is the following strong approximation theorem. It implies that, under the above assumptions, the ex–post efficient quantity and welfare are asymptotically normal for large markets. More precisely, the univariate distributions of $K_a$ and $W_a$ can be approximated by normal distributions with
respective means and variances \((nt_a, n\sigma_a^2)\) and \((n \int_0^{t_a} E_a(t) \, dt, n\varsigma_a^2)\). Moreover, the joint distribution of \((K_a, W_a)\) is asymptotically normal as well. We also give upper bounds for the convergence rates.

**Theorem 4.2.1.** Under assumptions \((A1)-(A3)\), for the net of markets \(\{\mathcal{R}_a\}_{a \in \mathcal{A}}\) there exist a net \(\{\mathcal{R}_a\}_{a \in \mathcal{A}}\) of realizations of these markets on a common probability space together with a net of bivariate normal random vectors \(\{(Z_a(1), Z_a(2))\}_{a \in \mathcal{A}}\) with zero means and

\[
\begin{align*}
\text{Var}(Z_a(1)) &= \sigma_a^2, \quad \text{Var}(Z_a(2)) = \varsigma_a^2, \\
\text{Cov}(Z_a(1), Z_a(2)) &= \kappa_a := -S_a(t_a)/E'(t_a),
\end{align*}
\]

such that for the ex–post efficient quantity \(K_a\) one has

\[
\lim_{n \to \infty} \sup_a \frac{n^{1/4}}{(\ln n)^{1/2}} \left| \frac{K_a - nt_a}{n^{1/2}} - Z_a(1) \right| < \infty \quad \text{a.s.} \quad (4.9)
\]

and for welfare \(W_a\) one has

\[
\lim_{n \to \infty} \sup_a \frac{n^{1/2}}{\ln n} \left| \frac{1}{n^{1/2}} \left( W_a - n \int_0^{t_a} E_a(t) \, dt \right) - Z_a(2) \right| < \infty \quad \text{a.s.}
\]

The proof uses the common probability space method and is deferred to Section 4.4. The idea is to consider, for a given \(a = (n, m)\), the empirical quantile functions associated with the samples of buyer valuations \(V_1, \ldots, V_n\) and seller costs \(C_1, \ldots, C_m\). The ex–post efficient quantity and welfare, defined in (4.1) and (4.2) respectively, can then be expressed as linear functionals of the respective empirical quantile processes. By the results of Csörgő and Révész [41], it is possible to construct a suitable probability space carrying a sequence of empirical quantile processes that converge almost surely to an appropriately weighted Brownian bridge process. We use this result to approximate \(K_a\) and \(W_a\) by linear functionals of the respective limiting Gaussian process. To complete the proof, we derive and apply an appropriate generalisation of the delta method and compute bounds for the approximation rate.

As an illustration to the assertion of the theorem, Figure 4.4 shows the scatterplot of \(10^4\) independent realizations of the centered and scaled (as per the statement of Theorem 4.2.1) random vector \((K_a, W_a)\) in the case when \(a = (500, 250)\), \(F(x) = x\) and \(G(x) = x^2\) for \(x \in [0, 1]\), together with the 0.25, 0.5 and 0.95 ellipsoidal quantiles for the sample. Figure 4.5 displays these empirical ellipsoidal quantiles together with the respective (theoretical) ellipsoidal quantiles for the approximating distribution of \((Z_a(1), Z_a(2))\). Both plots were generated using MATHEMATICA 10 (refer to Appendix A.1.2 for code).

Recall that, for a bivariate distribution \(P\) with finite second order moments, the ellipsoidal quantiles are defined as follows. Let \(\mathbf{\mu}\) and \(\Sigma\) be the
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Figure 4.4: Scatterplot of $10^4$ independently simulated scaled copies of $(K_a, W_a)$ and its ellipsoidal quantiles (see the paragraph following Theorem 4.2.1 for more detail).

mean vector and covariance matrix of $P$, respectively. The ellipsoidal quantile of level $u \in (0, 1)$ is defined as the boundary $\partial A$ of the smallest set of the form

$$A := \{y \in \mathbb{R}^2 : (y - \mu)' \Sigma (y - \mu) \leq \text{const} \}$$

such that $P(A) \geq u$. Empirical ellipsoidal quantiles for a bivariate sample are defined as the ellipsoidal quantiles of the respective empirical distribution.

Remark 4.2.2. Note that $n \asymp m$ ($n$ and $m$ are asymptotically equivalent) under assumption (A2), so one could state the assertions of the theorem in a similar way using $m$ rather than $n$ as well. Note also that $\kappa_a > 0$ (as one would expect, of course) since $S_a(t_a) > 0$ and $E_a'(t_a) < 0$.

Remark 4.2.3. The true rate of convergence of the distribution of $K_a$ to the normal distribution is most likely $n^{-1/2}$, as indicated by the bounds (4.10) below established in the special case when $F = G$. So the crudeness of the bound in (4.9) seems to be due to the method of proof employed.

In the interesting special case $F = G$, the distribution of $K_a$ is independent of $F$. Moreover, it is known to have the standard hypergeometric distribution (see the proof of part (ii) below for a more precise statement) and the parameters of the approximating normal law for $K_a$ admit simple explicit
Figure 4.5: The empirical ellipsoidal quantiles from Figure 4.4 together with the ellipsoidal quantiles of the approximating bivariate normal distribution.

representations as functions of \( a \). In addition, in that case one can establish a better convergence rate to the normal distribution than that claimed in Theorem 4.2.1.

The distribution of \( W_a \) (and hence that of \((K_a, W_a)\)) and the approximating normal law do depend on \( F \). However, one can also obtain simple closed formulae for the parameters of the approximating normal laws when the common distribution \( F = G \) is uniform on \([\underline{v}, \overline{v}]\). We will only deal here with the case \( \underline{v} = 0, \overline{v} = 1 \), the results in the general case following in a straightforward way, and state our results in the form of the following theorem.

**Theorem 4.2.4.** If \( F = G \) then:

(i) one has

\[
 t_a = \frac{\lambda_a}{1 + \lambda_a}, \quad \sigma_a^2 = \frac{\lambda_a^2}{(1 + \lambda_a)^3};
\]

(ii) there exist constants \( C_1, C_2 \in (0, \infty) \) such that

\[
 \frac{C_1}{n^{1/2}} \leq \sup_x \left| P \left( \frac{K_a - nt_a}{\sigma_a n^{1/2}} \leq x \right) - \Phi(x) \right| \leq \frac{C_2}{n^{1/2}}, \quad (4.10)
\]

where \( \Phi \) is the standard normal distribution function.
(iii) If, moreover, \(F(t) = G(t) = t\) for \(t \in (0, 1)\), then
\[
\int_0^t E_a(t) \, dt = \frac{\lambda_a}{2(1 + \lambda_a)}, \quad \varsigma_a^2 = \frac{\lambda_a(1 + 3\lambda_a + \lambda_a^2)}{12(1 + \lambda_a)^3}, \quad \kappa_a = \frac{\lambda_a^2}{2(1 + \lambda_a)^3}.
\]

Remark 4.2.5. Observe that, in the case from part (iii), the correlation coefficient between the components of the approximating normal distribution can be easily found to be equal to \(\sqrt{3/(\lambda_a^{-1} + 3 + \lambda_a)}\). This quantity attains its maximum value \(\sqrt{3/5}\) at \(\lambda_a = 1\) (that is, when \(m = n\)) and vanishes as \(\max\{\lambda_a, \lambda_a^{-1}\} \to \infty\).

4.3 Applications and extensions

For ease of exposition, we restricted attention to the ex-post efficient mechanism in Section 4.2. However, our analysis generalises to a much richer class of mechanisms. Indeed, let \(B\) and \(S\) be non-decreasing real functions and consider the transformed sets of buyer valuations and seller costs given by \(\{B(V_1), \ldots, B(V_n)\}\) and \(\{S(C_1), \ldots, S(C_m)\}\), respectively. Then our analysis immediately applies to the mechanism that induces the ex-post efficient outcome with respect to these transformed valuations and costs, provided the assumptions (A1)–(A3) hold for the composite functions \(F \circ B^{(-1)}\) and \(G \circ S^{(-1)}\). This is illustrated in Figure 4.6. In Chapter 3 we showed that this setup subsumes a large class of mechanisms, where the functions \(B\) and \(S\) depend on the objective of the intermediary and the equilibrium bidding strategies of buyers and sellers. For example, our analysis immediately applies to profit-maximising and \(\alpha\)-mechanisms.

For some mechanisms, such as \(k\)-double auctions (refer to, for example, Chatterjee and Samuels [29] and Satterthwaite and Williams [132]), the strategic behaviour of agents depends on \(n\) and \(m\). More precisely, the strategic behaviour of a given agent refers to the magnitude of the difference between the type of that agent and the type that agent reports to the intermediary. Our analysis will only apply to such mechanisms if the strategic behaviour of all buyers and sellers vanishes uniformly on the support of the respective distribution functions \(F\) and \(G\) at a rate faster than \(n^{-1} \ln n\) and \(m^{-1} \ln m\), respectively. For example, by the results of Rustichini, Satterthwaite, and Williams [131], the \(k\)-double auction satisfies this condition.

4.4 Proofs of the main results

The proofs will use the common probability space method and transformed uniform empirical quantile functions, so we will be using the basic notation and key facts introduced in Section 2.3.
Costs & Values

Figure 4.6: Our analysis applies to any mechanism that induces the ex–post efficient outcome with respect to non-decreasing transformations of the empirical quantile functions associated with buyer valuations and seller costs.

4.4.1 Proof of Theorem 4.2.1

Proof. It is easy to see from (4.1), (4.4) and (4.5) that

\[ \delta_a := K_a n^{-1} - t_a = \mathcal{H}(E_a) - \mathcal{H}(E_a), \]  

where

\[ E_a(t) := V_n (1 - t) - C_m (\lambda^{-1} a t), \quad t \in [0,1], \]  

and \( V_n \) and \( C_m \) denote the empirical quantile functions for the samples of buyer valuations \( V_1, \ldots, V_n \) and seller costs \( C_1, \ldots, C_m \), respectively. Now we will analyse the behaviour of \( \delta_a \) for “large” \( a \) when market realisations \( R_a \) are defined using the common probability space construction (2.6).

To that end, we will assume that

\[ V_i = V_{i,n} := F^{(-1)}(U_{i,n}^V) \quad \text{and} \quad C_j = C_{j,m} := G^{(-1)}(U_{j,m}^C), \]  

where \( \{U_{1,n}^V, \ldots, U_{n,n}^V\}_{n \geq 1} \) and \( \{U_{1,m}^C, \ldots, U_{m,m}^C\}_{m \geq 1} \) are independent triangular arrays of row-wise independent \( U[0,1] \)-distributed random variables, which are defined on a common probability space with Brownian bridges \( \mathbb{B}^V \) and \( \mathbb{B}^C \) that are independent of each other so that, as \( n,m \to \infty \), for the respective empirical quantile functions one has

\[ U_{n}^V(t) = t + n^{-1/2} B^V(t) + \theta_n^V(t)n^{-1} \ln n, \quad t \in [0,1], \]

\[ U_{m}^C(t) = t + m^{-1/2} B^C(t) + \theta_m^C(t)m^{-1} \ln m, \quad t \in [0,1], \]  

(4.15)
where \( \| \theta_n^V \|_\infty = O(1) \) and \( \| \theta_m^C \|_\infty = O(1) \) a.s.

In this construction, the empirical quantile functions \( V_n \) and \( C_m \) in (4.13) are those for the samples in (4.14), that is they are given by the compositions

\[
V_n(t) := \left(F^{(-1)} \circ U^V_n \right)(t) \quad \text{and} \quad C_m(t) := \left(G^{(-1)} \circ U^C_m \right)(t).
\]

(4.16)

So to analyse the asymptotic distribution of \( K_a \), we will now turn to the asymptotic behaviour of \( E_a \) specified in (4.13) where the terms on the righthand side are given by (4.16).

Using (4.16), (4.15) and conditions \( (A1), (A3) \) to take Taylor series expansions with two terms and Lagrange’s form of the remainder gives, after elementary transformations,

\[
V_n(1-t) = F^{(-1)} \left(1 - t + n^{-1/2} B^V(1-t) + \theta_n^V(1-t)n^{-1}\ln n \right)
= F^{(-1)}(1-t) + \frac{n^{-1/2} B^V(1-t) + \theta_n^V(1-t)n^{-1}\ln n}{f(F^{(-1)}(1-t))}, \quad t \in (0,1),
\]

(4.17)

and, for \( t \in (0,1 \land \lambda_a) \),

\[
C_m(\lambda_a^{-1}t) = G^{(-1)}(\lambda_a^{-1}t + m^{-1/2} B^C(\lambda_a^{-1}t) + \theta_m^C(\lambda_a^{-1}t)m^{-1}\ln m)
= G^{(-1)}(\lambda_a^{-1}t) + \frac{m^{-1/2} B^C(\lambda_a^{-1}t) + \theta_m^C(\lambda_a^{-1}t)m^{-1}\ln m}{g(G^{(-1)}(\lambda_a^{-1}t))},
\]

(4.18)

where \( \| \theta_m^C \|_\infty + \| \theta_n^V \|_\infty = O(1) \) a.s.

Substituting (4.17) and (4.18) into the representation for \( E_a \) in (4.13) and using (4.3) gives

\[
E_a(t) = E_a(t) + n^{-1/2} Z_a(t) + \varphi_a(t)n^{-1}\ln n,
\]

(4.19)

where \( \| \varphi_a \| = O(1) \) a.s. and

\[
Z_a(t) := \frac{B^V(1-t)}{f(F^{(-1)}(1-t))} - \frac{B^C(\lambda_a^{-1}t)}{\lambda_a^{-1/2} g(G^{(-1)}(\lambda_a^{-1}t))}, \quad t \in (0,1).
\]

(4.20)

One can see from (4.12) and (4.19) that if the functional \( H \) were differentiable in a suitable sense at the “point” \( E_a \), one could derive the desired asymptotic normality of \( K_a \) using a suitable version of the functional delta method (see, for example, Ch. 20 of van der Vaart [139]). So we will now turn to studying the local properties of \( H \) at \( E_a \).

Since \( f \), \( g \) and \( \lambda_a \) are bounded, we see from (4.6) that

\[
-\gamma := \sup_{\alpha \in \vartheta} \sup_{t \in (0, \lambda_a \land 1)} E'_a(t) < 0.
\]
Figure 4.7: A graphical illustration of inequalities (4.22), (4.23). Recall that

\[ K_a n^{-1} = \mathcal{H}(E_a(t)) = t_a + \delta_a. \]

As \( E_a(t_a) = 0 \), the above implies that

\[ E_a(t) \geq \gamma(t_a - t), \quad 0 \leq t < t_a; \quad E_a(t) \leq \gamma(t_a - t), \quad t_a < t \leq \lambda_a \wedge 1. \]

Hence, for any \( v \in D[0,1] \), setting \( \eta := \|v\|_\infty \), one has

\[
\inf_{0 \leq t < t_a - \eta/\gamma} (E_a(t_a) + v(t)) > \gamma \eta/\gamma - \eta = 0,
\]

\[
\sup_{t_a + \eta/\gamma < t \leq \lambda_a \wedge 1} (E_a(t) + v(t)) < -\gamma \eta/\gamma + \eta = 0,
\]

so that the plot of \( E_a(t_a) + v(t) \) must “dive” under zero within \( \eta/\gamma \) of \( t_a \):

\[ |\mathcal{H}(E_a + v) - \mathcal{H}(E_a)| \equiv |\mathcal{H}(E_a + v) - t_a| \leq \eta/\gamma = \|v\|_\infty / \gamma. \]

Thus, the functional \( \mathcal{H} \) is Lipschitz continuous in the uniform topology at the point \( E_a \). Using this, (4.12) and (4.19), we have

\[ \delta_a = O(n^{-1/2}) \quad \text{a.s.} \tag{4.21} \]

Further, since \( E_a \) is right continuous, it follows from the definition of \( \mathcal{H} \) that

\[ E_a(t_a + \delta_a) \leq 0. \tag{4.22} \]

On the other hand, using the order statistics \( V_{[i-1],n} \) (descending) and \( C_{(i),m} \) (ascending) for our samples (4.14) (see the definitions just before (4.1)), we obtain (see Figure 4.7) for an illustration of the first inequality

\[
E_a(t_a + \delta_a) \geq - \max_{2 \leq i \leq K_a} \left[ (V_{[i-1],n} - V_{[i],n}) + (C_{(i),m} - C_{(i-1),m}) \right]
\]

\[
\geq - \max_{2 \leq i \leq n} (V_{[i-1],n} - V_{[i],n}) - \max_{2 \leq i \leq m} (C_{(i),m} - C_{(i-1),m}). \tag{4.23} \]
From a well-known result regarding maximal uniform spacings (see, for example, Devroye [49]) and representation (4.14), we have
\[
\max_{2 \leq i \leq n} \left( V_{[i-1],n} - V_{[i],n} \right) \leq \max_{2 \leq i \leq n} \left( U_{[i-1],n}^V - U_{[i],n}^V \right) = O\left(n^{-1} \ln n\right) \quad \text{a.s.}
\]

Since the second term on the right-hand side of (4.23) has the same order of magnitude, (4.22) and (4.23) now yield
\[
E_a(t_a + \delta_a) = O\left(n^{-1} \ln n\right) \quad \text{a.s.}
\]

Therefore (4.19) implies that
\[
E_a(t_a + \delta_a) + n^{-1/2} Z_a(t_a + \delta_a) = O\left(n^{-1} \ln n\right). \quad (4.24)
\]

Under assumption (A3) we also have
\[
E_a(t_a + \delta_a) = E_a(t_a) + \delta_a E'_a(t_a) + O(\delta_a^2) = \delta_a E'_a(t_a) + O(\delta_a^2) \quad \text{a.s.} \quad (4.25)
\]

Combining (4.24) with (4.25) and using (4.21) gives
\[
\delta_a E'_a(t_a) = -n^{-1/2} Z_a(t_a + \delta_a) + O(n^{-1} \ln n)
\]
\[
= -n^{-1/2} Z_a(t_a) + n^{-1/2} \psi_a \omega_{Z_a}(\delta_a) + O(n^{-1} \ln n) \quad (4.26)
\]

with \( |\psi_a| \leq 1 \), where
\[
\omega_h(\delta) = \sup_{|t-s| \leq \delta} |h(t) - h(s)|
\]
denotes the modulus of continuity of the continuous function \( h \) on \([0,1]\). Recall that, for a Brownian bridge process \( B \), one has
\[
\limsup_{\delta \downarrow 0} \frac{w_B(\delta)}{\sqrt{2 \delta \ln \left(1/\delta\right)}} = 1 \quad \text{a.s.}
\]
(see, for example, Theorem 1.4.1 in Csörgő and Révész [42]). As this holds for both \( B^V \) and \( B^C \), and \( \lambda_a, f \) and \( g \) are bounded away from zero, it follows from (4.20) that
\[
\omega_{Z_a}(\delta_a) = O(\delta_a^{1/2} \ln^{1/2}(1/\delta_a)) = O(n^{-1/4}(\ln n)^{1/2}) \quad \text{a.s.}
\]
Hence (4.26) now yields
\[
\delta_a = - \frac{Z_a(t_a)}{n^{1/2} E'_a(t_a)} + O\left(n^{-3/4}(\ln n)^{1/2}\right) \quad \text{a.s.} \quad (4.27)
\]

Recall that, for a Brownian bridge process \( B \) and any \( t \in (0,1) \), \( B(t) \) is normally distributed with respective mean and variance \((0, t(1-t))\). Since the
processes $B^V$ and $B^C$ in (4.20) are independent, we immediately see that $Z_a(t_a)$ is normally distributed with zero mean and variance

$$\frac{t_a(1-t_a)}{f^2(F^{(-1)}(1-t_a))} + \frac{t_a(1-\lambda_a^{-1}t_a)}{\lambda_a^2 g^2(G^{(-1)}(\lambda_a^{-1}t_a))}.$$  

Setting

$$Z_a^{(1)} := -Z_a(t_a)/E_a(t_a),$$

it follows from (4.7) that $Z_a^{(1)}$ is normally distributed with respective mean and variance $(0, \sigma_a^2)$. Thus, (4.27) establishes the first assertion of Theorem 4.2.1.

For welfare, from (4.2), (4.12) and (4.13) we have

$$W_a = \sum_{i=1}^{K_a} (V_{i|n} - C_{i|m})$$

$$= n \int_{0}^{K_a/n} \left( V_n(1-t) - C_m(\lambda_a^{-1}t) \right) dt = n \int_{0}^{K_a/n} E_a(t) dt$$

$$= n \int_{0}^{t_a+\delta_a} \left( E_a(t) + n^{-1/2}Z_a(t) + \varphi_a(t)n^{-1} \ln n \right) dt.$$  

(4.28)

Now note that replacing $\int_{t_a}^{t_a+\delta_a}$ with $\int_{0}^{t_a}$ in the last line will only introduce an error $O(1)$ a.s. Indeed, since $E_a(t_a) = 0$ and $E'_a(t)$ is bounded in view of our assumptions (A1) and (A2) (cf. (4.6)), one has from (4.21) that

$$n \int_{t_a}^{t_a+\delta_a} E_a(t) dt = O \left( n \int_{t_a}^{t_a+|\delta_a|} (t-t_a) dt \right) = O \left( n\delta_a^2 \right) = O(1) \quad \text{a.s.}$$

Further, it is clear from (A1), (A2) and (4.20) that there exists a constant $c < \infty$ such that

$$\max_{t \in [0,1]} |Z_a(t)| \leq Y := c \left( \max_{t \in [0,1]} |B^V(t)| + \max_{t \in [0,1]} |B^C(t)| \right) < \infty \quad \text{a.s.}$$

Hence, again using (4.21),

$$\left| n^{1/2} \int_{t_a}^{t_a+\delta_a} Z_a(t) dt \right| \leq n^{1/2}|\delta_a|Y = O(1) \quad \text{a.s.}$$

This proves the desired claim since the contribution of the last term in the integrand in (4.28) to $\int_{t_a}^{t_a+\delta_a}$ will be even smaller in magnitude (recall that $\|\varphi_a\|_\infty = O(1)$ a.s. and one has (4.21)).

Thus we obtain from (4.28) that

$$W_a = n \int_{0}^{t_a} E_a(t) dt + n^{1/2} \int_{0}^{t_a} Z_a(t) dt + O(\ln n) \quad \text{a.s.}$$  

(4.29)
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It is clear from (4.20) that \( Z_a \) is a zero mean Gaussian process, and so the second integral on the right-hand side of (4.29), to be denoted by \( Z_a^{(2)} \), is zero mean normal as well. For brevity, set \( f^*(t) := f(F^{(-1)}(t)) \) and \( g^*(t) := g(G^{(-1)}(t)) \). Recalling the right relation in (4.5) and also that \( \mathbb{B}^V \) and \( \mathbb{B}^C \) are independent Brownian bridges, we compute the variance of \( Z_a^{(2)} \) as

\[
E \left[ \left( \int_0^t Z_a(t) \, dt \right)^2 \right]
= \int_0^t \int_0^t E [Z_a(s)Z_a(t)] \, ds \, dt = 2 \int_0^t \int_0^t E [Z_a(s)Z_a(t)] \, ds \, dt
= 2 \int_0^t \int_0^t \left( \frac{E [\mathbb{B}^V(1-s)\mathbb{B}^V(1-t)]}{f^*(1-s)f^*(1-t)} + \frac{E [\mathbb{B}^C(\lambda_a^{-1}s)\mathbb{B}^C(\lambda_a^{-1}t)]}{\lambda_a^2g^*(\lambda_a^{-1}s)g^*(\lambda_a^{-1}t)} \right) \, ds \, dt
= 2 \int_0^t \int_0^t \left( \frac{s(1-t)}{f^*(1-s)f^*(1-t)} + \frac{s(1-\lambda_a^{-1}t)}{\lambda_a^2g^*(\lambda_a^{-1}s)g^*(\lambda_a^{-1}t)} \right) \, ds \, dt
= 2 \int_0^t \left( \int_0^t \frac{s}{f^*(1-t)} \, ds + \frac{1-\lambda_a^{-1}t}{g^*(\lambda_a^{-1})} \int_0^t \frac{s}{\lambda_a^2g^*(\lambda_a^{-1}s)} \, ds \right) \, dt. (4.30)

Changing variables \( x := F^{(-1)}(1-s) \) and \( x := G^{(-1)}(\lambda_a^{-1}s) \), respectively, in the two integrals inside the square brackets in the last line in (4.30), we obtain the desired representation (4.8) for \( \varsigma_a^2 \). Now the second assertion of the theorem follows from (4.29).

To complete the proof of the theorem, it remains to compute the covariance

\[
E \left[ Z_a^{(1)} Z_a^{(2)} \right] = -E \left[ \frac{Z_a(t_a)}{E_a(t_a)} \int_0^t Z_a(t) \, dt \right]
= -\frac{1}{E_a(t_a)} \int_0^t E [Z_a(t_a)Z_a(t)] \, dt = -\frac{S_a(t_a)}{E_a(t_a)},
\]

where the last equality follows from evaluation of the inner integral in (4.30) and the definition of the function \( S_a \) (following (4.8)). Theorem 4.2.1 is proved.

\[\square\]

Remark 4.4.1. An argument similar to the second part of our proof of the asymptotic normality of \( K_a \) can be found in the first example in Section 8, Ch. I of Borovkov [22]. However, in our case the quantity \( t_a \) depends on \( a \) and does not converge to any value. Therefore, the first-order derivative of \( \mathcal{H} \) cannot be computed by directly applying a known general version of the delta method, such as the main result in Borovkov [23].

Remark 4.4.2. The reader is referred to Csörgő and Révész [43] for the minimal boundedness conditions that must be imposed on \( f' \) and \( g' \) for (4.18) and (4.17) to hold. It turns out that, for the approximation of \( K_a \), these boundedness conditions may be weakened further as they are required only in a neighbourhood of \( t_a \).
4.4.2 Prove of Theorem 4.2.4

Proof. (i) When $G = F$, the equation $E_a(t_a) = 0$ for $t_a$ (see (4.3), (4.5)) turns into $F^{-1}(\lambda_a^{-1}t_a) = F^{-1}(1 - t_a)$, which means that $\lambda_a^{-1}t_a = 1 - t_a$, yielding the desired representation for $t_a$.

Next, in this case for $t = t_a$ the value of (4.6) turns into

$$E'_a(t_a) = -\frac{1}{t_ah(F^{-1}(1 - t_a))},$$

so that (4.7) becomes

$$\sigma_a^2 = t_a^2 \left[ t_a(1 - t_a) + t_a^2 \lambda_a^{-2} \right] = t_a^2(1 - t_a) = \frac{\lambda_a^2}{(1 + \lambda_a)^3}.

(ii) As we already pointed out, in the special case when $F = G$ the exact distribution of $K_a$ does not depend on $F$. In fact, as shown in Chapter 3, $K_a$ has the standard hypergeometric distribution $Hg(n, m, n+m)$. It is well known that, as $m+n \to \infty$ and the quantity $\lambda_a$ remains bounded away from 0 and 1 (which is ensured by our assumption (A2)), that distribution can be approximated by a normal distribution (see, for example, Theorem 2.1 in Lahiri and Chatterjee [86]). The stated bounds (4.10) follow from the results established in Theorem 2.2 in Lahiri and Chatterjee [86].

(iii) First note that, in the case of the uniform distributions $F = G$ on $[0,1]$, one has $F^{-1}(t) = t$ on $[0,1]$, so that

$$\int_0^{t_a} E_a(t) \, dt = \int_0^{t_a} (1 - t - t/\lambda_a) \, dt = \int_0^{t_a} (1 - t/t_a) \, dt = \frac{t_a}{2} = \frac{\lambda_a}{2(1 + \lambda_a)}.

Next, since $f(t) \equiv g(t) \equiv 1$ on $[0,1]$, we also have

$$S_a(t) = (1 - t) \int_{1-t}^1 (1 - x) \, dx + (1 - \lambda_a^{-1}t) \int_0^{\lambda_a^{-1}t} x \, dx

= \frac{1}{2}t^2(1 - t) + \frac{1}{2} \left( \frac{t}{\lambda_a} \right)^2 \left( 1 - \frac{t}{\lambda_a} \right), \quad t \in (0, t_a).

Integrating this expression in $t$ from 0 to $t_a$ yields the second formula in (4.11). To get the last formula in (4.11), we just note that $E_a(t_a)' = -1/t_a$ and, as $t_a/\lambda_a = 1 - t_a$, one has $S_a(t_a) = \frac{1}{2}t_a^2(1 - t_a) + \frac{1}{2}(1 - t_a)^2t_a = \frac{1}{2}t_a^2(1 - t_a)$, and so $\zeta_a = -S_a(t_a)/E_a(t_a)' = \frac{1}{2}t_a^2(1 - t_a)$.

4.5 Conclusion

In this chapter we considered the ex–post efficient outcome of a canonical economic market model involving buyers and sellers with independent and identically distributed random valuations and costs, respectively. When the number
of buyers and sellers is large, we showed that the joint distribution of the equilibrium quantity traded and welfare is asymptotically normal. Moreover, we bounded the approximation rate. The proof proceeded by constructing, on a common probability space, a representation consisting of two independent empirical quantile processes, which in large markets can be approximated by independent Brownian bridges. The distribution of interest was then approximated by that of a functional of a Gaussian process. To complete the proof, we then used an appropriate generalisation of the delta method. We also showed that this methodology immediately applies to a variety of the market mechanisms (including the profit–maximising mechanism, $\alpha$–mechanisms and $k$–double auctions) considered in Chapter 3.

Future work could involve further generalising the results of our analysis. For example, it would be interesting to consider a non–trivial setup involving multi–unit demand and supply. In its current form, our analysis is not immediately applicable to computing quantities such as profit under the two–sided VCG mechanism. Although using the common probability space method allows us to deal with variable correlations, computing profit under the two–sided VCG mechanism would require estimating order statistic spacing which are of order smaller than $n^{-1} \ln n$. It would be interesting to investigate whether known results regarding asymptotic order statistic spacings could be used to adapt our analysis to such cases.
Chapter 5

Dynamic Market Making

We derive the class of Bayesian optimal mechanisms in a two-period extension of the classic model of Myerson and Satterthwaite.

5.1 Introduction

Markets are two-sided, bringing together buyers and sellers, and often dynamic in nature, as traders arrive and depart over time. Prominent examples include commodity markets (such as those for agricultural products, gold, coal and iron ore), stock exchanges, radio spectrum license exchanges and pollution permit markets. Moreover, in secondary markets for durable goods, or forward contracts over non-durable goods, dynamics play an important role. Durability of goods, persistence of preferences and the arrival of traders over time creates a tradeoff between the increase in market thickness that occurs as buyers and sellers accumulate versus the opportunity cost of delay associated with discounting. This is the tradeoff we are interested in investigating for the remainder of this thesis. A natural starting point is the two-period extension of the classic model of Myerson and Satterthwaite [117], which we study in this chapter.

In this chapter we consider a Bayesian mechanism design setup with two periods. In each period a single buyer and seller arrive. The market maker determines how frequently the market is cleared. That is, after period one agents have arrived and reported their types, the market maker can elect to clear the market or wait. We assume period one agents are patient and will be available to trade in period two if the market is not cleared in period one. We make the standard assumptions that agent types are private information but type distributions are common knowledge. We also assume that types are independently distributed.

Within this setup, we derive the class of Bayesian optimal mechanisms known as $\alpha$-mechanisms, which includes the first-best (welfare-maximising), second-best and profit-maximising mechanisms. These optimal dynamic mechanisms are sophisticated insofar as they require incentivised elicitation of information about types in both periods, making allocation and payment decisions non-trivial functions of period one reports. We compare our optimal
dynamic mechanisms to two benchmark static mechanisms. The first is one under which the market maker cannot wait in period one and must clear the market. In this case, period one and period two are simply the standard Myerson and Satterthwaite [117] setup, with a common discount factor applied to all agents present in period two. Under the second benchmark mechanism the mechanism design must wait in period one and clear the market in period two. This gives a Gresik and Satterthwaite [68] setup with two buyers and two sellers and a common discount factor applied to all agents.

Investigating the properties of the optimal dynamic mechanisms we find that, unsurprisingly, these mechanisms outperform the benchmark static mechanisms. We also find that relative to the repeated Myerson and Satterthwaite mechanism, dynamic mechanisms tend to increase the convexity of the region of trade in period one. This occurs because it is optimal to delay relatively “asymmetric” period one trades, in the hope that the more efficient period one agent can trade with an agent arriving in period two. Finally, we find that contrary to standard mechanism design intuition, the region of trade in period one does not necessarily decrease if the designer places a greater emphasis on rent extraction.

The complexity of the Bayesian optimal mechanisms derived in this chapter raises the question of whether there are simpler mechanisms that approximate the dynamically optimal outcome without requiring truthful type revelation in both periods. We answer this question affirmatively as follows. We look at a simple price–posting mechanism in period one (see Hagerty and Rogerson [70]). When both agents in period one agree to trade at the posted prices, they trade, and period two collapses to the setup of Myerson and Satterthwaite. If one of the period one agents decides to wait, period two collapses to the Gresik and Satterthwaite setup described above. We show that even with myopically optimal posted prices in period one, this simple dynamic mechanism outperforms both benchmark static mechanisms, provided \( \delta \) is sufficiently large.

Finally, in this chapter we find that when agents have continuous type spaces this leads to an implicit characterisation of the class of optimal dynamic mechanisms that is cumbersome to analyse. Therefore, it seems that our two–period extension of the model of Myerson and Satterthwaite lacks the tractability needed to consider further extensions and applications of economic interest. In particular, it seems necessary to study models with additional periods in order to determine which of the results in this chapter are driven by end–of–the–world effects. Motivated by this observation, in Chapter 6 we study an infinite–horizon dynamic market model in which agents have simple binary types.

The focus of much of mechanism design has been on first–best allocation rules. First–best is a natural objective, and is simple whenever the first–best
allocation coincides with ex–post efficiency.\footnote{For example, the vast literature on optimal partnership dissolution in the tradition of Cramton, Gibbons, and Klemperer \cite{Cramton2000} has almost exclusively focused on ex–post efficiency; see Loertscher and Wasser \cite{Loertscher2012} for up–to–date references and an exception.} However, in dynamic mechanism design setups such as ours, characterising the first–best allocation rule is non–trivial because the market maker must consider the dynamic impact of current allocation decisions on future expected payoffs. Therefore, in our setup even when the unique goal is to maximise expected discounted social surplus, deriving the first–best allocation involves trading off current period surplus against expected discounted gains in surplus from delay.\footnote{This differs from static setups such as Myerson and Satterthwaite \cite{Myerson1981} where the first–best allocation rule requires no derivation – trade if and only if the buyer’s value exceeds the seller’s cost. In such settings, allocation rules become non–trivial only once revenue consideration enter the objective.}

In a dynamic setup with discounting, it will never be optimal for a mechanism designer to delay trade unless market efficiency increases with market thickness. The notion that market efficiency increases with market thickness is formalised by Gresik and Satterthwaite \cite{Gresik1993} (who derive the rate at which a two–sided market converges to ex–post efficiency under a mechanism which maximises welfare subject to interim budget balance, incentive compatibility and individual rationality constraints), Rustichini, Satterthwaite, and Williams \cite{Rustichini1997} (who study $k$–double auctions in which agents’ signals are independent), Cripps and Swinkels \cite{Cripps1998} (who study $k$–double auctions and allow for some dependence between agents’ signals) and Tatur \cite{Tatur2015} (whose results imply large double auctions with transaction fees also converge to efficiency).

The vast literature concerning mechanism design has largely focused on static setups (see Section 1.3.1 for a brief overview of this literature), notwithstanding the recent upsurge of interest in dynamic mechanism design, with the contributions of Bergemann and Välimäki \cite{Bergemann2007}, Athey and Segal \cite{Athey2010}, Pavan, Segal, and Toikka \cite{Pavan2012}, Skrzypacz and Toikka \cite{Skrzypacz2013}, Parkes and Singh \cite{Parkes2013}, Gershkov and Moldovanu \cite{Gershkov2013}, Board and Skrzypacz \cite{Board2015}, Baccara, Lee, and Yariv \cite{Baccara2016} and Herbst and Schickner \cite{Herbst2017}. A detailed overview of this literature can be found in Section 1.3.2. In this chapter, we depart from the existing dynamic mechanism design literature by considering an environment in which agents have persistent types, explicitly addressing the optimal timing problem described by Pavan, Segal, and Toikka \cite{Pavan2012} and explicitly deriving the class of Bayesian optimal mechanisms (including the first–best mechanism, the second–best mechanism and the profit–maximising mechanism).

There are numerous papers outside of the mechanism design literature which consider a variety of two–sided market models, often motivated by the study of financial markets. Mendelson \cite{Mendelson2015} considers a model for a clearing house in which orders accumulate for a fixed time $T$ before the market is cleared. However, the impact of the parameter $T$ on market efficiency or agent incentives is not considered. There is also a vast literature concerning
limit order books (see, for example, Kelly and Yudovina [79] and references therein). In these continuous time models, orders are not stored and feasible transactions are immediately cleared from the market. Recently, Budish, Cramton, and Shim [25] proposed storing bids in limit order books, which is notionally similar to the optimal “storing” of traders that emerges from our analysis. However, the motivations for so doing are fundamentally different. Bid storing in Budish, Cramton, and Shim arises as a means to mitigate possibilities for arbitrage and socially wasteful investments in arbitrage. In contrast, “storing traders” is the solution to a dynamic optimisation problem in our setup, with the objective being to maximise expected discounted social surplus of the agents.

The remainder of this chapter is organised as follows. In Section 5.2, we describe the details of our model. We then derive the class of $\alpha$–mechanisms that apply to this setup in Section 5.3. We also discuss a variety of comparative statics relating to these mechanisms, including comparisons with appropriate static mechanisms. In Section 5.4 we discuss approximate implementation of the optimal mechanisms derived in Section 5.3 via a price–posting mechanism in the first period. Concluding remarks are provided in Section 5.5.

5.2 Setup

We consider a two–period, two–sided Bayesian mechanism design market model for a homogeneous, indivisible good. In each period, a pair consisting of a single buyer and seller arrives. In the first period, the market maker (or designer) must decide whether to clear the market or wait until the second period. If the market is not cleared in the first period, the period one buyer and seller pair wait until the second period. In the second period, the market must be cleared and a common discount factor of $\delta \in [0, 1]$ is applied to all agents.\footnote{In some papers (see, for example, Lauermann [88]) $\delta$ is interpreted as the probability that period one agents are forced to exit the market without trading after the mechanism designer elects to wait in period one.}

5.2.1 Agents and Information

We assume that each buyer demands at most one unit and each seller has the capacity to produce at most one unit. In every period $t \in \{1, 2\}$ one buyer $B_t$ and and one seller $S_t$ arrive and draw their values $V_t$ and costs $C_t$ independently from the absolutely continuous distributions $F$ and $G$, respectively. Let the support of $f$ and $g$ be given by $[v, \overline{v}]$ and $[c, \overline{c}]$ respectively and assume that $f$ and $g$ are differentiable with $f(v) > 0$ for $v \in [v, \overline{v}]$ and $g(c) > 0$ for $c \in [c, \overline{c}]$. We assume that $\overline{v} > c$ (to avoid the case in which no trade is always ex–post efficient) and $\underline{v} < \underline{c}$ (to avoid the case in which full trade is always
5.2. SETUP

Recall from Section 1.2.6 that the virtual valuation and cost functions,
\[ \Phi(v) = v - \frac{1-F(v)}{f(v)} \quad \text{and} \quad \Gamma(c) = c + \frac{G(c)}{g(c)} \]
play an important role in Bayesian mechanism design. For \( \alpha \in [0,1] \) we also introduce the weighted virtual valuation and cost functions,
\[ \Phi_\alpha(v) = v - \alpha \frac{1-F(v)}{f(v)} \quad \text{and} \quad \Gamma_\alpha(c) = c + \alpha \frac{G(c)}{g(c)} \]

We impose Myerson’s [116] regularity condition, that \( \Phi(v) \) and \( \Gamma(c) \) are strictly increasing functions on their respective supports. We also assume that these functions are differentiable. Let \( \Theta_t = (V_t, C_t) \) be the types of the agents who arrive in period \( t \) and let \( \Theta = (\Theta_1, \Theta_2) \). Given an agent \( l \in \{B_1, B_2, S_1, S_2\} \), we also let \( \Theta_{-l} \) denote the types of all agents other than \( l \). Let \( f(v) := f(v_1) f(v_2) \) and \( g(c) := g(c_1) g(c_2) \) be the respective joint densities of buyers’ and sellers’ types. We assume buyers’ types and sellers’ types are independent.

As was done in Section 1.2.6, we assume all agents are risk neutral and have quasilinear preferences. That is, suppose buyer \( B_t \) has valuation \( v_t \), receives a unit of the good with probability \( q_t \) and pays an expected transfer of \( m_t \). The expected payoff of buyer \( t \) is then given by
\[ u^{B_t}(v_t, q_t, m_t) := v_t q_t - m_t. \]
Similarly, if seller \( S_t \) with cost \( c_t \) produces the good with probability \( q_t \) and receives a transfer of \( m_t \), his expected payoff is given by
\[ u^{S_t}(c_t, q_t, m_t) := m_t - c_t q_t. \]

We assume that the value of agents’ outside option of not participating in the mechanism is zero.

5.2.2 Direct Mechanisms

By the revelation principle, we may restrict ourselves to direct mechanisms without loss of generality. Furthermore, we may also restrict ourselves to
Let \( M_2 \) given \( \hat{\theta}_2 \). A direct mechanism \( (Q, M) \) consists of an allocation rule \( Q = (Q_1, Q_2) \) and a payment rule \( M = (M_1, M_2) \). The period one allocation rule \( Q_1 : [\underline{\theta}, \overline{\theta}] \times [\underline{v}, \overline{v}] \rightarrow \{0, 1\}^2 \) maps the reports of the period one buyer and seller pair to the allocation of each agent (which is 1 if that agent trades and 0 otherwise). The period two allocation rule \( Q_2 : [\underline{\theta}, \overline{\theta}]^2 \times [\underline{v}, \overline{v}]^2 \times \{0, 1\}^2 \rightarrow \{0, 1\}^4 \) depends on the period one allocation and maps the reports of all buyers and sellers to the period two allocation. Similarly, we have a period one transfer rule \( M_1 : [\underline{\theta}, \overline{\theta}] \times [\underline{v}, \overline{v}] \rightarrow \mathbb{R}^2 \) and a period two transfer rule \( M_2 : [\underline{\theta}, \overline{\theta}]^2 \times [\underline{v}, \overline{v}]^2 \rightarrow \mathbb{R}^4 \).

Let \( Q_1^{B_1}(\hat{\theta}_1) \) and \( Q_1^{S_1}(\hat{\theta}_1) \) denote the respective period one allocations of buyer \( B_1 \) and seller \( S_1 \) given their reports \( \hat{\theta}_1 \). Feasibility requires that \( Q_1^{B_1}(\hat{\theta}_1) = Q_1^{S_1}(\hat{\theta}_1) \). Consequently, we can simplify notation and set

\[
Q_1(\hat{\theta}_1) := Q_1^{B_1}(\hat{\theta}_1) = Q_1^{S_1}(\hat{\theta}_1).
\]

Let \( Q_2^{B_2}(\hat{\theta}, Q_1) \) and \( Q_2^{S_2}(\hat{\theta}, Q_1) \) respective allocations of the buyer and seller who arrive in period two, given that they report \( \hat{\theta} \) when \( Q_1 = Q_1(\hat{\theta}_1) \). Let \( Q_2^{B_1}(\hat{\theta}) \) and \( Q_2^{S_1}(\hat{\theta}) \) be the period two allocations for period one agents when reported types are \( \hat{\theta} \).

Feasibility also requires that \( Q_2^{B_2}(\hat{\theta}, 1) = Q_2^{S_2}(\hat{\theta}, 1) \). That is, when the period one agents trade in period one, period two becomes the standard bilateral trade problem. Once again we can simplify notation by setting

\[
Q_2(\hat{\theta}_2) := Q_2^{B_2}(\hat{\theta}, 1) = Q_2^{S_2}(\hat{\theta}, 1).
\]

If \( Q_1(\hat{\theta}_1) = 0 \), the feasibility constraint in period two becomes

\[
Q_2^{B_1}(\hat{\theta}) + Q_2^{B_2}(\hat{\theta}, 0) = Q_2^{S_1}(\hat{\theta}) + Q_2^{S_2}(\hat{\theta}, 0).
\]

Similarly, let \( M_1^{B_1}(\hat{\theta}_1) \) and \( M_1^{S_1}(\hat{\theta}_1) \) denote the payments made by the period one agents in period one given \( \hat{\theta}_1 \). Let \( M_2^{B_2}(\hat{\theta}) \) and \( M_2^{S_2}(\hat{\theta}) \) denote the payments made by the period two agents in period two given \( \hat{\theta} \). Finally, let \( M_2^{B_1}(\hat{\theta}) \) and \( M_2^{S_1}(\hat{\theta}) \) denote the payments made by period one agents in period two given \( \hat{\theta} \).

\footnote{We are interested in the class of Bayesian optimal mechanisms known as \( \alpha \)-mechanisms. These mechanisms are deterministic up to tie-breaking and we may use a deterministic tie-breaking rule.}

\footnote{There is no need to make the dependence on \( Q_1 = 0 \) explicit because the allocations can only be equal to 1 when \( Q_1 = 0 \).}
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5.2.3 Incentive Compatibility and Individual Rationality

Assuming all other agents report truthfully, the interim discounted allocation probabilities for each agent are given by

\[
q^{B_1}(\hat{v}_1) = E_{\Theta-B_1} \left[ Q_1(\hat{v}_1, C_1) + \delta (1 - Q_1(\hat{v}_1, C_1))Q_2^{B_1}(\hat{v}_1, \Theta-B_1) \right],
\]

\[
q^{S_1}(\hat{c}_1) = E_{\Theta-s_1} \left[ Q_1(V_1, \hat{c}_1) + \delta (1 - Q_1(V_1, \hat{c}_1))Q_2^{S_1}(\hat{c}_1, \Theta-s_1) \right],
\]

\[
q^{B_2}(\hat{v}_2) = \delta E_{\Theta-S_2} \left[ Q_2(\hat{v}_2, C_2)Q_1(\Theta_1) + Q_2^{B_2}(\hat{v}_2, \Theta-B_2, 0)(1 - Q_1(\Theta_1)) \right],
\]

\[
q^{S_2}(\hat{c}_2) = \delta E_{\Theta-S_2} \left[ Q_2(V_2, \hat{c}_2)Q_1(\Theta_1) + Q_2^{S_2}(\hat{c}_2, \Theta-S_2, 0)(1 - Q_1(\Theta_1)) \right].
\]

Similarly, the expected interim discounted payments made by each agent are given by

\[
m^{B_1}(\hat{v}_1) = E_{\Theta-B_1} \left[ M_1^{B_1}(\hat{v}_1, C_1) + \delta M_2^{B_1}(\hat{v}_1, \Theta-B_1) \right], \tag{5.1}
\]

\[
m^{S_1}(\hat{c}_1) = E_{\Theta-s_1} \left[ M_1^{S_1}(V_1, \hat{c}_2) + \delta M_2^{S_1}(\hat{c}_2, \Theta-s_1) \right], \tag{5.2}
\]

\[
m^{B_2}(\hat{v}_2) = \delta E_{\Theta-S_2} \left[ M_2^{B_2}(\hat{v}_2, \Theta-B_2) \right], \tag{5.3}
\]

\[
m^{S_2}(\hat{c}_2) = \delta E_{\Theta-S_2} \left[ M_2^{S_2}(\hat{c}_2, \Theta-S_2) \right]. \tag{5.4}
\]

Recall that Bayesian incentive compatibility requires that for all buyers $B_t$ with values $v_t \in [\underline{v}, \overline{v}]$ and sellers $S_t$ with costs $c_t \in [\underline{c}, \overline{c}]$, their respective interim payoff functions $U^{B_t}$ and $U^{S_t}$ are given by

\[
U^{B_t}(v_t) = \max_{\hat{v}_t \in [\underline{v}, \overline{v}]} \left\{ v_t q^{B_t}(\hat{v}_t) - m^{B_t}(\hat{v}_t) \right\}, \tag{5.5}
\]

\[
U^{S_t}(c_t) = \max_{\hat{c}_t \in [\underline{c}, \overline{c}]} \left\{ m^{S_t}(\hat{c}_t) - c_t q^{S_t}(\hat{c}_t) \right\}. \tag{5.6}
\]

The interim individual rationality constraint applies when agents have the outside option of not participating in the mechanism. In this case, we require that agents' interim expected payoffs exceed their outside option of receiving a payoff of zero. That is, for all $v_t \in [\underline{v}, \overline{v}]$ and $c_t \in [\underline{c}, \overline{c}]$, we require

\[
v_t q^{B_t}(v_t) - m^{B_t}(v_t) \geq 0 \quad \text{and} \quad m^{S_t}(c_t) - c_t q^{S_t}(c_t) \geq 0.
\]

5.3 Bayesian Optimal Mechanisms

5.3.1 Expected Discounted Profit

Let $O = [\underline{v}, \overline{v}]^2 \times [\underline{c}, \overline{c}]^2$, $Q_2^{B_2}(\Theta, 0) = Q_2^{S_2}(\Theta)$ and $Q_2^{S_2}(\Theta, 0) = Q_2^{S_2}(\Theta)$. Expected discounted welfare under a direct Bayesian incentive compatible mech-
anism is given by
\[ W = \int \left\{ \left[ v_1 - c_1 + \delta(v_2 - c_2)Q_2(\theta)\right]Q_1(\theta_1) + \delta(1 - Q_1(\theta_1))\left[v_1Q_2^{B_1}(\theta) + \delta(v_2Q_2^{B_2}(\theta) - (c_1Q_2^{S_1}(\theta) + c_2Q_2^{S_2}(\theta)))\right] f(v)g(c) \, d\theta. \]

We begin by deriving expected discounted profit \( R \) under any direct mechanism which satisfies incentive compatibility and individual rationality constraints. We start by stating a useful lemma.

**Lemma 5.3.1.** An allocation rule is incentive compatible if and only if, for all \( v_t \in [\underline{v}, \bar{v}] \) and \( c_t \in [\underline{c}, \bar{c}] \), \( q^{B_t}(v_t) \) is non-decreasing and \( q^{S_t}(c_t) \) is non-increasing.

**Proof.** Once we have defined the interim expected payoffs of each agent, the argument presented in Section 1.2.4 immediately applies to the dynamic setup of this chapter.

Recall from Section 1.2.6 that an immediate consequence of the proof of Lemma 5.3.1 is an analogue of Myerson’s [116] revenue equivalence theorem.

**Lemma 5.3.2.** Under an incentive compatible mechanism
\[ U^{B_t}(v_t) = U^{B_t}(\bar{v}) + \int_{\underline{v}}^{v_t} q^{B_t}(y) \, dy \quad \text{and} \quad U^{S_t}(c_t) = U^{S_t}(\bar{c}) + \int_{c_t}^{\bar{c}} q^{S_t}(y) \, dy. \]

Since
\[ \int_{\underline{v}}^{v_t} q^{B_t}(y) \, dy \geq 0 \quad \text{and} \quad \int_{c_t}^{\bar{c}} q^{S_t}(y) \, dy \geq 0, \]

it follows from Lemma 5.3.2 that interim individual rationality is satisfied under a Bayesian incentive compatible mechanism if and only if \( U^{B_t}(\bar{v}) \geq 0 \) and \( U^{S_t}(\bar{c}) \geq 0 \).

Using the definitions of \( U^{B_t}(v_t) \) and \( U^{S_t}(c_t) \), the equations in Lemma 5.3.2 can now be rewritten
\[ m^{B_t}(v_t) = -U^{B_t}(\bar{v}) + v_tq^{B_t}(v_t) - \int_{\underline{v}}^{v_t} q^{B_t}(y) \, dy, \]
\[ m^{S_t}(c_t) = U^{S_t}(\bar{c}) + c_tq^{S_t}(c_t) + \int_{c_t}^{\bar{c}} q^{S_t}(y) \, dy. \]

We now wish to determine the ex-ante expected transfers \( \mathbb{E} \left[ m^{B_t}(V_t) \right] \) and \( \mathbb{E} \left[ m^{S_t}(C_t) \right] \). To do this, we integrate over the above equations and change the order of integration in the resulting double integrals. This gives
\[ \mathbb{E} \left[ m^{B_t}(V_t) \right] = -U^{B_t}(\bar{v}) + \int_{\underline{v}}^{\bar{v}} \Phi(y)q^{B_t}(y) f(y) \, dy, \]
\[ \mathbb{E} \left[ m^{S_t}(C_t) \right] = U^{S_t}(\bar{c}) + \int_{\underline{c}}^{\bar{c}} \Gamma(y)q^{S_t}(y) g(y) \, dy. \]
Thus, by (5.1) to (5.4), the expected discounted profit $R$ of the mechanism is given by

$$R = \sum_{t=1}^{2} E \left[ m^{R_t}(V_t) \right] - \sum_{t=1}^{2} E \left[ m^{S_t}(C_t) \right]$$

$$= \int O \left\{ \left[ \Phi(v_1) - \Gamma(c_1) + \delta(\Phi(v_2) - \Gamma(c_2)) \right] Q_2(\theta) \right\} Q_1(\theta_1) + \delta(1 - Q_1(\theta_1)) \times \left\{ \Phi(v_1)Q_2^{B_1}(\theta) + \Phi(v_2)Q_2^{B_2}(\theta) - (\Gamma(c_1)Q_2^{S_1}(\theta) + \Gamma(c_2)Q_2^{S_2}(\theta)) \right\} \times f(v)g(c) d\theta - \sum_{t=1}^{2} \left( U^{B_t}(\nu) + U^{S_t}(\bar{c}) \right).$$

### 5.3.2 Description of Mechanisms

Recall from Section 1.2.6 that $\alpha$–mechanisms maximise a convex combination $(1 - \alpha)W + \alpha R$, where $\alpha \in [0, 1]$, subject to Bayesian incentive compatibility and interim individual rationality constraints. We now describe the class of $\alpha$-mechanisms for our setup.

Using our results in the previous section, we have

$$(1 - \alpha)W + \alpha R$$

$$= \int O \left\{ \left[ \Phi_\alpha(v_1) - \Gamma_\alpha(c_1) + \delta(\Phi_\alpha(v_2) - \Gamma_\alpha(c_2)) \right] Q_2(\theta) \right\} Q_1(\theta_1) + \delta(1 - Q_1(\theta_1)) \times \left\{ \Phi_\alpha(v_1)Q_2^{B_1}(\theta) + \Phi_\alpha(v_2)Q_2^{B_2}(\theta) - (\Gamma_\alpha(c_1)Q_2^{S_1}(\theta) + \Gamma_\alpha(c_2)Q_2^{S_2}(\theta)) \right\} \times f(v)g(c) d\theta - \alpha \sum_{t=1}^{2} \left( U^{B_t}(\nu) + U^{S_t}(\bar{c}) \right).$$

Notice that it is optimal to set

$$\sum_{t=1}^{2} \left( U^{B_t}(\nu) + U^{S_t}(\bar{c}) \right) = 0. \quad (5.7)$$

Since interim individual rationality requires, for $t = 1$ and $t = 2$, $U^{B_t}(\nu) \geq 0$ and $U^{S_t}(\bar{c}) \geq 0$, it follows from (5.7) that $U^{B_t}(\nu) = 0$ and $U^{S_t}(\bar{c}) = 0$. To derive the $\alpha$–mechanism allocation rule, we now consider a relaxed version of the maximisation problem. That is, we maximise the above integral ignoring the local incentive compatibility constraints and then verify that the solution to this relaxed optimisation problem is also the solution to the original maximisation problem.

We start by considering the term $$(\Phi_\alpha(v_2) - \Gamma_\alpha(c_2))Q_2(\theta).$$ Maximising this term is simply an extension of the standard bilateral trade problem. We set
\( Q_2(\theta) = 1 \) if \( \Phi_\alpha(v_2) \geq \Gamma_\alpha(c_2) \) and \( Q_2(\theta) = 0 \) otherwise. Thus, we let \( Q_2^\alpha(\theta) \) be an indicator function which takes a value of 1 if and only if \( \Phi_\alpha(v_2) \geq \Gamma_\alpha(c_2) \).\(^6\)

Next, we consider the term \( \Phi_\alpha(v_1) Q_2^B_1(\theta) + \Phi_\alpha(v_2) Q_2^B_2(\theta) - (\Gamma_\alpha(c_1) Q_2^S_1(\theta) + \Gamma_\alpha(c_2) Q_2^S_2(\theta)) \). This is again a standard problem and the expression is maximised by inducing trade for agents who belong to the weighted virtual Walrasian sets (see 1.2.6). That is, we set \((Q_2^S)^\alpha\) to be an indicator function which takes a value of 1 if and only if agent \( k \) belongs to the Walrasian sets when buyers types are given by the set \( \{\Phi_\alpha(v_1)\}_{t=1,2} \) and sellers types are given by the set \( \{\Gamma_\alpha(c_1)\}_{t=1,2} \).

Finally, we must determine the optimal \( Q_1(\theta) \). Let

\[
\Pi_1^{\alpha,\delta}(\theta_1) = \Phi_\alpha(v_1) - \Gamma_\alpha(c_1) + \delta E[(\Phi_\alpha(v_2) - \Gamma_\alpha(c_2)) Q_2^\alpha(\theta_1)]
\]

and

\[
\Pi_2^\alpha(\theta_1) = E \left[ \Phi_\alpha(v_1)(Q_2^B_1)^\alpha(\theta) + \Phi_\alpha(v_2)(Q_2^B_2)^\alpha(\theta) - (\Gamma_\alpha(c_1)(Q_2^S_1)^\alpha(\theta) + \Gamma_\alpha(c_2)(Q_2^S_2)^\alpha(\theta)) \right].
\]

Then it is optimal to set \( Q_1^\alpha(\theta_1) \) to be the indicator function that takes a value of 1 if and only if \( \Pi_1^{\alpha,\delta}(\theta_1) \geq \delta \Pi_2^\alpha(\theta) \). When \( v_1 \geq \Phi_\alpha^{-1}(\Gamma_\alpha(c_1)) \), \( \Pi_1^{\alpha,\delta}(\theta_1) \) and \( \Pi_2^\alpha(\theta_1) \) are explicitly given by

\[
\Pi_1^{\alpha,\delta}(\theta_1) = \delta \int_{\Xi} \int_{\Xi} (\Phi_\alpha(v_2) - \Gamma_\alpha(c_2)) g(c_2) f(v_2) dc_2 dv_2 + \Phi_\alpha(v_1)
\]

\[
- \Gamma_\alpha(c_1) \tag{5.8}
\]

and

\[
\Pi_2^\alpha(\theta_1) = \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} (\Phi_\alpha(v_1) - \Gamma_\alpha(c_1)) g(c_2) f(v_2) dc_2 dv_2
\]

\[
+ \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} (\Phi_\alpha(v_1) + \Phi_\alpha(v_2) - \Gamma_\alpha(c_1) - \Gamma_\alpha(c_2)) g(c_2)
\]

\[
\times f(v_2) dc_2 dv_2 + \int_{v_1}^{\Xi} \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} (\Phi_\alpha(v_2) - \Gamma_\alpha(c_1)) g(c_2) f(v_2) dc_2 dv_2
\]

\[
+ \int_{v_1}^{\Xi} \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} (\Phi_\alpha(v_1) + \Phi_\alpha(v_2) - \Gamma_\alpha(c_1) - \Gamma_\alpha(c_2)) g(c_2)
\]

\[
\times f(v_2) dc_2 dv_2 + \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} (\Phi_\alpha(v_1) - \Gamma_\alpha(c_1)) g(c_2) f(v_2) dc_2 dv_2
\]

\[
+ \int_{\Xi} \int_{\Phi_\alpha^{-1}(\Gamma_\alpha(c_1))}^{v_1} (\Phi_\alpha(v_1) - \Gamma_\alpha(c_2)) g(c_2) f(v_2) dc_2 dv_2.
\]

\(^6\)Note that under an \( \alpha \)-mechanism, neither \( Q_2^\alpha(\theta) \) or the expected value of \( (\Phi_\alpha(v_2) - \Gamma_\alpha(c_2)) Q_2^\alpha(\theta) \) depend on \( \theta_1 \).
It follows that $\Pi_1^{\alpha,\delta}$ and $\Pi_2^\alpha$ are differentiable with respect to $v_1$ and $c_1$ since the functions $\Phi$, $\Gamma$, $f$ and $g$ are differentiable.

We have now solved the relaxed version of our optimisation problem. It remains to verify that this solution does not violate the Bayesian incentive compatibility constraints for period 1 agents.

**Proposition 5.3.3.** Under the optimal allocation rules,

\[
(q^{B_1})^\alpha(v_1) = E[Q_1^\alpha(\theta_1) + \delta(1 - Q_1^\alpha(\theta_1))(Q_2^{B_1})^\alpha(\theta)|v_1],
\]

\[
(q^{S_1})^\alpha(c_1) = E[Q_1^\alpha(\theta_1) + \delta(1 - Q_1^\alpha(\theta_1))(Q_2^{S_1})^\alpha(\theta)|c_1]
\]

are non-decreasing in $v_1$ and non-increasing in $c_1$ respectively.

**Proof.** We start by showing that

\[
(q^{B_1})^\alpha(v_1) = E[Q_1^\alpha(\theta_1) + \delta(1 - Q_1^\alpha(\theta_1))(Q_2^{B_1})^\alpha(\theta)|v_1]
\]

is non-decreasing in $v_1$. Recall that under the optimal allocation rule, $Q_2^\alpha(\theta_1)$ and $(Q_2^{B_1})^\alpha(\theta)$ are indicator functions. Thus $\delta(Q_2^{B_1})^\alpha(\theta) \leq 1$ and it suffices to show that $Q_1^\alpha(\theta_1)$ is non-decreasing in $v_1$. Furthermore, we have that $Q_1^\alpha(\theta_1)$ takes a value of 1 if and only if $\Pi_1^{\alpha,\delta}(\theta_1) \geq \delta \Pi_2^\alpha(\theta_1)$. Here, $\Pi_1^{\alpha,\delta}$ and $\Pi_2^\alpha$ are continuous since $F$ and $G$ are continuous. Thus, it suffices to show that $\Pi_1^{\alpha,\delta}$ and $\Pi_2^\alpha$ are non-decreasing in $v_1$ and

\[
\frac{\partial \Pi_1^{\alpha,\delta}(\theta_1)}{\partial v_1} \geq \frac{\partial \Pi_2^\alpha(\theta_1)}{\partial v_1}.
\]

Since $\Phi_\alpha(v_1)$ is increasing in $v_1$, $\Pi_1^{\alpha,\delta}$ is increasing in $v_1$. Furthermore, $\Pi_2^\alpha$ is equal to the maximum welfare generated when two buyers with values $\Phi_\alpha(v_1)$ and $\Phi_\alpha(v_2)$ trade with two sellers with costs $\Gamma_\alpha(c_1)$ and $\Gamma_\alpha(c_2)$. Thus $\Pi_2^\alpha$ is non-decreasing in $\Phi_\alpha(v_1)$. Since $\Phi_\alpha(v_1)$ is increasing in $v_1$, it follows that $\Pi_2^\alpha$ is non-decreasing in $v_1$. It remains to show the derivative condition. As was previously noted, $\Pi_1^{\alpha,\delta}$ and $\Pi_2^\alpha$ are differentiable with respect to $v_1$. Computing the derivative of $\Pi_1^{\alpha,\delta}$ we have

\[
\frac{\partial \Pi_1^{\alpha,\delta}(\theta_1)}{\partial v_1} = \Phi_\alpha'(v_1).
\]

The derivative of $\Pi_2^\alpha$ must be computed more carefully. We start by defining

\[
\tilde{\Pi}_2^\alpha(\theta) = \Phi_\alpha(v_1)Q_2^{B_1}(\theta) + \Phi_\alpha(v_2)Q_2^{B_2}(\theta) - \Gamma_\alpha(c_1)Q_2^{S_1}(\theta) - \Gamma_\alpha(c_2)Q_2^{S_2}(\theta),
\]

so that

\[
\frac{\partial \Pi_2^\alpha(\theta_1)}{\partial v_1} = \frac{\partial}{\partial v_1} E \left[ \tilde{\Pi}_2^\alpha(\theta) \right]. \tag{5.9}
\]
Recall that $Q^B_2(\theta)$ is an indicator function that takes a value of 1 if and only if $B_1$ is in the weighted virtual Walrasian set, given $\theta$. This implies $Q^B_2(\theta)$ is non-decreasing in $v_1$. Furthermore, whenever $Q^B_2(\theta) = 1$, an increase in $v_1$ has no effect on the allocation probabilities $Q^k_2(\theta)$. If $Q^B_2(\theta) = 0$, a decrease in $v_1$ also has no impact on the allocation probabilities. Let $\tilde{v}_1(c_1, v_2, c_2)$ denote the unique value of $v_1$ at which $Q^B_2(\theta)$ changes value from 0 to 1. This is the only point at which $\tilde{\Pi}^{a}_{2}(v_1, c_1, v_2, c_2)$ is not differentiable. Putting this together, we have

$$\frac{\partial \tilde{\Pi}^{a}_{2}(\theta)}{\partial v_1} = \begin{cases} 
\Phi'(v_1), & v_1 > \tilde{v}_1(c_1, v_2, c_2) \\
0, & v_1 < \tilde{v}_1(c_1, v_2, c_2).
\end{cases}$$

Since $\Phi_a(v_1)$ is increasing and has compact support $[\underline{v}, \bar{v}]$, it is bounded and has bounded derivative. Thus, the regularity conditions imply $0 \leq \frac{\partial \tilde{\Pi}^{a}_{2}(\theta)}{\partial v_1} \leq \Phi'(v_1)$ a.e. and we have (as a consequence of the dominated convergence theorem),

$$\frac{\partial \Pi^{a}_{1}(\theta_1)}{\partial c_1} \leq \frac{\partial \Pi^{a}_{2}(\theta_1)}{\partial c_1}. \quad (5.11)$$

A similar argument shows that $q^{S_{1}}(c_1)$ is non-increasing in $c_1$ and

$$\frac{\partial \Pi^{a,\delta}_{1}(\theta_1)}{\partial c_1} \leq \frac{\partial \Pi^{a}_{2}(\theta_1)}{\partial c_1}. \quad (5.11)$$

Now that the class of $\alpha$–mechanisms has been characterised we may express their allocation rules more explicitly and determine $(1 - \alpha)W + \alpha R$.

In particular, for given $\alpha$ and $\delta$, let $\eta^{a,\delta}(c_1)$ denote the lower bound of the region of trade in period under the optimal mechanism. That is, trade in period one occurs if and only if $v_1 \geq \eta^{a,\delta}(c_1)$. If trade occurs in period one, then period two trade simply occurs if $v_2 \geq \Phi^{-1}_a(\Gamma_a(c_2))$. This is illustrated in Figure 5.1. Finally, $\eta^{a,\delta}(c_1)$ is characterised by the equation

$$\Pi^{a,\delta}_{1}(\eta^{a,\delta}(c_1), c_1) - \delta \Pi^{a}_{2}(\eta^{a,\delta}(c_1), c_1) = 0.$$
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\[ \Phi^{-1}_\alpha(\Gamma_\alpha(c_1)) \]

\[ \eta^{\alpha,\delta}(c_1) \]

\[ \Phi^{-1}_\alpha(\Gamma_\alpha(c_2)) \]

Figure 5.1: The period one and period two allocation rules when \( \bar{v} = \bar{c} = 1 \), \( v = \bar{c} = 0 \) and \( v_1 \geq \eta^{\alpha,\delta}(c_1) \).

\[ \Gamma^{-1}_\alpha(\Phi_\alpha(v_1)) \]

\[ \Phi^{-1}_\alpha(\Gamma_\alpha(c_2)) \]

\[ B_1, S_2, B_2, S_1 \]

\[ \Phi^{-1}_\alpha(\Gamma_\alpha(v_1)) \]

Figure 5.2: The period two allocation rules when \( \bar{v} = \bar{c} = 1 \), \( v = \bar{c} = 0 \) and trade does not occur in period one.

5.3.3 Comparative Statics

The Effect of \( \delta \) We begin with a theorem that describes the comparative statics related to the discount factor \( \delta \).

**Theorem 5.3.4.** The function \( \eta^{\alpha,\delta}(c_1) \) is increasing in \( \delta \) and, for \( \delta \in [0, 1) \), in \( c_1 \). It further satisfies \( \eta^{\alpha,0}(c_1) = \Phi^{-1}_\alpha(\Gamma_\alpha(c_1)) \) and \( \eta^{\alpha,1}(c_1) = \bar{v} \). Finally, assuming \( \Phi''(v) = 0 \) and \( \Gamma''(c) = 0 \), the convexity of \( \eta^{\alpha,\delta}(c_1) \) is increasing in \( \delta \). That is,

\[
\frac{\partial}{\partial \delta} \left( \frac{d^2 \eta^{\alpha,\delta}(c_1)}{dc_1^2} \right) > 0.
\]

**Proof.** We start by showing that \( v_1 = \eta^{\alpha,\delta}(c_1) \) is increasing in \( \delta \). This function
is implicitly defined as the solution to the equation

$$
\Pi^\alpha,\delta(\theta_1) := \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) = 0. \tag{5.12}
$$

Differentiating (5.12) we obtain

$$
\frac{\partial \Pi^\alpha,\delta_1(\theta_1)}{\partial v_1} dv_1 + \frac{\partial \Pi^\alpha,\delta_1(\theta_1)}{\partial \delta} - \Pi^\alpha_2(\theta_1) - \delta \frac{\partial \Pi^\alpha_2(\theta_1)}{\partial v_1} dv_1 = 0,
$$

which implies that

$$
dv_1 = \frac{\Pi^\alpha_2(\theta_1) - \frac{\partial}{\partial \delta} \Pi^\alpha,\delta_1(\theta_1)}{\frac{\partial}{\partial v_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right]}.
$$

When \( \delta < 1 \), by (5.10) we have

$$
\frac{\partial}{\partial v_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right] > 0.
$$

We also have

$$
\Pi^\alpha_2(\theta_1) - \frac{\partial \Pi^\alpha,\delta_1(\theta_1)}{\partial \delta} > 0
$$

since

$$
\frac{\partial \Pi^\alpha,\delta_1(\theta_1)}{\partial \delta} = \int_{v_2}^{\bar{v}} \left( \Phi_\alpha(v_2) - \Gamma_\alpha(c_2) \right) g(c_2) f(v_2) dc_2 dv_2
$$

is the value of \( (1 - \alpha)W + \alpha R \) under the associated \( \alpha \)-mechanism when we have a static setup with a single Myerson–Satterthwaite pair, while \( \Pi^\alpha_2(\theta_1) \) is the same quantity but in a static setup with two Myerson–Satterthwaite pairs. It follows that \( dv_1/d\delta > 0 \) and \( \eta^\alpha,\delta(c_1) \) is increasing in \( \delta \).

We now show that, for \( \delta < 1 \), \( v_1 = \eta^\alpha,\delta(c_1) \) is increasing in \( c_1 \). Differentiating (5.12), we obtain

$$
\frac{\partial}{\partial v_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right] dv_1 + \frac{\partial}{\partial c_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right] = 0 \tag{5.13}
$$

which implies that

$$
dv_1 = \frac{\delta \frac{\partial}{\partial c_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right]}{\frac{\partial}{\partial v_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right]}.
$$

When \( \delta < 1 \), by (5.10) and (5.11) we have

$$
\frac{\partial}{\partial v_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right] > 0 \quad \text{and} \quad \frac{\partial}{\partial c_1} \left[ \Pi^\alpha,\delta_1(\theta_1) - \delta \Pi^\alpha_2(\theta_1) \right] > 0. \tag{5.15}
$$
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It immediately follows that the fraction on the right–hand side of (5.14) has a positive denominator that is decreasing in \( \delta \) and a positive numerator that is increasing in \( \delta \). Thus, the fraction on the right–hand side of (5.14) is positive and increasing in \( \delta \) and so too is \( dv_1/dc_1 \).

When \( \delta = 0 \), we have

\[
(1 - \alpha)W + \alpha R = \int_0^\infty [\Phi_\alpha(v_1) - \Phi_\alpha(c_1)] Q_1(\theta_1) f(v) g(c) d\theta.
\]

This is maximised by setting \( Q_1(\theta_1) \) to the indicator function which takes a value of 1 if and only if \( \Phi_\alpha(v_1) \geq \Gamma_\alpha(c_1) \). Hence, in this case we have

\[
\eta^{\alpha,0}(c_1) = \Phi_\alpha^{-1}(\Gamma_\alpha(c_1)).
\]

When \( \delta = 1 \), we have \( \Pi^{\alpha,\delta}(\theta_1) < \delta \Pi^\alpha(\theta_1) \) except for the special case \( v_1 = \bar{v} \) and \( c_1 = \bar{c} \). Thus, we have

\[
\eta^{\alpha,1}(c_1) = \bar{v}.
\]

Finally, we consider the curvature of \( \eta^{\alpha,\delta} \). Differentiating (5.13), we obtain

\[
\frac{\partial}{\partial v_1} [\Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1)] \frac{dv_1}{dc_1} + 2 \frac{\partial^2}{\partial c_1 \partial v_1} [\Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1)] \frac{dv_1}{dc_1} + \frac{\partial^2}{\partial v_1^2} [\Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1)] = 0,
\]

which implies that

\[
\frac{d^2 v_1}{dc_1^2} = - \frac{\frac{\partial^2}{\partial v_1^2} [\Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1)] \left( \frac{dv_1}{dc_1} \right)^2 + \frac{\partial^2}{\partial c_1 \partial v_1} [\Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1)]}{\frac{\partial}{\partial v_1} [\Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1)]}.
\]

Before we can determine the sign of this expression, we must compute the unknown derivatives. First, by (5.8) we have

\[
\frac{\partial^2 \Pi_1^{\alpha,\delta}(\theta_1)}{\partial v_1^2} = \Phi''_\alpha(v_1), \quad \frac{\partial^2 \Pi_1^{\alpha,\delta}(\theta_1)}{\partial c_1^2} = -\Gamma''_\alpha(c_1) \text{ and } \frac{\partial \Pi_1^{\alpha,\delta}(\theta_1)}{\partial c_1 \partial v_1} = 0.
\]

Second, returning to the proof of Proposition 5.3.3, we have

\[
\frac{\partial \Pi_2^\alpha(\theta_1)}{\partial v_1} = E \left[ \Phi'_\alpha(v_1) \mathbb{1}(v_1 > \tilde{v}_1(c_1, v_2, c_2)) \right].
\]
It then immediately follows from the definition of $\tilde{v}_1(c_1, v_2, c_2)$ (see the proof of 5.3.3) that
\[
\frac{\partial \Pi_3^2(\theta_1)}{\partial v_1} = \int_{\mathbb{V}_1} \Phi'_\alpha(v_1) f(v_2) \, dv_2 + \int_{\mathbb{V}} \int_{\mathbb{C}} \Gamma_{\alpha}^{-1} \Phi_\alpha(v_1) \Phi'_\alpha(v_1) f(v_2) g(c_2) \, dc_2 \, dv_2.
\]
Differentiating, we obtain
\[
\frac{\partial^2 \Pi_3^2(\theta_1)}{\partial v_1^2} = \int_{\mathbb{V}_1} \Phi''_\alpha(v_1) f(v_2) \, dv_2 + \Phi'_\alpha(v_1) f(v_1) - \int_{\mathbb{V}} \int_{\mathbb{C}} \Phi''_\alpha(v_1) f(v_2) g(c_2) \, dc_2 \, dv_2
\]
\[
\times f(v_1) g(c_2) \, dc_2 + \int_{\mathbb{V}_1} \frac{\partial}{\partial v_1} \int_{\mathbb{C}} \Gamma_{\alpha}^{-1} \Phi_\alpha(v_1) \Phi'_\alpha(v_1) f(v_2) g(c_2) \, dc_2 \, dv_2
\]
\[
= \int_{\mathbb{V}_1} \Phi''_\alpha(v_1) f(v_2) \, dv_2 + \int_{\mathbb{V}_1} \int_{\mathbb{C}} \Phi''_\alpha(v_1) f(v_2) g(c_2) \, dc_2 \, dv_2
\]
\[
+ (\Gamma_{\alpha}^{-1} \circ \Phi_\alpha)'(v_1) \int_{\mathbb{V}_1} \Phi'_\alpha(v_1) f(v_2) g(\Gamma_{\alpha}^{-1} \circ \Phi_\alpha(v_1)) \, dv_2
\]
\[
+ \int_{\Gamma_{\alpha}^{-1} \circ \Phi_\alpha(v_1)} \Phi'_\alpha(v_1) f(v_1) g(c_2) \, dc_2 = 0.
\]
Finally, using similar arguments to those presented in the proof of Proposition 5.3.3, it can also be shown that
\[
\frac{\partial \Pi_3^2(\theta_1)}{\partial c_1} = - \int_{c_1} \Gamma_{\alpha}'(c_1) g(c_2) \, dc_2 - \int_{\mathbb{V}} \int_{c_1} \Gamma_{\alpha}'(c_1) f(v_2) g(c_2) \, dc_2 \, dv_2.
\]
Differentiating, we obtain
\[
\frac{\partial^2 \Pi_3^2(\theta_1)}{\partial c_1^2} = - \int_{\mathbb{V}} \int_{c_1} \frac{\partial}{\partial c_1} \int_{\mathbb{C}} \Gamma_{\alpha}'(c_1) f(v_2) g(c_2) \, dc_2 \, dv_2
\]
\[
+ (\Phi^{-1}_{\alpha} \circ \Gamma_{\alpha})(c_1) \int_{c_1} \Gamma_{\alpha}'(c_1) f((\Phi^{-1}_{\alpha} \circ \Gamma_{\alpha})(c_1)) g(c_2) \, dc_2
\]
\[
- \int_{c_1} \Gamma''_{\alpha}(c_1) g(c_2) \, dc_2 + \Gamma_{\alpha}'(c_1) g(c_1)
\]
\[
= - \int_{c_1} \Gamma_{\alpha}'(c_1) g(c_2) \, dc_2 - \int_{\mathbb{V}} \int_{c_1} \Gamma''_{\alpha}(c_1) f(v_2) g(c_2) \, dc_2 \, dv_2
\]
\[
+ (\Phi^{-1}_{\alpha} \circ \Gamma_{\alpha})(c_1) \int_{c_1} \Gamma_{\alpha}'(c_1) f((\Phi^{-1}_{\alpha} \circ \Gamma_{\alpha})(c_1)) g(c_2) \, dc_2
\]
\[
+ \int_{\mathbb{V}} \Gamma_{\alpha}(c_1) f(v_2) g(c_1) \, dv_2.
\]
It follows from (5.17) and (5.19) that the second term on the right-hand side of (5.16) is zero. Thus, we have

\[
\frac{d^2 v_1}{dc_1^2} = -\frac{\partial^2}{\partial v_1^2} \left[ \Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^{\alpha}(\theta_1) \right] \left( \frac{dc_1}{v_1} \right)^2 - \frac{\partial^2}{\partial c_1^2} \left[ \Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^{\alpha}(\theta_1) \right].
\] (5.21)

Combining (5.17), (5.18) and (5.20) with \(\Phi''(v) = 0\), \(\Gamma''(c) = 0\), the definition of the virtual type functions (see Section 1.2.6) and the regularity conditions, it follows that the numerator of the right-hand side of (5.21) is positive. Furthermore, since \(dv_1/dc_1\) is increasing in \(\delta\), the numerator of the right-hand side of (5.21) is increasing in \(\delta\). By (5.15), the denominator of the right-hand side of (5.21) is positive. Furthermore, it is decreasing in \(\delta\).

To summarise, we have established that the numerator of the right-hand side of (5.21) is positive and increasing in \(\delta\) and the denominator is positive and decreasing in \(\delta\). Thus, the fraction on the right-hand side of (5.21) is positive for \(\delta > 0\) and increasing in \(\delta\). It immediately follows that

\[
\frac{\partial}{\partial \delta} \left( \frac{d^2 \eta^{\alpha,\delta}(c_1)}{dc_1^2} \right) > 0,
\]

whenever \(\Phi''(v) = 0\) and \(\Gamma''(c) = 0\).

We now describe the intuition behind the results presented in Theorem 5.3.4. The fact that \(\eta^{\alpha,\delta}(c_1)\) is increasing in \(c_1\) follows directly from the Bayesian incentive compatibility constraint. Furthermore, \(\eta^{\alpha,\delta}(c_1)\) is increasing in \(\delta\) due to the tradeoff faced by the market maker when deciding whether to clear the market in period one. If \(v_1 \geq \Phi^{-1}_\alpha(\Gamma_\alpha(c_1))\), the market maker can clear the market in period one and receive a payoff of \(\Phi_\alpha(v_1) - \Gamma_\alpha(c_1)\). If the market is not cleared in period one, the period one agents will increase market efficiency in period two. However, period two payoffs are discounted. As \(\delta\) increases, the value of the contribution of the period one agents to the market in period two increases. Thus, the market maker is less likely to clear the market in period one. When \(\delta = 0\) it is never optimal to wait at \(t = 1\). Thus, the setup collapses to a repeated Myerson–Satterthwaite setup. In this case, trade occurs if and only if \(\Phi_\alpha(v_1) \geq \Gamma_\alpha(c_1)\) and we have \(\eta^{\alpha,0}(c_1) = \Phi^{-1}_\alpha(\Gamma_\alpha(c_1))\). Note that when \(\alpha = 0\), trade occurs if and only if \(v_1 \geq c_1\) and \(\eta^{0,\delta}(c_1) = c_1\).

When \(\delta = 1\), the market will be cleared at \(t = 1\) only if \(v_1 = \bar{v}\) and \(c_1 = \underline{c}\) (an event that occurs with probability zero) and \(\eta^{\alpha,\delta}(c_1) = \bar{v}\).

Finally, the convexity result may also be explained intuitively. The market maker benefits from waiting in period one only if exactly one of the period one agents trades in period two. If neither or both of the period agents would trade in period two, it is ex-post inefficient to clear the market in period one. Next, suppose that \(\bar{v} = \bar{c} = 1\) and \(\underline{v} = \underline{c} = 0\) and consider buyer–seller pairs along the line \(v_1 = 1 - c_1\), shown in Figure 5.3. These pairs of agents are
symmetric in the sense that it is relatively less likely that exactly one of these agents will trade in period two. In contrast, buyer–seller pairings along the line \( v_1 = 0.25 - c_1 \) are asymmetric as they involve a relatively efficient seller paired with a relatively less efficient buyer. Thus, if the market maker does not clear the market in period one, it is relatively more likely that only the seller from period one will trade in period two. Thus, symmetric buyer–seller pairs are more likely to trade in period one and asymmetric buyer–seller pairs are less likely to trade in period one. As a result, dynamics tend to increase the convexity of the trading region in period one.

Under the assumption that \( \Phi''(v) = 0 \) and \( \Gamma''(c) = 0 \), the dynamic effect described in the previous paragraph is the only influence over the curvature of \( \eta^{\alpha,\delta} \). Thus, the curvature of \( \eta^{\alpha,\delta} \) is positive and increasing in \( \delta \). Following this intuition, if the curvature of \( \Phi^{-1}_\alpha \circ \Gamma_\alpha \) is negative, an increase in \( \delta \) will increase the curvature of \( \eta^{\alpha,\delta} \). This occurs because the influence of the negative curvature of \( \Phi^{-1}_\alpha \circ \Gamma_\alpha \) is reduced as \( \delta \) increases, and the dynamic effect which increases the curvature of \( \eta^{\alpha,\delta} \) becomes relatively more important. Finally, if the curvature of \( \Phi^{-1}_\alpha \circ \Gamma_\alpha \) is positive, it is not necessarily the case that an increase in \( \delta \) will result in an increase in the curvature of \( \eta^{\alpha,\delta} \). Whether this occurs depends on the curvature of \( \Phi^{-1}_\alpha \circ \Gamma_\alpha \) (whose influence decreases as \( \delta \) increases) relative to the curvature associated with the dynamic effect (whose influence increases as \( \delta \) increases).

**The Effect of \( \alpha \)** Based on intuition from static mechanism design, one might expect that \( \eta^{\alpha,\delta}(c_1) \) would be increasing in \( \alpha \). However, this is not the
5.3. BAYESIAN OPTIMAL MECHANISMS

Figure 5.4: For uniform types, \( \eta^{\alpha,\delta}(c_1) \) is decreasing in a certain range of \( \alpha \) values for \( \delta = 0.95 \).

... differentiation (5.12) we obtain

\[
\frac{d\eta^{\alpha,\delta}(c_1)}{d\alpha} = \frac{\frac{\partial}{\partial\alpha} \left[ \delta \Pi_2^\alpha(\theta_1) - \Pi_1^{\alpha,\delta}(\theta_1) \right]}{\frac{\partial}{\partial v_1} \left[ \Pi_1^{\alpha,\delta}(\theta_1) - \delta \Pi_2^\alpha(\theta_1) \right]}.
\]

When \( \delta < 1 \), the denominator of this derivative is positive by (5.10). To investigate the sign of the numerator, we consider the contribution from each agent. First, for \( B_1 \) we have

\[
\text{cont}(B_1) = -\frac{\partial \Phi_\alpha(v_1)}{\partial \alpha} + \delta \frac{\partial}{\partial \alpha} \left[ \int_{v_1}^{\nu} \int_{\xi}^{\Gamma_{\alpha}^{-1} \Phi_\alpha(v_1)} \Phi_\alpha(v_1)f(v_2)g(c_2)dc_2dv_2 \right]
\]

\[
+ \delta \frac{\partial}{\partial \alpha} \left[ \int_{v_1}^{\nu} \Phi_\alpha(v_1)f(v_2)dv_2 \right]
\]

\[
= \left( 1 - \delta \int_{v_1}^{\nu} f(v_2)dv_2 - \delta \int_{v_1}^{\nu} \int_{\xi}^{\Gamma_{\alpha}^{-1} \Phi_\alpha(v_1)} f(v_2)g(c_2)dc_2dv_2 \right)
\]

\[
\times \frac{1 - F(v_1)}{f(v_1)} + \delta \frac{\partial \Gamma_{\alpha}^{-1} \circ \Phi_\alpha(v_1)}{\partial \alpha} \int_{v_1}^{\nu} \Phi_\alpha(v_1)f(v_2)g(\Gamma_{\alpha}^{-1} \circ \Phi_\alpha(v_1))dv_2.
\]

For \( B_2 \) we have

\[
\text{cont}(B_2) = \delta \frac{\partial}{\partial \alpha} \left[ \int_{c_1}^{\nu} \int_{\xi}^{\Gamma_{\alpha}^{-1} \circ \Phi_\alpha(v_2)} \Phi_\alpha(v_2)f(v_2)g(c_2)dc_2dv_2 \right]
\]

\[
- \delta \frac{\partial}{\partial \alpha} \left[ \int_{v_1}^{\nu} \int_{\xi}^{\Gamma_{\alpha}^{-1} \circ \Phi_\alpha(v_2)} \Phi_\alpha(v_2)f(v_2)g(c_2)dc_2dv_2 \right]
\]

\[
+ \delta \frac{\partial}{\partial \alpha} \left[ \int_{v_1}^{\nu} \Phi_\alpha(v_2)f(v_2)dv_2 \right].
\]
Figure 5.5: When α increases, the probability that period 1 agents trade in period 2 decreases. This is a distortion that does not occur in static mechanism design setups.

Figure 5.6: The first panel shows profit under welfare maximisation, the second shows profit under the α-mechanism with α = 0.5 and the third shows welfare under profit–maximisation.

The contributions to these integrals from sellers are similar.

It can be seen that there are both positive and negative terms in each of these expressions. Thus, \( \eta_{\alpha, \delta}(c_1) \) is increasing in \( \alpha \) for low values of \( \delta \) but it may decrease in \( \alpha \) for large values of \( \delta \). Figure 5.4 provides a numerical example of this behaviour when buyer valuations and seller costs have a uniform \( U[0, 1] \) distribution (refer to Appendix A.2.1 for Mathematica 10 code).

Intuitively, when the value of α increases, this leads to a decrease in the function \( \Phi_\alpha \) and an increase in the function \( \Gamma_\alpha \). This direct effect tends to decrease trade in both period one and period two and is the usual effect seen in static mechanism design setups. However, for a fixed \( v_1 \) and \( c_1 \), an increase in \( \alpha \) leads to a decrease in the probability that \( B_1 \) and \( S_1 \) trade in period two. This effect (which does not occur in static setups) makes waiting in period one relatively less attractive and tends to increase trade in period one relative
5.4. APPROXIMATE IMPLEMENTATION

The complexity of the Bayesian optimal mechanisms raises the question of whether there are simpler mechanisms that approximate the outcomes of these mechanisms without requiring the designer to elicit types from agents in both periods. In this section, we consider approximate implementation via a price–posting mechanism in period one. We then show that with uniform $U[0,1]$ type distributions and myopically optimal posted prices, this mechanism out-

Comparison to Static Mechanisms  Since there are always an equal number of buyers and sellers in period one and period two, the results of Williams [142] show that the welfare–maximising $\alpha$-mechanism that applies to our setup (as specified in Section 5.2) will run an ex–ante expected deficit. We compare our dynamic mechanisms (and later our price–posting approximation of these mechanisms) to two benchmark static setups. The first is a repeated Myerson–Satterthwaite model, with a discount factor of $\delta$ applied to period two. The second is a Gresik–Satterthwaite model with two buyers and two sellers, with a discount factor of $\delta$ applied to all agents. Compared to these static setups, the welfare–maximising dynamic mechanism must generate more welfare compared to its static counter–parts. However, it also generates more profit under such a mechanism, subject to the constraint that the mechanism does not run a deficit. The profit–maximising dynamic mechanism also generates more welfare compared to its static counter–parts. These results are numerically illustrated in Figure 5.6 when buyer and seller types have a $U[0,1]$ distribution (refer to Appendix A.2.2 for MATHEMATICA 10 code). However, we can immediately see that these results hold for general $F$ and $G$. This follows from the fact that the relative positions of the endpoints of each of the profit and welfare schedules shown in Figure 5.6 are invariant under a change in $F$ and $G$ (of course, for profit schedules, we must condition on the $\alpha$–mechanisms running an ex–ante expected deficit or profit) and these schedules are monotone in $\delta$.

Finally, given any discount factor $\delta$, we let $\alpha^*(\delta)$ denote the value of $\alpha$ corresponding to the second–best mechanism in our dynamic setup. We also let $\alpha^*_{GS}$ and $\alpha^*_{MS}$ denote the values of $\alpha$ corresponding to the second–best mechanisms of the Gresik–Satterthwaite model and the Myerson–Satterthwaite model, respectively. It follows from the analysis of Gresik and Satterthwaite [68] that $\alpha^*_{GS} < \alpha^*_{MS}$. Thus, $\alpha^*(\delta)$ decreases in $\delta$ and we have $\alpha^*(0) = \alpha^*_{MS}$ and $\alpha^*(1) = \alpha^*_{GS}$. Figure 5.7 shows a numerical computation of $\alpha^*(\delta)$ for uniform types using Ridder’s method (see Flannery et al. [54] for Ridder’s method and Appendix A.2.3 for MATHEMATICA 10 code).

5.4 Approximate Implementation

The complexity of the Bayesian optimal mechanisms raises the question of whether there are simpler mechanisms that approximate the outcomes of these mechanisms without requiring the designer to elicit types from agents in both periods. In this section, we consider approximate implementation via a price–posting mechanism in period one. We then show that with uniform $U[0,1]$ type distributions and myopically optimal posted prices, this mechanism out-
The second–best mechanism is such that $\alpha^* (\delta)$ decreases in $\delta$, $\alpha^*(0) = \alpha^*_{MS}$ and $\alpha^*(1) = \alpha^*_{GS}$.

performs the static benchmark mechanisms, provided $\delta$ is sufficiently large. For ease of exposition, we focus on maximising expected discounted welfare. However, the analysis presented here generalises to other design objectives, provided appropriate posted prices are chosen in period one.

Suppose that the designer no longer elicits information from agents in period one. Instead, the designer posts period one prices of $p_B$ for the buyer and $p_S$ for the seller. If both period one agents accept the posted prices they trade and the designer runs the ex–post efficient Myerson–Satterthwaite mechanism in period two. Otherwise, period one agents wait until period two when the designer runs the ex–post efficient Gresik–Satterthwaite mechanism.

Let $\Pi^{B1}_1(v_1)$ denote the expected payoff received by the period one buyer in period two when her value is $v_1$. Similarly, let $\Pi^{S1}_1(c_1)$ denote the expected payoff received by the period one seller in period two when his cost is $c_1$. Since these functions are monotone in agent types, given $p_B$ and $p_S$, we can define cutoff types $\tilde{v}_1(p_B)$ and $\tilde{c}_1(p_S)$. These cutoff types are such that the period one buyer will accept $p_B$ whenever her valuation exceeds $\tilde{v}_1(p_B)$ and the period one seller will accept $p_S$ whenever his cost is below $\tilde{c}_1(p_S)$. The cutoff types $\tilde{v}_1(p_B)$ and $\tilde{c}_1(p_S)$ then satisfy the following equations

$$ \delta \Pi^{B1}_2(\tilde{v}_1(p_B)) = \tilde{v}_1(p_B) - p_B \quad \text{and} \quad \delta \Pi^{S1}_2(\tilde{c}_1(p_S)) = p_S - \tilde{c}_1(p_S). $$

We now suppose that agent types have a uniform $U[0,1]$ distribution and that myopically optimal posted prices of $p_B = p_S = 0.5$ are chosen (see Hagerty and Rogerson [70]). Then we may numerically compute the cutoff type functions and compare the performance of the price–posting mechanism to the welfare–maximising mechanism that applies to the Myerson–Satterthwaite and Gresik–Satterthwaite models described in the previous
5.4. APPROXIMATE IMPLEMENTATION

Figure 5.8: The performance of the price–posting mechanism relative to two static benchmarks.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>0.6</td>
<td>0.3</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Figure 5.9: The period one allocation under the price–posting mechanism.

(a) \( \delta = 0.85 \)  
(b) \( \delta = 0 \)

Following this logic, we may extend these results for general \( F \) and \( G \) by
characterising the endpoints and exploiting the monotonicity of the welfare schedules shown in Figure 5.8. The only qualification that applies to the more general case is that, with myopically optimal prices posted in period one, it is not necessarily the case that the price–posting mechanism will collapse to the Gresik–Satterthwaite model when \( \delta = 1 \). The Gresik–Satterthwaite mechanism may outperform the price–posting mechanism for sufficiently large \( \delta \). However, with general \( F \) and \( G \) and optimally chosen posted prices the price–posting mechanism will collapse to the Gresik–Satterthwaite model when \( \delta = 1 \).{\textsuperscript{7}}

5.5 Conclusion

The primary goal of this thesis is to develop tractable models of dynamic markets and in this chapter we considered a natural starting point – a two–period extension of the classic bilateral trade model of Myerson and Satterthwaite [117]. In this setup the market maker faces a tradeoff between the opportunity cost of delaying trade to period two and increasing market thickness in period two. We derived the class of Bayesian optimal \( \alpha \)–mechanisms and investigated the properties of these mechanisms. In particular, we showed that the Bayesian optimal dynamic mechanism outperforms two benchmark static mechanisms. The optimal dynamic mechanisms derived in this chapter are complex insofar as the optimal allocation and payment rules are non–trivial functions of agent reports in period one. This motivated us to consider a price–posting approximation of these mechanisms in Section 5.4.

In this chapter we solved our two–period dynamic model with continuous type distributions using backward induction, which required computing payoffs for every possible realisation of agent types. The resulting characterisation of the optimal mechanisms made investigating the properties of these mechanisms (such as proving Theorem 5.3.4) cumbersome. For example, we were unable to further generalise and formalise our discussion regarding the comparison of our dynamically optimal mechanisms to the benchmark static models and the price–posting mechanism. Furthermore, in its current form, our dynamic model cannot easily be generalised to include additional periods and agents. It appears that fixed point arguments could be used to prove the existence of an optimal mechanism in a model with a finite number of periods. However, the methodology we used to derive comparative statics results for the two period model does not appear to be sufficiently tractable to allow us to consider additional periods. Finally, our comparative static results relating to the parameter \( \alpha \) is counter–intuitive and it is not clear if this is driven by end–of–the–world effects. Motivated by these observations, in the next chapter we analyse a more tractable dynamic market model with countably many periods and binary agent type distributions.

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{\textsuperscript{7}}To see this, simply set \( p_B = 1 \) and \( p_S = 0 \) when \( \delta = 1 \).
Chapter 6

Optimal Market Thickness and Clearing

We determine the class of Bayesian optimal mechanisms in an infinite-horizon model in which agents have binary types.

6.1 Introduction

Notwithstanding the standard model of the market, in which trade is centralised and essentially static, trading institutions in the real world differ with respect to the degree of centralisation and the frequency and flexibility with which markets clear. Continuous-time double auctions, widely used in stock exchanges, execute compatible trades instantaneously. Some exchanges, such as the Global Dairy Trade and the IEX, clear at fixed intervals, while others run without a fixed schedule. Markets also differ with respect to the degree of centralisation with which they operate and clear as witnessed, for example, by the policy debates and legislative initiatives regarding clearing in financial markets in the wake of the Global Financial Crisis.

All else equal, larger centralised markets are better because, as Williamson [143] observed in a different context, in the integrated market the market maker can always replicate market outcomes in two, or more, smaller stand-alone markets, and can sometimes do strictly better by exploiting additional or superior opportunities for trade. Not surprisingly, therefore, increasing market thickness has been at the heart of economics, from Adam Smith’s farmers’ access to markets to the growth of modern cities, the emergence of financial markets, internet platforms, health care exchanges and contemporary market design initiatives such as the FCC’s ongoing incentive auction.\(^1\)

When traders arrive over time, a tradeoff arises between the opportunity cost of delay and the increase in market thickness due to delay. This raises the question of what the optimal market clearing policy is as well as whether the standard market model is an innocuous simplification or misses important

\(^1\)For a recent survey of two-sided market design, see Loertscher, Marx, and Wilkening [91].
elements, thereby clouding relevant insights. Assessing this tradeoff is challenging because the state space expands with increases in the time horizon, the number of agents and the richness of agents’ type spaces and because future states will be endogenous to the market clearing policy. Moreover, when agents have private information about their types, the analysis will also have to account for their incentive compatibility and individual rationality constraints. Some of these issues were highlighted in Chapter 5.

In this chapter, we set up and solve a dynamic model, overcoming the aforementioned challenges by imposing a binary type space, assuming that in every period exactly one buyer–seller pair arrives and by mapping the problem to a Markov decision process, which allows us to account for the endogeneity of future states and paths. We show that a particularly simple type of policy, a so-called threshold policy, is optimal.\(^2\) The mechanism that implements the dynamically efficient policy, while accounting for agents’ incentive compatibility and individual rationality constraints, is found by relating the optimal policy to an allocation rule and using mechanism design techniques to construct appropriate transfers. Profit–maximising and constrained efficient mechanisms have allocation rules that are isomorphic to that implied by the dynamically efficient policy and can hence be constructed analogously.

From our analysis emerge dynamic notions of efficiency and optimality that differ from their static counterparts in subtle but important ways. In contrast to efficiency in static settings, dynamic efficiency is neither a distribution–free concept nor does it coincide with ex–post efficiency. In the dynamic setting, efficiency requires trading off the benefits from inferior trades today against superior but uncertain trades in the future. Dynamic efficiency involves an element of speculation, which depends on the distributions of types. Moreover, for a fixed discount factor, executing fewer inferior trades is indicative of greater rent extraction by the market maker.

We distinguish between three forms of dynamic market clearing policies in a centralised clearing house: discriminatory (or unconstrained), uniform and fixed frequency market clearing. We contrast these with what we call decentralised trade, whereby we mean the institution in which every agent either trades in the period in which he arrives, provided the buyer’s value exceeds the seller’s cost in this period, or never trades. We show that a profit–maximising market maker under each form of centralisation – i.e. under discriminatory, uniform or fixed frequency market clearing – generates higher social welfare than decentralised trade. Furthermore, the main gains from of centralisation relative to a decentralised market are already achieved with fixed frequency market clearing, which is the least sophisticated form of centralisation, if the discount factor is sufficiently large. Moreover, we show

\(^2\)In Chapter 7 we will consider richer type spaces, where threshold policies are used to construct a finite partition of the state space. This allows the optimal market clearing policy to be determined using a simple implementation of the standard policy iteration algorithm (see Section 2.4).
that the tradeoff between increased market thickness and delayed trade can be
exploited to overcome the impossibility of (ex–post) efficiency without running
a deficit, provided the agents and the market maker are patient enough.

This chapter relates to the literature on market making and intermedi-
ation. There is a vast literature on intermediation in two–sided dynamic
markets, often motivated by the study of financial markets; see, for example,
Mendelson [107] or Kelly and Yudovina [79] and references therein.\textsuperscript{3} In con-
trast to this chapter, the aforementioned literature does not consider agents’
incentives in two–sided intermediated markets and leaves aside the possibility
that the market maker’s objective may entail generating profit. This, to-
gether with increasing market thickness under the guise of “getting both sides
on board”, looms large in the literature on two–sided markets in Industrial
Organisation such as Caillaud and Jullien [28, 27], Rochet and Tirole [127,
129], Armstrong [4], and Gomes [66]. With the notable exception of Gomes,
these papers take a reduced–form approach as to how the intermediary gen-
erates surplus. Typically, the setups analysed in this literature are one–shot.
Rent extraction is also at centre stage in the literature on static Bayesian
mechanism design with two–sided private information such as Myerson and
Satterthwaite [117], Baliga and Vohra [10], or Loertscher and Marx [90]. Our
model generalises the design setup to a dynamic, infinite horizon model while
narrowing the scope to discrete types to keep the state space tractable. Of
course, revenue considerations play an important role in the impossibility the-
orems of ex–post efficient trade such as Myerson and Satterthwaite [117] and
Gresik and Satterthwaite [68]. An implication of our analysis of dynamically
efficient market clearing is that the impossibility vanishes with enough pa-
tience.

This chapter is also related to the burgeoning literature on dynamic match-
ing. Apart from assuming quasilinear utility and allowing for transfers, our
model parallels that of Baccara, Lee, and Yariv [9], who determine an efficient
mechanism in a dynamic matching environment without transfers. Techni-
cally, the key difference between our model and theirs, besides transfers, is
that Baccara, Lee, and Yariv [9] incorporate an explicit per period waiting cost
while in our model all agents, including the market maker, have the same dis-
count factor.\textsuperscript{4} In a setup with one–sided matching in which the match–specific
surplus function induces an element of interdependent valuations, Herbst and
Schickner [73] also assess the tradeoff between the quality of matches and the

\textsuperscript{3}Malamud and Rostek [101] show that decentralised exchanges may outperform cen-
tralised ones. At first glance, this may appear to invalidate our introductory statements
that larger markets are better, all else equal. The contradiction is spurious because Mala-
mud and Rostek consider a fixed market mechanism, which is not optimal. Consequently,
arguments based on selective intervention in the tradition of Williamson [143] do not apply.

\textsuperscript{4}Baccara, Lee, and Yariv [9] study an infinite horizon Markov decision process without
discounting, while we consider an infinite horizon Markov decision process with discounting
(see 2.4).
cost of waiting. Our work is complementary to theirs in that it focuses on a
two–sided setup with private values and a homogenous good.

There has also been a recent upsurge of interest in dynamic mechanism
design (see Bergemann and Said [15] and Bergemann and Pavan [14] for re-
cent surveys). Much of this literature, notably Bergemann and Välimäki [17],
Athey and Segal [6], Pavan, Segal, and Toikka [121] and Skrzypacz and Toikka
[136], considers settings in which a static population of agents receives private
information over time. The aforementioned papers focus on constructing in-
centive compatible transfers and implicitly define the allocation rule as the
solution to a dynamic programming problem. A notable exception is Fersht-
man and Pavan [52], who explicitly compare outcomes under welfare and profit
maximisation in a two–sided many–to–many matching economy in which agent
valuations evolve dynamically. This chapter is part of the small but growing
strand of literature that considers a dynamic population of agents with per-
sistent types. In such setups, the current allocation decision determines the
set of feasible allocations in future periods and the designer faces the optimal
timing problem of deciding when to run a static mechanism. Recent contribu-
tions to this strand of literature include Parkes and Singh [120], Gershkov and
Moldovani [60] and Board and Skrzypacz [20]. However, these papers do not
explicitly address the optimal timing problem nor do they consider varying de-
grees of market centralisation or compare welfare and profit maximisation, all
of which are considered in this chapter. More recent contributions which ad-
dress the optimal timing problem in one-sided settings include Pai and Vohra
[119] and Mierendorff [110] (who study revenue–maximisation in a dynamic
environment with one-sided private information), as well as Mierendorff [111]
(who considers the efficient allocation of a single object).

The remainder of this chapter is structured as follows. Section 6.2 intro-
duces the setup. Section 6.3 derives the dynamically efficient policies under
discriminatory, uniform and fixed frequency market clearing. Section 6.4 ac-
counts for agents’ private information and derives the dynamically optimal
mechanisms. Section 6.5 concludes.

6.2 Setup

We consider an infinite–horizon setup with one market maker (which we also
refer to as the designer when accounting for agents’ private information) in
which a buyer $B_t$ and a seller $S_t$ arrive in each period $t \in \mathbb{N}$. All agents
and the market maker are risk neutral and have a common discount factor
$\delta \in (0, 1)$. All agents have quasilinear preferences. Each buyer demands at
most one unit and each seller has the capacity to produce at most one unit.
The value of agents’ outside option of not participating is zero.

Buyers are of type $\bar{v}$ or $v$ with probability $p$ or $1 – p$ respectively. Sellers
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Figure 6.1: In every period, a buyer–seller pair arrives. Buyers and sellers draw their values and costs independently from the distributions \( \{v, \bar{v}\} \) and \( \{c, \bar{c}\} \), respectively, with \( \bar{c} < v < \bar{v} < v, \) probability \( p \) on \( v \) and on \( c \) and the common discount factor \( \delta \). For \( \delta = 1 \), \( p \) is the ex–post efficient quantity.

are of type \( c \) with probability \( p \) and \( \bar{c} \) with probability \( 1 - p \). We assume

\[
\bar{v} > \bar{c} > v > c \geq 0
\]  

(6.1)

to obtain a non–trivial problem.\(^5\) Throughout this paper, we will refer to buyers of type \( \bar{v} \) and sellers of type \( \bar{c} \) as efficient. Similarly, we will refer to buyers of type \( v \) and sellers of type \( c \) as inefficient. When \( \delta = 1 \) the setup collapses to a static environment in which there is a continuum of traders and \( p \) is the proportion of sellers of type \( v \) and buyers of type \( c \). The ex–post efficient outcome for this setup is illustrated in Figure 6.1.

We assume that agents can only trade via the market maker’s platform and that agents who are cleared from the market exit forever. In practical markets market makers may face a variety of constraints, such as the requirement that the whole market be cleared at the time of clearing,\(^6\) and we distinguish between three forms of market clearing policies.

**Definition 6.2.1.** Under discriminatory market clearing the market maker determines which agents are cleared from the market in each period. Under Uniform market clearing the whole market is cleared at the time of clearing.

\(^5\)These conditions avoid the trivial cases \( \bar{v} < c \) (where it is efficient for no trade to take place), \( v > \bar{c} \) (where it is efficient for every pair of agents to trade) and \( \bar{v} < \bar{c} \) and \( v < c \) (where it is efficient for only agents of type \( \bar{v} \) and \( \bar{c} \) to trade). It also avoids the asymmetric cases where only one type of the \( v \) and \( c \) agents can trade.

\(^6\)For example, many farmers’ markets operate on fixed weekdays. Similarly, the Global Dairy Trading company clears the market at fixed intervals.
Finally, fixed frequency market clearing requires that, in addition to market clearing being uniform, the market is cleared at fixed intervals.

6.3 Dynamic Efficiency

To determine the dynamically efficient market clearing policies, we formulate our problem as a Markov decision process, which we then use to analyse, in turn, discriminatory, uniform and fixed frequency market clearing. Throughout this section, we assume the market maker is fully informed about the types of agents who have arrived. We relax this assumption in Section 6.4.

6.3.1 Markov Decision Processes

We now describe the market maker’s dynamic optimisation problem in terms of a Markov decision process, normalising \( v - c = 1 \) and imposing the symmetry assumption \( v - c = v - c =: \Delta \). Notice that (6.1) implies \( \Delta < 1/2 \) so if pairs of type \((\tau, \tau)\) and \((\nu, \nu)\) are present, an increase in welfare is achieved by rematching these pairs to create a \((\nu, \nu)\) pair. Under discriminatory market clearing, for example, the market maker’s problem is to determine which pairs to clear from the market following the arrival of a new pair in each period \( t \in \mathbb{N} \). When a pair of type \((\tau, \tau)\) or \((\nu, \nu)\) is present, the market maker has an incentive to wait (rather than clear the market) in the hope of eventually rematching the efficient agent in this pair with another efficient agent, creating a \((\nu, \nu)\) trade. In principle, this decision depends on the entire history of the arrival process. However, as shown below, the state space can be simplified considerably.

We call a \((\nu, \nu)\) pair efficient, pairs \((\tau, \tau)\) and \((\nu, \nu)\) suboptimal and a \((\tau, \tau)\) pair inefficient. Regardless of the form of market clearing, the underlying state at date \( t \) is identified as follows. First determine the number of efficient pairs present. Then determine the number of identical suboptimal \((\tau, \tau)\) or \((\nu, \nu)\) pairs present among the remaining set of agents. Notice that non–identical suboptimal pairs, \((\tau, \tau)\) and \((\nu, \nu)\), cannot be simultaneously present as these pairs can be split and rematched to form one efficient pair and one inefficient pair. Inefficient pairs can be ignored since these do not generate positive surplus and cannot be rematched to create efficient pairs. Thus, for every form of market clearing the binary state space of the market maker’s Markov decision process is given by \( \mathcal{X} := \{(x_E, x_I) : x_E, x_I \in \mathbb{Z}_{\geq 0}\} \), where \( x_E \) and \( x_I \) are the number of efficient pairs and suboptimal pairs present, respectively. Let \( \mathbf{X}_t \in \mathcal{X} \) denote the state of the market after the arrival of period \( t \) agents.

Denote by \( \mathcal{A}_x \) the set of actions available to the market maker in state \( x \) and let \( A = \bigcup_{x \in \mathcal{X}} \mathcal{A}_x \). Notice that unlike \( \mathcal{X} \), \( A \) depends on the market clearing constraints faced by the market maker. Under discriminatory market clearing, the market maker may partially clear the market. Since it is optimal to immediately clear all efficient pairs, under discriminatory market clearing
we restrict ourselves to the action space \( A_x = \{(a_E, a_I) : a_E, a_I \in \mathbb{Z}_{\geq 0}, a_E = x_E, a_I \leq x_I\} \), where \( a_E \) and \( a_I \) denote the respective number of efficient pairs and suboptimal pairs cleared from the market. Under uniform and fixed frequency market clearing, the entire market must be cleared, in which case we have \( A'_x = \{0, x\} \). Here we use the ‘prime’ notation to distinguish different objects that will eventually be used to construct Markov decision processes associated with each form of market clearing.

As in Section 2.4 we let \( A_t \) denote the action taken by the market maker in period \( t \in \mathbb{N} \), and denote by
\[
P_a(x, y) := \mathbb{P}(X_{t+1} = y \mid X_t = x, A_t = a)
\]
the transition probability that, if the market maker takes the action \( a \) in state \( x \) in period \( t \), the state in period \( t + 1 \) will be \( y \). For any action \( a = (a_E, a_I) \), we have
\[
P_a(x, (x_E - a_E + 1, x_I - a_I)) = p^2, \quad P_a(x, (x_E - a_E, x_I - a_I)) = (1 - p)^2.
\]
If \( x_I = 0 \) or \( a_I = x_I \) a suboptimal pair arriving in period \( t + 1 \) cannot be rematched. We have
\[
P_a(x, (x_E - a_E, 1)) = 2p(1 - p).
\]
Otherwise, if an identical suboptimal pair arrives, it cannot be rematched and if a non–identical suboptimal pair arrives, the efficient agents in each pair can be rematched to form one efficient pair. Consequently, we have
\[
P_a(x, (x_E - a_E, x_I - a_I + 1)) = P_a(x, (x_E - a_E + 1, x_I - a_I - 1)) = p(1 - p).
\]
We denote by
\[
r(a) = a_E + \Delta a_I
\]
the immediate reward when action \( a \in A \) is implemented.

To unify the analysis for all forms of market clearing, we assume that the market maker observes a state-dependent signal \( s \in S \). The timing in each period is then as follows. At the beginning of a given period a pair of agents arrives, the market maker then observes a signal regarding the state of the market and takes an action before the period ends.

Let \( S_t \) denote the signal observed by the market maker in period \( t \) and denote by \( O(x, s) \) the probability that the process is in state \( x \) given that the

\(^7\)Notice that under uniform or fixed frequency market clearing the market maker does not need to observe the state \( x \) to implement these actions. Here, \( x \) corresponds to clearing the entire market and \( 0 \) corresponds to waiting. We define the action space using this notation for convenience.
market maker observes the signal $s$. Under discriminatory and uniform market clearing, the market maker can make decisions contingent on the state of the market, in which case we have $S = \mathcal{X}$ and $O(x, s) = 1$. For fixed frequency market clearing, the signal $s$ is simply a scalar equal to the number of periods since the market was last cleared. Using $O'$ and $S'$ to denote, respectively, probabilities and the set of signals under fixed frequency market clearing, we therefore have $S' = \mathbb{N}$ and

$$O'(x, s) := \mathbb{P}(X_t = x \mid S_t = s) = \mathbb{P}(X_t = x \mid A_{t-s} = X_{t-s}, A_{t-s+1} = ... = A_{t-1} = 0),$$

since the market maker observes the signal $s \in S'$ if the market was last cleared in period $t - s$. Let $s \in S'$ be given. Then, for any $x \in \mathcal{X}$ such that $x_E + x_I > s$, $O'(x, s) = 0$. For $x \in \mathcal{X}$, with $x_I = 0$ and $x_E \leq s$ we have

$$O'(x, s) = \left(\frac{s}{x_E}\right)^{x_E} (1 - p)^{s - x_E},$$

which is the probability that $x_E$ efficient agents arrive on each side of the market in $s$ periods. Similarly, for $x \in \mathcal{X}$ with $x_I > 0$ and $x_E + x_I \leq s$ we have

$$O'(x, s) = 2 \left(\frac{s}{x_E + x_I}\right) \left(\frac{s}{x_E}\right)^{x_E + x_I} (1 - p)^{2s - 2x_E - x_I}.$$

This is the probability that $x_E + x_I$ efficient agents arrive on one side of the market and $x_E$ arrive on the other.

Recall from Section 2.4 that given a partially observable Markov decision process $\langle \mathcal{X}, A, P, r, S, O, \delta \rangle$, a policy $\pi : S \rightarrow A$ is such that $\pi(s) \in A$ specifies the action taken by the market maker upon observing the signal $s$. Under discriminatory market clearing, the market maker’s optimisation problem reduces to determining the optimal policy of $\langle \mathcal{X}, A, P, r, S, O, \delta \rangle$. Under uniform and fixed frequency market clearing, the market maker determines the optimal policy of $\langle \mathcal{X}, A', P, r, S', O', \delta \rangle$, respectively. 

Under a given policy, the state of the market at the end of each period evolves according to a Markov chain $\{Y_t\}_{t \in \mathbb{N}}$. We refer to this as the market order book. In order to characterise the evolution of the order book, we must determine the optimal policy under each form of market clearing.

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*While it is convenient to use one model to describe discriminatory, uniform and fixed frequency market clearing, it suffices to consider standard Markov decision processes for discriminatory and uniform market clearing. For a Markov decision process $\langle \mathcal{X}, A, P, r, \delta \rangle$, a policy $\pi : \mathcal{X} \rightarrow A$ is such that $\pi(x) \in A_x$ specifies the action taken by the market maker in state $x$. Then under discriminatory and uniform market clearing, the market maker’s optimisation problem is simply to determine the optimal policy of the Markov decision processes $\langle \mathcal{X}, A, P, r, \delta \rangle$ and $\langle \mathcal{X}, A', P, r, \delta \rangle$ respectively.*
6.3. DYNAMIC EFFICIENCY

6.3.2 Threshold Policies

We begin by defining a simple class of policies, which we call threshold policies. Under a threshold policy, the market maker chooses from two possible actions on the basis of a one-dimensional statistic that describes the state of the market. If the statistic is below some threshold, the market maker implements one action. If the threshold is achieved or exceeded, the market maker implements the other action.

Definition 6.3.1. Under uniform and fixed frequency market clearing a threshold policy \( \pi^\tau \) consists of a mapping \( T: S \rightarrow \mathbb{R}_{\geq 0} \) and an action space \( A^T_S = \{0, s\} \) and a threshold \( \tau \in \mathbb{R}_{\geq 0} \) such that

\[
\pi^\tau(s) = 0 \quad \text{if} \quad T(s) < \tau \quad \text{and} \quad \pi^\tau(s) = s \quad \text{if} \quad T(s) \geq \tau.
\]

Under discriminatory market clearing a threshold policy \( \pi^\tau \) consists of a mapping \( T: X \rightarrow \mathbb{R}_{\geq 0} \), an action space \( A^T_X = \{(x_E, 0), (x_E, a(x_I))\} \) and a threshold \( \tau \in \mathbb{R}_{\geq 0} \) such that

\[
\pi^\tau(x) = (x_E, 0) \quad \text{if} \quad T(x) < \tau \quad \text{and} \quad \pi^\tau(x) = (x_E, a(x_I)) \quad \text{if} \quad T(x) \geq \tau.
\]

It turns out that the optimal policy is a threshold policy under each form of market clearing. This is intuitive, given that the market maker essentially faces a binary choice in each period\(^9\) and that the arrival process is stationary. Threshold policies are analogous to policies induced by a Gittens [64] index, that apply to multi-armed bandit problems. Similar policies are optimal in the setups of Baccara, Lee, and Yariv [9] and Herbst and Schickner [73].

Theorem 6.3.2. The optimal policy under discriminatory, uniform and fixed frequency market clearing is a threshold policy with threshold \( \tau(p, \Delta, \delta) \). Specifically:

1. Under fixed frequency market clearing \( T(s) = s \).
2. Under uniform market clearing \( s = x \) and \( T(s) = r(x) \).
3. Under discriminatory market clearing \( T(x) = x_I \) and \( a(x_I) = \lceil x_I - \tau \rceil \).

Proof. Under discriminatory market clearing, the optimal policy must immediately clear all efficient pairs. Since the arrival process is stationary and the discount factor is constant, the optimal policy is stationary. Sample paths of the Markov decision process are such that if \( x_I \) suboptimal trades are stored in a given period, \( x_I - 1 \) trades must have been stored in some previous period.

\(^9\)Under discriminatory market clearing the market maker simply decides whether to clear or store after the arrival of an identical suboptimal pair. Under uniform and fixed frequency market clearing, the market maker decides whether to wait or clear the market in each period.
Thus by stationarity, if \( x_I \) suboptimal trades are stored under the optimal policy, it must be optimal to retain \( x_I - 1 \) trades.

An unbounded number of suboptimal pairs cannot be stored under the optimal policy. As the number of stored suboptimal pairs diverges to infinity, the expected number of periods until an additional stored suboptimal pair is rematched diverges to infinity. Thus, the benefit of storing an additional suboptimal pair converges to zero, while the benefit of immediately clearing a suboptimal pair is \( \Delta \). Therefore, there exists a maximum number \( \pi^* \) of suboptimal trades which can be optimally stored. Thus, the optimal policy \( \pi^* \) is a threshold policy.

We next consider uniform market clearing. If we use the notation of Blackwell [19] we can map our action space \( \mathcal{A}' \) to one containing only two actions (which correspond to clearing and waiting). Thus, the results of Blackwell apply and an optimal policy \( \pi^* \) must exist. Let \( \pi^* \) denote any optimal policy of the Markov decision process \( \langle X, \mathcal{A}', P, r, \delta \rangle \). The optimal policy is stationary (the arrival process is stationary and the discount factor is constant) and must clear the market whenever it is in a state of the form \( (x_E, 0) \), with \( x_E \in \mathbb{N} \). As the number of stored suboptimal pairs diverges to infinity, the expected time until each additional stored pair is rematched diverges to infinity. Therefore, the benefit of storing each additional suboptimal pair converges to zero, while the immediate reward for clearing a suboptimal pair from the market is fixed at \( \Delta \). Thus, for a given number of stored efficient pairs, the optimal policy cannot allow an unbounded number of identical suboptimal pairs to accumulate.

It follows that for every \( x^*_E \in \mathbb{Z}_{>0} \) there exists a state \( x^* = (x^*_E, x^*_I) \) such that \( \pi^*(x^*) = 0 \) and \( \pi^*(x^*_E, x^*_I + 1) = (x^*_E, x^*_I + 1) \). We call such states cut-off states. Denote the expected present value of being in the cutoff state \( x^* \) under the optimal policy by \( V_{\pi^*}^U(x^*) \), the total expected discounted reward earned by the designer in the subsequent period. It is finite because an unbounded number of pairs cannot accumulate under \( \pi^* \) and we are considering a discounted process. For any state \( x \), the benefit of waiting to clear the market is increasing in \( x_I \) and the benefit of clearing is increasing in \( r(x) \).

Since \( r(x^*_E + 1, x^*_I) > r(x^*) \) and \( r(x^*_E + 1, x^*_I - 1) > r(x^*) \) it follows that if \( \pi^*(x^*_E, x^*_I + 1) = (x^*_E, x^*_I + 1) \), we must also have \( \pi^*(x^*_E + 1, x^*_I) = (x^*_E + 1, x^*_I) \) and \( \pi^*(x^*_E + 1, x^*_I - 1) = (x^*_E + 1, x^*_I - 1) \). Finally, let \( V_{\pi^*}^U(0) \) denote the expected present value of being in the state 0 under the optimal policy. The Bellman equation which characterises \( V_{\pi^*}^U(x^*) \) is then given by

\[
V_{\pi^*}^U(x^*) = \delta \left[ p^2 (r(x^*) + 1 + V_{\pi^*}^U(0)) + p(1-p)(r(x^*) + \Delta + V_{\pi^*}^U(0)) \right]
+ p(1-p)(r(x^*) + 1 - \Delta + V_{\pi^*}^U(0)) + (1-p)^2 V_{\pi^*}^U(x^*). \tag{6.2}
\]

If the market is cleared in state \( x^* \), the payoff is the immediate reward \( r(x^*) \) plus the expected present value of being in the state 0. By the principle of
the optimality of dynamic programming,

\[ V_{π^*}(x^*) \geq r(x^*) + V_{π^*}(0). \]  \hfill (6.3)

Notice that the right-hand sides of (6.2) and (6.3) depend directly on \( x^* \) only through \( r(x^*) \). Replace \( r(x^*) \) with \( τ^* \) in (6.2) and (6.3) and suppose (6.3) holds with equality. Then, for every cutoff state \( x^* \), \( r(x^*) \leq τ^* \). Using the definition of \( τ^* \), substituting (6.3) into (6.2) and rearranging, it can be shown that \( τ^* \) satisfies

\[ τ^* + V_{π^*}(0) = \frac{δp}{1−δ}. \]  \hfill (6.4)

Thus, for any state \( x \in \mathcal{X} \setminus \{(x_E, 0) : x_E \in \mathbb{N}\} \), the market should be cleared if and only if \( x^*_E + \Delta x_j^* > τ^* \). Therefore, the optimal policy \( π^* \) is a threshold policy, where the threshold \( τ^* \in \mathbb{R}_{≥0} \) is characterised by (6.4).

Finally, the proof for fixed frequency market clearing is completely analogous to the discriminatory market clearing case.

Theorem 6.3.2 provides us with useful information about the structure and evolution of the market order book under each form of market clearing. Furthermore, we can now restrict attention to a small class of policies in order to determine the optimal market clearing policies. This gives rise to a tractable dynamic programming approach, which we exploit to characterise the optimal policy.

6.3.3 Evolution of the Market Order Book

First–Best: Discriminatory Market Clearing  Because discriminatory market clearing imposes no constraints above and beyond the uncertainty about future types inherent in the problem, we refer to the efficient policy under discriminatory market clearing as first–best. Under this policy, the market maker immediately clears efficient pairs from the market and stores identical suboptimal pairs. Each threshold policy \( τ \) induces a Markov chain \( \{Y_t\}_{t \in \mathbb{N}} \) over \( \{0, \ldots, τ\} \), the number of identical suboptimal pairs stored in the order book.

We now describe the transitions in this Markov chain. If \( Y_t = 0 \), we have \( Y_{t+1} = 0 \) with probability \( 1 - 2p(1−p) \), which corresponds to the arrival of an efficient or inefficient pair in period \( t + 1 \), and \( Y_{t+1} = 1 \) with probability \( 2p(1−p) \), which corresponds to the arrival of a suboptimal pair. Similarly, if \( Y_t \in \{1, \ldots, τ\} \), we have \( Y_{t+1} = Y_t \) with probability \( 1 - 2p(1−p) \), \( Y_{t+1} = Y_t - 1 \) with probability \( p(1−p) \), corresponding to the arrival of a non–identical suboptimal pair which is rematched, and \( Y_{t+1} = Y_t + 1 \) with probability \( p(1−p) \), which is the probability with which an identical suboptimal pair arrives. Finally, if \( Y_t = τ \) we have \( Y_{t+1} = Y_t \) with probability \( 1 − p(1−p) \) because this is the probability with which an efficient pair, an inefficient pair or an
Figure 6.2: The Markov chain over the number of stored suboptimal pairs induced by the optimal policy $\pi^*$ under discriminatory market clearing with $\lambda = p(1 - p)$.

Identical suboptimal pair arrive and $Y_{t+1} = Y_t - 1$ with probability $p(1 - p)$. Thus, $\{Y_t\}_{t \in \mathbb{N}}$ is a finite birth–and–death process, as illustrated in Figure 6.2.

Accounting for the immediate reward earned by the market maker when these transitions take place, we can write the Bellman equation associated with the Markov decision process. Take any $y \in \{0, 1, \ldots, \tau\}$ and let $V^D_\tau(y)$ denote the expected present value of having $y$ identical suboptimal pairs stored at the end of any period under the threshold policy with threshold $\tau \in \mathbb{N}$. Any such policy is, for $y \in \{1, \ldots, \tau - 1\}$, characterised by the Bellman equation

$$
V^D_\tau(y) = \delta \left[ p^2(1 + V^D_\tau(y)) + p(1 - p)(1 + V^D_\tau(y - 1) + V^D_\tau(y + 1)) \right. \\
+ (1 - p)^2V^D_\tau(y) \left. \right],
$$

with boundary conditions

$$
V^D_\tau(0) = \delta \left[ p^2(1 + V^D_\tau(0)) + 2p(1 - p)V^D_\tau(1) + (1 - p)^2V^D_\tau(0) \right],
\quad (6.6)
$$

$$
V^D_\tau(\tau) = \delta \left[ p^2(1 + V^D_\tau(\tau)) + p(1 - p)(1 + V^D_\tau(\tau - 1) + \Delta + V^D_\tau(\tau)) \right. \\
+ (1 - p)^2V^D_\tau(\tau) \left. \right].
\quad (6.7)
$$

The optimal threshold $\tau^*$ is then characterised by the stopping condition

$$
V^D_\tau(\tau^*) > \Delta + V^D_{\tau^*}(\tau^* - 1) \quad \text{and} \quad V^D_{\tau^*+1}(\tau^* + 1) \leq \Delta + V^D_{\tau^*+1}(\tau^*).
\quad (6.8)
$$

To compute the optimal threshold, one can start with the threshold policy given by $\tau = 1$, check condition (6.8) and iterate. Algorithm 6.3.3 formalises this procedure.

**Algorithm 6.3.3.** Begin with the threshold policy characterised by $\tau = 1$ and solve the linear system defined in (6.5), (6.6) and (6.7). If $V^D_1(1) > \Delta + V^D_1(0)$, proceed to step 2. Otherwise, return $\tau^* = 0$. At step $i$,

1. Solve (6.5) with $\tau = i$ to determine $V^D_i(i)$ and $V^D_i(i - 1)$. 


2. If \( V_i^D(i) > \Delta + V_i^D(i - 1) \), proceed to step \( i + 1 \). Otherwise, return \( \tau^* = i - 1 \).

This algorithm must terminate after finitely many steps since \( \tau^* \) is finite. Algorithm 6.3.3 is a simple example of what is known as policy iteration in the Markov decision process literature (see 2.4). We start with the policy \( \tau = 1 \) and compute the associated state values. We proceed to iterate over a set of test policies until the optimal policy is reached. With each iteration the test policy is updated based on the optimality condition for the values computed for that test policy. Policy iteration is simple in this case because the set of test policies (which must be the set of all possible optimal policies) has already been refined to the set of threshold policies by Theorem 6.3.2.

**Proposition 6.3.4.** Under the discriminatory market clearing threshold policy with threshold \( \tau \), the stationary distribution \( \kappa \) of the Markov chain \( \{Y_t\}_{t \in \mathbb{N}} \) is given by

\[
\kappa_0 = \frac{1}{2\tau + 1} \quad \text{and} \quad \kappa_i = \frac{2}{2\tau + 1}, \quad \forall i \in \{1, \ldots, \tau\}.
\]

The market maker’s stationary expected per period payoff is given by

\[
W_i^D(\tau) = \frac{p^2 + 2p(1-p)(\Delta + \tau)}{2\tau + 1}.
\]

**Proof.** The transition matrix \( P \) of the order book Markov chain \( \{Y_t\}_{t \in \mathbb{N}} \) is given by

\[
P = \begin{pmatrix}
1 - 2\lambda & 2\lambda & 0 & \cdots & 0 & 0 & 0 \\
\lambda & 1 - 2\lambda & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda & 1 - 2\lambda & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 - 2\lambda & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & 1 - 2\lambda & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & \lambda & 1 - \lambda
\end{pmatrix},
\]

where \( \lambda = p(1-p) \). The stationary distribution \( \kappa \) satisfies \( \kappa P = \kappa \) and we solve

\[
\sum_{i=0}^{\tau^*} \kappa_i = 1, \quad \kappa_0 = \kappa_0(1 - 2\lambda) + \kappa_1 \lambda, \quad \kappa_1 = 2\kappa_0 \lambda + \kappa_1(1 - 2\lambda) + \kappa_2 \lambda,
\]

\[
\forall i \in \{1, \ldots, \tau^* - 1\}, \quad \kappa_i = \kappa_{i-1} \lambda + \kappa_i(1 - 2\lambda) + \kappa_{i+1} \lambda,
\]

\[
k_{\tau^*} = \kappa_{\tau^* - 1} \lambda + \kappa_{\tau^*}(1 - \lambda).
\]

The last equation is a second order difference equation with constant coefficients. We substitute a trial solution \( \kappa_i = \psi^i > 0 \) into this equation and solve the characteristic equation of the recursion. This gives

\[
\psi^i = \psi^{i-1} \lambda + \psi^i(1 - 2\lambda) + \psi^{i+1} \lambda \quad \Rightarrow \psi^2 - 2\psi + 1 = 0 \quad \Rightarrow \psi = 1,
\]
so the general solution to the difference equation is \( \kappa = c_1 + c_2 i \). Here \( c_1, c_2 \in \mathbb{R} \) are unknown constants. Substituting the general solution into the equations for \( \kappa \), we obtain \( \kappa = \kappa_{\tau-1} \). Substituting this into the equation for \( \kappa_{\tau-1} \) gives \( \kappa_{\tau-1} = \kappa_{\tau-2} \). Thus we require \( c_1 + (\tau^* - 1) c_2 = c_1 + (\tau^* - 2) c_2 \) which implies that \( c_2 = 0 \). Substituting \( \kappa_1 = c_1 \) into the equations for \( \kappa_0 \) and \( \kappa_1 \) and solving these equations simultaneously gives \( \kappa_0 = c_1 / 2 \) and \( \kappa_1 = c_1 \).

Finally, using the normalisation equation, we have

\[
\frac{c_1}{2} + \sum_{i=1}^{\tau^*} c_1 = 1 \quad \Rightarrow c_1 + 2c_1 \tau^* = 2 \quad \Rightarrow c_1 = \frac{2}{2 \tau^* + 1}.
\]

Thus, the stationary distribution \( \kappa \) of the Markov chain \( \{Y_t\}_{t \in \mathbb{N}} \) is given by

\[
\kappa_0 = \frac{1}{2 \tau + 1}, \quad \forall i \in \{1, \ldots, \tau\}, \quad \kappa_i = \frac{2}{2 \tau + 1}.
\]

We now compute the market maker’s expected period \( t \) payoff when the market is stationary under the threshold policy with threshold \( \tau \). With probability \( p^2 \) and \( (1 - p)^2 \) a \((\overline{v}, \overline{c})\) pair and a \((\underline{v}, \underline{c})\) pair arrive respectively, creating respective payoffs of 1 and 0. A suboptimal pair arrives with probability \( 2p(1 - p) \). With probability \( \tau / (2 \tau + 1) \) this pair arrives to a market in which non–identical suboptimal pairs are stored. In this case, it is rematched to create an efficient trade which is immediately cleared. With probability \( \tau / (2 \tau + 1) \) the number of identical suboptimal pairs stored is less than \( \tau \) and the arriving pair is stored. Finally, with probability \( 1 / (2 \tau + 1) \) the maximum number of identical suboptimal pairs are stored and the one suboptimal pair is immediately cleared. Thus, assuming the market is stationary, the market maker’s expected period \( t \) payoff is

\[
W^D_t(\tau) = p^2 + \frac{2p(1 - p)(\Delta + \tau)}{2 \tau + 1}.
\]

Observe that the distributions \( \kappa \) is uniform except for the boundary \( \kappa_0 \). If we changed the state space of \( \{Y_t\}_{t \in \mathbb{N}} \) to distinguish the type of suboptimal pairs stored (i.e. \((\overline{v}, \overline{c})\) or \((\underline{v}, \underline{c})\)), which corresponds to the approach taken by Baccara, Lee, and Yariv [9], we would obtain a uniform stationary distribution for the order book.

The expression for \( W^D_t(\tau) \) has a simple and intuitive explanation. With probability \( p^2 \) an efficient pair arrives and trades, creating a welfare gain of 1. With probability \( 2p(1 - p) \) a suboptimal pair arrives and there are several possibilities. With probability \( (1/2)(1 - \kappa_0) = \tau / (2 \tau + 1) \) there is a positive number of stored suboptimal pairs of the opposite kind. This arrival and the stored traders permit the creation of an efficient pair, which trades and adds a welfare gain of 1. With probability \( (1/2)\kappa_\tau = 1 / (2 \tau + 1) \), \( \tau \) suboptimal pairs
of the same kind are stored, meaning that one suboptimal pair is cleared, generating a gain of $\Delta$. In all other cases, the arriving suboptimal pair is stored and no immediate reward is earned by the market maker.

Observe that $W^D_t(\tau)$ is increasing in $\tau$ because the expression for per period welfare under the stationary distribution does not account for the opportunity cost of achieving the stationary distribution. Furthermore,

$$\lim_{\tau \to \infty} W^D_t(\tau) = p^2 + p(1 - p) = p$$

since, as $\tau \to \infty$, the setup collapses to a static setting with a continuum of traders. However, for $\delta < 1$, the opportunity cost of accumulating so many suboptimal pairs will be excessive.10

Uniform Market Clearing Under uniform market clearing, the market maker stores both efficient and suboptimal pairs up to a threshold value of $\tau$. We now describe the structure of the order book Markov chain $\{Y_t\}_{t\in\mathbb{N}}$, as illustrated in Figure 6.3. One can think of the number of stored efficient pairs as the level of the Markov chain and the number of stored suboptimal trades as the state of the Markov chain within that level. We include an additional level for the state 0, denoted by level $\emptyset$. Under the threshold policy $\tau$, $\bar{y}_E = \lfloor \tau \rfloor$ is the maximum number of efficient pairs that can be stored. For $i \in \{0, 1, \ldots, \bar{y}_E\}$, the maximum number of suboptimal pairs stored at level $i$ is $k_i = \lfloor (\tau - i)/\Delta \rfloor$. Therefore, the order book Markov chain is a level–dependent quasi–birth–death process (see, for example, Latouche and Ramaswami [87]).

The transition matrix $P'$ of the order book Markov chain $\{Y_t\}_{t\in\mathbb{N}}$ under uniform market clearing has a block structure of the form

$$P' = \begin{pmatrix}
A_{\emptyset\emptyset} & A_{\emptyset 0} & 0 & 0 & \cdots & 0 \\
A_{0\emptyset} & A_{00} & A_{01} & 0 & \cdots & 0 \\
A_{1\emptyset} & 0 & A_{11} & A_{12} & \cdots & 0 \\
A_{2\emptyset} & 0 & 0 & A_{22} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
A_{\bar{y}_E\emptyset} & 0 & 0 & 0 & \cdots & A_{\bar{y}_E\bar{y}_E}
\end{pmatrix}.$$

Take $i \in \{0, 1, \ldots, \bar{y}_E\}$. The $A_{0\emptyset}$ blocks are $k_i \times 1$ matrices which encode the transitions from level $i$ to the market clearing state. The $A_{ii}$ blocks are

10When a suboptimal pair is stored behind $\tau$ other pairs, it will be cleared in $(\tau + 1)/\lambda$ periods in expectation. The market maker forgoes an immediate payoff of $\Delta$ and receives an expected discounted payoff of $\delta^{(\tau+1)/\lambda}$. Note that $\tau^*$ does not satisfy $\Delta = \delta^{(\tau^*+1)/\lambda}$ since we must account for the impact storing a trade has on the future evolution of the market. However, the formula $\tau^* \approx \frac{\ln(\Delta)}{\ln(\delta)} - 1$ provides an approximation, which, if used as the initial guess for $\tau^*$ in Algorithm 6.3.3 instead of 0, can increase the speed of the algorithm.
$\mathcal{K}_i \times \mathcal{K}_i$ matrices which encode the transitions from level $i$ to level $i$. The $A_{ii+1}$ blocks are $\mathcal{K}_i \times \mathcal{K}_{i+1}$ matrices which encode the transitions from level $i$ to level $i + 1$. Letting $\mu_0 = (1 - p)^2$ and $\mu_1 = p(1 - p)$ we have $A_{i0} = (\mu_1, 0, A_i)'$, where $A_i = (p^2, p^2 + \mu_1, \ldots, p^2 + \mu_1, p^2 + 2\mu_1)'$ is a vector of length $\mathcal{K}_i - \mathcal{K}_{i+1}$.

Furthermore,

$$A_{ii} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \mu_1 \\ 0 & 0 & \cdots & \mu_0 \end{pmatrix} \quad \text{and} \quad A_{ii+1} = \begin{pmatrix} p^2 & 0 & \cdots & 0 \\ \mu_1 & p^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & p^2 \\ 0 & 0 & \cdots & \mu_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Finally $A_{xE,0} = (p^2 + \mu_1, \ldots, p^2 + \mu_1, p^2 + 2\mu_1)'$ is a vector of length $\mathcal{K}_x - 1$ matrix, $A_{00} = (2\mu_1, 0)$ is a $1 \times \tau$ matrix and $A_{00} = (p^2 + \mu_0)$.

From Figure 6.3 and the structure of $P'$, it can be seen that the Markov chain $\{Y_t\}_{t \in \mathbb{N}}$ is similar in nature to a level–dependent quasi–birth–and–death process (see, for example, Latouche and Ramaswami [87]). The stationary distribution $\pi$ satisfies $\pi P' = \pi$. That is, we must solve the system of equations

$$\pi_0 + \sum_{i=0}^{\tau E} \pi_i \cdot e = 1, \quad \pi_0 A_{00} + \pi_0 A_{00} = \pi_0,$$

$$\forall i \in \{1, \ldots, \tau E\}, \quad \pi_{i-1} A_{i-1i} + \pi_i A_{ii} = \pi_i.$$

Here, $e$ is a vector of ones of the appropriate length. Since, the matrix $I - A_{ii}$ is invertible we have

$$\pi_0 = \pi_0 A_{00} (I - A_{00})^{-1} \quad \text{and} \quad \pi_i = \pi_{i-1} A_{i-1i} (I - A_{ii})^{-1}.$$

Thus, for all $i \in \{0, 1, \ldots, \tau E\}$, we have

$$\pi_i = \pi_0 A_{00} (I - A_{00})^{-1} \prod_{j=1}^{i} A_{j-1j} (I - A_{jj})^{-1}, \quad \text{with} \quad \pi_0 = 1 - \sum_{i=0}^{\tau} \pi_i \cdot e.$$

Note that the matrix $A_{i-1i} (I - A_{ii})^{-1}$ has a probabilistic interpretation. The $(j, k)$th entry of this matrix is the expected number of visits to state $k$ in level $i - 1$ before the process moves to level $i$, given that the matrix started in state $j$ in level $i - 1$.

Let $V_U^T(i)$ denote the vector of the expected values associated with being in each state in level $i$ under the threshold policy characterised by $\tau$. Then
using $P'$, the linear system defined in (6.9) and (6.10) can be rewritten in matrix form as

$$
(I - \delta P') \begin{pmatrix}
V^U_{\tau}(0) \\
V^U_{\tau}(1) \\
\vdots \\
V^U_{\tau}(x_E)
\end{pmatrix} = \delta \begin{pmatrix}
r^\prime_\tau(0) \\
r^\prime_\tau(1) \\
\vdots \\
r^\prime_\tau(x_E)
\end{pmatrix}.
$$

Here, $r^\prime_\tau(0) = p^2$ and for $i \in \{0, \ldots, x_E - 1\}$ we have $r^\prime_\tau(i) = (\mu_1(i + 1), 0, r^\prime_i)'$ where

$$
r^\prime_i = \begin{pmatrix}
p^2(i + 1 + \Delta(k_{i+1} + 1)) \\
p^2(i + 1 + \Delta(k_{i+1} + 2)) + \mu_1(i + 1 + \Delta(k_{i+1} + 1)) \\
\vdots \\
p^2(i + 1 + \Delta(k_i)) + 2\mu_1(i + 1 + \Delta(k_i - 1)) + \mu_1(1 + 2\Delta)
\end{pmatrix},
$$

is a $k \times 1$ column vector and

$$
r^\prime_\tau(x_E) = \begin{pmatrix}
p^2(1 + \tau_E + \Delta) + \mu_1(1 + \tau_E) \\
p^2(1 + \tau_E + 2\Delta) + \mu_1(1 + \tau_E + \Delta) \\
\vdots \\
p^2(1 + \tau_E + k \tau_E - 1) + \mu_1(1 + \tau_E + \Delta(k \tau_E - 1))) + \mu_1(1 + 2\Delta)
\end{pmatrix}.
$$

We now derive the Bellman equation of the associated Markov decision process. The notation $Z = \{(0,0), (0,1), (1,0), (1,-1)\}$, which captures the set of possible changes to the state $y = (y_E, y_I)$ following the next arrival, is convenient because it allows us to sum over all possible transitions of the order book Markov chain. Define the function $P_Z : Z \rightarrow [0, 1]$ by

$$
P_Z(1, 0) = p^2, \ P_Z(0, 1) = p(1 - p), \ P_Z(1, -1) = p(1 - p), \ P_Z(0, 0) = (1 - p)^2,
$$

which gives the probability of each of the changes captured in $Z$. For example, $(1, -1)$ corresponds to the arrival of a suboptimal pair that results in a stored suboptimal being rematched to create an efficient pair. This occurs with probability $p(1 - p)$, provided $y_I > 0$.

Let $V^U_\tau(y_E, y_I)$ denote the expected discounted present value of being in state $(y_E, y_I)$ under the threshold policy with threshold $\tau$. If the state of the market is $(y_E, 0)$ for some $y_E > 0$, the market maker will immediately clear and earn a reward of $y_E$ plus the expected present value of being in state $0$. Therefore, we have

$$
V^U_\tau(y_E, 0) = y_E + V^U_\tau(0).
$$

(6.9)
Next suppose the market is in any state \( y = (y_E, y_I) \) such that \( y_I > 0 \) and \( r(y) < \tau \), where \( r(y_E, y_I) = y_E + y_I \Delta \) denotes the immediate reward from clearing the market. Under the threshold policy \( \tau \), the market maker will earn an immediate reward only when the market reaches a state \( y' \) such that \( r(y') \geq \tau \). Consequently,

\[
V^U_\tau(y) = \delta \sum_{z \in \mathcal{Z}} P_Z(z) \left[ V^U_\tau(y + z) \mathbb{I}(r(y + z) < \tau) + (r(y + z) + V^U_\tau(0)) \mathbb{I}(r(y + z) \geq \tau) \right].
\]

(6.10)

Any threshold policy is characterised by this linear system. As with discriminatory market clearing, this Bellman equation can be used to to derive a stopping condition satisfied by \( \tau^* \).

By the proof of Theorem 6.3.2, the optimal threshold \( \tau^* \) is such that for any \( x^*_E > 0 \) there exists a cutoff state \( x^* = (x^*_E, x^*_I) \) with

\[
V^U_\tau^*(x^*) > r(x^*) + V^U_\tau^*(0) \quad \text{and} \quad V^U_\tau^*(x^*_E, x^*_I + 1) \leq r(x^*_E, x^*_I + 1) + V^U_\tau^*(0).
\]

That is, a cutoff state is such that the market is optimally cleared if an additional identical suboptimal pair arrives. Algorithm 6.3.5 using these stopping conditions to determine \( \tau^* \).
Algorithm 6.3.5. Begin with the threshold policy characterised by \( \tau = \Delta \), where \( \Delta \) is the value of a single suboptimal trade. Solve the linear system defined in (6.9) and (6.10). If \( V^U_\tau(0,1) \geq \Delta + V^U_\tau(0) \), proceed to step 2. Otherwise, return \( \tau^* = 0 \). At step i,

1. Solve (6.9) and (6.10) with \( \tau = i\Delta \) to determine \( V^U_\tau(0,i) \) and \( V^U_\tau(0) \).

2. If \( V^U_\tau(0,i) \geq i\Delta + V^U_\tau(0) \), proceed to step \( i+1 \). Otherwise, set \( \tau' = (i-1)\Delta \).

If \( \tau' + \Delta < 1 \), return \( \tau^* = \tau' \). Otherwise, for all \( j \in \mathbb{N} \) such that \( \tau' + \Delta < j \),

1. Set \( k = \lfloor (\tau' + \Delta - j) / \Delta \rfloor \) and solve (6.9) and (6.10) with \( \tau = j + k\Delta \) to determine \( V^U_\tau(j,k) \) and \( V^U_\tau(0) \).

2. If \( V^U_\tau(k,j) \geq j + k\Delta + V^U_\tau(0) \) and \( \tau' < j + k\Delta \) update \( \tau' = j + k\Delta \).

Return \( \tau^* = \tau' \).

This algorithm terminates after finitely many steps since \( \tau^* \) satisfies (6.4) and is finite. Alternatively, we can initially increase the candidate threshold by increments of 1 and then increments of \( \Delta \).

Fixed frequency market clearing Under fixed frequency market clearing, the state space of the limit order book Markov chain is given by \( \{(y_E, y_I) : 0 \leq y_E + y_I \leq \tau, y_E, y_I \in \mathbb{Z}_{\geq 0} \} \). If the market is cleared every \( \tau \) periods, the market maker’s expected discounted payoff is given by

\[
W^F(\tau) = \frac{\delta^{\tau-1}}{1 - \delta^\tau} \sum_{j=0}^{\tau} \sum_{k=0}^{\tau} \binom{\tau}{j} \binom{\tau}{k} (\min\{j,k\} + |j-k|\Delta) p^{j+k}(1-p)^{2\tau-j-k}.
\]

(6.11)

Algorithm 6.3.6 uses this formula to compute \( \tau^* \).

Algorithm 6.3.6. Begin with the threshold policy characterised by \( \tau = 2 \) and compute \( W^F(2) \) and \( W^F(1) \) using (6.11). If \( W^F(2) \geq W^F(1) \) proceed to step 2. Otherwise, return \( \tau^* = 1 \). At step i,

1. Compute \( W^F(i) \) using (6.11).

2. If \( W^F(i) \geq W^F(i-1) \), proceed to step \( i+1 \). Otherwise, return \( \tau^* = i-1 \).

Under each form of market clearing, the dynamic programming characterisation of the optimal policy can be used to prove the following proposition.

Proposition 6.3.7. Under the optimal policy, total expected discounted welfare is increasing in \( \delta \), \( p \) and \( \Delta \). The optimal threshold \( \tau^* \) is increasing in \( \delta \) and decreasing in \( \Delta \). Furthermore, under discriminatory market clearing, the optimal threshold \( \tau^* \) is increasing in \( p(1-p) \).
Proof. We begin by considering discriminatory market clearing. If the market designer changes the market clearing threshold from \( \tau \) to \( \tau + 1 \), the expected change in welfare under the stationary distribution is

\[
W^D(\tau + 1) - W^D(\tau) = \frac{1}{1 - \delta} \cdot \frac{2p(1 - p)(1 - 2\Delta)}{(2\tau + 3)(2\tau + 1)}
\]

Differentiating with respect to the problem parameters, we obtain

\[
\frac{\partial(W^D(\tau + 1) - W^D(\tau))}{\partial \delta} = \frac{1}{(1 - \delta)^2} \cdot \frac{2p(1 - p)(1 - 2\Delta)}{(2\tau + 3)(2\tau + 1)} > 0,
\]

\[
\frac{\partial(W^D(\tau + 1) - W^D(\tau))}{\partial p} = \frac{1}{1 - \delta} \cdot \frac{2(1 - 2\Delta)}{(2\tau + 3)(2\tau + 1)} > 0.
\]

\[
\frac{\partial(W^D(\tau + 1) - W^D(\tau))}{\partial \Delta} = -\frac{1}{1 - \delta} \cdot \frac{4p(1 - p)}{(2\tau + 3)(2\tau + 1)} < 0.
\]

Since the payoff associated with increasing \( \tau \) is increasing in \( \delta \) and \( p(1 - p) \) and decreasing in \( \Delta \), so too is \( \tau^* \).

Next, examining (6.5), (6.6) and (6.7), it can be seen that for \( x_I \in \{0, 1, \ldots, \tau^*\} \) an increase in \( \Delta \) and \( \delta \) leads to an increase in \( V^{D*}_{\tau^*}(x_I) \). Since the total expected discounted payoff is increasing for each state, total expected discounted welfare is increasing in \( \Delta \) and \( \delta \). For \( x_I \in \{1, \ldots, \tau^* - 1\} \), ranking the outcomes on the right-hand side of (6.5) by payoff gives \( 1 + V^D_{\tau^*}(x_I) > 1 + V^D_{\tau^*}(x_I - 1) > V^D_{\tau^*}(x_I + 1) > V^D_{\tau^*}(x_I) \). The outcomes \( 1 + V^D_{\tau^*}(x_I - 1) \) and \( V^D_{\tau^*}(x_I + 1) \) occur with equal probability and an increase in \( p \) leads to an increase in the probability of the best outcome and a decrease in the probability of the worst outcome. Since similar reasoning applies to (6.6) and (6.7), an increase in \( p \) increases the total expected discounted payoff for each state. Thus, total expected welfare is increasing in \( p \).

Under uniform market clearing we may proceed in exactly the same manner as discriminatory market clearing to show that for every state \( x \) such that \( r(x) \leq \tau^* \), \( V^{U*}_{\tau^*}(x) \) is increasing in \( \delta \) and \( p \) and decreasing in \( \Delta \). It immediately follows that total expected discounted welfare is increasing in \( \delta \) and \( p \) and decreasing in \( \Delta \). Next, in any state \( x \in \mathcal{X} \), the immediate reward from clearing the market is independent of \( \delta \) and given by \( r(x) \). The expected discounted reward earned by waiting is increasing in \( \delta \). Thus, \( \tau^* \) is increasing in \( \delta \). Thus, \( \tau^* \) is decreasing in \( \Delta \) by (6.4) since \( V^{U*}_{\tau^*}(0) \) is increasing in \( \Delta \).

Under fixed frequency market clearing, for every \( \tau \in \mathbb{N} \), \( W^F(\tau^*) \) is increasing in \( \delta \), \( p \) and \( \Delta \). It follows that \( W^F(\tau^*) \) is increasing in \( \delta \), \( p \) and \( \Delta \). Next, for any period \( t \), an increase in \( \delta \) and a decrease in \( \Delta \) increases the expected payoff from waiting to clear the market relative to the expected payoff from clearing the market. Therefore, \( \tau^* \) is increasing in \( \delta \) and decreasing in \( \Delta \). \[\square\]

Total expected discounted welfare is increasing in the discount factor, the value of suboptimal matches and the probability of efficient types arriving on
6.4. DYNAMIC OPTIMALITY

(a) Increasing in \( \delta \)
(b) Increasing in \( p(1 - p) \)
(c) Decreasing in \( \Delta \)

Figure 6.4: A numerical illustration of the comparative static results for \( \tau^* \) under discriminatory market clearing.

Note 6.3.8. The waiting time for an efficient agent depends on the queueing protocol used in the order book. The simplest case is one in which a first-come-first-served queueing protocol is used for the order book and identical suboptimal pairs arriving to a full order book are cleared immediately. In this case, efficient agents wait only if they arrive as part of a suboptimal pair to an empty market or a market with identical suboptimal pairs stored. For \( i \in \{1, \ldots, \tau^*\} \), let \( \Lambda_i \) denote the waiting time for the \( i \)th stored suboptimal pair. Since \( \Lambda_i \) is given by the sum of \( i \) geometrically distributed random variables with parameter \( \lambda = p(1 - p) \), we have \( \Lambda_i \sim \text{NB}(i, 1 - \lambda) \), where NB denotes the negative binomial distribution\(^{11}\). Thus, the waiting time for efficient types is 0 with probability \( (\tau^* + 1)/(2\tau^* + 1) \) and \( \Lambda_i \) with probability \( 1/(2\tau^* + 1) \).

6.4 Dynamic Optimality

We now extend the setup to account for the private information traders plausibly have and use mechanism design techniques to derive the interim incentive compatible and individually rational mechanisms that implement the dynamically efficient policy but otherwise maximises the market maker’s expected

\(^{11}\)Recall that \( X \sim \text{NB}(r, q) \) counts the number of trials until \( r \) failures occur.
discounted revenue. Then we derive the dynamically optimal (e.g. profit-maximising) mechanisms.

6.4.1 Private Information Setup

We now assume that agent types are private information while the parameters $\underline{\varepsilon}, \overline{\varepsilon}, \underline{\tau}, \overline{\tau}$ and $p$ are common knowledge. Information is elicited from period $t$ agents once, when they report their types upon arrival. The designer determines the period $t$ allocation on the basis of period $t$ reports (for period $t$ agents) and the reports of the previous $t - 1$ periods (for any additional agents present in period $t$). We assume agents observe the reports of agents that arrived in previous periods, prior to reporting their own type.

Without loss of generality, we restrict ourselves to direct, deterministic mechanisms. Let $\hat{\Theta}_t = (\hat{V}_t, \hat{C}_t)$ be the reports of period $t$ agents and denote the period $t$ complete history of reports by $\hat{H}_t = (\hat{\Theta}_1, \ldots, \hat{\Theta}_t) \in \mathcal{H}_t := \{\tau, \psi\}^t \times \{\underline{\varepsilon}, \overline{\varepsilon}\}^t$. A direct mechanism $(Q, M)$ consists of an allocation rule $Q = \{Q_i\}_{i \in \mathbb{N}}$ and a payment rule $M = \{M_i\}_{i \in \mathbb{N}}$. The period $t$ allocation rule $Q_t : \mathcal{H}_t \rightarrow \{0, 1\}^{2t}$ maps the period $t$ history of agents reports to the set of period $t$ outcomes. Similarly, we have a period $t$ transfer rule $M_t : \mathcal{H}_t \rightarrow \mathbb{R}^{2t}$.

12Of course, this means that the interim individual rationality constraints will hold with equality for the worst-off types. This approach is motivated by the prevalence of impossibility theorems (see, for example, Myerson and Satterthwaite [117] and Section 1.2.7) for two-sided settings like ours and parallels the approach taken in the partnership literature, where the question of whether efficient dissolution is possible is answered by maximising the revenue of a market maker subject to interim incentive compatibility and individual rationality and subject to the allocation rule being efficient; see, for example, Loertscher and Wasser [98] and the references therein.

13If the arrival time of each agent is private information and agents cannot observe the state of the market prior to reporting their arrival, it is a Bayesian Nash equilibrium for agents to truthfully report their arrival by virtue of discounting. If agents can observe the state of the market prior to reporting their arrival, agents must be incentivised to truthfully report their arrival. Under welfare-maximisation, the Online VCG mechanism of Parkes and Singh [120] (which generalises the VCG mechanism to dynamic settings with hidden arrivals) could be used to construct appropriate transfers. Solving this problem under revenue-maximisation would require generalising the results of Pai and Vohra [119] to two-sided settings.

14For direct mechanisms, no generality is lost because of the revelation principle. The restriction to deterministic mechanisms is without loss of generality because optimal mechanisms are deterministic up to the resolution of ties, which can be broken using deterministic rules.

15To consider random mechanisms, we would have to introduce more notation. Let $\psi_i^t \in \{0, 1\}^{2t}$ denote the period $\tau$ allocation, where $\psi_i^{B_t} \in \{0, 1\}$ and $\psi_i^{S_t} \in \{0, 1\}$ denote the period $\tau$ allocation for period $i \leq \tau$ agents. The period $t$ history $\hat{h}_t \in \mathcal{H}_t$ must now contain the complete history of reports (which includes the reports of period $t$ agents) and allocations (up to the period $t - 1$ allocation). The period $t$ allocation rule $Q_t : \mathcal{H}_t \rightarrow \mathcal{P}(\{\hat{B}_t, \ldots, B_t\} \cup \{S_t, \ldots, S_t\})$ maps the period $t$ history of agents reports to the set of probability distributions over $\{\hat{B}_t, \ldots, B_t\} \cup \{S_t, \ldots, S_t\}$. If we endow $\{\mathcal{H}_t\}_{t \in \mathbb{N}}$ with the natural filtration $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$, we require that $Q_t$ is $\mathcal{F}_t$-measurable. Feasibility requires that for all $t \in \mathbb{N}$ and $\hat{h}_t \in \mathcal{H}_t$, $\sum_{i=1}^{t} Q_i^{S_t}(\hat{h}_i) \leq \sum_{i=1}^{t} Q_i^{S_t}(\hat{h}_i)$. Furthermore, if $\psi_i^{B_t} = 1$ for some
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The probability that buyer and seller $i \in \{1, \ldots, t\}$ trade in period $t$ when the period $t$ report history is given by $\hat{h}_t$ is $Q^B_t(\hat{h}_t)$ and $Q^S_t(\hat{h}_t)$ respectively. Feasibility requires that, for all $t \in \mathbb{N}$ and $\hat{h}_t \in \mathcal{H}_t$,

$$\sum_{i=1}^{t} Q^B_i(\hat{h}_t) \leq \sum_{i=1}^{t} Q^S_i(\hat{h}_t)$$  \hspace{1cm} (6.12)

and, for all $i \in \{1, \ldots, t\}$, $\sum_{j=i}^{t} Q^B_j(\hat{h}_j) \leq 1$ and $\sum_{j=i}^{t} Q^S_j(\hat{h}_j) \leq 1$. Note that (6.12) will hold with equality under an optimal mechanism. Let the period $t-1$ report history $\hat{h}_{t-1}$ be given and denote by $V_t$ and $C_t$ the true types of the period $t$ buyer and seller, respectively. When $B_t$ reports $\hat{v}_t$ and $S_t$ reports $\hat{c}_t$ the respective interim discounted allocation probabilities, assuming all other agents arriving after period $t-1$ report truthfully, are given by

$$q^B_t(\hat{v}_t, \hat{h}_{t-1}) = \sum_{i=t}^{\infty} \sum_{h_i \in \mathcal{H}_i} \delta^{i-1} Q^B_i(h_i) \mathbb{P}(H_i = h_i|V_t = \hat{v}_t, H_{t-1} = \hat{h}_{t-1}),$$

$$q^S_t(\hat{c}_t, \hat{h}_{t-1}) = \sum_{i=t}^{\infty} \sum_{h_i \in \mathcal{H}_i} \delta^{i-1} Q^S_i(h_i) \mathbb{P}(H_i = h_i|C_t = \hat{c}_t, H_{t-1} = \hat{h}_{t-1}),$$

where $\mathbb{P}(H_i = h_i|V_t = \hat{v}_t, H_{t-1} = \hat{h}_{t-1})$ denotes the conditional probability that the period $i \geq t$ report history is $h_i$, given that the period $t-1$ report history is $\hat{h}_{t-1}$ and $B_t$ reports $\hat{v}_t$ in period $t$, and analogously for $\mathbb{P}(H_i = h_i|C_t = \hat{c}_t, H_{t-1} = \hat{h}_{t-1})$.

Let $M^B_t(\hat{h}_t)$ and $M^S_t(\hat{h}_t)$ denote the respective expected payments made by $B_t$ and $S_t$ in period $t$ given $\hat{h}_t$. When the period $t-1$ report history is given by $\hat{h}_t$ and $B_t$ reports $\hat{v}_t$ and $S_t$ reports $\hat{c}_t$ the respective expected interim discounted payments, assuming all other agents arriving after period $t-1$ report truthfully, are given by

$$m^B_t(\hat{v}_t, \hat{h}_{t-1}) = \sum_{i=t}^{\infty} \sum_{h_i \in \mathcal{H}_i} \delta^{i-1} M^B_i(h_i) \mathbb{P}(H_i = h_i|V_t = \hat{v}_t, H_{t-1} = \hat{h}_{t-1}),$$

$$m^S_t(\hat{c}_t, \hat{h}_{t-1}) = \sum_{i=t}^{\infty} \sum_{h_i \in \mathcal{H}_i} \delta^{i-1} M^S_i(h_i) \mathbb{P}(H_i = h_i|C_t = \hat{c}_t, H_{t-1} = \hat{h}_{t-1}).$$

Suppose $B_t$ and $S_t$ report types $\hat{v}_t$ and $\hat{c}_t$ when their true types are $v_t$ and $c_t$ respectively. Then interim expected discounted payoffs, assuming sellers produce in the period in which they trade, are given by

$$U^B_t(\hat{v}_t, v_t, \hat{h}_{t-1}) = v_t q^B_t(\hat{v}_t, \hat{h}_{t-1}) - m^B_t(\hat{v}_t, \hat{h}_{t-1}),$$

$$U^S_t(\hat{c}_t, c_t, \hat{h}_{t-1}) = m^S_t(\hat{c}_t, \hat{h}_{t-1}) - c_t q^S_t(\hat{c}_t, \hat{h}_{t-1}).$$

$i < t$ then $\psi_s^{B_i} = 0$ for all $i \leq s < t$. Similarly for $\psi_s^{S_i}$.
Substituting (6.13) into (6.14), we obtain
\[ v_t q_t^B(v_t, \hat{h}_{t-1}) - m_t^B(v_t, \hat{h}_{t-1}) \geq 0 \quad \text{and} \quad m_t^S(c_t, \hat{h}_{t-1}) - c_t q_t^S(c_t, \hat{h}_{t-1}) \geq 0. \]

It is well-known that, assuming revenue maximisation for a given allocation rule, interim individual rationality constraints bind for the worst-off types (i.e. buyers of type \( \underline{q} \) and sellers of type \( \underline{v} \)) and that, with binary types, the interim incentive compatibility constraints bind for the efficient types (i.e. \( \overline{v} \) and \( \overline{c} \)). The interim incentive compatibility constraints for the worst-off type are satisfied if
\[ q_t^B(\overline{v}, \hat{h}_{t-1}) \geq q_t^B(v_t, \hat{h}_{t-1}) \quad \text{and} \quad q_t^S(v_t, \hat{h}_{t-1}) \geq q_t^S(\overline{v}, \hat{h}_{t-1}). \]

These last two conditions are automatically satisfied, for example, by the dynamically efficient policies derived in the previous section.

Expected social welfare under any direct truthful mechanism is given by
\[
W = \sum_{t=1}^{\infty} \sum_{h_t \in \mathcal{H}_t} \delta^{t-1} \left( v_t Q_t^B(h_t) - c_t Q_t^S(h_t) \right) \mathbb{P}(H_t = h_t). \tag{6.16}
\]

We have already determined the allocations which maximise this objective function under various degrees of centralisation.

---

16 This means that
\[
U^B_t(v_t, \hat{h}_{t-1}) = \max_{v_t \in \{\underline{q}, \overline{q}\}} \{v_t q_t^B(v_t, \hat{h}_{t-1}) - m_t^B(v_t, \hat{h}_{t-1})\} \quad \text{and} \quad U^S_t(c_t, \hat{h}_{t-1}) = \max_{c_t \in \{\underline{q}, \overline{q}\}} \{m_t^S(c_t, \hat{h}_{t-1}) - c_t q_t^S(c_t, \hat{h}_{t-1})\}
\]

for all \( \hat{h}_{t-1} \in \mathcal{H}_{t-1}, v_t \in \{\overline{v}, \underline{v}\} \) and \( c_t \in \{\overline{c}, \underline{c}\} \).

17 Formally, the binding interim individual rationality constraints are
\[
m_t^B(v, \hat{h}_{t-1}) = v q_t^B(v, \hat{h}_{t-1}) \quad \text{and} \quad m_t^S(\overline{v}, \hat{h}_{t-1}) = \overline{v} q_t^S(\overline{v}, \hat{h}_{t-1}) \tag{6.13}
\]

and the binding interim incentive compatibility constraints are
\[
\overline{v} q_t^B(\overline{v}, \hat{h}_{t-1}) - m_t^B(v, \hat{h}_{t-1}) = \overline{v} q_t^B(\overline{v}, \hat{h}_{t-1}) - m_t^B(v, \hat{h}_{t-1}),
\]
\[
m_t^B(\overline{v}, \hat{h}_{t-1}) - \overline{v} q_t^S(\overline{v}, \hat{h}_{t-1}) = m_t^S(\overline{v}, \hat{h}_{t-1}) - \overline{v} q_t^S(\overline{v}, \hat{h}_{t-1}). \tag{6.14}
\]

Substituting (6.13) into (6.14), we obtain
\[
m_t^B(\overline{v}, \hat{h}_{t-1}) = \overline{v} q_t^B(\overline{v}, \hat{h}_{t-1}) - q_t^B(\overline{v}, \hat{h}_{t-1}) + v q_t^B(v, \hat{h}_{t-1}),
\]
\[
m_t^S(\overline{v}, \hat{h}_{t-1}) = \overline{v} q_t^S(\overline{v}, \hat{h}_{t-1}) - q_t^S(\overline{v}, \hat{h}_{t-1}) + v q_t^S(v, \hat{h}_{t-1}). \tag{6.15}
\]
6.4. DYNAMIC OPTIMALITY

Proposition 6.4.1. The dynamically efficient market clearing policies under discriminatory, uniform and fixed frequency market clearing can be implemented using a direct, interim incentive compatible and interim individually rational mechanism.

Proof. We prove the result for uniform market clearing. The other cases are similar. Start by constructing a direct allocation rule from the optimal market clearing policy. Let \( \hat{h} \in \{\tau, v\}^N \times \{\underline{c}, \tau\}^N \) be a realisation of the report process and \( \hat{h}_t \) denote \( \hat{h} \) restricted to its first \( 2t \) components. Let \( \{\tau^h_j\}_{j \in \mathbb{N}} \) denote the subset of periods such that the designer optimally chooses to clear the market under \( \pi^* \), given \( \hat{h} \) and set \( \tau^h_0 = 0 \) for convenience. For all \( i \in \mathbb{N} \) there exists \( j \in \mathbb{N} \) such that \( \tau^h_{j-1} < i \leq \tau^h_j \). The period \( \tau^h_j \) history of reports can be mapped to \( X^h_{j} \), the state of \( \langle X', A', P, r, \delta \rangle \) in period \( \tau^h_j \). Then if buyer \( i \) is part of an efficient or a suboptimal pair in period \( \tau^h_j \) we simply set \( Q^B_i(\hat{h}_{j-1}) = 1 \) and, for all \( k \in \mathbb{N} \setminus \{\tau^h_j\} \), \( Q^B_i(\hat{h}_k) = 0 \). Otherwise, we set \( Q^B_i(\hat{h}_k) = 0 \) for all \( k \in \mathbb{N} \). Similarly for seller \( i \).

Next, we verify the incentive compatibility constraints \( q^B_i(\tau, \hat{h}_{i-1}) \geq q^B_i(\tau, \hat{h}_{i-1}) \) and \( q^S_i(\underline{c}, \hat{h}_{i-1}) \geq q^S_i(\underline{c}, \hat{h}_{i-1}) \). These constraints hold under \( \pi^* \) since the arrival of a \( v \) or \( c \) agent cannot increase the expected number of periods until the next market clearing event (the Markov chain moves to a state with fewer expected transitions between it and the 0 state) and \( v \) and \( c \) agents are more likely to trade as part of any given market clearing event (these agents have rematching priority over \( v \) and \( \tau \) agents).

Proposition 6.4.1 is closely related to the main result of Parkes and Singh [120] which, while they do not explicitly determine the first-best allocation rule, shows that the first–best allocation can be implemented via an appropriate generalisation of the VCG mechanism. Proposition 6.4.1 shows that discriminatory market clearing (which is the solution of a relaxed version of the designer’s optimisation problem in which types are common knowledge) can be implemented via a direct, interim incentive compatible and interim individually rational mechanism. Thus, discriminatory market clearing is the first-best allocation rule.

The dynamically efficient policies are unique under uniform and fixed frequency market clearing. Under discriminatory market clearing, the efficient policy is unique only up to the queueing protocol for stored identical suboptimal pairs, which, from an efficiency perspective, is irrelevant.

6.4.2 Dynamically Optimal Mechanisms

We now determine the profit–maximising mechanism under discriminatory, uniform and fixed frequency market clearing. The problem proceeds in a similar manner to the continuous–types problem (see Sections 1.2.6 and 5.3.1)
and it is useful to begin by introducing the virtual types. Denote by \( \Phi(\tau) := \tau \) and \( \Gamma(\zeta) := \zeta \), respectively, the virtual value and virtual cost of the efficient types and by

\[
\Phi(\upsilon) := \upsilon - \frac{p}{1 - p}(\tau - \upsilon) \quad \text{and} \quad \Gamma(\tau) := \tau + \frac{p}{1 - p}(\tau - \zeta)
\]  

(6.17)

the virtual value and virtual cost of the inefficient buyer and seller types, respectively. Note that the discrete binary analogue of Myerson’s [116] regularity conditions, \( \Phi(\tau) > \Phi(\upsilon) \) and \( \Gamma(\tau) > \Gamma(\zeta) \), are satisfied. To obtain a non-trivial profit-maximisation problem we assume

\[
\tau > \Gamma(\tau) > \Phi(\upsilon) > \zeta.
\]  

(6.18)

If \( \tau \leq \Gamma(\tau) \) (and by symmetry \( \Phi(\upsilon) \leq \zeta \) were the case), then the market maker would only want to induce trade by buyers with high valuations and sellers with low costs, which makes the incentive problem degenerate: Buyers can simply be asked to pay \( \tau \) if they trade and sellers can be paid \( \zeta \) if they trade. (If \( \Gamma(\tau) \leq \Phi(\upsilon) \), were the case, full trade would be optimal, but this condition is ruled out by our assumptions of symmetry and \( \Delta < 1/2 \).) Under the normalisation \( \tau = 1 \) and \( \zeta = 0 \) and the symmetry assumption \( \zeta = 1 - \upsilon = 1 - \Delta \), (6.18) is easily seen to be equivalent to \( p < \Delta \).

**Proposition 6.4.2.** Expected profit under any direct, interim incentive compatible and individually rational mechanism that implements the allocation rule \( Q \) (and otherwise maximises revenue) is given by

\[
R = \sum_{i=1}^{\infty} \sum_{t=1}^{\infty} \sum_{h_t \in H_t} \delta^{t-1} \left( \Phi(v_i)Q_t^{B_i}(h_t) - \Gamma(c_i)Q_t^{S_i}(h_t) \right) \mathbb{P}(H_t = h_t).
\]  

(6.19)

**Proof.** For ease of exposition, we temporarily remove the explicit dependence on \( H_{t-1} \) from the mechanism. Denote the respective probability mass functions of \( v_t \) and \( c_t \) by \( f \) and \( g \). Using (6.13) and (6.15) we have

\[
R = \sum_{i=1}^{\infty} \mathbb{E} \left[ m^{B_i}(V_i) - m^{S_i}(C_i) \right]
\]

\[
= \sum_{i=1}^{\infty} \left\{ \left[ q^{B_i}(\upsilon) - \zeta(q^{S_i}(\zeta) - q^{S_i}(\tau)) - \tau q^{S_i}(\tau) \right] f(\upsilon)g(\zeta)
+ \left[ \bar{\upsilon} q^{B_i}(\bar{\upsilon}) - \bar{\upsilon} q^{S_i}(\bar{\upsilon}) \right] f(\bar{\upsilon})g(\bar{\upsilon}) \right\}
\]

\[
= \sum_{i=1}^{\infty} \left\{ \left[ q^{B_i}(\upsilon) \left( \upsilon + \frac{f(\bar{\upsilon})(\upsilon - \tau)}{f(\upsilon)} \right) - \zeta q^{S_i}(\zeta) \right] f(\upsilon)g(\zeta) \right\}
\]
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\[ R = \sum_{i=1}^{\infty} \sum_{v_i = \bar{v}} \sum_{c_i = \bar{c}} \left[ \Phi(v_i) q^{B_i}(v_i) - \Gamma(c_i) q^{S_i}(c_i) \right] f(v_i) g(c_i) \]

\[ = \sum_{i=1}^{\infty} \sum_{v_i = \bar{v}} \Phi(v_i) q^{B_i}(v_i) f(v_i) - \sum_{i=1}^{\infty} \sum_{c_i = \bar{c}} \Gamma(c_i) q^{S_i}(c_i) g(c_i). \]

Finally, using the definitions of the interim expected discounted trade probabilities gives

\[ R = \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \sum_{\ell \in H_t} \sum_{v_i = \bar{v}} \sum_{c_i = \bar{c}} \delta^{t-1} \Phi(v_i) Q^{B_i}(h_t) \mathbb{P}(H_t = h_t | V_t = v_i) f(v_i) \]

\[ - \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \sum_{\ell \in H_t} \sum_{c_i = \bar{c}} \delta^{t-1} \Gamma(c_i) Q^{S_i}(h_t) \mathbb{P}(H_t = h_t | C_t = c_i) g(c_i) \]

\[ = \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \sum_{\ell \in H_t} \delta^{t-1} \Phi(v_i) Q^{B_i}(h_t) \mathbb{P}(H_t = h_t) \]

\[ - \sum_{i=1}^{\infty} \sum_{t=i}^{\infty} \sum_{\ell \in H_t} \delta^{t-1} \Gamma(c_i) Q^{S_i}(h_t) \mathbb{P}(H_t = h_t) \]

\[ = \sum_{t=1}^{\infty} \sum_{\ell \in H_t} \delta^{t-1} \left( \Phi(v_i) Q^{B_i}(h_t) - \Gamma(c_i) Q^{S_i}(h_t) \right) \mathbb{P}(H_t = h_t) \]

as required.

Without much additional cost, we can generalise the analysis further to mechanisms that maximise social welfare subject to a profit constraint \( R \geq R \).

Recall from Section 1.2.6 that these mechanisms are equivalent to the class of \( \alpha \)-mechanisms, that maximise the convex combination \((1 - \alpha)W + \alpha R\) for some \( \alpha \in [0, 1]\). For \( \alpha \in [0, 1]\), the weighted virtual types for the inefficient buyers and sellers are given by

\[ \Phi_\alpha(v) := (1 - \alpha)v + \alpha \Phi(v) \quad \text{and} \quad \Gamma_\alpha(\tau) := (1 - \alpha)\tau + \alpha \Gamma(\tau), \quad (6.20) \]

18 Under the second–best mechanism, it is required that the mechanism does not run a deficit.
while for the efficient traders we simply have \( \Phi_\alpha(\tau) := \tau \) and \( \Gamma_\alpha(\zeta) := \zeta \).

By (6.17) we have
\[
\Phi_\alpha(\varpi) = \frac{(1 - p(1 - \alpha))\varpi - \alpha p\varpi}{1 - p} \quad \text{and} \quad \Gamma_\alpha(\varpi) = \frac{(1 - p(1 - \alpha))\varpi - \alpha p\varpi}{1 - p}.
\]

(6.21)

Mechanisms that maximise the convex combination
\[
(1 - \alpha)W + \alpha R
= \sum_{t=1}^{\infty} \sum_{i=1}^{t} \sum_{h_t \in H_t} \delta^{t-1} \left( \Phi_\alpha(\nu_i)Q_t^{B_i}(h_t) - \Gamma_\alpha(c_i)Q_t^{S_i}(h_t) \right) \mathbb{P}(H_t = h_t)
\]
subject to interim incentive compatibility and interim individual rationality constraints are thus constrained efficient mechanisms, with the additional constraint being some revenue threshold \( R \). For the sake of terminological clarity, we refer to such mechanisms as dynamically optimal or simply as optimal mechanisms. For a given form of market clearing, the profit–maximising mechanism corresponds to the special case of an optimal mechanism with \( \alpha = 1 \) while the dynamically efficient mechanism represents the special case with \( \alpha = 0 \).

Notice that the objective function for optimal mechanisms has the same functional form as the expression for expected discounted social welfare in (6.16), the only difference being that agents types are replaced with weighted virtual types. Furthermore, by (6.21) and the definition of \( \Delta \) we have
\[
\Phi_\alpha(\varpi) - \Gamma_\alpha(\varpi) = (1 - \alpha)\Delta + \frac{\alpha(\Delta - p)}{1 - p} = \Phi_\alpha(\varpi) - \Gamma_\alpha(\varpi).
\]

(6.22)

Assuming \( \Delta_\alpha > 0 \), which is the generalisation of condition (6.18) and easily seen to be satisfied if \( p < \Delta_\alpha / (\alpha(1 - \Delta_\alpha) + \Delta_\alpha) \), we can directly apply the results from Section 6.3. In this case, the first–best mechanism (i.e. the dynamically efficient mechanism under discriminatory market clearing) does not run a deficit. A general sufficient condition under which first–best is possible without running a deficit is provided in Proposition 6.4.3.

**Proposition 6.4.3.** If \( \delta \) is large enough, the first–best mechanism does not run a deficit.

**Proof.** Clearly, the position holds if \( \Delta \geq p \), as the profit–maximising allocation is given by the efficient allocation in this case. Suppose \( p > \Delta \) and consider the threshold mechanism with \( \tau = 1 \). In this case, we can analytically determine expected discounted profit under the threshold mechanism by solving (6.6) and (6.7), with \( \Delta \) replaced by \( (\Delta - p)/(1 - p) \). We find that
\[
R = \frac{p^2 \left( 2\Delta + \delta \Delta^2 - 2\delta \Delta p - 3\delta p + 1 \right)}{\left( \delta - 1 \right) \left( \delta + 3\delta p^2 - 3\delta p - 1 \right)}.
\]
Since profit is increasing in $\Delta$, we have $R < 0$ only if
\[ \frac{p^2 (\delta + \delta p^2 - 3\delta p + 1)}{(\delta - 1)(\delta + 3\delta p^2 - 3\delta p - 1)} < 0. \] (6.23)

Assuming that (6.23) holds we have
\[ \frac{\delta + \delta p^2 - 3\delta p + 1}{1 - \delta - 3\delta p^2 + 3\delta p} < 0 \Rightarrow 1 + \delta (1 - 3p + p^2) < 0. \]

Since $1 - 3p + p^2$ is minimised when $p = 1$, (6.23) requires
\[ 1 - \delta < 0, \] (6.24)
which is a contradiction. Thus, we have $R > 0$ when $\tau = 1$ for $p > \Delta$.

Furthermore, since the probability that suboptimal pairs trade decreases as $\tau$ increases, this implies that $R > 0$ for any threshold mechanism with $\tau > 0$.

Finally, by Proposition 6.3.7, $\tau^*$ is increasing in $\delta$. Thus, for sufficiently large $\delta$ we have $\tau^* > 0$ and $R > 0$.

The impossibility result of Myerson and Satterthwaite [117] does not hold, in general, for binary type distributions.\(^{19}\) However, Proposition 6.4.3 sheds new light on the impossibility of ex–post efficient trade along the lines of Myerson and Satterthwaite [117] (see Section 1.2.7) for dynamic environments. Assuming $p > 2\Delta$, the profit–maximising interim incentive compatible and individually rational efficient mechanism for $\delta = 0$ generates a deficit.\(^{20}\) From Proposition 6.4.3, we know that this deficit disappears when $\delta$ is sufficiently large. Thus, dynamics and optimally trading off gains from market thickness against costs of delay offer a way of overcoming the impossibility of ex–post efficient trade.\(^{21}\)

### 6.4.3 Comparative “Statics”

We say that the expected number of traders stored per period captures the patience of the market maker. Under discriminatory and fixed frequency

\(^{19}\)See Matsuo [103] for a treatment of the bilateral problem of Myerson and Satterthwaite [117] with binary types and Kos and Manea [81] for a version with general discrete types.

\(^{20}\)This static mechanism is readily derived. The interim individual rationality constraints for the inefficient traders are made binding by making interim expected payments for the buyer of type $\bar{v} = 1 - \Delta$ equal to $p_\bar{v}$ and the interim expected payment to the seller of type $\bar{c} = \Delta$ equal to $p\bar{c}$. Interim incentive compatibility for the efficient types then means that the buyer of type $\bar{v}$ pays no more than $(1 - p)\bar{v} + p_\bar{c}$ and the seller of type $\bar{c}$ is paid not less than $p\bar{c} + (1 - p)\bar{c}$. Substituting $\bar{v} = 1$ and $\bar{c}$, the expected revenue of the market maker is thus not more than $p(2\Delta - p)$, which is negative for $p > 2\Delta$.

\(^{21}\)Of course, this is related to the strand of literature in the tradition of Gresik and Satterthwaite [68] that investigates how quickly inefficiency disappears as markets that are constrained not to run a deficit grow large. The novelty of our setup and of Proposition 6.4.3 is the dynamic nature of this growth and the associated need for dynamic market mechanisms.
market clearing, the patience of the market maker is increasing in the optimal threshold. Since $\tau^*$ is decreasing in $\Delta$, the patience of the market maker is also increasing in $\tau^*$ under uniform market clearing.

**Proposition 6.4.4.** The patience of the market maker is increasing in $\alpha$.

**Proof.** Differentiating $\Delta_\alpha$ we obtain

$$\frac{\partial \Delta_\alpha}{\partial \alpha} = -\Delta + \frac{\Delta - p}{1 - p} = \frac{-p(1 - \Delta)}{1 - p} < 0.$$  

Recall that $\tau^*$ is decreasing in $\Delta$. Combining this with the previous result show that $\tau^*$ is increasing in $\alpha$. Thus, the patience of the market designer is increasing in the weight $\alpha$ on profit.

Interestingly, whenever $\Delta_\alpha < 0$, the designer is perfectly patient insofar as he does not execute any suboptimal trades.

**Corollary 6.4.5.** The patience of a profit–maximising market maker exceeds that of a welfare–maximising one.

**Proof.** For sufficiently $\delta$, $W^{D,0}(\delta) > W^{U,0}(\delta)$. Further,

$$\lim_{\delta \to 1} (W^{D,0}(\delta) - W^{D,1}(\delta)) = 0.$$  

Thus, by the results of Theorem 6.4.9, for sufficiently large $\delta$, $W^{D,1}(\delta) > W^{U,0}(\delta)$. The other cases are analogous.

Corollary 6.4.5 is reminiscent of Hotelling’s [75] finding that a monopolist extracts an exhaustible resource at a slower rate than a perfectly competitive industry. As is the case in static environments, these distortions arise under the optimal mechanism as a means of reducing the informational rents of agents. Although intuitive, Corollary 6.4.5 has important implications. In the perfectly patient limit (i.e. as $\delta \to 1$), which is equivalent to a static setup with a continuum of traders, all trades are efficient trades.

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22 Corollary 6.4.5 does not necessarily extend to finite horizon models with richer type spaces. For example, consider a two–period model in which a single Myerson and Satterthwaite [117] pair arrives in each period, with a common discount factor applied to period two. Based on static mechanism design intuition, one might expect the market designer to increase profit by restricting trade in each period. However, this leads to a decrease in the probability that period one agents trade in period two, which reduces the benefit of waiting in period one. Thus, in some cases it is optimal for the market designer to increase period one trade to raise additional profit. See Chapter 5.

23 Inefficiently few matches also take place under profit maximisation in the dynamic matching model of Pershman and Pavan [52].

24 In the limit as $\delta \to 1$, there is no opportunity cost associated with storing suboptimal pairs and we must have $\tau^* \to \infty$. In the limit, all efficient agents are eventually cleared from the market as part of an efficient trade and inefficient agents trade with probability 0.
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Illustrated in Figure 6.1 in Section 6.2, is efficient and the average quantity traded per period, the ex-post efficient quantity, is $p$. Consequently, in static setups suboptimal trades are indicative of inefficiency and possibly of rent extraction.\footnote{For example, Yavaş\cite{Yavaş2014} investigates whether profit seeking real-estate brokers have an incentive to maximise the number of trades, which is achieved by exclusively inducing suboptimal trades, rather than surplus because they earn a commission per trade.}

However, the efficient outcome is different when $\delta < 1$. Under the first-best mechanisms (i.e. the dynamically efficient policy under discriminatory market clearing), a suboptimal or inferior trade takes place in a given period if a suboptimal pair arrives to a market in which $\tau^*$ identical suboptimal pairs are stored. Thus, $(\bar{v}, \bar{c})$ and $(\bar{v}, c)$ trades take place in each period with probability $p(1 - p)/(2\tau^* + 1)$. This is illustrated in Figure 6.5. Therefore, what is inefficient in a static setting is an integral part of dynamic efficiency. Moreover, for a fixed discount factor, by Corollary 6.4.5 the fewer are such apparently inefficient suboptimal trades, the greater is the market maker’s rent extraction. Of course, this is no proof that suboptimal trades are efficient. However, it shows that efficiency and rent extraction have different, and at times, opposite implications in dynamic and in static settings.

Another fundamental difference to static setups is that in the dynamic setting efficiency is not a distribution-free concept. This is so because for any $\alpha \in [0, 1]$ the optimal mechanism depends on $p$ under any of the market clearing policies considered and has implications for implementation in “detail-free” environments as discussed in Section 7.2.3 below.

**Unraveling in Matching Markets** Early trading or matching has received a lot of attention in the matching literature, where it is typically perceived as something that is not desirable from society’s perspective. For example, Doval\cite{Doval2010} proposes an extension of static matching stability to dynamic matching environments and suggests that unraveling may be prevented if matching is facilitated by a clearing house which restricts agents from waiting for a better
match. This suggestion is in line with the findings of Baccara, Lee, and Yariv [9] but contrasts with our findings, which are based on mechanisms that are always interim incentive compatible and individually rational because monetary transfers provide additional means to align the objectives of individuals and the market makers. This prevents undesirable unraveling without violating voluntary participation.

6.4.4 Degrees of Market Centralisation

Because creating larger markets and employing increasingly sophisticated mechanisms may come at a cost (e.g. advertising, promotion and physical infrastructure investments), we now compare the different degrees of market centralisation – i.e. discriminatory, uniform and fixed frequency market clearing – to decentralised trade and quantify the benefits of increasingly sophisticated market design.

We start by considering a decentralised trade setup in which there is no mechanism for storing traders and agents can only trade in the period in which they arrive. Trades that increase social surplus are executed immediately. Total expected discounted welfare is

$$W^{0,0}(\delta) = \frac{1}{1-\delta} \left( p^2 + 2p(1-p)\Delta \right). \quad (6.25)$$

The outcome under decentralised trade coincides with the first–best outcome when $\delta = 0$ and can be implemented in dominant strategies. It is interesting to note that under continuous–time double auction mechanisms feasible trades are also executed immediately. Thus, the outcome of decentralised trade is the same as the outcome that would result under a continuous–time double auction with truthful bidding.

26 In contrast, the form of unraveling previously emphasised in matching literature is associated with agents matching prior to participating in a centralised platform which implements a static matching (see, for example, McKinney, Niederle, and Roth [106]).

27 Baccara, Lee, and Yariv [9] show that in their two–sided matching model agents wait inefficiently long for an improved match in a decentralised setting and assume that the central planner, or match maker, can force matches upon individuals in the centralised clearinghouse.

28 One can view this as a random search and match environment without search frictions within the current period but with infinitely costly search across periods. If one added additional search frictions within a period this would naturally increase the benefit of employing a centralised market mechanism relative to decentralised trade.

29 Continuous–time double auctions are not incentive compatible as the bid of a given trader affects both the probability of trade and, in the event that trade occurs, the market price. Under strategic bidding one would expect efficient types to bid shade in order to avoid trading with an inefficient type so that they receive a higher expected payoff. Although the equilibrium behaviour of a continuous–time double–auctions is difficult to characterise (see for example, Satterthwaite and Williams [133]), the outcome under the first–best mechanism provides an efficiency benchmark for evaluating the outcome of a continuous–time double auction.
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Figure 6.6: A numerical illustration of the performance of the optimal mechanism versus decentralised trade for a variety of parameter values.

Denote by $W^{D,\alpha}$ expected discounted welfare (starting from an empty market at $t = 0$) under discriminatory market clearing with a designer who maximises $(1 - \alpha) W + \alpha R$ with $\alpha \in [0, 1]$. Similarly, we use to notation $W^{U,\alpha}$, $W^{F,\alpha}$ and $W^{0,\alpha}$ for uniform market clearing, fixed frequency market clearing and decentralised trade, respectively.

Of course, welfare under the optimal welfare–maximising mechanism weakly exceeds welfare under decentralised trade. This is illustrated in Figure 6.6 (see Appendix B.1 for code). The following theorem compares welfare under the optimal profit–maximising mechanism to welfare under decentralised trade.

**Theorem 6.4.6.** There exist $\delta, \overline{\delta} \in [0, 1)$ with $\delta < \overline{\delta}$ such that

- $W^{D,1}(\delta) \leq W^{0,0}(\delta)$ for all $\delta \leq \overline{\delta}$,
- $W^{D,1}(\delta) < W^{0,0}(\delta)$ for all $\delta \in (\delta, \overline{\delta})$,
- $W^{D,1}(\overline{\delta}) = W^{0,0}(\overline{\delta})$ and $W^{D,1}(\delta) > W^{0,0}(\delta)$ for all $\delta > \overline{\delta}$.

As mentioned in the footnote preceding the theorem, we prove the statement in this footnote, which implies Theorem 6.4.6. To proceed, we first state and prove two lemmas.

**Lemma 6.4.7.** Under discriminatory market clearing, for any threshold policy with threshold $\tau$,

$$\frac{dV^D(\tau)}{d\delta} > \cdots > \frac{dV^D(1)}{d\delta} > \cdots > \frac{dV^D(0)}{d\delta}.$$  

30A more precise but not necessarily more transparent statement is: If $\Delta_1 > 0$, there exist $\delta_1, \delta_2 \in (0, 1)$ with $\delta_1 < \delta_2$ such that (i) $W^{D,1}(\delta) = W^{0,0}(\delta)$ for $\delta \in [0, \delta_1] \cup \{\delta_2\}$, (ii) $W^{D,1}(\delta) < W^{0,0}(\delta)$ for $\delta \in (\delta_1, \delta_2)$ and (iii) $W^{D,1}(\delta) > W^{0,0}(\delta)$ for $\delta \in (\delta_2, 1]$. If $\Delta_1 \leq 0$ there exists $\delta_3 \in (0, 1)$ such that (iv) $W^{D,1}(\delta) < W^{0,0}(\delta)$ for $\delta \in [0, \delta_3)$, (v) $W^{D,1}(\delta_3) = W^{0,0}(\delta_3)$ and (vi) $W^{D,1}(\delta_3) > W^{0,0}(\delta_3)$ for $\delta \in (\delta_3, 1]$. 

Proof. Let \( V^D_\tau \) denote the transpose of \((V^D_\tau(0), V^D_\tau(1), \ldots, V^D_\tau(\tau))\) and \( r \) denote the transpose of \((p^2, p, \ldots, p, p(1 + (1 - p)\Delta))\), which is a vector of length \( \tau + 1 \). Furthermore, let \( I \) denote the identity matrix. Using \( P \) we can rewrite (6.5) in matrix form,

\[
(I - \delta P)V^D_\tau = \delta r. \tag{6.26}
\]

This is a special case of Poisson’s equation for discounted discrete infinite horizon Markov decision processes, which is a property that we exploit in the Appendix.\(^{31}\)

Recall that for any threshold policy \( \tau \), \( V^D_\tau \) satisfies the Poisson equation given by

\[
(I - \delta P)V^D_\tau = \delta r.
\]

Here, \( V^D_\tau \) is such that, \( i, j \in \{0, 1, \ldots, \tau\} \) with \( i > j \), \( V^D_\tau(i) > V^D_\tau(j) \). This immediately follows from the fact that \( r_i \geq r_j \) (the expected immediate reward earned in the next period is weakly larger for state \( i \) compared to state \( j \)) and the structure of \( P \) (starting from state \( i \), the market transitions to a higher state in expectation compared to state \( j \)).

Differentiating the Poisson equation with respect to \( \delta \), we obtain

\[
(I - \delta P)\frac{dV^D_\tau}{d\delta} = r + PV^D_\tau = \frac{V^D_\tau}{\delta}. \tag{6.27}
\]

Therefore, aside from different right–hand side vectors, \( dV^D_\tau/d\delta \) and \( V^D_\tau \) satisfy the same Poisson equation.\(^{32}\) Since \( V^D_\tau(i) > V^D_\tau(j) \) we must have that \( dV^D_\tau(i)/d\delta > dV^D_\tau(j)/d\delta \).

Before stating the next lemma, we introduce the following notation. Let \( \tau^{D,0}(\hat{\delta}) \) and \( \tau^{D,1}(\hat{\delta}) \) denote the optimal thresholds under welfare–maximising and profit–maximising discriminatory market clearing, respectively.

**Lemma 6.4.8.** For any \( \hat{\delta} \in [0, 1] \) such that \( W^{D,1}(\hat{\delta}) \geq W^{0,0}(\hat{\delta}) \) and \( \tau^{D,1}(\hat{\delta}) > 0 \),

\[
\frac{dW^{D,1}(\delta)}{d\delta} \bigg|_{\delta = \hat{\delta}} > \frac{dW^{0,0}(\delta)}{d\delta} \bigg|_{\delta = \hat{\delta}}.
\]

\(^{31}\)This may be rewritten as the standard discrete–time Poisson equation (see, for example, Glynn and Meyn [65]) by adding to the Markov chain an absorbing state in which a reward of 0 is earned. We then suppose that in every period the Markov chain transitions to this absorbing state with probability \( 1 - \delta \) and scale all other transition probabilities by \( \delta \).

\(^{32}\)Equivalently, \( dV^D_\tau/d\delta \) is the vector of value functions corresponding to Markov decision process which, aside from the reward function, is identical to \( \langle X, A, P, r, \delta \rangle \).
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Proof. Suppose that \( \hat{\delta} \) is such that \( W^{D,1}(\hat{\delta}) \geq W^{0,0}(\hat{\delta}) \) and \( \tau^{D,1}(\hat{\delta}) > 0 \). We consider (6.6) with \( \tau = \tau^{D,1}(\hat{\delta}) \). Differentiating (6.6) gives

\[
d\frac{V_\tau^D(0)}{d\delta} = \frac{V_\tau^D(0)}{\delta(1 - \delta(1 - 2\lambda))} + \frac{2\delta\lambda}{1 - \delta(1 - 2\lambda)} \frac{dV_\tau^D(1)}{d\delta}.
\]

(6.28)

Since \( V_\tau^D(1) > V_\tau^D(0) \) by Lemma 6.4.7, (6.28) implies

\[
d\frac{V_\tau^D(0)}{d\delta} > \frac{V_\tau^D(0)}{\delta(1 - \delta)}.
\]

(6.29)

At the point \( \hat{\delta} \) we have \( \delta W^{D,1}(\hat{\delta}) = V_\tau^D(0, \hat{\delta}) = \delta W^{0,0}(\hat{\delta}) \). Furthermore,

\[
\delta W^{0,0}(\delta) = \frac{\delta(p^2 + 2p(1 - p)\Delta)}{(1 - \delta)} \Rightarrow \frac{d(\delta W^{0,0}(\delta))}{d\delta} = \frac{p^2 + 2p(1 - p)\Delta}{(1 - \delta)^2}.
\]

Combining this with (6.29) we obtain

\[
\frac{d}{d\delta} \left. \left( \delta W^{D,1}(\delta) \right) \right|_{\delta = \hat{\delta}} > \frac{W^{0,0}(\hat{\delta})}{1 - \hat{\delta}} = \frac{d}{d\delta} \left. \left( \delta W^{0,0}(\delta) \right) \right|_{\delta = \hat{\delta}}.
\]

This implies

\[
\frac{dW^{D,1}(\delta)}{d\delta} \bigg|_{\delta = \hat{\delta}} > \frac{dW^{0,0}(\delta)}{d\delta} \bigg|_{\delta = \hat{\delta}}
\]

as required.

We now prove Theorem 6.4.6. Recall that we let \( \tau^{D,0}(\delta) \) and \( \tau^{D,1}(\delta) \) denote the optimal thresholds under welfare–maximising and profit–maximising discriminatory market clearing, respectively.

Proof. We first deal with the \( \Delta_1 > 0 \) case. The functions \( W^{D,0} \), \( W^{D,1} \) and \( W^{0,0} \) are continuous (by Proposition 6.3.7) and satisfy \( W^{D,0}(0) = W^{D,1}(0) = W^{0,0}(0) \) and \( W^{D,0} \) and \( W^{0,0} \) are increasing in \( \delta \). By Proposition 6.3.7, \( \tau^{D,0}(\delta) \) and \( \tau^{D,1}(\delta) \) are increasing in \( \delta \) and by Corollary 6.4.5, \( \tau^{D,1}(\delta) \geq \tau^{D,0}(\delta) \). Thus, setting \( \hat{\delta} = \sup\{\delta \in [0, 1] : \tau^{D,1}(\delta) = 0\} \) we have, for sufficiently small \( \epsilon > 0 \), \( \tau^{D,0}(\hat{\delta} + \epsilon) = 0 \) and \( W^{0,0}(\hat{\delta} + \epsilon) = W^{D,0}(\hat{\delta} + \epsilon) > W^{D,1}(\hat{\delta} + \epsilon) \). Next, we set \( \overline{\delta} = \min\{\delta \in [\hat{\delta}, 1] : W^{D,1}(\delta) = W^{0,0}(\delta)\} \) so that \( W^{0,0}(\overline{\delta}) > W^{D,1}(\delta) \) for \( \delta \in (\hat{\delta}, \overline{\delta}) \). To see that \( \overline{\delta} \) exists, notice that under decentralised trade expected welfare per period welfare is constant and equal to \( p^2 + 2p(1 - p)\Delta \). As \( \delta \to 1 \) expected welfare per period under the profit–maximisation mechanism converges to \( p \). Since \( p > p^2 + 2p(1 - p)\Delta \), there exists \( \hat{\delta} \in (\hat{\delta}, 1] \) such that \( W^{D,1}(\hat{\delta}) = W^{0,0}(\hat{\delta}) \). That \( W^{D,1}(\delta) > W^{0,0}(\delta) \) for \( \delta > \overline{\delta} \) follows directly from Lemma 6.4.8. If \( \Delta_1 \leq 0 \) we may set \( \delta = 0 \) and the existence of \( \overline{\delta} \) follows directly from the previous argument, as well as the fact that in this case \( W^{D,0}(0) > W^{D,1}(0) \).
Theory 6.4.6 effectively says that for δ sufficiently large, welfare under profit-maximising discriminatory market clearing exceeds welfare under myopically, periodic ex-post efficient (see Section 1.2.8) decentralised trade. This is illustrated in Figure 6.7. To explain the intuition behind this result, we start with the case ∆₁ > 0. Here, no traders are stored under any mechanism for small values of δ. Thus, welfare under decentralised trade is equal to welfare under the profit-maximising mechanism. For intermediate values of δ, the number of traders stored under the dynamically efficient mechanism is closer to zero than to the larger number of trades stored under the profit-maximisation mechanism. As δ → 1 both the welfare-maximising and profit-maximising designer become perfectly patient and so for sufficiently large δ, welfare under the profit-maximising mechanism exceeds welfare under myopically, periodic ex-post efficient decentralised trade. The intuition for the ∆₁ ≤ 0 case is analogous, except that an inefficiently large number of traders are stored under profit-maximising discriminatory market clearing even for δ = 0.

To generalise Theorem 6.4.6, we first consider the change in welfare resulting from an increase in the degree of market centralisation.

**Theorem 6.4.9 (Gains from Centralisation).** The absolute gains from a higher degree of centralisation are increasing in δ and diverge as δ approaches 1. Formally, the functions \(W^{D,\alpha}(\delta) - W^{U,\alpha}(\delta)\), \(W^{U,\alpha}(\delta) - W^{F,\alpha}(\delta)\) and \(W^{F,\alpha}(\delta) - W^{0,\alpha}(\delta)\) with domain \((0,1)\) are positive, increasing in δ and diverge as δ → 1.

The relative gains from additional centralisation vanish while the relative gains from any degree of centralisation relative to decentralised trade remain strictly positive as δ approaches 1. Formally, we have

\[
\lim_{\delta \to 1} \frac{W^{D,\alpha}(\delta) - W^{U,\alpha}(\delta)}{W^{D,\alpha}(\delta)} = 0,
\]

\[
\lim_{\delta \to 1} \frac{W^{U,\alpha}(\delta) - W^{F,\alpha}(\delta)}{W^{U,\alpha}(\delta)} = 0,
\]
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\[
\lim_{\delta \to 1} \frac{W^{F,\alpha}(\delta) - W^{0,\alpha}(\delta)}{W^{F,\alpha}(\delta)} = (1 - p)(1 - 2\Delta).
\]

To prove this theorem, we start by stating and proving a useful lemma.

**Lemma 6.4.10.** Let \( V^D(0) \) and \( V^U(0) \) denote the expected discounted reward earned, starting from an empty market, under the optimal market clearing policy with discriminatory market clearing and uniform market clearing respectively. Then,

\[
\frac{dV^D(0)}{d\delta} > \frac{dV^U(0)}{d\delta}.
\]

**Proof.** By the proof of Lemma 6.4.7 we have

\[
(I - \delta P) \frac{dV^D}{d\delta} = \frac{V^D}{\delta},
\]

which shows that, aside from differing reward functions, \( dV^D/d\delta \) and \( V^D \) solve the same Markov decision process. Similarly, by the analysis in Section 6.3.3 we have

\[
(I - \delta P') \frac{dV^U}{d\delta} = \frac{V^U}{\delta}
\]

and, aside from differing reward functions, \( dV^U/d\delta \) and \( V^U \) also solve the same Markov decision process. Furthermore, for any \( x_I \in \{0, 1, \ldots, \tau^*\} \) we must have \( V^D(x_I) > V^U(0, x_I) \) since \( V^U \) solves a more constrained version of the same optimisation problem as \( V^D \). Therefore, \( dV^U/d\delta \) solves a more constrained optimisation problem for which smaller rewards are earned when the process transitions between states of the associated Markov decision process compared to \( dV^D/d\delta \). It immediately follows that \( dV^D(0)/d\delta > dV^U(0)/d\delta \). \( \square \)

We are now ready to prove Theorem 6.4.9.

**Proof.** Since \( W^{D,\alpha}(0) = W^{U,\alpha}(0), \ W^{D,\alpha}(\delta) = V^D(0)/\delta \) and \( W^{U,\alpha}(\delta) = V^U(0)/\delta \) the fact that \( W^{D,\alpha}(\delta) - W^{U,\alpha}(\delta) \) is positive, increasing in \( \delta \) and diverges as \( \delta \to 1 \) follows directly from Lemma 3. We may show that the function \( W^{U,\alpha}(\delta) - W^{F,\alpha}(\delta) \) has the required properties by adapting the arguments from the proof of Lemma 6.4.10 to uniform and fixed frequency market clearing. By (6.11) and (6.25) we have

\[
\frac{dW^{F,\alpha}}{d\delta} = \frac{\delta^* + \tau* - 1}{\delta(1 - \delta^*)} W^{F,\alpha}(\delta) \quad \text{and} \quad \frac{dW^{0,\alpha}}{d\delta} = \frac{1}{1 - \delta} W^{0,\alpha}(\delta).
\]

Since \( (\delta^* + \tau* - 1)/(1 - \delta^*) \geq 1/(1 - \delta) \) and \( W^{F,\alpha}(\delta) \geq W^{D,\alpha}(\delta) \) for all \( \delta \in (0, 1) \), it follows immediately that \( W^{F,\alpha}(\delta) - W^{0,\alpha}(\delta) \) is positive and increasing.
in $\delta$. The last statement of the theorem follows immediately from the fact that under discriminatory, uniform and fixed frequency market clearing per period welfare must converge to $p$ as $\delta \to 1$ and under discriminatory market clearing per period welfare must converge to $p^2 + 2p(1-p)\Delta$. This also shows that $W^{F,0}(\delta) - W^{0,0}(\delta)$ diverges as $\delta \to 1$.

Theorem 6.4.9 describes absolute and relative gains from (additional) centralisation and is illustrated in Figures 6.8 and 6.9 (refer to Appendices B.1, B.2 and B.3 for code that implements Algorithms 6.3.3, 6.3.5 and 6.3.6 in MATLAB R2016A, as well as a consistency checks created using value iteration). As $\delta$ becomes sufficiently large, the absolute gains from additional centralisation – e.g. from going from fixed frequency to uniform market clearing or from uniform to discriminatory market clearing – increase. Although the functions $W^{D,0}(\delta)$, $W^{U,0}(\delta)$, $W^{F,0}(\delta)$ converge to the same limit, the absolute gains diverge on $(0,1)$ as $\delta \to 1$ because of the differences in the speed of convergence. If additional centralisation came at a fixed cost that is unrelated to $\delta$, the first part of Theorem 6.4.9 would provide a powerful argument in favour of additional centralisation and sophistication. However, the cost of additional sophistication may plausibly be proportional to $\delta$ because additional centralisation comes at the cost of requiring additional investments (in infrastructure) that scales with the number of traders stored (and thus, indirectly, with $\delta$). In this case, the relative gains are the relevant benchmark and the gist of Theorem 6.4.9 is that the key is to depart from decentralised trade in favor of even the most moderate form of centralisation (i.e. fixed frequency market clearing).

Another pertinent issue in the design of two-sided markets is the need to “bring both sides of the market on board” (see, for example, Caillaud and Jullien [27] and Rochet and Tirole [129]). While a full analysis of this question requires a different model and is thus beyond the scope of this paper, the following corollary sheds new light on this question:

**Corollary 6.4.11.** There exist $\delta_D, \delta_U, \delta_F \in (0,1)$ such that $W^{D,1}(\delta) > W^{U,0}(\delta)$ for all $\delta > \delta_D$, $W^{U,1}(\delta) > W^{F,0}(\delta)$ for all $\delta > \delta_U$ and $W^{F,1}(\delta) > W^{0,0}(\delta)$ for all $\delta > \delta_F$. 
Figure 6.9: The relative gains of additional centralisation.

In words, Corollary 6.4.11 says that for sufficiently large discount factors, a profit–maximising platform will generate more welfare than an otherwise welfare–maximising platform in a market with a lower degree of centralisation. This is so because efficient types trade with relatively high probability under the profit–maximising platform, which is efficient for a sufficiently large discount factor. Therefore, if a profit–oriented centralised platform needs to attract buyers and sellers from, say, a welfare–maximising platform with a lower degree of centralisation, by Corollary 6.4.11 the profit–oriented platform can do so by offering a sufficiently high share of the trade surplus to efficient types while extracting all surplus from the inefficient types. By getting the key players on board – in our setting, these are the buyers of type $v$ and the sellers with cost $c$ – the others will have no choice but to follow suit.

6.5 Conclusion

In this chapter we analysed the tradeoff between the benefits of increased market thickness and the cost of delay in an infinite–horizon Bayesian mechanism design setup in which buyers and sellers with binary types arrive in pairs in every period. We showed that a threshold policy is optimal under discriminatory, uniform and fixed frequency market clearing. Exploiting dynamic programming and mechanism design techniques, we determined the optimal market clearing policy and the dynamically optimal mechanism in each case.

Our analysis revealed that a profit–maximising market maker under each form of centralisation – i.e. under discriminatory, uniform or fixed frequency market clearing – generates higher social welfare than a less centralised welfare–targeting market maker, provided the discount factor is sufficiently large. Indeed, with sufficient patience, the main gains from centralisation relative to a decentralised market are already achieved with fixed frequency market clearing, which is the least sophisticated form of centralisation. Moreover, we showed that the tradeoff between increased market thickness and delayed trade can be exploited to overcome the impossibility of (ex–post) efficiency without running a deficit, provided the agents and the market maker
are patient enough. As far as we know, this is the first instance that relates the (im)possibility of efficient trade to the dynamics of trade.

In the next chapter, we consider several extensions and applications of the models presented in this chapter.
Chapter 7

Extensions and Applications

We consider various extensions and applications of the dynamic market models derived in Chapter 6.

7.1 Introduction

In Chapter 6 we analysed the tradeoff between the opportunity cost of delay and increased market thickness in the future in an infinite horizon discrete-time Bayesian mechanism design setup in which buyers and sellers with binary types arrive in pairs in every period. We considered varying degrees of market centralisation, including discriminatory, uniform and fixed frequency market clearing. In each case, we derived the dynamically optimal market clearing policy and the mechanism that implements it by exploiting dynamic programming and mechanism design techniques. In this chapter we show that these dynamic market models provide the tractability required to explore several applications of economic interest and consider a variety of extensions.

Applying our model and analysis to the comparison of indirect taxes, which featured prominently in policy debates in the aftermath of the Global Financial Crisis, we show that specific taxes are distortionary whereas ad valorem taxes (that are levied on the profit of a profit-maximising market maker) are not. Augmenting the model to allow for in-house production by the market maker, we connect the incentives to integrate with the arrival process of buyers and sellers and show that the market maker is willing to incur a higher cost for in-house production the larger is the share of efficient traders. Finally, we show that it is possible to construct asymptotically optimal, detail-free, interim incentive compatible and individually rational mechanisms in environments in which the market maker does not know the distributions from which agents draw their types.

We may also consider a variety of extensions of the baseline model described in Chapter 6. It turns out that our methodology and the key results from Chapter 6 generalise, with minor qualifications, to a variety of alternative settings. In particular, we consider the arrival of unpaired agents, continuous-time arrival processes, group arrivals and richer type spaces. A sufficient condition for all our comparative “statics” results to generalise to richer type
spaces is a strengthening of Myerson’s regularity condition, which we call *dynamic regularity*. In addition to monotonicity of virtual type functions, dynamic regularity ensures that a profit–maximising market maker benefits at least as much from a trade that is superior from a social welfare perspective as does a benevolent social planner. With binary types, dynamic regularity is implied by regularity.

As previously stated, many of the extensions considered in this chapter increase the size of the state space of the underlying Markov decision process so that the optimal policy is no longer a simple threshold policy. However, threshold policies may still be used to construct a finite partition of the state space, which allows the optimal market clearing policies to be determined using a simple implementation of policy iteration. We provide a detailed exposition of this methodology for the case of unpaired arrivals.

The remainder of this chapter is structured as follows. In Section 7.2 we consider a variety of applications, including taxation policy, in–house production by the designer and robust mechanism design. We consider various extensions in Section 7.3. Section 7.4 details the methodology used to compute the optimal policies under the extensions which increase the state space of the underlying Markov decision process. Finally, concluding remarks are provided in Section 7.5.

### 7.2 Applications

Our model and analysis can be used to shed light on other questions of pertinent interest to economists such as in–house production, indirect taxation, and robust mechanisms.

#### 7.2.1 In–House Production

We start by considering an extension of the baseline model in which a profit–maximising designer has in addition the ability to produce in–house at unit cost \( \hat{c} \), which may alternatively be thought of as being vertically integrated. Under complete information, full surplus extraction is possible and the designer receives payoffs of \( \overline{v} - \zeta \) and \( \overline{v} - \overline{\tau} \) when trade is induced for efficient pairs and suboptimal pairs respectively. The designer will therefore optimally produce in–house at times if and only if \( \hat{c} < \overline{\tau} \). When types are the agents’ private information, the designer will optimally produce in–house at times if and only if \( \hat{c} < \Gamma(\overline{\tau}) \). Since \( \overline{\tau} < \Gamma(\overline{\tau}) \), the designer will produce more often in–house when agent types are private information, reflecting the notion that agency costs foster integration (see, for example, Williamson [143]). By the same token, under private information the designer will choose to produce
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in–house even when this is socially wasteful in the sense that \( \tau < \cd < \Gamma(\tau) \).\(^1\)

Interestingly, because \( \Gamma(\tau) \) increases in the share \( p \) of efficient traders, the incentives for vertical integration increase in this share. That is, the larger is \( p \) the larger is the social waste the designer is willing to accept while still producing in–house. While the probability that in–house production occurs decreases in \( p \), the threshold value for the in–house production cost of the profit–maximising, vertically integrated market maker increases in \( p \).

7.2.2 Indirect Taxes

The effect of different forms of indirect taxes on economic outcomes is another question of traditional interest in economics, and one that has received renewed attention in the debates following the Global Financial Crisis about alternative forms of transaction taxes for financial markets. For perfectly competitive and thick markets, which in our setup correspond to the limit case as \( \delta \to 1 \), it is well known that specific and ad valorem taxes are equivalent. It is, however, an open question how these tax instruments compare in markets whose thickness is endogenously determined.

To address this question, we now assume that the market maker is a profit maximiser and that authorities can observe and, under an ad valorem tax, tax the market maker’s revenue.\(^2\) This is analogous to the standard assumption in oligopoly models of indirect taxation that firms’ profits can be observed and taxed. The analysis applies equally to discriminatory, uniform and fixed frequency market clearing.

Under a specific tax \( \sigma \) per unit traded with \( \sigma > 0 \), the relative value of a suboptimal trade compared to an efficient trade decreases from \( \Delta_1 \) to \( (\Delta_1 - \sigma)/(1 - \sigma) \). If \( \sigma > \Delta_1 \), this will induce the market maker to become perfectly patient. If \( \Delta_1 > \sigma \), by the results of Section 3, this will induce the maker to increase the threshold \( \tau^* \). Thus, a specific tax distorts the relative value of suboptimal trades, inducing the market maker to create an excessively thick market and further reducing the welfare of buyers and sellers. In contrast, an ad valorem tax levied as a percentage on the market maker’s revenue will not affect the relative value of a suboptimal trade. Thus, the market clearing policy employed by the market maker will not change and an ad valorem tax can be levied without affecting social welfare. Consequently, we conclude that ad valorem taxes are superior to specific taxes in markets

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\(^1\) Whether profitable vertical integration by a profit–maximising market maker is indeed socially wasteful when compared to the equilibrium outcomes without integration (rather than to first–best) appears to depend on the size of \( \cd \).

\(^2\) We focus on profit–maximising market makers in this subsection to simplify the analysis. Otherwise, we would have to derive the optimal policies and mechanisms anew and impose an assumption as to how much the market maker cares for tax revenue relative to social surplus and his own profit.
whose thickness is endogenously determined by a profit–maximising market maker.\(^3\)

### 7.2.3 Robust Mechanisms

As noted, dynamic efficiency is not a distribution–free notion. A criticism of Bayesian mechanism design, often associated with Wilson [144], is that it depends on the fine details of the environment such as assumptions about distributions and on higher–order beliefs, which, in reality, designers and possibly the agents may be uncertain or agnostic about.\(^4\) In dynamic settings like ours, this creates a tension between efficiency and robustness as defined by Bergemann and Morris [13]. We now briefly sketch how this tension can be resolved or softened by estimating the distributions from agents’ reports, assuming for the remainder of this subsection that \(\alpha = 0\). Exactly because dynamic efficiency is not a distribution–free concept, estimation proves useful even when the designer puts zero weight on profit.\(^5\)

The basic idea is simple. With every additional trader arriving and reporting his type, an additional data point is generated, which, as time goes on and data accumulate, can be used to estimate the underlying distributional parameter \(p\) ever more precisely. For this purpose, it does not matter whether the designer and the agents are frequentists or Bayesian. The key is to maintain incentive compatibility for all agents and all types because otherwise reports induce biased estimates of the true distribution.\(^6\) Maintaining incentive compatibility is possible though a little trickier than it may appear at first glance.

---

\(^3\)Observe that the distorting effects of specific taxes vanish as \(\delta\) approaches 1 because in the limit suboptimal trades vanish.

\(^4\)In private values environments, this “Wilson” critique has led to the postulate that, for practicality, mechanisms endow the agents with dominant strategies and be free of the details of the underlying environment. In turn, these have led to the development of literatures on robust mechanism design and on mechanisms that estimate distributions; see, respectively, Bergemann and Morris [13] and Loertscher and Marx [90] and the references therein.

\(^5\)Most of the literature on mechanisms with estimation has focused on estimation to approximate Bayesian optimality (see, for example, Baliga and Vohra [10] and Segal [134]). The only exceptions we are aware of that use estimation in mechanism design with two-sided private information when the designer’s objective is efficiency without a deficit are Loertscher and Mezzetti [92], who use estimation to gauge aggregate demand and supply, and a special case in Loertscher and Marx [90], where estimation increases expected surplus and the speed of convergence in a double–clock auction. Gershkov and Moldovanu [61] consider achieving the efficient allocation in a dynamic environment with one-sided private information where the designer gradually learns the distribution of agents’ values. In a more recent paper, Gershkov, Moldovanu, and Strack [62] consider a related problem in which the designer known the distribution of agent types but must estimate the arrival process.

\(^6\)In light of the strand of literature that suggests that design problems be approached by violating (or getting rid of) incentive compatibility constraints (see, for example, the discussion in Milgrom [112]), it may be worth emphasising that in dynamic settings like the present one this approach would prove problematic because absent incentive compatibility distributions cannot be learned from bids or reports.
Suppose the designer shares the history of reports with the newly arriving agents. Even so, the agents’ estimates of distributions will typically not be the same because the agents are privately informed about their own types. This requires imposing some leeway in the incentive compatibility constraints, which is possible because of the discrete type space.\textsuperscript{7} Under discriminatory market clearing the designer can update the threshold $\tau^*(\cdot)$ as efficient, non–identical suboptimal and inefficient pairs are cleared from the market upon arrival. Only suboptimal pairs which arrive to an empty market and identical suboptimal pairs are stored. By employing a first–in first–out queueing protocol, the designer can ensure that stored agents are unaffected if the market clearing policy is updated on the basis of their reports.\textsuperscript{8}

7.3 Extensions

We now discuss how our methodology and main results generalise in a variety of extensions. Although a threshold policy will not be optimal in general, in all cases we consider threshold policies can be used to partition the state space, which makes the problem computationally tractable. This methodology is developed in detail in Section 7.4.

7.3.1 General Arrival Processes

Under discriminatory market clearing (see Definition 6.2.1), we can easily relax the assumption that buyers and sellers do not arrive in pairs. To that end, suppose in every period a buyer of type $v$ arrives with probability $p_1(p_2)$, and with probability $1 - p_1 - p_2$ no buyer arrives, and likewise for sellers.\textsuperscript{9} The designer will optimally store an unbounded number of unpaired efficient types and will store identical suboptimal pairs up to a threshold which can be computed using the methodology described in Section 6.3. Similarly, we can compute the optimal policy under fixed frequency market clearing by modifying (6.11).

With uniform market clearing (see Definition 6.2.1), dealing with unpaired agents is more problematic because from the market maker’s perspective, no immediate reward is earned when unpaired agents are cleared from the market.

\textsuperscript{7}Specifically, let $\hat{p}_t$ be the period $t$ estimate of $p$ by the designer and let $\hat{p}_t(\theta)$ be the estimate of $p$ of an agent of type $\theta$ with $\theta \in \{v, \tilde{v}, \tilde{c} \}$ who arrives in period $t$. Under uniform market clearing, if the market was last cleared in period $s$ with $s < t$, the market maker can use the threshold $\tau^*(\hat{p}_s)$ and use $\hat{p}_t(\theta)$ to determine expected discounted probabilities of trade and payments for an agent of type $\theta$ to maintain incentive compatibility and individual rationality constraints.

\textsuperscript{8}We will not deal with fixed frequency market clearing because it seems a little self–defeating to have market clearing occur in a fixed frequency while estimating distributions (without adjusting the thresholds and therefore the frequency of market clearing).

\textsuperscript{9}That is, in every period a seller of type $c$ (\tilde{c}) arrives with probability $p_1(p_2)$, and with probability $1 - p_1 - p_2$ no seller arrives.
but they are useful for forming pairs in the future. Thus, there is no upper bound on the number of unpaired agents stored under the optimal policy and allowing for unpaired agents significantly increases the size of the state space of the associated Markov decision process. We deal with this case in Section 7.4.

Our results immediately generalise to the case in which pairs of buyers and sellers arrive according to a Poisson process. Shortly, we will further generalise this to allow for more general renewal processes. In Section 7.4 we allow for buyers and sellers to arrive according to independent Poisson processes. Under every extension of the arrival process considered in this chapter, Proposition 6.4.4 and the vertical integration and taxation results of Sections 7.2.1 and 7.2.2 generalise because they depend directly on $\Delta_\alpha$ but not on the arrival process. Furthermore, once the state space and transition probabilities are determined for a given arrival process, the results regarding the degree of market centralisation in Corollary 6.4.11 also generalise. These results depend directly on the constraints applied to the optimisation problem and by virtue of the properties of the Poisson equation, the proof of Lemma 3 applies to any arrival process.

Renewal Process Arrivals  The results of Chapter 6 immediately generalise to the case in which pairs of buyers and sellers arrive according to a Poisson process. If the intensity of the arrival process is $\eta$ then the expected inter-arrival time is $1/\eta$ and the results in Section 6.3 apply if we simply use a discount factor of $\delta^{1/\eta}$. We can also consider the case in which pairs of buyers and sellers arrive according to a more general renewal processes. Let $A(s)$ denote the residual lifetime of the renewal process so that if an arrival occurred at time $t = 0$ and a second arrival has not occurred by time $t = s$, then $A(s)$ is the time between $s$ and the next arrival. If the renewal process exhibits the “new is better than used in expectation” property (that is, if $E[A(s)]$ is decreasing in $s$; see, for example, Barlow and Proschan [11]) then our methodology immediately generalises by simply using a discount factor of $\delta E[A(0)]$. If the renewal process does not have this property, at any time $s$ such that $E[A(s)] > E[A(0)]$, the market designer must recompute the optimal policy by applying a discount factor of $\delta E[A(s)]$ to the next arrival (but retaining a discount factor of $\delta E[A(0)]$ for all subsequent arrivals).

Group Arrivals  By appropriately updating the transition probabilities of the underlying Markov decision processes, our baseline model can easily be

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10 In Section 7.4 we further generalise this to the case in which buyers and sellers arrive according to independent Poisson processes.

11 This fact has previously been exploited in the literature on search with recall. See for, for example, Zuckerman [146], Gallien [58] (who actually requires the “increasing failure rate” property which implies the “new is better than used in expectation” property) and Gershkov and Moldovanu [59].
7.3. EXTENSIONS

denote the maximum number of pairs who arrive in a given period, where \(\xi \in \mathbb{Z} := \{(\xi_E, \xi_I) \in \mathbb{Z}_+^2 : \xi_E + \xi_I \leq N\}\). Under discriminatory market clearing, the optimal policy will be a threshold policy in which up to \(\tau\) identical suboptimal pairs are stored. For \(x_I \in \{0, \ldots, \tau\}\), the corresponding threshold policy is characterised by the linear system

\[
V^D_\tau(x_I) = \delta \sum_{\xi \in \mathbb{Z}} P(\Xi = \xi) \left[ (\xi_E + V^D_\tau(x_I + \xi_I))I(x_I + \xi_I \leq \tau) + (\xi_E + \Delta(x_I + \xi_I - \tau) + V^D_\tau(\tau))I(x_I + \xi_I > \tau) \right].
\]

Similarly, under uniform market clearing, the optimal policy will be a threshold policy in which trades accumulate in the order book up to a threshold value \(\tau\). For \((x_E, x_I) \in \mathbb{Z}_+^2\) be such that \(x_E + \Delta x_I < \tau\) and \(x_I > 0\), the corresponding threshold policy is characterised by

\[
V^U_\tau(x_E, x_I) = \delta \sum_{\xi \in \mathbb{Z}} P(\Xi = \xi) \left[ V^U_\tau(x_E + \xi_E, x_I + \xi_I)I(x_E + \xi_E + \Delta(x_I + \xi_I) < T) + (x_E + \xi_E + \Delta(x_I + \xi_I) + V^U_\tau(0, 0))I(x_E + \xi_E + \Delta(x_I + \xi_I) \geq T) \right].
\]

As before, fixed frequency market clearing can be dealt with by appropriately modifying (6.11). Considering group arrivals essentially enables us to consider non-uniform arrival processes. For example, this extension can be used to model markets in which large numbers of buyers and sellers tend to arrive together.

7.3.2 General Type Spaces

Our analysis also extends to models with richer, discrete type spaces. Assume that buyers draw their types independently from a discrete distribution \(F\) whose support is given by \(\{v_1, \ldots, v_n\}\) with \(v_1 < \cdots < v_n\) for \(n \in \mathbb{N}\) and that sellers draw their types independently from a discrete distribution \(G\) with support \(\{c_1, \ldots, c_m\}\), where \(c_1 < \cdots < c_m\) for \(m \in \mathbb{N}\). For \(i \in \{1, \ldots, n-1\}\) and \(j \in \{2, \ldots, m\}\), the virtual type functions are

\[
\Phi(v_i) = v_i - (v_{i+1} - v_i) \frac{1 - F(v_i)}{f(v_i)} \quad \text{and} \quad \Gamma(c_j) = c_j + (c_j - c_{j-1}) \frac{G(c_{j-1})}{g(c_j)},
\]

while for \(i = n\) and \(j = 1\), they are \(\Phi(v_n) = v_n\) and \(\Gamma(c_1) = c_1\).

Our methodology generalises immediately under the assumption \(v_2 > c_m > v_1 > c_{m-1}\), which serves the same purpose as the restriction \(\overline{\sigma} > \overline{\tau} > \overline{\tau}\).
Extensions and Applications

Figure 7.1: In order to consider general discrete type spaces, we must deal with two cases.

$v > c$ in the binary type setup (see Panel (a), Figure 7.1). One just has to expand the state space of the Markov decision process so that it includes each type of efficient and suboptimal trade. The main challenge in dealing with general type spaces when $v_2 > c_m > v_1 > c_{m-1}$ does not hold is that the market maker will now optimally store some types of infeasible trades which could prove useful for rematching in future periods (see Panel (b), Figure 7.1). Since there is no cost associated with storing such trades there is no bound on the number that can accumulate under the optimal policy. Thus, the existence of this type of infeasible trade substantially increases the size of the state space of the underlying Markov decision process. However, this can be dealt with by applying the same methodology used to solve the problem created by unpaired agents.\footnote{That is, analogous to the methodology developed in Section 7.4, we consider a related Markov decision process in which any infeasible trades containing an efficient agent are replaced by a trade containing the efficient agent paired with the least efficient agent that creates a feasible trade. By determining the optimal policy of the modified Markov decision process, we can identify a finite number of candidate optimal policies for the original Markov decision process.}

For the main results of Chapter 6 to generalise (in particular, Proposition 6.4.4, Corollary 6.4.11, and the vertical integration and taxation results of Sections 7.2.1 and 7.2.2), a new condition, which we call dynamic regularity, is sufficient whereas Myerson’s regularity condition is not. Distributions $F$ and $G$ are said to satisfy dynamic regularity if $\Phi$ and $\Gamma$ are non-decreasing and if, for $i \in \{1, \ldots, n-1\}$ and $j \in \{2, \ldots, m\}$,

$$\Phi(v_{i+1}) - \Phi(v_i) > v_{i+1} - v_i \quad \text{and} \quad \Gamma(c_j) - \Gamma(c_{j-1}) > c_j - c_{j-1} \quad (7.1)$$

holds. Dynamic regularity ensures that market makers who place a higher value on extracting rent always receive a higher payoff from rematching.
traders. Thus, if an efficiency–targeting market maker chooses to wait to clear the market in a particular state, so does a profit–maximising market maker. Like in static settings, Myerson’s regularity condition is sufficient for pointwise maximisation to be incentive compatible. However, in a dynamic setting, it no longer suffices for rent extraction and efficiency to be isomorphic because a rent extracting designer may have all sorts of interests in “reshuffling” trades. Dynamic regularity guarantees that the isomorphism extends.

Lastly, if we maintain the assumption of binary types, our analysis, including the mechanism design approach of Section 7.2, also generalises to agents with multi–unit demands and multi–unit capacities as discussed below. The key insight is that the design problem retains the property that agents’ private information pertains to a single dimension, namely the number of units a buyer values highly respectively the number of units a seller can produce at low costs.

Multi–Unit Traders With binary valuations and costs, it is also possible to extend the model to multi–unit traders without sacrificing its amenability to the mechanism design techniques. Specifically, assume that each buyer demands \( k \in \mathbb{N} \) units and each seller has the capacity to supply \( k \) units. A buyer’s type \( \theta_{B_t} \in \{0, \ldots, k\} \) is the number of units for which she has a marginal value of \( v \) while her marginal value for any of the additional units \( \max\{k - \theta_{B_t}, 0\} \) is \( c \). Similarly, seller’s type \( \theta_{S_t} \in \{0, \ldots, k\} \) is the number of units for which he has a marginal cost of \( c \) while his marginal cost for producing any of the additional units \( \max\{k - \theta_{S_t}, 0\} \) is \( c \). Assume that buyer types are distributed according to a discrete distribution \( F \) with \( \text{supp}(F) \subset \{0, \ldots, k\} \) and sellers types are distributed according to some discrete distribution \( G \) with \( \text{supp}(G) = \{0, \ldots, k\} \).

The arrival of the period \( t \) buyer and seller is equivalent to the arrival of \( \min\{\theta_{B_t}, \theta_{S_t}\} \) efficient pairs, \( |\theta_{B_t} - \theta_{S_t}| \) suboptimal pairs and \( k - \max\{\theta_{B_t}, \theta_{S_t}\} \) pairs which cannot trade. Thus, this problem is a special case of the group arrivals extension discussed above.

Demand Complementarity When buyer demand exhibits complementarities, this increases the relative benefit of storing traders in dynamic environments. Assume that a certain proportion of buyers demand two units of the good and derive zero utility from consuming a single unit. This increases the benefit of delaying market clearing since there are additional gains from rematching such traders. Specifically, assume that \( \beta \) is the proportion of buyers who demand two units with constant marginal value. Then when a given buyer arrives, with probability \( \beta p \) demand is \( (v, v) \), with probability \( \beta (1 - p) \)

\[ 14 \text{If, for example, buyers valuations for each unit are independent and equal to } v \text{ with probability } p \text{ and } \overline{v} \text{ with probability } 1 - p \text{ we obtain a binomial type distribution } Bn(k, p). \]
demand is \((v,v)\), with probability \((1 - \beta)p\) demand is \(\pi\) and with probability \((1 - \beta)(1 - p)\) demand is \(v\). We assume the type distribution for sellers is unchanged.

When \(\delta = 0\), buyers who demand two units cannot trade and the designer’s payoff is \((1 - \beta)(p^2 + 2p(1 - p)\Delta)\). When \(\delta = 1\), there is demand for \((1 + \beta)p\) units at valuation \(\pi\) so under the efficient allocation \(p\) sellers of type \(c\) and \(\beta p\) sellers of type \(\pi\) trade. For \(0 < \delta < 1\), we consider discriminatory market clearing and suppose the seller side of the market is short.\(^\text{15}\) After an arrival, the designer should clear any available \((\pi, \pi, \zeta, \xi)\) matchings, then clear any \((\pi, \xi)\) pairs and store \((\pi, \pi, \zeta, \xi), (\pi, \pi, \pi, \xi)\) and \((\pi, \pi)\) matches up to a threshold. The optimal threshold for each type of match decreases in the present value of that match. If a \((v, c)\) pair arrives, the seller should always be stored.\(^\text{16}\) Finally, notice that matches of type \((v, v, c)\) and \((v, v, \pi)\) can be stored at no cost and there is no bound on the number of matches of this type that can accumulate under the optimal mechanism. Thus, in the case of uniform market clearing the details of deriving the optimal mechanism are similar to the case in which unpaired agents arrive at the market (see Section 7.4).

### 7.4 Methodology for Dealing with Unpaired Agents

To deal with the problem of unpaired agents under uniform market clearing, we identify a finite partition of the state space which can be used to determine the optimal policy. One can accomplish this by computing the optimal threshold policy of a related Markov decision process. One can compute the optimal threshold \(\tau\) of this Markov decision process using the results of Section 6.3. Agents of type \(\pi\) and \(\zeta\) arrive with equal probability under this new and the original Markov decision processes. However, rematching frictions are reduced under the new Markov decision process because all agents must arrive in pairs. Thus, the threshold \(\tau\) provides an upper bound on the value of trades which can be stored under the optimal policy of the original Markov decision process. This allows a finite number of candidate optimal policies to be identified so that a simple policy iteration algorithm can be used to determine the optimal policy.

For ease of exposition, we now consider a continuous-time version of the model and suppose that buyers arrive according to the Poisson process \(\{N_t\}_{t \in \mathbb{R}_{\geq 0}}\) with rate \(\eta\) and sellers arrive according to the Poisson process \(\{M_t\}_{t \in \mathbb{R}_{\geq 0}}\) with rate \(\eta\). This ensures that a pair of buyers and sellers arrives with probability zero, which reduces the number of transitions which must

\(^{15}\)When the buyer side of the market is short, things are simple. We match \(\pi\) and \((\pi, \pi)\) types with the appropriate number of \(\xi\) types until the seller side of the market is short.

\(^{16}\)Sellers of type \(\zeta\) are useful since this type of seller will trade with non-zero probability even when \(\delta = 1\).
be considered in the Markov Decision Process. However, the methodology described here immediately applies to the setup described in Section 7.3.

We let \( \{B_i\}_{i \in \mathbb{N}} \) denote the set of buyers, so that the arrival time of \( B_i \) is given by \( \min \{ t \in \mathbb{R}_{>0} : N_t = i \} \). Similarly, let \( \{S_j\}_{j \in \mathbb{N}} \) denote the set of sellers, with the arrival of \( S_j \) given by \( \min \{ t \in \mathbb{R}_{>0} : M_t = j \} \). All other details of the setup remain unchanged.

### 7.4. METHODOLOGY FOR DEALING WITH UNPAIRED AGENTS

#### 7.4.1 Markov Decision Process

We start by defining the state space of the Markov decision process. We can think of the state space as having two branches, depending on whether there is a shortage of \( \pi \) or \( \zeta \) agents. Since buyers and sellers arrive at the same rate and agents of type \( \pi \) and \( \zeta \) arrive with equal probability, by symmetry, there is no need to distinguish these cases. Denote by \( Y_1 = \{(y_E, y_I, u_E, u_I, u'_I) : y_I + u_E \neq 0, u_E u_I = 0 \} \) the region of the state space in which there is a shortage of either \( \pi \) or \( \zeta \) agents. If we have a shortage \( \zeta \) agents, \( y_E \) denotes the number of \( (\pi, \zeta) \) pairs, \( y_I \) denotes the number of \( (\pi, \pi) \) pairs and \( u_E, u_I \) and \( u'_I \) denote the number of unpaired \( \pi \), \( \pi \) and \( \zeta \) agents, respectively. If we have a shortage of \( \pi \) agents, \( y_E \) denotes the number of \( (\pi, \zeta) \) pairs, \( y_I \) denotes the number of \( (\zeta, \zeta) \) pairs and \( u_E, u_I \) and \( u'_I \) denote the number of unpaired \( \pi \), \( \zeta \) and \( \pi \) agents, respectively.

There are two additional regions of the state space where these branches join. Let \( Y_0 = \{(w_I, w'_I) : w_I, w'_I \in \mathbb{Z}_{\geq 0} \} \) denote the region of the state space in which there are no \( \pi \) or \( \zeta \) agents present. Here, \( w_I \) counts the number of \( \zeta \) agents and \( w'_I \) counts the number of \( \pi \) agents. If the process is in \( Y_0 \) and a \( \pi \) or \( \zeta \) arrival occurs, the process will transition to the appropriate branch of \( Y_1 \). Finally, the process may transition from \( Y_1 \) to a region in which all \( \pi \) and \( \zeta \) agents are part of an efficient pair. We denote this region of the state space by \( Y_2 = \{(y_E, 0, 0, u_I, u'_I) : y_E > 0 \} \). For convenience, we assume that if a transition from \( Y_1 \) to \( Y_2 \) occurs, the labelling of the states is unchanged. If the process is in \( Y_2 \) and a \( \pi \) or \( \zeta \) arrival takes place, the process will transition to the appropriate branch of \( Y_1 \). The state space of the Markov decision process is given by \( Y = Y_0 \cup Y_1 \cup Y_2 \). Although this notation seems cumbersome, defining the state space in this manner will be convenient when we partition the state space and compute the optimal policy.

We now define the set of actions available in each state. For states \( y \in Y_0 \) we can set \( B_y = \{0\} \) since \( r(y) = 0 \) and it is not optimal for the designer to clear. For states \( y = (y_E, y_I, u_E, u_I, u'_I) \in Y_1 \cup Y_2 \) we set \( B_y = \{y, 0\} \).

Let \( P_b(y, y') \) denote the probability that if the designer takes action \( b \) in state \( y \), the state of the market following the next Poisson arrival is \( y' \). For \( y = (w_I, w'_I) \in Y_0 \), if a \( \pi \) or \( \zeta \) arrival occurs, the process will remain in \( Y_0 \) and we have

\[
P_b(y, (w_I + 1, w'_I)) = (1 - p)/2 \quad \text{and} \quad P_b(y, (w_I, w'_I + 1)) = (1 - p)/2.
\]
If a \( \pi \) or \( \zeta \) arrival occurs, the process will transition to \( \mathcal{V}_1 \) and we have
\[
P_b(y, (0, 0, 1, w_I, w'_I)) = p/2 \quad \text{and} \quad P_b(y, (0, 0, 1, w'_I, w_I)) = p/2.
\]
For \( y = (y_E, y_I, u_E, u_I, u'_I) \in \mathcal{V}_1 \cup \mathcal{V}_2 \) and \( b = y \), the process moves to \( 0 \) and the previous transition probabilities apply. We next consider \( y \in \mathcal{V}_1 \) and \( b = 0 \) with \( u_E > 0 \). If unpaired agents of type \( \pi \) are stored and a \( \pi \) or \( \zeta \) arrival takes place, no new pairs can be created. By symmetry, we have
\[
P_0(y, y + (0, 0, 1, 0, 0)) = p/2 \quad \text{and} \quad P_0(y, y + (0, 0, 0, 1)) = (1 - p)/2.
\]
If unpaired agents of type \( \pi \) are stored and a \( \zeta \) arrival takes place, an inefficient pair is created. If a \( \zeta \) arrival occurs, a suboptimal pair is created. By symmetry, we have
\[
P_0(y, y + (1, 0, -1, 0, 0)) = p/2 \quad \text{and} \quad P_0(y, y + (0, 1, -1, 0, 0)) = (1 - p)/2.
\]
Similar logic applies when \( u_I > 0 \) and we have
\[
P_0(y, y + (0, 0, 0)) = p/2, \quad P_0(y, y + (1, -1, 0, 0)) = p/2,
P_0(y, y + (0, 0, 1, 0)) = (1 - p)/2, \quad P_0(y, y + (0, 0, 0, 1)) = (1 - p)/2.
\]
Similarly, if \( u_E = u_I = 0 \) we have
\[
P_0(y, y + (0, 0, 0)) = p/2, \quad P_0(y, y + (1, -1, 0, 0)) = p/2,
P_0(y, y + (0, 0, 1, 0)) = (1 - p)/2, \quad P_0(y, y + (0, 0, 0, 1)) = (1 - p)/2.
\]
If \( y \in \mathcal{V}_2 \) and \( b = 0 \), we have
\[
P_0(y, y + (0, 0, 0, 1, 0)) = (1 - p)/2, \quad P_0(y, y + (0, 0, 0, 0, 1)) = (1 - p)/2.
\]
If \( u_I > 0 \) then it follows that \( P_0(y, (y_E, 1, 0, u_I - 1, u'_I)) = p/2 \) with \( P_0(y, (y_E, 0, 1, 0, u'_I)) = p/2 \) otherwise. Finally, \( P_0(y, (y_E, 1, 0, u'_I - 1, u_I)) = p/2 \) if \( u'_I > 0 \) and \( P_0(y, (y_E, 0, 1, 0, u_I)) = p/2 \) otherwise.

Finally, if \( y \in \mathcal{V}_1 \cup \mathcal{V}_2 \) and action \( b = y \) is implemented, the immediate reward is
\[
s(y) = y_E + \Delta y_I.
\]
Otherwise, the designer earns no reward. The market designer’s problem is to determine the optimal policy \( \pi^* \) of the Markov decision process \( (\mathcal{V}, \mathcal{B}, P, s, \delta) \). However, dealing with unpaired agents is problematic. From the designer’s perspective, no immediate reward is earned when unpaired agents are cleared from the market but they are useful for forming pairs in the future. Therefore, there is no upper bound on the number of unpaired agents stored under the optimal policy.
7.4. METHODOLOGY FOR DEALING WITH UNPAIRED AGENTS

7.4.2 Partitioning the State Space

To deal with the problem of unpaired agents, we identify a finite partition of the state space which can be used to determine the optimal policy. We accomplish this by computing the optimal threshold policy of a related Markov decision process.

Suppose temporarily that every buyer arrives paired with a seller of type \(\pi\) and every seller arrives paired with a buyer of type \(\nu\), so that agents arrive according to the Poisson process \(\{L_t\}_{t \geq 0}\) with rate \(2\eta\). Any given arrival is \((\nu, \pi)\) with probability \(p/2\), \((\pi, \nu)\) with probability \(p/2\) and \((\nu, \pi)\) with probability \((1 - p)\). The state space, the set of available actions and the reward function of this Markov decision process are the same as the Markov decision process \(\langle X, A', P, r, \delta \rangle\) considered in Section 6.3. However, we must update the transition probabilities defined. For our new process we have \(P_a(x, x - a) = 1 - p\).

If \(x = a\) or \(x_1 = 0\) we also have \(P_a(x, (x_E - a_E, x_1 - a_1 + 1)) = p\).

Otherwise,
\[
P_a(x, (x_E, x_1 + 1)) = p/2 \quad \text{and} \quad P_a(x, (x_E + 1, x_1 - 1)) = p/2.
\]

Thus, we have a Markov decision process \(\langle X, A', P, r, \delta \rangle\). Repeating the analysis from Section 6.3 allows us to determine the optimal threshold policy of this Markov decision process \(\langle X, A', P, r, \delta \rangle\). Let \(\tau\) denote the threshold corresponding to the optimal threshold policy.

**Theorem 7.4.1.** Under the optimal policy \(\pi^*\) of \(\langle Y, B, P, s, \delta \rangle\), \(\pi^*(y) = y\) for all \(y \in Y\) such that \(s(y) > \tau\).

**Proof.** Suppose the market is in state \(y \in Y\), which might have unpaired agents. We compare the optimal market clearing policy under the original Markov decision process \(\langle Y, B, P, s, \delta \rangle\) to the optimal market clearing policy when the arrival process is \(\{L_t\}_{t \in \mathbb{R}_{\geq 0}}\). If the market is in state \(y\) and the future arrival process is \(\{L_t\}_{t \in \mathbb{R}_{\geq 0}}\), it must be optimal to clear the market if \(s(y) > \tau\) because the threshold \(\tau\) applies regardless of whether unpaired agents are present (this follows directly from the proof of Theorem 6.3.2).

The reward earned when the market is cleared immediately is independent of the arrival process but the benefit of waiting is greater under the arrival process \(\{L_t\}_{t \in \mathbb{R}_{\geq 0}}\). Although agents of type \(\pi\) and \(\nu\) arrive at the same rate under each arrival process, these agents always arrive in a pair under \(\{L_t\}_{t \in \mathbb{R}_{\geq 0}}\) which improves future trading possibilities. It follows that \(\pi^*(y) = y\) for all \(y \in Y\) such that \(s(y) > \tau\). \(\square\)

We have already established that it is optimal to wait for all states \(y \in Y_0\) and it is optimal to clear for states \(y \in Y_1 \cup Y_2\) such that \(s(y) > \tau\). We now...
define the finite partition of the state space $Y$ which contains states for which
we must determine whether it is optimal to wait to clear the market. That
is, we use the upper bound $\tau$ to determine the finite partition over which we
must search for the optimal policy.

Suppose the market is in some state $y \in Y_1 \cup Y_2$ such that $r(y) \leq \tau$. Notice that under the optimal policy $\pi^*$

$$\nu_{yE} = 1 + \max\{i \in \mathbb{N} : yE + \Delta i \leq \tau\}$$

is the maximum number of unpaired agents that could feasibly trade when
the next market clearing event occurs. It follows that for $i, j \geq \nu_{yE}$,

$$V_{\pi^*}(yE, yI, i, u_I, u'_I) = V_{\pi^*}(yE, yI, j, u_I, u'_I).$$

Analogous arguments apply to the $u_I$ and $u'_I$ states. Intu-

itively, once $\nu_{yE}$ unpaired agents have accumulated, any additional unpaired
agents are of no benefit to the designer. Therefore, the set of states in $Y_1$ that
must be checked in order to determine the optimal policy is given by

$$Z_1 = \bigcup_{yE=0}^{\lfloor \tau \rfloor} \bigcup_{yI=0}^{\nu_{yE}-yI} \left( \bigcup_{i=0}^{\nu_{yE}+yI} \left( yE, yI, i, 0, j \right) \right)$$

$$\bigcup_{j=0}^{\nu_{yE}+yI} \left( yE, yI, 0, i, j \right).$$

The set of states in $Y_2$ that must be checked is given by

$$Z_2 = \bigcup_{yE=1}^{\lfloor \tau \rfloor} \bigcup_{i=0}^{\nu_{yE}} \bigcup_{j=0}^{\nu_{yE}} (yE, 0, 0, i, j).$$

**Theorem 7.4.2.** The optimal policy $\pi^*$ is uniquely determined by its values
on the finite subsets $Z_1 \subset Y_1$ and $Z_2 \subset Y_2$ defined in (7.3) and (7.4).

**Proof.** We have established that for $y \in Y$ such that $s(y) > \tau$, $\pi^*(y) = y$. For states $y \in Y_0$, we have $\pi^*(y) = 0$. Suppose $\pi^*(y)$ is known for all $y \in Z_2$. Then for any $y \in Y_2$,

$$V_{\pi^*}(y) = V_{\pi^*}(y') \quad \text{and} \quad \pi^*(y) = \pi^*(y'),$$

where $y' = (yE, 0, 0, \min\{u_I, \nu_{yE}\}, \min\{u'_I, \nu_{yE}\}) \in Z_2$. Analogous logic applies for states $y \in Y_1$. Thus, $\pi^*$ is completely characterised.

Since $Z$ contains a finite number of states we have a finite number of
candidate optimal policies for the Markov decision process $(Y, B, P, s, \delta)$. We
can therefore use a policy iteration algorithm to determine the optimal policy
$\pi^*$. We start by setting $Z_0 = \{(i, j, 0) \in Y_0 : 0 \leq i, j \leq \nu_0\}$, where $\nu_0 = 1 + \max\{i \in \mathbb{N} : \Delta i \leq \tau\}$.
Figure 7.2: In Panel (a), the methodology developed in Section 6.3 was used to compute the optimal threshold policy for the continuous–time version of the model with coupled Poisson arrival processes. Panel (b) shows welfare under the optimal threshold policy. Panel (c) shows welfare under the optimal policy when the arrival processes are independent, where the optimal policy was computed using Algorithm 7.4.3. This panel also shows welfare under the optimal policy of \( \langle X, A', P, r, \delta \rangle \) (the process used to compute the upper bound \( \tau \)).

**Algorithm 7.4.3.** Begin with the policy \( \pi_0(y) = 0 \ \forall \ y \in Z_0 \cup Z \). At step \( i \in \mathbb{N} \),

1. For every \( y \in Z_0 \cup Z \) compute \( V_{\pi_{i-1}}(y) \).

2. For every \( y \in Z \), if \( V_{\pi_{i-1}}(y) \leq r(y) + V_{\pi_{i-1}}(0) \) set \( \pi_i(y) = y \). Otherwise set \( \pi_i(y) = 0 \).

3. If, for every \( y \in Z \), \( \pi_i(y) = \pi_{i-1}(y) \), return \( \pi^* = \pi_i \). Otherwise proceed to step \( i + 1 \).

Since \( Z \) is finite, by Howard [76], this algorithm returns \( \pi^* \) in finitely many steps (see Section 2.4). For states in \( Y_0 \) it suffices to compute \( V_\pi(y) \) for \( y \in Z_0 \). To see this, notice that \( V_\pi(w_1, w'_1, 0) = V_\pi(\min\{w_1, v_0\}, \min\{w'_1, v_0\}, 0) \) since at most \( v_0 \) agents of type \( v \) or \( c \) can trade when the next market clearing event occurs. Similarly, for states in \( Y_1 \cup Y_2 \) it suffices to compute values for states in \( Z \).

To compute the value function \( V_\pi \) for a given policy \( \pi \), we may formulate and solve a finite linear system similar (this was done for \( \langle X, A, P, r, \delta \rangle \) in (6.9) and (6.10)). Alternatively, we can use value iteration to compute \( V_\pi \) numerically and recursively as follows. Start by setting \( V_{\pi,0}(y) = 1 \) for all \( y \in Z_0 \cup Z \). At step \( j \), we set

\[
V_{\pi,i+1}(y) = \sum_{y' \in Z_0 \cup Z} P_{\pi_i(y')}(y, y') \left( s(y') + \delta V_{\pi,i}(y') \right)
\]

until \( \|V_{\pi,i+1} - V_{\pi,i}\|_\infty < \varepsilon \), where \( \varepsilon \) is the prescribed tolerance (see Section 2.4).
Some numerical results produced by implementing Algorithm 7.4.3 in MATLAB R2016a can be found in Figure 7.2 (refer to Appendix B.4 for code).

7.5 Conclusion

In this Chapter, we considered various applications and extensions of the tractable baseline model developed in Chapter 6.

Comparing specific and ad valorem taxes in a setup in which market thickness is endogenous, we showed that specific taxes are distortionary whereas ad valorem taxes, levied on the profit of a profit–maximising market maker, are not. Applying our model to in–house production, we connected the incentives to integrate with the arrival process of buyers and sellers.

This chapter thus provided a novel answer to the question raised by Coase [32] as to why there are firms in the first place if markets are such an efficient means of allocating resources. Interpreting increases in the share of efficient traders as increases in the efficiency of the market, efficiency strengthens incentives for vertical integration (where strength is measured as the threshold cost below which the profit–maximising market maker will choose to produce in–house). Although a larger fraction of efficient traders makes in–house production less likely, the integrated firm is keener on in–house production insofar as it is willing to incur a larger cost. To the best of our knowledge, these predictions are novel.

Finally, the notion of dynamic efficiency developed in Chapter 6 is not distribution–free, which creates a tension between efficiency and robustness. We showed that these tensions can be resolved by constructing asymptotically optimal prior–free mechanisms, which estimate type distributions from agent reports.

We also considered a variety of extensions of the baseline model described in Chapter 6, showing that our methodology and key results generalise. In particular, we considered the arrival of unpaired agents, continuous–time arrival processes, group arrivals and richer type spaces. We showed that although many of these extensions increase the state space of the problem, simple threshold policies can be used to construct a finite partition of the state space. Finally, we illustrated that our results generalise to richer type spaces under a strengthening of Myerson’s regularity condition, which we call dynamic regularity.
Chapter 8

Conclusion

Some concluding remarks are provided and possibilities for future work are discussed

In this thesis we started by investigating the statistical properties of static market mechanisms. First, we introduced a general non-central hypergeometric distribution. We showed that under the assumption of independent private values, the equilibrium quantity traded has a general non-central hypergeometric distribution for a variety of market mechanisms. Second, for the same class of mechanisms, we developed a general methodology for approximating outcomes in large markets. In particular, we showed that the joint distribution of the equilibrium quantity traded and welfare is asymptotically normal, and computed the parameters of the approximating normal distribution. In future work, it would be interesting to see whether we can verify and exploit the statistical predictions made the results of these chapters using real market data. For example, suppose we had data concerning the number of market participants and units traded for an appropriate market. Depending on the application, we could use the results of Chapter 3 to shed light on the distribution of buyer valuations relative to seller costs, the strategic behaviour of market participants or rent extraction by the market maker. The difficulty of achieving this in practice would be finding an appropriate data set containing information concerning the number of market participants.

The second part of this thesis considered dynamic markets in which traders arrive over time and optimally assessed the tradeoff between the benefits of market thickness stemming from accumulating traders and the cost of delay. We started by considering a two-period extension of the classic model of Myerson and Satterthwaite [117] and derived the class of Bayesian optimal α-mechanisms within this setup. We found that dynamics tends to increase the convexity of the period one allocation so that more “symmetric” trades take place in period one, while “asymmetric” trades are delayed in the hope that the more efficient period one agent can be rematched in period two. We also considered approximate implementation in the form of a price-posting mechanism and found that even when myopically optimal prices are used, this mechanism performs well relative to benchmark static mechanisms. Un-
CONCLUSION

fortunately, the computational complexity of this model made it difficult to
derive further results, which motivated us to consider models with simpler
type spaces.

The final chapters of this thesis focused on an infinite horizon dynamic
market model in which a buyer–seller pair arrives in each period and agents
have binary types. This model provided us with the tractability needed to con-
sider varying degrees of market centralisation, as well as a variety of extensions
and applications. We found that, provided the discount factor is sufficiently
high, a profit–maximising two–sided platform generates more welfare com-
pared to a less centralised, welfare–targeting market maker. Further, most of
the benefits from dynamic mechanisms are captured by clearing markets at
fixed, optimally chosen frequencies. We introduced notions of dynamic effi-
ciency and optimality and explored their implications for in–house production
by the market maker, taxation policy and robust mechanism design. Finally,
we showed that with minor qualifications, our analysis and results extend to
more general arrival processes and type spaces. In particular, we found that
under a dynamic generalisation of Myerson’s [116] regularity condition, we
can consider more general discrete type spaces.

Future research could involve further developing the applications and ex-
tensions considered in Chapter 7. For example, it would be interesting to
further develop our model of in–house production because this could provide
a novel model of the firm and a novel perspective on the boundary between the
firm and the market. There is scope to further investigate models involving
demand complementarity in dynamic markets, as it appears that the gains
from centralisation would be more significant in such an environment. Some
preliminary research considering correlated agent types has already been con-
ducted and it appears that the results of McAfee and Reny [105] could be
applied to achieve full surplus extraction in the two–period model considered
in Chapter 5. Extending these results could be a fruitful avenue for future
research. Given the connection identified between this thesis and the Industrial
Organisations literature on two–sided platforms, it would be interesting to
explore the connection between market thickness and the market power of
a two–sided platform or consider a model involve competing two–sided plat-
forms. However, based on previous work in the literature concerning two–sided
platforms, it appears that tractability would be of significant concern for this
latter problem.

We also intend to further develop the price–posting mechanism from Chap-
ter 5 and consider similar mechanisms for the models of Chapters 6 and
7. Specifically, we have already conducted preliminary research considering
a price–posting implementation under discriminatory market clearing. We
found that provided the discount factor is sufficiently high (so that it is not
optimal to clear the market in every period), a mechanism that posts prices
prior to the arrival of traders in each period can be used to implement the
optimal mechanism. This fact appears to be at the heart of the possibility
result we derived in Chapter 6. Furthermore, we found that the volatility of
the optimal posted prices (and the price impact of the report of a given agent)
is decreasing in the discount factor and market thickness. Thus, it seems that
we could study price–posting mechanisms in order to connect our definition of
market thickness with those more commonly used in finance. In future work
we would like to formalise and generalise these results.

Finally, we would also like to investigate whether the mathematical
methodology we developed to solve dynamic market models is useful for
solving other problems in which similar dynamic tradeoffs arise. In par-
ticular, of interest are potential problems in organisational economics (for
example, knowledge accumulation and dissemination) and political economy
(for example, optimal term limits).

\footnote{Recall that we found that the dynamically efficient mechanism does not run an expected
deficit, provided it is not optimal to clear the market in every period. It appears that this
result is driven by the fact that we can find an implementation that allows the market maker
to post prices prior to the arrival of traders in each period.}
Appendix A

Mathematica Code

A.1 Chapter 4

A.1.1 Figure 4.3

ListPlot[
  MapThread[{{#1, #2} &, {Table[i/n, {i, 1, n}],
    Sort[RandomVariate[UniformDistribution[{0, 1}], n], Greater]}],
  MapThread[{{#1, #2} &, {Table[i/m*λa, {i, 1, m}],
    Sort[RandomVariate[BetaDistribution[2, 1], m]]}],
  Table[{{i/n, InvF[1 - i/n]}, {i, 1, n}],
  Table[{{i/m*λa, InvG[i/m]}, {i, 1, m}}, Joined -> True,
  AxesLabel -> {"t"},
  PlotStyle -> {{Gray, Dashed}, {Gray}, {Blue, Dashed}, {Red}},
  PlotLegends -> Placed[{{"F-1(1-t)", "G-1(t/λa)"}, {0.75, 0.95}},
  BaseStyle -> {FontSize -> 20},AspectRatio -> 0.8]

A.1.2 Figures 4.4 and 4.5

(* Set problem parameters *)
Needs["MultivariateStatistics"]

n = 500;
m = 250;
λa = m/n;
F[x_] := x;
f[x_] := 1;
InvF[x_] := x;
G[x_] := x^2;
g[x_] := 2*x;
InvG[x_] := Sqrt[x];
\[ Ex[x_] := \text{InvF}[1 - x] - \text{InvG}[x/\lambda] \]
\[ t_0 = t /. \text{Solve}[Ex[t] == 0, t][1] \]
\[ a2x = \text{Simplify} \left( \frac{1}{(D[Ex[y], y] /. y \to t_0)^2} \left( t_0 \ast (1 - t_0) + \frac{(1 - t_0) \ast g[\text{InvG}[t_0/\lambda]]}{\sqrt{\text{InvG}[t_0/\lambda]^2}} \right) \right) \]
\[ \text{IntEx} = \text{Integrate}[Ex[t], \{t, 0, t_0\}] \]
\[ So[t_] := (1 - t) \text{Integrate}[(1 - F[x]), \{x, \text{InvF}[1 - t], 1\}] + \frac{1 - t/\lambda}{g[\text{InvG}[t/\lambda]]} \text{Integrate}[G[x], \{x, 0, \text{InvG}[t/\lambda]\}] \]
\[ \rho2x = \text{Simplify}[2 \ast \text{Integrate}[So[t], \{t, 0, t_0\}] \]
\[ \text{Covx} = \text{Simplify}[-So[t0] / (D[Ex[y], y] /. y \to t0)] \]
\[ Q[data_] := \text{Total}[Boole[\{x1 \geq 0\} \& (@data[1] - data[2])]] \]
\[ W[data_] := \text{Total}[\text{Select}[\{data[1] - data[2], \# > 0\}]] \]
\[ \text{data}[\text{Nsim}, n, m] := \text{Module}[\{data\}, \text{Tdata} = \text{Table}[\{i, 1, \text{Nsim}\}] \]
\[ \text{Do}[] \]
\[ \text{data} = \{\text{Take}[\text{Sort}[\text{RandomVariate}[\text{UniformDistribution}[\{0, 1\}], n], \text{Greater}], \text{Min}[n, m], \text{Take}[\text{Sort}[\text{RandomVariate}[\text{BetaDistribution}[2, 1], m]], \text{Min}[n, m]]\} \]
\[ \text{Tdata}[1] = \{Q[data] - n \ast t_0] / \text{Sqrt}[n], (W[data] - n \ast \text{IntEx}) / \text{Sqrt}[n]\}; \]
\[ i, 1, \text{Nsim}]\}; \text{Tdata} \]
\[ \text{(Run 10,000 simulations)} \]
\[ \text{Nsim} = 10000; \]
\[ \text{dataset} = \text{data}[\text{Nsim}, n, m] \]
\[ \text{(Compute empirical ellipsoidal quantiles for the simulations)} \]
\[ \text{q25} = \text{EllipsoidQuantile}[\text{dataset}, 0.25]; \]
\[ \text{q75} = \text{EllipsoidQuantile}[\text{dataset}, 0.75]; \]
\[ \text{q95} = \text{EllipsoidQuantile}[\text{dataset}, 0.95]; \]
\[ \text{(Compute theoretical ellipsoidal quantiles)} \]
\[ \text{tq25} = \text{EllipsoidQuantile}[\text{MultinormalDistribution}[\{0, 0\}], \{a2x, Covx\}, \{Covx, \rho2x\}], 0.25]; \]
\[ \text{tq75} = \text{EllipsoidQuantile}[\text{MultinormalDistribution}[\{0, 0\}], \{a2x, Covx\}, \{Covx, \rho2x\}], 0.75]; \]
\[ \text{tq95} = \text{EllipsoidQuantile}[\text{MultinormalDistribution}[\{0, 0\}], \{a2x, Covx\}, \{Covx, \rho2x\}], 0.95]; \]
\[ \text{(Create plots)} \]
\[ \text{lg1} = \text{PointLegend}[\text{Blue}, \{\text{"Simulated Data"}\}]; \]
\[ \text{lg2} = \text{LineLegend}[\text{Pink}, \{\text{"Simulated Ellipsoidal Quantiles"}\}]; \]
\[ \text{lg3} = \text{LineLegend}[\text{Dashed}, \{\text{"Theoretical Ellipsoidal Quantiles"}\}]; \]
A.2 Chapter 5

A.2.1 Figure 5.4

(* Define type distributions *)
\[ y = 0; \]
\[ v = 1; \]
\[ c = 0; \]
\[ σ = 1; \]
\[ F[v_] := v; \]
\[ f[v_] := 1; \]
\[ G[c_] := c; \]
\[ g[c_] := 1; \]

(* Define virtual valuation and cost functions *)
\[ g[v_] := v - \frac{1-F[v]}{f[v]}; \]
\[ Γ[c_] := c + G[c]; \]
\[ g[c] \]

(* Define weighted virtual valuation and cost functions *)
\[ \rho_0[v_c, c_] := (1-α) \cdot v + α \cdot G[v]; \]
\[ \rho_1[v_c, c_] := (1-α) \cdot c + α \cdot Γ[c]; \]
\[ \text{Inverse}[v_c, c_] := \text{InverseFunction}[\rho_0, 1, 2][v, c]; \]
\[ \text{Inverse}[c, c_c] := \text{InverseFunction}[\rho_0, 1, 2][c, c]; \]
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(* Determine expected weighted surplus if the market designer waits at $t = 1$ *)
Integrate[(\$\$[v1, a] + \$\$[v2, a] - Ra[c1, a] - Ra[c2, a]) f[v2] g[c2],
{v2, v1, v}, {c2, c, InverseRA[\$\$[v1, a], a]}] +
Integrate[(\$\$[v1, a] + \$\$[v2, a] - Ra[c1, a] - Ra[c2, a]) f[v2] g[c2],
{v2, InverseRA[Ra[c1, a], a], v1}, {c2, c, InverseRA[\$\$[v2, a], a]}] +
Integrate[(\$\$[v1, a] - Ra[c2, a]) f[v2] g[c2],
{v2, InverseRA[Ra[c1, a], a], v1}, {c2, c, c}] +
Integrate[(\$\$[v1, a] - Ra[c1, a]) f[v2] g[c2],
{v2, v, InverseRA[Ra[c1, a], a]}, {c2, c1, c}] +
Integrate[(\$\$[v1, a] - Ra[c2, a]) f[v2] g[c2],
{v2, v, InverseRA[Ra[c2, a], a]}, {c2, c, c}] +
Integrate[(\$\$[v2, a] - Ra[c2, a]) f[v2] g[c2], {v2, v1, v},
{c2, InverseRA[\$\$[v1, a], a], c}] // Simplify

\[
\frac{1}{9 (1 + \alpha)^2} \left(3 + 3 \alpha - \alpha^3 - 6 (1 + \alpha)^3 c_1 + 3 \alpha (1 + \alpha)^2 c_1^2 + 2 (1 + \alpha)^3 c_1^3 - 6 \alpha (1 + \alpha)^2 v_1 + 3 (1 + \alpha)^2 (2 + 3 \alpha) v_1^2 - 2 (1 + \alpha)^3 v_1^3\right)
\]

(* Compute expected revenue associated with clearing at $t = 1$ *)
\$\$[v1, a] - Ra[c1, a] +
\delta \* Integrate[(\$\$[v2, a] - Ra[c2, a]) f[v2] g[c2], {v2, v, v},
{c2, c, InverseRA[\$\$[v2, a], a]}] // Simplify

\[
\frac{-6 \alpha - 6 \alpha^2 + \delta - \alpha \delta + \alpha^2 \delta - 6 (1 + \alpha)^2 c_1 + 6 (1 + \alpha)^2 v_1}{6 (1 + \alpha)}
\]

(* Determine the period 1 allocation rule *)
Solve[
\[
\frac{1}{1 + \alpha} \left(3 + 3 \alpha - \alpha^3 - 6 (1 + \alpha)^3 c + 3 \alpha (1 + \alpha)^2 c^2 + 2 (1 + \alpha)^3 c^3 - 6 \alpha (1 + \alpha)^2 v_1 + 3 (1 + \alpha)^2 (2 + 3 \alpha) v_1^2 - 2 (1 + \alpha)^3 v_1^3\right) =
\left(-6 \alpha - 6 \alpha^2 + \delta - \alpha \delta + \alpha^2 \delta - 6 (1 + \alpha)^2 c + 6 (1 + \alpha)^2 v_1\right) = 0, v_1\] // Simplify

(* Using this output determine the period 1 allocation rule and create plot *)

\textbf{A.2.2 Figure 5.6}

(* Use $\alpha$ to specify the identity of the $\alpha$ mechanism. Use $\beta$ to specify the payoff function. *)
\$\$[v, c, \beta] := (1 - \beta) \* f[v] + \beta \* \$\$[v];
\$\$[v, c, \beta] := (1 - \beta) \* G[c] + \beta \* f[c];

(* Write a function that numerically computes the period 1 allocation *)
Int[\$\$, \$\$, \beta] :=
Module[{z}, z /. FindRoot[clearpolicy[z, \$\$, \$\$, \beta] == v, \{z, (c + v) / 2\}]][1]];

(* Divide (c,v) space into five integration regions to compute payoff under given $\alpha$ mechanism *)
A.2. CHAPTER 5

(* Determine total payoff in the MS setup with no option to wait *)
\[ W_{12}[\delta, \alpha, \beta] := \]
\[ \text{Integrate}\left[\left(\rho^2[v_1, \beta] - \rho^2[c_1, \beta]\right) f[v_1] g[c_1],\right.\]
\[ \left\{c_1, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\right\}, \{v_1, \text{Inverse}\left[\text{Out}\left[c_1, \alpha\right], \alpha\right], v\}\right] + \]
\[ \delta^* \text{Integrate}\left[\left(\rho^2[v_2, \beta] - \rho^2[c_2, \beta]\right) f[v_2] g[c_2],\right.\]
\[ \left\{c_2, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\right\}, \{v_2, \text{Inverse}\left[\text{Out}\left[c_2, \alpha\right], \alpha\right], v\}\right]\]

(* Determine total payoff in the GS setup with all agents present at t = 2 *)
\[ W_{12}[\delta, \alpha, \beta] := \]
\[ \delta^* \]
\[ \text{NIntegrate}\left[\left(\rho^2[v_1, \beta] - \rho^2[c_1, \beta]\right) f[v_1] g[c_1],\right.\]
\[ \left\{v_1, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\right\}, \{c_2, \alpha, \text{Inverse}\left[\text{Out}\left[v_2, \alpha\right], \alpha\right]\}\right] + \]
\[ \text{Integrate}\left[\left(\rho^2[v_1, \beta] - \rho^2[c_1, \beta]\right) f[v_1] g[c_1],\right.\]
\[ \left\{v_1, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\right\}, \{c_2, \alpha, \text{Inverse}\left[\text{Out}\left[v_2, \alpha\right], \alpha\right]\}\right] + \]
\[ \delta^* \text{Integrate}\left[\left(\rho^2[v_2, \beta] - \rho^2[c_2, \beta]\right) f[v_2] g[c_2],\right.\]
\[ \left\{v_2, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\right\}, \{c_2, \alpha, \text{Inverse}\left[\text{Out}\left[v_2, \alpha\right], \alpha\right]\}\right]\]

(* Now compute the payoff for the region: \(\text{In}(c) \text{Out}(v) \text{Out}(\alpha) \text{Out}(\beta)\) *)
\[ W_{1}[\delta, \alpha, \beta] := \]
\[ \text{NIntegrate}\left[\left(\rho^2[v_1, \beta] - \rho^2[c_1, \beta]\right) \right.\]
\[ \left. + \delta^* \text{Integrate}\left[\left(\rho^2[v_2, \beta] - \rho^2[c_2, \beta]\right) f[v_2] g[c_2],\right.\]
\[ \left\{v_2, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\right\}, \{v_1, \text{Inverse}\left[\text{Out}\left[c_2, \alpha\right], \alpha\right], v\}\right] + \]
\[ \delta^* \text{Integrate}\left[0, \{c_1, \alpha, \text{Inverse}\left[\text{In}\left[v, \alpha\right], \alpha\right]\}, \{v_1, \text{clearpolicy}[c_1, \delta, \alpha], v\}\right]\]
(** Compute the payoff for the region: $\Delta^{2}(\Gamma_{c}(c)) \cap \Gamma^{4}(\Delta(c))$ **)  

```
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(* There are four cases for t = 2: 1) two trades take place, 2) buyer 1 and seller 1 trade, 3) buyer 1 and seller 2 trade, 4) buyer 2 and seller 1 trade *)

\[
\delta \partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial\partial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A.2. CHAPTER 5

(*) Compute the payoff for the region: $\mathcal{R}(c) \cap \mathcal{Y} \cap \mathcal{S}(\mathcal{X}(c))$

(*) There are four cases for $t = 2$: 1) no trades take place, 2) buyer 2 and seller 2 trade, 3) buyer 1 and seller 2 trade, 4) buyer 2 and seller 1 trade *)

$\mathbb{W}(\delta, \xi, \beta) :=$

$\delta \ast \text{Integrate}[\text{Integrate}[0, \{v_2, v_1\}, \{c_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}] +$

$\text{Integrate}[0, \{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, \xi, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{v, c_2, \xi, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, v_1, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +\}

$\delta \ast$

$\text{Integrate}[\{v_2, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}, \{c_2, \text{Inverse}[\mathcal{R}(v_2, a, a, c)\}] +$

$\text{Integrate}[\{v_2, v_1, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}, \{c_2, \xi, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}, \{v, c_2, \xi, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +\}

$\delta \ast$

$\text{Integrate}[\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{c_2, \text{Inverse}[\mathcal{Y}(v_2, a, a, c)\}] +$

$\text{Integrate}[\{v_2, v_1, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +\}

$\delta \ast$

$\text{Integrate}[\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{c_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}] +$

$\text{Integrate}[\{v_2, v_1, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +$

$\text{Integrate}[\mathcal{Y}(v_2, \beta - \mathcal{Y}(c_2, \beta)) f[v_1] g[c_1] f[v_2] g[c_2],$

$\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{v, c_2, c_1\}] +\}

$\delta \ast$

$\text{Integrate}[\{v_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{c_2, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}] +$

$\text{Integrate}[\{v_2, v_1, \text{Inverse}[\mathcal{X}(v_2, a, a, c)\}, \{c_2, c_1\}] +$
(* Compute the payoff for the region: \( y_1 < V \) \( V < y_2 \) *\)
(* There are four cases for \( t = 2 \): 1) no trades take place, 2)
       buyer 2 and seller 2 trade, 3) buyer 1 and seller 2 trade, 4)
       buyer 2 and seller 1 trade *)

\[ W_5[\sigma, \alpha, \beta, \gamma] := \]
\[ = \text{Integrate}[\text{Integrate}[0, \{v_2, y, v_1\}, \{c_2, \text{InverseLog}[\text{Div}[v_1, \alpha], \alpha], \alpha\}, \gamma] + \]
\[ \text{Integrate}[0, \{v_2, v_1, y\}, \{c_2, \text{InverseLog}[\text{Div}[v_2, \alpha], \alpha], \alpha\}, \gamma] + \]
\[ \text{Integrate}[\{\beta + \gamma - \beta\}, \{c_2, \beta\}, \gamma] f[v_1] g[c_1] f[v_2] g[c_2], \]
\[ \{v_2, v_1, y\}, \{c_2, \gamma, \text{InverseLog}[\text{Div}[v_2, \alpha], \alpha]\}, \gamma] + \]
\[ \text{Integrate}[\{\beta + \gamma - \beta\}, \{c_2, \beta\}, \gamma] f[v_1] g[c_1] f[v_2] g[c_2], \]
\[ \{v_2, v_1, y\}, \{c_2, \gamma, \text{InverseLog}[\text{Div}[v_2, \alpha], \alpha]\}, \gamma] + \]
\[ \{c_1, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha], \alpha\}, \{v_1, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha], \gamma]\} + \]
\[ \delta * \]
\[ \text{Integrate}[\text{Integrate}[0, \{c_2, \gamma, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha]\}, \gamma] + \]
\[ \{v_2, y, \text{InverseLog}[\text{Div}[c_1, \alpha], \alpha]\}, \gamma] + \]
\[ \text{Integrate}[0, \{c_2, \gamma, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha], \gamma\}, \gamma] + \]
\[ \{v_2, y, \gamma\} + \]
\[ \text{Integrate}[\{\beta + \gamma - \beta\}, \{c_2, \beta\}, \gamma] f[v_1] g[c_1] f[v_2] g[c_2], \]
\[ \{c_2, \gamma, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha\}, \gamma\}, \{v_2, \text{InverseLog}[\text{Div}[c_1, \alpha], \alpha], \gamma\}, \gamma\}, \gamma] + \]
\[ \{c_1, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha], \alpha\}, \{v_1, y, \text{InverseLog}[\text{Div}[\gamma, \alpha], \alpha]\}, \gamma\} + \]
\[ W_5[\sigma, \alpha, \beta] \]

(* Determine total payoff under the \( \alpha \) mechanism *)

\[ W[\sigma, \alpha, \beta] := W_1[\sigma, \alpha] + W_2[\sigma, \alpha, \beta] + W_3[\sigma, \alpha, \beta] + W_4[\sigma, \alpha, \beta] + \]
\[ W_5[\sigma, \alpha, \beta] \]

(* Create plots *)

A.2.3 Figure 5.6

(* Use Ridder's method to determine the \( \alpha \)- mechanism that gives zero revenue for a given value of \( \delta \) *)

\[ \text{Ridder}[\sigma, \epsilon] := \text{Module}[\{a = 0, b = 1, c = 1/2, k = 1, d, fa, fb, fc, fd\}, \]
\[ \text{While}[\text{Abs}[k] > \sigma, fa = W[\sigma, a, 1]; \]
\[ fb = W[\sigma, b, 1]; \]
\[ fc = W[\sigma, c, 1]; \]
\[ d = c + (c-a)/(\text{Sign}[fa-fb]) \times \text{Sqrt}[fc^2-2 \times fa \times fb]; \]
\[ fd = W[\sigma, d, 1]; \]
\[ \text{If}[\text{Sign}[fd] \neq \text{Sign}[fc], \text{If}[c < d, a = c; \]
\[ b = d, a = d; \]
\[ b = c], \text{If}[\text{Sign}[fa] \neq \text{Sign}[fd], b = d, a = d]; \]
\[ c = (a + b) / 2; \]
\[ k = b - a]; \]
\[ \text{Return}[c] \]
A.2.4 Figures 5.8 and 5.9

(* We need to compute cutoff types under the price-posting mechanism *)
(* Compute buyer's discounted expected payoff from waiting. The first case is v₁ ≥ c₁ and the second is v₁ < c₁ *)

\[ \delta \cdot \text{Integrate[Integrate}[v₁ - c₁, {c₂, 0, c₁}, {v₂, c₁, 1}] + \]
\[ \text{Integrate}[v₁ - c₂, {c₂, c₁, v₁}, {v₂, c₂, 1}] + \]
\[ \text{Integrate}[v₁ - v₂, {v₂, c₂, v₁}, {c₂, v₂, 1}] + \]
\[ \text{Integrate}[v₂ - c₁, {c₂, v₂, 0}, {c₁, c₂, 1}] + \]
\[ \text{Integrate}[v₂ - v₁, {v₁, c₂, 0}, {c₁, 0, v₁}] + \]
\[ \delta \cdot \text{Integrate}[\text{Integrate}[v₂, v₂, v₂, v₂, v₁, v₁, v₁, 1]] // \]
\[ \text{Simplify} \]

\[ -\frac{1}{2} \delta (-2 + v₁) v₁ \]

(* Compute seller's discounted expected payoff from waiting. The first case is v₁ ≥ c₁ and the second is v₁ < c₁ *)

\[ \delta \cdot \text{Integrate[Integrate}[v₁ - c₁, {v₂, v₁, 1}, {c₂, 0, v₁}] + \]
\[ \text{Integrate}[v₂ - c₁, {v₂, c₁, v₁}, {c₂, 0, v₂}] + \]
\[ \text{Integrate}[c₂ - c₁, {v₂, c₁, v₁}, {v₂, 0, c₂}] + \]
\[ \text{Integrate}[c₂ - v₂, {v₂, v₁, 1}, {c₂, v₂, 0}, {v₁, v₂, 1}] + \]
\[ \text{Integrate}[v₂ - c₁, {c₂, v₁, 1}, {c₁, v₂, 1}, {v₁, 1}] + \]
\[ \delta \cdot \text{Integrate[Integrate}[c₂ - c₁, {v₂, c₁, 1}, {c₂, v₂, 1}] + \]
\[ \text{Integrate}[v₂ - c₁, {v₂, c₁, 1}, {c₂, v₂, 1}, {v₁, 0, c₁}] // \]
\[ \text{Simplify} \]

\[ -\frac{1}{2} \delta (-1 + c₁)³ (1 + c₁) \]

(* Given δ and posted prices, compute cutoff types *)

\[ \delta = 0.75; \]
\[ \text{ps} = 0.5; \]
\[ \text{pb} = 0.5; \]
\[ \text{starvalues} = \]
\[ \text{Solve}[[\left(-\frac{1}{2} (-2 + v) v^3 \delta = v - \text{pb}, \right. \]
\[ \left. -\frac{1}{2} (-1 + c)³ (1 + c) \delta \equiv \text{ps} - c \land c ≥ 0 \land v ≥ 0 \land c ≤ 1 \land v ≤ 1 \right], \{v, c\}]; \]
\[ \text{vstar} = v /. \text{starvalues}[1]; \]
\[ \text{cstar} = c /. \text{starvalues}[1]; \]
MATHEMATICA CODE

(* Determine welfare with price posting in period 1 *)
(* This expression is valid only for v\text{star} \geq 0.5 *)

\text{W}[\delta, \text{cstar}_\text{r}, \text{v\text{star}}_\text{r}] :=
\text{Integrate}[\text{v}_1 - \text{c}_1 + \delta \times \text{Integrate}[\text{v}_2 - \text{c}_2, \{\text{v}_2, 0, 1\}, \{\text{c}_2, 0, \text{v}_2\}],
\{\text{c}_2, 0, \text{c\text{star}}, \text{v}_2, \text{v\text{star}}, 1\}] +
\delta \times \text{Integrate}[\text{Integrate}[\text{v}_1 - \text{c}_1 + \text{v}_2 - \text{c}_2, \{\text{v}_2, \text{v}_1, 1\}, \{\text{c}_2, 0, \text{v}_1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_1 + \text{v}_2 - \text{c}_2, \{\text{v}_2, \text{c}_1, \text{v}_1\}, \{\text{c}_2, 0, \text{v}_2\}] +
\text{Integrate}[\text{v}_1 - \text{c}_1, \{\text{v}_2, \text{c}_1, \text{v}_1\}, \{\text{c}_2, \text{v}_2, 1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_1, \{\text{v}_2, 0, \text{c}_1\}, \{\text{c}_2, \text{c}_1, 1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_2, \{\text{v}_2, 0, \text{c}_1\}, \{\text{c}_2, 0, \text{c}_1\}] +
\text{Integrate}[\text{v}_2 - \text{c}_1, \{\text{v}_2, \text{v}_1, 1\}, \{\text{c}_2, \text{v}_1, 1\}, \{\text{c}_1, \text{c\text{star}}, \text{v}_1, \text{v\text{star}}, 1\}] +
\delta \times \text{Integrate}[\text{Integrate}[\text{v}_1 - \text{c}_1 + \text{v}_2 - \text{c}_2, \{\text{v}_2, \text{v}_1, 1\}, \{\text{c}_2, 0, \text{v}_1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_1 + \text{v}_2 - \text{c}_2, \{\text{v}_2, \text{c}_1, \text{v}_1\}, \{\text{c}_2, 0, \text{v}_2\}] +
\text{Integrate}[\text{v}_1 - \text{c}_1, \{\text{v}_2, \text{c}_1, \text{v}_1\}, \{\text{c}_2, \text{v}_2, 1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_1, \{\text{v}_2, 0, \text{c}_1\}, \{\text{c}_2, \text{c}_1, 1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_2, \{\text{v}_2, 0, \text{c}_1\}, \{\text{c}_2, 0, \text{c}_1\}] +
\text{Integrate}[\text{v}_2 - \text{c}_1, \{\text{v}_2, \text{v}_1, 1\}, \{\text{c}_2, \text{v}_1, 1\}, \{\text{c}_1, 0, \text{c\text{star}}, \text{v}_1, \text{v\text{star}}, 1\}] +
\delta \times \text{Integrate}[\text{Integrate}[0, \{\text{v}_2, 0, \text{v}_1\}, \{\text{c}_2, \text{v}_1, 1\}] +
\text{Integrate}[0, \{\text{v}_2, \text{v}_1, \text{c}_1\}, \{\text{c}_2, \text{v}_2, 1\}] +
\text{Integrate}[\text{v}_2 - \text{c}_2, \{\text{v}_2, \text{v}_1, \text{c}_1\}, \{\text{c}_2, 0, \text{v}_2\}] +
\text{Integrate}[\text{v}_2 - \text{c}_2, \{\text{v}_2, \text{c}_1, 1\}, \{\text{c}_2, 0, \text{c}_1\}] +
\text{Integrate}[\text{v}_1 - \text{c}_2, \{\text{v}_2, 0, \text{v}_1\}, \{\text{c}_2, 0, \text{v}_1\}] +
\text{Integrate}[\text{v}_2 - \text{c}_1, \{\text{v}_2, \text{c}_1, 1\}, \{\text{c}_2, \text{c}_1, 1\}, \{\text{c}_1, 0, 1\},
\{\text{v}_1, 0, \text{c}_1\}];

(* Create plots *)
Appendix B

Matlab Code

B.1 Discriminatory Market Clearing

B.1.1 Policy Iteration

% Write a function that computes the value function for a
% threshold policy with threshold x
function [output] = policyV(x,d,p,D)

a1 = -d*p*(1-p);
a0 = 1 - d - 2*a1;
a2 = 1 - d - a1;

P = full(gallery('tridiag',x+1,a1,a0,a1));
P(1,2) = 2*a1;
P(x+1,x+1) = a2;
r = d*p*ones(x+1,1);
r(1) = d*p^2;
r(x+1) = p*d*(1+(1-p)*D);
output = linsolve(P,r);
end

% Write a second function that uses the previous function to
% compute the optimal threshold
function [output,payoff,R] = optimalx(d,p,D)

sol = policyV(1,d,p,D);

if sol(2) < sol(1) + D
    output = 0;
payoff = (p^2+2*p*(1-p)*D)/(1-d);
    R = (p^2+2*p*(1-p)*(D-p)/(1-p))/(1-d);
else
stop = 0;
i = 1;

while stop == 0
    payoff = sol(1)/d;
sol = policyV(i,d,p,D);
    if sol(i+1) < sol(i) + D
        stop = 1;
    end
    output = i-1;
i = i + 1;
end

R = policyV(output,d,p,(D-p)/(1-p));
R = R(1);
end

B.1.2 Value Iteration

% Write a function that iteratively computes the optimal value function, provided the optimal threshold is set above the true optimal threshold function [output] = valueV(x,d,p,D)

output = ones(x+1,1);
newoutput = zeros(x+1,1);

while max(abs(output-newoutput)) > 10^(-15)
    output = newoutput;
    newoutput = zeros(x+1,1);
    newoutput(1) = d*p^2*(1+output(1))+2*p*(1-p)*max(D+
        output(1),output(2))+(1-p)^2*output(1);
    newoutput(x+1) = d*p^2*(1+output(x+1))+p*(1-p)*(1+output
        (x))+p*(1-p)*(D+output(x+1))+(1-p)^2*output(x+1));
    for i = 2:x
        newoutput(i) = d*p^2*(1+output(i))+p*(1-p)*(1+output
            (i-1))+p*(1-p)*max(D+output(i),output(i+1))+(1-p)
            ^2*output(i));
    end
end

B.2 Uniform Market Clearing

B.2.1 Policy Iteration

% Write a function that computes the value function for a threshold policy with threshold x function [output] = policyV(T,d,p,D)
% When using the floor function, need to add a number smaller than Matlab's
% working precision
eps = 10^{-12};

if T <= 1
    x = floor((T+eps)/D);
    a1 = -d*p*(1-p);
    a0 = 1 - d*(1-p)^2;
    P = full(gallery('tridiag',x+1,0,a0,a1));
    P(:,1) = -d*(p^2+p*(1-p))*ones(x+1,1);
    P(1,1) = 1 - d - 2*a1;
    P(1,2) = 2*a1;
    P(x+1,1) = -d*(p^2+2*p*(1-p));
    r = d*p^2*(1+D*(0:x-1))+d*p*(1-p)*(1+D*(-1:x-1));
    r(1) = d*p^2;
    r(x+1) = r(x+1)+d*p*(1-p)*(x+1)*D;
    output = linsolve(P,r);
else
    % Start by creating variables which specify the structure of the state
    % of the state
    % If T is an integer, reduce T by a small amount to prevent a zero entry
    % being created in the xI vector
    xE = floor(T-eps);
    xI = ones(xE+1,1);
    k = ones(xE,1);

    % Need to add a small number larger than Matlab's working precision to T to
    % prevent errors with floor
    for i = 1:xE+1
        xI(i) = floor((T+eps-(i-1))/D);
    end

    for i = 1:xE
        k(i) = xI(i)-xI(i+1);
    end

    % Create the diagonal of P (including last column)
    A = cell(xE+2,1);
for i = 1:xE+1
    A{1} = full(gallery('tridiag', xI(i), 0, (1-p)^2, p*(1-p)));
end
A{xE+2} = p^2+(1-p)^2;
P = blkdiag(A{:});

% create the upper diagonal of P (excluding last column)
P(1:xI(2), xI(1)+1:xI(1)+xI(2)) = full(gallery('tridiag', xI(2), p*(1-p), p^2, 0));
P(xI(2)+1, xI(1)+xI(2)) = p*(1-p);
for i = 2:xE
    P(sum(xI(1:i-1)) + 1:sum(xI(1:i)) + xI(i+1), sum(xI(1:i+1))) =
    full(gallery('tridiag', xI(i+1), p*(1-p), p^2, 0));
P(sum(xI(1:i-1)) + xI(i+1)+1, sum(xI(1:i+1))) = p*(1-p);
end

% create bottom left entry of P
P(sum(xI)+1, 1) = 2*p*(1-p);

% create right column of P (excluding bottom right entry)
P(1, sum(xI)+1) = p*(1-p);
P(xI(1) - k(1) + 1:sum(xI)+1) = (p^2 + p*(1-p))*ones(k(1), 1);
P(xI(1) - k(1) + 1, sum(xI)+1) = p^2;
P(xI(1), sum(xI)+1) = p^2 + 2*p*(1-p);
for i = 2:xE
    P(sum(xI(1:i-1)) + 1, sum(xI)+1) = p*(1-p);
P(sum(xI(1:i)) - k(i) + 1:sum(xI(1:i)), sum(xI)+1) = (p^2 + p*(1-p))*ones(k(i), 1);
P(sum(xI(1:i)) - k(i) + 1, sum(xI)+1) = p^2;
P(sum(xI(1:i)), sum(xI)+1) = p^2 + 2*p*(1-p);
end
P(sum(xI(1:xE)) + 1:sum(xI) - 1, sum(xI)+1) = (p^2 + p*(1-p))*ones(xI(xE+1)-1, 1);
P(sum(xI), sum(xI)+1) = p^2 + 2*p*(1-p);

% create the last entry of r
r = zeros(sum(xI)+1, 1);
r(sum(xI)+1) = p^2;

% create first vector of r
r(1) = p*(1-p);
r(xI(1) - k(1) + 1) = p^2*(1+D*(xI(1) - k(1) + 1));
r(xI(1) - k(1) + 2:sum(xI(1))) = p^2*(1+D*((xI(1) - k(1) + 2):xI(1)) +
    p*(1-p)*(1+D*((xI(1) - k(1) + 1):(xI(1)-1)));
r(xI(1)) = r(xI(1)) + p*(1-p)*D*(xI(1)+1);
for i = 2:xE
B.2. UNIFORM MARKET CLEARING

\[ r(\sum(xI(1:i-1)+1) = p(1-p)(1+i-1); \]
\[ r(\sum(xI(1:i)-k(i)+1) = p\cdot2^*(i+D*(xI(i)-k(i)+1)); \]
\[ r(\sum(xI(1:i)-k(i)+2:sum(xI(1:i))) = p\cdot2^*(i+D*((xI(i)-k(i)+2):xI(i)))+p*(1-p)*(i+D*((xI(i)-k(i)+1)):xI(i-1))); \]
\[ r(\sum(xI(1:i))) = r(\sum(xI(1:i)))+p*(1-p)*(i+D*(xI(i)+1)); \]
\[ end \]
\[ r(\sum(xI(1:xE)+1:sum(xI))) = p\cdot2^*(xE+1+D*(1:xI(xE+1)))+ p*(1-p)*(xE+1+D*(0:(xI(xE+1)-1)))); \]
\[ r(\sum(xI)) = r(\sum(xI))+p*(1-p)*(xE+D*(xI(xE+1)+1)); \]

% solve linear system (Poisson equation) to determine value functions

\[ output = \text{linsolve(eye(sum(xI)+1)-d*P,d*r);} \]
\[ end \]
\[ end \]

% Write a second function that uses the previous function to compute the optimal threshold

function [output, payoff, R] = optimalT(d, p, D)

eps = 10^(-12);

T = D;
sol = policyV(T,d,p,D);

if sol(2) < D + sol(1)
    output = 0;
    payoff = (p\cdot2^2*p*(1-p)*D)/(1-d);
    R = (p\cdot2^2*p*(D-p))/(1-d);
else
    stop = 0;
    i = 1;

    while stop == 0 && i*D - eps < 1
        payoff = sol(1)/d;
        sol = policyV(D*i,d,p,D);
        if sol(i+1) < sol(1) + D*i
            stop = 1;
            output = D*(i-1);
            R = revenue(output,d,p,D);
            R = R(end)/d;
        else
            i = i + 1;
        end
    end
end
if \( i \cdot D - \text{eps} > 1 \)

\[
T = 1 + D;
\]

\[
sol = \text{policyV}(T, d, p, D);
\]

\[
xI = \text{floor}((T + \text{eps})/D);
\]

if \( \text{sol}(xI) < D \cdot xI + \text{sol}(\text{end}) \)

\[
\text{output} = 1;
\]

\[
sol = \text{policyV}(1, d, p, D);
\]

\[
\text{payoff} = \text{sol}(1)/d;
\]

\[
R = \text{revenue}([\text{output}, d, p, D]);
\]

\[
R = R(\text{end})/d;
\]

else

\[
\text{stop} = 0;
\]

\[
i = 2;
\]

while \( \text{stop} == 0 \)

\[
\text{payoff} = \text{sol}(\text{end})/d;
\]

\[
T = 1 + i \cdot D;
\]

\[
sol = \text{policyV}(T, d, p, D);
\]

\[
xI = \text{floor}((T + \text{eps})/D);
\]

if \( \text{sol}(xI) < D \cdot xI + \text{sol}(\text{end}) \)

\[
\text{stop} = 1;
\]

\[
\text{output} = 1 + D \cdot (i - 1);
\]

\[
R = \text{revenue}([\text{output}, d, p, D]);
\]

\[
R = R(\text{end})/d;
\]

end

\[
i = i + 1;
\]

end

end

end

\textbf{B.2.2 Value Iteration}

\% Write a function that iteratively computes the optimal
\% value function, provided the optimal threshold is set
\% above the true optimal threshold
\% function [output] = valueV(T, d, p, D)

\% when using the floor function, need to add a number smaller
\% than matlab's
\% working precision
\% eps = 10^{-12};

if \( T <= 1 \)

\[
x = \text{floor}((T + \text{eps})/D);
\]
output = ones(x+1,1);
newoutput = zeros(x+1,1);

while max(abs(output-newoutput)) > 10^(-15)
    output = newoutput;
    newoutput = zeros(x+1,1);

    newoutput(1) = d*(p^2*(1+output(1)) + 2*p*(1-p)*max(D+output(1),output(2)) + (1-p)^2*output(1));
    newoutput(x+1) = d*(p^2*(1+D*x+output(1)) + ps*(1-p)*(1+D*(x-1)+output(1)) + p*(1-p)*(D*(x+1)+output(1)) + (1-p)^2*output(x+1));
    for i = 2:x
        newoutput(i) = d*(p^2*(1+(i-1)*D+output(1)) + p*(1-p)*(1+(i-2)*D+output(1)) + p*(1-p)*max(D*i+output(1),output(i)+1) + (1-p)^2*output(i));
    end
end

else
    % start by creating variables which specify the structure of the state
    % space

    % if T is an integer, reduce T by a small amount to prevent a zero entry
    % being created in the xI vector
    xE = floor(T-eps);
    xI = ones(xE+1,1);
    k = ones(xE,1);

    % need to add a small number larger than matlab's working precision to T to
    % prevent errors with floor
    for i = 1:xE
        xI(i) = floor((T+eps-(i-1))/D);
    end

    for i = 1:xE
        k(i) = xI(i)-xI(i+1);
    end

    output = ones(sum(xI)+1,1);
    newoutput = zeros(sum(xI)+1,1);

    while max(abs(output-newoutput)) > 10^(-15)
        output = newoutput;
    end
newoutput = zeros(sum(xI) + 1, 1);

% start with the market clearing state
newoutput(end) = d*(p^2*(1+output(end)) + 2*p*(1-p)*max(D+output(end), output(1)) + (1-p)^2*output(end));

% do each level except for the last level
for i = 1:xE
    entry = sum(xI(1:i-1));
    % do the first entry
    newoutput(entry+1) = d*(p^2*max(i+D+output(end), output(sum(xI(1:i-1)))+1)+p*(1-p)*max(i
        +(j-1)*D+output(end), output(sum(xI(1:i-1)))+j
        -1)) + p*(1-p)*max(i-1+(j+1)*D+output(end), output(entry+j+1)+(1-p)^2*output(entry+j))
    ;
end

% do remaining entries in non-market clearing section
for j = 2:xI(i)-k(i)
    newoutput(entry+j) = d*(p^2*max(i+j*D+output(entry), output(sum(xI(1:i-1)))+j)+p*(1-p)*max(i
        +(j-1)*D+output(end), output(sum(xI(1:i-1)))+j
        -1)) + p*(1-p)*max(i-1+(j+1)*D+output(end), output(entry+j+1)+(1-p)^2*output(entry+j))
    ;
end

% do middle entries in market clearing section
for j = xI(i)-k(i)+2:xI(i)-1
    newoutput(entry+j) = d*(p^2*(i+j*D+output(entry)) + p*(1-p)*max(i+(j-1)*D+output(entry)+p*(1-p)
        *max(i-1+(j+1)*D+output(end), output(entry+j+1)+(1-p)^2*output(entry+j))
    ;
end

% do first and last entry in market clearing section
j = xI(i)-k(i)+1;
newoutput(entry+j) = d*(p^2*(i+j*D+output(entry)) + p*(1-p)*max(i+(j-1)*D+output(end), output(sum(xI
    (1:i)+j-1)+p*(1-p)*max(i-1+(j+1)*D+output(end), output(entry+j+1)+(1-p)^2*output(entry+j))
    );
newoutput(xI(i)) = d*(p^2*(i+xI(i)*D+output(entry)) + p*(1-p)*(i+xI(i)-1)*D+output(end)+p*(1-p)*
    *(i-1+(xI(i)+1)*D+output(end))+(1-p)^2*output(xI(i)));
end
B.3. FIXED FREQUENCY MARKET CLEARING

% do the last level entries
entry = sum(xI(1:xE));
for j = 1:xI(end)-1
    newoutput(entry+j) = d*(p^2*(xE+1+j*D+output(end)) + p*(1-p)*
    max(xE+(j+1)*D+output(end), output(entry+j+1))
    +(1-p)^2*output(entry+j));
end
newoutput(end-1) = d*(p^2*(xE+1+xI(end)*D+output(end)) + p*(1-p)*
    xE+((xI(end)+1)*D+output(end))+(1-p)^2*output(end-1));
end
end

B.3 Fixed Frequency Market Clearing

% Write a function that computes the optimal market clearing frequency
function [output, payoff, R] = optimalfreq(d,p,D)
output = 0;
payoff = (p^2+2*p*(1-p)*D)/(1-d);
stop = 0;
k = 1;
newpayoff = 0;
while stop == 0
    for i = 0:k
        for j = 0:k
            newpayoff = newpayoff + d*(k-1)*abs(i-j) + min(i, j)*abs(i-j)
            *D)*nchoosek(k,i)*nchoosek(k,j)*p^i*(1-p)^j
            *(2*k-i-j)/(1-d^k);
        end
    end
    if newpayoff < payoff
        stop = 1;
        output = k - 1;
    else
        k = k + 1;
        payoff = newpayoff;
        newpayoff = 0;
    end
end
MATLAB CODE

\[ D = (D-p)/(1-p); \]
\[ R = 0; \]
\[ k = \text{output}; \]
\[ \text{for } i = 0:k \]
\[ \text{for } j = 0:k \]
\[ R = R + d^{(k-1)}*(\min(i,j)+abs(i-j)*D)*\text{nchoosek}(k,i)*\text{nchoosek}(k,j)*p^{(i+j)}*(1-p)^{(2*k-i-j)/(1-d*k)}; \]
\[ \text{end} \]
\[ \text{end} \]
\[ \text{end} \]

B.4 Arrival of Unpaired Agents

B.4.1 Computing the Upper Bound

% We write a function that computes the value function of the policy associated with a candidate upper bound process.
function [output] = upperboundpolicyV(T, d, p, D)

% when using the floor function, need to add a number smaller than matlab's working precision

x = floor((T+eps)/D);

a1 = -d*p/2;
a0 = 1 - d*(1-p);

P = full(gallery('tridiag',x+1,0,a0,a1));
P(:,1) = -d*(0+p/2)*ones(x+1,1);
P(1,1) = 1 - d - 2*a1;
P(1,2) = 2*a1;
P(x+1,1) = -d*(0+p);

r = d*(1+D*(0:x'))+d*(p/2)*(1+D*(-1:x-1'));
r(1) = 0;
r(x+1) = r(x+1)+d*(p/2)*(x+1)*D;

output = linsolve(P,r);
end

% Write a second function that uses the previous function to compute the upper bound.
function [output, payoff] = upperbound(d, p, D)

T = D;
sol = upperboundpolicyV(T,d,p,D);
if \( \text{sol}(2) < D + \text{sol}(1) \)
\[
\text{output} = 0; \\
\text{payoff} = (0+p*\text{D})/(1-d);
\]

else
\[
\text{stop} = 0; \\
i = 1;
\]

while \( \text{stop} == 0 \)
\[
\text{payoff} = \frac{\text{sol}(1)}{d}; \\
\text{sol} = \text{upperboundpolicyV}(D*i,d,p,D);
\]

if \( \text{sol}(i+1) < \text{sol}(1) + D*i \)
\[
\text{stop} = 1; \\
\text{output} = D*(i-1);
\]

else
\[
i = i + 1;
\]
end
end
end
end

\section*{B.4.2 Computing the Optimal Policy}

% Write a function that computes the value function for a candidate optimal policy

\textbf{function} \([\text{output}] = \text{optimalpolicyvalueV}(\text{policy},d,p,D)\)

\[\text{eps} = 10^{-12};\]
\[\text{upperT} = \text{floor}(\text{upperbound}(d,p,D)/D+\text{eps});\]
\[\text{output} = \text{cell}(\text{upperT}+1,1);\]
\[\text{newoutput} = \text{cell}(\text{upperT}+1,1);\]

\[\text{output}\{1\} = \{\text{ones}(\text{upperT}+2,\text{upperT}+2),\text{ones}(\text{upperT}+1,1)\};\]
\[\text{newoutput}\{1\} = \{\text{zeros}(\text{upperT}+2,\text{upperT}+2),\text{zeros}(\text{upperT}+1,1)\};\]

for \( i = 2: \text{upperT}+1 \)
\[
\text{output}\{i\} = \text{ones}(2*(\text{upperT}+2-i)+1,1); \\
\text{newoutput}\{i\} = \text{zeros}(2*(\text{upperT}+2-i)+1,1); \\
\]
end

distance = 1;

while \( \text{distance} > 10^{-8} \)
\[
\text{output} = \text{newoutput}; \\
\text{newoutput}\{1\} = \{\text{zeros}(\text{upperT}+2,\text{upperT}+2),\text{zeros}(\text{upperT}+1,1)\};
\]

for \( i = 2: \text{upperT}+1 \)
\[
\text{newoutput}\{i\} = \text{zeros}(2*(\text{upperT}+2-i)+1,1); \\
\]
end

% an inefficient agent arrives on the long side of the market
MATLAB CODE

% (vL, empty) (1−p)/2

for i = 1:upperT+2
    newoutput{1}{1}(i,:) = newoutput{1}{1}(i,:) + d*((1−p)/2)*output{1}{1}(i,min(i+1,upperT+2));
end

newoutput{1}{2} = newoutput{1}{2} + d*((1−p)/2)*output{1}{2};

for i = 2:upperT+1
    newoutput{i} = newoutput{i} + d*((1−p)/2)*output{i};
end

% an efficient agent arrives on the long side of the market
% (vH, empty) p/2

% have no pairs stored
newoutput{1}{1}(1,:) = newoutput{1}{1}(1,:) + d*(p/2)*output{1}{1};
newoutput{1}{2}(1,:) = newoutput{1}{2}(1,:) + d*(p/2)*output{2}{1};

for i = 3:upperT+2
    newoutput{1}{1}(1,:) = newoutput{1}{1}(1,:) + d*(p/2)*output{2}{1}(upperT+i−1);
end

for i = 1:upperT+1
    newoutput{1}{2}(i) = newoutput{1}{2}(i) + d*(p/2)*output{1}{2}(min(i+1,upperT+1));
end

% have unpaired efficient agents
for i = 2:upperT+1
    for j = 1:upperT+3−i
        newoutput{i}{j}(j) = newoutput{i}{j}(j) + d*(p/2)*output{i}{j}(min(j+1,upperT+3−i));
    end
end

% have unpaired inefficient agents
for i = 2:upperT
    % move to zero unpaired agents when one stored unpaired agent is
    % rematched
    if policy{i+1}(1) == 1
        newoutput{i}(upperT−i+4) = newoutput{i}(upperT−i+4) + d*(p/2)*output{i+1}(1);
    else
        % further code...
    end
end
B.4. ARRIVAL OF UNPAIRED AGENTS

\[
\text{newoutput}_{i}(\text{upperT} - i + 4) = \text{newoutput}_{i}(\text{upperT} - i + 4) + d \ast (p/2) \ast (D \ast \text{output}_{i+1}(1)(1, 1)) ;
\]

end

for \( j = \text{upperT} - i + 5:2 \ast (\text{upperT} - i + 2)+1 \)

if \( \text{policy}_{i+1}(j-2) == 1 \)

\[
\text{newoutput}_{i}(j) = \text{newoutput}_{i}(j) + d \ast (p/2) \ast \text{output}_{i+1}(1)(1, 1) ;
\]

else

\[
\text{newoutput}_{i}(j) = \text{newoutput}_{i}(j) + d \ast (p/2) \ast (D \ast \text{output}_{1}(1)(1, 1)) ;
\]

end

\text{end} \quad \% \text{go to } j-2 \text{ since the vector for the next level is 2 shorter}

end

\text{newoutput}(\text{end})(3) = \text{newoutput}(\text{end})(3) + d \ast (p/2) \ast (D \ast (\text{upperT} + 1)+\text{output}_{1}(1)(1, 1)) ;

\%
\text{a deficit inefficient agent arrives without a surplus efficient agent}

\%
(vL, cH) (empty, cH) (1-p)/2

\%
when pairs are stored

for \( i = 2: \text{upperT}+1 \)

\[
\text{newoutput}_{i}(1) = \text{newoutput}_{i}(1) + d \ast ((1-p)/2) \ast \text{output}_{i}(\text{upperT} - i + 4) ;
\]

for \( j = \text{upperT} - i + 4:2 \ast (\text{upperT} - i + 2)+1 \)

\[
\text{newoutput}_{i}(j) = \text{newoutput}_{i}(j) + d \ast ((1-p)/2) \ast \text{output}_{i}(\min(j+1, 2 \ast (\text{upperT} - i + 2)+1)) ;
\]

end

end

for \( i = 2: \text{upperT} \)

for \( j = 2: \text{upperT}+3-i \)

if \( \text{policy}_{i+1}(j-1) == 1 \)

\[
\text{newoutput}_{i}(j) = \text{newoutput}_{i}(j) + d \ast ((1-p)/2) \ast \text{output}_{i+1}(j-1) ;
\]

else

\[
\text{newoutput}_{i}(j) = \text{newoutput}_{i}(j) + d \ast ((1-p)/2) \ast (D \ast \text{output}_{1}(1)(1, 1)) ;
\]

end

end

\text{newoutput}(\text{end})(2) = \text{newoutput}(\text{end})(2) + d \ast ((1-p)/2) \ast (D \ast (\text{upperT}+1)+\text{output}_{1}(1)(1, 1)) ;

\%
when no pairs are stored

for \( i = 1: \text{upperT}+1 \)

\[
\text{newoutput}_{1}(2)(i) = \text{newoutput}_{1}(2)(i) + d \ast ((1-p)/2) \ast \text{output}_{1}(1)(1, 1) ;
\]

end
% an efficient agent arrives on the short side of the market

% (vL,cL) (empty,cL) p/2

for i = 2:upperT+1
    newoutput{i}(1) = newoutput{i}(1)+d*(p/2)*(1+D*(i-2)+
    output{i}(1));
for j = 2:upperT-i+3
    newoutput{i}(j) = newoutput{i}(j)+d*(p/2)*(1+D*(i
    -1)+output{i}(1));
end
for j = upperT-i+4:2*(upperT-i+2)+1
    newoutput{i}(j) = newoutput{i}(j)+d*(p/2)*(1+D*(i
    -2)+output{i}(1));
end
end

% when pairs are not stored

newoutput{1}(i,:,:) = newoutput{1}(i,:)+d*(p/2)*
    output{1}(i,:);
for i = 2:upperT+2
    newoutput{1}(i,:) = newoutput{1}(i,:)+d*0*
    output{2}(i);
end
for i = 1:upperT+1
    newoutput{1}(i) = newoutput{1}(i)+d*(p/2)*(1+
    output{1}(i));
end

distance = 0;
for k = 2:upperT+1
    distance = max(distance,max(abs(output{k}-newoutput{k 
    })))
end

distance = max(distance,max(max(abs(output{1}(1)-
    newoutput{1}(1,:))))

newoutput[1]{1}));
distance = max(distance, max(max(abs(output[1]{2} -
newoutput[1]{2})));
end
end

% Write a second function that uses the previous function and
the upper bound to compute the optimal policy
function [policy, payoff] = optimalpolicy(d,p,D)

values = valueV(d,p,D);
eps = 10^(-12);
upperT = floor(upperbound(d,p,D)/D+eps);
lowerT = 0;

policy = cell(upperT+1,1);
policy{1} = {ones(upperT+2,upperT+2),ones(upperT+1,1)};
for i = 2:lowerT+1
    policy{i} = ones(2*(upperT+2-i)+1,1);
end
for i = lowerT+2:length(policy)
    policy{i} = values{i}-(i-1)*D-values{1}{1}(1,1)>0;
end
payoff = optimalpolicyvalueV(policy,d,p,D);
payoff = payoff{1}{1}(1,1)/d;
end

B.4.3 Value Iteration

% Write a function that iteratively computes the optimal
value function, provided we start with the policy
associated with the upper bound
function [output] = valueV(d,p,D)

eps = 10^(-12);
upperT = floor(upperbound(d,p,D)/D+eps);
output = cell(upperT+1,1);
ewoutput = cell(upperT+1,1);

output{1} = {ones(upperT+2,upperT+2),ones(upperT+1,1)};
ewoutput{1} = {zeros(upperT+2,upperT+2),zeros(upperT+1,1)};
for i = 2:upperT+1
    output{i} = ones(2*(upperT+2-i)+1,1);
ewoutput{i} = zeros(2*(upperT+2-i)+1,1);
end
distance = 1;

while distance > 10^(-8)
    output = newoutput;
    newoutput{1} = {zeros(upperT+2,upperT+2),zeros(upperT+1,1)};
    for i = 2:upperT+1
        newoutput{i} = zeros(2*(upperT+2-i)+1,1);
    end

% an inefficient agent arrives on the long side of the market
% (vL, empty) (1−p)/2
    for i = 1:upperT+2
        newoutput{1}{1}{1,:,:} = newoutput{1}{1}{1,:,:}+d*((1−p)/2)*output{1}{1}{1,:,:}+min(i+1,upperT+2);
    end
    newoutput{1}{2} = newoutput{1}{2}+d*((1−p)/2)*output{1}{2};
    for i = 2:upperT+1
        newoutput{i} = newoutput{i}+d*((1−p)/2)*output{i};
    end

% an efficient agent arrives on the long side of the market
% (vH, empty) p/2

% have no pairs stored
    newoutput{1}{1}{1} = newoutput{1}{1}{1}+d*(p/2)*output{1}{2}(1);
    newoutput{1}{1}{2} = newoutput{1}{1}{2}+d*(p/2)*output{2}(1);
    for i = 3:upperT+2
        newoutput{1}{1}{1}(i,:) = newoutput{1}{1}{1}(i,:)+d*(p/2)*output{2}(upperT+i-1);
    end

    for i = 1:upperT+1
        newoutput{1}{2}{i} = newoutput{1}{2}{i}+d*(p/2)*output{1}{2}(min(i+1,upperT+1));
    end

% have unpaired efficient agents
    for i = 2:upperT+1
        for j = 1:upperT+3−i
            newoutput{i}(j) = newoutput{i}(j)+d*(p/2)*output{i}(min(j+1,upperT+3−i));
        end
    end
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end
end

% have unpaired inefficient agents
for i = 2:upperT
  % move to zero unpaired agents when one stored unpaired agent is rematched
  newoutput{i}(upperT−i+4) = newoutput{i}(upperT−i+4)+d*(p/2)*output{i+1}(1);
  for j = upperT−i+5:2*(upperT−i+2)+1
    newoutput{i}{j} = newoutput{i}{j}+d*(p/2)*output{i+1}{j−2};
    % go to j−2 since the vector for the next level is 2 shorter
  end
end
newoutput{end}{3} = newoutput{end}{3}+d*(p/2)*(D*(upperT +1)+output{1}{1});

% a deficit inefficient agent arrives without a surplus efficient agent
% (empty, ch) p/2

% when pairs are stored
for i = 2:upperT+1
  newoutput{i}{1} = newoutput{i}{1}+d*((1−p)/2)*output{i}{upperT−i+4};
  for j = upperT−i+4:2*(upperT−i+2)+1
    newoutput{i}{j} = newoutput{i}{j}+d*((1−p)/2)*output{i}{min(j+1,2*(upperT−i+2)+1)};
  end
end
for i = 2:upperT
  for j = 2:upperT+3−i
    newoutput{i}{j} = newoutput{i}{j}+d*((1−p)/2)*output{i+1}{j−1};
  end
end
newoutput{end}{2} = newoutput{end}{2}+d*((1−p)/2)*(D*(upperT+1)+output{1}{1});

% when no pairs are stored
for i = 1:upperT+1
  newoutput{1}{2}(i) = newoutput{1}{2}(i)+d*((1−p)/2)*output{2}(i);
end
for i = 1:upperT+2
    for j = 1:upperT+2
        newoutput{1}{1}(i,j) = newoutput{1}{1}(i,j) + d*0*output{1}{1}(min(i+1,upperT+2),min(j+1,upperT+2)) + d*((1-p)/2)*output{1}{1}(min(i+1,upperT+2),j);
    end
end

% an efficient agent arrives on the short side of the market
% (empty,cL) p/2

% when pairs are stored
for i = 2:upperT+1
    newoutput{i}{1}(1) = newoutput{i}{1}(1) + d*(p/2)*(1+D*(i-2) + output{1}{1}(1,1));
    for j = 2:upperT-i+3
        newoutput{i}{j}(j) = newoutput{i}{j}(j) + d*(p/2)*(1+D*(i-1) + output{1}{1}(1,1));
    end
    for j = upperT-i+4:2*(upperT-i+2)+1
        newoutput{i}{j}(j) = newoutput{i}{j}(j) + d*(p/2)*(1+D*(i-2) + output{1}{1}(1,1));
    end
end

% when pairs are not stored
for i = 1:upperT+1
    newoutput{1}{2}(i) = newoutput{1}{2}(i) + d*(p/2)*(1+output{1}{1}(1,1));
end

distance = 0;
for k = 2:upperT+1
    distance = max(distance, max(abs(output{k} - newoutput{k})));
end
distance = max(distance, max(max(abs(output{1}{1} - newoutput{1}{1})))),
distance = max(distance, max(max(abs(output{1}{2} - newoutput{1}{1})))),
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newoutput\{1\}\{2\});

end
end
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