Extinction in branching processes with countably many types

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Abstract

Multitype branching processes describe the evolution of populations in which individuals give birth independently according to a probability distribution that depends on their type. In this thesis, we consider the extinction of branching processes with countably infinitely many types.

We begin by developing a class of iterative methods to compute the global extinction probability vector $q$. In particular, we construct a sequence of truncated and augmented branching processes with finite but increasing sets of types. A pathwise approach is then used to show that, under some sufficient conditions, the corresponding sequence of extinction probability vectors converges to the infinite vector $q$. Besides giving rise to a family of algorithmic methods, our approach leads to new sufficient conditions for $q < 1$.

When then turn our attention to a specific class of branching processes which we refer to as lower Hessenberg branching processes. These are branching processes with typeset $\mathcal{X} = \{0, 1, 2, \ldots\}$, in which individuals of type $i$ may give birth to offspring of type $j \leq i + 1$ only. For this class of processes, we study the set of fixed points of the progeny generating vector. In particular, we highlight the existence of a continuum of fixed points whose minimum is the global extinction probability vector $q$ and whose maximum is the partial extinction probability vector $\tilde{q}$. We derive a computationally efficient partial extinction criterion. In the case where $\tilde{q} = 1$, we derive a global extinction criterion, and in the case where $\tilde{q} < 1$, we develop necessary and sufficient conditions for $q = \tilde{q}$.

Finally, we consider a more general class of structured branching processes which we refer to as block lower Hessenberg branching processes. For these processes, we derive partial and global extinction criteria, and we study the probability of extinction $q(A)$ in any set of types $A$. In particular, we develop conditions for $q(A)$ to be different from the global and partial extinction probability vectors, we present an iterative method to compute the
vectors $q(A)$, and we investigate their location in the set of fixed points of the progeny generating vector.
Declaration

This is to certify that:

i. the thesis comprises only my original work towards the PhD except where indicated in the Preface,

ii. due acknowledgment has been made in the text to all other material used,

iii. the thesis is fewer than 100 000 words in length, exclusive of tables, maps, bibliographies and appendices.

Signed

[Signature]

Peter Timothy Braunsteins
Preface

This thesis contains original research in Chapters 3, 4, and 5. Chapter 3 is based on the following paper, which is currently in revision.


Chapter 4 is based on the following paper, which is currently in revision.


Chapter 5 is based on the following working paper.


All co-authorship has taken place in accordance with the Graduate Research Training Policy of the University of Melbourne.
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### Notation

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<td>( 1(\cdot) )</td>
<td>Indicator function</td>
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<td>( a.s. )</td>
<td>Equal almost surely</td>
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<tr>
<td>( \overset{d}{=} )</td>
<td>Equal in distribution</td>
</tr>
<tr>
<td>( \bar{A} )</td>
<td>Complement of the set ( A )</td>
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<tr>
<td>( \mathcal{E}(A) )</td>
<td>Event of extinction in the set ( A )</td>
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<td>( \mathcal{E}_g )</td>
<td>Global extinction event</td>
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<tr>
<td>( e_i )</td>
<td>Vector with ( i )-th entry 1 and all other entries 0</td>
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<td>( \mathcal{E}_n(A) )</td>
<td>Event of extinction in set ( A ) by generation ( n )</td>
</tr>
<tr>
<td>( { E_n^{(i)}(Z) }_{n \geq 0} )</td>
<td>Type-( i ) branching process embedded with respect to ( { Z_n } )</td>
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<td>( \mathcal{E}_p )</td>
<td>Partial extinction event</td>
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<td>( f \circ g )</td>
<td>Composition of ( f ) and ( g )</td>
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<td>( G(\cdot) )</td>
<td>Progeny generating vector</td>
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<td>( J )</td>
<td>Set of virtual individuals</td>
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<td>( m_{E_n^{(i)}(Z)} )</td>
<td>Expected number of offspring in the type ( i ) process embedded with respect to ( { Z_n } )</td>
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<td>( M )</td>
<td>Mean progeny matrix</td>
</tr>
<tr>
<td>( N_I )</td>
<td>Random offspring vector of virtual individual ( I )</td>
</tr>
<tr>
<td>( \mathbb{N}_0 )</td>
<td>Set of non-negative integers</td>
</tr>
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<td>( \mathbb{N} )</td>
<td>Set of natural numbers</td>
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<td>( \nu(\cdot) )</td>
<td>Convergence norm</td>
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<tr>
<td>( (\Omega, \mathcal{F}, \mathbb{P}) )</td>
<td>Probability space</td>
</tr>
<tr>
<td>( \varphi_0 )</td>
<td>Type of the single individual in generation 0</td>
</tr>
<tr>
<td>( p_j(\mathbf{r}) )</td>
<td>Probability that a type-( j ) individual has offspring vector ( \mathbf{r} )</td>
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\( q(A) \) Probability of extinction in the set \( A \) given any initial type
\( q_n(A) \) Probability of extinction in the set \( A \) by generation \( n \) given any initial type
\( q_n \) Probability of global extinction by generation \( n \) given any initial type
\( \tilde{q} \) Partial extinction probability vector
\( q \) Global extinction probability vector
\( \mathbb{R}_{>0} \) Real numbers greater than 0
\( \rho(\cdot) \) Perron-Frobenius eigenvalue
\( \mathbb{R} \) Real numbers
\( \{S_k\} \) Seed process
\( S \) Set of solutions of \( s = G(s) \)
\( \mathcal{X} \) Set of types
\( X \) Set of virtual individuals that appear in the population
\( Y_k \) Embedded Galton–Watson process in a varying environment
\( Z_n^{(k)} \) Population size vector of the process modified so that all types greater than \( k \) are immortal
\( \tilde{Z}_n^{(A)} \) Population size vector of the process modified so that all types in \( \bar{A} \) are sterile
\( \tilde{Z}_n^{(k)} \) Population size vector of the process modified so that all types greater than \( k \) are sterile
\( Z_n \) Population size vector
List of examples

Example 1 illustrates the difference between partial and global extinction (page 25).

Example 2 describes a parametric family of processes where the total population size can stabilise at a finite value. In this example $q < \lim_{k \to \infty} \tilde{q}^{(k)} < \tilde{q}$ (page 48).

Example 3 demonstrates that $\lim_{k \to \infty} \tilde{q}^{(k)}$ does not necessarily exist (page 61).

Example 4 proves that it is possible to have $q < \lim_{k \to \infty} \tilde{q}^{(k)} = \tilde{q}$ (page 63).

Example 5 describes a process whose progeny generating vector has countably many fixed points greater than $\tilde{q}$ and uncountably many fixed points less than $\tilde{q}$ (page 85).

Example 6 presents a parametric family of processes with a common mean progeny matrix $M$, which contains some processes with $q = 1$ and others with $q < 1$ (page 98).

Example 7 demonstrates that it is possible to have $q < \tilde{q} < 1$ in an irreducible branching process. In addition, this example also exhibits non-exponential growth (page 100).

Example 8 illustrates a parametric family of irreducible processes whose members can have either one, two, or four distinct extinction probability vectors (page 127).
Example 9 is a non-symmetric block lower Hessenberg branching process that is locally isomorphic to a lower Hessenberg branching process (page 130). In this example there are three distinction probability vectors.
Chapter 1

Introduction

1.1 Motivation

The survival or extinction of a population is subject to random chance. A population that begins with a few very strong, prolific individuals may die out, while a population that begins with the same number of their weaker, less reproductive cousins may establish itself. Branching processes are models that help us to understand why. For example, the simple branching process, proposed by Galton and Watson in 1875, was designed to illustrate how family names once prominent in European aristocracy were disappearing, while the population of Europe as a whole increased. Galton and Watson’s original model has since led to a number of generalisations which now form the active research field of branching processes. In these models the probability that the population eventually becomes empty, or *extinct*, remains one of the central topics of interest.

In this thesis we consider a variant of Galton and Watson’s original model in which individuals take one of several *types*. A multitype Galton–Watson branching process (MGWBP) describes the evolution of a population of independent individuals who live for a single generation, and at death, give birth to a random number of offspring that may be of various types. An individual gives birth according to an *offspring distribution* that depends only on that individual’s type. From a modelling perspective, it makes sense to introduce multiple types if individuals display attributes that remain fixed throughout their lifetime and influence their offspring distribution. In a biological population an individual’s type may correspond to its genotype, whereas in a nuclear reaction, where we model a population of atoms, an atom’s type may corre-
spond to its isotopic number. These are examples of populations that can be modelled with a finite number of types: biological populations generally contain only finitely many genotypes and atoms have only finitely many stable isotopes.

Multitype Galton–Watson processes with a finite number of types have received significant attention in the literature. Reference books such as Harris [31], Athreya and Ney [2], and Jagers [44] each devote a chapter to these processes. In addition, Mode [57] is specifically dedicated to multitype branching processes. In these texts many of the fundamental questions about the extinction probabilities of MGWBPs with finitely many types are resolved. Letting \( q \) where \( q_i \) denotes the probability of extinction given that the process starts with a single individual of type \( i \), we observe that classical reference books such as [31] demonstrate how to derive an exact expression for \( q \), compute \( q \) algorithmically if no exact expression can be found, and determine whether \( q = 1 \) using only the expected number of offspring from individuals of each type. Specifically, if \( G(\cdot) \) denotes the probability generating vector whose entries correspond to each type’s reproduction law, and \( M \) is a matrix whose entries \( M_{ij} \) record the mean number of offspring of type \( j \) born to a parent of type \( i \), then they show that

\[
(i) \quad q \text{ satisfies the fixed point equation } s = G(s), \tag{1.1}
\]

which, under regularity assumptions, has at most two solutions \( q \) and \( 1 \).

\[
(ii) \quad q \text{ can be computed algorithmically through repeated application of } G(\cdot) \text{ to a vector } s < 1, \text{ and}
\]

\[
(iii) \quad q = 1 \text{ if and only if the Perron-Frobenius eigenvalue of } M = (M_{ij}) \text{ is less than or equal to } 1.
\]

Assertions (i)–(iii), which are explained in greater detail in Chapter 2, summarise the most well known properties of the extinction probability vector \( q \) for MGWBPs with finitely many types.

During the lifetime of some populations, the number of types that may appear is unbounded. For these populations a model which allows an infinite number of types is necessary. The most striking difference between finite and
1.1. Motivation

Infinite type branching processes, is that in an infinite-type branching process there are multiple ways to define extinction, each potentially leading to a different extinction probability. The two most commonly studied extinction events are *global extinction*, the event that the number of individuals in the population eventually hits 0, and *partial extinction*, the event that every type eventually disappears from the population. The distinction between partial and global extinction, which may at first appear arbitrary, is best illustrated through an example. Suppose we were to use an multitype branching process to model the population of all living beings. It would then be reasonable to consider each species as a separate type. Because there is no finite limit on the number of new species that may appear in the future, we would use an infinite-type model. We would uncover the estimate of Raup [63] that 99.9% of all species that have ever existed are extinct. The logical result would be a model in which there is a positive probability of global survival, as life on Earth is persisting, but with partial extinction, as each species appears doomed to extinction. Thus, rather than being merely a mathematical inconvenience, the possibility of partial extinction without global extinction encapsulates a fate that is experienced by many populations.

A number of authors capitalise on the unique attributes of infinite-type branching processes in an applied setting: Yule [70] models speciation rates in biological populations, Kimmel and Stivers [41] model unstable gene amplification in malignant tumours, and Barbour and Luczak [5] model the movement of parasites through a host population. Each of these authors study behaviour that would not be properly captured by a model with finitely many types. Chapter 7 of Kimmel and Axelrod [4] is dedicated specifically to the biological applications of branching processes with infinitely many types.

In comparison to MGWBPs with finitely many types, significantly less is known about MGWBPs with infinitely many types. In fact, attempts to generalise (i)–(iii) to the infinite-type setting have led to a number of open questions: the set of solutions to Equation (1.1) is still not well understood, the computational aspects of \( q \) have received little attention, and an easily applicable criterion for \( q = 1 \) is still lacking. The primary goal of this thesis is to make progress toward a solution to each of these problems.

The analysis of MGWBPs with infinitely many types is often a balancing act: restrictive assumptions must be made in order to make the processes amenable to analysis, but these assumptions should not be so restrictive that
the behaviours which make branching processes with infinitely many types unique are completely suppressed. In this thesis we attempt to conduct our analysis in the most general setting possible. In most cases, when a restrictive assumption is made we provide an example that illustrates the behaviour that the assumption guards against. This informs the secondary goal of this thesis, which is to illustrate the unique behaviour of infinite-type branching processes that makes them more difficult to analyse than their finite-type counterparts.

1.2 Outline

We now give an outline of this thesis and summarise its contributions:

- In Chapter 2 we rigorously define MGWBPs with countably many types and introduce the tools we use to study them. We summarise the contributions of authors who have attempted to extend (i)-(iii) to the infinite-type setting, and we give a literature review of the topic as a whole.

Chapters 3, 4 and 5, which make up the original contributions of the thesis, all follow a similar pattern. That is, in each chapter properties of branching processes with infinitely many types are deduced from an intermediary branching process which is simpler to study.

- In Chapter 3 we establish a link between the extinction probability of MGWBPs with finitely many types and the global extinction probability of MGWBPs with infinitely many types. In particular, we define a sequence of finite-type MGWBPs whose extinction probability vectors converge to $q$. This enables us to use established results for finite-type processes to derive new results in the infinite-type setting. In particular, we develop new algorithms for computing $q$, and we derive sufficient conditions for $q = 1$ and $q < 1$. This work is an extension of [35].

- In Chapter 4 we consider a class of branching processes with countably many types which we refer to as lower Hessenberg branching processes. These are MGWBPs with typeset $\{0, 1, 2, \ldots\}$, in which individuals of type $i$ may give birth to offspring of type $j \leq i + 1$ only. We establish a connection between this class of processes and single type Galton–Watson processes in a varying environment. By exploiting this link we study the set of solutions to Equation (1.1). In particular, we highlight
the existence of a continuum of solutions whose minimum is the global extinction probability vector and whose maximum is the partial extinction probability vector. In the case where there is almost sure partial extinction, we derive necessary and sufficient conditions for $q = 1$.

- In Chapter 5 we extend the analysis of Chapter 4 to block lower Hessenberg branching processes. These are MGWBP$\text{s}$ in which there exists $d \geq 1$ such that an individual of type $i$ may give birth to offspring of type $j \leq i + d$ only. Following arguments similar to Chapter 4 we establish necessary and sufficient conditions for $q = 1$. We then study the probability of extinction $q(A)$ in any set of types $A$. In particular, we develop conditions for $q(A)$ to be different from the global and partial extinction probability vectors, we present an iterative method to compute the extinction probability vectors $q(A)$, and we investigate their location in the set of solutions to Equation (1.1). The results in this chapter motivate a number of open questions.

### 1.2.1 Conventions

In this thesis we take the following conventions.

- We let $e_i$ denote a vector with $i$-th entry 1 and all other entries 0.
- We let $0$ denote a vector with all entries 0, and we let $1$ denote a vector with all entries 1; the size of these vectors will be dictated by the context.
- For any vectors $x$ and $y$, we let $x^T$ denote the transpose of $x$, we write $x \leq y$ if $x_i \leq y_i$ for all $i$, and we write $x < y$ if $x \leq y$ and $x_i < y_i$ for at least one $i$.
- For any set $X$ and any countable set $I$, we let
  $$X^I = \{x = (x_i)_{i \in I} : x_i \in X, \forall i \in I\}.$$ 
- Finally, we let $\mathbb{1}(\cdot)$ denote the indicator function.
Chapter 2

Branching processes with countably many types

2.1 Basic definitions

In this thesis we consider multitype Galton–Watson branching processes (MG-WBPs) with a countable typeset. The typeset, which we denote by $\mathcal{X}$, may be any countable set of abstract elements. However, when $\mathcal{X}$ contains only $k$ elements we may define a bijection between it and the set $\{0, 1, \ldots, k - 1\}$, and when $\mathcal{X}$ contains countably infinitely many elements then we may do the same with the set $\{0, 1, 2, \ldots\}$. Unless stated otherwise we therefore restrict our attention to these integer-valued sets.

Throughout this thesis we assume that the process initially contains a single individual whose type is denoted by $\varphi_0 \in \mathcal{X}$. The MGWBP then updates according to the following rules:

(i) each individual lives for a single generation, and

(ii) at death, gives birth to $\mathbf{r} = (r_j)_{j \in \mathcal{X}}$ offspring, that is, $r_0$ individuals of type 0, $r_1$ individuals of type 1, etc., where the vector $\mathbf{r}$ is chosen independently from that of all other individuals according to a probability distribution, $p_j(\cdot)$, specific to the parental type $j \in \mathcal{X}$.

We define the MGWBP on the Ulam-Harris space (see [31, Ch. VI]). To do this, we define a set of virtual individuals, made up of labels that identify every individual that could potentially appear in the population. Let $\mathcal{J} = \bigcup_{n \geq 0} \mathcal{J}_n$ denote the set of virtual individuals, where $\mathcal{J}_n$ describes the
virtual \( n \)-th generation. Each virtual individual is defined recursively using its string of ancestors. That is, let \( \mathcal{J}_0 = \mathcal{X} \), where \( \varphi_0 \in \mathcal{J}_0 \) specifies the type of the single individual in generation 0 (the root), and for \( n \geq 1 \), let \( \mathcal{J}_n = \mathcal{X} \times (\mathbb{N} \times \mathcal{X} \times \mathbb{N})^n \), where \((\varphi_0; i_1, j_1, y_1; \ldots; i_n, j_n, y_n)\) denotes the \( i_n \)-th child of type \( j_n \) born to \((\varphi_0; i_1, j_1, y_1; \ldots; i_{n-1}, j_{n-1}, y_{n-1})\). In each triple \((i_n, j_n, y_n)\) the third entry \( y_n \) contains the individual’s unique identification number. While we do not need the identification number \( y_n \) to define the branching process with countably many types, we make use of it in Section 2.4. We describe \( y_n \) in further detail in the next paragraph. The sequence of triples \((\varphi_0; i_1, j_1, y_1; \ldots; i_n, j_n, y_n)\), which represents the past history of an individual, is referred to as a reproduction trail \([8, 62]\).

The set \( \mathcal{J} \) of virtual individuals is countable and therefore its elements can be ordered. For instance, we can order the virtual individuals in such a way that the sums of their \( i_\cdot \) and \( j_\cdot \) entries forms an increasing sequence, with ties broken arbitrarily. The unique identification number \( y_n \) may then be the virtual individual’s position in this ordered sequence. For example:

\[
(\varphi_0), \\
(\varphi_0; 1, 0, 1), \\
(\varphi_0; 2, 0, 2), (\varphi_0; 1, 1, 3), (\varphi_0; 1, 0, 1; 1, 0, 4), \\
(\varphi_0; 3, 0, 5), (\varphi_0; 2, 1, 6), (\varphi_0; 1, 2, 7), (\varphi_0; 2, 0, 2; 1, 0, 8), (\varphi_0; 1, 1, 3; 1, 0, 9), \ldots
\]

Not every virtual individual will actually appear in the population. Nonetheless, to each virtual individual, we assign potential offspring vectors containing the number of children of each type the individual has if it appears in the population. Assigning children to every virtual individual leads to redundancies, but this makes things simpler from a measure theoretic perspective, as it allows the branching process to be defined as a countable sequence of independent random vectors.

We denote the random potential offspring vector of virtual individual \( I \in \mathcal{J} \) by \( \mathbf{N}(I) = (N_j(I))_{j \in \mathcal{X}}, \) where \( N_j(I) \) gives the number of type \( j \) offspring born to individual \( I \). For each \( I \in \mathcal{J} \) the random vector \( \mathbf{N}(I) \) is defined on the probability triple \((\Omega_I, \mathcal{F}_I, P_I)\), where

\[
\Omega_I = R_{\mathcal{X}} := \left\{ \mathbf{r} \in (\mathbb{N}_0)^{\mathcal{X}} : \sum_{j \in \mathcal{X}} r_j < \infty \right\},
\]
2.1. Basic definitions

\( \mathcal{F}_I \) is the corresponding Borel sigma algebra, and \( \mathbb{P}_I \) is a probability measure such that \( \mathbb{P}_I(B) = \sum_{r \in B} p_j(r) \) if \( I \) is of type \( j \), for all \( B \in \mathcal{F}_I \). The sequence of probability spaces \( \{ (\Omega_I, \mathcal{F}_I, \mathbb{P}_I) \}_{I \in J} \) induces the product probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) on which the Galton–Watson branching process with countably many types is defined. Note that each outcome \( \omega \in \Omega \) is simply a list of the potential offspring born to each virtual individual, that is, \( \omega = \{ r_I \}_{I \in J} \).

A virtual individual \( I = (\varphi_0; i_1, j_1, 1; \ldots; i_n, j_n, n) \) appears in the population if and only if its parent, \( (\varphi_0; i_1, j_1, 1; \ldots; i_{n-1}, j_{n-1}, n-1) \) appears in the population and has at least \( i_n \) children of type \( j_n \). That is, given the sequence of offspring vectors \( \{ r_I \}_{I \in J} \), we can define the set of individuals that appear in the population, \( X(\omega) = \bigcup_{n \geq 0} X_n(\omega) \), recursively:

\[
X_0(\omega) = \{ \varphi_0 \}, \quad X_n(\omega) = \{ x = (\tilde{x}; i_n, j_n, n) \in J_n : \tilde{x} \in X_{n-1}(\omega), \quad i_n \leq r_{\tilde{x}, j_n} \},
\]

where \( X_n(\omega) \) is the set of generation-\( n \) individuals that appear in the population. The sets \( X_n(\omega) \), for \( n \geq 0 \), are sufficient to describe the evolution of the population, but in most cases, they contain more information than required. We are thus led to define the \textit{population size vector} \( Z_n(\omega) \), whose entries

\[
Z_{n,j}(\omega) = \sum_{I \in J_n} 1(I \in X_n(\omega), j_n = j), \quad \omega \in \Omega, \ j \in \mathcal{X},
\]

count the number of individuals of type \( j \) that appear in generation \( n \). In this thesis we often refer to branching processes by their sequence of random population size vectors, \( \{ Z_n \}_{n \geq 0} \).

The sequence \( \{ Z_n \} \) forms a Markov chain with state space \( \mathbb{R}^X \), whose dependence from one generation to the next can be expressed neatly. Indeed, the next generation is always made up of the independently generated offspring of the individuals in the current generation; thus if \( \{ \xi_{i,j} : i \in \mathbb{N}, j \in \mathcal{X} \} \) is a sequence of independent random vectors with distribution \( p_j(\cdot) \), then

\[
Z_{n+1} = \sum_{j \in \mathcal{X}} \sum_{i=1}^d \xi_{i,j}.
\]

Equation (2.3) is referred to as the \textit{branching process equation}.

We now introduce the two main tools in the study of multitype Galton–Watson branching processes. The first is the \textit{progeny generating vector}, \( G : [0, 1]^\mathcal{X} \rightarrow [0, 1]^\mathcal{X} \), whose \( i \)th entry is defined from the offspring distribution of
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Type $i$:

$$G_i(s) := \sum_{r \in \mathcal{R}_X} p_i(r) s^r = \sum_{r \in \mathcal{R}_X} p_i(r) \prod_{k=1}^{\infty} s_{k}^{r_k}, \quad i \in \mathcal{X}. \quad (2.4)$$

Observe that $G(\cdot)$ is monotone increasing, that is, if $s \leq u$ then $G(s) \leq G(u)$. In addition, using the branching process equation, we can show that the generating vector of the population size $Z_n$ at generation $n$, is obtained by iterating $G(\cdot)$ $n$ times (see for instance [57]),

$$\left( \mathbb{E}_i \left( s_{Z_n^i} \right) \right)_{i \in \mathcal{X}} = G^{(n)}(s) := G \circ G \circ \cdots \circ G(s) = G(G^{(n-1)}(s)), \quad (2.5)$$

where $\mathbb{E}_i(\cdot) := \mathbb{E}(\varphi_0 = i)$. We will see that (2.5) is a particularly useful property of the progeny generating vector.

The second tool we introduce to study $\{Z_n\}$ is the mean progeny matrix, $M$, whose entries are given by

$$M_{ij} := \left. \frac{\partial G_i(s)}{\partial s_j} \right|_{s=1}, \quad \text{for } i, j \in \mathcal{X},$$

where $M_{ij}$ can be interpreted as the expected number of type-$j$ children born to a parent of type $i$. We let $M_{ij}^{(n)} = (M^n)_{ij}$ which, using the branching process equation, can be shown to give the expected number of type $j$ descendants in generation $n$ from a type-$i$ root, $\mathbb{E}(Z_{n,j}^i | \varphi_0 = i)$. We assume that the row sums of $M$ are finite so that the expected total number of direct offspring from individuals of any type is finite. It is sometimes convenient to associate a graph $(\mathcal{X}, E_{\mathcal{X}})$, to the mean progeny matrix whose set of vertices corresponds to the set of types $\mathcal{X}$, and in which there is an oriented edge between vertices $i$ and $j$ with weight $M_{ij}$ if and only if $M_{ij} > 0$. We shall refer to this graph as the mean progeny representation graph. We say that there is a path from type $i$ to type $j$ if such a (directed) path exists in the mean progeny representation graph. The process $\{Z_n\}$ is said to be irreducible if there is a path between every pair of vertices, that is, if $(\mathcal{X}, E_{\mathcal{X}})$ is connected. For any $i, j \in \mathcal{X}$, let $f_{ij}^{(0)} = 0$ and $f_{ij}^{(1)} = M_{ij}$, and for $n \geq 2$ let

$$f_{ij}^{(n)} := \sum_{i_1, i_2, \ldots, i_{n-1} \neq j} M_{i_1i_2} \cdots M_{i_{n-1}j}. \quad (2.6)$$

The term $f_{ij}^{(n)}$ represents the weighted sum of $n$-step first passage paths from $i$ to $j$ in $(\mathcal{X}, E_{\mathcal{X}})$. In addition, we define the generating function

$$F_{ij}(s) := \sum_{n=0}^{\infty} f_{ij}^{(n)} s^n, \quad s \geq 0, \quad (2.7)$$
where $F_{ij}(1)$ gives the weighted sum of all first passage paths from $i$ to $j$ in $(\mathcal{X}, E_\mathcal{X})$.

Another immediate consequence of the branching process equation (2.3) is that if the population becomes empty at some generation $n$, $Z_n = 0$, then it will remain empty forever. In other words, the Markov chain $\{Z_n\}$ has an absorbing state at $0$. When this absorbing state is reached we say the branching process has become \textit{globally extinct}. We define the global extinction event,

$$E_g = \{\omega \in \Omega : \lim_{n \to \infty} Z_n(\omega) = 0\},$$

and the \textit{global extinction probability vector}, $q := (q_i)_{i \in \mathcal{X}}$, where

$$q_i := \mathbb{P}_i(E_g),$$

and $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | \varnothing_0 = i)$. The global extinction probability vector is one of the main quantities of interest throughout this thesis, and we now list some of its elementary properties. First note that in an irreducible branching process all entries of $q$ are equal to 1, or all entries of $q$ are strictly less than 1. To see why, suppose instead that there exists $i, j \in \mathcal{X}$ such that $q_i = 1$ and $q_j \neq 1$. Then because the process is irreducible, a type-$i$ individual has a positive chance of having a type-$j$ descendant, which in turn leads to the conclusion that $q_i < 1$, a contradiction. In addition, if we set $q_n := (q_{n,i})_{i \in \mathcal{X}}$ where

$$q_{n,i} = \mathbb{P}_i(Z_n = 0),$$

then, since the state $Z_n = 0$ is absorbing, the sequence $\{q_n\}_{n \geq 0}$ is monotone increasing. Thus, by the monotone convergence theorem,

$$q = \lim_{n \to \infty} q_n.$$

In addition, if we let $s = 0$ in Equation (2.5), we see that the vectors $q_n$ satisfy the recursion $q(1) = G(0)$, and

$$q_n = G(q(n-1)) = G^{(n)}(0),$$

so that $q = \lim_{n \to \infty} G^{(n)}(0)$. If we take $n$ to infinity in Equation (2.8) then, by the continuity of $G(\cdot)$, we obtain

$$q = G(q).$$

Thus, the global extinction probability vector belongs to the set

$$S := \{s \in [0, 1]^{\mathcal{X}} : s = G(s)\},$$
which we shall study throughout this thesis. In particular, \( q \) corresponds to
the unique componentwise minimal non-negative element in this set, for if
there exists \( s \in S \) such that \( 0 \leq s_i < q_i \) for some \( i \in X \), then
\[
\lim_{n \to \infty} G^{(n)}_i(s) = s_i < q_i = \lim_{n \to \infty} G^{(n)}_i(0),
\]
which contradicts the fact that \( G(\cdot) \) (and therefore its iterates) are monotone
increasing in each variable.

We have highlighted two important facts about \( q \). First, it is obtained
through repeated application of the progeny generating vector to the vector \( 0 \),
and second, it is the minimal non-negative solution to the fixed point equation
(2.9).

## 2.2 Finite-type branching processes

At present the properties of the global extinction probability vector \( q \) are
much better understood in the finite-type setting than in the infinite-type
setting. For MGWBPs with a finite number of types, many of the fundamental
questions concerning \( q \) are resolved in classical texts such as Harris
[31]. We summarise these classical results now as they set out a blueprint for
the questions we should address in the infinite-type setting, and they suggest
techniques that might be applied to answer them.

In this section, let \( X = \{0, 1, \ldots, k - 1\} \) and assume

- the mean progeny matrix \( M \) is **positive regular**, that is, irreducible and
  aperiodic, and

- the process is **non-singular**, that is, individuals of at least one type can
  have a total number of offspring different from 1.

Note that the issues that arise by replacing positive regularity by irreducibility
of \( M \) can, in most cases, be remedied following the arguments in [57, Section
2.2]. The reducible case is treated in [34]. We do not go into detail here.

The first result concerns the stability or instability of the total population
size, \( |Z_n| := \sum_{i=0}^{k-1} Z_{n,i} \), as \( n \to \infty \). On the surface there are three possible
destinations for \( |Z_n| \) as \( n \to \infty \). Either the population experiences global
extinction, \( |Z_n| \to 0 \), or the population increases without bound, \( |Z_n| \to \infty \), or the population returns to some finite set, \( |Z_n| \in \{1, \ldots, K\} \), infinitely
many times. The following theorem demonstrates that the latter outcome has
probability 0, which means that the population either dies out or becomes arbitrarily large. This phenomenon is referred to as the dichotomy between extinction and unbounded growth [30].

**Theorem 1** For any $i \in \mathcal{X}$, $\mathbb{P}_i(|Z_n| \to 0) + \mathbb{P}_i(|Z_n| \to \infty) = 1$.

The ideas behind Theorem 1 are most clearly demonstrated under the additional assumption that $q_i > 0$ for all $0 \leq i \leq k - 1$. In this case, every time the total population size falls below $K$ there is some (perhaps very small) chance of global extinction. If the total population size were to fall below $K$ infinitely often, then there are infinitely many chances for the population to experience global extinction. Eventually one of these chances will be taken and the population must become globally extinct, leading to the result. Without the additional assumption that $q_i > 0$ for all $0 \leq i \leq k - 1$, the arguments become more technical. We refer the reader to [31, Ch. II, Theorem 6.1] for further details.

The dichotomy between extinction and unbounded growth leads to the following result concerning the set of fixed points $S$.

**Theorem 2** If $s \in [0, 1)^X \setminus \{1\}$, then

$$\lim_{n \to \infty} G^{(n)}(s) = q.$$ 

Moreover,

$$S = \{q, 1\},$$

where $S$ is defined in (2.10).

Note that the first assertion implies the second. We sketch the proof under the stronger assumption that there exists $\varepsilon > 0$ such that $s_i < 1 - \varepsilon$ for all $0 \leq i \leq k - 1$. In that case, for any $i \in \mathcal{X}$ and $K \geq 1$ we have

$$G^{(n)}_i(s) = \mathbb{P}_i(Z_n = 0) + \sum_{r:1 < |r| \leq K} \mathbb{P}_i(Z_n = r)s^r + \sum_{|r| > K} \mathbb{P}_i(Z_n = r)s^r,$$

where the first term is $q_{n,i}$, the second is dominated by $\mathbb{P}_i(0 < |Z_n| \leq K)$, and the third is dominated by $(1 - \varepsilon)^K$. As $n \to \infty$, the first term converges to $q_i$, and by Theorem 1 the second converges to 0 regardless of the value of $K$. The result then follows by taking $K$ arbitrarily large. Without the additional assumption that each entry of $s$ is bounded away from 1, the proof becomes
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more technical. We refer the reader to [31, Ch. II, Theorem 7.2] for further details.

According to Theorem 2, the set $S$ contains one or two elements: one if there is almost sure global extinction, $q = 1$, and two if there is a positive chance of global survival, $q < 1$. It is then natural to classify each branching process into one of these two categories. As we demonstrate in Theorem 3, it is possible to classify any finite-type branching process using only its mean progeny matrix $M$, with no other information about its offspring distribution. If such a classification rule exists, based solely on $M$, then we call it a *global extinction criterion*.

To give context to Theorem 3 note that if $\rho(M)$ denotes the eigenvalue of $M$ with maximal absolute value, then the Perron-Frobenius theorem (see [65, Sec. 1.1]) ensures that it is real, positive and such that

$$
\frac{M_{ij}^{(n)}}{(\rho(M))^n} \rightarrow v_i u_j \quad \text{as } n \rightarrow \infty,
$$

where $u$ and $v$ are the corresponding (strictly positive) left and right eigenvectors, normalised so that $u^T 1 = 1$ and $u^T v = 1$. In short, the expected total population size $E_i(|Z_n|)$ either grows exponentially, converges to a positive constant, or decays exponentially, depending on whether $\rho(M) > 1$, $\rho(M) = 1$ or $\rho(M) < 1$, respectively. The following result states that there is a positive chance of global survival if and only if there is exponential growth.

**Theorem 3** The global extinction probability vector $q = 1$ if and only if $\rho(M) \leq 1$.

To sketch the proof of Theorem 3, we first suppose that $\rho(M) \leq 1$. In this case for any $i$, by the Perron-Frobenius theorem, $E_i(|Z_n|)$ is uniformly bounded in $n$. That is, there exists some $K > 0$ such that for any $n, N > 0$

$$
K \geq \sum_{u=1}^{\infty} u P_i(|Z_n| = u) \geq NP_i(|Z_n| > N).
$$

Because this holds for all $n$ and arbitrarily large $N$, we have $P_i(|Z_n| \rightarrow \infty) = 0$ for any $i \in \mathcal{X}$. Thus, by Theorem 1 we have $q = 1$.

Now suppose $\rho(M) > 1$. For any $n, N > 0$ we have $q_{n+N} = G^{(n)}(q_N)$. Taking a multivariate Taylor expansion around $1$ we obtain

$$
q_{n+N} = 1 - M^n(1 - q_N) + o(|1 - q_N|),
$$
where the remainder $o(1 - q_N)$ also depends on $n$. By the Perron-Frobenius theorem we may choose $n$ such that $|M^n s| > c|s|$ for any $s > 0$ and $c > 0$. If we assume that $q = 1$, we can let $c = 2$ and choose $N$ large enough to ensure the remainder $o(1 - q_N)$ is sufficiently small so that

$$|q_{n+N}| < |q_N|,$$

which contradicts the fact that $\{q_n\}$ is a monotonically increasing sequence. This implies that $q < 1$. We refer the reader to [31, Ch. II, Theorem 7.1] for the detailed arguments.

When $\rho(M) > 1$ (and therefore $q < 1$) we would often like to know the precise value of $q$. By Theorem 2, $q$ is the unique solution with all entries strictly less than 1 to the fixed point equation $s = G(s)$. However, in all but a few special cases the fixed point equation cannot be solved analytically, and $q$ must instead be computed algorithmically. We have already encountered the most well known algorithm for computing $q$: the functional iteration algorithm, which is the recursive calculation of $q_n$ described by (2.8). This algorithm converges to $q$ linearly, that is, as $n \to \infty$

$$\frac{|q - q_n|}{|q - q_{n-1}|} \to c,$$

where $0 < c < 1$. Taking a Taylor expansion of $q_n = G(q_{n-1})$ about $q$ it is possible to determine that $c = \rho(M(q))$, where $(M(q))_{i,j} = \frac{G_q}{s_j}$. There are also more efficient computational algorithms, such as Newton’s method [23, 36].

Much more is known about MGWBPs with finitely many types. One particularly significant contribution, which is not directly relevant to this thesis, is the Kesten-Stigum theorem [40]. For any initial type $i$ this theorem states that, under second moment assumptions, $Z_n/(\rho(M))^n$ converges in distribution to $W_iu$, where $W_i$ is a scalar random variable with $P(W_i = 0) = q_i$ and $u$ is the left Perron-Frobenius eigenvector. This implies that, if the population survives globally, then with probability 1 the total population size $|Z_n|$ grows exponentially with rate $\rho(M)$, and with probability 1, the proportion of each type $Z_{n,j}/|Z_n|$ converges to the deterministic value $u_j$.

To summarise, the fundamental results on global extinction in branching processes with finitely many types directly relevant to this thesis are:

1. The set $S$ can be characterised completely,
2. There is an established global extinction criterion, and

3. There are a number of algorithms for computing \( q \) numerically.

In the next section we review the corresponding results available in the infinite-type setting.

## 2.3 Infinite-type branching processes

Several attempts have been made to extend the results on global extinction of MGWBPs with finitely many types to the infinite-type setting. In this section, we summarise some of the significant contributions towards this goal. Each theorem that we present has a direct analogue in the previous section. A sketch of proof is provided only if it differs significantly from that of the corresponding theorem in the previous section. Throughout this section we assume that the process \( \{Z_n\} \) is irreducible.

The first result generalises Theorem 1 on the dichotomy between global extinction and unbounded growth:

**Theorem 4** If

\[
\inf_{i \in \mathcal{X}} q_i > 0
\]  

then,

\[
P_i(|Z_n| \to 0) + P_i(|Z_n| \to \infty) = 1 \quad \text{for all } i \in \mathcal{X}.
\]

In an irreducible MGWBP with finitely many types, non-singularity of the process is necessary and sufficient for (2.12). In the infinite-type setting Theorem 4 gives only a sufficient condition. We can verify that (2.11) is not necessary for (2.12) by considering an example where individuals of every type \( i \in \mathcal{X} \) have exactly two offspring with probability 1; in this case the probability of global extinction is 0, which means \( \inf_{i \in \mathcal{X}} q_i = 0 \), and with probability 1, \(|Z_n| = 2^n\), which means \( P_i(|Z_n| \to \infty) = 1 \). Fortunately, in many cases (2.11) can be verified easily, for instance by observing that it holds if the types have a uniformly positive chance of having no offspring, that is, if \( \inf_{i \in \mathcal{X}} P_i(0) > 0 \).

In Chapter 3 we give an example of a non-singular irreducible MGWBP with infinitely many types that does not satisfy (2.12). We refer the reader to [71, Example 4.5] for another similar example.

A more general statement of Theorem 4 can be found in [46, Theorem 2] (see also [68]). Other equivalent (but less general than [46, Theorem 2]) results
are given in [31, Ch. III, Theorem 11.2], [60, Lemma 3.3], and [69, Condition 2.1]. The author of [60] also gives a second more technical condition in [60, Lemma 3.4] and then remarks that he is not completely satisfied with either of the conditions he provides. We are unaware of anyone who has addressed this remark.

We now provide an analogue of Theorem 2 in the infinite-type case.

**Theorem 5** If (2.11) holds, then for any \( s \in [0,1]^X \) such that \( \inf_{i \in \mathcal{X}} s_i < 1 \), we have

\[
\lim_{n \to \infty} G^{(n)}(s) = q.
\]

Moreover, in this case \( S \) contains at most a single element with all entries uniformly bounded away from 1, which corresponds to \( q \).

By irreducibility, any solution \( s \in [0,1]^X \setminus \{1\} \) to the fixed point equation \( s = G(s) \), different from \( 1 \) is such that \( s_i < 1 \) for all \( i \in \mathcal{X} \) (see for instance [66, Theorem 2]). Theorem 5 states that there may only be one such solution with \( \sup_i s_i < 1 \) which, when it exists, corresponds to \( q \), but it leaves open the possibility that there may be some additional solutions \( s \) such that \( \lim \sup s_i = 1 \). The existence of these additional solutions is ruled out in [66, Corollary 4], which states that \( S = \{q,1\} \), (i.e. the same as in the finite-type case). However, a small mistake in the proof of this theorem is pointed out by Bertacchi and Zucca in [9], where it is shown that such additional elements can exist. We provide an intuitive explanation of why these additional elements can exist in the next section. In a follow up paper [10], the same authors give an example where \( S \) contains uncountably many solutions \( s \) such that \( s_i \to 1 \).

The exact composition of the set \( S \) is still an open problem.

We now summarise several attempts to generalise the global extinction criterion in Theorem 3 to the infinite-type setting. This task has proven difficult. To make the problem more manageable, authors generally present sufficient conditions for \( q = 1 \) and \( q < 1 \) separately. We list next sufficient conditions for almost sure global extinction. These conditions correspond to [60, Theorem 3.2], [71, Theorem 4.1], [35, Proposition 4.5], and [69, Proposition 2], respectively.

**Theorem 6** If \( \inf_{i \in \mathcal{X}} q_i > 0 \), then each of the following conditions are sufficient for \( q = 1 \):

(i) \( E_i |Z_1| \leq 1 \) for all \( i \in \mathcal{X} \)
(ii) \( \lim \inf_i (\mathbb{E}_i (|Z_n|)^{1/n}) < 1 \) for some \( i \in \mathcal{X} \)

(iii) there exists \( \lambda \leq 1 \) and \( u \in (\mathbb{R}_{> 0})^X \) such that \( |u| < \infty \) and \( uM \leq \lambda u \)

(iv) \( \lim \inf_i \mathbb{E}_i (|Z_n|) < \infty \) for some \( i \in \mathcal{X} \).

Each of (i), (ii) and (iii) can be derived from (iv). In other words, each of these sufficient conditions is implied by the fact that if the expected total population size does not increase to infinity then, under condition (2.11), there must be almost sure global extinction. Verifying condition (iv) is often difficult and each of (i)-(iii) can be viewed as an attempt to package this simple idea in a way that is user friendly but still sufficiently general. We shall provide another alternate packaging in Chapter 3. Using contrasting methods, both [71] and [35] demonstrate that (iv) is not a necessary condition for \( q = 1 \). The approach in [71] is of particular interest to us because the author constructs two examples with the same mean progeny matrix \( M \), the first with \( q < 1 \) and the second with \( q = 1 \), thus verifying that even when (2.11) holds, no extinction criterion based solely upon \( M \) exists. To construct necessary and sufficient conditions for almost sure global extinction, the author then suggests that second moment conditions should be imposed. We are unaware of anyone who has followed up on this suggestion. This is something we explore in Chapter 4.

There have been comparatively fewer attempts to derive sufficient conditions for \( q < 1 \). This is perhaps due to the difficulties we have just described. We list some of these conditions now. Conditions (v) and (vi) correspond to [60, Theorem 3.2] and [66, Theorem 4], respectively. Condition (vii) has been stated by a number of authors; we give the corresponding references in the next section.

**Theorem 7** Each of the following conditions are sufficient for \( q < 1 \):

(v) \( \inf_{i \in \mathcal{X}} \mathbb{E}_i |Z_1| > 1 \) and \( \sup_{i \in \mathcal{X}} \mathbb{E}_i |Z_1|^2 < \infty \)

(vi) (A) There exists \( K > 0 \) such that \( \mathbb{P}_i (|Z_1| \leq K) = 1 \) for all \( i \in \mathcal{X} \);

(B) there exists \( r > 1 \) and \( u \in \mathbb{R}^X \) with \( 0 < \inf_i u_i \leq \sup_i u_i < \infty \) such that \( rM \mathbb{u} = \mathbb{u} \)

(vii) \( \nu(M) := \lim \sup_{n \to \infty} (M_{ij}^{(n)})^{1/n} > 1 \)
Both Conditions (v) and (vi) are sufficient for the stronger assertion that \( \sup_i q_i < 1 \). Condition (vii) is less restrictive than (vi), but it is not sufficient for this stronger assertion. We will explain Condition (vii) in detail in the next section. Conditions (v) and (vi) are derived through a second order Taylor expansion argument that is reminiscent of the first order Taylor expansion argument used to prove Theorem 3. We shall apply a similar second order argument in Chapter 3 to obtain another sufficient condition for \( q < 1 \).

If none of conditions (i)-(vii) can be applied, we must then turn to other methods to determine whether \( q = 1 \). Recall that there is a positive chance of global survival if and only if there exists \( s \in [0, 1)^X \) such that \( s = G(s) \). An explicit solution to this fixed point equation can rarely be found, but using the fact that \( G(\cdot) \) is monotone increasing in its elements we can obtain a more manageable criterion.

**Theorem 8** We have \( q < 1 \) if and only if there exists \( s \in [0, 1)^X \) such that \( G(s) \leq s \).

Nonetheless, even for relatively simple examples, the application of Theorem 8 is still challenging, see for instance [8].

Even when it is known that \( q < 1 \), it still remains to compute its precise value. In theory, this can again be achieved through the functional iteration algorithm, that is, by using the recursion

\[
q_n = G(q_{n-1})
\]

(2.13) to compute \( q_n \) for large \( n \). However, this is impractical because in the infinite-type case, (2.13) corresponds to an infinite system of equations. A logical solution is to truncate this infinite system of equations, and then iterate on the resulting finite system of equations instead. The authors of [35] demonstrate that unless this truncation is done carefully, unexpected results may be obtained. Until the recent works of [35] and [64], very little had been done concerning the computational aspects of \( q \). We will detail the results of these papers in Sections 2.6 and 2.7 respectively.

In this section we have seen that many results for finite-type branching processes do not extend easily to the infinite-type setting. There are however some restrictive assumptions that can be enforced to make infinite-type branching processes more amenable to analysis. Along with regularity assumptions Moy [58, 59] assumes that \( M \) is \( R \)-positive recurrent (for the definition of \( R \)-positive
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recurrence see [65, Definition 6.2]). With these assumptions Moy derives a result similar to the Kesten-Sigum theorem. That is, for all $i \in X$ he proves that $\mathbb{Z}_n/(\nu(M))^n$ converges in distribution to $W_i u$, where $W_i$ is a scalar random variable, $u$ is the unique solution to $1/\nu(M)u^\top M = u^\top$ normalised so that $|u| = 1$, and $\nu(M)$ is the convergence norm of $M$, which we describe in greater detail next section. Harris [31, Ch. III, Theorem 14.1] gives a similar result that applies to processes with a general (possibly uncountable) type space. In his case these results are obtained under [31, Ch. III Condition 10.1], which in our notation is: there exists $n_0$ such that

$$0 < a \leq M_{i,j}^{(n_0)} \leq b < \infty \quad \text{for all } i, j \in X.$$

Condition 10.1 (when written in its more general form) is more suited to compact continuous type spaces (i.e. $X = [0, 10]$) as it is particularly restrictive when applied to branching processes with a countably infinite typeset. Through these results, Moy and Harris in effect demonstrate that under certain assumptions infinite-type branching processes mirror the behaviour of finite-type ones.

To summarise the analogy to 1–3 given at the end of the previous section for general MGWBPs with a countably infinite number of types:

1. The set $S$ is still not well understood,
2. There is no established global extinction criterion, and
3. The computational aspects of $q$ have received little attention.

When there are infinitely many types, open questions remain. It is the goal of this thesis to make progress toward finding answers to these three questions. But why does transitioning from finitely many types to infinitely many types cause so many difficulties? In addition to a number of technical challenges that arise, this transition alters questions about extinction on a more fundamental level.

2.4 Partial extinction

2.4.1 Extinction in sets

In an irreducible MGWBP with finitely may types, $q$ is the only extinction probability vector, whereas in an irreducible MGWBP with infinitely many
2.4. Partial extinction

types, there may be multiple other distinct extinction probability vectors. To explain this statement, for a MGWBP with either finitely or infinitely many types we define extinction, more generally, in any set of types \( A \subseteq \mathcal{X} \). For any \( n \geq 0 \) we let

\[
E_n(A) := \left\{ \omega \in \Omega : \sum_{\ell=n}^{\infty} \sum_{i \in A} Z_{\ell i} = 0 \right\}
\]

denote the event that no type-\( i \in A \) individuals appear in the population from generation \( n \). Observe that \( E_n(A) \subseteq E_{n+1}(A) \) for all \( n \geq 0 \). We let

\[
E(A) = \lim_{n \to \infty} E_n(A)
\]

represent the event of extinction in the set of types \( A \). We define the vector \( q_n(A) := (q_{n,i}(A))_{i \in \mathcal{X}} \) whose \( i \)-th element is given by

\[
q_{n,i}(A) := \mathbb{P}_i(E_n(A)).
\]

Using Equation (2.3) and a single step conditioning argument, we determine that the vectors \( \{q_n(A)\}_{n \geq 0} \) form a non-decreasing sequence that satisfies the equation

\[
q_{n+1}(A) := G(q_n(A)),
\]

see also [10]. By the monotone convergence theorem, for any \( A \subseteq \mathcal{X} \), the extinction probability vector \( q(A) := \lim_{n \to \infty} q_n(A) \) exists. In addition, by the continuity of \( G(\cdot) \), \( q(A) \) satisfies the fixed point equation

\[
q(A) = G(q(A)), \quad (2.14)
\]

and is therefore an element of the set \( S = \{ s \in [0,1]^\mathcal{X} : s = G(s) \} \). Note also that for any \( A \subseteq B \subseteq \mathcal{X} \) we have \( E(B) \subseteq E(A) \), which implies

\[
q(B) \leq q(A). \quad (2.15)
\]

The definition of these more general extinction probability vectors leads to redundancies. For example, take any two finite sets \( A, B \subseteq \mathcal{X} \). For any irreducible branching process the probability that a type-\( i \in A \) individual has a type-\( j \in B \) descendent is strictly positive. If the process survives in the set \( A \), it must return to this set infinitely many times. This means the process has infinitely many chances to return to the set \( B \), which implies that it also almost surely survives in the set \( B \) as well. Repeating this argument with \( A \) and \( B \) interchanged we see that

\[
q(A) = q(B).
\]
We prove this rigorously and in a more general form in Theorem 26. One consequence of this result is that in an irreducible MGWBP with finitely many types we have \( q(A) = q \) for all \( A \subseteq \mathcal{X} \), that is, there is only a single distinct extinction probability vector corresponding to \( q \).

In the case of an irreducible MGWBP with infinitely many types, when \( A \) is a finite set, the discussion in the previous paragraph does not necessarily imply that \( q(A) \) coincides with \( q = q(\mathcal{X}) \). This is because \( \mathcal{X} \) contains infinitely many elements. Indeed, in the infinite-type case there may be a positive chance that every type eventually disappears from the population while the total population size increases without bound, leading to \( q < q(A) \). We give an example illustrating this in the next subsection. Thus, when \( A \) and \( B \) are finite, \( q(A) \) and \( q(B) \) coincide, and when \( A \) is infinite and \( B \) is finite \( q(A) \) and \( q(B) \) do not necessarily coincide. A natural question is then: do \( q(A) \) and \( q(B) \) coincide when \( A \) and \( B \) are infinite? We defer an investigation of this question until Chapter 5. In the meantime, in addition to global extinction, we will focus only on the event that there is extinction in every finite set, which is referred to as partial extinction.

2.4.2 Partial extinction

We define the partial extinction event,

\[
\mathcal{E}_p := \bigcap_{A \subseteq \mathcal{X} : |A| < \infty} \mathcal{E}(A),
\]

and the partial extinction probability vector, \( \tilde{q} = (\tilde{q}_i)_{i \in \mathcal{X}} \), where

\[
\tilde{q}_i := \mathbb{P}_i(\mathcal{E}_p).
\]

By a one step conditioning argument we have

\[
\tilde{q}_i = \sum_{r \in R_X} p_i(r) \prod_{i \in \mathcal{X}} (\tilde{q}_i)^{r_i} = G_i(\tilde{q}),
\]

and therefore \( \tilde{q} \in S \). In addition, by (2.15), we have

\[
q \leq \tilde{q}.
\]

According to the discussion at the end of the previous section, in an irreducible branching process \( \tilde{q} = q(A) \) for any finite \( A \subseteq \mathcal{X} \), however, this is not necessarily the case in a reducible branching process. We choose to study
2.4. Partial extinction

Partial extinction rather than extinction in specific finite sets because in later chapters this enables us to say more in the reducible setting.

Partial extinction has been referred to by many names. For example, in the context of random measures (see Kallenberg [38]), partial extinction is analogous to vague convergence of \(\{Z_n\}\) to \(0\), and in the context of branching random walks (see Section 2.7) partial extinction is analogous to local extinction at every location. By referring to \(E_p\) as the partial extinction event we follow the terminology of [35].

During our investigation of \(\tilde{q}\) we seek to answer the same questions we asked about \(q\). More specifically, we would like to locate \(\tilde{q}\) in the set \(S\) (recall \(q\) is the minimal nonnegative element), use \(M\) to develop necessary and sufficient conditions for \(\tilde{q} = 1\), and derive an algorithm to compute \(\tilde{q}\).

There has already been significant progress toward these goals. We begin with a summary of existing partial extinction criteria.

Central to the analysis of partial extinction criteria is a single type Galton–Watson process embedded within \(\{Z_n\}\). We refer to this process as the type-\(i\) branching process embedded with respect to \(\{Z_n\}\) and denote it by \(\{E_n^{(i)}(Z)\}_{n \geq 0}\). This embedded process is considered in [18], [26], and [55] among others. The process \(\{E_n^{(i)}(Z)\}\) is defined on \((\Omega, \mathcal{F}, \mathbb{P})\). Its sample paths are constructed from those of \(\{Z_n : \varphi_0 = i\}\), by taking all type-\(i\) individuals that appear in \(\{Z_n\}\) and defining the direct descendants of these individuals as their closest (in generation) type-\(i\) descendants in \(\{Z_n\}\). More specifically, we define a function \(\tilde{f}^{(i)} : \mathcal{J} \rightarrow \mathcal{J}\) that takes a line of descent \((\varphi_0; i_1, j_1, y_1; i_2, j_2, y_2; \ldots; i_n, j_n, y_n)\) and deletes all triples \((i_k, j_k, y_k)\) whose type, \(j_k\), is not equal to \(i\). In Figure 2.1 we illustrate \(\{Z_n\}\) and \(\{E_n^{(0)}(Z)\}\) for a particular outcome \(\omega \in \Omega\). When the function \(\tilde{f}^{(i)}\) is applied to \(X\), the resulting random family tree evolves as a single-type Galton–Watson process.

To construct the family tree of \(\{E_n^{(0)}(Z)\}\), we require the third entry in the triples \((i_k, j_k, y_k)\), corresponding to the unique identification number. To understand why, suppose we removed it, then for \(\omega\) illustrated in Figure 2.1 we would have

\[
\tilde{f}^{(0)}(0; 1, 1; 1, 0) = \tilde{f}^{(0)}(0; 1, 3; 1, 0) = (0; 1, 0),
\]

that is, both individuals in generation one of \(\{E_n^{(0)}(Z)\}\) would have the same label. This would make the construction of the lineages in the second generation of \(\{E_n^{(0)}(Z)\}\) unclear. Thus, we would be unable to recover the family tree of \(\{E_n^{(0)}(Z)\}\).
We are interested in the process \( \{E_n^{(i)}(Z)\} \) because its extinction probability is equivalent to the probability that type \( i \) becomes extinct in \( \{Z_n\} \), see [26]. By Theorem 3 this means that type \( i \) faces almost sure extinction in \( \{Z_n\} \) if and only if the mean number of offspring from individuals in \( \{E_n^{(i)}(Z)\} \), which we denote by \( m_{E_n^{(i)}(Z)} \), is less than or equal to 1, that is,

\[
q_i(\{i\}) = 1 \iff m_{E_n^{(i)}(Z)} \leq 1. 
\] (2.16)

The first generation of \( \{E_n^{(i)}(Z)\} \) is formed by counting the lines of descent \((i; i_1, j_1, y_1; \ldots; i_n, i, y_n) \in X \) such that \( j_k \neq i \) for all \( 1 \leq k \leq n - 1 \), that is, by summing over the first return paths to \( i \). Consequently,

\[
m_{E_n^{(i)}(Z)} = F_{ii}(1),
\] (2.17)

where \( F_{ii}(s) \) is defined in (2.7). Combining (2.17) and (2.16) we obtain

\[
\tilde{\eta} = 1 \iff m_{E_n^{(i)}(Z)} = F_{ii}(1) \leq 1
\] (2.18)

when \( \{Z_n\} \) is irreducible.

This partial extinction criterion in more commonly expressed using the convergence norm,

\[
\nu(M) = \sup_{i,j \in \mathcal{X}} \left( \limsup_{n \to \infty} \left( M_{ij}^{(n)} \right)^{1/n} \right),
\]

that we have already encountered in Theorem 7. Observe that, by [65, Theorem 6.1], when \( M \) is irreducible, \( \limsup_n \left( M_{ij}^{(n)} \right)^{1/n} \) is independent of \( i \) and
In addition note that, if the typeset $X$ is finite, then the convergence norm is equivalent to the Perron-Frobenius eigenvalue, that is, $\nu(M) = \rho(M)$. The following theorem can be found in [7] and [26] among others.

**Theorem 9** If $\{Z_n\}$ is irreducible, then the following two statements are equivalent:

(i) $\bar{q} = 1$

(ii) $\nu(M) \leq 1$.  \hspace{1cm} (2.19)

To demonstrate that criteria (2.18) and (2.19) are equivalent, we introduce the generating function

$$T_{ii}(s) := \sum_{k=0}^{\infty} M_{ii}^{(k)} s^k,$$

for which

$$\nu(M) = 1/\sup\{s \geq 0 : T_{ii}(s) < \infty\}.$$

For each $k \geq 1$, observe that $M_{ii}^{(k)}$ can be expressed as a sum of convolutions of the weighted first passage paths,

$$M_{ii}^{(k)} = f_{ii}^{(k)} + \sum_{j=0}^{k} f_{ii}^{(j)} f_{ii}^{(k-j)} + \sum_{j=0}^{k} \sum_{\ell=0}^{k-j} f_{ii}^{(j)} f_{ii}^{(\ell)} f_{ii}^{(k-\ell-j)} + \ldots,$$

where $f_{ii}^{(k)}$ is defined in (2.6). Using the fact that convolutions translate into products of generating functions, we obtain that for any $s \geq 0$

$$T_{ii}(s) = 1 + F_{ii}(s) + (F_{ii}(s))^2 + (F_{ii}(s))^3 + \ldots,$$

which converges if and only if $F_{ii}(s) < 1$. The fact that $F_{ii}(s)$ is strictly increasing in $s$ then implies that $\nu(M) \leq 1$ if and only if $F_{ii}(1) \leq 1$, which gives the result.

It is often difficult to obtain an analytic expression for $\nu(M)$. In later chapters, rather than Criterion (2.19), we often use Criterion (2.18). Nonetheless we now demonstrate the applicability of Criterion (2.19) through an example.

**Example 1.** Consider a process whose progeny generating vector contains entries

$$G_i(s) = \begin{cases} \frac{b}{u}s_0^i + \frac{c}{u}s_1^i + 1 - \frac{b+c}{u}, & \text{if } i = 0, \\
\frac{a}{u}s_{i-1} + \frac{b}{u}s_i^u + \frac{c}{u}s_{i+1}^u + 1 - \frac{a+b+c}{u}, & \text{if } i \geq 1,
\end{cases}$$
Chapter 2. Branching processes with countably many types

Figure 2.2: The mean progeny representation graph corresponding to Example 1.

for some $a, b, c > 0$, where $u = [a + b + c] + 1$. The corresponding mean progeny matrix is

$$
M = \begin{bmatrix}
    b & c & 0 & 0 & 0 & \ldots \\
    a & b & c & 0 & 0 & \\
    0 & a & b & c & 0 & \\
    0 & 0 & a & b & c & \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \\
\end{bmatrix},
$$

(2.20)

whose mean progeny representation graph is illustrated in Figure 2.2. By [35, Proposition 5.1] the convergence norm of (2.20) is

$$
\nu(M) = b + 2\sqrt{ac}.
$$

We consider two different sets of values for $a$, $b$ and $c$:

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>1/5</td>
<td>1/2</td>
<td>7/20</td>
</tr>
<tr>
<td>(ii)</td>
<td>1/20</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

the first with $\nu(M) \approx 1.03$, and the second with $\nu(M) \approx 0.82$. By Theorem 9, we then have $\tilde{q} < 1$ and $\tilde{q} = 1$ respectively. In the upper and lower panels of Figure 2.3 we plot a single outcome of the process for parameter values (i) and (ii). We observe that in both cases the total population size increases exponentially. In the upper panel the populations of each specific type also grow exponentially, whereas in the lower panel these populations reach a peak value before eventually decreasing to zero. This indicates that the process illustrated in the upper panel will experience neither partial nor global extinction, while the process illustrated in the lower panel will experience partial extinction without global extinction. We will later demonstrate that for parameter values (i) and (ii) we have $q = \tilde{q} < 1$ and $q < \tilde{q} = 1$, respectively.
2.4. Partial extinction

2.4.3 Strong and non-strong local survival

Through Theorem 9 we have a neat rule for partitioning branching processes with infinitely many types into the classes $\tilde{q} = 1$ and $\tilde{q} < 1$, but if we consider both the partial and global extinction probability vectors simultaneously, then we can partition further. Indeed, every irreducible MGWBP with infinitely many types falls into one of the following categories:

\begin{align*}
(i) & \quad q = \tilde{q} = 1 \\
(ii) & \quad q = \tilde{q} < 1 \\
(iii) & \quad q < \tilde{q} = 1 \\
(iv) & \quad q < \tilde{q} < 1.
\end{align*}

Figure 2.3: A simulation of the the total population size and the population size of specific types for parameter values (i) (upper panel) and (ii) (lower panel).
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Following the terminology of [9] branching processes that fall into (ii) or (iv) are said to satisfy strong local survival and non-strong local survival, respectively. The topic of strong and non-strong local survival is considered in a number of papers including [9], [10], [26], [27], and [55]. In Chapter 4 we complement this work by developing a method of partitioning a special class of branching processes into (i) – (iv) using $M$.

Observe that the existence of irreducible MGWBPs that exhibit non-strong local survival, $q < \tilde{q} < 1$, provides insight into the set of fixed points $S$. First, they imply that $S$ can contain elements other than $q$ and $1$, and second they imply that $\tilde{q}$ may be neither the minimal element of $S$, which is $q$, nor the maximal element of $S$, which is $1$. This leads us to the following two questions:

- given the set $S$ can we identify which element corresponds to $\tilde{q}$? And,
- given $S$ can we identify in which of the four classes (i) – (iv) a process falls?

To the best of our knowledge these questions have not been considered in the literature. We address them in Chapters 4 and 5. On the other hand, if we are able to characterise the set $S$, then we may gain further insight into the extinction probability vectors $q$ and $\tilde{q}$. For instance, we have the following corollary to Theorem 5.

Corollary 1 Suppose $\inf_{i \in \mathcal{X}} q_i > 0$. If $q < \tilde{q} < 1$ then $\sup_{i \in \mathcal{X}} \tilde{q}_i = 1$.

Examples of irreducible MGWBPs that exhibit non-strong local survival can be found in [9, Example 4.2] and [18, Example 1]. We construct another in Chapter 4.

The partial extinction probability vector $\tilde{q}$ helps us to isolate the work that remains to derive a global extinction criterion. Through the partial extinction criterion in Theorem 9, we can separate the categories $q = \tilde{q} < 1$ and $q < \tilde{q} < 1$ from $q = \tilde{q} = 1$ and $q < \tilde{q} = 1$; a global extinction criterion requires the further separation of $q < \tilde{q} = 1$ from $q = \tilde{q} = 1$. To derive this partial extinction criterion we applied the global extinction criterion (Theorem 3) for a single type process. To derive a global extinction criterion for MGWBPs with infinitely many types, a different approach is required and this has proven difficult. There is however a notable exception: a class of processes for which, under minor conditions, the mean progeny matrix can be used to separate
the categories \( q < \bar{q} = 1 \) and \( q = \bar{q} = 1 \), and therefore, for which, a global extinction criterion exists. We now detail this class of processes.

2.5 Galton–Watson processes in a varying environment

A Galton–Watson Process in a varying environment (GWPVE) is a generalisation of a single type Galton–Watson process that allows the offspring distribution to vary deterministically with the generation. GWPVEs form a special subclass of MGWBPs with infinitely many types: they correspond to the reducible subclass of infinite-type branching processes in which a type-i \( \in X \) individual can give birth to type-(i + 1) offspring only. Each generation in a GWPVE thus corresponds to a type. Type i therefore becomes extinct by generation \( i + 1 \), which implies that \( \bar{q} = 1 \). For such processes we are able to reproduce two of the results we stated for finite-type processes: (i) we can derive necessary and sufficient conditions for the population to experience the dichotomy between extinction and unbounded growth, and (ii) derive can a global extinction criterion. In this section, we assume without loss of generality that the population begins with a single type-0 individual.

We introduce some new notation. We denote the probability that an individual in generation \( n \) has \( k \) offspring by \( p_{nk} := p_n(ke_{n+1}) \), and we denote the corresponding progeny generating function by

\[
g_n(s) := \sum_{k=0}^{\infty} p_{nk}s^k, \quad s \in [0, 1].
\]

The expected number of offspring from an individual in generation \( n \) is \( \mu_n := g'_n(1) \). If \( Z_n \) denotes the number of individuals in generation \( n \), then \( Z_n \) has probability generating function

\[
f_n(s) = g_0 \circ g_1 \circ \cdots \circ g_{n-1}(s), \quad s \in [0, 1].
\]

The probability of extinction by generation \( n \) is \( \mathbb{P}(Z_n = 0) = f_n(0) \). Thus, the extinction probability \( q_0 \) may be computed algorithmically through the sequential application of \( g_{n-1}, g_{n-2}, \ldots, g_0 \) to 0 for large \( n \). The rate at which \( f_n(0) \) converges to \( q_0 \) was explored in [24] under the assumption that \( q_0 = 1 \). In the case when \( q_0 < 1 \) little is known about this convergence rate.
We now give necessary and sufficient conditions for the population to experience the dichotomy between extinction and unbounded growth. The following result can be found in [17], [43], and [50].

**Theorem 10** If $p_{n0} < 1$ for all $n \geq 0$ then $\mathbb{P}(Z_n \to 0) + \mathbb{P}(Z_n \to \infty) = 1$ if and only if $\sum_{n=0}^{\infty} (1 - p_{n1}) = \infty$.

We sketch the proof of Theorem 10 under the additional assumption that $p_{n1} > 0$ for all $n \geq 0$. The arguments generalise without it, see [43, p.174]. Under this assumption, we note that $\sum_{n=0}^{\infty} (1 - p_{n1}) = \infty$ is necessary and sufficient for $\prod_{n=0}^{\infty} p_{n1} = 0$. To see why, observe that when $\liminf_{n} p_{n1} < 1$ this equivalence is immediate, and that when $\liminf_{n} p_{n1} = 1$ the population can never be made up of a finite number of individuals whose descendants each have exactly one offspring. Roughly speaking, this means the population can never completely stabilise. To obtain the stronger assertion in Theorem 10 this argument must be expanded upon. We refer the reader to [17] for these details.

We now turn our attention to the derivation of a global extinction criterion. In the context of GWPVE, this problem is addressed in a number of papers, including [39], [47], and [54]. We summarise here the results of Agresti [1]. He establishes the following bounds on the probability generating function $f_n(s)$,

$$1 - \left( (m_{0\to n}(1 - s))^{-1} + \sum_{j=0}^{n-1} \frac{g_{j}''(0)}{\mu_{j}m_{0\to j+1}} \right)^{-1} \leq f_n(s) \leq 1 - \left( (m_{0\to n}(1 - s))^{-1} + \sum_{j=0}^{n-1} \frac{g_{j}''(1)}{\mu_{j}m_{0\to j+1}} \right)^{-1}, \quad (2.21)$$
2.5. Galton–Watson processes in a varying environment

where \( m_{0\to n} := \prod_{k=0}^{n-1} \mu_k \). The lower bound is derived by taking a second order Taylor expansion of \( f_n(s) \) around \( s = 1 \), and the upper bound is derived via a comparison to a specific class of GWPVEs referred to as linear fractional (see Section 2.7). Letting \( s = 0 \) in (2.21) Agresti obtains

\[
1 - \left( m_{0\to n}^{-1} + \frac{1}{2} \sum_{j=0}^{n-1} \frac{g_j''(0)}{\mu_j m_{0\to j+1}} \right)^{-1} \leq \mathbb{P}(Z_n = 0) \leq 1 - \left( m_{0\to n}^{-1} + \sum_{j=0}^{n-1} \frac{g_j''(1)}{\mu_j m_{0\to j+1}} \right)^{-1}. \tag{2.22}
\]

Note that, by the same arguments we used to prove Theorem 3, if the expected total population size does not converge to infinity then, by Theorem 10, there is almost sure global extinction if and only if \( \sum_n (1 - p_n) = \infty \). The interesting case thus occurs when \( \lim_n m_{0\to n} = \infty \), for which (2.22) yields

\[
1 - \left( \frac{1}{2} \sum_{j=0}^{\infty} \frac{g_j''(0)}{(\mu_j m_{0\to j+1})} \right)^{-1} \leq q_0 \leq 1 - \left( \sum_{j=0}^{\infty} \frac{g_j''(1)}{\mu_j m_{0\to j+1}} \right)^{-1}.
\]

This leads Agresti to the following necessary and sufficient conditions for almost sure global extinction.

**Theorem 11** If \( \inf_{j \geq 0} \left( \frac{g_j''(0)}{\mu_j} \right) > 0 \) and \( \sup_{j \geq 0} \left( \frac{g_j''(1)}{\mu_j} \right) < \infty \) then \( q_0 = 1 \) if and only if \( \sum_{j=0}^{\infty} m_{0\to j+1} = \infty \).

Theorem 11 differs from the global extinction criterion we derived for MG-WBPs with finitely many types (Theorem 3) in a couple of notable ways. Recall that, by the Perron-Frobenius theorem, the expected total population size in an irreducible MGWBP with finitely many types grows exponentially at rate \( \rho(M) \), and that there is almost sure global extinction if and only if \( \rho(M) \leq 1 \). In contrast, the expected total population size in a GWPVE may experience non-exponential (algebraic) growth. For instance, if \( \mu_n = 1 + \frac{1}{n+1} \) then the expected total population size grows linearly, that is, \( m_{0\to n} = n \). A global extinction criterion for GWPVEs must be sensitive enough to classify these instances. Roughly speaking, Theorem 11 implies that the boundary between potential survival and almost sure extinction is linear growth, for which \( q_0 = 1 \). Furthermore, the criterion in Theorem 11 applies only under additional conditions: the condition \( \inf_{j \geq 0} \left( \frac{g_j''(0)}{\mu_j} \right) > 0 \) ensures that individuals in the process have a positive chance of having two offspring, which makes \( \{Z_n\} \)
sufficiently variable, for instance, by ruling out cases where \( \sum_n (1 - p_{n1}) < \infty \), whereas the condition \( \sup_{j \geq 0} (g_j^*(1) / \mu_j) < \infty \) ensures that the offspring distributions of individuals have finite variance, which means \( \{Z_n\} \) is not too variable. We will see precisely what ‘too variable’ means when we investigate the consequences of violating \( \sup_{j \geq 0} (g_j^*(1) / \mu_j) < \infty \) in Chapter 4.

For GWPVEs we gave necessary and sufficient conditions for both the dichotomy between extinction and unbounded growth, and almost sure global extinction. The fact that we were able to do this for a class of processes for which \( \mathbf{q} \) and \( \tilde{\mathbf{q}} \) do not necessarily coincide leads us to ask: what can GWPVEs teach us about extinction in irreducible MGWBPs with infinitely many types? This is the idea behind Chapter 4.

## 2.6 Computational techniques

In this section we summarise the two algorithmic methods for computing \( \mathbf{q} \) and \( \tilde{\mathbf{q}} \) which were developed in [35]. These methods have a physical interpretation based on two sequences of truncated finite-type branching processes, and play a key role in the remainder of this thesis.

### 2.6.1 Partial extinction probability

For \( A \subseteq \mathcal{X} \), the process \( \{\tilde{Z}_n^{(A)}\} \) is constructed on \((\Omega, \mathcal{F}, \mathbb{P})\) by removing the descendants of all individuals of type \( i \in \bar{A} \), where \( \bar{A} \) denotes the complement of \( A \). That is, for each \( \omega \in \Omega \) and individual \( I \in J \) of type \( j \), we let

\[
\tilde{N}^{(A)}(\omega, I) = N(\omega, I) \mathbb{1}(j \in A),
\]

where \( \tilde{N}^{(A)}(\omega, I) \) denotes the potential offspring of individual \( I \) in \( \{\tilde{Z}_n^{(A)}\} \). The condition of appearance of an individual in the truncated branching process is then the same as (2.1), replacing \( r_{ij} \equiv N_j(\omega, I) \) by \( i_{ij}^{(A)} = \tilde{N}_j^{(A)}(\omega, I) \), and the definition of the population size vector \( \tilde{Z}_n^{(A)}(\omega) = (\tilde{Z}_{n,i}^{(A)}(\omega))_{i \in \mathcal{X}} \) is analogous to (2.2). Consequently, in \( \{\tilde{Z}_n^{(A)}\} \),

1. \( i \) all types in \( A \) have the same progeny distribution as the corresponding types in \( \{Z_n\} \), and
2. \( ii \) all types in \( \bar{A} \) die with no offspring; these types are said to be **sterile**.

The process \( \{\tilde{Z}_n^{(A)}\} \) performs two roles that have parallels in the study of Markov chains with state space \( \mathcal{X} \). First, the **restriction** of \( \{\tilde{Z}_n^{(A)}\} \) to the set
2.6. Computational techniques

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{An illustration of \(\{Z_n\}, \{\tilde{Z}_n^{(1)}\}\) and \(\{Z_n^{(1)}\}\) for an outcome \(\omega \in \Omega\).}
\end{figure}

\(A, \{(\tilde{Z}^{(A)}_n)_{i \in A}\}_{n \geq 0}\), is the branching process formed from \(\{Z_n\}\) by immediately killing all offspring whose type belongs to \(\tilde{A}\), that is, the process with the taboo subset \(\tilde{A}\). Second, \((\tilde{Z}^{(A)}_n)_{i \in \tilde{A}}\) is the vector that counts the lines of descent that first enter \(\tilde{A}\) in generation \(n\), that is, the vector of \(n\)-step first passage paths to \(\tilde{A}\). We let \(M^{(A)} := (M_{ij})_{i, j \in A}\) be the mean progeny sub-matrix restricted to the types in \(A\), and we denote by \(\tilde{q}^{(A)}\) the global extinction probability vector of \(\{\tilde{Z}_n^{(A)}\}\). The vector \(\tilde{q}^{(A)}\) is the minimal solution to the fixed point equation

\begin{equation}
\tilde{s} = \tilde{G}^{(A)}(\tilde{s}), \quad \tilde{s} \in [0, 1]^X,
\end{equation}

where \(\tilde{G}^{(A)}(\tilde{s}) := (\tilde{G}^{(A)}_i(\tilde{s}))_{i \in \tilde{A}}\) contains entries

\begin{equation}
\tilde{G}^{(A)}_i(\tilde{s}) = \begin{cases} 
G_i(s_0 I^{(0 \in \tilde{A})}, s_1 I^{(1 \in \tilde{A})}, \ldots), & i \in A \\
1, & i \in \tilde{A}.
\end{cases}
\end{equation}

When \(A\) is finite, \(\tilde{q}^{(A)}\) can be computed using established methods for MGWBPs with finitely many types. For example, when \(T_k = \{0, 1, \ldots, k\}\), the unknown entries of \(\tilde{q}^{(T_k)} = (\tilde{q}^{(T_k)}_0, \ldots, \tilde{q}^{(T_k)}_k, 1, 1, \ldots)\) correspond to the minimal nonnegative solution of the finite system of equations

\begin{equation}
s_i = \tilde{G}^{(T_k)}_i(s_0, s_1, \ldots, s_k), \quad \text{for } 0 \leq i \leq k,
\end{equation}

which can be computed using the functional iteration algorithm. Throughout this thesis, the process \(\{\tilde{Z}_n^{(T_k)}\}\) and its extinction probability vector \(\tilde{q}^{(T_k)}\) will be of particular interest. With a slight abuse of notation, we shall denote them by \(\tilde{Z}_n^{(k)}\) and \(\tilde{q}^{(k)}\), respectively. An illustration of \(\{Z_n\}\) and \(\{\tilde{Z}_n^{(1)}\}\) for a specific \(\omega \in \Omega\) is given in Figure 2.4. The following theorem corresponds to [35, Lemma 3.2].
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Figure 2.5: The extinction probabilities $\tilde{q}_0^{(k)}$ and $q_0^{(k)}$ in Example 1 (page 25).

Theorem 12 The sequence of vectors $\{\tilde{q}^{(k)}\}_{k \geq 0}$ is monotonically decreasing and converges pointwise to the partial extinction probability vector $\tilde{q}$.

Observe that, by construction, for any $k \geq 0$,

$$\{\omega \in \Omega : \lim_{n \to \infty} \tilde{Z}_n^{(k+1)}(\omega) = 0\} \subseteq \{\omega \in \Omega : \lim_{n \to \infty} \tilde{Z}_n^{(k)}(\omega) = 0\},$$

leading to $\tilde{q}^{(k+1)} \leq \tilde{q}^{(k)}$. The proof that $\tilde{q}^{(k)} \to q$ as $k \to \infty$ is an application of the monotone convergence theorem. We refer the reader to [35] for the detailed arguments.

Theorem 12 leads to an algorithm for computing $\tilde{q}$, that is, through successive evaluation of $\tilde{q}^{(k)}$ for large values of $k$. In Figure 2.5 we plot $q_0^{(k)}$ (bold dashed) for Example 1 on page 25 with parameter values (i) (upper panel) and (ii) (lower panel). We see that $q_0 \approx 0.976$ and $q_0 = 1$ for (i) and (ii), respectively. Note that the rate at which the entries of $\tilde{q}^{(k)}$ converge to those of $q$ remains an open problem.
Theorem 12 also leads an alternate proof of the partial extinction criterion: 
\( \tilde{q} = 1 \) if and only if \( \nu(M) > 1 \) (see Theorem 9). Let \( \tilde{M}^{(k)} = \tilde{M}^{(T_k)} \) be the \( k \times k \) north-west truncation of the infinite matrix \( M \). If we assume that \( \tilde{M}^{(k)} \) is irreducible for all \( k \geq 1 \), then, by Theorems 3 and 12, we have

\[
\tilde{q} = 1 \iff \forall k \geq 1, \quad \tilde{q}^{(k)} = 1 \iff \forall k \geq 1, \quad \rho(\tilde{M}^{(k)}) \leq 1.
\]

Through [65, Theorem 6.8], which states that \( \rho(\tilde{M}^{(k)}) \leq \nu(M) \), we then obtain the partial extinction criterion. In [71, Theorems 5.1 and 5.2] the same result is derived without the additional assumption that \( \tilde{M}^{(k)} \) is irreducible for each \( k \). Note that [65, Theorem 6.8] suggests an algorithm for testing the criterion \( \nu(M) \leq 1 \), that is, through the repeated evaluation of \( \rho(\tilde{M}^{(k)}) \). We will present other more efficient algorithms that apply to a narrower class of branching processes in Chapters 4 and 5.

### 2.6.2 Global extinction probability

To compute \( q \), we introduce the family of branching processes \( \{Z_n^{(k)}\}_{n \geq 0} \), for \( k \geq 0 \). For each fixed value of \( k \), the process \( \{Z_n^{(k)}\} \) is constructed on \( (\Omega, \mathcal{F}, \mathbb{P}) \) from the realisations of \( \{Z_n\} \) by removing all individuals of type \( i \geq k \) and their descendants, and then replacing each pruned branch with an infinite line of descent made up of type-\( \Delta \) individuals. More formally, we augment the type-set \( \mathcal{X} \) to carry an additional type-\( \Delta \), and extend the set of potential individuals \( \mathcal{J} \) to include this type. For each \( \omega \in \Omega \) and \( I \in \mathcal{J} \) we define \( N^{(k)}(I, \omega) \) with entries

\[
N_i^{(k)}(\omega, I) := \begin{cases} 
N_i(\omega, I), & i \in \{1, \ldots, k\} \\
0, & i \in \{k + 1, k + 2, \ldots\} \\
\sum_{j=k+1}^{\infty} N_j(\omega, I), & i = \Delta,
\end{cases}
\]

if \( I \) has type \( j \in \{1, \ldots, k\} \) and we let \( N^{(k)}(\omega, I) = e_\Delta \) if \( I \) has type \( j \in \Delta \cup \{k + 1, k + 2, \ldots\} \). The condition of appearance of an individual in the population and the definition of the population size vector \( Z_n^{(k)} \) are again analogous to those of \( \{Z_n\} \). As a consequence, in \( \{Z_n^{(k)}\} \),

\( (i) \) all types in \( \{1, \ldots, k\} \) have the same progeny distribution as the corresponding types in \( \{Z_n\} \), and
all types in \{k + 1, k + 2, \ldots \} are instantaneously replaced by the absorbing type \( \Delta \), which at each generation produces a single type-\( \Delta \) progeny with probability one.

Once a type-\( \Delta \) individual is born, \( \{Z_n^{(k)}\} \) does not become extinct. In this sense, individuals of type \( \Delta \) can be thought of as \textit{immortal}. An illustration of \( \{Z_n^{(k)}\} \) and \( \{Z_n^{(1)}\} \) for a specific \( \omega \in \Omega \) is given in Figure 2.4. The extinction probability vector of \( \{Z_n^{(k)}\} \), which we denote by \( q^{(k)} \), can then be interpreted as the probability that \( \{Z_n\} \) becomes extinct before a type \( i > k \) enters the population. For convenience, we omit the entry \( q_{\Delta}^{(k)} = 0 \) in \( q^{(k)} \).

Like \( \tilde{q}^{(k)} \), the vector \( q^{(k)} \) contains only finitely many unknown entries,

\[
q^{(k)} = (q_1^{(k)}, \ldots, q_k^{(k)}, 0, 0, \ldots),
\]

which, in this case, correspond to the minimal non-negative solution of the finite system of equations

\[
s_i = G_i^{(k)}(s_1, s_2, \ldots, s_k), \quad 1 \leq i \leq k,
\]

where \( G_i^{(k)}(s_1, \ldots, s_k) := G(s_1, \ldots, s_k, 0, 0, \ldots) \). This solution can again be computed using established algorithms for MGWBPs with finitely many types. The next theorem corresponds to [35, Lemma 3.1].

**Theorem 13** The sequence of vectors \( \{q^{(k)}\}_{k \geq 0} \) is monotonically decreasing and converges pointwise to the global extinction probability vector \( q \).

The proof of Theorem 13 uses the monotone convergence theorem. Let \( N_e := \inf\{n : Z_n = 0\} \) and \( \tau_k := \inf\{n : \sum_{i=k+1}^{\infty} Z_{n,i} > 0\} \). Observe that \( \{\omega : N_e < \tau_k\} \subseteq \{\omega : N_e < \tau_{k+1}\} \). Therefore, for any \( i \in \mathcal{X} \),

\[
\lim_{k \to \infty} q_i^{(k)} = \lim_{k \to \infty} \mathbb{P}_i(N_e < \tau_k)
= \mathbb{P}_i(N_e < \lim_{k \to \infty} \tau_k)
= \mathbb{P}_i(N_e < \infty)
= q_i,
\]

leading to the result.

Theorem 13 implies that \( q \) can be computed numerically through successive evaluation of \( q^{(k)} \) for large values of \( k \). In Figure 2.5 we plot \( q^{(k)} \) (solid line) for Example 1 on page 25 with parameter values (i) (upper panel) and
(ii) (lower panel). The upper panel shows that $q_0 \approx 0.976$ for parameter values (i) and the lower panel shows that $q_0 \approx 0.956$ for parameter values (ii). There is again no established method to determine the rate at which the entries of $q^{(k)}$ converge to those of $q$. However, combining Theorems 12 and 13 with the fact that $q \leq \tilde{q}$ we see that for any $k$, $\tilde{q}$ and $q$ are both caught between $\tilde{q}^{(k)}$ and $q^{(k)}$. Therefore, in examples where $q = \tilde{q}$, we can often obtain tight bounds on the value of $q$. In this thesis however, we primarily focus on processes where $q < \tilde{q}$. In this case we cannot verify the accuracy of our numerical illustrations, hence we support them with theoretical results.

It is unclear how to use Theorem 13 to make progress toward a global extinction criterion. This is due to the presence of the immortal type $\Delta$ in $\{Z_n^{(k)}\}$. In our current formulation, $\{Z_n^{(k)}\}$ is reducible and contains a singular subclass of types $\{\Delta\}$, which precludes the application of Theorem 3 (page 14), which assumed positive regularity and non-singularity. The formulation of $\{Z_n^{(k)}\}$ can be altered to satisfy the assumptions of the theorem. We could, for instance, define $\{Z_n^{(k)}\}$ so that type-$\Delta$ individuals have at least one offspring of each type in $T_k \cup \Delta$. However, due to the presence of the immortal type, the Perron-Frobenius eigenvalue of the corresponding mean progeny matrix will always be greater than 1, regardless of whether $q = 1$ or $q < 1$, making it difficult to gain insight from these values.

2.6.3 An intermediate case

To compute $\tilde{q}$ and $q$ we have considered two rules for replacing individuals of type $i > k$ in $\{Z_n\}$: replacement by a sterile type, which can be considered infinitely weak, and replacement by an immortal type, which can be considered infinitely strong. It is then natural to question what happens in-between, that is, what happens if we replace each individual of type $i > k$ with a type selected randomly from the set $\{1, \ldots, k\}$. And, if we are able to demonstrate that the corresponding extinction probability vectors converge to $q$ as $k$ increases, can we then leverage our understanding of finite-type branching processes to progress toward a global extinction criterion? We investigate these questions in Chapter 3.
2.7 Special classes of infinite-type branching processes

In this chapter we have highlighted a number of open questions about branching processes with infinitely many types. Rather than tackling these problems in full generality, a common approach is to consider a restricted class of infinite-type branching processes, for which more results can be derived. Although specific, these special classes often illuminate properties of more general infinite-type branching processes. To add context to the remaining chapters, we now summarise a selection of these classes, underscoring the comparative advantages of each.

2.7.1 Continuous-time branching random walks

In a continuous-time branching random walk, type-\(i\) individuals live for an exponentially distributed length of time with mean 1, and during their lifetimes, they give birth to type-\(j\) individuals according to independent Poisson processes with rate \(M_{ij}\). With regard to extinction, these continuous-time processes can be viewed in discrete generations, that is, we need only to consider the total number of offspring of each type born to each individual. In their discrete time analogue, type-\(i\) individuals give birth to a geometric total number of offspring with mean \(\sum_{j \in \mathcal{X}} M_{ij}\), each offspring being assigned type \(j\) independently with probability \(M_{ij} / \sum_{j \in \mathcal{X}} M_{ij}\).

These processes have been studied by a number of authors. Pemantle and Stacey [62], and Stacey [67] let \(M = \lambda K\), where \(\lambda \in \mathbb{R}_{\geq 0}\) and \(K := (K_{ij})_{i,j \in \mathcal{X}}\), and examine properties of \(K\) for which there exists \(\lambda > 0\) such that \(q < \tilde{q} = 1\). More relevant to this thesis is the work of Bertacchi and Zucca [6, 7, 8] who investigate global extinction criteria. Observe that because the offspring distribution of a continuous time branching random walk is completely specified by \(M\), an extinction criterion based solely on \(M\) must exist. The following result corresponds to [8, Corollary 4.8].

**Proposition 1** The global extinction probability vector \(q < 1\) if and only if there exists \(v \in [0, 1]^\mathcal{X}\) such that

\[
\sum_{j \in \mathcal{X}} M_{ij} v_j \geq \frac{v_i}{1 - v_i}, \quad \forall i \in \mathcal{X}.
\]
This criterion lacks a clear physical interpretation and is often difficult to verify in practice. This leads the authors to define the \textit{global extinction parameter}:

\[ \xi(M) := \sup_{i \in \mathcal{X}} \left( \liminf_{n \to \infty} \left( \sum_{j \in \mathcal{X}} M_{ij}^{(n)} \right)^{1/n} \right), \]

which, roughly speaking, is the asymptotic exponential growth rate of the mean total population size. They establish that, under some additional regularity conditions (see \cite[Proposition 4.6]{7}), \( \xi(M) < 1 \) and \( \xi(M) > 1 \) implies \( q < 1 \) and \( q > 1 \), respectively.

The authors then demonstrate that when \( \xi(M) = 1 \) it is possible to have both \( q = 1 \) and \( q < 1 \). This illustrates that, unlike the finite-type case where \( \xi(M) = \rho(M) \), when there are infinitely many types \( \xi(M) \) does not provide a suitable global extinction criterion.

### 2.7.2 Linear fractional branching processes

The key property of a linear fractional branching process is that it has a progeny generating vector which, when iterated, retains the same functional form. The definition of these processes, which is well known in the single type case (see \cite[Ch. I Sec 7.1]{31}), was extended to the infinite-type case by Sagitov and Lindo \cite{49} and Sagitov \cite{64}. Their processes are represented by the triplet \((H, g, m)\), where \((H_{ij})_{i,j \in \mathcal{X}}\) is a sub-stochastic matrix, \(g := (g_i)_{i \in \mathcal{X}}\) a non-defective distribution, and \(m\) a positive constant. In these processes, a type \(i\) individual has no offspring with probability \(1 - \sum_{j \in \mathcal{X}} h_{ij}\). Given that it has at least one offspring, the type of its first offspring is \(j\) with probability \(h_{ij}/\sum_{j \in \mathcal{X}} h_{ij}\), and the number of potential subsequent offspring has a geometric distribution with mean \(m\). These subsequent offspring are assigned a type independently according to the distribution \(g\), which is independent of the parental type. The resulting progeny generating vector has entries

\[ G_i(s) = 1 - \sum_{j \in \mathcal{X}} h_{ij} + \frac{\sum_{j \in \mathcal{X}} h_{ij}s_j}{1 + m - m \sum_{j \in \mathcal{X}} g_j s_j}, \]

which indeed has iterates that retain the same functional form, see \cite[Theorem 3]{64}.

\footnote{In their papers \(\nu(M)\) is referred to as the local extinction parameter.}
With regard to extinction, these processes display a limited range of behaviour. For example, observe that if \( m > 0 \), then \( \varrho = \tilde{\varrho} \). The advantage of considering linear fractional branching processes is that \( G(\cdot) \), and therefore \( P_i(Z_n = 0) \), has a neat explicit expression for each \( n \geq 0 \) and \( i \in \mathcal{X} \). This is of particular interest to us because other than \([35]\), we are unaware of other authors who have considered computational aspects of the infinite vectors \( \varrho \) and \( \tilde{\varrho} \).

Aside from computational advantages, we note that many results for general single type Galton–Watson processes are derived via comparison with linear fractional branching processes (recall Agresti’s derivation of Theorem 11). Generalising the definition of a linear fractional branching process to the infinite-type setting thus presents an opportunity to extend these arguments to the infinite-type case. This is another promising approach, which we do not pursue in this thesis.

### 2.7.3 Branching random walks in a random environment

The primary feature of a branching random walk in a random environment (BRWRE) is that the entries of the progeny generating vector \( G(\cdot) \) are determined independently at random. Unlike a branching process in a random environment, in a BRWRE, after the entries of \( G(\cdot) \) are selected, they remain fixed throughout the lifetime of the process, which then evolves according to standard rules. These processes are generally studied under the assumption of an *independent and identically distributed random environment for each type* (see \([18]\)). This assumption is satisfied for example if \( \mathcal{X} = \mathbb{Z} \) and

\[
G_i(s) = \begin{cases} 
  s_{i-1} & \text{w.p. } 1/2 \\
  s_i s_{i+1} & \text{w.p. } 1/2,
\end{cases}
\]

independently of all other entries \( G_j(s) \), \( j \in \mathcal{X} \). Note that, in a BRWRE, the vectors \( \varrho \) and \( \tilde{\varrho} \) are defined conditionally on the outcome of \( G(\cdot) \), and are therefore random.

These processes have been studied by a number of authors. Comets and Popov \([18]\), and Müller \([61]\) prove that

\[
\tilde{\varrho} < 1 \text{ a.s. } \Rightarrow \tilde{\varrho} = \varrho \text{ a.s.}
\]

Comets and Popov \([18]\) prove that either \( \tilde{\varrho} < 1 \text{ a.s.} \) or \( \tilde{\varrho} = 1 \text{ a.s.} \) and derive a partial extinction criterion that depends only on the support of the
environmental law. Under additional conditions, Gantert et al. [27] prove that either \( q < 1 \) a.s. or \( q = 1 \) a.s. and derive a global extinction criterion that is expressed as an infinite product of random matrices.

In the remainder of this thesis we introduce a number of assumptions, and through examples we show that the purpose of these assumptions is to guard against pathologies displayed by infinite-type branching processes. The fact that these pathologies play no role in the analysis of branching random walks in a random environment indicates these pathologies have probability 0 of arising. Thus, by randomising \( G(\cdot) \) the authors are, in some sense, imposing a number of regularity conditions on the process.

### 2.7.4 Branching random walks with an absorbing barrier

A branching random walk with an absorbing barrier can be defined as an infinite-type branching process whose progeny generating vector \( G(\cdot) \) satisfies the system of equations

\[
G_{i-1}(s_0, s_1, s_2, \ldots) = G_i(1, s_0, s_1, s_2, \ldots) \quad \text{for all } i \geq 1. \tag{2.25}
\]

To interpret Equation (2.25), it is helpful to think of each type as a location on the non-negative integers. Relative to its location, Equation (2.25) implies that each location has the same offspring distribution with the additional condition that individuals with a negative location are immediately killed. For this reason we say that there is an absorbing barrier at \(-1\). These processes are introduced in [11], where a partial extinction criterion is derived. The more recent papers [25] and [42] construct a global extinction criterion when the location of the absorbing barrier is a deterministic function of the generation.

Our interest in branching random walks with an absorbing barrier centres around Example 1 on page 25, which satisfies (2.25). In this thesis, we consider a number of variants of Example 1, each emphasising a different property that an infinite-type branching process can display. In these examples we make use of the following results, which we are unable to find in the literature.

**Proposition 2** In a branching random walk with an absorbing barrier \( \{q_i\}_{i \geq 0} \) and \( \{\tilde{q}_i\}_{i \geq 0} \) are non-increasing sequences.

**Proof:** Let \( A = \{1, 2, 3, \ldots\} \). By (2.25), for any fixed \( i \geq 1 \), we have \( G_{i-1}(s) = G_i(1, s_0, s_1, s_2, \ldots) \), and by Equation (2.24) we have \( \tilde{G}_i^{(A)}(s) = G_i(1, s_1, s_2, s_3, \ldots) \). Thus, up to a relabelling of types, the processes \( \{Z_n\} \)
and \( \{(\tilde{Z}_{n}^{(A)})_{i\in A}\} \) are stochastically equivalent, that is, if we consider the processes \( \{Z_n : \varphi_0 = i\} \) and \( \{\tilde{Z}_n(A) : \varphi_0 = i + 1\} \) then
\[
Z_{n,i} \overset{d}{=} \tilde{Z}_{n,i+1}^{(A)}
\]
for all \( n \geq 0 \) and \( i \geq 0 \). If \( \tilde{q}^{(p,A)} \) and \( \tilde{q}^{(A)} \) denote the partial and global extinction probability vectors of \( \tilde{Z}_{n,i+1}^{(A)} \), respectively, then \( \tilde{q}_i = \tilde{q}_{i+1}^{(p,A)} \leq \tilde{q}_{i+1} \) and \( q_i = \tilde{q}_{i+1}^{(A)} \leq q_{i+1} \).

\[\text{Corollary 2}\] In an irreducible branching random walk with an absorbing barrier, if \( \inf_{i\in\chi} q_i > 0 \) then \( \tilde{q} < 1 \) implies \( q = \tilde{q} \).

**Proof:** By Proposition 2, when \( \tilde{q} < 1 \) the entries of \( \tilde{q} \) are uniformly bounded from 1. Thus, by Corollary 1 on page 28 we have \( q = \tilde{q} \). \[\Box\]
Chapter 3

Random replacement

3.1 Introduction

In the previous chapter we stated that for multitype Galton-Watson branching processes (MGWBPs) with countably infinitely many types there is no established global extinction criterion and the computational aspects of the global extinction probability vector $\mathbf{q}$ have received little attention. In this chapter we address the two problems in parallel by defining two new probabilistic tools: to each MGWBP with countably many types $\{Z_n\}$, we associate

(i) a sequence of truncated and augmented finite-type branching processes $\{\tilde{Z}_n^{(k)}\}_{k \geq 1, n \geq 0}$, which themselves naturally define

(ii) an embedded branching process $\{S_k\}$ referred to as the seed process. The next two paragraphs provide an intuitive description of these two tools and their benefits.

For each $k \geq 1$, the $k$th finite-type branching process $\tilde{Z}_n^{(k)}$ is constructed pathwise on the same probability space as the original process $\{Z_n\}$ by replacing all types larger than $k$ with a type randomly selected from the set $\{1, \ldots, k\}$ according to some distribution $\alpha^{(k)}$. The corresponding (finite) extinction probability vector is denoted by $\tilde{\mathbf{q}}^{(k)}$. In the main theorem of this chapter (Theorem 14), we prove that, under some sufficient conditions on $\{Z_n\}$ (closely related to the dichotomy between extinction and unbounded growth) and on the sequence of replacement distributions $\{\alpha^{(k)}\}_{k \geq 1}$ (similar to a tightness condition), the sequence $\{\tilde{\mathbf{q}}^{(k)}\}_{k \geq 1}$ converges to the global extinction probability $\mathbf{q}$. This result establishes a link between the probability of extinction of non-singular irreducible MGWBPs with finitely many types and the probability of global extinction in the infinite-type setting. It has several implications. First, Theorem 14 extends the work in [35] (detailed in
Section 2.6, in which two monotone sequences of extinction probability vectors, \( \{q^{(k)}\}_{k \geq 1} \) and \( \{\tilde{q}^{(k)}\}_{k \geq 1} \), are shown to converge respectively to \( q \) and \( \tilde{q} \) via the monotone convergence theorem. In contrast, the new sequence \( \{\bar{q}^{(k)}\} \) is not necessarily monotone, and a completely different approach is required. Second, as a direct consequence of Theorem 14, we are able to apply known results on MGWBPs with a finite number of types to derive new sufficient conditions for \( q = 1 \) and \( \tilde{q} < 1 \) in the infinite-type setting. For the reasons given in Section 2.6, such results could not be obtained using the sequences \( \{q^{(k)}\} \) and \( \{\tilde{q}^{(k)}\} \).

The seed process \( \{S_k\} \) is an MGWP evolving in a varying environment, that arises naturally when exploring the asymptotic behaviour of \( \{\bar{q}^{(k)}\} \). It is constructed pathwise from the family of finite-type processes \( \{Z_n^{(k)}\} \) as follows: the individuals (or seeds) in the \( k \)th generation of \( \{S_k\} \) correspond to the individuals in \( \{Z_n\} \) which are replaced by a random type according to \( \alpha^{(k)} \) to form \( \{\bar{Z}_n^{(k)}\} \). The seed process is the fundamental ingredient in the proof of Theorem 14 and in addition, enjoys several interesting properties on its own. For example, \( \{S_k\} \) almost surely becomes extinct if and only if global and partial extinction of the original process coincide. While in the present chapter our interest in the seed process remains its application to the sequence \( \{\bar{q}^{(k)}\} \), we lay the foundations for Chapter 4, in which properties of the seed process are exploited further, to yield, among other results, a global extinction criterion that applies to a class of branching processes referred to as lower Hessenberg.

For completeness we also investigate the convergence properties of the sequence \( \{\bar{q}^{(k)}\} \) when the conditions in Theorem 14 on the replacement distributions \( \{\alpha^{(k)}\} \) are not met. We consider (a) replacement by the last type, that is, \( \alpha^{(k)} = e_k \), and (b) replacement by a uniformly distributed type, that is, \( \alpha^{(k)} = 1/k \) and, one particular example that focuses on each case, Examples 3 and 4, respectively. In Example 3, we prove that the limit of the sequence \( \{\tilde{q}^{(k)}\} \) does not always exist, and in Example 4, the limit does exist but may correspond to the partial extinction probability \( \tilde{q} \). Example 3 highlights the sensitivity of the limit of \( \{\tilde{q}^{(k)}\} \) under (a), whereas Example 4 demonstrates how alternative choices of \( \{\alpha^{(k)}\} \) may lead to contrasting asymptotic behaviour in \( \{\bar{q}^{(k)}\} \).

This chapter is organised as follows. In Section 3.2 we focus on the construction of the branching process \( \{Z_n^{(k)}\} \) on \( (\Omega,\mathcal{F},\mathbb{P}) \). In Section 3.3 we
establish sufficient conditions for the convergence of \( \{ \bar{q}^{(k)} \} \) to \( q \), we study properties of the related seed process, and we prove the main theorem on the convergence of \( \{ \bar{q}^{(k)} \} \). In Section 3.4 we derive sufficient conditions for \( q = 1 \) and \( q < 1 \). In Section 3.5 we study the asymptotic behaviour of \( \{ \bar{q}^{(k)} \} \) for replacement distributions that do not satisfy the conditions of our main theorem and provide some numerical illustrations. In Section 3.6 we prove the results given in Section 3.5 on Examples 3 and 4. Finally, in Section 3.7 we provide the pseudo-code for the computation of the global and partial extinction probabilities and discuss the computational efficiency of our algorithms.

Unless stated otherwise, in this chapter we let \( \mathcal{X} = \{1, 2, 3, \ldots \} \) and \( T_k = \{1, 2, \ldots , k\} \).

### 3.2 Pathwise construction

For each \( k \geq 1 \), we construct recursively the truncated and augmented branching process \( \{ \bar{Z}_{n}^{(k)} \}_{n \geq 0} \) for which

1. all types in \( T_k \) have the same progeny distribution as the corresponding types in \( \{ \bar{Z}_n \} \), and
2. all types in \( \bar{T}_k \) are instantaneously, and independently of each other, replaced by type \( \mathcal{X} \in \{1, \ldots , k\} \) which is selected using the probability distribution \( \alpha^{(k)} \). The replaced individuals then generate new individuals according to the progeny distribution of their type.

To construct the sample paths of \( \{ \bar{Z}_{n}^{(k)} \}_{n \geq 0} \) we first augment the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) to carry the sequence of independent random variables \( \{X_{l}^{(k)}(I)\}_{l \in \mathbb{N}, I \in \mathcal{I}} \). For each \( k \geq 1 \), these random variables take values in \( \{1, \ldots , k\} \) and have probability distribution \( \alpha^{(k)} = (\alpha_{1}^{(k)}, \alpha_{2}^{(k)}, \ldots , \alpha_{k}^{(k)}) \), where \( \alpha_{i}^{(k)} = \mathbb{P}(X_{l}^{(k)}(I) = i) \). We interpret \( X_{l}^{(k)}(\omega, I) \) as the replacement type of the \( l \)th offspring of type strictly larger than \( k \) born to \( I \) for the realisation \( \omega \). Let \( N_{(k, \infty)}(\omega, I) = \sum_{j=k+1}^{\infty} N_{j}(\omega, I) \) and define \( \bar{N}^{(k)}(\omega, I) \) with entries

\[
\bar{N}_{i}^{(k)}(\omega, I) = \begin{cases} 
N_{i}(\omega, I) + \sum_{j=1}^{N_{(k, \infty)}(\omega, I)} 1\{X_{j}^{(k)}(\omega, I) = i\}, & 0 \leq i \leq k \\
0, & i > k.
\end{cases}
\]

The condition of appearance of an individual in the truncated branching process is then the same as (2.1), replacing \( r_{lj} = N_{j}(\omega, I) \) by \( r_{lj} = \bar{N}_{j}^{(k)}(\omega, I) \),
Lemma 1 For any $k \geq 1$ and replacement distribution $\alpha^{(k)}$, 

$$q^{(k)} \leq \bar{q}^{(k)} \leq \hat{q}^{(k)}.$$
3.3. Convergence to global extinction

Proof. From the pathwise construction of the branching processes, it is clear that
\[ \{ \omega : \lim_{n \to \infty} |Z_n^{(k)}(\omega)| = 0 \} \subseteq \{ \omega : \lim_{n \to \infty} |\hat{Z}_n^{(k)}(\omega)| = 0 \} \subseteq \{ \omega : \lim_{n \to \infty} |\tilde{Z}_n^{(k)}(\omega)| = 0 \}, \]
and the result follows. \[\square\]

Corollary 3  For any sequence \( \{ \alpha^{(k)} \} \) of replacement distributions,
\[
q \leq \liminf_{k \to \infty} \tilde{q}^{(k)} \leq \limsup_{k \to \infty} \tilde{q}^{(k)} \leq \hat{q}.
\]

Proof. By Theorems 12 and 13 we have \( q^{(k)} \to q \) and \( \tilde{q}^{(k)} \to \tilde{q} \). The result then follows from Lemma 1. \[\square\]

A consequence of Corollary 3 is that, when it exists, the limit of the sequence \( \{ \tilde{q}^{(k)} \} \) can only overestimate the probability of global extinction. In Section 3.3.1, we illustrate a situation where \( q < \lim_{k \to \infty} q^{(k)} < \tilde{q} \).

3.3 Convergence to global extinction

In this section, we assume that the sequence of replacement distributions \( \{ \alpha^{(k)} \} \) satisfies a property slightly more general than tightness, that is,

Assumption 1 There exist constants \( N_1, N_2 \geq 1 \) and \( a > 0 \), all independent of \( k \), such that
\[
\min\{N_1,k\} \sum_{i=1}^{\min\{N_1,k\}} \alpha_i^{(k)} \geq a \quad \text{for all } k \geq N_2.
\]

Replacement distributions that do not satisfy Assumption 1 include replacement by type \( k \), \( \alpha^{(k)} = e_k \), and replacement by a uniformly distributed type, \( \alpha^{(k)} = 1/k \). These special cases will be considered in Section 3.5. Replacement with a fixed type, however, satisfies Assumption 1; for example, when \( \alpha^{(k)} = e_1 \), it holds with \( N_1 = 1, N_2 = 1 \) and \( a = 1 \). An example of sequence of replacement distributions satisfying Assumption 1 but which is not tight is \( \alpha^{(k)} = (a, 0, \ldots, 0, 1 - a) \) for some \( 0 < a < 1 \).
3.3.1 A motivating example

In the next example we demonstrate that Assumption 1 alone is not sufficient to ensure $\bar{q}^{(k)} \to q$ as $k \to \infty$.

**Example 2.** Consider a two-parameter irreducible branching process $\{Z_n\}$ with countably many types where type-1 individuals produce a single type-2 individual with probability $a > 0$ and no offspring with probability $1 - a$, and each type-$i \in \{2, 3, \ldots\}$ individual produces a single type-$(i + 1)$ offspring with probability one and a further Poisson($b^{i-1}$) type-1 individuals, where $0 < b < 1$. The progeny generating vector of this process contains entries

$$G_1(s) = a s_2 + 1 - a,$$

and

$$G_i(s) = \sum_{k \geq 0} \frac{(b^{i-1})^k}{k!} e^{-b^{i-1}} s_1^k s_{i+1} = \exp\{b^{i-1}(s_1 - 1)\} s_{i+1},$$

for $i \geq 2$. The corresponding mean progeny representation graph is illustrated in Figure 3.2.

From the definition of the process it is clear that the global extinction probability vector $q$ contains entries $q_1 = 1 - a$ and $q_i = 0$ for all $i \geq 2$. We assume that the population initially contains a single individual of type 1. Because $\{Z_n\}$ is irreducible, the probability of partial extinction $\bar{q}_1$ is equal to the extinction probability of the embedded type-1 process, $\{E^{(1)}_n(Z)\}$ (see page 24 for the definition of $\{E^{(1)}_n(Z)\}$). With reference to Figure 3.2, we observe that in this embedded process individuals have no offspring with probability $1 - a$, otherwise they give birth according to the sum of countably infinitely many independent Poisson random variables with respective means $b^{i-1}$, for $i \geq 2$. Consequently, $\bar{q}_1$ is the minimal nonnegative solution of

$$x = 1 - a + aF(x), \quad \text{for } 0 \leq x \leq 1,$$
3.3. Convergence to global extinction

Figure 3.3: Differences in the extinction probabilities related to Example 2, plotted as a function of the parameters $a$ and $b$.

where $F(\cdot)$ is given by

$$F(x) = \prod_{i=2}^{\infty} \exp\{b^{i-1}(x-1)\} = \exp\{b(1-b)^{-1}(x-1)\}. \quad (3.1)$$

Note that $F(\cdot)$ is the probability generating function of a Poisson random variable with mean $b/(1-b)$. The corresponding mean offspring $m_{E_1^{(1)}(Z)} = ab/(1-b)$ indicates that $\hat{q}_1 = 1$ if and only if $a \leq (1-b)/b$, in which particular cases $1-a = q_1 < \hat{q}_1$. The top panel in Figure 3.3 shows the difference $\hat{q}_1 - q_1$ as a function of the parameter values.

Now consider the process $\{Z_n^{(k)}\}$ with replacement distribution $\alpha^{(k)} = e_1$, and its conditional extinction probability $q_1^{(k)}$, given that $\varphi_0 = 1$. This irreducible branching process has finitely many types, hence it becomes extinct if and only if type 1 becomes extinct. Thus, $q_1^{(k)}$ corresponds to the extinction probability of the type-1 process embedded with respect to $\{Z_n^{(k)}\}$. The progeny generating function of $\{E_n^{(1)}(Z^{(k)})\}$, that we denote by $G_{1,k}(\cdot)$, is
given by

\[ G_{1,k}(x) = 1 - a + aF_k(x)x, \quad \text{for } 0 \leq x \leq 1, \]

where

\[ F_k(x) = \prod_{i \geq 2} \exp\{b^{-1}(x - 1)\} = \exp\{b(1 - b^{k-1})(1 - b)^{-1}(x - 1)\} \]

is the probability generating function of a Poisson random variable with parameter \( b(1 - b^{k-1})(1 - b)^{-1} \). Note that here we multiply \( aF_k(x) \) by \( x \) to account for the type-(\( k-1 \)) descendant (instantaneously replaced by type 1) of each type-1 individual that has a type-2 offspring. By continuity of \( G_{1,k}(\cdot) \), the limit \( \bar{q}_1 \) is the minimal nonnegative solution of

\[ x = 1 - a + aF(x)x, \]

where \( F(x) \) is given in (3.1). The corresponding mean progeny \( m_{1,\infty} = G_{1,\infty}^1(1) = a(1 + b/(1 - b)) = a + m_{E_k(1)}(z) \) indicates that \( \bar{q}_1 = 1 \) if and only if \( a \leq 1 - b \), in which particular case \( \bar{q}_1 > q_1 = 1 - a \). The middle panel in Figure 3.3 shows the difference \( \bar{q}_1 - q_1 \) as a function of the parameter values. This highlights the fact that the sequence \( \{q^{(k)}\} \) does not always converge to the global extinction probability \( q \).

For completeness, in the bottom panel of Figure 3.3 we plot the difference \( \bar{q}_1 - \bar{q}_1 \) as a function of the parameter values. From the arguments above, if \( a \leq 1 - b \) then \( q_1 < \bar{q}_1 = \bar{q}_1 = 1 \), so the sequence \( \{q^{(k)}\} \) can potentially converge to the partial extinction probability.

3.3.2 The seed process

Example 2 illustrates the need to further explore the conditions under which \( \{\bar{q}^{(k)}\} \) converges to \( q \) as \( k \to \infty \). Observe that, for any \( k \geq 1 \), we have

\[
\bar{q}^{(k)} = P_i \left( \lim_{n \to \infty} |\bar{Z}^{(k)}_n| = 0 \right) \\
= P_i \left( \lim_{n \to \infty} |\bar{Z}^{(k)}_n| = 0, \lim_{n \to \infty} |Z^{(k)}_n| = 0 \right) \\
+ P_i \left( \lim_{n \to \infty} |\bar{Z}^{(k)}_n| = 0, |S_k| \geq 1 \right) \\
= q^{(k)} + \sum_{x \geq 1} \left( \alpha^{(k)} q^{(k)} \right)^x P_i (|S_k| = x), \tag{3.2}
\]

where \( |S_k| \) denotes the (finite) number of sterile types produced over the lifetime of \( \{\bar{Z}^{(k)}_n\} \), which are replaced by some random types in \( \{Z^{(k)}_n\} \) and
immortal types in \( \{ Z_n^{(k)} \} \). To understand Equation (3.2) one may think of simulating the branching processes with \( \varphi_0 = i \) in two stages: by first constructing the path of \( \{ \tilde{Z}_n^{(k)} \} \), and then constructing those of \( \{ Z_n^{(k)} \} \) and \( \{ \tilde{Z}_n^{(k)} \} \) by taking the outcome of \( \{ \tilde{Z}_n^{(k)} \} \), replacing the sterile individuals, and simulating their daughter processes according to the respective replacement and updating rules. Conditional on the first stage of simulation, there are two ways in which \( \{ Z_n^{(k)} \} \) can die, either: (i) \( \{ \tilde{Z}_n^{(k)} \} \) dies before producing a sterile type, in which case \( \{ Z_n^{(k)} \} \) also dies (this occurs with probability \( q_i^{(k)} \)), or (ii) \( \{ \tilde{Z}_n^{(k)} \} \) dies after producing \( 1 \leq x < \infty \) sterile individuals, in which case \( \{ Z_n^{(k)} \} \) dies with probability \( (\alpha^{(k)} \bar{q}^{(k)})^x \) in the second stage of simulation.

Because \( q^{(k)} \rightarrow q \), this generally indicates that in order for \( \{ \bar{q}^{(k)} \} \) to converge to \( q \), we need to avoid cases where there is a positive probability that the number of sterile individuals produced over the lifetime of \( \{ \tilde{Z}_n^{(k)} \} \) remains positive and uniformly bounded for all \( k \). This is not satisfied in Example 2 as, for any \( a < 1 \), \( 0 < b < 1 \) and for all \( k \geq 1 \),

\[ \mathbb{P}_1 (|S_k| = 1) > aF(0) > 0. \]

We defer a formal statement of this idea until Lemma 4 and now formally introduce the seed process \( \{ S_k \}_{k \geq 0} \), defined from the paths of \( \{ \tilde{Z}_n^{(k)} \} \).

**Definition 1** The seed process \( \{ S_k = (S_{k,1}, S_{k,2}, ...) \}_{k \geq 0} \) defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) is such that for any \( \omega \in \Omega \), \( S_0(\omega) = Z_0(\omega) \), and for \( k \geq 1 \), if \( \lim_{n \rightarrow \infty} \tilde{Z}_n^{(k)}(\omega) = 0 \), then

\[
S_{k,i}(\omega) = \begin{cases} 
0, & \text{if } i \leq k, \\
\sum_{n=1}^{\infty} \tilde{Z}_n^{(k)}(\omega), & \text{if } i > k,
\end{cases}
\]

(3.3)

otherwise,

\[
S_{k,i}(\omega) = 0 \quad \text{for all } i \geq 1.
\]

(3.4)

We take the convention that \( |S_k| = 0 \) when \( \{ \tilde{Z}_n^{(k)} \} \) does not become extinct in order to ensure that \( \mathbb{P}_1 (|S_k| < \infty) = 1 \) for all fixed \( i \) and \( k \). It is not hard to show that \( \{ S_k \} \) forms a Markov chain on \( R_X \) (see page 8 for the definition of \( R_X \)). More precisely, \( \{ S_k \} \) is a branching process with countably many types, in which the progeny distribution depends on the generation, and which is subject to total catastrophe, where every individual in the population is killed in a single generation. Such a total catastrophe happens at generation \( k + 1 \) in the seed process for some \( \omega \in \Omega \), if \( \tilde{Z}_n^{(k)}(\omega) \) becomes extinct while \( \tilde{Z}_n^{(k+1)}(\omega) \)
survives. For a given $\omega \in \Omega$, the sequence $\{S_k(\omega)\}$ admits a unique tree representation that can be constructed from the tree of $\{Z_n(\omega)\}$. This is illustrated in Figure 3.4. The nodes of the tree of the seed process will be called seeds. Observe that in the specific setting of a nearest neighbour branching random walk with $\tilde{q} = 1$, the seed process reduces to the first modified process used in the proof of [27, Theorem 2.9]. In our generalised construction, when $\{Z_n\}$ dies, individuals in the $k$th generation of the seed process with type strictly greater than $k$ produce only one exact copy of themselves, and the number of generations $\{S_k\}$ lives is equivalent to the largest type produced in $\{Z_n\}$.

The seed process enjoys several other properties which will be exploited to prove Theorem 14 on the convergence of $\{q^{(k)}\}$ to $q$ stated in the next subsection.

**Lemma 2** The state $0$ is absorbing for the seed process $\{S_k\}$.

**Proof.** Suppose $S_k(\omega) = 0$ for some $\omega \in \Omega$. Then either $\liminf_{n \to \infty} |\tilde{Z}_n^{(k)}(\omega)| > 0$ or $\lim_{n \to \infty} |Z_n^{(k)}(\omega)| = 0$. In addition, by construction,

$$\{\omega : \liminf_{n \to \infty} |Z_n^{(k)}(\omega)| > 0\} \subseteq \{\omega : \liminf_{n \to \infty} |Z_n^{(k+1)}(\omega)| > 0\}$$

and

$$\{\omega : \lim_{n \to \infty} |Z_n^{(k)}(\omega)| = 0\} \subseteq \{\omega : \lim_{n \to \infty} |Z_n^{(k+1)}(\omega)| = 0\},$$

which implies $S_{k+1}(\omega) = 0$. 

\[\square\]
Additionally, we obtain an expression for the probability that \( \{S_k\} \) has reached the absorbing state by generation \( k \) in terms of the extinction probabilities of \( \{Z_n^{(k)}\} \) and \( \{Z_n^{(k)}\} \):

**Lemma 3** \( P_i(\{|S_k| = 0\}) = 1 - q_i^{(k)} + q_i^{(k)} \).

**Proof.** By the same argument as in the proof of Lemma 2,

\[
P_i(\{|S_k| = 0\}) = P_i \left( \{\liminf_{n \to \infty} |Z_n^{(k)}| > 0\} \cup \{\lim_{n \to \infty} |Z_n^{(k)}| = 0\} \right),
\]

where the two events are mutually exclusive.

This provides us with a condition for the global and partial extinction probabilities to coincide,

**Corollary 4** For all \( i \geq 1 \), the following two statements are equivalent

(i) \( q_i = \tilde{q}_i \)

(ii) \( P_i \left( \lim_{k \to \infty} |S_k| = 0 \right) = 1 \).

We rewrite equation (3.2) as

\[
q_i^{(k)} - \tilde{q}_i^{(k)} = E_i \left( (\alpha_i^{(k)} q_i^{(k)})^{|S_k|} - 1 \{|S_k| = 0\} \right). \tag{3.5}
\]

The next lemma formalises the discussion preceding Definition 1.

**Lemma 4** Assume that, for some \( i \in S \) and \( \pi > 0 \) there exists \( B < \infty \) such that

\[
\liminf_{k \to \infty} P_i(0 < |S_k| < B) = \pi.
\]

If, in addition, \( \{\alpha_i^{(k)}\} \) satisfies Assumption 1 for some \( N_1 \) such that \( q_j > 0 \) for all \( j \in \{1, \ldots, N_1\} \), then \( \liminf_{k \to \infty} \tilde{q}_i^{(k)} > q_i \).

**Proof.** Because \( q_j^{(k)} \to q_j > 0 \) for all \( j \in \{1, \ldots, N_1\} \), there exists \( \beta > 0 \) and \( K \in \mathbb{N} \) such that, for all \( k > K \) and \( j \in \{1, \ldots, N_1\} \), \( q_j^{(k)} \geq \beta \). By Lemma 1, we also have \( \tilde{q}^{(k)} > q^{(k)} \) for all \( k \in \mathbb{N} \). Hence

\[
\tilde{q}_j^{(k)} \geq \beta \text{ for all } k > K \text{ and all } j \in \{1, \ldots, N_1\}.
\]

It follows from Assumption 1 that for any \( k > \max\{K, N_1, N_2\} \),

\[
\alpha^{(k)} q^{(k)} \geq \sum_{j=1}^{N_1} \alpha_j^{(k)} \tilde{q}_j^{(k)} \geq \beta a > 0.
\]
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Then, by (3.5),
\[
\liminf_{k \to \infty} \left( q_i^{(k)} - q_i^{(k)} \right) = \liminf_{k \to \infty} \mathbb{E}_i \left( \left( \alpha^{(k)} q^{(k)} \right) |S_k| - 1 \{ |S_k| = 0 \} \right) \\
\geq \liminf_{k \to \infty} \mathbb{E}_i \left( \left( \alpha^{(k)} q^{(k)} \right) |S_k| \right) |S_k| < B \right) \mathbb{P}_i \left( 0 < |S_k| < B \right) \\
\geq (\beta a)^B \pi > 0,
\]

which completes the proof. □

Lemma 4 suggests that the conditions we impose for \( q^{(k)} \to q \) should also be sufficient for \( \{|S_k|\} \) to display the dichotomy between extinction and unbounded growth, that is, with probability 1 either \( |S_k| \to 0 \) as \( k \to \infty \), or there exists an integer \( n(\omega) \) for which \( |S_k(\omega)| = 0 \) for all \( k \geq n(\omega) \). We impose a condition similar to, but more general than, the well known sufficient condition ‘\( \inf \delta_i > 0 \)' for \( \{|S_k|\} \) to display the dichotomy between extinction and unbounded growth that we gave in Theorem 4 on page 16.

**Assumption 2** \( \liminf_i q_i > 0 \).

Observe that Assumption 2 is satisfied when \( \liminf_i p_i(0) > 0 \).

**Lemma 5** Suppose Assumption 2 holds, then for all \( i \in S \)
\[
\mathbb{P}_i \left( |S_k| \to 0 \text{ or } \infty \right) = 1.
\]

**Proof.** By Assumption 2 there exist \( N_3 \in \mathbb{N} \) and \( \beta > 0 \) such that \( q_i > \beta \) for all \( i > N_3 \), and by Lemma 3, \( \mathbb{P}_i \left( \lim_{k \to \infty} |S_k| = 0 \right) \geq q_i \) for all \( i \). Thus for all \( 1 \leq i \leq N_3 \),
\[
\mathbb{P}_i \left( \lim_{k \to \infty} |S_k| = 0 \right) \geq \mathbb{E}_i \left( \prod_{j=1}^{\infty} q_j^{S_k-N_3} \right) \geq \mathbb{E}_i \left( \beta^{S_k-N_3} \right) := \delta_i > 0,
\]

and for all \( i > N_3 \), \( \mathbb{P}_i \left( \lim_{k \to \infty} |S_k| = 0 \right) \geq \beta \). Up to the possibility of a total catastrophe, the individuals in \( \{S_k\} \) behave independently, hence for any \( |s_0| \leq x \), we have
\[
\mathbb{P} \left( \lim_{k \to \infty} |S_k| = 0 \mid |S_0| = s_0 \right) \geq \left( \min_{1 \leq i \leq N_3} \delta_i, \beta \right)^x > 0. \quad (3.6)
\]

Combining Lemma 2 and Equation (3.6) with the fact that \( \{S_k\} \) is a Markov chain, the result then follows from [46, Theorem 2]. □
3.3. Convergence to global extinction

In specific cases, the extinction probability of the seed process can be easier to analyse than that of the original branching process. In the next chapter we consider one such subclass of branching processes called lower Hessenberg where, by building upon the results of the present section, we are able to analyse the set of fixed points of the original process and derive necessary and sufficient conditions for its almost sure global extinction.

3.3.3 Convergence to global extinction

In this section, we state our result on the pointwise convergence of the sequence \( \{q^{(k)}\} \) to the global extinction probability \( q \). To obtain convergence, Equation (3.5) suggests that, conditionally on \( \mathbb{P}(|S_k| \to \infty) > 0 \), one must show that \( \alpha^{(k)} q^{(k)} \) is bounded away from 1 for all sufficiently large \( k \). To prove this, we use a regenerative argument, which may break down for some replacement distributions \( \alpha^{(k)} \), such as the ones presented in the next section.

For a fixed \( k \), each seed in \( S_k \) corresponds to a sterile individual produced over the lifetime of \( \tilde{Z}_n^{(k)} \). To obtain \( \tilde{Z}_n^{(k)} \), these seeds are replaced, independently of each other, by new individuals whose types follow the distribution \( \alpha^{(k)} \), and whose daughter processes themselves may be thought of as producing an i.i.d. number of new seeds, and so on. Thus, the process formed by taking all replaced individuals from \( \tilde{Z}_n^{(k)} \) which correspond to seeds, and connecting each of these individuals to its nearest replaced seed ancestor in \( \tilde{Z}_n^{(k)} \), is a multitype Galton-Watson process on \( \Omega, \mathcal{F}, \mathbb{P} \) with type space \( T_k = \{1, \ldots, k\} \). We refer to this process as the embedded replacement process, and denote it as \( \tilde{Z}_n^{(e,k)} \). Each child’s type is chosen independently of the type of its parent and other siblings, and therefore the corresponding progeny generating function \( G^{(e,k)} : [0, 1]^{T_k} \to [0, 1]^{T_k} \) is such that

\[
G_i^{(e,k)}(s) = \sum_{x \geq 0} \left( \alpha^{(k)} x \right)^s \mathbb{P}_i(|S_k| = x), \quad i \in T_k. \tag{3.7}
\]

We use the convention that \( Z_0^{(e,k)} = \tilde{Z}_0^{(k)} \), that is, we include the initial individual in \( Z_0^{(e,k)} \) regardless of whether it has been replaced. The embedded replacement process can be constructed pathwise for each \( \omega \in \Omega \), but we omit the details here. Conditional on the initial type \( \varphi_0 \in T_k \), for each \( \omega \in \Omega \) we have \( |Z_1^{(e,k)}(\omega)| = |S_k(\omega)| \). Figure 3.5 gives an illustration of the construction of \( \{Z_n^{(e,k)}\} \) when compared to the corresponding realisation of \( \{\tilde{Z}_n^{(k)}\} \) when \( k = 4 \) and \( \alpha^{(4)} = e_1 \). In Figure 3.5 the type-2 root is common to both
Figure 3.5: A realisation of \( \{\tilde{Z}_n^{(4)}\} \) (left) with the corresponding realisation of the \( \{Z_n^{(e,4)}\} \) (right) when \( \alpha^{(k)} = e_1 \). The black nodes represent individuals with type greater than 4 which are immediately replaced with type 1.

processes and the black type-1 nodes represent individuals that have been replaced.

It is clear that if the embedded replacement process does not become extinct then neither does \( \{\tilde{Z}_n^{(k)}\} \). Thus,

\[
\tilde{q}_j^{(k)} \leq q_j^{(e,k)} \quad \text{for all } j \in T_k.
\]

(3.8)

where \( q_j^{(e,k)} \) is the extinction probability vector of \( \{Z_n^{(e,k)}\} \). We are now in a position to prove our result on the pointwise convergence of the sequence \( \{\tilde{q}^{(k)}\} \) to the global extinction probability \( q \).

**Theorem 14** Suppose Assumption 2 holds. If the sequence of replacement distributions \( \{\alpha^{(k)}\} \) satisfies Assumption 1 with \( N_1 \) such that either

(i) \( \tilde{q}_j < 1 \) for all \( j \in \{1, \ldots, N_1\} \), or

(ii) \( \tilde{q}_j = 1 \) for all \( j \in \{1, \ldots, N_1\} \), and there is a path from any \( j \in \{1, \ldots, N_1\} \) to the initial type \( i \),

then

\[
\lim_{k \to \infty} \tilde{q}_i^{(k)} = q_i.
\]

In particular, if \( \{Z_n\} \) is irreducible, then under Assumptions 1 and 2,

\[
\lim_{k \to \infty} q^{(k)} = q.
\]
3.3. Convergence to global extinction

Proof. By (3.5), we have for any fixed \( i \geq 1, k \geq 1, \) and for any arbitrary integer \( K \geq 1, \)
\[
\hat{q}_i^{(k)} - \tilde{q}_i^{(k)} = 
\E_i \left( \left( \alpha^{(k)} \tilde{q}_i^{(k)} \right)^{|S_k|} \right) \Pr_i \left( 0 < |S_k| < K \right) \Pr_i \left( 0 < |S_k| < K \right) \Pr_i \left( |S_k| < K \right) (3.9)
+ \E_i \left( \left( \alpha^{(k)} \tilde{q}_i^{(k)} \right)^{|S_k|} \right) \Pr_i \left( |S_k| \geq K \right) \Pr_i \left( |S_k| \geq K \right). (3.10)
\]

Under Assumption 2, by Lemma 5 we get that for any \( K \geq 1, \)
\[
\Pr_i \left( |S_k| \geq K \right) \Pr_i \left( |S_k| \geq K \right) \Pr_i \left( |S_k| \geq K \right) (3.11)
\]

and the result follows from Corollary 3 (page 47) by choosing \( K \) large enough.

Assume that \( (i) \) holds. Then there exists \( \varepsilon > 0 \) and \( L \geq 1 \) such that for all \( j \in \{1, \ldots, N_1\} \) and \( k \geq L, \) we have \( \tilde{q}_j^{(k)} < 1 - \varepsilon. \) By Lemma 1 (page 46), for all \( j \in \{1, \ldots, N_1\} \) and \( k \geq L, \) we have \( \tilde{q}_j^{(k)} < 1 - \varepsilon. \) It follows that for any \( k > \max\{L, N_1\}, \)
\[
\alpha^{(k)} \tilde{q}^{(k)} = \sum_{j=1}^{N_1} \alpha_j^{(k)} \tilde{q}_j^{(k)} + \sum_{j=N_1+1}^{k} \alpha_j^{(k)} \tilde{q}_j^{(k)}
< (1 - \varepsilon) \sum_{j=1}^{N_1} \alpha_j^{(k)} + \sum_{j=N_1+1}^{k} \alpha_j^{(k)}
= \sum_{j=1}^{N_1} \alpha_j^{(k)} - \varepsilon \sum_{j=1}^{N_1} \alpha_j^{(k)} \leq 1 - \varepsilon a,
\]
where the last inequality follows from Assumption 1. With this, (3.11) becomes
\[
\limsup_{k \to \infty} \left( \hat{q}_i^{(k)} - \tilde{q}_i^{(k)} \right) < c_i \left( 1 - \varepsilon a \right)^K,
\]
and the result follows from Corollary 3 (page 47) and by choosing \( K \) large enough.

Assume that \( (ii) \) holds. First observe that if \( c_i = 0 \) for all \( i \geq 1 \) in (3.11), then the result immediately follows. In the remainder of the proof we
assume that there exists $i \geq 1$ such that $c_i > 0$, and we first show that this implies that $c_j > 0$ for all $j \in \{1, \ldots, N_1\}$. Indeed, let $\theta_i$ be the first generation an individual of type $i$ is born in $\{Z_n\}$. Then, by assumption, $\mathbb{P}_j(\theta_i < \infty) > 0$ for all $j \in \{1, \ldots, N_1\}$. Next, we have

$$
\mathbb{P}_j(|S_k| \geq K) \geq \mathbb{P}_j(|S_k| \geq K | \theta_i < \tau_k) \mathbb{P}_j(\theta_i < \tau_k),
$$

where $\tau_k = \inf\{n \geq 0 : \sum_{i=k+1}^{\infty} Z_{n,i} > 0\}$ denote the first passage time to $\tilde{T}_k = \{k + 1, k + 2, \ldots\}$. In addition, if the process starts with one type-$j$ individual and generates a type-$i$ individual before a seed, then the total number of seeds would be larger than $K$ if the type-$i$ individual itself generates more than $K$ seeds, that is,

$$
\mathbb{P}_i(|S_k| \geq K) \leq \mathbb{P}_j(|S_k| \geq K | \theta_i < \tau_k),
$$

so that

$$
\mathbb{P}_j(|S_k| \geq K) \geq \mathbb{P}_i(|S_k| \geq K) \mathbb{P}_j(\theta_i < \tau_k).
$$

Note that here we use the assumption $\tilde{q}_j = 1$ to avoid the possibility of total catastrophe in the seed process. As $k \to \infty$, the last inequality becomes $c_j \geq c_i \mathbb{P}_j(\theta_i < \infty) > 0$, as required.

Now that we have shown $c_j > 0$ for some $i \geq 1$ implies $c_j > 0$ for all $j \in \{1, \ldots, N_1\}$, it remains to show that $c_i > 0$ implies $\alpha^{(k)} q^{(k)}$ is bounded away from 1 for $k$ sufficiently large. Since $c_j > 0$, it follows that for any $\varepsilon > 0$ there exists an integer $W_j$ depending on $K$ such that for all $k > W_j$,

$$
\mathbb{P}_j(|S_k| \geq K) > c_j - \varepsilon. \tag{3.12}
$$

Let $W = \max_{1 \leq j \leq N_1} \{W_j\}$, and $c = \min_{1 \leq j \leq N_1} \{c_j\} > 0$. With reference to (3.7) and (3.12), we observe that for any $k \geq \{W, N_1, N_2\}$, the process $\{Z^{(e,k)}_n : \varphi_0 \in \{1, \ldots, N_1\}\}$ is then stochastically larger than the branching process $\{Z^{(e,k,2)}_n\}_{n \geq 0}$ with type set $T_{N_1} := \{1, \ldots, N_1\}$ and progeny generating function $G^{(e,k,2)} : [0,1]^{T_{N_1}} \to [0,1]^{T_{N_1}}$ such that, for any $i \in T_{N_1}$,

$$
G^{(e,k,2)}_i(s) = \left(\sum_{j=1}^{N_1} \alpha^{(k)}_j s_j + 1 - \sum_{j=1}^{N_1} \alpha^{(k)}_j \right)^K (c - \varepsilon) + 1 - (c - \varepsilon).
$$

This corresponds to the branching process in which each individual has $K$ offspring with probability $c - \varepsilon$ and 0 offspring otherwise, where the types of
the offspring are assigned independently according to the possibly defective distribution \((\alpha_1^{(k)}, \ldots, \alpha_{N_1}^{(k)})\) and individuals not assigned a type are immediately killed. Since \(G_i^{(e,k,2)}(\cdot)\) is independent of \(i\), \(|Z_n^{(e,k,2)}|\) behaves like a single-type Galton-Watson process, that is, it is locally isomorphic to a single-type branching process (see [71, Definition 4.2]). Combining this with the fact that by Assumption 1, \(\sum_{i=1}^{N_1} \alpha_i^{(k)} \geq a\) for all \(k \geq N_2\), we see that, for all \(k \geq \max\{W, N_1, N_2\}\), \(|Z_n^{(e,k,2)}|\) is stochastically larger than the single-type branching process \(Z_n^{(e,3)}\) with progeny generating function

\[
G^{(e,3)}(s) = (as + 1 - a)K(c - \varepsilon) + 1 - (c - \varepsilon).
\]

By taking \(K > 2/(a(c - \varepsilon))\) in order to bound the mean progeny of \(Z_n^{(e,3)}\) away from 1, we obtain \(q_j^{(e,k)} \leq q_j^{(e,k,2)} \leq q^{(e,3)} < 1 - \gamma\) for any \(k \geq \max\{W, N_1, N_2\}\), \(j \in \{1, \ldots, N_1\}\), and for some \(\gamma > 0\). Using the same argument as the one used when assuming (i) holds, we obtain \(\alpha^{(k)}q^{(e,k)} < 1 - \gamma a\), and therefore by (3.8),

\[
\alpha^{(k)}q^{(k)} \leq \alpha^{(k)}q^{(e,k)} < 1 - \gamma a
\]

for \(k\) sufficiently large, which proves the result. \(\square\)

### 3.4 Conditions for extinction and survival

Theorem 14 establishes a relationship between extinction of finite-type branching processes and global extinction of infinite-type branching processes. We now directly exploit this link and well-known results on finite-type branching processes in a first attempt to derive sufficient conditions for \(q = 1\) and \(q < 1\). Throughout this section we assume that \(\{Z_n\}\) and \(\{\alpha^{(k)}\}\) satisfy the conditions of Theorem 14.

Recall that \(\tilde{M}^{(k)}\) denotes the \(k\)th north-west truncation of the mean progeny matrix \(M\), and let \(x^{(k)}\) be the \(k \times 1\) vector such that \(x_i^{(k)} = \sum_{j \geq k} M_{ij}\). The mean progeny matrix of \(\{Z_n^{(k)}\}\) is then \(\tilde{M}^{(k)} := \tilde{x}^{(k)} + \alpha^{(k)}\), and by Theorem 3 we have \(\bar{q}^{(k)} = 1\) if and only if \(\rho(\tilde{M}^{(k)}) \leq 1\). This leads to a neat sufficient condition for almost sure global extinction.

**Corollary 5** If \(\rho(\tilde{M}^{(k)}) \leq 1\) for infinitely many \(k\) then \(q = 1\).

Corollary 5 implies that if \(\liminf_k \rho(\tilde{M}^{(k)}) < 1\) then \(q = 1\). Conversely, one may expect that \(\liminf_k \rho(\tilde{M}^{(k)}) > 1\) implies \(q < 1\), however, this is not
necessarily the case. Indeed, [71, Example 4.4] corresponds to a case where \( \liminf_k \rho(\bar{M}(k)) \geq 2 \) and \( q = 1 \). Additional higher moment conditions are therefore required. We impose the following condition.

**Assumption 3** There exists \( B_1 < \infty \) such that \( \mathbb{E}_i(|Z_1|^2) < B_1 \) for all \( i \geq 0 \).

Let \( \bar{v}^{(k)} \) denote the right Perron-Frobenius eigenvector of \( \bar{M}(k) \) and \( \bar{A}_{i,j,k}^{(k)} := \partial G_i^{(k)}(s)/(\partial s_i \partial s_j \partial s_k) \big|_{s=1} \), for \( 1 \leq i, j, k \leq q \), where \( G^{(k)}(s) \) is the progeny generating function corresponding to \( \{\bar{Z}_n^{(k)}\} \). We now provide sufficient conditions for \( q < 1 \).

**Proposition 3** Under Assumption 3, if \( \{\alpha^{(k)}\} \) is such that

(i) there exists \( B_2 < \infty \) independent of \( i, j, k \) such that \( \bar{v}_i^{(k)}/\bar{v}_i^{(k)} < B_2 \) whenever \( \bar{M}_{ij}^{(k)} > 0 \), and

(ii) there exists \( i \geq 1 \) such that \( \liminf_{k \to \infty} (\bar{v}_i^{(k)}/\sup_j \{\bar{v}_j^{(k)}\}) = b > 0 \),

then \( \liminf_k \rho(\bar{M}(k)) > 1 \) implies \( q_i < 1 \).

**Proof.** Observe that if there exists \( s^{(k)} \) such that \( G^{(k)}(s^{(k)}) \leq s^{(k)} \) then \( \bar{q}^{(k)} \leq s^{(k)} \). Let \( c^{(k)} := 1/\sup_j \{\bar{v}_j^{(k)}\} \). By the Taylor expansion formula in [3, Corollary 3], we have for any \( 1 \leq i \leq k \) and \( 0 < \theta < 1 \),

\[
G_i^{(k)}(1 - \theta \bar{v}_i^{(k)} c^{(k)}) < 1 - \theta c^{(k)} \sum_j \bar{v}_j^{(k)} \bar{M}_{ij}^{(k)} + (\theta c^{(k)})^2 \sum_{j \neq k} \bar{v}_j^{(k)} \bar{v}_k^{(k)} \bar{A}_{i,j,k}^{(k)} \\
< 1 - \theta \rho(\bar{M}(k)) \bar{v}_i^{(k)} c^{(k)} + (\theta B_2 \bar{v}_i^{(k)} c^{(k)})^2 \sum_{j \neq k} \bar{A}_{i,j,k}^{(k)},
\]

where \( \sum_{j \neq k} \bar{A}_{i,j,k}^{(k)} = \mathbb{E}_i(|\bar{Z}_1^{(k)}|^2) = \mathbb{E}_i(|Z_1|^2) \leq B_1 \). Thus, for any \( 1 < a < \liminf_k \rho(\bar{M}(k)) \) there exists \( K < \infty \) such that

\[
G_i^{(k)}(1 - \theta \bar{v}_i^{(k)} c^{(k)}) < 1 - \theta \bar{v}_i^{(k)} c^{(k)}(a - \theta B_2 B_1),
\]

for all \( k > K \). If \( \theta < \frac{a-1}{B_1 B_2} \), then \( \bar{q}^{(k)} < 1 - \theta \bar{v}_i^{(k)} c^{(k)} \) for all \( k \geq K \). By Theorem 14 and (ii) we then obtain \( q_i < 1 - \theta b \). 

Observe that if there exists \( \varepsilon > 0 \) such that \( \bar{M}_{ij}^{(k)} > 0 \) implies \( \bar{M}_{ij}^{(k)} > \varepsilon \) for all \( k \geq 0 \), then \( \bar{v}_j^{(k)}/\bar{v}_i^{(k)} < \rho(\bar{M}(k))/\varepsilon \). This means that if, in addition, \( \limsup_k \rho(\bar{M}(k)) < \infty \) then (i) holds.

Proposition 3 leads naturally to sufficient conditions for the entries of \( q \) to be uniformly bounded away from 1.
Corollary 6 Under Assumption 3, if \( \{\alpha^{(k)}\} \) is such that \( 0 < b \leq \bar{v}_i^{(k)} \leq c < \infty \) for all \( k \geq 0 \) and \( 1 \leq i \leq k \), then \( \liminf_k \rho(\bar{M}^{(k)}) > 1 \) implies \( \sup_i q_i < 1 \).

Proof. Following the arguments in the proof of Lemma 3, there exists \( \theta > 0 \) such that \( q_i < 1 - \theta b/c \) for all \( i \geq 1 \).

Theorem 4 of [66] (see (vi) Theorem 7 on page 18) is similar to Corollary 6, however it requires the convergence norm \( \nu(M) \) of \( M \) to be strictly larger than 1, which is known to be sufficient for \( \tilde{q} < 1 \). Note that repeating the same arguments with the sequence \( \{q^{(k)}\} \) instead of \( \{\tilde{q}^{(k)}\} \) leads to a result similar to [66, Theorem 4].

The primary difference between Corollary 6 and [66, Theorem 4] therefore lies in the conditions 'lim inf \( k \to \infty \), \( \rho(\bar{M}^{(k)}) > 1 \)' and '\( \nu(M) > 1 \)'. In both Examples 3 and 4, when \( q < \tilde{q} = 1 \), the former is satisfied but the latter is not.

3.5 Examples and relaxations of Assumption 1

Theorem 14 proves that \( \tilde{q}^{(k)} \to q \) for a large class of replacement distributions \( \{\alpha^{(k)}\} \). In this section we demonstrate that when \( \{\alpha^{(k)}\} \) is chosen so that Assumption 1 does not hold, the sequence \( \{\tilde{q}^{(k)}\} \) exhibits a range of asymptotic behaviours. Indeed, we show that its limit does not necessarily exist (Example 3), or does not necessarily converge to \( q \) (Example 4). The proofs of the results pertaining to these examples are gathered in Section 3.6. These results are related to those in [28, 29, 37], where the algorithmic computation of the stationary distribution of a recurrent infinite state Markov chain is considered.

Example 3 (Replacement with type \( k \)). Let \( \alpha^{(k)} = e_k \) and consider a modified version of the example of [35, Section 5.1], in which the odd types are stronger than the even types. That is, we assume \( a, c > 0, d > 1 \) and define

\[
G_1(s) = \frac{cd}{t} s_2 + 1 - \frac{cd}{t},
\]

and for \( i \geq 2 \),

\[
G_i(s) = \begin{cases} 
\frac{cd}{u} s_{i+1} + \frac{ad}{u} s_{i-1} + 1 - \frac{d(a + c)}{v} & \text{when } i \text{ is odd,} \\
\frac{c}{d} s_{i+1} + \frac{a}{d} s_{i-1} + 1 - \frac{(a + c)}{d} & \text{when } i \text{ is even,}
\end{cases}
\]

where \( t = [dc] + 1, u = [d(c + a)] + 1 \) and \( v = [(c + a)/d] + 1 \).
When $k$ is odd $\{\tilde{Z}_n^{(k)}\}$ has the mean progeny representation graph given in Figure 3.6; when $k$ is even, there is an equivalent graph. We consider the type-$k$ process $\{E_n^{(k)}(\tilde{Z}(k))\}$ embedded with respect to $\{Z_n^{(k)}: \varphi_0 = k\}$, with mean progeny $m^{(k)}_{E_n^{(k)}}(\tilde{Z}(k))$ that we denote by $\tilde{m}^{(k)}$ for short. The limit of the sequence $\{\tilde{m}^{(k)}\}$ does not generally exist, however its limit superior and inferior are finite when $ac \leq 1/4$, as we show in the next lemma.

**Lemma 6** The mean progeny of $\{E_n^{(k)}(\tilde{Z}(k))\}$ described in Example 3 satisfies
\[
\lim_{k \to \infty} \tilde{m}^{(2k+1)} = cd + \frac{1}{2} \left( 1 - \sqrt{1 - 4ac} \right) \quad \text{and} \quad \lim_{k \to \infty} \tilde{m}^{(2k)} = c/d + \frac{1}{2} \left( 1 - \sqrt{1 - 4ac} \right)
\]
when $ac \leq 1/4$, and $\lim_{k \to \infty} m^{(k)} = +\infty$ when $ac > 1/4$.

As a consequence of Lemma 6, $\lim_{k \to \infty} m^{(2k+1)} - m^{(2k)} = c(d-d^{-1})$, which indicates it is possible to choose $a$, $c$ and $d$ so that as $k \to \infty$, $\tilde{m}^{(k)}$ oscillates between values less than 1 and greater than 1. This observation leads us to the following result.

**Proposition 4** Consider the branching process described in Example 3. Assume that $d > 1$ and that $\alpha^{(k)} = e_k$. Then $\lim_{k \to \infty} \tilde{q}^{(k)} = q$ when $ac > 1/4$. Additionally, $\tilde{q} = 1$ if and only if $ac \leq 1/4$, and when this is satisfied,

(i) if $d^{-1} > (1 + \sqrt{1 - 4ac})/2c$ then $\lim_{k \to \infty} \tilde{q}^{(k)} = q$,

(ii) if $d^{-1} \leq (1 + \sqrt{1 - 4ac})/2c < d$ then
\[
\lim_{k \to \infty} \tilde{q}^{(2k+1)} = q \quad \text{and} \quad \lim_{k \to \infty} \tilde{q}^{(2k)} = \tilde{q} = 1,
\]

(iii) if $d \leq (1 + \sqrt{1 - 4ac})/2c$ then $\lim_{k \to \infty} \tilde{q}^{(k)} = \tilde{q} = q = 1$. 

Figure 3.6: The mean progeny representation graph corresponding to $\{Z_n^{(k)}\}$ when $k$ is odd in Example 3.
3.5. Examples and relaxations of Assumption 1

In Figure 3.7 we plot $\bar{q}_1^{(k)}$ (red dashed), $\hat{q}_1^{(k)}$ (blue dashed) and $\tilde{q}_1^{(k)}$ for $\alpha^{(k)} = e_1$ (solid yellow bold), $\alpha^{(k)} = 1/k$ (solid green bold) and $\alpha^{(k)} = e_k$ (solid black fine). In the top two plots we let $a = 1/6$ and $c = 7/8$, in which case $ac < 1/4$ and $(1 + \sqrt{1 - 4ac})/2 \approx 0.94$. With this in mind, we choose $d^{-1} = 0.95$ (panel (a)) and $d^{-1} = 0.93$ (panel (b)). In agreement with Proposition 4, for $\alpha^{(k)} = e_k$ we observe that $\hat{q}_1^{(k)} \to q_1$ when $d^{-1} = 0.95$, whereas $\bar{q}_1^{(2k+1)} \to q_1$ and $\tilde{q}_1^{(2k)} \to \tilde{q}_1 = 1$ when $d^{-1} = 0.93$. For these values of $a$, $c$ and $d$ it appears that $q_1 < 1$, which lead us to conclude that $\lim_{k \to \infty} \hat{q}_1^{(k)} \neq \limsup_{k \to \infty} \bar{q}_1^{(k)}$ when $d^{-1} = 0.93$ and $\alpha^{(k)} = e_k$. In panel (c) of Figure 3.7 we let $a = 1/3$ and $c = 13/16$, in which case $ac > 1/4$ so that $\hat{q}_1 < 1$ for any value of $d$. We choose $d = 2$ and observe that all sequences converge to $q_1 = \hat{q}_1$. In panel (d) of Figure 3.7 we let $a = 1/6$, $c = 13/16$ and $d = 2$, which means $ac < 1/4$ and $d^{-1} < (1 + \sqrt{1 - 4ac})/2 \approx 1.03 < d$, which entails that for $\alpha^{(k)} = e_k$, $\bar{q}_1^{(2k+1)} \to q_1$ and $\tilde{q}_1^{(2k)} \to \tilde{q}_1$. However, in this case $\hat{q}_1 = q_1 = 1$ and thus the limit of $\bar{q}^{(k)}$ exists.

Observe that for the branching process described in Example 3, Proposition 4 implies that when $\alpha^{(k)} = e_k$ we have $\lim_{k \to \infty} \bar{q}^{(k)} = q$. Provided Assumption 2 holds, we conjecture that this is true in general.

The next example demonstrates that when $\alpha^{(k)} = 1/k$ and $\lim_{k \to \infty} q_i > 0$, there is not always a subsequence of $\bar{q}^{(k)}$ that converges to $q$.

**Example 4 (Replacement with a uniform type).** For ease of notation, in this example we use the type set $\mathcal{X} = \{2, 3, 4, \ldots \}$. Suppose $p, \varepsilon \in (0, 1)$ and $3p\varepsilon^2 < 1$, and consider the following progeny distribution:

$$G_2(s) = ps_3 + (1 - p),$$

and for $i \geq 3$,

$$G_i(s) = \begin{cases} 
\varepsilon p(1 - 3p\varepsilon^{i/2}) s_{i-1} s_3^3 \\
+ p(1 - 3p\varepsilon^{i/2})(1 - \varepsilon) s_3^2 \\
+ \varepsilon(1 - p(1 - 3p\varepsilon^{i/2})) s_{i-1} \\
+ (1 - \varepsilon - p(1 - 3p\varepsilon^{i/2})(1 - \varepsilon)), \quad \text{if } i \in \{2^j\}_{j \geq 2}, \\
\varepsilon s_{i-1} + (1 - \varepsilon), \quad \text{otherwise}.
\end{cases}$$

The corresponding mean progeny representation graph is shown in Figure 3.8.

**Proposition 5** For the branching process described in Example 4,
Figure 3.7: Sequences of extinction probabilities $q^{(k)}_1$, $q^{(k)}_1$ and $q^{(k)}_1$ for different replacement distributions and different parameter values, corresponding to Example 3. Details are given in the text.
3.6. Proofs related to the examples

Proof of Lemma 6. We calculate $\bar{m}^{(k)}$ by taking the weighted sum of all first return paths to $k$ in the mean progeny representation graph,

$$\bar{m}^{(k)} = M_{k,k}^{(k)} + M_{k,k-1}^{(k)}M_{k-1,k}^{(k)} + M_{k,k-1}^{(k)}M_{k-1,k-1}^{(k)}M_{k-2,k-1}^{(k)} + \cdots.$$

Observe that the number of these paths, $k \to (k-1) \to \cdots \to (k-1) \to k$, with length $2(l+1) \leq 2(k-1)$ is given by the Catalan number,

$$C_l = \frac{1}{l+1}\binom{2l}{l}, \quad l \geq 0, \quad (3.14)$$
Chapter 3. Random replacement

Figure 3.9: Sequences of extinction probabilities $\tilde{q}_1^{(k)}$, $\check{q}_1^{(k)}$, and $\bar{q}_1^{(k)}$ for different replacement distributions and different parameter values, corresponding to Example 4. Details are given in the text.

whereas, because no path can fall below type 1, the number of paths is less than $C_l$ when $2(l + 1) > 2(k - 1)$. In these expressions, $(l + 1)$ can be interpreted as the total number of negative increments in the paths. Furthermore, the length of each first return path is even, the total number of positive and negative increments of each first return path is equal, and each first return path alternates between odd and even states. Hence,

$$\bar{M}_{k,k}^{(k)} + \sum_{l=0}^{k-2} C_l (ac)^{l+1} \leq \tilde{m}^{(k)} \leq \check{M}_{k,k}^{(k)} + \sum_{l=0}^{\infty} C_l (ac)^{l+1}.$$ 

The infinite series converges when $ac \leq 1/4$ and diverges when $ac > 1/4$. In addition, when $ac \leq 1/4$,

$$\sum_{l=0}^{\infty} \frac{1}{l+1} \binom{2l}{l} (ac)^l = \frac{1}{2} \left( 1 - \sqrt{1 - 4ac} \right),$$

which gives the result. □
3.6. Proofs related to the examples

Proof of Proposition 4. Following an approach analogous to the proof of Proposition 5.1 in [35], we can show that the convergence norm of the mean progeny matrix $M$ is $\nu(M) = 2\sqrt{ac}$. By Proposition 4.1 in [35], we see that $\tilde{q} = 1$ if and only if $ac \leq 1/4$.

We first turn our attention to cases (i) and (ii). Observe that the number of first return paths to $k$ of any fixed length in the mean progeny representation graph is monotone increasing with $k$. This means that the sequences $\{\tilde{m}_i^{(2k+1)}\}$ and $\{\tilde{m}_i^{(2k)}\}$ are monotonically increasing with respect to $k$. Due to the repetitive structure of the progeny distributions and the relative weakness of type 1 with respect to other odd types, we also have

$$\tilde{q}^{(2k+1)}_{2k+1} \geq \tilde{q}^{(2k+3)}_{2k+3}, \quad \text{and} \quad \tilde{q}^{(2k)}_{2k} \geq \tilde{q}^{(2k+2)}_{2k+2},$$

for all $k \geq 1$. This can be proved rigorously following arguments analogous to those in Proposition 2. If $\lim_{k \to \infty} \tilde{m}_i^{(2k+1)} > 1$, then there exists $\varepsilon_1 > 0$ and an integer $k_1$ such that for all $k \geq k_1$,

$$\tilde{q}^{(2k+1)}_{2k+1} < 1 - \varepsilon_1,$$

(3.15)

whereas, if $\lim_{k \to \infty} \tilde{m}_i^{(2k+1)} \leq 1$ then, for all $k \geq 1$,

$$\tilde{q}^{(2k+1)}_{2k+1} = 1.$$

(3.16)

An equivalent result holds when we take the limit over the even values of $k$.

Next, we have $\inf_i q_i \geq \inf_i p_i(0) > 0$ which, by Lemma 5, implies that $\mathbb{P}_i(|S_k| \to 0) + \mathbb{P}_i(|S_k| \to 0) = 1$ for all $i \geq 1$. Therefore, for any arbitrary integer $K \geq 1$,

$$\limsup_{k \to \infty} \left( \tilde{q}_i^{(k)} - q_i^{(k)} \right) = c_i \limsup_{k \to \infty} \mathbb{E}_i \left( \left( \frac{q_k^{(k)}}{q_k^{(k)} \tilde{q}_k} \right) |S_k|, |S_k| > K \right),$$

where $c_i = \tilde{q}_i - q_i = \lim_{k \to \infty} \mathbb{P}_i(|S_k| > K)$ by Lemmas 2, 3 and 5, with the same holding when $\limsup$ is replaced by $\liminf$. In combination with (3.15) and (3.16) we then obtain

$$\lim_{k \to \infty} \left( \tilde{q}_i^{(2k+1)} - q_i^{(2k+1)} \right) = \begin{cases} 0, & \text{if } \lim_{k \to \infty} \tilde{m}_i^{(2k+1)} > 1, \\ \tilde{q}_i - q_i, & \text{if } \lim_{k \to \infty} \tilde{m}_i^{(2k+1)} \leq 1 \end{cases}$$

and

$$\lim_{k \to \infty} \left( \tilde{q}_i^{(2k)} - q_i^{(2k)} \right) = \begin{cases} 0, & \text{if } \lim_{k \to \infty} \tilde{m}_i^{(2k)} > 1, \\ \tilde{q}_i - q_i, & \text{if } \lim_{k \to \infty} \tilde{m}_i^{(2k)} \leq 1 \end{cases}.$$
Chapter 3. Random replacement

Figure 3.10: A visual representation of the mean progeny matrix in Example 4. Bold edges in (b) have weight $\frac{3p(1-3pe^2)}{2}$.

Use of the fact that $\lim_{k \to \infty} q^{(k)} = q$ and Lemma 6 then provides the result.

Consider now case (iii). We apply Proposition 4.5 in [35], which states that if there exists $\lambda \leq 1$ and a row vector $x > 0$ such that $x1 < \infty$ and $xM \leq \lambda x$, then $q = 1$. We let $x = (x_i)_{i \geq 1}$ with $x_i = (\sqrt{d})^{(-1)} \cdot x^{1-i}$ for some $x > 0$. In this case, $xM \leq x$ is equivalent to $cx^2 + a \leq x$, that is, $x$ belongs to the interval $[(1 - \sqrt{1-4ac})/2c, (1 + \sqrt{1-4ac})/2c]$. Moreover, $x > 1$ ensures $x1 < \infty$. It follows that whenever $1 < (1 + \sqrt{1-4ac})/2c$, there exists an $x$ satisfying both conditions, which implies that $\tilde{q} = q = 1$.

**Proof of Proposition 5.** To prove (i), we consider the type-$2^k$ process embedded with respect to $\{Z_n^{(2^k)} : \varphi_0 = 2^k\}$, and calculate its mean number of offspring, denoted by $\tilde{m}^{(2^k)}$, for $k \geq 1$. We tackle this by computing the weighted sum of all first return paths to node $2^k$ in the mean progeny representation graph illustrated in Figure 3.8, which we alter by removing all nodes greater than $2^k$ to account for the corresponding types being sterile in $\{Z_n^{(2^k)}\}$. Observe that when $k = 2$, $(2^k = 4)$ there is a single first return path $4 \to 3 \to 2 \to 4$, thus, $\tilde{m}^{(4)} = 3pe^2$. When $k > 2$, we calculate $\tilde{m}^{(2^k)}$ recursively as follows. First observe that each first return path to $2^k$ begins with the sequence of edges $2^k \to (2^k - 1) \to \cdots \to (2^k - 1 + 1) \to 2^{k-1}$, and ends with the edge $2^{k-1} \to 2^k$. Additionally, the remainder of each path (or the midsection) can be partitioned into the first return paths that were summed to obtain $\tilde{m}^{(2^{k-1})}$. That is, for the purpose of calculating $\tilde{m}^{(2^k)}$, in the mean progeny matrix representation graph, nodes of type $< 2^{k-1}$ can then be replaced by a loop to type $2^{k-1}$ with weight $\tilde{m}^{(2^{k-1})}$. See Figure 3.10(a) for an
illustration when \( k = 3 \). It can then be shown that,
\[
\tilde{m}^{(2^k)} = \varepsilon^{2^{k-1}} \left[ 1 + \tilde{m}^{(2^{k-1})} + (\tilde{m}^{(2^{k-1})})^2 + \ldots \right] 3p(1 - 3p\varepsilon^{2^{k-2}}).
\]

We can then prove by induction that \( \tilde{m}^{(2^k)} = \varepsilon^{2^{k-1}} 3p \), which leads to \( \tilde{m}^{(2^k)} < 1 \) for all \( k \geq 1 \) since, by assumption, \( 3p\varepsilon^{2^{k-1}} \leq 3p\varepsilon^2 < 1 \). Combining this with the fact that for all \( k \geq 2 \), \( \{ \tilde{Z}_n^{(2^k)} \} \) is irreducible, we obtain \( \tilde{q}^{(2^k)} = 1 \) for all \( k \geq 1 \). Since \( \{ \tilde{2}^k \}_{k \geq 1} \) is an infinite subsequence of \( \mathbb{N} \), the result then follows from the fact that \( \lim_{k \to \infty} \tilde{q}^{(k)} = \tilde{q} \).

We now prove (ii). If \( p > 1/3 \), then there exists \( \gamma > 0 \) such that \( p = 1/3 + \gamma \), and there exist an integer \( N \) and a constant \( 0 < C < 3\gamma \) such that \( 3p(1 - 3p\varepsilon^{i/2}) = (1 + 3\gamma)(1 - 3p\varepsilon^{i/2}) > 1 + C \) for all \( i \geq 2^N \) (since \( 3p\varepsilon^{i/2} \) becomes arbitrarily close to 0 as \( i \) increases). By disregarding all types \( j \) such that \( j \leq 2^N \) or \( j \notin \{ \tilde{2}^k \}_{k \geq 2} \) it can be shown that \( \{|Z_n| : \varphi_0 = 2^N\} \) is stochastically greater than the Galton-Watson process with progeny generating function \( G(s) = (1/3 + C/3)s^3 + (2/3 - C/3) \). Since \( G'(1) > 1 \), we have \( q_{2^N} < 1 \). The result then follows from irreducibility.

To prove (iii) we consider the type-2\(^k\) process embedded in \( \{ \tilde{Z}_n^{(2^k)} : \varphi_0 = 2^k \} \) and calculate its mean number of offspring \( \tilde{m}^{(2^k)} \). To account for the instantaneous replacement of all individuals of type \( > 2^k \) with a type uniformly distributed on \( \{2, \ldots, 2^k\} \), the graph illustrated in Figure 3.8 is altered by removing all nodes greater than \( 2^k \) and adding an edge of weight \( 3p(1 - 3p\varepsilon^{2^{k-1}})/(2^k - 1) \) from node \( 2^k \) to all the remaining nodes. See Figure 3.10(b) for an illustration when \( k = 3 \). We then calculate the weighted sum of all first return paths to \( 2^k \). Note that these paths include those involved in the computation of \( \tilde{m}^{(2^k)} \). More specifically, if we let \( f_{i,k} \) be the weighted sum of all first passage paths from \( 2^i \) to \( 2^k \), then
\[
\tilde{m}^{(2^k)} = \tilde{m}^{(2^k)} + \frac{3p(1 - 3p\varepsilon^{2^{k-1}})}{2^k - 1} \left( 1 + \sum_{j=1}^{k-1} f_{k-j,k} \left( \frac{2^{k-j-1}}{\varepsilon^j} \right) \right).
\]

By applying a recursive argument analogous to the proof of (i), it can then be shown that \( f_{i,k} = (3p)^{k-i} \). Assuming \( p \neq 1/3 \) we have,
\[
\tilde{m}^{(2^k)} = 3p\varepsilon^{2^{k-1}} + \frac{3p(1 - 3p\varepsilon^{2^{k-1}})}{2^k - 1} \left( 1 + \frac{1}{1 - \varepsilon} \sum_{j=1}^{k-1} (3p)^j(1 - \varepsilon^{2^{k-j}}) \right)
\]
\[
\leq 3p\varepsilon^{2^{k-1}} + \frac{3p}{2^k - 1} \left( \frac{1 - (3p)^k}{(1 - 3p)(1 - \varepsilon)} + \varepsilon \right) \to 0 \quad \text{as } k \to \infty
\]
when $p < 2/3$. When $p = 1/3$ it can be shown that an equivalent result holds. This demonstrates that $\lim_{k \to \infty} \tilde{m}^{(2^k)} = 0$ when $p < 2/3$. The proof that $\tilde{m}^{(k)} \to 0$ is an extension of the same method and is omitted. Therefore, there exists an integer $N$ such that for all $k > N$, $\tilde{m}^{(k)} < 1$ hence $\tilde{q}^{(k)} = 1$, which shows that $\tilde{q}^{(k)} \to 1$ as $k \to \infty$.

\section*{3.7 Computational aspects}

\subsection*{3.7.1 Pseudo code}

The three sequences of extinction probabilities \{\(q^{(k)}\), \(\tilde{q}^{(k)}\), and \(\bar{q}^{(k)}\)} defined in Section 3.2 are easy to implement in practice, as we show now. Since the convergences $q^{(k)} \to q$, $\tilde{q}^{(k)} \to \tilde{q}$, and $\bar{q}^{(k)} \to \bar{q}$ (under the assumptions of Theorem 14) are pointwise, in order to evaluate the $i$th entry of the desired extinction probability vector, the chosen sequence of approximating vectors should be computed for $k \geq i$.

For $k \geq 1$, let $s^{(k)} = (s_1, \ldots, s_k)^\top \in [0,1]^k$ and $G^{(k)}(s) := (G_1(s), \ldots, G_k(s))^\top$, and let $\varepsilon$ be some predetermined tolerance error. For any $i \in S$, the pseudo-code for the numerical computation of $q_i$ or $\tilde{q}_i$ depends on the function $u(s^{(k)}) \in [0,1]^\infty$ as described below in (3.17)-(3.19), which determines which of the three sequences is used:

\begin{itemize}
  \item Set $x^{(old)}_i := 2$, $k := i$
  \item Compute $x^{(k)}$ as the minimal non-negative solution of $s^{(k)} = G^{(k)}(s^{(k)}, u(s^{(k)}))$
  \item While $|x^{(k)}_i - x^{(old)}_i| > \varepsilon$
    \begin{itemize}
      \item $k := k + 1$
      \item Set $x^{(old)} := x^{(k)}$
    \end{itemize}
  \item Compute $x^{(k)}$ as the minimal non-negative solution of $s^{(k)} = G^{(k)}(s^{(k)}, u(s^{(k)}))$
\end{itemize}

Return $x_i := x^{(k)}_i$,

where

\begin{equation}
  u(s^{(k)}) = 0 \quad \Rightarrow \quad x^{(k)} = q^{(k)} \text{ and } x_i \approx q_i, \tag{3.17}
\end{equation}

\begin{equation}
  u(s^{(k)}) = 1 \quad \Rightarrow \quad x^{(k)} = \tilde{q}^{(k)} \text{ and } x_i \approx \tilde{q}_i, \tag{3.18}
\end{equation}

\begin{equation}
  u(s^{(k)}) = \sum_{j=1}^k \alpha^{(k)}_j s_j \quad \Rightarrow \quad x^{(k)} = \bar{q}^{(k)} \text{ and } x_i \approx q_i. \tag{3.19}
\end{equation}
3.7. Computational aspects

Note that the linear functional iteration algorithm or the quadratic Newton algorithm can be applied to compute the minimal non-negative solution of the finite system \( s^{[k]} = G^{[k]}(s^{[k]}, u(s^{[k]})) \) for each value of \( k \).

3.7.2 Comments on computational efficiency

In this chapter we have proposed a class of algorithms for computing \( q \) that depend on the sequence of replacement distributions \( \{\alpha^{(k)}\} \). We have focused on the eventual convergence of these algorithms but have thus far neglected questions about the rate at which they converge. We briefly discuss this topic through a numerical example.

In panels (a) and (c) of Figure 3.11, for \( k = 1, \ldots, 50 \) we plot \( \tilde{q}^{(k)}_1 \), \( \tilde{q}^{(k)}_1 \), and \( \bar{q}^{(k)}_1 \) with \( \alpha^{(k)} = e_1 \), \( \alpha^{(k)} = 1/k \), and \( \alpha^{(k)} = e_k \), corresponding to Example 1 on page 25 with parameter values (i) and (ii), respectively. Roughly speaking, without considering \( \tilde{q}^{(k)}_1 \), in both plots we see that \( \bar{q}^{(k)}_1 \) with \( \alpha^{(k)} = e_k \) converges to \( q_1 \) the fastest, and \( q^{(k)}_1 \) converges to \( q_1 \) the slowest.

Recall that a sequence \( \{x_k\} \) is said to converge linearly at rate \( 0 < h < 1 \) if \( \lim_k |x - x_{k+1}|/|x - x_k| = h \). In panels (b) and (d) we plot the ratio \( |x - x_{k+1}|/|x - x_k| \) for each of the sequences plotted in (a) and (c) with \( x \) taken to be the value of the corresponding sequence at \( k = 50 \). These panels suggest that \( \{q^{(k)}_1\} \), and \( \{\bar{q}^{(k)}_1\} \) with \( \alpha^{(k)} = e_1 \) and \( \alpha^{(k)} = 1/k \), converge linearly at the same rate, \( h \approx 0.8 \) and \( h \approx 0.9 \) for parameter values (i) and (ii) respectively, whereas they suggest that \( \{\tilde{q}^{(k)}_1\} \) with \( \alpha^{(k)} = e_k \) converges at a faster linear rate, \( h \approx 0.25 \) and \( h \approx 0.08 \) for parameter values (i) and (ii) respectively. Note that for both sets of parameter values, when \( \alpha^{(k)} = e_k \), there exists \( k \leq 30 \) such that \( \tilde{q}^{(50)}_1 - \tilde{q}^{(1)}_1 < 2^{-16} \) causing this difference to be recorded as 0. We conclude that for this particular example, although the replacement distribution \( \alpha^{(k)} = e_k \) does not satisfy Assumption 2, it leads to convergence, and this convergence happens at a faster rate than the other replacement distributions.

More generally, the flexibility in the choice of replacement distributions \( \{\alpha^{(k)}\} \) motivates the search for an optimal choice minimising the convergence rate \( h \). In addition, it invites us to question which sequence \( \{\alpha^{(k)}\} \) (if any) leads to a convergence rate that differs from that of \( q^{(k)} \). This is a potential direction of future research.
Chapter 3. Random replacement

Figure 3.11: Panels (a) and (c): $\tilde{q}_1^{(k)}$, $q_1^{(k)}$, and $\overline{q}_1^{(k)}$ for different replacement distributions, corresponding to Example 1 parameter values (i) and (ii) respectively. Panels (b) and (d): $\frac{q_1^{(50)} - q_1^{(k+1)}}{q_1^{(50)} - q_1^{(k)}}$, $\frac{q_1^{(50)} - q_1^{(k+1)}}{q_1^{(50)} - q_1^{(k)}}$, and $\frac{q_1^{(50)} - q_1^{(k+1)}}{q_1^{(50)} - q_1^{(k)}}$, corresponding to the values in (a) and (c) respectively.
Chapter 4

Extinction in lower Hessenberg branching processes

4.1 Introduction

In this chapter we study a class of multitype Galton–Watson branching processes (MGWBPs) with countably infinitely many types which we refer to as lower Hessenberg branching processes (LHBPs). In these processes, the only constraint, apart from usual regularity conditions, is that type-\(i\) individuals can produce offspring of type no larger than \(i + 1\), for all \(i \geq 0\); as a consequence their mean progeny matrix has a lower Hessenberg form. Special cases of LHBPs include skip-free branching random walks and nearest-neighbour branching random walks on the nonnegative integers, see for instance [27, 9, 10].

To study the properties of LHBPs we use a probabilistic approach that relies on a single pathwise argument: we reduce the study of the LHP to that of a Galton-Watson process in a varying environment (GWPVE) embedded within the LHP. The embedded GWPVE is equivalent to the seed process introduced in the previous chapter with one important difference: whenever the seed process experiences a total catastrophe, where every individual in the population is killed in one generation, the embedded GWPVE experiences explosion. For this reason, we say that the embedded GWPVE is explosive, in the sense that individuals may have an infinite number of offspring in one
generation. With this adjusted convention we show the equivalence between
global extinction in the LHBP and extinction in the embedded GWPVE, and
between partial extinction in the LHBP and the event that all generations
in the embedded GWPVE are finite. Based on this relationship, we obtain
several results for LHBPs:

(i) We prove the existence of a continuum of fixed-point solutions \( s \in S = \{u \in [0,1]^\mathcal{X} : u = G(u)\} \) with \( q \leq s \leq \hat{q} \), whose componentwise minimum and maximum are the global and partial extinction probability vectors, respectively. In the irreducible case, the set \( S \) consists of this continuum and the fixed point \( 1 \). In the reducible case, under the assumption that \( M_{i,i+1} > 0 \) for all \( i \in \mathcal{X} \), we show that there may be an additional countable number of fixed points \( s \) such that \( \hat{q} \leq s \leq 1 \).

(ii) We establish a connection between the growth rates of the embedded
GWPVE and the convergence rate of \( s_i \) to \( 1 \) as \( i \to \infty \) for any \( s \in S \setminus \{q\} \); this yields a physical interpretation for the fixed points lying in between \( q \) and \( \hat{q} \).

(iii) In the non-trivial case where \( \hat{q} = 1 \), we provide a necessary and sufficient
condition for global extinction which holds under some second moment
conditions. We illustrate the broad applicability of the criterion through
some examples.

(iv) Finally, under additional assumptions, we build on the global extinction
criterion to derive necessary and sufficient conditions for strong local
survival.

While there is a vast literature on GWPVEs, the explosive case is yet to
be considered in the context of a varying environment. In order to prove our
main theorems for LHBPs, we both apply known results on GWPVEs and
develop new ones.

The chapter is organised as follows. After some preliminaries, in Section
4.3 we construct the embedded GWPVE with explosion, we derive a relation-
ship between it and the LHBP, and we analyse its generation-dependent
offspring distribution. In Section 4.4 we develop (i) and (ii), and in Sections
4.5 and 4.6 we deal with (iii) and (iv), respectively. The proofs related to the
illustrative examples are gathered in Section 4.7.
4.2 Preliminaries

In this chapter we let $X = \mathbb{N}_0 = \{0,1,2,\ldots\}$ and $T_k = \{0,1,\ldots,k\}$. We assume that \{Z_n\} is lower Hessenberg, that is, we assume that if virtual individual $I \in \mathcal{J}$ is of type $i$, then its offspring vector belongs to the set $\Omega_i = (\mathbb{N}_0)^{\mathcal{J}_{i+1}}$. We also assume that $M_{i,i+1} > 0$ for all $i \in X$. In this setting, the immortal type-$\Delta$ individuals in the process \{Z_n^{(k)}\} (defined in Section 2.6.2) always correspond to individuals of type $k+1$. We therefore retain the notation ‘$k+1$’ for the immortal types. This means that the progeny generating vector of \{Z_n^{(k)}\}, denoted by $G^{[k]}(s_0, \ldots, s_{k+1})$, contains entries

$$G^{[k]}(s_0, \ldots, s_{k+1}) = \begin{cases} G_i(s), & 0 \leq i \leq k \\ s_{k+1}, & i = k+1. \end{cases} \quad (4.1)$$

We let

$$G^{[k,n]}(s_0, \ldots, s_{k+1}) = G^{[k]} \circ G^{[k]} \circ \cdots \circ G^{[k]}(s_0, \ldots, s_{k+1}). \quad (4.2)$$

4.3 An embedded GWPVE with explosions

4.3.1 A link between extinction and explosion

The embedded GWPVE, which we denote by \{Y_k\}, is constructed from the sample paths of \{Z_n\} as follows: we define a function $f_g : \mathcal{J} \rightarrow \mathcal{J}$ that takes each line of descent ($\varphi_0; i_1, j_1, y_1; \ldots; i_n, j_n, y_n$) and deletes each triple $(i_k, j_k, y_k)$ whose type is not strictly larger than all its ancestors. For each $\omega \in \Omega$ the family tree of \{Y_k\} is then given by $f_g(X(\omega))$, where $X(\omega)$ denotes the family tree of \{Z_n\}. Note that in this construction, for the same reasons described on page 23, we require the unique identification numbers $y_k$. A trajectory of \{Y_k\} (right) compared to the corresponding realisation of \{Z_n\} (left) is given in Figure 4.1. We take the convention that \{Y_k\} starts at the generation number corresponding to the initial type $\varphi_0$ in \{Z_n\}. Due to the lower Hessenberg assumption, the $k$th generation of \{Y_k\} then contains type-$k$ individuals only.

To emphasise the connection between the embedded GWPVE and the processes \{\tilde{Z}_n^{(k)}\} and \{Z_n^{(k)}\}, we give the following equivalent definition of $Y_k$. 
Definition 2 For any $\omega \in \Omega$, the $k$-th generation of the embedded GWPVE is defined as

$$Y_k(\omega) = \sum_{n=0}^{\infty} \tilde{Z}^{(k-1)}_{n,k}(\omega) = \lim_{n \to \infty} Z^{(k-1)}_{n,k}(\omega), \quad k \geq \varphi_0.$$  \hspace{1cm} (4.3)

Observe that for any outcome $\omega \in \Omega$, each sterile type-$k$ individual that appears during the lifetime of $\{\tilde{Z}^{(k-1)}_n\}$ is a descendant of a sterile type-$(k-1)$ individual that appears during the lifetime of $\{\tilde{Z}^{(k-2)}_n\}$. In $\{\tilde{Z}^{(k-1)}_n\}$ the daughter processes of these type-$(k-1)$ individuals are independent and identically distributed, which means that $Y_k$ satisfies a branching processes equation of the form

$$Y_k \overset{d}{=} \sum_{i=1}^{Y_{k-1}} \xi_{k,i},$$  \hspace{1cm} (4.4)

where $\{\xi_{k,i}\}_{i \geq 1}$ is a sequence of i.i.d. random variables such that, $\xi_{k,i} \overset{d}{=} \sum_{n=0}^{\infty} \tilde{Z}^{(k-1)}_{n,k}$ when $\varphi_0 = k - 1$. Thus, $\{Y_k\}$ is indeed a Galton–Watson process in a varying environment. It is, however, not a standard GWPVE. Indeed, individuals in $\{Y_k\}$ may have a positive chance of giving birth to an infinite number of offspring. Following the terminology of [49] we say that if $Y_k = \infty$ then $Y_k$ has experienced explosion by generation $k$. The next lemma states that this occurs if and only if $\{\tilde{Z}^{(k-1)}_n\}$ survives globally, and connects the extinction event $\{Y_k = 0\}$ to the extinction of $\{Z^{(k-1)}_n\}$.
Lemma 7 For any $k \geq \varphi_0$,

$$\{\omega \in \Omega : Y_k(\omega) < \infty\} \overset{a.s.}{=} \{\omega \in \Omega : \lim_{n \to \infty} \tilde{Z}_n^{(k-1)}(\omega) = 0\},$$

and

$$\{\omega \in \Omega : Y_k(\omega) = 0\} \overset{a.s.}{=} \{\omega \in \Omega : \lim_{n \to \infty} Z_n^{(k-1)}(\omega) = 0\}.$$ 

Proof. To prove the first assertion, first suppose that $\omega \in \{\lim_n \tilde{Z}_n^{(k-1)} = 0\}$. In this case there exists a generation $N < \infty$ such that $\tilde{Z}_n^{(k-1)}(\omega) = 0$ for all $n \geq N$. By definition, individuals in $\{\tilde{Z}_n^{(k-1)}\}$ have only a finite number of offspring, therefore $\sum_{n=0}^{\infty} \tilde{Z}_n^{(k-1)}(\omega) = \sum_{n=0}^{N} \tilde{Z}_n^{(k-1)}(\omega) < \infty$, which implies $\omega \in \{Y_k < \infty\}$. It then remains to prove $P(Y_k < \infty, \lim inf_n |\tilde{Z}_n^{(k-1)}| > 0) = 0$.

Because $M_{i,i+1} > 0$ for any $i \in T_{k-1}$, we have

$$P \left( \sum_{n=0}^{k} \tilde{Z}_n^{(k-1)} = 0 \right) \leq 1 - \varepsilon_i,$$

for some $\varepsilon_i > 0$. Thus, if $\varepsilon = \min_{i \in T_k} \{\varepsilon_i\} > 0$ then for any initial population vector $\tilde{z}_0^{(k-1)} \in (N_0)^{T_k}$, we have

$$P \left( \sum_{n=0}^{k} \tilde{Z}_n^{(k-1)} = 0, |\tilde{Z}_k^{(k-1)}| > 0 \right) \left| \tilde{Z}_0^{(k-1)} = \tilde{z}_0^{(k-1)} \right| \leq 1 - \varepsilon.$$

By the Markov property, we then have

$$P(Y_k < \infty, \lim inf_{n \to \infty} |\tilde{Z}_n^{(k-1)}| > 0)$$

$$\leq P \left( \exists N \geq 0 : \sum_{m=N}^{\infty} \tilde{Z}_m^{(k-1)} = 0, |\tilde{Z}_n^{(k-1)}| > 0 \forall n \right)$$

$$= P \left( \exists N \geq 0 : \sum_{\ell=0}^{N+(\ell+1)k} \tilde{Z}_{m,k}^{(k-1)} = 0, |\tilde{Z}_{N+(\ell+1)k}^{(k-1)}| > 0 \right)$$

$$\leq \sum_{N=0}^{\infty} \prod_{\ell=0}^{\infty} (1 - \varepsilon) = 0,$$

leading to the first assertion. The second assertion follows from the same arguments. \hfill \Box

By construction,

$$\{\omega \in \Omega : \lim inf_n Z_n^{(k-1)}(\omega) > 0\} \subseteq \{\omega \in \Omega : \lim inf_n \tilde{Z}_n^{(k)}(\omega) > 0\},$$

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and

\[ \{ \omega \in \Omega : \lim_{n} Z_n^{(k-1)}(\omega) = 0 \} \subseteq \{ \omega \in \Omega : \lim_{n} Z_n^{(k)}(\omega) = 0 \}. \]

Thus, Lemma 7 implies that \( \{Y_k\} \) has two absorbing states, 0 and \( \infty \). The next corollary formalises the equivalence between the events that \( \{Z_n\} \) experiences

(i) both partial and global extinction,

(ii) partial extinction but not global extinction,

(iii) neither partial nor global extinction,

and the events that \( \{Y_k\} \) respectively reaches

(i) the absorbing state 0,

(ii) neither absorbing state 0 nor \( \infty \),

(iii) the absorbing state \( \infty \).

**Corollary 7** The global extinction event \( \mathcal{E}_g \) a.s. \( \equiv \{ \omega \in \Omega : \lim_k Y_k(\omega) = 0 \} \) and the partial extinction event \( \mathcal{E}_p \) a.s. \( \equiv \{ \omega \in \Omega : Y_k(\omega) < \infty, \forall k \geq \phi_0 \} \).

**Proof.** The result follows from Lemma 7 and the arguments in the proofs of [35, Lemmas 3.1 and 3.2] respectively.

By Corollary 7 we can express any question about the extinction probability vectors \( \mathbf{q} \) and \( \tilde{\mathbf{q}} \) in terms of the process \( \{Y_k\} \).

**Corollary 8** For any \( k \geq 0 \) and \( 0 \leq i \leq k \),

\[ q_i^{(k)} = \mathbb{P}_i (Y_{k+1} = 0) \quad \text{and} \quad \tilde{q}_i^{(k)} = \mathbb{P}_i (Y_{k+1} < \infty), \]

and for any \( i \geq 0 \),

\[ q_i = \mathbb{P}_i (\lim_{k \to \infty} Y_k = 0) \quad \text{and} \quad \tilde{q}_i = \mathbb{P}_i (\forall k \geq i, Y_k < \infty). \]

**Proof:** The results are immediate consequences of Lemma 7 and Corollary 7.
4.3. An embedded GWPVE with explosions

4.3.2 The defective generating functions of the embedded GWPVE

To make use of the links between \{Z_n\} and \{Y_k\} that we derived in Section 4.3.1, we need to characterise the progeny generating function corresponding to each generation of the embedded GWPVE. For \( k \geq 0 \), we let

\[
g_k(s) := \mathbb{E}_k(s^{Y_{k+1}} \mathbf{1}\{Y_{k+1} < \infty\}) = \sum_{\ell \geq 0} \mathbb{P}\left( \sum_{n=1}^{\infty} \tilde{Z}_{n,k+1}^{(k)} = \ell \mid \varphi_0 = k \right) s^{\ell}, \quad (4.5)
\]

where \( s \in [0, 1] \) and \( \mathbb{E}_k(\cdot) = \mathbb{E}(\cdot | Y_k = 1) \). Because individuals in \( \{Y_k\} \) may have an infinite number of offspring, these progeny generating functions may be defective, that is, \( g_k(1) = \mathbb{P}_k(Y_{k+1} < \infty) = \hat{q}_k^{(k)} \) may be less than 1.

By the branching processes equation (4.4), the generating function of \( Y_{k+1} \), conditional on \( Y_i = 1 \) for \( i \leq k \), is given by

\[
g_{i\rightarrow k}(s) := g_i \circ g_{i+1} \circ \cdots \circ g_k(s), \quad s \in [0, 1].
\]

Consequently, by Corollary 8, we have \( \hat{q}_i^{(k)} = g_{i\rightarrow k}(0), \hat{q}_k^{(k)} = g_{i\rightarrow k}(1), g_i = \lim_{k \to \infty} g_{i\rightarrow k}(0) \), and \( \hat{q}_i = \lim_{k \to \infty} g_{i\rightarrow k}(1) \).

The next two lemmas provide respectively an explicit and an implicit relation between the sequence of progeny generating functions \{\( g_k(\cdot) \)\} and the progeny generating function \( G(\cdot) \) of the original LHBP. The first lemma makes use of the following technical assumption:

**Assumption 4** For all \( k \geq 0 \),

\[
\mathbb{P}_k \left( \lim_{n \to \infty} \sum_{i \in T_k} Z_{n,i}^{(k)} \to 0 \right) + \mathbb{P}_k \left( \lim_{n \to \infty} \sum_{i \in T_k} Z_{n,i}^{(k)} \to \infty \right) = 1. \quad (4.6)
\]

Note that \( Z_{n,i}^{(k)} = \tilde{Z}_{n,i}^{(k)} \) for all \( n \geq 0 \) and \( i \in T_k \). Sufficient conditions for (4.6) can be found in Section 2.2.

**Lemma 8** If Assumption 4 holds, then for all \( k \geq \varphi_0 \), the progeny generating function of \( \{Y_k\} \) at generation \( k \) is given by

\[
g_k(s) = \lim_{n \to \infty} G^{[k,n]}_k(s_0, s_1, \ldots, s_k, s), \quad s \in [0, 1],
\]

where \( (s_0, s_1, \ldots, s_k) \in [0, 1)^T_k \) and \( G^{[k,n]}(\cdot) \) is defined in (4.2).
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Proof. By (4.3) and (4.5),

\[ g_k(s) = \mathbb{E}_k \left( s^{ \lim_{n \to \infty} Z_{n,k+1}^{(k)} } \mathbb{1} \left\{ \lim_{n \to \infty} Z_{n,k+1}^{(k)} < \infty \right\} \right). \quad (4.7) \]

By Assumption 4 and the fact that \((s_0, \ldots, s_k) \in [0, 1)^{T_k}\),

\[ P_k \left( \lim_{n \to \infty} \prod_{i=0}^{k} s_i^{Z_{n,i}^{(k)}} = 0 \right) + P_k \left( \lim_{n \to \infty} \prod_{i=0}^{k} s_i^{Z_{n,i}^{(k)}} = 1 \right) = 1, \]

that is, \(\lim_{n \to \infty} \prod_{i=0}^{k} s_i^{Z_{n,i}^{(k)}} \) is an indicator function. In addition, Lemma 7 implies

\[ \{ \omega \in \Omega : \lim_{n \to \infty} Z_{n,k+1}^{(k)}(\omega) < \infty \} = \{ \omega \in \Omega : \lim_{n \to \infty} \tilde{Z}_n^{(k)}(\omega) = 0 \} \]

\[ = \{ \omega \in \Omega : \lim_{n \to \infty} \prod_{i=0}^{k} s_i^{Z_{n,i}^{(k)}}(\omega) = 1 \} \]

\[ = \{ \omega \in \Omega : \lim_{n \to \infty} \prod_{i=0}^{k} s_i^{Z_{n,i}^{(k)}}(\omega) = 1 \}. \]

Thus, (4.7) can be rewritten as

\[ g_k(s) = \mathbb{E}_k \left( \lim_{n \to \infty} s^{Z_{n,k+1}^{(k)} \prod_{i=0}^{k} s_i^{Z_{n,i}^{(k)}}} \right) \]

\[ = \lim_{n \to \infty} G_k^{(k,n)}(s_0, s_1, \ldots, s_k, s), \quad (4.8) \]

where (4.8) follows from the dominated convergence theorem. \( \square \)

Lemma 9 For any \( k \geq 0 \), the progeny generating function \( g_k(\cdot) \) satisfies

\[ g_k(s) = G_k(0, g_{0,k}(s), g_{1,k}(s), \ldots, g_k(s), s). \quad (4.9) \]

Proof. By conditioning on the offspring of a type-\( k \) individual in \( \{ \tilde{Z}_n^{(k)} \} \),

\[ g_k(s) = \mathbb{E} \left[ \sum_{z \geq 0} \tilde{Z}_n^{(k)} \mathbb{1} \left\{ \sum_{z \geq 0} \tilde{Z}_n^{(k)} < \infty \right\} \right] \mathbb{1} \left\{ \varphi_0 = k \right\} \]

\[ = \sum_{z \geq 0} \mathbb{E} \left[ \sum_{z \geq 0} \tilde{Z}_n^{(k)} \mathbb{1} \left\{ \sum_{z \geq 0} Z_n^{(k)} < \infty \right\} \right] \mathbb{1} \left\{ \varphi_0 = k, \tilde{Z}_1^{(k)} = z \right\} \mathbb{P}[\tilde{Z}_1^{(k)} = z | \varphi_0 = k]. \]
4.4 Fixed points and extinction probabilities

Then, by the Markov property and the independence between the daughter processes of individuals from the same generation,

\[
\mathbb{E} \left[ \sum_{n=0}^{\infty} \frac{\tilde{Z}_{n,k+1}^{(k)}}{Z_{n,k+1}^{(k)}} \right] \mid \varphi_0 = k, \tilde{Z}_1^{(k)} = (z_0, \ldots, z_k, z_{k+1})
\]

\[
= s^{z_{k+1}} \prod_{i=0}^{k} \mathbb{E} \left[ \sum_{n=0}^{\infty} \frac{\tilde{Z}_{n,k+1}^{(k)}}{Z_{n,k+1}^{(k)}} \right] \mid \varphi_0 = i \right] z_i
\]

\[
= s^{z_{k+1}} \prod_{i=0}^{k} g_{i \rightarrow k}(s)^{z_i},
\]

(4.10)

where (4.10) follows from the branching process equation (4.4). This leads to

\[
g_k(s) = \sum_{z \geq 0} s_{z+k} \prod_{i=0}^{k} g_{i \rightarrow k}(s)^{z_i} s^{z_{k+1}} \mathbb{P}[\tilde{Z}_1^{(k)} = z \mid \varphi_0 = k],
\]

which completes the proof. \(\square\)

4.4 Fixed points and extinction probabilities

In this section we characterise the set \(S = \{s \in [0,1]^\mathcal{X} : s = G(s)\}\). Due to the lower Hessenberg structure, \(S\) is one-dimensional: indeed, the fixed point equation \(s = G(s)\) can be written as

\[
s_0 = G_0(s_0, s_1)
\]

\[
s_1 = G_1(s_0, s_1, s_2)
\]

\[
s_2 = G_2(s_0, s_1, s_2, s_3)
\]

\[
\vdots
\]

Thus, because \(G(\cdot)\) is a monotone increasing function in each of its variables, for any \(s \in S\), each entry \(s_i\) is uniquely determined by \(s_0\). We therefore define the one-dimensional projection sets of \(S\),

\[
S_i = \{x \in [0,1] : \exists s \in S, \text{ such that } s_i = x\}, \quad i \in \mathcal{X}.
\]

As most of the results in this section somehow rely on Lemma 8, we shall assume that Assumption 4 generally holds here.
4.4.1 Fixed points of the embedded process

The main results in this section rely on the relation between the set of fixed points $S$ and the set

$$\mathcal{S}^e = \{ s \in [0,1]^X : s_k = g_k(s_{k+1}) \forall k \geq 0 \},$$

which can be seen as the set of fixed points of the embedded GWPVE. Because each $g_k(\cdot)$ is a monotone increasing function, like $S$, the set $\mathcal{S}^e$ is one-dimensional. The next two lemmas establish the relationship between $S$ and $\mathcal{S}^e$, dealing separately with the irreducible and reducible cases. For any vector $s \in S$, we shall write $\bar{s}^{(k)} := (s_0, s_1, \ldots, s_k)$ for the restriction of $s$ to its first $k + 1$ entries.

**Lemma 10** If $\{Z_n\}$ is irreducible then $S = \mathcal{S}^e \cup \{1\}$.

**Proof.** Suppose $s \in S$ and $s \neq 1$. For all $k, n \geq 0$, $\bar{s}^{(k+1)}$ satisfies the fixed point equation

$$\bar{s}^{(k+1)} = G[k](\bar{s}^{(k+1)}),$$

which implies that $\bar{s}^{(k+1)} = G[k,n](\bar{s}^{(k+1)})$ for any $n \geq 0$. To see why, observe that (4.11) is a system of $k + 2$ equations where the first $k + 1$ equations correspond to those of $s = G(s)$, and the $(k + 2)$-nd is $s_{k+1} = s_{k+1}$, which is satisfied trivially. Because $\{Z_n\}$ is irreducible and $s \neq 1$ we have $s_i < 1$ for all $i \in X$ (see [66, Theorem 2]). Thus, by Lemma 8 we have $g_k(s_{k+1}) = \lim_{n \to \infty} G_k[n](\bar{s}^{(k+1)}) = s_k$ for all $k \geq 0$ leading to $s \in \mathcal{S}^e$.

Now suppose $s \in \mathcal{S}^e$, then by Lemma 9, for all $k \geq 0$

$$s_k = g_k(s_{k+1}) = G_k(0, s_{k+1}, g_{1-k}(s_{k+1}), \ldots, g_k(s_{k+1}), s_{k+1}) = G_k(s),$$

therefore $s \in S$.  

We now assume that $\{Z_n\}$ is reducible, but retain the assumption that $M_{i,i+1} > 0$ for all $i \geq 0$. In this case, there is a non-empty set $R$ that contains all types $k$ such that there is no path from $k + 1$ to $k$ in the mean progeny representation graph of $\{Z_n\}$. We order the elements of $R = \{r_1, r_2, \ldots\}$ such that $r_i < r_{i+1}$, and we let $R_i = \{r_{i-1} + 1, r_{i-1} + 2, \ldots, r_i\}$ for all $i \geq 1$, where we set $r_0 = -1$. Note that by the construction of $R$, the sub-matrices $\tilde{M}^{(R_i)} = (M_{ij})_{i,j \in R_i}$ are irreducible. We define the subset $R^* \subseteq R$ such that

$$R^* = \{ r_i \in R : \rho(\tilde{M}^{(R_i)}) > 1 \}.$$  \hspace{1cm} (4.12)
4.4. Fixed points and extinction probabilities

Lemma 11 If \( \{Z_n\} \) is reducible then the set of fixed points is \( S = S^{[c]} \cup \tilde{S} \cup \{1\} \), where

\[
\tilde{S} := \{ \tilde{q}^{(r_i)} : r_i \in R^s \}
\]

contains finitely or countably infinitely many distinct elements.

Proof. By the arguments in the proof of Lemma 10, any \( s \in S \) such that \( s_i < 1 \) for all \( i \in \mathcal{X} \) belongs to \( S^{[c]} \). Thus, for any \( s \in S \setminus S^{[c]} \) there exists \( j \in \mathcal{X} \) such that \( s_j = 1 \). In this case for all \( n \geq 0 \) we have

\[
1 = s_j = G_j^{(n)}(s) = \sum_{z} p_j(Z_n = z) \prod_{i=0}^{\infty} s_i^z, \tag{4.13}
\]

which implies that \( s_i = 1 \) for all \( i \in \mathcal{X} \) such that there exists a path from \( j \) to \( i \) in the mean progeny representation graph of \( \{Z_n\} \). By the assumption, \( M_{i,i+1} > 0 \) for all \( i \geq 0 \), this implies \( s_\ell = 1 \) for all \( \ell > j \). As a consequence, any \( s \in S \setminus S^{[c]} \) can be written as \( (\tilde{s}^{(k)}, 1) \), for some \( k \geq 0 \) where \( \tilde{s}^{(k)} \in [0,1)^{T_k} \).

In addition, because the matrices \( \tilde{M}^{(R_i)} \) are irreducible, we have \( k = r_i \) for some \( i \geq 1 \). The vector \( \tilde{s}^{(r_i)} \) satisfies the finite system of equations \( \tilde{s}^{(r_i)} \) has a unique solution to this system of equations with all entries strictly less than 1, which, when it exists corresponds to \( (\tilde{q}^{(r_i)})_{\ell \in T_{r_i}} \). Because \( M_{i,i+1} > 0 \) for all \( i \geq 0 \), such a solution exists if and only if \( \tilde{q}^{(r_i)} \) has a unique solution to this system of equations with all entries strictly less than 1, which, when it exists corresponds to \( (\tilde{q}^{(r_i)})_{\ell \in T_{r_i}} \). By Theorem 3, this is the case if and only if \( \rho(\tilde{M}^{(R_i)}) > 1 \). □

Remark 1 The vector \( \tilde{q} \) can belong to both \( S^{[c]} \) and \( \tilde{S} \), but otherwise the two sets are mutually exclusive.

Remark 2 Because, for any \( i \geq 1 \), there is no path from the types in \( T_{r_i} \) to those in \( T_{r_i} \) in the mean progeny representation graph, we have \( \tilde{q}^{(r_i)} = q(T_{r_i}) \) (see page 21 for the definition of \( q(A) \)). Thus, each additional fixed point corresponds to the probability of extinction in a specific set of types.

4.4.2 One dimensional projection sets

We are now in a position to completely characterise the one-dimensional projection sets \( S_i \) and identify which elements of \( S \) correspond to the global and partial extinction probability vectors.
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Figure 4.2: A visual representation of possible sets $S_i$ in the irreducible case.

**Theorem 15** If $S = \{1\}$ then $q = \hat{q} = 1$; otherwise

$$q = \min S \quad \text{and} \quad \hat{q} = \sup S \setminus \{1\}.$$  

In particular,

$$S_i = [q_i, \hat{q}_i] \cup \{q_i^{(r_j)}\}_{r_j \in R^*} \cup 1 \quad \text{for all } i \geq 0.$$  

**Proof.** We show that

$$q = \min S^{[e]} \quad \text{and} \quad \hat{q} = \max S^{[e]},$$

and for any $i \geq 0$, $S_i^{[e]} = [q_i, \hat{q}_i]$, where

$$S_i^{[e]} = \{x \in [0,1] : \exists s \in S^{[e]} \text{ such that } s_i = x\}.$$  

These results follow from the fact that $g_i(\cdot)$ and $g_i^{-1}(\cdot)$ are monotone increasing functions, and therefore so are $g_{i\to j}(\cdot)$ and $g_{i\to j-1}(\cdot) := g_{j-1}^{-1} \circ \cdots \circ g_i^{-1}(\cdot)$ for $j > i$. Let $s \in S^{[e]}$, then for all $0 \leq i < k,$

$$q_i^{(k-1)} = g_{i\to k-1}(0) \leq s_i = g_{i\to k-1}(s_k) \leq g_{i\to k-1}(1) = \hat{q}_i^{(k-1)}.$$  

Taking the limit as $k \to \infty$ we obtain $q_i \leq s_i \leq \hat{q}_i$ for all $i \geq 0$, which shows (4.14). Now suppose $q_i \leq s_i \leq \hat{q}_i$. For any $j < i$, define $s_j := g_{j\to i-1}(s_i)$; then

$$q_j = g_{j\to i-1}(q_i) \leq s_j \leq g_{j\to i-1}(\hat{q}_i) = \hat{q}_j.$$  

Similarly, for any $j > i$, define $s_j := g_{i\to j-1}^{-1}(s_i)$; then

$$q_j = g_{i\to j-1}^{-1}(q_i) \leq s_j \leq g_{i\to j-1}^{-1}(\hat{q}_i) = \hat{q}_j.$$
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This shows that for any \( i \geq 0 \) and for any \( s_i \in [q_i, \tilde{q}_i] \), it is possible to construct a vector \( s \) belonging to \( S^{(c)} \).

Theorem 15 essentially states that, in the irreducible case, \( S \) either contains 1, 2 or uncountably many elements. More specifically, it states that \( q \) is the componentwise minimal element of \( S \) which is the beginning of a continuum of elements whose supremum is reached by \( \tilde{q} \). An illustration is given in Figure 4.2.

In the reducible case, the number of fixed points in between \( \tilde{q} \) and 1 is at most the cardinality of the set \( R^* \), which is countable. Because each element of \( \tilde{S} \) corresponds to a sub-process causing \( \{Y_k\} \) to have a positive probability of explosion, we may question whether it is possible to have

\[
q < \tilde{q} \quad \text{and} \quad |\tilde{S}| = \infty
\]  \hspace{1cm} (4.15)

simultaneously. In Example 5 we demonstrate that this is indeed possible.

**Example 5.** Suppose \( \{Z_n\} \) has the progeny generating function given by

\[
G_i(s) = \left( \frac{1}{3^{i+1}} s_1^{3^{i+2}} + 1 - \frac{1}{3^{i+1}} \right) \left( \frac{2}{3} s_1^{2} + \frac{1}{3} \right), \quad i \geq 0.
\]

The corresponding mean progeny representation graph is illustrated in Figure 4.3. In this example type-\( i \) individuals can only give birth to type-\( i \) and type-(\( i + 1 \)) offspring. They give birth to two type-(\( i + 1 \)) offspring with probability \( 2/3 \) and no type-(\( i + 1 \)) offspring with probability \( 1/3 \). In addition, they independently give birth to \( 3^{i+2} \) type-\( i \) offspring with probability \( 1/3^{i+1} \) and no type-\( i \) offspring with probability \( 1 - 1/3^{i+1} \). Consequently, the expected number of type-\( i \) offspring born to a type-\( i \) parent is constant in \( i \), however the probability that a type-\( i \) parent has any type-\( i \) offspring decays geometrically with \( i \).

The proof of the next proposition is given in Section 4.7.

**Proposition 6** For Example 5 (4.15) holds.
For this example, there are therefore countably infinitely many fixed points larger than $\tilde{q}$ and uncountably many fixed points smaller than $\tilde{q}$. Note that by Theorem 12 on page 34 the fixed points larger than $\tilde{q}$ form a decreasing sequence that converges to $\tilde{q}$. In Figure 4.4 we give an illustration of the projection set $S_0$ computed numerically using the methods described in Section 2.6.

### 4.4.3 The infinite dimensional set $S$

Now that we have thoroughly explored the properties of the one-dimensional projection sets $S_i$ we move on to deriving results for the infinite-dimensional set $S$. We begin by providing a sufficient condition for

$$\lim_{i \to \infty} s_i = 1, \quad \text{for all } s \in S\setminus\{q\}, \quad (4.16)$$

that is, for $S$ to contain at most a single element $q$ whose entries do not converge to 1. Note that we could still have $\lim_{i \to \infty} q_i = 1$. For LHBPs this generalises Theorem 5 on page 17. Let

$$p_i^{(1)} := \sum_{v : |v| = 1} p_i(v)$$

be the probability that an individual of type $i$ in $\{Z_n\}$ has exactly one offspring.

**Theorem 16** If

$$\sum_{i=0}^{\infty} (1 - p_i^{(1)}) = \infty \quad (4.17)$$

then (4.16) holds.

The proof of Theorem 16 uses the following lemma. We state this lemma separately because, for LHBPs, it implies that Assumption 2 in Theorem 14 on page 56, can be replaced by (4.17).

**Lemma 12** If (4.17) holds then $\mathbb{P}(Y_k \to 0) + \mathbb{P}(Y_k \to \infty) = 1$. 

""
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**Proof.** By Theorem 10 on page 30 we have \( \mathbb{P}(Y_k \to 0) + \mathbb{P}(Y_k \to \infty) = 1 \) if and only if
\[
\sum_{k=0}^{\infty} (1 - g_k'(0)) = \infty, \tag{4.18}
\]
where \( g_k'(0) \) can be identified as the probability that individuals in generation \( k \) of the embedded process have exactly one offspring. While this result was proven in the setting of non-defective generating functions \( \{g_k(\cdot)\} \), it still holds in the defective case. It is thus sufficient to prove that (4.17) implies (4.18).

Suppose (4.17) holds, and assume that
\[
\mathbb{P}(Y_k = 0, \forall k \geq 0) > 0. \tag{4.19}
\]
In this case, there can be only finitely many \( k \) such that \( g_k'(0) = 0 \). Thus, repeating the arguments that immediately follow Theorem 10, (4.19) holds if and only if there exists \( \ell \geq 0 \) such that
\[
\prod_{k=\ell}^{\infty} g_k'(0) = \mathbb{P}_\ell(Y_k = 1, \forall k \geq \ell) > 0. \tag{4.20}
\]
In addition, because \( M_{i,i+1} > 0 \) for all \( i \geq 0 \), in every generation of the embedded process (including any for which \( g_k'(0) = 0 \)) individuals have a positive chance of giving birth to at least one offspring. In combination with (4.20) this implies that there exists \( c > 0 \) such that for any \( \ell \geq 0 \),
\[
\mathbb{P}_\ell(Y_k \geq 1, \forall k \geq \ell) \geq c. \tag{4.21}
\]
Now, recall that each individual in \( Y_k \) corresponds to an individual in \( \{Z_n\} \) (see Figure 4.1). Note that if the corresponding individual in \( \{Z_n\} \) has no offspring then the individual in \( Y_k \) also has no offspring. On the other hand, if the corresponding individual in \( \{Z_n\} \) has two or more offspring, and two of these offspring have at least one descendant each that appears in \( Y_{k+1} \) (which, by (4.21) happens with probability greater than or equal to \( c^2 \)), then the individual in \( Y_k \) has at least two offspring. Thus, for all \( \ell \geq 0 \),
\[
1 - g_k'(0) \geq c^2 (1 - p_k^{(1)}),
\]
which implies
\[
\sum_{k=0}^{\infty} (1 - g_k'(0)) \geq c^2 \sum_{k=0}^{\infty} (1 - p_k^{(1)}) = \infty.
\]
Proof of Theorem 16. For any \( s \in \tilde{S} \cup \{1\} \) the result is immediate, so by Lemma 11 we need only to prove the result when \( s \in S^c \). In this case, for all \( k \geq 0 \), we have

\[
s_0 = g_{0 \to k-1}(s_k) = \mathbb{E}_0 \left( s_k^{Y_k} \mathbb{I}\{Y_k < \infty\} \right) = q_0^{(k-1)} + \mathbb{E}_0 \left( s_k^{Y_k} \mathbb{I}\{0 < Y_k < \infty\} \right).
\]

(4.22)

Suppose \( \liminf_k s_k < 1 \). In this case there exists an infinite sequence \( \{k_1, k_2, \ldots\} \) such that \( s_{k_i} < 1 - \varepsilon \) for all \( i \geq 1 \) and some \( \varepsilon > 0 \). For each \( i \geq 1 \) and \( K \geq 1 \), we have

\[
\mathbb{E}_0 \left( s_{k_i}^{Y_{k_i}} \mathbb{I}\{0 < Y_{k_i} < \infty\} \right) \leq \mathbb{P}_0(0 < Y_{k_i} < K) + (1 - \varepsilon)^K.
\]

By Lemma 12 for any \( K \geq 1 \) we have \( \mathbb{P}_0(0 < Y_{k_i} < K) \to 0 \) as \( i \to \infty \). Thus, by letting \( K \) increase to infinity, we obtain

\[
\liminf_k \mathbb{E}_0 \left( s_{k_i}^{Y_{k_i}} \mathbb{I}\{0 < Y_{k_i} < \infty\} \right) = 0,
\]

and, because \( q_0^{(k)} \to q_0 \) as \( k \to \infty \), from (4.22) we obtain \( s_0 = q_0 \). This shows that the only element \( s \in S^c \) such that \( \liminf_k s_k < 1 \) is \( s = q \). □

Now that we have general sufficient conditions for \( 1 - s_i \to 0 \), we investigate properties of this convergence. The next two theorems use the following lemma.

Lemma 13: If \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) are sequences of non-negative real numbers such that \( a_n \in (0, 1) \) for all \( n \geq 0 \), and \( b_n \to \infty \), then

\[
\limsup_n a_n^{b_n} = \exp\{-\liminf_n b_n(1 - a_n)\} \quad \text{and} \quad \liminf_n a_n^{b_n} = \exp\{-\limsup_n b_n(1 - a_n)\}.
\]

(4.23)

(4.24)

Proof. For any \( n \geq 0 \) we have

\[
a_n^{b_n} = \left( 1 - \frac{b_n(1 - a_n)}{b_n} \right)^{b_n}.
\]

The result then follows from the identity \( \lim_{n \to \infty} (1 - c/n)^n = e^{-c} \), for any \( c \in \mathbb{R} \). □
Theorem 17 If Condition (4.17) holds then, for any \( s \in S \setminus \{ \tilde{S} \cup 1 \} \) we have

\[
\lim_{k \to \infty} \frac{1 - q_i}{1 - s_i} = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{1 - \tilde{q}_i}{1 - s_i} = 0.
\]

Proof. Suppose \( s \in S \setminus \{ \tilde{S} \cup 1 \} \). In that case, by Theorem 15, \( q \leq s \leq \tilde{q} \). In addition, by Lemma 11, for all \( k \geq 0 \), we have on one hand

\[
s_0 = E_0(s_k Y_k \mathbf{1}\{Y_k < \infty\}) = q_0^{(k-1)} + E_0( s_k \mathbf{1}\{0 < Y_k < \infty\} )
\]

and, on the other hand,

\[
s_0 = E_0(s_k Y_k \mathbf{1}\{Y_k < \infty\}) = \tilde{q}_0^{(k-1)} + E_0( s_k \mathbf{1}\{0 < Y_k < \infty\} ) \tag{4.25}
\]

Without loss of generality we assume that \( q_0^{(k)} \to q_0 \) and \( \tilde{q}_0^{(k)} \to \tilde{q}_0 \), we have

\[
s_0 = q_0 \iff \lim_{k \to \infty} E_0( s_k \mathbf{1}\{0 < Y_k < \infty\} ) = 0
\]

\[
s_0 = \tilde{q}_0 \iff \lim_{k \to \infty} E_0( s_k \mathbf{1}\{0 < Y_k < \infty\} ) = 0
\]

Because \( s_k Y_k \) is nonnegative and uniformly bounded by 1 these statements are equivalent to

\[
s_0 = q_0 \iff \mathbb{P}_0 \left( s_k Y_k \to 0 \mid \forall k \geq 0, 0 < Y_k < \infty \right) = 1
\]

\[
s_0 = \tilde{q}_0 \iff \mathbb{P}_0 \left( s_k Y_k \to 1 \mid \forall k \geq 0, 0 < Y_k < \infty \right) = 1
\]

By Lemma 12 we may then apply Lemma 13 to obtain

\[
s_0 = q_0 \iff \mathbb{P}_0 \left( Y_k (1 - s_k) \to \infty \mid \forall k \geq 0, 0 < Y_k < \infty \right) = 1
\]

\[
s_0 = \tilde{q}_0 \iff \mathbb{P}_0 \left( Y_k (1 - s_k) \to 0 \mid \forall k \geq 0, 0 < Y_k < \infty \right) = 1,
\]

which leads to the result. \( \square \)

We now give a physical interpretation to the fixed points \( s \) such that \( q < s < \tilde{q} \). The next theorem demonstrates that the rate at which \( 1 - s_i \)}
decays is closely linked to the asymptotic growth of \( \{Y_k\} \). In this context, we define a *growth rate* to be a sequence of real numbers, \( \{C_k\}_{k \geq 0} \) such that
\[
\lim_{k \to \infty} \frac{Y_k}{C_k} = W(\{C_k\}) \quad \text{exists a.s.,}
\]
where \( W(\{C_k\}) \) is a non-negative, potentially defective, random variable with \( \mathbb{P}(0 < W(\{C_k\}) < \infty) > 0 \). We denote the generating function of \( W(\{C_k\}) \) by
\[
g_{W(\{C_k\})}(z) = \mathbb{E}_0 \left( z^{W(\{C_k\})} 1 \{ W(\{C_k\}) < \infty \} \right).
\]
Growth rates of non-defective GWPVEs (\( \tilde{q} = 1 \)) have been studied by a number of authors. Although it is natural to assume that \( \{m_{0\rightarrow k}\} \) is a growth rate, it may not always be the case. Sufficient conditions for \( \{m_{0\rightarrow k}\} \) to be a growth rate are given in [43] and conditions for it to be the only distinct growth rate are discussed in [12, 21]. Examples of GWPVEs with multiple growth rates can be found in [22, 52]. We now show how this feature transfers over to the decay rates of \( 1 - s_i \).

**Theorem 18** Suppose Condition (4.17) holds. If \( s \in S \setminus \{\tilde{S} \cup 1\} \) and there exists some growth rate \( \{C_k\} \) such that
\[
g_{W(\{C_k\})}(0) < s_0 < g_{W(\{C_k\})}(1),
\]
then
\[
\lim_{k \to \infty} (1 - s_k)C_k = c \in (0, \infty),
\]
where \( c \) is such that
\[
s_0 = g_{W(\{C_k\})}(e^{-c}).
\]

**Proof.** By the arguments in the proof of Lemma 10, any \( s \in S \setminus \{\tilde{S} \cup 1\} \) is such that \( s_i < 1 \) for all \( i \in \mathcal{X} \), and \( s \in S^{[e]} \). Therefore, for all \( k \geq 1 \),
\[
s_0 = \mathbb{E}_0 \left( s_k^{Y_k} 1 \{ Y_k < \infty \} \right) = \mathbb{E}_0 \left( s_k^{Y_k} \right),
\]
which can be rewritten as
\[
s_0 = \mathbb{E}_0 \left( s_k^{Y_k/C_k} 1 \{ W(\{C_k\}) = 0 \} \right) + \mathbb{E}_0 \left( s_k^{C_k/C_k} 1 \{ 0 < W(\{C_k\}) < \infty \} \right) + \mathbb{E}_0 \left( s_k^{C_k/C_k} 1 \{ W(\{C_k\}) = \infty \} \right).
\]
By assumption we have
\[
g_{W(\{C_k\})}(0) = \mathbb{P}_0 (W(\{C_k\}) = 0) < s_0 < g_{W(\{C_k\})}(1) = \mathbb{P}_0 (W(\{C_k\}) < \infty).
\]
If \( \liminf_k (s_k)^{C_k} = 0 \), then taking \( \liminf_k \) in (4.28) gives
\[
s_0 \leq \mathbb{P}_0(W(\{C_k\}) = 0),
\]
which contradicts (4.29). Similarly, if \( \lim \sup_k (s_k)^{C_k} = 1 \), then taking \( \lim \sup_k \) in (4.28) gives
\[
s_0 \geq \mathbb{P}_0(W(\{C_k\}) < \infty),
\]
which also contradicts (4.29). Consequently
\[
0 < \lim \inf_k s_k^{C_k} \leq \lim \sup_k s_k^{C_k} < 1. \tag{4.30}
\]
By (4.27) we then have
\[
\begin{align*}
    s_0 &= \lim \sup_{k \to \infty} \mathbb{E}_0 \left( (s_k)^{Y_k/C_k} \right) \\
         &= \mathbb{E}_0 \left( \lim \sup_{k \to \infty} (s_k)^{Y_k/C_k} \right) \tag{4.31} \\
         &= \mathbb{E}_0 \left( \lim \sup_{k \to \infty} s_k^{C_k} W(\{C_k\}) \right), \tag{4.32}
\end{align*}
\]
where (4.31) follows from the dominated convergence theorem, and (4.32) requires (4.30). If we repeat the same argument with \( \lim \sup \) replaced by \( \lim \inf \), we finally obtain
\[
s_0 = g_W(\{C_k\})(\lim \sup_k s_k^{C_k}) = g_W(\{C_k\})(\lim \inf_k s_k^{C_k}).
\]
By (4.17) and Theorem 16 we have \( s_k \to 1 \) and thus through (4.30) we obtain \( C_k \to \infty \). Lemma 13 then gives \( \lim_{k \to \infty} (s_k)^{C_k} = e^{-c} \), where \( c = \lim_{k \to \infty} (1 - s_k) C_k \). This means \( s_0 = \mathbb{E}_0 \left( e^{-W(\{C_k\})} \right) = g_W(\{C_k\}) (e^{-c}). \)

We conclude this section with a summary of our findings on the set \( S \). The set \( S \) contains a continuum of elements whose minimum is \( \tilde{q} \) and whose maximum is \( \tilde{q} \). When \( \{Z_n\} \) is irreducible, \( S \) is made up entirely of this continuum of elements, with the additional fixed point \( 1 \), whereas when \( \{Z_n\} \) is reducible it may have a countable number of extra elements which are greater than \( \tilde{q} \). Each of these additional elements correspond to an extinction probability vector \( q(T_k) \), for some \( k \geq 0 \). With the possible exception of \( q \), for any \( s \in S \) we have \( 1 - s_i \to 0 \) as \( i \to \infty \). The decay rates of \( 1 - q_i \) and \( 1 - \tilde{q}_i \) are unique, whereas the intermediate elements \( q < s < \tilde{q} \) may share one or several decay rates, which have a one-to-one correspondence with the growth rates of \( \{Y_k\} \).

Furthermore, these intermediate elements completely specify the generating function \( g_W(\{C_k\})(\cdot) \) and thereby the distribution of \( W(\{C_k\}) \). This gives a physical meaning to the intermediate elements: in short, they describe the
evolution of \( \{Y_k\} \) when there is partial extinction without global extinction. While this physical interpretation is in terms of the growth of \( \{Y_k\} \), we expect that it is closely related to the growth of \( \{|Z_n|\} \).

### 4.5 Extinction Criteria

In Section 2.4 we provided a partial extinction criterion, and we stated that to derive a global extinction criterion it remains to develop a rule for partitioning the categories \( q < \tilde{q} = 1 \) and \( q = \tilde{q} = 1 \). In other words, it remains to construct a global extinction criterion that applies when \( \tilde{q} = 1 \). In this section we derive such a criterion that is valid under additional second moment assumptions. We highlight the importance of these additional assumptions in Example 6 where we describe a family of processes with a common mean progeny matrix, in which \( q = 1 \) for some processes and \( q < 1 \) for others.

#### 4.5.1 Factorial moments

Under the assumption \( \tilde{q} = 1 \), the embedded GWPVE is non-explosive, and the results described in Section 2.5 apply. In particular, Equation (2.22) and Theorem 11 (see page 31), which use the first and second factorial moments

\[
\mu_k := g'_k(1) \quad \text{and} \quad a_k := g''_k(1).
\]

In the next lemma we derive recursive expressions for these quantities in terms of the first and second factorial moments of the offspring distribution in \( \{Z_n\} \).

We use the shorthand notation

\[
G'_{k,i}(s) := \frac{\partial G_k(u)}{\partial u_i} \bigg|_{u=s} \quad \text{and} \quad G''_{k,ij}(s) := \frac{\partial^2 G_k(u)}{\partial u_i \partial u_j} \bigg|_{u=s},
\]

and we let \( A_{k,ij} = G''_{k,ij}(1) \). We also take the convention that \( \prod_{i=k}^{k-1} = 1 \) and \( g_{k+1\rightarrow k}(s) = s \).

**Lemma 14** Suppose \( \tilde{q} = 1 \), then

\[
\mu_0 = \frac{M_{01}}{1 - M_{00}} \quad \text{and} \quad a_0 = \frac{\mu_0^2 A_{0,00} + A_{0,11} + 2\mu_0 A_{0,01}}{1 - M_{00}}, \quad (4.33)
\]

and for \( k \geq 1 \),

\[
\mu_k = \frac{M_{k,k+1}}{1 - \sum_{i=0}^{k} M_{ki} \prod_{j=i}^{k-1} \mu_j}, \quad (4.34)
\]
4.5. Extinction Criteria

and

\[
a_k = \frac{\sum_{i=0}^{k} M_{ki} \sum_{j=1}^{k-1} \alpha_j \left( \prod_{\ell=i}^{j-1} \mu_{\ell} \right) \left( \prod_{\ell=j+1}^{k} \mu_{\ell}^2 \right)}{1 - \sum_{i=0}^{k} M_{ki} \prod_{\ell=i}^{k-1} \mu_{\ell}} + \frac{\sum_{i=0}^{k+1} \sum_{j=0}^{k+1} \left( \prod_{\ell=i}^{j} \mu_{\ell} \right) \left( \prod_{\ell=j+1}^{k} \mu_{\ell} \right) A_{k,ij}}{1 - \sum_{i=0}^{k} M_{ki} \prod_{\ell=i}^{k-1} \mu_{\ell}}.
\]  

(4.35)

**Proof.** By Lemma 9, for any \( k \geq 0 \),

\[
g_k'(s) = \frac{d}{ds} [G_k(g_0 \rightarrow k(s), \ldots, g_{k+1} \rightarrow k(s))] = \sum_{i=0}^{k+1} g_i'(s) G_{k,i}(g_0 \rightarrow k(s), \ldots, g_{k+1} \rightarrow k(s)),
\]  

(4.36)

where \( g_i'(s) = \prod_{j=1}^{k} g_j'(g_{j+1} \rightarrow k(s)) \). The assumption \( \tilde{q} = 1 \) implies \( g_i \rightarrow k(1) = 1 \) for all \( i, k \), and therefore

\[
\mu_k = g_k'(1) = \sum_{i=0}^{k} M_{ki} \prod_{j=i}^{k} \mu_j + M_{k,k+1},
\]

which leads to the expression for \( \mu_0 \) and the recursive Equation (4.34).

Next, by differentiating (4.36) with respect to \( s \), we obtain

\[
g_k''(s) = \sum_{i=0}^{k+1} g_i''(s) G_{k,i}(g_0 \rightarrow k(s), \ldots, g_{k+1} \rightarrow k(s))
\]

\[
+ \sum_{i=0}^{k+1} g_i'(s) \sum_{j=0}^{k+1} g_j'(s) G_{k,ij}(g_0 \rightarrow k(s), \ldots, g_{k+1} \rightarrow k(s)),
\]

where, for \( 0 \leq i \leq k \),

\[
g_i''(s) = \sum_{j=1}^{i-1} \left( \prod_{\ell=i}^{j-1} g'_{\ell}(g_{\ell+1} \rightarrow k(s)) \right) g_j''(g_{j+1} \rightarrow k(s)) \left( \prod_{\ell=j+1}^{k} g'_{\ell}(g_{\ell+1} \rightarrow k(s)) \right)^2.
\]

This implies

\[
a_k = g_k''(1) = \sum_{i=0}^{k} M_{ki} \sum_{j=1}^{k} \alpha_j \left( \prod_{\ell=i}^{j-1} \mu_{\ell} \right) \left( \prod_{\ell=j+1}^{k} \mu_{\ell}^2 \right) + \sum_{i=0}^{k+1} \sum_{j=0}^{k+1} \left( \prod_{\ell=i}^{j} \mu_{\ell} \right) \left( \prod_{\ell=j+1}^{k} \mu_{\ell} \right) A_{k,ij},
\]

which gives,

\[
a_k \left( 1 - \sum_{i=0}^{k} M_{ki} \prod_{\ell=i}^{k-1} \mu_{\ell} \right) = \sum_{i=0}^{k} M_{ki} \prod_{j=i}^{k-1} \alpha_j \left( \prod_{\ell=i}^{j-1} \mu_{\ell} \right) \left( \prod_{\ell=j+1}^{k} \mu_{\ell}^2 \right) + \sum_{i=0}^{k+1} \sum_{j=0}^{k+1} \left( \prod_{\ell=i}^{j} \mu_{\ell} \right) \left( \prod_{\ell=j+1}^{k} \mu_{\ell} \right) A_{k,ij},
\]
leading to the expression for $a_0$ and the recursive Equation (4.35). 

When $\tilde{q} = 1$ is not assumed in Lemma 14, the recursive expressions (4.34) and (4.35) can still be used to compute two sequences, which may not correspond to the first and second factorial moments of the progeny distributions of $\{Y_k\}$, but which we shall even so denote by $\{\mu_k\}$ and $\{a_k\}$. For these sequences to correspond to well defined moments, their elements must be non-negative and finite, that is, the denominator common to (4.34) and (4.35) must be strictly greater than 0 for all $k \geq 0$. Thus, if we let

$$x_k := \sum_{i=0}^{k} M_{ki} \prod_{j=i}^{k-1} \mu_j,$$

we require

$$0 \leq x_k < 1 \quad \text{for all } k \geq 0.$$

By giving a physical interpretation to $x_k$ we now show that, in the irreducible case, (4.37) holds if and only if $\tilde{q} = 1$.

**Theorem 19** If $\{Z_n\}$ is irreducible then $\tilde{q} = 1$ if and only if $0 \leq x_k < 1$ for all $k \geq 0$.

**Proof.** Recall that for any $k \geq 0$ and $0 \leq i \leq k$ the process $\{E_n^{(i)}(\tilde{Z}_n^{(k)})\}$, is constructed from $\{\tilde{Z}_n^{(k)} : \varphi_0 = i\}$ by taking all type-$i$ individuals that appear in $\{\tilde{Z}_n^{(k)}\}$ and defining the direct descendants of these individuals as the closest (in generation) type-$i$ descendants in $\{\tilde{Z}_n^{(k)}\}$; the process $\{E_n^{(i)}(\tilde{Z}_n^{(k)})\}$ forms a single type Galton-Watson process that becomes extinct if and only if type $i$ becomes extinct in $\{\tilde{Z}_n^{(k)}\}$. Because $M_{i,i+1} > 0$ for all $i \geq 0$, the extinction of type $k$ in $\{\tilde{Z}_n^{(k)}\}$ is almost surely equivalent to the extinction of the whole process $\{\tilde{Z}_n^{(k)}\}$. Hence, for any $k \geq 0$,

$$\tilde{q}^{(k)} < 1 \quad \text{if and only if} \quad m_{E_n^{(k)}(\tilde{Z}_n^{(k)})} > 1,$$

where $m_{E_n^{(k)}(\tilde{Z}_n^{(k)})}$ is the mean number of offspring born to an individual in $\{E_n^{(k)}(\tilde{Z}_n^{(k)})\}$. The value of $m_{E_n^{(k)}(\tilde{Z}_n^{(k)})}$ is obtained by taking the weighted sum of all first return paths to $k$ in the mean progeny representation graph of $\{\tilde{Z}_n^{(k)}\}$. Because $M$ is lower Hessenberg, the weighted sum of all first passage paths from $i$ to $k$ is $\prod_{\ell=1}^{k-1} \mu_\ell$. By conditioning on the progeny of an individual of type $k$ in $\{\tilde{Z}_n^{(k)}\}$ we then obtain

$$m_{E_n^{(k)}(\tilde{Z}_n^{(k)})} = M_{k,0} \mu_0 \mu_1 \cdots \mu_{k-1} + M_{k,1} \mu_1 \mu_2 \cdots \mu_{k-1} + \cdots + M_{k,k} = x_k.$$
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Thus, if $0 < x_k < 1$ for all $k \geq 0$ then $\tilde{q}^{(k)} = 1$ for all $k$, and therefore $\tilde{q} = \lim_{k \to \infty} \tilde{q}^{(k)} = 1$. Similarly, if there exists $k$ such that $x_k > 1$, then $\tilde{q} \leq \tilde{q}^{(k)} < 1$. Now suppose there exists $k$ such that $x_k = 1$. Then by the irreducibility of $\{Z_n\}$ there exists $k^* > k$ such that there is a first return path with strictly positive weight of the form $k \rightarrow k + 1 \rightarrow \ldots \rightarrow k^* \rightarrow \ldots \rightarrow k$ in the mean progeny representation graph of $\{\tilde{Z}_n^{(k^*)}\}$. This implies

$$m_{E_n^{(k)}}(\tilde{Z}_n^{(k^*)}) > m_{E_n^{(k)}}(\tilde{Z}_n^{(k)}) = 1,$$

and hence

$$\tilde{q} \leq \tilde{q}^{(k^*)} < 1.$$  

\[\square\]

In the reducible case the same result applies with the notable exception of instances where there exists $k \geq 0$ such that $x_k = 1$ and there is no path from $k + 1$ to $k$ in $\{Z_n\}$. Every time this occurs, we can then form a new LHBP by ignoring types $\{0, \ldots, k\}$, which almost surely die out, and consider only types $\{k + 1, k + 2, \ldots\}$. This new process then becomes globally (partially) extinct almost surely if and only if $\{Z_n\}$ becomes globally (partially) extinct almost surely.

Theorem 19 demonstrates that in computing the sequences $\{\mu_k\}$ and $\{a_k\}$ we are also implementing a new partial extinction criterion for LHBP's, that is, $\{\mu_k\}$ and $\{a_k\}$ take finite nonnegative values if and only if $\tilde{q} = 1$. This criterion has high practical relevance: it is indeed more efficient to compute the sequences $\{\mu_k\}$ and $\{a_k\}$ using the recursion provided in Lemma 14 than to evaluate the convergence norm of $M$ as the limit of the sequence of spectral radii of the north-west truncations of the mean progeny matrix $M$ (see [65, Theorem 6.8]).

In the sequel we use the shorthand notation

$$m_{i \rightarrow k} := E_i[Y_{k+1}] = g_i'_{i \rightarrow k}(1) = \prod_{j=i}^k \mu_j.$$

We note that $m_{i \rightarrow k} = F_{ik}(1)$, where $F_{ik}(\cdot)$ is defined on page 10.

4.5.2 Global extinction criterion

We are now in a position to apply Equation (2.22) and Theorem 11 (see page 31) respectively.
Chapter 4. Extinction in lower Hessenberg branching processes

Theorem 20 If $\tilde{q} = 1$ and $A_{k,ij} < \infty$ for all $k, i, j \geq 0$, then for any $1 \leq i < k$,

$$1 - \left[ \frac{1}{m_{i\rightarrow(k-1)}} + \frac{1}{2} \sum_{j=i}^{k-1} \frac{g_j'(0)}{\mu_j m_{i\rightarrow j}} \right]^{-1} \leq q_i^{(k)} \leq 1 - \left[ \frac{1}{m_{i\rightarrow(k-1)}} + \frac{1}{2} \sum_{j=i}^{k-1} \frac{\alpha_j}{\mu_j m_{i\rightarrow j}} \right]^{-1}.$$

Theorem 21 Suppose $\tilde{q} = 1$. If $\sup_{k \geq 0} (\alpha_k/\mu_k) < \infty$ and $\inf_{k \geq m_0} (g_k'(0)/\mu_k) > 0$ for some finite $n_0$, then the following two statements are equivalent

(i) $q = 1$,

(ii) $\sum_{j=0}^{\infty} \frac{1}{m_{0\rightarrow j}} = \infty$.

Note that although we have no simple way to compute $g_k'(0)$ we can often verify that $\inf_{k \geq m_0} (g_k'(0)/\mu_k) > 0$ for instance by using the fact that $g_k'(0) \geq 2p_k(2)e^{k+1}$.

Roughly speaking, Theorem 21 states that the boundary between extinction and potential survival is the expected linear growth of $\{Y_k\}$, that is, $E_0(Y_k) = m_{0\rightarrow k-1} = ck$, for some constant $c > 0$. It is however not immediately clear how to interpret this criterion in terms of the expected growth of $\{Z_n\}$. The next theorem develops a link between condition (ii) in Theorem 21 and the exponential growth rate of the mean total population size $\xi(M) = \lim \inf_n \sqrt[n]{E_0|Z_n|}$.

Theorem 22 Assume $\nu(M) \leq 1$. If $\xi(M) > 1$, then

$$\lim \sup_n \sqrt[n]{m_{0\rightarrow n}} \geq \xi(M), \quad (4.38)$$

and if $\xi(M) < 1$, then

$$\lim \inf_n \sqrt[n]{m_{0\rightarrow n}} \leq \lim \sup_n (E_0|Z_n|)^{1/n}. \quad (4.39)$$

Proof: We have $m_{0\rightarrow(n-1)} = \sum_{k=0}^{n} E_0(Z_{n,k}) m_{k\rightarrow(n-1)}$, where $m_{n\rightarrow(n-1)} := 1$, which gives

$$E_0|Z_n| \inf_{0 \leq k \leq n} m_{k\rightarrow(n-1)} \leq m_{0\rightarrow(n-1)} \leq E_0|Z_n| \sup_{0 \leq k \leq n} m_{k\rightarrow(n-1)}. \quad (4.40)$$

Now suppose $\xi(M) > 1$. In order to prove (4.38) we need to show that

$$\exists n_0 < \infty \text{ such that } \inf_{0 \leq k \leq n} m_{k\rightarrow(n-1)} < 1 \text{ for all } n > n_0. \quad (4.41)$$
Indeed, if (4.41) holds, because \( m_{n \to (n-1)} := 1 \) we have \( \lim sup_n \inf_{0 \leq k \leq n} m_{k \to (n-1)} = 1 \), and thus by (4.40),

\[
\lim sup_n \sqrt[n]{m_{0 \to (n-1)}} \geq \lim sup_n \left( \mathbb{E}_0 |Z_n| \inf_{0 \leq k \leq n} m_{k \to (n-1)} \right)^{1/n} \\
\geq \lim inf_n (\mathbb{E}_0 |Z_n|)^{1/n} \lim sup_n \left( \inf_{0 \leq k \leq n} m_{k \to (n-1)} \right)^{1/n} \\
= \xi(M).
\]

To show (4.41) assume there exists \( n_0 := \sup \left\{ n : \inf_{0 \leq k \leq n} m_{k \to (n-1)} = 1 \right\} < \infty \), and observe that for any \( n \geq 0 \) the recursion

\[
\inf_{0 \leq k \leq n} m_{k \to (n-1)} = \min \left\{ \left( \inf_{0 \leq k \leq n-1} m_{k \to (n-2)} \right)^{\mu_{n-1}}, \mu_{n-1}, 1 \right\}
\]

holds. This implies that for all \( n > n_0 \),

\[
\inf_{0 \leq k \leq n} m_{k \to (n-1)} = \left( \inf_{0 \leq k \leq n-1} m_{k \to (n-2)} \right)^{\mu_{n-1}},
\]

and \( \inf_{0 \leq k \leq n} m_{k \to (n-1)} = m_{n_0 \to (n-1)} \), which gives

\[
m_{0 \to n} \left( \inf_{0 \leq k \leq n} m_{k \to n} \right)^{-1} = m_{0 \to (n_0-1)}, \quad \text{for all } n > n_0.
\]

By Equation (4.40) we then have \( \mathbb{E}_0 |Z_n| \leq m_{0 \to (n_0-1)} \), for all \( n > n_0 \), which contradicts the fact that \( \xi(M) > 1 \) and shows (4.41). When \( \xi(M) < 1 \) a similar argument can be used to obtain (4.39).

By Theorem 22, if both \( \lim_n \sqrt[n]{m_{0 \to n}} \) and \( \lim_n (\mathbb{E}_0 |Z_n|)^{1/n} \) exist (which is the case in both examples in Section 4.5.3), then by the root test for convergence,

\[
\xi(M) > 1 \Rightarrow \sum_{j=0}^{\infty} \frac{1}{m_{0 \to j}} < \infty \quad \text{and} \quad \xi(M) < 1 \Rightarrow \sum_{j=0}^{\infty} \frac{1}{m_{0 \to j}} = \infty.
\]

Thus, if \( \xi(M) \neq 1 \) then condition (ii) in Theorem 21 may be replaced by \( \xi(M) < 1 \). One contribution of Theorem 21, which is motivated by the examples in [7], is to provide an extinction criterion applicable even when \( \xi(M) = 1 \), as we illustrate in the next section.
4.5.3 Illustrative examples

We now consider two examples. Example 6 demonstrates that the mean progeny matrix $M$ is not sufficient to determine whether $q < 1$ or $q = 1$. This fact was observed by Zucca in [71, Example 4.4]. However, we note that in his example $\sum_{j \neq i+1} M_{ij} \to 0$ as $i \to \infty$, which means the process behaves asymptotically as a GWPVE. In addition, in treating this example he uses Theorem 8 which requires an explicit expression for the progeny generating function. Through Example 6 we consider a broader class of processes, and prove our results using a simpler argument that does not require an explicit expression for the progeny generating function.

In Example 7 we apply Theorem 21 to a LHBP with $\xi(M) = 1$. This example also motivates Section 4.6 on strong and non-strong local survival.

All proofs related to the results in this section are collected in Section 4.7.

**Example 6.** Consider a LHBP $\{Z_n\}$ with mean progeny matrix

\[
M = \begin{bmatrix}
  b & c & 0 & 0 & 0 & \ldots \\
  a & b & c & 0 & 0 \\
  0 & a & b & c & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix},
\]

and progeny generating vector $G(\cdot)$. We assume that $a, c > 0$ and that $A_{k,ij} \leq B$ for all $k, i, j \geq 0$, for some positive constant $B$. Note that apart from these assumptions we impose no other condition on the progeny generating vector $G(\cdot)$. This family of processes therefore contains that of Example 1 on page 25.

We now consider a modification of $\{Z_n\}$, which we denote by $\{Z_n^{(u)}\}$ for some parameter $u \geq 1$, whose progeny generating vector, $G^{(u)}(s)$, is given by

\[
G^{(u)}_i(s_{i-1}, s_i, s_{i+1}) = \frac{1}{u^i} G_i(s_{i-1}, s_i, s_{i+1}^{[u^i]}) + \left(1 - \frac{1}{u^i}\right) G_i(s_{i-1}, s_i, 1), \ i \geq 0.
\]

This modification decreases the probability that a type-$i$ individual has any type-$(i+1)$ offspring by a factor of $1/[u^i]$ but when the type-$i$ individual does have type-$(i+1)$ offspring, their number is increased by a factor of $[u^i]$, which causes the mean progeny matrix to remain unchanged. Before providing results on the extinction of $\{Z_n^{(u)}\}$ we require the following lemma on branching processes with the tridiagonal mean progeny matrix (4.42).
Lemma 15 Suppose \( \{Z_n\} \) has a mean progeny matrix given by (4.42), then \( \tilde{q} = 1 \) if and only if

\[
\begin{align*}
    b < 1 \quad \text{and} \quad (1 - b)^2 - 4ac & \geq 0, \tag{4.44} \\
\end{align*}
\]

and when (4.44) holds

\[
    \mu_k \not\rightarrow \mu := \frac{1 - b - \sqrt{(1 - b)^2 - 4ac}}{2a} \ \text{as} \ k \to \infty. \tag{4.45}
\]

Note that the partial extinction criterion (4.44) was given previously in [35] and is implied by [11, Theorem 1].

We are now in a position to characterise the global extinction probability of \( \{Z_n^{(u)}\} \).

Proposition 7 Consider the branching processes \( \{Z_n^{(u)}\} \) defined in Example 6, and suppose \( b < 1 \) and \( (1 - b)^2 - 4ac > 0 \). If \( \mu < 1 \) then \( q = 1 \), whereas if \( \mu \geq 1 \), then

\[
    u > \mu \ \Rightarrow \ q = 1 \quad \text{and} \quad u < \mu \ \Rightarrow \ q < 1,
\]

where \( \mu \) is given in (4.45).

Roughly speaking, Proposition 7 states that if the probability that a type-\( k \) individual has any type-(\( k + 1 \)) offspring, which is approximately \( 1/u^k \), decays faster than \( 1/E_0(Y_k) = 1/\mu^k \) as \( k \to \infty \), then \( q = 1 \), whereas if the probability that a type-\( k \) individual has any type-(\( k + 1 \)) offspring decays slower than \( 1/E_0(Y_k) \), then \( q < 1 \). Proposition 7 does not treat the critical case \( u = \mu \).

An important sub-case of Example 6, where this critical case can be treated, is \( u = 1 \), to which Example 1 belongs. Note that this is the only case where the second moments of the offspring distributions are uniformly bounded. For this subclass, when combined with Lemma 12, Proposition 7 yields

Corollary 9 If \( u = 1 \) and (4.17) holds then \( q = 1 \) if and only if \( \mu \leq 1 \).

Through Corollaries 9 and 2, and Lemma 15, we have now proved that in Example 1, parameter values \((i)\) lead to \( q = \tilde{q} < 1 \), and parameter values \((ii)\) lead to \( q < \tilde{q} = 1 \), as stated on page 26.
Example 7. Let \( \{Z_n\} \) have a mean progeny matrix \( M \) such that \( M_{01} = 1 \), and for \( i \geq 0 \),

\[
M_{i,i-1} = \gamma \frac{i + 1}{i} \quad \text{and} \quad M_{i,i+1} = (1 - \gamma) \frac{i + 1}{i}, \quad 0 \leq \gamma \leq 1, \quad (4.46)
\]

with all remaining entries being 0. The mean progeny representation graph corresponding to this process is illustrated in Figure 4.5. We further assume that \( \inf_k p_k(2e_{k+1}) > 0 \) and there exists \( B < \infty \) such that \( A_k,ij \leq B \) for all \( i, j, k \geq 0 \).

In this example, the proportion of type-\((i-1)\) offspring that a type \( i \) parent has is \( \gamma \), and the proportion of type-\((i+1)\) offspring that a type \( i \) parent has is \( 1 - \gamma \). In addition, as the type of the parent increases, they have progressively less children on average. Thus, in some sense, we expect the branching process to be stronger as \( \gamma \) increases, because the population will linger around the more prolific, smaller types for longer.

It is not difficult to show that for this example \( \xi(M) = 1 \) if and only if \( \nu(M) \leq 1 \), which is the case for a range of values of \( \gamma \), as we shall see.

Proposition 8 For the set of branching processes described in Example 7 \( q = 1 \) if and only if \( \gamma = 0 \).

Proposition 8 states that the process experiences almost sure global extinction if and only if type-\( i \) individuals can only have type-\((i+1)\) offspring so that it coincides exactly with the embedded GWPVE.

We choose

\[
G_k(s) = \begin{cases} 
\frac{1}{2} s_1^2 + \frac{1}{2}, & k = 0 \\
\frac{k+1}{4k} (\gamma s_{k-1} + (1 - \gamma) s_{k+1})^2 + \frac{3k-1}{4k}, & k \geq 1,
\end{cases}
\]

which satisfies (4.46), and in Figure 4.6 we plot \( q_0^{(8000)} \approx q_0 \) and \( \tilde{q}_0^{(8000)} \approx \tilde{q}_0 \) for \( \gamma \in [0, 1] \). Although we proved that \( q_0 = 1 \) when \( \gamma = 0 \), we observe that...
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Figure 4.6: The extinction probabilities $q_0^{(8000)}$ (solid) and $q_0^{(8000)}$ (dashed) for $\gamma \in [0, 1]$.

$q_0^{(8000)} \approx 0.95$ for this value of $\gamma$. This is because, when $\gamma = 0$, Theorem 20 implies

$$q_0 - q_0^{(k)} \approx \left( \sum_{\ell=0}^{k} \frac{1}{\ell} \right)^{-1} \sim \log^{-1}(k),$$

so the convergence of $q^{(k)}$ to $q = 1$ is slow. For GWPVEs with $q < 1$, little attention has been paid to this convergence rate in the literature, so for this example not much can be said when $\gamma > 0$. Using Lemma 14 and Theorem 19 we numerically determine that $\tilde{q} = 1$ if and only if $\gamma \leq \gamma^*$ where

$$\gamma^* = \max\{\gamma : 0 < \mu_k < \infty \ \forall k \geq 0\} \approx 0.1625. \quad (4.47)$$

Note that in this particular example, due to properties of the sequence $\{\mu_k\}$, a sufficient condition for $\tilde{q} = 1$ is the existence of some $k$ such that $\mu_k < \mu_{k-1}$ (see the proof of Proposition 8), therefore $\gamma^*$ can be evaluated particularly efficiently. Given $q_0^{(8000)} < q_0 < \tilde{q}_0 < \tilde{q}^{(8000)}$, by visual inspection, the curves of partial and global extinction seem to merge from some value of $\gamma$, however the cut-off is not clear and further analysis is required to pinpoint the precise value. We are also interested in understanding whether this value depends only on the mean progeny matrix or whether other offspring distributions lead to different values. We address these questions in the next section.
4.6 Strong local survival

In Section 2.4.3 we stated that each irreducible infinite-type branching process falls into one of the four categories \( q = \tilde{q} = 1 \), \( q < \tilde{q} = 1 \), \( q < \tilde{q} < 1 \) or \( q = \tilde{q} < 1 \). The results in the previous section deal with the classification of LHBP with \( \tilde{q} = 1 \). In the present section we build on these results to establish a method for determining whether LHBP with \( \tilde{q} < 1 \) experience strong local survival (\( q = \tilde{q} < 1 \)), or non-strong local survival (\( q < \tilde{q} < 1 \)).

Recall that \( \tilde{M}^{(A)} = (M_{ij})_{i,j \in A} \), \( T_k = \{0, 1, \ldots, k\} \), \( \bar{T}_k = \{k+1, k+2, \ldots\} \), \( q(A) \) contains the probability that the set of types \( A \) becomes extinct given the initial type, the process \( \{\tilde{Z}^{(A)}_n\} \) is constructed from \( \{Z_n\} \) by making all types in \( A \) sterile, and the global extinction probability vector of \( \{\tilde{Z}^{(A)}_n\} \) is denoted by \( \tilde{q}^{(A)} \) (which is not to be confused with its partial extinction probability vector). For any \( k \geq 0 \), we partition \( M \) into four components

\[
M = \begin{bmatrix}
\tilde{M}^{(T_k)} & \tilde{M}_{12} \\
\tilde{M}_{21} & \tilde{M}^{(\bar{T}_k)}
\end{bmatrix},
\]

where \( \tilde{M}^{(T_k)} \) is of dimension \((k+1) \times (k+1)\) and the other three submatrices are infinite.

**Theorem 23** If there exists \( k \geq 0 \) such that

(i) \( \tilde{q}_i > 0 \) for all \( i \in T_k \),

(ii) \( \tilde{M}_{21} \) contains a finite number of strictly positive entries,

(iii) \( \rho\left(\tilde{M}^{(T_k)}\right) > 1 \), and

(iv) \( \nu\left(\tilde{M}^{(\bar{T}_k)}\right) \leq 1 \),

then \( \tilde{q} < 1 \), and

\[
q = \tilde{q} \quad \text{if and only if} \quad \tilde{q}^{(T_k)} = 1.
\]

**Proof:** Assume (i)-(iv) holds. If \( \{\tilde{Z}^{(T_k)}_n\} \) endures almost sure global extinction, that is, \( \tilde{q}^{(T_k)} = 1 \), then the process \( \{Z_n\} \) can avoid global extinction only through the presence of the types in \( T_k \), which implies \( q = q(T_k) \). On the other hand, by (iii) we have \( \tilde{q} \leq \tilde{q}^{(k)} < 1 \), and by (iv), the process \( \{\tilde{Z}^{(T_k)}_n\} \) becomes partially extinct with probability 1. Thus, the process \( \{Z_n\} \) can avoid partial extinction only through the presence of the types in \( T_k \), which implies \( \tilde{q} = q(T_k) \). So we have \( q = \tilde{q} < 1 \).
4.6. Strong local survival

Now suppose \( \tilde{q}(T_k) < 1 \). Let the initial type \( \tilde{\varphi}_0 > k \) be such that \( \tilde{q}(T_k) < 1 \). Define \( \ell = \max\{i \in \mathcal{X} : \sum_{j \in T_k} M_{ij} > 0\} \), which by (ii) is finite. By (iv), with probability 1 there exists a generation \( N \) such that \( \tilde{Z}_{n,i} = 0 \) for all \( n \geq N \) and \( 0 \leq i \leq \ell \). This implies that with probability 1 there are finitely many sterile type-\( j \in T_k \) individuals produced over the lifetime of \( \{\tilde{Z}_n\} \).

Like those in Chapter 3, the sterile individuals of \( \{\tilde{Z}_n\} \) can be thought of as seeds in \( \{Z_n\} \), that is, each can be thought of as giving the process \( \{Z_n\} \) an independent chance to avoid partial extinction. Indeed, the process \( \{Z_n\} \) becomes partially extinct if and only if the daughter processes of these seeds in \( \{Z_n\} \) become partially extinct. By (i), regardless of the outcome of \( \{\tilde{Z}_n\} \), there is a positive chance that all these daughter processes become partially extinct. Thus, there is a positive chance that the process becomes partially extinct but not globally extinct. Because global extinction implies partial extinction, \( E_g \subseteq E_p \), we then have \( P_{\tilde{\varphi}_0}(E_p) = P_{\tilde{\varphi}_0}(E_g) + P_{\tilde{\varphi}_0}(E_p \setminus E_g) \), where \( P_{\tilde{\varphi}_0}(E_p \setminus E_g) > 0 \). This implies \( q < \tilde{q} < 1 \).

When \( \nu \left( \bar{M}(\tilde{T}_k) \right) \leq 1 \), we may apply Theorem 21 to determine whether \( \tilde{q}(T_k) < 1 \). This means that, under additional second moment assumptions, the necessary and sufficient condition for strong local survival (\( q = \tilde{q} < 1 \)) in Theorem 23 can be restated in terms of the mean progeny matrix \( M \).

We are now in a position to answer the questions posed at the end of the previous section. The next result is proved in Section 4.7.

**Proposition 9** For the branching processes described in Example 7,

\[
\begin{align*}
\gamma = 0 & \implies q = \tilde{q} = 1 \\
\gamma \in (0, \gamma^*) & \implies q < \tilde{q} = 1 \\
\gamma \in (\gamma^*, 1/2) & \implies q < \tilde{q} < 1 \\
\gamma \in (1/2, 1] & \implies q = \tilde{q} < 1,
\end{align*}
\]

where \( \gamma^* \) is given in (4.47).

Proposition 9 demonstrates that the curves for partial and global extinction represented in Figure 4.6 merge at \( \gamma = 1/2 \) and that this value is independent of the particular offspring distributions. At the critical value \( \gamma = 1/2 \) there exists no \( k \) satisfying Condition (iv) causing this case to remain untreated.
Chapter 4. Extinction in lower Hessenberg branching processes

Recall that in Corollary 2 (page 42) we proved that in Example 1 (page 25) there exists no combination of parameter values \( a, b, \) and \( c \) such that \( q < \tilde{q} < 1 \). Theorem 23 suggests that a minor modification to this example can lead to \( q < \tilde{q} < 1 \). In particular, it suggests that if we take parameter values \((ii)\) in this example, for which \( q < \tilde{q} = 1 \), and then increase the expected number of type-0 offspring born to a type-0 parent by a large enough value, we obtain \( q < \tilde{q} < 1 \). Indeed, in this case, then the conditions of Theorem 23 hold with \( k = 0 \), and by Corollary 9 we have \( \tilde{q}^{(T_0)} < 1 \). In Figure 4.7 we illustrate the mean progeny representation graph of a modification of Example 1 with parameter values \((ii)\) for which \( q < \tilde{q} < 1 \).

4.7 Proofs related to the examples

**Proof of Proposition 6.** From Figure 4.3 we observe that \( R^* = X \), hence there are countably infinitely many fixed points greater than \( \tilde{q} \). Now if \( \tilde{q}_0 > q_0 \) then there are also uncountably many fixed points less than \( \tilde{q} \). A sufficient condition for \( \tilde{q}_0 > q_0 \) is that there is a positive probability of global survival without any type-\( i \) individual ever having a type-\( i \) offspring. More specifically, if we let

\[
A_i = \{ \omega \in \Omega : Z_{i,i-1} = 0 \}, \quad \text{and} \quad B = \{ \omega \in \Omega : \liminf_{n \to \infty} |Z_n| > 0 \},
\]

then we have

\[
\tilde{q}_0 - q_0 \geq \mathbb{P}_0 \left( (\cap_{i=1}^\infty A_i) \cap B \right) = \mathbb{P}_0 \left( \cap_{i=1}^\infty A_i \mid B \right) \mathbb{P}_0(B) \geq \left( \prod_{i=0}^\infty \left( 1 - \frac{1}{3^{i+1}} \right) \right)^{2^i} q^*,
\]

where \( q^* = 1/2 \) is the extinction probability of the single-type Galton-Watson process with progeny generating function \( f(s) = (2/3)s^2 + (1/3) \), which we can
observe is stochastically dominated by $\{ |Z_n| \}$. Thus, if \( \prod_{i=0}^{\infty} (1 - \frac{1}{3^{i+1}})^{2^i} > 0 \) then \( \tilde{q}_0 > q_0 \). Now,
\[
\prod_{i=0}^{\infty} \left(1 - \frac{1}{3^{i+1}}\right)^{2^i} = \prod_{i=0}^{\infty} \left(1 - \frac{1}{3^{i+1}}\right)^{3^i} (2/3)^i \leq \prod_{i=0}^{\infty} c^{(2/3)^i},
\]
where \( c = 2/3 \), and \( \prod_{i=0}^{\infty} c^{(2/3)^i} = c^{\sum_{i=0}^{\infty} (2/3)^i} = c^3 > 0 \).

**Proof of Lemma 15.** Because (4.44) holds, Lemma 14 gives
\[
\mu_0 = \frac{c}{1 - b}, \quad \text{and} \quad \mu_k = \frac{c}{1 - b - a\mu_{k-1}}, \quad \text{for all } k \geq 0. \quad (4.48)
\]
Because \( \mu_1 > \mu_0 \) and
\[
\mu_k - \mu_{k-1} = \frac{c}{1 - b - a\mu_{k-1}} - \frac{c}{1 - b - a\mu_{k-2}},
\]
by induction the sequence \( \{\mu_k\}_{k \geq 0} \) is strictly positive and increasing. Therefore, under the assumption that \( a > 0, \tilde{q} = 1 \) implies that \( \{\mu_k\} \) converges to a finite limit \( \mu \), where \( \mu \) satisfies the equation \( ax^2 - (1 - b)x + c = 0 \), which has real solutions
\[
x_{\pm} = \frac{1 - b \pm \sqrt{(1 - b)^2 - 4ac}}{2a},
\]
since (4.44) holds. When (4.44) holds we have \( \mu_0 \leq x_- \) which, combined with (4.48) and the fact that
\[
x_- = \frac{c}{1 - b - ax_-},
\]
implies \( \mu_k \leq x_- \) for all \( k \geq 0 \), hence \( \mu_k \not\geq \mu = x_- \).

**Proof of Proposition 7.** Let \( \Delta = (1 - b)^2 - 4ac > 0 \). First, suppose \( u > \mu \). In this case we have
\[
1 - q_0^{(k)} = \frac{E_0(Y_k)}{E_0(Y_k | Y_k > 0)} \leq \frac{\mu^k}{u^{k-1}},
\]
where \( E_0(Y_k) \leq \mu^k \) follows from Lemma 15 and \( E_0(Y_k | Y_k > 0) \geq u^{k-1} \) follows from the fact that the minimum number of type-k offspring born to a type-(\( k - 1 \)) parent is \( u^{k-1} \). This then implies
\[
1 - q_0 = 1 - \lim_{k \to \infty} q_0^{(k)} \leq \lim_{k \to \infty} \frac{\mu^k}{u^{k-1}} = 0,
\]
and therefore \( \mathbf{q} = 1 \) by irreducibility.
Now suppose $1 \leq u < \mu$. Observe that $A^{(u)}_{k;j} = A_{k;j}$ for all $i$ and $j$ with the exception of $A^{(u)}_{k,(k+1)(k+1)} = [u^k]A_{k,(k+1)(k+1)} + c([u^k] - 1)$. By Lemma 14 we then have

$$a_k = \frac{a_{k-1}^2 + [u^k]A_{k,(k+1)(k+1)} + O(1)}{1 - b - a\mu_k} \leq \frac{a_{k-1}^2 + B[u^k] + O(1)}{1 - b - a\mu} = a_{k-1}\mu_k \frac{a \frac{1-b-\Delta^{1/2}}{2a} + B^*u^k + O(1)}{1 - b - \Delta^{1/2}} = a_{k-1}\mu_k \frac{1 - b - \Delta^{1/2}}{1 - b + \Delta^{1/2}} + B^*u^k + O(1),$$

for all $k \geq 0$ and some $B^* < \infty$, which implies

$$a_k = O \left( \left[ \max \left\{ u, \mu \frac{1 - b - \Delta^{1/2}}{1 - b + \Delta^{1/2}} \right\} \right]^k \right).$$

By assumption, $\Delta > 0$ and $u < \mu$ which gives $\max \left\{ u, \mu \frac{1 - b - \Delta^{1/2}}{1 - b + \Delta^{1/2}} \right\} < \mu$. Using the fact that $\mu_k \not\sim \mu$, through application of the root test we then obtain

$$\sum_{k=0}^{\infty} \frac{a_k}{\mu_k m_0 \to_k} < \infty,$$

which, by the right-hand-side of Theorem 20, gives $q_0 < 1$. 

**Proof of Corollary 9.** It remains to show $q = 1$ when $\mu = 1$. Lemma 12 implies $\mathbb{P}_0(Y_k \to 0) + \mathbb{P}_0(Y_k \to \infty) = 1$ and Lemma 15 implies $\mathbb{E}_0(Y_k) = \prod_{i=0}^{k-1} \mu_i \leq 1$ for all $k$, leading to $\mathbb{P}_0(Y_k \to \infty) = 0$ and the result. 

**Proof of Proposition 8.** If $\gamma = 0$ then $\mu_0 = M_0 = 1$, and for $k \geq 1$, $\mu_k = \frac{k+1}{k}$. This gives $m \to k = k + 1$, and therefore

$$\sum_{k=0}^{\infty} \frac{1}{m_0 \to k} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

which by Theorem 21, whose conditions are easily shown to be satisfied, implies $q = 1$.

Now suppose $\gamma > 0$. By Lemma 14, for $k \geq 1$,

$$\mu_k = f_k(\mu_{k-1}) := \frac{k+1}{k} \frac{(1 - \gamma)}{1 - \frac{k+1}{k} \gamma \mu_{k-1}}. \quad (4.49)$$
4.7. Proofs related to the examples

If there exists \( k \) such that \( \mu_k > 1/\gamma \), then by Theorem 19 we have \( q \leq \tilde{q} < 1 \). Assume from now on that \( \tilde{q} = 1 \), which implies that \( 0 \leq \mu_k \leq 1/\gamma < \infty \) for all \( k \), and \( \gamma < 1/2 \). Since \( \mu_0 = M_{01} = 1 \), using Equation (4.49) we can inductively show that \( \mu_k \geq 1 \) for all \( k \geq 0 \). We then have, for any \( k \geq 1 \),

\[
\mu_k \geq \frac{k+1}{k}(1-\gamma) \geq 1 + \frac{1}{k(1-\gamma)}.
\]

(4.50)

The Raabe-Duhamel test for convergence ensures that \( \sum_{k=0}^{\infty} (1/m_{0-k}) < \infty \), since for \( k \geq 1 \),

\[
k(\left(\frac{1/m_{0-(k-1)}}{1/m_{0-k}}\right) - 1) = k(\mu_k - 1) \geq \frac{1}{1-\gamma} > 1.
\]

To complete the proof we must then verify that the conditions of Theorem 21 are satisfied. First, we have \( \inf g''_k(0)/\mu_k \geq \inf 2p_k(2e_{k+1})\gamma > 0 \). Next, by Lemma 14, for all \( k \geq 1 \) and for some constant \( K < \infty \) independent of \( k \),

\[
a_k = a_{k-1} M_{k,k-1} \mu_k^2 + \frac{A_{k,(k-1)(k-1)} \mu_k^2 + 2A_{k,(k-1)(k+1)} \mu_{k-1} \mu_k + A_{k,(k+1)(k+1)}}{1 - M_{k,k-1} \mu_k} \\
\leq a_{k-1} \frac{k+1}{k} \mu_k^2 + K,
\]

since, as \( \tilde{q} = 1 \), the denominator is uniformly bounded away from 0 and we assumed there exists \( B < \infty \) such that \( A_{k,i,j} \leq B \) for all \( i, j, k \geq 0 \). If \( \mu_k \to 1 \), then for large \( k \),

\[
a_k \leq \left(\frac{\gamma}{1-\gamma} + o(1)\right) a_{k-1} + K.
\]

Since \( \gamma < 1/2 \), we have \( \gamma/(1-\gamma) < 1 \), which means that \( \{a_k\} \) is a uniformly bounded sequence. Combining this with the fact that \( \mu_k \geq 1 \) for all \( k \) implies \( \sup a_k/\mu_k < \infty \) and Theorem 21 completes the proof.

It remains to prove that \( \mu_k \to 1 \). Observe that (4.49) implies that if \( \mu_k < \mu_{k-1} \) for some \( k \), then \( \mu_{k+1} < \mu_k \), and thus \( \mu = \lim_{k \to \infty} \mu_k \) exists since \( 1 \leq \mu_k \leq 1/\gamma \) for all \( k \). Taking \( k \to \infty \) in (4.49) we obtain that \( \mu \) satisfies

\[
\mu = \frac{1 - \gamma}{1 - \gamma\mu} = f(\mu),
\]

which means \( \mu \) is either 1 or \((1-\gamma)/\gamma \geq 1 \). The function \( f(x) \) is convex, thus \( f(x) > x \) for all \( x > (1-\gamma)/\gamma \); in addition, by (4.49), \( \mu_{k+1} > f(\mu_k) \) for all \( k \geq 0 \). These imply that if \( \mu_k > (1-\gamma)/\gamma \) for some \( k \), then \( \mu_{k+\ell} \) becomes negative for some \( \ell > 1 \), which is a contradiction. So the sequence \( \{\mu_k\} \) lives
in the open interval \((1, (1 - \gamma)/\gamma)\). Let

\[
v^{(k)}_{\pm} = \frac{1}{2\gamma} \left[ \frac{k}{k+1} \pm \sqrt{\left( \frac{k}{k+1} \right)^2 - 4\gamma(1 - \gamma)} \right]
\]

be the solutions of the equation \(x = f_k(x)\). By the convexity of \(f_k(x)\) for all \(k\), if there exists \(K \geq 1\) such that \(v^{(K+1)}_{-} < \mu_K < v^{(K+1)}_{+}\) then \(\{\mu_k\}_{k \geq K}\) is a decreasing sequence which converges to 1. Suppose \(\mu = (1 - \gamma)/\gamma\). Then \(\mu_K < v^{(K+1)}_{+}\) for some \(K\). We can then construct a LHBP, \(\{Z^n\}_n\), stochastically smaller than \(\{Z^n\}_n\) by selecting a sufficiently large type \(K\) and independently killing each type-\((K + 1)\) child born to a type-\(K\) parent with a probability carefully chosen to ensure \(v^{*(K+1)}_{-} < \mu^*_K < v^{*(K+1)}_{+}\). For this modified process we have \(\mu^* = 1\), and repeating previous arguments, we obtain \(q < q^* < 1\). \(\Box\)

**Proof of Proposition 9.** Given Proposition 8, Lemma 14, and Theorem 19, it remains to show that \(q < q^*\) when \(\gamma \in (\gamma^*, 1/2)\), and \(q = q^*\) when \(\gamma \in (1/2, 1]\). Note that, in either case, because \(q < 1\), there exists \(K_1\) such that \(p(\bar{M}(T_{k_{1}})) > 1\) for all \(k \geq K_1\). In addition, for any \(x > 1\) there exists \(K(x)\) such that \(M_k, k+1 < x(1 - \gamma)\) and \(M_k, k-1 < x\gamma\) for all \(k \geq K(x)\). Because \(\gamma \neq 1/2\), we may choose \(\bar{x} > 1\) small enough so that \(1 - 4\bar{x}^2(1 - \gamma)\gamma > 0\). By Lemma 15, this implies that \(\nu(\bar{M}(T_{k_{1}})) < 1\) for all \(k \geq K := K(\bar{x})\), and

\[
\hat{\mu}_k^{(T_{K})} \leq 1 - \sqrt{1 - 4\bar{x}^2(1 - \gamma)\gamma} \\
2\bar{x}\gamma
\]

for all \(k \geq 0\), where \(\{\hat{\mu}_k^{(T_{K})}\}_{k \geq 0}\) corresponds to \(\{\mu_k\}\) except it is computed using \(\bar{M}(T_{K})\) rather than \(M\).

Assume first that \(\gamma \in (1/2, 1]\). This implies that

\[
1 - \sqrt{1 - 4(1 - \gamma)\gamma} < 1,
\]

so we may choose \(x^* \leq \bar{x}\) small enough, corresponding to \(K^* := K(x^*) \geq K\), so that \(\hat{\mu}_k^{(T_{K^*})} < 1 - \varepsilon\) for all \(k \geq 0\) and some \(\varepsilon > 0\). Hence there exists \(K = \max\{K_1, K^*\} < \infty\) satisfying the conditions of Theorem 23 with \(q^{(T_K)} = 1\).

Now suppose \(\gamma \in (c, 1/2)\). If for any \(K > \bar{K}\) there exists \(k_1 \geq 0\) such that \(\hat{\mu}_k^{(T_{K})} \geq 1\), then by the recursion (4.49) we have

\[
\hat{\mu}_k^{(T_{K})} \geq 1 + \frac{1}{k(1 - \gamma)}, \quad \forall k > k_1,
\]
4.7. Proofs related to the examples

and the result is derived by repeating the steps that follow Equation (4.50) in the proof of Theorem 8. Suppose instead that there exists $K > \bar{K}$ such that

$$\tilde{\mu}^{(T_K)}_k < 1$$

for all $k \geq 0$. Then by Equation (4.49) we have

$$\tilde{\mu}^{(T_K)}_{k-1} < 1 - \frac{1}{\gamma (k + 1)},$$

which implies

$$\prod_{i=0}^{\infty} \tilde{\mu}^{(T_K)}_i = 0.$$  \hspace{1cm} (4.51)

To demonstrate that this leads to a contradiction we compare $M$ to a matrix $M^*$ with strictly smaller entries than $M$. The matrix $M^*$ is such that $M^*_{0,1} = 1 - \gamma$ and

$$M^*_{k,k-1} = \gamma \quad \text{and} \quad M^*_{k,k+1} = 1 - \gamma, \quad \forall k \geq 1,$$

with all other entries 0. The value of $\prod_{i=0}^{\infty} \mu^*_i$, with $\{\mu^*_i\}_{i \geq 0}$ computed using $M^*$, then has a probabilistic interpretation. In particular, $\prod_{i=0}^{\infty} \mu^*_i$ is the probability that a simple random walk on the integers, with transition probabilities $p_+ = 1 - \gamma > p_- = \gamma$, whose initial value is 0, never hits $-1$. When $1 - \gamma > \gamma$ it is well known that this value is non-zero. By the fact that $\tilde{\mu}^{(T_K)} > M^*$ we then have $\tilde{\mu}^{(T_K)}_i > \mu^*_i$ for all $i \geq 0$ which implies $\prod_{i=0}^{\infty} \tilde{\mu}^{(T_K)}_i > \prod_{i=0}^{\infty} \mu^*_i > 0$, contradicting (4.51). \qed
Chapter 5

Extinction in block lower Hessenberg branching processes

5.1 Introduction

In the previous chapter we assumed that \( \{Z_n\} \) was a lower Hessenberg branching process (LHBP), that is, we assumed that individuals of type \( i \) could not give birth to type \( j > i + d \) offspring, where \( d = 1 \). In the present chapter we relax this assumption by allowing \( d \) to be any finite positive integer. With this more general assumption, the mean progeny matrix of \( \{Z_n\} \) can be represented as a block lower Hessenberg matrix, with block size \( d \). We call a multitype Galton Watson branching process (MGWBP) whose mean progeny matrix has this specific structure a block lower-Hessenberg process, or block LHBP for short.

We begin this chapter by extending the partial and global extinction criteria given in Theorems 19 and 21, respectively, to block LHBP. In doing this, we demonstrate that if we embed a Galton Watson process in a varying environment (GWPVE) with \( d \) types, in place of the single-type embedded GWPVE of the previous chapter, then the arguments leading to these extinction criteria generalise naturally.

Extending the results on the set of fixed points \( S \) turns out to be more challenging. This is because unique behaviour arises from the more general block structure. Recall that for any \( A \subseteq \mathcal{X} \), the vector \( q(A) \) gives the proba-
bility that the types in $A$ eventually die out, conditional on the initial type. In addition, recall that $q(A)$ is an element of $S$ (see Section 2.4.1). In an irreducible LHBP ($d = 1$), if the set $A$ contains finitely many types ($|A| < \infty$) then $q(A) = \tilde{q}$, whereas if the set $A$ contains infinitely many types ($|A| = \infty$) then $q(A) = q$. Consequently, in LHBP there are at most two distinct extinction probability vectors corresponding to $q$ and $\tilde{q}$. In a block LHBP, as soon as $d > 1$, more distinct extinction probability vectors come into the picture. Indeed, in an irreducible block LHBP there may exist $A \subseteq \mathcal{X}$ such that $q < q(A) < \tilde{q}$. Thus, if we are to understand the set $S$ in this more general setting, we must be able to characterise the extinction probability vectors $q(A)$. This is our primary objective here.

Properties of the vectors $q(A)$ are typically difficult to derive for general MGWBPs. By considering the specific class of block LHBP, we are able to derive more results; in particular,

(i) we provide sufficient conditions for $q = q(A)$, $q < q(A) < \tilde{q}$ and $q(A) = \tilde{q}$.

(ii) we develop an iterative method to compute $q(A)$ for any set $A$, and

(iii) we make progress towards locating the vectors $q(A)$ in the set $S$.

These results enable us to treat an example where, by varying a single parameter, we can transition smoothly between situations where there exists one, two and four distinct extinction probability vectors. This example provides a unique insight that motivates a number of open questions.

Apart from the recent work in [9, 10], it appears that the study of the extinction probability vectors $q(A)$ for possibly infinite sets $A$ has received little attention in the literature. There are however parallels in the literature on Markov chains with state space $\mathcal{X}$, that is, on singular MGWBPs with countably many types. Two notable contributions come from Blackwell [13], who gives necessary and sufficient conditions for the existence of $A \subseteq \mathcal{X}$ such that $q < q(A) < \tilde{q}$ (see [13, Theorem 2]), and Van Doorn [20], who demonstrates that for the sets $A$ we shall focus on, $q < q(A) < \tilde{q}$ is equivalent to asymptotic periodicity in a related Markov chain.

The chapter is organised as follows. After a brief preliminary section, in Section 5.3 we derive partial and global extinction criteria. In Section 5.4, we develop methods to determine whether there exists $q(A)$ such that
In Section 5.5 we present an iterative method to compute \( q(A) \).
In Section 5.6 we locate the global and partial extinction probability vectors in the set \( S \) for a particular class of block LHBPs. Finally, in Section 5.7 we illustrate our main theorems through two examples, and motivate some open questions.

In order to ease notation, in this chapter we make the following change in convention: the \((i, j)\)-th entry of any matrix \( H \) will be denoted by \( H_{ij} \) rather than \( H(i, j) \) as in the previous chapters. The \( i \)-th entry of a vector \( v \) will continue to be denoted by \( v_i \). In addition, in previous chapters we have let \( 1 \) be a vector whose length is implied by the context. In this chapter we give the reader additional assistance by letting \( 1 \) denote an infinite vector of 1’s, and \( 1_x \) denote an \( x \times 1 \) vector of 1’s. For any vector \( x \) or matrix \( H \), we write \( x < \infty \) and \( H < \infty \) if all of their entries are finite.

5.2 Preliminaries and notation

In this chapter we consider the two-dimensional typeset

\[
\mathcal{X}_d = \{ \langle k, i \rangle : k \in \mathbb{N}_0, i \in \{1, \ldots, d\} \}
\]

for some \( 1 \leq d < \infty \), which naturally arises from the structure of block LHBPs. Observe that we now index the typeset \( \mathcal{X} \) by \( d \). It will be implicitly assumed that the types in any subset \( A \subseteq \mathcal{X}_d \) are ordered lexicographically. We say a type-\( \langle k, i \rangle \) individual is in level \( k \) and phase \( i \). We partition \( \mathcal{X}_d \) in two ways: by level, \( \mathcal{X}_d = \bigcup_{k \geq 0} \ell_k \), where \( \ell_k = \{ \langle k, 1 \rangle, \langle k, 2 \rangle, \ldots, \langle k, d \rangle \} \); and by phase, \( \mathcal{X}_d = \bigcup_{i=1}^d A_i \), where \( A_i = \{ \langle 0, i \rangle, \langle 1, i \rangle, \ldots \} \). Here we let \( T_k = \bigcup_{i=0}^k \ell_i \) be set of types whose level is at most \( k \).

The primary assumption in this chapter is that an individual in level \( k \) cannot have any level \( j > k + 1 \) offspring. In other words, the offspring vector from a level-\( k \) individual belongs to the set

\[
R_{k,d} = \left\{ r \in (\mathbb{N}_0)^{\mathcal{X}_d} : r_j = 0 \quad \forall j \in \bigcup_{i=k+2}^\infty \ell_i \right\}.
\]

While this assumption is made throughout, many of our results hold without it. A consequence of this assumption is that the mean progeny matrix \( M \) has
a block lower Hessenberg structure,

\[
M = \begin{bmatrix}
M_{00} & M_{01} & 0 & 0 & 0 & \ldots \\
M_{10} & M_{11} & M_{12} & 0 & 0 \\
M_{20} & M_{21} & M_{22} & M_{23} & 0 \\
M_{30} & M_{31} & M_{32} & M_{33} & M_{34} \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

where for \( k, l \geq 0 \), \( M_{kl} := (M(i,j))_{i \in \ell_k, j \in \ell_l} \) are square matrices of order \( d \) (recall the change in convention mentioned the in final paragraph of Section 5.1).

In this chapter we apply [9, Theorem 3.3] on a number of occasions. For completeness, we now state and prove this theorem.

Recall that \( q_0(A) = (q_{0,i}(A))_{i \in X_d} \), where \( q_{0,i}(A) \) represents the probability that, given \( \varphi_0 = i \), no type-\( j \in A \) individuals ever enter the population.

**Theorem 24** For any \( A \subseteq X_d \) the following statements are equivalent:

(i) \( q(A) > q \),

(ii) there exists \( i \in X_d \) such that \( q_{0,i}(A) > q_i \),

(iii) there exists \( i \in X_d \) such that there is a positive chance of global survival with \( \varphi_0 = i \) without visiting \( A \).

**Proof.** We obtain (ii)\( \Rightarrow \) (i) from the inequalities \( q(A) \geq q_0(A) \) and \( q(A) \geq q \). The implication (i)\( \Rightarrow \) (ii) follows from the monotonicity of \( G^{(n)}(\cdot) \) for all \( n \): if \( q_0(A) \leq q \), then \( q_n(A) = G^{(n)}(q_0(A)) \leq G^{(n)}(q) = q \) for all \( n \), which implies \( q(A) \leq q \). The relations (ii)\( \Rightarrow \) (iii) and (iii)\( \Rightarrow \) (i) are immediate. \( \square \)

### 5.3 Partial and global extinction criteria

We begin our analysis by deriving a partial and global extinction criteria for block LHBP. The criteria we give can be viewed as the matrix equivalents of Theorems 19 and 21 on pages 94 and 96 respectively.

Both our global and partial extinction criteria are based on the sequence of \( d \times d \) matrices \( \{M_k\}_{k \geq 0} \) recursively defined as

\[
M_k = \sum_{n \geq 0} \left( M^{(k)} \right)^n M_{k,k+1}, \quad k \geq 0,
\]
5.3. Partial and global extinction criteria

where

\[ M^{(0)} = M_{00}, \quad M^{(k)} := \sum_{i=0}^{k} M_{ki} M_{i\rightarrow k-1}, \quad k \geq 1, \quad (5.2) \]

with \( M_{i\rightarrow j} := M_i M_{i+1} \cdots M_j \) for \( i \leq j \) and \( M_{i\rightarrow j} = I \) for \( i > j \) (where \( I \) denotes the identity matrix). We set \( M_k := \infty \) if the series does not converge, that is, if and only if \( \rho(M^{(k)}) \geq 1 \), where \( \rho(\cdot) \) denotes the Perron-Frobenius eigenvalue. If the series converges, then

\[ M_k = \left[ I - M^{(k)} \right]^{-1} M_{k,k+1}, \quad (5.3) \]

and we can compute \( M_k \) recursively. We refer to the matrices \( M_k \) as step-up matrices because of their similarity to the step-down probability matrices \( G^{(k)} \) in [48] defined for level-dependent quasi-birth-and-death processes. The term “step-up” comes from the fact that, when \( M^j \ll 1 \) for \( j = 1, \ldots, k \), the matrix \( M_k \) records the expected number of first passage paths to \( \ell_{k+1} \) that descend from a single individual in \( \ell_k \), or more specifically,

\[ M_k(i, j) = E_{(k,i)} \left( \sum_{n \geq 0} \frac{\mathcal{Z}_n(T_k)}{n(k+1,j)} \right), \]

where \( E(\cdot) := E(|\phi_0 = i) \). We give a proof of this in Lemma 17. In addition to the step up matrices \( \{M_k\} \), our global extinction criterion makes use of three regularity assumptions:

(A1) \( \inf_i p_i(0) > 0 \),

(A2) \( \inf_{k \geq 0, i,j \in \{1, \ldots, d\}} p_{(k,i)}(2e_{(k,j)}) > 0 \),

(A3) \( \sup_k \|A_k\|_{\infty} < \infty \), where the \( d \times d^2 \) matrices \( A_k \) satisfy the recursion

\[ A_0 = \left[ I - M_{00} \right]^{-1} \left[ V_{0,00}(M_{00} \otimes M_{00}) + V_{0,01}(M_{00} \otimes I) + V_{0,10}(I \otimes M_{00}) + V_{0,11} \right], \]

\[ A_k = \left[ I - \sum_{i=0}^{k} M_{ki} M_{i\rightarrow k-1} \right]^{-1} \cdot \left\{ \sum_{i=0}^{k+1} \sum_{j=0}^{k+1} V_{k,ij} \left( M_{i\rightarrow k} \otimes M_{j\rightarrow k} \right) \right. \]

\[ + \sum_{i=0}^{k-1} M_{ki} M_{i\rightarrow j-1} A_j \left( M_{j+1\rightarrow k} \otimes M_{j+1\rightarrow k} \right) \right\}, \quad k \geq 1, \]

the \( d \times d^2 \) matrices \( V_{k,ij} \) have entries

\[ V_{k,ij}(a; b, c) := \left. \frac{\partial^2 G_{(k,a)}(s)}{\partial s_{(i,b)} \partial s_{(j,c)}} \right|_{s = 1}, \]
and \( \otimes \) denotes the Kronecker product.

Assumptions (A1) and (A2) are simple to check in practice, whereas assumption (A3) is more challenging. However, we can gain insight into whether (A3) holds by using the above recursion to compute each \( A_k \) numerically.

**Theorem 25** If \( \{Z_n\} \) is irreducible, then

\[
\mathcal{M}_k < \infty \text{ for all } k \geq 0 \iff \bar{q} = 1, \tag{5.4}
\]

and if \( \bar{q} = 1 \), then under Assumptions (A1)–(A3),

\[
\sum_{k=0}^{\infty} (\mathbf{1}_d^\top \mathcal{M}_{0\rightarrow k} \mathbf{1}_d)^{-1} = \infty \iff q = 1. \tag{5.5}
\]

Before turning to the proof of Theorem 25, we introduce the branching process \( \{Y_k\} \) embedded within \( \{Z_n\} \), whose sample paths are constructed from those of \( \{Z_n\} \) as follows: we define a function \( f_g(\cdot) : \mathcal{J} \rightarrow \mathcal{J} \) that takes each (possibly infinite) line of descent \( (\varphi_0; i_1, j_1, y_1; i_2, j_2, y_2; \ldots) \) and deletes each triple \( (i_k, j_k, y_k) \) whose level is not strictly larger than all its ancestors. For each \( \omega \in \Omega \) the family tree of \( \{Y_k\} \) is then given by \( f_g(X(\omega)) \), where \( X(\omega) \) denotes the family tree of \( \{Z_n\} \); see the middle panel in Figure 5.1 for an example. Observe that generation \( k \) of \( \{Y_k\} \) contains individuals in level \( k \) only. Specifically, it contains the individuals that are the first to enter level \( k \) in their line of descent. To avoid confusion we take the convention that \( \{Y_k\} \) starts at the generation corresponding to the level of the initial type in
5.3. Partial and global extinction criteria

\{Z_n\}. The embedded process \{Y_k\} evolves as a \(d\)-type Galton-Watson process whose offspring distributions vary deterministically with the generation: an individual’s phase corresponds to its type and an individual’s level corresponds to its generation. The process \{Y_k\} is therefore a multitype Galton-Watson process in a varying environment (see for instance [32]). In addition, for the reasons laid out in Section 4.3.1, individuals in \{Y_k\} may have an infinite number of offspring; in this case, we say that \{Y_k\} is explosive. Following the same arguments as those in Corollary 7 (see page 78) we obtain,

\[
\mathcal{E}(X_d) \overset{a.s.}{=} \{ \lim_{k \to \infty} |Y_k| = 0 \},
\]

(5.6)

and

\[
\bigcap_{k=0}^{\infty} \mathcal{E}(T_k) \overset{a.s.}{=} \{ |Y_k| < \infty, \forall k \geq 0 \}.
\]

(5.7)

In other words, \{Z_n\} experiences global extinction if and only if \{Y_k\} experiences extinction, and \{Z_n\} experiences local survival (avoids partial extinction) if and only if \{Y_k\} experiences explosion. This enables us to evaluate whether partial or global extinction occurs in \{Z_n\} simply by observing \{Y_k\).

We denote the progeny generating function of \(Y_k\) at generation \(k\) by \(g_k(s^{(d)}) = (g_k_i(s^{(d)}))_{1 \leq i \leq d}\), where \(s^{(d)} \in [0,1]^d\). For \(\ell \leq k\) we let \(g_{\ell-k}(s^{(d)}) := g_{\ell} \circ g_{\ell+1} \circ \cdots \circ g_k(s^{(d)})\).

**Lemma 16** For any \(k \geq 0\), the progeny generating function \(g_k(\cdot)\) satisfies

\[
g_{k,i}(s^{(d)}) = G_{(k,i)}(g_0 \circ \cdots \circ g_k(s^{(d)}), \ldots, g_k(s^{(d)}), s^{(d)}).
\]

(5.8)

**Proof.** The proof follows the same conditioning argument as that of Lemma 9 on page 80 but in the multitype setting.

We now show that the matrices \{\mathcal{M}_k\} and \{\mathcal{A}_k\} correspond to the first and second factorial moment progeny matrices in \{Y_k\}.

**Lemma 17** If \(\tilde{q} = 1\), then for any \(k \geq 0\),

\[
\mathcal{M}_k(i,j) = \left. \frac{\partial^2 g_{k,i}(s^{(d)})}{\partial s_j^{(d)}} \right|_{s^{(d)}=1}, \quad \text{and} \quad \mathcal{A}_k(i,j,l) = \left. \frac{\partial^2 g_{k,i}(s^{(d)})}{\partial s_j^{(d)} \partial s_l^{(d)}} \right|_{s^{(d)}=1}.
\]

**Proof.** By (5.7) and the assumption \(\tilde{q} = 1\), we have \(|Y_k| < \infty\) almost surely for all \(k\). Thus, \(g_{\ell-k}(1) = 1\) for all \(\ell \leq k\) and \(k \geq 0\). The statement then follows by successive differentiations of (5.8).

\(\square\)
Proof of Theorem 25. By Lemma 17, (5.5) follows from (5.6) and [19, Theorem 2.3]. To obtain (5.4) we embed a second process in \( \mathcal{Z}_n \), this time with the mean progeny matrix \( M^{(k)} \) defined in (5.2). To do this we introduce a function \( f_{p,k}(\cdot) : \mathcal{J} \to \mathcal{J} \) that takes a (possibly infinite) line of descent \((\varphi_0; i_1, j_1, y_1; i_2, j_2, y_2, \ldots)\), and operates in two stages: first, it deletes the descendants of all triples \((i_\ell, j_\ell, y_\ell)\) whose level is strictly larger than \( k \), to obtain the corresponding line of descent in \( \{ \tilde{Z}_n^{(T_k)} \} \); and second, it deletes all remaining triples whose level differs from \( k \) to obtain the restriction (see [48, p118]) of \( \{ \tilde{Z}_n^{(T_k)} \} \) to level \( k \). When the function \( f_{p,k}(\cdot) \) is applied to the random family tree \( X \), the result is a random tree which evolves as a \( d \)-type Galton Watson process; see the right tree in Figure 5.1 as an example. In addition, if \( M_j < \infty \) for all \( j = 0, \ldots, k - 1 \), the mean progeny matrix of this embedded process is indeed given by \( M^{(k)} \). By irreducibility, with probability 1 this embedded process endures extinction if and only if \( \{ \tilde{Z}_n^{(T_k)} \} \) does as well. Thus, invoking the extinction criterion for finite-type processes (see Theorem 3 on page 14), \( \tilde{q}^{(T_k)} = 1 \) if and only if \( \rho(M^{(k)}) \leq 1 \). Thus if \( \rho(M^{(k)}) < 1 \) for all \( k \geq 0 \), then \( \tilde{q}^{(T_k)} = 1 \) for all \( k \), and according to [35, Lemma 3.2] we then have \( \tilde{q} = 1 \). Similarly, if there exists \( k \) such that \( \rho(M^{(k)}) > 1 \) then \( \tilde{q}^{(T_k)} < 1 \) and \( \tilde{q} \leq \tilde{q}^{(T_k)} < 1 \). Finally, if there exists \( k \) such that \( \rho(M^{(k)}) = 1 \), then by irreducibility there exists a path from level \( k \) to itself via a maximum level \( \ell > k \) in the mean progeny representation graph of \( M \), which again leads to \( \tilde{q} \leq \tilde{q}^{(T_k)} < 1 \). \( \square \)

5.4 Extinction in a set of types

We now turn our attention to the more general extinction probability vectors \( q(A) \), in particular, we investigate how to determine when \( q(A) \) differs from \( q \) and \( \tilde{q} \). We begin with a general result that allows us to use \( q_{0,i}(A) \), the probability that a type-\( i \) individual has no descendants in \( A \), to compare extinction probability vectors.

**Theorem 26** Let \( A, B \subseteq \mathcal{X}_d \). If \( \sup_{i \in B} q_{0,i}(A) < 1 \) then \( q(A) \leq q(B) \).

**Proof.** We follow a similar argument to [46, Theorem 2] which was developed in a different context. Let \( \mathcal{F}_n \) denote the history of the process up to generation \( n \). By Lévy’s 0-1 law, for any fixed \( \ell \geq 0 \),

\[
\mathbb{P}(\mathcal{E}_\ell(A)|\mathcal{F}_n) \to \mathbb{1}(\mathcal{E}_\ell(A)) \quad \text{as } n \to \infty
\]  

(5.9)
on a subset $\Omega^*_t$ of the sample space that has probability 1. Let $\Omega^* = \bigcap_{t \geq 0} \Omega^*_t$.

For any outcome $\omega \in \hat{\mathcal{E}}(B) \cap \Omega^*$ (such that $\{Z_n(\omega)\}$ contains individuals with types in $B$ for infinitely many $n$), we have $\mathbb{P}(\mathcal{E}(A) | \mathcal{F}_n)(\omega) < 1 - \varepsilon$ for infinitely many $n$, and for some $\varepsilon > 0$. Thus, by (5.9), $\mathbf{1}(\mathcal{E}(A))(\omega) < 1 - \varepsilon$, that is, $\omega \in \hat{\mathcal{E}}_t(A)$. Since this holds for all $\ell$, $\omega \in \bigcup_{\ell \geq 0} \hat{\mathcal{E}}_\ell(A) = \hat{\mathcal{E}}(A)$. Hence $\mathcal{E}(A) \cap \hat{\mathcal{E}}(B) \subseteq \Omega^*$, leading to

$$P_i(\mathcal{E}(A)) = P_i\left(\mathcal{E}(A) \cap \hat{\mathcal{E}}(B)\right) + P_i\left(\mathcal{E}(A) \cap \hat{\mathcal{E}}(B)\right) \leq P_i(\mathcal{E}(B))$$

for any $i \in \mathcal{X}_d$.

**Corollary 10** Let $A \subseteq \mathcal{X}_d$. If $\{Z_n\}$ is irreducible then $q(A) \leq \bar{q}$, and if in addition $|A| < \infty$ then $q(A) = \bar{q}$.

**Proof.** We first show that if $|A| < \infty$, then $q(A) = \bar{q}$. By irreducibility, the condition of Theorem 26 is satisfied for any finite sets $A$ and $B$. Thus, letting $B = T_k$, we have $q(A) = q(T_k)$ for all $k \geq 0$. Because $\mathcal{E}(T_{k+1}) \subseteq \mathcal{E}(T_k)$, by the monotone convergence theorem,

$$q(A) = \lim_{k \to \infty} q(T_k) = \mathbb{P}\left(\lim_{k \to \infty} \mathcal{E}(T_k)\right) = \bar{q}.$$ 

Now, for any $A \subseteq \mathcal{X}_d$ (not necessarily finite) and $i \in A$ we have $q(A) \leq q(\{i\})$, and by what precedes, $q(\{i\}) = \bar{q}$, therefore $q(A) \leq \bar{q}$. □

Given Corollary 10 we will focus on extinction in infinite sets $A$. In particular, we shall consider sets $A$ belonging to the sigma algebra generated by the phase partition $\{A_i\}$, which we denote by $\sigma(A_1, \ldots, A_d)$. As we will see, even with just two phases ($d = 2$), it is possible for a process to survive in phase one, $A_1$, while enduring extinction in phase two, $A_2$, and vice versa. A concrete example is provided in Section 5.7. Nonetheless, the following result states that if the phases are sufficiently intertwined, then the probability of extinction in any set $A \in \sigma(A_1, \ldots, A_d)$ coincides with the global extinction probability.

**Corollary 11** If $\sup_{\ell \in A} q_{0, \ell}(A_j) < 1$ for all $i, j \in \{1, \ldots, d\}$ then $q(A) = q(\{i\})$ for any $A \in \sigma(A_1, \ldots, A_d)$.

**Proof.** Because $d < \infty$, the condition $\sup_{\ell \in A} q_{0, \ell}(A_j) < 1$ for all $i, j \in \{1, \ldots, d\}$ implies that $\sup_{\ell \in \mathcal{X}_d} q_{0, \ell}(A) < 1$ for any $A \in \sigma(A_1, \ldots, A_d)$. The statement then follows from Theorem 26. □
Corollaries 10 and 11 indicate that, under quite general conditions, \( q(A) = \tilde{q} \) if \(|A| < \infty \) and \( q(A) = q \) if \(|A| = \infty \), the same as in the single-phase LHBP case we studied in Chapter 4. So, when do we have \( q < q(A) < \tilde{q} \)? We begin with a necessary condition.

Recall that \( \{Z_n^{(A)}\} \) is the process formed by making all types in \( A \) sterile, and \( \tilde{q}^{(A)} \) is the corresponding global extinction probability vector.

**Proposition 10** If \( q < q(A) \) then

\[
\tilde{q}^{(A)} < 1. \tag{5.10}
\]

**Proof.** The result is a direct consequence of Theorem 24.

Proposition 10 states that to have \( q < q(A) \), it must be possible for \( \{Z_n\} \) to survive in the types \( \bar{A} \) without any outside assistance from the types in \( A \). To verify (5.10), we can observe that when \( A \in \sigma(A_1, \ldots, A_d) \), \( M^{(A)} \) is block lower Hessenberg; we can then compute the corresponding sequence \( \{\tilde{M}_k^{(A)}\}_{k \geq 0} \) using (5.1) with \( \tilde{M}^{(A)} \) substituted for \( M \), and apply Theorem 25. The matrices \( \{\tilde{M}_k^{(A)}\} \) are also a fundamental ingredient in Theorem 27. In preparation for this theorem, for each level \( k \geq 0 \), we let \( \bar{A}(k) = \bar{A} \cap \ell_k \), and we define

- the vector \( t_k^{(\bar{A})} = (t_{k,i}^{(\bar{A})})_{i \in \bar{A}(k)} \), where
  \[
t_{k,i}^{(\bar{A})} := \sum_{j \in A} M(i, j)
\]
  is the expected total number of direct descendants in \( A \) from a parent of type \( \langle k, i \rangle \in \bar{A} \), and

- the matrix \( \tilde{F}_k^{(\bar{A})} = (\tilde{F}_k(i,j))_{i,j \in \bar{A}(k)} \), where \( \tilde{F}_k^{(\bar{A})}(i,i) := 1 \), and where for \( i \neq j \),
  \[
  \tilde{F}_k^{(\bar{A})}(i,j) := M^{(\bar{A})}(i,j) + \sum_{i_1,i_2,\ldots,i_k \neq j} \tilde{M}^{(\bar{A})}(i,i_1)\tilde{M}^{(\bar{A})}(i_1,i_2)\cdots\tilde{M}^{(\bar{A})}(i_k,j)
\]
  is the weighted sum of first passage paths from \( i \) to \( j \) in level \( k \) in the mean progeny progeny representation graph of \( \tilde{M}^{(A)} \).

We also let \( \tilde{M}_{0 \to k-1}^{(\bar{A})} := \tilde{M}_0^{(\bar{A})} \tilde{M}_1^{(\bar{A})} \cdots \tilde{M}_{k-1}^{(\bar{A})} \), and \( v \) be the number of phases in \( \bar{A} \) so that \( v = |\bar{A}(k)| \) for all \( k \).
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**Theorem 27** Let $A \in \sigma(A_1, \ldots, A_d)$, and assume $\bar{q}^{(\bar{A})} < 1$ and $\nu(M^{(\bar{A})}) < 1$. If, in addition,

(A) $\sum_{k=0}^{\infty} (1_v^T \bar{t}_k^{(\bar{A})}) \bar{M}_0^{(\bar{A})} < \infty$, and

(B) there exists $K < \infty$ such that $\bar{F}_k^{(\bar{A})} \leq K \textbf{1}_v \cdot \textbf{1}_v^T$ for all $k \geq 0$,

then $\bar{q} < q(A)$ and $\bar{q}(\bar{A}) < \bar{q}$.

**Proof.** We first demonstrate that, under the conditions of the theorem, the expected number of sterile individuals produced over the lifetime of $\{\bar{Z}_n^{(\bar{A})}\}$ (those with type in $A$) is finite. Without loss of generality we assume that the process starts with an individual of type $i \in \bar{A}(0)$. Let $\bar{M}^{(\bar{A},n)}(i,j)$ denote the $(i,j)$th entry of the $n$th power of $\bar{M}^{(\bar{A})}$. The expected number of sterile types produced throughout the lifetime of $\{\bar{Z}_n^{(\bar{A})}\}$ is then given by

$$E_i \left( \sum_{n=0}^{\infty} \sum_{l \in A} \bar{Z}_n^{(\bar{A})} \right) = \sum_{k=0}^{\infty} \sum_{j \in A(k)} \sum_{n=0}^{\infty} \bar{M}^{(\bar{A},n)}(i,j) \bar{t}_k^{(\bar{A})}. \quad (5.11)$$

Observe that for any $k \geq 0$, $i \in \bar{A}(0)$, and $j \in \bar{A}(k)$, we have

$$\sum_{n=0}^{\infty} \bar{M}^{(\bar{A},n)}(i,j) = \sum_{l \in A(k)} \bar{M}_0^{(\bar{A})} \bar{F}_k^{(\bar{A})}(i,l) \sum_{n=0}^{\infty} \bar{M}^{(\bar{A},n)}(j,l)$$

$$\leq \left( \frac{1}{1 - \nu(M^{(\bar{A})})} \right) \left( \bar{M}_0^{(\bar{A})} \bar{F}_k^{(\bar{A})} \right)(i,j), \quad (5.12)$$

where (5.12) follows from [65, Theorem A4]. By (5.11), (5.12), and the assumptions of the theorem, we then have in matrix form

$$E_{\bar{A}(0)} \left( \sum_{n=0}^{\infty} \sum_{l \in A} \bar{Z}_n^{(\bar{A})} \right) \leq \left( \frac{1}{1 - \nu(M^{(\bar{A})})} \right) \sum_{k=0}^{\infty} \bar{M}_0^{(\bar{A})} \bar{F}_k^{(\bar{A})} \bar{t}_k^{(\bar{A})}$$

$$\leq \left( \frac{K}{1 - \nu(M^{(\bar{A})})} \right) \sum_{k=0}^{\infty} (\textbf{1}_v^T \bar{t}_k^{(\bar{A})}) \bar{M}_0^{(\bar{A})} < \infty. \quad (5.13)$$

Because this expectation is finite, with probability 1 there exists a generation $n$ after which a sterile type in $A$ never appears in the population. Thus, under the assumption $\bar{q}^{(\bar{A})} < 1$, there exists a type $i \in \bar{A}$ such that starting from $i$ there is a positive chance of global survival without entering the set $A$. Thus by Theorem 24 we have $q < q(A)$. In addition, by the assumption
\( \nu(\tilde{M}(\tilde{A})) < 1, \) if \( \{Z_n\} \) survives in \( \tilde{A} \) but not in \( A \), then it becomes partially extinct with probability one, leading to \( q(\tilde{A}) < \tilde{q} \). \( \square \)

Observe that if for some set \( B \in \sigma(A_1, \ldots, A_d) \) the conditions of Theorem 27 hold with both \( A = B \) and \( A = \tilde{B} \), then \( q < q(B) < \tilde{q} \) and \( q < q(\tilde{B}) < \tilde{q} \).

Condition (A) of Theorem 27 can be verified easily if there are only finitely many edges between \( A \) and \( \tilde{A} \) in the mean progeny representation graph of \( M \), because in that case there are only finitely many values of \( k \) such that \( t_k^{(A)} \) is non-zero. Condition (B) of is of a more technical nature. It holds for example if \( \tilde{A} \) contains a single phase, or if the phases in \( \tilde{A} \) are sufficiently intertwined, or if there is some symmetry between the phases. The next lemma formalises this.

**Lemma 18** Let \( A \in \sigma(A_1, \ldots, A_d) \). If \( \nu(\tilde{M}(\tilde{A})) < 1 \) then each of the following conditions are sufficient for Condition (B) of Theorem 27:

(B1) \( \tilde{A} = A_i \) for some \( i \in \{1, \ldots, d\} \);

(B2) There exists \( \varepsilon > 0 \) such that \( \tilde{F}_k^{(\tilde{A})}(i, j) > \varepsilon \) for all \( i, j \in \tilde{A}(k) \) and \( k \geq 0 \);

(B3) For any \( k, \ell \geq 0 \), \( i \in \tilde{A}(k) \) and \( j \in \tilde{A}(\ell) \), we have \( \tilde{M}(\tilde{A})(i, j) = \tilde{M}(\tilde{A})(j, i) \).

**Proof.** The sufficiency of (B1) is trivial because in this case \( \tilde{F}_k^{(\tilde{A})} = 1 \) for all \( k \geq 0 \). The sufficiency of (B2) and (B3) both follow from the fact that when \( \nu(\tilde{M}(\tilde{A})) < 1 \), for any \( k \geq 0 \) and \( i, j \in \tilde{A}(k) \), \( \tilde{F}_k^{(\tilde{A})}(i, j)\tilde{F}_k^{(\tilde{A})}(j, i) \) is bounded above by the weighted sum of first return paths from \( i \) to \( i \), which, according to the arguments that immediately follow Theorem 9 on page 25, is strictly less than 1. \( \square \)

In the specific case where \( \{Z_n\} \) is singular, that is, each individual produces exactly one offspring with probability one, the process survives with probability 1 (\( q = 0 \)), and the process \( \{Z_n\}_{n \geq 0} \), where \( Z_n := i \Leftrightarrow Z_{n,i} > 0 \), corresponds to an irreducible Markov chain on the state space \( X_d \). The arguments in the proof of Theorem 27 then lead to the following corollary, which can be viewed as the algorithmic complement to the more theoretical result of [20, Corollary 8]. Roughly speaking, this corollary provides a necessary and sufficient condition for \( \{Z_n\} \) to be asymptotically reducible.
5.5. Computational methods

Corollary 12 Let \( A \in \sigma(A_1, \ldots, A_d) \). If \(|Z_n| = 1\) a.s. for all \( n \geq 0 \), then

\[
q(A) > 0 \iff \lim_{k \to \infty} \mathbf{1}_v^\top \left( \prod_{j=k}^{\infty} \tilde{M}_j^{(A)} \right) \mathbf{1}_v > 0. \tag{5.13}
\]

In addition, if \( \sum_{x \in \tilde{A}(k+1)} M_j^{(A)}(\langle k, i \rangle, x) > 0 \) for all \( \langle k, i \rangle \in \tilde{A} \), then the right hand side of (5.13) may be replaced by \( \mathbf{1}_v^\top \left( \prod_{j=0}^{\infty} \tilde{M}_j^{(A)} \right) \mathbf{1}_v > 0. \)

**Proof.** By Theorem 24, \( q(A) > 0 \) if and only if there exists \( i \in \tilde{A} \) such that the probability that the chain \( \{Z_n\} \) never visits \( A \) starting from \( i \) is strictly positive, which is equivalent to the right hand side of (5.13). The additional condition \( \sum_{x \in \tilde{A}(k+1)} M_j^{(A)}(\langle k, i \rangle, x) > 0 \) for all \( \langle k, i \rangle \in \tilde{A} \) ensures that there is no null factor in the product \( \prod_{j=0}^{\infty} \tilde{M}_j^{(A)} \). \( \square \)

5.5 Computational methods

Given the existence of extinction probability vectors \( q(A) \) different from \( \tilde{q} \), we now develop a method to compute them.

For any \( k, l \geq -1 \), we define the finite-type branching process \( \{Z_n^{(k,l)}(A)\} \) on the same probability space as \( \{Z_n\} \) with progeny generating vector \( G^{(k,l)}(A)(s) \) such that

\[
G_i^{(k,l)}(A)(s) = \begin{cases} 
0, & \text{if } i \in A \cap \tilde{T}_k \\
1, & \text{if } i \in \tilde{A} \cap \tilde{T}_l \\
G_i(s) & \text{otherwise},
\end{cases}
\]

and we denote by \( q^{(k,l)}(A) \) its extinction probability vector. In other words, \( q^{(k,l)}(A) \) is the probability that the branching process \( \{\tilde{Z}_n^{(A \cup T_1)}\} \) becomes extinct before producing a type in \( A \cap \tilde{T}_k \) (with \( T_{-1} = \emptyset \) and \( \tilde{T}_{-1} = \lambda_d \)). Following the steps described in Section 3.7.1, \( q^{(k,l)}(A) \) can then be computed for any \( k, l \) using established techniques for finite-type branching processes.

**Theorem 28** If \( \{Z_n\} \) is irreducible then

\[
\lim_{k \to \infty} \lim_{l \to \infty} q^{(k,l)}(A) = q(A).
\]

**Proof.** By Lemma 3.2 of [35], for any fixed value of \( k \), \( \lim_{l \to \infty} q^{(k,l)}(A) \) is the partial extinction probability of the original process modified so that types in
Chapter 5. Extinction in block lower Hessenberg branching processes

$A \cap \tilde{T}_k$ are immortal. Let

$$N(A) := \inf \left\{ N : \sum_{n=N}^{\infty} \sum_{i \in A} Z_{n,i} = 0 \right\}$$

be the first generation from which type-$i \in A$ individuals no longer appear in the population, and let

$$\tau_k(A) := \inf \left\{ k : \sum_{n=0}^{k} \sum_{i \in A \cap \tilde{T}_k} Z_{n,i} > 0 \right\}$$ (5.14)

be the first generation at which a type in $A \cap \tilde{T}_k$ appears in the population. By Corollary 10 and the fact that $|A \cap \tilde{T}_k| < \infty$ for all $k$, we have, for all $i \in \mathcal{X}_l$,

$$\lim_{l \to \infty} \mathbb{P}_i(\{N(A) < \tau_k(A)\}) = \mathbb{P}_i(\{N(A) < \tau_k(A)\}).$$

By the monotone convergence theorem we then have

$$\lim_{k \to \infty} \mathbb{P}_i(\{N(A) < \tau_k(A)\}) = \mathbb{P}_i(\{N(A) < \lim_{k \to \infty} \tau_k(A)\})$$

$$= \mathbb{P}_i(\{N(A) < \infty\})$$

$$= \mathbb{P}_i(\mathcal{E}(A)).$$

While Theorem 28 may be applied in a general setting, it requires both $k$ and $l$ to be increased to infinity separately. From a computational perspective it would be more efficient to set $l = k$ and let them both increase to infinity together. We now derive a sufficient condition ensuring convergence of the resulting sequence.

**Theorem 29** Suppose $\sup_{i \in A} q_i^{(A)} < 1$, then

$$\lim_{k \to \infty} q^{(k,k)}(A) = q(A).$$

**Proof.** First suppose that $\{Z_n\}$ becomes extinct in the set $A$. In this case there exists $K$ such that $\tau_k(A) = \infty$ for all $k > K$, where $\tau_k(A)$ is defined in (5.14). In addition, by the arguments that lead to Corollary 10 there is
5.6. Fixed Points

almost sure partial extinction. This implies that, for \( k > K \) there is almost sure global extinction in \( \{ Z_n^{(k,k)}(A) \} \), leading to \( q(A) \leq \lim \inf_k q^{(k,k)}(A) \).

Now suppose that \( \{ Z_n \} \) survives in the set \( A \). At any generation \( n \) consider the daughter processes of each individual in \( Z_n \) truncated so that all types in \( A \) have no offspring. Note that if one of these truncated daughter processes survives globally then there exists \( K \) such that \( \{ Z^{(k,k)}(A) \} \) survives globally for all \( k > K \). This is because for these values of \( k \) an immortal type \( A \) must eventually be born into the population. Let \( D \) be the event that, throughout the life of \( \{ Z_n \} \) there exists an individual that has a truncated daughter process which survives globally. By assumption there exists \( \varepsilon > 0 \) such that whenever \( (Z_{n,i}) \in A \) is non-empty we have \( \mathbb{P}(D|\mathcal{F}_n) > \varepsilon \). Because \( \{ Z_n \} \) survives in the set \( A \), \( (Z_{n,i}) \in A \) is non-empty for infinitely many values of \( n \), therefore, following the same arguments as in the proof of Theorem 26, the event \( D \) occurs with probability 1. This then implies \( q(A) \geq \lim \sup_k q^{(k,k)}(A) \).

To understand why we impose the sufficient condition \( \sup_{i \in A} q_i^{(A)} < 1 \) in Theorem 29, let us consider an example with two phases where this condition is not satisfied. Assume \( G(s) \) contains entries

\[
G_{(k,i)}(s) = \begin{cases} 
  s_{(k+1,1)} s_{(0,2)}, & \langle k, i \rangle \in A_1 \\
  s_{k+1} s_{(k+1,2)} + \frac{1}{k+2}, & \langle k, i \rangle \in A_2.
\end{cases}
\]

In this case \( q_{(0,1)}(A_2) = 0 \) but \( q_{(0,1)}^{(k,k)}(A_2) = (1 - \frac{1}{k+2})^{k+1} \to e^{-1} \). While this is a reducible example, it highlights a pathology that can also occur in the irreducible setting.

5.6 Fixed Points

We now briefly turn our attention to the set \( S \), which contains the extinction probability vectors \( q(A) \). To allow the results for (single phase) LHBPs derived in Section 4.4.2 on page 83 to be applied directly, we introduce the concept of a block LHPB that is locally isomorphic to a (single phase) LHPB. Roughly speaking, this is a block LHPB which is stochastically equivalent to a LHPB when we sum the number of individuals in each phase, that is, \( (\sum_{j \in A} Z_{n,j})_{k \geq 0} \overset{d}{=} \hat{Z}_n \), for all \( n \geq 0 \), where \( \{ \hat{Z}_n \} \) is a LHPB. More specifically, \( \{ Z_n \} \) is locally isomorphic to a LHPB \( \{ \hat{Z}_n \} \) if for each level \( k \geq 0 \), there exists
a probability distribution \( \hat{p}_k(\cdot) : R_{k,1} \to [0,1] \) such that for any \( u \in R_{k,1} \) and \( j \in \ell_k, \)
\[
\hat{p}_k(u) = \sum_{r \in R_{k,d} \text{ s.t. } \sum_{x \in \ell_i} r_x = u_i \forall i} p_j(r). \tag{5.15}
\]
We define the projection of \( S \) onto the \( k \)th level,
\[
S_k := \{ u \in [0,1]^{\ell_k} : \exists s \in S \text{ with } (s_j)_{j \in \ell_k} = u \}.
\]
The next proposition is the irreducible block analogue of Theorem 15 on page 84.

**Proposition 11** Suppose \( \{Z_n\} \) is locally isomorphic to a LHBP which satisfies the conditions of Theorem 15, then \( q = \min S, \) \( \bar{q} = \sup S \setminus \{1\}, \) and
\[
(xq_{(k,1)} + (1-x)\bar{q}_{(k,1)})1_d \in S_k, \quad \text{for all } x \in [0,1]. \tag{5.16}
\]

**Proof.** Let \( s = [0,1]^{\ell_1} \) and suppose \( s = \hat{G}(s) \). Then for any \( k \geq 0 \) and any \( j \in \ell_k, \)
\[
G_j(s_01_d, s_11_d, \ldots) = \sum_{u \in R_{k,d}} (s_01_d, s_11_d, \ldots)^u p_j(u)
\]
\[
= \sum_{u \in R_{k,1}} s^u \left( \sum_{v \in \ell_i} p_j(v) \right)
\]
\[
= \sum_{u \in R_{k,1}} s^u \hat{p}_k(u) = s_k.
\]
Therefore,
\[
(s_01_d, s_11_d, \ldots) = \hat{G}(s_01_d, s_11_d, \ldots),
\]
and the result follows as a direct consequence of Theorem 15. \( \square \)

In the next section we demonstrate that, for block LBHPs locally isomorphic to a LHBP, not only does \( S_k \) contain the affine function between \( (q_i)_{i \in \ell_k} \) and \( (\bar{q}_i)_{i \in \ell_k}, \) described in (5.16), in some cases this function makes up \( S_k \) in its entirety.

### 5.7 Illustrative examples

We now illustrate our main theorems through two examples, and motivate some open questions.
Example 8. We consider a block LHBP with two phases \((d = 2)\) whose progeny generating vector \(G(s)\) contains entries

\[
G_{(k,i)}(s) = \begin{cases} 
\frac{b}{u} s^{u}_{(0,i)} + \frac{c}{u} s^{u}_{(1,i)} + \frac{y}{u} s^{u}_{(0,1+(-1)^{i+1})} + 1 - \frac{c+b+y}{u}, & (k,i) \in \ell_0 \\
\frac{a}{u} s^{u}_{(k-1,i)} + \frac{b}{u} s^{u}_{(k,i)} + \frac{c}{u} s^{u}_{(k+1,i)} + \frac{y}{u} x s^{u}_{(k,1+(-1)^{i+1})} + 1 - \frac{c+b+y}{u}, & (k,i) \in \hat{\ell}_0,
\end{cases}
\]

where \(a, b, c, y > 0, x \geq 1\) and \(u = [a + b + c + y + 1]\). The corresponding mean progeny representation graph is illustrated in Figure 5.2.

Roughly speaking, Example 8 is constructed by taking two identical copies of Example 1 on page 25, and then letting these copies correspond to the processes \(\tilde{Z}_n^{(A_1)}\) and \(\tilde{Z}_n^{(A_2)}\) restricted to phases 1 and 2 respectively. We then give individuals the chance to have offspring within the same level but in a different phase, according to a probability that decays geometrically at rate \(x\) with the individual’s level.

For this example we can verify that \((5.15)\) holds. Hence the process is locally isomorphic to a LHBP \(\{\tilde{Z}_n\}\). The next proposition highlights the contrasting asymptotic behaviour of the branching process as we vary the geometric decay rate \(x\).

Proposition 12 Suppose \(b + 2\sqrt{ac} < 1\) and

\[\mu := \left(1 - b - \sqrt{(1-b)^2 - 4ac}\right)/2a > 1.\]

We have
(i) if \( x = 1 \) and \( b + y + 2\sqrt{ac} \leq 1 \), then \( q = q(A_1) = q(A_2) < \tilde{q} = 1 \);
(ii) if \( x = 1 \) and \( b + y + 2\sqrt{ac} > 1 \), then \( q = q(A_1) = q(A_2) = \tilde{q} < 1 \);
(iii) if \( x > 1 \), then \( q < \tilde{q} \);
(iv) if \( x > \mu \), then \( q < q(A_1) < \tilde{q} \) and \( q < q(A_2) < \tilde{q} \).

**Proof.** (i) and (ii). Suppose \( x = 1 \). By Corollary 11, \( q = q(A_1) = q(A_2) \).

Note that there is partial (global) extinction in \( \{Z_n\} \) if any only if there is partial (global) extinction in its local isomorphism \( \{\tilde{Z}_n\} \). Denote the mean progeny matrix of \( \{\tilde{Z}_n\} \) by \( \tilde{M} \). This is a tridiagonal matrix with entries 
\[ \tilde{M}(i, i - 1) = a1 \{i \geq 1\}, \tilde{M}(i, i) = b + y, \tilde{M}(i, i + 1) = c \] for \( i \geq 0 \), and 0 otherwise. By [35, Proposition 5.1], \( \nu(\tilde{M}) = b + y + 2\sqrt{ac} \), which means
\[ \tilde{q} = 1 \iff b + y + 2\sqrt{ac} \leq 1, \]
yielding (i). Observe that, when \( x = 1 \), \( \{\tilde{Z}_n\} \) is a branching random walk with an absorbing barrier. Thus, by Proposition 2 (page 41) the extinction probability \( \tilde{q}_{(k,i)} \) is decreasing in \( k \). Consequently, when \( b + y + 2\sqrt{ac} > 1 \) the entries of \( \tilde{q} \) are uniformly bounded away from 1. By Theorem 5 (page 17), \( S \) contains only one such element, which, when combined with the fact that \( q \leq \tilde{q} \), yields (ii).

(iii). Suppose \( x > 1 \). Let \( \{\tilde{Z}_n^{(T_k)}\} \) denote the process \( \{Z_n\} \) taboo on \( T_k \), and let \( \tilde{M}^{(T_k)} = (\tilde{M}^{(T_k)}(i, j))_{i,j \geq 1} \) be its mean progeny matrix, with entries relabelled for convenience. We have \( \tilde{M}^{(T_k)}(i, i) = b + y/x^{k+1} \), \( \tilde{M}^{(T_k)}(i, i+1) = c \) for \( i \geq 1 \), and \( \tilde{M}^{(T_k)}(i, i-1) = a \) for \( i \geq 2 \). In addition, \( b \leq b + y/x^{k+1} \leq b + y/x^{k+1} \) for all \( k \), \( i \). Therefore, by definition of the convergence norm, and by [35, Proposition 5.1], \( b + 2\sqrt{ac} \leq \nu(\tilde{M}^{(T_k)}) \leq b + y/x^{k+1} + 2\sqrt{ac} \) for all \( k \), leading to
\[ \lim_{k \to \infty} \nu(\tilde{M}^{(T_k)}) = b + 2\sqrt{ac} < 1. \]

By Corollary 9 (page 99), for any \( k \geq 0 \) and initial type \( i \in T_k \), \( \{\tilde{Z}_n^{(T_k)}\} \) has a positive chance of global survival. When \( \tilde{q} = 1 \) we then clearly have \( q < \tilde{q} \), whereas when \( \tilde{q} < 1 \) the assertion follows through direct application of Theorem 23 (page 102).

(iv). Suppose \( x > \mu \). We have \( t_k(A_1) = y/x^k \), and by Lemma 15 (page 99) \( \tilde{M}_k^{(A_1)} \to \mu \). Thus,
\[ \lim_{k \to \infty} \left( \tilde{M}_0^{(A_1)} t_k^{(A_1)} \right)^{1/k} = \mu/x < 1. \]
5.7. Illustrative examples

The root test for convergence then gives $\sum_{k=0}^{\infty} M_{0-k}^{(A_1)} t_k^{(A_1)} < \infty$. By Corollary 9 we have $\tilde{q}^{(A_1)} < 1$, and by [35, Proposition 5.1] we have $\nu(\tilde{M}^{(A_1)}) = b + 2\sqrt{ac} < 1$. Thus, we may apply Theorem 27 (page 121) to obtain $q < q(A_2)$ and $q(A_1) < \tilde{q}$. The result then follows by repeating the same arguments with $A_2$ in place of $A_1$.

We choose parameter values $a = 1/5$, $b = 0$, $c = 1$ and $y = 1/5$, and study the extinction probabilities for different values of the parameter $x$. In this case,
\[ b + y + 2\sqrt{ac} \approx 1.09 \text{ and } \mu \approx 1.38. \] Note that, in combination, Corollary 9 and Proposition 2 imply that \( \sup_{i \in A_j} \tilde{q}_i(A_j) < 1 \) for \( j = 1, 2 \). We can then use Theorem 29 to compute \( q(A_1) \) and \( q(A_2) \). The top graph in Figure 5.3 depicts the extinction probabilities \( q(0,1), q(0,1)(A_1), q(0,1)(A_2) \) and \( \tilde{q}(0,1) \) for \( 1 \leq x \leq 3 \). As Proposition 12 suggests, we observe two phase transitions, the first at \( x = 1 \), where the number of distinct extinction probability vectors increases from one to two (even if it only becomes clear slightly after \( x = 1 \)), and the second at \( x = \mu \), where the number of distinct extinction probability vectors increases from two to four. By visual inspection, there exists an infimum value of \( x \) for which \( \tilde{q} = 1 \). Using Theorem 25 we numerically estimate that this value is 1.09.

The bottom nine graphs in Figure 5.3 illustrate the set \( S_0 \) (\( S \) projected on to level 0) for nine values of \( x \) ranging from \( x = 1 \) to \( x = 20 \). In particular, we plot the boundary of \( S_0 \) (the interior also belongs to \( S_0 \)). The projected extinction probabilities \( (q(0,1), q(0,2)), (q(0,1)(A_1), q(0,2)(A_1)), (q(0,1)(A_2), q(0,2)(A_2)), \) and \( (\tilde{q}(0,1), \tilde{q}(0,2)) \) are marked by bold discs. We observe that for small values of \( x \) (i.e. \( x = 1, 1.05, 1.1 \)) the elements in \( S_0 \) cling tightly to the straight line of fixed points connecting \( q \) and \( \tilde{q} \) that we identified in Proposition 11. As \( x \) increases, the set \( S_0 \) inflates until it visibly contains area. Noticeably, this occurs when \( x \leq \mu \) (i.e. \( x = 1.2 \)) as well as when \( x > \mu \). For large values of \( x \) (i.e. \( x = 5, 20 \)), the extinction probabilities \( (q(0,1)(A_1), q(0,2)(A_1)) \) and \( (q(0,1)(A_2), q(0,2)(A_2)) \) appear to correspond to vertices in \( S_0 \), while this is less clear for smaller values of \( x > \mu \) (i.e. \( x = 1.4, 1.6 \)).

Due to the symmetry of the progeny distributions between phases 1 and 2, the level projection sets \( S_k \) are symmetric with respect to the diagonal. The next example considers an asymmetric modification of Example 8.

**Example 9.** We modify Example 8 so that it now has the mean progeny representation graph given in Figure 5.4. In Figure 5.5 we plot the set \( S_0 \) for \( a = 1/5, b = 1/20, \) and \( c = 1 \), and observe that there are only three distinct extinction probability vectors, \( q, q(A_2) \) and \( \tilde{q} \). Indeed, by Corollary 11, \( q(A_1) = q \). Despite the lack of symmetry, this branching process is still locally isomorphic to a single phase LHBP. Thus, by Proposition 11, the set \( S_0 \) still contains the linear segment that connects the global and partial extinction probabilities, \( (q(0,1), q(0,2)) \) and \( (\tilde{q}(0,1), \tilde{q}(0,2)) \). On inspection of Figure 5.5 we see that this linear segment now sits on the boundary of \( S_0 \).
Examples 1 and 2 motivate several questions, which to our knowledge remain open. In particular:

(i) We have only focused on sets $A \in \sigma(A_1, \ldots, A_d)$, leading to a maximum of $2^d$ potentially distinct extinction probability vectors $\mathbf{q}(A)$; we may then ask how many distinct vectors are possible if we consider more general sets of types, and if more than $2^d$ distinct vectors can exist.

(ii) Given the set $S$, we may question whether it is possible to identify which elements correspond to extinction probability vectors.

Inspired by the specific shapes of the projection sets displayed in Figures 5.3 and 5.5, we now conjecture an answer to (ii).

**Conjecture 1** If $\mathbf{q} = \tilde{\mathbf{q}}$ then $S = \{\mathbf{q}, \mathbf{1}\}$, whereas if $\mathbf{q} < \tilde{\mathbf{q}}$ then $S$ contains a continuum of elements, whose minimum is $\mathbf{q}$, and whose maximum is $\tilde{\mathbf{q}}$. In addition, the boundary of any projection set is differentiable everywhere.
except at each point that corresponds to an extinction probability vector $q(A)$ for some $A \subseteq \mathcal{X}_d$. 
Chapter 6

Conclusion

In the introduction of this thesis we listed three fundamental results on the extinction of multitype Galton–Watson branching processes (MGWBPs) with a finite number of types. We then observed that attempts to generalise these results to the infinite-type setting had led to a number of open questions. In particular, (i) the set of fixed points $S$ was not well understood, (ii) the computational aspects of the global extinction probability vector $q$ had received little attention, and (iii) there existed no easily applicable global extinction criterion. We then suggested a reason why the transition from finitely many types to infinitely many types has proven so difficult: when a process contains finitely many types, there is a single distinct extinction probability vector, whereas when a process contains infinitely many types, there can be multiple distinct extinction probability vectors, the two most important being the global extinction probability vector $q$ and the partial extinction probability vector $\tilde{q}$.

In this thesis we have progressed towards a solution to each of the three open questions by focussing on the relationship between $q$ and $\tilde{q}$. We addressed (i) by demonstrating that, for lower Hessenberg branching processes (LHBPs), the set $S$ is made up of a continuum of elements whose minimum is $q$ and whose maximum is $\tilde{q}$—this continuum only exists when $q < \tilde{q}$ (Chapter 4). In addition, we extended this analysis to block LHBPs, and we found that in this more general setting, new extinction probability vectors can arise—these new vectors only exist when $q < \tilde{q}$ (Chapter 5). We addressed (ii) in a general setting by developing a new class of algorithms for computing $q$—proving the convergence of these algorithms is only non-trivial when $q < \tilde{q}$ (Chapter 3). Finally, we addressed (iii) by deriving global extinction criteria...
for LHBPs and block LHBPs—to do this we first needed to assume that $\bar{q} = 1$ (Chapters 4 and 5).

Our approach also suggests a number of future directions of research. To completely resolve (i), it remains to fully characterise $S$ for block LHBPs, and then extend these results to a general setting. With regard to (ii), it remains to study the computational efficiency of the class of algorithms proposed in Chapter 3. Concerning (iii), it remains to extend the global extinction criteria given in Chapters 4 and 5 to the general setting.

One particularly interesting direction of future research is Conjecture 1 that we postulated on page 131 for block LHBPs. We believe that this conjecture actually applies more generally to any irreducible MGWBP with countably many types. If true, this would be a broad extension of the well established fact that $q$ is the minimal nonnegative solution of the fixed point equation $s = G(s)$. 
Bibliography


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