The Isoperimetric Problem in Block Designs

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Submitted in total fulfilment of
the requirements of the degree of
Doctor of Philosophy

July 2018

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To my children, Musa and Muna.

Abstract

This thesis focuses on the problem of determining the isoperimetric numbers of Levi graphs of block designs. We study two versions of this: the vertex-isoperimetric number and the edge-isoperimetric number. In the general case, we manage to obtain strong upper and lower bounds on these numbers for Levi graphs of arbitrary block designs. We also obtain stronger or even exact values for some families of graphs, particularly those arising from finite geometries or difference sets.

More specifically, we obtain the exact value of the vertex-isoperimetric number for Levi graphs of unitals, complements of unitals, complements of the point-hyperplane design $\mathrm{PG}(n, q)$ for all n and q, as well as $\mathrm{PG}(2, q)$ for q up to 16. We also obtain the exact value of the edge-isoperimetric number for Levi graphs arising from difference sets whose host group contains a subgroup of index 2. This includes, for example, all $\mathrm{PG}(n, q)$ for which n and q are both odd.

As a byproduct of this research, we also investigate two additional tools in the process. The first one is the minimum shadow problem, which we use to construct stronger bounds for the vertex-isoperimetric number. This was already studied by Harper [25] and Ure [48] for the projective plane, and includes more general theorems like the Kruskal-Katona theorem. We develop this theory in more depth specifically for block designs, though much of theory can likely be generalised to incidence structures.

The second tool is the sequence $d_{a,1}$ arising from the generalised tangent numbers, which we use to obtain improved isoperimetic inequalities for certain families of graphs. Since this sequence is highly number-theoretical in nature, we are able to exploit this to construct stronger bounds on the edge-isoperimetric number for graphs arising from quadratic residues. We are successful in applying this to $\mathrm{PG}(2, q)$ when q is odd, but not so much for $\mathrm{PG}(n, q)$ for even n larger than 2. We also succeed in using $d_{a,1}$ on Paley difference sets as well as on Paley graphs, answering in the affirmative a conjecture by Cramer et al. [15] regarding the latter.

Declaration

This is to certify that:

(i) the thesis comprises only my original work towards the PhD except where indicated in the preface;

(ii) due acknowledgement has been made in the text to all other material used; and

(iii) the thesis is fewer than 100,000 words excluding tables, maps, bibliographies, and appendices.

Muhammad Adib Surani
July 2018

Preface

In accordance with the Graduate Research Training Policy of the University of Melbourne, some of the research in this thesis was carried out in collaboration with others. My contributions to the results in each chapter are as follows:

- **Chapter 3:** 80%.
- **Chapter 4:** 90%.
- **All other chapters:** 100%.

For each of the following sections, I acknowledge the important contributions of the following people:

- **Sections 3.3, 3.4, 4.4 and 4.5**: Andrew Elvey Price and Sanming Zhou.
- **Sections 3.6 and 4.7**: Alice M. W. Hui and Sanming Zhou.

The following is a list of publications prepared and submitted during my PhD candidature for which I have co-authorship on. These appear as [22] and [34] respectively in the bibliography:

1. Andrew Elvey Price, **Muhammad Adib Surani** and Sanming Zhou. The isoperimetric number of the incidence graph of $\mathrm{PG}(n, q)$. *The Electronic Journal of Combinatorics,* in press, 2018.
2. Alice M. W. Hui, **Muhammad Adib Surani** and Sanming Zhou. The vertex-isoperimetric number of the incidence and non-incidence graphs of unitals. *Designs, Codes and Cryptography,* in press, https://doi.org/10.1007/s10623-018-0498-x, 2018.

Acknowledgements

I am highly indebted to my supervisor Sanming Zhou, whose great insights directed me to a problem that was fairly challenging but just manageable for me to solve. You first taught me the definition of a graph eight years ago in an undergraduate course, and have since guided me through my honours thesis, and now my Ph.D. Thank you for all these years of supervision, and for introducing me to the wonderful yet eccentric world of combinatorics.

I would also like to thank my co-supervisor Charl Ras, as well as the chair of my advisory committee Lawrence Reeves, for their general advice and guidance in seeing me through my candidature.

More specifically, a number of results in this thesis would not have existed, or would have existed in a weaker form without the help of the following people. Firstly, I would like to thank Andrew Elvey Price whose discussion led to improving the coefficient of $p\sqrt{p}$ in Theorem 3.21 from 0.45 to 0.5, back when I had relegated this to a mere conjecture. Secondly, I would also like to thank Alice Hui for her invaluable discussion on all things related to unitals. Our collaboration has led to the exact vertex-isoperimetric numbers of both incidence and non-incidence graphs of unitals, as seen in Section 4.7.

I am also grateful to `mercury` from Math.StackExchange, as well as the On-Line Encylopedia of Integer Sequences, for leading me to discover the intriguing sequence $d_{a,1}$, which forms the basis of Chapter 6. Many thanks in general to Math.StackExchange, MathOverflow, and TeX.StackExchange for always having answers to almost anything I would want to ask.

Finally, I would like to thank my family and friends for their support. I would like to single out my wife Nadiah in this regard – you always kept me believing in myself.

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Nomenclature

Roman Symbols

$a(\Gamma)$	The algebraic connectivity of a graph Γ, page 3
$\mathcal{B}(I)$	The block set of an incidence structure I, page 9
$\mathrm{BM}(\alpha, \beta, q)$	A Buekenhout-Metz unital (BM unital), page 14
$bw(\Gamma)$	The bisection width of a graph Γ, page 61
D	A difference set, page 12
$\mathcal{D}$	A balanced independent block design (BIBD), page 10
$\mathcal{D}_{n,q}$	The point-hyperplane incidence design of $\mathrm{PG}(n, q)$, page 12
$\mathrm{dev}(D)$	The development of a difference set D, page 12
$E(\Gamma)$	Edge set of a graph Γ, page 11
$E(S)$	Edges with both endpoints in S, page 11
$E(S, T)$	Edges with one endpoint in each of S and T, page 11
$\mathbb{F}_q$	The finite field of order q, page 12
$H(q)$	The classical unital of order q, page 13
I	An incidence structure, page 9
$\overline{I}$	The complement of an incidence structure I, page 10
I^*	The dual of an incidence structure I, page 9
$i_E(\Gamma)$	The edge-isoperimetric number of a graph Γ, page 61

$i_V(\Gamma)$	The vertex-isoperimetric number of a graph Γ, page 33
$P(I)$	The point set of an incidence structure I, page 9
$P(q)$	The Paley graph of order q, page 92
$\mathrm{PD}(q)$	The Paley design of order q, page 91
$\mathrm{PG}(n,q)$	The projective space of dimension n and order q, page 12
$\mathrm{PG}_d(n,q)$	The incidence design of points and d-planes of $\mathrm{PG}(n,q)$, page 12
U	A unital, page 13
$V(\Gamma)$	Vertex set of a graph Γ, page 11

Greek Symbols

$\overline{\alpha}(I)$	The incidence-free number of an incidence structure I, page 16
$\partial(S)$	The shadow of S, page 15
$\delta(S)$	The vertex boundary of S, page 32
$\Phi_E(\Gamma,n)$	The edge-isoperimetric parameter of a graph Γ, page 59
$\Phi_S(I,n)$	The minimum shadow parameter of an incidence structure I, page 15
$\Phi_V(\Gamma,n)$	The vertex-isoperimetric parameter of a graph Γ, page 33
Γ	A graph, page 11
$\Gamma_{n,q}$	The Levi graph of $\mathcal{D}_{n,q}$, page 12
$\Gamma(I)$	The Levi graph of an incidence structure I, page 11

Chapter 1

Introduction

1.1 Background

The *classical isoperimetric problem* asks the following question: Out of all closed curves on the Euclidean plane with a given perimeter, which curve encloses the largest area? This problem dates back to antiquity, and in fact it was a common misconception in ancient times that the area enclosed by a curve was determined by its perimeter.

It was the ancient Greeks who first "knew" that the answer must be a circle, though this would not be proven rigorously until much later. Steiner published five proofs of this fact in 1841, though all of them assumed the existence of a solution; it was only in 1879 that Weierstrass rigorously showed that a solution exists, thus completing the proof. The reader is invited to read Blåsjö's excellent expository [9] on the history of this problem.

This problem has since been generalised in various ways, such as in $\mathbb{R}^n$ and other metric spaces, simply by replacing the term "area" with a general measure and "perimeter" with a general boundary. Of particular interest to us is the discrete form of the isoperimetric problem, where the question is to minimise the boundary of a subset of vertices on a graph, over all subsets with a given cardinality. This minimum value is usually denoted by $\Phi(\Gamma, n)$ for a graph Γ and a fixed cardinality n, and a subset of vertices satisfying this extremal property is called an *isoperimetric set*.

There are numerous ways to define a boundary in the context of a graph, and we briefly describe the two that are most studied in the literature. The *vertex boundary* of a subset of vertices S is the set of vertices outside S that are adjacent to at least one vertex in S. The *edge boundary* of S is the set of edges with one endpoint in S and other endpoint outside S. Then the *vertex-isoperimetric problem* (thereafter

VIP) asks to find a set S that minimises the vertex boundary for a fixed $|S|$, while the *edge-isoperimetric problem* (thereafter EIP) asks to find the set that minimises the edge boundary.

One of the earliest, and most fundamental, examples of the discrete isoperimetric problem is on the n-dimensional hypercube Q_n. This is the graph with vertex set $V = \{0,1\}^n$ and two vertices are adjacent if their Hamming distance is one, that is, they differ in exactly one coordinate. Harper [28] showed that taking the first k elements of V under the lexicographical ordering of V always formed an vertex-isoperimetric set. Similarly, but in a different paper, Harper [26] also showed that taking the first k elements of V under the simplicial ordering produces an edge-isoperimetric set.

What is remarkable about these two problems is that they exhibit the *nested solutions* property, that is, that there exists a linear ordering on the vertex set such that taking the first n elements always forms an isoperimetric set. This allows beautiful theories like stabilising symmetries and compression to be used to obtain these solutions [38]. The idea behind stabilising symmetries is that given a set, you can perform operations like “move the points as far left as possible” without increasing its boundary. Then it suffices to only find the stable solutions. Compression on the other hand can be used for graphs that naturally factor into smaller graphs, such as Cartesian products of graphs; see Harper’s monograph [27] for a general treatment into using global morphisms to solve isoperimetric problems. Such techniques can, and have been used successfully to solve the VIP and EIP in a large number of graphs. In addition to Harper’s monograph, we also refer the reader to [8] for a more combinatorial perspective, and [38] for a more probabilistic perspective of the problem.

It is not true in general that nested solutions must exist for the isoperimetric problem. We will show in this thesis that the family of graphs we are studying (the Levi graphs of block designs) do not generally admit nested solutions for the vertex or edge isoperimetric problems. That it contains a large automorphism group is insufficient – the graph needs a much stronger form of symmetry, such as the stabilising symmetry we described above. Since we do not have such strong symmetries, we will usually have to be content with ad-hoc approaches rather than a nice global framework that can easily extend to other families of graphs.

In practice, especially when the nested solutions property is lacking, we will not be able to obtain $\Phi(\Gamma, n)$ exactly. In this case, we usually have to be content with bounds on $\Phi(\Gamma, n)$ instead, in what is known as *isoperimetric inequalities*. For a graph Γ, define the isoperimetric number of Γ to be the minimum value of $\Phi(\Gamma, n)/n$ over all positive integers n of at most half the number of the vertices. This gives, in some sense, a measure of how quickly the boundary is expanding.

We can use the two notions of boundary we have earlier to define the vertex-isoperimetric number i_V and edge-isoperimetric number i_E (also known as the Cheeger constant) of a graph. Compared to the more general VIP/EIP, the isoperimetric numbers arise more naturally in the context of understanding the overall "connectivity" of a graph.

One of the most interesting applications of the isoperimetric numbers come from expander graphs. In turn, expander graphs themselves see applications in probabilistic combinatorics, pure and applied mathematics, theoretical computer science, cryptography, and many others; see [33] for a recent survey. Roughly speaking, expander graphs are increasing families of fixed-degree graphs admitting a "large" isoperimetric number (that is, its limit inferior is strictly positive). From this perspective, the notions of i_E and i_V are equivalent as they differ by at most a constant factor [42].

Spectral graph theory can be used to obtain bounds on the isoperimetric numbers using adjacency or Laplacian eigenvalues [17]. Of particular interest is the inequality

$$i_E(\Gamma) \geq \frac{a(\Gamma)}{2}, \tag{1.1}$$

where $a(\Gamma)$ refers to the algebraic connectivity of Γ, or equivalently the second-smallest Laplacian eigenvalue of Γ. There is a characterisation by Bezrukov [7] of graphs known to attain equality in (1.1), but few explicit families of graphs are known that have this property. This includes the complete graphs, the complete bipartite graphs and the hypercubes, etc. In this thesis, we show that Levi graphs of block designs arising from certain types of difference sets also admit this property.

Another interesting area of research in this direction are the Ramanujan graphs, which are regular graphs whose algebraic connectivity is the "best possible". These were introduced by Lubotzky, Philips and Sarnak [39], who constructed an infinite family of k-regular Ramanujan graphs. The notion of Ramanujan graph has since been extended to (k, r)-biregular graphs [32], and this definition is compatible with the first definition.

As we have discussed, a large body of work has been done in establishing bounds on the isoperimetric numbers (both vertex and edge variants) of various families of graphs. On our part, we will be mainly focusing on the Levi graphs of balanced incomplete block designs (thereafter BIBDs). There are at least two good motivations for this. The first is that these graphs have very strong combinatorial properties, and can even be characterised as the extremal bipartite graphs with respect to some of these properties [32]. For example, Levi graphs of BIBDs are always Ramanujan graphs. This is quite a strong property, because it suggests that these graphs have good connectivity in general and are likely to come up as good examples in many applications. Along the same lines, the Laplacian eigenvalues of block designs are also well-studied, and we get an immediate lower bound on the edge-isoperimetric number from this. That is, if Γ is the Levi graph of a symmetric (v, k, λ)-BIBD, then we know from [32] that $a(\Gamma) = k - \sqrt{k - \lambda}$, so that

$$i_E(\Gamma) \geq \frac{k - \sqrt{k - \lambda}}{2}, \tag{1.2}$$

which we will call the *spectral lower bound* of $i_E(\Gamma)$. There are corresponding spectral bounds for the upper bound of the EIP, and also for the VIP [41], but these are generally not very strong and we do not present them here.

Secondly, symmetry in BIBDs is quite well-studied in the literature, so we can use highly symmetrical BIBDs to great effect in our investigation. In fact, part of our initial motivation was to study a conjecture by Babai [1] regarding the isoperimetric problem in arc-transitive graphs. To that end, a significant portion of this thesis studies the point-hyperplane incidence and non-incidence graphs of $\mathrm{PG}(2, q)$, which are known to be 2-arc-transitive [20].

In tackling the VIP, a third variant of the isoperimetric problem arises naturally. Instead of a graph, we can take subsets of points S on an incidence structure, and have its *shadow* be the set of blocks incident with at least one point in S. This gives us another variant of the isoperimetric problem, called the *minimum shadow problem* (thereafter MSP). The most celebrated instance of this problem would likely be the Kruskal-Katona Theorem [35, 36]. Harper [27] also notes that the EIP is really just an extension of the MSP, making the MSP itself a problem worth studying.

The MSP on $\mathrm{PG}(2, q)$ was first studied by Harper and Hergert [25], although they simply called it "the isoperimetric problem"; in this paper they solved the MSP for $q \in \{2, 3, 4\}$. This result was later extended by Ure [48], who solved the MSP for $q \in \{5, 7\}$, and also showed that the MSP on $\mathrm{PG}(2, 8)$ did not admit nested solutions.

As with the isoperimetric numbers earlier, it proves useful to define a single value that encapsulates the most important properties of the MSP. The invariant we choose here is the *incidence-free number* $\overline{\alpha}$, first introduced in [18], which roughly measures the size of a large number of points and blocks which are non-incident. For a symmetric design, this is the largest value n for which there are n points and n blocks which are non-incident, and this value (along with some variants of it) has been studied by [18] for $\mathrm{PG}(n, q)$, and by [47] for $\mathrm{PG}(2, q)$ and Steiner triple systems.

On our part, we can view the MSP as a "one-sided" variant of the VIP. In doing this, we are able to construct a bound on the VIP using the MSP, and in many cases this bound even turns out to be sharp. It makes sense then, to structure the thesis in a way that discusses the MSP as the "starting point" of the VIP, rather than as being a third unrelated variant of the isoperimetric problem.

The rest of the thesis will be organised as follows. The remaining section of Chapter 1 summarises our main results and explains how it fits into the existing literature. Chapter 2 introduces the notation and definitions to be used throughout the rest of the thesis. Chapter 3 discusses the MSP in BIBDs, and establishes bounds on the incidence-free number. Chapter 4 analyses the VIP for Levi graphs of BIBDs, and determines bounds on the vertex-isoperimetric number. Chapter 5 deals with the EIP for Levi graphs of BIBDs, and obtains bounds on the edge-isoperimetric number. Chapter 6 takes a slight detour and discusses the generalised tangent numbers, and in particular the sequence $d_{a,1}$. This sequence is number theoretical in nature, but we exploit some of its properties to obtain stronger bounds on the edge-isoperimetric number for some families of graphs. Finally, Chapter 7 wraps up the thesis with some concluding remarks and further questions.

1.2 Summary of main results

For ease of reference, we will summarise the main results of the thesis in this section.

Roughly speaking, the core of this thesis is to establish bounds on the vertex-isoperimetric number (Chapters 3 and 4) and the edge-isoperimetric number (Chapters 5 and 6) of Levi graphs of BIBDs. To that end, we will only mention the results that specifically bound (or obtain exact values for) the isoperimetric numbers i_E and i_V, and defer the exact lemmas and machinery we have used to the subsequent chapters.

Our first major result is the following bounds on the isoperimetric number of general Levi graphs of BIBDs. The bounds on $i_V(\Gamma)$ are new, while the bounds on $i_E(\Gamma)$ are an improvement over the current best known bounds in [37]. This result is a consolidation of (4.3) and Theorems 4.15, 5.5 and 5.11.

Theorem 1.1. *Let* Γ *be the Levi graph of a BIBD with parameters* (v, b, r, k, λ)*. Then its vertex-isoperimetric number satisfies the bounds*

$$\frac{2v}{v+b}\left(1-\frac{\sqrt{r^2-\lambda b}}{r}\right) \leq i_V(\Gamma) \leq \frac{2v}{v+b-1}$$

and its edge-isoperimetric number satisfies the bounds

$$\frac{vr}{v+b}\left(1-\frac{\sqrt{r^2-\lambda b}}{r}\right) \leq i_E(\Gamma) \leq \frac{vr}{v+b-1}.$$

We also obtain the exact vertex-isoperimetric numbers for Levi graphs of three (infinite) families of BIBDs, that is, the complements of unitals, the unitals admitting sufficiently large k-arcs, and the point-hyperplane non-incidence designs of $\mathrm{PG}(n, q)$. For the former two BIBDs, we obtain the following result, which is a consolidation of Theorem 4.32 and Corollary 4.35.

Theorem 1.2. *Let* U *be a unital of order* n*. Then*

$$i_V(\Gamma(\overline{U})) = \begin{cases} \frac{4}{5}, & \text{if } n = 2 \\ \frac{2(n^3+1)}{n^4+n^2}, & \text{if } n \geq 3. \end{cases}$$

Furthermore, if U *admits* k*-arcs for sufficiently large* k*, then*

$$i_V(\Gamma(U)) = \frac{2\left\lfloor n^3 - n^2 + \frac{\sqrt{8n^2+9}-1}{2}\right\rfloor}{n^4+n^2}.$$

The third family of graphs for which we have determined i_V exactly is the family of point-hyperplane non-incidence graphs of $\mathrm{PG}(n, q)$. The following result is from Theorem 4.27.

Theorem 1.3. *Let* q *be a prime power, and* $n \geq 2$ *an integer. Let* $\mathcal{D}$ *be the point-hyperplane incidence design of* $\mathrm{PG}(n, q)$*. Then*

$$i_V(\Gamma(\overline{\mathcal{D}})) = 1 - \frac{q^{\left\lfloor \frac{n+1}{2}\right\rfloor} - 1}{q-1}.$$

For the edge-isoperimetric number, we establish the exact values for developments of some difference sets; this family of BIBDs include, for example, the point-hyperplane incidence design of PG(n, q) when n and q are both odd. This adds to the list of graphs for which $i_E(\Gamma) = \frac{a(\Gamma)}{2}$ hold; see [7]. This result is explored in Theorem 5.31.

Theorem 1.4. *Let G be a group that contains a subgroup of index 2. If D is a (v, k, λ)-difference set of G, then*

$$i_E(\Gamma(\mathrm{dev}(D))) = \frac{k - \sqrt{k-\lambda}}{2}.$$

Remarkably, we do not know the exact value of either i_E or i_V for the "simplest" case of $\Gamma_{2,q}$, that is, the point-hyperplane incidence graph of PG$(2, q)$. The above results hold for complements of this, or by taking PG(n, q) for higher q, but not PG$(2, q)$ itself. Note that if n or q is even, then v is odd, so that the cyclic group of order v does not have an index 2 subgroup.

Since PG$(2, q)$ is a very important family of graphs, we make a focused effort in this thesis to calculate its isoperimetric numbers for small q. The vertex-isoperimetric numbers are computed in Section 4.4, and the edge-isoperimetric numbers in Section 5.3. The results are summarised in Table 1.1. In doing so, we also prove in Sections 4.4.3 and 5.3.3 that neither the VIP nor the EIP on BIBDs have nested solutions in general.

Table 1.1 Exact isoperimetric numbers of $\Gamma_{2,q}$ for small q

q	$i_V(\Gamma_{2,q})$	$i_E(\Gamma_{2,q})$
2	5/7	7/7
3	10/13	16/13
4	15/21	33/21
5	24/31	60/31
7	44/57	156/57
8	57/73	
9	72/91	
11	105/133	
13	147/183	
16	221/273	

We do not have exact values for any larger q than those in Table 1.1, but we do establish stronger asymptotic bounds than the ones presented for the general case. The vertex-isoperimetric bound is implied by the results of [18], though we derive a stronger coefficient. The edge-isoperimetric bound is new, and is a significant improvement over the previous best known bounds in [37] for projective planes. This result is a consolidation of Theorems 4.26 and 6.19.

Theorem 1.5. *Let q be a prime power, and $n \geq 2$ an integer. Then*

$$i_V(\Gamma_{n,q}) = 1 - \Theta\left(q^{\frac{1-n}{2}}\right)$$

and

$$i_E(\Gamma_{2,q}) = \frac{q}{2} - \Theta(\sqrt{q}).$$

Finally, we also establish new bounds for Paley designs and Paley graphs. The bounds on the former is a new result, while for the Paley graphs we improve the upper bound from [15]. This result is a consolidation of Theorems 6.26 and 6.30.

Theorem 1.6. *Let p be a prime. Then*

$$i_E(\Gamma(\mathrm{PD}(p))) = \frac{p}{4} - \Theta(\sqrt{p})$$

and

$$i_E(P(p)) = \frac{p}{4} - \Theta(\sqrt{p}).$$

We also confirm a suspicion in [15] about the asymptotic growth of a bound they constructed, and in so doing, extend the theory of generalised tangent numbers in Chapter 6 in order to prove this growth. We explore this bound in Section 6.7.

Chapter 2

Preliminaries and Definitions

2.1 Incidence structures

Most of the notation and definitions seen in this thesis are standard; but the reader is invited to read [6] for more details on design theory, and [31] for more details on projective geometries over finite fields.

Definition 2.1. *An* incidence structure *I is a triple $(P, \mathcal{B}, \partial)$, where P and $\mathcal{B}$ are disjoint finite sets, and $\partial \subseteq P \times \mathcal{B}$ is the incidence relation. The elements of P, $\mathcal{B}$, and ∂ are called points, blocks, and flags respectively.*

It is often more convenient to think of ∂ as a function from P to subsets of $\mathcal{B}$, and then additively extend its domain to all subsets of P. That is, if $S \subseteq P$ is a subset of points, we write the *shadow* $\partial(S)$ of S as

$$\partial(S) = \{B \in \mathcal{B} : (p, B) \in \partial \text{ for some } p \in S\}.$$

This slight abuse of notation will not cause any confusion.

The *intersection number* of a block B with respect to S is the number of points in S incident with B. The *intersection spectrum* of S is the multiset of intersection numbers of all blocks with respect to S.

Often, we will just let $\mathcal{B}$ be a set of subsets of P, so that the incidence relation is given by containment. Such incidence structures are called *simple*, and we will simply write $I = (P, \mathcal{B})$. In this case, the intersection number of a block B is given directly by $|B \cap S|$.

The *dual* of I is given by $I^* = (\mathcal{B}, P, \partial^*)$, where ∂^* is the inverse relation of ∂, that is, $\partial^* = \{(y, x) : (x, y) \in \partial\}$. It is clear that $I^{**} = I$. We say that I is *self-dual* if I is isomorphic to its dual. If $I = (P, \mathcal{B})$ is a simple incidence structure, then its *complement* is given by $\overline{I} = (P, \overline{\mathcal{B}})$ where $\overline{\mathcal{B}} = \{P \setminus B : B \in \mathcal{B}\}$.

The two main types of incidence structures that we will be concerned with are block designs and graphs, both of which we will define here.

Definition 2.2. *A* balanced incomplete block design *(thereafter BIBD) is an incidence structure $\mathcal{D} = (P, \mathcal{B})$ with v points and b blocks, such that every point is contained in exactly r blocks, every block contains exactly k points, and every 2-subset of points is contained in exactly λ blocks. We call (v, b, r, k, λ) the* parameters *of $\mathcal{D}$. For non-degeneracy, we will require that $v > k > \lambda$.*

It is well-established that the parameters of a BIBD must satisfy

$$vr = bk$$

and

$$\lambda(v-1) = r(k-1),$$

so that any three parameters determine the remaining two. Because of this, we also say that $\mathcal{D}$ is a (v, k, λ)-BIBD.

Fisher's inequality also tells us that $v \leq b$ for every BIBD. If equality is obtained, we say that $\mathcal{D}$ is *symmetric*[1]. If $\mathcal{D}$ is symmetric, the value

$$n = k - \lambda$$

is called the *order* of $\mathcal{D}$. It is not difficult to see that we must have

$$4n - 1 \leq v \leq n^2 + n + 1.$$

The lower bound is obtained by Hadamard designs, and the upper bound by projective planes. The dual of a symmetric BIBD is itself a BIBD with the same parameters as the primal.

The complement of a BIBD $\mathcal{D}$ with parameters (v, b, r, k, λ) is also itself a BIBD $\overline{\mathcal{D}}$ with parameters $(v, b, b-r, v-k, b-2r+\lambda)$.

Definition 2.3. *A (k, d)-arc of a BIBD, for some integer k, is a set of k points of which no $d+1$ lie in the same block. Equivalently, a (k, d)-arc is a set of points with*

[1]We personally prefer the term *square* as introduced by [12], but unfortunately this term has not caught on, and we will stick with the more conventional "symmetric".

no intersection number larger than d. If $d = 2$, it is conventional to omit the d and simply call it a k-arc. We also simply call it an arc *if its size k is not a concern.*

There are at least two properties of arcs that are of particular interest to us. A (k, d)-arc is *maximal* if a $(k+1, d)$-arc does not exist, and *complete* if it is not contained in any $(k + 1, d)$-arc. It is clear that maximal arcs must be complete, but complete arcs may not necessarily be maximal.[2]

We will denote by $m_d(\mathcal{D})$ the largest value of k for which a (k, d)-arc exists in $\mathcal{D}$.

Definition 2.4. *A* graph Γ *is a pair (V, E), where V is a finite set of* vertices*, and E is a set of 2-subsets of V (called edges). For convenience, we often write uv to mean the edge $\{u, v\}$.*

If S and T are two disjoint subsets of V, then $E(S, T)$ is the set of edges with one endpoint in S and the other endpoint in T. Similarly, we denote by $E(S)$ the set of edges induced by S, that is, the edges with both endpoints in S.

If $S \subseteq V$ is a subset of vertices, we denote by $\overline{S}$ the set of vertices outside S, or equivalently $\overline{S} = V \setminus S$. The *vertex boundary* $\delta(S)$ of S is the set of vertices in $\overline{S}$ adjacent to some vertex in S.

The *vertex-isoperimetric number* i_V of a graph $\Gamma = (V, E)$ is the quantity

$$i_V(\Gamma) = \min_S \frac{|\delta(S)|}{|S|},$$

where S is taken over all non-empty subsets of V of cardinality at most $\frac{|V|}{2}$. The *edge-isoperimetric number* i_E of Γ is similarly defined as

$$i_E(\Gamma) = \min_S \frac{|E(S, \overline{S})|}{|S|}$$

over the same domain.

Definition 2.5. *Let $I = (P, \mathcal{B}, \partial)$ be an incidence structure. The* Levi graph *(or incidence graph) of I, denoted by $\Gamma(I)$, is the bipartite graph with vertex set $V = P \cup \mathcal{B}$ and edge set $E = \{uv : (u, v) \in \partial\}$. For any $S \subseteq V$, we say that S* is of type (x, y) *if $|S \cap P| = x$ and $|S \cap \mathcal{B}| = y$.*

[2] The term "maximal" has persisted in the literature for historic reasons; it really refers to an arc of maximum size. Likewise, the term "complete" is used for arcs that have the maximal property.

Note that the graphs $\Gamma(I)$ and $\Gamma(I^*)$ are isomorphic. We can similarly define the *non-incidence graph* of I with the same vertex set, but with the edges between non-incident pairs of points and blocks instead. This is the same as the Levi graph of its complement and hence is denoted by $\Gamma(\overline{I})$.

Next we will define a difference set, since they give rise to many interesting BIBDs.

Definition 2.6. *Let G be a group of order v. A (v,k,λ)*-difference set *D is a k-subset of G such that every non-identity element of G can be written in the form $d_i d_j^{-1}$ in exactly λ ways, where d_i and d_j are elements of D. The* development $\mathrm{dev}(D)$ *of D is the incidence structure with point set G, and whose blocks are the translates of D, that is, the set $\{gD : g \in G\}$.*

It is clear that if D is a difference set, then $\mathrm{dev}(D)$ must be a symmetric BIBD.

2.2 Finite geometry

Let q be a prime power. We will denote by $\mathbb{F}_q$ the finite field with q elements.

Definition 2.7. *The* projective geometry $\mathrm{PG}(n,q)$ *is the set of linear subspaces of $\mathbb{F}_q^{n+1}$, ordered by containment.*

We can construct a nice family of BIBDs from this geometry. For a positive integer $d \leq n$, denote by $\mathrm{PG}_d(n,q)$ the BIBD whose point and block sets are the 1-dimensional and $(d+1)$-dimensional linear subspaces of $\mathbb{F}_q^{n+1}$, with incidence given by containment.

Of these BIBDs, $\mathrm{PG}_{n-1}(n,q)$ is particularly nice because it is a self-dual symmetric BIBD, so we place special emphasis on this family of BIBDs. Because of its prevalence in this thesis, we will use the shorter notation $\mathcal{D}_{n,q}$ and $\Gamma_{n,q}$ to refer to the BIBD $\mathrm{PG}_{n-1}(n,q)$ and its Levi graph respectively. It is easy to check that $\mathcal{D}_{n,q}$ is a $\left(\frac{q^{n+1}-1}{q-1}, \frac{q^n-1}{q-1}, \frac{q^{n-1}-1}{q-1}\right)$-BIBD.

We know that $\mathcal{D}_{n,q}$ is the development of a difference set of a cyclic group [45]. As such, this gives us two representations of $\mathcal{D}_{n,q}$, both of which have their benefits and drawbacks, and we use both representations throughout this thesis.

1. **Homogeneous coordinates:** In this system, we represents points of $\mathcal{D}_{n,q}$ as 1-dimensional subspaces of $\mathbb{F}_q^{n+1}$, that is

$$P = \{\langle u \rangle : u \in \mathbb{F}_q^{n+1} \setminus 0\}.$$

Because the blocks are n-dimensional subspaces of $\mathbb{F}_q^{n+1}$, the orthogonal complement of a block is simply a 1-dimensional subspace, so we can write the blocks as

$$\mathcal{B} = \{\langle u \rangle^{\perp} : u \in \mathbb{F}_q^{n+1} \setminus 0\}.$$

2. **Difference sets:** As per [45], we can obtain a (v, k, λ)-difference set D of the cyclic group $\mathbb{Z}_v$, such that $\mathcal{D}_{n,q} = \mathrm{dev}(D)$. Then we write

$$P = \mathbb{Z}_v \quad \text{and} \quad \mathcal{B} = \{x + D : x \in \mathbb{Z}_v\}.$$

For $\mathcal{D}_{2,q}$, a difference set representation is given in Appendix A for each $q \leq 16$. A larger catalogue of difference sets can be found in [13].

For the most part, the homogeneous coordinates are useful when we are treating $\mathcal{D}_{n,q}$ as a geometric object, and the difference set representation is useful when we wish to concisely represent a set of points as a witness for an isoperimetric inequality.

There exists an orthogonal polarity in $\mathrm{PG}(2, q)$ for all q, for which there are $q + 1$ absolute points, that is, points incident with their polar. The Erdős-Renyi graphs $\mathrm{ER}(q)$ is the graph whose vertex set is the set of points of $\mathrm{PG}(2, q)$, and two distinct vertices (x_0, x_1, x_2) and (y_0, y_1, y_2) are adjacent if $x_0y_0 + x_1y_1 + x_2y_2 = 0$. We can also think of it as the quotient graph of $\Gamma_{n,q}$ obtained by identifying each pole in $\Gamma_{2,q}$ with its polar. Note that we lose graph regularity this way due to the absolute points; to rectify this issue we can also define the graph $\mathrm{ER}^{\circ}(q)$ with the same edges plus an additional loop on each absolute point.

If q is square, then $\mathrm{PG}(2, q)$ also contains a unitary polarity, of which there are then $q\sqrt{q} + 1$ absolute points. A BIBD can be formed by taking these absolute points as the point set, and the non-absolute lines as the block set. This motivates the following definition.

Definition 2.8. *A* unital *of order n is a $(n^3 + 1, n + 1, 1)$-BIBD; its parameters are $(n^3 + 1, n^4 - n^3 + n^2, n^2, n + 1, 1)$.*

The unital derived from a unitary polarity of $\mathrm{PG}(2, q^2)$ as before is called a *classical unital*, and is denoted by $H(q)$. The points of $H(q)$ are the points of a Hermitian curve, which is projectively equivalent to the curve

$$\left\{ \langle (x, y, z) \rangle : x, y, z \in \mathbb{F}_{q^2}, x^{q+1} + y^{q+1} + z^{q+1} = 0 \right\}.$$

Another family of unitals embeddable in $\mathrm{PG}(2,q^2)$ consists of *Buekenhout-Metz unitals* (thereafter BM unitals) arising from two constructions [11, 40] of Buekenhout. These unitals are called *parabolic* or *hyperbolic*, depending on the number of unital points on the line at infinity of $\mathrm{PG}(2,q^2)$. When q is odd, the point set of a parabolic BM-unital is of the form

$$\left\{\left\langle(x, \alpha x^2 + \beta x^{q+1} + r, 1)\right\rangle : x \in \mathbb{F}_q^2, r \in \mathbb{F}_q\right\} \cup \{\langle(0,1,0)\rangle\}, \tag{2.1}$$

where $\alpha, \beta \in \mathbb{F}_{q^2}$, and $(\beta^q - \beta)^2 + 4\alpha^{q+1}$ is a non-square in $\mathbb{F}_q$ [3]. When q is even, the point set of a parabolic BM-unital is of the form (2.1), where $\alpha, \beta \in \mathbb{F}_{q^2}, \beta \notin \mathbb{F}_q$, and $\alpha^{q+1}/(\beta^q + \beta)^2$ has trace 0 over $\mathbb{F}_2$ [21]. In both cases we denote the unital defined in (2.1) by $\mathrm{BM}(\alpha, \beta, q)$.

Our main interest lies in those unitals which admit large arcs. In fact, it was proved in [23] that for any prime power q, $H(q)$ admits a complete $(q^2 - q + 1)$-arc. In [2, 30] it was proved that for an odd prime power q, if $\alpha \in \mathbb{F}_{q^2}$ and $\beta \in \mathbb{F}_q$ are such that $(\beta^q - \beta)^2 + 4\alpha^{q+1}$ is a non-square in $\mathbb{F}_q$, then the unital $\mathrm{BM}(\alpha, \beta, q)$ admits a complete $(q^2 + 1)$-arc.

Chapter 3

The Minimum Shadow Problem

In this chapter, we will discuss the minimum shadow problem (MSP) on BIBDs. In Section 3.1, we provide the basic definitions and notation to be used throughout the rest of the chapter, and also discuss some of the properties of the MSP. In Section 3.2, we establish some general isoperimetric inequalities for the MSP that hold for all (or many) BIBDs.

The remaining sections will focus on specific families of BIBDs. Section 3.3 looks at exact solutions of the MSP on $\mathcal{D}_{2,q}$ for small q. We generalise this in Section 3.4 by looking at $\mathcal{D}_{n,q}$ for general n and q, before looking at its complement in Section 3.5. Finally, in Section 3.6 we will study the MSP on unitals, and establish the exact incidence-free number for unitals admitting sufficiently large arcs.

3.1 Basic definitions and properties

Definition 3.1. *Let $I = (P, \mathcal{B}, \partial)$ be an incidence structure, and let $S \subseteq P$ be a subset of points. The* shadow *$\partial(S)$ of S is the set of blocks containing some element of S. The* minimum shadow parameter *is the function*

$$\Phi_S(I, n) = \min\{|\partial(S)| : S \subseteq P, |S| = n\}.$$

Note that the subscript S in Φ_S refers to the shadow as the boundary function, and has nothing to do with the subset S. We say that S is an *MSP-set* if $|\partial(S)| = \Phi_S(I, |S|)$. It is fairly clear that $\Phi_S(I, n)$ increases with n.

We say that the MSP on I has *nested solutions* if there exists an ordering of P such that the first n points of P in this ordering always form an MSP-set for all n.

An MSP-set S is *closed* if $|\partial(S')| > |\partial(S)|$ for all $|S'| > |S|$. The term "closed" here comes from Harper [25], and can be slightly misleading since it really describes a property of the isoperimetric function rather than the set itself.

We will also define an "inclusive" variant of this parameter. That is,

$$\Phi'_S(I, n) = \min\{|S \cup \partial(S)| : S \subseteq P, |S| = n\}.$$

It is also clear that $\Phi'_S(I, n)$ is strictly increasing with n, and that

$$\Phi'_S(I, n) = \Phi_S(I, n) + n.$$

The term "incidence-free" was first used in [18] to describe sets of points and lines that are not incident to each other. We generalise the concept to incidence structures.

Definition 3.2. *Let I be an incidence structure with v points and b blocks. The* incidence-free number $\overline{\alpha}(I)$ *of I is the largest integer n for which $\Phi'_S(I, n) \le \frac{v+b}{2}$.*

For a symmetric design $\mathcal{D}$, there is another perspective of this number. Consider the bipartite graph $\Gamma = \Gamma(\mathcal{D})$. Call an independent set of Γ *balanced* if it contains the same number of vertices in both bipartite parts. Then $\overline{\alpha}(\mathcal{D})$ is half the size of a maximum balanced independent set in Γ.

3.2 General isoperimetric inequalities

In this section we will prove some general isoperimetric inequalities relating to the MSP. We start off with an integer analogue of the Cauchy-Schwarz inequality.

Lemma 3.3. *Let $m, a_1, a_2, \ldots, a_n$ be integers, for some positive integer n. Then*

$$\sum_{i=1}^{n} a_i(a_i - 1) \ge 2m \sum_{i=1}^{n} a_i - m(m+1)n.$$

Furthermore, equality holds if $a_i \in \{m, m+1\}$ for all i.

Proof. Since $x^2 \ge x$ for all integers x, we necessarily have

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} \Big((a_i - m)^2 - (a_i - m) \Big) \qquad (3.1) \\ &= \sum_{i=1}^{n} a_i\,(a_i - 1) - 2m \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} m(m+1), \end{aligned}$$

and the result follows from rearranging. If $a_i \in \{m, m+1\}$ for all i, then the right hand side of (3.1) is zero and we have equality. □

The following theorem was already known to Ure [48] for $\mathcal{D}_{2,q}$. We extend it to all BIBDs.

Theorem 3.4. *Let $\mathcal{D} = (P, \mathcal{B}, \partial)$ be a BIBD with parameters (v, b, r, k, λ), and Γ its Levi graph. Let m be a positive integer. If $n \in \{0, 1, \dots, v\}$, then*

$$\Phi_S(\mathcal{D}, n) \geq \frac{(2rm - \lambda(n-1))n}{m(m+1)}. \tag{3.2}$$

Furthermore, equality holds in (3.2) if there exists an n-set of points such that every block has intersection number 0, m, or $m+1$ with respect to this set.

Proof. Let S be an n-set of point vertices. Let F be the set of edges in Γ with one end-vertex in S and the other in $\partial(S)$. By double counting, we have

$$rn = |F| = \sum_{B \in \partial(S)} |B \cap S|.$$

Denote by Q the set of paths of length two in Γ with both end-vertices in S. By double counting again, we obtain

$$\lambda n(n-1) = |Q| = \sum_{B \in \partial(S)} |B \cap S|(|B \cap S| - 1).$$

By applying Lemma 3.3, this becomes

$$\begin{aligned}
\lambda n(n-1) &= \sum_{B \in \partial(S)} |B \cap S|(|B \cap S| - 1) \\
&\geq 2m \sum_{B \in \partial(S)} |B \cap S| - m(m+1)|\partial(S)| \\
&= 2rmn - m(m+1)|\partial(S)|,
\end{aligned}$$

so that

$$|\partial(S)| \geq \frac{(2rm - \lambda(n-1))n}{m(m+1)},$$

and the result follows from the arbitrary choice of S.

Now, suppose every block in $\partial(S)$ has intersection number m or $m+1$. Then we get equality in Lemma 3.3 and therefore equality in (3.2). □

It is evident from Theorem 3.4 that arcs are necessarily MSP-sets, since their intersection numbers are always in $\{0, 1, 2\}$.

We can get a uniform lower bound by picking the "best" possible m that maximises the right hand side of Theorem 3.4. This gives us the following result, which can be seen as a generalisation of [47, Theorem 3.1], where they bound the number of lines non-incident with a set of points in a projective plane of order q.

Corollary 3.5. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ). If $n \in \{0, 1, \dots, v\}$, then*

$$\Phi_S(\mathcal{D}, n) \geq \frac{r^2 n}{r + \lambda(n-1)}.$$

Proof. Let $m = \left\lfloor \frac{\lambda(n-1)}{r} \right\rfloor + 1$, so that

$$\left(\frac{\lambda(n-1)}{r} + 1 - m\right)\left(m - \frac{\lambda(n-1)}{r}\right) \geq 0.$$

That is,

$$\left(\frac{\lambda(n-1)}{r} + 1\right)\left(2m - \frac{\lambda(n-1)}{r}\right) \geq m(m+1).$$

Rearranging this inequality yields

$$\frac{2rm - \lambda(n-1)}{m(m+1)} \geq \frac{r^2}{r + \lambda(n-1)}.$$

The result follows from this and Theorem 3.4. □

Lemma 3.6 ([25, Theorem 3])**.** *S is a closed MSP-set of I if and only if $\mathcal{B} \setminus \partial(S)$ is a closed MSP-set of I^*.*

Note that Lemma 3.6 also implies the more general bound

$$\Phi_S(I, v - \Phi_S(I^*, n)) \leq b - n, \tag{3.3}$$

where v and b are the number of points and blocks of I respectively.

Theorem 3.7. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ). If $n \in \{0, 1, \dots, b\}$, then*

$$\Phi_S(\mathcal{D}^*, n) \geq \frac{rkn}{r^2 - \lambda(b-n)}.$$

Proof. Let $s = v - \Phi_S(\mathcal{D}^*, n)$. From Corollary 3.5 and (3.3) we have

$$\frac{r^2 s}{r + \lambda(s-1)} \le \Phi_S(\mathcal{D}, s) \le b - n.$$

This reduces to $s \le \frac{(b-n)(r-\lambda)}{r^2-\lambda(b-n)}$, and thus

$$\begin{aligned}\Phi_S(\mathcal{D}^*, n) &\ge v - \frac{(b-n)(r-\lambda)}{r^2-\lambda(b-n)} \\ &= \frac{vr^2 - (b-n)(\lambda(v-1)+r)}{r^2-\lambda(b-n)} \\ &= \frac{rkn}{r^2-\lambda(b-n)}.\end{aligned}$$

□

Lemma 3.8. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ). Then*

$$\overline{\alpha}(\mathcal{D}) = 0 \textit{ if and only if } r \ge \left\lfloor \frac{v+b}{2} \right\rfloor,$$

and

$$\overline{\alpha}(\mathcal{D}) = 1 \textit{ if and only if } r + 1 \le \left\lfloor \frac{v+b}{2} \right\rfloor \le 2r - \lambda + 1.$$

Proof. It is an easy check that $\Phi'_S(\mathcal{D}, 0) = 0$, $\Phi'_S(\mathcal{D}, 1) = r+1$, and $\Phi'_S(\mathcal{D}, 2) = 2r-\lambda+2$. The result follows immediately from these and the definition of $\overline{\alpha}$. □

Theorem 3.9. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ), and let $\mu = \sqrt{k-\lambda}$. If $\mathcal{D}$ is symmetric or $\mu \ge 2$, then*

$$\overline{\alpha}(\mathcal{D}) \le \frac{\mu v}{k+\mu}.$$

Proof. Let f be given by

$$f(x) = k\left(\frac{2}{\frac{k}{\mu}+1} - 1\right) - x\left(1 - \frac{2}{\frac{x-\lambda}{x\mu}+1}\right).$$

The first step is to prove that $f(r) \ge 0$. Notice that $f(k) = 0$, so that if $\mathcal{D}$ is symmetric then $f(r) = 0$. Otherwise, we now wish to show that f is increasing, so we take its derivative to obtain

$$f'(x) = \frac{x\mu(x\mu - 2\lambda) - (x-\lambda)^2}{(x-\lambda+x\mu)^2}.$$

Let g be the numerator of $f'(x)$, that is,

$$g(x) = x\mu(x\mu - 2\lambda) - (x - \lambda)^2.$$

Expanding $g(x)$ and replacing λ with $k - \mu^2$, we obtain

$$\begin{aligned} g(x) &= k^2\mu(\mu - 2) + \mu^3(2k - \mu) + (x - k)(\mu - 1)((x - k)(\mu + 1) + 2\mu(k + \mu)) \\ &\geq 0 \end{aligned}$$

for all $x \geq k$, and thus it follows that $f(r) \geq 0$.

Now let $n = \frac{\mu v}{k+\mu}$. We can use the identity $\lambda(v - 1) = r(k - 1)$ to write

$$\lambda n = \frac{\mu}{k + \mu}(rk - r + \lambda),$$

which yields

$$\begin{aligned} \frac{r^2 n}{r + \lambda(n - 1)} + n &= \frac{r^2 \frac{\mu v}{k+\mu}}{r - \lambda + \frac{\mu}{k+\mu}(rk - r + \lambda)} + \frac{\mu v}{k\mu} \\ &= v\left(\frac{1}{\frac{k}{\mu} + 1}\right) + \frac{rv}{k}\left(\frac{1}{\frac{r-\lambda}{r\mu} + 1}\right) \\ &= \frac{v}{2k}(f(r) + r + k) \\ &\geq \frac{v + b}{2}. \end{aligned}$$

It follows from this and Corollary 3.5 that $\overline{\alpha}(\mathcal{D}) \leq n$ as required. □

3.3 The MSP on $\mathcal{D}_{2,q}$ for small q

The study of the MSP on $\mathcal{D}_{2,q}$ was first initiated by Harper and Hergert [25]. They concluded that nested solutions exist for $q \in \{2, 3, 4\}$, albeit without a formal proof. This was confirmed and extended by Ure [48] in her thesis, where she provided explicit nested solutions for all $q \leq 7$. It was also proven in her thesis that the MSP on $\mathcal{D}_{2,8}$ does not have nested solutions. We will use her results to evaluate $\overline{\alpha}(\mathcal{D}_{2,q})$ for $q \leq 8$, and then build on top of that to obtain $\overline{\alpha}(\mathcal{D}_{2,q})$ for q up to 16.

3.3.1 The case $2 \leq q \leq 7$

Let $q \in \{2, 3, 4, 5, 7\}$. The MSP on $\mathcal{D}_{2,q}$ has been solved in these cases, but we can still establish their incidence-free numbers.

We have the following values of $\Phi_S(\mathcal{D}_{2,q}, n)$ from Ure [48]:

$$\begin{aligned} \Phi_S(\mathcal{D}_{2,2}, 2) &= 5 & \Phi_S(\mathcal{D}_{2,2}, 3) &= 6 \\ \Phi_S(\mathcal{D}_{2,3}, 3) &= 9 & \Phi_S(\mathcal{D}_{2,3}, 4) &= 10 \\ \Phi_S(\mathcal{D}_{2,4}, 6) &= 15 & \Phi_S(\mathcal{D}_{2,4}, 7) &= 17 \\ \Phi_S(\mathcal{D}_{2,5}, 7) &= 23 & \Phi_S(\mathcal{D}_{2,5}, 8) &= 24 \\ \Phi_S(\mathcal{D}_{2,7}, 13) &= 44 & \Phi_S(\mathcal{D}_{2,7}, 14) &= 45 \end{aligned}$$

which leads to the following corollary.

Corollary 3.10. *For $q \in \{2, 3, 4, 5, 7\}$, the incidence-free numbers of $\mathcal{D}_{2,q}$ are*

- $\overline{\alpha}(\mathcal{D}_{2,2}) = 2$*;*
- $\overline{\alpha}(\mathcal{D}_{2,3}) = 3$*;*
- $\overline{\alpha}(\mathcal{D}_{2,4}) = 6$*;*
- $\overline{\alpha}(\mathcal{D}_{2,5}) = 7$*; and*
- $\overline{\alpha}(\mathcal{D}_{2,7}) = 13$*.* □

3.3.2 The case $8 \leq q \leq 16$

We first prove an upper bound that works for many values of q.

Lemma 3.11. *Let q be a prime power, and x a positive integer. Let $m = \left\lfloor \frac{x}{q+1} \right\rfloor + 1$. If*

$$\frac{x+1}{m+1}\left(2(q+1) - \frac{x}{m}\right) > q(q+1) - x,$$

then $\overline{\alpha}(\mathcal{D}_{2,q}) \leq x$.

Proof. We can apply Theorem 3.4 to obtain

$$\Phi_S(\mathcal{D}_{2,q}, x+1) \geq \frac{x+1}{m+1}\left(2(q+1) - \frac{x}{m}\right) > q^2 + q - x,$$

and thus

$$\Phi'_S(\mathcal{D}_{2,q}, x+1) > q^2 + q + 1.$$ □

We have immediately the following corollary.

Corollary 3.12. *For $q \in \{8, 9, 11, 13, 16\}$, we have the following upper bounds:*

- $\overline{\alpha}(\mathcal{D}_{2,8}) \leq 16$;
- $\overline{\alpha}(\mathcal{D}_{2,9}) \leq 21$;
- $\overline{\alpha}(\mathcal{D}_{2,11}) \leq 28$;
- $\overline{\alpha}(\mathcal{D}_{2,13}) \leq 36$; *and*
- $\overline{\alpha}(\mathcal{D}_{2,16}) \leq 52$. □

We will see shortly that all these upper bounds except the second one are sharp.

Lemma 3.13. *If S is a $(17, 3)$-arc of $\mathcal{D}_{2,9}$, then $|\partial(S)| \geq 72$.*

Proof. In [14] it is shown that there are only four $(17, 3)$-arcs in $\mathcal{D}_{2,9}$ up to isomorphism. Using the difference set representation in Appendix A, these are the point sets

- $S_1 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 20, 25, 34, 41, 46, 67, 69, 71\}$;
- $S_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 21, 34, 37, 41, 42, 47, 66, 67, 72\}$;
- $S_3 = \{0, 1, 2, 3, 4, 5, 6, 7, 34, 39, 40, 43, 44, 54, 68, 74, 76\}$; and
- $S_4 = \{0, 1, 2, 3, 4, 5, 6, 7, 34, 39, 43, 44, 48, 67, 68, 74, 86\}$.

It is straightforward to check that $|\partial(S_i)|$ evaluates to 73, 74, 73 and 72 respectively. □

Theorem 3.14. $\overline{\alpha}(\mathcal{D}_{2,9}) \leq 19$.

Proof. Suppose otherwise. Let $\mathcal{D}_{2,9} = (P, \mathcal{B}, \partial)$. Then there exists $S \in P$ with $|S| = 20$ and $|\partial(S)| \leq 71$. Using the same technique as in Theorem 3.4 we have

$$\sum_{B \in \partial(S)} |B \cap S| = 200$$

and

$$\sum_{B \in \partial(S)} |B \cap S|(|B \cap S| - 1) = 380.$$

Then

$$\begin{aligned}
\sum_{B \in \partial(S)} (|B \cap S| - 2)(|B \cap S| - 3) &= \sum_{B \in \partial(S)} |B \cap S|(|B \cap S| - 1) - 4|B \cap S| + 6 \\
&= 380 - 800 + 6|\partial(S)| \\
&\leq 6.
\end{aligned}$$

This implies that there is no block with intersection number larger than 5, and the spectrum must fall into one of the following cases.

Case 1. There is a unique block with intersection number 5, and no blocks with intersection number 4.

Case 2. There is no block with intersection number 5, and at most three blocks with intersection number 4.

In both cases, we can easily construct a $(17,3)$-arc S' by removing three points from S. But this procedure also yields $|\partial(S')| \leq |\partial(S)| \leq 71$, directly contradicting Theorem 3.14. □

For the lower bound, it suffices to simply produce a witness set. As before, we give the elements of the set using the difference set representation in Appendix A. It is then a simple matter of checking, by hand or by computer, that these sets provide lower bounds that match the upper bounds we have established. We provide sample code that performs this verification in Appendix B.

Theorem 3.15. *For $q \in \{8, 9, 11, 13, 16\}$, the incidence-free numbers of $\mathcal{D}_{2,q}$ are*

- $\overline{\alpha}(\mathcal{D}_{2,8}) = 16$*;*
- $\overline{\alpha}(\mathcal{D}_{2,9}) = 19$*;*
- $\overline{\alpha}(\mathcal{D}_{2,11}) = 28$*;*
- $\overline{\alpha}(\mathcal{D}_{2,13}) = 36$*; and*
- $\overline{\alpha}(\mathcal{D}_{2,16}) = 52$*.* □

Proof. Use the following sets as witnesses (lower bounds) for $\Gamma_{2,q}$:

$q = 8$: $\{0, 1, 2, 3, 4, 5, 6, 21, 22, 25, 26, 27, 38, 42, 47, 67\}$

$q = 9$: $\{0, 1, 2, 3, 4, 5, 6, 7, 17, 22, 35, 47, 48, 67, 68, 69, 71, 87, 88\}$

$q = 11$: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 54, 58, 59, 62, 63, 64, 98, 116, 120, 121, 125, 126, 129, 130, 131\}$.

$q = 13$: $\{0, 1, 2, 3, 4, 5, 6, 7, 16, 17, 18, 20, 24, 31, 34, 47, 56, 59, 76, 77, 81, 89, 91, 94, 96, 103, 106, 108, 127, 129, 131, 143, 151, 165, 173, 179\}$

$q = 16$: $\{0, 1, 2, 3, 4, 5, 6, 7, 13, 20, 25, 38, 43, 62, 77, 84, 86, 87, 96, 101, 103, 104, 107, 108, 109, 124, 134, 137, 138, 140, 150, 155, 161, 164, 166, 169, 171, 172, 175, 177, 186, 189, 211, 214, 215, 216, 224, 241, 250, 263, 266, 269\}$.

The upper bounds follow from Corollary 3.12 and Theorem 3.14. □

3.4 The MSP on $\mathcal{D}_{n,q}$ for general n and q

We already know that maximal arcs are MSP-sets. As it turns out, we are in fact able to characterise exactly when maximal arcs exist in $\mathcal{D}_{n,q}$. For odd q, we know from [4] that no non-trivial maximal arcs exist, and for even $q = 2^h$, we know from [19] that there exist maximal arcs of every degree $2^t, 1 \leq t \leq h$ in $\mathcal{D}_{2,2^h}$.

Aside from maximal arcs however, not much is known about the general MSP in $\mathcal{D}_{n,q}$, so we will limit our scope and focus largely on the incidence-free number in this section. In general, we do not have a better upper bound on the incidence-free number than the one obtained in Theorem 3.9, but we present two simplifications of it here with more explicit values.

Corollary 3.16. *Let n be a positive integer, and q a prime power. Then*

$$\overline{\alpha}(\mathcal{D}_{n,q}) \leq \frac{1 + q^{\frac{n+1}{2}}}{1 + q^{\frac{1-n}{2}}}.$$

Proof. This follows immediately from Theorem 3.9. □

Corollary 3.17. *If q is a prime power, then $\overline{\alpha}(\mathcal{D}_{2,q}) \leq q\sqrt{q} - q + \sqrt{q}$.*

It is clear that equality cannot hold in Corollary 3.17 unless q is square. In fact, we can say something a lot stronger from our discussion of maximal arcs earlier.

Theorem 3.18. *Let q be a prime power. Then $\overline{\alpha}(\mathcal{D}_{2,q}) = q\sqrt{q} - q + \sqrt{q}$ if and only if q is an even power of 2.*

Note for example, that Theorem 3.18 gives us $\overline{\alpha}(\mathcal{D}_{2,16}) = 52$, a result which we obtained in Theorem 3.15 by explicit construction.

In [43], Mubayi and Williford obtain bounds on the independence number of ER_q. This is almost equivalent to our problem of finding the incidence-free number of $\mathcal{D}_{2,q}$. The only difference that needs to be potentially considered is the case when absolute points appear in the set. Fortunately, none of their constructions need this extra consideration and they all carry over perfectly to the MSP, as noted in [18].

The following lemma is a consolidation of multiple lemmas and corollaries in [18].

Lemma 3.19. *Let p be a prime, and q a prime power. Then*

$$\overline{\alpha}(\Gamma_{1,q}) \geq \frac{q}{2}; \tag{3.4}$$
$$\overline{\alpha}(\Gamma_{2,p}) \geq \frac{120}{73\sqrt{73}}p^{\frac{3}{2}}; \tag{3.5}$$
$$\overline{\alpha}(\Gamma_{2,p^{2k}}) \geq \frac{p^{3k}}{2} \text{ for all positive integers } k; \tag{3.6}$$
$$\overline{\alpha}(\Gamma_{2,p^{2k+1}}) \geq p^{3k}\overline{\alpha}(\Gamma_{2,p}) \text{ for all positive integers } k\text{; and} \tag{3.7}$$
$$\overline{\alpha}(\Gamma_{n+2,q}) \geq q\overline{\alpha}(\Gamma_{n,q}) \text{ for all positive integers } n. \tag{3.8}$$

As before, note that (3.7) gives us $\overline{\alpha}(\mathcal{D}_{2,8}) \geq 16$, a result which we obtained in Theorem 3.15 by explicit construction.

Now, we will proceed to improve the lower bound of (3.5). To do so, we will use some known results on the well-known circle problem and its primitive variant. For any real number $r > 0$, define

$$C(r) = \left\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 \leq r\right\}$$

and

$$C'(r) = \left\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 \leq r, (x, y) = 1\right\}.$$

Lemma 3.20. *Let $r > 0$ and $\epsilon > 0$ be real numbers. Then*

$$|C(r)| = \pi r + \mathcal{O}\left(r^{\frac{1}{2}}\right);$$
$$|C'(r)| = \frac{6}{\pi}r + \mathcal{O}\left(r^{\frac{1}{2}+\epsilon}\right);$$
$$\sum_{(x,y)\in C'(r)} \sqrt{x^2 + y^2} = \frac{4}{\pi}r\sqrt{r} + \mathcal{O}\left(r^{1+\epsilon}\right).$$

Proof. Since $C(\lfloor r\rfloor) \subseteq C(r) \subseteq C(\lceil r\rceil)$ and $C'(\lfloor r\rfloor) \subseteq C'(r) \subseteq C'(\lceil r\rceil)$, it suffices to prove these equalities for positive integers r.

Let $r > 0$ be an integer. The first two equalities are well-known in the literature as the Gauss circle problem and the primitive Gauss circle problem respectively; see [29] and [49]. The third one follows from the first two because

$$\begin{aligned}\sum_{(x,y)\in C'(r)} \sqrt{x^2+y^2} &= \sum_{i=1}^{r} \sqrt{i}\,(|C'(i)| - |C'(i-1)|) \\ &= \sum_{i=1}^{r} \frac{6}{\pi}\sqrt{i} + \sum_{i=1}^{r-1}\left(\sqrt{i}-\sqrt{i+1}\right)\left(|C'(i)| - \frac{6}{\pi}i\right) + \sqrt{r}\left(|C'(r)| - \frac{6}{\pi}r\right) \\ &= \frac{4}{\pi}r\sqrt{r} + \mathcal{O}\left(r^{1+\epsilon}\right),\end{aligned}$$

where the last line follows from the fact that $\sqrt{i}-\sqrt{i+1} = \mathcal{O}\left(\frac{1}{\sqrt{i}}\right)$. □

Theorem 3.21. *Let $\epsilon > 0$ be a real number. If p is prime, then*

$$\overline{\alpha}(\mathcal{D}_{2,p}) \geq \frac{1}{2}p\sqrt{p} - \mathcal{O}\left(p^{\frac{5}{4}+\varepsilon}\right).$$

Proof. We prove this by construction. Let

$$S = \left\{\langle(x,y,1)\rangle : (x,y) \in C\left(\frac{p\sqrt{p}}{2\pi}\right)\right\}$$

and

$$T = \left\{\langle(a,b,c)\rangle^{\perp} : a,b,c \in \mathbb{Z}, \left|c - \frac{p}{2}\right| < \frac{p}{2} - \frac{p^{\frac{3}{4}}}{\sqrt{2\pi}}\sqrt{a^2+b^2}\right\}.$$

We first claim that $S \cup T$ is an independent set of $\Gamma_{2,p}$. Indeed, for any combination of x, y, a, b, c as above, we have $0 < (x, y, 1) \cdot (a, b, c) < p$, because

$$\begin{aligned}\left|ax+by+c-\frac{p}{2}\right| &\leq |ax+by| + \left|c-\frac{p}{2}\right| \\ &< \sqrt{x^2+y^2}\sqrt{a^2+b^2} + \frac{p}{2} - \frac{p^{\frac{3}{4}}}{\sqrt{2\pi}}\sqrt{a^2+b^2} \\ &\leq \frac{p}{2}.\end{aligned}$$

Thus $S \cup T$ is an independent set of $\Gamma_{2,p}$.

It follows directly from Lemma 3.20 that $|S| = \frac{p\sqrt{p}}{2} + \mathcal{O}\left(p^{\frac{3}{4}}\right)$. We can get a lower bound for $|T|$ by only picking the points where $(a, b) = 1$ and identifying (a, b, c) with $(-a, -b, p - c)$.

Using this and Lemma 3.20, we obtain

$$\begin{aligned}|T| &\geq \frac{1}{2}\sum_{(x,y)\in C'\left(\frac{\pi}{2}\sqrt{p}\right)}\left(p-1-2\frac{p^{\frac{3}{4}}}{\sqrt{2\pi}}\sqrt{x^2+y^2}\right)\\ &= \frac{p-1}{2}\left|C'\left(\frac{\pi}{2}\sqrt{p}\right)\right| - \frac{p^{\frac{3}{4}}}{\sqrt{2\pi}}\sum_{(x,y)\in C'\left(\frac{\pi}{2}\sqrt{p}\right)}\sqrt{x^2+y^2}\\ &= \frac{p-1}{2}\left(3\sqrt{p}+\mathcal{O}\left(p^{\frac{1}{4}+\epsilon}\right)\right) - \frac{p^{\frac{3}{4}}}{\sqrt{2\pi}}\left(\sqrt{2\pi}p^{\frac{3}{4}}+\mathcal{O}\left(p^{\frac{1}{2}+\epsilon}\right)\right)\\ &= \frac{p\sqrt{p}}{2} - \mathcal{O}\left(p^{\frac{5}{4}+\epsilon}\right).\end{aligned}$$

The desired result follows from this and the definition of $\overline{\alpha}$. ☐

A natural question to ask at this point is whether it is possible to get rid of the $\mathcal{O}\left(p^{\frac{5}{4}+\epsilon}\right)$ term. We do not have an answer for that, but have checked that it is possible to get rid of the error term for small primes.

Lemma 3.22. *Let $p \leq 5000$ be prime. Then $\overline{\alpha}(\mathcal{D}_{2,p}) \geq \frac{1}{2}p\sqrt{p}$.*

Proof. We use a similar construction of S and T as in Theorem 3.21, but we modify the radius of the circle slightly. That is, let

$$S = \{\langle(x,y,1)\rangle : (x,y) \in C\left(r^*\right)\},$$

where r^* the smallest positive real number that guarantees $|S| \geq \frac{1}{2}p\sqrt{p}$. Then it suffices to simply check that $|\partial(S)| \leq p^2+p+1-\frac{1}{2}p\sqrt{p}$. We perform the verification by computer on all primes up to at least 5000, and the result is always true. Our sample code that does this can be found in Appendix B. ☐

We also extend Theorem 3.21 to a more general result.

Theorem 3.23. *Let $\epsilon > 0$ be a real number, and $q = p^e$ a prime power. Then*

$$\overline{\alpha}(\mathcal{D}_{2,q}) \geq \left(\frac{1}{2}-\mathcal{O}\left(p^{\varepsilon-\frac{1}{4}}\right)\right)q^{\frac{3}{2}}.$$

Proof. The case $e = 1$ was already proven in Theorem 3.21. If $e = 2k+1$ for some positive integer k, then we know from (3.7) that

$$\overline{\alpha}(\mathcal{D}_{2,q}) \geq p^{3k}\overline{\alpha}(\mathcal{D}_{2,p}) = \left(\frac{1}{2}-\mathcal{O}\left(p^{\varepsilon-\frac{1}{4}}\right)\right)q^{\frac{3}{2}}.$$

Finally, if e is even, then the result follows directly from (3.6). □

Theorem 3.24. *Let $\epsilon > 0$ be a real number, n a positive integer, and q a prime power. Then*

$$\overline{\alpha}(\mathcal{D}_{n,q}) \geq \left(\frac{1}{2} - \mathcal{O}\left(p^{\varepsilon-\frac{1}{4}}\right)\right) q^{\frac{n+1}{2}}.$$

Proof. We prove this by induction on n, splitting it into odd and even cases. The result is trivially true when $n = 1$ due to (3.4), and we have already established the result for $n = 2$ in Theorem 3.23.

Finally, if $n \geq 3$, we use (3.8) and the induction hypothesis to show that

$$\overline{\alpha}(\mathcal{D}_{n,q}) \geq q\overline{\alpha}(\mathcal{D}_{n-2,q}) \geq q\left(\frac{1}{2} - \mathcal{O}\left(p^{\varepsilon-\frac{1}{4}}\right)\right) q^{\frac{n-1}{2}} \geq \left(\frac{1}{2} - \mathcal{O}\left(p^{\varepsilon-\frac{1}{4}}\right)\right) q^{\frac{n+1}{2}}.$$ □

3.5 The MSP on $\overline{\mathcal{D}_{n,q}}$

In this section, we will completely solve the MSP on $\overline{\mathcal{D}_{n,q}}$, the point-hyperplane non-incidence BIBD of $\mathrm{PG}(n, q)$. This turns out to be significantly easier than on its complement $\mathcal{D}_{n,q}$. Recall that $\overline{\mathcal{D}_{n,q}}$ is a symmetric BIBD with the parameters

$$(v, k, \lambda) = \left(\frac{q^{n+1} - 1}{q - 1}, q^n, q^n - q^{n-1}\right).$$

We start with a basic linear algebraic result.

Lemma 3.25. *Let X and Y be non-empty subsets of $\mathbb{F}_q^n$ such that $x \cdot y = 0$ for all $x \in X, y \in Y$. Then*

$$\left\lceil \log_q |X| \right\rceil + \left\lceil \log_q |Y| \right\rceil \leq n.$$

Proof. It is clear that $\mathrm{span}(X)$ and $\mathrm{span}(Y)$ are orthogonal subspaces of $\mathbb{F}(q)^n$, and thus it follows that the sum of their dimensions must be no more than n. □

Theorem 3.26. *Let $n \geq 2$ be an integer, and q a prime power. Then*

$$\Phi_S(\overline{\mathcal{D}_{n,q}}, x) = \frac{q^{n+1} - q^{n+1-m(x)}}{q - 1},$$

where $m(x) = \left\lceil \log_q ((q-1)x + 1) \right\rceil$.

Proof. We prove the lower bound first. Let $\overline{\mathcal{D}_{n,q}} = (P, \mathcal{B})$ and let $X \subseteq P$ be an x-subset of points. Let $Y = \mathcal{B} \setminus \partial(X)$, and let $y = |Y|$.

By treating elements of X as a 1-dimensional subspace of $\mathbb{F}_q^{n+1}$, we can "unproject" it into $(q-1)x+1$ points of $\mathbb{F}_q^{n+1}$, and similarly for Y (taking its orthogonal poles). It follows from Lemma 3.25 that

$$m(x) + m(y) \leq n+1.$$

This rearranges to

$$(q-1)y + 1 \leq q^{n+1-m(x)},$$

from which it follows that

$$|\partial(X)| = |\mathcal{B}| - y \geq \frac{q^{n+1} - q^{n+1-m(x)}}{q-1},$$

and the lower bound for $\Phi_S(\overline{\mathcal{D}_{n,q}}, x)$ follows from the arbitrary choice of X.

For the upper bound, let S be an $m(x)$-dimensional subspace of $\mathbb{F}_q^{n+1}$, and let X be the set of 1-dimensional subspaces of S, so that $X \subseteq P$ and $|X| \geq x$. It follows that

$$\Phi_S(\overline{\mathcal{D}_{n,q}}, x) \leq \Phi_S(\overline{\mathcal{D}_{n,q}}, |X|) \leq |\partial(X)| = \frac{q^{n+1} - q^{n+1-m(x)}}{q-1}$$

as required. □

Theorem 3.27. *Let $n \geq 2$ be an integer, and q a prime power. Then*

$$\overline{\alpha}(\overline{\mathcal{D}_{n,q}}) = \frac{q^{\lfloor \frac{n+1}{2} \rfloor} - 1}{q-1}.$$

Proof. This follows directly from Theorem 3.26, since we can easily check that

$$\Phi_S'\left(\overline{\mathcal{D}_{n,q}}, \frac{q^{\lfloor \frac{n+1}{2} \rfloor} - 1}{q-1}\right) = \frac{q^{n+1} - q^{\lceil \frac{n+1}{2} \rceil} + q^{\lfloor \frac{n+1}{2} \rfloor} - 1}{q-1} \leq \frac{q^{n+1}-1}{q-1},$$

and

$$\Phi_S'\left(\overline{\mathcal{D}_{n,q}}, \frac{q^{\lfloor \frac{n+1}{2} \rfloor} - 1}{q-1} + 1\right) = \frac{q^{n+1} - q^{\lceil \frac{n+1}{2} \rceil - 1} + q^{\lfloor \frac{n+1}{2} \rfloor} - 1}{q-1} + 1 > \frac{q^{n+1}-1}{q-1}.$$ □

3.6 The MSP on unitals

Theorem 3.28. *Let U be a unital of order n. Then*

$$\min\{\lfloor c(n)\rfloor, m_2(U)\} \leq \overline{\alpha}(U) \leq \lfloor c(n)\rfloor,$$

where

$$c(n) = n^2 - \frac{\sqrt{8n^2+9}-3}{2}. \tag{3.9}$$

Proof. First we motivate where $c(n)$ comes from. Let $g : [0, n^2+1] \to \left[0, \frac{(n^2+1)(n^2+2)}{2}\right]$ be given by

$$g(z) = (n^2+1)z - \frac{z(z-1)}{2}.$$

It is easily verified that g is increasing, concave, and bijective, so that g^{-1} exists. Then it is an easy check that

$$c(n) = g^{-1}\left(\frac{n^2(n^2+1)}{2}\right).$$

Now, for any integer x we have $\Phi'_S(U,x) \geq g(x)$ from Theorem 3.4 (by letting $m = 1$). So if $x > c(n)$ then $\Phi'_S(U,x) > \frac{v+b}{2}$, and it follows from this that $\overline{\alpha}(U) \leq \lfloor c(n)\rfloor$.

For the lower bound, let $x = \min\{\lfloor c(n)\rfloor, m_2(U)\}$. Let X be an x-arc of U; this always exists as we can just take an x-subset of a $m_2(U)$-arc.

Now, every block in $\partial(X)$ is incident with exactly one or two points in X. Then Theorem 3.4 (with $m = 1$) gives us $\Phi'_S(U,x) = g(x)$. Since $x \leq c(n)$, it follows that

$$\Phi'_S(U,x) \leq g(c(n)) = \frac{n^2(n^2+1)}{2} = \left\lfloor \frac{v+b}{2} \right\rfloor,$$

and thus $\overline{\alpha}(U) \geq x$ as required. □

Corollary 3.29. *Let U be a unital of order n. If $m_2(U) \geq \lfloor c(n)\rfloor$, then $\overline{\alpha}(U) = \lfloor c(n)\rfloor$.*

Corollary 3.29 holds, for example, on the classical unitals $H(q)$ for any prime power q, as well as on the BM-unitals $\mathrm{BM}(\alpha,\beta,q)$ with q an odd prime power, $\alpha \in \mathbb{F}_{q^2}$ and $\beta \in \mathbb{F}_q$ such that $(\beta^q - \beta)^2 + 4\alpha^{q+1}$ is a non-square in $\mathbb{F}_q$.

Finally, we wrap up the chapter by looking at complements of unitals.

Theorem 3.30. *If U be a unital design of order n, then*

$$\overline{\alpha}(\overline{U}) = \begin{cases} 1, & \text{if } n = 2 \\ 0, & \text{if } n \geq 3. \end{cases}$$

Proof. This follows from Lemma 3.8 since $\overline{U}$ has the parameters

$$(v, b, r, k, \lambda) = (n^3 + 1, n^4 - n^3 + n^2, n^4 - n^3, n^3 - n, n^4 - n^3 - n^2 + 1).$$

The case $n = 2$ can be verified manually, and if $n \geq 3$ we have $n^4 \geq 3n^3 \geq 2n^3 + n^2 + 1$, so that $r = n^4 - n^3 \geq \frac{n^4+n^2+1}{2} \geq \frac{v+b}{2}$ as required. □

Chapter 4

The Vertex-Isoperimetric Problem

In this chapter, we will discuss the vertex-isoperimetric problem (VIP) on Levi graphs of BIBDs. In Section 4.1, we provide the basic definitions and notation to be used throughout the rest of the chapter, and also discuss some of the properties of the VIP. In Section 4.2, we establish some general isoperimetric inequalities for the VIP that hold for Levi graphs of all (or many) BIBDs. We also obtain a good general lower bound on the vertex-isoperimetric number in Section 4.3.

The remaining sections will focus on the VIP for specific families of graphs, particularly those related to $\mathrm{PG}(n,q)$ or Levi graphs of unitals.

Section 4.4 looks at solutions of the VIP on $\Gamma_{2,q}$ for small q. We also establish the fact that $\Gamma_{2,2}$ and $\Gamma_{2,3}$ both admit nested solutions for the VIP, while $\Gamma_{2,4}$ does not. In addition, we also obtain the exact vertex-isoperimetric number of $\Gamma_{2,q}$ for all $q \leq 16$ in this section. We generalise this in Section 4.5 by looking at $\mathcal{D}_{n,q}$ for general n and q, before looking at its complement in Section 4.6.

Finally, in Section 4.7 we will study the VIP on unitals, and establish the exact vertex-isoperimetric number for unitals with sufficiently large arcs. This class of unitals includes, for example, the classical unitals. We also determine the exact vertex-isoperimetric number for Levi graphs of complements of unitals.

4.1 Basic definitions and properties

Definition 4.1. *Let $\Gamma = (V, E)$ be a graph. If $S \subseteq V$ is a subset of vertices, recall that the vertex boundary $\delta(S)$ is the set of vertices in $\overline{S}$ that are adjacent to some vertex in*

S. The vertex-isoperimetric parameter is the function

$$\Phi_V(\Gamma, n) = \min\{|\delta(S)| : S \subseteq V, |S| = n\}.$$

Any set S satisfying $\Phi_V(\Gamma, |S|) = |\delta(S)|$ is called a *VIP-set* of Γ. The VIP on Γ is said to have *nested solutions* if there exists an ordering on V such that the first n vertices of V in this ordering always form a VIP-set for all n.

A VIP-set S is *closed* if $|\delta(S')| > |\delta(S)|$ for all $|S'| > |S|$.

We are mostly interested in considering the VIP on Levi graphs of incidence structures. Indeed, if $I = (P, \mathcal{B}, \partial)$ is an incidence structure, then ∂ and ∂^* are simply restrictions of the vertex boundary δ of $\Gamma(I)$. It is thus useful in this case to consider a variant of this parameter that considers the size of both bipartite parts. We write this variant as

$$\Phi_V(I, x, y) = \min\{|\delta(S)| : S \subseteq V(\Gamma(I)), S \text{ is of type } (x, y)\}.$$

It is clear from this relation that

$$\Phi_V(\Gamma(I), n) = \min_{x+y=n} \Phi_V(I, x, y).$$

Additionally, restricting x or y to zero makes it an instance of the MSP, that is

$$\Phi_V(I, x, 0) = \Phi_S(I, x)$$

and

$$\Phi_V(I, 0, y) = \Phi_S(I^*, y).$$

Definition 4.2. *Let $\Gamma = (V, E)$ be a graph. The vertex-isoperimetric number $i_V(\Gamma)$ of Γ is the quantity*

$$i_V(\Gamma) = \min\left\{\frac{\Phi_V(\Gamma, n)}{n} : 1 \leq n \leq \frac{|V|}{2}\right\}.$$

Lemma 4.3. *Let $I = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ). Then*

$$i_V(\Gamma(I)) \leq \frac{v - \overline{\alpha}(I)}{\left\lfloor \frac{v+b}{2} \right\rfloor}. \tag{4.1}$$

Proof. Let X be an MSP-set of I of cardinality $\overline{\alpha}(I)$. Take Y with $\partial(X) \subseteq Y \subseteq \mathcal{B}$ such that $|Y| = \left\lfloor \frac{v+b}{2} \right\rfloor - |X|$, and let $S = X \cup Y$. Then $|S| = \left\lfloor \frac{v+b}{2} \right\rfloor$ and $\delta(S) = P \setminus X$, so that

$$|\delta(S)| \leq v - \overline{\alpha}(I). \tag{4.2}$$

The desired inequality follows by diving both sides of (4.2) by $|S|$. □

Note that it follows immediately from Lemma 4.3 that

$$i_V(\Gamma(I)) \leq \frac{2v}{v+b-1}. \tag{4.3}$$

We say that I is *$\overline{\alpha}$-sharp* if equality holds in (4.1). It turns out that many interesting BIBDs are in fact $\overline{\alpha}$-sharp, and we will see many examples in this chapter.

As with the MSP before, we can similarly introduce the "inclusive" variants

$$\Phi'_V(\Gamma, n) = \min\{|S \cup \delta(S)| : S \subseteq V, |S| = n\}$$

and

$$\Phi'_V(I, x, y) = \min\{|S \cup \delta(S)| : S \subseteq V(\Gamma(I)), |S \cap P| = x, |S \cap \mathcal{B}| = y\}.$$

These functions turn out to be easier to reason with due to the following lemma.

Lemma 4.4. *Let $\Gamma = (V, E)$ be a graph. Then $\Phi'_V(I, n)$ is increasing with n.*

Proof. Let S be a non-empty VIP-set of Γ. Let $T \subseteq S$ with $|T| = |S| - 1$. Any vertex in the boundary of T must be in S or its boundary. Thus

$$\Phi'_V(\Gamma, |T|) \leq |T \cup \partial(T)| \leq |S \cup \partial(S)| = \Phi'_V(\Gamma, |S|).$$ □

4.2 General isoperimetric inequalities

We start off with a fairly obvious inequality.

Lemma 4.5. *Let $I = (P, \mathcal{B})$ be an incidence structure, and x and y be non-negative integers with $x \leq |P|$ and $y \leq |\mathcal{B}|$. Then*

$$\Phi_V(I, x, y) \geq \max\{\Phi_S(I, x) - y, 0\} + \max\{\Phi_S(I^*, y) - x, 0\}.$$

Proof. Let $S \subseteq V$ be a subset of vertices of type (x, y), and let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. Then $\delta(S) = (\partial(X) \setminus Y) \cup (\partial(Y) \setminus X)$. It follows that

$$\begin{aligned} |\delta(S)| &\geq |\partial(X) \setminus Y| + |\partial(Y) \setminus X| \\ &\geq \max\{\Phi_S(I, x) - y, 0\} + \max\{\Phi_S(I^*, y) - x, 0\}, \end{aligned}$$

and the result follows from the arbitrary choice of S. □

Lemma 4.6. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ) and $G = (V, E)$ its Levi graph. Let $S \subseteq V$ be a subset of vertices of type (x, y), and suppose $\delta(S)$ is of type (x', y'). Then*

$$\lambda x(v - x - x') \leq \frac{k^2}{4} y'.$$

Proof. Let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. Let A be the set of paths of length two in G with one end-vertex in X and the other end-vertex in $P \setminus (X \cup \delta(Y))$. Since each pair of points in $X \times (P \setminus (X \cup N(Y)))$ are contained in exactly λ blocks, we have

$$|A| = \lambda x(v - x - x'). \tag{4.4}$$

Consider a block B that is the middle vertex of a path in A. Since B is adjacent to a point in X, we have $B \in \delta(X)$. Similarly, B is adjacent to a point outside $\delta(Y)$, and so $B \notin Y$. Hence $B \in \delta(X) \setminus Y$. Denote by a_B the number of points in X adjacent to B in G, and b_B the number of points in $P \setminus (X \cup \delta(Y))$ adjacent to B in G. Then $a_B + b_B \leq k$ and so $a_B b_B \leq \left(\frac{k}{2}\right)^2$. It follows that each such block B can contribute at most $\left(\frac{k}{2}\right)^2$ paths to A. Since there are at most y' such blocks, it follows that

$$|A| \leq \frac{k^2}{4} y'.$$

The result follows from this together with (4.4). □

Theorem 4.7. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ). Let $x \leq v$ and $y \leq b$ be non-negative integers. If $x \leq \frac{k^2}{4\lambda}$, then*

$$\Phi_V(\mathcal{D}, x, y) \geq \left(\frac{4\lambda v}{k^2} - 1\right) x + \left(1 - \frac{4\lambda x}{k^2}\right) \Phi_S(\mathcal{D}^*, y).$$

Proof. Let $S \subseteq V$ be a subset of vertices of type (x, y), and let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. Let $x' = |\delta(Y) \setminus X|$ and $y' = |\delta(X) \setminus Y|$. Applying Lemma 4.6, we obtain

$$\begin{aligned}
|\delta(S)| &\geq x' + y' \\
&= \left(\frac{4\lambda x}{k^2} x' + y'\right) + \left(1 - \frac{4\lambda x}{k^2}\right) x' \\
&\geq \frac{4\lambda x}{k^2}(v - x) + \left(1 - \frac{4\lambda x}{k^2}\right) (\Phi_S(\mathcal{D}^*, y) - x) \\
&= \left(\frac{4\lambda v}{k^2} - 1\right) x + \left(1 - \frac{4\lambda x}{k^2}\right) \Phi_S(\mathcal{D}^*, y),
\end{aligned}$$

and the result follows from the arbitrary choice of S. □

Theorem 4.8. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ). Let $x \leq v$ and $y \leq b$ be non-negative integers. If $x \geq \frac{k^2}{4\lambda}$, then*

$$\Phi_V(\mathcal{D}, x, y) \geq v - x + \left(1 - \frac{k^2}{4\lambda x}\right) (\Phi_S(\mathcal{D}, x) - y) .$$

Proof. Let $S \subseteq V$ be a subset of vertices of type (x, y), and let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. Let $x' = |\delta(Y) \setminus X|$ and $y' = |\delta(X) \setminus Y|$. Applying Lemma 4.6, we obtain

$$\begin{aligned}
|\delta(S)| &\geq x' + y' \\
&= \left(x' + \frac{k^2}{4\lambda x} y'\right) + \left(1 - \frac{k^2}{4\lambda x}\right) y' \\
&\geq v - x + \left(1 - \frac{k^2}{4\lambda x}\right) (\Phi_S(\mathcal{D}, x) - y)
\end{aligned}$$

and the result follows from the arbitrary choice of S. □

Lemma 4.9. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ), and $\Gamma = (V, E)$ its Levi graph. Let m be a positive integer. If $x \in \{0, 1, \ldots, v\}$, then*

$$\Phi_V(\mathcal{D}, x, 1) \geq \frac{2rmx - \lambda x(x-1)}{m(m+1)} + k - \frac{(m+2)(m+3)}{4}.$$

Proof. Let $S \subseteq V$ be a subset of vertices of type $(x, 1)$. Let $X = S \cap P$, and denote by B^* the unique block vertex in S.

By double counting, we have

$$\sum_{B \in \partial(X)} |B \cap X| = rx$$

and

$$\sum_{B \in \partial(X)} |B \cap X|(|B \cap X| - 1) = \lambda x(x-1).$$

Let $Z = \partial(X) \setminus \{B^*\}$. It follows from Lemma 3.3 that

$$\begin{aligned}
\lambda x(x-1) - |B^* \cap X|(|B^* \cap X| - 1) &= \sum_{B \in Z} |B \cap X|(|B \cap X| - 1) \\
&\geq 2m \sum_{B \in Z} |B \cap X| - m(m+1)|Z| \\
&= 2m(rx - |B^* \cap X|) - m(m+1)|Z|.
\end{aligned}$$

Writing $s = |B^* \cap X|$, we have $|Z| \geq \frac{1}{m(m+1)}\left(2rmx - \lambda x(x-1) + s(s-2m-1)\right)$. Now, since $\frac{m(m+3)}{2}$ is an integer, we must have

$$\left(\frac{m(m+3)}{2} - s\right)\left(\frac{m(m+3)}{2} - s + 1\right) \geq 0,$$

from which it follows that

$$s(s-2m-1) \geq m(m+1)\left(s - \frac{(m+2)(m+3)}{4}\right),$$

and thus

$$|Z| \geq \frac{2rm - \lambda x(x-1)}{m(m+1)} + s - \frac{(m+2)(m+3)}{4}.$$

Finally, note that $\delta(S) = Z \cup (B^* \setminus X)$, so that

$$|\delta(S)| = |Z| + k - s \geq \frac{2rm - \lambda x(x-1)}{m(m+1)} + k - \frac{(m+2)(m+3)}{4},$$

and the result follows from the arbitrary choice of S. $\square$

4.3 General lower bound on i_V

For the rest of this section, given parameters (v, b, r, k, λ) of a 2-design, we let $u : [0,1] \to [0,1]$ be given by

$$u(x) = \frac{rkx}{r - \lambda + \lambda vx}.$$

It is immediately clear that $u(0) = 0$ and $u(1) = 1$. Furthermore, it is an easy check that for all $x \geq 0$, we have

$$u'(x) = \frac{rk(r-\lambda)}{(r-\lambda+\lambda vx)^2} \geq 0$$

and

$$u''(x) = \frac{-2vrk\lambda(r-\lambda)}{(r-\lambda+\lambda vx)^3} \leq 0,$$

so that u is increasing, concave, and bijective.

Now we also let $f : [0, v] \to [0, v+b]$ and $g : [0, b] \to [0, v+b]$ be given by

$$f(x) = \frac{r^2x}{r+\lambda(x-1)} + x \qquad \text{and} \qquad g(y) = \frac{rky}{r^2 - \lambda(b-y)} + y. \tag{4.5}$$

This is equivalent to defining them as

$$f(x) = bu\left(\frac{x}{v}\right) + x \qquad \text{and} \qquad g(y) = vu\left(\frac{y}{b}\right) + y. \tag{4.6}$$

Lemma 4.10. *Let f and g be as defined as in* (4.6)*. Then the following properties hold:*

1. *f and g are increasing, concave, and bijective;*
2. *f^{-1} and g^{-1} exist and are increasing and convex;*
3. *$f(x) \geq g(x)$ for all $x \in [0, v]$;*
4. *$f'(x) \geq g'(x)$ for all $x \in [0, v]$;*
5. *$f^{-1}(x) \leq g^{-1}(x)$ for all $x \in [0, v+b]$;*
6. *$f^{-1}(x) + g^{-1}(x) \leq x$ for all $x \in [0, v+b]$; and*
7. *$f(x - g^{-1}(x)) \geq g(x - f^{-1}(x))$ for all $x \in [0, v+b]$.*

Proof. The first two properties follow immediately from the properties of u.

The third inequality follows from

$$f(x) = bu\left(\frac{x}{v}\right) + x \geq vu\left(\frac{x}{v}\right) + x \geq vu\left(\frac{x}{b}\right) + x = g(x).$$

For the fourth inequality, we have

$$f'(x) - 1 = \frac{b}{v}u'\left(\frac{x}{v}\right) = \frac{rk(r-\lambda)}{\frac{v}{b}(r-\lambda+\lambda x)^2} \geq \frac{rk(r-\lambda)}{\frac{v}{b}\left(\frac{b}{v}(r-\lambda)+\lambda x\right)^2} = \frac{v}{b}u'\left(\frac{x}{b}\right) = g'(x) - 1.$$

The fifth inequality follows from

$$g(f^{-1}(x)) \leq f(f^{-1}(x)) = g(g^{-1}(x)).$$

The sixth inequality is immediately true when $x = 0$. If $x \neq 0$, then we have

$$\frac{f^{-1}(x)}{x} \leq \frac{f^{-1}(v+b)}{v+b} = \frac{v}{v+b}$$

and

$$\frac{g^{-1}(x)}{x} \leq \frac{g^{-1}(v+b)}{v+b} = \frac{b}{v+b},$$

so that

$$\frac{f^{-1}(x) + g^{-1}(x)}{x} \leq 1.$$

For the final claim, fix an x and let $h : [f^{-1}(x), x - g^{-1}(x)] \to [0, v+b]$ be given by $h(y) = g(y + g^{-1}(x) - f^{-1}(x))$. Then for any y in the domain of h, we have

$$f'(y) \geq g'(y) \geq g'(y + g^{-1}(x) - f^{-1}(x)) = h'(y)$$

and thus $(f - h)$ is increasing. This means that

$$\begin{aligned} f(x - g^{-1}(x)) - g(x - f^{-1}(x)) &= (f-h)(x - g^{-1}(x)) \\ &\geq (f-h)(f^{-1}(x)) \\ &= 0, \end{aligned}$$

which proves our claim. □

Theorem 4.11. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ), and $\Gamma = (V, E)$ its Levi graph. If $S \subseteq V$ is a subset of vertices, then*

$$\Phi'_V(\Gamma, n) \geq g(n - f^{-1}(n)),$$

where the functions f and g are given as in (4.5).

Proof. Let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. For conciseness, write $x = |X|$ and $n = |S|$. Then

$$\begin{aligned}|\delta(S)| &= |\delta(X) \setminus Y| + |\delta(Y) \setminus X| \\ &\geq \max\{|\delta(X)| - |Y|, 0\} + \max\{|\delta(Y)| - |X|, 0\} \\ &\geq \max\{f(|X|) - |X| - |Y|, 0\} + \max\{g(|Y|) - |Y| - |X|, 0\} \\ &= \max\{f(x) - n, 0\} + \max\{g(n - x) - n, 0\}.\end{aligned}$$

We can now split the proof into three cases.

1. If $x \leq f^{-1}(n)$ then

$$|\delta(S)| \geq g(n - x) - n \geq g(n - f^{-1}(n)) - n.$$

2. If $f^{-1}(n) \leq x \leq n - g^{-1}(n)$ then the function mapping z to $f(z) + g(n - z)$ is convex and so must obtain its minimum and either endpoint. More specifically,

$$\begin{aligned}|\delta(S)| &\geq f(x) - n + g(n - x) - n \\ &\geq \min\{f(f^{-1}(n)) + g(n - f^{-1}(n)), f(n - g^{-1}(n)) + g(g^{-1}(n))\} - 2n \\ &= \min\{g(n - f^{-1}(n)), f(n - g^{-1}(n))\} - n \\ &= g(n - f^{-1}(n)) - n.\end{aligned}$$

3. Finally, if $x \geq n - g^{-1}(n)$ then

$$|\delta(S)| \geq f(x) - n \geq f(n - g^{-1}(n)) - n \geq g(n - f^{-1}(n)) - n. \qquad \square$$

Lemma 4.12. *Let $\Gamma = (V, E)$ be the Levi graph of a BIBD with parameters (v, b, r, k, λ). Let m be a positive integer, and n a positive real number, satisfying $m \leq n \leq \frac{v+b}{2}$. Then*

$$\frac{\Phi_V(\Gamma, m)}{m} \geq -1 + \frac{1}{n} g(n - f^{-1}(n)),$$

where the functions f and g are given as in (4.5).

Proof. First, note that for any $y \in [0, b]$ and $\alpha \in [0, 1]$, we necessarily have

$$g(\alpha y) = g(\alpha y + (1-\alpha)0) \geq \alpha g(y) + (1-\alpha)g(0) = \alpha g(y).$$

Similarly, it is also true that $f^{-1}(\alpha x) \leq \alpha f^{-1}(x)$ for any $x \in [0, v+b]$ and $\alpha \in [0, 1]$. Then

$$\begin{aligned}\frac{\Phi_V(\Gamma, m)}{m} &\geq -1 + \frac{1}{m} g\left(m - f^{-1}(m)\right) \\ &\geq -1 + \frac{1}{m} g\left(\frac{m}{n}\left(n - f^{-1}(n)\right)\right) \\ &\geq -1 + \frac{1}{n} g\left(n - f^{-1}(n)\right)\end{aligned}$$

as required. □

Lemma 4.13. *Let Γ be the Levi graph of a BIBD with parameters (v, b, r, k, λ). Then*

$$i_V(\Gamma) \geq \frac{2}{v+b}\, g\left(\frac{v+b}{2} - f^{-1}\left(\frac{v+b}{2}\right)\right) - 1,$$

where the functions f and g are given as in (4.5).

Proof. Let $S \subseteq V(\Gamma)$ with $1 \leq |S| \leq \frac{v+b}{2}$. Then from Lemma 4.12 we have

$$\frac{|\partial(S)|}{|S|} \geq \frac{\Phi_V(\Gamma, |S|)}{|S|} \geq \frac{2}{v+b}\, g\left(\frac{v+b}{2} - f^{-1}\left(\frac{v+b}{2}\right)\right) - 1,$$

and the result follows from the arbitrary choice of S. □

Lemma 4.14. *Let Γ be the Levi graph of a BIBD with parameters (v, b, r, k, λ). Then*

$$i_V(\Gamma) \geq \frac{2}{s^{-1} + st + \sqrt{(st)^2 - 2t + 1}},$$

where $s = \frac{2r^2}{\lambda b} - 1$ and $t = \frac{v+b}{2v}$.

Proof. We can rewrite the functions f and g from (4.5) as

$$f(x) = \frac{b(s+1)\frac{x}{v}}{s - 1 + 2\frac{x}{v}} + x \quad \text{and} \quad g(y) = \frac{v(s+1)\frac{y}{b}}{s - 1 + 2\frac{y}{b}} + y.$$

It is then easy, albeit tedious, to verify that

$$\frac{f^{-1}(vt)}{v} = \frac{1 - st + \sqrt{(st)^2 - 2t + 1}}{2}.$$

Then

$$\begin{aligned} i_V(\Gamma) &\geq \frac{1}{vt} g(vt - f^{-1}(vt)) - 1 \\ &= \frac{1}{vt}\left(\frac{v(s+1)\frac{vt - f^{-1}(vt)}{b}}{s - 1 + 2\frac{vt - f^{-1}(vt)}{b}} + vt - f^{-1}(vt)\right) - 1 \\ &= \frac{1}{t}\left(\frac{(s+1)\left(t - \frac{f^{-1}(vt)}{v}\right)}{(s-1)(2t-1) + 2\left(t - \frac{f^{-1}(vt)}{v}\right)} - \frac{f^{-1}(vt)}{v}\right) \\ &= \frac{1}{t}\left(\frac{2t}{s^{-1} + st + \sqrt{(st)^2 - 2t + 1}}\right), \end{aligned}$$

where the final simplifying equality can be verified purely by a straightforward, albeit tedious, algebraic manipulation of s and t (that is, without using v, b, r, k or λ). □

Theorem 4.15. *Let* Γ *be the Levi graph of a BIBD with parameters* (v, b, r, k, λ)*. Then*

$$i_V(\Gamma) \geq \frac{2v}{v+b}\left(1 - \frac{\sqrt{r^2 - \lambda b}}{r}\right).$$

Proof. This is an extension of the previous result. Using the same s and t as in Lemma 4.14, we have

$$\begin{aligned} i_V(\Gamma) &\geq \frac{2}{s^{-1} + st + \sqrt{(st)^2 - 2t + 1}} \\ &\geq \frac{2}{1 + st + \sqrt{(s^2-1)t^2 + (t-1)^2}} \\ &\geq \frac{2}{1 + st + t\sqrt{s^2 - 1} + t - 1} \\ &= \frac{1}{t}\left(1 - \sqrt{1 - \frac{2}{s+1}}\right) \\ &= \frac{2v}{v+b}\left(1 - \frac{\sqrt{r^2 - \lambda b}}{r}\right). \end{aligned}$$

□

Finally, we write down a simplification of this result for symmetric BIBDs.

Theorem 4.16. *Let* Γ *be the Levi graph of a symmetric* (v,k,λ)*-BIBD, and let* $\mu = \sqrt{k-\lambda}$*. Then*

$$i_V(\Gamma) \geq (k-\mu)\frac{k^2+\mu^2}{k^3+\mu^3}.$$

Proof. It is an easy check that we have $s = \frac{k^2+\mu^2}{k^2-\mu^2}$ and $t = 1$ in the statement of Lemma 4.14. Then

$$\begin{aligned} i_V(\Gamma) &\geq \frac{2}{s^{-1}+s+\sqrt{s^2-1}} \\ &= \frac{2}{\frac{k^2-\mu^2}{k^2+\mu^2}+\frac{k+\mu}{k-\mu}} \\ &= (k-\mu)\frac{k^2+\mu^2}{k^3+\mu^3} \end{aligned}$$

as required. □

4.4 The VIP on $\Gamma_{2,q}$ for small q

We will show in this section that $\mathcal{D}_{2,q}$ is $\overline{\alpha}$-sharp for all prime powers $q \leq 16$.

4.4.1 Nested solutions for $q = 2$

Let $\mathcal{D} = \mathcal{D}_{2,2}$, and Γ its Levi graph. We will demonstrate that the VIP on Γ has nested solutions, given by the ordering

$$V = \{0, 3{+}D, 5, 1, 2, D, 4{+}D, 5{+}D, 6, 4, 2{+}D, 3, 6{+}D, 1{+}D\},$$

where D is the difference set of $\mathcal{D}$ as given in Appendix A.

Let S_n denote the set obtained by taking the first n elements of V in this ordering, $0 \leq n \leq 14$. We summarise the vertex boundary sizes of S_n in Table 4.1.

Table 4.1 Nested solutions for the VIP on $\Gamma_{2,2}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lvert\delta(S_n)\rvert$	0	3	4	5	6	6	5	5	4	4	3	3	2	1	0

In what follows, we will prove the lower bounds on the VIP and show that they match the upper bounds in Table 4.1.

For $n \leq 3$, we have from Theorem 4.11 that

$$\Phi_V(\Gamma, 1) \geq 3; \quad \Phi_V(\Gamma, 2) \geq 4; \quad \text{and} \quad \Phi_V(\Gamma, 3) \geq 5.$$

For $n = 4$, we split it up into three smaller cases:

$$\left.\begin{array}{l}\text{Lemma 4.5 yields } \Phi_V(\mathcal{D}, 4, 0) = 6 \\ \text{Theorem 4.8 yields } \Phi_V(\mathcal{D}, 3, 1) \geq 6 \\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D}, 2, 2) \geq 6\end{array}\right\} \implies \Phi_V(\Gamma, 4) \geq 6.$$

This gives us closed MSP-sets for $n \in \{0, 1, 2, 3, 4\}$, which then yield dual closed MSP-sets for $n \in \{6, 8, 10, 14\}$. The remaining values of n do not admit closed MSP-sets, and thus satisfy $\Phi_S(\Gamma, n) = \Phi_S(\Gamma, n+1) + 1$.

By evaluating $\frac{\Phi_S(\Gamma,n)}{n}$ for all $1 \leq n \leq 7$, we obtain the following result.

Theorem 4.17. $i_V(\Gamma_{2,2}) = \frac{5}{7}$, *and thus* $\Gamma_{2,2}$ *is* $\overline{\alpha}$*-sharp.*

4.4.2 Nested solutions for $q = 3$

Let $\mathcal{D} = \mathcal{D}_{2,3}$, and Γ its Levi graph. We will demonstrate that the VIP on Γ has nested solutions, given by the ordering

$$\begin{aligned} V = \{&0, 0{+}D, 3, 8, 4, 8{+}D, 11, 1, 5, 10, 7{+}D, 4{+}D, 9{+}D, 11{+}D, \\ &12, 2{+}D, 12{+}D, 1{+}D, 2, 9, 7, 6, 3{+}D, 5{+}D, 10{+}D, 6{+}D\}, \end{aligned}$$

where D is the difference set of $\mathcal{D}$ as given in Appendix A.

Table 4.2 Nested solutions for the VIP on $\Gamma_{2,2}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\delta(S_n)\rvert$	0	4	6	8	10	11	11	11	10	11	11	11	10	10

n	14	15	16	17	18	19	20	21	22	23	24	25	26
$\lvert\delta(S_n)\rvert$	9	8	8	7	6	6	5	4	4	3	2	1	0

In what follows, we will prove the lower bounds on the VIP and show that they match the upper bounds in Table 4.2.

For $n \in \{1, 2\}$, we have $\Phi_V(\Gamma, 1) \geq 4$ and $\Phi_V(\Gamma, 2) \geq 6$ from Theorem 4.11.

For $n \in \{3,4,5,6,7\}$, we need a bit more work to obtain the following:

$$\left.\begin{aligned}\text{Lemma 4.5 yields } \Phi_V(\mathcal{D},3,0) = 9\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},2,1) \geq 8\end{aligned}\right\} \implies \Phi_V(\Gamma,3) \geq 8;$$

$$\left.\begin{aligned}\text{Lemma 4.5 yields } \Phi_V(\mathcal{D},4,0) = 10\\ \text{Lemma 4.9 yields } \Phi_V(\mathcal{D},3,1) \geq 10\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},2,2) \geq 10\end{aligned}\right\} \implies \Phi_V(\Gamma,4) \geq 10;$$

$$\left.\begin{aligned}\text{Lemma 4.5 yields } \Phi_V(\mathcal{D},5,0) = 11\\ \text{Lemma 4.9 yields } \Phi_V(\mathcal{D},4,1) \geq 11\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},3,2) \geq 11\end{aligned}\right\} \implies \Phi_V(\Gamma,5) \geq 11;$$

$$\left.\begin{aligned}\text{Lemma 4.5 yields } \Phi_V(\mathcal{D},6,0) = 11\\ \text{Lemma 4.9 yields } \Phi_V(\mathcal{D},5,1) \geq 11\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},4,2) \geq 11\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},3,3) \geq 12\end{aligned}\right\} \implies \Phi_V(\Gamma,6) \geq 11; \text{ and}$$

$$\left.\begin{aligned}\text{Lemma 4.5 yields } \Phi_V(\mathcal{D},7,0) = 12\\ \text{Theorem 4.8 yields } \Phi_V(\mathcal{D},6,1) \geq 11\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},5,2) \geq 11\\ \text{Lemma 4.5 yields } \Phi_V(\mathcal{D},4,3) \geq 12\end{aligned}\right\} \implies \Phi_V(\Gamma,7) \geq 11.$$

In addition, we also have $\Phi_V(\Gamma,8) \geq \Phi_V(\Gamma,7) - 1 \geq 10$.

This gives us closed MSP-sets for $n \in \{0,1,2,3,4,5,6,8\}$, which then yield dual closed MSP-sets for $n \in \{9,10,12,15,18,21,26\}$. The remaining values of n do not admit closed MSP-sets, and thus satisfy $\Phi_S(\Gamma,n) = \Phi_S(\Gamma,n+1)+1$.

By evaluating $\frac{\Phi_S(\Gamma,n)}{n}$ for all $1 \leq n \leq 13$, we obtain the following result.

Theorem 4.18. $i_V(\Gamma_{2,3}) = \frac{10}{13}$, *and thus* $\Gamma_{2,3}$ *is* $\overline{\alpha}$*-sharp.*

4.4.3 No nested solutions for $q = 4$

We show in this subsection that the VIP on $\Gamma_{2,4}$ does not have nested solutions.

Lemma 4.19. $\Phi_V(\Gamma_{2,4},2) = 8$. *Furthermore, such a VIP-set must be of type* $(1,1)$.

Proof. We have $\Phi_V(\Gamma_{2,4}, 1, 1) = 8$ from Lemma 4.5. The result then follows from this and the fact that $\Phi_V(\Gamma_{2,4}, 2, 0) = \Phi_S(\mathcal{D}_{2,4}, 2) = 9$. □

Lemma 4.20. $\Phi_V(\Gamma_{2,4}, 5) = 15$. *Furthermore, such a VIP-set must be of type* $(0, 5)$ *or* $(5, 0)$.

Proof. We know that

$$\Phi_V(\Gamma_{2,4}, 5, 0) = \Phi_S(\mathcal{D}_{2,4}, 5) = 15.$$

Using Lemma 4.9 with $r = k = 5, \lambda = m = 1, x = 4$, we have

$$\Phi_V(\Gamma_{2,4}, 4, 1) \geq 16.$$

Finally, we also have from Lemma 4.5 that

$$\Phi_V(\Gamma_{2,4}, 3, 2) \geq 16.$$

□

Corollary 4.21. *The VIP on* $\Gamma_{2,4}$ *does not have nested solutions.*

Theorem 4.22. $i_V(\Gamma_{2,4}) = \frac{5}{7}$, *and thus* $\Gamma_{2,4}$ *is* $\overline{\alpha}$*-sharp.*

Proof. If $1 \leq n \leq 19$, then by Lemma 4.12, we have

$$\frac{\Phi_V(\Gamma_{2,4}, n)}{n} \geq \frac{17690 - 783\sqrt{101}}{12977} > \frac{5}{7},$$

and if $n = 20$, then $\Phi_V(\Gamma_{2,4}, n) \geq \frac{9077-203\sqrt{401}}{356} > 14$, so that $\frac{\Phi_V(\Gamma_{2,4},n)}{n} \geq \frac{15}{20} > \frac{5}{7}$.

So assume $n = 21$. Over all pairs (x, y) in $(0, 21), (1, 20), \ldots, (10, 11)$ except $(6, 15)$, we can use Lemma 4.5 to obtain $\Phi_V(\mathcal{D}_{2,4}, x, y) \geq 15$. Finally, we have from Theorem 4.7 that $\Phi_V(\mathcal{D}_{2,4}, 6, 15) \geq \frac{374}{25} > 14$, so that $\Phi_V(\Gamma_{2,4}, 21) \geq 15$.

It follows that $i_V(\Gamma_{2,4}) = \frac{5}{7}$, and thus $\Gamma_{2,4}$ is $\overline{\alpha}$-sharp as required. □

4.4.4 The case $5 \leq q \leq 16$

In this subsection, we show that all $\Gamma_{2,q}$ with $5 \leq q \leq 16$ is $\overline{\alpha}$-sharp. The solution seems to be difficult to obtain analytically, and is essentially a case-by-case brute force proof.

Because of this, it becomes more useful to run a program to compute the vertex-isoperimetric number. Since testing out all subsets takes exponential time, we weaken some of the constraints and give a polynomial time program for the relaxed problem.

Define $h : [0, v] \to [0, 2v]$ by

$$h(x) = \frac{(q+1)^2 x}{q+x} + x$$

and $g : \{0, 1, \ldots, v\} \to \{0, 1, \ldots, v\}$ by

$$g(x) = \begin{cases} 0, & \text{if } x = 0 \\ \dfrac{x(2m(x)(q+1) - x + 1)}{m(x)(m(x)+1)}, & \text{if } x \neq 0, \end{cases}$$

where $m(x) = \left\lfloor \frac{q+x}{q+1} \right\rfloor$.

Theorem 4.23. *Let q be a prime power and $v = q^2+q+1$. Let $\mathcal{D} = \mathcal{D}_{2,q} = (P, \mathcal{B})$, and $\Gamma = (V, E)$ its Levi graph. Suppose the value of $\overline{\alpha}(\mathcal{D})$ is already explicitly known. Then the optimal value of the following program is a lower bound for $i_V(\Gamma)$. Furthermore, this problem has a time complexity of $\mathcal{O}(q^{12})$.*

$$\begin{aligned}
&\textit{minimize} && \frac{c+d}{a+b} \\
&\textit{subject to} && a, b, c, d, e, f \in \{0, 1, \ldots, v\} && (4.7) \\
& && a + b \leq v && (4.8) \\
& && a + c + e = b + d + f = v && (4.9) \\
& && \frac{h(a + b - h^{-1}(a+b))}{a+b} - 1 \leq 1 - \frac{\overline{\alpha}(\Gamma_{2,q})}{v} && (4.10) \\
& && a \leq b && (4.11) \\
& && b + d \geq g(a) && (4.12) \\
& && a + c \geq g(b) && (4.13) \\
& && c + e \geq g(f) && (4.14) \\
& && \textit{If } e \geq 1 \textit{ then } qd \geq a && (4.15) \\
& && \textit{If } e = 2 \textit{ then } q(d+1) \geq 2(a+1) && (4.16) \\
& && \textit{If } q = 5 \textit{ and } a = 9 \textit{ then } b + d \geq 25 && (4.17) \\
& && \textit{If } a > \overline{\alpha}(\Gamma_{2,p}) \textit{ then } b + d \geq v - \overline{\alpha}(\Gamma_{2,p}) && (4.18)
\end{aligned}$$

Proof. Let $S \subseteq V$ be a subset of vertices satisfying $|S| \leq v$ and $|\delta(S)|/|S| = i_V(\Gamma)$. Let $A = S \cap V_1, B = S \cap V_2, C = \delta(S) \cap V_1, D = \delta(S) \cap V_2, E = V_1 \setminus (S \cup \delta(S))$ and

$F = V_2 \setminus (S \cup \delta(S))$. Without loss of generality we may assume that $|A| \leq |B|$. Let a, b, c, d, e, f be the cardinalities of A, B, C, D, E, F respectively. It suffices to check that all the twelve conditions are satisfied.

Conditions (4.7),(4.8), and (4.9) are trivially true. Condition (4.10) follows from Lemma 4.3 and Lemma 4.12. Condition (4.11) follows from our assumption that $|A| \leq |B|$. Condition (4.12) follows from Theorem 3.4 and the fact that $\delta(A) \subseteq B \cup D$, and conditions (4.13) and (4.14) are similar.

Note that if $e \geq 1$ then there is a common line in D for each pair of points in $A \times E$. Each such line in D then contains at most q points in A. So $qd \geq a$ and condition (4.15) follows.

Condition (4.16) uses a similar combinatorial argument, but we also take into account the fact that the two points in E can have at most one line in D joining them. By counting the number of 2-arcs from A to E (keeping in mind that all other lines in D contain at most one point in E), we obtain the inequality $2(q-1) + q(d-1) \geq 2a$.

Condition (4.17) is a sporadic case and is covered in [48, Section 4.3.1]. Finally, condition (4.18) follows from the definition of $\overline{\alpha}$ and the fact that $\delta(A) \subseteq B \cup D$.

We obtain the optimal value in polynomial time (with respect to q) by enumerating, in the worst case, all $(q^2+q+1)^6$ combinations of a, b, c, d, e, f. □

By running the program in Theorem 4.23 (see Appendix B for the accompanying MAGMA code and output), we obtain the following result.

Theorem 4.24. *Let q be a prime power with $q \leq 16$. Then $\Gamma_{2,q}$ is $\overline{\alpha}$-sharp.*

4.5 The VIP on $\Gamma_{n,q}$ for general n and q

In general, we do not have a better upper bound on $i_V(\Gamma_{n,q})$ than the default $\overline{\alpha}$ bound, that is,

$$i_V(\Gamma_{n,q}) \leq 1 - \frac{\overline{\alpha}(\Gamma_{n,q})}{\frac{q^{n+1}-1}{q-1}},$$

but we can simplify it. Combining this with the bound in Theorem 3.24 yields

$$i_V(\Gamma_{n,q}) \leq 1 - \left(\frac{1}{2} - \mathcal{O}\left(p^{\varepsilon - \frac{1}{4}}\right)\right) \frac{q^{\frac{n+1}{2}}(q-1)}{q^{n+1}-1},$$

The following theorem rewrites the lower bound of Theorem 4.16 in a form that matches this upper bound.

Theorem 4.25. *Let $n \geq 2$ be an integer and q a prime power. Then*

$$i_V(\Gamma_{n,q}) > 1 - \frac{q^{\frac{n+1}{2}}(q-1)}{q^{n+1}-1}.$$

Proof. If $n = 2$ and $q \leq 3$, we know that $i_V(\Gamma_{2,2}) = \frac{5}{7}$ and $i_V(\Gamma_{2,3}) = \frac{10}{13}$, which are strictly larger than $1 - \frac{2\sqrt{2}}{7}$ and $1 - \frac{3\sqrt{3}}{7}$ respectively.

For the rest of the proof assume $n \geq 3$ or $q \geq 4$. Let $(v, k, \lambda) = \left(\frac{q^{n+1}-1}{q-1}, \frac{q^n-1}{q-1}, \frac{q^{n-1}-1}{q-1}\right)$ be the parameters of $\mathcal{D}_{n,q}$, and let $\mu = \sqrt{k-\lambda} = q^{\frac{n-1}{2}}$.

In this case we have $k - \lambda = \mu^2 = q^{n-1} \geq 4$ and so $k^2 \geq k(4+\lambda) > 4k - 4\lambda = 4\mu^2$. Thus $k - \mu > \mu \geq 2$. Therefore,

$$\frac{k^2 - k\mu + 2\mu^2}{\mu(k-\mu)^2} = \frac{1}{\mu} + \frac{1}{k-\mu} + \frac{2\mu}{(k-\mu)^2} < 2 \leq q.$$

This together with Theorem 4.16 implies

$$\begin{aligned} i_V(\Gamma_{n,q}) &\geq (k-\mu)\frac{k^2+\mu^2}{k^3+\mu^3} \\ &= 1 - \frac{\mu}{k + \frac{\mu(k-\mu)^2}{k^2-k\mu+2\mu^2}} \\ &> 1 - \frac{\mu}{k+\frac{1}{q}} \\ &= 1 - \frac{q^{\frac{n+1}{2}}(q-1)}{q^{n+1}-1}. \end{aligned}$$

□

Theorem 4.26. *Let $n \geq 2$ be an integer, $q = p^e$ a prime power, and $\varepsilon > 0$ a real number with $0 < \varepsilon < \frac{1}{4}$. Then*

$$i_V(\Gamma_{n,q}) = 1 - c_{n,q}\frac{q^{\frac{n+1}{2}}(q-1)}{q^{n+1}-1}$$

for some real number $c_{n,q}$ with $\frac{1}{2} - \mathcal{O}\left(p^{\varepsilon-\frac{1}{4}}\right) \leq c_{n,q} < 1$.

For $\mathrm{PG}(2,q)$, the result in Theorem 4.26 simplifies to $i_V(\Gamma_{2,q}) = 1 - \Theta\left(\frac{1}{\sqrt{q}}\right)$.

4.6 The VIP on $\Gamma(\overline{\mathcal{D}_{n,q}})$

Theorem 4.27. *Let $n \geq 2$ be an integer and q a prime power. Then $\overline{\mathcal{D}_{n,q}}$ is $\overline{\alpha}$-sharp.*

Proof. For conciseness, we write $\mathcal{D} = \overline{\mathcal{D}_{n,q}}$. Recall that $\mathcal{D}$ is a symmetric BIBD with parameters $(v,k,\lambda) = \left(\frac{q^{n+1}-1}{q-1}, q^n, q^n - q^{n-1}\right)$. Write $m(x) = \left\lceil \log_q((q-1)x+1) \right\rceil$ as per Theorem 3.26. Let $S \subseteq V$ be a subset of vertices of type (x,y) with $1 \leq |S| \leq v$, and assume without loss of generality that $x \leq y$. We split the proof into three cases.

Case 1. If $m(y) = n+1$ and $m(x) \geq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$, then $\Phi_S(\mathcal{D}, y) = v$ from Theorem 3.26. We also have $n+1-m(x) \leq n - \left\lfloor \frac{n+1}{2} \right\rfloor \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, so that

$$\begin{aligned}\Phi_V(\mathcal{D}, x, y) &\geq \Phi_S(\mathcal{D}, x) + \Phi_S(\mathcal{D}, y) - x - y \\ &\geq \Phi_S(\mathcal{D}, x) \\ &\geq \frac{q^{n+1} - q^{\left\lfloor \frac{n+1}{2} \right\rfloor}}{q-1} \\ &= v - \overline{\alpha}(\mathcal{D}).\end{aligned}$$

Case 2. If $m(y) = n+1$ and $m(x) \leq \left\lfloor \frac{n+1}{2} \right\rfloor$, then $\Phi_S(\mathcal{D}, y) = v$ and $x \leq \frac{q^{\left\lfloor \frac{n+1}{2} \right\rfloor}-1}{q-1}$, so that

$$\Phi_V(\mathcal{D}, x, y) \geq \Phi_S(\mathcal{D}, y) - x \geq v - \overline{\alpha}(\mathcal{D}).$$

Case 3. If $m(y) \leq n$ then $y \leq \frac{q^n-1}{q-1}$, so that

$$\Phi_S(\mathcal{D}, y) \geq \frac{k^2 y}{k + \lambda(y-1)} \geq qy \geq 2y.$$

Since $x \leq y$, it follows similarly that $\Phi_S(\mathcal{D}, x) \geq 2x$, so that

$$\frac{\Phi_V(\mathcal{D}, x, y)}{x+y} \geq \frac{\Phi_S(\mathcal{D}, x) + \Phi_S(\mathcal{D}, y) - x - y}{x+y} \geq 1.$$

It follows from the arbitrary choice of S that $i_V(\Gamma_{n,q}) \geq \frac{v - \overline{\alpha}(\mathcal{D})}{v}$, and thus $\mathcal{D}$ is $\overline{\alpha}$-sharp by definition. □

4.7 The VIP on unitals

4.7.1 General unitals

Throughout this section we assume that $U = (P, \mathcal{B}, \partial)$ is a unital of order $n \geq 2$ and $\Gamma = (V, E)$ is its Levi graph. We will also assume $c(n)$ is the function defined in (3.9).

Recall that the parameters of U are given by

$$(v, b, r, k, \lambda) = (n^3 + 1, n^4 - n^3 + n^2, n^2, n + 1, 1).$$

Let $S \subseteq V$ be a subset of vertices of type (x, y) with $1 \leq |S| \leq \frac{v+b}{2}$, and suppose that $\delta(S)$ is of type (x', y'). Note that these conditions imply $x + y \leq \frac{n^2(n^2+1)}{2}$.

We then obtain the following bounds from Theorem 3.4 (with $m = 1$), Corollary 3.5, Theorem 3.7, and Lemma 4.6 respectively.

$$y + y' \geq \frac{x(2n^2 + 1 - x)}{2}; \tag{4.19}$$

$$y + y' \geq \frac{n^4 x}{n^2 - 1 + x}; \tag{4.20}$$

$$x + x' \geq \frac{n^2(n + 1)y}{n^2(n - 1) + y}; \text{ and} \tag{4.21}$$

$$y' \geq \frac{4x(n^3 + 1 - x - x')}{(n + 1)^2}. \tag{4.22}$$

For the rest of this section, define the function

$$f(x, y) = \frac{4x}{(n + 1)^2}(n^3 + 1) + \left(1 - \frac{4x}{(n + 1)^2}\right) \frac{n^2(n + 1)y}{n^2(n - 1) + y} - x.$$

It is readily seen that $f(x, y)$ is linear in x for a fixed y, and monotonic in y for a fixed x. We exploit this property in the following result.

Lemma 4.28. *If either*

$$x \leq \frac{(n + 1)^2}{4} \quad \text{and} \quad \frac{f(x, y)}{x + y} \geq \frac{2(n^3 + 1 - \lfloor c(n) \rfloor)}{n^4 + n^2}, \tag{4.23}$$

or

$$x > \frac{(n + 1)^2}{4} \quad \text{and} \quad x - \left(1 - \frac{(n + 1)^2}{4x}\right) y' < \lfloor c(n) \rfloor + 1, \tag{4.24}$$

then

$$\frac{x' + y'}{x + y} \geq \frac{2(n^3 + 1 - \lfloor c(n) \rfloor)}{n^2(n^2 + 1)}. \tag{4.25}$$

Proof. If (4.23) holds, then

$$\begin{aligned} x'+y' &= \left(\frac{4x}{(n+1)^2}x'+y'\right)+\left(1-\frac{4x}{(n+1)^2}\right)x' \\ &\geq \frac{4x}{(n+1)^2}\left(n^3+1-x\right)+\left(1-\frac{4x}{(n+1)^2}\right)\left(\frac{n^2(n+1)y}{n^2(n-1)+y}-x\right) \\ &= f(x,y), \end{aligned}$$

which together with (4.23) implies (4.25).

On the other hand, if (4.24) holds, then

$$\begin{aligned} x'+y' &= \left(x'+\frac{(n+1)^2}{4x}y'\right)+\left(1-\frac{(n+1)^2}{4x}\right)y' \\ &\geq n^3+1-x+\left(1-\frac{(n+1)^2}{4x}\right)y' \\ &> n^3-\lfloor c(n)\rfloor. \end{aligned}$$

Thus, $x'+y' \geq n^3+1-\lfloor c(n)\rfloor$, which together with $x+y \leq \frac{n^2(n^2+1)}{2}$ yields (4.25). □

We first consider the lower bound in the case when $n=2$.

Theorem 4.29. *Let U be the unital of order $n=2$. Then*

$$i_V(U) \geq \frac{2(n^3+1-\lfloor c(n)\rfloor)}{n^2(n^2+1)} = \frac{7}{10}.$$

Proof. In this case we have $x+y \leq \frac{n^2(n^2+1)}{2} = 10$. Moreover, condition (4.23) is reduced to

$$x \leq 2 \quad \text{and} \quad \frac{f(x,y)}{x+y} \geq \frac{7}{10},$$

and condition (4.24) is reduced to

$$x \geq 3 \quad \text{and} \quad x-\left(1-\frac{9}{4x}\right)y' < 3.$$

By Lemma 4.28 it suffices to verify that one of these conditions holds. In fact, if $x=0$, then $\frac{f(x,y)}{x+y} = \frac{12}{4+y} > \frac{7}{10}$. If $x=1$, then $y \leq 9$, so that $\frac{f(x,y)}{x+y} = \frac{29y+36}{3y^2+15y+12} > \frac{7}{10}$. If $x=2$, then $x'+y' \geq \frac{4x}{9}x'+y' \geq \frac{4x}{9}(9-x) > 6$, so that $\frac{x'+y'}{x+y} \geq \frac{7}{10}$. Finally, if $x \geq 3$, then

$$\begin{aligned} x - \left(1 - \frac{9}{4x}\right) y' &= x - \left(1 - \frac{9}{4x}\right)(x + (y+y') - (x+y)) \\ &\leq x - \left(1 - \frac{9}{4x}\right)\left(x + \frac{16x}{3+x} - 10\right) \\ &= \frac{84}{3+x} - \frac{45}{2x} - \frac{15}{4} \\ &< 3. \end{aligned}$$

□

Next we proceed to consider the lower bound in the $n \geq 3$ case.

Theorem 4.30. *Let U be a unital of order $n \geq 3$. Then*

$$i_V(U) \geq \frac{2(n^3 + 1 - \lfloor c(n) \rfloor)}{n^2(n^2+1)}.$$

Proof. Since $n \geq 3$, it follows that $8n^2 + 9 \leq (3n)^2$ and hence

$$c(n) \geq n^2 - \frac{3}{2}(n-1) \geq \frac{n(n+1)}{2} \geq \frac{(n+1)^2}{4}.$$

We consider the following six cases one by one. In each case we show that either (4.23) or (4.24) is satisfied and so the lower bound follows immediately from Lemma 4.28 and the arbitrary choice of $S \subseteq V$ with $1 \leq |S| \leq \frac{|V|}{2}$.

Case 1. Assume $x \leq \frac{(n+1)^2}{4}$ and $y = 0$.

Then

$$\begin{aligned} \frac{f(x,y)}{x+y} &= \frac{4}{(n+1)^2}(n^3+1) - 1 \\ &\geq \frac{2}{n} \\ &\geq \frac{2\left(n^3 + 1 - \lfloor c(n) \rfloor\right)}{n^2(n^2+1)}, \end{aligned}$$

so that condition (4.23) is satisfied.

Case 2. Assume $x \leq \frac{(n+1)^2}{4}$ and $1 \leq y \leq \frac{n^4 - n^3 + n^2}{2}$.

Since $f(x,y)$ is linear in x, we have

$$\begin{aligned}\frac{f(x,y)}{x+y} &\geq \min\left\{\frac{f(0,y)}{y}, \frac{f\left(\frac{(n+1)^2}{4}, y\right)}{\frac{(n+1)^2}{4}+y}\right\}\\ &= \min\left\{\frac{n^2(n+1)}{n^2(n-1)+y}, \frac{n^3+1-\frac{(n+1)^2}{4}}{\frac{(n+1)^2}{4}+y}\right\}\\ &\geq \min\left\{\frac{2(n+1)}{n^2+n-1}, \frac{(n+1)(4n^2-5n+3)}{(n+1)^2+2(n^4-n^3+n^2)}\right\}\\ &\geq \frac{2}{n}\\ &\geq \frac{2\left(n^3+1-\lfloor c(n)\rfloor\right)}{n^2(n^2+1)}.\end{aligned}$$

Hence condition (4.23) is satisfied.

Case 3. Assume $x \leq \frac{(n+1)^2}{4}$ and $y > \frac{n^4-n^3+n^2}{2}$.

Since $f(x,y)$ increases with y, we have

$$\begin{aligned}f(x,y) &\geq f\left(x, \frac{n^4-n^3+n^2}{2}\right)\\ &= \frac{4x}{(n+1)^2}(n^3+1) + \left(1-\frac{4x}{(n+1)^2}\right)\frac{n^2(n^3+1)}{n^2+n-1} - x\\ &= \left(\frac{4(n^3+1)}{(n+1)^2} - \frac{4n^2(n^3+1)}{(n+1)^2(n^2+n-1)} - 1\right)x + \frac{n^2(n^3+1)}{n^2+n-1}\\ &= \left(\frac{4(n^3+1)(n-1)}{(n+1)^2(n^2+n-1)} - 1\right)x + \frac{n^2(n^3+1)}{n^2+n-1}.\end{aligned}$$

Since $n \geq 3$, one can verify that the coefficient of x in the last line is positive. Hence

$$f(x,y) \geq \frac{n^2(n^3+1)}{n^2+n-1} \geq n^3+1-\lfloor c(n)\rfloor.$$

This together with $x+y \leq \frac{n^2(n^2+1)}{2}$ yields

$$\frac{f(x,y)}{x+y} \geq \frac{2\left(n^3+1-\lfloor c(n)\rfloor\right)}{n^2(n^2+1)}.$$

That is, condition (4.23) is satisfied.

Case 4. Assume $\frac{(n+1)^2}{4} < x \leq c(n)$.

Then

$$x - \left(1 - \frac{(n+1)^2}{4x}\right) y' \le x \le c(n) < \lfloor c(n) \rfloor + 1$$

and so condition (4.24) is satisfied.

Case 5. Assume $c(n) < x \le n^2$.

Since $c(n) \ge \frac{n(n+1)}{2}$, we have

$$\frac{c(n)}{n^2} \ge \frac{(n+1)^2}{4c(n)} \ge \frac{(n+1)^2}{4x}.$$

And since g is concave, we have

$$\frac{g(x) - g(c(n))}{x - c(n)} \ge \frac{g(n^2) - g(c(n))}{n^2 - c(n)} = \frac{1}{1 - \frac{c(n)}{n^2}} \ge \frac{1}{1 - \frac{(n+1)^2}{4x}}.$$

On the other hand,

$$\begin{aligned} y' &\ge \frac{x(2n^2 + 1 - x)}{2} - y \\ &\ge \frac{x(2n^2 + 1 - x)}{2} + x - \frac{n^4 + n^2}{2} \\ &= g(x) - g(c(n)). \end{aligned}$$

So

$$x - \left(1 - \frac{(n+1)^2}{4x}\right) y' \le x - (x - c(n)) = c(n) < \lfloor c(n) \rfloor + 1$$

and (4.24) is satisfied.

Case 6. Assume $x > n^2$.

Let

$$h(z) = z - \left(1 - \frac{(n+1)^2}{4z}\right) \left(\frac{n^4 z}{n^2 - 1 + z} + z - \frac{n^4 + n^2}{2}\right).$$

Then

$$x - \left(1 - \frac{(n+1)^2}{4x}\right) y' \le h(x).$$

Thus, to show that (4.24) is satisfied, it suffices to prove $h(x) \le c(n)$. We achieve this by showing that $h(n^2) \le c(n)$ and that the function $h(z)$ is decreasing with z.

To justify the first claim, we notice that $(3n^2-1)(3n+1) \geq 12(2n^2-1)$. Hence

$$\begin{aligned} h(n^2) &= \left(1 - \frac{(n+1)^2}{4n^2}\right)\left(\frac{n^6}{2n^2-1} + n^2 - \frac{n^4+n^2}{2}\right) \\ &= n^2 - \frac{(3n^2-1)(3n+1)(n-1)}{8(2n^2-1)} \\ &= n^2 - \frac{3n^2-1}{2n^2-1}\frac{3n+1}{8}(n-1) \\ &\leq n^2 - \frac{3}{2}(n-1) \\ &\leq c(n). \end{aligned}$$

To prove the second claim, we take the derivative

$$\begin{aligned} \frac{dh}{dz} &= 1 - \frac{(n{+}1)^2}{4z^2}\left(\frac{n^4 z}{n^2-1+z} + z - \frac{n^4{+}n^2}{2}\right) - \left(1 - \frac{(n{+}1)^2}{4z}\right)\left(\frac{n^4(n^2{-}1)}{(n^2-1+z)^2} + 1\right) \\ &= \frac{n^2(n+1)}{8}\left(\frac{(n^2+1)(n+1)}{z^2} - \frac{2n^2(5n-3)}{(n^2-1+z)^2}\right). \end{aligned}$$

It can be easily verified that $n^2(5n-3) \geq 2(n^2+1)(n+1)$. Thus

$$\frac{2n^2(5n-3)}{(n^2+1)(n+1)} \geq 4 \geq \left(\frac{n^2-1}{z} + 1\right)^2$$

and therefore $\frac{dh}{dz} \leq 0$. So $h(z)$ is decreasing with z.

In summary, in each case above either (4.23) or (4.24) holds for any $S \subseteq V$ with $1 \leq |S| \leq \frac{|V|}{2}$. Thus, by Lemma 4.28, we obtain the desired lower bound. □

We can combine these two cases into a single result.

Corollary 4.31. *Let U be a unital of order n. Then*

$$i_V(U) \geq \frac{2(n^3+1-\lfloor c(n) \rfloor)}{n^2(n^2+1)}.$$

In the event that U has large arcs, namely, $m_2(U) \geq \lfloor c(n) \rfloor$, we can obtain an exact value for the vertex-isoperimetric number.

Theorem 4.32. *Let U be a unital of order n. If $m_2(U) \geq \lfloor c(n) \rfloor$, then U is $\overline{\alpha}$-sharp, and*

$$i_V(U) = \frac{2(n^3+1-\lfloor c(n) \rfloor)}{n^2(n^2+1)}.$$

Proof. This follows immediately from Corollaries 3.29 and 4.31 and Lemma 4.3. □

As with Corollary 3.29 previously, Theorem 4.32 also holds on the classical unitals $H(q)$ for any prime power q, as well as on the BM-unitals $\mathrm{BM}(\alpha, \beta, q)$ with q an odd prime power, $\alpha \in \mathbb{F}_{q^2}$ and $\beta \in \mathbb{F}_q$ such that $(\beta^q - \beta)^2 + 4\alpha^{q+1}$ is a non-square in $\mathbb{F}_q$.

4.7.2 Complements of unitals

Recall from Theorem 3.30 that if U is a unital design of order $n \geq 2$, then

$$\overline{\alpha}(\overline{U}) = \begin{cases} 1, & \text{if } n = 2 \\ 0, & \text{if } n \geq 3. \end{cases}$$

We split up the following results into two cases similarly.

Theorem 4.33. *Let U be the unital design of order 2. Then $\overline{U}$ is $\overline{\alpha}$-sharp.*

Proof. Recall that the parameters of $\overline{U}$ are $(v, b, r, k, \lambda) = (9, 12, 8, 6, 5)$. Let $\overline{U} = (P, \overline{\mathcal{B}})$, and $\Gamma = (V, \overline{E})$ its Levi graph. Let $S \subseteq V$ be a subset of vertices of type (x, y) with $1 \leq |S| \leq 10$, and suppose $\delta(S)$ is of type (x', y').

Then Corollary 3.5 and Theorem 3.7 reduce to $y + y' \geq \frac{64x}{3+5x}$ and $x + x' \geq \frac{48y}{4+5y}$ respectively. In order to prove that $\overline{U}$ is $\overline{\alpha}$-sharp, we need to show that $i_V(\Gamma) \geq \frac{4}{5}$, for which it suffices to show one of $\frac{x'+y'}{x+y} \geq 1$ or $x' + y' \geq 8$. We split this into four cases.

Case 1. If $x \leq 1$ and $y \leq 4$, then $y + y' \geq \frac{64x}{3+5x} \geq 2x$ and $x + x' \geq \frac{48y}{4+5y} \geq 2y$ so that $\frac{x'+y'}{x+y} = \frac{(x+x')+(y+y')}{x+y} - 1 \geq 1$.

Case 2. If $x \leq 1$ and $y \geq 5$, then $x + x' \geq \frac{240}{29} > 8$. Since $x + x'$ is an integer, it follows that $x + x' \geq 9$ and thus $x' \geq 8$.

Case 3. If $x \geq 2$ and $y \leq 1$, then $y + y' \geq \frac{128}{13} > 10$. This implies that $y + y' \geq 11$, from which it follows that $y' \geq 10$.

Case 4. Finally, if $x \geq 2$ and $y \geq 2$, then we still have $y + y' \geq 11$ from before. Additionally, we also have $x + x' \geq \frac{96}{14} > 6$, from which it follows that $x + x' \geq 7$ and thus $x' + y' = (x + x') + (y + y') - (x + y) \geq 8$. □

Theorem 4.34. *Let U be a unital design of order $n \geq 3$. Then $\overline{U}$ is $\overline{\alpha}$-sharp.*

Proof. Recall that the parameters of $\overline{U}$ are

$$(v, b, r, k, \lambda) = (n^3 + 1, n^4 - n^3 + n^2, n^4 - n^3, n^3 - n, n^4 - n^3 - n^2 + 1).$$

Let $\overline{U} = (P, \overline{\mathcal{B}})$, and $\Gamma = (V, \overline{E})$ its Levi graph. Let $S \subseteq V$ be a subset of vertices of type (x, y) with $1 \leq |S| \leq \frac{v+b}{2}$, and suppose $\delta(S)$ is of type (x', y').

As before, Corollary 3.5 and Theorem 3.7 reduce to

$$y + y' \geq \frac{r^2 x}{r + \lambda(x - 1)} \qquad \text{and} \qquad x + x' \geq \frac{rky}{r^2 - \lambda(b - y)}$$

respectively. In order to prove $i_V(\Gamma) \geq \frac{4}{5}$, it suffices to show that $\frac{x'+y'}{x+y} \geq 1$ or $x'+y' \geq 8$.

Since $\frac{2(n^3+1)}{n^2(n^2+1)} < 1$ and $x + y \leq \frac{n^2(n^2+1)}{2}$, in order to prove that $\overline{U}$ is $\overline{\alpha}$-sharp, we need to show that $i_V(\Gamma) \geq \frac{2(n^3+1)}{n^2(n^2+1)}$, for which it suffices to show one of $\frac{x'+y'}{x+y} \geq 1$ or $x' + y' \geq n^3 + 1$. We split this into four cases.

Case 1. If $y = 0$, then $\frac{x'+y'}{x+y} \geq \frac{y+y'}{x} \geq \frac{r^2}{r+\lambda(x-1)} \geq \frac{r^2}{r+\lambda(v-1)} = \frac{r}{k} \geq 1$.

Case 2. If $x = 0$ and $y \leq n^2$, then $\frac{x'+y'}{x+y} \geq \frac{x+x'}{y} \geq \frac{rk}{r^2-\lambda(b-y)} \geq \frac{rk}{r^2-\lambda(b-n^2)} = n \geq 1$.

Case 3. If $x = 0$ and $y > n^2$, then $x + x' \geq \frac{rky}{r^2-\lambda(b-y)} \geq \left(\frac{rk}{r^2-\lambda(b-n^2)}\right) y = ny > n^3$. Since x' is an integer, it follows that $x' = n^3 + 1$.

Case 4. If $x \geq 1$ and $y \geq 1$, then $y + y' \geq r$ and $x + x' \geq k$, so that

$$x' + y' = (x + x') + (y + y') - (x + y) \geq n^4 - n - \frac{n^4 + n^2}{2} \geq n^3 + 1. \qquad \square$$

Finally, we consolidate the two results here in a single statement.

Corollary 4.35. *Let U be a unital design of order $n \geq 2$. Then $\overline{U}$ is $\overline{\alpha}$-sharp, and*

$$i_V(\Gamma(\overline{U})) = \begin{cases} \frac{4}{5}, & \text{if } n = 2 \\ \frac{2(n^3+1)}{n^2(n^2+1)}, & \text{if } n \geq 3. \end{cases}$$

Chapter 5

The Edge-Isoperimetric Problem

In this chapter, we will discuss the edge-isoperimetric problem (EIP) on Levi graphs of BIBDs. In Section 5.1, we provide the basic definitions and notation to be used throughout the rest of the chapter, and also discuss some of the properties of the EIP. In Section 5.2, we establish some general isoperimetric inequalities for the EIP that hold for Levi graphs of all (or many) BIBDs.

We only focus on one specific family of graphs in this chapter, that is, in Section 5.3 where we investigate the EIP on $\Gamma_{2,q}$ when $q \leq 8$. We also establish the fact that $\Gamma_{2,2}$ and $\Gamma_{2,3}$ both admit nested solutions for the EIP, while $\Gamma_{2,4}$ does not. We do revisit some other families later in Chapter 6, but only after developing a further theory of generalised tangent numbers. Before doing so, we wrap up this chapter in Section 5.4 by establishing the exact edge-isoperimetric number for graphs arising from certain difference sets. These graphs include, for example, all $\Gamma_{n,q}$ when n and q are both odd.

5.1 Basic definitions and properties

Definition 5.1. *Let $\Gamma = (V, E)$ be a graph. If $S \subseteq V$ is a subset of vertices, the edge boundary of S, denoted by $E(S, \overline{S})$, is the set of edges with one endpoint in S and the other endpoint not in S. The edge-isoperimetric parameter is the function*

$$\Phi_E(\Gamma, n) = \min\{|E(S, \overline{S})| : S \subseteq V, |S| = n\}.$$

It is clear from this definition that $\Phi_E(\Gamma, n) = \Phi_E(\Gamma, |V| - n)$.

The set S is called an *EIP-set* if $\Phi_E(\Gamma, |S|) = |E(S, \overline{S})|$. The EIP on Γ is said to have *nested solutions* if there exists an ordering on V such that the first n vertices of V in this ordering always form an EIP-set for all n.

As with the VIP before, it is useful to consider the incidence structure variant:

$$\Phi_E(I, x, y) = \min\{|E(S, \overline{S})| : S \subseteq V(\Gamma(I)), |S \cap P| = x, |S \cap \mathcal{B}| = y\}.$$

Again, we have the obvious relation

$$\Phi_E(\Gamma(I), n) = \min_{x+y=n} \Phi_E(I, x, y).$$

In general, the "boundary" version of the EIP can be difficult to work with, so it is preferable in proofs to work with the induced EIP instead, whereby we consider the set of induced edges rather than the edge boundary. That is, we define

$$\Phi'_E(\Gamma, n) = \max\{|E(S)| : S \subseteq V, |S| = n\},$$

where $E(S)$ is the set of edges in the subgraph of Γ induced by S. Equivalently, $E(S)$ is the set of edges with both endpoints in S. Similarly, we define the incidence structure variant

$$\Phi'_E(I, x, y) = \max\{|E(S)| : S \subseteq V(\Gamma(I)), S \text{ is of type } (x, y)\}.$$

If $\mathcal{D}$ is a BIBD with parameters (v, b, r, k, λ), it is clear that

$$\Phi_E(\mathcal{D}, x, y) = rx + ky - 2\Phi'_E(\mathcal{D}, x, y), \tag{5.1}$$

so that it is sufficient to consider only the induced version. Furthermore, if Γ is k-regular, for example, if it is the Levi graph of a symmetric BIBD, then

$$\Phi_E(\Gamma, n) = kn - 2\Phi'_E(\Gamma, n). \tag{5.2}$$

There exists a third natural variant as well, which is to consider the set of edges incident to any vertex of S. In fact, this is exactly the MSP on Γ, that is, if we treat $\Gamma = (V, E)$ as an incidence structure. If Γ is k-regular, we simply have

$$\Phi_S(\Gamma, n) = kn - \Phi'_E(\Gamma, n).$$

This analogue has been noted by Harper [27], but otherwise we will not be looking at this variant at all.

Definition 5.2. *Let* $\Gamma = (V, E)$ *be a graph. The edge-isoperimetric number* $i_E(\Gamma)$ *of* Γ *is the quantity*

$$i_E(\Gamma) = \min\left\{\frac{\Phi_E(\Gamma, n)}{n} : 1 \leq n \leq \frac{|V|}{2}\right\}.$$

A very closely related invariant of a graph is its bisection width. Let $\Gamma = (V, E)$ be a graph. The *bisection width* $bw(\Gamma)$ of Γ is the minimum number of edges between two parts of a bipartition of V into equal or almost equal sizes (that is, differing by one if the number of vertices is odd).

We can define it precisely in terms of our edge-isoperimetric parameter:

$$bw(\Gamma) = \Phi_E\left(\Gamma, \left\lfloor \frac{|V|}{2} \right\rfloor\right).$$

We know from spectral graph theory [17] that if Γ is k-regular and $a(\Gamma)$ is the algebraic connectivity of Γ (that is, the second-smallest Laplacian eigenvalue of Γ), then

$$\frac{a(\Gamma)}{2} \leq i_E(\Gamma) \leq \frac{bw(\Gamma)}{\left\lfloor \frac{|V|}{2} \right\rfloor}.$$

In general, this lower bound of $\frac{a(\Gamma)}{2}$ is difficult to improve, and is in fact sharp for several important families of graphs. For this reason, this chapter will focus largely on the construction of good upper bounds on the bisection width instead.

5.2 General isoperimetric inequalities

We start off with a simple observation for the induced EIP.

Lemma 5.3. *Let* $\mathcal{D}$ *be a BIBD with parameters* (v, b, r, k, λ)*, and* Γ *its Levi graph. Then for* $n \leq 4$*, we have*

$$\Phi'_E(\Gamma, n) = \begin{cases} 0, & \textit{if } n = 0 \textit{ or } n = 1 \\ 1, & \textit{if } n = 2 \\ 2, & \textit{if } n = 3 \\ 3, & \textit{if } n = 4 \textit{ and } \lambda = 1 \\ 4, & \textit{if } n = 4 \textit{ and } \lambda \geq 2. \end{cases}$$

Proof. The case $n \leq 3$ is trivial, so assume $n = 4$. Since Γ is bipartite, the only way to induce four edges is to have a 4-cycle, which exists exactly when $\lambda \geq 2$, as required. □

We proceed to demonstrate a simple upper bound for the EIP, or equivalently a lower bound for the induced EIP.

Theorem 5.4. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ). Let x and y be non-negative integers with $x \leq v$ and $y \leq b$. Then*

$$\Phi'_E(\mathcal{D}, x, y) \geq \frac{rxy}{b},$$

and consequently

$$\Phi_E(\mathcal{D}, x, y) \leq \frac{vr}{2} - \frac{2r}{b}\left(\frac{v}{2} - x\right)\left(\frac{b}{2} - y\right).$$

Proof. Let $X \subseteq P$ be any x-subset of points. Then there are rx edges from X to $\mathcal{B}$. By the generalised pigeonhole principle, there must exist $Y \subseteq \mathcal{B}$ with $|Y| = y$ such that Y is incident with at least $\frac{rxy}{b}$ of those edges. It follows that $\Phi'_E(\mathcal{D}, x, y) \geq \frac{rxy}{b}$.

Using the equivalence noted in (5.1), we obtain

$$\begin{aligned} \Phi_E(\mathcal{D}, x, y) &= rx + ky - \Phi'_E(\mathcal{D}, x, y) \\ &\leq rx + ky - \frac{2rxy}{b} \\ &= \frac{vr}{2} - \frac{2r}{b}\left(\frac{v}{2} - x\right)\left(\frac{b}{2} - y\right) \end{aligned}$$

as required. □

Theorem 5.5. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ), and Γ its Levi graph. Then*

$$i_E(\Gamma) \leq \frac{vr}{v + b - 1}.$$

Proof. We first consider the case where at least one of v and b is even.

Pick $x = \left\lfloor \frac{v}{2} \right\rfloor$ and $y = \left\lfloor \frac{b}{2} \right\rfloor$, so that $\Phi_E(\mathcal{D}, x, y) \leq \frac{vr}{2}$ from Theorem 5.4. Since $x + y \geq \frac{v+b-1}{2}$, it follows that

$$i_E(\Gamma) \leq \frac{\Phi_E(\mathcal{D}, x, y)}{x + y} \leq \frac{vr}{v + b - 1}.$$

Otherwise both v and b are odd, so pick $x = \frac{v-1}{2}$ and $y = \frac{b+1}{2}$. It follows from Theorem 5.4 that $\Phi_E(\mathcal{D}, x, y) \leq \frac{vr}{2} + \frac{r}{2b}$. Then

$$\begin{aligned}\frac{\Phi_E(\mathcal{D}, x, y)}{x+y} &\leq \frac{\frac{vr}{2} + \frac{r}{2b}}{\frac{v+b}{2}} \\ &= r\left(\frac{v + \frac{1}{b}}{v+b}\right) \\ &= r\left(\frac{v}{v+b-1} - \frac{(v-1)(b-1)}{b(v+b)(v+b-1)}\right) \\ &\leq \frac{vr}{v+b-1}\end{aligned}$$

as required. □

The following result is an analogue of Theorem 3.4 for the EIP.

Lemma 5.6. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ), and Γ its Levi graph. Let $X \subseteq P$ and $Y \subseteq \mathcal{B}$ be subsets of the point vertices and block vertices respectively. Let $x = |X|$, $y = |Y|$, and $e = |E(X, Y)|$. Let m and n be integers. Then*

$$2me + 2n(rx - e) \leq m(m+1)y + n(n+1)(b-y) + \lambda x(x-1).$$

Proof. Let $\mathcal{B} = \{B_1, B_2 \dots, B_b\}$ and let $Y = \{B_1, B_2, \dots, B_y\}$. For each $i = 1, 2, \dots, b$, let $a_i = |B_i \cap X|$. By counting the paths of length 2 from X to X in two different ways we have

$$\begin{aligned}\lambda x(x-1) &= \sum_{i=1}^{b} a_i(a_i - 1) \\ &= \sum_{i=1}^{y} a_i(a_i-1) + \sum_{i=y+1}^{b} a_i(a_i-1) \\ &\geq 2me - m(m+1)y + 2n(rx-e) - n(n+1)(b-y),\end{aligned}$$

where the last line follows by two applications of Lemma 3.3. □

Lemma 5.7. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ), and $\Gamma = (V, E)$ its Levi graph. Let $S \subseteq V$ be a subset of vertices of type (x, y), and let $e = |E(S)|$. Then*

$$(be - rxy)^2 \leq xy(v-x)(b-y)(r^2 - \lambda b).$$

Proof. Let $m = \left\lfloor \frac{e}{y} \right\rfloor$ and $n = \left\lfloor \frac{rx-e}{b-y} \right\rfloor$, so that

$$m \leq \frac{e}{y} \leq m+1$$

and

$$n \leq \frac{rx-e}{b-y} \leq n+1.$$

Then by rearranging the inequalities we get

$$m(m+1) \leq \frac{e}{y}\left(2m+1-\frac{e}{y}\right)$$

and

$$n(n+1) \leq \frac{rx-e}{b-y}\left(2n+1-\frac{rx-e}{b-y}\right).$$

Combining this with Lemma 5.6 yields

$$\begin{aligned} 2me + 2n(rx-e) &\leq m(m+1)y + n(n+1)(b-y) + \lambda x(x-1) \\ &\leq e\left(2m+1-\frac{e}{y}\right) + (rx-e)\left(2n+1-\frac{rx-e}{b-y}\right) + \lambda x(x-1), \end{aligned}$$

which rearranges to form

$$\frac{e^2}{y} + \frac{(rx-e)^2}{b-y} \leq \lambda x(x-1) + rx.$$

Finally, we can manipulate this to get

$$\begin{aligned} (be-rxy)^2 + r^2x^2y(b-y) &= be^2(b-y) + b(rx-e)^2y \\ &\leq bxy(b-y)(\lambda x + r - \lambda) \\ &= xy(b-y)(\lambda bx + v(r^2-\lambda b)) \\ &= xy(b-y)(r^2-\lambda b)(v-x) + r^2x^2y(b-y), \end{aligned}$$

from which the desired inequality follows. □

A simple rearrangement of Lemma 5.7 yields the following corollary.

Corollary 5.8. *Let $\mathcal{D}$ be a BIBD with parameters (v, b, r, k, λ). Let $x \leq v$ and $y \leq b$ be non-negative integers. Then*

$$\Phi'_E(\mathcal{D}, x, y) \leq \frac{rxy + \sqrt{xy(v-x)(b-y)(r^2 - \lambda b)}}{b}.$$

Lemma 5.9. *Let $\mathcal{D}$ be a symmetric (v, k, λ)-BIBD, and Γ its Levi graph. Then*

$$\Phi'_E(\Gamma, n) \leq \frac{kn^2 + n(2v-n)\sqrt{k-\lambda}}{4v}.$$

Proof. Let x and y be non-negative integers summing to n. It suffices to show that

$$\Phi'_E(\mathcal{D}, x, y) \leq \frac{kn^2 + n(2v-n)\sqrt{k-\lambda}}{4v}.$$

But this follows from Corollary 5.8 since $xy \leq \frac{n^2}{4}$ and $(v-x)(v-y) \leq \left(v - \frac{n}{2}\right)^2$. □

Lemma 5.10. *Let x, y, v, b be real values with $1 \leq v \leq b$ and $0 \neq x + y \leq \frac{v+b}{2}$. Then*

$$\frac{bx + vy - 2xy}{x+y} \geq \frac{v+b}{vb}.$$

Proof. First assume that $x > \frac{v}{2}$ or $y > \frac{b}{2}$. Only one of those conditions can be true since $x + y \leq \frac{v+b}{2}$. Then we necessarily have

$$(2x - v)(2y - b) \leq 0,$$

and thus

$$\begin{aligned}\frac{bx + vy - 2xy}{x+y} &= \frac{vb - (2x-v)(2y-b)}{2(x+y)} \\ &\geq \frac{vb}{v+b}.\end{aligned}$$

Otherwise, we have $x \leq \frac{v}{2}$ and $y \leq \frac{b}{2}$, so that

$$\begin{aligned} \frac{bx + vy - 2xy}{x+y} &= v - 2x + \frac{bx - vx + 2x^2}{x+y} \\ &\geq v - 2x + \frac{bx - vx + 2x^2}{x + \frac{b}{2}} \\ &= \frac{vb}{b+2x} \\ &\geq \frac{vb}{v+b}. \end{aligned}$$

□

Finally, the following result gives us a good lower bound on the edge-isoperimetric number of Levi graphs of arbitrary BIBDs.

Theorem 5.11. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ), and $\Gamma = (V, E)$ its Levi graph. Then*

$$i_E(\Gamma) \geq \frac{v}{v+b}\left(r - \sqrt{r^2 - \lambda b}\right).$$

Proof. Let $S \subseteq V$ be a subset of vertices with $1 \leq |S| \leq \frac{v+b}{2}$, and suppose S is of type (x, y). Let $X = S \cap P$ and $Y = S \cup \mathcal{B}$.

Then

$$\begin{aligned} |E(S, \overline{S})| &= rx + ky - 2|E(X, Y)| \\ &\geq rx + ky - \frac{2}{b}\left(rxy + \sqrt{xy(v-x)(b-y)(r^2 - \lambda b)}\right) \\ &\geq rx + ky - \frac{2}{b}\left(rxy + \frac{x(b-y) + y(v-x)}{2}\sqrt{r^2 - \lambda b}\right) \\ &= (bx + vy - 2xy)\frac{r - \sqrt{r^2 - \lambda b}}{b}. \end{aligned}$$

This gives us

$$\frac{|E(S, \overline{S})|}{x+y} \geq \frac{v}{v+b}\left(r - \sqrt{r^2 - \lambda b}\right)$$

as required. □

Note that if $\mathcal{D}$ is symmetric, then Theorem 5.11 reduces to

$$i_E(\Gamma) \geq \frac{k - \sqrt{k - \lambda}}{2}, \tag{5.3}$$

which is exactly the spectral lower bound we saw in (1.2).

Lemma 5.12. *Let $\mathcal{D}$ be a symmetric (v, k, λ)-BIBD, and e a real number. Let x, y, and n be positive integers with $x + y = n \leq v$. If $\Phi'_E(\mathcal{D}, x, y) \geq e$, then*

$$(x - y)^2 \leq n^2 - \frac{2}{\lambda}(A - \sqrt{A^2 - 4\lambda v e^2}),$$

where $A = 2ke + (v - n)(k - \lambda)$.

Proof. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given by

$$f(w) = \frac{kw + \sqrt{w(v^2 - vn + w)(k - \lambda)}}{v} - e,$$

so that we know from Corollary 5.8 that $e \leq \Phi'_E(D, x, y) \leq f(xy) + e$, noting that $(v - x)(v - y) = v^2 - vn + xy$. It is also clear that f is strictly increasing. It is then easy, albeit tedious, to check that f has a (unique) root at $w^* = \frac{A - \sqrt{A^2 - 4\lambda v e^2}}{2\lambda}$.

It follows that $xy \geq w^*$. Hence

$$(x - y)^2 = (x + y)^2 - 4xy \leq n^2 - \frac{2}{\lambda}(A - \sqrt{A^2 - 4\lambda v e^2}). \qquad \square$$

Finally, we wrap up this section by relating the EIP on a BIBD with that of its complement.

Lemma 5.13. *Let $\mathcal{D} = (P, \mathcal{B})$ be a BIBD with parameters (v, b, r, k, λ). Let $x \leq v$ and $y \leq b$ be non-negative integers. Then*

$$\Phi'_E(\overline{\mathcal{D}}, x, y) = x(y - r) + \Phi'_E(\mathcal{D}, x, b - y).$$

Proof. Let $\Gamma = (V, E)$ be the Levi graph of $\mathcal{D}$, and write $\Gamma(\overline{\mathcal{D}}) = (V, \overline{E})$. Let $S \subseteq V$ be a subset of vertices of type (x, y), and suppose $\Phi'_E(\overline{\mathcal{D}}, x, y) = |E(\overline{S})|$. Write $X = S \cap P$ and $Y = S \cap \mathcal{B}$.

Then $|E(X, Y)| = xy - |\overline{E}(X, Y)|$, from which it follows that

$$\begin{aligned} |E(X, \mathcal{B} \setminus Y)| &= |E(X, \mathcal{B})| - |E(X, Y)| \\ &= rx - xy + |\overline{E}(X, Y)| \\ &= x(r - y) + |\overline{E}(X, Y)|. \end{aligned}$$

It follows that

$$\Phi'_E(\mathcal{D}, x, b-y) \geq |E(X, \mathcal{B} \setminus Y)| = x(r-y) + \Phi'_E(\overline{\mathcal{D}}, x, y). \tag{5.4}$$

Similarly, we can apply (5.4) to $\overline{\mathcal{D}}$ to obtain

$$\Phi'_E(\overline{\mathcal{D}}, x, y) \geq x(b-r-(b-y)) + \Phi'_E(\mathcal{D}, x, b-y). \tag{5.5}$$

It follows from (5.4) and (5.5) that $\Phi'_E(\overline{\mathcal{D}}, x, y) = x(y-r) + \Phi'_E(\mathcal{D}, x, b-y)$. □

5.3 The EIP on $\Gamma_{2,q}$ for small q

In this section, we will obtain some solutions to the EIP on $\Gamma_{2,q}$ for small q. We will work with the induced variant Φ'_E since it is equivalent to the EIP, as per (5.2). Note that if Γ is the Levi graph of a symmetric (v, k, λ)-BIBD, then $\Phi_E(\Gamma, n) = \Phi_E(\Gamma, v-n)$ by symmetry, so that it suffices to only solve $\Phi_E(\Gamma, n)$ for $n \leq v$.

5.3.1 Nested solutions for $q = 2$

Let $\mathcal{D} = \mathcal{D}_{2,2}$, and Γ its Levi graph. We will demonstrate that the EIP on Γ has nested solutions, given by the ordering

$$V = \{0, 3{+}D, 4, 0{+}D, 1, 6{+}D, 2, 5{+}D, 3, 1{+}D, 5, 2{+}D, 6, 4{+}D\},$$

where D is the difference set of $\mathcal{D}$ as given in Appendix A.

Let S_n denote the set obtained by taking the first n elements of V in this ordering, $0 \leq n \leq 14$. We summarise the edge boundary sizes of S_n in Table 5.1.

Table 5.1 Nested solutions for the induced EIP on $\Gamma_{2,2}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\lvert E(S_n)\rvert$	0	0	1	2	3	4	6	7	9	10	12	14	16	18	21

In what follows, we will prove the upper bounds on the induced EIP and show that they match the lower bounds in Table 5.1. It turns out that Lemma 5.9 is sufficient to prove this for all $0 \leq n \leq 14$, and we are done.

By evaluating $\frac{\Phi_E(\Gamma,n)}{n}$ for all $1 \leq n \leq 7$, we obtain the following result.

Theorem 5.14. $i_E(\Gamma_{2,2}) = 1$.

5.3.2 Nested solutions for $q = 3$

Let $\mathcal{D} = \mathcal{D}_{2,3}$, and Γ its Levi graph. We will demonstrate that the EIP on Γ has nested solutions, given by the ordering

$$V = \{0, 0{+}D, 1, 1{+}D, 2, 12{+}D, 3, 2{+}D, 4, 3{+}D, 12, 4{+}D, 5,$$
$$5{+}D, 6, 6{+}D, 7, 11{+}D, 8, 7{+}D, 9, 8{+}D, 11, 9{+}D, 10, 10{+}D\},$$

where D is the difference set of $\mathcal{D}$ as given in Appendix A.

Let S_n denote the set obtained by taking the first n elements of V in this ordering, $0 \leq n \leq 26$. We summarise the vertex boundary sizes of S_n in Table 5.2.

Table 5.2 Nested solutions for the induced EIP on $\Gamma_{2,3}$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert E(S_n)\rvert$	0	0	1	2	3	4	6	7	9	10	12	14	16	18

n	14	15	16	17	18	19	20	21	22	23	24	25	26
$\lvert E(S_n)\rvert$	20	22	24	26	29	31	34	36	39	42	45	48	52

In what follows, we will prove the upper bounds on the induced EIP and show that they match the lower bounds in Table 5.2. If $n \leq 4$, then Lemma 5.3 gives us the matching upper bounds, so assume $5 \leq n \leq 13$.

If $n \in \{6, 8, 11, 12, 13\}$, then Lemma 5.9 provides us with matching upper bounds, so assume now that $n \in \{5, 7, 9, 10\}$.

Write $e_n = |E(S_n)|$, that is, $e_5 = 4$, $e_7 = 7$, $e_9 = 10$, and $e_{10} = 12$. We wish to show that $\Phi'_E(\Gamma, n) \leq e_n$.

Suppose there exists a counterexample, that is, there exist positive integers x and y summing to n such that $\Phi'_E(\mathcal{D}, x, y) \geq e_n + 1$, for some $n \in \{5, 7, 9, 10\}$. Then by testing the four values of n with Lemma 5.12, we find that we must necessarily have $(x - y)^2 \leq 3$ in all four cases. This means that $|x - y| \leq 1$, and hence x and y are automatically determined if we assume $x \leq y$. Finally, we just perform a case-by-case check to see that all of them fail Lemma 5.6, namely:

- $\Phi'_E(\mathcal{D}, 2, 3) \leq 4$, by letting $m = 1$ and $n = 0$;
- $\Phi'_E(\mathcal{D}, 3, 4) \leq 7$, by letting $m = 1$ and $n = 0$;

- $\Phi'_E(\mathcal{D}, 4, 5) \leq 10$, by letting $m = 1$ and $n = 0$; and
- $\Phi'_E(\mathcal{D}, 5, 5) \leq 12$, by letting $m = 2$ and $n = 0$.

This completes all cases $n \leq 13$, and the upper half of the values can then be evaluated easily. We thus conclude that the EIP on $\mathcal{D}_{2,3}$ does have nested solutions.

By evaluating $\frac{\Phi_E(\Gamma,n)}{n}$ for all $1 \leq n \leq 13$, we obtain the following result.

Theorem 5.15. $i_E(\Gamma_{2,3}) = \frac{16}{13}$.

5.3.3 No nested solutions for $q = 4$

Let $\mathcal{D} = \mathcal{D}_{2,4}$, and $\Gamma = (V, E)$ its Levi graph. We will show in this section that the EIP on Γ does not have nested solutions.

Lemma 5.16. $\Phi'_E(\Gamma_{2,4}, 14) = 21$. *Furthermore, the subgraph of* $\Gamma_{2,4}$ *induced by such an EIP-set is isomorphic to* $\Gamma_{2,2}$.

Proof. The lower bound of 21 is clear since PG$(2, 4)$ contains a Fano subplane, and the upper bound of 21 follows from Lemma 5.9.

Now, let S be an EIP-set of type (x, y) with $x + y = 14$, and assume without loss of generality that $x \leq y$. Applying Lemma 5.12 yields $x = y = 7$, so that S must be of type $(7, 7)$.

Let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. Let Q be the set of paths of length two going from X to Y and then back to X. By double-counting the elements of Q, we get the bounds

$$42 = 21\left(\frac{21}{y} - 1\right) \leq |Q| \leq x(x-1) = 42.$$

All inequalities are tight, so it follows that every block in Y must have intersection number 3, and that every pair of points in X is mutually incident to a block in Y. Hence (X, Y) forms a Fano subplane of $\mathcal{D}_{2,4}$ as required. □

Lemma 5.17. $\Phi'_E(\mathcal{D}_{2,4}, 21) = 36$. *Furthermore, the subgraph of* $\Gamma_{2,4}$ *induced by such an EIP-set is isomorphic to* $\Gamma(H(2))$*, the Levi graph of the unital of order 2.*

Proof. The proof follows similarly to that of Lemma 5.16. The lower bound of 36 is clear since PG$(2, 4)$ contains a $H(2)$ subdesign, and the upper bound of 36 follows from Lemma 5.9.

Let S be an EIP-set of type (x, y) with $x + y = 21$, and assume without loss of generality that $x \leq y$. Applying Lemma 5.12 yields $(x - y)^2 \leq 9$, so that S must be of type $(9,12)$ or $(10,11)$. However, applying Lemma 5.6 with $m=3$ and $n=1$ yields $\Phi'_E(\mathcal{D},10,11) \le 35$, so that S must be of type $(9,12)$.

Let $X = S \cap P$ and $Y = S \cap \mathcal{B}$. Let Q be the set of paths of length two going from X to Y and then back to X. By double-counting the elements of Q, we get the bounds

$$72 = 36\left(\frac{36}{y} - 1\right) \le |Q| \le x(x-1) = 72.$$

All inequalities are tight, so it follows that every block in Y must have intersection number 3, and that every pair of points in X is mutually incident to a block in Y. By symmetry, it also follows that each point in X is incident with exactly four blocks in Y. Hence (X,Y) forms a $H(2)$ subdesign of $\mathcal{D}_{2,4}$ as required. □

Corollary 5.18. *The EIP on $\mathcal{D}_{2,2}$ does not have nested solutions.*

Proof. This follows from Lemmas 5.16 and 5.17, and the fact that $H(2)$ does not contain a Fano subplane. □

5.3.4 The case $5 \le q \le 8$

In this section, we determine the exact values of $i_E(\Gamma_{2,q})$ for $q \in \{5,7\}$, and narrow it down to one of two possible values when $q = 8$. Point sets are listed using their representation as Singer difference sets; see Appendix A for the exact difference sets being used.

Lemma 5.19. $\Phi'_E(\mathcal{D}_{2,5},15,16) \ge 63$.

Proof. For the lower bound, let $X = \{0,2,3,4,5,6,7,8,9,15,16,17,18,19,20\} \subseteq \mathbb{Z}_{31}$. Then X has the intersection spectrum $\{1^3, 2^{12}, 3^1, 4^{15}\}$. Letting Y be the 16 blocks at the top of the spectrum gives us $|E(X,Y)| = 63$ as required. □

Lemma 5.20. *If $1 \le n \le 31$ then $\Phi'_E(\Gamma_{2,5},n) \le \frac{63n}{31}$.*

Proof. If $1 \le n \le 30$ then we know from Lemma 5.9 that

$$\frac{\Phi'_E(\Gamma_{2,5},n)}{n} \le \frac{45 + 8\sqrt{5}}{31} < \frac{63}{31}.$$

If $n = 31$ then Lemma 5.9 yields

$$\Phi'_E(\Gamma_{2,5},31) \le \left\lfloor \frac{186 + 21\sqrt{5}}{4} \right\rfloor = \frac{63n}{31}.$$ □

Theorem 5.21. $i_E(\Gamma_{2,5}) = \frac{60}{31}$.

Proof. This follows from Lemmas 5.19 and 5.20. □

Lemma 5.22. $\Phi'_E(\mathcal{D}_{2,7}, 27, 30) \geq 150$.

Proof. For the lower bound, let X be the $(27, 5)$-arc

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 22, 27, 30, 31, 39, 42, 44, 47, 48, 53, 54\}$$

in $\mathbb{Z}_{57}$, which has the intersection spectrum $\{2^{15}, 3^{12}, 5^{30}\}$. Letting Y be the 30 blocks at the top of the spectrum gives us $|E(X, Y)| = 150$ as required. □

Lemma 5.23. *If* $1 \leq n \leq 57$, *then* $\Phi'_E(\Gamma_{2,7}, n) \leq \frac{50n}{19}$.

Proof. If $1 \leq n \leq 55$, then we know from Lemma 5.9 that

$$\frac{\Phi'_E(\Gamma_{2,7}, n)}{n} \leq \frac{440 + 59\sqrt{7}}{228} < \frac{50}{19}.$$

If $n = 56$, then Lemma 5.9 yields

$$\Phi'_E(\Gamma_{2,7}, n) \leq \left\lfloor \frac{6272 + 812\sqrt{7}}{57} \right\rfloor = 147 < \frac{50n}{19}.$$

Now assume $n = 57$. Suppose for now that $\Phi'_E(\Gamma_{2,7}, n) \geq 151$. Then there must be a corresponding witness EIP-set S, and suppose without loss of generality that S is of type (x, y) with $x \leq y$. Then we know from Lemma 5.12 that $(x - y)^2 < 16$, from which it follows that S is of type $(27, 30)$ or $(28, 29)$.

But Lemma 5.6 (with $m = 5$ and $n = 2$) yields

$$\Phi_E(\mathcal{D}_{2,7}, 28, 29) \leq 149$$

and

$$\Phi_E(\mathcal{D}_{2,7}, 27, 30) \leq 150.$$

It follows that $\Phi'_E(\Gamma_{2,7}, 57) \leq 150 = \frac{50n}{19}$ as required. □

Theorem 5.24. $i_E(\Gamma_{2,7}) = \frac{52}{19}$.

Proof. This follows from Lemmas 5.22 and 5.23. □

Finally, we look at the case when $q = 8$. In this case, we have been unable to obtain the exact value for $i_E(\Gamma_{2,8})$, but we narrow it down to one of two possible values here.

Lemma 5.25. $\Phi'_E(\mathcal{D}_{2,8}, 36, 37) \geq 213$.

Proof. For the lower bound, let X be the $(36, 6)$-arc

$$\{0, 1, 2, 3, 4, 5, 7, 9, 10, 13, 16, 18, 19, 21, 22, 23, 24, 25, 28, 37\\ 41, 45, 49, 51, 54, 56, 57, 58, 59, 62, 63, 64, 65, 66, 67, 69\} \subseteq \mathbb{Z}_{73},$$

which has the intersection spectrum $\{2^2, 3^{29}, 4^6, 5^7, 6^{29}\}$. Letting Y be the 37 blocks at the top of the spectrum gives us $|E(X,Y)| = 213$ as required. □

Theorem 5.26. $\Phi'_E(\Gamma_{2,8}, 73) \in \{213, 214\}$.

Proof. Suppose that $\Phi'_E(\Gamma_{2,8}, 57) \geq 215$, and let S be its corresponding witness EIP-set of type (x, y) with $x \leq y$. Then we know from Lemma 5.12 that $(x - y)^2 < 22$, so that S is of type $(35, 38)$ or $(36, 37)$.

But Lemma 5.6 yields $\Phi'_E(\mathcal{D}_{2,8}, 35, 38) \leq 213$ (using $m = 5$ and $n = 2$) and $\Phi'_E(\mathcal{D}_{2,8}, 36, 37) \leq 214$ (using $m = 5$ and $n = 3$). Thus $\Phi'_E(\Gamma_{2,8}, 73) \leq 214$, and the lower bound of 213 comes from Lemma 5.25. □

Lemma 5.27. *If* $1 \leq n \leq 72$, *then* $\Phi'_E(\Gamma_{2,8}, n) \leq \frac{213n}{73}$.

Proof. If $1 \leq n \leq 71$, then we know from Lemma 5.9 that

$$\frac{\Phi'_E(\Gamma_{2,8}, n)}{n} \leq \frac{639 + 150\sqrt{2}}{292} < \frac{213}{73}.$$

So assume that $n = 72$. We claim that $\Phi'_E(\Gamma_{2,8}, n) \leq 210$. Indeed, suppose that $\Phi'_E(\Gamma_{2,8}, 72) \geq 211$, and let S be its corresponding witness EIP-set of type (x, y) with $x \leq y$. Then we know from Lemma 5.12 that $(x - y)^2 < 10$, so that S is of type $(35, 37)$ or $(36, 36)$.

But Lemma 5.6 yields $\Phi'_E(\mathcal{D}_{2,8}, 35, 37) \leq 209$ (using $m = 5$ and $n = 2$) and $\Phi'_E(\mathcal{D}_{2,8}, 36, 36) \leq 210$ (using $m = 5$ and $n = 3$). Thus

$$\Phi'_E(\Gamma_{2,8}, n) \leq 210 < \frac{213n}{73}$$

as required. □

Theorem 5.28. $i_E(\Gamma_{2,8}) = \frac{1}{73}\Phi_E(\Gamma_{2,8}, 73)$.

Proof. It suffices to show that if $1 \leq n \leq 72$, then $\Phi_E(\Gamma_{2,8}, n) \geq \frac{n}{73}\Phi_E(\Gamma_{2,8}, 73)$. We know by applying (5.1) to Theorem 5.26 that

$$\Phi_E(\Gamma_{2,8}, 73) \in \{229, 231\}.$$

Then the result follows by manipulating the statement of Lemma 5.27 to obtain

$$\begin{aligned}\Phi_E(\Gamma_{2,8}, n) &= 9n - 2\Phi'_E(\Gamma_{2,8}, n)\\ &\geq 9n - \frac{426n}{73}\\ &= \frac{231n}{73}\\ &\geq \frac{n}{73}\Phi_E(\Gamma_{2,8}, 73).\end{aligned}$$

□

Corollary 5.29. $i_E(\Gamma_{2,8}) \in \left\{\frac{229}{73}, \frac{231}{73}\right\}$.

5.4 The EIP on graphs arising from difference sets

In this section, we look at BIBDs arising from developments of a specific subfamily of difference sets. In particular, we will derive the exact bisection widths and edge-isoperimetric numbers of the Levi graphs of such BIBDs.

Theorem 5.30. *Let G be a group admitting a (v, k, λ)-difference set D and a subgroup H of index 2. Then*

$$|D \cap H| = \frac{k \pm \sqrt{k - \lambda}}{2}.$$

Proof. Let $n = |D \cap H|$. Then

$$\begin{aligned}n^2 + (k-n)^2 &= \sum_{g \in D \cap H} |D \cap H| + \sum_{g \in D \setminus H} |D \setminus H|\\ &= \sum_{g \in D \cap H} |D \cap Hg| + \sum_{g \in D \setminus H} |D \cap Hg|\\ &= \sum_{g \in D} |D \cap Hg|\\ &= \sum_{h \in H} |D \cap hD|\\ &= k + \lambda\left(|H| - 1\right)\\ &= \frac{k^2 + k - \lambda}{2}.\end{aligned}$$

This is a quadratic equation in n, with solutions

$$n = \frac{k \pm \sqrt{k-\lambda}}{2}$$

as required. □

Theorem 5.31. *Let G be a group of order v, admitting a (v, k, λ)-difference set D and a subgroup H of index 2. Let $\mathcal{D}$ be the development of D, and Γ its Levi graph. Then*

$$bw(\Gamma) = \frac{v(k - \sqrt{k-\lambda})}{2}$$

and

$$i_E(\Gamma) = \frac{k - \sqrt{k-\lambda}}{2}.$$

Proof. Let $n = |D \cap H|$. We can assume without loss of generality that $n \geq \frac{k}{2}$, since we can otherwise just translate D by some element of $G \setminus H$. Then from Theorem 5.30 we have $n = \frac{k+\sqrt{k-\lambda}}{2}$.

Now let $X = H$ and $Y = \{hD : h \in H\}$. This gives us $|E(X, Y)| = \frac{nv}{2}$, so that

$$bw(\Gamma) \leq kv - \frac{nv}{2} = \frac{v(k - \sqrt{k-\lambda})}{2}.$$

The result follows immediately from this and (5.3), that is,

$$\frac{k - \sqrt{k-\lambda}}{2} \leq i_E(\Gamma) \leq \frac{bw(\Gamma)}{v} \leq \frac{k - \sqrt{k-\lambda}}{2}.$$

□

As mentioned in Section 1.1, graphs satisfying $bw(\Gamma) = \frac{|V(\Gamma)|a(\Gamma)}{4}$ have been studied by Bezrukov et al. [7], where they provide two other equivalent definitions of this family of graphs (in terms of eigenvectors and cut edges respectively). The members of this family include complete graphs and hypercubes etc., but to our knowledge the Levi graphs of developments of difference sets are a new addition to this list.

Of course, it helps to see an immediate application. We have as an easy consequence the following result.

Theorem 5.32. *Let n be a positive odd integer, and let q be an odd prime power. Then*

$$i_E(\Gamma_{n,q}) = \frac{1}{2}\left(\frac{q^n - 1}{q - 1} - q^{\frac{n-1}{2}}\right).$$

Proof. By [45], $\mathcal{D}_{n,q}$ is the development of a (v, k, λ)-difference set of $\mathbb{Z}_v$ with

$$v = \frac{q^{n+1} - 1}{q - 1} = (q + 1)\frac{q^{n+1} - 1}{q^2 - 1}.$$

Then v must be even, so that $\mathbb{Z}_v$ has a subgroup of index 2 and we can apply Theorem 5.31. □

Note that this result does not hold if n or q is even, since v would necessarily be odd. We will investigate the edge-isoperimetric number of $\Gamma_{2,q}$ more closely in the next chapter.

Chapter 6

Generalised Tangent Numbers and their Applications to the EIP

In this chapter, we will be investigating a particular sequence of numbers $d_{a,1}$, and demonstrate a number of remarkable properties that it exhibits. The first few numbers of this sequence are $1, 1, 2, 4, 4, 6, 8, \ldots$ (OEIS A000061 [46]).

It is quite surprising that these numbers have not been given much attention in the literature, so we hope this brief exposition brings some light into why these numbers might be interesting to study.

The first three sections of this chapter will be a brief review of basic number-theoretical concepts building up to the construction and properties of the sequence $d_{a,1}$, and we defer the proofs and more technical details to [24] and [44]. The latter four sections of this chapter will then apply these results to the EIP. In particular, we will use $d_{a,1}$ to construct strong upper bounds on the edge-isoperimetric numbers of $\Gamma_{2,q}$, $\Gamma(\overline{\mathcal{D}_{2,q}})$, the Levi graphs of developments of Paley difference sets, as well as the Paley graphs (Sections 6.4–6.7 respectively).

6.1 Background

If q is a prime power, we will use the notation $\mathbb{F}_q$ to denote the finite field (or Galois field) of order q. The *absolute trace* of an element in $\mathbb{F}_q$, where $q = p^e$ is a prime power, is

$$\mathrm{Tr}(x) = x + x^p + \cdots + x^{p^{e-1}},$$

and is an element of $\mathbb{F}_p$. In this chapter we will always use the term *trace* to mean the absolute trace.

We say that an integer n is a *quadratic residue* mod p if there exists an integer x such that $n \equiv x^2 \pmod{p}$.

We then define the Legendre symbol as follows, for odd prime p:

$$\left(\frac{a}{p}\right) = \begin{cases} 0, & \text{if } a \equiv 0 \pmod{p} \\ 1, & \text{if } a \not\equiv 0 \pmod{p} \text{ and } a \text{ is a quadratic residue mod } p \\ -1, & \text{if } a \text{ is not a quadratic residue mod } p. \end{cases}$$

Since this is a useful symbol, we extend this to finite fields of odd order in the obvious way:

$$\left(\frac{a}{\mathbb{F}_q}\right) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0 \text{ and } a = x^2 \text{ for some } x \in \mathbb{F}_q \\ -1 & \text{if } a \neq x^2 \text{ for all } x \in \mathbb{F}_q. \end{cases}$$

Note that this is consistent with the regular Legendre symbol when q is prime, that is, $\left(\frac{a}{p}\right) = \left(\frac{a}{\mathbb{F}_p}\right)$ for all odd primes p.

However, this is not to be confused with the Jacobi symbol, which we also use. For any positive odd integer n with prime factorisation $n = \prod_i p_i^{e_i}$, we define the Jacobi symbol as

$$\left(\frac{a}{n}\right) = \prod_i \left(\frac{a}{p_i}\right)^{e_i}.$$

Again, this is consistent with the Legendre symbol when n is prime, but it is not true for general odd prime powers q that $\left(\frac{a}{q}\right) = \left(\frac{a}{\mathbb{F}_q}\right)$. Following the convention for empty products, we let $\left(\frac{a}{1}\right) = 1$.

The Jacobi symbols are known to satisfy to the following useful property.

Theorem 6.1 (Law of quadratic reciprocity [24])**.** *If m and n are odd positive coprime integers, then*

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{(m-1)(n-1)}{4}}.$$

A Dirichlet character modulo m is a group homomorphism from $\mathbb{Z}_m^*$ to $\mathbb{C}^*$. A Dirichlet L-series is a function of the form

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character and s is a complex number with $\Re(s) > 1$.

Theorem 6.2 (Euler product)**.** *Let χ be a Dirichlet character. If $s \in \mathbb{C}$ with $\Re(s) > 1$, then*

$$L(s, \chi) = \prod_{prime\ p} \left(1 - \chi(p)p^{-s}\right)^{-1}.$$

Finally, we have built enough definitions and can begin to describe the actual sequence. In [44], Shanks generalised the Euler numbers and the tangent numbers in various ways. For the purpose of this thesis, we are only interested in one of them, that is, the sequence $d_{a,1}$.

Let a be an integer. We define the Dirichlet L-series L_a by

$$L_a(s) = \sum_{k=0}^{\infty} \left(\frac{-a}{2k+1}\right)(2k+1)^{-s}.$$

If $a \geq 2$ then we define

$$d_{a,1} = \frac{4a\sqrt{a}}{\pi^2} L_{-a}(2).$$

It is particularly notable that $d_{a,1}$ is always an integer. Note also that it is convention to define $d_{1,1}$ to be equal to one, though we will not use this anywhere. In any case, Shanks' motivation was to evaluate $d_{a,1}$ by transforming an infinite series into a finite sum. We, on the other hand, are interested in the reverse – we have a finite sum that we wish to convert back to an infinite series in order to obtain good bounds.

Theorem 6.3 (Shanks [44])**.** *Let $a \geq 2$ be a square-free integer. Then*

$$d_{a,1} = \begin{cases} \sum_{k=1}^{(a-1)/2} \left(\frac{k}{a}\right)(a-4k), & \text{if } a \equiv 1 \pmod 4 \\ \sum_{k=1}^{\lceil (a-1)/2 \rceil} \left(\frac{a}{2k-1}\right)(a-2k+1), & \text{if } a \not\equiv 1 \pmod 4. \end{cases}$$

Furthermore, if m is an integer with $m \geq 2$, then

$$d_{m^2,1} = \frac{1}{2} m^3 \prod_{p \mid m} \left(1 - p^{-2}\right)$$

and

$$d_{am^2,1} = m^3 \prod_{p \mid m} \left(1 - \left(\frac{a}{p}\right) p^{-2}\right) d_{a,1},$$

where the products are taken over all odd primes p (if any) that divide m.

We can use this to get a good asymptotic estimate for $d_{a,1}$.

Lemma 6.4. *Let $a \geq 2$ be an integer. Then*

$$\frac{a\sqrt{a}}{3} \leq d_{a,1} \leq \frac{a\sqrt{a}}{2}.$$

Proof. For the upper bound, we can use the estimate

$$L_{-a}(2) \leq \sum_{k=0}^{\infty}(2k+1)^{-2} = (1-2^{-2})\zeta(2) = \frac{\pi^2}{8}. \tag{6.1}$$

For the lower bound, let χ be the character (mod $4a$) associated with L_{-a}. From the Euler product we have

$$\begin{aligned} L_{-a}(2) &= \prod_{\text{prime } p} (1-\chi(p)p^{-2})^{-1} \\ &\geq (1+2^{-2}) \prod_{\text{prime } p} (1+p^{-2})^{-1}, \end{aligned}$$

where the last equality follows since $\chi(2) = 0$ and $\chi(p) \geq -1$ for all odd p. This yields

$$L_{-a}(2) \geq \frac{5}{4}\frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{12}.$$

This combined with (6.1) gives the required bounds

$$\frac{a\sqrt{a}}{3} \leq d_{a,1} \leq \frac{a\sqrt{a}}{2}. \qquad \square$$

We are particularly interested in values of a for which $d_{a,1}$ is close to the upper bound. We can state this more precisely.

Definition 6.5. *Let n and k be positive integers. We say that n is a k-pseudosquare if n is not a square and $\left(\frac{n}{p}\right) = 1$ for all odd primes $p \leq k$.*

We know from Dirichlet's theorem about primes in arithmetic progressions that k-pseudosquares must exist for all k. It is clear then, that $\frac{d_{a,1}}{a\sqrt{a}}$ can be taken arbitrarily close to $\frac{1}{2}$ by letting a be a k-pseudosquare for sufficiently large k.

Example 6.6. *Let $p = 1364915449155829880328 1$. Then p is a prime number that is also a 336-pseudosquare. We have*

$$\frac{d_{p,1}}{p\sqrt{p}} = \frac{4}{\pi^2}L_{-p}(2) \approx 0.4997530\ldots$$

to seven decimal places.

Of course, it is much more efficient to just let q be the square of a prime.

Example 6.7. *Let $q = 1018081 = 1009^2$ be the square of a prime. Then*

$$\frac{d_{q,1}}{q\sqrt{q}} = \frac{1}{2}\left(1 - \frac{1}{q}\right) \approx 0.4999995\ldots$$

to seven decimal places.

6.2 Finite sums relating to $d_{p,1}$ for prime p

For prime p, we introduce the convenient notation

$$K_p = \left\{1, 2, \ldots, \frac{p-1}{2}\right\}.$$

Lemma 6.8. *Let p be a prime with $p \equiv 1 \pmod 4$. Then the following sums hold:*

1. $\displaystyle\sum_{k \in K_p} \left(\frac{k}{p}\right) = 0$;
2. $\displaystyle\sum_{k \in K_p} k \left(\frac{k}{p}\right) = -\frac{d_{p,1}}{4}$; *and*
3. $\displaystyle\sum_{s,t \in K_p} \left(\frac{s-t}{p}\right) = \frac{d_{p,1}}{2}$.

Proof. The first equation follows from

$$0 = \sum_{k \in \mathbb{F}_p} \left(\frac{k}{p}\right) = 2 \sum_{k \in K_p} \left(\frac{k}{p}\right).$$

We can simplify Theorem 6.3 to get

$$d_{p,1} = \sum_{k \in K_p} \left(\frac{k}{p}\right)(p - 4k) = -4 \sum_{k \in K_p} k \left(\frac{x}{p}\right),$$

from which the second equation follows.

Finally, we have

$$\begin{aligned}\sum_{s,t\in K_p}\left(\frac{s-t}{p}\right) &= \sum_{k\in K_p}\left(\left(\frac{k}{p}\right)+\left(\frac{-k}{p}\right)\right)\left(\frac{p-1}{2}-k\right)\\ &= \sum_{k\in K_p}\left(\frac{k}{p}\right)(p-1-2x)\\ &= -2\sum_{k\in K_p}k\left(\frac{k}{p}\right)\\ &= \frac{d_{p,1}}{2}\end{aligned}$$

as required. □

Lemma 6.9. *Let p be a prime with $p\equiv 3 \pmod 4$. Then the following sums hold:*

1. $\displaystyle\sum_{k\in K_p}\left(\frac{1-4k}{p}\right)=0$;
2. $\displaystyle\sum_{k\in K_p}\left(\frac{1+4k}{p}\right)=-1$;
3. $\displaystyle\sum_{k\in K_p}k\left(\left(\frac{1-4k}{p}\right)+\left(\frac{1+4k}{p}\right)\right)=-\frac{d_{p,1}}{2}$; *and*
4. $\displaystyle\sum_{s,t\in K_p}\left(\frac{4(s-t)+1}{p}\right)=\frac{d_{p,1}}{2}$.

Proof. For the first equation we have

$$\sum_{y=\frac{p+5}{4}}^{\frac{p-1}{2}}\left(\frac{1-4x}{p}\right)=\sum_{x=1}^{\frac{p-3}{4}}\left(\frac{4x-1-2p}{p}\right)=-\sum_{x=1}^{\frac{p-3}{4}}\left(\frac{1-4x}{p}\right).$$

This sum reduces to

$$\sum_{k\in K_p}\left(\frac{1-4k}{p}\right)=\sum_{k=1}^{\frac{p-3}{4}}\left(\left(\frac{1-4k}{p}\right)-\left(\frac{1-4k}{p}\right)\right)=0.$$

For the second equation we have

$$0=\sum_{k\in K_p}\left(\frac{1+4x}{p}\right)=\left(\frac{1}{p}\right)+\sum_{k\in K_p}\left(\left(\frac{1+4k}{p}\right)+\left(\frac{1-4k}{p}\right)\right)=1+\sum_{k\in K_p}\left(\frac{1+4k}{p}\right).$$

Thirdly, let $\Omega = \sum_{k\in K_p} k\left(\left(\frac{1-4k}{p}\right)+\left(\frac{1+4k}{p}\right)\right)$.
We have from quadratic reciprocity that

$$\begin{aligned}\Omega &= \sum_{k\in K_p} k\left(\left(\frac{p}{4k-1}\right)+\left(\frac{p}{4k+1}\right)\right)\\ &= \sum_{k=1}^{p}\left\lfloor\frac{k}{2}\right\rfloor\left(\frac{p}{2k-1}\right)\\ &= \sum_{k=1}^{\frac{p-1}{2}}\left\lfloor\frac{k}{2}\right\rfloor\left(\frac{p}{2k-1}\right)+\sum_{k=\frac{p+3}{2}}^{p}\left\lfloor\frac{k}{2}\right\rfloor\left(\frac{p}{2k-1}\right).\end{aligned}$$

Now, for the latter sum, we make the change of variable $n = p+1-k$, so that

$$\left\lfloor\frac{k}{2}\right\rfloor = \frac{p+1}{2}-\left\lceil\frac{n}{2}\right\rceil \text{ and } \left(\frac{p}{2k-1}\right)=\left(\frac{p}{2p-2n+1}\right)=-\left(\frac{p}{2n-1}\right).$$

This gives us the full sum

$$\begin{aligned}\Omega &= \sum_{k=1}^{\frac{p-1}{2}}\left\lfloor\frac{k}{2}\right\rfloor\left(\frac{p}{2k-1}\right)-\sum_{n=1}^{\frac{p-1}{2}}\left(\frac{p+1}{2}-\left\lceil\frac{n}{2}\right\rceil\right)\left(\frac{p}{2n-1}\right)\\ &= \sum_{k\in K_p}\left(k-\frac{p+1}{2}\right)\left(\frac{p}{2k-1}\right)\\ &= -\frac{d_{p,1}}{2}.\end{aligned}$$

Finally, for the last equation we have

$$\begin{aligned}\sum_{s,t\in K_p}\left(\frac{4(s-t)+1}{p}\right) &= \frac{p-1}{2}+\sum_{k\in K_p}\left(\left(\frac{1+4k}{p}\right)+\left(\frac{1-4k}{p}\right)\right)\left(\frac{p-1}{2}-k\right)\\ &= -\sum_{k\in K_p} k\left(\left(\frac{1+4k}{p}\right)+\left(\frac{1-4k}{p}\right)\right)\\ &= \frac{d_{p,1}}{2}\end{aligned}$$

as required. □

Theorem 6.10. *Let p be an odd prime. Then there exist $S, T \subseteq \mathbb{F}_p$ such that*

$$\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{p}\right)=\frac{d_{p,1}}{2}.$$

Proof. If $p \equiv 1 \pmod 4$, then letting $S = T = K_p$ is sufficient as per Lemma 6.8. If $p \equiv 3 \pmod 4$, then let $S = \{4k+1 : k \in K_p\}$ and $T = \{4k : k \in K_p\}$ as per Lemma 6.9. □

6.3 Finite sums relating to $d_{q,1}$ for a prime power q

The aim is to generalise Theorem 6.10 to something that works for all prime powers.

First, we have the following quadratic Gauss sum due to Hasse–Davenport [16]. The original paper is in German, but a more accessible proof of the result can be found, for example, in [5].

Theorem 6.11 (Quadratic Gauss sum [16])**.** *Let $q = p^e$ be a prime power. Then*

$$\sum_{x\in\mathbb{F}_q} \left(\frac{x}{\mathbb{F}_q}\right) \exp\left(\frac{2\pi i \operatorname{Tr}(x)}{p}\right) = \begin{cases} -(-1)^e\sqrt{q}, & \text{if } p \equiv 1 \pmod 4 \\ -(-i)^e\sqrt{q}, & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

This theorem is actually more powerful than we need, in that it gives the sign of the Gauss sum as well as its magnitude. In what follows we will only need the magnitude of the result, that is,

$$\left|\sum_{x\in\mathbb{F}_q} \left(\frac{x}{\mathbb{F}_q}\right) \exp\left(\frac{2\pi i \operatorname{Tr}(x)}{p}\right)\right| = \sqrt{q}.$$

That said, we note that it is possible to get more precise versions of the following results by keeping the sign information, even if we do not actually do that.

For a prime power $q = p^e$, we introduce the sum

$$A_q(n) = \sum_{x\in\operatorname{Tr}^{-1}(n)} \left(\frac{x}{\mathbb{F}_q}\right), n \in \mathbb{F}_p,$$

where Tr denotes the absolute trace from $\mathbb{F}_q$ to $\mathbb{F}_p$.

Lemma 6.12. *Let $q = p^e$. Then $|A_q(1)| = p^{\lfloor (e-1)/2\rfloor}$. Furthermore, for each $n \in \mathbb{F}_p$ we have*

$$\frac{A_q(n)}{A_q(1)} = \begin{cases} \left(\frac{n}{p}\right), & \text{if } e \text{ is odd} \\ 1 - p[n=0], & \text{if } e \text{ is even,} \end{cases}$$

where the Kronecker delta $[n = 0]$ takes the value 1 if $n = 0$, and zero otherwise.

Proof. We prove the latter claim first, assuming that $A_q(1)$ is non-zero. If $n \neq 0$, then

$$A_q(n) = \sum_{x \in \mathrm{Tr}^{-1}(n)} \left(\frac{x}{\mathbb{F}_q}\right) = \sum_{x \in \mathrm{Tr}^{-1}(1)} \left(\frac{nx}{\mathbb{F}_q}\right) = \left(\frac{n}{\mathbb{F}_q}\right) A_q(1).$$

Now, if e is odd then $\left(\frac{n}{\mathbb{F}_q}\right) = \left(\frac{n}{p}\right)$, and otherwise if e is even we have $\left(\frac{n}{\mathbb{F}_q}\right) = 1$. The value of $A_q(0)$ then follows from the simple observation that $\sum_{n \in \mathbb{F}_p} A_q(n) = 0$.

For the former claim, let $S = \sum_{n \in \mathbb{F}_p} A_q(n)\xi^n$, where $\xi = \exp\left(\frac{2\pi i}{p}\right)$ is a pth root of unity. Then $S = \sum_{x \in \mathbb{F}_q} \left(\frac{x}{q}\right) \xi^{\mathrm{Tr}(x)}$, so that $|S| = \sqrt{q}$ from Theorem 6.11. If e is odd, we have

$$S = A_q(1) \sum_{n \in \mathbb{F}_p} \left(\frac{n}{p}\right) \xi^n,$$

from which it follows that $\sqrt{q} = |A_q(1)|\sqrt{p}$ and thus $|A_q(1)| = p^{(e-1)/2}$. Similarly, if e is even, then

$$S = A_q(1) \left(-p + \sum_{n \in \mathbb{F}_p} \xi^n\right),$$

from which it follows that $\sqrt{q} = |A_q(1)|p$ and thus $|A_q(1)| = p^{(e-2)/2}$ as required. □

Lemma 6.13. *Let $q = p^{2e+1}$ be an odd power of an odd prime, $e \geq 1$. Then there exist $S, T \subseteq \mathbb{F}_q$ such that*

$$\left|\sum_{s \in S} \sum_{t \in T} \left(\frac{s-t}{p}\right)\right| = \frac{d_{q,1}}{2}.$$

Proof. Let $X, Y \subseteq \mathbb{F}_p$ such that

$$\sum_{s \in X} \sum_{t \in Y} \left(\frac{s-t}{\mathbb{F}_q}\right) = \frac{d_{p,1}}{2}.$$

Let $S = \mathrm{Tr}^{-1}(X)$ and $T = \mathrm{Tr}^{-1}(Y)$, so that

$$\begin{aligned}
\sum_{s' \in S} \sum_{t' \in T} \left(\frac{s'-t'}{\mathbb{F}_q}\right) &= \sum_{s \in X} \sum_{s' \in \mathrm{Tr}^{-1}(s)} \sum_{t \in Y} \sum_{t' \in \mathrm{Tr}^{-1}(t)} \left(\frac{s'-t'}{\mathbb{F}_q}\right) \\
&= \sum_{s \in X} \sum_{s' \in \mathrm{Tr}^{-1}(s)} \sum_{t \in Y} A_q(s-t) \\
&= A_q(1) \sum_{s \in X} \sum_{s' \in \mathrm{Tr}^{-1}(s)} \sum_{t \in Y} \left(\frac{s-t}{p}\right) \\
&= A_q(1) \cdot \frac{q}{p} \sum_{s \in X} \sum_{t \in Y} \left(\frac{s-t}{p}\right)
\end{aligned}$$

Since $|A_q(1)| = p^e$, it follows that

$$\left|\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right)\right| = p^e \cdot \frac{q}{p} \cdot \frac{d_{p,1,}}{2} = \frac{d_{q,1}}{2}$$

as required. □

Lemma 6.14. *Let $q = p^{2e}$ be an even power of an odd prime. Then there exist $S, T \subseteq \mathbb{F}_q$ such that*

$$\left|\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right)\right| = \frac{d_{q,1}}{2}.$$

Proof. Let $X = \mathrm{Tr}^{-1}(K_p)$. Then

$$\begin{aligned}
\sum_{s',t'\in K_p}\left(\frac{s'-t'}{\mathbb{F}_q}\right) &= \sum_{s\in K_p}\sum_{s'\in \mathrm{Tr}^{-1}(s)}\sum_{t\in K_p}\sum_{t'\in\mathrm{Tr}^{-1}(t)}\left(\frac{s'-t'}{\mathbb{F}_q}\right)\\
&= \sum_{s\in K_p}\sum_{s'\in \mathrm{Tr}^{-1}(s)}\sum_{t\in K_p} A_q(s-t)\\
&= A_q(1)\sum_{s\in K_p}\sum_{s'\in \mathrm{Tr}^{-1}(s)}\sum_{t\in K_p}(1-p[s=t])\\
&= A_q(1)\sum_{s\in K_p}\sum_{s'\in \mathrm{Tr}^{-1}(s)}(|K_p|-p)\\
&= A_q(1)\cdot\frac{p-1}{2}\cdot\frac{q}{p}\cdot\left(\frac{p-1}{2}-p\right).
\end{aligned}$$

Since $|A_q(1)| = p^{e-1}$, it follows that

$$\left|\sum_{s,t\in X}\left(\frac{s-t}{\mathbb{F}_q}\right)\right| = p^{e-1}\cdot\frac{p^2-1}{4}\cdot\frac{q}{p} = \frac{d_{q,1}}{2}$$

as required. □

Theorem 6.15. *Let $q = p^e$ be an odd prime power. Then there exist subsets $S, T \subseteq \mathbb{F}_q$ such that*

$$\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) = \frac{d_{q,1}}{2}.$$

Proof. From Theorem 6.10 and Lemmas 6.13 and 6.14, we can always pick $S, T \subseteq \mathbb{F}_q$ such that

$$\left|\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right)\right| = \frac{d_{q,1}}{2}.$$

If this sum is negative, then pick $\overline{S} = \mathbb{F}_q \setminus S$, so that

$$\sum_{s\in\overline{S}}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) = \sum_{s\in\mathbb{F}_q}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) - \sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) = \frac{d_{q,1}}{2}. \qquad \square$$

6.4 The EIP on $\Gamma_{2,q}$ for odd q

Finally, we have the first real application of $d_{a,1}$ to the EIP.

Theorem 6.16. *Let q be an odd prime power. Then*

$$\Phi'_E\left(\mathcal{D}_{2,q}, \frac{q(q+1)}{2}, \frac{q(q+1)}{2}\right) \geq q\left(\frac{q+1}{2}\right)^2 + \frac{(q-1)d_{q,1}}{2}.$$

Proof. For a subset $A \subseteq \mathbb{F}_q$, let $g_A : \mathbb{F}_q^2 \to \{0,1\}$ be given by

$$g_A(x,s) = \left[\left(\frac{x}{\mathbb{F}_q}\right) \geq 0\right] - \left(\frac{x}{\mathbb{F}_q}\right)[s \in A],$$

where $[I]$ is the Iverson bracket and takes the value of 1 or 0 depending on whether the condition I is true or false respectively.

As per Theorem 6.15, let S and T be subsets of $\mathbb{F}_q$ satisfying

$$\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) = \frac{d_{q,1}}{2}.$$

Writing the point and block sets of $\mathcal{D}_{2,q}$ as P and $\mathcal{B}$ respectively, let

$$X = \{\langle(-1,x,s)\rangle \subseteq P : x,s \in \mathbb{F}_q, g_S(x,s) = 1\}$$

and

$$Y = \{\langle(t,y,1)\rangle^\perp \subseteq \mathcal{B} : y,t \in \mathbb{F}_q, g_T(y,t) = 1\}.$$

We first note that for any $A \subseteq \mathbb{F}_q$, we must have

$$\begin{aligned}\sum_{x,s\in\mathbb{F}_q} g_A(x,s) &= \sum_{s\in\mathbb{F}_q}\sum_{x\in\mathbb{F}_q}\left[\left(\frac{x}{\mathbb{F}_q}\right) \geq 0\right] - \sum_{s\in A}[s\in\mathbb{F}_q]\sum_{x\in\mathbb{F}_q}\left(\frac{x}{\mathbb{F}_q}\right)\\ &= \frac{q(q+1)}{2}.\end{aligned}$$

It follows from this that $|X| = |Y| = \frac{q(q+1)}{2}$.

We now need to show that $|E(X,Y)| = q\left(\frac{q+1}{2}\right)^2 + \frac{(q-1)}{2}d_{q,1}$. To do so, we first claim that if $x, z \in \mathbb{F}_q$ then

$$\sum_{y\in\mathbb{F}_q}\left(\frac{y}{q}\right)[xy = z] = \left(\frac{x}{\mathbb{F}_q}\right)\left(\frac{z}{\mathbb{F}_q}\right).$$

This is easily seen to be true if $x = 0$ since

$$\sum_{y\in\mathbb{F}_q}\left(\frac{y}{q}\right)[xy = z] = [z = 0]\sum_{y\in\mathbb{F}_q}\left(\frac{y}{q}\right) = 0,$$

and otherwise if $x \neq 0$ we have

$$\sum_{y\in\mathbb{F}_q}\left(\frac{y}{q}\right)[xy = z] = \left(\frac{x^{-1}z}{\mathbb{F}_q}\right) = \left(\frac{x}{\mathbb{F}_q}\right)\left(\frac{z}{\mathbb{F}_q}\right).$$

Now, for some fixed $x, s \in \mathbb{F}_q$, we have

$$\begin{aligned}
&\sum_{y,t\in\mathbb{F}_q} g_T(y,t)[(-1,x,s)\cdot(t,y,1) = 0] \\
&\quad= \sum_{y,t\in\mathbb{F}_q}\left(\left[\left(\frac{y}{\mathbb{F}_q}\right)\geq 0\right] - \left(\frac{y}{\mathbb{F}_q}\right)[t\in T]\right)[xy = s-t] \\
&\quad= \sum_{y\in\mathbb{F}_q}\left[\left(\frac{y}{\mathbb{F}_q}\right)\geq 0\right]\sum_{t\in\mathbb{F}_q}[xy = s-t] - \sum_{t\in T}\sum_{y\in\mathbb{F}_q}\left(\frac{y}{\mathbb{F}_q}\right)[xy = s-t] \\
&\quad= \frac{q+1}{2} - \sum_{t\in T}\left(\frac{x}{\mathbb{F}_q}\right)\left(\frac{s-t}{\mathbb{F}_q}\right).
\end{aligned}$$

Finally, putting it all together we get

$$\begin{aligned}
|E(X,Y)| &= \sum_{x,s\in\mathbb{F}_q} g_S(x,s)\sum_{y,t\in\mathbb{F}_q} g_T(y,t)[(-1,x,s)\cdot(t,y,1) = 0] \\
&= \sum_{x,s\in\mathbb{F}_q} g_S(x,s)\left(\frac{q+1}{2}\right) - \sum_{x,s\in\mathbb{F}_q} g_S(x,s)\sum_{t\in T}\left(\frac{x}{\mathbb{F}_q}\right)\left(\frac{s-t}{\mathbb{F}_q}\right) \\
&= q\left(\frac{q+1}{2}\right)^2 - \sum_{x,s\in\mathbb{F}_q}\left(\frac{x}{\mathbb{F}_q}\right)\left(\left[\left(\frac{x}{\mathbb{F}_q}\right)\geq 0\right] - \left(\frac{x}{\mathbb{F}_q}\right)[s\in S]\right)\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) \\
&= q\left(\frac{q+1}{2}\right)^2 - \sum_{x\in\mathbb{F}_q}\left[\left(\frac{x}{\mathbb{F}_q}\right)=1\right]\sum_{t\in T}\sum_{s\in\mathbb{F}_q}\left(\frac{s-t}{\mathbb{F}_q}\right) + \sum_{x\in\mathbb{F}_q}\left(\frac{x^2}{\mathbb{F}_q}\right)\sum_{s\in S}\sum_{t\in T}\left(\frac{s-t}{\mathbb{F}_q}\right) \\
&= q\left(\frac{q+1}{2}\right)^2 + \frac{(q-1)d_{q,1}}{2}.
\end{aligned}$$

□

Corollary 6.17. *If q is an odd prime power, then*

$$i_E(\Gamma_{2,q}) \leq \frac{q+1}{2} - \frac{d_{q,1}}{2(q+3)}.$$

Proof. This follows easily from Theorem 6.16, since

$$\begin{aligned} i_E(\Gamma_{2,q}) &\leq \frac{\Phi_E\left(\mathcal{D}_{2,q}, \frac{q(q+1)}{2}, \frac{q(q+1)}{2}\right)}{q(q+1)} \\ &= q+1 - \frac{2\Phi'_E\left(\mathcal{D}_{2,q}, \frac{q(q+1)}{2}, \frac{q(q+1)}{2}\right)}{q(q+1)} \\ &\leq q+1 - \frac{2}{q(q+1)}\left(q\left(\frac{q+1}{2}\right)^2 + \frac{q-1}{2}d_{q,1}\right) \\ &= \frac{q+1}{2} - \frac{(q-1)d_{q,1}}{q(q+1)} \\ &\leq \frac{q+1}{2} - \frac{d_{q,1}}{q+3}, \end{aligned}$$

where the last inequality follows from the fact that $(q+3)(q-1) \geq q(q+1)$. □

Of course, since we have good bounds on $d_{q,1}$ we have the following corollary.

Corollary 6.18. *If q is an odd prime power, then*

$$i_E(\Gamma_{2,q}) \leq \frac{q+1}{2} - \frac{\sqrt{q}}{3} + \frac{1}{\sqrt{q}}.$$

Proof. This follows immediately from the previous corollary since

$$\frac{d_{q,1}}{q+3} \geq \frac{q\sqrt{q}}{3(q+3)} \geq \frac{q-3}{3\sqrt{q}}.$$ □

Theorem 6.19. *If q is an odd prime power, then $i_E(\Gamma_{2,q}) = \frac{q}{2} - \Theta(\sqrt{q})$.*

Proof. The upper bound is from Corollary 6.18 and the lower bound is from (5.3). □

6.5 The EIP on $\Gamma(\overline{\mathcal{D}_{2,q}})$

In this section we will derive bounds on the edge-isoperimetric number of $\Gamma(\overline{\mathcal{D}_{2,q}})$, using results from the previous section.

Theorem 6.20. *Let q be an odd prime power. Then*

$$\Phi'_E\left(\overline{\mathcal{D}_{2,q}}, \frac{q(q+1)}{2}, \frac{q(q+1)}{2}+1\right) \geq \frac{q(q+1)(q^2+1)}{4} + \frac{(q-1)d_{q,1}}{2}.$$

Proof. Let $\mathcal{D} = \mathcal{D}_{2,q}$ and $x = \frac{q(q+1)}{2}$. We know from Theorem 6.16 that

$$\Phi'_E(\mathcal{D}, x, x) \geq q\left(\frac{q+1}{2}\right)^2 + \frac{(q-1)d_{q,1}}{2}.$$

Substituting $r = q+1$ and $b = q^2+q+1$ into Lemma 5.13 then yields

$$\begin{aligned}\Phi'_E(\overline{\mathcal{D}}, x, b-x) &= x(b-x-r) + \Phi'_E(\mathcal{D}, x, x)\\ &\geq \frac{q(q+1)(q^2+1)}{4} + \frac{(q-1)d_{q,1}}{2}\end{aligned}$$

as required. □

Corollary 6.21. *Let q be an odd prime power. Then*

$$i_E(\Gamma(\overline{\mathcal{D}_{2,q}})) \leq \frac{q^2}{2} - \frac{\sqrt{q}}{3} + \frac{2}{3\sqrt{q}}.$$

Proof. Using the same technique as in Corollary 6.17, we get

$$\begin{aligned}i_E(\Gamma(\overline{\mathcal{D}_{2,q}})) &\leq q^2 - \frac{2\Phi'_E\left(\mathcal{D}_{2,q}, \frac{q(q+1)}{2}, \frac{q(q+1)}{2}+1\right)}{q^2+q+1}\\ &= \frac{q^2}{2} - \frac{q+2(q-1)d_{q,1}}{2(q^2+q+1)}.\end{aligned}$$

The desired result follows by noticing that

$$\frac{q+2(q-1)d_{q,1}}{2(q^2+q+1)} \geq \frac{2(q-1)d_{q,1}}{2\frac{q^3-1}{q-1}} \geq \frac{(q-1)^2}{q^3}\frac{q\sqrt{q}}{3} = \frac{\sqrt{q}}{3} - \frac{2}{3\sqrt{q}} + \frac{1}{3q\sqrt{q}}.$$ □

Theorem 6.22. *If q is an odd prime power, then $i_E(\Gamma(\overline{\mathcal{D}_{2,q}})) = \frac{q^2}{2} - \Theta(\sqrt{q})$.*

Proof. The upper bound is from Corollary 6.21 and the lower bound is from (5.3). □

6.6 The EIP on $\Gamma(\mathrm{PD}(p))$

Let q be a prime power with $q \equiv 3 \pmod 4$, and D be the set of non-zero squares of the additive group $(\mathbb{F}_q, +)$. Then D is a $\left(q, \frac{q-1}{2}, \frac{q-3}{4}\right)$-difference set, called a *Paley difference set*. The *Paley design of order* q, denoted by $\mathrm{PD}(q)$, is the development of D. Thus $\mathrm{PD}(q)$ is a $\left(q, \frac{q-1}{2}, \frac{q-3}{4}\right)$-BIBD.

Theorem 6.23. *Let p be an odd prime, and let D be the difference set of $\mathbb{F}_p$ formed by the non-zero squares. Let $\mathcal{D} = \mathrm{PD}(q) = (P, \mathcal{B})$ be the development of D. Then*

$$\Phi'_E\left(\mathcal{D}, \frac{p-1}{2}, \frac{p+1}{2}\right) \geq \frac{p(p-1)}{8} + \frac{d_{p,1}-1}{4}.$$

Proof. Let $S = \{4k+1 : k \in K_p\} \subseteq P$ and $T = \{4k + D : k \in \{0\} \cup K_p\} \subseteq \mathcal{B}$. It is clear that a point $s \in P$ is incident with a block $t + D \in \mathcal{B}$ if and only if $\left(\frac{s-t}{p}\right) = 1$. This implies that

$$|E(S,T)| = \sum_{s\in S}\sum_{t\in T}\left[\left(\frac{s-t}{p}\right) = 1\right].$$

Manipulating this sum in the usual way, we obtain

$$|E(S,T)| = \sum_{s,t\in K_p}\left[\left(\frac{4(s-t)+1}{p}\right) = 1\right] + \sum_{s\in K_p}\left[\left(\frac{4s+1}{p}\right) = 1\right]. \tag{6.2}$$

The latter term reduces to

$$\sum_{s\in K_p}\left[\left(\frac{4s+1}{p}\right) = 1\right] = \frac{1}{2}\sum_{s\in K_p}\left(\left(\frac{1+4s}{p}\right) + 1\right) = \frac{p-3}{4}, \tag{6.3}$$

while the former term simplifies to

$$\begin{aligned}\sum_{s,t\in K_p}\left[\left(\frac{4(s-t)+1}{p}\right) = 1\right] &= \frac{1}{2}\left(\sum_{s,t\in K_p}\left(\frac{4(s-t)+1}{p}\right) + 1 - \left[\left(\frac{4(s-t)+1}{p}\right) = 0\right]\right) \\ &= \frac{1}{2}\left(\frac{d_{p,1}}{2} + \left(\frac{p-1}{2}\right)^2 - \frac{p-3}{4}\right).\end{aligned} \tag{6.4}$$

Finally, substituting (6.3) and (6.4) back into (6.2) yields

$$\Phi'_E\left(\mathcal{D}, \frac{p-1}{2}, \frac{p+1}{2}\right) \geq |E(S,T)| = \frac{p(p-1)}{8} + \frac{d_{p,1}-1}{4}$$

as required. □

We get an upper bound on the bisection width as a relatively easy corollary.

Corollary 6.24. *Let p be a prime with $p \equiv 3 \pmod 4$. Then*

$$bw(\Gamma(\mathrm{PD}(p))) \leq \frac{p(p-1)}{4} - \frac{d_{p,1}-1}{2}.$$

Proof. Let $\mathcal{D} = \mathrm{PD}(p)$, and Γ its Levi graph. The result follows directly from Theorem 6.23 since

$$\begin{aligned} bw(\Gamma) &= \frac{p(p-1)}{2} - 2\Phi'_E(\Gamma, p) \\ &\leq \frac{p(p-1)}{2} - 2\Phi'_E\left(\mathcal{D}, \frac{p-1}{2}, \frac{p+1}{2}\right) \\ &= \frac{p(p-1)}{4} - \frac{d_{p,1}-1}{2}. \end{aligned}$$ □

Corollary 6.25. *Let p be a prime with $p \equiv 3 \pmod 4$. Let Γ be the Levi graph of* $\mathrm{PD}(p)$*. Then*

$$i_E(\Gamma) \leq \frac{p-1}{4} - \frac{\sqrt{p}}{6} + \frac{1}{2p}.$$

Proof. This follows directly from the previous corollary since

$$i_E(\Gamma) \leq \frac{bw(\Gamma)}{p} \leq \frac{p-1}{4} - \frac{\frac{p\sqrt{p}}{3} - 1}{2p}.$$ □

Theorem 6.26. *If p is an odd prime with $p \equiv 3 \pmod 4$, then*

$$i_E(\Gamma(\mathrm{PD}(p))) = \frac{p}{4} - \Theta(\sqrt{p}).$$

Proof. The upper bound is from Corollary 6.25 and the lower bound is from (5.3). □

6.7 The EIP on $P(p)$

It is worth noting that the $\mathrm{PD}(q)$ structure from the previous section can only be a BIBD if $q \equiv 3 \pmod 4$, due to the antisymmetric distribution of its quadratic residues. If $q \equiv 1 \pmod 4$, we have a symmetric distribution of quadratic residues instead. We can define a similar, yet very different graph this way.

Let q be a prime power with $q \equiv 1 \pmod 4$, and let D be the set of non-zero squares of $\mathbb{F}_q$. The *Paley graph of order q*, denoted by $P(q)$ is the Cayley graph of the additive group $(\mathbb{F}_q, +)$ with respect to D. Thus $P(q)$ is a $\left(\frac{q-1}{2}\right)$-regular graph on q vertices. In fact, it is a strongly regular graph, that is, any pair of adjacent vertices has $\frac{q-5}{4}$ exactly common neighbours, and pair of non-adjacent vertices has exactly $\frac{q-1}{4}$ adjacent neighbours.

The edge-isoperimetric number of Paley graphs of prime order p was studied by Cramer et al. [15], who gave the bounds

$$\frac{p-\sqrt{p}}{4} \leq i_E(P(p)) \leq \frac{p-1}{4}. \tag{6.5}$$

Note that the lower bound is exactly the spectral lower bound. In the same paper, they also construct a different upper bound, called the α-bound, using sums of quadratic residues. They suspect that it provides an upper bound stronger than $\frac{p-1}{4}$, but are unable to obtain an asymptotic bound for it. We will prove this suspicion in this section. More specifically, they describe the α-bound in the following way.

Theorem 6.27 ([15, Theorem 7])**.** *Let p be a prime with $p \equiv 1 \pmod 4$. Let $k = \frac{p-1}{4}$, and let the non-zero squares of $\mathbb{F}_p$ be given by $\{\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_k\}$, with $1 = \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq \frac{p-1}{2}$. Then*

$$i_E(P(p)) \leq \frac{1}{k}\sum_{i=1}^{k} \alpha_i.$$

It turns out this is exactly the kind of sum that we have developed a framework for. We give the following simplification.

Theorem 6.28. *Let p be a prime with $p \equiv 1 \pmod 4$, and α_i be as defined in Theorem 6.27. Then*

$$\sum_{i=1}^{\frac{p-1}{4}} \alpha_i = \frac{p^2-1}{16} - \frac{d_{p,1}}{8}.$$

Proof. The idea is to change the language from summing the α_i to summing over some Legendre symbols instead.

That is,

$$\begin{aligned}\sum_{i=1}^{\frac{p-1}{4}} \alpha_i &= \sum_{k\in K_p} k\left[\left(\frac{k}{p}\right)=1\right]\\ &= \frac{1}{2}\sum_{k\in K_p} k\left(1+\left(\frac{k}{p}\right)\right)\\ &= \frac{1}{2}\left(\frac{\frac{p-1}{2}\left(\frac{p-1}{2}+1\right)}{2}-\frac{d_{p,1}}{4}\right)\\ &= \frac{p^2-1}{16}-\frac{d_{p,1}}{8}.\end{aligned}$$ □

Corollary 6.29. *Let p be a prime with $p \equiv 1 \pmod 4$. Then*

$$i_E(P(p)) \le \frac{p+1}{4} - \frac{\sqrt{p}}{6}.$$

Proof. This is a direct application of the previous theorem with

$$i_E(P(p)) \le \frac{\frac{p^2-1}{16}-\frac{d_{p,1}}{8}}{\frac{p-1}{4}} \le \frac{p+1}{4} - \frac{p\sqrt{p}}{6(p-1)} \le \frac{p+1}{4} - \frac{\sqrt{p}}{6}.$$ □

Theorem 6.30. *If p is an odd prime with $p \equiv 1 \pmod 4$, then*

$$i_E(P(p)) = \frac{p}{4} - \Theta(\sqrt{p}).$$

Proof. The upper bound is from Corollary 6.29 and the lower bound is from (6.5). □

Chapter 7

Concluding Remarks

7.1 The relation between the VIP and the EIP

In Chapter 4, we drew a link between the VIP and the MSP. Later in Chapter 5, we drew another link between the EIP and the MSP. But the VIP and EIP, which at first glance appear to be very highly related problems, do not appear to have any obvious connection. We do know that i_V is bounded above and below by constant factors of i_E, so that they can even be considered "equivalent" from the perspective of expander graphs. But aside from that, they appear to describe completely different things, and in general VIP-sets are not EIP-sets, and vice-versa.

To that end, it intrigues us that our bounds on i_E and i_V take very similar forms, even though they were derived using completely different methods. Furthermore, both the upper and lower bounds differ by a factor of exactly $\frac{r}{2}$. We recall the following isoperimetric inequalities on the Levi graph of a BIBD with parameters (v, b, r, k, λ):

$$\frac{2v}{v+b}\left(1-\frac{\sqrt{r^2-\lambda b}}{r}\right) \leq i_V(\Gamma) \leq \frac{2v}{v+b-1}; \tag{7.1}$$

$$\frac{vr}{v+b}\left(1-\frac{\sqrt{r^2-\lambda b}}{r}\right) \leq i_E(\Gamma) \leq \frac{vr}{v+b-1}. \tag{7.2}$$

We have investigated this similarity briefly, but have found no immediately obvious reason as to why the bounds differ exactly by a factor of $\frac{r}{2}$ that way. The next best thing then, is to compute some actual quotients of $\frac{i_E(\Gamma)}{i_V(\Gamma)}$ and observe how they differ.

Looking at $\Gamma_{2,q}$ for $q \in \{2,3,4,5,7,8\}$, we find that we always have

$$\frac{q}{2} \leq \frac{i_E(\Gamma_{2,q})}{i_V(\Gamma_{2,q})} < \frac{r}{2},$$

where we get equality in the lower bound when $q = 5$. We do not yet have a theory as to why this is true, or whether it even continues to be true for all q. The exact values for $q \leq 7$ are given in Table 7.1.

Table 7.1 Ratios between isoperimetric numbers of $\Gamma_{2,q}$ for $q \leq 7$

q	$i_V(\Gamma_{2,q})$	$i_E(\Gamma_{2,q})$	i_E/i_V	$q/2$	$r/2$
2	5/7	7/7	1.4	1	1.5
3	10/13	16/13	1.6	1.5	2
4	15/21	33/21	2.2	2	2.5
5	24/31	60/31	2.5	2.5	3
7	44/57	156/57	≈3.55	3.5	4

Another perspective is to determine when the upper and lower bounds of (7.1) and (7.2) are sharp. The upper bound in (7.1) is sharp – indeed, we saw that this is the case for the Levi graphs of complements of classical unitals. It is not known if any non-degenerate graph obtains the lower bound in (7.1), but it is asymptotically tight in the sense that for any $\varepsilon > 0$, there exists a BIBD with parameters (v, b, r, k, λ) whose Levi graph Γ satisfies

$$i_V(\Gamma) \leq \frac{2v}{v+b}\left(1 - \frac{(1-\varepsilon)\sqrt{r^2 - \lambda b}}{r}\right).$$

This happens, for example, for the family of graphs $\Gamma_{2,q}$ where q is a power of 4 (as a consequence of Theorem 3.18 and Lemma 4.3).

Interestingly, the situation is reversed for the edge-isoperimetric number. It is the lower bound in (7.2) that is sharp – indeed, we proved exactly this in Theorem 5.31. On the other hand, it is not known if any non-degenerate graph obtains the upper bound of (7.2). There are degenerate cases where equality can be obtained, such as the complete bipartite graph $K_{n,n+1}$.

7.2 Corrigenda to a paper on the EIP of Levi graphs of BIBDs

The edge-isoperimetric number of Levi graphs of block designs was first studied by Lanphier et al. in [37], where they give asymptotically matching lower and upper bounds for the incidence graphs of block designs, with particular emphasis on projective planes and Hadamard designs. However, we have found at least three different errors in this paper. We make a note of the errors here.

Claim 7.1 ([37, Theorem 1]). *Let Γ be the Levi graph of a BIBD with parameters (v, b, r, k, λ). Then*

$$i_E(\Gamma) \leq \frac{\left\lceil \frac{b}{2} \right\rceil k}{\left\lfloor \frac{v}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil}.$$

The proof uses the property that $\left\lfloor \frac{v}{2} \right\rfloor + \left\lceil \frac{b}{2} \right\rceil \leq \frac{v+b}{2}$, but this is not true when v is even and b is odd, which is the case for unital designs of odd order, for example.

They also demonstrated the following bound:

Claim 7.2 ([37, Theorem 2]). *Let Γ be the Levi graph of a symmetric (v, k, λ)-BIBD. Then*

$$i_E(\Gamma) \geq \frac{2\lambda^2(v-k)\binom{k}{2}}{3\left((k-1)(k-2)(\lambda-1)^2 + \lambda^2(k-1)(v-k)\right)}. \tag{7.3}$$

However, there is an error in their proof, whereby the coefficient of 3 in the denominator should actually be 7, thus decreasing the lower bound by a constant factor. This can also be demonstrated using the counterexample $\Gamma = \Gamma_{2,3}$, where we know from Theorem 5.15 that $i_E(\Gamma) = \frac{16}{13}$, but the right hand side of (7.3) evaluates to the larger $\frac{4}{3}$. Regardless, our lower bound obtained in Theorem 5.11 is (asymptotically) an improvement over this incorrect bound anyway.

The third error is in their application of the bound to projective planes.

Claim 7.3 ([37, Corollary 2]). *Let q be a prime power. Then*

$$\frac{q+1}{3} \leq i_E\left(\Gamma_{2,q}\right) \leq \frac{q+1}{2}\left(\frac{q^2+q}{q^2+q+1}\right).$$

The upper bound here is an incorrect application of [37, Theorem 1], whereby the numerator of the term in parentheses should be q^2+q+2 rather than q^2+q. It should also be noted that the lower bound of $\frac{q+1}{3}$ is also incorrect anyway, as a consequence of the earlier error in [37, Theorem 2].

7.3 Future work

Some of the results obtained in this thesis might possibly be generalised to a more general class of incidence structures. Likewise, some of the BIBD-specific results might also be extended to work with strongly regular graphs, since they have similar combinatorial properties.

Here we discuss future possible extensions to this project.

1. In Theorem 3.21, we showed that if p is prime and $\varepsilon > 0$, then

$$\overline{\alpha}(\Gamma_{2,p}) \geq \frac{p\sqrt{p}}{2} - \mathcal{O}\left(p^{\frac{5}{4}+\varepsilon}\right).$$

   Is it possible to remove the big O term? We know from Lemma 3.22 that if a counterexample exists, that is, a prime p exists for which $\overline{\alpha}(\Gamma_{2,p}) < \frac{p\sqrt{p}}{2}$, then $p > 5000$.

2. Can we obtain good bounds for $i_E(\Gamma_{n,q})$ when n is an even integer greater than 2? It almost looks like we should be able leverage the results for $i_E(\Gamma_{2,q})$, perhaps in a similar way in which we related $\overline{\alpha}(\mathcal{D}_{n+2,q})$ with $\overline{\alpha}(\mathcal{D}_{n,q})$ in Lemma 3.19.

3. How about bounds for $i_E(\Gamma_{n,q})$ for when q is even?

4. We defined the BIBD $\mathrm{PG}_d(n, q)$ but have not studied them except in the specific case when $d = n - 1$. Can any of our results on $\mathcal{D}_{n,q}$ be generalised to these BIBDs?

5. Can we obtain a stronger correlation between $\overline{\alpha}$ and i_V? In particular, can we generalise the program in Theorem 4.23? We would at least want something strong enough to give the exact value of $i_V(\Gamma_{2,q})$ when q is a power of 4.

6. Can we characterise the BIBDs that are $\overline{\alpha}$-sharp? We note here that we do not actually know of any BIBDs that are *not* $\overline{\alpha}$-sharp.

7. Is it possible to extend the results on $P(p)$ and $\mathrm{PD}(p)$ to general prime powers? Our bounds currently only work when p is an odd prime.

In saying all this, we feel that we have only barely scratched the surface of the isoperimetric problems in block designs. While the above questions are individually interesting, it would be elegant as a whole to be able to develop a more general theory for this class of problems, similar to Harper's framework of global morphisms in [27].

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Appendix A

Difference sets for $\mathcal{D}_{2,q}$

Here we provide the Singer difference sets for $\mathcal{D}_{2,q}$, $q \leq 16$, as described in Section 2.2. These sets are far from unique, so we follow convention and use the exact difference sets given in [13, Table 18.73]. We use these to more efficiently double-check the claims in Lemma 3.13, Theorem 3.15, and Sections 4.4 and 5.3.

Table A.1 Difference sets for $\mathcal{D}_{2,q}$, $q \leq 16$

Design	(v, k, λ)	Difference set
$\mathcal{D}_{2,2}$	$(7, 3, 1)$	$\{1, 2, 4\}$
$\mathcal{D}_{2,3}$	$(13, 4, 1)$	$\{0, 1, 3, 9\}$
$\mathcal{D}_{2,4}$	$(21, 5, 1)$	$\{3, 6, 7, 12, 14\}$
$\mathcal{D}_{2,5}$	$(31, 6, 1)$	$\{1, 5, 11, 24, 25, 27\}$
$\mathcal{D}_{2,7}$	$(57, 8, 1)$	$\{1, 6, 7, 9, 19, 38, 42, 49\}$
$\mathcal{D}_{2,8}$	$(73, 9, 1)$	$\{1, 2, 4, 8, 16, 32, 37, 55, 64\}$
$\mathcal{D}_{2,9}$	$(91, 10, 1)$	$\{0, 1, 3, 9, 27, 49, 56, 61, 77, 81\}$
$\mathcal{D}_{2,11}$	$(133, 12, 1)$	$\{1, 11, 16, 40, 41, 43, 52, 60, 74, 78, 121, 128\}$
$\mathcal{D}_{2,13}$	$(183, 14, 1)$	$\{0, 2, 3, 10, 26, 39, 43, 61, 109, 121, 130, 136, 141, 155\}$
$\mathcal{D}_{2,16}$	$(273, 17, 1)$	$\{1,2,4,8,16,32,64,91,117,128,137,182,195,205,234,239,256\}$

Appendix B

Sample code

1. Program for upper bound of $i_V(\Gamma_{2,q})$. The following MAGMA [10] code solves the program in Theorem 4.23 by brute forcing through the entire sample space. It is designed to run in under 2 minutes, so that it is possible to verify it using the online calculator at https://magma.maths.usyd.edu.au/calc/.

```
for tup in [<2,2>, <3,3>, <4,6>, <5,7>, <7,13>,
            <8,16>, <9,19>, <11,28>, <13,36>, <16,52>] do
    q := tup[1];
    alph := tup[2];
    v := q^2 + q + 1;
    upperbound := 1 - alph/v;

    f := func<x | (q+1)^2 * x / (q + x) + x>;
    finv := func<x | (-b+Sqrt(b^2+4*q*x))/2
    where b is q^2+3*q+1-x>;
    sizes := {x : x in {1..v} |
    f(x-finv(x))/x - 1 le upperbound};

    g := func<x | x eq 0 select 0 else
                  Ceiling(x*(2*M*(q+1)-x+1)/(M^2+M))
                  where M is Floor((q + x) / (q + 1))>;

    "q =", q, ": i_v is in [",
    Min({(c+d)/(a+b) :
         c in {0..v-a},
         d in {0..v-b},
```

```
            a in {0..b} meet {x-b : x in sizes},
            b in {1..v} |
            (b + d ge g(a))
            and (a + c ge g(b))
            and (c + e ge g(f))
            and ((a le alph) or (b + d ge v - alph))
            and ((q ne 5) or (a ne 9) or (b + d ge 25))
            and ((e eq 0) or (d * q ge a))
            and ((e ne 2) or (d * q + q - 2 ge 2 * a))
            where e is v - a - c
            where f is v - b - d}),
        ",", upperbound, "]";
end for;
```

2. Program to generate intersection spectra of arcs. This is C# code for verifying the claims in Sections 3.3.2 and 5.3.4.

   This script can be run, for example, in LINQPad (https://www.linqpad.net/) or in Visual Studio.

```
var adj = new Dictionary<int,int[]> {
    [2] = new[]{1,2,4},
    [3] = new[]{0,1,3,9},
    [4] = new[]{3,6,7,12,14},
    [5] = new[]{1,5,11,24,25,27},
    [7] = new[]{1,6,7,9,19,38,42,49},
    [8] = new[]{1,2,4,8,16,32,37,55,64},
    [9] = new[]{0,1,3,9,27,49,56,61,77,81},
    [11] = new[]{1,11,16,40,41,43,52,60,74,78,121,128},
    [13] = new[]{0,2,3,10,26,39,43,61,109,121,130,136,
                141,155},
    [16] = new[]{1,2,4,8,16,32,64,91,117,128,137,182,
                195,205,234,239,256}
};

void PrintSpectrum(int q, params int[] set)
{
    var v = q * q + q + 1;
    var spectrum = from a in Enumerable.Range(0, v)
```

```
                  let cnt = set.Count(x => adj[q]
                      .Contains((v - a + x) % v))
                  orderby cnt
                  group cnt by cnt into g2
                  select $"{g2.Key}^{g2.Count()}";
    Console.WriteLine(string.Join(", ", spectrum));
}

// The four (17;3)-arcs of PG(2,9)
PrintSpectrum(9,0,1,2,3,4,5,6,7,8,20,25,34,41,46,67,69,71);
PrintSpectrum(9,0,1,2,3,4,5,6,7,21,34,37,41,42,47,66,67,72);
PrintSpectrum(9,0,1,2,3,4,5,6,7,34,39,40,43,44,54,68,74,76);
PrintSpectrum(9,0,1,2,3,4,5,6,7,34,39,43,44,48,67,68,74,86);

// Vertex-isoperimetric values
PrintSpectrum(8,0,1,2,3,4,5,6,21,22,25,26,27,38,42,47,67);
PrintSpectrum(9,0,1,2,3,4,5,6,7,17,22,35,47,48,67,68,
                69,71,87,88);
PrintSpectrum(11,0,1,2,3,4,5,6,7,8,9,10,11,12,54,58,59,62,
                 63,64,98,116,120,121,125,126,129,130,131);
PrintSpectrum(13,0,1,2,3,4,5,6,7,16,17,18,20,24,31,34,47,
                 56,59,76,77,81,89,91,94,96,103,106,108,
                 127,129,131,143,151,165,173,179);
PrintSpectrum(16,0,1,2,3,4,5,6,7,13,20,25,38,43,62,77,84,
                 86,87,96,101,103,104,107,108,109,124,134,
                 137,138,140,150,155,161,164,166,169,171,
                 172,175,177,186,189,211,214,215,216,224,
                 241,250,263,266,269);

// Edge-isoperimetric values
PrintSpectrum(5,0,2,3,4,5,6,7,8,9,15,16,17,18,19,20);
PrintSpectrum(7,0,1,2,3,4,5,6,7,8,9,10,11,12,15,16,17,
                22,27,30,31,39,42,44,47,48,53,54);
PrintSpectrum(8,0,1,2,3,4,5,7,9,10,13,16,18,19,21,22,
                23,24,25,28,37,41,45,49,51,54,56,57,
                58,59,62,63,64,65,66,67,69);
```

3. C# code to verify Lemma 3.22, that is, $\overline{\alpha}(\mathcal{D}_{2,p}) \geq \frac{1}{2}p\sqrt{p}$ for all $p \leq 5000$.

```
bool AtLeastHalf(int p)
{
    int ans = 0, count = 0, n = p * p + p;
    var lines = new bool[n];

    var circle = from i in Enumerable.Range(0, p * p)
                 let dx = i / p - p / 2
                 let dy = i % p - p / 2
                 orderby dx * dx + dy * dy
                 select i;
    var points = (int)Math.Ceiling(0.5 * Math.Pow(p, 1.5));
    foreach (var item in circle.Take(points))
    {
        int x = item % p, y = item / p;
        for (int m = 0; m <= p; m++)
        {
            int z = m == 0 ? x : m * p + (y + m * x) % p;
            if (!lines[z])
            {
                count++;
                lines[z] = true;
            }
        }

        if (ans++ + count > n) return false;
    }
    return true;
}

for (int p = 2; p < 5000; p++)
{
    // skip any values of p that are not prime
    if (Enumerable.Range(2, (int)Math.Sqrt(p) - 1)
        .Any(q => p % q == 0)) continue;
    Console.WriteLine($"{p}) {AtLeastHalf(p)}");
}
```