Statistical evidence of an asymptotic geometric structure to the momentum transporting motions in turbulent boundary layers

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The turbulence contribution to the mean flow is reflected by the motions producing the Reynolds shear stress \(\langle -uv \rangle\) and its gradient. Recent analyses of the mean dynamical equation, along with data, evidence that these motions asymptotically exhibit self-similar geometric properties. The present study discerns additional properties associated with the \(uv\)-signal, with an emphasis on the magnitudes and length scales of its negative contributions. The signals analysed derive from high resolution multi-wire hotwire sensor data acquired in flat plate turbulent boundary layers. Space filling properties of the present signals are shown to reinforce previous observations, while the skewness of \(uv\) suggests a connection between the size and magnitude of the negative excursions on the inertial domain. Here, the size and length scales of the negative \(uv\)-motions are shown to increase with distance from the wall, whereas their occurrences decreases. A joint analysis of the signal magnitudes and their corresponding lengths reveals that the length scales that contribute most to \(\langle -uv \rangle\) are distinctly larger than the average geometric size of the negative \(uv\) motions. Co-spectra of the streamwise and wall-normal velocities, however, are shown to exhibit invariance across the inertial region when their wavelengths are normalised by the width distribution, \(W(y)\), of the scaling layer hierarchy, that renders the mean momentum equation invariant on the inertial domain.
1. Introduction

Studies of turbulent boundary layer flows often associate turbulent transport with the action of organized vortical motions. Consequently, the identification and characterisation of the geometric aspects of the turbulence are often synonymous with searching for and quantifying the vortical motions within the flow. Such motions include lambda/hairpin vortices, counter-rotating vortices, vortical fissures, and streaks, to name a few. In the mean momentum equation, however, the net effect of the turbulent motions is manifest as the gradient of Reynolds shear stress, \( d(-uv)/dy \), where \( u \) and \( v \) are the streamwise and wall-normal fluctuation velocities, and \( y \) is the wall-normal distance. Therefore, in contrast to the vortical motions, it is the \( uv \)-motions, and, more specifically their average variation in \( y \), that are of primary significance. Here we do not necessarily question the relevance of vortical motions to the dynamics, indeed, the Reynolds stress gradient can be related to the difference of velocity vorticity products, but rather to emphasize that \( d(-uv)/dy \) plays the direct determining role in establishing the distribution of mean momentum. Given this, the purpose of this paper is to investigate geometric features of the \( uv \)-motions, and to relate these features to the dynamics of turbulent boundary layers (TBLs).

At this point, we mention that the inclusion of geometric features in the study of wall-turbulence has a long history, particularly after the discovery of coherent motions in boundary layers [e.g., Kline et al., 1967, Robinson, 1990]. Early modelling efforts based on a self-similar eddy hierarchy, primarily for the log-region of the TBLs [e.g., Perry and Chong, 1982, Townsend, 1976] has met with considerable success in the recent past [e.g., Marusic et al., 2013]. Along a slightly different line of inquiry, tools from flow instability have been utilised to understand the recurring motions in turbulent flows. The most notable being the physical mechanism of the self-sustaining near wall cycle [e.g., Waleffe, 1997]. In the log-region, there has been suggestions of a similar self-sustaining process [e.g., Hwang, 2015]. Furthermore, the tools from the ‘non-modal’ stability analysis [e.g., Schmid, 2007] has been used to plausibly explain the large-scale structures observed in turbulent boundary layers [e.g., Del Álamo and Jimenez, 2006], and also to model more detailed features of wall-turbulence [e.g., Sharma and McKeon, 2013].

The impetus for the present effort to consider the geometric features in the \( uv \)-motions is provided by the findings in the study by Klewicki et al. [2014]. Building upon the analysis framework of Fife and co-workers [e.g., Fife et al., 2005, Klewicki, 2010, Klewicki et al., 2014, Wei et al., 2015], this study exploits the dynamic self-similarities admitted by the mean momentum equation to expose a complementary geometric structure. The theoretical framework leverages the leading balances of terms in the mean dynamical equation, and via variable transformations, the exchange of leading balance as a function of the increasing scale of the momentum transporting motions with wall-normal distance. These analyses evidence the existence of an underlying hierarchy (distribution) of scaling layers, \( W(y) \), whose widths physically represent the characteristic size of the turbulent motions responsible for momentum transport, which is dominated in the wallward i.e., by the negative \( uv \) motions, or \( wu< \), herein. The \( y \) variation of \( W \), \( dW/dy \), is of central importance, as it is analytically shown to approach a constant on the domain where the leading order balance solely involves inertial terms. In the context of the mean momentum equation, this is the origin of distance-from-the-wall-scaling which underlies a logarithmic mean velocity profile, and recent empirical studies at high Reynolds number support the onset of logarithmic dependence on inertial domain as specified by the present theory [e.g., Marusic et al., 2013, Vincentí et al., 2013].

The features of the theory most relevant to the present study relate to the fact that the coordinate stretching function, \( \phi \), that yields an invariant form of the mean momentum equation across the entire hierarchy of scaling layers is analytically given by \( \phi = dy/dW \), and thus this scaling parameter approaches constancy, \( \phi \rightarrow \phi_\infty \) as the Reynolds number tends to infinity. Indeed, the theory specifies that \( \phi_\infty^2 \equiv 1/\kappa \), where \( \kappa \) is the leading coefficient in the logarithmic mean profile equation (i.e., the von Kármán constant). By employing a valid but discrete construction of the layer hierarchy Klewicki et al. [2014] show that the quantity \( \phi_\infty^2/(\phi_\infty + 1) \)
symbol & $\delta^+$ & $U_\infty$ (m/s) & $u_\tau$ (m/s) & $\nu/u_\tau \times 10^{-6}$ m & $l^+$ & Facility \\
\hline
$\triangle$ & 2400 & 10.1 & 0.37 & 43 & 11.5 & HRNBLWT \\
$\triangle$ & 3300 & 10.1 & 0.36 & 45 & 11.1 & HRNBLWT \\
$\square$ & 3400 & 4.4 & 0.16 & 95 & 5.3 & FPF \\
$\triangledown$ & 3700 & 15.2 & 0.54 & 30 & 16.9 & HRNBLWT \\
$\Diamond$ & 4500 & 6.6 & 0.23 & 67 & 7.5 & FPF \\
$\triangledown$ & 4700 & 15.1 & 0.52 & 31 & 16.3 & HRNBLWT \\
$\triangle$ & 4800 & 10.0 & 0.34 & 47 & 10.7 & HRNBLWT \\
$\bigcirc$ & 5400 & 8.8 & 0.30 & 52 & 9.6 & FPF \\
$\square$ & 6100 & 4.2 & 0.14 & 100 & 5.0 & FPF \\
$\triangle$ & 7000 & 10.0 & 0.33 & 48 & 10.5 & HRNBLWT \\
$\Diamond$ & 7700 & 6.6 & 0.22 & 68 & 7.3 & FPF \\
$\triangledown$ & 7800 & 15.3 & 0.51 & 31 & 16.3 & HRNBLWT \\
$\square$ & 9500 & 4.3 & 0.14 & 108 & 4.7 & FPF \\
$\bigcirc$ & 9700 & 8.8 & 0.29 & 53 & 9.4 & FPF \\
$\triangledown$ & 10100 & 15.3 & 0.50 & 32 & 15.6 & HRNBLWT \\
$\Diamond$ & 13100 & 6.6 & 0.21 & 71 & 7.1 & FPF \\
$\bigcirc$ & 16400 & 8.8 & 0.28 & 55 & 9.1 & FPF \\

Table 1. Experimental parameters. $U_\infty$ is the free stream velocity and $l$ is the representative length of the sensor.

approaches a constant $\alpha$ on the inertial domain, and through a number of empirical measures show that $\alpha$ equals unity to within a couple of percent, i.e., within the measurement uncertainty. Remarkably, the condition $\alpha = 1$ indicates that $\phi_c = \Phi$, where $\Phi = (1 + \sqrt{5})/2$ is the golden ratio. Geometrically, $\alpha = 1$ corresponds to an extended version of distance-from-the-wall scaling. It requires both a proportionality between each layer width and its distance from the wall (derived from the theory), and the same proportionality between adjacent layers; rationally expected as the number of hierarchy layers, and thus Reynolds number, tends to infinity. The analysis then extends these findings to surmise that the fraction of time that the $uv$-signal is negative on the inertial domain should equal $\Phi^{-1} = 0.618...$, while direct measurements revealed an approximately a constant value close to 0.62 at the highest Reynolds number explored. Overall, these findings suggest some kind geometrically self-similar arrangement of $uv$-motions on the inertial domain, and the aim of present work is to further clarify the geometric structure of these negative $uv$-motions.

2. Experimental database

For the purposes of this paper we employ the experimental database of $u$ and $v$ signals acquired at the High Reynolds Number Boundary Layer Wind Tunnel (HRNBLWT) and at the Flow Physics Facility (FPF) located at the University of Melbourne and at the University of New Hampshire, respectively, using hot-wire anemometry with four sensing elements. The details of the experimental procedure are presented in Morrill-Winter et al. [2015] and Morrill-Winter et al. [2016], and table 1 shows the relevant experimental parameters, where the friction Reynolds number $\delta^+ = u_\tau \delta/\nu$ range from 2400 to 16400. Here, the friction velocity $u_\tau = \sqrt{\nu dU/dy|_{wall}}$ is constructed from the wall shear (with $U$ the total streamwise velocity) and the kinematic viscosity $\nu$, and $\delta$ is the boundary layer thickness based on the composite velocity profile [Chauhan et al., 2009]. Note that the superscript $+$ will represent normalisation by viscous scales; for example, $\langle uv \rangle^+ = \langle uv \rangle / u_\tau^2$, and $l^+ = l/(\nu/u_\tau)$. Also, since we intend to evaluate the lengths of the zero-crossings in the $uv$-signal, which is susceptible to noise, we have filtered all the $uv$-signals using a second order Butterworth filter with a cut-off frequency of $1/(3\nu/u_\tau^2)$. 

3. Geometric features of negative $uv$-motions

The time series of $uv$-signal is converted into a binary signal, with $-1$ for negative $uv$ magnitudes and $+1$ for the positives. The length (or the time period) of the $-1$ portion of the signal divided by the total length of the signal is the negative time fraction. Figure 1(a) shows this time fraction for different Reynolds numbers (presented by different symbols) as a function of the normalised wall-distance $y^+ / \sqrt{\nu} = y / \sqrt{\delta^+ (\nu/u_\tau)}$. The normalisation (by $\sqrt{\delta (\nu/u_\tau)}$ - the intermediate length scale) is to highlight the fact that in boundary layers the beginning of the inertial region starts at $\approx 3 \sqrt{\nu}$. The averaged time fraction within the inertial region (ending near $0.15 \delta^+$) is presented in figure 1(b) for increasing $\delta^+$, where it is evident that the time fraction approaches a constant $\simeq 1/\Phi = 0.618...$ with increasing Reynolds number. Here we note that the start of the inertial layer may be different for boundary layers than in pipes and channels. That is, earlier results that are largely based on channel flow analyses put the onset of the inertial domain at $y^+ \simeq 2.6 \sqrt{\nu}$, while our more recent measurements [in Morrill-Winter et al., 2016] for the boundary layer (used herein) indicate that that this onset is near $y^+ \simeq 3.6 \sqrt{\nu}$. Furthermore, we point out that the precise values of the constants $2.6$ and $3.6$ are obtained empirically from the experimental data, even though the theory predicts these constants to be $O(1)$. Even though a slight difference in the beginning of the inertial region is noted for the two flows, the analysis of the $uv$-motions, however, is plausibly similar in both flows, especially in the inertial regions where we expect to see self-similar behaviours.

Insights regarding the behavior of the time fraction is gained by considering the lengths of the negative $uv$-motions, say, $L_{<}$. Here the negative time segments ($t_{<}$) are converted into spatial lengths by employing the local mean velocity at the corresponding wall location, i.e., $L_{<} = t_{<} U(y)$. Note that the time fraction is the same as the length fraction, and is independent of the convection velocity used. Additionally, the relevant $L_{<}$ in the present case is observed to be less than $\delta/10$, and hence is hardly affected by the use of Taylor’s frozen turbulence hypothesis [Del Alamo and Jiménez, 2009] – even if it were an issue. The average of these lengths in inner units ($L_{<}^{+}$) is presented in figure 2 on log-log axes, where the lighter (gray) symbols are outside the inertial domain. Clearly, $\langle L_{<}^{+} \rangle$ increases with distance from the wall. In the inertial region the average length increases approximately as $\langle L_{<}^{+} \rangle \sim y^{+0.382}$. In fact the power law exponent is curiously close to $1/\Phi^2$, as indicated by the solid line in the figure. Note that the power law has a range that seems to extend beyond $y/\delta = 0.15$.

As an interesting aside, we note that the average lengths of $u$-signal both positive and negative have been investigated by Sreenivasan and co-workers [e.g., Kailasnath and Sreenivasan, 1993, Sreenivasan et al., 1983]. It turns out that the average length is proportional to the Taylor microscale, and this is true for any random signal that obeys a Gaussian probability density function. In the inertial region or log-region of the turbulent boundary layer, the Taylor microscale is proportional to $y^{+1/2}$. Figure 2 also shows a line of slope $1/2$ using dashed line for comparison. Evidently, $\langle L_{<}^{+} \rangle$ follows a different scaling. In fact, it will be made clear later (in figure 3) that the probability density function (PDF) of $uv$-signal is far from being Gaussian, ruling out the expectation of a $y^{+1/2}$-scaling. Furthermore, we observe that the signal spends a large portion of its time at values close to zero. This is likely to cause some uncertainty in determining $\langle L_{<}^{+} \rangle$; however, the signals were obtained from two different experimental facilities and both show highly similar behaviour.

The above discussion shows that, with close approximation the negative time fraction, i.e., $\langle L_{<}^{+} \rangle N_{<}/L_{total}^{+} = 1/\Phi$ in the inertial region, where $N_{<}$ is the number of negative $uv$-segments within the total length of the signal $L_{total}$. Taking $\langle L_{<}^{+} \rangle = Cy^{+1/\Phi^2}$, where $C$ is possibly a weak function of $\delta^+$,

$$\frac{N_{<}}{L_{total}^{+}} = \frac{1}{C \Phi} y^{+1/\Phi^2}. \quad (3.1)$$

This suggests that with increasing wall height the average number of segments decreases, which of course is a consequence of the increasing lengths with a constant time fraction.
4. Magnitude of $uv$-motions

Discerning a self-similar structure that connects the geometry and dynamics of the momentum transporting motions naturally leads to consideration of the signal magnitudes. Features of the magnitudes of $uv$-motions are represented by its PDF, and an example is given in figure 3. Figure 3(a) shows the PDF of the $-uv^+$ signal for $y^+$ near the start of the inertial domain at $\delta^+ = 10100$ with a dark (coloured) solid line, whereas the light (grey) line is the cumulative density function (CDF). The hatched regions show the percentage of data about the mean. The highly non-Gaussian nature of PDF and the high concentration of the data around the zero-magnitude is clear. Figure 3(b) presents the PDFs as a function of $y^+$. Interestingly, the shape of the PDF is nearly invariant over most of the boundary layer, with an obvious reduction in width in the outer region. This reduction in the PDF size in the outer part is a consequence of the non-turbulent region interspersed with the turbulent $uv$-containing region. In fact, the (externally) intermittent region starts at a $y^+$ of about $(1/3)\delta^+$ [Chauhan et al., 2014], which is roughly the location where the width of the PDF size starts to decrease.

To highlight the non-Gaussian character of the PDFs, we show skewness of $-uv$ profiles in figure 4. Not only is the skewness non-zero, rather it also attains a convincingly constant value of about 1.62 on the inertial domain. This value is intriguingly (surprisingly) close to $\Phi$, as shown by the horizontal solid black line in figure 4.
5. Connection between the length scales and the magnitude of $uv$-motions

The apparent appearance of $\phi_c \simeq \Phi$ in the magnitude and the geometry of $uv$-motions is intriguing, and, at present, the structural properties leading to these observations are not clear. It seems quite likely, however, that the magnitude of the $uv$-motions and the length scale of these motions are inherently related, and this motivates further enquiry. To this end, we use two techniques: (a) joint PDF of negative area of the $uv$ signal and the corresponding lengths, and (b) spectral analysis.

(a) Joint PDF of negative areas and lengths

In an attempt to understand any dependence of the $uv$-magnitude and the corresponding length scale we construct a box-signal, which has the same zero-crossing positions and the same areas of the positive and negative excursion of the $uv$-signal. An example is shown in figure 5(a), where the grey region has the same area in both the top and the bottom panels. This construction therefore assigns an area (positive or negative) to every corresponding zero-crossing length. Here we consider only the negative areas $A_\leq := \int_{L_\leq}^0 u v_- \, dL_\leq$ and the corresponding lengths $L_\leq$. A joint PDF of $A_\leq$ and $L_\leq$ is presented in figure 5(b), the diagonal shape of which indicates that larger lengths correlate with larger areas or $uv$-magnitudes. We note that the mean value of $A_\leq$, $\langle A_\leq \rangle$, is related to the negative contribution to $\langle uv \rangle$, $\langle uv_\leq \rangle$, because,

$$\langle uv_\leq \rangle = \frac{\sum (A_\leq)}{L_{total}} = \frac{\Sigma (A_\leq)}{N_\leq} \frac{N_\leq}{L_{total}},$$

$$\langle A_\leq \rangle = \langle A_\leq \rangle \frac{1}{C_{\Phi} y^{1/\Phi^2}}. \quad (5.1a)$$

Furthermore, to understand the dependence of $L'_{\leq}$ on $A_\leq$, we evaluate $A_\leq$ conditioned on $L'_{\leq}$, using,

$$\langle A_\leq | L'_{\leq} \rangle \, P(L'_{\leq}) = \int P(L'_{\leq}, A_\leq) \, A_\leq \, dA_\leq$$

(5.2)
Figure 3. PDF and CDF of $-uv^+$ at $\delta^+ = 10100$. (a) At $y^+ = 2.6\sqrt{\delta^+}$. (b) At all $y^+$ locations measured, and presented as a contour plot. The hatched areas on the CDF correspond to confidence bounds about the median, specifically $0.475 \leq \text{CDF}(-uv^+) \leq 0.525$ (or 95% confidence), $0.25 \leq \text{CDF}(-uv^+) \leq 0.75$ (or 50% confidence), and $0.1 \leq \text{CDF}(-uv^+) \leq 0.9$ (or 20% confidence). The PDF magnitudes are given by the color bar. Note the color increments are logarithmically spaced.

Figure 4. The skewness of $-uv$ versus wall-normal position normalized by the intermediate length scale. Symbols are tabulated in 1.
which follows from the usual conditional probability equation, \( P(A_{x}^{+}|L_{z}^{+}) P(L_{z}^{+}) = P(L_{z}^{+}, A_{x}^{+}) \) [e.g. Papoulis, 1991]. The expression \( \langle A_{x}^{+}|L_{z}^{+} \rangle P(L_{z}^{+}) \) is plotted in figure 5(c) as a solid line, which shows the contribution \( L_{z}^{+} \) makes to the \( \langle A_{x}^{+} \rangle \). Here \( \langle A_{x}^{+} \rangle \) is the area under the curve as seen by integrating (5.2) over all \( L_{z}^{+} \). From (5.1a) it is clear that the \( L_{z}^{+} \) that makes maximum contribution to \( \langle A_{x}^{+} \rangle \) also contributes maximally to \( \langle uw|_{L_{z}^{+}} \rangle \). This maximum \( L_{z}^{+} \) is also shown on the same figure with a horizontal grey solid line. If, however, \( A_{x}^{+} \) is independent of \( L_{z}^{+} \), then \( P(L_{z}^{+}, A_{x}^{+}) = P(L_{z}^{+}) P(A_{x}^{+}) \), and (5.2) is equal to \( \langle A_{x}^{+} \rangle P(L_{z}^{+}) \), which is also plotted in figure 5(c) using a dashed line. It is evident that the dashed and the solid lines are markedly different, representing the degree of dependence of the magnitude of the negative \( uw \) motion on their associated length scales. A horizontal dashed line also shows \( \langle L_{z}^{+} \rangle \), which indeed differs from the maximum \( L_{z}^{+} \) location of \( \langle A_{x}^{+}|L_{z}^{+} \rangle P(L_{z}^{+}) \). This suggests that while \( L_{z}^{+} \) might connect to an underlying geometric structure, a length scale larger than \( L_{z}^{+} \) is dynamically more relevant. A preliminary analysis shows that the maximum \( L_{z}^{+} \) location increases with \( y \) according to a power
law dependence which seems greater than $1/\Phi^2$. Further analysis is, however, required before more firm conclusions can be drawn.

(b) Spectral analysis

Typical analyses that utilize the co-spectra of $u$ and $v$ take into account the contributions to the magnitude of $\langle uv \rangle$ at particular length scales. Such analyses do not relate directly to the $uv$ signal, and cannot distinguish between the negative and positive regions in the $uv$ time series.

Nevertheless, spectral analysis provides useful information, and as an example the (pre-multiplied) co-spectrogram of $-u$ and $v$, $G$, is presented in figure 6 for $\delta^+ = 7000$. The time ($t$) signal has been converted to space using the local mean velocity ($x = t U$) with the corresponding wavenumber $k_x$, and the ordinate in the figure is presented as the wavelength ($\lambda_x = 2\pi/k_x$) normalised by viscous scales. We note that the pre-multiplied or weighted co-spectrum is independent of $U$. The figure presents contour shades (or colour) of the pre-multiplied co-spectra magnitudes. The dark solid line which is $\lambda_x \sim y$ seems to pass roughly through the middle of the contours.

In fact, the proportionality of the length scale with wall-distance is a well known empirical result in wall-turbulence; especially in the inertial region [Perry and Chong, 1982, Townsend, 1976]. More recently, and as discussed at the outset, the analysis that starts from the mean momentum equation of the wall-bounded turbulent flow has placed this distance-from-the-wall scaling on a more rigorous footing [e.g., Fife et al., 2005, Klewicki, 2010, Klewicki et al., 2007, Wei et al., 2005], and most recently for the zero pressure gradient turbulent boundary layer as well [Morrill-Winter et al., 2016]. As described previously, a central element of the theory is the finding of a length scale distribution ($W(y)$) that makes the mean momentum equation self-similar, where $W$ is defined as:

$$W := \left(-\frac{d^2U+}{dy^2}\right)^{-1/2}, \quad (5.3)$$

and on the inertial region $W \sim y$.

The efficacy of $W$ in scaling the data is evidenced in a slightly different fashion in figure 7 using the same data presented in figure 6. The different lines correspond to varying wall locations, and the darker (black) lines are in the inertial region. The abscissa is the wavelength normalised by $W$, where the constant $C_T$ is arbitrarily chosen so as to allow the peaks in the co-spectra to be nominally centered around one. The important feature of figure 7 is the near collapse of all the dark lines within the inertial region. This provides further evidence of the distance-from-the-wall scaling of the $uv$-motions that directly relate to the wall-normal distribution of mean streamwise velocity and its logarithmic dependence.

A more direct (however, experimentally more challenging) way to show the same scaling is to differentiate the co-spectrum $G$ with $y$. The differentiated co-spectrum is presented in figure 8. Here we also show the distribution of $W$ obtained using the relation (5.3) as white circular symbols. We note that, $d(-uv)/dy = \int (dG/dy) \, dk_x$ [e.g., Chin et al., 2014] which makes the differentiated co-spectrogram particularly relevant for the mean equation. It is clear from figure 8 that the location where the differentiated co-spectrogram passes through zero coincides remarkably well with the slope of the $W(y)$ profile, that is, the theoretically reasoned $W \sim y$ scaling. Figure 8 also shows the mean vorticity $|\Omega_z| = dU/dy$ as $C_T \Phi/|\Omega_z|$, which in the inertial region should be directly proportional to $y$ at sufficiently high Reynolds number.

(c) Qualitative example of the wall-scaling in $uv$-motions

Finally, we present a qualitative example of $W$ scaling of $uv$-motions from the Direct Numerical Simulation (DNS) database of turbulent boundary layer by Sillero et al. [2013] at $\delta^+ \approx 2400$. Figure 9 shows streamwise-spanwise planes of $uv$-contours at four different wall locations presented as four columns. The blue colour denotes negative $uv$-regions, whereas red denotes the positive $uv$-regions, with grey regions close to zero. The top panel shows the actual planes from the DNS,
Figure 6. The frequency pre-multiplied co-spectrogram of \(-u\) and \(v\), \(G\), for \(\delta^+ \simeq 7000\). The solid line indicated \(\lambda_x \sim y\).

Figure 7. Frequency pre-multiplied co-spectra of \(-u\) and \(v\) versus wavelength normalized by \(C_T W(y^+)\) for \(\delta^+ = 7000\) \((C_T = 20)\). The \(\ldots\) represent \(f^+ G^+\) over the domain \(3.6 \sqrt{\delta^+} \leq y^+ \leq 0.2 \delta^+\), and \((\ldots)\) represent co-spectra at every other wall location measured.

while the bottom panel is a zoomed-in view of the grey boxed regions of the top panel. The sizes of the grey boxes are scaled with the local \(W\), which increases linearly with increasing \(y\). The boxes have been selected to have the same proportion of positive and negative \(uv\)-areas. The top panel qualitatively shows that the length scales become larger with increasing \(y\), whereas the number of regions decrease. The scaled bottom panel, however, shows roughly the same structure across the \(y\)-locations due to its scaling with \(W\).
6. Summary and concluding remarks

Motivated by previous observations, herein we have sought to focus directly on the \( uv \)-motions (rather than the usual vortical motions) due to its presence in the mean momentum equation of turbulent boundary layers. Analyses of the data over a decade in friction Reynolds number range show that the negative-uv \((uv^-)\) motions constitute about 62\% of the overall time. The time fraction is almost a constant within the inertial region, and, consistent with the extended version of distance-from-the-wall scaling noted at the outset, seems to approach a value near 1/Φ = 0.618... with increasing \( Re \). On the other hand, the average length (or time) associated with \( uv^- \), \( \langle L^- \rangle \), increases with wall-normal distance, showing a power law dependence of the form \( \langle L^- \rangle \sim y^+1/2^{2/} \). Consequently, within the inertial region, the number of occurrences of \( uv^- \) decrease with increasing wall distance, whereas their length scales increase. The fact that the power law exponent of 1/Φ² is different from 1/2 is a consequence of a highly non-Gaussian \( uv^- \) PDF with a non-zero skewness. Remarkably, this skewness attains a value very close to Φ in the inertial region across all Reynolds numbers, leading us to surmise an underlying relationship between the magnitude and length of the negative \( uv^- \) excursions.

The apparent occurrence of quantities close to Φ in various statistics of the \( uv^- \)-motions is intriguing, and our understanding of this behavior remains incomplete. In general, however, there is strong empirical and analytical evidence for a self-similar structure across the inertial region. As just noted, we suspect a link between the magnitude of \( uv \) and a length scale associated with it. We observe that the \( \langle L^- \rangle \) profile follows a power law exponent \( \sim 1/Φ² \), whereas from a straightforward distance-from-the-wall based argument one would expect this value to be unity. In search of a link between magnitude and length, we then constructed a measure of negative \( uv^- \)-magnitude, \( A^-\), and examined the joint PDF between \( A^-\) and the corresponding \( L^- \). The conditional \( A^-\) showed that the dynamically significant length, which has a maximum contribution to the magnitude of \( \langle A^-\rangle \), is noticeably larger than \( \langle L^- \rangle \). Here it is relevant to note that the net Reynolds shear stress,

\[
\langle -uv \rangle = \frac{\Sigma(A^+_v) + \Sigma(A^+_u)}{L^+_\text{total}} = \frac{N^-}{L^+_\text{total}} + \frac{N^+}{L^+_\text{total}},
\]

(6.1)
where the positive counterparts $A_>$ and $N_>$ have analogous meanings to the negative ones. Also, $d(-uv)/dy$ is now simply related to the positive and negative wall-normal gradients of the magnitude and number densities of the $uv$-motions. In this paper we have only considered the first term in the last expression of (6.1), and perhaps the behavior of the second term needs to be clarified. A broader analysis of the $uv$ PDFs across Reynolds numbers and wall-normal positions also seems warranted.

Notwithstanding the limitations of the above described analyses, the $uv$ co-spectra shows a clear distance-from-the-wall scaling in the inertial region. This scaling follows from multiscale analysis of the mean momentum equation (that underlies and motivates the present research), and thus is distinct from other theoretical developments that simply assume wall scaling. The scaling is evidenced by a convincing coalescence of all of the $uv$ co-spectra on to a single curve when the wavelength is normalised by the scaling layer width distribution, $W(y)$, that yields an invariant
form of the mean momentum equation. What remains unresolved is whether the extended version of wall scaling (analytically yielding $\phi_c \equiv \Phi$, as described in the Introduction) becomes operative at high Reynolds number, and if so, what are the geometric and dynamical relationships that explain how the self-similar structure of the mean flow is reflected (as empirically suggested) in the motions responsible for the wallward flux of momentum?

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