A Justification for Deduction and Its Puzzling Corollary

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Abstract

This thesis is about how deduction is analytic and, at the same time, informative. In the first two chapters I am after the question of the justification of deduction. This justification is circular in the sense that to explain how deduction works we use some basic deductive rules. However, this circularity is not trivial as not every rule can be justified circularly. Moreover, deductive rules may not need suasive justification because they are not ampliative. Deduction preserves meaning, that is, the meaning of non-logical vocabulary of any theory which is developed by deductive reasoning remains unchanged. It means that deduction adds no information to what we already had in our premises. This is why deduction is analytic.

However, there are many ways deduction can be informative. In the next three chapters, I will pick a specific kind of deductive reasoning, namely arithmetical reasoning, and will attempt to understand the nature of information we obtain by this kind of reasoning. There is a difference between simple deductive moves such as inferring 'Socrates is mortal' from 'All human are mortal' and 'Socrates is human', and inferring that a relation is reflexive given that it is directed, symmetric and transitive. The latter is more complicated and not as easy to prove as the former. Therefore it is informative and the proof we construct to prove it puts us in an epistemic position that we were not in before having the proof. To be more specific, I show that concepts we need to confirm the conclusion are made in the process of proving the conclusion.

This is to certify that:

1. the thesis comprises only my original work towards the PhD except where indicated in the Preface;
2. due acknowledgement has been made in the text to all other material used;
3. the thesis is fewer than 100,000 words in length, exclusive of tables, maps, and bibliographies.

Salman Panahy

The benefit I have received from being part of the Melbourne's vibrant logic community and communicating with its open and generous members is extraordinary.

I am also vary thankful for the staff of the Philosophy Department at the University of Melbourne for their friendly support; specifically I should name Colin Marshall and Laura Schroeter for their useful comments of parts of this thesis. I am also in Australian people's debt without their generosity, via Australian Postgraduate Award, this study was not possible for me.

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Contents

Introduction

The question of the justification of deduction is a matter of concern for many who are interested in philosophy of logic. Attempting to answer this question has created a variety of debates. For example, seeking to define the core notion of logic has divided philosophers in to two main camps. One group think that logic is the science of truth and how it is preserved.[1]The other camp think of the consequence relation as the central notion of logic. Similarly, questioning the nature of methods used in developing and evaluating logics has led to a debate over whether these methods are a priori or a posteriori (also known as exceptionalist/anti-exeptionalist).[2] The first chapter begins with the question of the justification of deduction and continues by delving into the disagreement between Dummett and Haack over the nature of this justification. Dummett argues that the justification of deduction does not have a suasive nature,[3]this justification just explains how logical connectives behave. By the justification of deduction he does not mean justification of a specific logical system, or a specific derivation, but justification of deduction in its most general sense, that is relying on deduction as a method of gaining knowledge. On the other hand, Haack argues that the justification of deduction, in the above sense, is as in need of persuasion as induction.[4]

In the first section of this chapter, it shall be argued that what leads Haack to draw such a comparison between deduction and induction is her presumption that truth and truth preservation are the core notions of logic.[5] This section continues by taking Dummett's side in this debate and developing his ideas about the distinction between suasive and explanatory justifications.

[1]This thought can be traced back to some of Frege's writings but [Priest, 2005], Chapter 11, is the example I have in mind.

[2][Hjortland, 2017]

[3][Dummett, 1978], p.296

[4][Haack, 1976]

[5]Of course it does not mean that to be sceptical about justification of deduction, one needs to take truth as the central notion of logic, there are philosophers who take the cosequence relation as the central notion of logic and are still sceptical about any a priori justification for deduction; an example can be found in [Hjortland, 2017].

The argument, that is developed throughout this chapter, is that the task of identifying logical forms and discerning them is essential even for a sceptic such as Haack and given these forms, logical arguments are not ampliative, therefore they do not need suasive justification.

The second section of Chapter One looks at the justificatory role of soundness and completeness as two main features of any formal system. The position which is argued for here is that soundness and completeness play a justificatory role in so far as they define the meaning of logical vocabulary. In other words, their justificatory weight depends on the weight we give them in defining the meaning of logical constants. For Dummett, for instance, this weight is neither zero nor particularly profound, to adopt his terminology, soundness and completeness provide us a justification on some 'level'.[6]

The next section deals with a famous challenge in regard to justifying logical rules, namely the circular nature of these justifications. Any argument for justifying basic logical rules, such as Modus Ponens (MP), endorses the very logical rule in one way or another. This circularity not only rubs off persuasiveness of the justification in question, but also, justifies unwanted inference rules such as Haack's Modus Morons (MM) and Prior's Tonk. Also in this section we shall see that one way to respond to this challenge is to divide circularity into two kinds, rule circularity and gross circularity. Then it is shown how Dummett's attempt to reject grossness of rule circularity fails.

The last section of the first chapter is dedicated to introducing meaning theoretic concerns that help us to evaluate logical inference rules. The idea is that to break the circle of justification for logical rules we need to bring some relevant meaning theoretical concerns into account. This idea is not new and has been discussed by Dummett himself in various contexts, particularly in his book *The Logical Basis of Metaphysics*.[7] However, a different representation of these meaning theoretic concerns in the context of inferentialism is provided here. As is stated in this section, any plan to make these meaning theoretic concerns work, i.e. to exclude unintended inference rules, is an ambitious one. For instance, there are difficulties in the way the notion of harmony leads to a conservative extension of a language.

Chapter Two starts with examining problems in relation to the claim that harmony leads to conservative extension of a language. Dummett uses harmony in two different senses, he talks about intrinsic harmony which,

[6]We shall see that he considers three levels of justification and soundness and completeness provide a justification in the second level. Soundness assures us that the consequence relation is not a trivial relation. And completeness guarantees that it fully represents what semantically follows from what.

[7][Dummett, 1991b]

technically speaking, means normalizability or the ability to level a local peak (removing a complex formula in a proof).[8] Dummett believes that this gives us total harmony or conservative extension.[9] That is, inability to infer anything more than what we already have been given on a global level. One problem is that, as Read has shown,[10] intrinsic harmony cannot rule out Tonk, as Tonk is normalizable (although Read believes his rules for Tonk do not lead to non-conservative extension).[11]And another problem is that, even putting Tonk aside and only considering the usual logical vocabulary, intrinsic harmony will not lead to conservative extension. Considering Natural Deduction (ND), there are theorems of classical logic, such as $\vdash ((A \to B) \to A) \to A$ that are not provable without using rules for negation. They become provable only once we add negation to our vocabulary. So either we should rule out negation, or accept Tonk too.

In the first section we also shall see other ways meaning theoretic concerns can affect schematic aspects of logical rules, including, criteria such as simplicity, single-endedness, and purity. We shall see that none of the usual classical and intuitionistic candidate rules for negation satisfy all of these criteria. The agenda then is to search for a possible way to build a conservatively extendible system with usual logical vocabulary that preferably meets all these criteria. The existence of such a conservatively extendible system makes us able to rule out Tonk as a logical connective because it leads to non-conservative extension.

In a quest to find such rules, we consider Sequent Calculus (SC) as a formal system that provides guidelines. The suggested solution for having pure, single-ended, and simple rules, inspired by SC, is adopting multi-conclusion inference rules. This solution, of course, is not new and has been previously suggested by Read;[12] what is new is applying the same method to obtain such rules for negation.

The third section is a defence of multiple conclusions; in this section two objections toward multiple conclusion are examined and responded to. The first one, raised by Steinberger,[13] is that arguments in our usual day to day practice of reasoning are single conclusion. And the second one, raised by Rumfitt,[14] is that multiple conclusions are too demanding when it comes to failed inferences. For instance, in case of Sorites paradoxes,

[8][Dummett, 1991b], Chapter 11.
[9]Ibid. Chapter 9.
[10][Read, 2000]
[11][Read, 2010], p.564
[12][Read, 2000], p.145
[13][Steinberger, 2011b]
[14][Rumfitt, 2008b]

multiple conclusions demand that there is a certain point in the chain of arguments that the transition ceases to be correct. In response to the first argument, it has been argued that Steinberger's approach takes the meaning of logical connectives as primitive. While if the context of deducability is taken to be primitive, then multiple conclusions are not identical to a single disjunctive conclusion. In regard to the second objection, it has been argued that Rumfitt's concerns are excessive. That is, although Rumfitt's point is correct, the commitment to the existence of a breaking point in the chain of reasoning is weaker than what it seems at first glance.

The fourth section is about meaning preservation and how logic preserves the meaning of non-logical expressions in reasoning where the inferential relations define the meaning of a sub-atomic expression. The meaning of, say, a predicate such as 'x is a mammal', is defined according to what can be inferred from 'x is mammal' and from what we can infer 'x is a mammal'. This section introduces three ways in which the meaning of non-logical expressions can be preserved in a deductive reasoning. Two of these versions are based on bilateralism. Bilateralism, in contrast to unilateralism, takes both assertability and deniability conditions into account in regard to defining the meaning of logical and non-logical expressions. Unilateralism, in contrast, takes assertability conditions as defining the meaning of logical and non-logical expressions. The third version which is developed in this chapter, is based on unilateralism and multiple conclusions.

The last section introduces what Dummett[15] calls the tension between legitimacy and the usefulness of deductive reasoning as the puzzling result of accepting that logical reasoning is meaning preserving. The idea is that if we accept that deductive reasoning preserves meaning, then how might deductive reasoning expand our knowledge, where deductive reasoning is the only known legitimate way to expand our knowledge? This clarification is important because deductive reasoning may help us to gather knowledge where deductive reasoning is not the only legitimate way to have that knowledge. Dummett's point is about the first case. Arithmetical knowledge is a good example of where the accepted legitimate way of gathering knowledge is deduction. And its similarities to, and differences with, logic is one of the central themes in the three remaining chapters.

Chapter Three is the beginning of a search to find an answer for the puzzle introduced in chapter two: how deductive reasoning might simultaneously preserve meaning and expand knowledge. What is assumed in this investigation is that theorems of arithmetic are informative, that is they increase or expand our knowledge about arithmetic. We also assume that in a legiti-

[15] [Dummett, 1978], p.297

mate logical deduction, we gain nothing more than what is already given in the premises. A straightforward hypothesis is that proofs in arithmetic are different from proofs in logic. This idea has a history in philosophy, Kant[16] thought that arithmetic proofs are synthetic while logical proofs are analytic. So the afore mentioned hypothesis has been examined via probing Kantian ideas about the difference between arithmetic and logical reasoning.

The reason why Kant thought that arithmetical reasoning is synthetic was that he thought our intuitions are involved in this kind of reasoning. In the first section we attempt to understand what intuition means for Kant. For him, intuition is a faculty different from understanding. Taking a functionalistic approach, we try to grasp the input/output of this faculty as well as how it works. This helps us to see possible roles intuition might play in deductive reasoning, if any. The characteristics of intuition, as a faculty, are that the inputs of intuition are individuals, they have been introduced to intuition immediately, and the function of intuition as a faculty is synthesis.

In the second section we shall see that synthesis as function of intuition means unifying a manifold under a unit. Adopting this understanding of synthesis, the third section is an attempt to understand how Kant distinguished analytic judgements from synthetic ones. To do so, different accounts of judgement have been considered and in each case we see the possible forms that the analytic/synthetic distinction in judgements might take. In the end, judgements with non-conceptual content are introduced as possible candidates for what makes arithmetical reasoning intuitive.

The fourth section is about possible roles intuition might play in an arithmetical proof according to Kant's writings. It shall be argued that if one accepts the role of non-conceptual contents in arithmetical proofs, then intuition plays a justificatory role in some inferential steps in an arithmetical proof. The other possibility is that intuition plays a role in constructing a proof. In this scenario intuitions do not have a justificatory role in arithmetical reasoning. However, as has been explained in this section, the latter approach does not have enough textual support.

The focus, in the last section, is on Frege[17] as a critic of Kant's claim that arithmetical proofs involve intuition and that arithmetic theorems are synthetic. Frege's idea is that an exact and developed language of reasoning makes appealing to intuition unnecessary in arithmetical reasoning. Following an overview of Frege and Kant disagreement over the nature of logic, their different understandings of conceptual analysis is diagnosed as the reason for their disagreement about arithmetical proofs. It shall be shown that Frege's

[16] [Kant, 1781], p.144, B16
[17] [Frege, 1884]

understanding of the conceptual structure of a formal language is insensitive to what Kant might consider as two different concepts. This difference becomes more vivid when it comes to relations. For instance, 'the Greeks defeated the Persians at Plataea' and 'the Persians have been defeated by the Greeks at Plataea' have the same conceptual structure while for someone with a Kantian understanding of concepts, they have two different conceptual structures.

Chapter Four aims at clarifying the distinction between analytic truth and analytic justification. This clarification is important since in the legitimacy-fruitfulness tension, introduced at the end of Chapter Two, the legitimacy condition had been stated as 'not going further than what is given in premises' and now we need to define what that means. The chapter starts with Frege's definition of analyticity and Dummett's modification of it.[18]For Frege, analyticity means analytic justification and this account has nothing to say in regard to atomic sentences. At best, one might say there is a similarity between the way an analytic sentence is verified and the way a proof step is justified.

Firstly we review analyticity as a property of truth and Quine's objections to it as truth by convention. Then we examine two more recent attempts to overcome Quine's objection: one called 'Metaphysical Analyticity' (MA), developed by Gillian Russell,[19] and the other known as 'Epistemic Analyticity' (EA) introduced and advocated by Paul Boghossian.[20] Before a detailed study of MA and EA, in section two we contemplate the idea of facts about language. It is argued that although this Carnapean idea cannot be understood as part of usual logic as we know it, a more detailed account of meaning might help us to clarify how 'facts about how we use language' contribute to defining the truth value of a sentence.

MA is discussed in detail in section Three; Gillian Russell distinguishes four different aspects of meaning: character, content, reference determiner, and referent. She also distinguishes three contexts–the contexts of introduction, utterance, and evaluation. Then she defines analyticity as truth in virtue of reference determiner when, and only when, the proposition expressed in contexts of introduction and utterance, is true in the context of evaluation. Similarities and differences between Russell's and Carnap's project shall be discussed in this section as well.

In the next section we shall investigate the connection between Russell's account of analyticity and logic. Russell believes that the reference of a

[18][Dummett, 1991a], Chapter Three, p.26
[19][Russell, 2014],[Russell, 2008]
[20][Boghossian, 1997],[Boghossian, 2001]

proposition is a truth value, therefore an argument is valid, according to her, when the truth conditions of the conclusion is contained in the truth conditions of the premises. Considering her understanding of containment, she has problem justifying some inferential moves such as the one from A to $B \to A$ or from A and $\neg A$ to B. Moreover although $\forall x(x = x)$ is analytic, $a = a$ is not although it follows form the latter analytically.

Boghossian's definition of Epistemic Analyiticty, also known as EA, is the subject matter of the fifth section. First, we review the idea of EA in the context of inferential role semantics, then we examine how it responses to Quine's truth by convention objection. Finally we consider some concerns raised about a priori nature of EA. Although it is hard to point at anything non-empirical in EA, it can be said that it relies, at least partly, on how the human cognitive system works and that it can be characterized as a priori in this sense.

The last section of the fourth chapter addresses the connection between EA and logic. It is argued that the Fregeian idea, developed by Dummett, that the method of the verification of analytic truths is similar to the method of verification of logical inferences, can be maintained in this account of analyticity. It also responds to three concerns in regard to this idea. However, it is concluded that this is just a similarity since logical reasoning preserves meaning (in an inferential context), while an analytic inference might change the meaning of expressions (in an inferential context).

After considering two notions of analyticity in chapter Four, in Chapter Five, we analyse proofs, both in logic and basic arithmetic, to see whether or not proof steps in these proofs are analytic (have analytic justification). Where analytic justification means being justified by the meaning of those words that play a pivotal role in a proof step. As discussed at the end of chapter Three, one way of understanding the appeal to intuition in reasoning is accepting the existence and involvement of non-conceptual contents in reasoning. Now we want to check whether the involvement of notions such as identity and order, in form of relations like 'x is bigger than y, in arithmetical reasoning makes this kind of reasoning intuitive.

The first section is devoted to defining formal proof steps that involve notions such as identity and order. The idea is to see whether it is possible to say there are some inferential steps in a proof that are purely based on notions like identity or order. In other words, are there steps that are justified merely by the meaning of these operators? The answer provided for this question is 'yes'. We shall see that they genuinely play an inferential role and that their role cannot be covered by usual logical vocabulary.

Now that we accept these notions are inevitable, what is the effect of them in generality of a proof, where generality means being valid in more

models? This is the question that section two attempts to answer. It is argued that if any non-conceptual content is involved in inference steps, then the validity of an argument is model-sensitive. The argument is developed by rejecting such model-sensitivity for identity and leaving such possibility for the notion of order. The term used for expressions such as 'bigger than' that might be model-sensitive is 'expressions with object-sensitive meaning'. That is expressions with meanings which become fully clear when we know in which context and about which objects they have been used.

The aforementioned points about object-sensitive meaning and its relation to generality raises curiosity about taking an approach in the opposite direction, that is, attempting to see if abstraction from objects leads to any more generality. This is the topic of our third section. Here we review different notions of formality of logic in Kant and Frege. The result of this investigation is that more abstraction from objects does not lead to more generality, where generality means being applicable in general.

In chapter Three, we see that Hinttikka[21] argues for the appeal to intuition in predicate logic and formal language richer than that. This is so because, according to him, we deal with individuals in reasoning in these languages. Section Four of chapter Five prepares the scene for having another understanding of the appeal to intuition in formal reasoning. The one in which reasoning with individuals, on its own, is not what makes a reasoning intuitive, rather what makes us to appeal to intuition is the conceptual complexity which is the result of dealing with individuals in reasoning. In this section also we see that reasoning with individuals in predicate logic creates a complicated conceptual framework in a proof that is useful to show Frege-Kant disagreement over conceptual analysis.

The importance of individuals in predicate logic takes a more formal shape in the next section where we study an observation by Hazen.[22] He draws our attention to some arguments in predicate logic with normal proofs (normal proofs are defined shortly after this section) which have a certain number of object variables and any other proof of these arguments with less object variable are not normal. This observation is a support to the claim that reasoning with individuals might sometimes create a conceptually complex structure.

Section six aims at supporting two main claims; the first claim is that conceptual complexity created by reasoning with individuals can be used to distinguish proofs that, in some sense, go beyond their conceptual resources (we call them synthetic proofs) froms. And the second claim is that there is

[21] [Hintikka, 1973]
[22] [Hazen, 1999]

a sense in which we can talk about appeal to intuition for proving the conclusion from premises in some arguments. We start with reviewing Gentzen's notion of normal proof as a non-roundabout proof and then will expand that notion to define analytic proofs as proofs that do not use more that what is given in their premises in establishing the conclusion. In the next step, the notion of conceptual resource, which is provided by the premises and the conclusion of an argument, is defined. Then, by giving some formal definitions, we show how some proofs use more than what is provided by the conceptual resource of premises and the conclusion. In the next step the connection between this phenomenon and complexity of establishing a proof is discussed.

Two different measures for complexity of proofs are introduced in the last section–one is based on the variety of inference rules used in a proof and the other is based on our new definition of concepts appeared in a proof. It is argued that the latter measure of complexity does a better job in addressing challenges of establishing a proof.

Chapter 1

Justification of Deduction

It is common to justify logical rules by saying that they preserve truth, that is whenever premises of an argument are true so is the conclusion. However, there are serious doubts about the justificatory power of this claim. Thinkers such as Susan Haack[1] have given reasons in favour of this scepticism, others, like Michael Dummett,[2] have attempted to justify logical rules, although not directly as truth-preserving, but as meaning-preserving, in the sense that it preserves the meaning of non-logical fraction of the language it is part of.[3] One other related issue is that philosophers such as Haack and Gilbert Harman[4] hold that logic as a formal practice is different from reasoning in general, and reasoning in natural language in particular. And therefore, we should be cautious about applying formal logic in everyday reasoning. As an instance, Haack separates deductive implication from deductive inference.[5] By contrast, philosophers such as Michael Dummett,[6] Greg Restall[7] and Paul Boghossian,[8] think that formal logic is about forms that reasoning in natural language takes and their validity. In other words, there is a strong enough connection between deductive implication and deductive inference. One way to understand these different positions is as a difference in their conception of the subject matter of logic. The first group hold that 'truth' is the subject matter of logic as a science, while the second group believe 'what follows from what' is the main concern of logic. This difference may lead to different

[1] [Haack, 1976]

[2] [Dummett, 1978],[Dummett, 1991b]

[3] In the next chapter we shall give a detailed account of what does it mean to preserve meaning.

[4] [Harman, 1986]

[5] [Haack, 1982]

[6] [Dummett, 1978],[Dummett, 1991b]

[7] [Restall, 2005]

[8] [Boghossian, 2001]

accounts of the justification of deduction. Where justification of deduction can be understood as 'what logical laws are good for?', 'what are criteria to evaluate logical laws or rules?', and 'what are possible reasons to revise logical laws?'. We approach the issue of how to justify deduction by way of an examination of disagreement between Dummett and Haack over this topic. We then expand the investigation to more recent ideas in this chapter and the next chapter.

1.1 How Similar Are Justifications of Deduction and Induction?

In her classic paper, Haack claims that justification of deductive arguments (she prefers 'argument' rather than 'inference' as being more precise) is as problematic as justification of inductive ones. By inductive argument she means generalization over a reasonable number of observation.[9] An inductive reasoning, after gathering enough evidence, is strong. And a deductive argument is any argument that is logically valid. For instance, any argument in the form of Modus Ponens (MP). She portrays the challenge as follows:

Induction Dilemma (ID): Deductive justification is too strong for justifying induction and inductive justification is circular.

Deduction Dilemma (DD): Inductive justification is too weak for justifying deduction and deductive justification is circular.

Haack sees ID on a par with DD, since she finds suggested characteristics to distinguish between deductive and inductive inferences quasi metaphoric. That is, if we take these characteristics literally, then they turn out to be false. She names a specific characteristic:

Amplication: Deductive arguments are non-ampliative (explicative), while inductive arguments are ampliative.

Her definition of the non-ampliative character of deductive arguments is similar to analyticity of deductive argument: "containing nothing in the conclusion not already contained in the premises"[10] (We will see that although there is a difference between justifying logical rules based on the meaning of

[9] [Haack, 1976], p.113
[10] Ibid. p.112

logical words appearing in them and the claim that they are non-ampliative, they are closely connected). Haack does not give any example of cases in which literal understanding of this characteristic turns out to be false. However, arithmetic reasoning may be a generic case. Rarely does anyone think of arithmetic theorems as non-ampliative, yet many people do hold that they are examples of deductive reasoning. Another difficulty that Haack might have been trying to mention here is that finding a literal reading of the criterion is not an easy task on its own, let alone checking its falsehood. What does containment mean? If it means meaning containment, then meaning as a speaker understands it, or as it is? What is contained in premises? Is this containment similar to the way that a bunch of unassembled IKEA pieces of a table contain a table? and so on.

So far, Haack has rejected the claim that there is a characteristic difference between deductive and inductive inferences. Now it is time to point to a similar difficulty we face in justifying good rules of deductive/inductive argumentation. Suppose that we are dealing with a sceptic about deductive/inductive rules. To justify rules we need evidence. Of course one may say there is also another option, namely to give up the demand for evidence. Nonetheless, in the case of a conversation with a sceptic, this approach either terminates the conversation or leads to a change of subject. Because either she accepts that there are things we do not give evidence for and that deductive/inductive rules are not among these things. Or she denies that we should accept anything without evidence, and then the topic of our conversation shifts to whether we should accept anything without evidence.

Let us get back to the challenge of giving evidence for deductive/inductive rules to a sceptic. Usually evidences are divided into two kinds, inferential and non-inferential. There no commonly accepted distinction between inferential and non-inferential evidences or justifications. One way of defining this distinction is to say that non-inferential justifications are direct while inferential justifications are indirect evidences for a belief. A simplified example can be given as follows: we saw a possum in the corridor last night, and this is our evidence for believing 'there was a possum in the corridor last night'. On another occasion, we pass through the corridor in the morning, and see some foot prints, a messy garbage bin, and some waste of a certain shape and size, and these are our evidence for believing 'there was a possum in the corridor last night'. In the first case, the fact that supports the truth of the sentence 'there is a possum in the corridor', is that there is a possum in the corridor. In the second case, when we list the facts that support the truth of 'there is a possum in the corridor', the sentence itself does not appear in the list. The first scenario is an instance of direct, non-inferential evidence for a claim. And the second one is an example of indirect or inferential evidence

for the same claim. If we use p,q,r, and s to express sentences, and turnstile to show validity in general (that is both formal and non-formal validity), then the schematic expressions of the two scenarios are as (1) and (2) respectively:

1. $p \vdash p$
2. $p, q, r \vdash s$

Now to be able to decide whether inferential or/and non-inferential justifications are possible in the case of justification of deductive/inductive inference, we need to know what exactly our sceptic is doubtful about. According to Haack,[11] our sceptic is suspicious about deductive arguments being truth preserving and inductive arguments being strong. It means that our sceptic is sceptical about some rules of inference; she wants to know:

a. Are deductive/inductive rules of inference doing what we expect them to do?

If we express the expectation as a claim, then it would be something like (b):

b. Deductive/inductive inferences are truth preserving/often truth preserving (and we have reason to believe that they do so).

How easy or hard it is to convince a sceptic in the above sense, partly depends on their understanding of truth. If they accommodate conventional truths or, generally speaking, some kind of a priori justification of some truths, then there are possible ways to convince them that deduction and even induction reliably preserve truth.[12] However, sceptics about deductive or inductive inference, usually doubt a priori truths too. In the rest of this section, we assume that our hypothetical sceptic rejects any kind of conventional truth. This is specifically relevant because it has implications in justification of deduction.

In case of deduction, it is common among logicians to define a syntax and a semantics for a formal language. And then establish a consequence relation for syntactic derivability (usually shown by $\vdash$), and a semantic validity (usually shown by $\vDash$). It is also common to think of truth as 'satisfaction in at least one model' or 'being true in at least one model'. Now, if there is a way to show that every syntactically derivable argument is semantically valid (known as soundness theorem) and vice versa (known as completeness theorem), then it has been established that rules of inference, in the language at hand, are truth-preserving. And that there is no valid argument without

[11] Ibid. p.114

[12] For a recent view on truth by convention see [Warren, 2015]

syntactic proof. Therefore, in one sense, the question expressed in (a) and defined precisely in (b), has a positive answer in the case of deduction.

However, if someone resists against this way of understanding truth, then soundness and completeness do not have much justificatory role beyond technical purposes. This is so because soundness and completeness do not produce any evidence for claim (b) as truth in the sense of satisfiability in a model does not fully capture truth per se. A hard line sceptic might reason that in the practice of formal logic we actually build models, therefore if we understand truth as 'satisfiability in a model' then what we have is a truth by convention. That is, the truth in 'true in a model' has an artificial or conventional nature. While in its common sense meaning, which has appeared in (b), 'truth' does not have a conventional nature, for it is not something defined based on our stipulation. Our hard line sceptic thinks that we do not lay down truths, but we discover them. Whether or not there was a possum dwelling in the corridor last night is something to be discovered not to be decided. So truth by convention does not do the job of justifying deduction, because 'truth' in (b) is not a matter of convention. There are also arguments for truth by convention to explain plausibility of truth by convention, [Syverson, 2002] is an instance of such attempts.

Tenability of conventional truths is not our concern here, for the sake of argument we grant our sceptic to hold their opinion on truth by convention. Therefore, we are still at our starting point; looking for possible ways of justifying deductive/inductive rules of inference. The next step is to figure out whether the possible justification is non-inferential (direct) or inferential (indirect). The advantage of appealing to non-inferential justifications for reasoning is that we do not need to use any inferential movement to justify inferences. Consequently, we avoid any sort of circularity.

Considering the claim expressed in (b), there are different candidates for a non-inferential (direct) way of justifying deductive/inductive inferential rules: one is to check every instance of deductive/inductive inferences to see if they preserve truth. This way of non-inferential justification will lead to an inductive inference. Here is why: a direct way of justifying inference rules would be to check all instances of them such that we were able to say 'this inference is correct is a good reason to believe that this inference is correct'. However, there are infinitely many of these instances, therefore it is impossible for us to check all of them. The best we can do is to check as many of them as we can and then make an induction based on our observation. In this case our justification will have an inductive form. Consequently, we will face the second horn of ID in the attempt to justify inductive inference rules, and the first horn of DD in the attempt to justify deductive inference rules.

At least two more non-inferential ways of justifying deduction have been

suggested in the literature.[13] One appeals to a rational quasi-perceptual intuition to discern forms and the other appeals to what is called 'default reasonable beliefs' to justify deductive inferences. In the first account, following our example, the relation between discerning the form of deductive inference and considering that as truth-preserving is somehow similar to seeing a possum in the corridor and considering this as sufficient justification to hold that there is a possum in the corridor. This understanding of direct justification for deductive inferences does not lead to an inductive inference, because we do not need to justify a movement from seeing a possum to believing that we see a possum. We discern the form in something which is true as a matter of fact, or we understand truth via forms. So there is no need to infer the connection between forms on the one hand and truths on the other. Of course a sceptic can be sceptical about perception too. However, in common sense moving from perceiving something to believing it is a justified move.

One obvious difficulty with this account of direct justification for deduction is that forms are not as concrete as possums. And perceptual accessibility to them is not as inter-subjective or objective as to possums. Therefore, many other subjective, and possibly mistaken, perceptions also can be justified in the same way. Moreover, as Boghossian mentions,[14] this approach does not provide a satisfactory explanation on how it bridges the gap between grasping form and realizing truth. An inductive move, in the first account of direct justification, fills this gap much better. Consequently, 'intuition' or 'clear and distinct perception' are merely names for the mystery we are addressing rather than solutions for it.[15]

The remaining possibility is that there might be beliefs that are based on neither observation nor have been inferred from other beliefs, and still are reasonable. There are various accounts of reasonable beliefs that need no justification what so ever. The 'Default reasonable beliefs' account is an account that holds there are such beliefs. Boghossian's position looks reasonable here. He holds that if such an idea is to contribute to any theory of knowledge, then "there has to be a way of saying which beliefs are default-reasonable and why".[16] That is, there must be some criteria for determining a belief as default and an explanation for why satisfaction of those criteria is sufficient for our purpose.

There are two ways of introducing criteria for a default belief in the literature. One gives an explanation of reasonableness by-default via introducing

[13] [Boghossian, 2001], [Wright, 2000]
[14] [Boghossian, 2001], p.6
[15] Ibid. p.6
[16] Ibid. p.7

kinds of concepts that possessing them requires specific conditions or has consequences. For instance, self-fulfilling beliefs are beliefs such that merely having them guarantees their truth (and not their reasonableness). Or self-evident beliefs are, in a similar way, plausible based on the way they are understood (we may think of them as based on intuition). The other criterion, according to Wright[17] is that any attempt to make a belief unjustified has to presuppose it. His example is 'I am capable of rational thought'.

Two points about 'default belief' strategy: although it has been categorized as non-inferential justification, it is not a direct justification either. We take a belief as default only after an unsuccessful series of attempts to reject its justification without presupposing it. Only then we, perforce, have to conclude that the belief in question is a default belief. This process is not circular, but is not direct either. The other point, as Wright correctly mentions,[18] is that this strategy is not suitable for justifying that deductive reasoning is truth-preserving. Even if we show that basic rules of deductive inference are the default of our reasoning, this in itself, does not say anything about those inferences being truth preserving.

Having non-inferential justifications considered, let us now turn to inferential ones. Any inferential way to justify deductive/inductive inference rules does so by giving an argument. Namely an argument to justify that schematic deductive/inductive inference rules are truth-preserving or usually truth-preserving. This way is indirect in the sense that it relies on something which has been justified for schemas or forms and not for each instance of inference. The problem with inferential justification is that the inference used for justifying a deductive/inductive inference schema, itself, is either deductive or inductive. And according to ID and DD, such a justification is either circular or too strong or too weak. It is clear why a weak justification is not sufficient to justify deductive rules. And although a strong justification for inductive rules still justifies those rules, at the same time it casts doubt on the deduction/induction distinction.

The remaining point is circularity. The next part of this chapter is an investigation into circularity and what, if anything, is wrong with it. Our concern here is to check whether there is any significant difference between deductive and inductive inferences, such that effects acceptable methods of justifying them.

More illustration on the indirectness of inferential justification may help us to find an answer for the above mentioned question. Let us examine a famous example. Lewis Carroll's has a well-known dialogue between Achilles

[17][Wright, 2000], p.47
[18]Ibid. p.47

and the Tortoise.[19] There are different readings of the moral of the story, which is not our concern here.[20] What helps us in the topic at hand (showing the indirectness of inferential justification) is the way the challenge is raised. The story can be summarized in this way: the Tortoise accepts the truth of some antecedent conditions and a conditional with those antecedents and a consequence. However she refuses to accept the consequence based on the conditional and the antecedent of the conditional. In other words, she rejects Modus Ponens as a valid rule of inference and asks for more grounds to derive the conclusion. Here are two ways of formalizing this story: in order to accept (3), the Tortoise asks for justification of (4) and (5), or in order to accept (3), she asks that (6) and (7) be also justified.

3. $\dfrac{A \to B \quad A}{B}$

4. $\dfrac{\dfrac{A \to B \quad A}{B} \quad A \to B \quad A}{B}$

5. $\dfrac{\dfrac{\dfrac{A \to B \quad A}{B} \quad A \to B \quad A}{B} \quad \dfrac{A \to B \quad A}{B} \quad A \to B \quad A \quad \cdots}{B}$

6. $\dfrac{A \quad A \to B \quad (A \wedge (A \to B)) \to B}{B}$

7. $\dfrac{A \quad A \to B \quad (A \wedge (A \to B)) \to B \quad (A \wedge (A \to B) \wedge ((A \wedge (A \to B)) \to B)) \to B \quad \cdots}{B}$

The story ends as follows: "The Tortoise was saying "Have you got that last step written down? Unless I've lost count, that makes a thousand and one. There are several millions more to come." " Well, if we check all the premises in ... (998 written premises and all the one million coming ones), they all have a common form. In case of (5) the common form is (8) and in case of (7) the common form is (9). Where α and β are variables that can be replaced by formulas or inferences, in case of (8).

8. $\alpha \qquad \dfrac{\alpha}{\beta} \qquad \dfrac{\dfrac{\alpha}{\beta} \quad \alpha}{\beta}$

9. $(\alpha \wedge (\alpha \to \beta)) \to \beta$

[19] [Carroll, 1895]

[20] There is a list of different readings in [Engel, 2007]

More specifically, in the case of (5), we can replace ... with infinitely many premises and each three of them has the common form shown in (8).[21] And, in case of (7), each of the infinitely many premises in ... has the common form shown in (9).[22] Now if we justify anything for a schema, say $\frac{\alpha}{\beta}$, then we may have justified something for some formulas or inferences indirectly. Because if we express what has been justified for a schema as a proposition, say p, then the proposition that expresses the same justification for an instance of that schema is another proposition, say q. This makes the justification similar to (2), which is an indirect justification. Now if our sceptic thinks of direct justification as 'the ultimate' justification to hold a belief, then an inferential justification hardly has that justificatory power for them. In the other words, the gap between direct and indirect justification can hardly be filled for an inference sceptic. For such a sceptic, as soon as a justification is indirect it is not fully granted regardless of being deductive or inductive. It can be said that the epistemic status of every inference is similar to the epistemic status of inductive inferences for this kind of radical scepticism.

However, if we accept indirect, that is inferential, justification for inference rules. And accept that the objects of inferential justification of inferential rules are schemas, then there could be ways to distinguish deductive inferences from inductive ones. In general, it is common to distinguish between deduction and induction by referring to certainty of the conclusion given the premises. It is also possible to attribute certain forms to inductive reasoning as well as deductive reasoning. For instance inferring a future oriented statement from some past oriented statements. Or inferring a general claim from a number of particular claims. Of course these characterisations are arguable unless we have an exact schematic presentation of induction and deduction. Nonetheless, we should note that committing to a certain number of schemas as the inductive schemas is not necessary; what matters is to accept that there are such schemas. The following schemas are commonly accepted as examples of deductive and inductive inferences:

10. Deductive Inference: $\dfrac{\textit{all As are Bs} \qquad \textit{a is A}}{\textit{a is B}}$

11. Inductive Inference: $\dfrac{\textit{a is A and is B too} \qquad \textit{b is A and is B too} \qquad \cdots}{\textit{all As are Bs}}$

If our ultimate concern is how certain we can be about the truth of a conclusion once we apply each of these inferences, then surely deductive and

[21] To be exact in substitution, we need to use index for α and β in each substitution.
[22] To be exact in substitution, we need to use index for α and β in each substitution.

inductive inferences are on a par. Since most of the time we are inductively justified in believing 'all As are Bs', so we consider this a true premise, and therefore the truth of 'a is B' will not be more certain than its grounds. Of course, if we understand 'truth preservation' as 'whenever we consider premises as true the conclusion is true too', then the justification of deduction and induction expressed in (b) will become (c).

c. Whenever/most of the time that we apply deductive/inductive inferences to true premises we arrive at a true conclusion.

If we understand the justification of deduction/induction as (c), then these two justifications have more similarity than difference. As a matter of fact, if we think of (c) as the justification of deduction/induction, then practically the possibility of being in a position to check the correctness of (c) is next to zero. Because, practically the possibility of being in a position to check the truth of 'all As are Bs' is next to zero. In the other words, if we understand the justification of deduction/induction as (c), then most likely we are seeking a direct justification of the first kind for deductive/inductive inferences. And we just have seen that such a justification leads to an inductive inference eventually. It is therefore not difficult to see why the justification problem, as articulated in (c), makes justification of both deduction and induction more similar than different.

However, this is not the only way of understanding (b). If we accept inferential (indirect) justification for inferential rules, and we accept that objects of inferential (indirect) justification in the case of inferential rules are inference schemas, then there is an important difference between the schemas expressed in (10) and (11). What we need to see this difference is that the meaning of some words in a sentence has some constant characteristics that does not change in different sentences. That is we need to accept a version of meaning molecularism. The difference between (10) and (11) is that (10) is not ampliative while (11) is. To accept that if we are in a position to know truth of premises in (10), then knowing truth of conclusion does not add to our knowledge, we are required to accept that 'all' has a sufficiently precise meaning on its own. And the very meaning of 'all' also makes the inference (11) ampliative.

It seems that there is a connection between what just has been said and the point Dummett tries to make by referring to suasive/explanatory distinction.[23] He argues that there is an asymmetry between deductive and inductive reasoning. While "in the case of induction, we appear to have a

[23][Dummett, 1978] Though the original paper is written in 1973.

quite unconvincing argument that there could not in principle be a justification, but we lack any candidate for a justification,"[24] in the case of deduction "... we have excellent candidates, in the soundness and completeness proofs, for arguments justifying particular logical systems ..."[25] and the reason that soundness and completeness proofs have a justificatory role for deductive inferences despite circularity is that "our problem is not to persuade anyone, not even ourselves, to employ deductive arguments: it is to find a satisfactory explanation of the role of such arguments in our use of language ... Such an argument will, of course, be deductive in character, but that will not rob it of its explanatory power ...".[26] According to these quotations, Dummett heavily rests on the assumption that justification of deduction at this level (soon we will see that Dummett considers three different levels of justification for deduction) has an explanatory role, while justification of induction has a suasive role. For he then argues that circularity does not have a negative effect on the explanatory power of a justification but destroys its suasive nature. Nonetheless, he does not give any sufficient reason for this assumption.

Why should we not need a suasive argument for deductive rules of inference? What are the unwritten grounds of Dummett's assumption? Haack nicely challenges the claimed asymmetry between deductive and inductive inferences. It is the main topic of her 'Justification of Deduction',[27] and what we have done in the present chapter is actually a reflection on her reasonable scepticism towards Dummett's assumption. This reflection has been along Dummett's line of thought. We need to say more in order to complete this Dummettian response to Haack's scepticism. So far we have seen that the roots of Haack's scepticism, about the asymmetry of deductive and inductive inferences, are in scepticism about meaning molecularism, while taking truth as a primitive non-conventional notion. The remaining step is to show that Dummett's ground for assuming that deductive inference does not need suasive justification is his faith in meaning molecularism.

According to Dummett: "explanation often takes the form of constructing a deductive argument , the conclusion of which is a statement of the fact needing explanation: but, unlike what happens in a suasive argument, in an explanatory argument the epistemic direction may run counter to the direction of logical consequence."[28] Now if someone, like Haack, were to ask Dummett why he thinks one would accept Modus Ponens without being sceptical about this deductive inferential rule, then Dummett's answer might

[24]Ibid. p.295
[25]Ibid. p.295
[26]Ibid. p.296
[27][Haack, 1982]
[28][Dummett, 1978], p.296

be something in this line:

d. The meaning of logical words occurring in the schema alone is enough to make us sure Modus Ponens is a correct inference rule.

This answer also has a consequence for Lewis Carroll's puzzle. We do not need more than two premises A and $A \to B$ to conclude B. Or on the other formulation, if inferring β (conclusion) from α (major and minor premises) is based on the meaning of the logical words involved in the inference, then that creates enough ground for the inference. There is no need for more premises, or more premises do not change the facts about the inferability of β from α.

There are three problems with this answer, according to Haack. First of all it leaves the person sceptical of deductive inferences "in need of a suasive argument for deduction."[29] Why don't we need to be persuaded about deductive rules of inference? Haack doubts that "people do antecedently believe in the justifiability of deduction but not in the justifiability of induction".[30] Secondly, the same strategy is applicable about any commonly accepted idea. She thus believes that Dummett's strategy makes it "too easy to 'explain' any accepted belief ... it is indiscriminating".[31] And finally, reasonable interpretations of Dummett's position "quite casually and falsely assume that there is a unique practice, 'deduction', which we can somehow use in its own justification."[32]

In regard to the first problem, there is a noteworthy point in Dummett's distinction that he leaves unsaid. Although he makes a reasonable claim that circularity does not 'rub off' explanatory power of the justification, he does not explain why a deductive justification does not need a suasive justification. We could say more by relying on non-ampliative features of deductive schemas. A sceptic could say 'Modus Ponens is truth preserving' is as ampliative as 'there was a possum in the corridor last night', and stands in need of a suasive argument. However, if we accept what has been said about direct and indirect ways of justifying inference rules such as the Modus Ponens schema, then this schema does not represent an ampliative inferential move. Therefore there is no need to introduce a suasive argument to justify that inferential move. Later in this chapter we will examine the question whether deductive rules 'analytic' and 'non-ampliative' are the same claims.

To be able to examine Haack's second and third objections to the Dummettian approach, we need to explore Dummett's position on justification of deduction further. This is the topic of the following part.

[29] [Haack, 1982], p.220
[30] Ibid. p.220
[31] Ibid. p.220-221
[32] Ibid. p.221

1.2 Can Soundness And Completeness Serve as Justification of Deduction?

Dummett starts his 'Justification of Deduction' by mentioning a common practice among logicians: they run "two parallel notions of logical consequence, one syntactic and the other semantic, and then attempt to establish ... extensional equivalence [between them]."[33] The ultimate aim of soundness is to make sure that whenever a syntactic consequence relation holds there is a semantic explanation for its validity, or whenever there is a semantic explanation for invalidity of a consequence relation (the counterexample), there is no syntactic demonstration for it. When we have soundness we can be assured that the syntactic consequence relation is not a trivial one. That is, it does not hold between any random expressions in our language that are subject to truth or falsehood, or does not hold between any of them at all. And completeness assures us that our syntactic consequence relation is capable of demonstrating any semantic consequence relation; or else from a disproof we can find a counterexample. When we have an answer for each of the questions of soundness and completeness, then, from a logical point of view, we have settled the most important issues. Specifically, once we have soundness we can claim for the meaningfulness of our inference rules, and non-triviality of the consequence relation in our object language. This seems to be the reason why Dummett holds that soundness and completeness have a justificatory role for deductive reasoning. However, it should be admitted that the role of completeness is not so clear in this picture. If someone like Dummett accepts that soundness is a necessary condition for meaningfulness, then one proposal is that completeness makes us sure that the meaning in question (meaning of logical vocabulary) has been fully expressed; but given Dummett's meaning theoretic concerns, this proposal is not appealing. We shall become more familiar with his meaning theoretic concerns in this and the next chapter. What can be said firmly is that if one, like Dummett, holds (d) then anything that contributes to the meaningfulness of logical words has a justificatory role, if only partially.

Dummett then mentions two other approaches to the issue of justification of deduction according to those truth preserving is the ultimate justification of deduction. He attributes one of these views to philosophers: the view is that truth is a primitive and non-conventional notion and, as we mentioned in the previous part, semantic definitions lead to a conventional truth. Therefore, soundness and completeness, though important, are not ultimate justifications of deductive reasoning where the justification of deduction is

[33] [Dummett, 1978]

held to be as expressed in (b). Haack represents this point of view nicely when she writes: "Soundness and completeness proofs are ... results purely about [language] L, one might say, not about the relation of L to anything outside it ..."[34] or "Soundness and completeness proofs establish something important about internal cohesiveness of a logical system."[35]

The other approach that Dummett introduces is one in which "the syntactic notion of logical consequence is required for proving positive results ... [and] the semantic notion is required for negative results ..."[36] In this approach "in order to guarantee that a demonstration of semantic invalidity really does show that the argument in question cannot be reduced to the rules of inference we have taken as primitive, we require a soundness proof. In order to satisfy our selves that the semantics we have adopted is adequate in the sense that any form of argument not reducible to the primitive rules is semantically invalid in our sense, we need a completeness proof."[37] As can be seen, "the interest of the semantic notion would then lie entirely in its use to demonstrate failure of logical consequence [which has been defined syntactically]."[38] Again, if we think of justification of deduction as (b), then on this view, soundness and completeness justify inference rules trivially. This can be understood in two ways: in one sense, the syntactic notion of the consequence relation is not responsive to semantic notion of it. The second sense can be expressed as Haack has done: "... there are too many of them ... there are different logical systems, systems which their proponents take to be rivals of each other, each of which can be shown to be sound and complete."[39]

Dummett's diversion from the first point of view is that soundness and completeness play a partial role in justification of deductive inferences since they (specifically soundness) are part of the meaningfulness conditions for logical words. While if someone thinks of truth in its non-conventional sense and thinks of (b) as justification of deduction, then soundness and completeness have no justificatory role for them. This is because preserving a conventional truth does not say much about preserving a non-conventional one. This position can be attributed to Haack. However, it is not the only reason Haack challenges Dummett for considering a justificatory role for soundness and completeness. Her second, and even strongest, argument against the justificatory role of soundness and completeness is that proofs of soundness and completeness are not only circular, but can also justify wrong inferences.

[34][Haack, 1982], p.223
[35]Ibid. p.224
[36][Dummett, 1978], p.292
[37]Ibid. p.292
[38]Ibid. p.293
[39][Haack, 1982], p.224

And even if we accept that deductive arguments do not require suasive justification, and therefore accept the explanatory role of circular proofs, the real issue is that these proofs are "indiscriminating".[40] That is, not only they justify rival logics, but also justify incorrect inferential rules. We refer to this problem as 'the Bad Company' (BC) problem, and this brings us to the subject of the next section.

Dummett's objection to the second point of view is that meaning in the sense that he has in mind (that justifies inference schemas on his account) is a richer notion than what can be captured by merely semantic (truth preserving) concerns. In other words, simply meeting semantic requirements, truth preserving in this context, does not make a schema a suitable inference rules. This is because he thinks that the meaning of a logical word should also tell us about how to use it.[41] For example, considering a truth table for the conditional, there are different inferential moves that are safe, but only two of them are suitable for expressing the meaning of the conditional. He writes: "It is, indeed, open to argument, not merely whether, for example, the two-valued truth-tables give a correct account of certain sentential operators of natural language, but whether they constitute a legitimate form for the explanations of the meanings of any possible sentential operators whatever ... It is thus quite impossible that it should be an utter illusion that semantic accounts of the logical constants supply an explanation of their meanings, ...".[42] We will see more of Dummett's meaning theoretic ideas about inferential schemas in the second next section.

1.3 Rule Circularity And The Bad Company Problem

Now it is time to examine Haack's second point against the justificatory role of soundness and completeness proofs. This argument is more challenging than the first one(which argues circularity rubs off the justificatory power of an argument), since it requires neither any presupposition about truth nor any scepticism about meaning. She uses a style similar to how logicians introduce the semantics of the object language. Consider the truth table of the conditional (table I):

[40]Ibid. p.220-221

[41]His example in [Dummett, 1978], p.294-295 is Frege's point about 'and' and 'but' which have the same sense, given that truth tables are defining sense, but we use them in different ways, or, as Dummett says, they have different tones.

[42][Dummett, 1978], p.294-295

I.

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

And here are two inference schemas, one our famous Modus Ponens (MP) and the other Haack's infamous Modus Morons (MM):

$$\frac{A \quad A \to B}{B}\ \text{MP} \qquad\qquad \frac{B \quad A \to B}{A}\ \text{MM}$$

And here are two arguments, (e) and (f) to explain why MP and MM are correct inferences respectively; they both are based on the truth table of the conditional:

e. Suppose that A is true, and $A \to B$ is true. Then according to the truth table for $\to$, if A is true and $A \to B$ is true, then B is true. So B must be true.[43]

f. Suppose that B is true and $A \to B$ is true.Then according to the truth table for $\to$, if A is true, then B is true and $A \to B$ is true. So A must be true.[44]

Argument (e) is according to MP when A *is true, and* $A \to B$ *is true* is replaced with α, and B *is true* is replaced with β. Also (f) is according to MM when A *is true* is replaced with α, and B *is true and* $A \to B$ *is true* is replaced with β.

$$\frac{\alpha \quad \alpha \to \beta}{\beta}\ (e)\ \text{schema} \qquad\qquad \frac{\beta \quad \alpha \to \beta}{\alpha}\ (f)\ \text{schema}$$

If (f) looks somewhat odd,[45] then (g) is an alternative for it. It also has the form of MM when we replace *if* B *then* $A \to B$ with β, and A with α.

g. According to truth table of $\to$, if B then $A \to B$. Now if A then (still) if B then $A \to B$. Therefore, A.

In all cases the reasoning for justifying or explaining an inferential schema has the very form of the schema it is supposed to justify. It is usually referred to as rule circularity and arguments like (e) and (f) are rule circular arguments. They are called so to distinguish them from circular arguments like (h).

[43] [Haack, 1976], p.119

[44] Ibid. p.120, with a bit of rewording.

[45] The second sentence in (f) can be understood better in this version: Then according to the truth table for $\to$, under the condition mentioned in the first sentence, if A is true and $A \to B$ is true, then B is true.

h. There was a possum in the corridor last night, and some foot prints, and a massy garbage bin, and some waste of a certain shape and size, therefore there was a possum in the corridor last night.

There are different ways to indicate this difference; prima facie, the difference is between believing in the truth of a claim that is expressed by a proposition, and believing in the truth of a claim that acting according to a set of rules will lead to a specific result. In the first case, if we presume the intended conclusion, then it is easy to conclude it. As can be seen, unlike (e) and (f), in (h) the intended conclusion is among the premises and this makes the whole argument (h) pointless. So if this sort of triviality does not hold about rule circularity then there is a genuine difference between the kind of circularity occurring in (h) and rule circularity. However, how to check this point about rules is a tricky question.

In contrast to (h), Dummett raises a case in which what is accepted as a logical truth in the logic of the meta-theory does not have demonstration in the object language.[46] He considers a case in which we accept that every proposition is either true or false in the metatheory, while we have no rule for introducing or eliminating negation as such. What we have are rules for dealing with negated complex expressions with the other logical connectives, but not for expressions with negation alone. That is, we do not deal with negation as a primitive logical word. In this situation, although the law of excluded middle is a truth of the metatheory, it is not a theorem of our theory. For instance, if we use tableaux method, the negated excluded middle does not lead to close branches, although the closure rule is that a branch is closed when an expression and its negation both appear in it. In the other words, once the logical apparatus of object language is not complete, that is, cannot demonstrate all the truths of metalanguage, then it is not guaranteed that what is true in metalanguage can be demonstrated in object language. And if we show that we can demonstrate every truth of metalanguage in object language, then we have shown "something".[47]

Although Dummett points to a real distinction between the kind of circularity that he call's 'gross circularity' (the case in (h)) and the kind of circularity that he call's 'pragmatic circularity' (and which we refer to as 'Rule Circularity'(RC)), it does not address Haack's objection correctly. To be able to show that there is a genuine difference between circularity and rule circularity, we need to check whether every schematic rule can be justified by presuming the very rule schema in question or not. And here justification of rule schemas is being truth preserving. For an inferential rule R, it mainly

[46][Dummett, 1991b], p.203
[47]Ibid. p.203

includes an argument that starts by assuming the premises of R as true and ends by showing the truth of the conclusion of R. This argument should use the inference rule R, but there is no requirement to use only R. If we show 'A is true' as $T(A)$, then in case of MM the argument would be like (12).

12. $$\frac{T(B) \qquad T(A \to B)}{T(A)}\ \text{MM}$$

And here is one way to demonstrate how accepting the original MM as a correct inference leads proving (12).

13. $$\dfrac{\dfrac{\dfrac{T(B)}{B} \qquad \dfrac{T(A \to B)}{A \to B}}{A}\ \text{MM}}{T(A)}$$

The only rules used in (13) except MM are (14) and (15) known as Disquotation and Semantic Ascent respectively.[48]

14. $$\frac{T(\phi)}{\phi}$$

15. $$\frac{\phi}{T(\phi)}$$

To summarise, it seems that given any connective #, if we accept (14) and (15) in their general sense, where ϕ can be substituted by any expression regardless of its complexity, then any arbitrary inference rule can be justified using the very inference rule in question and (14) and (15). In this sense, rule circularity is as trivial as what Dummett calls gross circularity. This means that the ultimate grounds of inference rules, if there are such things, should be looked for somewhere outside of schemas themselves. Does it mean that schemas play no role in their own justification? The answer is 'no' for some logicians including Dummett. They think there are some meaning theoretic concerns that can help us to distinguish good inferential rules from a number of possible rules.

[48] I have taken those from [Tennant, 2005].

1.4 Meaning Theoretic Concerns as a Guide to Defining The Form of Inference Rules

When we think about meaning it is important to define what we mean by 'meaning theoretic concern'. These concerns are not just semantic concerns. That is, they are not just truth-related concerns, they cover broader issues. For instance, the question 'is there any correct or incorrect way of introducing an element to a language?' indicates a more general concern than truth, although there might be a truth theoretic answer to it as well.

Therefore, one of the questions we need to address, when we aim to find a solution for the BC problem, is whether there is any semantic factor that helps us to rule out bad inference rules. One attempt to do so can be seen in Boghossian's rejection of Tonk rules.[49] Tonk has been introduced to the literature of philosophy of logic by Arthur Prior. In his classical two page paper, *The runabout inference ticket*,[50] Prior introduced a constant called 'Tonk' with the following introduction and elimination rules to bankrupt the idea that introduction and elimination rules are enough to define the meaning of a logical constant:

16. $$\frac{A}{ATonkB}\ TonkI \qquad \frac{ATonkB}{B}\ TonkE$$

Boghossian formulates an argument to justify 'Tonk introduction' as follows.[51]

17. $$\dfrac{\dfrac{\dfrac{\dfrac{p}{T(p)}}{T(p)TonkT(q)} \quad (T(p)TonkT(q)) \longleftrightarrow (T(pTonkq))}{T(pTonkq)}}{pTonkq}$$

Where p and q are atomic sentences and the biconditional is a meaning postulate. His strategy to reject (17) is attempting to reject the meaning postulate by arguing that Tonk rules (introduction and elimination) while giving Tonk a conceptual role, do not give it meaning. This is because the composite expression, formed by Tonk as connective, does not have truth conditions. The starting point is granting that "we aim to have true beliefs and that we attempt to satisfy that aim by having justified beliefs".[52] Then,

[49] [Boghossian, 2001]
[50] [Prior, 1967]
[51] A critique of Boghossians's argument can be found in [Tennant, 2005].
[52] [Boghossian, 2001], p.1

by accepting RC, he tries to answer this question: how can the rule circularity of an argument warrant its own conclusion?[53] Here are conditions under which an argument transmits warrant from its premise p to its conclusion q for subject S ideally, according to Boghossian:

i. S is justified in believing that p.

j. p and q are logically related in such a way that if p is true, that is a good reason for supposing that q is (at least likely to be) true.

k. S knows, or is justified in believing that the logical relation between p and q is as specified in (j).

l. S infers q from p because of his belief specified in (k).

To maintain rule circular warrant transition, he rejects the necessity of (k) for warrant transition. The reason is that, if we had (k), then there would not be any problem such as RC in justifying logical rules. For RC happens when we want to justify believing in the logical relation between p and q. Now if (k) is not necessary for transmitting warrant, then we need a link from external condition (j) to the subject S. That is, to relate (j) to what S knows. We need a link here because as Boghossian correctly mentions, the mere fact that a particular inference is truth-preserving bears no link to the thinker's entitlement to it. The next step is making sense of "the idea that a thinker is entitled to reason in a particular way, without this involving-incoherently- that the thinker knows something [here that the rule is truth-preserving] about the rule involved in his reasoning".[54] His suggestion for this sense-making step is:

m. "Our logical words ... mean what they do by virtue of their inferential role ... in some inferences and not in others".[55]

To have a good understanding of what (m) means, we need to take a brief but somewhat closer look at the inferential approach to semantics. Inferential (conceptual) Role Semantics (IRS) is usually considered either as a completion for referential semantics or an alternative approach to it. The idea is that for IRS, unlike referential semantics, the meaning of an expression is determined according to the role of that expression in thinking and not merely what it refers to. The other way of articulating the difference between referential semantics and IRS is that, in the former, the meaning

[53]Ibid. p.12
[54]Ibid. p.28
[55]Ibid. p.28

of an expression is the matter of what it denotes to or what it stands for, while in the latter it is partially the matter of use, or in some readings of IRS, solely the matter of use.

Since the IRS is an idea about meaning in general, we need to narrow this idea down to the case of logical words. Here, in applying IRS to logical words, there is a difference between thinking of meaning as partially the matter of use or wholly the matter of use. Logical words usually are used in the process of exchanging arguments, so it would be appropriate to think of their role in arguments. If we grant that the role of logical words in an argument is determined by inferential rules, such as Introduction (I) and Elimination (E) rules, then it can be said that the meanings of logical words are determined, at least partially, by specific inferential rules.

If we accept (m), then to accept transferring warrant from premises to the conclusion, where inference rules are supposed to preserve truth, we need to explain this coincidence (that logical words mean what they do, and inferential rules are truth preserving). Boghossian responds to this need by putting forward the view that having truth conditions is a precondition of having meaning. He holds that a single or compound proposition needs to be 'thinkable' to have meaning.[56] So $A\#B$ should be thinkable if we want to attribute meaning to it; where '#' is a candidate for being a logical connector. According to him, a single or compound proposition is thinkable if there is "a way the world is when the proposition [expression] is true".[57]

Boghossian therefore thinks of this inferential role as partially constituting the meaning of logical words.[58] Truth conditions also matter in determining the meaning of these words. His account of truth conditions for a complex sentence with Tonk as principal operator suggests truth functionality. He writes: "But we can readily see that there can be no consistent assignment of truth value to sentences of the form 'A Tonk B' given the introduction and elimination rules for Tonk".[59] So his argument is that the meaning postulate used in (17) is not legitimate since Tonk has no meaning. This argument will be convincing if one more detailed question, expressed in (n), has a positive answer. Given that the meaning postulate in (17) actually postulates distribution of truth predicate over Tonk and what Boghossian actually shows is that the connective Tonk is not truth functional:

n. Is truth functionality a necessary condition for distribution of truth

[56]Ibid. p.32

[57]Ibid. p.32

[58]He has a compelling reason for denying that 'every conceptual role determines a meaning' in [Boghossian, 2001], p.33. To see an account of how inferential role fully determines meaning once meaning is truth conditions look at [Peregrin, 2006].

[59][Boghossian, 2001], p.32

over a connective?

The development of Possible World semantics suggests that the answer to this question is 'no'. However, Boghossian might also say that by 'consistent assignment of truth' he does not mean truth functionality, and that Possible Worlds semantics also can maintain another way of consistently assigning truth.

To sum up Boghossian's meaning theoretic argument, he has done a good job of explaining the epistemic grounds of what we may call the 'self-justifying' nature of deductive inference rules. However, in a crucial part of his argument, expressed in (m), he talks about a specific fragment of inferences, but says not much about features of that fragment of inference rules that defines the meaning of logical words.

There are, however, more detailed discussions of the distinctive features of meaning constitutive inferential rules in Dummett's writings. Dummett holds that once we come across challenges similar to the one that Haack puts forward, we are engaged in a different level of justification. According to him, there are three levels of justification for deduction:[60] in the first level, we appeal to some rules to justify the others. For instance we appeal to some basic laws of inference such as MP to justify some more complex logical theorems. At the second level we appeal to soundness and completeness to show that our logical systems are not doing something trivial (soundness)given the premises. Actually they separate a fragment of formulas from all possible formulas as consequence. And also that we can rely upon them to fully spell out what we mean by using our language, because there is nothing that is implied by our semantic system which is not demonstrable in our formal system (completeness). To respond to the challenges of the third level, he thinks, we need to think about the conditions of the possibility of deduction. That is, we need to think how it happens that we use deductive reasoning successfully. This leads us to think of meaning not merely as truth conditions, but also as a set of broader concerns about using language. Here are his main points:[61]

o. To explain the fact that we learn bits of language and we can then use these bits to make new fragments of language, we need to accept there are contents of terms that remain unchanged. On the other hand establishing new connections between different bits of language might change their meaning.

[60] [Dummett, 1978], p.297 and [Dummett, 1991b], chapters 9 and 10.

[61] Can be seen in [Dummett, 1978], p.302-303 and in [Dummett, 1991b], chapter 8.

For instance, considering a language, without logical words, which includes just sentences, if three people (or three theories) A, B, and C use a fragment L of language in the following ways (where turnstile roughly means implication):

18. $L : p, q, r, s \qquad A : p \vdash q \qquad B : p \vdash q,\ r \vdash q \qquad C : p \vdash q,\ p \vdash r$

then it seems that A and B are using q and r in different ways. Or that these sentences mean different things to them. A and C also are using p and r in different senses. And B and C use p, q, and r in different meanings. Now if A expands her language to L' with the following connections between sentences:

19. $L' : p, q, r, s, t \qquad A : p \vdash q,\ q \vdash t,\ t \vdash r$

then introducing the new sentence t has changed the meanings of p, q, and r for the same reason mentioned to explain (18). In the logical terminology L' is a non-conservative extension of L since there was no inference from p to r for the person A before introducing t. While now if we stick the inferences in (19) together, there is one inference from p to r. On the other hand, in regard to (18), if we think of q as a sentence that expresses, say, a mathematical thought, which has a proof from, say p, once we have another proof for it from r, do we have a proof for the same thought or do we have a proof for a different thought, say q'? Common sense says that we have two proofs for the same thought. It can be said that q has two different meanings for the person (or in theory) A to the person (or in theory) B. However, q needs to have the same meaning for the same person (within the same theory) unless a new sentence, like t connects some of the other sentences and creates a new meaning for some old expressions for person (or theory) A. The idea is how, if possible at all, to make sure that introducing logical words to a language do not change the meaning of non-logical fraction of a language such as L.

p. If we think of logical words as a fragment of language and of inferential rules as introduction and elimination of these words to language, then by appeal to the concept of harmony between introduction rules as conditions for assertability or entitlement on the one hand, and elimination rules as consequences of asserting on the other, we come to possible ways to explain conservative expansion of a language.

There are many points in regard to (p) that we shall probe in detail in the next chapter. The intuition here is that harmonious inference rules help us to save the meaning of non-logical words where saving meaning is understood as conservative extension. Dummett here is inspired by a technical notion

of normalized proof. The idea is that if rules are harmonious in the sense mentioned in (p) and they are used in the correct order, then any proof made by using inferential rules no matter how long, will be harmonious. And if harmony means saying no more than what we are entitled to say, then we have expanded our language while saving the meanings of the existing non-logical expressions. Moreover, if deductive inferences are these harmonious introduction and elimination rules, then deduction just helps us to expand our language without committing us to any additional bit of information. If this is the case, then it is a further evidence for the thought that deductive rules do not need suasive justification since they do not add to our knowledge in the way that 'there was a possum in the corridor last night' does.

Holding this idea in addition to the undeniable informativeness of at least some deductive reasonings such as mathematical proofs, leads Dummett to admit there is a tension between the legitimacy and usefulness of deduction. There is evidence to doubt the connection between the technical notion of normal proof and conservative extension of language (we shall consider this evidence in the next chapter). However, Dummett's position is an interesting angle to look at the epistemic function of mathematical proofs. This will be the main concern of the last chapter. Let us get back to our business in this chapter, the concerns about learning segments of language and using them to build new segments. This is usually referred to as 'Compositionality'. Now we need to consider these general linguistic concerns in the particular context of logical vocabulary.

q. For meaning theoretic concerns, chiefly compositionality, introduction rules that are single-ended, simple, and pure are self-justifying rules. Elimination (E) rules are justified once they are in harmony with Introduction (I) rules.

This is where meaning theoretic concerns have formal interpretations. The first implication of (q) to Haack's MM challenge is that to be able to judge this inference rule, we need to know how her logical word has been introduced to the context. That is, we cannot make any judgement about the appropriateness of an inferential E rule without considering its corresponding I rule. We can also find an answer to the question raised by reviewing Boghossian, namely, what are the characteristics of meaning constitutive inference rules.

Let us start by introducing the key terms in (q). Given that the logical word # will be introduced to a context by an inference with a conclusion that # is its principle operator and it will be eliminated from a context once it appears in the major premise of an inference:

r. A rule is single ended if it is an I rule but not an E one (of possibly

different logical words) or an E rule but not an I one.[62]

As an example, the rule schema (20) is not single ended, because we have conditional both in the premises and the conclusion.

20. $\dfrac{A \to B \quad B \to C}{A \to C}$

s. A rule schema is pure when only one logical word appears in it.[63]

As an instance, Modus Tollendo Ponens (21) is not a pure rule, because we have two logical words appearing in the schema.

21. $\dfrac{A \to B \quad \neg B}{\neg A}$

t. A rule schema is simple if every logical word appearing in it occurs as principal operator.[64]

For instance, rule (22), known as double negation elimination is not simple, because the nested negation is not the principal operator.

22. $\dfrac{\neg\neg A}{A}$

Of course none of these conditions are essential on their own. What makes them important is "minimal demand" of the "principle of compositionality" which is: application of a rule should lead to a higher logical complexity.[65] This demand is called the 'complexity condition'.

An example may help us to understand what Dummett means by harmony here.[66] Consider a language L with connections between atomic sentences (Dummett calls these connections as 'boundary rules')[67] as mentioned in (23). Adding each of the logical words individually, except Tonk, with I and E rules as is mentioned in (24) leads to a conservative extension of L, given that the conclusion lines in the inference rules of (24) have the same behaviour as the turnstile in (23).

23. $L' : p, q, r, s, t \qquad A : p \vdash r,\ q \vdash r,\ r \vdash s,\ p, q \vdash t$

[62] [Dummett, 1991b], p.256
[63] Ibid. p.257
[64] Ibid. p.257
[65] Ibid. p.258
[66] Ibid. p.250
[67] Ibid. p.254

24.

$$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A \to B}\ {\to}I \qquad \dfrac{A \to B \quad A}{B}\ {\to}E \qquad \dfrac{A \quad B}{A \wedge B}\ {\wedge}I \qquad \dfrac{A \wedge B}{A}\ {\wedge}E \ \dfrac{A \wedge B}{B}\ {\wedge}E$$

$$\dfrac{A}{A \vee B}\ {\vee}I \ \dfrac{B}{A \vee B}\ {\vee}I \qquad \dfrac{A \vee B \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}\ {\vee}E \qquad \dfrac{A}{A\mathrm{Tonk}B}\ Tonk\mathrm{I} \ \dfrac{A\mathrm{Tonk}B}{B}\ Tonk\mathrm{E}$$

For instance, it does not matter which one of these connectives we are adding to L (except Tonk), we can never start a proof from p alone and end up with t. Or start from r and end up with s. However, by adding Tonk we make the consequence relation a trivial one. We can start from any proposition in L and conclude any other one. What this approach would say about MM is that for (25) to be a good E rule, its I rule should be something like (26).

25. $\dfrac{A\#B \quad A}{B}\ \#E$

26. $\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{A\#B}\ \#I$

There are serious concerns about the ideas expressed in (q) and (p). In (q), the idea of harmonious rules of inference is connected to conservative extension. If we are interested in meaning preservation as a justification for deductive inferences, then conservative extension is the notion that gives us what we want. Nonetheless, the connection between harmony and conservative extension depends on how we understand harmony. There is a natural reason to think of I rules as self-justifying rules, in line with Boghossian's argument. We give the meaning of a logical word by the way of introducing it to a context which has not had it before. However, are introduction rules enough for fixing the meaning of all ordinary logical words? There are three specific cases to doubt that I rules can give us all we need to know about how to use logical words: disjunction, the universal quantifier, and negation. Here are I rules for these three logical words as they appear in Gentzen's *Investigations into Logical Deduction*:[68]

[68]There are two reasons to use Gentzen's original formalization[Gentzen, 1964]. First of all, it helps us to follow the development of the idea from the beginning by following how these rules have been changed, and secondly, Dummett has been inspired by Gentzen in his understanding of justificatory role on I rules.

27.

$$\frac{A}{A \vee B}\ \vee I \quad \frac{B}{A \vee B}\ \vee I \qquad \frac{Fa}{\forall x Fx}\ \forall I \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\ \neg I$$

In the case of disjunction, although there are ways to explain how an I rule can justify the relevant E rule (28),[69] Dummett's Fundamental Assumption (FA) is not a good explanation for it. According to FA, whenever we are entitled to assert a complex statement, we could have arrived at it by means of an argument terminating with the I rule governing the principal operator.[70] Dummett, himself is aware of this point and is not very happy with the status of FA in the case of disjunction.[71] As a matter of fact, there is no point in disjunction I rules but justifying relevant E rules, since whenever we are entitled to assert A or assert B, asserting $A \vee B$ is pointless, and we usually assert a disjunction only when we are not entitled to assert A or assert B. In this case, I rules justify the useful related E rule.

28. $$\frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ \vee E$$

One reasonable answer to this concern (that I rules are not enough to fix the meaning of the logical word) in the case of universal quantifier, is that the I rule for it is not suitable to express the meaning of the universal quantifier. It should be introduced like (29), where the side condition to introduce the universal quantifier is that y must not occur free either in $\forall x Ax$ or in any assumption $A(x/y)$ is relied upon.[72]

29. $$\frac{A(x/y)}{\forall x Ax}\ \forall I$$

The situation is further complicated when we come to negation: there is no I rule that fits Dummett's criteria. It is true that the I rule for negation in (27) has no negation in the upper side of the conclusion line, but applying the E rule for negation in Gentzen's account will make it appear in the upper side of the negation I rule. Here is Gentzen's rule:

[69] For a more detailed account of different approaches to justification of disjunction take a look at [Ripley,]

[70] [Dummett, 1991b], p.257

[71] Ibid. p.265-272

[72] [Negri and Plato, 2001]

30. $$\frac{A \quad \neg A}{\bot}\ \neg E$$

And the negation I rule might be stated in this way:

31. $$\frac{\dfrac{B \quad \begin{array}{c} A \\ \vdots \\ \neg B \end{array}}{\bot}}{\neg A}$$

It is true that (30) is an E rule for negation and not an I rule for $\bot$, that is, we may come across $\bot$ not via (30) but via a false equation like 5=4 . However, when we come to $\bot$ in a way like the one is shown in (31), then the negation has already appeared in the upper side of the conclusion line. In the case of the E rule for negation, the rule expressed in (32) looks simpler, and closer to the Dummettian virtue of defining the meaning of logical words directly by rules and not with the assistance of semantic theory.[73]

32. $$\frac{\begin{array}{c} [\neg A] \\ \vdots \\ \bot \end{array}}{A}$$

The problem with these rules is that almost anywhere outside the realm of mathematics, $\bot$ has a logical nature. That is, usually some process like (31) happens once we are not dealing with mathematical equations. And if we do not want to use $\bot$, then our I rule is not going to be single ended. Yet more importantly, it will not meet the complexity condition, which is the very reason that I rules are supposed on the above account to be self-justifying. A question that naturally arises is whether the formal points in (q) are the only ways of applying meaning theoretic concerns about the self-justifying nature of logical inference laws. A negative answer to this question will reduce the shininess of Dummett's account of the justification of deduction. Also it casts doubt on claims about the analyticity of deductive rules of inference when we understand analyticity in terms of correctness in virtue of meaning of the logical words.

Now we come to some final remarks about harmony and conservative extension. Harmony can be understood as one way of explaining the non-ampliative nature of deductive inferences. This is why philosophers such as Dummett[74] and Tennant[75] try to define the notion of disharmony by strong

[73] [Dummett, 1991b], p.299
[74] Ibid. chapter 11.
[75] [Tennant, 2013]

and weak inference. Part of the literature around harmony is an attempt to answer the question whether an E rule must be as strong as an I rule in order to be in harmony with it. Or the red line is 'not being stronger' than the I rule, so that a weaker elimination rule may still be in harmony with the corresponding I rule. We will examine what weak and strong might mean in the next chapter.

Another avenue to define the non-ampliative nature of deductive reasoning is the idea of conservative extension. Given a body of information expressed in a set of atomic sentences, with possible boundary connections among them, introducing logical words to the language will not change the status of the information. We also mentioned the importance of conservative extension for the claim that logical reasoning preservers meaning. These two points alone are suffient to make conservative extension an interesting topic.

Therefore, the relation between conservativeness and harmony is a genuine concern. Does harmony lead to conservative extension and visa-versa?

u. harmony $\leftrightarrows$ conservative extension

We have seen that the requirement of conservativeness rules out Tonk as a logical word. Does it also rule out an important logical word such as negation? Logicians have of course suggested examples of harmonious rules for a higher-order notion like truth, which does not lead to a conservative extension.[76] One example is adding (14) and (15) (introducing and eliminating truth as a predicate) to arithmetics, which leads to Godel's sentence that was not derivable before introducing the truth predicate. There is also an example of the introduction of classical negation, for which the I rule (the principle of double negation) does not meet Dummett's criteria. This in turn leads to a conservative extension once added to a language of atomic propositions together with disjunction and conjunction, while leading to a non-conservative extension added to the same language with conditional.[77] This happens while disjunction, conjunction, and the conditional, all have harmonious rules that satisfy Dummettian criteria. These examples are related to the connection between harmony and conservative extension in (u). The question regarding the contrary direction is similarly interesting, namely, is any conservative extension possible in the absence of harmonious rules?

The next chapter is dedicated to examining the mentioned account of harmony and conservative extension as well as alternative understandings of them. We also consider how these alternatives can serve to explain analyticity and the non-ampliative nature of deductive reasoning.

[76] [Read, 2000], p.127 though he mentions that he has token it from Prawitz.
[77] [Steinberger, 2011a]p.625 footnote no.15

Chapter 2

Preserving Meaning as Justification of Deduction

The main focus in this chapter is on evaluating the implications of meaning theoretic concerns with regard to choosing deductive rules in particular and resolving logical disputes in general. The previous search for the justification of deduction ended up with Dummett's reading of the implications of these meaning theoretic concerns regarding logical debates. In the more recent literature, the debate has been developed between opponents and proponents of the claim that intuitionistic logic is preferable over classical logic as long as we take meaning theoretic concerns into consideration. By the end of this chapter, we will see that meaning theoretic concerns would do not much in favour of intuitionistic logic in the dispute among logicians over rival logics.

We shall start with examining the connection between harmony and conservative extension in the light of issues raised by Stephen Read.[1] Following these points shall take us to compare Natural Deduction (ND) and Sequent Calculus (SC) as mediums of dealing with meaning theoretic issues. Consequently, debates over single or multiple conclusions shall become a matter of concern since it is this which distinguishes Intuitionistic logic from classical logic in the context of SC. And finally, an alternative reading of meaning preservation, namely the bilateralistic approach, will be probed. This shall be done via considering Greg Restall's[2] and Ian Rumfitt's[3] ideas about bilateralism.

[1] [Read, 2000]
[2] [Restall, 2005]
[3] [Rumfitt, 2000], [Rumfitt, 2015]

2.1 Harmony and Conservative Extension

There are concerns about Dummett's claim that what he later calls intrinsic harmony implies conservative extension,[4] since to connect intrinsic harmony to conservative extension, Dummett has to make a number of conceptual shifts as shown in (a).

a. Intrinsic harmony $\rightarrow$ total harmony $\rightarrow$ conservative extension

And each of these conceptual shifts has their own critiques. Dummett interprets the notion of intrinsic harmony as normalisability in the context of proof theory. Normalization has a specific function; it is supposed to serve as a support for Gentzen's thesis about normal proofs. According to this thesis, which is called Hauptsatz, in a normal proof "No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result".[5] This is a claim about possibly several inference steps. Gentzen used SC to support his claim although he had started with formalizing simpler steps of inferences in proofs in ND, where he introduced I and E rules and had mentioned that "The introductions represent ... the "definitions" of the symbols concerned and the eliminations are no more, in the final analysis, than the consequences of these definitions."[6]

Later, Dag Prawitz attempted to give an account of Gentzen's idea that in ND, I rules fix the meaning of logical words and E rules are a kind of consequence of them, by introducing the Inversion Principle (IP). Here is his definition of IP: "Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premises of α, when combined with deductions of the minor premises of α (if any), already 'contain' a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α".[7] This idea can be formalized in ND (in a specific case of a connective) as (1) where upper case letters stand for formulas, possibly atomic, and indexed πs stand for proofs. And (2) is the representation of the same idea in SC where upper case letters stand for formulas, possibly atomic, and Greek capital letters stand for multisets (sets with indexes for repeated formulas) of formulas. These multisets can be understood as undischarged leaves of a proof tree in ND.

[4] [Dummett, 1991b], p.250
[5] [Gentzen, 1964], p.289
[6] Ibid, p.295
[7] [Prawitz, 1965], p.33

1. $$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A\#B\end{array}\quad\begin{array}{c}\vdots\,\pi_2\\ \Delta\end{array}}{C}\,\#E \quad\rightsquigarrow\quad \begin{array}{c}\vdots\,\pi_1\\ \vdots\,\pi_2\\ \overline{C}\end{array} \quad or \quad \begin{array}{c}\vdots\,\pi_2\\ \vdots\,\pi_1\\ \overline{C}\end{array}$$

2. $$\dfrac{\Gamma \vdash A\#B \qquad A\#B, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

In formalization (1), the minor premise (the right hand side above the elimination line) might be empty, like a conjunction elimination, or not, like a conditional elimination. In the case of a disjunction elimination, we even have more than one minor premise. Also except in the case of a conditional, 'deductions sufficient for deriving the major premises' of an argument (shown as π_1), come first. What happens in (1) is that a logical word, #, has been introduced and immediately removed from the context of a proof, or as (2) shows $A\#C$, which has been introduced in both branches, has just been omitted from the proof. This is perhaps what inspired Dummett to write "Normalizability implies that, for each logical constant #, the full language is a conservative extension of that obtained by omitting # from its vocabulary"[8] and this connects normalizability to conservative extension.

The notion of conservative extension was introduced in Belnap's well-known *Tonk, Plonk and Plink*[9] where he argues that 'the key solution' to the Tonk problem is that it does not match up with our 'antecedently given context of deducibility'.[10] The way that Belnap cashes out the mismatch of Tonk rules with the antecedently given context of deducibility is that adding Tonk to our context of deducibility will end up with a non-conservative extension of this context; where conservative extension has been defined as follows:

b. Deducibility statement $A_1, \cdots, A_n \vdash B$ not containing Tonk is not provable in the deducability context unless that statement is already provable in the absence of Tonk rules and axioms.

Now if we consider $A\#B$ in (2) as '$A\,Tonk\,B$', then it seems that Dummett should not be wrong in connecting these two notions (normalization and conservative extension). Of course to guarantee that in the upper left hand side of (2) we have reached at $A\#B$ in a proper way, Dummett employs a Fundamental Assumption (FA). The FA is that whenever we are entitled

[8] [Dummett, 1991b], p.250
[9] [Belnap, 1962]
[10] Ibid. p.131

to assert a complex statement, we could have arrived at it by means of an argument terminating with the I rule governing the principal operator.[11] Also in what we derive from $A\#B$ we need to be *stable*, that is, willing to assert a statement whenever its consequences are warranted.[12]

However, there is an important difference between (b) and (2) on the one hand and (1) on the other hand. In both (2) and (b) we are dealing with formulas, or sentences (Γ and Δ are undischarged assumptions), while in (1) πs are proofs or derivations. Considering (2), let us assume that we have a pair of harmonious I and E rules for #, is that enough to conservatively introduce and eliminate #? The answer to this question is negative if the existing context of deducibility behaves differently with and without $A\#B$. That is, the turnstile in the upper right hand side of (2) can behave differently from the turnstile in the upper left hand side. Dummett thinks that if every other I and E rules involved in the context completely determine the meaning of the relevant logical word, then the answer to the aforementioned question is Yes. He writes "[intrinsic] Harmony between logical rules cannot, in general, be demanded: it can be demanded only when those rules are held completely to determine the meanings of the logical constants."[13]

While accepting the very same thought about harmony, Stephen Read rejects the claim that harmony can prevent us from non-conservative extension.[14] Read's strategy involves accepting inversion, but rejecting Dummettian criteria for logical rules being self-justifying (purity, simplicity, and single-endedness). By accepting inversion as a logical requirement, he defines the Autonomy of logical constants. They are autonomous in the sense that their meanings are fully defined by their I rules,[15] without the need for Dummett's criteria that a rule to be self-justified. In other words, he gives a different understanding of self-justification of I rules. Then he rejects Dummett's criteria as too demanding. Also he accepts that "one is entitled to infer from a formula containing it [the logical constant] no more and no less than one can infer from grounds for its introduction. All indirect proofs reduce to direct proofs."[16] However, as a result of rejecting some of Dummett's meaning theoretic concerns such as purity, this process of inversion does not necessarily include decreasing the degree of complexity of maximum formula ($A\#C$).[17]

[11] [Dummett, 1991b], p.257
[12] Ibid. p.287
[13] Ibid. p.286
[14] [Read, 2000],[Read, 2010]
[15] [Read, 2000], p.131
[16] Ibid. p.131
[17] [Read, 2010], p.564

This is how Read gives meaning to Gentzen's idea about I rules: I rules define the meaning of logical words and E rules are defined based on the guide line quoted in the previous paragraph. These E rules are in general form. For instance he accepts the following rules to govern the use of negation (we have seen them in the previous chapter too, they are suggested by Gentzen[18] and justified by Prawitz)[19]:

$$3.\quad \dfrac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\,\neg I \qquad \dfrac{\bot}{D}\,\bot E$$

Where $\bot$, called as absurdity by Read, is treated as a constant for falsehood. In the last chapter we saw that if $\bot$ is not treated as a constant the introduction rule for negation would not be single ended, that is, it would not be an I rule only or an E rule only. Where a rule of inference is an I rule for a logical constant # if the principle connective of the conclusion is #, and it is an E rule once # is the principle connective of one of the premises.[20] This is because negation appears both in premises and the conclusion if $\bot$ is not treated as a constant. Even if $\bot$ is considered as a constant which connects zero formulas together,[21] by Dummett's criteria for I and E rules, rules in (3) still are not single ended because the rule is an E rule for $\bot$ ($\bot$ is the constant in the premises) and an I rule for the negation (negation is the principle logical constant in the conclusion).

One way to overcome this problem is to argue that the left hand side rule in (3) is not an E rule for $\bot$ because it is not a general way to eliminate $\bot$. The general elimination rule for $\bot$ is the right hand side rule in (3). Read justifies harmony between rules in (3) as follows: since there is no I rule for $\bot$ then its E rule is in harmony with the I rule vacuously.[22] There is one problem with this approach; locally speaking, the I rule in (3) is a specific case of the E rule. However in global level the I rule includes discharging A, while the E rule is a local rule and therefore silent about global affairs. So it is not quite right to say that the E rule is the general version of the I rule in (3). Although, if one accepts Read's suggestion that the right hand side

[18] [Gentzen, 1964]

[19] [Prawitz, 1973]

[20] [Dummett, 1991b], p.256

[21] One way to explain it is that absurdity can be a meaning absurdity such as 4=6. Of course this kind of absurdity can also be understood as connecting a number of statements together. For instance 4=6 can be understood as 'a is equal to b', in presence of 'a is smaller than c' and 'c is smaller than b'.

[22] [Read, 2000], p.139

rule in (3) is more general than the left hand side rule, then his point sounds appealing.

The upshot here is that absurdity or falsehood is a more general notion than negation and does not need I rule because it is a zero-place constant. Negation is defined according to falsehood and then there is a general E role for $\perp$.[23] And that is enough for meeting Inversion Principle requirements. Dummett's meaning theoretic concerns are not met in this explanation as they are evaluated unnecessary for the notion of harmony Read advocates. For instance, Read acknowledges that when we consider absurdity ($\perp$) as a constant, then the I rule for negation is not pure. That is, there is more than one constant playing a prominent role in this schema. This impurity is unavoidable when we define a logical constant, here negation, based on other notion, namely falsehood.

The rules for negation and absurdity give Read enough space to introduce what he calls the Proof-Conditional Liar where $\neg\bullet$ abbreviates $\bullet \rightarrow \perp$.[24]

4. $$\dfrac{\begin{array}{c}\vdots\\ \neg\bullet\end{array}}{\bullet}\,\bullet I \qquad \dfrac{[\bullet] \quad \begin{array}{c}\vdots\\ \neg\bullet\end{array}}{A}\,\bullet E$$

Adding $\bullet$ to the vocabulary amounts to triviality, in other words inconsistency; but Read does not believe that excluding inconsistency is the job of harmony.[25] He writes: "Perhaps in the end we would conclude $\cdots$ that $\bullet$ I and $\bullet$ E were not good rules to have. But in coming to that conclusion, we need to be able to express $\bullet$ I and $\bullet$ E and to understand their (harmonious) relationship."[26]

Moreover, Read argues that appeal to harmony is not going to rule out Tonk. His reason for this claim is that we can have a general E rule for Tonk which is in harmony with its I rule. Here are the usual rules for Tonk:

5. $$\dfrac{A}{A\,Tonk\,B}\,TonkI \qquad \dfrac{A\,Tonk\,B}{B}\,TonkE$$

Read introduces a new E rule for Tonk which is more general.

[23]We can have an E rule for negation, which is only one way of introducing falsehood.

[24]Formalization in (4) is not Read's, he uses SC system, (4) is a ND version of his rules. To see his rules check [Read, 2000], p.141

[25]Ibid. p.142

[26]Ibid. p.142

$$6.\quad \dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A\end{array}}{A\,Tonk\,B}\,TonkI \qquad \begin{array}{c}[A]\\ \vdots\,\pi_2\\ C\end{array}}{C}\,TonkE \quad \rightsquigarrow \quad \begin{array}{c}\vdots\,\pi_1\\ A\\ \vdots\,\pi_2\\ C\end{array}$$

The general rule in (6) meets the criterion of harmony introduced by Read, that is saying no more and no less than what can be said by the introduction rule. Although there might be a $ATonkB$ (otherwise known as the maximum formula) trapped in π_2 the degree of it lower than the $A\,Tonk\,B$ in the left hand schema. Therefore the maximum formula can be levelled, that is the proof is normalizable. So harmony in this sense, that is autonomy of the meaning of the logical words, does not rule out Tonk. Although Read is correct in saying that the possible Tonk formula in A has a lower rank than $A\,Tonk\,B$, it does not guarantee that the proof in the right hand side in (6) has been levelled as $A\,Tonk\,B$ can still appear in π_2.

Tonk is not the only constant which might not get levelled in the inversion process. Read emphasises that every inversion process does not involve decreasing the degree of complexity of the maximum formula in case of negation.[27] Negation is the logical word which its conversion to a normal proof does not decrease the degree of the maximum formula.[28] Therefore, if one wants to rule out Tonk as a logical connective because its inversion process does not result in levelling its maximum formula, then one should be prepared to rule out negation as well.

Although harmony, in a Readian sense, does not prevent inconsistency, it allows us to extend our language conservatively even by adding Tonk.[29] This is so because the E rule Read introduces for Tonk is different form the one given by Prior. As can be seen, Prior's E role for Tonk, in (5), can be a specific case of the general rule in (6) when B is derivable from A.

Similarly, Read argues that one cannot reject the double negation elimination rule as non-harmonious because it leads to non-conservative extension of the positive fragment of propositional logic. Read makes this point as a reflection on Dummett's accusation against the double negation elimination as E rule for negation in Classical ND (CND). Dummett mentions the fact that in CND without rules for negation(7) and (8) are not provable, while there is no negation in these formulas.[30]

[27][Read, 2010], p.564-565

[28]Ibid. p.569

[29]Inconsistency and conservative extension sometimes have been referred to as Theoretic and Systematic conservativeness. This is a way to address their differences, [Steinberger, 2011a] is an example of this literature.

[30][Dummett, 1991b], p.291-292

7. $\vdash (A \to (B \vee C)) \to ((A \to B) \vee (A \to C))$

8. $\vdash ((A \to B) \to A) \to A$

They have proof only with the assistance of rules for negation and the negation will disappear by the double negation elimination rule. However, as Read correctly mentions, considering this fact in isolation can be misleading.[31] The other fact is that (7) and (8) are provable in Classical Sequent Calculus (CSC). The reason is that in CSC conclusions can be multiple. That is, in the context of SC, adding a negation does not lead to a non-conservative extension. Considering these facts, Read believes that Dummett gives the wrong address when talking about the difference between Intuitionistic and Classical logic. The difference is not in two different negations but is in two different conditionals. He suggests (9) as an alternative I rule. Where ',' should be read as disjunction and X is a multiset (a set with index for repetition).

9. $$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B, X\end{array}}{A \to B, X} \to I'$$

Then he shows that having this new conditional, if we add Intuitionistic rules for negation, we will end up proving classical theorems. In response to the possible objection from anyone who insists on the Intuitionistic idea that to introduce $A \to B$ we need a proof from A to B, Read emphasises that in a debate between a Classical and an Intuitionist logician, this claim is a sort of question begging.[32] In other words, he pushes back against Dummett's suggestion in *the Logical Basis of Metaphysics* [33] that in considering purely meaning theoretic concerns, the disagreement between a Classical and an Intuitionist logician will be solved in favour of the Intuitionist since he believes none of Dummett's criteria do a favour to the Intuitionistic logician. CND is normalizable, its rules are harmonious in the sense that Read understands them, that is, general harmony. And the positive fragment of it (CND) will be conservatively extended once negation will be introduced.

In Read's idea, both Tonk and classical negation can be added to our logic conservatively when conservative is understood in the systematic level and not the theoretic level. That is we can have inconsistent sentences in our theory, but we cannot infer more than the grounds of introducing a complex

[31] [Read, 2000], p.151
[32] Ibid. p.145
[33] [Dummett, 1991b]

expression by eliminating its principal connector.[34] According to this line of thought, classical negation is saved from being non-conservative, as well as Tonk. Therefore we still cannot rule out Tonk as non-logical connective.

However, notwithstanding, Read's harmonious rules for negation and multiple conclusions, two key elements of Read's approach, are in tension with meaning molecularism, one of the really important notions for Dummett. It can be said that meaning molecularism does not have any logical force. And having meaning molecularism as a meaning concern is too demanding with regard to formal logic. Nonetheless, as we have seen in the previous chapter, there is evidence that Dummett's justification of deduction is heavily reliant on molecularism. This means that by giving it (molecularism) up his justification of deduction does not have the previous force.

Let us start with explaining why multiple conclusions and Read's rules for negation are in tension with meaning molecularism, at least as Dummett understands it. If we consider (10), the first relation from the left is the clearest relation between two atomic sentences or two molecules of language in the context of inferentialism, then by moving towards the right the ambiguity in the relation among atomic sentences increases respectively.

10. $p \vdash q \qquad p, r \vdash q \qquad p \vdash q, r \qquad p, r \vdash q, s$

In (10), the comma on the left hand side of the turnstile should be understood as 'putting together' and on the right hand side as 'branching'. Of course there is no general agreement about the second and third relations from the left; some logicians believe that the third connection is more ambiguous than the second[35] and some other logicians hold that there is no difference between the second and the third in terms of ambiguity.[36] The common sense view is that if a number of premises imply a single conclusion the relation among the sentences being involved in the inference is less ambiguous than them implying several conclusions. As a matter of fact, we usually call the first group of premises as 'conclusive' and not the second one. However, it does not mean that being conclusive is a logical demand, but at the very least, it can be said that the outcome of multiple conclusions is in tension with the idea of meaning molecularism in the sense pictured in (10). The advocates of no different between the second and third cases in (10), of course in terms of ambiguity in the relation among the sentences, argue that in case of moving from right to left the second relation is more ambiguous. The argument goes like this: consider a situation in which q is false, then it is not clear which one

[34]For a more detailed definition see[Steinberger, 2011a].
[35][Steinberger, 2011b]
[36][Restall, 2005]

(or probably both) of the premises are false, but if both of q and r are false we know that p is false. There is other evidence for such a tension; we shall consider it in a separate section dedicated to single and multiple conclusions.

Also the fact that there are no I and E rules for negation without the assistance of the other constant - in one reading only conditional and in the other both conditional and absurdity - means that negation cannot be defined independently of at least one other logical constant. Independent definability usually is referred to as Separability. This does not create any direct tension with meaning molecularism. However if this dependence on the meaning of another logical constant leads to non-conservative extension, then inseparability is indirectly in tension with meaning molecularism.

Of course as we have seen in the previous chapter, the intuitionistic rules for negation that Dummett endorses, do not meet the requirement of being single-ended in the literal sense. A degree of concession is needed for any rule of negation to be considered pure or single ended. That is, these two notions need to be interpreted in such a way that they fit the rules of negation. For instance, Milne defines purity as follows:[37]

c. A rule is pure if only one connective plays a distinguished role in it; it is simple if any connective playing a distinguished role occurs only as main connective in any formulas containing its distinguished occurrence.

Dummett allows impure, non-simple rules given that the justification of that rule does not become cyclic. He writes "An impure **c**-introduction rule will make the understanding of **c** depend on the prior understanding of the other logical constants figuring in the rule. Certainly we do not want such a relation of dependence to be cyclic; but there would be nothing objectionable in principle if we could so order the logical constants that the understanding of each depended only on the understanding of those preceding it in the ordering. Given such an ordering, we could not demand that each rule be simple, either. The introduction rules for **c** might individually provide for the derivation of sentences of different forms with **c** as principal operator, according to the other logical constants occurring in them: together they would provide for the derivation of any sentence with **c** as principal operator".[38] Now we need to check whether there is any connection between inseparability and non-conservative extension. In other words, we want to check if the fact that a logical word such as negation cannot be defined without the assistance of other logical words has anything to do with extending a language non-conservatively. If we accept what has been said in the last two paragraphs

[37] [Milne, 2002], p.522
[38] [Dummett, 1991b], p.257-258

about avoiding a literal reading of purity and simplicity and try to interpret them in the light of 'distinguished role' as in (c), then rules in accordance with compositionality are more useful for defining the distinguished role of each logical word. And ND proofs usually can be formed better in this regard while they are formed based on the SC rule. Before following the implications of molecularism for the Classic and Intuitionist dispute, if any, it would be useful to take a brief look at ND and SC as two formal systems and at their capacities and possible differences with regard to the above-mentioned debate.

2.2 Sequent Calculus (SC) and Natural Deduction (ND)

If we remain faithful to the common interpretation of Gentzen's intention to introduce two formal systems ND and SC,[39] then it should be said that SC is more capable than ND when it comes to the study of general features of deductive reasoning. Therefore it is not surprising if a number of logicians appeal to SC to judge the dispute over Classical and intuitionistic logic.[40] Two features of SC make it a good medium to discuss the above matter; Subformula property and Structural rules. Here are Gentzen's original SC rules for conditional where Greek capital letters stand for sequents of formulas and Latin capital letters for formulas:

11. $$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ R\to \qquad\qquad \frac{\Gamma \vdash A, \Delta \qquad \Theta, B \vdash \Lambda}{\Gamma, \Theta, A \to B \vdash \Delta, \Lambda}\ L\to$$

As can be seen, SC rules are in the form of introduction to the right and left hand side of the turnstile. This gives them the subformula property, that is, if we consider formulas above the conclusion line and under it, formulas which appear above the line are subformulas of the formulas which appear under the line. Moreover, inferences may have multiple conclusions.

Rules for introducing logical operators to the right and left hand side of the turnstile are called Operational Rules in SC. There are also rules of SC with no logical operators appearing in them; they are called Structural rules. Gentzen himself has used these rules, together with the Operational rules, to prove Hauptsatz, according to which every proof in ND can be normalized. Considering the fact that I and E rules in ND are in correspondence with the Operational rules in SC (I rules are very similar to right introduction rules

[39] An instance of this common interpretation can be seen in [Prawitz, 1974]

[40] Good examples of such attempts can be seen in [Read, 2000] and [Milne, 2002]

and E rules can be obtained by left introduction rules from basic sequents), what give the extra space to Gentzen to prove Hauptsatz are the Structural rules. A natural understanding of Structural rules is that they define general rules for combining proofs.[41] Here are Structural rules; their names are Thinning or Weakening (Wk), Counteraction (Con), Interchange (Int), and Cut. Except for Cut, the rest have right and left hand side versions:

12. $\dfrac{\Gamma \vdash \Theta}{\Gamma, A \vdash \Theta}\ LWk \quad \dfrac{\Gamma \vdash \Theta}{\Gamma \vdash A, \Theta}\ RWk \quad \dfrac{\Gamma, A, A \vdash \Theta}{\Gamma, A \vdash \Theta}\ LCon \quad \dfrac{\Gamma \vdash \Theta, B, B}{\Gamma \vdash \Theta, B}\ RCon$

$$\dfrac{\Gamma, A, B \vdash \Theta}{\Gamma, B, A \vdash \Theta}\ LInt \quad \dfrac{\Gamma \vdash \Theta, A, B}{\Gamma \vdash \Theta, B, A}\ RInt \quad \dfrac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Theta'}{\Gamma, \Delta \vdash \Theta, \Theta'}\ Cut$$

To be more exact, it can be said that these rules define the behaviour of the turnstile, or define the properties of deducibility.[42] Dropping one or some of these rules has an immediate effect on theorems provable in our logical apparatus. For instance, without RWk, (8) is not provable in the positive fragment of SC. However, this does not necessarily mean that the meaning of the conditional will change by dropping some of these Structural rules. If we accept that the meaning of the conditional is defined by (11), and rules in (12) define the behaviour of the turnstile (the context of deducibility), then by dropping some of the rules in (12) the extension of theorems changes. While the form of rules in (11) do not. This picture changes when it comes to ND; since in ND changes happen in our discharge policy for the conditional I rule. That is the way we use the conditional changes. Therefore, it is really difficult to reject a change of the meaning of the conditional in ND by a change of deducibility when we think of meaning as use.

The subformula property in SC makes it possible to use proofs in SC as a guide line to build proofs in ND.[43] Also we have mentioned that structural rules, specifically right structural rules give us a much stronger apparatus. Here is a proof for (8) in SC:

[41] Such a reading can be found in [Restall, 2014]

[42] I have come across this idea reading [Restall, 2005], and [Restall, 2010], especially the latter.

[43] A systematic attempt with such an approach can be seen in [Negri and Plato, 2001].

$$
13.\ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}\,RWk}{A \vdash B, A}\,RInt}{\vdash A \to B, A}\,R\to \qquad A \vdash A}{(A \to B) \to A \vdash A, A}\,L\to}{(A \to B) \to A \vdash A}\,RCon}{\vdash ((A \to B) \to A) \to A}\,R\to
$$

The proof is magnificently beautiful with many virtues we were seeking; there are only structural rules and rules for conditional. Two head branches are basic sequents getting more complicated towards the bottom. The ND version of proof of (8) is not as tidy though, here is one version:

$$
14.\ \dfrac{\dfrac{\dfrac{\dfrac{[\neg A]2 \qquad \dfrac{[(A \to B) \to A]3 \qquad \dfrac{\dfrac{[A]1 \quad [\neg A]2}{B}\,\bot E}{A \to B}\,\to I(1)}{A}\,\to E}{\bot}\,\neg E}{\neg\neg A}\,\neg I(2)}{A}\,DNE}{((A \to B) \to A) \to A}\,\to I(3)
$$

This proof exhibits several issues we have been following through the meaning theoretic concerns in the context of ND. In the proof (14), negation appears and disappears to help us with proving theorem (8) which has no negation in it. Moreover, it is not only I and E rules for negation, but also an elimination rule for $\bot$. Falsehood does not have I rule since (15) is not the only way to introduce falsehood; as we saw in the first chapter, a formula such as $4 = 5$ also can be an instance of falsehood as well.

$$
15.\ \dfrac{A \quad \neg A}{\bot}
$$

This is why the inferential move in the fifth line of (14) is a negation E rule. This very fact makes it rather hard to accept a definition of negation based on falsehood. The problem is not harmony, since there is no I rule any E rule is in harmony with it. And falsehood is a more general notion than contradiction. However, if we can have I and E rules for negation, then isn't it a more logical notion than falsehood (even though falsehood is more primitive)? If we want to use only I and E rules for negation, here is the resulting proof:

16.
$$
\dfrac{\dfrac{\dfrac{\dfrac{[\neg A]3 \qquad \dfrac{[(A \to B) \to A]4 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg B]1 \qquad \dfrac{[A]2}{\neg B \to A}\,{\to}I}{A}\,{\to}E \qquad \dfrac{[\neg B]1 \qquad \dfrac{[\neg A]3}{\neg B \to \neg A}\,{\to}I}{\neg A}\,{\to}E}{\neg\neg B}\,{\neg}I1}{B}\,{\neg}E}{A \to B}\,{\to}I2}{A}\,{\to}E}{A}\,{\neg}I3}{\neg\neg A}\,{\neg}E}{((A \to B) \to A) \to A}\,{\to}I4
$$

In (16), Double Negation Elimination (DNE) in (14) is considered as the classical E rule for negation. And vacuous conditional introductions have been applied for exactly the same technical purpose that falsehood elimination had been applied in (14); namely introducing B to the context of proof. In the SC version of the proof, that is proof (13), this bit has been done by the weakening on the right (RWk). The second proof (16) uses pure negation and conditional rules, at the expense of a bit longer proof, but we still have to use negation to prove a theorem with only conditionals as its connectors.

At least two reactions come to mind in regard to these proofs; one is that, the number of theorems in a logical system is not a good criterion to take or leave its rules. We take or leave some rules based on some concerns and it does not matter whether (8) is a theorem of the outcome system or not. The other reaction is that actually cases like (8) may change our understanding about the previous concerns. To evaluate these two positions we need to take a closer look at SC and ND as formal systems. After all, SC has useful properties like separability which serve some of our meaning theoretic concerns. So we might ask - can we just change ND rules by getting directions from SC, as Read did in his new I rule for conditional in (9)? Does this approach help us to make pure, single ended, simple rules in ND? If these goals can be achieved, then having multiple conclusions might be an affordable price for those with mainly meaning theoretic concerns.

In this regard there are two noteworthy points; first of all, there is one aspect that ND saves which disappears in SC. The major premise and minor premises distinction is not applicable in SC. The second point is that the notion of introduction changes from ND to SC. This is specifically important for, doing some adjustments become necessary to make sense of single-endedness in ND. With these two points in mind, we are going to check if we can rebuild the rules for negation. By reinterpreting meaning theoretic concerns, we then can determine if the result is satisfactory.[44] In doing so, we resist the

[44]Two different attempts to do so, by different approaches, can be seen in [Milne, 2002]

temptation of reading commas on both sides of the turnstile, or in renewed ND rules, as conjunction or disjunction. The reason is that, as we shall see in the next section, it is arguable if commas are the same as connectives. If one thinks that commas are the same as connectives, then this casts a thick sceptical shadow over the possibility of defining the meaning of a logical word independent of the others (separability). We shall discuss this in the next section.

Here are Gentzen's rules for negation in SC:

17. $$\frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\ \neg RI \qquad \frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\ \neg LI$$

And to recall, here are definitions of Single-Ended (SE), Pure (P), and Simple (S) rules:

d. A rule is SE if it is only I rule or only E rule (the logical word appears only in premises or only in conclusion).

e. A rule is P if only one logical word appears in it.

f. A rule is S if any logical word appearing in it occurs as principal operator.

Considering these definitions, Milne changes the left hand side rule in (18) to the right hand side and correctly considers it as P and S.

18. $$\frac{\begin{array}{c}[A]\\ \vdots \\ B \wedge \neg B\end{array}}{\neg A} \quad \rightsquigarrow \quad \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[A]\\ \vdots \\ \neg B\end{array}}{\neg A}$$

Usual intuitionistic rules for handling negation are:

19. $$\frac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A}\ \neg I \qquad \frac{A \quad \neg A}{\bot}\ \neg E \qquad \frac{\bot}{D}\ \bot E$$

They are SE but neither P nor S once we consider $\bot$ as a logical word. There are philosophical and logical points for these rules, such as avoiding double negation elimination and priority and generality of the notion of falsehood over negation. However Dummett's *The Logical Basis of Metaphysics* tries to give a meaning theoretic justification for these rules. This can be challenged. Inspired by Read's suggested rules for the conditional in (9) and Milne's Classical negation I rule in (20), we can suggest two SE, P, and S rules for negation as in (21).

---

and [Read, 2000]

20. $$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{\neg A \vee B}\ \vee I$$

21. $$\dfrac{\begin{array}{c}[A]\\ \vdots\\ B\end{array}}{\neg A, B}\ \neg I \qquad \dfrac{A \quad \neg A}{B}\ \neg E$$

Obviously the I rule has multiple conclusions, but it nonetheless meets all the sought-after meaning theoretic criteria, namely SE, P, and S. Besides, it resembles the sort of connection a conditional has with the context of deducibility. Now if these rules together with other ND rules, specifically Read's multiple conclusion way for introducing a conditional in (7), meet the condition of conservative extension as well, then we have met many of our meaning theoretic concerns.[45] Here is Read's proof for (8) in ND with rule (9):

22. $$\dfrac{\dfrac{\dfrac{[(A \to B) \to A]2 \quad \dfrac{\dfrac{[A]1}{A, B}\ Wk}{A, A \to B}\ \to I1}{A, A}\ \to E}{A}\ Con}{((A \to B) \to A) \to A}\ \to I2$$

Actually it is a fact that adopting multiple conclusion rules lets us expand the positive fragment of propositional logic in ND conservatively.[46] Moreover, these new rules for negation have subformula property. In the second next section we will see how this property helps us to keep negation as a logical constant while excluding Tonk. Recall, in the previous section we saw that Read saved Tonk by giving it a harmonious general E rule. Before that, let us focus on multiple conclusions as it is the crucial point for the view which is promoted here. And of course accepting or rejecting multiple conclusions deserves a separate section.

[45] A proof of soundness and completeness of a multiple conclusion system for classical logic can be found in the appendix at the end of this thesis.

[46] For instance, Read has shown that in [Read, 2000], other logicians as well.

2.3 Single Conclusion or Multiple Conclusions

Perhaps the first point that comes to mind comparing schema (20) and I rule schema in (21) is that actually SE, P, and S in the I rule in (21) are achieved by cheating. The disjunction in (20) is hidden in (21). As a matter of fact, in the context of ND there is an asymmetry between the upper and lower side of the conclusion line. In the upper side, we have a major premise and a minor premise or even premises. It looks natural to us to see them as a way of combining bits of information or formulas. However, this can be seen as conjoining these information or formulas. If this is accepted, then it is hard to see why we cannot use a comma on the lower side of the conclusion line and understand it disjointly.

In spite of the aforementioned point, in this section two lines of argument will be presented to justify deductions with multiple conclusions. First we will see an argument against the idea that the meaning of a logical constant, here disjunction, is primitive to the context of deducability. And then a disjunction-free account of multiple conclusion deduction shall be reviewed. These arguments serve as a response to Florian Steinberger[47] who raises some points against multiple conclusion deduction from an inferentialist point of view.

Ian Rumfitt also raises another point against multiple conclusion deduction.[48] He argues that adopting a multiple conclusion consequent relation is semantically demanding. Specifically when it comes to deduction on vague situations, Rumfitt suggests that multiple conclusion deduction makes us to commit to the existence of a sharp point. This commitment usually is not what we want in vague circumstances. Moreover, he believes that this phenomena would take away a useful property of deduction which is ability to concatenate deductions together. In response to his objections, it will be proposed that our commitment to the sharp point is not that strong. And that we still can splice together deductions with multiple conclusions.

Steinberger's point is that a conclusion in day to day reasoning, is single. Therefore if ND deserves the adjective 'natural' the conclusion should remain single. He introduces a principle:[49]

g. Principle of answerability: only such deductive systems are permissible as can be seen to be suitably connected to our ordinary deductive inferential practices.

He criticises Restall's and Read's examples of natural multiple conclusion

[47] [Steinberger, 2011b]
[48] [Rumfitt, 2008b]
[49] [Steinberger, 2011b], p.335

reasoning. Here are Read's and Restall's reasoning schemas (of course these schemas are not primitive and they explain why these are valid):[50]

23. $$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B \vee C\end{array}}{(A \to B) \vee C}\ \mathit{Read} \qquad \dfrac{\forall x(Ax \vee Bx) \quad \begin{array}{c}[Aa \vee Ba]\\ \vdots \\ \forall xAx \vee \exists xBx\end{array}}{\forall xAx \vee \exists xBx}\ \mathit{Restall}$$

The motivation for introducing these proof schemas is to show that subformulas of a disjunction can be treated separately as if they are two different formulas. Steinberger argues that introducing a logical operator into subordinate positions is "not generally acceptable from a constructive point of view... can only be justified by making essential use of specifically classical modes of inference".[51] Both Read[52] and Restall,[53] would and did respond to this objection by saying that in a dispute between a Classical logician and an Intuitionist, as much as Steinberger is right to reject 'specifically classical modes of inference', classical logicians are right to consider 'constructive modes of inference' as too demanding. The Dummettian strategy of beginning from a common starting point and convincing the other side by what is held in common does not work here, for there is no common ground in this specific dispute. And if we appeal to meaning theoretical concerns as the higher court, then it depends upon how we cash out these meaning concerns. If we understand these meaning theoretic concerns in a formal way, as Dummett himself suggests that is as purity, simplicity and single-endedness, then as we just saw, there are reasons in favour of multiple conclusions.

More can be said about the above mentioned debate, but let us introduce a different argument in favour of multiple conclusion deduction and in reflect to Steinberger's Principle of Answerability. Although it already has been mentioned that single conclusion inferences are more common in ordinary inferential practices, and usually we call a number of premises conclusive once a single conclusion can be inferred from them. However, being conclusive is not a necessary requirement for a deduction. Whether a number of premises are conclusive or not is a matter of practical purposes. For instance, from (h) we might infer (i) and (or) infer (j).

h. x is human.

i. x is a living being.

[50] [Read, 2000], p.145, [Restall, 2005], p.200
[51] Ibid. p.345
[52] [Read, 2000]
[53] [Restall, 2005]

j. x can be a female, x can be a transgender, x can be a male.

If an inference from (h) to (i) is deductive, why is not an inference from (h) to (j)? In other words, given a number of premises, depending on practical matters, we might have conclusive or inconclusive information. And in everyday life we come across circumstances in which we need to make deductions based on insufficient information.

Another example that helps to see the naturalness of reasoning towards multiple conclusions is a game called Master Mind. The interesting point about Master Mind is that it resembles scientific reasoning based on insufficient information. The game is between two parties, one fills m holes by pins which come in n colours. This creates a matrix of $m \times n$ possible arrangements of pins. The other party, who is not able to see the arrangement, starts guessing by filling m holes. Each time a guess is made, the first person gives feedback by black and white pins. White pins indicate the right choice of colour and black pins indicate the right position. In this way, each time a guess is made a number of possibilities will be ruled out. And based on the feedback the second party needs to deduce another guess. This process of guess and feedback continues until the second party makes a correct guess or loses the game.Each step of this guessing process is an example of deduction based on insufficient information.

Deductive inference based on insufficient information happens in many research projects as well as everyday investigations and reasonably can be considered as a natural deductive practice. Indeed proponents of single conclusion deduction, such as Steinberger, admit these examples as natural deductive practices. However, they think these examples are instances of single conclusion deductions with a disjunctive conclusion.[54] If defenders of single conclusion deductions are right, then one of the significant consequences of their position is that the rules given in a multiple conclusion system are not pure. This is because, according to proponents of the single conclusion deduction, any logical word has been introduced to the context (to the right in SC or below the conclusion line ND) actually has been introduced into a "subformula within the scope of disjunction".[55] In other words, rules with multiple conclusions "license introduction of operators into subordinate positions with respect to the disjunction operator".[56]

However, the version of vernacularism at work in the Prinsiple of Answerability has an implicit assumption, namely that the meaning of logical

[54]Steinberger follows this strategy when he rejects Restall's example of natural multiple conclusion reasoning. [Steinberger, 2011b], p341-346

[55]Ibid. p.344

[56]Ibid. p.344

constants are more primitive than the context of deducability. And in fact this is the meaning of logical constants that defines the behaviour of the consequence relation. That is why Steinberger writes: "multiple-conclusion systems make it impossible in general to characterise the meaning of any one logical constant in isolation. Someone unacquainted with the meanings of the logical constants would not be able to fully comprehend the (logically relevant) meaning of, say, 'and' without already having mastered the meaning of 'or'".[57]

The assumption that the meaning of logical constants are more primitive than the behaviour of the consequence relation can be challenged. It sounds reasonable to think that we learn about 'and' and 'or' before learning about what follows from what, but it does not mean that our understanding of the latter does not affect our understanding of the former. Let us consider our deductive inferences with insufficient information again, in many cases, if we understand them disjunctively, they will take forms such as $p \vdash q \vee r \vee s$. We will have a hard time justifying their validity by appealing to the meaning of 'or', if we take the disjunction introduction rules as defining the meaning of 'or'. According to introduction rules for 'or', the meaning of 'or' is such that we can introduce it when we have one of the disjuncts. An easier way to justify validity of the mentioned inconclusive deduction is that if one accepts the premise p, then one cannot reject q, r, and s all together.[58] The same justification serves well for justifying disjunction introduction too.

Generally speaking, the idea that the context of deducability affects the meaning of logical constants, is not an unfamiliar position among logicians and philosophers of logic.[59] Although it might be less popular among inferentialists as they might think that giving priority to the meaning of logical constants is more realistic when we are analysing vernacular reasoning. Nonetheless, if a more abstract reasoning does a better job in explaining the meaning of a more concrete or primitive notion, then why we should not take it into account.

Let us get back to the disjunction introduction rule again. Here are three justifications for disjunction introduction in ND: someone in position of asserting A/B is in position to assert $A \vee B$. The second one: someone in position of asserting A/B cannot deny $A \vee B$. The third one: let us understand $A \vdash A, B$ as while asserting A, one cannot deny A together with another expression such as B. Therefore asserting A, makes $A \vee B$

[57][Steinberger, 2011b] p.345

[58]The same is correct for multiple premise deduction, but since no one challenges that, we are not going to talk about that.

[59]For an example of an extreme position, in which the context of deducability defines the meaning of logical constants, look at [Boričić, 1985].

undeniable. Even if we can explain why does someone already in position of asserting A/B want to assert $A \vee B$ in the first justification, the second explanation sounds better than the first one. This is because we do not need to deal with the awkwardness of justifying the fist explanation. Although the second explanation does not explain why we might be interested in denying $A \vee B$. The third explanation gives us a better clue about how B might be added to the context and from there we can see why denying $A \vee B$ matters.

The above argument, if accepted, shows how we can justify a meaning giving rule for a logical constant by appealing to the context of deducability. This should be enough to cast some doubts on the idea that meaning of logical constants are more primitive than the context of deducability. The same also is applicable to multiple premise arguments, where 'considering them together' can be seen as 'conjoining' them. The SC right and left hand side of the turnstile look similar. The asymmetry between above and below of the conclusion line in ND is the main evidence for proponents of a single conclusion to consider it as more natural. Above the conclusion line we may have more than one premise. That is what makes Milne able to change the impure and non-simple rule for introducing negation, in (18), to a pure and simple one and it looks natural. However, explaining downward branching is not easy without the assistance of 'or'. This must be admitted. Even admitting this asymmetry, there is at least one way to justify multiple conclusion deductions without any appeal to 'or'.

We can have a disjunction-free reading of multiple conclusion which helps us to understand the right hand side of the turnstile in the same conjoining way that we understand the left hand side of it. This understanding has already been applied in the third justification given for disjunction introduction. In this justification, instead of asserting multiple conclusions disjointly, we consider undeniability of them conjointly. This approach employs a version of bilateralism and we will examine it in more details in the next session. For now, it is worth noting that applying this approach to justifying multiple conclusion deduction does two jobs; provides us with a disjunction-free justification of multiple conclusion deduction. And enables us to consider multiple conclusions conjointly.

Dealing with Rumfitt's objection to multiple conclusions needs background information. Let us consider an example:

k. There are nineteen ancient Roman soldiers and one Persian one in a junction between two ancient tunnels, the Roman side shallower than the Persian side, there are traces of sulphur and bitumen all along the walls, considering what laws of physics tell us about smoke and what laws of chemistry tell us about sulphur and bitumen, these soldiers died

> because of forming sulphuric acid in their lungs, and this is an early use of chemical warfare.

This argument can take a deductive form with a bit of fiddling, and what happens in this deductive reasoning is that we splice bits of information coming from different disciplines such as archaeology, physics, and chemistry to come to a conclusion in field of archaeology. This is an example of what Rumfitt says about deduction: "So, in the first instance, what mastery of a deductive capacity provides is not a new reliable method for forming beliefs per se, but rather a means of combining reliable methods of belief-formation that one already possesses so as to yield a new method which has a wider range than its components".[60]

To explain how this combination happens, he defines the conceptual content of a statement in any given context as "live possibilities that the statement is understood to exclude".[61] Then he argues that to know the conclusion of a deductive inference, one who already knows the premises of that inference, is involved in a conceptual work. This conceptual work is realizing that for every possibilities that the conclusion of the inference excludes one knows something, by knowing premises, that precludes that possibility.[62] Rumfitt holds that this "conceptual work" is the reason why our deductive capacity "goes beyond anything that is strictly implicated in ... knowledge of the premises" and from here concludes that "actual knowledge of the conclusion need not be contained in knowledge of the premises. We need to find the appropriate partition of the excluded cases. Deductive rules are, in effect, rules for doing this".[63]

So far so good. Rumfitt has shown that an important aspect of deductive reasoning is that we can knit different methods of belief-forming together. If we accept this point then transitivity of deductive inferences is important since if they are transitive then we can connect arguments together. Common understanding of transitivity in a logical deduction is admissibility of the structural rule 'Cut'. Rumfitt takes a definition of multiple conclusions version of cut, given by Shoesmith and Smiley,[64] and argues that given this version of cut, it is too demanding once a chain of deductions does not lead to a valid argument. Here are the mentioned definition of multiple conclusions cut(l) and its formal version suitable for this context (24):[65]

[60][Rumfitt, 2008a], p.63
[61]Ibid. p.69
[62]Ibid. p.74-75
[63]Ibid. p.75
[64][D.J.Shoesmith and Smiley, 1978]
[65]The additive or non-shared version of cut have been presented here so that shows the use of deduction mentioned by Rumfitt.

l. If there exists a set of statements Z such that $X, Z_1 \vDash Z_2, Y$ for each partition $\langle Z_1, Z_2 \rangle$ of Z , then $X \vDash Y$.[66]

24. $$\frac{\Gamma \vdash \Theta, A \qquad A, \Delta \vdash \Theta'}{\Gamma, \Delta \vdash \Theta, \Theta'}\ Cut$$

The cut rule (24) does not have a very straightforward correspondence in ND. We should note that cut, understood as the general form of combining derivations, is a much broader notion than normalizability. The reason is that the top left hand side sequent of the cut rule in (24) does not require to be an introduction rule. Neither does the top right hand side sequent of (24) require to be an elimination rule. And they are not required to be rules of the same logical word. The closest idea to SC cut in ND is usually referred to as *closure under derivation composition* or closure under *substitution*, shown in (25) where the comma should be understood as branching:[67]

25. $$\begin{array}{c} \Gamma \\ \vdots\ \pi_1 \\ \overbrace{A, \Theta} \end{array} \quad \begin{array}{c} \underbrace{\Delta \quad A} \\ \vdots\ \pi_2 \\ \Theta' \end{array} \qquad \rightsquigarrow \qquad \begin{array}{c} \Gamma \\ \vdots\ \pi_1 \\ \Delta \quad \overbrace{A, \Theta} \\ \vdots\ \pi_2 \\ \Theta' \end{array}$$

Which means, if the result of substituting derivation π_1 from Γ to A, Θ with A in derivation π_2 from A and Δ to Θ' forms a correct derivation in ND, then derivation π_1 can be substituted to A in π_2.[68] The new derivation does not rely on A any more. However, A is not omitted from this derivation.

Given the material sequent relation (m), in cases such as vague terms, or in general Sorites paradoxes, (n) is the logical equivalence of (l).

m. If a set of statements Y follows from another set of statements X, then a thinker who accepts X and rejects Y will be making at least one mistake to the facts.

[66] [Rumfitt, 2008b], p.82

[67] This point has been mentioned in [Negri and Plato, 2001] and [Francez, 2014].

[68] Note that the notion of validity for a multiple conclusion derivation does not require either of the formulas in the conclusion to be provable separatly. One way to understand multiple conclusion proofs is that they are called so because they cannot provide a proof for either of specific formulas in the conclusion, but provide a proof for the conclusions *jointly*. Just like the premises that jointly, and not separatly, provide the ground for the conclusions. For instance, consider the proof from $p \vee q, r$ to $q, p \wedge r$ by combining the proof from $p \vee q$ to p, q and the proof from p, r to $p \wedge r$ and cutting p.

n. If $X \nvDash Y$ then for every set Z of statements, there exists a partition $\langle Z_1, Z_2 \rangle$ of Z such that $X, Z_1 \nvDash Z_2, Y$.

According to (n), in case of vague terms, there should be a sharp point in which inferring conclusions do not follow from premises. For instance, if we consider a colour strip from red to orange, there should be a point in which 'if this bit is red then that bit is red too' does not hold. And since multiple conclusions imply (n), it is too demanding, therefore a single conclusion serves better whenever we want to splice arguments together.

Two points are worthy to note in reflecting on this argument. First of all, it is not clear what exactly is the role of multiple conclusions in the described situation. Rumfitt, arguably,[69] shows how applying multiple conclusions in vague situation leads to commitment to the existence of a sharp point, but he does not show that the single conclusion will not. There is nothing in Rumfitt's argument that cannot be said about single conclusion, we only need to consider the set Y as having one member. Here is a single conclusion version of cut:

26. $$\frac{\Gamma \vdash A \qquad A, \Delta \vdash C}{\Gamma, \Delta \vdash C}$$

Given (26), if C does not follow from Γ and Δ, then either A does not follow from Γ or C does not follow from A and Δ. Putting it in the context of vagueness, if the first horn is the case then there is a sharp point, if the second horn is the case then we need to repeat our bottom to top method to check at which point in Δ it does not follow from z_i that z_{i+1}. And considering his point about splicing arguments, (27) and (28) respectively are formalizations of the single and multiple conclusion applied to a vague situation:

27. $$\frac{r_1 \vdash r_2,\ r_2 \vdash r_3\ , ...,\ r_{n-1}, \vdash r_n}{r_1 \vdash r_n}$$

28. $$\frac{r_1 \vdash r_2, ..., r_n\ \cdots\ r_1, ..., r_i \vdash r_{i+1}, ..., r_n\ \cdots\ r_1, ..., r_{n-1} \vdash r_n}{r_1 \vdash r_n}$$

According to Rumfitt if $r_1 \nvdash r_n$, that is, generally speaking, if the result of splicing a chain of deductions went wrong. Then we can fix the problem by taking care of the faulty bit and saving the rest of reasoning. And he thinks that it is not the case about multiple conclusion deduction. Given (28), why the same cannot be done for the multiple conclusion version?

[69] Very shortly we will see why it is arguable.

Secondly, (n) is not the only correct formulation of what is logically equivalent to (l). A more formal articulation of (l) will illustrate the logical structure of it:

o. $(\exists Z = \{z_1, z_2 \cdots, z_n\} | \forall \langle z_i, z_{i+1} \rangle \in Z\ X, z_i \vDash z_{i+1}, Y) \Rightarrow (X \vDash Y)$

The logical equivalent of (o), then can take either the form of (p) or (q):

p. $(X \nvDash Y) \Rightarrow (\nexists Z = \{z_1, z_2 \cdots, z_n\} | \forall \langle z_i, z_{i+1} \rangle \in Z\ X, z_i \vDash z_{i+1}, Y)$

q. $(X \nvDash Y) \Rightarrow (\forall Z = \{z_1, z_2 \cdots, z_n\} | \exists \langle z_i, z_{i+1} \rangle \in Z\ X, z_i \nvDash z_{i+1}, Y)$

What is expressed in (q), which is the intuitionistic understanding of proving, is a stronger claim than what is expressed in (p), which is the classical understanding of proving. (p) Indicates that once Y does not follow from X then there is no set Z such that for each partition $\langle z_i, z_{i+1} \rangle$ of it $z_i \vDash z_{i+1}$. However, (q) claims that once Y does not follow from X then every set Z that we consider, there is at least a partition $\langle z_i, z_{i+1} \rangle$ of it that $z_i \nvDash z_{i+1}$. Commitment to the existence of a sharp point in a vague context comes with (q), and not with (p). Regardless of adopting a single or multiple conclusion(s), if we choose (p) as logically equivalent to (o), then there is no sharp point, and if we take (q) as logically equivalent to (o), then there should be a sharp point.

To sum up this section, arguments raised by the opponents of multiple conclusions are not forceful enough to convince us to abandon it if it gives us all that our meaning theoretic concerns demand.

2.4 How Logic Preserves Meaning

Getting back to our original question regarding the justification of deduction, in the first chapter, Dummett suggests the appeal to meaning theoretic concerns for justifying rules of logical deduction. And by meaning theoretic concerns, he has 'harmony' in mind, harmony between I and E rules. The guideline for his account of harmony is the Inversion Principle according to which grounds for the major and minor premises of an E rule with B as conclusion are enough to conclude B directly. Besides, the meanings of our logical words are fixed by these I and E rules. Intuitively, it means that we neither gain nor lose anything in introducing and immediately eliminating a logical word. And if it is the case about every single logical word, then it is the case about using all of them together too.

This has implications on the role of logic in language as a whole. The claim is that logic preserves meaning where meaning has been understood in an inferential way. Let us spell it out bit by bit.

r. The meaning of non-logical expressions of a language, such as predicates, change whenever the grounds of inferring them, or what can be inferred from them changes.

For example, the meaning of the term 'mammal' changed when the following inferences ceased to be correct:

s. x is warm blooded, has vertebrate, hair or fur, milks her kid, and gives birth alive, therefore x is a mammal

t. x is mammal, therefore x is not laying eggs.

Or the meaning of human would change if the following inference was correct:

u. x is human, therefore x is right handed.

Given this definition of change of meaning, if we consider a theory as a set of sentences connecting certain predicates by what, following Dummett, can be called boundary inferences (connections between atomic sentences like $p \vdash q$), then it can be said that these boundary connections fix the meaning of the vocabulary appeared in that theory. Now we can define meaning preservation as no change in meaning:

w. The meaning of non-logical expressions of a language, such as predicates, is preserved so long as there is no change in their meaning in the sense mentioned in (r).

If we understand the Inversion Principle as there is no loss or gain of information in introducing and eliminating a logical word to a context, then there is a way to show that applying logic, say to develop the theory, will preserve meaning in our theory. This happens in the sense that these boundary conditions would remain untouched while the theory can be expanded using logical vocabulary. If it happened that a new piece of information connects two other bits of information, then it would appear in our boundary conditions as a change in our theory (for instance the boundary condition $p \vdash q$, $r \vdash s$ would expand into the new boundary condition $p \vdash q$, $r \vdash s$, $p \vdash t$, $t \vdash r$). In this scenario any change of meaning, in the inferential sense, would appear in the boundary conditions. This is in tune with the familiar thought that empirical evidences might change the meaning of our non-logical expressions such as predicates, while applying logic to an existing fixed boundary would not change the boundary conditions and so preserves the meaning of those non-logical expressions. It also seats coherently with the idea that logical rules do not need suasive justification.

The problem with this Dummettian plan then is the inseparability of some logical rules, namely rules for introducing and eliminating negation. That is, rules for negation are not introducible without the assistance of at least one other logical word; it could be any of other usual connectives or Falsehood. And at the same time we have a fact that some formulas are not provable in the positive segment of Classical Natural Deduction (CND) while they are provable in CND plus negation; and there is no negation in these formulas (our example was $((A \to B) \to A) \to A$). The reason why this is a problem for the claim that logic preserves meaning is that adding negation is a non-conservative extension of our language because it changes the inferability or deducability of the positive fragment of our language. So it needs to be fixed. Based on his account of harmony, Dummett blamed double negation elimination as being responsible for this fact (non-conservative extension of CND).

As we saw, two sections ago, Read gives an alternative account of harmony, he calls it autonomy of logical words. According to this understanding of harmony, I rules give meaning to the logical word and the E rules meet the conditions of that meaning in a general sense. Recalling Dummett's interpretation of schema (2) helps to understand the difference between Read's and Dummett's position. Dummett considers the harmony between I and E rules of each logical word in isolation, while Read holds that once we introduce a logical word into the context of deducibility (an argument) then what is derivable depends on the other logical words engaged in the context as well. Therefore, it is enough to make sure that what we derive from a formula is not going any further than what the relevant I rule gives us together with what we already have in the context. Perhaps what makes Read take such a stance is the difficulty of defining negation without the assistance of the other logical words.

And then we saw Read argues that in this new sense, harmony neither guarantees conservative extension nor consistency. He accepts Tonk as its general version of E rule is harmonious in the Readian sense. As a matter of fact, he thinks that if we rule out Tonk, then we have to rule out negation as well. As in both cases there is no guarantee that the maximum formula can be levelled in the normalization process. Here are his reductions for Tonk and negation:

29. $$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ A\end{array}}{A\,Tonk\,B}\,TonkI \qquad \begin{array}{c}[A]\\ \vdots\,\pi_2\\ C\end{array}}{C}\,TonkE \quad \rightsquigarrow \quad \begin{array}{c}\vdots\,\pi_1\\ A\\ \vdots\,\pi_2\\ C\end{array}$$

30. $$\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\,\pi_1\\ B\end{array} \quad \begin{array}{c}[A]\\ \vdots\,\pi_2\\ \neg B\end{array}}{\neg A}\,\neg I \qquad \begin{array}{c}\vdots\,\pi_3\\ A\end{array}}{C}\,\neg E \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ A\\ \vdots\,\pi_1\\ B\end{array} \quad \begin{array}{c}\vdots\,\pi_3\\ A\\ \vdots\,\pi_2\\ \neg B\end{array}}{C}\,\neg E$$

In (29) π_2 actually might include $A\,Tonk\,B$ and in (30) the steps token to introduce $\neg B$ must have been involved $\neg$ given the I rule schema. However, considering the suggested pair of rules for negation in (21), this problem is solved as the I rule is having the subformula property because of being pure, simple and single-ended. Therefore, we can rule out Tonk without being subject to holding double standards towards Tonk and negation. Here is the reduction process for negation rules in (21):

31. $$\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\,\pi_1\\ B\end{array}}{\neg A, B}\,\neg I \qquad \begin{array}{c}[\neg A]\\ \vdots\,\pi_2\\ B\end{array}}{B}\,\neg E \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\,\pi_3\\ A\\ \vdots\,\pi_1\\ B\end{array} \quad \begin{array}{c}\vdots\,\pi_4\\ \neg A\\ \vdots\,\pi_2\\ B\end{array}}{B}\,\neg E$$

Also the other problem in the way of logic preserving meaning had been ratified by Read's own suggestion. We just saw that with regard to adopting the multiple conclusion approach, adding negation to the positive fragment of logic will not lead to a non-conservative extension. So Read's two concerns about how meaning (from the inferentialist point of view) could be preserved by logic are both ratified. Of course all of this is possible at the expense of accepting multiple conclusions.

The inferential reading of logical consequence is not the only way of understanding how logic preserves meaning. At least one more account exists; Bilateralism. Bilateralism is a branch of 'meaning as use' approach to meaning that emphasizes both assertion and denial conditions, rather than only assertion conditions to fix the meaning of expressions of a language. Of course the original thought, first expressed in *"Yes" and "No"*[70] was meant to be about the meaning of the logical works only, but as we shall see it can be expanded to non-logical expressions as well. In *"Yes" and "No"* Rumfitt proposes a Natural Deduction (ND) system with signs for assertion and denial in which plus (+) stands for assertion and minus (–) stands for denial, the motivation for doing so is that denying p, as a speech act, is different from asserting $\neg p$.

[70] [Rumfitt, 2000]

For Rumfitt, natural deduction proofs have speech acts (of assertion and denial) as components, not just the contents of those assertions and denials; a proof from $+A$ to $+B$ is different from a proof from $-B$ to $-A$. And the meaning of logical constants is fixed when we know how to assert and deny them (especially capturing the bilateral meaning of negation is central for him). If we want to expand the idea to the non-logical expressions of language, then we might have the following version of (r):

r′. The meaning of non-logical expressions of a language, such as predicates, change whenever the conditions of asserting/denying them, or what can be followed from asserting/denying them changes.

Bilateralism also has another version introduced by Greg Restall.[71] Let us assume that we start with the boundary condition $p \vdash q, r$, that is, we start with the boundary condition that p implies either q or r.[72] Then according to Restall's theory any state in which we assert p while denying both q and r, is out of bounds. As another example, if we start with the boundary condition $p, q \vdash r$, then any position in which both p and q are asserted while r is denied is out of bounds. So the Restall's bilateralist version of (r) can be articulated as follows:

r″. The meaning of non-logical expressions of a language, such as predicates, change whenever what can be denied when they are asserted, or what can be asserted when they are denied changes.

Comparing these three versions of meaning preservation, as justification of deduction, would be an interesting topic, though is beyond the scope of this thesis's concern as we are about to investigate a mind bugling consequence of this justification in the rest of this writing. However, it must be acknowledged that logical constants in a Rumfitt-style signed language are, in some sense, separable and the language is conservatively extendible. Here are inferential rules for logical constants in a signed language (also known as operational rules) where Φ stands for either a signed formula or $\bot$:

32. $$\frac{+A \qquad +B}{+(A \wedge B)}\,{+\wedge I} \qquad \frac{+(A \wedge B)}{+A}\,{+\wedge E} \qquad \frac{+(A \wedge B)}{+B}\,{+\wedge E}$$

$$\frac{-A}{-(A \wedge B)}\,{-\wedge I} \qquad \frac{-B}{-(A \wedge B)}\,{-\wedge I} \qquad \frac{-(A \wedge B) \qquad \begin{array}{c}[-A]\\ \vdots \\ \Phi\end{array} \qquad \begin{array}{c}[-B]\\ \vdots \\ \Phi\end{array}}{\Phi}\,{-\wedge E}$$

[71] [Restall, 2005]

[72] As a matter of fact, Restall defines his notion of 'out of bounds' based on logical consequence relation. Here we start with a boundary condition such as $p \vdash q, r$ which is not a logically valid inference.

$$\dfrac{+A}{+(A \vee B)}\,{+\vee I} \qquad \dfrac{+B}{+(A \vee B)}\,{+\vee I} \qquad \dfrac{+(A \vee B) \quad \begin{array}{c}[+A]\\ \vdots \\ \Phi\end{array} \quad \begin{array}{c}[+B]\\ \vdots \\ \Phi\end{array}}{\Phi}\,{+\vee E}$$

$$\dfrac{-A \quad -B}{-(A \vee B)}\,{-\vee I} \qquad \dfrac{-(A \vee B)}{-A}\,{-\vee E} \qquad \dfrac{-(A \vee B)}{-B}\,{\vee E}$$

$$\dfrac{\begin{array}{c}[+A]\\ \vdots \\ +B\end{array}}{+(A \to B)}\,{+\to I} \qquad \dfrac{+(A \to B) \quad +A}{+B}\,{+\to E}$$

$$\dfrac{+A \quad -B}{-(A \to B)}\,{-\to I} \qquad \dfrac{-(A \to B)}{+A}\,{-\to E} \qquad \dfrac{-(A \to B)}{-B}\,{-\to E}$$

$$\dfrac{-A}{+\neg A}\,{+\neg I} \qquad \dfrac{+\neg A}{-A}\,{+\neg E} \qquad \dfrac{+A}{-\neg A}\,{-\neg I} \qquad \dfrac{-\neg A}{+A}\,{-\neg E}$$

Obviously, every rule has assertion and denial sign and no inference rule for any logical constant is expressible without these signs, but inference rules of all the usual logical constants, including negation, has been expressed without the assistance of each other. Moreover, there are two structural rules governing assertion and denial where α is a signed formula and $\alpha*$ is the same form with swapped sign. That is if a formula is asserted then it becomes denied and vice versa.

33. $$\dfrac{\begin{array}{c}\Gamma,[\alpha]\\ \vdots \\ \bot\end{array}}{\alpha*}\,{C1} \qquad \dfrac{\Gamma \quad \alpha \quad \alpha*}{\bot}\,{C2}$$

And applying conditional rules, only, and structural rules gives us a proof of $\vdash +(((A \to B) \to A) \to A)$:

34. $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[+A]2 \quad [-A]3 \quad [-B]1}{\bot}\,{C1(1)}}{+B}\,{C2}}{+(A \to B)}\,{+\to I(2)} \quad [-A]3}{-((A \to B) \to A)}\,{-\to I} \quad +((A \to B) \to A)}{\bot}\,{C1}}{+(((A \to B) \to A) \to A)}\,{+\to I(3)}\,{C2}$$

2.5 A Puzzling Corollary

Given meaning preservation as justification of deduction, boundary conditions such as $p \vdash q$ can change by changing our state of knowledge. The source

of this knowledge can be experiment or testimony or some other sources than logic, as logic is supposed to preserve meaning exactly in the sense that applying it (logic) does not change any boundary condition that we start with. This must be the source of what Dummett refers to as the tension between the legitimacy and fruitfulness of deduction.[73]

Intuitively speaking, the idea is that in a legitimate deductive reasoning we cannot infer anything more than what we already have as premises (the idea behind the Inversion Principle). On the other hand, a deductive reasoning is fruitful when teaches us something we did not know before having the deductive reasoning or proof. How a deductive reasoning can be both legitimate and fruitful? One possible answer to this question is that proofs change the meaning of the concepts involved in the premises of the argument, so what we learn is a new meaning for the old concepts. This way is closed to anyone who holds that deductive reasoning preserves meaning.

An example will illustrate the tension. Here is a theorem of basic arithmetic many of us have come across during our study:

x. Any number which its sum of digits is divisible by three, itself is divisible by three.

If we hold that all the steps of a proof of (x) is deductive, then we are forced to admit that there are such cases for which logic alone is capable of establishing boundary conditions such as $p \vdash q$. Even if we accept that this apparently new bit of information has been hidden in the accepted axioms of arithmetic, there are many mathematical proofs in which lemmas play the role of middle terms. That is, proposition t connects two propositions p and r when these have not been connected before ($p \vdash t, t \vdash r$). And all of this process happens deductively. This is not a puzzle for any philosopher who thinks logic is not informative, or who thinks propositions are not fixed units of meaning but that their meaning changes by the development of a theory they appear in. This implies that the meaning of a mathematical conjecture changes once it becomes a theorem, which is counter-intuitive. However these are neither Dummett's options, nor those of anyone who thinks that the justification of deduction is preserving meaning.

This Dummettian concern apparently has been misunderstood by Susan Haack,[74] when she accuses Dummett of confusing deductive implication with deductive inference. She believes that "Deductive implication is necessary; deductive inference is informative".[75] In (k) we saw a case in which we

[73] [Dummett, 1978], p.297
[74] [Haack, 1982], p.225-226
[75] Ibid. p.226

gained information about a historical event in a deductive manner. This is what Haack means by information gathered by deductive inference. To point at the difference between Dummett's concern and Haack's understanding of his concern, in the (k) case, there are other non-deductive ways to gather information about that historical event, but there is no non-deductive way to establish mathematical truths.

One possible response to this puzzle is that mathematics or more specifically arithmetic is not reducible to logic. The classical proponent of this idea is Kant. Famously, he argues that arithmetic proofs appeal to our intuition as well as logic. From the other side of the table there are philosophers who believe arithmetic proofs are reducible to logic; this idea is called Logicism. The well-known figure of logicism is Frege. There is a huge literature for both sides of the debate. We shall probe the main ideas of both sides in the coming chapters.

If one takes the Kantian path, which is thinking that our intuitions are involved in either accepting premises or justifying the inferential steps on a proof in arithmetic, then one way to answer the legitimacy/fruitfulness question is that proofs in arithmetic are not purely logical. Therefore even if we accept that logic preserves meaning (nothing is gained or lost in a logical reasoning) it does not mean that arithmetic proofs also preserve meaning. However, if one takes the Fregean path, that is logicism, and also accepts that the justification of deduction is preserving meaning, then needs to explain what is the nature of the information gained by a proof in arithmetic.

In our definition of meaning preservation, we took it almost as a result of legitimacy condition (nothing is gained or lost in a logical reasoning), and this legitimacy condition sometimes has been referred to as analyticity. Analyticity also is known as 'truth in virtue of meaning', these two notions of analyticity are not quite the same. In the second next chapter we distinguish between these two notions of analyticity by distinguishing between 'analytic justification' and 'analytic truth'. Then in the last chapter we shall tackle the question whether arithmetic proofs are analytic, in the sense that they have analytic justification, or not. Is any account of logic that allows us to derive arithmetical proofs still analytic just in this sense we introduced? If yes, then is there any sense in which these proofs go beyond their premises? On the other hand, if there is a difference between logical proofs and arithmetical proofs then can this difference be articulated in terms of the analytic and synthetic distinction? What syntheticity might mean, if any, as non-analytic where analyticity is understood as having analytic justification?

One possible way of reconciling the legitimacy and fruitfulness of deductive reasoning is to claim that deductive reasoning, as it appears in arithmetical reasoning, teaches us new facts about language (the language of

arithmetic to be more specific) and not the world. This idea usually has been attributed to Carnap. The other way, not necessarily different from the first one, is to argue that what happens in a proof of arithmetic is connecting concepts that are not connected together when we consider them in isolation, though they are logically connected. And with an arithmetical proof we just discover these connections and this is what makes such a proof informative.

For instance, the concept of 'divisible by three' for a two digit decimal number such as ab can be formalized as $\exists x((a \times 10) + b = 3 \times x)$ and the concept 'the sum of digits divisible by three' can be formalized as $\exists y(a + b = 3 \times y)$. Since $(a * 10) + b$ is not identical with $a + b$ the connection between these two concepts is not obvious, to establish this connection (being divisible by three) we need a proof. Although this connection can be claimed to be embedded in axioms of arithmetic. Such an account should explain whether or not in a proof of arithmetic this connection is established via introducing new concepts that have not been explicit in the premises and the conclusion of the argument. If yes, how we can recognize them? If not, then in which sense, if any, does an arithmetical proof go beyond its grounds in the premises and the conclusion? The next three chapters attempt to find an answer for these questions.

Chapter 3

The Development of the Analytic-Synthetic Distinction

Given that arithmetic theorems are informative and logic, at least logical words operating over propositions, preserves meaning, then several questions come to mind. What, if anything, separates a proof in arithmetic from a proof in logic? Does predicate logic preserve meaning in the same sense that propositional logic does? If it does, then what is the nature of the information that we get from an arithmetical proof that is different from a logical one? Can this difference between arithmetical and logical proofs be articulated in terms of the analytic/synthetic distinction? If so, then how can we address this difference? Examination of the historical disagreement between Kant and Frege over analyticity of arithmetic proofs is a potentially fruitful way of approaching these questions.

The debate can be roughly sketched as follows: for Kant, we need both logic and intuition to be able to establish a proof of arithmetic in general (this is a different claim from the claim that every arithmetic proof appeals to intuition, the latter is stronger than the former). On the other hand, Frege argues that with a sufficiently developed logic, arithmetic is reducible to logic in the sense that there is no need to appeal to intuition. This chapter starts by discussing possible meanings of intuition in Kant's philosophy. Then we shall see possible ways in which a logical system, as developed as Frege's, could or could not be intuition free. Both topics will then be investigated in detail in the subsequent chapters.

In this chapter, also we shall examine the connection between the appeal to intuition and the analytic/synthetic distinction. The analytic/synthetic division has been understood in different ways since Kant has drawn our attention to this idea. There have been attempts to clarify the distinction by appeal to metaphysical ideas about essence, necessity and contingency, etc.

or epistemic properties such as the a priori/a posteriori distinction. Some philosophers, like Frege, have endorsed the distinction and tried to clarify it. And some other philosophers, such as Quine, have rejected the distinction due to the lack of clarity in the criteria to define these terms (analytic and synthetic).

After Frege the majority of attempts to deal with this issue have been formed in the context of logic and philosophy of language. Some have tried to understand analyticity as truth in virtue of meaning, and some as truth which is the result of conceptual analysis. Also there are ideas that propose analyticity is a property of verification methods and not the structure of an expression or judgement.[1] The discussion has established specific vocabulary and a number of key ideas; terms such as analysis, synthesis, concept, intuition, triviality, being informative, a priori, a posteriori, necessarily true, contingently true, arbitrarily true, being about the world, being about language and so on. In this chapter and the next two chapters we examine some ideas and propose some other ones, with an emphasis on the logical and epistemic aspects of the topic. The goal of the investigation is to have a better understanding of the disagreement between Kant and Frege on analyticity of arithmetic. This chapter will prepare the scene for the next two chapters by pointing to some pivotal ideas in Kant and Frege.

3.1 Kant on Intuition and Understanding

It is accepted among Kant commentators that he believes in two separate cognitive faculties, intuition and understanding. The following table summarizes the properties, arguably, attributed to each of these faculties, their inputs, their functions, and their outputs:[2]

	Name of the faculty	Object/matter/ input of the faculty	Properties of the input	Properties of the function	Output of the faculty	Properties of the output
1.	**Intuition**	Sense data	Singular (raw)	Immediate Receptive	Intuitive representations (intuitions)	Singular (ordered)
	Intellect	Representations (intuitive or intellectual)	Universal or Singular (ordered)	Mediated Spontaneous Discursive demonstrable	Intellectual representations (concepts)	Universal of higher order

Commentators usually can be divided into two streams in regard to what is the main character of intuition. One group emphasises singularity as the

[1] [Dummett, 1978]

[2] In drawing this table I have been influenced mostly by [Falkenstein, 2004].

main characteristic of Kant's account of intuition,[3] and the other group hold that immediacy is the key feature of his ideas on intuition.[4] A range of different reasons have been given by proponents of the singular reading, mostly influenced by Kant's approach to mathematical reasoning. For instance, Jaakko Hintikka argues that for Kant the connection between intuition and sensibility is not obvious. He argues that Kant gives his reasons in favour of this connection (intuition and sensibility) in the Transcendental Aesthetic. Therefore this connection should be attributed to him only after the writing of the Transcendental Aesthetic; whereas Kant talks about appeal to intuition in mathematical reasoning before the Transcendental Aesthetic.[5]

Separating intuition from sensibility, Hintikka argues that dealing with individuals in constructive mathematical reasoning and predicate logic requires what Kant calls 'singular representations'. And this appeal to singular representations actually is what makes Kant thinking that appeal to intuition in mathematical reasoning in needed. He also argues that appeal to the Cartesian analytic geometrical method is why Kant holds that arithmetic equations are immediate and indemonstrable.[6] Analytic geometry uses the Cartesian coordinate system to represent points, lines, and surfaces. Putting it in modern terms, analytic geometry is a model for numbers. Now if we think of this model as justification for our number theory, then geometrical imaginations that are immediate and, at least logically, indemonstrable for Kant are justifications of our number theory.

So singularity is the criterion of 'being intuitive', for Hintikka, rather than being the immediate result of sensation. This idea is not in tension with Kant's use of singular representations; as he contrasts those (singular representations) with general or reflected representations (concepts). And there is no entitlement to sensation in the distinction between singular and general representations as Kirk Dallas Wilson proposes.[7] What is the criteria of 'being singular' then? One answer is that it concerns dealing with individuals or having individuals as content. The other is that it is the result of an act of synthesis. The first idea is supported by Hintikka[8] and Parsons,[9] and the second by Wilson (we shall return to this topic soon).

[3]Examples of such reading can be found in [Hintikka, 1992b] and [Howell, 1973].

[4]A detailed defence of immediacy as the characteristic of Kant's intuitionism can be found in the part one of [Falkenstein, 2004]. And for a different kind of support look at [Parsons, 1992].

[5][Hintikka, 1992a], p.23

[6]Ibid. p.32

[7][Wilson, 1975], p.1

[8][Hintikka, 1969]

[9][Parsons, 1992]

Proponents of prioritizing immediacy, on the other hand, have a variety of explanations. Charles Parsons holds that we have immediate grasp of forms. He argues that what makes a mathematical argument intuitive is its immediacy. We have an immediate access to the form of singular representations via what he calls 'intellectual intuition'.[10] I will address Parson's ideas on arithmetic in the chapter five, for now, it can be said that he is most likely referring to what is named 'singular ordered' representations in table (1). Another argument for immediacy and against singularity has been given by Loren Falkenstein. He argues that putting emphasis on the singularity of intuitions makes us blind to an important change of position in Kant's ideas. According to Falkenstain, from the *Inaugural Dissertation* (ID) to the *Critique of Pure Reason* (CPR) Kant's position about intuitive and intellectual representations (intuitions and concepts) takes a notable turn. In the former, both intuitions and concepts are objects of knowledge separately, while in the latter they form the object of knowledge together but not separately. His evidence for this idea is Kant's famous quotation at the beginning of Transcendental Logic "Thoughts without content are empty, intuitions without concepts are blind" (A51-B75).[11]

Arguably, it can be said that from this point of view, immediacy is understood as 'happening rapidly and imperceptibly' or 'on the subconscious level'. Any function that includes a conscious process or inference is in tension with this meaning of immediacy. Falkenstain, defends 'order as the form of intuition' against 'form as mechanism' and 'form as representation' for this very reason.[12] A visual thought experiment might be illustrative to understand 'order as form of intuition'. Let us consider a situation in which our sight is limited, say in a heavy fog or sand storm, such that we cannot realize what the two shadows approaching us are. That is, we cannot put them under any category. We even do not know whether or not they are sentient beings. Even in such a situation, we still can say that one is on the right hand side of the other or one is bigger than the other. The thought experiment can be repeated in a normal sight situation but in a very restricted time, say in a portion of second. The same applies here too, we may not be able to realise details, but we get a rough picture which perhaps includes some ordering. To put it in a more specific way, what is shared in all of our sensual experiences is a kind of orderedness in time or space. This orderedness is understood as form of intuition.

Let us conclude this discussion about intuition with three points: as given

[10]Ibid. p.47
[11][Falkenstein, 2004], Chapter 1
[12]Ibid. Chapter 2

in the table (1), intuitive representations, which are products of the intuition as a cognitive faculty, are also called intuitions. It is helpful to have this distinction (between these two different senses of intuition) in mind in the rest of the chapter since this naming can cause confusion. As detailed in the table, for Kant, only the mental activities that happen in the intellectual level are demonstrable. Most likely this means that only intellectual activities are expressible in language. This last point is important in order to understand the analytic – synthetic distinction. And finally, the table (1) just refers to inputs and outputs of intuition as a cognitive faculty. It does not give us any information about what this faculty actually does to inputs to give the particular outputs. This is the topic of the next section.

The other faculty of cognition is intellect, according to Kant. As shown in the table (1), the function of this faculty is mediated in the sense that it does not deal with raw sense data. However, whether this means that every function of this faculty is mediated by concepts or not, is a matter of debate between Conceptualists and Non-Conceptualists.[13] Since Kant holds that the intellect functions via judgements, being conceptualist or non-conceptualist affects one's account of judgement too. Judgements then can be divided into judgements with conceptual content and judgements without conceptual content.

It might seem puzzling as to how non-conceptualists could accept that the intellect works with judgements but not necessarily with concepts, but it should be noted that non-conceptual content is possible. And at least some non-conceptualists hold that this content (non-conceptual content) sometimes has a significant semantic role. For example, Robert Hanna writes: "Non-conceptualism holds that non-conceptual content exists and is representationally significant (i.e. meaningful in the 'semantic' sense of describing or referring to states-of-affairs, properties, or individuals of some sort). More precisely however, non-conceptualism says (a) that there are cognitive capacities which are not determined (or at least not fully determined) by conceptual capacities, and (b) that the cognitive capacities which outstrip conceptual capacities can be possessed by rational and non-rational animals alike, whether human or non-human."[14]

One way of making sense of non-conceptual content is to think of intuitions (singular ordered representations) as the output of the faculty named intuition, and have elements that are not conceptualizable or at least properly conceptualizable. Hanna addresses this point by saying that "non-conceptual

[13]For an account of how Kant can be connected to conceptualism look at [McDowell, 1994] and for his connection to non-conceptualism look at [Hanna, 2005].

[14][Hanna, 2005]

cognitive content in the contemporary sense is, for all philosophical intents and purposes, identical to intuitional cognitive content in Kant's sense." [15] Holding any sort of non-conceptualism would not challenge the discursiveness of the act of the intellect, but it might challenge demonstrably of it. The reason is that demonstrably also depends on syntactic limitations. For instance considering analytic geometry again, π has a clear geometrical definition that can be used in any discursive reasoning, but it cannot be represented in numbers without a degree of approximation.

According to Kant, we understand by our faculty of intellect. And this happens with the mediation of concepts (and intuitions, as output of the faculty of intuition, according to non-conceptualists) and by making judgements. That is we understand what things are by putting them under concepts. However, concepts, as such, are not innate to the faculty of intellect (at least not all of them) in the sense that they do not exist prior to experience. They are made in the faculty of intellect, as a result of activities of this faculty. Forming concepts is one of the functions of intellectual activities according to Kant. Concepts get formed as result of interactions we have with the world around us. This interaction is mediated by making judgements. Discovery observations are good examples showing how concepts get formed on the basis of empirical experiences and pre-existing concepts via acts of making judgements. Let us try to represent what is going on in the mind of someone who has seen a peacock for the first time while it is displaying its fully fanned-out tail, but who has no words to describe this phenomenon:

a. Am I seeing a weird kind of bird standing in front of a colourful wall-like thing? Oh no, it is actually a bird with very unique tail feathers.

b. It has very long tail feathers with eye-like markings on them. So colourful as well.

c. It is definitely a bird but it is not a rooster, nor a turkey, nor a pheasant. Although it has similarities with them.

At (a), the person's visual organs have been stimulated by some entities. From what was perceived, our subject was able to form some judgements. Of course it is a tough question whether the cognitive act of intuition happens via making judgements or not. If we understand the immediacy of the act of intuition as a sort of sub-conscious act and at the same time think of making judgement as a conscious act, then (a) cannot happen by making a judgement. However, if like non-conceptualists, we believe in sub-conscious judgements, then there is no tension in considering (a) as a judgement. The

[15] Ibid. p.248

main feature of this stage is that our subject could put the receptive data into a sort of order (here it is a whole-part order but other kinds of orderings are possible such as 'bigger than' or 'on the left hand side of' and so on). This is what enables our agent to realize the animal across different visual experiences, say for example, seeing it from different angles.[16] Most likely, this is what Kant refers to as the synthesis of apprehensions in intuition.

In the next stage, (b), the subject is able to discern (is conscious of, has discursive access to) certain visual properties such as colour and shape. This enables our subject to form imaginations, say, imagining the tail feathers off the animal and stuck on a long stick. Or imagining an elephant-like creature with those feathers growing from its forehead. According to Kant, our subject is able to do so since they are able to abstract certain features of the tail feathers such as length, colour, and pattern. This kind of act of imagining, most likely, is what Kant calls synthesis of reproduction in a representation of imagination. A noteworthy point is that there is no tension in considering an act of imagination as making a judgement since both are conscious acts. Of course we might not be able to verbalize an imagining.

And finally the last step, (c), is the birth of a new concept, namely a new species-concept. Our agent starts to compare the abstracted features of the animal with other similar animals belonging to an already known species. And since none of them captures all the significant features of the animal in question, our subject needs to coin a new species-concept to reflect or capture all the important properties of the animal in question. By doing so, either just mentally or mentally and verbally, the agent is able to recognize the other instances of this animal. This is what Kant names synthesis of recognition in a concept.

Of course there are several important points in need of clarification in order to explain the activity of cognitive faculties (intuition and intellect) in the act of understanding. Our example is concerned with a very specific case of concept formation; it does not cover two extremes cases of the act of cognition, namely very concrete and very abstract ones. In this example, our subject was able to use concepts like 'bird' and 'tail' from the very first step, however, as we saw earlier, the engagement of concepts is not necessary for the act of intuiting a sense data (at least according to non-conceptualists). As a matter of fact sometimes it is not easy at all to talk about the first step of cognition in intuition with the assistance of concepts. At the other end of the spectrum, it is not difficult to follow how the process of abstraction, reflection, and comparison ends up by coining a new concept[17] in this example, but

[16]Thanks to Colin Marshall for a very helpful discussion on this point.

[17]For a detailed explanation of the process look at [Longuenesse, 1998], ch.5.

it can be really complicated in more abstract cases such as studying relations among some abstract concepts. For instance, whether coining a new concept to represent a set of conceptual entities or not is a common philosophical debate known as being 'reductionist' or 'non-reductionist' about those entities (for instance if we claim that knowledge is nothing but true justified belief, then we are reductionist since we think knowledge is definable with the assistance of only three other notions).

This section concludes with two more points about the function of the intellect in two extremes. At the abstract end, whether the faculty of intellect is empty prior to experience or not is an open question. Kant deals with this question in the Transcendental Deduction where he argues that all of our judgements are governed by highly abstract concepts called categories. By deduction, here, of course he does not mean a rigorous syllogistic reasoning, but pointing at key reasons in a brief though compelling manner like that practised in the courts at that time.[18] Therefore, not surprisingly it is not easy to figure out whether making judgements in accordance with categories is a cognitive limitation similar to our sensory limitations or if the categories are actively governing any single act of cognition. In the first case, we can hold that the intellect is an empty faculty prior to experience, but on the second reading categories are there prior to experience actively contributing in the recognizing of empirical experiences.

In regard to the intuitive end of cognition, three positions can be distinguished. The first is that the immediate act of intuiting sense data does not happen via forming any kind of judgement. The main reason for this position is that if we consider immediacy as a sub-conscious function of intuition, then it will be hard to think that it happens via any act of judging, as judging is a conscious act. The second position is that every act of intuiting sense data happens via an act of judgement; however consciousness and being conceptual are not necessary for judgements (non-cognitivists). The main motivation for this position is to explain cognitive content of intuition (the output of the faculty of intuition) without stretching the notion of concept. Because as soon as we start to propose criteria to define the notion of concept, there are some contents that fall short of being a concept while having semantic significance. And finally the third position that concepts are engaged in every act of judging (conceptualists) contradicts the fact that the mental activity of making judgements cannot always be articulated. Perhaps this may be due to lack of enough conceptions or even vocabulary to address the mental activity of judging. The last approach, as we shall see, works better once it comes to probing and demonstrating discursive reasoning.

[18]For a paper with detailed reasoning look at [Henrich, 1989].

More specifically, once we consider arithmetical reasoning which is a form of discursive reasoning, holding the third position, then we are able to define concepts. Concepts that might not have properties of well-known and developed concepts (properties such as, say, position condition), but they can be defined formally. That is, we can point at them and keep track of them in formal reasoning. In chapter five, we will see what this actually means. On the other hand, holding the first position rules out the possibility of appeal to intuition in any discursive reasoning as discursive reasoning if nothing else is a conscious process. It is hard to see how appeal to something sub-conscious can be pointed at in a conscious activity.

3.2 Intuition and Synthesis

In an insightful paper, Kirk Dallas Wilson argues that the function of intuition as a faculty is synthesis;[19] where synthesis is an act of unifying a manifold under a mereological sum. That is, recognizing a manifold as one thing such that elements in the manifold are understood as parts of a whole. He believes that what makes a representation singular is a whole-part relation between elements of that representation. In contrast, a representation is universal if it represents more than one thing. This indicates the membership relation between representation (concept) and what it represents (objects falling under that concept).[20]

However, Kant uses the word synthesis in describing three mental processes: synthesis of apprehension in intuition, synthesis of reproduction in a representation of imagination, and synthesis of recognition in a concept.[21] The common theme in all of these processes is the unifying of a manifold or plurality of mental entities (apprehensions, representations, and concepts respectively) under another mental entity of the same nature, in the case of apprehension and representation, or a different nature, in case of the synthesis of recognition in a concept.

For instance, apprehending a certain number of wooden pieces as one object is an example of the synthesis of apprehension in intuition which produces an ordered singular representation. Representing this singular representation in different imaginations, say seeing that object from different angles, is an instance of synthesis of reproduction in a representation of imagination. And finally, putting several imaginations together, with certain sorts of similar im-

[19] [Wilson, 1975]

[20] On the other hand, Wilson argues that immediacy, understood as extensional isomorphism, is extensionally identical with singularity.

[21] [Longuenesse, 1998], Chapter 2.

ages, to form the concept of 'table' and putting several objects that share those aspects while, perhaps, differing in other irrelevant aspects, under the title of 'table' is an example of synthesis of recognition in a concept.

This means that synthesis is not restricted to intuitions; therefore we cannot take synthesis as the measure of singularity and singularity as the measure of being intuitive. So, although Wilson's point about synthesis and singularity is interesting, it is not successful in backing up singularity against immediacy in the debate over which one is the criterion for being a product of intuition (the faculty) since synthesis is not just a characteristic of intuitions. However, if we accept that the act of synthesis leads to a whole-part relation between a singular representation and its manifold of elements in intuitions, then in today's context, we will be able to explain some of the inferential implications that stem from intuition.

For instance, if someone tells us that "I was at home yesterday" we know that their hand and also their fingers were at home too. Or if we have been told that "Freddie has broken his hand" we would not infer that "Freddie has broken his fingers" because we know that hand has other parts and the former sentence does not give us enough reason to infer the latter one. It can be said that our mental process of recognizing things in the process of concept forming helps us to make correct inferences. In the example mentioned , how we have formed concepts of 'hand' and 'finger' is such that one is part of the other. So once we hear someone's hand is broken we know that it is not enough information that allows us to infer the one's finger is broken.

In this section we saw a persuasive definition of synthesis, namely unifying under whole-part relation. And we saw interesting possible outcomes of this definition. However, as we shall see in the next part, reducing synthesis to unification under whole-part relation leaves us unable to explain some aspects of what we call synthetic judgements.

3.3 Analytic and Synthetic Judgements

Based on what we have discussed so far, Kant's analytic/synthetic distinction can be understood at least in two different contexts; judgements as mental activities and judgements as linguistic entities. As a mental activity, we saw that synthesis is the act of unification of manifolds. Analysis, as a mental activity, is the other way around - once a unit or an entity has become the object of cognition, it can be broken into a number of maybe smaller or simpler mental entities. For instance sometimes we put a concept, say 'incontinence' in the context of Aristotelian ethics, as the topic of our investigation for a better understanding. To understand this concept, we may break it into

other concepts such as 'voluntary act', 'acting against a value', and 'values that one holds'.

The purpose of analysis, however, is something many of Kant critiques label as 'psychological'; and this will not be our concern here. Kant has a standard definition of synthetic and analytic judgements which is in accordance with Aristotelian logic. We might call this definition 'semantical' and not 'psychological'. The reason is that it is about judgements as linguistic entities and we can think of the content of these judgements as semantic content. And what makes the definition standard is its location in The *Critique of Pure Reason* (CPR). At A7-B11 under the title of 'On the difference between analytic and synthetic judgements' he writes "In all judgements in which the relation of subject to the predicate is thought $\cdots$ this relation is possible in two different ways. Ether the predicate B belongs to the subject A as something that is contained in this concept A; or B lies entirely outside the concept A $\cdots$ In the first case I call the judgement analytic, in the second synthetic."[22] And what makes it in accordance with Aristotelian logic is the vocabulary Kant uses in that definition. According to Aristotelian tradition, a judgement is decomposed into three more elementary or three simpler linguistic parts: subject, predicate and a copula (is or is not) that defines the relation between the concept of subject and the concept of predicate.[23]

In agreement with this definition, Kant mentions some other distinctions between analytic and synthetic judgements. For instance, he mentions that in analytic judgements the connection of predicate to the subject (the copula in affirmative judgements) is conceived through identity while it is not the case for synthetic judgements. Or that one can call analytic judgement 'judgements of clarification' and synthetic judgement 'judgement of amplification'.[24] Also elsewhere (CPR A151/B191) he allows "the principle of contradiction to count as the universal and completely sufficient principle of all analytic cognition."[25] However, I shall stick to the standard definition as our guideline since the other definitions are not necessarily the same as the standard one and Kant derives them from the standard definition and a number of other assumptions about the mental activity of judging.

An obvious feature of the standard definition is that it is based on analyticity. That is, it gives a positive definition of what an analytic judgement is and sweeps the rest of judgement away under the title of synthetic judgement. If in a judgement we have a concept as subject and a concept as predicate,

[22][Kant, 1781], p.141

[23]In Chapter Five we shall probe Aristotelian logic in a more detailed fashion to make sense of the logical nature of relation between two concepts.

[24][Kant, 1781], p.141

[25]Ibid. p.280

then if the concept which is predicated is already part of the concept of the subject, then our judgement is analytically true. In the frame of Aristotelian judgement, the other remaining possibility is that the subject concept does not include the predicate concept, or in other words the predicate concept does not belong to the subject concept. These kinds of judgement are true because individual instances of the subject concept fall under the predicate judgement, but why then are they called synthetic? Is it because a synthesis of individuals is involved in making them? As our knowledge is empirical, the synthesis of individuals is involved even in forming analytic judgement. How we could know (d) if at least someone had not been apprehending an individual peacock via synthesis at intuition?[26]

d. Peacocks are birds.

e. Peacocks fly.

And yet (d) is an arguably analytic judgement, in the standard Kantian sense,[27] whereas (e) is synthetic. However, it can be said that the truth of (e) relies on the act of synthesis of individuals in a way that (d) does not. In the second case synthesis of the individual includes not only perceiving the individual, but also its situation among the other things. Synthesis, in (e), does not include only the act of unifying an individual under whole-part relation. The same holds in case of (f) and (g) Kant's original examples.

f. All bodies are extended.

g. All bodies are heavy.

What does an act of synthesis of individuals include then? Well this is not an easy question to answer. At the very least it includes unifying under whole-part relation but realizing that a body has weight needs more than this. Either we feel the weight of something or we see it is dropping or we know about force between two masses. All of these cases include perceiving a kind of relation. So let us say synthesis includes but is not restricted to unification under whole-part relation. If we take this idea as the core of the analytic synthetic distinction, then in order to stretch it to cases that remain out of Aristotelian logic, we need to adjust the definition of synthetic. For instance we do not just attribute properties to individuals that fall under

[26] Of course the claim is not that everyone who knows what peacock is, has been gathering that knowledge by apprehending a token peacock. Most likely, our first piece of information about peacocks come from a photo or a drawing of this animal.

[27] Because of the empirical nature of the subject of the judgement, it is always subject to change by our best knowledge of the time.

a concept (that is what Aristotelian judgement are for), but we try to put individuals under a category or concept, or we compare them or we want to study relations among them. These activities are conscious and intentional. Therefore it is reasonable to hold that Kant counts them as judgement-laden.

Now if we want to summarize all the possible ways a judgement may or may not rely on an act of synthesis we shall end up with table (2). According to the standard definition, hardly any act of mental analysis happens without an act of judging. Whether all the mental activities of synthesis happen via judgement depend on our position about judgement and concept. Detailed in the table below are possibilities where conceptual analysis and synthesis refers to the act of putting an individual instance of a concept under another concept (Aristotelian judgement) and the rest covers all other possible acts of synthesis that a non-Aristotelian judgement in general may rely on:

		judgement at intellect, No judgement at intuition	**Non-conceptual judgement at intuition Conceptual judgement at intellect**	**Conceptual judgement is possible at both intuition and intellect**
	Analytic judgement	Conceptual containment as a result of conceptual analysis	Conceptual containment as a result of conceptual analysis	Conceptual containment as a result of conceptual analysis
2.	**Synthetic judgement**	Result of conceptual synthesis at intellect Result of non-conceptual non-judgemental synthesis at intuition	Result of conceptual synthesis at intellect Result of non-conceptual judgemental synthesis at intuition Result of non-conceptual non-judgemental synthesis at intuition	Result of conceptual synthesis at intellect Result of conceptual synthesis at intuition Result of non-conceptual judgemental synthesis at intuition Result of non-conceptual non-judgemental synthesis at intuition.

Starting from left to the right, proponents of no judgement at the intuitive level hold that there are instances of synthesis that are not judgement-laden. For instance, according to them, evidence of the synthesis of imaginations being guided by judgement is rare (occasions such as B151-152 of CPR).[28] Not surprisingly, lack of explanation for some synthetic judgement is one result of adopting this approach.[29] This is because if we think of forming

[28]Thanks to Colin Marshall for mentioning the section.

[29]For a convincing defence of this claim (not all synthetic judgement have explanation) with extensive textual evidence look at [Marshall, 2014].

judgement as a criterion for conscious acts, then no judgement means no conscious act. And it is hard to see how we can explain a non-conscious or sub-conscious act of ourselves.

Of course there are good reasons to believe in the existence of non-explainable synthetic judgement. One, for the sake of example, is the regress problem. According to this problem, to explain a synthetic a priori judgement we need to appeal to another synthetic a priori judgement.[30] This is because an analytic judgement, though useful for clarification purposes, falls short of explaining a synthetic judgement. And a synthetic a posteriori judgement cannot be used to explain a synthetic a priori one since the latter would not be a priori any more. So no matter how deep we go, there would be some synthetic a priori judgement left without justification (at least theoretic justification as they may have normative or pragmatic justifications). An interesting example of such basic synthetic a priori claims is Kant's version of the 'principle of contradiction' or the 'principle of identity'.[31]

As we have seen before, non-conceptualists hold that there are non-conceptual contents that have semantic significance. And the suggested connection of this view to Kant's philosophy was that at least some of these non-conceptual contents are the result of our non-conceptual judgement at intuition.[32] This of course does not reject the possibility of non-judgemental functions of intuition, but rejects the possibility of intuitive (immediate) conceptual judgement. That is, judgement that have been formed in the faculty of intuition, and include concepts. The reason is that notions need to meet some conditions in order to count as concepts and not every notion involved in intuition meets those criteria.

It makes total sense to demand a certain definitions for a concept and also to consider a category for those judgement contents that do not fit to those definitions. For instance Hanna considers concepts as "intensionally-structured mental representation types".[33] And defines non-conceptual contents as a "cognitive content that either (i) lacks concepts either globally [$\cdots$] or (ii) does not require the correct application of concepts even if it requires concepts, or (iii) does not require concepts even if it happens to include concepts that correctly apply [$\cdots$] or else (iv) requires both concepts and also their correct application but does not require the possession or self -conscious rational grasp of those concepts by the user of those concepts [$\cdots$]

[30] I assume that every judgement its truth is not obvious, needs an explanation to justify it; a justification, then needs to have an explanatory role.

[31] For textual evidence, look at [Marshall, 2014], p.554.

[32] [Hanna, 2005]

[33] Ibid. p.252

".[34] However, this does not ratify the possibility of conceptual judgement in intuition. And to map it onto Kant's philosophy we need to be careful.

For example Hanna attempts to explain how Kant's philosophy of mathematics relies on non-conceptual contents.[35] He correctly argues for the independence of Kant's position on syntheticity of arithmetic from his undeveloped account of logic.[36] And then argues for the syntheticity of arithmetic in accordance with Kant's philosophy of arithmetic. He extracts Kant's conditions for a truth of arithmetic to be synthetic a priori as follows: "(1) consistently deniable, (2) semantically dependent on pure intuition and (3) necessarily true in the restricted sense that it is true in every experience-able world."[37] Considering criterion number two together with Hanna's position about non-conceptuality of intuitive judgement, we are allowed to conclude that the truths of arithmetic semantically rely on non-conceptual judgement.[38]

This is in accordance with Kant's idea here: "The arithmetical proposition is therefore always synthetic; one becomes all the more distinctly aware of that if one takes somewhat larger numbers, for it is then clear that, *twist and turn our concepts* as we will, without getting help from intuition we could never find the sum by means of the mere analysis of our concepts."[39] This quotation is in accordance with the idea that Kant did not believe in manipulating concepts in an arithmetical proof. This should be enough to confirm excluding concepts from the faculty of intuition according to Kant. However, we will see possible ways in which concepts can be turned and twisted in a proof.

Let us wrap up our Kantian meditations; we started with intuition and understanding, and in an attempt to separate these two faculties we saw a criterion for the act of intuition, immediacy against a mediated act of understanding. Also a criterion for the result or output of an act of intuition has been considered, namely singularity, against the generality of concepts. Immediacy has been understood as a sub-conscious act of ordering versus understanding as the conscious act of putting an object under a concept. This account of immediacy is mainly concerned with our mental activities and

[34]Ibid, p.252. For an explanation of terms used in the quotation, please check the footnote of the relevant page.

[35][Hanna, 2002]

[36]Ibid. p.329-330

[37]Ibid. p.332

[38]In Chapter Five we shall see an account of 'semantically relying on non-conceptual judgement' where it is defined as 'objective meaning' for basic mathematical relations such as 'bigger that'.

[39][Kant, 1781], B16, p.144. Emphasis is mine.

does not have a formal demonstration or counterpart in a formal language. So the way in which it may contribute to explain the role of intuition in a symbolic or formal practice such as mathematical reasoning is perhaps by saying something about our mental activities during reasoning and not any formal representation of immediacy. This is, at best, indirect in the sense that even if it says something about immediacy in the process of reasoning, what is said is not traceable in formal reasoning.

Therefore, thinkers with formal or symbolic concerns have attempted to explain the appeal to intuition in mathematical reasoning by relying on singularity. In this approach, any formal reasoning with symbols that refer to an individual entity requires intuition. Any proponent of this view needs to deal with at least two challenges if they want to attribute it to Kant. One is that all the predicate logic, including a big portion of Aristotelian logic, would count as intuitive and Kant does not hold that Aristotelian logic is intuitive.[40] The other is the difficulty of explaining the syntheticity of mathematical truths, precisely arithmetical truths (the connection between being intuitive and being synthetic).

One way of overcoming the latter difficulty, as we saw, is to think of synthesis as an act of synthesis of individuals where synthesis includes but is not merely unifying under whole-part relations. In this way singularity, which is the result of synthesis, is the criterion of being the output of intuition and, so any reasoning concerning individual or singular entities is intuitive and synthetic. However, the first problem remains untouched. In the next part we are going to see a way of using immediacy as the criterion for being intuitive that provides us a better understanding of the syntheticity of arithmetic by saving the benefits of singularity (as criterion of being intuitive) and removing its problems (keeping Aristotelian logic analytic).

If we do not restrict the synthesis of an individual to merely putting its elements into a whole-part relation, then it can be claimed that discerning relations between or among individuals also is part of the synthesis of individuals. And relations are immediate in a straightforward Kantian sense; we study objects without putting them under concepts. That is, without attributing properties to them or determining their kind or genus or species. Also judgement of relations meet all the other conditions of being immediate; as previously discussed, most of the examples that we saw to explain immediacy were relations such as 'X is on right hand side of Y' or 'X is bigger than Y'. Of course not all the relations can be grasped sub-consciously, quickly and at the first glance, but it is the same for the unity of an object too. Most of us would have this experience facing a cage of snakes; it takes a

[40]We shall pay detailed attention to this matter in chapter Five.

while to figure out who is who. Putting the phenomenological aspects aside, immediacy has a clear formal demonstration in this reading; two or more place predicates in predicate logic. And this keeps all the Aristotelian logic analytic as they (Aristotelian syllogism) are the monadic portion of predicate logic.

3.4 Intuition and Syntheticity in the Context of Kant's Philosophy of Arithmetic

The main concern in this part is how to understand Kant's claim that arithmetic judgement are synthetic. Applying the standard definition (containment of the concept of predicate in the concept of subject) to an arithmetical case such as $5+7=12$ is tricky. The most straightforward way of assimilating $5+7=12$ to 'A is B' is to think of the concept of $5+7$ and 12 as A and B respectively and identity as 'is'. However, it should not be the right way of interpreting what Kant means. Here is the reason: at B17 in CPR, he considers $X+Y>X$ as an instance of 'the whole is greater than its parts' and therefore analytic.[41] So if we consider $5+7$ as a whole, then $5+7=12$ is an instance of 'whole is equal to itself' and therefore analytic. However, Kant explicitly says $5+7=12$ is synthetic, that is he sees it as $X+Y=Z$. This formalization is not exact enough to show the connection between these numbers. By adopting a more exact formalization it is possible to turn the judgement into an analytic one.

Another path to pursue is to pay attention to what he says in order to explain why $5+7=12$ is synthetic. According to Kant we confirm the truth of $5+7=12$ by imagining a process of adding seven things to five things (of course it is one possible way of confirming the equation for us). He generalizes his point by saying that although analytic judgement such as $X=X$ and $X+Y>X$ (as instances of 'the whole is equal to itself' and 'the whole is greater than its parts', respectively) are "valid in accordance with mere concepts, [these] are admitted in mathematics only because they can be exhibited in intuition."[42] Therefore the appeal to intuition is what makes a specific arithmetic judgement synthetic not a schema like $X+Y>X$.

Why we should appeal to intuition to confirm a specific arithmetic judgement then? At least two answers come to mind; its content or its form (precisely speaking its lack of logical form). According to the first answer, expressions with relations such as 'greater' or 'bigger', as well as identity,

[41] [Kant, 1781], p.145
[42] Ibid. p.145

can be analytic or synthetic depending on which objects they ar applied to. If they are applied to concepts, then they make analytic judgement. And if they are applied to objects, then they make synthetic judgement. Therefore form does not matter in a discussion concerning the analytic/synthetic distinction.

However, if we think in line with the second answer, the appeal to intuition is needed because of the lack of formality. It is worthy of note that what Kant knew as General logic, which deals with 'logical forms', is not enough for mathematical reasoning, because logical forms are for reasoning with concepts while mathematics deals with objects. So he thinks for reasoning with objects we need what he calls 'transcendental logic' and there is no talk of logical form in transcendental logic. Of course talking about the form of intuition is possible in the context of transcendental logic but forms of intuition are not logical forms, they are forms of time and space. Perhaps that is why conceptually correct judgement such as $X + Y > X$, although acceptable considering their conceptual status might not necessarily remain correct being applied to objects.[43]

If we adopt the first answer to the question about necessity to appeal to intuition, we should admit the justificatory role of intuition in mathematics, in general, and in arithmetic, in particular. Since only by appeal to intuition we can figure out if the reasoning is correct. Accepting the second answer however makes room for two possibilities; one is the possibility of intuition-free mathematical reasoning, or at least arithmetical reasoning. And the other, the possibility of appeal to intuition even in a purely logical (that is without arithmetical vocabulary) reasoning.

An argument for the possibility of intuition-free arithmetical reasoning goes like this: Kant considers $X + Y > X$ conceptually valid because it is an instance of 'the whole is greater than its parts' and not because of its form. It is true that $X + Y > X$ is an instance of 'the whole is greater than its parts' because of its form, but this form is not a logical form, it is an intuitive form. Kant does hold that $X + Y > X$ has form; since he thinks that "pure intuition contains merely the form under which something is intuited".[44] However, most likely, he does not hold that a pure intuition can be true because of its form. Therefore if there is a way to show that a pure intuition can be true because of its form, then it can be said that intuition-free reasoning is possible; in the sense that intuition does not play a justificatory role in reasoning any more.

[43]For instance if X and Y are replaced by 2 and -1 the judgement is not correct any more.

[44]Ibid. A51-B75, p.193

On the other hand, intuitive logical reasoning becomes possible if we accept the reading according to which immediacy, as the criterion for being intuitive, is understood as relation (two or more place predicate). In the previous section we saw that Kant talks about 'conceptual turns and twists' and how they are not enough to justify a mathematical judgement. In chapter five we shall see how and in which sense it is possible to turn and twist concepts. However, keeping track of these turns and twists might need appealing to intuition. In this case, appeal to intuition will not be restricted to mathematical or arithmetical reasoning, but any reasoning that includes relation might need an appeal to intuition.

Let us finish this section with a possible objection to considering relations as immediate cognition and a response to this objection. If immediate cognition of objects means cognition without putting them under concepts (attributing a property to them) and relation means two or more place predicates in predicate logic, then every such relation can also be understood as a property or one place predicate too. For instance, 'X is bigger than Y' as a relation, can be understood as 'being bigger than Y' which is a property for X. Therefore it is not immediate as such.

One possible answer to this objection is that predicates are different from concepts. Every concept can be expressed as a predicate, but every predicate is not a concept necessarily. Specifically once we define concepts with criteria such as possession condition (this is perhaps one of the reasons why some philosophers talk about non-conceptual content). Moreover, if we think of Kantian concepts as expressible in the form of one place predicates and also think of relations as expressible in the form of two or more place predicates, then concepts representing essential properties. For instance represent the species or property. And some of the concepts represent what might be called 'arbitrary properties'. Relations usually refer to the latter sort of properties that does not have that much to do with the object itself, but expresses its relations to other objects.

Therefore considering a relation between two objects makes a reasonable difference with attributing a property to one object. This is coheres with Kant's idea at the introduction of CPR, where he writes: "Mathematics gives us a splendid example of how far we can go with a priori cognition independently of experience."[45] An example might be illustrative; there are inferential moves that we can make relying on relations among objects, individuals or some entities. Let us imagine that we have two object-like things and we do not know what they are. That is, we cannot put them under any known kind of things (even we cannot say are these solid, liquid, or gas).

[45]Ibid. p.129

We still can make some statements about the relation between their sizes or their position in relation to each other and so on. Here is an example:

h. a is to the west of b, therefore b is to the east of a.

The point about relations is that they can produce what Kant calls 'arbitrary concepts'.[46] In the case of our example, 'being to the west of b', and 'being to the east of a' are arbitrary concepts. These arbitrary concepts may not play any important epistemic role in cases in which we have visual or other sensory access to the entities under investigation. However, in cases in which we do not have that access, very large or small numbers for instance, these arbitrary concepts may play a salient epistemic role. Moreover, these arbitrary concepts are usually rather contingent as the relation between objects or entities are, but if the relation between objects is necessary, these arbitrary concepts become necessary too. An example of such necessary relation is 'orderedness' among numbers.

Syntheticity of arithmetic statements can be explained by so-called 'arbitrary concepts' which occur during the process of arithmetic reasoning. If we accept that arbitrary (say new) concepts born in the process of reasoning and individuals fall under them, then we might learn new things about individuals once they fall under these arbitrary concepts during a proof. For instance, given that number 'seven' is an entity, knowing that it is the biggest prime number smaller than ten, is learning something new about this entity.

These arbitrary concepts might provide the conceptual ground needed to affirm the arithmetic claims; a conceptual ground that might in some sense go beyond the conceptual structure of the original claim. For instance if to prove (i) we argue that there is some x such that $(a > x) \wedge (x > b)$ then, in one sense, we have gone further than the conceptual structure of the original claim.

i. $a > b$

Sometimes it is in explaining the process of recognising sense data that we can think of some relations among objects that have inferential implications for us. In chapter five, I demonstrate that this aspect of mental activity and its non-reducibility to logic (at least in logical systems we are dealing with these days) can be seen as a ground for defining what the possible role of intuition is in a formal reasoning.

[46] [Longuenesse, 1998], p.108

3.5 Frege

Frege does not share Kant's idea about syntheticity of arithmetical judgement; he believes that they are analytic. However there are good reasons to think that they do not hold contradictory ideas.[47] As a matter of fact, in some sense, Frege attempts to develop the Kantian agenda of studying rules of discursive reasoning. We shall see that Frege diverges from Kant's standard definition of the analytic/synthetic distinction. Then it will be shown that Frege's attempts can be seen as rejecting syntheticity of arithmetical judgement in the sense of arithmetic judgement lacking a logical form.[48] And finally Kant and Frege's different understanding of the notion of concept will be studied.

Here I shall focus on Frege's reaction to Kant's analytic/synthetic division. In the *Grundlagen der Arithmetik* (*Grundlagen*) Frege states that he is not intending to give a new meaning to these terms, but he wants to make them more accurate.[49] However, he starts with an important shift from what seems to be Kant's definitions of analyticity. We saw that Kant's measure for analyticity is only applicable to categorical judgement and is based on the relation between concept of subject and concept of predicate. Frege emphasises that there is a difference between "how we arrive at the content of a statement" and the source we "derive the justification for its assertion".[50] Then he states that the analytic/synthetic division concerns the justification of making a statement and not the content of that statement. At least in the specific case of arithmetic statements, Frege means a deductive justification, when he talks about justification.

In the preface of *Begriffsschrift* Frege writes: "The most reliable way of carrying out a proof, obviously, is to follow pure logic, ... Accordingly, we divide all truths that require justification into two kinds, those for which the proof can be carried out purely by means of logic and those for which it must be supported by facts of experience ... Now when I came to consider the question to which of these two kinds the judgement [here judgement are not mental activities but are statements] of arithmetic belong, ... I first had to ascertain how far one could proceed in arithmetic by means of inference alone... To prevent anything intuitive from penetrating here unnoticed, I

[47]For an early defence of this idea look at [Kirk, 1986], and for a more advanced approach see [MacFarlane, 2002].

[48]This was the second way of the three previously mentioned possible ways of reading Kant's position on syntheticity of arithmetical judgement.

[49][Frege, 1884], p.3, footnote.

[50]Ibid. p.3

had to bend every effort to keep the chain of inference free of gap."[51] From this quotation, it seems that at least one of Frege's motivations to invent his ideographic formal language involves rejecting Kant's claim about the necessity of appeal to intuition in an arithmetical reasoning. He hopes that with a logical language and an apparatus more developed than syllogism, that do not leave any gaps between inference steps, he would be able to prove statements that Kant believes cannot be proven without appealing to intuition.

Frege's concern about deductive justification in *Begriffsschrift* does not explain how the analytic/synthetic division connects to the justification of making a statement. It is not clear why. If we could make an intuition-free chain of reasoning as justification of our conclusion, it would make the conclusion or its justification an analytic one. As a matter of fact, Frege has other presuppositions about a proof and what it does in *Grundlagen*. In this text, he bridges the gap between 'a justification for holding a statement to be true' and the 'analytic/synthetic' distinction: "The problem becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths. If in carrying out this process, we come only on general logical laws and on definitions, then the truth is an analytic one ... if, however, it is impossible to give the proof without making use of truths which are not of a general logical nature, but belong to the sphere of some special science, then the proposition is a synthetic one."[52]

There is a difference between Frege's approach to proof in *Begriffsschrift* and in *Grundlagen*. In one, proof is a move from premise to a conclusion. In the other, proof starts from a claim and goes back to grounds of that claim. The second claim is stronger than the first one, and a bit more precise. His position in *Grundlagen* can be understood in this way: considering any statement, either there is a proof for that statement in Frege's logical system or not. If not, then the analytic/synthetic distinction is not applicable to that statement. If there is a proof, then either the statement is derivable from a logical truth, that is his three axioms that express logical truths, or not. If it is, then the proof is analytic. If we need more postulation to derive the statement, then if they are definitions, the statement is analytic. If they are facts, then the statement is synthetic.

According to this definition, the analytic/synthetic division is not exhaustive. That is we have three kinds of propositions, analytic, synthetic, and neither. The group of propositions for which there is no proof to their ultimate grounds are neither analytic nor synthetic. For instance, it is not clear

[51] [Hejienoort, 1976], p.5
[52] [Frege, 1884], p.4

what is the status of his own three logical axioms. What is interesting in this definition is not the outcome of Frege's definition but his presumptions about a proof. In his definition of the analytic/synthetic division, Frege has some implicit assumptions about logical proofs. One is that logical proofs take us back to more primitive propositions. And also in transforming premises to the conclusion, nothing which is not held general (in its broadest sense) will be added to the ultimate grounds of the proposition in conclusion. It is not clear whether or not he holds that any appeal to intuition in a proof can be expressed as 'truths which are not of a general logical nature' (I shall start the next chapter with this distinction between Kant and Frege).

As can be seen, Kantian conceptual containment has been replaced by derivability from purely logical truths via gap-free steps. This means that Frege hopes to demonstrate analyticity of arithmetic by a different formalization. Kant's and Frege's different notions of formality will be studied in detail in chapter five; here I consider other aspects of their difference on arithmetical judgement. In an amusing paper, George Boolos evaluates the success of Frege in giving an intuition-free or gap-free account of arithmetic reasoning.[53] He diagnoses one weak-spot in Frege's chain of reasoning that could be vulnerable to a Kantian objection. There is a substitution case in which Frege substitutes a relation with a formula under a second order assumption and to justify such an assumption we need to appeal to intuition. Then in the rest of the paper he attempts to explain why we do not need intuition to justify that assumption. If we accept Boolos's argument, then Frege is effectively defended against the accusation of an intuitive justification for arithmetic. However, his logic stays intuitive in another sense; it is heavily based on relations. We have seen that, according to one reading, what makes arithmetical judgement synthetic for Kant is appeal to non-logical forms of intuition. These forms usually can be considered as relations. For Frege appearance of these relations in an arithmetical reasoning is not a matter of importance. To see why, we should take a closer look at his account of concept and conceptual analysis.

Frege's hint about the difference between "how we arrive at the content of a statement" and the source we "derive the justification for its assertion" has been understood as an attempt to draw a sharp distinction between "the psychological" and "the logical", or "the subjective" and "the objective" or "subjective representation" and "objective representations" and that the latter is "the same for all".[54] It seems that this hint is a response to Kant's phenomenological method to explain the process of cognition. Kant believed

[53] *Reading the Begriffsschrift* in [Boolos, 1998]
[54] [Coffa, 1991], p.66

that the process of cognition functions according to three main operations: comparing, reflection, and abstraction. Pursuing the approach Frege attributes to Kant, logical features of statements are guides to our mental capacities of cognition. We compare things, objects by objects, concepts by concepts, objects by concepts, and so on. Kant drew this from the fact that we have subject and predicate in our statements. And he thought that we have concepts because we can reflect similarities in plural representations by abstraction, which is a process of omitting irrelevant details. For Frege, these are "how we arrive at the content of a statement" and not the "justification to assert them".[55]

Frege has two main objections to this approach. One is that attributing the same mental process to everyone, which is the source of objectivity (sameness for all) of judgement or statements for Kant, is idealistic. And perhaps there is no ground for that. The other is that abstraction as a process is not exact enough to give a precise account of concept forming. Frege also thought of 'sameness for all' as the measure for objectivity, though unlike Kant he did not think that the source of this objectivity is sameness of cognition as mental process or similar capacity of cognition. It is not easy to attribute a positive account of the source of objectivity to Frege. What he clearly asserts is that Kant's approach to objectivity is idealistic.[56]

His second objection about the arbitrary nature of abstraction as a mental activity for generalization and specifically for concept formation is more precise and well reported. In the Grundlagen he writes: "He[Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, this is one of the least fruitful".[57] The similar point has been repeated in reflecting on Husserl who had tried to draw the notion of numbers as the property of an aggregate or an arbitrary set. The importance of the notion of concept for Frege is because of its crucial role in his theory of quantification. His answer to the question 'in the sentence 'there are 7 Fs' what exactly does the number 7 count?' is that the number 7 "ascribes something of a concept".[58] In the other words, we count things that fall under concept F. Here I am not concerned with Frege's theory of numbers, but his points about concepts.

We have seen that, according to Kant, the last step of recognizing things is labelling or putting them under concepts. And we do this by making judgement. Frege understands this process as giving a list of characteristics, such as - being extended, being solid, having such and such geometrical form,

[55][Frege, 1884], p.3
[56]Ibid. p.37
[57]Ibid. p.100
[58][Dummett, 1991a], p.88

etc., say, for concept 'table'. He correctly notes that the process of forming all concepts is not like that. For instance, we do not make a list of characteristics for concepts that denote colours. The fact that the concept 'being red' has been created via reflection, abstraction and comparison, does not have any logical importance.

Another way to address the difference between these two approaches towards concept is that for Kant the mental process of forming a concept matters once it comes to the content of a concept. Whereas Frege believes that they are irrelevant to the semantic content of a concept. Everything, so far whether it is a thing or a collection of things (and not parts of things), can fall under every concept and the result is a true or false judgement. For Frege, concept is equal to predication, but for Kant it is not.

To avoid confusing names used to refer to an aggregate (arbitrary set) with them being used to refer to concepts, Frege makes the distinction between the relation membership and part/whole relation. We use the word bicycle to refer to it as a concept or to refer to a number of objects assembled together. This can be explained by saying that bicycles are members of an extension while elements of a bicycle are parts of a whole referred to as 'the bicycle'. That is an individual. The other point is that concepts do not need to refer to more than one thing or anything at all. Definitions that only refer to one thing or to nothing that fall under them are concepts too. 'The biggest prime number before 10' and 'the largest natural number' are two examples of such concepts respectively.

Although Frege's account of a statement is different from Kant's categorical judgement, a piece of writing in *On Concept and Object* would help us to compare these two: "a concept is the [Bedeutung] of a predicate; an object is something that can never be the whole [Bedeutung] of a predicate, but can be the [Bedeutung] of a subject."[59] So, unlike Kant who holds both subject and the predicate of a categorical statement are concepts, Frege holds that a concept cannot be subject of a statement. Actually, for Frege, the only function of categorical statements is putting objects under concepts.

The above-mentioned idea looks simple at first glance and makes sense in formal language, but it raises difficulties in natural language.[60] The problem raises when we want to talk about concepts. In the Fregean account, he is actually talking about things that fall under those concepts and it makes it difficult to talk about the concepts themselves. Treating concepts as names for extensions of them makes it possible that an extension falls under another concept. In other words, the class of objects (In the *Grundlagn* literature

[59] [Frege, 1951], p.173

[60] Some of these difficulties have been discussed in detail in [Russinoff, 1992].

the extension of a concept is called class)[61] that are named by a concept or fall under that concept, are an extension of another concept. Dummett attributes this definition of membership to Frege: "*a* is a member of *b* if there is some concept F such that *a* falls under F and *b* is the extension of F".[62] For instance, every token even number falls under concept 'evenness' and the set of even numbers is the extension of this concept (evenness), therefore, every token of even numbers is a member of the set of even numbers. Likewise, all the even numbers fall under the concept of 'being a rational number' and the set of rational numbers is the extension of this concept (being a rational number), therefore the set of even numbers is a member of the set of rational numbers.

In the preface of *Begriffsschrift*, Frege mentions "inadequacy" of natural language to show logical structure of judgements[63] and that he has replaced subject and predicate with argument and function respectively.[64] So it is not surprising that his analysis of categorical judgements is different from Kant's. In both the language he has invented, and in the first order predicate logic we use these days, there is no distinction such as subject and predicate. He clearly states that the distinction is not a matter of importance, for what he cares about are the consequences derivable from a judgement when it is combined with other certain judgements.[65] In his account there is no difference between conceptual content of (j) and (k).

j. "The Greeks defeated Persians at Plataea"

k. "The Persians were defeated by the Greeks at Plataea"[66]

This is another illustration showing the difference between the two approaches to the notion of concept. From a Kantian point of view, 'defeating' and 'being defeated' are two different concepts as every concept should be expressed as a one place predicate. Frege may accept this difference though rejecting its importance from the logical point of view. That is logical derivability does not represent conceptual differences necessarily.

However, Frege is not right here. To see why let us consider the following two sentences:[67]

l. Most of the Greek soldiers defeated most of the Persian soldiers at Plataea.

[61] [Dummett, 1991a], p.91
[62] Ibid. p.92
[63] [Hejienoort, 1976], p.5-6
[64] Ibid. p.7
[65] Ibid. p.12
[66] Ibid. p.12
[67] Sentences are inspired by Greg Restall's course examples.

m. Most of the Persian soldiers were defeated by most of the Greek soldiers at Plataea.

Now consider a model of three Greek and three Persian soldiers, and consider that G_1 has defeated P_1 and P_2, G_2 has defeated P_2 and P_3 , and P_3 has defeated G_3. In this model (l) is true but not (m). If we accept that nothing is different between (l) and (m) but involved concepts of 'defeating' and 'being defeated', then we are forced to accept that they are the reason of difference in truth conditions between these two sentences. Therefore, Kant and Frege's disagreement over analyticity of arithmetic statements can be explained in another angle. The notion of concept is rigid for Kant such that no conceptual turns and twists helps us prove arithmetical judgements. We need intuitive forms to prove arithmetical judgements. That is why these judgements are synthetic for Kant. While Frege's notion of concept is more flexible; such that conceptual turns and twists are enough for proving arithmetic judgements. Therefore, these are analytic.

In the next two chapters I shall explore many of the themes raised in this chapter in a more detailed manner. Chapter Four focuses more on the development of the notion of analyticity from conceptual containment to truth in virtue of the meaning. And chapter five is devoted to a better understanding of analytic/synthetic distinction in context of proofs, justificatory role of intuition in it, and a different approach to appeal to intuition in logical proofs.

Chapter 4

Analytic Truth, Analytic Justification

In the previous chapter we reviewed Kant's and Frege's understanding of analyticity in the context of a selection of their works. This chapter probes their ideas in the context of more developed logic and philosophy of language. We shall examine the possibility of marrying Kant's and Frege's account of analyticity by seeking a unifying feature for truths and proofs. To recap, here are Kant's and Frege's positions on analyticity:

Kant: a. If the concept of the subject contains the concept of the predicate, then the statement is analytic; otherwise it is synthetic.[1]

Frege: b. The Analytic/Synthetic (and a priori/a posteriori) division concerns the justification of making a statement and not the content of that statement.

c. If a proof (justification) of a statement rests on logical truths and definitions, then the statement is analytic. If it rests on non-logical facts (if we consider theorems of logic as facts), then the statement is synthetic. If facts are not general, then the statement is a posteriori.

d. The Analytic/Synthetic (a priori/a posteriori) division is not applicable once there is no proof for a statement.

A closer look at the Fregean account of the analytic/synthetic distinction reveals a tension within the definition. On one hand, in (d) it states that the distinction is not applicable in cases there is no justification available in form

[1] One may think that it is not a definition because it does not have the form of iff. Although it seems that both Kant and Frege were about defining the notion of analyticity.

of proof. On the other hand, judgement about analyticity or syntheticity of an statement, in case there is a proof, is defined based on the basic assumption statements which does not have proof (and therefore neither analytic nor synthetic). If we treat Frege's approach in a strict way, that is if we take proofs as a formal entities subject to validity in different logical systems, then there are a number of things that are neither analytic nor synthetic. For example, logically false statements relevant to any specific logical system, logical axioms, and presupposed facts. On its own, not being exhaustive is not a big failure for a distinction. The thing is that the definition does not provide any useful explanation why some statements derived from some non-analytic, non-synthetic statements are themselves analytic or synthetic.

To see the problem more clearly, let us consider classical logic just as an example of a logical framework. There is no proof for $p \wedge \neg p$, and according to (d), this is neither analytic nor synthetic.[2] However, intuitively speaking $p \wedge \neg p$ is analytically false. One response to this point is that, within the context of the classical logic, we have a disproof for $p \wedge \neg p$ that is we have a proof for its negation $(\neg p \vee p)$, so it makes sense to say $p \wedge \neg p$ is analytically false. This solves part of the problem, but does not help us with logical or factual axioms. Formally speaking, no atomic proposition is analytic or synthetic because no atomic proposition is a logical theorem. It does not matter if we come across P by observation or we derive it from $p \wedge q$. Also it does not matter what p means. For instance, let us consider the statement 'all bodies are extended'. This statement is analytic according to Kant, because the concept 'body' is contained in the concept 'extended'. Frege's account does not provide a straightforward answer in this case though. If 'body' is defined as 'something extended in space' then one can argue that 'all bodies are extended' logically follows from the mentioned definition and therefore it is analytic. However, if 'all bodies are extended' is taken as a general truth, then anything derived from it is synthetic a priori, while the statement itself remains neither analytic nor synthetic. This account does not explain why a proposition derived from a set of definitions, general truths, particular truths is analytic, synthetic a priori, and synthetic a posteriory respectively.

One possible solution is to extend the definition of analytic/synthetic distinction to cover truths expressed by atomic propositions. Dummett has such a suggestion to fix the problem.[3] He thinks since Frege does not consider the content of a statement but its status, the proof that he talks about is a proof in propositional logic. And if we were able to form proofs that, as Frege proposes, take us back to ultimate grounds of holding the statement true,

[2]Thanks to Andrew Parisi for his contribution here.

[3][Dummett, 1991a], p.23-24

then these ultimate grounds should be atomic propositions like P. These atomic propositions do not have any proof in propositional logic, that is, their truth cannot be established by any proof. It is because they are the smallest elements of language in propositional logic. Therefore, we need to extend Frege's definition and say if the ultimate grounds of a statement are particular contingent facts then the statement is a posteriori, if they are general laws, then the statement is synthetic a priori. And if they are general logical laws, such as say $a = a$ then the statement is analytic. This means that if we want to make sense of Frege's idea, we need to introduce the analytic/synthetic division for atomic propositions too.

However, if we define the analytic/synthetic division for atomic sentences, then we will lose the connection between logic and analyticity as Frege had in mind (his definition works on propositional logic as we saw). Actually all of Frege's emphasis on logic to define analyticity will lose point once we start looking for a theory of analyticity based on relations among sub-sentential elements of language. Then, what could he mean by stating that only if a statement rests on logical facts it is analytic? What are logical facts about sub-sentential elements? One way is to take the definition of sub-sentential terms as part of logic. However this way says nothing about justification of those definitions. A slight change in reading Frege's criterion will be illustrative; we can think that Frege actually wanted to say something about justification. He meant to point to the idea that justification of logical truths and inference rules used in proofs share similar properties that make both of these analytic. This approach seems to be suggested by Dummett.[4]

What has been mentioned already suggests two possible paths to take: one is to keep thinking of analyticity as a property of truth and the other is to think of it as a property of justification. Each one of these lines of thought has its own consequences as analyticity will become a property of two different things, truths in one and justifications in the other. In both cases definitions need to be exhaustive, that is, applicable to every statement of a language even atomic ones. If we think of analyticity as a property of truth, then the agenda will be to give a satisfying account of analyticity that covers anything that can be true or false. And if analyticity is taken to be a property of justification the theory needs to explain what is in common between the justification of analytically true or false atomic propositions and logical laws of inference.

If we take the first path, then to build a connection between analyticity and logic, we need to explain how logical truths are analytic. Now if we also adopt a compositional approach to language, it seems natural to define

[4]Ibid. Chapters 3 and 4

analyticity for atomic propositions first and then to expand it to proposition operators. This is not the way that Frege insisted analyticity should be defined as just has been discussed. He wanted to define analyticity based on logical truth. In other words, it seems that he wanted to define analyticity with the aid of some properties of logical truths.

If we take the other path and think of analyticity as a property of justifications, then to show the connection between analyticity and logic, the agenda will be to explain how justification of logical laws of inference are similar to justification of some atomic propositions. This way does not give any priority to atomic propositions and is capable of adopting a Fregean approach, that is, by giving priority to logic to define analyticity. To pursue this approach we also need to read Frege and Kant in such a way that shows the connection between this new notion of analyticity and their understanding of analyticity (to justify our use of the word 'analytic' for this new notion).

Each of these ways also needs to address syntheticity as well. That is, the definition of analyticity should be such that it gives a reasonable account of what is synthetic truth or justification. Specifically, it is good to know how each of these different positions helps us to deal with arithmetic truths. After all, Frege's definition of the analytic/synthetic distinction is meant to work such that arithmetic truths are considered analytic. And there are several issues concerning whether arithmetic truths or proofs are purely logical or not. So it would be a bonus for any account of analyticity that either explains how arithmetic truths or proofs are analytic or shows how they are synthetic.

Dummett suggests a reading of Frege that takes the second approach. It can be said that he thinks of proofs (in general) as a priori ways of verifying statements (atomic or complex propositions) and logical proofs as analytic.[5] As we saw, Frege was thinking of proof as justification of holding some statements true. Among the different logical systems developed in the twentieth century there are families of logical systems that do Frege's expected job, that is, giving the ultimate grounds of holding a statement true, Sequent Calculus and Tableaux systems. Relevant formal matters of the logical system shall be examined in the next chapter in detail. For the issue at hand, the existence of such systems can be taken for granted. Any statement that has a complex logical structure can be analysed to the level of atomic propositions in propositional sequent calculus. These atomic propositions can be arithmetic equations or general laws of a specific science or a particular contingent fact.

Reliance on logical truths which was Frege's criterion for analyticity, was

[5]In [Dummett, 1991a] Dummett mostly talks about arithmetic equations as a priori truths perhaps because we need proof for them, also in the first sentence of page 28 he call a logical theorem analytic because what renders it true are logical rules.

due to Frege's axiomatic system. Now that logical truths usually have been replaced by logical rules Dummett's interpretation of Frege can be seen as an attempt to meet a two-folded aim; explaining the difference between arithmetical equations and other sorts of atomic propositions. It also shows the similarity between arithmetic equations and logical laws (to save Frege's intention of raising the definition expressed in (b), (c), and (d)). One possible way of reading Dummett here is that he is appealing to the general notion of proof as a sort of justification. The notion of justification has been broadened such that logical proofs are not the only source of justification. In this way atomic propositions also may have justification as specifically arithmetic equations. Although their justifications are not logical proofs they may have common properties with logical proofs, namely being a priori. It seems that this is the line that Dummett has followed without explicitly stating that.

As a matter of fact, Frege gives his definition to define both the analytic/synthetic and the a prior/a posteriori distinction. Perhaps because of finding Frege's definition unsatisfying for the analytic/synthetic distinction, Dummett tries to read Frege's definition by emphasising the a priori/a posteriori distinction. This reading contains a noteworthy epistemological explanation. He thinks that the epistemology implicit in Frege's redefinition of the analytic/synthetic distinction is more interesting.[6] According to him, Frege holds that statements can be known either in an a priori way or in an a posteriori one (ignoring the possibility of statements that cannot be known at all). Although these two ways are not mutually exclusive, Dummett believes that in each case there is one intrinsic way for the verification of truths of a statement in accordance with the meaning of the statement. For instance, the intrinsic way of verifying the truth of "$13! = 6227020800$"[7] and "April comes after March"[8] is a priori while the intrinsic way of verifying 'the gravity force for any mass that is too lighter than earth is approximately 9.8 anywhere near the earth surface' is a posteriori. Therefore, once there are both a priori and a posteriori ways to verify the truth value of a statement, only one of them is the intrinsic or essential way of verification according to the meaning of the statement. And the other ways are accidental or non-essential in this regard.[9] For example, let us consider two ways of knowing that Fermat's conjecture is actually a theorem. One way is by the testimony of a mathematician, and the other by understanding a proof given for that.

[6]The title 'A finer classification' that he uses in [Dummett, 1991a] page 26 suggests that.

[7]Ibid. p.28

[8]Ibid. p.27

[9]He does not say it explicitly though the way that he talks about the intrinsic or accidental way of realizing the truth of a statement suggests it.

The first way is an a posteriori way and non-essential for our grasp of the meaning of the statement that expresses the theorem. And the second way is an a priori way and essential for our grasp of meaning of the statement that expresses the theorem. Here is his idea that connects the meaning of a proposition to the verification of that proposition:

e. The meaning of a proposition must be given in terms of what renders it true (independent of how we recognise it as true).[10]

Dummett does not define the a priori way of knowing a statement, that is, the way that we verify the statement as true, but he gives an example of an a priori way of knowing a statement. Once we consider a statement that constitutes the meaning of its elements (the subject of an atomic proposition in the simplest case), we know the truth value of that statement in an a priori way. His examples are "there are seven days in a week" and "April comes after March".

Among all possible a priori and a posteriori ways of knowing the truth value of a statement, deductive reasoning is an a priori one because similar to the case of "there are seven days in a week" logical rules constitute the meaning of logical words. And logical words are elements that determine the truth value of the statement. For instance, if we consider natural deduction systems, or sequent calculus systems, introduction and elimination rules give the meaning of logical words. In other words, among all sorts of justification for holding a statement true, deductive reasoning is an a priori justification. However all sort of statements cannot be verified as true by deductive reasoning. Specifically if a statement is an atomic proposition it does not have deductive justification in the sense that atomic propositions are not theorems of logic. Arithmetic equations can be instances of these atomic propositions, so they do not have logical proofs at least in propositional logic. But Dummett wanted to defend Frege's idea that arithmetic equations have very similar properties to logical rules (mainly, a priority).

To do so he coins the notion of 'claimant to apriority'[11] to refer to those atomic propositions that the intrinsic way of verifying their truth value is an a priori way. He claims that this notion separates arithmetic equations from theorems of propositional logic in the sense that the meaning of atomic propositions in a random instance of a logical theorem has no role in the truth of the theorem while the meanings of numbers on two sides of an equation do. That is meaning of sub-sentential elements is engaged in the process of verifying the truth of an arithmetic sentence while it is not the case for

[10]Ibid. p.24
[11]Ibid. p.28

a theorem of logic. He compares a theorem of propositional logic with a mathematical equation. If we take an instance of a theorem as our example of a statement, then people may know that (the instance of a theorem) not because they know logical rules, but because they know the meaning of the propositions which appear in the instance. The following is an example of such a situation: based on her experience in fixing cars, an automotive electrician may hold that 'if either the battery is discharged or the wires are disconnected the lights do not work' and if we ask her why she holds this as true she argues that 'if the battery is discharged the lights do not turn on, also if the connecting wires are disconnected the lights do not turn on'. According to Dummett, the electrician knows the following statement, which is an instance of a logical theorem, in an a posteriori way:

1. If the battery is discharged the lights do not turn on, also if the connecting wires are disconnected the lights do not turn on, so if either the battery is discharged or the wires are disconnected the lights do not work.[12]

He holds that in the case of instances of logical theorems knowing the meaning of atomic propositions that are a posteriori propositions (the intrinsic way of verifying them is an a posteriori way) may give us a way to verify the instance (though it is not the intrinsic way of verifying the truth value of the statement). While when it comes to arithmetic equations, the meanings of numbers are intrinsic for knowing the truth value of the equation. And the way of verifying equations is also a priori. In other words, they are 'claimant to apriority' or the intrinsic way of verifying their truth value is an a priori way. In these cases, even if we know them in an a posteriori way (using a calculator, or by testimony) we know that there is an a priori way to verify their truth value (they have proof or disproof). This suggests that arithmetic equations are similar to sentences like "there are seven days in a week" and logical laws and logical theorems in the sense that all of them are 'claimant to apriority'. However, arithmetic equations and sentences like "there are seven days in a week" differ from logical theorems in the sense that the meaning of subatomic elements are relevant to the verification of their truth (they need to be analysed at the predicate level).

Let us sum up Dummett's point: He divides statements into three categories. Some statements are not or cannot be known regardless of verification method (Dummett does not give any example of them, so it is not clear that statements with infinite elements are what he has in mind or statements that their truth value cannot be determined classically in certain circumstances,

[12]Dummett does not give this example, the example is mine.

that is, they are not true or false). Some statements are such that according to their meaning an a priori way is the intrinsic way of knowing their truth value; let us call them a priori statements. Arithmetic equations, statements that constitute the meaning of their subatomic elements, and logical theorems are examples of this sort. Some statements are such that the intrinsic way of knowing their truth value is an a posteriori way; let us call them a posteriori statements. Atomic propositions that express facts about particular objects are good examples of this type of statements.

However, we may come across verification of an a priori statement in an a posteriori way or the other way around. Atomic propositions belong to one of these three categories. Logical theorems are analytic because they are true based on logical laws; therefore they are a priori since logical proofs are a priori ways of verification. Also Dummett wants to make a difference between cases in which atomic propositions of a logical theorem are arithmetic equations and those in which atomic propositions of a logical theorem are particular contingent facts or general rules (those propositions that the intrinsic way of verifying their truth value is a posteriori). The difference is that although particular contingent facts may be inferred in an a priori way, their intrinsic verification method is a posteriori while arithmetic equations even if they can be known in an a posteriori way, the intrinsic way of verifying them is a priori. This forces Dummett to admit the role of the meaning of subatomic elements in arithmetic proofs and expansion of the notion of proof further than logical proofs where logical rules are introduction and elimination rules in propositional Logic. To put it in a straightforward way, logical rules are analytic, and in case of mathematical sciences and particularly arithmetic statements, the starting point of a proof is a number of a priori truths while in other sciences the starting point is a number of a posteriori truths.

One issue with Dummetts's thought is that his idea about an intrinsic way of verifying a statement is controversial. The controversy may come in two shapes; rejecting a priori/a posteriori ways of verification or accepting the distinction while rejecting the idea that one way of verification is intrinsic in accordance with the meaning of statements or propositions. We have examined a priori/a posteriori ways of justification in chapter one. To recap, we ended up considering a priori justification as any justification that roughly relays on understanding the meaning of expressions of a language. Where understanding an expression can be measured by the inferential role that expression plays for the speakers of a language. Applying this criterion to logical theorems plus accepting the claim that inferential rules are meaning constitutive forces us to consider logical proofs as a priori ways of verification.

Let us focus on the second concern. If there are different ways of verify-

ing an atomic proposition, then can we take one of these ways as intrinsic to the meaning of the proposition? For instance, Dummett holds that the intrinsic way of verifying "there are seven days in a week" is a priori, that is, understanding the meaning of 'week', 'seven days' and 'there are' is enough for verifying the proposition. Now let us imagine someone's understanding of week is based on holding the view that 'a week starts with a Monday and finishes with a Sunday' and she starts counting days to discover the number of days in a week which is a rather a posteriori way to verify "there are seven days in a week"[13] and it is still based on a correct grasp of the meaning of 'week'.

However, in the case of arithmetic knowledge Dummett's distinction is not easy to challenge. Although many people might think that we have learnt mathematics empirically, it is far from the actual practice of mathematicians. We may affirm that based on our experience every even number can be written as the sum of two prime numbers, but it is generally accepted that it is not a good reason for accepting this fact as a verification for 'every even number can be written as sum of two odd numbers'. What we need is a proof; and if such a proof can be found or constructed, most likely we hold that it is an a priori one. Also in the case of propositions that express particular contingent facts, we can consider cases in which a particular contingent fact (that the intrinsic way of verifying its truth value is an a posteriori way) follows from a number of premises that are particular contingent facts or general rules (the intrinsic way of verifying their truth value is an a posteriori way) by use of logical rules. In these cases the process of deriving the particular contingent fact (conclusion) from other particular contingent facts or general rules (premises) is a priori. That is the theorem that can be made by the conjunction of premises as antecedent and the conclusion as consequent is an a priori truth (the intrinsic way of verifying its truth value is a priori). Although it still sounds quite reasonable to think of the a posteriori way as the standard way of verifying such statements.

The moral of examining Dummett's reading of Frege is that taking analyticity as the property of logical laws (at the level of propositional logic) does not relieve us from defining analytic/synthetic at the level of atomic propositions. And it does not help us to draw such a distinction among atomic propositions. For instance, we cannot distinguish between arithmetic truths and particular contingent facts. That is why Dummett commits to broadening the notion of justification, from logical proofs to any proof that is a priori in a similar sense logical proofs are a priori. An example would help to illustrate this point: what is called the basic sequent in Sequent Calculus

[13]Thanks to Andrew Parisi for drawing my attention to this example.

has the form $A \Rightarrow A$ that in specific case of atomic propositions it becomes $p \Rightarrow p$. By this level of analysis we cannot distinguish the following three cases (where a stands as a name or constant):

2. $p : a = a$

3. p : Everything that is red is coloured.($\forall x(Rx \rightarrow Cx)$)

4. p : Everything that has mass has weight. ($\forall x(Mx \rightarrow Wx)$)

Though if we extend our logic (and therefore the notion of proof) to the level of predicates, then we can see the difference between an arithmetic basic truth and other basic truths. There is a formal triviality in (2) that is not in (3) and (4). Frege's hope was to show that in an arithmetic proof, the proof has a ground in the form of (2) though the vocabulary is more expanded than propositional logic (identity). So if we could argue for analyticity of $a = a$, then it would support Frege's point about analyticity of arithmetic truths, albeit some of those. Dummett has reasons to not extend the notion of analyticity to the level of predicate logic and just consider it as an a priori justification. The reason is the difficulties with Frege's attempt to show analyticity of arithmetic (we shall deal with some of these concerns in the next chapter).

One way of connecting logic and analyticity is to use a logical system such as Sequent Calculus to reach at the atomic ingredients of an expression and then explain how the truth of those ingredients are similar to the correctness of rules of the applied logical apparatus. However, even if we call arithmetic equations analytic still there is no explanation for (3), which is analytic in a Kantian sense, and how it differs from (4). We should note that, in (2), there is an interpretation of a form that convinces us about analyticity, there is no such a formal difference between (3) and (4). It seems that even if we could give a clear account of analyticity in terms of proof, there are still cases left in need of explanation.

There is also another way to connect proof (as justification) and analyticity (having natural deduction systems in mind): if we define analyticity as a property of proof (given it is done), then analytic arguments are those that have analytic proofs, and arguments with non-analytic (synthetic) proofs are synthetic. In this account, analyticity is attributed to arguments and not atomic propositions. The motivation is that we can consider inferential moves such as the move from $A\&B$ to A as analytic while if we come across A by experience, then the justification of A synthetic. We shall see such an attempt to define analytic arguments in the next chapter.

None of the aforementioned accounts of analyticity can explain the difference between (3) and (4).[14] Therefore it can be said that the advantage of an account of analyticity in terms of justification is that, in the ideal scenario, the connection between logic and analyticity can be clarified formally. The drawback is that it is not covering every intended case. That is, if we care about the historical context in the discussion, then there are cases that are analytic regarding analyticity as the property of truth that are not analytic in the former sense. In the rest of this chapter we shall examine attempts to give a definition of analyticity as a property of truth. In each case we will also try to see if there is any connection between analyticity as a property of truth and logic as a formal practice.

4.1 Analyticity as a Property of Truth

One way to understand Kant's definition of the analytic statement given in (a) (If the concept of the predicate contains in the concept of the subject of the statement is analytic, otherwise it is synthetic) is as follows:

f. If the concept of the subject in some way contains the concept of the predicate then truth value of the statement can be determined by solely analysing sub-statement elements, such a statement is analytic.

Although Kant's definition suggests a way for defining the truth value of an atomic statement, the idea can be (and has been) expanded to the case of more complex statements too. As we have seen in the previous chapter, by conceptual analysis, Kant refers to a mental process that can be traced by its linguistic demonstrations. We shall refer to this approach to conceptual analysis as the Cognitive approach. Also we saw that Frege has tried to change the subjective nature of Kant's conceptual analysis to a more objective one by giving a more detailed and rigorous account of conceptual analysis as an activity in language. From now on we will refer to this attitude toward conceptual analysis as the Objective approach. One way of explaining why the shift from 'conceptual analysis as a mental activity' to 'conceptual analysis as a linguistic activity' can be considered as 'giving a more objective account of analyticity' is that language reflects or represents what is called reality or facts, or the state of affairs. While mental activity, though truth apt, is a process that may differ from one individual to another. A possible

[14] In the previous chapter we saw that one possible explanation in regard to the difference between (3) and (4), in line with Kant's notion of concept, is that colour is a certain sort of property while weight is a relational notion. This makes sense once we have more than one mass.

way of following Kant's and Frege's approaches to conceptual analysis in a more modern philosophical context is to examine what has connected mental activities to linguistic ones, namely meaning.

The idea expressed in (f) connects to the notion of meaning in a straightforward way: we can define truth or falsehood of the analytic statements by examining the meaning of sub-statement elements that in case of atomic sentences are the subject and the predicate of the sentence. If mere conceptual analysis is not enough for defining the truth value of the sentence, it is synthetic. This definition relies heavily on 'meaning' and attacking meaning would affect analyticity. Here are some attempts to define meaning as a body of facts about language against scepticism with regard to meaning:

g. There are facts about language in terms of how we use it to refer to things, facts, and possible states of affairs. This can be considered as part of the meaning of the words in that language and analyticity can be explained in terms of these facts about language.

h. There are facts about language in terms of how we use it to infer what is true. These facts form part of the meaning of the words in that language and analyticity can be explained in terms of these facts about language.

Both of these themes have been developed as reflections on Quine's well-known scepticism about the existence of any fact of the matter about the meaning of linguistic expressions. Quine argued that since there is no fact of the matter about meaning, the analytic/synthetic distinction, that is based on assuming such facts, is a metaphysical claim.[15] Here metaphysical means mysterious (unexplainable by empirical science) features of the world. Both approaches to analyticity expressed in (g) and (h) can be seen as theories that explain (not necessarily empirically) meaning as facts about how we use language. One of them relies on features of the world, and the other relies on features of our understanding. Roots of the idea expressed in (g) can be found in Frege's departure from Kant. Following the used terms in the literature, we label the approach expressed in (g) as the Metaphysical approach to Analyticity (MA).[16] Here is one explanation for this choice: it

[15] "..., a boundary between analytic and synthetic statements simply has not been drawn. That there is such a distinction to be drawn at all is an unempirical dogma of empiricists, a metaphysical article of faith". [Quine, 1953], p.37

[16] the term "metaphysically analytic" used by Boghossian in his 1996 "Analyticity reconsidered" and many who have written on this topic. Gillian Russell reused that in [Russell, 2008]p.15. Apparently the naming refers to Quine's "Two Dogmas of Empiricism" in that Quine argues that 'truth in virtue of meaning' is relying on a(n) (unjustified believe) faith without any empirical evidence.

seems that there is a metaphysical assumption or hope there; namely that reality or states of affairs have a structure understandable for us and this reality actually is the source of the objectivity of the meaning of expressions of our language.

Rival ideas also have been developed to show how analytic truths can be defined avoiding metaphysical claims. Objectivity of meaning also can be cashed out in terms of facts about how we understand and infer things as linguistic activities which are truth apt. There are roots of the latter idea back to Kant's views on cognition as we mentioned in the previous chapter. From now on, this line of thought about analyticity will be addressed as the Epistemic approach to Analyticity (EA) (this labelling also has been taken from the same literature). As can be seen (g) and (h) have been expressed using the notion of the referential theory of meaning and the inferential theory of meaning respectively. The distinction specifically may become significant to understand the notion of 'belonging' or 'containment' used in (a) or (f) to express the relation between two concepts. Before examining the two mentioned themes, we shall have a short and selective look at the history behind the 'facts about a language'.

4.2 Facts About Language

Frege's attempt to give a linguistic character to the definition of analyticity is a part of a much bigger picture about language that was about to be drawn with more details in the first half of the twentieth century. In this picture a successful theory of meaning is one that defines what every single word refers to: either the object or properties of them, or the relation between or among them. These words make atomic propositions and each single proposition has a truth value. Among all words, logical connectives do not refer to any object or property of objects or relation among them, though they have a strong connection with the truth value of atomic propositions (if we do not want to commit to the claim that they are the truth function). So it can be said that logical words (at the level of propositional logic) are not referring to anything similar to what the other elements of the language refer to. Now if we use a language to talk about objects and their properties and relations in the world, say one of our natural languages such as English, then it can be said that logical words do not refer to anything in the world. They relate to truth value and truth value is the property of linguistic entities such as sentences and propositions. From this fact it can be concluded that logical words are about linguistic entities and facts about them are 'facts about language'. Of course it is not in accordance with Frege's ideas necessarily since from 'truth

value is the property of linguistic entities' it does not necessarily follow that 'truth has a linguistic nature'.

The idea that 'logical words are referring to linguistic entities' can be developed by saying that once a complex sentence is true because of the meaning of logical words appearing in that sentence, the truth of the sentence depends upon the words that are not referring to anything in the world. The meanings of these words are facts about the language itself. This line of thought leads us to another reading of Frege's attempt to define analyticity in terms of logic (though not the one he himself necessarily accepts). Here it is:

i. A sentence is analytically true if the truth value of the sentence only is determined by (depends on) the meaning of those elements that are not referring to anything in the world.

The definition goes well once we consider these two sentences:[17]

5. Brutus killed Caesar.

6. Brutus killed Caesar or did not kill Caesar.

Truth value of (5) depends on what has happened in the past, if there is an event in the past that makes (5) true then it is true, it is false otherwise. If someone accepts this analysis, then the very way of assessing the truth value of (5) guarantees that (6) is true without enquiring the past events. So far we have an understanding, though sketchy, about what does it mean to be true because of the meaning of logical words like 'or' and 'not'. There are at least three problems with this understanding of analyticity given we buy the story that has been told so far: it does not cover all the classic examples of analytic truths. For instance, this is Kant's example of an analytic sentence:

7. All bodies are extended.

Also these classical examples of analyticity differ from logical truths in an important way. To explain this difference let us consider another example:

8. All bodies are heavy.

If we concede (5) and (6), on one hand and (7) and (8) on the other, we see that a substantial difference between (5) and (6) is that to evaluate (5) sub-atomic elements of language are important , though they do not matter in evaluating (6). This property usually has been addressed as the 'topic

[17]The examples are Quine's in [Quine, 1963].

neutrality' of logical words. However, it is not the case about (7) and (8). They both have an identical logical (syntactical) structure, but (7) is analytic while (8) is synthetic according to Kant. This is exactly the same difficulty that we had faced a few pages ago where we could not explain the difference between (3) and (4). Words like 'body', 'heavy', 'extended' are not logical words and they refer to something in the world but they play a significant role in evaluating (7) and (8).

And the last and most important problem, the idea (the referential story about logical words and that they do not refer to anything in the world expressed in (i)) is not as neat as it looks at first glance. This point can be shown more clearly when we consider logical words in predicate logic. In the case of a language with predicates, quantifiers, and identity it is hard to say quantifiers and identity do not refer to anything outside the language (at least in the sense that an n-place predicate refers to the world). Quine's clever example to clarify the issue is this:[18]

9. Everything is self-identical $\forall x(x = x)$.

We may have really mixed intuitions about (9); it is difficult to say that it is not referring to anything out of the language. Merely saying that it is due to the meaning of '=' is not enough to clarify the situation. Either adopting this meaning is due to linguistic convention or 'being self-identical' is a property of things out there. What is expressed in (9) is referring to things outside of the language. This concern can be extended back to propositional logic as well. The fact that logical connectives are 'connected to truth' (they have a truth condition or are a truth function) and therefore are 'about language', is not mutually exclusive with them being 'about the world'. The point is that referring or not referring to anything in the world is not a good measure for being 'about language' or 'about the world'. Any theory that wants to consider meaning as 'facts about language' needs to have a clearer account of meaning.

Another way of understanding 'facts about language' is to consider it as facts about how we use language. This idea has been practiced to its extremes by Carnap. As a matter of fact, the way that Carnap dealt with language helps us to point to one important fact at least about formal language. The fact that we cannot overcome the limitations of a language even if we have explicit ideas about how we use it. Limitations such as those have been shown by Godel's incompleteness theorem.[19] On the other hand, the Carnapean approach to analyticity gives insights into the topic that can be seen in more recent accounts of analyticity. We shall see them shortly.

[18] Ibid. p.390

[19] A good discussion in this regard can be found in [Friedman, 1988].

The idea is that in the same way that we define the meaning of logical words (by introduction and elimination rules, say, in a Natural deduction system or axioms in an axiomatic system), the connection between some concepts that are represented by predicates can be explicitly expressed by means of what Carnap calls meaning postulations. For instance, we can always add a formal version of (7) to any formal system that is aiming to represent a portion of natural language or a scientific theory. There have been discussions over Carnap's terminology; what he calls syntax partly includes what is called semantic these days. Also he has a broader understanding of logic. For instance, meaning postulations can be part of his logical transformation rules. His formal definitions, though interesting, are complicated but, overall,[20] his measure for counting an expression as logical is that the content of the terms appeared in the expression is fully defined by language rules. Now if the meanings of terms are such that one can infer one statement from the other and the contents of terms are fully defined in the language, then this can be expressed in a logical transformation rule. So a meaning postulation can be a logical transformation rule in a certain language.

However, as Quine has mentioned[21] there is no syntactical feature that separates, say (3), as other expressions in a classical predicate logic. That is if one uses (3) as a transition from Rx to Cx in a line of a proof, no one can recognize that it is a meaning postulation unless we write it down in front of it, as a justification, that it is a meaning postulation. Formally speaking any proof containing (3) as a meaning postulation is not a theorem of logic in any ordinary use of term 'logic' as a formal practice.

Following a Carnapean line of thought and to avoid the three previously mentioned objections the idea expressed in (i) can be modified as follows:

j. A sentence is analytically true if the truth value of the sentence is only determined by (depends on) the language rules.

Regarding our purpose (giving an explanation for analytic truth) the definition over-produces analytic sentences. Actually, (j) per se is a trivial definition of analytic sentences unless we have a more restricted understanding of language rules. There are countless ways of categorizing rules for a language. One of the most reasonable ones concerns ways of concatenation of the smallest elements of a language to make more complex linguistic expressions. These rules are called Formation Rules in Carnap's literature. Some of these formation rules have an interesting property of being generative (most

[20]For more details look at his explanation about 'indefinite terms in syntax', 'transformation rules', 'content', 'logical and descriptive expressions', and specifically an example at p.178, all in [Carnap, 1937], p.165 to p.182.

[21][Quine, 1963], p.404-405

of them recursively). That is they give a particular structure to terms such that some relations hold among them. For instance, natural numbers can be produced as terms by formation rules. It is true that all the facts about relations among these terms cannot be proved (facts such as every even number can be written as a sum of two prime numbers), but the content of these terms are defined by formation rules. And those facts that can be proven (any theorem of arithmetic) have been proven relying on these contents entirely. According to Carnap that is why mathematical truths are analytic truths. It is true that these days we consider the language of proofs as a level higher that the language that produces numbers and the truth is not syntactical in a more exact technical sense, but the concept that Carnap has in mind is interesting and useful. Namely the idea of the degrees that content of a term can be defined in a language.

Other sorts of rules that Carnap considers are Transition Rules. Transformation rules are inferential rules, that is, rules of safe inference. An inference can be safe because of our observations. For instance from X has AIDS we can infer X has been infected by the HIV virus. This inference is safe according to our best knowledge so far. Or an inference can be safe because of how we have formed language expressions. For instance, from X is bachelor to X is male. The latter transformation rules are logical transformation rules for Carnap. If we could reduce all Carnapean logical transformation rules to classical logical inference rules and those that could be justified by logical inference rules and formation rules, then the following revision of (j) could be the case:

k. A sentence is analytically true if it is logically true or can be transformed to a logical truth replacing non-logical elements by their synonyms expressed in meaning postulations.

It works well in the following case:

10. All bachelors are male.

11. 'Bachelor' has been formed as an abbreviation of 'unmarried adult male'.

12. All unmarried adult males are male.

Alas not all Carnap's logical transition rules can be reduced to classical logical inference rules applied to formation rules. That is why he considers meaning postulation and takes them as logical transition. For example he would consider (3) and (7) as logical transition rules because they are meaning postulations.

Without appeal to formation rules, it is hard to see how an exact sense of 'facts about language' can be cashed out. We can make a list of safe inferences or transitions by adding the following to the existing list of (3) and (7):

13. X is north of Y, therefore Y is south of X.

14. X is the friend of Y; therefore Y is the friend of X.

And some non-safe inferences:

16. X loves Y, therefore Y loves X.

17. X is part of Y, therefore Y is part of X.

Perhaps if we ask Carnap, he would say that the meaning of 'north', 'south', and 'friend' are such that the complex expressions have fully defined meanings in the language of a particular geometrical theory and a theory about friendship respectively. And these fully defined meanings guarantee the transitions. And the meaning of 'love' and 'part' are such that the complex expressions have fully defined meanings in relevant theories, and transitions in (15) and (16) are not guaranteed based on these theories. Carnap's response can be interesting only if each of these theories is formed such that the claimed compositionality is demonstrable in a clear way (this is what we shall see in the next part). Although even then it is not clear how we can call them 'facts about language' because these languages are theories of something. They are not topic neutral and facts about them are facts about something, not language per se. Facts about the general use of language or in other words facts about ordinary use of linguistic elements can count as 'facts about language'. In the next part we shall see a recent attempt to give an account of analyticity based on general aspects of our use of language by giving a more complex account of meaning.

4.3 Metaphysical Analyticity (MA)

In the more recent philosophical context, many things have changed since Carnap's *Logical Syntax of Language*.[22] Syntax, semantic distinctions have become developed and well-established. Particularly developments in possible world semantics, which is a powerful tool to articulate truth conditions of expressions, has created significant changes in our understanding and analysis of sentences and their content. A well known example of such affects

[22][Carnap, 1937]

is Kripke's ground-breaking work on analysis of names using a particular notion of possible worlds.[23] Kripke understands names as rigid designators. That is, names designate an object and it is a crucial criterion in defining a possible states of affairs as 'metaphysically possible'. This understanding of names' semantics makes a big difference compared to definite description or cluster theory of the semantics of names. For instance, to evaluate truth conditions of 'Aristotle was fond of dogs'[24] metaphysically possible worlds to evaluate are those in which there is Aristotle. The worlds in which there is no Aristotle and someone else has done most of what is attributed Aristotle is not a metaphysically possible world.

This analysis challenged many classical ideas. Most significantly, it distinguished between the metaphysical notion of necessity and the epistemic notion of a priori. In Kripke's analysis there are necessary a posteriori truths such as 'Hesperus is Phosphorus' and 'Water is H_2O' as well as contingent a priori truths such as 'Stick S is one meter long at t_0'.[25] This also affected theories of meaning and consequently the notion of analyticity. Because analysis of the content of the concepts changed after this new understanding of the semantics of the ordinary language. It should not be far from truth to say that Kripke's contribution shed a new light to the discussion on facts about language/facts about the world distinction. Possible worlds semantics has been used widely to articulate both metaphysical and epistemic possibilities, as well as their connections to each other, since Kripke's work.

Frege distinguished between what he called *Sinn* or sense and *Bedeutung* or reference. The distinction is applicable to sub-sentential expressions as well as sentences. For Frege, sense of a sentence is a thought, while its reference is a truth value. Sense of a predicate is, arguably, a function that picks instances of the property expressed by the predicate. As Dummett has suggested this property is decidable, that is for every object we can say if that object has the property in question or not.[26] And the reference of a predicate is the set of its instances. Sense of a name or a definite description is a function that picks an individual while the reference of a name or definite description is an individual.[27] Carnap distinguished between *Extension* and *Intension*. He considered extension almost the same as reference and intension almost the same as sense. According to Carnap two terms are co-extensions when their extensions are the same. And two terms are co-intensions when their extensions are the same and they are *L-equivalent*

[23] [Kripke, 1980]
[24] Ibid. p.6
[25] Ibid. p.56
[26] [Dummett, 1973], p.216-217
[27] [Chalmers, 2002]

where L-equivalent terms are defined by meaning postulations.[28] For instance, 'equilateral triangle' and 'equiangular triangle' are co-extension expressions (intensional isomorphic in Carnap's terminology) while 'bachelor' and 'unmarried man' are co-intension expressions (synonymous in Carnap's terminology).[29]

It is now a common sense view among many philosophers that sentences are things like natural language sentences and propositions are the things that sentences express. Sentences may have ambiguity or, in Frege's and Carnap's terminology, be unsaturated or not interpreted, while propositions are closed or fully interpreted. As we saw in the previous section, one shortcoming of Carnap's view was that his account of analyticity was so theory-dependent. Now if a theory can give us an account of sentence assessment that relies on general aspects of our use of language, and in that theory it becomes clear that how sometimes the way that we use language can fully determine the truth value of the statement (even when it is not fully interpreted, or still its referent is ambiguous), then we may have a more general account of analyticity as truth in virtue of meaning.

Recent attempts to provide a theory of analyticity usually use possible world semantics to capture some familiar features of Fregean sense or Carnapean intension. For example, Chalmers formulates rigid designation as Kripkean intension. For the word 'water' Kripkean intension can be understood as 'what is the extension of water in the actual world'. While intension can be understood as 'what is the extension of water in this world'. Now if we consider a table with a vertical dimension as 'epistemic possibilities' or 'contexts of utterance' and a horizontal dimension as 'metaphysical possibilities' or 'contexts of evaluation' then the rows represent Kripkean intentions and the diagonals represent intensions.[30] Chalmers then analyses the diagonal intension, or as he calls it *two-dimensional intension.* A two-dimensional intension includes a *primary* and a *secondary* intension. A primary intension is a function from the context or possible world of utterance to the extension in that world. And a secondary intension is a function from the context or possible world of evaluation to the extension in that world. One can think of the context of utterance as an epistemic scenario or situation and of the context of evaluation as a metaphysical possibility in a Kripkean sense of metaphysics. Frank Jackson provides a similar analysis with actual and counter-factual dimensions.[31]

[28] [Parsons, 2016]
[29] Examples taken from [Jackson, 1998], p.34.
[30] [Chalmers, 2008]
[31] [Jackson, 1998], Chapter Two.

Gillian Russell gives an analogous analysis by a different terminology.[32] She attempts to spell out the role of each 'the world' and 'meaning' in determining the truth value of a sentence where meaning connects sentence S to proposition P and P is satisfiable but does not have meaning (in this line of thought a proposition is the meaning of a sentence; prepositions do not have meaning themselves). The following table is a possible way of understanding what S and P could comprise :

S	Name	Definite Description	Properties	Relations	Demonstratives
P	A set of one Object (O)	A set of one Object(O)	(Decidable) Set of Objects (SO)	(Decidable) Set of Ordered Tuples (SOT)	O,SO,SOT

A sentence is analytic when its truth value is determined fully by its meaning, that is, what connects S to P. Russell distinguishes four aspects of meaning, but she relates analyticity only to one of them. According to her, one may refer to four different things when talks about the meaning; though analyticity can be defined based on the Reference Determiner alone. Here are Russell's distinct meaning aspects:

Character: The thing speakers must know to count as understanding an expression.

Content: What the word contributes to what a sentence containing it says (the proposition P).

Reference Determiner: A condition which an object must meet in order to be the referent of, or fall in the extension of, an expression.

Referent/extension: The set of objects to which the term applies.

As can be seen Russell's 'reference determiner' is very similar to Carnap and Chalmers intension. According to Russell's theory, an analytically true sentence is one for which its truth value is fully determined by merely considering conditions that must be met by objects in order to be referents of sub-sentential elements of a sentence (the definition is designed for names and properties and not relations, but can be extended to relations as well). To have a clear understanding of what this means, Russell distinguishes three contexts: the context of introduction, the context of utterance, and the context of evaluation. We use different sorts of expressions in our language; for

[32] [Russell, 2008]

some of them 'the condition' that an object needs to meet to be the referent of an expression is the matter of the states of the world. For instance, whether something meets the conditions of being an extension or referent of H_2O or 'the shortest spy' is a matter of the states of the world. In other words, the referent of some terms is sensitive to the context of evaluation (which is the context in which truth values are assigned).

Also there are some other terms for them 'the condition' should be met by something to be the referent of them is 'insensitive to the world in a certain way'. That is these terms stick to their referent (if there is any) regardless of the context of evaluation, so they are insensitive to the context of evaluation. This property is called 'rigidity' by Russell following Kripke. Famous examples of these kinds of words are names and expressions we use for direct referring which have no meaning but their referent (so the content that they share in the expressed proposition is sensitive to the context of introduction, because they have no content to contribute in the context of evaluation if they don't succeed in referring to anything). Gillian Russell believes that there are two other sorts of rigid designators which are not directly referential; first, descriptions which pick the same object in every possible world because of what she calls 'substantial modal facts'. Mathematical equations are instances of these descriptions. The second rigid but not directly referential designators are what she calls 'rigidified descriptions'. The example of this kind of description, which was first mentioned by Kaplan, is 'the agent of this context' or 'I'. Given these two new sorts of rigidity, there are aspects of 'character' of some words that justify their rigidity (like 'I'). Since these two latter kinds of rigid designators are not directly referential, their reference determiner, that is the condition that they should meet to be the referent of the word, is different from their content, the thing that they share with the proposition (their referent).

To sum up, truth value of an atomic sentence S is determined based on P in the context of evaluation. The contents of sub-sentential elements contribute in P. If their reference determiner (the condition needed to be met to be the referent of an expression) is sensitive to the context of evaluation, that is, P in the context of evaluation is different from P in the other two contexts (because content differs from one context to the other), then the world determines the content of sub-sentential elements and hence the truth value of P. Also if the expression does not pick any object in the contexts of introduction or utterance and the referent is the only thing that the expression contributes to P, then there is no P in the context of evaluation to be true or false. Now if we could have the same P in all contexts and P was true (false) the sentence S could be called analytically truth (false) in the sense that the content of P is independent of the world. Of course if P is

true or false this is always the matter of the world. Here is the definition of truth in virtue of meaning (analyticity):[33]

l. A sentence S is true in virtue of meaning (reference determiner) if and only if for all pairs of context of introduction and context of utterance, the proposition expressed by S with respect to those contexts is true in the context of evaluation.

Analytic truths are not identical with necessary truths; Russell explains that this confusion happens because of considering the 'content' of an expression as its meaning, if one (like Russell) thinks of 'analyticity' as 'truth in virtue of reference determiner' then analytic and necessary truths do not coincide anymore because the reference determiner and the content are not the same. For instance, 'I am here now' is an analytic truth because, according to the reference determiner aspect of the meaning, it expresses: 'the object which meets the conditions of being the utterance of P in the context of uttering P is at the place (that meets the conditions of being) of uttering P at the time (that meets the conditions of being) of uttering P while the proposition P that it expresses (which is: Salman is in the Old Quad at 5pm Sunday April 14th 2014) is the same in contexts of utterance and evaluation and at the same time is a matter of pure contingency.

In this section we had a brief look at a family of theories that provide an analysis of analyticity as a property of truth. What is in common among all of these theories is appealing to the notion of sense or intension. However, more recent theories of sense have a significant different from Frege's account of sense. The source of informativity of a statement is the furniture of the external world in these recent theories of analyticity. As a result, they have no explanation for the source of informativity of, say, arithmetic equations. Chalmers states this point nicely:

> This [a posteriori] understanding of cognitive significance differs from Frege's. On Frege's account, a priori knowledge can be cognitively significant: the knowledge that 59+46 is 105 is cognitively significant, for example, because this knowledge requires some cognitive work. It is very hard to articulate this notion of cognitive significance precisely, however, and it is not clear that there is a useful precise notion nearby. The notion of apriority can serve at least as a useful substitute.[34]

At the end of chapter Two, we stated that our main goal in the rest of this investigation is how deductive reasoning is informative, or in Chalmers

[33] [Russell, 2008], p.56
[34] [Chalmers, 2002], p.150

terminology, cognitively significant when deduction is the only legitimate way of verifying an statement. In this section we saw two modern theories of analyticity as a property of truth. In the next section we will see why theories of analytic truth are not very helpful means to serve our end in this research.

4.4 MA and Logic

Let us start with Chalmers' two-dimentional semantics and his account of cognitive significance. He defines two kinds of intension; a primary intension which is a function from rationally or epistemically possible worlds to the extension of a term or a sentence. And a secondary intension which is a function from metaphysically possible worlds to the extension of a term or a sentence.[35] Something cognitively significant enters when there is a possibility of deference between these two intensions. For instance, 'Hesperus is Hesperus' is not cognitively significant while 'Hesperus is Phosphorus' is.

This theory of intension works well to capture the epistemic gain when there is a possibility of change in the extension of our terms. Such a theory, however, cannot capture the epistemic gain of identity statements with no possibility of change in the extension of the terms. As we just saw in the last section, arithmetic equations are, arguably, examples of such informative identity expressions. Hesperus could or could not be identical with Phosphorus, but the sum of 59 and 46 could not be any number other than 105 in any arithmetically possible world (which definitely includes Kripkean metaphysically possible worlds). Chalmers' two-dimensional semantics also creates an account for the a priori information which we shall review briefly in the next section.

Gillian Russell has stated tentatively that her account of analyticity can lead to an explanation of analyticity of logic; here is her definition of analytic validity:[36]

m. "A pair $\langle \Gamma, A \rangle$, where Γ is a set of sentences and A is a sentence, is analytically valid just in case the reference determiner for the conclusion is contained in the reference determiner for the premises."

If we adjust her definition of reference determiner for a sub-sentential expression[37] while considering her position that sentences refer to propositions and

[35] [Chalmers, 2008], p.586

[36] [Russell, 2008], p.102

[37] "A reference determiner for an expression is a non-circular condition, such that any object which meets it satisfies the expression and does so in virtue of meeting that condition..." [Russell, 2008], p.98

propositions are true or false, then the reference determiner of a proposition is its truth conditions. Therefore (m) changes to (n):

n. "A pair $\langle \Gamma, A \rangle$, where Γ is a set of sentences and A is a sentence, is analytically valid just in case the truth conditions of the conclusion is contained in the truth conditions of the premises."

Also logical constants are truth functions and these truth functions are rigid designators[38] and refer to the same thing in all the contexts. Logical rules (she calls them rules of implication to emphasise that logical consequence is about implication) are the reference determiners of these connectives which we accept by stipulation.[39] While, for epistemic considerations,[40] she may accept that logical truths (valid arguments), owe their truth (validity) to stipulations about the logical connectives, she holds that the ultimate justification of logical rules is truth preservation. Consequently, logical rules may change by changes in our theory of truth (logical truths or rules may change with changes in semantics and semantic may change for a better grasp of reality).[41] For instance, having more than two truth values changes some logical truths (like the excluded middle). Or might change some aspects of deduction such as admissibility of cut.[42]

Based on what has been said, Russell's position on the epistemology of logic is like this: logical truths (or rules) are not a priori where a priori means logical truths are justified by logical rules themselves or logical rules do not need justification. For Russell, they do need justification and they are subject to change according to what she calls 'inference to the best explanation'.[43]. Where the best explanation defines our theory of truth and a logical system is evaluated according to congruency with our best theory of truth.

Containment of truth conditions, as stated in (n), does not fit neatly with some basic logical rules. For instance let us consider the following two inferences:

18. $\dfrac{A}{A \vee B}$

[38]Ibid. p.60-61

[39][Russell, 2014] p.165-166

[40]Ibid. p.165-166

[41]Ibid. p.173

[42]If in the case of vague terms such as colour terms we think of truth in degrees, then cut is not admissible.

[43]In [Russell, 2008], p.170, after a discussion about the challenge of epistemic regress by, referring to the Achilles/the tortoise paradox, Russell regrets that she does not have any answer for rejecting the tortoise argument against the force of logic. In [Russell, 2014],p. 174 she concludes that since justification of logical rules is 'inference to the best explanation' and not logical rules themselves, logical rules are not circular.

19. $\dfrac{A}{B \to A}$

The truth conditions of the conclusions in these two inferences are not contained in the truth conditions of the premises as the truth conditions of those complex expressions could be defined based on the truth conditions of the elements which are absent in the premises. Russell is aware of this point And before giving her definition of analytic validity, she asserts that such inferences as (18) and (19) do not fit into her definition properly.[44]

Another point about her position is that although the reference determiner of an atomic sentence, that is the truth condition of it, is determined by the reference determiner (and not the content) of sub-sentential elements, what contributes to the truth value of the sentence is its content (the proposition).[45] This gap between the reference determiner as meaning and content as meaning makes Russell hold that logical truths are not epistemically analytic. That is, there is a gap between knowing the truth value of a sentence in virtue of the meaning of it (in the sense of knowing its truth condition) and knowing the sentence in the sense of knowing the truth value of the proposition it expresses. One may know "all vixens are vixens"[46] is a logical truth without knowing the truth value of "all vixens are vixens". It happens, say, when someone does not know what 'vixen' means or knows the meaning but suspends believing in the existence of vixens.

One consequence of Russell's distinction between sense (reference determiner) and content is that one can accept (20) as an analytically valid inference while considering the premise as analytically true and, at the same time, rejecting the conclusion as analytically true.[47] That is an analytically valid inference from an analytic truth does not lead to an analytic truth necessarily. Likewise, one can hold that an analytic truth can be inferred from a non-analytic truth in an analytically valid inference like (21). Of course the step in (21) is analytic only if the $a = a$ has been inferred from $\forall x(x = x)$ earlier in a proof.

20. $\dfrac{\forall x(x = x)}{a = a}$

[44]She gives ex falso quodlibet as an example but we have replaced it by disjunction introduction as ex falso is a controversial rule for many non-classical logicians who think similar to Russell about the justification of deduction.

[45]Ibid. p.166

[46][Russell, 2014], p.174

[47]The inference in (20) is not exactly her example, her example is $\dfrac{\forall x(Fx)}{Fa}$ in which both premise and conclusion are not analytic but the argument is. [Russell, 2008], p.102.

21. $\dfrac{a = a}{\forall x(x = x)}$

Based on the aforementioned distinction between sense and content, Russell does not accept 'Hesperus is Hesperus' as analytic because names are sensitive to the context of introduction. As a result, Russell confirms that the set of analytic sentences is not closed under logical consequence.[48] She uses this fact to introduce the notion of wide analyticity.

o. "A sentence is widely analytic just in case it follows by analytically valid rules from a set containing only analytic sentences".[49]

This is a wide notion of analyticity, for Russell, because it includes non-analytic truths that are the conclusion of analytically valid inferences from analytic sentences. Inferences from analytic truths in which a constant is substituted by a variable could be examples of such analytic inferences. However, it is hard to see how the same concerns that rise about the success of ordinary names in referring could rise about constants in cases like (20).

Although Russell's concerns about the content of names sounds like a not very interesting point when it comes to constants in an artificial language, the distinction between the content and the reference determiner (intension or sense) of an expression in general can be interesting. We saw that Chalmers' theory of intension could not provide a helpful explanation for the kind of information we gain from an arithmetic equation. The reason was that Chalmers' theory has not much to say about the process of finding the extension of an expression. Whether the extension of 'this expression' could be different from the extension of 'that expression' is the only source of information in Chalmers' theory. While the distinction between the content and the intension of expressions will be helpful for defining the epistemic gain in cases we do not know if the extension of 'this expression', whatever it is, is the same as the extension of 'that expression' but we know that if these expressions are co-extensional, they are so necessarily.

Russell has distinguished between the content an the reference determiner of an expression for the similar reasons to Chalmers. Chalmers is concerned about whether or not the extension of two expressions could be different and Russell concerns whether or not an expression has any extension at all. Therefore she admits that arithmetic equations are analytic, because even if we treat numbers as names, they do refer to something in every possible world and they refer to the same thing in every possible world. However we can explain the epistemic gain from an arithmetic equation by distinguishing

[48]Ibid. p.103
[49]Ibid. p.103

between the content an the reference determiner of an expression. Let us understand $5 + 7 = 12$ as 'the extension of $5 + 7$ is identical to the extension of 12'. Now to check if it is true we need to define the extension of $5 + 7$. To do so we need to consider the content of $5 + 7$ which includes an operation to define the extension in question. Without doing this operation, namely adding 7 to 5, we cannot confirm that $5 + 7$ and 12 are co-extensive. The process of operating is informative because it would make us able to confirm the truth of equation.

Interestingly, if we define containment as containment of reference determiners, then Kantian analyticity can be defined as containment of the reference determiner of the predicate in the subject. Now if we think of the above-mentioned arithmetic equation, the reference determiners of 5 and of 7 on their own do not contain the reference determiner of '12'. And if we think of an equation like a categorical statement, then Russell's distinction between content and reference determiner justifies the Kantian approach to arithmetic equations while asserting its metaphysical analyticity. However, there are serious doubts about identifying equations and categorical judgement. For instance, equations are two sided or symmetric while categorical judgement are one sided or asymmetric, that is from 'a is equal to b' one can infer 'b is equal to a', but from 'a is b' we cannot infer 'b is a' (we cannot infer 'animals are human' from 'humans are animal').

In the last two sections we examined accounts of analyticity that explain analyticity as a property of truth. The theories have a compositional approach to language and define analyticity for atomic bits of a language and then expand it to the molecular level. Therefore they has a methodological angle with Frege's attempt to define analyticity based on logic. However they have adopted a version of 'linguistic facts' that the truth of analytic expressions depends upon and this gives these theories a Fregean flavour. On the other hand, as we saw, some analytically valid inferences in one of these theories of analyticity do not coincide at least with classically valid logical inferences. Also logically valid inferences do not necessarily preserve analyticity in the sense explained in that theory. Moreover, one of the accounts presented here can accommodate an explanation for informativeness of arithmetic equations which is based on the gap between the reference determiner and the content of expressions.

4.5 Epistemic Analyticity (EA)

Another way of responding to Quine's scepticism about meaning is to understand meaning as a number of facts about language in terms of how we use

language to infer what is true. These facts form at least part of the meaning of the words and analyticity can be explained in terms of these facts about language. For instance, we may think of 'having extension' as part of the meaning of 'body' because the inference from 'x is body' to 'x is extended' is always a correct inference (we restrict ourselves to non-metaphoric cases that these words are used to talk about space). This is familiar to us; we have examined conceptual (inferential) role semantics, for usual logical words, in chapter two. Now we shall see one possible response according to this theory of meaning with regard to the issue of analytic truths. The theory is committed to the Quinean idea that truth has a firm connection to the world but also wants to leave space for a priori analyticity by saying that anyone who understands the meaning of the analytic sentence is justified in thinking of that as true.[50] As Boghossian says, this is an attempt to make sense of holding these two points together:

p. Propositions are mind-independent language-independent abstract objects that have their truth conditions essentially.[51]

q. An analytic explanation of the a priori is possible, where one understands 'a priori' as warrant for holding truth of a sentence (and not proposition) independent of sensory experience.[52]

Where 'a priori' or 'independent of sensory' means:

r. Mere grasp of the meaning of a sentence by an agent sufficed for the agent's being justified in holding the sentence true.[53]

Boghossian's concern is to keep propositions separate from sentences and truth from justification; most likely because, following Quine, he holds that the content of true propositions can be fixed only by an ideal complete theory. That is the one that has the truth value of every atomic proposition. There is no such a theory at hand, so the contents of true propositions are fixed just until 'further notice'. However, unlike Quine, Boghossian does not give up meaning because of unfixed content (note that content of a sub-atomic expression is what it contributes to the proposition it appears in. Since we do not have a complete theory we do not have all the propositions and hence content of our sub-atomic expressions are not fixed). He believes to find facts about meaning we need to consider the functions of language. At the end

[50] [Boghossian, 2003], p.15
[51] [Boghossian, 1997], p.332
[52] Ibid. p.333
[53] Ibid. p.334

of the day, these facts about meaning may not give us true propositions but may give us justification to hold some propositions as true.

We can think of different kinds of things that we do with language; accepting, rejecting, ruling, promising, and so on. Also sometimes we create some functions for words either by implicit or by explicit stipulations (in a few pages we shall see implicit/explicit stipulations). So far nothing is weird; we may stipulate how we use words to picture a fact. Things may look weird once it comes to the notion of a priori truth, because once a sentence is capable of being true or false, it means that it is factual i.e. if it is true it expresses a fact. This may cause two concerns; the general concern is whether we can make facts with stipulation. Or equally, whether we can gain knowledge by stipulation. In other words, can we think of truth (knowledge) by convention? The specific concern is - can we establish facts (truths) with defining functions for words? To ratify the general concern, it can be said that finding stipulated facts is not difficult. It is a fact that people drive on the left or right hand side of roads, but what makes it a fact except a convention?

Let us deal with the specific concern; it is true that finding linguistic stipulations is not difficult; we use 'I' to refer to the person who is the agent of the context of uterance, or we call whatever is extended 'body', and so on. But do these stipulations make our sentence true? To be more specific, is the very fact that a sentence S means proposition P enough for the truth of S? Or do we need P to be the case as well? As we have seen in the previous section, MA gave us an account of meaning (reference determiner) in which the fact that S means P is enough for S to be analytic even if we do not know what P is and hence what its truth value is. However, the fact that S means P sometimes can be known a posteriori because the conditions need to be met by the object to be the referent of an expression (reference determiner) may not be the same things that one needs to know to understand that expression (character).[54] Boghossian's claim is that the fact that S means P is enough justification for holding S as true a priori but this fact does not make P true. So he is not committed to a priori knowledge if we think knowledge includes truth. He is committed to a priori justification.

What is in common between Boghossian's (EA) and Russell's (MA) theory of analyticity is that the truth value of P plays no role in S being analytic. The three main differences between them (EA and MA) are that EA rests on meaning as character while MA rests on meaning as reference determiner. And that EA is about justification while MA is about truth. Moreover, EA claims on a priority but MA makes no epistemic claim on analyticity.

A priori is a tricky issue either as a property of knowledge or justification.

[54]One may understand 'water' without knowing its referent, that is H_2O.

In the case of stipulated or conventional facts (knowledge) like driving rules, although they are conventional facts, they are neither knowable a priori, nor to be held as a priori justifiable. For instance, the fact that people drive on the left hand side of the roads in Australia is neither knowable nor justifiable a priori. In the case of linguistic facts, if we give a certain role to some words, then does it seem plausible to think that if a user of a language can define the truth value of a sentence by just considering what she knows about these roles, then she knows the sentence a priori (or is justified a priori in holding that sentence as true)?

The issue that is raised here is that even if we give up a priori truth and claim a priori justification, learning a language that includes learning its stipulations too, is not an a priori process; actually it is very empirical. Then what does it mean to call a truth that we hold according to our knowledge of language as a priori? Here is one explanation: it is true that the process of learning a language is empirical, but once we learn it, it is an a priori knowledge in the sense that it is now part of our background knowledge. Now if based on this background knowledge of language we could define the truth value of a sentence, then we are justified in holding that sentence as true. And of course this knowledge is neither necessary nor non-refutable. However, the thing is that natural language is a very complicated phenomenon and it is really hard to think of knowledge of this phenomenon as coherent and ambiguity-free background knowledge. To articulate linguistic knowledge clearly, it is common to consider two kinds of linguistic definitions, explicit and implicit. (22) Expresses an explicit definition of 'bachelor' while (23) and (23) together give an implicit meaning to 'bachelor'.

22. Bachelor means an unmarried adult male.

23. Inference from 'x is bachelor' to 'x is an adult male' is always a correct inference.

24. Inference from 'x is bachelor' to 'x is unmarried' is always a correct inference.

It is difficult to accept (23) and (24) and reject (25) and (26).

25. All bachelors are adult males.

26. All bachelors are unmarried.

In the example at hand, the meaning of the word 'bachelor' is connected to the meaning of the words 'adult male' and 'unmarried'. It seems that the meaning of the complex expression 'adult male' is settled, but the meaning of

the word 'unmarried' is not. To settle the meaning of the word 'unmarried' we need to specify what we mean by 'marriage'. Are we emphasising the legal meaning or not? This chain of interconnection among words leads to Quine's well-known 'meaning holism' idea that casts doubt on any non-postulated fact of the matter about the meaning of the words. That is, Quine is sceptical about the existence of facts that connect S to P, because this connection rests on the notion of synonymy. And the synonymy connection among words in natural language is not exact unless we use implicit or explicit postulations of meaning. Quine's problem with postulated meaning was that we cannot have conventional truths.

Boghossian's responses to this Quinean worry is that first of all these conventional truths connect sentence S to proposition P. Our conventions are not responsible for making P true. Quine's consideration makes sense in so far as our stipulations make P true. That is, our stipulations cannot make Quineian 'facts of the matter' true. Secondly, we just select among the existing facts that connect S to P. Therefore it is not a stipulation as such, to say that our linguistic rules that connect S to P are mere stipulations requaires investigations that are outside of the scope of philosophy. So Quine's 'truth by convention' objection loses some teeth against Carnap as the Carnapean meaning postulations can be seen as matters of 'fact about inferential roles of words' without them intervening on the truth of P or being theory-base transition rules.

Boghossian's idea may meet Quine's concern about conventional truths, although some philosophers, perhaps including Quine, do not find this a priori account of analyticity attractive, interesting, or important philosophically speaking. The reason is that we have made the a priori reasonable at the expense of, firstly, disconnecting the a priori from interesting truths, that is, truths about the world. And secondly, being selective about less interesting truths, that is, truths about meaning, because if we do not select among facts about meaning, then the connection between S and P cannot be established (since there is no specific fact about synonymy). Once a priority is the property of justification and not knowledge, being refutable therefore not necessary, and with no interesting connection to the meaning (analyticity) then what is this distinction (a priori/a posteriori) for? Therefore, the distinction is either incorrect or useless.

One way of thinking of 'refutable knowledge' or 'non-necessary truths' as 'interesting or important' is having an alternative understanding of 'a priori'. The word 'a priori' was coined in relation to our mental or cognitive dispositions that make us able to analyse and understand our experience. If we add ideas such as (s) and (t) to the picture of a priori analyticity, then we may be able to give a different understanding of inferential theory of

meaning.

s. The empirical process of learning language is a process of tuning into, or calibrating our way of using words in thinking with other thinkers.

t. The meaning of linguistic expressions has meaningful connections with the mental activities of cognition and understanding.

It seems that 'the mere grasp of the meaning' that 'is sufficed for the agent's being justified in holding the sentence true' in (r) presupposes (s) and (t). Otherwise it is hard to justify using selective facts (implicit meaning stipulations) to ratify the Quinean 'meaning holism' objection about local meaning connections as a philosophically interesting matter.

No doubt arguing for EA needs more than what is said here; we just saw a sketch of a possible understanding of analyticity as epistemic justification. A well-developed theory about a priori and the connection between a priori and the way our cognitive capacities are can be found in Chalmers' two-dimentional semantics[55] and Jackson's account of conceptual analysis.[56] Very briefly, Chalmers defines a priori sentences as a sentences their first intensions are true in all rationally possible rational scenarios.[57] Two sections a go we saw that the first intension of an expression is a function from a rationally possible scenario to extension of that expression. To justify this definition Chalmers defines *qualitative* vocabulary as a vocabulary without terms such as names and natural kind terms (roughly speaking terms that make room for necessary a posteriori or a priori contingent truths).[58] Intuitively speaking, Chalmers' qualitative vocabulary includes terms of in a certain level of abstraction which are necessary product of the way we understand the world. For instance, 'Jones is taller than Smith' is entailed by the fact that Jones is six foot tall and Smith is five foot ten tall.[59] Then he explains how sentences including these vocabulary could be true in all possible rational scenarios. Jackson has a similar think in mind by conceptual analysis.

Our interest, from now, will be directed towards one interesting point about EA. The point is that analyticity as epistemic justification can serve as a notion that unifies logical rules of inference with analytically true sentences without any commitment to compositionality. Let's recall, at the beginning

[55]For an informative but brief introduction to two-dimentional semantics with lots of reference to the previous work look at [Chalmers, 2008]

[56][Jackson, 1998], the first three chapters.

[57][Chalmers, 2008], p.586

[58]Ibid. p.590

[59]The example is Jacson's in [Jackson, 1998], p.5.

of this chapter we mentioned Frege's attempt to explain analyticity with the assist of logical truths. And we saw how that method is unsuccessful in explaining analyticity. Also we saw how analyticity as a property of truth is independent of logic. That is, there is no need to appeal to logic to explain analyticity as a property for some truths. However, once we understand analyticity as an epistemic justification we can make sense of a Fregean attempt which was an attempt to explain analyticity with the assistance of properties of good reasoning. In this way of thinking, Frege's appeal to logical truths (instead of properties of good reasoning) is because of his logical apparatus and not a necessity for his project. But what are these properties of good reasoning and how does this idea work?

4.6 EA and Logic

In the section Four of this chapter we saw that one possible way to cash out epistemic fruitfulness of a sentence is that its truth value is not definable with merely considering the meaning of its elements (there are possible circumstances in which the sentence is false). We know that this way is not open to anyone who wants to explain the epistemic gain we receive from an analytic truth. The epistemic starting point looks the same for someone who does not know whether Hesperus is Phosphorus and someone who is wondering whether 13! is co-extensional with 6227020800. A significant difficulty for any attempt to explain the epistemic gain in the second case is finding an objective measure for defining when someone learns something new. Because whether or not an arithmetic equation is trivially true to someone or not depends on one's arithmetical skills. If anyone who knows basic rules of arithmetic could process as fast as a calculator, then it seems that all equations such as the above would be trivial for them. The same applies to any complicated theorem of predicate logic.

Understanding analyticity as analytic justification of inference moves opens up the opportunity to explore how far we can go based on the meaning of expressions alone. In the next chapter we shall see some possible ways in which we can provide some objective measures for conceptual complexity of an analytic reasoning. Before doing so, in the last section of this chapter we will consider three possible objections to the idea of analytic justification.

At least from Aristotle to Kant's time, in a good argument, subject matter does not change and we derive the conclusion from nothing but the accepted premises. Given these two features, and following the Fregean dream of defining analyticity with properties of a good reasoning, we are faced with two challenges: firstly, if we consider logical inferences (inferences based on

introduction and elimination of logical words) with analytic inferences (inferences based on the meaning of some non-logical words that have inferential influence), it seems that sometimes our subject matter changes in the latter cases. And secondly, logical inferences do not meet the second condition of good reasoning.

To explain the first point let us consider (3), (7), and (10); we may distinguish between sentences like (3) (all red things are coloured things), (10) (all bachelors are male), and (7) (all bodies are extended). The reason is that if we consider these as inferential rules that permit inferring 'X is coloured' from 'X is red' and so on, then the inferential rule corresponds to (7) will change the subject of the argument. As has been expressed in (k), (3) and (10) can be reformulated as logical truths in the form of:

27. $\forall x((Px \wedge Qx) \rightarrow Px)$

28. $(P \wedge Q) \rightarrow P$

While (7) cannot be reformulated in the same way. This difference can also be mentioned as the distinction between definition and postulation. Once we are talking about something red or someone as a bachelor, we are talking about a colour or a man respectively. This is not the case with (7); it seems that we are changing the subject from 'body' to 'extended'.

Inferentialists might accept this point, but reject it as a problem as it should not be a problem for an inferentialist who thinks that sub-atomic expressions do not have a fixed content. The content of sub-atomic expressions is fixed in accordance with the context of inference. 'Body' could mean a 'bunch of' in another context. Hence, this does not sound like an appropriate way of distinguishing logical inferences from analytic ones for an inferentialist. That is, if we justify logical rules appealing to inferential role semantics (like the approach that we saw in chapter one and two) it is not consistent to accept the 'subject matter' objection against (7).

About the second difficulty, not using any resource other than premises, it should be clear what these resources are and what it means to not using anything further than premises. As we have seen before in this chapter, inference moves such as (18) (disjunction introduction) and (19) (vacuously discharging a premise) are formally against this idea. There are branches of logic that ban these moves, but the majority of logics allow such moves. However there is one way of understanding this requirement of good reasoning that keeps (18) and (19) inside the area of good reasoning. The idea is that what appears in the premises or the conclusion appears in the proof from premises to the conclusion as well. The idea is that a good proof is not a

roundabout. This idea has been put forward by Gentzen[60] and it has had a huge influence on logical studies since then. How to make sense of this interpretation of analyticity requires a more detailed examination which is the focus of the next chapter.

So far in this section two possible difficulties in the way of finding similarity between logical inferences and analytic inferences have been considered. The first problem does not look like a knock down hit to the idea of finding similarities between formal or logical inferences and non-formal ones. And the second one needs more examination. Another issue that might indicate an important difference between logical (formal) and analytic (non-formal) inferences is the existence of structural rules. Putting emphasis on structural rules like weakening as distinctive features of logical inferences is a way of distinguishing logical inferences from analytic inferences.[61]

However, at least in the case of inference from 'x is body' to 'x is extended', any possible additional information about x that may turn the inference to a bad one actually changes the meaning of 'body' as well. For instance, one way of invalidating the inference from 'x is body' to 'x is extended' by adding more information about x is to add 'x is data'. So it can be said that while (29) is valid (30) is not.

29. x is body, therefore x is extended.

30. x is body and x is data, therefore x is extended.

What makes (30) invalid is understanding 'body' in its metaphoric meaning and understanding 'extended' in its literal meaning, where talking about space is considered as a literal way of using the words 'body' and 'extended'. It is interesting that if we understand 'extended' as a metaphor, then there is no reason to consider (30) as an invalid inference. In general, one possible explanation for monotonicity of an analytic inference is that if the additional information does not change facts about the way that we use language, then the inference which has been valid based on the meaning, will remain valid. The other structural rules such as contraction and exchange hold for EA analytic inferences too. Although theoretically it is possible that a predicate like 'body' has two different meanings in two different inferences and becomes the middle term of these two different inferences, but we consider 'body' as the middle term only when it has the same meaning in both inferences, therefore it can be said that EA analytic inferences are transitive as well.

[60] [Gentzen, 1964]

[61] Mark Sainsbury in [Sainsbury, 2001], p.11, regards Thinning and monotonicity as a distinctive feature of deductive logic. Also he considers Cut or transitivity as the other important property of deductive logic (p.26-7).

None of the three mentioned angles of approach to the difference between logical and analytical inferences looks promising. This must give us enough reason to think that there are more similarities between them than differences. Of course this claim cannot go anywhere more than similarity because at least one sharp distinction is there. As we saw in the second chapter, logical inferences extend a theory conservatively.This is not the case about analytic (non-formal) inferences. To recall, if we had a theory with boundary rules such as $p \vdash q$, $r \vdash s$, then using only usual logical words to expand the theory preserves the meaning of vocabulary of the theory by not allowing any atomic inference that is not in our boundary rules.[62] While in the case of analytic inferences, if a term becomes the middle term for two inferences, then it might lead to an increase of boundary rules. This validates the standard way of distinguishing a logical inference from an analytic inference which is not appealing to logical words. In this case, logical inferences are those that are valid because of the meaning of logical words (like introduction and elimination rules in natural deduction or sequent calculus systems).

We started this chapter with Dammett's verificationist idea of a priori as the shared property between logical, analytic and mathematical truths. And saw its shortcomings. Then we examined the development of the idea of 'facts about language' since the early twentieth century in a very selective manner. And we ended up with two more recent accounts of analyticity as a property of truth and justification respectively. It can be seen that both approaches separate sentences from propositions, and allow inferring sentences from sentences without considering the truth value of propositions that these sentences express. For instance, according to MA inferring 'Hesperus is Hesperus' from $\forall x(x = x)$ is valid because the truth condition of the latter, whatever it is, contains the truth conditions of the former. While 'Hesperus is Hesperus' is not an analytic truth in the metaphysical theory of analyticity. The mentioned inference is valid according to the epistemic theory of analyticity (EA) because the meaning of the logical words in $\forall x(x = x)$ in an a priori way, that is independent of any additional information and solely based on the meaning of the logical words, justifies inferring 'Hesperus is Hesperus'.

This makes us able to see how the notion of analytic justification can be an umbrella covering both grounds for holding an expression as true and an inference as valid. However, as we saw, there are other ways to separate logical inferences from non-logical analytic inferences. In the next chapter we shall focus on the notion of proof as an analytic justification by trying to

[62]We saw details in Chapter Two.

understand what it means to use only the resources in the premises. We shall examine the idea of analytic proof and will check whether or not it helps us to separate mathematical proofs from the purely logical ones. Ultimately, we shall attempt to provide an objective criterion for how proofs can be informative while being analytically justified and a priori.

Chapter 5

Towards a Definition of Analytic Proof

In the last two chapters we have been reviewing some ideas about the analytic-synthetic distinction both in a classical and modern context in the hope to create enough background to tackle the question whether there is a difference between arithmetic arguments and logical arguments. And also to see if we can explain the epistemic gain in a deduction. The assumption is that things like arithmetic equations are paradigm cases of informative deductions while basic logical inferences are paradigm cases of obvious deductions. The agenda of this chapter is to see if intuition, in one of the senses explained in chapter Three, is involved in arithmetic reasoning. And if the answer is yes, then does it in any sense affect the generality of arithmetic reasoning. Clearly this approach is very charitable towards a probably not very popular school in history and philosophy of mathematics. As many mathematicians and philosophers of mathematics are inclined to say that accepting axioms or rules of inference basically is something like accepting definitions for practical reasons and does not rely on any a priori intuition as such.

The issue of generality can be understood in connection with the informativity of arithmetic reasoning. In chapter Four we saw that 'Hesperus is Phosphorus' is informative because it gives us information about the actual world. The statement could be false, the reason is that there are metaphysically possible worlds in which 'Hesperus is Phosphorus' is false. Now one way of arguing that arithmetic truths are truths about something more specific than logic, and therefore informative in a similar way to 'Hesperus is Phosphorus', is to say that appeal to intuition in arithmetic reasoning makes these truths less general than logical truths. In other words, models in which arithmetic statements are true has a metaphysically significant difference with models in which logical statements are true. For instance, let us

consider identity. We saw that in MA containment of truth conditions could be served as justification for an inference. More specifically, an inference is valid once the truth conditions of the premise or premises contain the truth conditions of the conclusion. applying this idea to identity, the truth conditions of $\forall x(x = x)$ whatever it is, contains the truth conditions of $a = a$. Nonetheless, $a = a$ is not analytically true according to the given account of MA. It seems that for this account of analyticity being self identical is a property of objects in the world, we cannot just assume that every object, what so ever, is self identical.[1] A part of our investigate in this chapter is directed to see if appearance of arithmetic vocabulary such as identity affects generality of arithmetic reasoning in a significant way.

Our guideline in the aforementioned investigation of generality is analyticity as a property of justification. In the previous chapter also we saw a move from analyticity as a property of truth to analyticity as a property of justification. In case of EA, we saw that the meaning of the premises justifies the inference to the conclusion. For instance, the meaning of 'x is a body' justifies the inference to 'x is extended' whereas it does not justify 'x has weight'. In this chapter we explore the idea of analyticity by considering proofs, more specifically, formal proofs. We start with this guiding idea that the analyticity of a proof step can be understood in terms of possible justifications available for that step. Specifically in regard to the meaning of words that play a crucial role in the inference. Following this guideline, we shall search for any possible source of justification engaged in the process of deriving a conclusion from premises. And we will check whether this potential justification can be attributed to the meaning of important words in an inference step or there are other alternative explanatory stories available for that inference. Since our basic motivation is to search for possible sources of difference, if any, between logical and arithmetic proofs we shall closely examine the different ideas expressed in the following questions: are logical proofs and arithmetic proofs different in terms of their sources of justification for inferential steps? Are they different in terms of the generality of proof? Is there any way to address the difference between proofs at all? How can the disagreement between Kant and Frege over the analyticity of arithmetic proof be understood in a more modern context?

5.1 Proof and Logic

If we think of logic as a formal system with different methods such as - natural deduction, sequent calculus, axiomatic, or tableaux systems - then

[1] Quine clearly states this position in [Quine, 1963], p.390.

proof is a broader notion than logical derivation (by logical derivations we mean derivations based on logical words only). At the very least there are two ways to articulate this fact (and they are not mutually exclusive of course): in terms of vocabulary and in terms of the rules of inference. Considering the vocabulary, symbols used in arithmetical proofs are more than symbols used in a logical proof. Therefore arithmetical languages are syntactically richer than logical ones. So, logical proofs are a specific kind of proof with regard to syntax. If we consider any logical system as an inferential apparatus, that becomes richer by adding arithmetical vocabulary (in the language and not the apparatus), then we can say that arithmetical proofs are very similar to logical proofs in the sense that they share the same inferential rules. However, pursuing this idea does not that smooth considering the actual practice of both logic and arithmetic. Some vocabulary that we add to the language of arithmetic have inferential forces similar to logical connectives. For instance, if we add '=' or '<' to a natural deduction system, they have inferential forces. Let us focus on '=' for, as we shall see, inferential rules for identity are almost reducible to predicate logic. Possible inferential rules for identity are as follows, however they do not fully characterize identitiy as they are not general enough:[2]

1. $\dfrac{}{a = a}$
2. $\dfrac{a = b}{b = a}$
3. $\dfrac{a = b \quad b = c}{a = c}$

Or they can be expressed as axioms like the following in an axiomatic system:

4. $\forall x(x = x)$
5. $\forall x \forall y((x = y) \rightarrow (y = x))$
6. $\forall x \forall y \forall z(((x = y) \wedge (y = z)) \rightarrow (x = z))$

There are two strategies to overcome this difficulty for anyone who wants to reduce reasoning with identity to a logical one. The first one is to show that the rules governing the use of '='are the same as the rules for logical vocabulary. The other one is to show that '='is reducible to logical vocabulary. For instance, if we choose a natural deduction system, to follow the first way, it means that we need to show that inferences in (1), (2), and (3) are derivable using only the rules of the logical vocabulary such as '$\exists$','$\forall$', '$\neg$' and '$\vee$'. And

[2]Examples are from chapter one of [Negri and Plato, 2011].

following the second way means to define identity using the above-mentioned vocabulary.

To pursue the first strategy, one cannot assume anything about identity but that everything is self-identical and that if something holds for a and $a = b$ then it also holds for b. This has been expressed in (7)[3] and more generally in (8).[4]

7. $$\frac{Pa \qquad a = b}{Pb}$$

8. $$\frac{\Gamma, Pa \vdash \Delta}{\Gamma, a = b, Pb \vdash \Delta} \qquad \frac{\Gamma, Pb \vdash \Delta}{\Gamma, a = b, Pa \vdash \Delta}$$

P here does not need to be a simple one-place predicate necessarily, we may also consider it as a complex property, say, a complex proposition in which a occurs. According to this line of thought identity has been treated as substitution condition for a two place predicate or a relation. It can be shown that by accepting (1) and (7), (2) and (3) are derivable.

9. $$\frac{\overline{a = a} \qquad a = b}{b = a}$$

10. $$\frac{a = b \qquad \dfrac{b = c}{c = b}}{a = c}$$

Here is how to read (9): assume a stays in the relation of 'being equal to a' which is our Px, that is x is substituted by a and we have a proposition like Pa. Now if $a = b$ then b can take the place of the first a in the relation of 'being equal to a' (to make the proposition Pb). Also (10) can be read as: assume b stays in the relation of 'being equal to a' (it can be expressed in a proposition like Pb). Now if $b = c$ then b can be replaced by c in the relation of 'being equal to a' (to make the proposition Pc).

As can be seen, the first strategy is not completely successful since appealing to the notion of self-identity would be inevitable. We shall probe the notion of identity shortly. The second strategy is even less successful. A famous attempt to define the notion of identity using logical concepts was pursued by Leibnitz. Two things a and b are identical once every property and relation that holds for a, also holds for b (this is a weaker claim than, b is replaceable with a in any context). Or once there is a property or relation that holds for one and not the other, then a and b are not identical. It

[3]Ibid. p.45

[4][Restall, 2013],p.92

is not difficult to realise that formalizing this criteria needs quantifying over properties and relations. Here are two formalizations of the above-mentioned ideas:

11. $\forall x \forall y((x = y) \longleftrightarrow \forall G(Gx \longrightarrow Gy))$

12. $\forall x \forall y((x \neq y) \longleftrightarrow \exists F(Fx \wedge \neg Fy))$

We shall not dwell on the second strategy any further;[5] instead, we will focus on the consequences of having '=' in our logical vocabulary. This is because if we accept the story that has been told about following the first strategy, then we are also forced to accept that the occurrence of '=' in our proofs is inevitable. At the very least, either we should accept (1) as a rule or accept (4) as an axiom. Now we want to see if the notion of identity has any effect on features of a logical proof such as generality and abstractness of reasoning. From this point , it will become easier to judge whether or not analyticity in the senses that we discussed in the previous chapter is a good measure of distinguishing logical reasoning from arithmetical reasoning.

5.2 Intuition and the Generality of Proofs

One idea that has been expressed by different thinkers to separate an arithmetical proof from a logical one is that arithmetical proofs appeal to intuition. The literature mostly has been shaped around the Kant and Frege disagreement about the analyticity of arithmetic proofs. In the previous chapter we have seen one reading of this disagreement (that 5 and 7 on their own does not contain 12) there is also another understanding of the disagreement, namely, that to prove 5+7=12 we need to appeal to our intuitions.

Some philosophers[6] believe that the source of justification for arithmetic proofs, as well as proofs in predicate logic, is intuition and that is what separates these proofs from proofs in propositional logic. Some other philosophers [7] reject the appeal to intuition as source of justification for these proofs, while accepting the appeal to intuition as the difference between arithmetical proofs (as well as predicate logic) and propositional logic. They define any proof in predicate logic and any richer language as a method of proof that deals with individuals rather than concepts, and think of this difference as

[5]The reason merely is that it has been done before, so we want to see if it is possible to work out identity without appealing to the second order logic. The curiosity is purely technical.

[6][Parsons, 1992] [Friedman, 1990]

[7]Such as Hintikka in [Hintikka, 1991]

a characteristic that separates predicate logic and arithmetical proofs from proofs in propositional logic. We shall examine these two thymes to have a better understanding of the nature of the difference that they offer.

Charles Parsons, a proponent of the first idea, argues that objects of intuition are directly represented to the mind, and this claim is based on Kant's distinction between concept and intuition as expressed below in (a).

a. The intuition is a singular representation, the concept a general or *reflected* representation.[8]

Here Kant opposes the direct intuition with the reflected representation. One may think of this direct representation as a visual accessibility to objects, perhaps without being able or willing to attribute any properties to them. In a Kantian sense, the very way that we receive the sense data, before fully perceiving the data (that is putting them under categories or concepts), allows us to infer different pieces of information. Parsons refers to this way of receiving data as intellectual intuition. A thought experiment would be illustrative: we can easily, by a visual check, judge which of the two objects covered separately, is bigger than the other without apprehending what they are. And if someone asks for the reason for holding this judgement we cannot say anything more than 'take a look'.[9] In other words, we do not argue for this claim; it is not demonstrable. No further explanation can help a sceptic to accept this claim.

In general, the fact that something is bigger than something else is a matter of contingency. However, the order of numbers is not contingent in this sense. Once we accept axioms about numbers, then the structure of the numbers follow from those axioms. Nonetheless, what makes us able to form and understand those axioms, according to Parsons, is called forms of intuition. And these forms of intuition, as he mentions, are not logically necessary.[10] One way of understanding Parson's point is that imagining a possible world in which we would not perceive the order and relations among objects as we do now, is easier than imagining a possible world in which we would and would not perceive thing as we do. If one accepts this idea, then it seems that there is a sense in which logical rules (rules for the core logical vocabulary) are more general than rules that define the order of numbers. So two levels of generality are distinguishable here; on one hand predicate logic allows us to talk about objects with no need to categorizing them. This makes a proof in predicate logic, to some extent, general because it is correct

[8] [Parsons, 1992], p.44

[9] We also have discussed it at Chapter Three.

[10] Ibid. p.50

regardless of the categories objects belong to. On the other hand, the possible worlds or models in which basic logical rules are not held seem to be harder to access (conceive or imagine or understand) than the possible worlds or models in which axioms of orderedness are not.

Let us focus on the first notion of generality first. In this notion of generality, any inferential move in a proof in any language richer than predicate logic is correct for any object. There are thinkers who believe that what makes these kinds of inferential moves correct are objective facts. For instance, one (like Quine as discussed in the previous chapter) may hold that what makes (4) true are objective facts about the world. And we saw that appealing to one version of (4) is unavoidable in our arithmetic proofs. Now if we appeal to any of these inferential moves in reasoning about objects that we have no direct access to, then the worry arises that we are applying our intuitions where there is no guarantee that they can be upheld. Because if what make our intellectual intuitions correct are objective facts (perhaps about the world), then appealing to them in reasoning about objects without having any idea about the facticity of these objects might not produce safe inferences in our reasoning.

However, as MacFarlane correctly argues, after Frege who breaks the Kantian link between conceptual content and sensibility, the above-mentioned worry is dealt with.[11] This means that objects that give content or object-sensitive meaning to concepts are not given to us by sensation necessarily. They may be introduced by, say, descriptions like 'the thing that stands in relation R with object y and has property F'. The generality of this way of introducing objects can be seen in this way: considering a formula such as $(\forall x \exists y(Rxy \wedge Fx))$, it quantifies over the types of things that stand in relation R (which holds between two things) to each other and the thing that stands on the left hand side of this relation also has the property F. Here we are faced with another level of generality. It is possible to consider, reasoning about objects possibly dissimilar to objects we perceive via our sensory system and it will not cause any problem.

One way to challenge this possibility is the way Parsons uses to explain Kant's position on arithmetical proofs and Michael Friedman articulates neatly.[12] They suggest that Kant holds that transcendental logic that deals with pure forms of intuition includes terms with object-sensitive meaning. Here is one way of understanding the idea: we use 'bigger than' at least in two contexts - size and number. One may grasp this distinction by considering the objects to which we attribute the relation. And although there are

[11] [MacFarlane, 2002], p.58
[12] [Friedman, 1990]

some basic similarities between these two uses, such as transitivity and anti-symmetry, their inferential force may differ. For instance, natural numbers are such that if a number is bigger than the other, in some sense, the smaller number is part of the bigger one, but this is not the case with regard to the size. The other possible way of thinking about 'standing in need of objects to complete or fix the meaning of an expression' is that objects may have non-conceptual properties. That is, non conceptual properties or qualities that cannot be captured by concepts, have non-conceptual contents as we discussed in chapter three (to be more precise, relations that form arbitrary concepts).

In a modern interpretation of this Kantian thought, expressions of predicate logic do not have meaning unless they are considered in a structure or frame (a class of models). In this reading, what we are able to write down using quantifiers in predicate logic has meaning; but the meaning is not totally defined prior to being interpreted in a model. Why it is so? Both Friedman and Parsons propose that the constructive method that uses existential quantifiers nested in universal ones wasn't practicable unless supported by thought experiments. And moreover, there was no systematic way of demonstrating any thought experiment using a symbolic language. Friedman considers our favourite arithmetic equation: he explains that applying something very similar to the axioms of Peano for arithmetic to justify correctness of an arithmetic equation, for Kant, could not be done without the notion of time. This does not imply that objects of arithmetic, namely numbers, are temporal, but that the temporality comes from the act of the successive iteration of producing numbers. The operation that can be represented via the expression of rules and the iteration of them couldn't be accomplished due to a lack of vocabulary.[13] Therefore the notion of an iterated succession could not be expressed without appealing to the notion of time. However, this does not mean that what can be written in the symbolic language of predicate logic now does not get its object-sensitive meaning from our formal intuitions.

Friedman takes the existence of unintended models for arithmetical theories as a support for the idea that intuitions are needed to pin down arithmetical notions. In other words, the object-sensitive meaning of theories in predicate logic needs a specific model to be fixed. This is also in line with Parsons' position that forms of intuition are not logically necessary. With these ideas about intuition and generality in mind, there are different ways of considering the appeal to intuition in a proof. Sometimes we may start with a number of premises which we need to appeal to intuition to justify

[13]For a good explanation how an argument (a logical one) can rely on thought experiment and spatial intuitions see [Hazen, 1999], p.86,87.

them as true, but none of the inferential steps has intuitive justification. And sometimes we may have inferential steps with intuitive justification too. Formally speaking, (13) expresses the first idea, where we need to appeal to our intuition to accept Γ and $\vdash_L$ stands for the consequence relation in a deductive apparatus with inferential rules for usual logical words without identity. Also (14) Expresses the second idea, $\vdash_A$ stands for the consequence relation in a deductive apparatus with extra inferential rules for notions such as identity and order[14] (whether we appeal to intuition to accept Γ' or not does not matter here).

13. $\Gamma \vdash_L A$

14. $\Gamma' \vdash_A A$

Both in this chapter and the previous chapter we have come across some vocabulary, other than the usual logical ones, with inferential power. Expressions such as 'X is red', 'X is body', 'X is bigger than Y'. In the previous chapter we saw that some of them are different from logical words in the sense that adding them to our theory might lead to a non-conservative extension of our theory. In this chapter we saw that some of them could have object-sensitive meaning, that is, their meaning does not become fixed until they are applied to objects. One possible aspect of having object-sensitive or unsaturated meaning is that to uphold a sentence containing these words or to accept inferential rules for them we might appeal to intuition whatever the appeal to intuition means. This can mean observational or any kind of non-demonstrable justification. Now, considering identity, does it make sense to say the appearance of identity in a sentence or in an inferential rule calls for intuition?

Let us start by applying the first idea on the notion of identity. We have just seen that, under the inspiration of Kant, Parsons takes intuitive justification as an immediate, self-evident, ground for accepting a truth or a rule. To recall, we cannot give words to someone who demands evidence for the claim that this thing is bigger than that thing. We may also not be able to say much to someone who asks for evidence for the correctness of modus ponens. However, the difference between these two cases is that there is a particular visual evidence for the first case, while none for the second. Even if in both cases we respond that our evidence for holdin the truth or the rule is the meaning of the conditional and 'bigger than', respectively; there is something more abstract, about the meaning of the conditional and there is something more concrete about 'bigger than'. Having this notion of appeal to intuition in mind, is there anything intuitive about identity?

[14]We have seen identity rules, and we shall see rules for order shortly.

According to the very Kantian notion of meaning, it depends on how we use the notion of identity. If we think of identifying a unit then as we have seen in chapter three, some Kantians would say we rely on our intuition to perceive the plurality of a row sense data as a unit. We do not give reason why we consider a specific combination of a wooden structure and cloth as a chair and another piece of cloth on top of that as a coat hung upon that chair and not as a part of the chair itself (we may do, but by appealing to concepts less obvious than what we are about to justify). So in some contexts the notion of identity is associated with cognition of a plurality as a unit. In such contexts identity is associated with intuition. However, if we use the notion of identity not to define the notion of unity but to define the notion of the empty set as in (15), then there is no act of cognition going on in this use of the notion of identity. And it is not easy to see the function of intuition in the justification of holding (15) as true or justified.

15. $\varnothing = \{x \mid x \neq x\}$

Now if someone uses this idea, as Frege did, to define the number zero and without appealing to the notion One or unit, and defines the other numbers with the assistance of succession, then it can be said that there is no need to appeal to intuition, in the noted sense, to deal with numbers as objects of arithmetic. And therefore we do not need to appeal to intuition, in the noted sense, in justifying expressions with identity occurring in them.[15] However, Parsons[16] argues that if we define the notion of order using numbers produced in that way, then although we do not rely on our intuitions to acquaintance with numbers, we appeal to intuition to understand the relations among them. This means that to understand expressions with the notions such as 'bigger than' in them, we need to appeal to intuition, or in other words, we rely on intuition to justify their truth value. In reflect to Parson's idea, it is hard to see how expressions with object-sensitive meaning may behave differently when applied on different objects, here a different string of numbers, when these objects are not given to us by intuition.

What has been said suggests that if we have a deductive apparatus with only the usual logical vocabulary ($\vdash_L$) and some axioms using the notion of identity, then we have not appealed to intuition in holding these axioms neither in justifying the inferential rules inasmuch as we are dealing with numbers. This was one sense of appealing to intuition that could make arithmetical proofs less general than logical proofs and we have seen that we

[15] In dealing with relations, we still may appeal to intuition in another sense that we will see shortly.

[16] [Parsons, 2008], Chapter Six.

seldom use that sense of intuition in using identity to define the notion of number. Of course we have not shown that arithmetic proofs do not entirely rely on intuition, because the notion of identity is not the only notion needed to build an arithmetic theory. And, the above-mentioned way of appealing to intuition is not the only possible way of understanding the appeal to intuition in a proof.

The other way of arguing for the appeal to intuition in arithmetic reasoning is introducing inferential rules for notions such as 'bigger than' with intuitive justification. There are some formal differences between ordinary logical words and arithmetical words such as 'bigger than'. Comparing regular logical vocabulary such as conjunction, disjunction, and conditional, according to many logicians, it is possible to define one word according to the other. While considering two notions such as 'identity' and 'bigger than', there is no such a connection. For instance, we may have (16) and (17) inferences just considering the ordinary meanings of 'identity' and 'bigger than' for numbers[17] but nothing like a bi-conditional or two-sided inference exists.

16. $a > b \vdash a \neq b \qquad a < b \vdash a \neq b$

17. $a \neq b \vdash a > b \vee a < b$

This leaves the existence of intuitive justification for inference rules including terms such as 'bigger than' an open possibility while rules including identity are intuition free. Moreover, the fact that inference rules with identity and order are not inter-derivable is not even a formal indication for the difference between the logical and arithmetic inference rules. If we think of a logical system such as natural deduction or sequent calculus with separate rules for each logical word, then inter-derivability is not a characteristic of logical words any more. However, the point that our formal intuitions complete the meaning of object-sensitive specific predicates (that have inferential force in an argument) could be a characteristic difference between logical and arithmetical vocabulary.

This is because if we think that the meaning of predicates such as 'bigger than', which are object-sensitive, is fixed when the objects this predicate applies to are fixed, then the fact that number theories are not categorical might become an issue. A theory is categorical when all of its models are identical up to isomorphism. However, arithmetic theories are not categorical as they have unintended interpretations, also referred to as non-standard models. To eliminate unintended interpretations, second order language is

[17]The inference (17) does not hold if, for instance, we consider different shapes with the same area.

required. That is, the difference between standard and non-standard models of arithmetics is expressible in a second order language and not in a first order level. Therefore, if we think of meaning inferentially, as we do in this thesis, difference in meaning needs to be visible in the first order level. Consequently, even one who thinks object-sensitive meaning justifies inferential rules should admit that arithmetical rules are as general as logical rules despite the existence of non-standard models for axiomatic arithmetics. Having an object-sensitive meaning makes a difference only in a second order context.[18]

Moreover, there are good reasons to have inferential rules for arithmetical notions, given that forms of intuition are not logically necessary. It is thus reasonable that inferential rules with intuitive justification (if intuition is the only source of justification for them) cannot be reduced to logical inferences. In other words, if we take a proposition or statement as logically necessary only if it follows from logical rules, then the fact that formal intuitions are not logically necessary means that the former is not derivable from the latter. And if arithmetical inferences have intuitive justifications then it makes sense that they cannot be rewritten using logical words. Therefore, if there is no proof for arithmetical facts without intuitive movements, then it is good to separate intuitive inferences from logical ones. And that is why it is good to keep track of intuitive inferences by separating them from logical inferences.

Let us conclude the issue of appeal to intuition as a source of justification by considering the case of predicate logic with identity. We saw that a deductive apparatus with predicate logic and identity can be built both as (18) and (19) where $\vdash_{LI}$ is a consequence relation for a deductive system with separate inferential rules for identity, and $\vdash_{L}$is a consequence relation for a deductive system without separate inferential rules for identity and with an axiom for substitution of identical things (to have a better sense of (19) take a look at what we had in (7) and (8)).

18. $\Gamma \vdash_{LI} A$

19. $\Gamma, a = b \vdash_{L} B$

This is true that A in (18) is more general than B in (19), because A in (18) holds in models in which $a = b$ may not be true while B in (19) does not hold in such models, but proof of B in (19) is more general than proof of A in (18) since the inferential rules it relies on does not need intuitive justification in the sense we discussed above.

[18]Points about the non standard models of arithmetic has been made in reflection to this thesis [DiGiorgio, 2010].

The only way intuitions may contribute in a proof for an argument with Γ and $a = b$ as premises and A as conclusion is by justifying $a = b$. In applying identity to natural numbers, once there is no appeal to intuition in the definition of numbers (in the sense that we saw earlier in this section), then it is hard to see the role of intuition in justifying $a = b$. One may say the same holds for the notion of 'bigger than'. That is, we may have inferential rules for governing the use of 'bigger than' or adding a premise or postulation to do the same job, like (20).

20. $\Gamma, \forall x \forall y \forall z(((x > y) \wedge (y > z)) \rightarrow (x > z)) \vdash_L C$

The thing is that the postulation in (20) has the function of an inferential rule. As can be seen, the difference between (20) and (19) is that the inferential force of 'bigger than' has been expressed with the assistance of other logical words; while in (19) we just have identity to justify substitution. After the next part we shall see more about relations such as 'bigger than', but if one accepts what has already been discussed about the difference between (19) and (20), then most likely one will accept that the proofs of A, B , and C are as general as each other when it comes to objects like numbers. However, when it comes to reasoning in general, it seems that A and B are more general than C. One could consider proofs as safe bridges from some premises to a conclusion and logical rules as some kind of pre-made blocks to make that bridge with certain standards of safety. Nonetheless, sometimes these blocks are not enough to bridge the gap between premises and the conclusion, and pre-made blocks of a different kind are needed to complete the desired bridge.[19]

Parson's claim about the intuitive justification of predicate logic with identity does not seem plausible to those who accept the account that has been articulated about the connection between identity and intuition. Neither, the point about the dependence of object-sensitive meaning of certain predicates (with inferential force such as 'bigger than') upon the class of models that a constructive proof creates sounds plausible. However, we shall see that although the use of intuition in the sense that Parsons has in mind, that is, direct justification for inferential rules is not plausible, there are other senses in which a proof in predicate logic with binary relations might appeal to intuition.

The above-mentioned outcome in regard to predicate logic is more in line with the other approach to intuition that does not give a justificatory force to intuitions. An advocate of such a view is Jaakko Hintikka, who considers

[19]This idea has been developed in [Negri and Plato, 2011] We shall see more about this idea in discussing simplicity of proofs.

the use of individuals in a proof as a measure of applying intuition. Unlike Parsons, he thinks there is no necessary connection between intuition and sensation.[20] And we do not need to adopt more than logical laws to deal with individuals in a proof.[21] He claims that reasoning with individuals stands versus reasoning with concepts. In the next part we will explore this idea in detail, but for now, reasoning with concepts can be understood as deriving judgement based only on relations among concepts. Whereas reasoning with objects means deriving propositions using objects that fall under concepts. For instance, one may conclude from 'humans are rational' that 'some animals are rational' merely because of the relation between concepts 'animal' and 'human', namely that one includes the other. Also one can conclude 'X is taller than Y' from 'there is a Z who is shorter than X and taller than Y'. In the second case although we employ conceptual analysis, we apply this to individuals but not to concepts. The conceptual analysis applied to individuals can be expressed in a more general way too: inferring something is taller than the other thing from the existence of a third thing which is shorter than the first one and taller than the second one is always a correct inference.

We may wonder why we should label reasoning with individuals as intuitive. Specifically when we are not ready to give any necessary sensory flavour to intuition. Specifically, if we think of numbers as individuals, then it is reasonable for us to accept that our grasp of these individuals can be established without sensation. An answer to this question is that sometimes we appeal to a sort of trick in reasoning to solve some problems and this is usually referred to as insight or intuition. For instance, any student may be faced with a question like (21) in a mathematics exam:

21. $\frac{2x^2(x+3)-2(x+3)}{x^2+2x-3}$

And there are four other formulas as possible answers for the other side of the equation. The intended answer is $2x+2$. This test is designed to check basic algebraic skills. Here is a possible way to the intended answer:

22. $\frac{2x^2(x+3)-2(x+3)}{x^2+2x-3} = \frac{2(x+3)(x^2-1)}{x^2+3x-x-3} = \frac{2(x+3)(x^2-1)}{x(x+3)-(x+3)} = \frac{2(x+3)(x^2-1)}{(x+3)(x-1)} = \frac{2(x+3)(x-1)(x+1)}{(x+3)(x-1)} = 2x+2$

Although we have applied basic rules of algebra in this proof, none of these rules force us to this proof. That is, there are various possibilities for applying rules that may not lead to the intended answer. The ability to find proofs in

[20] [Hintikka, 1991], p.130
[21] Ibid. p.131

arithmetic is something more than knowing the rules that we are allowed to apply. This is because rules are not fully prescriptive, in the sense that they leave a variety of possible paths open, even those not useful for the proof. In other words, they do not single out only ways that lead to the conclusion. In case of our example, algebraic rules just tell us the permitted moves, but they do not help us to discern potential common forms between upside and downside of the division line. One more example will be illustrative. A good case for comparing a more prescriptive set of rules with a less prescriptive one is the case of tableaux systems for propositional logic and the very same systems for predicate logic. In the case of propositional logic, there are only a few ways of applying rules and those that help us to close a tree. The order of applying the rules does not matter and there are a number of possible ways that all of them will give us the intended result. However, in the case of predicate logic, there are a variety of choices and we need something more than knowing the rules to be able to complete a tree. For instance a tree with the starting formulas expressed in (23) may conclude reasonably fast or it may go on and on by applying rules blindly.

23.

$$\forall x \forall y (Rxy \rightarrow Ryx)$$
$$\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$$
$$\forall x \exists y Rxy$$
$$\therefore \neg \forall x ((Rxx)$$

The common characteristic of these two examples is that rules leave open several choices and some choices are good while some are not. So merely knowing the rules is not enough for making a proof, but rather correct cognition also is needed (the same can be said comparing axiomatic propositional logic and Tableaux propositional logic, although appeal to intuition is not necessary in propositional logic). We can explain why $x^2 + 2x - 3$ is replaced by $x^2 + 3x - x - 3$ in (22) but this explanation is not part of the formal proof although it requires the capacity to discern something formal and abstract. We may think of the appeal to intuition as a characteristic of some proofs in this sense. Once we have a proof for an argument we can give an algorithm for the proof, but if there is no proof there is no algorithm for a proof. The use of intuition in this sense, in forming proofs, neither means to bring any implicit assumption apart from our explicit assumptions into the context of that proof (it may bring new formal entities into the context of a proof like the second step in (22)); nor using extra rules to reach the conclusion.

If we accept the line of argument provided in this part, then we may also accept that intuition as the justification of inferential rules might lead to less general proofs, although it does not lead to any exiting or significant difference between logical and arithmetical reasoning. The point of those who support justificatory role of intuition was that if we hold that some expressions have object-sensitive meaning (predicates or operators) and inferential force and this object-sensitive meaning is the justification for inferential rules with these expressions appearing in them, then we should accept that objects in the model of proofs using those inference rules plays a role in justifying inferences. However, as we saw, this point does not change the inferential role of expressions such as 'bigger than'. Nor it has any impact on the generality of the proofs that employ inference rules for such expressions. Also we showed that intuition as direct access to objects does not imply sensory cognition of objects. And finally it has been argued that there is at least one understanding of appealing to intuition in a formal reasoning that does not imply appealing to anything more than rules of reasoning in the formal reasoning in question. This notion of appeal to intuition is suggested by Hintikka. However, Hintikka's point about the difference between reasoning by conceptual analysis versus reasoning by individuals raises new questions: What does it mean to reason by individuals rather than concepts? And what are the possible impacts of this kind of reasoning? These are questions that we will be able to explore in more detail after the next part.

5.3 Form, Abstraction, and Generality

What has been said about the object-sensitive meaning of relations such as 'bigger than' and generality of reasoning with such inferential roles might have convinced us about generality of such proofs where generality is defined with the class of models. Another sense of generality is also thinkable, namely generality as applicability in a larger number of cases. One might think that in a language without signs for individuals, the object-sensitive meaning of words at least does not rely on objects of a model. And consequently, an inference in a piece of reasoning or a proof in this language is more general than a language with signs that refer to individuals, and therefore, more applicable to different situations. An early modern idea about how to form such a language was through abstraction. Roughly speaking, in this line of thought, sensory data collected from empirical experiences needs to be grasped via putting elements of these data under concepts. And concepts are made through a process called abstraction. In this understanding, the more abstract the more general. There are reasons to think that Kant also

had such an understanding of generality. At the beginning of the preface to the second edition of The Critique of Pure Reason (CPR), Kant endorses Aristotelian logic as the science that "presents and strictly proves nothing but the formal rules of all thinking".[22] A bit further on the same page he also describes logic as "a completed science that is not changing with psychological, metaphysical, or anthropological findings".

Around the same time as Kant, it had been established that there are good proofs that cannot be demonstrated by Aristotelian syllogism but they are still able to guarantee their conclusion. They are established by rigorous thought and reasoning but the subjects of these thoughts are not concepts as such but rather objects and relations among them. Also in the preface to the second edit of CPR, Kant discusses universal and a priori cognitions and categorizes arithmetics as a priori but intuition-base science; he writes: "Arithmetics gives us a splendid example of how far we can go with a priori cognition independently of experience [...] it is occupied [...] with objects and cognitions only in so far as these can be exhibited in intuition".[23]

According to one view, the distinction between proofs given in the form of Aristotelian syllogism and proofs using rigorous thought about objects and their relations for Kant is that the former is more formal than the latter and therefore more general.[24] The names that Kant uses for these two reasoning methods support this idea. He calls Aristotelian syllogism as General Logic and methods used in arithmetical reasoning as Transcendental Logic. According to this view, Kant thinks of Generality as abstraction from sensibility. And since, for him, concepts have content only if they are applicable to objects of sensible intuition, therefore if logic wants to be general it needs to deal with the form of concepts and not their content. In this way formality is a precondition for generality. And formality means no application of concepts to objects because concepts do not have content until they have been applied to an object. Using today's common logical syntax, it means that anything like Pa is not general because it is not formal since the content of the concept, and not only its form, is engaged. It is tempting to say that Px is general due to the fact that here the concept has been applied to nothing specific, and perhaps if Kant was aware of such a possibility (the possibility of having a calculus system using variables), then he would have changed his mind. However, the form suggests applying a concept to an object, even though an unknown object. It seems that P is the formal appearance of a concept for Kant's favourite logic which is Aristotle's. According to Kant,

[22] [Kant, 1781], p.106
[23] Ibid. p.107
[24] [MacFarlane, 2002]

concepts are the results of abstraction. The process of considering an individual and omitting everything but one aspect of that individual is called abstraction.

Analysis of concepts in Aristotelian syllogism can be translated into a language which is formal in the Kantian sense. The language is only able to express one specific relation between two concepts, namely inclusion. We have chosen the word inclusion instead of containment, to emphasize the extensional nature of the idea. In the following formalization these semantic rules hold:

b. $R_{A,B}$ If and only if concept A includes concept B.

c. $\neg A$ Is called the complement concept of A.

d. $\neg A$ Includes everything that A does not include.[25]

e. $\sim R$ Holds between two concepts or the complement concepts if and only if R does not hold between them.

Then applying these semantic rules here are translations of four possible judgement forms in Aristotelian syllogisms:

f. 'All B is A' (everything that belongs to or falls under concept B also belongs to or falls under concept A) can be translated as$R_{A,B}$.

g. 'No B is A' (nothing that belongs to or falls under concept B also belongs to or falls under concept A) can be translated as $R_{\neg A,B}$.

h. 'Some B is A' (there is at least one thing that belongs to or falls under concept B and also belongs to or falls under concept A) can be translated as $\sim R_{\neg A,B}$.

i. 'Some B is not A' (there is at least one thing that belongs to or falls under concept B and does not belong to or fall under concept A) can be translated as $\sim R_{\neg B,\neg A}$.

As can be seen, there is a correspondence between Aristotle's universal judgement and R. Also between his particular judgement and $\sim R$. Here are some of the syllogisms as examples: Barbara (the first form of the first figure), Ferio (another example of the first figure), Baroco (an example of the second figure), and Bocardo (an example of the third figure) respectively:

24. $R_{A,B}, R_{B,C} \vdash R_{A,C}$

[25]Here A should be understood as predicate.

25. $R_{\neg A,B}, \sim R_{\neg B,C} \vdash \sim R_{\neg A,\neg C}$

26. $R_{A,B}, \sim R_{\neg C,\neg A} \vdash \sim R_{\neg C,\neg B}$

27. $\sim R_{\neg C,\neg A}, R_(B,C) \vdash \sim R_{\neg B,\neg A}$

Meta-theoretic results of Aristotelian syllogisms can be found here too. The most vivid ones (once all of the inferences rewritten in the new language) are that there is no inference with two particular judgement. That is, two particular premises are not conclusive. At least one premise should be universal. Also, universal conclusions cannot be derived when one of the premises is particular.

Presumably this is what Kant meant by conceptual analysis and arguments that can be made applying the laws of general logic to such an analysis. Once such a relation holds between concepts the conclusion of any proof based on this relation is guaranteed. This method of reasoning is general in the sense that it is applicable to any field . However, syllogism cannot prove that much. And Kant asserted that we have rigorous proofs that cannot be established by mere syllogism. Good examples of careful reasoning that need more than syllogism are arithmetic proofs. As we have seen, the source of careful reasoning in arithmetic, to Kant, is intuition. In chapter three we saw that, for Kant, our way of cognition has strong links with logic since the act of cognition includes making judgement. Moreover, relations among objects that can be represented in intuition, such as 'being bigger than' have inferential force. We saw that it has led philosophers such as Parsons to think of intuitions as justification for some inferential steps. Nevertheless, the occurrence of individuals, that for Hintikka is the measure of any appeal to intuition, makes arithmetical proofs informal in a Kantian sense of formality. So, one may attribute the following idea to Kant: analytic proofs are those in which we come to the conclusion by mere conceptual analysis that can be achieved by Aristotelian syllogism. Both analytic and arithmetical proofs are guaranteed, but unlike analytic ones, arithmetical proofs are not that general since they are not formal. They are not formal because in proving arithmetical theorems we have proved something for objects and concepts, since applied to objects, have content and therefore are not formal any more.

Frege registered his disagreement with the formality which is the result of any abstraction and cannot "express a content through written signs".[26] He mentioned that a formal language is useful when "the content is not merely indicated but constructed out of its constituents by means of the same logical signs that are used in the computation".[27] It can be seen that

[26][Frege, 1884], p.65
[27]Ibid. p.65

Frege's notion of form is less abstract than Kant's in the sense that the formulas which are the output of this formalization give us more information about the connections among elements of the though has been formalized.[28] Does this make the proofs in this language less general in the sense of general applicability? We are going to see reasons why the answer is 'No'.

One thing is that whatever is formulated using syllogism can be formulated in the language of predicate logic as well. For instance, if we add (j), which is a definition which uses objects to define the notion of inclusion of concepts in (b), then we have the connection needed to transfer any of (24)-(27) to predicate logic (in (f) to (i) the notion of inclusion of a concept to another concept already has been expressed by using what is expressed in (j) for readers who are familiar with predicate logic more than Aristotelian logic). Therefore, any use of syllogism can be transformed to predicate logic.

j. Concept A includes concept B if and only if everything that falls under concept B also falls under concept A.

Moreover, one may think of the general applicability of logical rules in two ways. In one sense, generality means applicability of inferential rules (or logical rules in general) to any reasoning about every object whatsoever. In another sense, general applicability means that the way of reasoning in logical is the correct way of reasoning in general, therefore it should be applied whenever we do reasoning. The former account of generality is the one that we have been dealing with in the last two sections. MacFarlane labels that as the Descriptive account of generality.[29] He labels the second approach to generality as the Normative[30] account of generality and argues that Frege and Kant are in agreement about the generality of logic in this sense. However, they disagree over the notion of formality. Kant's notion of formality is more abstract than Frege's in the sense mentioned at the beginning of this section.

In the light of this understanding of Frege and Kant, both logical and arithmetic proofs are general in the normative sense. In Kantian vocabulary, the correct way of reasoning with concepts, or conceptual analysis is syllogism and the correct way of reasoning with objects can be found in arithmetic reasoning. In Fregean vocabulary, there is no such difference between reasoning with objects and reasoning with concepts; both include conceptual analysis. Frege's account of conceptual analysis is different from Kant's due to the fact that his account of formality and conceptual analysis, gives us much more detail about individuals in a thought. Therefore, in this section

[28]Two different approaches to logic as calculus and as language discussed in [Hejienoort, 1976] is one way of addressing Frege's position here.

[29][MacFarlane, 2002], p.33

[30]Ibid. p.35

we showed that there is no difference between logical and arithmetical reasoning in terms of general applicability. In the next section we shall follow Kant's and Frege's disagreement over the analyticity of arithmetic reasoning (proofs in arithmetic reasoning) in the light of implications of reasoning with objects.

5.4 Reasoning With Individuals

Kant's and Frege's disagreement over conceptual analysis can be addressed with the assistance of proof methods that use individuals in the process of proving a theorem or proving a conclusion from a number of premises. In B16 of CPR Kant argues that arithmetic equations are synthetic because to realize their truth no amount of "twist and turn" in our concepts is enough and "without getting help from intuitions we could never find the sum by means of the mere analysis of our concepts".[31] Frege, on the other hand, in his conclusion of *The Foundation of Arithmetic* criticizes Kant's approach to conceptual analysis by a "geometrical illustration", He writes: "If we represent the concepts (or their extensions) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do – in terms of our illustration- is to use the lines already given in a new way for the purpose of demarcating an area. Nothing essentially new, however, emerges in the process."[32]

One way of understanding these two different accounts of conceptual analysis is that in the language of Frege's predicate logic, we are able to establish proofs using objects that fall under concepts and that stand in relation with the other objects. Let us stipulate that being an argument of a one place predicates means falling under a concept and being an argument of a two or more place predicate means standing in a relation. For instance, (28) expresses that for every object there is another object that stands in a relation with that object and our first object also has a property (of course one object with two names also is a model of (28)). Now if we consider that formula (28) is a premise among other possible premises in an argument or even is part of a premise, then eliminating quantifiers and breaking the formula into its parts can be considered as conceptual analysis.

28. $\forall x \exists y (Rxy \wedge Fx)$

[31] [Kant, 1781], p.144
[32] [Frege, 1884], p.100

To break (28) into components, in a natural deduction system, we need to eliminate quantifiers. This requires replacing variables with constants. That is, we need to consider individuals that fall under concepts and stand in relation. Let us call this method of reasoning, reasoning with individuals. This stipulation makes more sense when we note that sometimes individuals play an important role in the process of reasoning. For instance, sometimes to prove that $a > b$ where a and b are numbers, we appeal to another number c which is $a > c$ and $c > b$. It is plausible to wonder about possible ways in which reasoning with individuals may differ from reasoning with concepts. One difference might be holding hidden assumptions about individuals or objects that we use in reasoning. These assumptions about objects do not become explicit because either they are non-conceptual (non-articulable) properties or they are so obvious that we take them for granted. We saw a bit about this version of intuition earlier in this chapter. It also has been raised that some thinkers hold that our intuitions bridge the gap between our formal reasoning and models that justify them. Also we saw that although there is no exact answer to this concern, our presuppositions about the objects we use in reasoning are too minimal, for instance we assume these objects are self-identical.

Moreover, since our acquaintance with some of these objects (grammatical objects at least) is not via sensation but via definitions, then the possibility of considering non-conceptual properties is very low if not next to impossible. In this case we must think reflective (opposite of direct) or representational non-conceptual properties are possible. This is a stronger claim than believing in non-articulable properties for objects that we just may grasp by our sensation. Because it requires us to reflect on those non-articulable properties in our abstract activity of giving definitions. It is also plausible to think that Kant's and Frege's accounts of generality are almost the same, since it is hard to see how logical reasoning can be held as the correct way of reasoning in general while rejecting its applicability in general. So none of these differences look like a characteristic (or radical) difference between their accounts of logic and logical reasoning, and also between logical reasoning and arithmetical reasoning.

One candidate for a distinctive difference between reasoning by objects and reasoning by concepts can be given with the assistance of the differing Kantian and Fregean accounts of conceptual analysis. If we interpret R in (29) as 'being after' then the relation Rxy, for Kant, breaks into two concepts: 'being after y' and 'being before x'. And any time we substitute variables with constants we introduce a new concept. So, once we substitute y with a, we change 'being after y' in to 'being after a'. And by repeating the substitution by b, we introduce a different concept, namely 'being after b'.

To illustrate the idea, 'being after 12' is a different concept from 'being after 15' with different extensions. It is not the case with properties however. The occurrence of Fa and Fb in a proof does not increase the number of concepts; the concept is still F. Therefore, if for instance, (29) and (30) are premises of an argument such that establishing the conclusion relies on them, that is, we need to consider them (use them) in the proof to reach the conclusion, then while for Frege, this process is just applying logical rules (analysis) to two formulas or two thoughts, for Kant the proof actually includes the introduction of new concepts.[33]

29. $\forall x \exists y Rxy$

30. $\forall x \forall y (Rxy \rightarrow Ryx)$

This looks like a good explanation for why Kant used to think that no amount of 'twists and turns' of concepts occurred in premise that we start with them, would suffice for proving an arithmetic theorem. And most likely Frege had been aware of that, because he embraces these new concepts as part of what makes us able to prove what we want to prove. This matches with his geometrical metaphor. He thinks that a proof, to be informative, needs to introduce us to something new. Most likely these new concepts have been the missing elements for confirming the truth of the theorem. This also explains why Kant thinks of arithmetic equations or theorems as synthetic because we confirm them with the mediation of new concepts. For instance, thinking of theorem (k) for the simple case of a two digit number like xy (x and y range over one digit numbers)leads us to the formula (31).

k. Every number with the sum of digits divisible by three is divisible by three.

31. $\forall x \forall y (\exists z (x + y = 3z) \rightarrow \exists w (10x + y = 3w))$

If we formalize the 'sum of digits' as $x+y$ and the number itself as $(10 \times x)+y$, then we cannot affirm the theorem straight away. The process of proof, then, should include the introduction of new concepts such as $(9x + x) + y$ and $9x + (x + y)$ and $(3 \times 3)x + 3z$ and then $3(3x) + 3(z)$ and finally $3(3x + z)$ to allow us to affirm the theorem in the case of two digit numbers. In the following argument, Fregean notion of concept would allow us to see the process of using the relation 'next to' as conceptual 'twists and turns', while a Kantian analysis would consider it as introducing new concepts: Y was

[33]In the last section of Chapter Three we saw Frege's understanding of relations and how they should be analysed.

standing on the left hand side of X such that anyone on her right hand side could see the fireworks, therefore X could see the fireworks. The similarity between these two examples is that in both cases a substitution plays the main role in deriving the conclusion. Whoever understands 10 affirms the correctness of replacing it with 9+1. Likewise, everyone who knows what 'Y was on the left hand side of X' affirms substituting it with 'X was on the right hand side of Y' and together with 'everyone on the right hand side of Y could see the fireworks ' would be able to derive the conclusion. An other way of cashing out the similarity between these two cases is that in both cases the individuals and relations between them play a crucial role in reasoning.

Two noteworthy points about these examples: one is that different individuals might stand in a relation and if someone, like Kant, gets used to thinking of only properties as concepts, then different individuals standing in a relation in a proof appears as different and even sometimes as new concepts. In everyday life relations are contingent and produce new but arbitrary Kantian concepts. For instance there are many things that may happen to be on our right hand side. However once it comes to arithmetic, given the structure of the numbers, relations among numbers are not contingent and many of these relations are not arbitrary in the sense that in one way or another give us conceptual ground to infer useful information about numbers. The other point is that substitution can be a creative and crucial part of reasoning by choosing the right substitute. Nonetheless, it is not something we can capture by stating rules of substitution. This can be understood as the important role of individuals in a proof. However, it also can be understood as the importance of substitution more generally.

If we think of constants not as names for objects, but as place-holders in a proof (their only difference with variables is that they are not substitutable with another constant in a proof), this increase of concepts can then be understood as a sign of the complexity of the structure of a proof (we shall see more in this regard in the last part of this chapter). Metaphorically, if we think of proofs as structures, individuals as nodes, and relations as links, then the number of links used in the structure is one measure of the complexity of a structure (proof). Another measure of complexity could be the number of nodes, however we shall see why the latter is a better measure for complexity. We are going to probe the idea of the similarity of arithmetic reasoning and logical reasoning in predicate logic to see to what extent this similarity holds.

5.5 Hazen's Observation

Now that we have a grasp of reasoning with individuals its time to have a look at some aspects of it. For instance, are these individuals necessary for the reasoning, in the sense that reasoning without them is not possible? If they are inevitable, given a certain argument, can we reduce the number of them in a proof? In an influential paper Allen Hazen has addressed these questions.[34] He takes the following argument and examines different possible proofs for it:[35]

32.

$$\begin{gathered}\forall x\forall y(Rxy \to \exists z\exists w(Sxz \wedge Szw \wedge Swy))\\ \forall x\forall y(Sxy \to \exists z\exists w(Txz \wedge Tzw \wedge Twy))\\ \forall x\forall y(Txy \to \exists z\exists w(Uxz \wedge Uzw \wedge Uwy))\\ \forall x\forall y((Uxy \wedge Fx) \to Fy)\\ \therefore \forall x\forall y((Rxy \wedge Fx) \to Fy)\end{gathered}$$

Technical details will be dealt with in the next section, here we just explain the facts roughly. The above argument includes 16 quantifiers, while the normal proof[36] of the argument in a natural deduction system involves 28 constants or object-variables. Hazen notes that there are other proofs for this argument which have less than 28 constants, but these proofs involve formulas such as $\forall x\forall y((Txy \wedge Fx) \to Fy)$ or $\forall x\forall y((Sxy \wedge Fx) \to Fy)$ that do not appear in either the premises or the conclusion of the argument in question.[37] We shall call these proofs Non-Normal. This inflation in the number of constants supports Hintikka's idea that reasoning in predicate logic is reasoning with individuals. Considering a system of predicate logic, say natural deduction, in which we introduce constants (individuals or names) to the context of a proof while eliminating quantifiers; it might come across as surprising if the number of constants exceeds the number of quantifiers occurring within the argument.However, we still have no clue about how to calculate the number of individuals implied by our premises; soon we say more in this regard.

What actually happens in proving the argument (32) is that we apply elimination rules to the premises line two and three respectively three and six times. So if we repeat these formulas each time we apply elimination

[34] [Hazen, 1999]

[35] Ibid. p.87. He mentions that it is a "toy version" of Boolos's example in [Boolos, 1984].

[36] We shall see the definition of normal proof in the next section.

[37] [Hazen, 1999], p.91-92

rules to them, then the number of constants does not exceed the number of quantifiers. However, if, following Frege, we think of 'applying logical rules to premises of an argument to prove the conclusion' as analysis of them (which actually leads to breaking formulas into its constituents), then Hazen's point is that proving some conclusions from a group of premises requires analysing premises in a repetitive manner which involves introducing constants to the proof. Of course it can be said that these objects or individuals or names that appear in the proof, though are new in the sense that they have not been appeared in our premises, have been nested in the premises. That is applying logical rules on premises requires these constants. This may convince us to accept that these constants are not new, but Hintikka and Hazen's main point still remains, that in some proofs constants play an important role. As we saw, Hazen shows this point by mentioning that if we want no logical formula to appearance in the premises or conclusions in our proof, then there is no way to reduce the number of constants in the proof.

We should resist against the temptation to think there is a direct relation between the increase in the number of constants and concepts (in the Kantian sense of concept, that is, just properties) in a proof. For instance, if we show constants fall under relation Rxy as ordered pairs then table (34) shows ordered pairs occur in the proof of argument (33).

33.

$$\forall x \forall y (Rxy \rightarrow Ryx)$$
$$\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$$
$$\forall x \exists y Rxy$$
$$\therefore \forall x (Rxx)$$

34. $Rxy : \langle a, b \rangle, \langle b, a \rangle, \langle a, a \rangle$

As can be seen, while relation Rxy has been instantiated three times and it introduces four concepts (properties) to the context of proof of argument (33), namely $Ra-$, $Rb-$, $R - a$ and $R - b$, the number of constants introduced to the proof is Two (a, b). On the other hand, the number of constants introduced in the proof of argument (35) is Three but no concept other than Px has been introduced to the proof of the argument where Px attributes property P to the individual x.

35.

$$\forall x (Px \rightarrow \exists y \exists z (\neg Py \wedge \neg Pz))$$
$$\exists x Px$$
$$\therefore \exists x \exists y \exists z (Px \wedge \neg Py \wedge \neg Pz)$$

What has been said about Kant's and Frege's different understandings of conceptual analysis and analyticity or syntheticity of arithmetical proofs, besides Hazen's observation, provides enough material to attempt to discern analytic and synthetic proofs in the context of modern predicate logic which is the aim of the next part of this chapter. This also helps us to see to what extent arithmetical and logical reasoning (predicate logic) are similar.

5.6 Analytic/Synthetic Proofs

Our definition of analytic and synthetic proof in predicate logic is inspired by Hintikka's idea. In *An Analysis of Analyticity*[38] he defines a valid proof step as analytic if "it does not increase the number of individuals one is considering in their relation to each other".[39] In this section we provide an account of analytic proof which is inspired by Hintikka's criterion for analytic steps in natural deduction. To do so, let us start with a similar notion, namely normal proof. Both of these ideas (normal proof and analytic proof) are concerned about what appears in a proof, as we shall see, one is conceren with which formulas appear in a proof (normal proof), and one is concered with which individuals appear in a proof (analytic proof).

In *Investigations into Logical Deduction*,[40] Gentzen introduced Natural Deduction (ND) to formalize the actual way of mathematical reasoning. He believed that a good proof should not be a roundabout. That is every formula which appears in a proof either already has to be in the premises or has to appear in the conclusion of the argument. This was a reaction to axiomatic systems of logic in which, to meet standard formal axioms' requirements, proofs of an arguments may include formulas that never appear in the premises or the conclusion. For instance, in an axiomatic system with axioms expressed in (36)-(38) where upper case letters stand for complex formulas and not just atomic propositions, and Modus Ponens (MP) as inferential rule, the proof of $p \to p$ is as shown in (39).

36. $A \to (B \to A)$

37. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

38. $(\neg B \to \neg A) \to (A \to B)$

39.

$$1 : p \to ((p \to p) \to p) \tag{36}$$

[38] [Hintikka, 1973]
[39] ibid, p.141
[40] [Gentzen, 1964]

$$2 : (p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p)) \qquad (37)$$

$$3 : ((p \to (p \to p)) \to (p \to p) \qquad 1, 2, MP$$

$$4 : p \to (p \to p) \qquad (36)$$

$$5 : p \to p \qquad 3, 4, MP$$

The formulas in the first and second lines of the proof (39) are there only to meet formal requirements of a proof in this particular axiomatic system. Since our axiomatic system has only one inference rule and only three axioms as mentioned in (36)-(38). A proof such as (39) is a roundabout proof according to Gentzen, because it contains a formula that is not really necessary for achieving the intended conclusion. Instead, he introduced a system which is close to natural ways we usually do reasoning, he named it Natural Deduction (ND). Here is a much simpler proof for $p \to p$ in ND:

$$40. \quad \dfrac{\dfrac{[p]1 \qquad \dfrac{[p]1}{p \to p} \to I}{p} \to E}{p \to p} \to I(1)$$

As can be seen, the proof, though still is odd, do not include any formula like those in line one and two of (39). The proof is odd because contains instances of both vacuous and multiple discharge for conditional introduction rule.

To be able to show what he means by a roundabout proof in ND, Gentzen introduced another system called Sequent Calculus (SC). SC system rules are divided into Structural and operational rules and operational rules are divided to introduction of a logical word to the right and left hand side of the turnstile. Structural rules do not involve any logical word, they are rules for how to combine premises and conclusions (the conclusion).[41] The rule schemas in (41) are rules for introducing conditional to the right and left hand side of the turnstile respectively. Unlike ND, there is no elimination rule in SC. This system has a property called the Subformula property which means in each of its rules the principle formulas (expressed by upper case Roman alphabet) on the upper side of the conclusion line are subformulas of the principle formulas on the lower side of the conclusion line.

$$41. \quad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} R \to \qquad \dfrac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C} L \to$$

Because of the subformula property, any sequent can be traced back to its basic sequent ($A \vdash A$). Proofs (42) and (43) are two samples of proofs in SC:

[41] We have seen them in Chapter Two.

42.
$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,Wk}{A \vdash B \to A}\,R\to}{\vdash (A \to B) \to A}\,R\to$$

43.
$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad \dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B}\,L\to}{A \to (A \to B), A, A \vdash B}\,L\to}{A \to (A \to B), A \vdash B}\,Ctr}{A \to (A \to B) \vdash A \to B}\,R\to}{\vdash (A \to (A \to B)) \to (A \to B)}\,R\to$$

As can be seen, what happens from the bottom to the top of these arguments is breaking a complex expression into its basic elements (although A, B ... can be complex expressions, they still can be singleton propositions too. The argument will be followed to the singleton level if there are complex expressions). A proof in SC gives us a complete picture about how a complex formula is built over its basic constituents. This also includes how many times these constituents have been assumed.

ND systems with mere rules of inference do not have a subformula property because any elimination rule actually behaves against the subformula property. However, it is possible to have proofs in ND in accordance with SC. The idea is to try to rebuild ND proofs in accordance with SC proofs. Here is a proof in SC:

44.
$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p \qquad \dfrac{q \vdash q}{r \to q, q \vdash q}\,Wk}{p \to q, r \to q, p \vdash q}\,L\to \qquad \dfrac{r \vdash r \qquad \dfrac{q \vdash q}{p \to q, q \vdash q}\,Wk}{p \to q, r \to q, r \vdash q}\,L\to}{p \to q, r \to q, p \vee r \vdash q}\,L\vee}{(p \to q) \wedge (r \to q), p \vee r \vdash q}\,L\wedge}{(p \to q) \wedge (r \to q) \vdash (p \vee r) \to q}\,R\to}{\vdash ((p \to q) \wedge (r \to q)) \to ((p \vee r) \to q)}\,R\to$$

And here is a proof in ND made in accordance with the SC one:

45.
$$\dfrac{\dfrac{\dfrac{[p \vee q]2 \qquad \dfrac{\dfrac{[(p \to q) \wedge (r \to q)]3}{P \to q}\,\wedge E \qquad [p]1}{q}\,\to E \qquad \dfrac{\dfrac{[(p \to q) \wedge (r \to q)]3}{r \to q}\,\wedge E \qquad [r]1}{q}\,\to E}{q}\,\vee E(1)}{(p \vee r) \to q}\,\to I(2)}{((p \to q) \wedge (r \to q)) \to ((p \vee r) \to q)}\,\to I(3)$$

In the ND proof, elimination rules have been used prior to introduction rules. This proof is really straightforward; it could be made longer; for instance the right hand side of the branch of the proof (45) could be like this:

46. $$\dfrac{\dfrac{\dfrac{\dfrac{(p \to q) \wedge (r \to q)}{r \to q}\ \wedge E \qquad r}{q}\ \to E \qquad r}{q \wedge r}\ \wedge I}{q}\ \wedge E$$

Considering mere rules, nothing is wrong with (46), but there is a detour in this piece of proof. That is in the last line of this short proof we regain q that we already had two lines ago, at the expense of the appearance of a formula $(q \wedge r)$ in our proof which neither is part of our premises nor part of the conclusion. This detour has happened by introducing and then immediately eliminating a logical word. What has been said, suggests that the rules of ND can be used in a certain way which forms direct and detour-free arguments. These kinds of arguments have been called Normal in the literature. Here is one definition of normal proofs:

l. A proof is normal if and only if the concluding formula of an introduction step is not at the same time the major premise of an elimination step of the same connector.[42]

To transform a non-normal proof in ND to a normal one, all detours in the proof should be removed. This can be done by a guideline provided by the Inversion principle. Intuitively speaking the Inversion principle says that "whatever follows from grounds of a proposition must follow from that proposition".[43] The formal definition that guides one to form a normal poof is: "Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premises of α, when combined with deductions of the minor premises of α (if any), already 'contain' a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α".[44] To form a normal proof from the beginning, a corresponding SC argument can be used. What happens if we use a SC argument to make a ND one is that all related elimination rules should be applied first and then introduction rules can be applied to reach the intended conclusion. An example of such ND proof, as we have seen, is (45). What happens in proofs such as (45) is

[42] [Dummett, 1977], p.158
[43] [Negri and Plato, 2001], p.6
[44] [Prawitz, 1965], p.35

that some rules have a larger scope than one step of a proof and such rules are sometimes called global. For instance, in (45), the scope of the disjunction elimination rule is from the second line to the fourth line and the scope of one of the conditional introductions is from the third line to the fifth line while the scope of the other conditional is from the first line to the last line. Here are some examples of the rule schemata with a global scope:

47. $$\dfrac{A \wedge B \qquad \begin{array}{c}[A \quad B]\\ \vdots \\ C\end{array}}{C}\ \wedge E \qquad \dfrac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ \vee E \qquad \dfrac{A \to B \qquad A \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ \to E$$

The elimination rules in (47) are called general elimination rules and definition of normal proofs with general elimination rules is given in (m).

m. A derivation in ND with general elimination rules is in normal form if all major premises of elimination rules are assumptions.[45]

ND normal proofs do not have the subformula property in the sense that SC arguments have. That is they are not mere breaking formulas into basic sequents. However, they have the subformula property in the sense that every formula which appears in the proof is the subformula of either the premises or the conclusion. It is not difficult to see why this is the case. Once we apply elimination rules we are just breaking formulas in the premises into their parts. Then when we start using introduction rules, we are just making more complex formulas, so whatever appears in the argument either has been part of the premises or will appear in the conclusion since the last applied rules are all introduction rules.

Normal proofs can also be called analytic proofs in the sense that in forming them we do not go any further than the resources that we have been given by the argument. It is worth noting that by argument we mean premises and the conclusion. The point is that without this amendment there are some logical moves that are not analytic if we consider premises only. For instance, in case of SC rules, weakening on both the right and left hand side would not be analytic if we only consider the above the conclusion line formulas. Also disjunction introduction and vacuous conditional introduction would not be analytic inference moves in ND if we only consider premises. The fact that there is nothing in proofs beyond what is in the premises and the conclusion has a syntactic manifestation in propositional logic proofs. However, the situation is a bit tricky when it comes to predicate logic. This is because defining the resources given to us by an argument is more complicated than propositional logic.

[45] [Negri and Plato, 2001], p.9

Let us consider a quantified formula such as $\exists xAx$ or $\forall xAx$, it is syntactically acceptable to say Ax is a subformula of any of these formulas; but how about Aa or Ab? to justify the claim that, say Aa is a subformula of, say $\exists xAx$, we need to appeal to something further than syntax. There are at least two justifications for subformula property in quantification rules, one syntactic and the other semantic. The syntactic justification, given by Gentzen and Prawitz, reasons that a is an unbound object variable (UOV) and x is a bound object variable (BOV) therefore, if Ax is a subformula of $\exists xAx$ or $\forall xAx$, then so is Aa. Because both x and a are variables. The semantic way to justify quantification rules is to say that the truth of Ba is guaranteed or contained in the truth of $\forall xAx$. A similar story can be told about $\exists xAx$.

In regard to the syntactic justification, the fact is that the behaviour of UOVs is more like names than variables. Hazen's observation, discussed in the previous section, helps us to address this point. As Hazen showed, sometimes the number of UOVs in a proof exceeds the number of BOVs in its relevant argument. And any attempt to reduce the number of UOVs leads to the appearance of formulas that are not subformulas of the premises or the conclusion.[46] If UOVs are just variables, then we should be able to replace them and reduce them. And if they are place-holders in a proof, we need to define which places do they hold. Because it is not the case that they hold a unique position in the whole proof. This is so because it is not the case that every time we apply a quantifier elimination rule we use a fresh (unused) UOV. We might use an UOV that have been appeared in former steps of a proof. This means that some of these UOVs might reappear in a proof; therefore they cannot hold a specific place in a proof globally. That is they cannot specify a position in the whole of the proof.

Moreover, there is no correspondence or clearly observable connection between the number of BOVs in arguments and the number of UOVs in their proofs. For instance, the argument from $\forall x\forall yRxy$ to $\exists x\exists y\exists z\exists w(Rxy\vee Ryx)$ can be proven with only two UOVs despite appearing four BOVs in the conclusion while proving $\forall x\forall y(Rxy\vee Ryx)$ from $\forall x\forall yRxy$ requires two UOVs which is the same number as BOVs that appear in the conclusion. So if we want to say that they hold a place, we need a more fine-grained theory about which place UOVs do hold. This ambiguity in justifying subformula property in predicate logic, makes it difficult to define the resources given to us by an argument in predicate logic. Now if we accept Hintikka's criterion for analytic proofs as proofs that in them the number of individuals considered in relation to each other do not increase, then not all proofs with subformula

[46]To check these formulas see[Hazen, 1999],p.92

property, that is normal proofs, are analytic. Where analytic means not going further than the resources given by an argument, and resources are understood as the number of relations between individuals appearing in an argument. This might be due to an increase in the number of individuals considered in a proof, or an increase in the number of considered tokens of a particular relation.

In regard to the second justification, it should be said that what is happening in a proof cannot be explained only in terms of semantics. Precisely speaking, there are steps in a proof that cannot be justified only by requirements of soundness. Some substitutions in a proof in predicate logic need to meet some *formal requirements* which have epistemic justification as well. UOVs play a different role compared to BOVs when it comes to this kind of justification. For instance, if at the middle of a proof from $\exists x(Px \to Qx)$ and $\forall xPx$ to $\exists Qx$, we have $Pa \to Qa$ and we need to eliminate the universally bound variable x in $\forall xPx$ in order to apply Modus Ponens (MP), we need to substitute x by a in order to apply MP. We could substitute x with, say, b and it was still a sound move, but then we could not apply MP any more because a in $Pa \to Qa$ is a particular individual, although arbitrary. This is so because a has been substituted for an existentially bounded quantifier earlier in the proof. So only the requirement of soundness for one particular inference step cannot justify our moves in a proof; an attention to the individuals that are evolved in the process of reasoning does matter too. Therefore, the semantic justification for subformula property, like the syntactic one, is not helpful enough to define the resources given by an argument. Consequently, none of the syntactic and semantic justifications for subformula property are sufficient for defining the notion of analytic proof in predicate logic, if we adopt Hintikka's idea about analyticity.

As we just saw in the algebraic example two sections ago, some proofs include introducing concepts that are derivable from the premises. Nonetheless, they play a different conceptual role from the premises. We saw that specific numbers, numbers 10,9,3, and 1 in that specific example, played a crucial role in forming concepts required for confirming the conclusion. Something similar to that can happen in predicate logic. Let us consider the following argument:

48.

$$\forall x\exists y\exists z(Sxy \wedge Sxz)$$
$$\forall x\forall y\forall z((Sxy \wedge Syz) \to Szx)$$
$$\forall x\forall y\forall z((Sxy \wedge Sxz) \to Syz)$$
$$\therefore \exists x\exists y(Sxy \wedge Syx)$$

If we read Sxy as 'S relates x to y', then the individuals that both relate to and are related by S, or in other words, stand in a symmetric relation after analysing the premises (applying quantifier elimination rules) in a not only valid but also suitable way, play a crucial role in confirming the conclusion. In both of these cases specific individuals play a salient role in forming concepts required for confirming the conclusion.

These considerations lead us to think that if we could expand the proof theoretic concern about keeping track of what premises provide and how we use them in a proof then this more detailed picture of proofs would help us to understand aspects of reasoning in predicate logic in a more detailed manner. The details that cannot be captured in the existing schematic rules for predicate logic in a ND system, expressed in (48).[47] For instance, A in schemata shown in (48) can be a complex expression with several relations and properties.

48. $$\frac{A(y\backslash x)}{\forall x Ax}\,\forall I \qquad \frac{\forall x Ax \qquad \begin{array}{c}[A(t\backslash x)]\\ \vdots \\ C\end{array}}{C}\,\forall E \qquad \frac{A(t\backslash x)}{\exists x Ax}\,\exists I \qquad \frac{\exists x Ax \qquad \begin{array}{c}[A(y\backslash x)]\\ \vdots \\ C\end{array}}{C}\,\exists E$$

An example will illustrate what cannot be captured in the above schemas. Having the 'concept as property' stipulation in mind, if we consider a formula such as $\forall x \forall y Rxy$ and compare it with $\forall x Bx$, then what is missing shows up. Conceptually speaking, there is no much difference between Ba and Bb, whereas Ray and Rby are two different concepts. Standing in relation R with a is one concept and standing in relation R with b is another. Of course these concepts might be very arbitrary, per se, 'standing in relation R with a, and a can be anything' is not substantially different from 'standing in relation R with b, and b can be anything'. However, when, say a, has been substituted for an existential quantifier somewhere before in the context of a proof, then Ray is not as *general* a concept as is Rby in that proof given that b has been substituted for a universally quantified variable. Or if keeping track of a position in a relation is important for an argument, then this might give a specific role to those arbitrary individuals that fill those positions while such role is not reflected in requirements of soundness for quantification rules. The idea here is to realize and distinguish the conceptual structure of arguments as part of the logical structure.

Therefore we need to introduce new notions to be able to refer to predicate formulas in a more detailed manner. Before doing so, the idea that formal definitions try to capture is this:

[47] [Negri and Plato, 2001], p.64-65

n. An analytic proof in predicate logic is a normal proof that does not use more conceptual resources than the argument provides for it.

To make sense of this idea we need to define what the conceptual resources are that an argument gives us and what does it mean to go further than that. Then the next step is to check under which circumstances proofs of arguments use any conceptual resources further than those provided by the argument. To capture these conceptual resources we rely on a simple idea that any n-place predicate that appears in the premises or the conclusion of an argument contributes n to the context of proof where $n > 1$. For example, a Two-place predicate such as Rxy appears in an argument, it contributes two concepts into the context of that argument, namely $Rx-$ and $R-y$. The thought behind this idea, has been borrowed from Kant. If we want to address these concepts formally, they are 'being the first, second, third...place of a specific relation'. These concepts are constant-sensitive that is each blank represents a different concept once the other blanks are occupied by different constants. For instance, the first place in $Rxab$ represents a different concept from $Rxcd$ or from $Rxyz$. That is, say 'being between a and b' is a different concept from 'being between c and d' or 'being between two things' and they might have different extensions.

This conceptual difference might become crucial in some arguments. For instance, in argument (49) it is crucial to distinguish between two positions in relation R this leads to an increase in the number of ordered pairs appearing in that proof. In this argument, only one pair of UOVs is required for a sound elimination, however, we need two of them to be able to confer the conclusion. The point is that even applying elimination rules to the conclusion will not show the required extra pair of UOVs as we can eliminate quantifiers in the conclusion formula soundly, only with one UOV.

49.

$$\forall x \forall y Rxy$$
$$\therefore \forall x \forall y (Rxy \vee Ryx)$$

While in argument (50), there is no such need, and therefore the proof of it requires the same number of ordered pairs that is given in the argument.

50.

$$\forall x \forall y Rxy$$
$$\therefore \forall x Rxx$$

Let us articulate the points have been said about the conceptual structure of expressions in predicate logic so far. This helps us to clarify the vocabulary we are going to use in the rest of this part. Considering the language of predicate logic:

o. Any blank in any n-place predicate of the language of the first order logic is called a position.

p. Two positions are *identical* if (and only if) they are located in the same blank of the same predicate.

q. Each individual (constant) that occupies a position in a n-place predicate falls under a concept. In other words, each position represents a concept.

r. A concept is defined once all the other positions of the n-place predicate are occupied by constants. Otherwise it is undefined.

s. Two identical positions represent the same defined concept only if all the other positions of the n-place predicate are occupied by the same corresponding constants.

These definitions give us enough vocabulary to keep the track of concepts in the process of proving an argument in predicate logic once the notion of 'concept' is used as a property that constants might fall under. Now we can examine whether anything similar to the algebraic example (k) happens in a proof in predicate logic or not. That is, is there any sense in which we go further than the conceptual resources that a premise provides us? From (r) it is clear that all the monadic predicates express defined concepts regardless of what occupies their position (a constant or variable). For the sake of ease, we shall consider relations (two place predicates) but every point mentioned here can be extended to more than two place predicates.

Considering $\forall x \exists y Rxy$ in (33), Rxy in the premise, provides two undefined concepts $Rx\_$ and R_y. So, if we apply the universal elimination rule to $\forall x \exists y Rxy$ then we have$\exists y Ray$. Now we have come across a defined concept Ra and the formula $\exists y Ray$ tells us that there is at least an individual (a constant) that falls under it ($Ra\_$). The fresh constant b which is introduced to eliminate the existential quantifier introduces another defined concept, namely R_b. That is the ordered pair $\langle a, b\rangle$ that instantiates Rxy and introduces two defined concepts to the context of proof. As we continue the proof, another ordered pair, namely $\langle b, a\rangle$, will appear as another instance of Rxy once we apply elimination rules to (51).

51. $\forall x \forall y (Rxy \rightarrow Ryx)$

This increases the number of defined concepts as it introduces two other ones, $Rb\_$ and R_a. And finally, the third instance, that is namely $\langle a, a \rangle$, will appear by applying the elimination rules to (52). However, this one does not introduce any other defined concepts since $Ra\_$ and R_a both have been introduced to the proof before.

52. $\forall x \forall y \forall z ((Rxy \rightarrow Ryz) \rightarrow Rxz)$

A relevant question here is whether we should consider introducing defined concepts as introducing new concepts to the context of proof or not. That is whether introducing defined concepts is actually going further than conceptual resources of the argument or not. As we have seen, a defined concept is a different concept from an undefined one ('being after something' is different from 'being after 10'); but, there is a real concern with considering a defined concept as a new one. The issue is that it would not explain the idea of going beyond the conceptual resources of an argument. There is no reason to accept that the concept represented by the first place in $Rxab$ goes any further than the concept represented by the first place in $Rxyz$ despite the fact that they are two different concepts, perhaps with two different epistemic roles in a proof. Specifically if the constants substituted with variables are not fresh in the context of that proof. However, there is no genuine difference between them and the variables in terms of, let's say, expansion of truth conditions.

A more plausible path to pursue is to understand the number of defined concepts, which are obtained from undefined concepts, as a measure for the complexity of a proof. In this way of thinking, it is true that all valid arguments are analytic in the sense that the defined concepts that are introduced to the context of proof, by applying the elimination rules to the formulas, are not new concepts. Nonetheless, the complication of the proof process and the different roles that defined concepts play in the proof makes the conclusion less self-evident than arguments such as 'all humans are mortal, Socrates is human, therefore Socrates is mortal'. To justify the idea, if we consider the definition (n) of an analytic proof, when choosing an efficient strategy for the substitution process is a crucial part of establishing a proof, then it sounds fair to say in that the proof we have used more than what conceptual resources of the argument offer us. The argument (33) is a good witness for this claim. So, we may understand the analytic/synthetic distinction in terms of different levels of complication of proof and not as forming a new concept.

Another reason for the epistemic significance of individuals, the defined concepts created with them, is that, as we have seen concerning intuition,

logical rules of predicate logic leave several possible ways to prove the conclusion from the premises. This is the case for propositional logic as well (we just saw a part of (45) could be proven as in (46)), though in a more restricted manner, but the point is that all the different paths of proofs in propositional logic will end with the conclusion sooner or later. In the case of predicate logic it is more complicated. We know that the process of proof search in monadic predicate logic (predicate logic with only one-place predicates), will end since predicate logic with monadic predicates is decidable. That is, given any formula, we can decide whether it is a theorem of predicate logic or not. Whereas proof searching process in a language with n-place predicates ($n > 1$) might not end. We have seen an example of such an argument (23) in the section on intuition. So one may think in cases that there are proofs for such arguments finding them does not only rely on making sound logical inferences.

For instance, the argument given in (33) has a reasonably short proof. The proof, as mentioned in the latter part, includes only two UOVs, but the proof also includes three ordered pairs as instances of relation R in the argument. And we saw that once the number of ordered pairs that stand in a relation increases (which is a different notion from increase in the number of individuals or names) the number of concepts also increases. One way of understanding the situation is that the introduced concepts though logically derivable from the premises, are required to assert the conclusion. Without their mediation and only by considering the premises, the conclusion is not perceivable. These introduced concepts might play a conceptual role that suggests to us a way of proving and this is the thing that general logical rules cannot provide. They may help us to find a proof for an argument (23) while blindly applying tableaux rules, to have proof of it in a tableaux system, might never end up at the intended conclusion. This further clarifies the sense in which we go beyond logical rules in reasoning with individuals.

The salient part of the task now is to give a formal account of how complicated a proof is. The guideline is that the more the paths that the logical structure of the argument leave open for us to follow, the more we need to go further than the conceptual resources we have been given, to establish the proof. In case of existential quantifiers, substituting variables that are bound by them is not that complicated; there is one way to go, namely substitute the variable by a fresh constant (in ND or tableaux systems). While in substituting variables that are bound by universal quantifiers there are more ways to go. When it comes to introduction rules, existential quantifiers can be trickier than universal quantifiers. This fact can be reflected in calculating the number of undefined concepts that an argument provides for a proof. This calculation can be done based on definition (t).

t. Two identical positions represent two different undefined concepts if (and only if) they introduce two different defined concepts to the context of a proof by applying the elimination rules (of quantifiers) to them.

Comparing two cases would be helpful; expression (53) includes four undefined concepts just concerning a specific relation R, $Rx\_$, R_y, R_z and R_w. The undefined concepts $Ry\_$ and $Rx\_$ can be the same since both x and y are bound by universal quantifiers and we do not have to use two different constants to substitute for them merely considering logical rules. If we need to do so it is because of the conceptual demands of the proof and not logical necessity. And R_y, R_z and R_w are going to introduce three different defined concepts since both variables z and w are bound by existential quantifiers that are in the scope of the universal quantifier that bounds y. This means that no matter which constant is substituted for y, the constants substituting for z and w are going to be different from that (in ND or tableaux systems).

53. $\ldots\forall x\ldots\forall y\ldots\exists z\ldots\exists w(\ldots Rxy\ldots Rxz\ldots Ryw\ldots)$

Though the expression (54) includes three undefined concepts only considering R, $Rx\_$,R_y, and R_w. The undefined concept $Rz\_$ can be the same as $Rx\_$ and the undefined concept R_z may or may not be different from R_y, as z is bound by a universal quantifier which is in the scope of the existential quantifier that bounds y and can be substituted by the same constant that has already been substituted by y.

54. $\ldots\forall x\ldots\exists y\ldots\forall z\ldots\exists w(\ldots Rxy\ldots Rxz\ldots Rzw\ldots)$

Now that we know how to calculate the number of undefined concepts in an argument, if the number of defined concepts occurring in a proof exceeds the number of undefined concepts in an argument then, in one sense, the proof needs more than the conceptual resources that its argument provides (to establish the conclusion). In other words, it means that the proof demands more than the conceptual resources that the argument contributes in the first place. Besides, the way that we have defined the number of undefined concepts represents the concern about complication of the proof process as well. Every proof which demands more than the conceptual resources given by its argument is called a synthetic proof. So the definition of analyticity given in (n) can transform to a more concrete one in (u) by actually cashing out what 'conceptual resource' is.

u. A normal proof is analytic if the number of defined concepts occurring in it does not exceed the number of undefined concepts either in its premises or in its conclusion, otherwise it is synthetic.

v. An argument is analytic if it has an analytic proof, and it is synthetic if all its possible proofs are synthetic.

These definitions also capture Hintikka's point about analytic proofs as proofs that does not increase the number of objects one needs to consider in relation to each other. This is because the definition (u) picks up the same proofs that the definition (w) picks.

w. A normal proof is analytic if the number of ordered pairs occurring in it does not exceed the number of ordered pairs either in its premises or in its conclusion

Arguments (55) and (56) are examples of analytic arguments and argument (57) is an example of a synthetic one. Since in (55) we have two undefined concepts (one ordered pair) in the premise ($Sx\_, S\_y$) and four undefined concepts (two ordered pairs) in the conclusion ($Sx\_, Sy\_, S\_z, S\_u$), and as proof (58) shows, the proof includes only two defined concepts ($Sa\_, S\_a$ that is one ordered pair).

55.

$$\forall x \forall y Sxy$$
$$\therefore \exists x \exists y \exists z \exists u (Sxy \vee Szu)$$

56.

$$\forall x \forall y Rxy$$
$$\therefore \forall x Rxx$$

57.

$$\forall x \forall y Pxy$$
$$\therefore \forall x \forall y (Pxy \vee Pyx)$$

58.

$$\dfrac{\forall x \forall y Sxy \qquad \dfrac{[\forall y Say]2 \qquad \dfrac{\dfrac{\dfrac{[Saa]1}{Saa \vee Saa}\,\vee I}{\exists x(Sxa \vee Saa)}\,\exists I \\ \vdots \\ \exists x \exists y \exists z \exists u (Sxy \vee Szu)}{\exists x \exists y \exists z \exists u (Sxy \vee Szu)}\,\forall E(1)}{\exists x \exists y \exists z \exists u (Sxy \vee Szu)}\,\forall E(2)$$

In the case of argument (56), the premise and the conclusion contain two undefined concepts each ($Rx\_$ and R_y, that is one ordered pair, in the premises and $Rx\_$ and R_x, that is one ordered pair, in the conclusion) and the proof for the argument also includes two defined concepts ($Ra\_$ and R_a, that is one ordered pair). In the case of argument (57) though, while each of the premises and the conclusion contain two undefined concepts ($Px\_$ and P_y, that is one ordered pair, for the premise and $Px\_$ and P_y, that is one ordered pair, for the conclusion), the proof contains four defined concepts ($Pa\_$,$Pb\_$,P_a,P_b, which means having two ordered pairs). The fact that an argument such as (57) is synthetic according to this account is coherent with discerning the conceptual structure of an argument as a part of its whole logical structure. And also is coherent with the idea that a defined concept that is introduced to a proof by applying the logical rules on an undefined concept (by applying elimination rules or other to a formula), is a different kind of information and does not increase our information necessarily. However, we shall see that sometimes synthetic proofs can be genuinely informative. The role played by individuals is not predominantly important in argument (57), the evidence for this claim is that argument (59), which is logically equivalent to argument (57) is analytic.

59.

$$\forall x \forall y Pxy$$
$$\therefore \neg \exists x \exists y \neg (Pxy \vee Pyx)$$

One point needs to be mentioned about calculating the number of undefined concepts in arguments with more than one premises. To calculate the number of concepts in an argument with more than one premises we can write the conjunction of them in the normal prenex form. This enables us to check that the order of applying the elimination rules to the premises does not have any effect on the number of undefined concepts in an argument. An example explains the point; the argument (60) is a mini model of argument (34).

60.

$$\forall x \forall y (Rxy \rightarrow \exists z (Sxz \wedge Szy))$$
$$\forall x \forall y ((Sxy \wedge Fx) \rightarrow Fy)$$
$$\therefore \forall x \forall y ((Rxy \wedge Fx) \rightarrow Fy)$$

And here are two different prenex forms of this argument:

61. $\forall x \forall y \exists z \forall w \forall v(((Rxy \rightarrow (Sxz \wedge Szy)) \wedge ((Swv \wedge Fw) \rightarrow Fv)) \rightarrow ((Rxy \wedge Fx) \rightarrow Fy))$

62. $\forall w \forall v \forall x \forall y \exists z(((Rxy \rightarrow (Sxz \wedge Szy)) \wedge ((Swv \wedge Fw) \rightarrow Fv)) \rightarrow ((Rxy \rightarrow Fx) \rightarrow F))$

The difference between (61) and (62) is that the order of applying the elimination rules to the premises is different . The prenex form (61) represents a proof in which we start with applying the elimination rules to the first premise and then we do so for the second premise. On the other hand, the prenex form (62) represents a proof in which we apply the elimination rules to the second premise and then to the first premise. We have used different variables for the second premise to be able to show the different steps of applying elimination rules to premises. The proof that the prenex form (61) represents includes four undefined concepts for $S(Sx\_$ and $Sw\_$ can be the same although both should be different from $Sz\_$. Also S_y and S_v can be the same though should be different from S_z), also the proof that the prenex form (62) represents includes four undefined concepts ($Sx\_$ and $Sw\_$ can be the same although both should be different from $Sz\_$. Also S_y and S_v can be the same though should be different from S_z). If different prenex forms had different numbers of undefined concepts then it would mean that a different order of analysing the premises would make a difference.

According to definitions (u), (v), and (w), arguments (32), (33), and (48) are synthetic since in the case of (32), as an instance, the proof includes 54 defined concepts (that is 27 ordered pairs) in the case of relation U, while the argument includes only six undefined concepts of U (which means three ordered pairs). Argument (33) includes two undefined concepts (one ordered pairs) in premises and two undefined concepts (one ordered pairs) in the conclusion, while its proof has four defined concepts (two ordered pairs). And in case of argument (48), the premises and the conclusion include three and four undefined concepts respectively(two ordered pairs each) while the proof includes eight defined concepts (four ordered pairs). The last point is an observation about argument (33); as can be seen in schema (63) what we are able to infer from $\forall x \exists y(Rxy)$ is that there is something that falls under $Ra\_$, where a can be anything. However, later in the process of proof we will become able to justify the claim that a itself also can fall under $Ra\_$ as the schema in (64) shows[48], because the variable that represents individuals or names that fall under $Ra\_$ is bound by a universal quantifier.

[48]In schema (64) Φxyz stands for $(Rxy \wedge Ryz) \rightarrow Rxz$ and Ψxx stand for Rxx.

63.
$$\dfrac{\forall x \exists y Rxy \qquad \dfrac{[\exists y Ray]2 \qquad \begin{array}{c}[Rab]1\\ \vdots \\ \forall x(Rxx)\end{array}}{\forall x(Rxx)}\,\exists E(1)}{\forall x(Rxx)}\,\forall E(2)$$

64.
$$\dfrac{\forall x \forall y \forall z \Phi xyz \qquad \dfrac{[\forall y \forall z \Phi ayz]3 \qquad \dfrac{[\forall z \Phi abz]2 \quad \begin{array}{c}[\Phi aba]\\ \vdots \\ \forall x \Psi xx\end{array}}{\forall x \Psi xx}\,\forall E(1)}{\forall x \Psi xx}\,\forall E(2)}{\forall x \Psi xx}\,\forall E(3)$$

If we consider schema (63) as the starting point of our proof and schema (64) as its end point (where Φxyz stands for $(Rxy \wedge Ryz) \rightarrow Rxz$ and Ψxx stand for Rxx), it can be said that we have come across new information during the process of proof. We knew that for everything, there is something that stands in relation R with it, but we couldn't confirm that everything can stand in relation R with itself. This cannot be inferred from any other premises on their own either. Also we saw that something similar happens in proving argument (48), These are just two examples of synthetic argument which are informative too.

To sum up this part, we saw that unlike the algebraic proofs of (22) and (31), in a proof in predicate logic we do not encounter a new concept either as a new form or as a piece of information which has not been nested in the premises or the conclusion. Nonetheless, what is common between algebraic proofs and some proofs in predicate logic is the important and sometimes creative role of individuals. Rules of formal reasoning, say substitution rules in predicate logic, do not tell us enough about what to substitute. This can be taken as the intuitive bit of formal reasoning.

The connection between our definition of synthetic proof and the intuition-lad acts of substitution in predicate logic can be summarised as follows: existential elimination and universal introduction procedures are fully dictated by requirements of soundness of an inference. While universal elimination and existential introduction procedures are not fully determined, though restricted by soundness requirements. We know that there is no restriction for substituting universally bound variables. It then remains all to our grasp on argument to choose a suitable substitute for a universally bound variable. Moreover, as we saw in proof (58), efficient existential introduction policy can reduce the number of individuals we need in a proof.

The proposed definition of analytic/synthetic proofs is an attempt to capture why in some cases based on mere understanding of the logical form of the premises and the conclusion we cannot confirm the correctness of an argument. The method provided to count the number of undefined concepts is an attempt to show how far we can go without actually doing the proof. And defined concepts can be calculated only after building the proof, so they represent conceptual resources that we have after having proofs. The difference between these two shows that the proof, which gives us the needed ground to confirm the correctness of an argument, relies on something more than the concepts which appear in the argument. This shows how a logical proof can be informative, even if the information gathered is only the correctness of the argument.

However, this new conceptual analysis does not help us to define a strategy for proving the conclusion from the premises since it cannot distinguish between different orders of applying logical rules to the premises. As we saw, different prenex forms of an argument have the same number of undefined concepts. If different prenex forms had different undefined concepts then this method of conceptual analysis might help us to choose a simpler method to prove the argument. Here is the reason: we saw that if the number of defined concepts exceed the number of undefined concepts, then the proof is synthetic and most likely needs a more complicated substitution. Given that a normal proof has a certain number of defined concepts, the prenex form that had more undefined concepts most likely would require a simpler substitution.

5.7 Measures For The Complexity of a Proof

This chapter was a quest to investigate possible differences between logical reasoning and arithmetic reasoning. One way of articulating this difference was that arithmetic reasoning has a richer set of inferential rules, in the sense that the number of words with inferential power in arithmetic reasoning is more than logical reasoning because we deal with particular objects in arithmetic. We also saw that a number of philosophers believe that some of those words with inferential force, such as 'bigger than', have an object-sensitive meaning. That is, their content is not completed until we know how and to which objects they have been applied. For instance, in the case of 'bigger than' we need to know whether it refers to the size or the quantity. The position we take on this issue affects our judgement about the analyticity of inferential steps in the sense that if we accept only the inferential steps based on the meaning of the words their meaning is totally fixed regardless of

the objects they apply to, then inferential steps based on, say 'bigger than', will not be analytic steps. Regardless of our position about the analyticity of arithmetic reasoning, the variety of inferential rules can be a measure of complexity of a proof. That is the more inferential rules (tools) applied in a proof, the more complicated it is.

Another possible measure for complication of a proof is the number of individuals we need to consider in relation to each other in proving a conclusion. We saw that this number increases specifically when individuals play a specifically significant role in a proof. With the new account of conceptual analysis, introducing constants also increases the number of concepts involved in the reasoning. However, the number of constants in a proof, per se, does not show the complication of establishing a proof necessarily. The reason is that the difficulty that may arise in applying elimination rules to quantifiers is to choose a constant which makes establishing the conclusion possible or easier. Actually, the substitution process would not be that difficult if we knew that for any substitution a new constant should be introduced to the context of proof. The complexity enters in performing a correct and at the same time effective substitute when we have more that one option. And it is not the case that every substitution increases the number of constants in a proof. Therefore if we consider the number of constants as a measure for complexity of a proof then we have failed to address the source of complexity.

Conclusion

We commenced our investigation by questioning the justification of deduction, that is, what are measures for evaluating logical rules where logical rules are expressed as inference rule schemas. We saw that if we take truth preservation as the justification of deduction, then as Susan Haack argued,[49] finding a suasive justification for deduction is no easier than justifying induction. We reviewed a common strategy to counter Haack, based on differentiating between gross circularity and rule circularity. The central trait of this argument was that rule circularity sometimes can fail in justifying the intended rule, while gross circularity justifies the intended claim trivially. Then we demonstrated that in case of the claim that logical inference rules are truth preserving, rule circularity is as trivial as gross circularity. This is because in a language with a truth predicate and the two rules of 'discotation' and 'semantic ascent', we can demonstrate that a certain rule preserves truth, using the very rule in question.

Further, considering the forms of deductive rules, we argued that since they are not ampliative, they do not need suasive justification– explanatory justifications are enough for justifying them. And circularity will not rub off the explanatory power of this explanatory justification. It is worth recalling this argument with an example, to clarify what it means to say that 'considering the form of a deduction it is not ampliative'. It is a law of physics, in ordinary speeds, that the distance travelled by an entity, including waves, in a certain time is equal to the quantity of time multiplied by the quantity of speed, given that speed is constant. It can be shown as $X = V.T$ where X stands for the distance, V stands for the speed, and T stands for the time. Now in some occasions, we know the speed and the distance, but we are keen to know the time needed for an object to travel that distance, and some times we know the speed and time and we are keen on knowing the distance. If we use X,V, and T as variables and a to f, as constants then here is the representation of the relevant arguments:

1. $X = V \times T, V = a, T = b \vdash X = a \times b = c$

[49] [Haack, 1976]

2. $X = V \times T, X = d, V = e \vdash T = d \div e = f$

In both of these cases, deduction has helped us to expand our knowledge, however, if we take our state of knowledge out of the equation, in each individual instance of using deduction what remains is as the following:

3. $c = a \times b, a, b \vdash c = a \times b$

4. $d = e \times f, d, e \vdash f = d \div e$

Now, if we consider each of the premises or the conclusions, they are bits of information on their own, but the deduction, as a whole does not add up any information (given we know the meaning of $\times$ and $\div$). This is what is meant when it is said that deduction is not ampliative. To cover all the instances, we say: 'given the form of deduction, it is not ampliative'. By this definition, it must be clear why deduction does not need suasive justification, despite the fact that it helps us to expand our knowledge in certain occasions.[50]

However, not being suasive is not the only problem a circular justification suffers from. We saw that although, as Dummett argued, circular justifications are not always trivial and they play a role,[51] they cannot exclude all the bad rules. Rules such as Modous Morons (MM) and Tonk introduction and elimination. Then we argued that to rule out these rules we need to appeal to some meaning related notions, where meaning related have a heavy practical flavour. For instance for a rule to purely spell out the meaning of a logical constant criteria such as purity, being simple, and single ended became desirable. An important criterion suggested by Dummett, who was influenced by Belnap, was the demand for harmony between introduction and elimination rules of a given logical constant.

Belnap demanded that if an inferentially important word was considered as a logical constant this should not change the notion of deducability in the context in which that inferentially important word is introduced to. Roots of the philosophical motivation for demanding a logical constant to do not change the notion of deducability, can be traced back to the idea that deduction is not ampliative. In other words, being non-ampliative is, at least partly, constitutive of the notion of deducability. Dummett formulated this

[50] In the last section of chapter two, we noted that perhaps this is the venue that Haack misunderstands Dummett and accuses him to confuse deductive inference with deductive implication.

[51] For instance, according to Dummett, soundness and completeness are examples of circular justifications that are not trivial and give us very important information about our formal system.

demand as the demand for inference rules to be harmonious. His formal interpretation of this idea was that in order, for introduction and elimination rules, to be in harmony in a formal system, they must behave in accordance to the Inversion Principle (IP).

As we saw, Dummett's suggestion faced difficulties–some philosophers, such as Stephen Read, argued that it is too much to ask the IP to achieve what Dummett demands. Read's point was that we accuse Tonk of changing the context of deducability, or in other words, non-conservatively expanding any given theory we add it to, because it validates $A \vdash B$ where there is no Tonk appeared in A or B and inferring B from A had not been valid if Tonk has not been introduced to the context. For the same reasons we should rule out negation; because given the classical and intuitionistic single conclusion rules for the conditional and negation in ND, there is no proof from zero number of premises to, say, $((A \to B) \to A) \to A$ without negation appearing in the proof, and yet, there is no negation appearing in this formula.

Then, inspired by Sequent Calculus (SC), we saw that it is possible to have inference rules that, besides meeting desirable properties such as purity, simpleness, and single-endedness (which makes them separable, that is, makes their meaning definable without the assistance of any other logical word), do not change the context of deducability. This includes negation too, therefore we were able to rule out Tonk without having to rule out negation as well. The cost for this benefit was accepting that conclusion can be multiple.

We examined two potential objections to multiple conclusions and responded to these objections. The first objection was raised by Florian Steineberger, arguing that multiple conclusion reasoning is not natural way of reasoning. We countered this objection by emphasizing that in circumstances involving a lack of information our deductive reasoning takes the multiple conclusion form and it is natural for those circumstances. The second objection, raised by Ian Rumfitt, was that a failed multiple conclusion deduction is too demanding. To be more specific, he argued that in case of vague predicates, applying multiple conclusion forces us to accept that there is a 'sharp point' in the string of deduction. That is there is a specific point in which our inference goes wrong. We rejected this objection by arguing that the same can be said about single conclusion deductions. Moreover, we demonstrated that whether there exists a sharp point, depends on how we understand proofs and this is a separate issue from adopting a single or multiple conclusion in our syntax.

Bringing Belnap's 'context of deducability', and Dummett's idea about deduction being 'non-ampliative' into the context of inferential role seman-

tics, we defined a notion of 'preserving meaning' where meaning of an expression, logical or non-logical, was understood as what follows from asserting (denying) a statement that the expression in question appears in that, and under which conditions we can assert (deny) that statement. And then we claimed that deduction preserves meaning. And here is how it makes sense: if we consider atomic propositions as the simplest and smallest elements of language in which meaning of subatomic expressions is fixed, then none of the formulas such as $p \to q$, $\forall x(Px \to Qx)$, are theorems of logic. So if we are developing a theory in our language, we need a set of such sentences to begin our theory with. Let us call them boundary conditions of our theory. These boundary conditions define the meaning of the non-logical vocabulary that appear in our theory. Whenever we use logic to expand our theory, in the sense of making more complex expressions, then based on what we just said about logic, the meaning of our vocabulary is preserved in the sense that our boundary conditions stay unchanged.

Then we introduced a challenge for this justification of deduction. The challenge has been noted by Dummett as the tension between legitimacy and fruitfulness of deduction. A legitimate deduction, for Dummett, was the one that conservatively extends any given theory in the language deduction happens in. In our understanding it is a deduction that preserves meaning. Intuitively speaking, in both cases it means that it does not produce any new knowledge. Knew knowledge here needs a bit of clarification. Not all the knowledge we gain using deduction is new in the sense we are interested in. A few paragraphs a go we saw that we can gather knowledge about distance, or time it take for something to travel, or the speed of a travelling object and so on. They have not been produced by deduction in the sense that there are other possible methods to actually measure these quantities. For instance, we usually use scales for measuring short distances or timers to measure time and so on and so forth. What, following Dummett, we consider as new knowledge, is the one that legitimately can be gathered only by deduction. Gathering this sort of knowledge is what Dummett, and us, are taking as fruitfulness of deduction. Examples of this sort of knowledge can be found in arithmetic.

We showed, via exapmles in chapters Two and Five, how legitimacy criterion stays in tension with the fruitfulness measure. Here is one of our examples: given any number in decimal system of numeration, we can infer 'x is divisible by three' from 'x's sum of digits is divisible by three'. Any proof that validates this inference, changes the meaning of both 'being divisible by three' and 'sum of digits being divisible by three' according to inferential role semantics account of meaning. What makes the situation even more interesting is that, unlike a theory about mammals, the acceptable way to confirm

and add this inference to our boundary inferences about decimal number, is having a proof for it.

What we did in the next three chapters was a quest to explain this phenomenon. So let us see what our study of proofs have to say about this phenomenon. Should we abandon the idea of deductive reasoning being meaning preserving? Or Should we reject that proofs such as the proof from 'x's sum of digits is divisible by three' to 'x is divisible by three' are essentially logical deductions? What would be the possible way to reconcile legitimacy and fruitfulness of deduction, if any?

In chapter Three we went through some suggestions about the necessity of appeal to intuition, specifically in arithmetical reasoning, via a historical review over the idea that the difference between the arithmetical reasoning and the logical reasoning is that we appeal to our intuitions in arithmetical reasoning. The historical figure supporting this idea was Kant and the opponent of this line of thought was Frege. The motivation was to see if this debate gives us any clue about any possible difference between the arithmetical reasoning and the logical reasoning that explains the above mentioned tension. Kant's position was that intuition, as a faculty of cognition which differs from understanding, helps us to have an immediate grasp of what we experience. Here immediate is used against a grasp with the mediation of putting objects of sense data under concepts or, in other words, understanding our experience. He believed that space and time are forms of our intuitions and without appealing to these, we cannot prove theorems of arithmetic.

On the other hand, Frege believed that if we could make our reasoning exact and explicit enough to leave no inferential gap to be filled by intuition, then there would not be any need to appeal to intuition. Our deliberations in chapter Three and Five suggested that Frege's exactness policy does not necessarily lead to an intuition-free reasoning. Intuition can reappear as the justification of some of our inference rules. Another noteworthy point was Frege and Kant's different notions of concept, specifically, we saw that Frege's conceptual structure is insensitive to some of the conceptual differences in the Kantian account of concept.

In the next stage in our quest to find some answer to the legitimacy-fruitfulness tension we examined different notions of analyticity as a possible legitimacy condition. We considered two accounts of analyticity–one as a property of truth and the other as a property for justification. According to one of the recent accounts of analyticity as a property of truth, provided by Gillian Russell, a logical step is analytic if the truth conditions of the conclusion of that step contains in the truth conditions of its premises. This version of analyticity had difficulties in explaining some classical inferences such as inferring B from A and $\neg A$ and inferring $B \to A$ from A. Analytic

justification on the other hand, justifies such inferences by appeal to the meaning of logical constants. Another recent account of analyticity, which also provides an account of a priori, has been considered too. It has been discussed that this account, provided by David Chalmers, does a very good job in explaining the fruitfulness of a posteriori truths, however cannot help us in cashing out the fruitfulness of a priori truths such as arithmetic ones.

By taking analyticity as a property of justification, and considering proofs as justifications, in chapter Five we sought to define analytic proofs. The guideline was that an analytic proof does not go further than what is given in premises and the conclusion. The result was that if logical rules alone were sufficient for constructing a proof, then it is analytic; otherwise it is synthetic. The formal outcome was an instruction to calculate the conceptual resources that an argument provides and the actual resources a proof actually needs where by proof we meant normal proof. This definition of analytic proofs also provides a different sense of appeal to intuition in formal reasoning from the Kantian notion of the appeal to intuition. Unlike the Kantian version of the appeal to intuition, in the recent account intuition has no justificatory role in proof steps; rather it acts like a guide for constructing proofs when logical rules are not sufficiently suggestive to do so. There is no guarantee for the correctness of our intuitions though.

Let us get back to our essential question about the tension between the legitimacy and the fruitfulness of deduction. It is hard to reject that inferences such as the inference from 'x's sum of digits is divisible by three' to 'x is divisible by three' increase our knowledge of numbers in a decimal system. At the same time, it is difficult to accept that the proof from the premise to the conclusion changes the meaning of predicates 'x is divisible by three' and 'x's sum of digits is divisible by three'. As a matter of fact, formalizing the arithmetic theorem in question as $\forall x(Px \to Qx)$, where x ranges over the set of natural numbers, does not do the justice here as the connection between the two predicates in our arithmetical example has not been captured in this formalization. Therefore, there is no reason to reject the claim that deduction is meaning preserving. However, we owe an explanation about the nature of information we gather from proofs of the theorem in question, because, as we just mentioned, it is hard to reject that the theorem is informative.

From what we have seen in chapters Three and Five, there was at least one ground to argue that proofs in arithmetic are essentially different from logical deductions. There are words, such as 'bigger than' with inferential force in proofs of arithmetic theorems that have object-sensitive meaning, or they are model-sensitive in the sense that their meaning does not fully defined unless we know on which objects they have been applied. But we saw that this possible difference in meaning cannot be expressed in a first

order context when it comes to non-standard models of arithmetic. Therefore the claimed difference in meaning is not of any interest for any inferential account of meaning. Nonetheless, some philosophers believe the existence of non-standard models is an evidence for appealing to intuition in arithmetic reasoning. Where intuitions plays a justificatory role in inferences based on words like 'bigger than'. If one accepts this view, then they would be hospitable to the idea that our access to the structure of numbers is intuitive. Of course, according to what we said in section two of chapter five, it does not necessarily mean that our access to numbers themselves is intuitive. This sense of appeal to intuition in arithmetic reasoning, separates arithmetic proofs from logical ones. Based on this distinction, one may argue that arithmetic theorems expand our knowledge about numbers and that is why they are informative, but the same thing cannot be said about logical proofs.

However, even if we accept everything that has been said about the appeal to intuition and structure of the numbers, it does not provide a granular formal account of how this new knowledge about numbers have been produced in the context of a proof. What we have seen in the 'Analytic/Synthetic Proofs' section gives us a granular understanding of the informative role a proof can play. Our definition of synthetic proofs has been designed such that spells out exactly what happens during the process of proving an argument that cannot be seen by only examining the argument itself. Put more specifically, examining an argument means calculating the number of undefined concepts that an argument provides us. In this process, we calculate the number of undefined concepts based on what logical accuracy requires. This must be a good enough interpretation of examining an argument. Then when the number of defined concepts that actually appear in the context of proof exceeds the number of calculated undefined concepts, it means that establishing the epistemic ground to confirm the conclusion, given premises, has needed more conceptual resource than logical accuracy demands. This epistemic ground which is needed to confirm the argument, is the information we gain from proof, and it is new because we have not been able to see that by merely examining the argument.

As can be seen, our explanation of new information gained from proof is not restricted to arithmetic proofs, it also includes proofs in predicate logic. This means that the new knowledge does not need to be about numbers only, whenever the process of deductive reasoning creates an epistemic ground, needed to confirm an argument and this epistemic ground has not been clear enough to be recognized from the argument itself, then proofs provide new information. Now if one asks why a proof might need more conceptual resource than what is demanded by logical accuracy, the answer is that what actually builds the epistemic ground that makes us able to con-

firm the argument is the conceptual structure of the proof, logical accuracy is necessary to make sure that the conceptual structure is built correctly. In other words, logical accuracy allows us to make certain moves to build the conceptual structure we need to affirm an argument, it does not necessarily give us the recipe to build a proof from premises to the conclusion.

In regard to this account of logical deduction being fruitful or informative and its connection to meaning preserving it should be said that of course the inferential definition we gave in chapter Two is not applicable on predicate logic. However, intuitively speaking, when proof of an argument is complicated in the sense that we cannot be sure about how to prove it by mere examination of the argument, then it can be said that the conclusion of the argument adds something to the meaning of the expressions which appear in it when we understand meaning in an inferential way. For instance, we really learn something new about relation R in the following argument, namely that it is reflexive.

5.

$$\begin{gathered}\forall x \forall y (Rxy \rightarrow Ryx) \\ \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \\ \forall x \exists y Rxy \\ \therefore \forall x (Rxx)\end{gathered}$$

At the same time, the meaning of expressions involved in a logical deduction remain unchanged in a significant sense; 'Fred is a friend of Suzy' does not change the meaning of 'x being friend with y'. However, ' x being friend of Suzy', in some context, might fill a conceptual gap that 'x being friend with y' or ' x being friend of Alex' cannot fill.

Of course not all the synthetic proofs are informative and synthetic proofs are not the only way deduction can be informative. As discussed in chapter Five, the following argument is synthetic though not that informative:

6.

$$\begin{gathered}\forall x \forall y Pxy \\ \therefore \forall x \forall y (Pxy \vee Pyx)\end{gathered}$$

The existence of synthetic proofs provides us a way to explain why some deductions are informative. As we saw, deduction can also be informative in other ways, for instance, the examples about speed and distance in this

chapter and some other examples in chapters Two and Five demonstrate alternative ways that deductive reasoning can be informative.

The fact that sometimes proving an argument demands more conceptual resources than what we see or expect from the argument itself is a possible angle to approach the decidability problem for predicate logic, and any language richer than it. We know that there are truths of mathematics for which we can provide proof, but prior to having them proved, there is no way to decide that they are provable. Examining these truths and their proof to identify possible reasons for this phenomenon is topic which deserves further research. This would be a way of expanding the investigation about the complexity of proofs, and the uncertain nature of concepts within these truths, which continue to evade computation in a straightforward manner.

Appendix

In this appendix we prove soundness and completness for the multiple conclusions presentation of classical logic defended in chapter Two. The derivation rules, presented in ND style, are as follows:[52]

$\wedge$:

$$\frac{A, \Delta \qquad B, \Delta}{A \wedge B, \Delta}\ \wedge I \qquad \frac{A \wedge B, \Delta}{A, \Delta}\ \wedge E \qquad \frac{A \wedge B, \Delta}{B, \Delta}\ \wedge E$$

$\vee$:

$$\frac{A, \Delta}{A \vee B, \Delta}\ \vee I \qquad \frac{B, \Delta}{A \vee B, \Delta}\ \vee I \qquad \frac{A \vee B, \Delta}{A, B, \Delta}\ \vee E$$

$\rightarrow$:

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B, \Delta\end{array}}{A \rightarrow B, \Delta}\ \rightarrow I \qquad \frac{A \rightarrow B, \Delta \qquad A, \Delta}{B, \Delta}\ \rightarrow E$$

$\neg$:

$$\frac{\begin{array}{c}[A]\\ \vdots \\ \Delta\end{array}}{\neg A, \Delta}\ \neg I \qquad \frac{A, \Delta \qquad \neg A, \Delta}{\Delta}\ \neg E$$

For doing so we adopt the following definitions:

D1: Atomic propositions such as p and q are formulas.

D2: If A and B are formulas then so are $A \wedge B$, $A \vee B$, $A \rightarrow B$, and $\neg A$ ($\neg B$).

[52] These rules are taken from [Francez, 2014], p.251. Also Weakening and Contraction rules used in proof (22) Chapter Two can be obtained by applying disjunction introduction and elimination rules, introduced here, and construing the lines of a deduction as sets of formulas respectively.

D3: In the following definitions Γ and Δ refer to possibly empty sets of formulas. Also Γ, A refers to the union of Γ with $\{A\}$. The same applies to A, Δ, B, Δ, and Γ, B.

D4:

1. A set of formulas such as Δ is deducible from a possibly empty set of formulas such as Γ if at least one member of Δ is also a member of Γ.

2. If $\Gamma \vdash A, \Delta$ and $\Gamma' \vdash B, \Delta$, then $\Gamma, \Gamma' \vdash A \wedge B, \Delta$.

3. If $\Gamma \vdash A \wedge B, \Delta$ then $\Gamma \vdash A, \Delta$. Also if $\Gamma \vdash A \wedge B, \Delta$, then $\Gamma \vdash B, \Delta$.

4. If $\Gamma \vdash A, \Delta$ then $\Gamma \vdash A \vee B, \Delta$. Also if $\Gamma \vdash B, \Delta$, then $\Gamma \vdash A \vee B, \Delta$.

5. If $\Gamma \vdash A \vee B, \Delta$ then $\Gamma \vdash A, B, \Delta$.

6. If $\Gamma, A \vdash B, \Delta$ then $\Gamma \vdash A \rightarrow B, \Delta$.

7. If $\Gamma \vdash A \rightarrow B, \Delta$ and $\Gamma \vdash A, \Delta$, then $\Gamma \vdash B, \Delta$.

8. If $\Gamma, A \vdash \Delta$ then $\Gamma \vdash \neg A, \Delta$.

9. If $\Gamma \vdash A, \Delta$ and $\Gamma \vdash \neg A, \Delta$ then $\Gamma \vdash \Delta$.

D5:

1. $\Gamma \vDash \Delta$ (to be read as Γ implies Δ) if and only if in every model (interpretation) that all members of Γ are true at least one member of Δ is true.

2. $A \wedge B$ is true if and only if A is true and B is true.

3. $A \vee B$ is true if and only if either A or B is true.

4. $A \rightarrow B$ is true if and only if either A is not true or B is true.

5. $\neg A$ is true if and only if A is not true.

Soundness: If $\Gamma \vdash \Delta$ then $\Gamma \vDash \Delta$.
To prove soundness we need to show that if there is a proof, that is finite steps of inferences each of them is an instance of the above mentioned inference rules, from Γ to Δ, then Γ implies Δ. That is there is no model or interpretation in which all the members of Γ are true and no member of Δ is true.

Proof: If no logical connective appears in Γ or Δ, then either there is a common atomic proposition between Γ and Δ or not. If there is no atomic proposition in common between Γ and Δ, then we are out of the scope of propositional logic. If there is, then in every model that all members of Γ are true the condition required in $D5-1$, that is the truth of at least one member of Δ, is met in any consistent assignment of values. If Γ or Δ include complex formulas then each step of any proof of $\Gamma \vdash \Delta$ is in correspondence with one of the $D4-2$ to $D4-9$. If the applied rule is $D4-2$, then in the antecedent of $D4-2$ either A is true or one member of Δ and either B is true or one member Δ. If one member of Δ is true, then the condition in $D5-1$ is met in the consequent of $D4-2$. And if A and B are true, then so is $A \wedge B$ according to $D5-2$. Therefore $D5-1$ is held in the consequent of $D4-2$ either way. The similar reasoning can be given for $D4-3$.

If the applied rule is $D4-4$, then either A is true or one member of Δ in the antecedent of $D4-4$. If one member of Δ is true, then the condition in $D5-1$ is met in the consequent of $D4-4$. If A is true, then so is $A \vee B$ according to $D5-3$. Therefore the $D5-1$ condition is met ether way in the consequent of $D4-4$. Very similar reasoning proves that the $D4-5$ rule meets the requirement of validity expressed in $D5-1$.

If the applied rule is $D4-6$, then in case one member of Δ is true in the antecedent of $D4-6$, the validity condition is met in the consequent of $D4-6$. Otherwise B is true in the antecedent of $D4-6$ and if B is true so is $A \to B$ in the consequent of $D4-6$ according to $D5-4$. Therefore $D5-1$ condition is met either way in the consequent of $D4-6$. If the applied rule is $D4-7$, then in case a member of Δ is true in the antecedent of $D4-7$ the validity requirement is met in the consequent of $D4-7$. Otherwise $A \to B$ is true when A is true too in the antecedent of $D4-7$. According to $D5-4$, the only possible way this can be the case is that B is true. Therefore the validity condition is met either way in the consequent of $D4-7$.

Finally, if the applied rule is $D4-8$, then according to $D5-1$, in the antecedent of $D4-8$, if A is true one member of Δ needs to be true too. In this case the deduction in the consequent of $D4-8$ meets the $D5-1$ condition. If A is not true, then $\neg A$ is true according to $D5-5$ and this makes the deduction in the consequent of $D4-8$ meets the requirement of validity. If the applied rule is $D4-9$, then the only possible way for both

deductions in the antecedent of $D4-9$ to be valid is that one member of Δ be true. In such a case the deduction in the consequent of $D4-9$ also meets the condition of validity stated in $D5-1$.

Completeness: If $\Gamma \vDash \Delta$ then $\Gamma \vdash \Delta$.
To prove completeness we prove its contrapositive, that is $\Gamma \nvdash \Delta$ then $\Gamma \nvDash \Delta$, because it is easier to prove. We need to show that in every case in which there is no proof from Γ to Δ, there is at least one model in which each member of Γ is true while no member of Δ is. To do so we start by showing how any (possibly empty) given set of premises Γ and any (possibly empty) given set of conclusions Δ such that $\Gamma \nvdash \Delta$ can be expanded to a partition Γ', Δ' of the entire language such that $\Gamma' \nvdash \Delta'$. Then we show that there is a valuation which assigns all members of Γ' true and all members of Δ' false.

Proof: given any (possibly empty) sets of formulas Γ and Δ such that $\Gamma \nvdash \Delta$, let us call them Γ_0 and Δ_0. And enumerate the entire language as A_1, A_2, and so on. starting with A_1, if $\Gamma_0 \nvdash A_1, \Delta_0$, then $\Gamma_1 = \Gamma_0$ and $\Delta_1 = \Delta_0 \cup \{A_1\}$. And if $\Gamma \vdash A_1, \Delta$, then $\Gamma_1 = \Gamma_0 \cup \{A_1\}$ and $\Delta_1 = \Delta_0$. More generally, for A_n, if $\Gamma_{n-1} \nvdash A_n, \Delta_{n-1}$, then $\Gamma_n = \Gamma_{n-1}$ and $\Delta_n = \Delta_{n-1} \cup \{A_n\}$ and if $\Gamma_{n-1} \vdash A_n, \Delta_{n-1}$, then $\Gamma_n = \Gamma_{n-1} \cup \{A_n\}$ and $\Delta_n = \Delta_{n-1}$. We continue like this for the entire language. In this way we partition the entire language into two parts Γ' and Δ' such that no finite subset of Δ' is deducible from any finite subset of Γ'. That is there is no proof, using schematic inferential rules mentioned at the beginning of this appendix, from any finite subset of Γ' to any finite subset of Δ'. Now we want to show that all members of Γ' can be evaluated true while all members of Δ' are evaluated false. This happens consistently, that is without assigning a given formula A both as true and false at the same time.

To do so, we define a valuation function v. For any given formula A, $v(A) = T$ if $A \in \Gamma'$ and $v(A) = F$ if $A \in \Delta'$. If A is an atomic formula, then it appears either in Δ' or in Γ' it cannot appear in both because if it would, then $\Gamma' \vdash \Delta'$ according to D4-1 and this is against our assumption. Therefore it can be assigned true if it appears in Γ' and false if it appears in Δ' consistently, that is without any clash in truth value.

In case of complex formulas, we need to show that $D5$ holds, in case of conjunction it is: $v(A \wedge B) = T$ if and only if $v(A) = T$ and $v(B) = T$. If $v(A \wedge B) = T$ then it is in Γ', and so are A and B since if they were in Δ', then $\Gamma' \vdash \Delta'$ according to D4-3. And this is against our assumption. On the other hand, if $v(A) = T$ and $v(B) = T$, then they are in Γ'. And so should be $A \wedge B$, as if it was in Δ', then $\Gamma' \vdash \Delta'$ according to D4-2. And this is against our assumption.

If $v(A \vee B) = T$, then it is in Γ'. And so is at least one of A or B, since if they both would appear in Δ' then $\Gamma' \vdash \Delta'$ according to D4-5. And this is against our assumption. If $v(A) = T$ or $v(B) = T$, that is at least one of them is in Γ'. And so should be $A \vee B$, as if it was in Δ', then $\Gamma' \vdash \Delta'$ according to D4-4. And this is against our assumption.

If $v(A \to B) = T$, then it appears in Γ'. And so is B when A is in Γ' too. Because if B was in Δ' while $(A \to B)$ and A are in Γ', then $\Gamma' \vdash \Delta'$ according to D4-7 and this is against our assumption. If $v(A) = F$ that is, it appears in Δ', or $v(B) = T$ that is, it appears in Γ', then $(A \to B)$ should appear in Γ', because otherwise $\Gamma' \vdash \Delta'$ according to D4-6. And this is against our assumption.

Finally, if $v(\neg A) = 1$, then it appears in Γ', in this case A cannot appear in Γ' too, because if it was, then $\Gamma' \vdash \Delta'$ according to D4-9, and this is against our assumption. And if $v(A) = F$, then it appears in Δ', in this case, $\neg A$ should appear in Γ' since if it appears in Δ' then $\Gamma' \vdash \Delta'$ according to D4-8 and this is against our assumption.

We have just shown how when there is no deduction from any given set of premises Γ to any given set of conclusions Δ, in a language with all logical connectives in propositional logic, there are valuations in which all members of Γ' are true while all members of Δ' are false without any conflict in truth value of any members of either set. This guarantees that there are models in which all members of Γ' are true while all members of Δ' are false. So we have proven the completeness of multiple conclusion classical propositional logic.

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