Moderated Mediation in Multilevel SEM:
Decomposing Effects of Race on Math Achievement
Within versus Between High Schools in the United States

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Single.Level.ModMed.zip
MonteCarlo.CI.zip
Multilevel.ModMed.zip
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At roughly similar times in the 1980s, social scientists formalized what have become enduring interests in multilevel modeling (e.g., Raudenbush & Bryk, 1986) and moderation and mediation (e.g., Baron & Kenny, 1986). Today, moderation and mediation models have been synthesized so that these effects can be combined and estimated in a wide variety of cases (Edwards & Lambert, 2007; Preacher, Rucker, & Hayes, 2007), including with latent variables and latent interactions (Cheung & Lau, 2015; Sardeshmukh & Vandenberg, 2016). In the multilevel arena, approaches now exist that allow assessing multilevel mediation (e.g., Preacher, Zyphur, & Zhang, 2010; Preacher, Zhang, & Zyphur, 2011) and multilevel moderation with latent variables (e.g., Preacher, Zhang, & Zyphur, 2016).

What remains to be offered, however, is a synthesis of these interests in a way that allows estimating moderation and mediation at multiple levels of analysis. Our chapter addresses this by first describing the logic of moderated mediation, including how to formalize it as a structural equation model (SEM). We then extend this logic to multilevel SEM (MSEM) to estimate level-specific moderated mediation. Our approach allows the typical random coefficient prediction method for estimating cross-level moderation with random slopes (as outcomes), but our approach can also use a latent moderated structural equations (LMS) approach to estimate moderation, which requires latent variable interactions (see Preacher et al., 2016).

In order to avoid the high-dimensional numerical integration that often accompanies these interactions, we describe a Bayesian ‘plausible values’ approach that multiply-imputes latent variable scores in first step, then allowing researchers to form product terms as if they were observed to estimate moderated effects in a second step. This approach can be used for
any model wherein latent interactions or power polynomials otherwise require numerical integration, and therefore it is also applicable in single-level models (e.g., Sardeshmukh & Vandenberg, 2016). In the context of MSEM, our plausible values approach has the benefit of comparatively fast estimation while still allowing higher-level product terms to be treated as if they were measured with error (unlike, for example, Leite & Zuo, 2011).

We offer a worked example using the well-known High School and Beyond (HS&B) dataset (e.g., Raudenbush & Bryk, 1986, 2002), with 7,185 students nested in 160 schools. The HS&B data and Mplus program code for all models that we estimate can be downloaded from quantpsy.org. With these additional materials, the reader can estimate the models that we specify and modify them to experiment with multilevel moderated mediation.

To help the reader keep track of the parameters in our models, we use familiar mediation notation (as in Baron & Kenny, 1986), with regression coefficients as follows: $a$ is the first path in a mediation relationship; $b$ is the second path in a mediation relationship; $c$ is the total effect of a predictor on an ultimate outcome without controlling for a mediator; and $c'$ is the direct effect of a predictor on an ultimate outcome while controlling for a mediator. Furthermore, where appropriate we use subscripts that indicate the outcome variable and the predictor variables associated with a regression coefficient. We illustrate this notation below, but recommend that the unfamiliar reader first consult primary texts such as Baron and Kenny (1986), MacKinnon (2008), Preacher and Hayes (2004), Preacher et al. (2007), and Hayes (2013).

Our models estimate some effects of being black in the United States (US). Given long-running racism and racial segregation in the US (Bonilla-Silva, 2006), and because race is social and relational (see Lucal, 1996; Smedley & Smedley, 2005; Tsui, Egan, & O'Reilly, 1992), we treat individuals’ self-identification as being black or non-black as indicating an important racial categorization in society—it matters. To be very clear, as with feminist
approaches to sex or gender (Haraway, 2006), our use of these terms and this categorization for our analyses is not meant to reify or otherwise reproduce racism. Instead, by showing negative effects of identifying as black on math achievement both directly and indirectly via socio-economic status (SES) within and between schools, our goal is to show racial inequalities so that they can be taken seriously and addressed.

In our Discussion, we describe the benefits of our MSEM approach. They include an improved ability to conceptually reason and hypothesize about multilevel moderated mediation effects. Furthermore, the flexibility of MSEM allows random intercepts and random slopes that can be used as predictors, outcomes, indicators, mediators, or moderators at higher levels of analysis. With this expanded toolbox, researchers can better conduct research that addresses worldly problems of concern, such as racism.

**Moderated Mediation**

To preface our discussion of multilevel moderated mediation, we first introduce basic concepts associated with moderation and mediation. We then treat moderated mediation in a single-level SEM framework and offer an empirical example using the HS&B dataset.

**Moderation**

‘Moderation’ refers to an interaction or a conditional effect, wherein the effect of a predictor variable $x$ on an outcome variable $y$ varies across the levels of another predictor $w$ (Cohen, Cohen, West, & Aiken, 2003). This kind of effect is usually modeled by forming a product term $xw$ among the two predictor variables as follows (see a conceptual model of this effect in Figure 1a; see a more statistically accurate depiction in Figure 1b):

$$y_i = \nu_y + c_{yx}x_i + c_{yw}w_i + c_{xw}x_iw_i + \varepsilon_{y,i}$$

wherein $i$ is a unit of observation (e.g., an individual student); $\nu$ is an intercept; each $c$ is a regression coefficient; and $\varepsilon$ is a residual. The conditional nature of the effects can be shown
by rearranging Eq. 1, which we do to illustrate the example of $x$’s effect on $y$ across varying levels of $w$:  

$$y_i = (ny + cyw_i) + (cyx + cyxw_i)x_i + ey,i$$  

(2).

Here, the first parenthetical term is a ‘simple intercept’, which equals the expected value of $y$ when $w$ takes on a specific value; whereas the second parenthetical term is a ‘simple slope’ of $x$, which equals the expected value of $y$ when $w$ takes on a specific value (Preacher et al., 2007). To test for moderation, researchers typically examine the statistical significance of $c_{yxw}$, which is sensible because only if $c_{yxw} \neq 0$ will the coefficient on $x$ in Eq. 2 detectably deviate from $c_x$. Furthermore, the statistical significance of a given simple slope can be computed for any given value of $w$ or it can be computed continuously across a range of observed $w$ values. All of this is typically done based on the SE of $c_{yxw}$ as computed under normal theory.

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INSERT FIGURES 1a-1b HERE
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Mediation

By the term ‘mediation’ we mean an indirect effect, such as the effect of $x$ on $y$ that is carried by a mediator $m$ (Cohen et al., 2003). This kind of effect can be shown in equations for $y$ and $m$ as follows (see Figure 2):

$$m_i = m + amx_i + m_i$$  

(3)

$$y_i = y + b_{ym}m_i + c_{yx}x_i + ey_i$$  

(4)

so that with substitution the indirect effect of $x$ is shown to be a product term as follows:

$$y_i = y + b_{ym}(y + amx_i + em_i) + c_{yx}x_i + ey_i$$  

(5)

which can be rearranged to show a traditional regression model structure as follows:

$$y_i = (y + b_{ym}y_i) + (amx_i + c_{yx}x_i + b_{ym}em_i) + ey_i$$  

(6)
wherein the first parenthetical term is an intercept; the second parenthetical term is the total
effect of $x$, composed of an indirect effect $a_{mx}b_{ym}$ and a direct effect $c'_{xy}$; and the term $b_{ym}e_{m,i}$
is the direct effect of $m$ that is independent of $x$. To test for mediation, researchers often
modify traditional tests of statistical significance by estimating a confidence interval around
$a_{mx}b_{ym}$—although the literature on mediation has historically distinguished between ‘partial’
and ‘full’ mediation (e.g., Baron & Kenny, 1986), we do not reproduce this distinction here
and instead focus on indirect effects. Because estimates of such effects are not normally
distributed, normal theory does not apply and therefore bootstrapping, Monte Carlo, or Bayes
procedures are typically used (e.g., Preacher & Hayes, 2008; Preacher & Selig, 2012; Wang

Moderated Mediation

The term ‘moderated mediation’ refers to the dependence of an indirect effect on at
least one moderator variable $w$, such that indirect effects are made conditional on values of a
moderator(s). Many specifications produce moderated mediation (Hayes, 2013), but a
general type can be shown by combining the logic of Eqs. 1, 3, and 4 as follows (see Figures
3a-3b; see Model 59 in Hayes, 2012-2016):

$$m_i = \beta_{m} + a_{mx}x_i + a_{mwx}w_i + a_{mwx}x_iw_i + \epsilon_{m,i}$$ \hspace{1cm} (7)

$$y_i = \beta_{y} + b_{ym}m_i + b_{ymw}m_iw_i + c'_{xy}x_i + c'_{ywm}w_i + c'_{ywm}x_iw_i + \epsilon_{y,i}$$ \hspace{1cm} (8)
such that $w$ moderates the effects of $x$ on $m$ and $m$ on $y$ (i.e., the ‘first stage’ and ‘second
stage’ moderated mediation from Edwards & Lambert, 2007), and $w$ moderates the direct
effect of $x$ on $y$ (i.e., the traditional form of moderation). The effects involved in this model
can be shown by substitution as follows:
\[ y_i = \nu_y + b_{ym}(\nu_m + a_{mx}x_i + a_{mw}w_i + a_{mwx}x_i w_i + \varepsilon_{m,i}) + \]
\[ + b_{ymw}(\nu_m + a_{mx}x_i + a_{mw}w_i + a_{mwx}x_i w_i + \varepsilon_{m,i})w_i + c'_{yx}x_i + c'_{ym}w_i + c'_{mwx}x_i w_i + \varepsilon_{y,i} \] (9)

which can be rearranged as follows:

\[ y_i = \nu_y + b_{ym}\nu_m + (b_{ymw}\nu_m + a_{mw}b_{ym} + a_{mwy}b_{ym}w_i + c'_{ym})w_i \]
\[ + [(a_{mx} + a_{mwx}w_i)(b_{ym} + b_{ymw}w_i) + (c'_{yx} + c'_{ymw})]x_i \]
\[ + (b_{ym} + b_{ymw}w_i)\varepsilon_{m,i} + \varepsilon_{y,i} \] (10)

which has a similar interpretation to Eq 6, such that the first bracketed term is the simple intercept of \( y \), which includes indirect and direct effects of \( w \); the second bracketed term is the total effect of \( x \), which is composed of first the indirect effect \((a_{mx} + a_{mwx}w_i)(b_{ym} + b_{ymw}w_i)\) and then the direct effect \((c'_{yx} + c'_{ymw})\); the first parenthetical term on the third line is the direct effect of \( m \), which is moderated by \( w \); and the final term is the residual of \( y \).

\[ \text{-----------------------------} \]
\[ \text{INSERT FIGURES 3a-3b HERE} \]
\[ \text{-----------------------------} \]

In order for the uninitiated reader to fluently understand moderated mediation in Eq. 10, some explanation is in order. Focusing on the effect of \( x \) in the second bracketed term, the moderation coefficients are \( a_{mx} \) and \( b_{ymw} \). Both of these are multiplied by \( w \), so that when both coefficients are equal to zero, moderation is not present and Eq. 10 is more like Eq. 6 because the indirect effect of \( x \) reduces to \( a_{mx}b_{ym} \). However, if \( a_{mx} \neq 0 \) and/or \( b_{ymw} \neq 0 \) then moderation is present. Specifically, \( a_{mxw} \) allows for moderation of the path linking the independent variable and the mediator (i.e., the \( a_{mx} \) path) and \( b_{ymw} \) allows for moderation of the path linking the mediator and the dependent variable (i.e., the \( b_{ym} \) path). But, because \( b_{ym} \) is part of the indirect effects involving paths \( a_{mx} \) and \( a_{mxw} \), both of these paths can be moderated by \( w \) when multiplied by \( b_{ymw} \).
As the reader may intuit, there are many ways to specify moderated mediation (see Hayes, 2013, 2012-2016, 2015). Instead of describing the many cases that are possible, we want to offer a general model structure for understanding moderated mediation that can be extended to the multilevel case. We now do this with a general SEM specification (e.g., Edwards & Lambert, 2007; Hayes & Preacher, 2013). Here and in our multilevel models, we use variants and simplifications of the model in Muthén and Asparouhov (2008) as implemented in Mplus (see Muthén & Muthén, 1998-2016; see also Preacher et al., 2010, 2016).

A General SEM Specification and Estimation

To begin, we show an “all y” SEM specification as follows:

\[
\mathbf{y}_i = \mathbf{v} + \mathbf{A} \mathbf{\eta}_i + \mathbf{e}_i, \tag{11}
\]

\[
\mathbf{\eta}_i = \mathbf{\alpha} + \mathbf{B} \mathbf{\eta}_i + \mathbf{\zeta}_i, \tag{12}
\]

wherein \( \mathbf{y}_i \) is a vector of observed scores on the dependent variables (often called observed indicators); \( \mathbf{v} \) is a vector of intercepts capturing the mean structure of the data; \( \mathbf{A} \) is a matrix of factor loadings representing the strength and direction of relationships among latent variables and their observed indicators; \( \mathbf{e}_i \) is a vector of residuals with covariance matrix \( \Theta \) (typically with unrestricted diagonal elements, i.e., estimated variances); \( \mathbf{\eta}_i \) is typically a vector of latent variables that are believed to cause the covariance structure of observed indicators, but it may also be used to reflect the actual observed variables if \( \mathbf{A} \) contains unities that link each observed variable with a single latent variable and elements in \( \mathbf{v} \) and \( \Theta \) are fixed at zero; \( \mathbf{\alpha} \) is a vector of intercepts or means corresponding to a latent variable mean structure (typically restricted to zero); \( \mathbf{B} \) is a matrix of regression coefficients often used to model causal effects among latent variables; and \( \mathbf{\zeta}_i \) is a matrix of residuals with covariance matrix \( \Psi \) (typically with unrestricted diagonal elements, i.e., variances). In the
In the case of latent interactions, \( \eta \) can be used to stack products of latent variables, so that all observed, latent, and product-term variables can be understood as existing in \( \eta \)—this is useful for concision, reducing the complexity of our equations (for more technical treatments, see Klein & Moosbrugger, 2000; Klein & Muthén, 2007; Preacher et al., 2016).

The result is that, in Eqs. 11 and 12, all moderation, mediation, and moderated mediation effects are either contained in \( B \) or they can be constructed from elements in \( B \).

For example, Eqs. 7 and 8 can be shown in the form of Eqs. 11 and 12 as follows:

\[
y_i = \begin{bmatrix} m_i \\ y_i \\ x_i \\ w_i \\ xw_i \\ mw_i \end{bmatrix} = \mathbf{v} + \Lambda \eta_i + \mathbf{\epsilon}_i = \]

\[
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_m \\ \eta_y \\ \eta_x \\ \eta_w \\ \eta_{xw} \\ \eta_{mw} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

and
\[ \eta_i = \begin{bmatrix} \eta_m \\ \eta_y \\ \eta_s \\ \eta_w \\ \eta_{yw} \\ \eta_{mw} \end{bmatrix} = \alpha + B \eta_i + \zeta_i = \]

\begin{bmatrix} v_m \\ v_y \\ \alpha_s \\ \alpha_w \\ \alpha_{sw} \\ \alpha_{mw} \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_{mx} & a_{mw} & a_{mxw} & 0 \\ b_{ym} & 0 & c'_{yx} & c'_{yw} & c'_{xw} & b_{ymw} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_m \\ \eta_y \\ \eta_s \\ \eta_w \\ \eta_{sw} \\ \eta_{mw} \end{bmatrix} + \begin{bmatrix} \epsilon_{m,i} \\ \epsilon_{y,i} \\ \zeta_{s,i} \\ \zeta_{w,i} \\ \zeta_{sw,i} \\ \zeta_{mw,i} \end{bmatrix} \tag{14}.

Here, by constraining the values of \( \mathbf{v}, \mathbf{A}, \) and \( \mathbf{e}_i \), Equation 13 equates the terms in \( \eta_i \) to the observed variables in \( \mathbf{y}_i \). Furthermore, the constraints imposed in Equation 14 result in any variables that are both predictors of the same outcome and are not related in \( \mathbf{B} \) having an unrestricted relationship in \( \Psi \). Given the above ordering of variables in \( \eta_i \), for example, the matrix \( \Psi \) might be shown as:

\[ \Psi = \begin{bmatrix} \psi_{m,m} \\ 0 & \psi_{j,y} \\ 0 & 0 & \psi_{x,x} \\ 0 & 0 & \psi_{x,w} & \psi_{w,w} \\ 0 & 0 & \psi_{x,xw} & \psi_{x,w} & \psi_{w,xw} & \psi_{w,w} \\ \psi_{m,mw} & 0 & \psi_{x,mw} & \psi_{w,mw} & \psi_{xw,mw} & \psi_{mw,mw} \end{bmatrix} \tag{15} \]

wherein the diagonal elements are variances or residual variances, and off-diagonal elements are covariances or residual covariances.

Estimation can be accomplished with a variety of tools. SEM is typically estimated with maximum likelihood, which may be robust to non-normality and the non-independence
of observations (as in Muthén & Muthén, 1998-2016). However, Bayesian approaches are also possible (see Muthén & Asparouhov, 2012). In all cases, estimators of uncertainty such as confidence intervals for indirect effects should not be based on normal theory.

**Single-Level Moderated Mediation: Race in High School and Beyond**

Using SEM, any combination of parameters can be estimated and used to examine moderated mediation (e.g., using Mplus features like ‘model indirect’ and ‘model constraint’; Muthén & Muthén, 1998-2016). To show this, we use the HS&B data (see online Mplus files in “Single.Level.Modmed.zip”). For illustrative purposes, we treat the data as single-level and use a sandwich estimator to adjust SEs for non-independence (with ‘Type=Complex’ and ‘Cluster=school’ in Mplus). To derive all CIs reported in the text below we use a Monte Carlo approach that allows estimating CIs in the presence of clustered data (which presents difficulty for more common nonparametric bootstrapping; see Preacher & Selig, 2012; see online Mplus files in “MonteCarlo.CI.zip”). Throughout, when SEs conform to normal theory, we present p-values rather than CIs. We would not normally recommend a single-level approach for these data, but working with this example allows us to contrast single-level results with the multilevel results that we present later.

The variables we use are: student math achievement, with higher scores implying greater achievement on a standardized test; student socio-economic status (SES), with higher scores meaning higher parental income, education, occupational attainment, and education-related possessions such as books; student race, wherein 1 = black and 0 = other; and student gender, wherein 0 = male and 1 = female. More details about the variables can be found in the publication on the HS&B data by National Opinion Research Center (1980).

We estimate a model wherein math achievement \( y \) is a function of SES \( m \) and race \( x \), with SES \( m \) also being a function of race \( x \). This model allows estimating some effects of being black in the US, which can negatively affect math achievement directly and indirectly
via SES (Altonji & Blank, 1999; Bertrand & Mullainathan, 2004; Bonilla-Silva, 2006; Steele & Aronson, 1995). Similar negative effects are known in the epidemiology literature, pointing to negative effects of being black in the US on health and other outcomes both directly and via SES (Navarro, 1990; Ren, Amick, & Williams, 1998). Furthermore, this is a sensible model in terms of causality because we assume that changes in student math achievement cannot influence parental SES, and that changes in math achievement or parental SES cannot influence individual race.

We allow gender $w$ to moderate all relationships (as in Eqs. 7-10, 13, and 14). This moderator is sensitive to different forms of race-based differences for black men versus black women (see Galinsky, Hall, & Cuddy, 2013; Hall, Hall, & Perry, 2016; Thomas, Witherspoon, & Speight, 2008; Tomaskovic-Devey, 1993; Wingfield, 2007). Because of such gender differences, the direct and indirect effects of race may differ for males versus females, and any direct effect of SES may differ. As with race, changes in other study variables cannot influence gender.

Table 1 displays model parameters. Based on Eq. 10 we can define multiple effects that will be of interest. These are specified under ‘model constraint’ in the Mplus input and shown in Table 2. To understand their construction, we formally define them and explain their substantive meaning. Although $w$ is categorical, our logic also works for continuous moderators by choosing relevant values of $w$ (e.g., 1 standard deviation above and below the mean) to make comparisons like those we describe.

First, we define the indirect effect for females, which is the simple slope of the indirect effect when the moderator gender = 1 (i.e., $w=1$). This is:

$$(a_{mt} + a_{mtw}w_i)(b_{mj} + b_{mjiw}w_i) = (a_{mt} + a_{mtw})(b_{mj} + b_{mjiw})$$

which is the indirect effect of race on math achievement via SES for females, and is -1.255 with 95% CI [-1.634, -.867], indicating that black females have lower math achievement due
to the effect of their race on their SES. The equivalent conditional indirect effect for males is when gender = 0 (i.e., \( w = 0 \) ), or:

\[
(a_m + a_m w_i)(b_m + b_m w_i) = a_m b_m
\]  

(17)

which defines the indirect effect of race on math achievement via SES for males, and is -1.27 with 95% CI [-1.604, -0.954], indicating that black males have lower math achievement due to the effect of their race on their SES. In turn, the difference in the indirect effects for men versus women is defined as follows:

\[
a_m b_m (a_m + a_m w_i)(b_m + b_m w_i) = (a_m b_m y_{m} + a_m b_{m w} + a_{m w} b_{m y} + a_{m w y m w})
\]  

(18)

This difference is interpreted as the moderating effect of gender on the indirect effect of race on math achievement via SES, while holding the direct effect of race constant. This difference is -.015 with 95% CI [-.444, .397], indicating no statistically significant difference between males and females in the indirect effect of race on math achievement via SES—although the point estimate of -.015 means the indirect effect of race appears to be slightly more negative for males, the CI centers almost on zero.

Comparing the direct effects of race on math achievement shows a similar pattern.

For females (\( w = 1 \)) this effect is:

\[
c_{y x} + c_{y x w} w_i = c_{y x}^w + c_{y x w}^w
\]  

(19)

which describes the direct effect of race on math achievement for females, and is -2.694 (\( p < .001 \)), indicating that black females have lower math achievement. The effect for males (\( w = 0 \)) is:

\[
c_{y x} + c_{y x w} w_i = c_{y x}^w
\]  

(20)

which describes the direct effect of race on math achievement for males, and is -3.012 (\( p < .001 \)), indicating that black males have lower math achievement. In turn, the difference in male versus female direct effects is:
\[ c'_{yx} - (c'_{yx} + c'_{ywx}) = -c'_{ywx} \]  

This difference is interpreted as the conditional interaction between race and gender (because SES is controlled for), and is \(-.318 \ (p = .509)\), indicating no statistically significant difference between males and females in the direct effect of race on math achievement.

For total effects the same logic applies, so that the effect for females \((w = 1)\) is:

\[
(a_{mx} + a_{mwx}w_i)(b_{ym} + b_{ymw}w_i) + (c'_{yx} + c'_{ywx}w_i) = (a_{mx} + a_{mwx})(b_{ym} + b_{ymw}) + (c'_{yx} + c'_{ywx})
\]

which is the overall effect of race on math achievement for females, and is \(-3.949\) with 95% CI \([-4.926, -2.947]\), indicating that for females, race has an overall negative effect on math achievement. For males \((w = 0)\) this is:

\[
(a_{mx} + a_{mwx}w_i)(b_{ym} + b_{ymw}w_i) + (c'_{yx} + c'_{ywx}w_i) = a_{mx}b_{ym} + c'_{yx}
\]

which describes the overall effect of race on math achievement for males, and is \(-4.282\) with 95% CI \([-5.117, -3.432]\), indicating that for males race has an overall negative effect on math achievement. In turn, the difference in these effects for males versus females is:

\[
(a_{mx}b_{ym} + c'_{yx} - [(a_{mx} + a_{mwx})(b_{ym} + b_{ymw}) + (c'_{yx} + c'_{ywx})]) =
- (a_{mx}b_{ym} + a_{mwx}b_{ym} + a_{mwx}b_{ymw} + c'_{ywx})
\]

This difference is interpreted as the moderating effect of gender on the effect of race on math achievement. This difference is \(-.333\) with 95% CI \([-1.396, .719]\), indicating no statistically significant difference between males and females in the overall effect of race on math achievement.

The direct effect of SES can also be estimated. For females \((w=1)\), this is:

\[
b_{ym} + b_{ymw}w_i = b_{ym} + b_{ymw}
\]

which describes the effect of SES on math achievement for females while holding race constant. This effect is \(2.827 \ (p < .001)\), indicating that higher SES for females leads to higher math achievement. The same effect for males \((w = 0)\) is:
which describes the effect of SES on math achievement for males while holding race constant. This effect is $2.515 (p < .001)$, indicating that higher SES for males leads to higher math achievement. In turn, their difference is simply

$$b_{ym} - (b_{ym} + b_{ymw}) = -b_{ymw}$$  \hspace{1cm} (27)$$

which describes the moderating effect of gender on the effect of SES on math achievement while holding race constant. This is $-.312 (p = .232)$, indicating no statistically significant difference between males and females in the effect of SES on math achievement while holding race constant.

Overall, the pattern of results for race is consistent with the long-running history of racism in the US. The effects of being black on math achievement are negative, both indirectly via SES and directly. This is predictable given the substantial literature on racism and its effects both generally (e.g., Bonilla-Silva, 2006) and on standardized test scores specifically (e.g., Steele & Aronson, 1995). However, contrary to what some research may suggest (e.g., Galinsky et al., 2013), we find no moderating effect of gender, even when testing direct effects of SES.

**Multilevel Moderated Mediation in MSEM**

Unfortunately, the above models and analyses are insensitive to the clustering in our data except that they adjust SEs. When children are nested in schools or data are otherwise grouped, different variances and effects are mixed: between-group and within-group (Cronbach, 1976; Cronbach & Snow, 1977; Cronbach & Webb, 1975; Preacher et al., 2010, 2016; Zhang, Zyphur, & Preacher, 2009). Between-group variances/effects are related to group means, whereas within-group variances/effects are related to deviations away from the means (as in ANOVA). In turn, when analyzing data and making inferences, within-group
terms represent individuals (e.g., students) and between-group terms represent groups (e.g., schools). To motivate this style of representation, we first justify it as follows.

**Motivating the Study of Race Between- and Within-Schools**

Our position (e.g., Preacher et al., 2010, 2011, 2016; Zhang et al., 2009) is that by ignoring clustering, single-level analyses create “uninterpretable blends” of variances and effects that are attributable to different kinds of things—students versus schools (Cronbach, 1976, p. 9.20). Furthermore, differences in the magnitudes of these terms across levels of analysis “occur with considerable regularity” (Raudenbush & Bryk, 2002, p. 140), and these differences have critical implications for substantive interventions that might be designed to target entire groups (e.g., schools) versus the individuals residing within them. The job of the data scientist who endeavors to inform intervention or policy planning is to decompose variances and effects so that inferences about different kinds of entities can be unambiguously made in light of the clustered structure of a dataset.

Despite the long recognition of the need to decompose level-specific effects (e.g., Dansereau & Yammarino, 2000), there is some debate regarding this point, especially for mediation and moderation analyses (e.g., Pituch & Stapleton, 2012; Tofighi & Thoemmes, 2014). Therefore, we clarify our views and then connect them to the case of multilevel moderated mediation to study race in the HS&B data. First, consider a dataset with information from students in a single school $j = 1$, and a simple bivariate model:

$$y_{i1} = \nu + \beta x_{i1} + \epsilon_{i1}$$

(28)

Here, $\beta$ is the effect of $x$ on $y$ for all students in school $j = 1$. As with all single-level regression models, we and most other researchers would refer to $\beta$ as an individual-level effect. However, what researchers really mean is that $\beta$ is a within-school effect, which becomes clear by rewriting Eq. 28 in terms of variances, covariances, and means as follows:
\[ y_{ij} = \mu + \frac{\sigma_{xy}}{\sigma_x^2} (x_{ij} - \mu_x) + \epsilon_{ij} \]  

(29)

wherein \( \mu \) terms are means, \( \sigma_{xy} \) is \( x-y \) covariance, and \( \sigma_x^2 \) is the variance of \( x \).

As Eq. 29 shows, group means play no part in deriving \( \beta \) in single-level analyses—the means for \( y \) and \( x \) could be changed by any value and \( \beta \) would be unchanged. Also, statistical inference with \( SEs \) is a function of \( \epsilon_{ij} \), which is a within-school term. In turn, so-called ‘individual-level’ analyses with data from a single group actually estimate within-group effects, meaning that when researchers make inferences about individuals, they have all-along been making within-group inferences (i.e., inferences about individuals that are always relative to the mean of the group). Therefore, consistent with typical regression practices, we recommend using within-group terms to make inferences about individuals.

To further illustrate this point, consider the HS&B data with \( N \) students in \( J \) schools. To make inferences about students, any confounding ‘unobserved heterogeneity’ associated with schools should be controlled. This kind of ‘fixed effects’ model—in econometrics terms—can be constructed by creating \( J-1 \) indicator variables (e.g., dummy codes) in a vector \( z_j \) as follows:

\[ y_{ij} = \nu + \beta x_{ij} + \delta_j z_j + \epsilon_{ij} \]  

(30)

here, the effect of school membership is accounted for by the coefficients in the vector \( \delta_j \) and the effect of \( x \) on \( y \) for students is still \( \beta \), which is a within-group term, just as is \( \epsilon_{ij} \) (which is used for statistical inference with \( SEs \)). Furthermore, by accounting for the ‘school effects’ \( \delta_j \), what is really occurring is that the school \textit{means} are being entirely accounted for, which is to say that \( \delta_j \) is accounting for all \textit{between-school} variance, which is associated with schools—to emphasize, in this kind of fixed-effects model there is no remaining between-school variance that can be accounted for with any additional predictors. In other
words, when attempting to make inferences about individuals while controlling for group effects, researchers focus on within-group variance to make inferences about individuals and control for between-group variance to model the effect of groups, just as is done when ‘within-group centering’ data by eliminating group means (Preacher et al., 2010).

Moreover, to motivate inferences about groups by using between-group variances and effects, consider an experiment wherein researchers randomly assign participants to a control group \( x_{j=1} \) or an experimental group \( x_{j=2} \). To make an inference about the effect of interest, researchers must model the effect of group membership, which can be shown as follows:

\[
y_j = \nu + \beta x_j + \epsilon_{ij} \tag{31}
\]

wherein the model now reflects terms for an individual \( i \) and a group \( j \), with \( x \) coding for group membership. Here, the effect of interest is \( \beta \), which in an ANOVA framework is well-known as a between-group effect, capturing the difference between the two group means. Here, researchers do not substantively care about \( \epsilon_{ij} \) because this within-group term is typically regarded as being due to individual or subject-specific effects.

In all cases described above—typical regression models, fixed-effects regression controlling for group effects, and ANOVA—researchers always make inferences about individuals using within-group variances or effects that model deviations away from group means, and inferences are made about groups using between-group variances or effects that model group means. Therefore, in all of our models we decompose the between- and within-group parts of any observed variables measured at the “individual” level, so that accurate inferences can be made about the appropriate kinds of things that are being assessed (e.g., students, schools, communities). Such decomposition of level-specific effects has long been recommended in the literature (e.g., Cronbach, 1976; Cronbach & Webb, 1975; Dansereau & Yammarino, 2000; Kozlowski & Klein, 2000), but this work is often overlooked.
For studying race, separating student versus school effects is key because different processes can influence students within schools versus schools as wholes (Benner & Graham, 2013), especially because schools are typically defined by local environments such as neighborhoods. Although black students in a school may experience individualized forms of racism as noted previously (motivating a focus on within-school effects), there is evidence that collective ‘institutional’ racism has profound effects. For example, formal and informal segregationist agendas in the US drove black individuals into poor and blighted neighborhoods (Massey & Denton, 1993; Seitles, 1998; Williams & Collins, 2001)—an infamous example is the design of low bridge overpasses to keep black bus passengers from crossing into upper-class white neighborhoods (Caro, 1974). In addition to being excluded from important social capital, institutional racism has had profound effects, including poorer nutrition, education, and employment rates, as well as community problems that make life unstable, stressful, and emotionally hard (Seaton & Yip 2009; Umaña-Taylor, 2016; Williams, 1999; Williams & Williams-Morris, 2000).

In turn, institutional racism causes covariance between racial composition, such as the proportion of black students in a school, and the collective testing outcomes at a school. This effect may operate directly, but it can also function indirectly through collective SES, which further reflects the problems of poorer neighborhoods and schools (e.g., Pickett & Pearl, 2001). As a moderator, gender composition could influence these effects because of the different ways that black males and females are collectively treated (Hall et al., 2016; Wingfield, 2007).

Therefore, it is reasonable to decompose the between- versus within-school parts of observed variables to examine collective versus individual effects of race. To do so, we now introduce multilevel approaches to moderation, mediation, and moderated mediation.

**Multilevel Moderation**
To understand MSEM for the purposes of multilevel moderated mediation analyses, we begin by extending moderation, mediation, and moderated mediation models from Eqs. 1, 3, 4, 7, and 8 to the multilevel case. In these models, observed variables such as \( y \) will typically reflect within- and between-group components when data are clustered or otherwise nested. These components can be decomposed as follows:

\[ y_{ij} = y_{Bj} + y_{Wij} \]  
\[ n \]

wherein a \( B \) subscript indicates a between-group part (e.g., a school mean, sometimes referred to as a “random intercept”) and a \( W \) subscript indicates a within-group part (e.g., a student’s relative standing after subtracting the school mean).

In turn, the moderation model in Eq. 1 can be reformulated by decomposing the \( B \) and \( W \) parts of all relevant variables as follows (for concision, we omit random slopes as regression coefficients that vary across groups, which are possible in our MSEM approach):

\[ y_{Bj} = \mu_{Bj} + c_{Byx}x_{Bj} + c_{Byw}w_{Bj} + c_{Byxw}x_{Bj}w_{Bj} + \epsilon_{B_{ij}} \]  
\[ n \]

\[ n \]

\[ y_{Wij} = c_{Wyx}x_{Wij} + c_{Wyx}w_{Wij} + c_{Wyxw}x_{Wij}w_{Wij} + \epsilon_{Wy_{ij}} \]  
\[ n \]

wherein all terms are as before, but \( B \) terms denote between-school variables or effects and \( W \) terms denote within-school variables or effects. Notice that the intercept for \( y \) in Eq. 33 is a \( B \) term, which is consistent with our arguments related to Eqs. 28-31. Notice also that the product terms \( x_{Bj}w_{Bj} \) and \( x_{Wij}w_{Wij} \) are not \((xw)_{Bj}\) and \((xw)_{Wij}\), because the latter implies first multiplying \( x \) and \( w \) and then decomposing the \( B \) and \( W \) parts of the product term, which is not the same as multiplying the \( B \) and \( W \) components (Preacher et al., 2016).

The point of Eqs. 33 and 34 is that the \( B \) coefficients can be used to make inferences about groups (e.g., schools) and the \( W \) coefficients can be used to make inferences about individuals, who are by design nested in the groups. The moderation effects \( c_{Byxw} \) and \( c_{Wyxw} \)
have the same interpretation as above, except they apply to moderation of $B$ and $W$ effects—similar operations as in Eq. 2 can define $B$ and $W$ moderation (Preacher et al., 2016).

**Multilevel Mediation**

The same is true for the mediation model in Eqs. 3 and 4, which in a multilevel framework would be:

$$m_{Bj} = \gamma_{Bm} + \alpha_{Bmx}x_{Bj} + \epsilon_{Bm,j} \quad (35)$$

$$y_{Bj} = \gamma_{By} + b_{Bm}m_{Bj} + c'_{Bmx}x_{Bj} + \epsilon_{By,j} \quad (36)$$

$$m_{Wij} = \alpha_{Wmx}x_{Wij} + \epsilon_{Wm,ij} \quad (37)$$

$$y_{Wij} = h_{Wym}m_{Wij} + c'_{Wyx}x_{Wij} + \epsilon_{Wy,ij} \quad (38)$$

wherein all terms are as before and with similar interpretations, except the $B$ parts apply to groups (e.g., schools) and the $W$ parts apply to individuals, who are nested within groups. Furthermore, the same logic of mediation exists, with $B$ and $W$ indirect effects being $a_{Bmx}b_{Bym}$ and $a_{Wmx}b_{Wym}$, respectively—the reader can perform the same operations as in Eqs. 5 and 6 for both $B$ and $W$ mediation (Preacher et al., 2010).

**Multilevel Moderated Mediation**

Multilevel moderated mediation implies the same straightforward extension to the $B$ and $W$ case, so that Eqs. 7 and 8 become (again, for concision, we omit random slopes):

$$m_{Bj} = \gamma_{Bm} + \alpha_{Bmx}x_{Bj} + \alpha_{Bmwx}w_{Bj} + \alpha_{Bmxx}x_{Bj}w_{Bj} + \epsilon_{Bm,j} \quad (39)$$

$$y_{Bj} = \gamma_{By} + b_{Bym}m_{Bj} + b_{Bmwy}m_{Bj}w_{Bj} + c'_{Bmy}x_{Bj} + c'_{Bmwy}x_{Bj}w_{Bj} + c'_{Bmxx}x_{Bj}w_{Bj} + \epsilon_{By,j} \quad (40)$$

$$m_{Wij} = \alpha_{Wmx}x_{Wij} + a_{Wmwx}w_{Wij} + a_{Wmxx}x_{Wij}w_{Wij} + \epsilon_{Wm,ij} \quad (41)$$

$$y_{Wij} = h_{Wym}m_{Wij} + b_{Wymw}m_{Wij}w_{Wij} + c'_{Wyx}x_{Wij} + c'_{Wmwy}x_{Wij}w_{Wij} + c'_{Wmxx}x_{Wij}w_{Wij} + \epsilon_{Wy,ij} \quad (42)$$

wherein moderated mediation has the same familiar form, except with separate $B$ and $W$ parts to allow inference to groups and individuals. In turn, the same operations we used to explain
moderated mediation after Eq. 8 apply to Eqs. 39-42, with \( B \) indirect and direct effects for groups as \((a_{\text{mix}} + a_{\text{mono}} w_{yj})(b_{\text{mono}} + b_{\text{mono}} w_{yj})\) and \((c'_{yj} + c'_{yj} w_{yj})\), respectively, and \( W \) indirect and direct effects for individuals as \((a_{\text{mix}} + a_{\text{mono}} w_{w_{ij}})(b_{\text{mono}} + b_{\text{mono}} w_{w_{ij}})\) and \((c_{yj} + c_{yj} w_{w_{ij}})\), respectively. Here, the reader can apply the same logic as with single-level analyses, but keeping in mind that \( B \) effects apply to groups and \( W \) effects apply to individuals.

A General MSEM Specification and Estimation

To estimate terms in Eqs. 33-42 with \( B \) and \( W \) moderated mediation parameters that mirror those in Eqs. 16-27 (and Table 1 and 2), we extend the SEM in Eqs. 11-15 to the multilevel case. We do this succinctly as follows, but we note that the interested reader can consult complementary treatments in Preacher et al. (2010, 2011, 2016):

\[
y_{ij} = \Lambda \eta_{ij} \tag{43}
\]

\[
\eta_{ij} = \alpha_{j} + B_{j} \eta_{ij} + \zeta_{ij} \tag{44}
\]

\[
\eta_{ij} = \mu + \beta \eta_{ij} + \zeta_{ij} \tag{45}
\]

wherein the meaning of terms differs from Eqs. 11-15.

In Eq. 43, \( y_{ij} \) is a vector of observed variables; \( \Lambda \) is a matrix indicating whether variables vary within-groups, between-groups, or both; and \( \eta_{ij} \) is a vector of latent variables that vary either within- or between-groups. For example, Eq. 32 can be formulated as Eq. 43 to clarify its meaning as follows:

\[
y_{ij} = \Lambda \eta_{ij} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} y_{Bj} \\ y_{wij} \end{bmatrix} = y_{Bj} + y_{wij} \tag{46}
\]

wherein the two elements in \( \Lambda \) for \( y_{ij} \) indicate that it has \( W \) and \( B \) parts, contained in \( \eta_{ij} \).

In turn, Eq. 44 contains: the \( B \) part of observed variables in an intercept vector \( \alpha_{j} \) (again, these are sometimes referred to as “random intercepts”); the \( W \) effects among the \( W \)
parts of observed variables in a matrix $\mathbf{B}_j$; and $W$ residuals in a vector $\mathbf{\zeta}_j$. In other words, the $W$ structural model is in $\mathbf{B}_j$, and therefore $\mathbf{B}_j$ will contain $W$ moderated mediation terms, such as those in Eqs. 41 and 42.

Alternatively, Eq. 44 contains $B$ model parts as follows: $\mathbf{\eta}_j$ contains all $B$ variables of interest from $\mathbf{\eta}_j$, but this is done by stacking the $B$ intercepts in $\mathbf{\alpha}_j$ as well as any random slopes from $\mathbf{B}_j$; $\mathbf{\mu}$ is a vector of intercepts or grand means; $\mathbf{\beta}$ is a matrix of $B$ effects; and $\mathbf{\zeta}_j$ is a vector of $B$ residuals. In other words, the $B$ structural model is in $\mathbf{\beta}$, and therefore $\mathbf{\beta}$ will contain $B$ moderated mediation terms, such as those in Eqs. 39 and 40.

Hopefully, by now the reader can infer that multilevel moderated mediation merely requires applying familiar single-level concepts to the $W$ and $B$ model parts in Eqs. 44 and 45. However, before proceeding, there are a few caveats to mention. First, MSEM allows for more flexibility than we can cover here, such as random slopes (e.g., for $W$ effects in $\mathbf{B}_j$, which would be stacked in $\mathbf{\eta}_j$). Such slopes allow cross-level interactions, which may be a useful complement to the multilevel moderated mediation we examine here. The interested reader can easily pursue this using our logic and that in Preacher et al. (2010, 2016), which discusses special issues related to the use of random slopes and cross-level interactions.

Second, because the $B$ and $W$ parts of an observed variable are latent, they must be estimated. This estimation can be done by calculating school averages, but as in most multilevel models (e.g., Raudenbush & Bryk, 2002), MSEM does this using an empirical Bayes approach to account for sampling error. However, in some cases this may be unwarranted and researchers may prefer to compute group means for $B$ parts as if they were observed (see Lüdtke et al., 2008; Lüdtke, Marsh, Robitzsch, & Trautwein, 2011; Marsh et al., 2009; Preacher et al., 2016). For HS&B data, both individuals and schools were randomly sampled, and therefore empirical Bayes estimation is warranted.
Third, in Eqs. 33 and 34, we noted that $x_{ij}^Bw_{ij}$ and $x_{ij}^Ww_{ij}$ are not $(xw)_B^j$ and $(xw)_W^j$. The implication for moderated mediation in MSEM is that the reader cannot simply compute observed product terms for interacting variables such as $x_{ij}^Wy_{ij}$ and then specify these in $y_{ij}$ from Eq. 43. The reason is that MSEM decomposes the $W$ and $B$ parts of observed variables in order to account for uncertainty associated with sampling error (as implied by Eq. 46).

To estimate $W$ and $B$ moderation requires computing product terms for latent $B$ and $W$ variables separately, as in $x_{ij}^Bw_{ij}$ and $x_{ij}^Ww_{ij}$ (Preacher et al., 2016). This is done in Mplus by putting a latent variable ‘behind’ a set of $B$ and $W$ parts of observed variables and then forming product terms as latent variable interactions—which Preacher et al. (2016) specify in their online supplemental material. Figure 4 shows this, wherein the interacting $B$ and $W$ parts of observed variables are treated as latent, with the variances of their $B$ and $W$ parts fixed to .01 to facilitate convergence (as in Preacher et al., 2016). This allows estimating the product terms required for multilevel moderation of various kinds.

Fourth, the recommended approach to latent interactions from Preacher et al. (2016) uses LMS, which is implemented in Mplus (see Klein & Moosbrugger, 2000; Klein & Muthén, 2007). However, this approach can encounter serious difficulties with convergence in the case of high-dimensional numerical integration. For example, the model in Figure 4 using the HS&B data was not estimable with adequate dimensions of numerical integration, and we could not achieve convergence using the ‘Integration=Montecarlo’ approach to this integration as shown in Preacher et al.’s (2016) online supplemental material (the reader is invited to experiment with the Mplus input in “Multilevel.ModMed.zip”). This is because our model has many latent variables (i.e., three $B$ terms, three $W$ terms, as well as their associated latent product terms), which is not surprising when combining the approaches of Preacher et al. (2016) with that of Preacher et al. (2010). Such complexity is to be expected with
multilevel moderated mediation (consider multilevel versions of Hayes, 2012-2016, which could even include many random slopes of W model parts). Indeed, such complexity is to be expected even in the single-level case of moderated mediation with latent variables.

Therefore, to avoid numerical integration, we use a Bayesian ‘plausible values’ approach to latent variable interactions in Mplus (see Asparouhov & Muthén, 2010a, 2010b, 2010c). This approach uses Bayesian estimation with default “diffuse” or “uninformative” prior probabilities to approximate maximum-likelihood estimation (see Muthén & Asparouhov, 2012). The key to this estimation is that it allows generating a Bayesian analogue of factor scores for latent variables by sampling from their posterior distribution some number of times (20 in our case; see Mislevy et al., 1992, and von Davier et al., 2009). Interestingly, this is equivalent to a multiple imputation method with latent variables treated as missing data, which overcomes the need for estimating latent variables and their interactions directly, which requires computationally difficult numerical integration.

Although research on multiple imputation shows that interactions or non-linear effects should be used for imputing observed data (e.g., Seaman, Bartlett, & White, 2012; Bartlett et al., 2015; von Hippel, 2009), including in the multilevel case (Goldstein, Carpenter, & Browne, 2014), our approach imputes missing unobserved variables rather than observed variables, and therefore our method should capture some of the interaction/non-linear patterns that are a function of the observed, non-missing data. In the multilevel case, we expect latent W and B scores can be accurately estimated even without including latent products in the model—although there are conditions for this being the case, including having no missing data (or very little missing data) along observed variables (for insight see the previous citations). After generating the plausible values in a first step, a maximum likelihood procedure is used to estimate parameters and compute model fit in a second step (Asparouhov & Muthén, 2010a, 2010b; Enders, 2010), with estimates averaged across the
plausible values and SEs adjusted for the uncertainty they indicate (as in Rubin, 1987; Schafer, 1997).

Our approach has two steps. Step 1: plausible values are generated in Mplus by estimating the model of interest without latent interactions, using a Bayes estimator with default diffuse/uninformative prior probability distributions (see Mplus files in “Multilevel.ModMed.Plausible.Values.zip”). Step 2: a model is estimated using plausible values as if they were multiple imputations (e.g., Rubin, 1987; Schafer, 1997), using a typical maximum-likelihood based approach, with product terms computed for the plausible values to approximate latent interactions (see Figure 5). This allows treating all B and W model parts as if they were observed, with uncertainty in latent variable values treated as variation across the multiple imputations (i.e., differences in the plausible values). To capture this uncertainty we use 20 imputations, which is a common number for multiple imputations (e.g., Rubin, 1987; Schafer, 1997). As the reader can grasp by experimenting with the full multilevel model and the two-step plausible values approach, the latter drastically simplifies the estimation of models involving latent interactions.

Fifth, and finally, because testing mediation with indirect effects cannot use SEs derived from normal theory, alternative approaches are recommended that we previously described. In the multilevel case with latent interactions, the situation is also complicated (see Zyphur, Zammuto, & Zhang, 2016). Therefore, we use a Monte Carlo approach wherein parameter estimates and their asymptotic covariance matrix are used to generate 10,000 estimates of effects (as in Zyphur et al., 2016; see online Mplus files in “MonteCarlo.CI.zip”) which are then used to empirically estimate CIs.

**Between- and Within-School Race in High School and Beyond**

Tables 3 and 4 display B and W model parameters, respectively (see also Figure 5). Based on Eq. 10, B and W moderated mediation effects can be defined for Eqs. 16-27 (as
applied to Eqs. 39-42). These are specified under ‘model constraint’ in our online Mplus files and shown in Tables 5 and 6 for $B$ and $W$ parameters, respectively. Because of the many results that exist in these tables, we summarize them for the sake of concision as follows.

The first thing to observe is that the $W$ effects for students follow the same pattern as the single-level results—the direct and indirect effects of race have CIs that do not contain zero, and there is no moderation by gender. This similarity is expected because single-level analyses allow $W$ parameters to dominate when there are many more units of observation at the $W$ level as compared with the $B$ level, and when $W$ variances are larger than $B$ variances (Preacher et al., 2010, 2016). For findings on the effects of race, the implication is that for any given student, being black in the US is harmful for math achievement directly and by having a negative effect on SES—again, both effects are consistent with the long-running history of racism in the US. However, gender seems to make little difference in these effects (although Table 3 shows that gender does have a sizable effect on math achievement, but this effect is smaller than the effect of race).

The $B$ effects for schools, however, tell a very different story. The upper panel of Table 6 shows results for all-female schools. The indirect effect of race on math achievement via SES for a school is -2.553 when comparing a school of black females versus a school of non-black females (with a CI not containing zero). However, the direct effect of race on math achievement for a school is -1.664 when comparing a school of black females versus a school of non-black females (with a CI containing zero). We have two important notes regarding this result. First, because variables will often have distinctive meanings, varying measurement metrics, and different reliabilities at $W$ versus $B$ levels (Bliese, 2000; Kowslowski & Klein, 2000), their path coefficients and the associated indirect effects are not readily comparable across levels. For example, the $B$ indirect effect of -2.553 is not directly comparable to the same $W$ indirect effect of -.652 because the units of analysis are different. This said, at each
level the same variable’s direct and indirect effects can be compared and the relative magnitude of these effects can be informative (e.g., the W direct effect for females is -2.412 and the W indirect effect is -.652). Also, the level-specific ratios of direct versus indirect effects are not influenced by measurement metrics and can be compared across B and W levels. Second, we used theoretically extreme values for the B-level moderator’s high versus low conditions, i.e., a school’s gender composition can be 100% female or 100% male. We similarly used extreme values for the school-level race variable such that a school can be 100% black or 100% non-black. Using extreme values helps our interpretation of the conditional indirect effects given the categorical predictor and moderator variables. When a continuous moderator is examined, the more conventional approach is to use 1 level-specific standard deviation above and below its level-specific mean to calculate indirect effects at B and W levels. Overall, the above findings show that for all-female schools, school-level race appears to influence school-level math achievement via school-level SES. This finding might be driven by neighborhood-level and school-level variables related to poverty. However, there is no direct effect of school-level race on school-level math achievement.

The same pattern of findings holds for black males in all-male schools. As shown in the middle panel of Table 6, the indirect effect of school-level race on school-level math achievement via school-level SES for a school comprised solely of black males is -5.077 when comparing a school of black males versus a school of non-black males (with a CI not containing zero). However, the direct effect yields a different conclusion, namely that the effect of school-level race on school-level math achievement for a school composed solely of black males is -.875 when compared to a school of non-black males (with a CI including zero). Therefore, again, for an entire male-only school, the effect of race appears to operate only indirectly via SES. Comparing the all-male school vs. all-female school conditions, the difference in indirect effects has a confidence interval including zero (i.e., [-5.937, .206]).
Therefore, school-level gender does not moderate the indirect effect of race on math achievement via school-level SES.

In sum, these results indicate that being a black student in the US has negative effects on math achievement, both directly and indirectly via SES, with the direct effects being stronger than those via SES. However, for schools as wholes, this is not the case. Consistent with centuries of institutional racism and other forms of injustice that cause a correlation between race and SES at the neighborhood and community levels (Bonilla-Silva, 2006), lower test scores for all-black schools (regardless of gender) appear to be entirely due to SES. The implication is that, at the school level, the poverty and socio-economic exclusion associated with racial differences may explain the effect of race on a school’s math achievement. Overall, in order to address differences in test scores, the US—like other countries—must do more to create racial equality by reducing the relationship between racial categories and important outcomes and socio-economic resources.

**Discussion**

We have described a novel approach for investigating multilevel moderated mediation using MSEM. Both conceptually and by example, we explore just one of the many possibilities for estimating these models, including a novel plausible values approach that avoids numerical integration—which would have otherwise derailed our analyses above. Future work can explore specific cases that would include random slopes and variables that vary only at the between-groups levels of analysis (see thorough treatments in Preacher et al., 2010, 2016). In all cases, the between- and within-groups parts of observed variables can be used to make inferences to higher- versus lower-level entities.

In terms of our plausible values approach, this can be used for any model wherein latent interactions or power polynomials would otherwise require numerical integration and observed data provide adequate information to estimate latent standings. Therefore, this
might be useful for single-level models that rely on LMS or other computationally intensive methods (e.g., Sardeshmukh & Vandenberg, 2016). Our plausible values approach has the benefit of allowing comparatively fast estimation when the number of multiple imputations is not overly large, but it does require various conditions being met that we will explore in a future paper. However, we do offer a word of caution when using asymptotic (co)variances of parameter estimates to compute CIs with a Monte Carlo procedure, as we use here. Because these covariances are meant to be *asymptotic* with similar estimation assumptions as a maximum likelihood estimator, researchers may be motivated to produce many multiple imputations, perhaps 1,000 or 10,000. For the purpose of our example and to make our online supplemental materials easier to download, we limited the number of imputations to 20, which is common in the multiple imputation and plausible values literatures (e.g., Mislevy et al., 1992; Rubin, 1987; Schafer, 1997; von Davier et al., 2009).

Beyond concerns regarding the number of imputations, there are additional limitations with our approach and inferences that should be recognized. First, our data are from 1979 and therefore may no longer be a good representation of the US in various ways, including the black and non-black composition of schools. Second, related to this composition, the inferences we make at the school level are based on some extrapolations from our data (i.e., theoretically extreme values for school-level race and school-level gender). Although there are schools in our sample and in the US wherein students are almost all black or all non-black (indeed, the overall trend of this segregation is getting worse rather than better in public schools; see Frankenberg & Lee, 2002), no schools in our sample were composed of all-black males or all-black females. Therefore, our inferences regarding moderated mediation warrant some caution at the school level because they cannot be clearly mapped onto observed ranges in the data (this is a general problem for interpreting and reporting effects that involve moderation; see Hayes, 2013; Hayes & Preacher, 2013).
In conclusion, whether estimating multilevel moderated mediation or other effects, we hope that we have shown the potential power of statistical modeling to produce images of people and society that can motivate practical action. In terms of race, it is clear that the US and other nations have a long way to go before justice or equality will be realized, and therefore additional steps should be taken to eliminate racial inequalities. In our view, this should be the goal of statistical analyses: to motivate changes that make a difference.
References


Table 1

Single-Level Moderated Mediation Model Parameters

<table>
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<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>SE</th>
<th>p-value</th>
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</thead>
<tbody>
<tr>
<td><strong>SES Parameters</strong></td>
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<td></td>
<td></td>
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<tr>
<td>( \nu_m ) (SES intercept)</td>
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<td>---</td>
<td>---</td>
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<td>.064</td>
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<td>.002</td>
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<td>( b_{ymw} ) (SES*gender→MA)</td>
<td>.312</td>
<td>.261</td>
<td>.232</td>
</tr>
<tr>
<td>( c'_{sx} ) (race→MA)</td>
<td>-3.012</td>
<td>.368</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>( c'_{yw} ) (gender→MA)</td>
<td>-1.466</td>
<td>.251</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>( c'_{sxw} ) (race*gender→MA)</td>
<td>.318</td>
<td>.481</td>
<td>.509</td>
</tr>
</tbody>
</table>

Note. MA = math achievement; SES = socio-economic status.
Table 2

**Single-Level Moderated Mediation Model’s Further Calculated Parameters**

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Lower 2.5%</th>
<th>Estimate</th>
<th>Upper 97.5%</th>
<th>SE</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Female</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-1.634</td>
<td>-1.255</td>
<td>-.867</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-2.694</td>
<td>---</td>
<td>.412</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-4.926</td>
<td>-3.949</td>
<td>-2.947</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>SES effect</td>
<td>---</td>
<td>2.827</td>
<td>---</td>
<td>.173</td>
<td>&lt;.001</td>
</tr>
<tr>
<td><strong>Male</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-1.604</td>
<td>-1.27</td>
<td>-.954</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-3.012</td>
<td>---</td>
<td>.368</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-5.117</td>
<td>-4.282</td>
<td>-3.432</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>SES effect</td>
<td>---</td>
<td>2.515</td>
<td>---</td>
<td>.211</td>
<td>&lt;.001</td>
</tr>
<tr>
<td><strong>Difference in Effects (Male – Female)</strong></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-0.444</td>
<td>-.015</td>
<td>.397</td>
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<td>---</td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-.318</td>
<td>---</td>
<td>.481</td>
<td>.509</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-1.396</td>
<td>-.333</td>
<td>.719</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>SES effect ( b_{\text{gen}} )</td>
<td>---</td>
<td>-.312</td>
<td>---</td>
<td>.261</td>
<td>.232</td>
</tr>
</tbody>
</table>

Note. Where parameters involve products of coefficients, CIs are generated by Monte Carlo using parameter estimates and their asymptotic covariance matrix with 10,000 draws.
Table 3

*Within-Level Moderated Mediation Model Parameters*

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>SE</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SES Effects</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_{Wmx}$ ($race \rightarrow SES$)</td>
<td>-.34</td>
<td>.037</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$a_{Wmw}$ ($gender \rightarrow SES$)</td>
<td>-.083</td>
<td>.018</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$a_{Wmxw}$ ($race \times gender \rightarrow SES$)</td>
<td>.06</td>
<td>.068</td>
<td>.377</td>
</tr>
<tr>
<td><strong>Math Achievement Effects</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_{Wy}$ ($SES \rightarrow MA$)</td>
<td>1.916</td>
<td>.118</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$b_{Wymw}$ ($SES \times gender \rightarrow MA$)</td>
<td>.415</td>
<td>.277</td>
<td>.134</td>
</tr>
<tr>
<td>$c'_{Wy}$ ($race \rightarrow MA$)</td>
<td>-2.946</td>
<td>.259</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$c'_{Wy}$ ($gender \rightarrow MA$)</td>
<td>-1.161</td>
<td>.183</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$c'_{Wyxw}$ ($race \times gender \rightarrow MA$)</td>
<td>.534</td>
<td>.514</td>
<td>.299</td>
</tr>
</tbody>
</table>

*Note.* MA = math achievement; SES = socio-economic status.
Table 4

*Between-Level Moderated Mediation Model Parameters*

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>SE</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>SES Effects</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_{Bm}$ (SES intercept)</td>
<td>.324</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$a_{Bmx}$ (race $\rightarrow$ SES)</td>
<td>-.782</td>
<td>.222</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$a_{Bmw}$ (gender $\rightarrow$ SES)</td>
<td>-.291</td>
<td>.166</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$a_{Bmxw}$ (race*gender $\rightarrow$ SES)</td>
<td>.234</td>
<td>.39</td>
<td>.149</td>
</tr>
<tr>
<td>Math Achievement Effects</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\nu_{By}$ (MA intercept)</td>
<td>13.761</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$b_{Bym}$ (SES $\rightarrow$ MA)</td>
<td>6.505</td>
<td>.99</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>$b_{Bymw}$ (SES*gender $\rightarrow$ MA)</td>
<td>-1.859</td>
<td>1.46</td>
<td>.203</td>
</tr>
<tr>
<td>$c'_{Byx}$ (race $\rightarrow$ MA)</td>
<td>-.875</td>
<td>1.279</td>
<td>.494</td>
</tr>
<tr>
<td>$c'_{Byw}$ (gender $\rightarrow$ MA)</td>
<td>-1.440</td>
<td>1.085</td>
<td>.184</td>
</tr>
<tr>
<td>$c'_{Byxw}$ (race*gender $\rightarrow$ MA)</td>
<td>-.788</td>
<td>2.105</td>
<td>.708</td>
</tr>
</tbody>
</table>

*Note.* MA = math achievement; SES = socio-economic status.
Table 5

Within-Level Moderated Mediation Model’s Further Calculated Parameters

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Lower2.5%</th>
<th>Estimate</th>
<th>Upper97.5%</th>
<th>SE</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Female</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-1.06</td>
<td>-0.652</td>
<td>-.305</td>
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<td></td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-2.412</td>
<td>---</td>
<td>.507</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-4.218</td>
<td>-3.064</td>
<td>-1.915</td>
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<td></td>
</tr>
<tr>
<td>SES effect</td>
<td>---</td>
<td>2.332</td>
<td>---</td>
<td>.300</td>
<td>&lt;.001</td>
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<tr>
<td><strong>Male</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-.824</td>
<td>-.651</td>
<td>-.497</td>
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</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-2.946</td>
<td>---</td>
<td>.259</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-4.164</td>
<td>-3.597</td>
<td>-3.032</td>
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</tr>
<tr>
<td>SES effect</td>
<td>---</td>
<td>1.916</td>
<td>---</td>
<td>.118</td>
<td>&lt;.001</td>
</tr>
<tr>
<td><strong>Difference in Effects (Male – Female)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-.319</td>
<td>.001</td>
<td>.367</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-.534</td>
<td>---</td>
<td>.514</td>
<td>.299</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-1.563</td>
<td>-.533</td>
<td>.476</td>
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</tr>
<tr>
<td>SES effect (b_{\text{WMM}})</td>
<td>---</td>
<td>.415</td>
<td>---</td>
<td>.277</td>
<td>.134</td>
</tr>
</tbody>
</table>

Note. Where parameters involve products of coefficients, CIs are generated by Monte Carlo using parameter estimates and their asymptotic covariance matrix with 10,000 draws.
Table 6

Between-Level Moderated Mediation Model's Further Calculated Parameters

<table>
<thead>
<tr>
<th>Parameter Name</th>
<th>Lower2.5%</th>
<th>Estimate</th>
<th>Upper97.5%</th>
<th>SE</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Female</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-4.145</td>
<td>-2.553</td>
<td>-.893</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-1.664</td>
<td>---</td>
<td>1.105</td>
<td>.132</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-6.512</td>
<td>-4.217</td>
<td>-1.475</td>
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<td>---</td>
</tr>
<tr>
<td>SES effect</td>
<td>---</td>
<td>4.646</td>
<td>---</td>
<td>.713</td>
<td>&lt;.001</td>
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<tr>
<td><strong>Male</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-7.357</td>
<td>-5.077</td>
<td>-2.913</td>
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<td>---</td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>-.875</td>
<td>---</td>
<td>1.279</td>
<td>.494</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-8.452</td>
<td>-5.953</td>
<td>-3.151</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>SES effect</td>
<td>---</td>
<td>6.505</td>
<td>---</td>
<td>.99</td>
<td>&lt;.001</td>
</tr>
<tr>
<td><strong>Difference in Effects (Male – Female)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Indirect race effect</td>
<td>-5.907</td>
<td>-2.524</td>
<td>.704</td>
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<td>---</td>
</tr>
<tr>
<td>Direct race effect</td>
<td>---</td>
<td>.788</td>
<td>---</td>
<td>2.105</td>
<td>.708</td>
</tr>
<tr>
<td>Total race effect</td>
<td>-6.09</td>
<td>-1.736</td>
<td>2.698</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>SES effect (b_{sym})</td>
<td>---</td>
<td>-1.859</td>
<td>---</td>
<td>1.46</td>
<td>.203</td>
</tr>
</tbody>
</table>

Note. Where parameters involve products of coefficients, CIs are generated by Monte Carlo using parameter estimates and their asymptotic covariance matrix with 10,000 draws.
Figures 1a

A conceptual diagram of a moderation model.

Figure 1b

A path diagram of a moderation model, wherein covariances among predictors are accounted for when deriving coefficients rather than explicitly estimated.

Figure 2

Path diagram of a mediation model.
Figures 3a

A conceptual diagram of our single-level moderated mediation model.

Figure 3b

A path diagram of our single-level moderated mediation model, wherein predictor covariances are explicitly estimated, including a covariance among m and mw. Path coefficients are labeled as in our equations, with ‘a’ terms indicating initial paths in a mediation/indirect effects equation, ‘b’ terms indicating second paths in a mediation/indirect effects equation, and ‘c’ paths indicating direct effects.
Figure 4

A path diagram of our multilevel moderated mediation model
Figure 5

A path diagram of our multilevel mediation model with all latent variables as plausible values (i.e., multiply imputed), which we represent as octagons.
Author/s:
Zyphur, M; Zhang, Z; Preacher, KJ; Bird, LJ

Title:
Moderated mediation in multilevel structural equation models: Decomposing effects of race on math achievement within versus between high Schools in the United States

Date:
2019

Citation:

Persistent Link:
http://hdl.handle.net/11343/221192

File Description:
Accepted version