Selected Topics on Massive MIMO Networks

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Abstract

Massive Multiple-Input Multiple-Output (MIMO) technology promises to deliver substantial gains in spectral efficiency and is set to play an important role in next generation mobile networks. This thesis examines the key challenges of pilot contamination, admission control and channel estimation in massive MIMO networks, and derives contributions which are focused on the new paradigm of the large antenna array at the base station (BS).

Firstly, motivated by current literature which holds pilot contamination as the dominant impairment in massive MIMO networks, we investigate the uplink signal-to-interference ratio (SIR) of two common receivers, the maximum ratio combining (MRC) and zero-forcer (ZF), under more general pilot allocation schemes. Our analysis reveals that the limiting performances of the MRC and ZF under general forms of pilot allocation are in fact quite different. By employing a more general pilot allocation where every BS uses a different orthogonal set of pilot sequences, we uncover the further counter-intuitive large antenna behaviour of the ZF detector being limited by forms of intra-cell interference as a result of pilot contamination.

Secondly, the next generation of networks will need to service an unprecedented leap in the number of devices requiring cellular access, particularly from the influx in smart and machine-to-machine devices operating under the Internet of Things (IoT) paradigm. Motivated by the need to service this increase in connected devices, we investigate the user-load capacity of cellular massive MIMO networks. We compare the pilot allocation scheme widely adopted in massive MIMO literature where all BSs reuse the same pilot orthogonal set, to a more general pilot set allocation, where all BSs use different orthogonal pilot sets. We show that the later is superior in the context of user-load capacity by allowing more connected users on the network for a given quality of service (QoS). Employing the different pilot set allocation, under system models that account for log-normal shadowing and best cell selection, we define efficient and simple user
admission regions that are able serve as the basis of a practical network user admission policy.

Finally, the density of cellular deployments is constantly increasing to deal with increases in user demand. Additionally, 5G networks will also make use of the large amount of available spectrum in the millimetre wave band to meet these increases in demand. These measures will ultimately result in wireless channels that are more likely to exhibit stronger line-of-sight (LoS) components. Consequently, we are motivated to investigate the estimation of wireless channels with LoS components using methods which can exploit the large antenna array at the massive MIMO BS. We design a novel assisted linear minimum mean square error (A-LMMSE) estimator which uses a-priori angle-of-arrival (AoA) information available under channels with LoS components in order to exploit the large antenna array at the BS. We show that the A-LMMSE estimator has a channel estimation performance close to that of the optimal minimum mean square error (MMSE) estimator under pure LoS channels, while only introducing a small amount of additional complexity to the highly efficient linear minimum mean square error (LMMSE) estimator. We present low complexity variants of the A-LMMSE estimator, and obtain bounds on the channel estimation performance in the limit of the number of the BS antennas under pure LoS and Rician channels. The superior performance of the A-LMMSE estimator is demonstrated through analysis and simulation of the resulting normalised mean square error (NMSE) and symbol error rate (SER) under single and multi-cell scenarios.
Declaration

- This thesis comprises only of my original work towards the PhD.
- Due acknowledgement has been made in the text to all other material used.
- This thesis is fewer than the maximum word limit of 100,000 words in length, exclusive of tables, bibliographies and appendices.

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Preface

This thesis describes the main areas of work that were completed during my PhD candidature. The original contributions of this thesis are in Chapters 2 to 5. I completed the work presented in these chapters, inclusive of problem formulation, mathematical analysis, implementation of the simulation environments, generation and analysis of the results. Prof. Jamie Evans has supervised the work presented in this thesis. Co-supervisor Dr. Rajitha Senanayake has provided feedback and comments for the work in Chapter 2, and co-supervisor Dr. Hazer Inaltekin has provided feedback and comments for the work in Chapter 4 and Chapter 5.

The work in Chapter 3.3 is adapted from the work published in the proceedings of the 2016 IEEE Global Communications Conference, which was co-authored with Professor Jamie Evans. The work in Chapter 4 was presented, and the recipient of the Best Student Poster Award at the Australian Communications Theory Workshop in February 2018 in Newcastle, Australia.

I would like to acknowledge the funding provided by the Australian Postgraduate Award and the additional funds provided by Monash University, where I completed the first year of my candidature. In addition, I acknowledge the funding provided by the studentship received over the remaining three years at the University of Melbourne. Furthermore, I would like to acknowledge the funding provided by the University of Melbourne to present my work at the IEEE Global Communications Conference in Washington DC, USA in 2016, and the Australian Communications Theory Workshop in 2018 in Newcastle, Australia.
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A final and special thankyou to Celia Konstas for her unconditional compassion, love and selflessness through some of the tough times of my candidature. The person I could always count on. I am forever grateful.
# Contents

List of Abbreviations ........................................... xxiii
List of Symbols .................................................. xxiii

## 1 Introduction .................................................. 1
  1.1 Motivation .................................................. 5
    1.1.1 Chapter 2 ............................................. 5
    1.1.2 Chapter 3 ............................................. 6
    1.1.3 Chapters 4 and 5 ..................................... 8
  1.2 Summary of Chapters and Contributions .................. 9
  1.3 List of Publications ...................................... 11

## 2 Performance Limits of Linear Detectors .................. 13
  2.1 Introduction ............................................. 13
  2.2 System Model ............................................. 17
    2.2.1 Pilot allocation .................................... 18
    2.2.2 The Path Loss Model ................................ 18
    2.2.3 Channel Estimation .................................. 19
      2.2.3.1 Reused Pilot Sets ............................ 20
      2.2.3.2 Different Pilot Sets ......................... 20
    2.2.4 Data Detection ...................................... 21
  2.3 The Large $M$ SIR expressions ......................... 22
    2.3.1 Maximal Ratio Combining ............................ 23
      2.3.1.1 Reused Pilot Sets ............................ 23
      2.3.1.2 Different Pilot Sets ......................... 24
    2.3.2 Zero-Forcing ....................................... 24
      2.3.2.1 Reused Pilot Sets ............................ 25
      2.3.2.2 Different pilot sets ......................... 26
  2.4 Comparison of the Large $M$ SIR expressions .......... 27
  2.5 Uplink Power Control ................................... 29
  2.6 Numerical Results ...................................... 29
  2.7 Conclusions and Future Work ............................ 34
4.4 Channel Estimation under the Rician Channel model .......................... 109
  4.4.1 Minimum Mean Squared Estimation .................................... 110
  4.4.2 Linear Minimum Mean Square Estimation ............................... 111
    4.4.2.1 Large M Analysis ................................................. 113
  4.4.3 Assisted Linear Minimum Mean Squared Estimation .................... 114
    4.4.3.1 Large M Analysis ............................................... 116
4.5 The Complex Rician Channel model ........................................... 117
  4.5.1 The LMMSE Estimator .................................................. 118
  4.5.2 The A-LMMSE Estimator ............................................... 119
4.6 Conclusions and Future Work .................................................. 121

5 Channel Estimation in Multi-user Massive MIMO ............................... 123
  5.1 Introduction ............................................................. 123
  5.2 Single Cell MU Massive MIMO ............................................. 126
    5.2.1 Orthogonal Pilots .................................................. 126
    5.2.2 Discussion : Extensions for Non-orthogonal Pilots ................. 132
  5.3 Multi Cell MU Massive MIMO .............................................. 133
    5.3.1 System Model ....................................................... 134
      5.3.1.1 A Distance Based Pathloss Model .............................. 137
    5.3.2 LMMSE Estimation ................................................... 138
      5.3.2.1 LMMSE Estimator ............................................... 138
      5.3.2.2 Cooperative LMMSE Estimator ................................. 138
    5.3.3 A-LMMSE Estimation ................................................ 139
      5.3.3.1 A-LMMSE Estimator .............................................. 139
      5.3.3.2 Cooperative A-LMMSE Estimator ............................... 140
    5.3.4 Numerical Results and Discussion .................................. 141
      5.3.4.1 Worst Case Inter-cell Interference ............................ 143
      5.3.4.2 K-factors of the Interfering User Channel .................... 144
    5.3.5 Conclusions and Future Work ...................................... 147

6 Conclusion ................................................................................. 151
  6.1 Future Work ................................................................. 152
    6.1.1 Two-Dimensional Antenna Arrays .................................... 152
    6.1.2 Downlink Massive MIMO .............................................. 153
    6.1.3 Low resolution ADCs .................................................. 153

A Appendix A .................................................................................. 155
  A.1 Uplink SIR approximation (3.22) ......................................... 155

B Appendix B .................................................................................. 159
  B.1 Single User LMMSE Estimator .............................................. 159
  B.2 MMSE Estimator for M antennas in LoS ................................ 161
  B.3 MMSE Estimator for M antennas in Rician fading .................... 163
  B.4 LMMSE Estimator Large M NMSE ........................................ 166
  B.5 A-LMMSE Estimator Large M NMSE ..................................... 169
  B.6 A-LMMSE Static Estimator Large M NMSE ............................. 171
List of Figures

1.1 Urban Macro-cell Massive MIMO BS serving a variety of users. 3
1.2 Simplified TDD slot. 4
2.1 Uplink Pilot Contamination - Pilot reception from user of interest in connected to BS $j$ is contaminated with the pilot transmission from interfering user connected to neighbouring BS $l$. 16
2.2 The finite $M$ and large $M$ SIR CDFs for the MRC and ZF detectors under the reused and different pilot set allocation schemes, under the simulation parameters in Table 2.1. 31
2.3 CDF of the SIR difference under different set scheme, under the simulation parameters in Table 2.1 with scenario dependent values for $K$. 33
2.4 SIR CDF for MRC and ZF detectors under Scenario II. 33
3.1 Tier 1 interferers (dark blue) and tier 2 interferers (light blue) to the BS of interest, with intra-cell tier 0 interferers (red), under different frequency reuse factors $w \in \{1, 3, 7\}$. 40
3.2 Circular cell approximation with user in cell $l$ interfering with the uplink reception at BS $j$. 44
3.3 The SIR CDF under the reused and different pilot set allocation scheme under the simulation parameters in Table 3.1, as $M \to \infty$. 46
3.4 Maximum number of users per cell, $k_{\text{max}}$ vs. QoS, under the simulation parameters in Table 3.1. 50
3.5 Maximum number of users per cell, $k_{\text{max}}$ and unregulated maximum number of users $k_{u}$ vs. SIR under fixed outage probability $\alpha = 0.05$ and simulation parameters in Table 3.1. 51
3.6 SIR CDF for $a \in [0, 0.1]$ under $w = 7$, i.e., worst-case Gaussian approximation of the SIR compared to SIR evaluated by Monte Carlo simulation, under large $M$ and simulation parameters in Table 3.1. 52
3.7 Comparison of SIR expressions, under (worst-case) fully loaded network under the different pilot scheme, under the simulation parameters in Table 3.3. 60
3.8 The BS admission regions and approximations under the simulation parameters in Table 3.3. 62
3.9 Example BS-dimensioned network admission region for simple 3 cell network example, shown in red, $K = 42$, $T = 1$, $L = 3$. 65
3.10 Tier-dimensioned BS admission regions for a particular BS for various QoS requirements, under the simulation parameters in Table 3.3, where $M = 500$. The admission region is the grey hashed area, defined by the number of admissible tier 1 and 2 interferers, $n_1$ and $n_2$, with fixed $n_0 = \tau / 2$. The orange squares are the regions that result from the uplink training constraint (3.2) under $w \in \{ 1, 3, 7 \}$.

4.1 A single user wavefront impinging on a uniform linear array (ULA) at the BS with azimuth AoA $\theta$. 81

4.2 The resulting NMSE for increasing antenna lengths, where $M$ is the number of equally spaced antenna elements, $\eta = -3dB$. 84

4.3 The resulting NMSE(dB) for the estimation of each individual channel $h_m$ for $M = 64$, where $NMSE_h_m = \frac{E\{ |h_m - \hat{h}_m|^2 \}}{E\{ |h_m|^2 \}}$ and $\eta = -3dB$. For the LMMSE estimator, the resulting NMSE from the analytical expression $\overline{MSE}$, i.e., the diagonal elements of the trace argument in (4.27), is also plotted. 86

4.4 The resulting NMSE from a genie-assisted LMMSE estimator with covariance terms (4.30), as the range of $\theta^{(p)}$ is reduced, $M = 64$, $\eta = -3dB$. 92

4.5 The resulting NMSE from a genie-assisted LMMSE estimator with covariance terms (4.35), as the range of $f_s^{(p)}$ is reduced, $M = 64$, $\eta = -3dB$. 93

4.6 Block Diagram of the A-LMMSE estimator. The Assistant provides imperfect a-priori AoA information, in the form of a spatial frequency estimate and estimated error variance, to the LMMSE based estimator. 94

4.7 The resulting NMSE from a genie-assisted LMMSE estimator with covariance terms (4.35), for different $\mu_f$ with increasing $M$, $R_f = 1/20$, $\eta = 0dB$. 95

4.8 The large $M$ NMSE expression in (4.36) for the genie-assisted LMMSE alongside finite $M$ simulation results with increasing $M$. Several values of $\overline{R}$ are shown, for a fixed $f_s = 0.25$ and $\eta = 0dB$. 97

4.9 The NMSE for the LMMSE and A-LMMSE and A-LMMSE static estimator, for $\eta = -3dB$, $\theta \sim \mathcal{U}[0, \pi / 2]$. 107

4.10 The SER for BSPK data decoding using perfect estimates compared to estimates from the LMMSE and A-LMMSE and A-LMMSE Static estimators, for $\eta = -3dB$, $\theta \sim \mathcal{U}[0, \pi / 2]$. 108

4.11 The NMSE for the Rician Channel model ($K_f = 3$) from Monte-Carlo simulation for the MMSE, LMMSE and variations of A-LMMSE, $\eta = -3dB$. 112

4.12 The NMSE for the Rician Channel model for LMMSE and variations of A-LMMSE as $K_f$ varies, $M = 64$, $\eta = -3dB$. 116

4.13 The NMSE for the single cell scenario for LMMSE and variations of A-LMMSE under the complex Rician channel model, $K_f = 3$, $\eta = -3dB$. 120

5.1 The QPSK SER in the single cell scenario, $K = 24$, $K_f = 3$, $\eta = -3dB$. 129

5.2 The beamforming patterns under the single cell scenario with orthogonal pilots for all users $K = 24$, $M = 64$, $K_f = 3$, $\eta = -3dB$. 130

5.3 Beamforming pattern, two users with LoS channels, with desired user AoA $\theta_k = \pi / 6$, and interfering user AoA $\theta_k' = 5\pi / 6$ using non-orthogonal pilots $M = 32$, $\eta = -3dB$. 134
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>The multi-cell scenario which models a desired user (red dot), and 3 neighbouring BSs, each with a single pilot contaminator (blue dots).</td>
<td>135</td>
</tr>
<tr>
<td>5.5</td>
<td>The NMSE and SER of the LMMSE and A-LMMSE under various propagation conditions, $K = 24, \eta = -3dB$, under the simulation parameters in Table 5.1.</td>
<td>142</td>
</tr>
<tr>
<td>5.6</td>
<td>The NMSE and SER of the LMMSE and A-LMMSE when all users are positioned at cell boundaries, $\sqrt{\beta_k} = \sqrt{\beta_{k'}} = 1, \eta = -3dB$, under the simulation parameters in Table 5.1.</td>
<td>145</td>
</tr>
<tr>
<td>5.7</td>
<td>Two user case, with resulting SER and channel inner product magnitude, $\frac{1}{M}</td>
<td>h_k^d h_{k'}^d</td>
</tr>
<tr>
<td>5.8</td>
<td>The SER under network conditions where interfering users are separated spatially from desired user $</td>
<td>\theta_k - \theta_{k'}</td>
</tr>
</tbody>
</table>
List of Abbreviations

AoA    Angle of Arrival
ADC    Analog-to-Digital Converter
AWGN   Additive White Gaussian Noise
BEP    Bit Error Probability
BPSK   Binary Phase Shift Keying
BS     Base Station
CDF    Cumulative Distribution Function
CRC    Cyclic Redundancy Check
CSI    Channel State Information
CTFT   Continuous Time Fourier Transform
DFT    Discrete Fourier Transform
DTFT   Discrete Time Fourier Transform
FDD    Frequency Division Duplex
FNBW   First Null Beamwidth
i.i.d.  Independently and Identically Distributed
LMMSE  Linear Minimum Mean Squared Error
LoS Line-Of-Sight
LTE Long Term Evolution
MAP Maximum A-Posteriori Probability
MGF Moment Generating Function
ML Maximum Likelihood
MPSK M-ary Phase Shift Keying
MRC Maximal Ratio Combining
MIMO Multiple Input Multiple Output
MMSE Minimum Mean Squared Error
MSE Mean Squared Error
MU Multi-user
nLoS Non line-of-sight
NMSE Normalised Mean Squared Error
M-QAM M-ary Quadrature Amplitude Modulation
PDF Probability Density Function
QoS Quality of Service
QPSK Quadrature Phase Shift Keying
SEP Symbol Error Probability
SER Symbol Error Rate
SINR Signal-to-Interference-Plus-Noise Ratio
SIR Signal-to-Interference Ratio
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
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<tr>
<td>TDD</td>
<td>Time Division Duplex</td>
</tr>
<tr>
<td>ZF</td>
<td>Zero Forcing</td>
</tr>
</tbody>
</table>
List of Symbols

$(\cdot)^H$  Hermitian transpose

$(\cdot)^T$  Transpose

$(\cdot)^*$  Conjugatation

$\mathbb{P}(A)$  Probability of event $A$

$E\{\cdot\}$  Expectation operator

$\mathcal{O}(\cdot)$  Big O runtime complexity

$\mathcal{F}\{\cdot\}$  Continuous Time Fourier Transform

$\mathcal{F}_{2\pi}\{\cdot\}$  Discrete Time Fourier Transform

$f_X(x)$  Probability Distribution Function of random variable $X$

$F_X(x)$  Cumulative Distribution Function of random variable $X$

$\sim \mathcal{T}$  Distributed according to $T$
Chapter 1
Introduction

Each generation in modern day mobile networks strives to meet the increases in wireless traffic demand from expanding cellular access in combination with the technological advances of human controlled devices. While the imminent fifth generation (5G) networks that are set to be introduced as early as 2019 are no exception to this, they require yet a further leap in capacity to serve the new surge in deployment of machine-to-machine devices. Further to this, 5G networks will offer an improved end user latency as low as 0.5 milliseconds [1] enabling a range of completely new real time applications that will further increase demand. Given that the frequency spectrum used for radio communications is an expensive, finite resource, increasing the spectrum efficiency, i.e., the data rate achievable per unit of bandwidth (bits/second/Hz), will always be a key driver to the evolution of wireless networks. Massive MIMO is a technology which is set to play a significant role in serving this increase in data rate demand in tomorrow’s 5G networks by providing significant gains in spectral efficiency.

While definitions vary amongst the literature, common to all definitions, Massive MIMO is the adoption of significantly more antenna elements at the base-station (BS) than the maximum of 8 currently employed (based on current standards such as 4G Release 10 of LTE-Advanced [2]). For several years now, a handful of massive MIMO test beds have been in existence, where installations involving 128 antennas are able to demonstrate spectral efficiency gains up to 50 times compared to that of Release 10 of LTE-Advanced [3, 4]. Meanwhile, early commercial BS solutions between telecommunication carrier companies and device manufactures have already demonstrated huge increases in spectral efficiency and achievable peak data rate [5]. These leaps in spectral efficiency are achievable through the adoption of multi-user (MU) schemes in massive MIMO, where all users communicate over the same time and frequency resources. Such
schemes function by exploiting spatial multiplexing, i.e., where the space dimension is used to separate the individual user data streams. Increasing the length of the antenna array increases its angular resolution, and under suitable channel conditions this results in more degrees of freedom in the system, consequently allowing a higher degree of spatial multiplexing [6]. As such, this form of multiplexing can be well utilised by the large antenna array at the massive MIMO BS, enabling the efficient communication with individual users despite the use of shared frequency and time resources amongst network users. Figure 1.1 depicts a macrocell BS exploiting spatial multiplexing during precoding and data detection in order to direct communications to a variety of devices, which include traditional mobile users along with newer automotive and device-to-device networks. What often, but not necessarily distinguishes massive MIMO from conventional MIMO systems is an operating scenario where the BS has an order(s) of magnitude more antenna elements than users i.e., $M \gg K$ [7, 8]. Such conditions increase the likelihood of supporting more aggressive spatial multiplexing and consequently higher spectral efficiencies [9].

The benefits of massive MIMO technology can be realised with a varying number of array geometries, several of which have been demonstrated in practice. In a practical setting the physical dimensions of the array are of concern and therefore elements arranged in rectangular grids or in a circular fashion as depicted in Figure 1.1 are favourable. Furthermore, such arrangements permit more enhanced methods of spatial multiplexing. Given the potential size and increased processing required by the large antenna arrays, such technology in 5G is initially aimed for deployment in urban macro BSs, and consequently we focus on this case. While these are the assumed conditions in this thesis, most aspects of this work can be readily applied to smaller scale environments, e.g., those utilising microcells or picocells.

Modern day communication systems primarily use coherent decoders, i.e., the data transmissions are decoded using knowledge of the wireless channel which separates the transmitter and receiver. In practice the channel is unknown and is estimated through the transmission of known pilot sequences. The estimation of the channel is the main focus of Chapters 4 and 5. While such communication between transmitter and receiver ultimately takes places over frequencies in the passband, this thesis focuses on baseband processing in such communication systems, where functions such as symbol detection and channel estimation are performed. Therefore the concepts and discussions in this thesis are agnostic to the bands of frequency spectrum used.
Given the large number of antenna elements in massive MIMO, there are significantly more channels between the terminal and BS to estimate compared to conventional MIMO systems. As a result of this overhead, time division duplex (TDD) systems are far more popular than frequency-division duplex in massive MIMO, given that the former is able to exploit the reciprocity of the wireless channel in the uplink and downlink. That is, the channel estimates formed at the BS can be used to aid the reception of data from the user, and the transmission of data to the user. Therefore, in contrast to a frequency division duplex (FDD) system, the resources and processing required for estimating the channels scales linearly with the number of connected users $K$, and is independent on the number of BS antennas $M$. Such features reduce the system overhead significantly given the large number of antenna elements in use at the BS under massive MIMO. In a further effort to reducing channel estimation overhead, the literature in massive MIMO often makes use of simple single antenna terminals, which we also employ in this work while pointing out that the technology does not prevent the use of more antennas at the terminal. Given that the majority of changes that come with massive MIMO are required at the BS where the function of channel estimation is performed under TDD, the work in this thesis focuses purely on the uplink direction, i.e., the transmission direction from the user to the BS.

An estimate of the channel is only able to effectively assist with reception and transmission over the resources restricted by the coherence time and coherence bandwidth of the channel, i.e., over the time and frequencies with which the channel does not change significantly (also known as the coherence interval). We therefore consider a simple narrowband block fading model, where the coherence time is divided amongst $\tau_{coh}$ symbols within our simplified TDD slot, as shown in
Figure 1.2. While the model in Figure 1.2 and consequently the work in this thesis is not specific to a modulation scheme or wireless standard, where relevant, some sections of the thesis adopt parameters based on the LTE uplink for numerical results.

TDD systems rely on some level of time synchronisation between the network BSs and terminals. We assume a fully synchronised MU-Massive MIMO TDD system, where all users of the network transmit their pilot and data symbols simultaneously. Figure 1.2 describes the purpose of the symbols communicated within the slot, and their related processing at the BS (in brackets). The uplink processing involves using the channel estimates derived from the processing of the first $\tau$ symbols, in order to perform the detection of the data symbols transmitted by the desired users. The channel estimates are also used to generate precoding matrices to enable the transmissions of data in the remaining $\tau_{dl}$ symbols to the desired users. Given the simultaneous transmission of the pilot and data symbols by all users of the network, the network has to be able to separate the pilot and subsequent data transmissions of its users. In the MU Massive MIMO TDD scenario, distinguishing pilot transmissions is a challenge given pilot transmissions of neighbouring users can significantly contaminate the pilot signals received at the BS for a desired user, for example when neighbouring users transmit the same pilot sequence as the desired user. This problem, known as pilot contamination, naturally results in the poorer detection of the desired users’ data symbols, and is considered a dominant impairment under massive MIMO networks. The detection of the desired user’s data transmission in the presence of the interfering users’ transmissions is effectively the function of spatial de-multiplexing, and therefore the user data detection (also known as receive combining) can be seen as the dual of the transmit precoding operation. Under channels where there is a dominant line-of-sight (LoS) component, the result of the precoding (and user data detection at the BS) is the forming of a beam in the direction of the desired user. Under channels which are modelled as Rayleigh fading channels, i.e., the superposition of many multipath compo-
nents, the result of the precoding is to generate waveforms transmitted by all antenna elements that sum constructively at the users location. Therefore, as also highlighted in [31], we adopt the terms receive data detection and transmit precoding, as opposed to receive and transmit beamforming, so that they can be used independently of the underlying wireless channel. Under both channels, the effectiveness of the user precoding and data detection, i.e., the amount of energy that can be effectively focused on or from a particular user, increases with the number of antennas, and hence advantageous for the large array at the Massive MIMO BS. When $M$ is large, forms of linear precoding and detection at the BS are nearly optimal [9] and consequently very attractive due to their reduced complexity. Consequently, we make use of two popular linear detectors in this thesis, the maximum ratio combiner (MRC) and zero-forcer (ZF), which are studied and compared in detail in Chapter 2.

## 1.1 Motivation

### 1.1.1 Chapter 2

It has been shown in the seminal paper [7] that as the number of base station antennas $M$ becomes large under i.i.d. Rayleigh channels and MRC detection, the effects of noise and uncorrelated intra and inter-cell interference disappear, and only inter-cell interference due to pilot contamination remains, i.e., the interference resulting from channel estimates which are contaminated from pilot transmissions of users in neighbouring cells. This result holding pilot contamination responsible for a finite capacity limit has attracted much attention in the research community. The works of [8, 10, 11] investigate the limits of uplink performance under i.i.d. Rayleigh channels while employing more sophisticated types of minimum mean squared error (MMSE) detectors. Even more sophisticated multi-cell (M-MMSE) detectors are employed in [12–14]. While the multi-cell detectors are more effective at suppressing inter-cell interference and result in significant performance gains even under variants that do not require BS cooperation, the results from the above-mentioned works demonstrate a finite spectral efficiency limit as $M$ grows to infinity. In contrast, however, it has been shown that while pilot contamination is a dominant impairment, the limits on spectral efficiency are largely a result from assuming unrealistic fully spatially uncorrelated channels, and in fact the spectral efficiency grows with $M$ unboundedly when some degree
of spatial correlation is considered [15].

The effects of pilot contamination are naturally also a function of how such pilots are allocated amongst the BSs of the network, and consequently there are several pilot allocation strategies that attempt to mitigate the effects of pilot contamination. Under spatially uncorrelated channels, more traditional approaches examine the levels of pilot reuse amongst cells [16], or through fractional pilot re-use which effectively avoids reusing pilot sequences amongst neighbouring cell edge users [17]. Under spatially correlated channels, there are several cooperative approaches which are able to utilise spatial information of the desired and interfering users to more effectively assign pilot sequences to try minimise the effects of pilot contamination [18–20].

Given that pilot contamination is a dominant impairment on the UL performance in massive MIMO networks, the work in Chapter 2 is motivated to investigate the effects of pilot contamination from two contrasting forms of network pilot allocations, and the resulting UL data detection performance of two widely used low complexity detectors.

1.1.2 Chapter 3

It is evident that multi-user massive MIMO technology deployed on future cellular networks will bring significant gains to network sum rate data capacity. Along side the usual increase in the amount of high data rate mobile users, we expect an explosion in the number of wireless sensor networks or smart devices requiring gateways to the internet. Under the “Internet of Things” paradigm the number of connected devices is estimated to be more than 25 billion by 2020 [21], and it is clear we additionally need to focus on providing sufficient user-load capacity. Under such a paradigm, the influx of such devices that will fall outside the traditional pedestrian or fixed broadband access generate yet another class of network user, with different reliability, latency and data rate requirements. These requirements in part can be captured as a quality-of-service (QoS) metric, where each class of user will have a different required QoS. Given our work is focused on the physical layer of massive MIMO networks, we consider a single class of user, with a target QoS that is defined by a minimum SIR ratio and outage probability. This is in contrast to the average SINR QoS used in related works such as [22], which is applicable given their limited focus on downlink single cell user-load capacity.
The primary function of an admission control policy is to ensure the relevant QoS for all users, upon admission of a user to the network via a given BS. The deployment of small cells along with macro cells in ultra dense networks will not only increase the complexity of evaluating the admission control policy, but also increase the frequency at which it needs to be evaluated. Further, in the context of upcoming 5G networks, it will also put more stringent requirements on the speed of its evaluation during hand-off given the aggressive sub millisecond user latency targets [27]. The rise of dense networks has also further motivated the investigation into better maximising the sum network user-load capacity by jointly considering load balancing and admission control algorithms [28].

Therefore a user admission policy must be able to efficiently evaluate the current state of the network, the burden introduced by the new user, and the boundaries where the QoS for the other users are compromised, known as the admission region. Using a QoS defined by a minimum SIR, an admission region could be defined by the region where all SIR requirements of the networks users are met, for example as employed by the downlink user-load capacity works in [22, 23]. However, evaluating whether the network falls inside such regions is generally more problematic in the uplink given the nature of the SIR resulting from random user locations. Therefore, we are motivated to define simple uplink admission regions which can be efficiently evaluated by an uplink admission policy.

Given our QoS is defined by SIR and outage probability, and our operation in an interference limited regime with uplink power control (ULPC), we are essentially interested in the additional interference generated by admission of the new user. In the uplink, the amount of interference generated will be a function of the received power of the new user to all the BSs of the network, and due to pilot contamination, it will also be a function of the pilot sequence allocation. If we focus on a particular BS, the amount of interference experienced at that BS will intuitively be inversely proportional to the distance the new user is from the particular BS. We exploit the fact that some groups, or tiers of users which surround the BS under consideration will ultimately generate similar types of interference, and by employing statistical multiplexing, we facilitate a simple definition of the admission regions.

As mentioned, the effects of pilot contamination mean that the pilot sequences allocated to the BSs of the network will have effects on the generated interference. While nearly all of the works
which examine the uplink interference in massive MIMO assume the reuse of pilot sequences amongst BSs [8, 12, 15, 24–26], we are motivated by the results of Chapter 2 to also examine the user capacity under the alternate allocation where each BS employs an independent random pilot sequence set.

1.1.3 Chapters 4 and 5

In addition to employing massive MIMO technology, future networks will start to make use of the large amounts of frequency spectrum available in the millimetre wave band, i.e., above 30GHz, to meet demands on network capacity. This spectrum has largely remained unused due to several unfavourable physical properties, namely the increased free space path loss due to the increase in carrier frequency, and a vulnerability to shadowing, resulting in high outage probabilities under non-LoS channels [30]. While the former can be compensated for by the array gain offered by large antenna arrays under massive MIMO, the later poses harder constraints on having to ensure channels with LoS components to support reliable millimetre wave communications [29, 30].

At the BS, the LoS (or otherwise known as the specular) components of the channel manifest as a form of spatial correlation across the array i.e., in some correlation between the channels observed at each antenna element of the array. This is in contrast to the widely adopted Rayleigh channel model which models the highly scattered environment by completely independent channels, resulting in zero correlation between the channels observed at the antenna elements of the array. However, even in bands commonly used today, in general fully spatially uncorrelated channels only exist under unrealistic assumptions [15, 36]. Furthermore, the deployment of small cells and increased BS density in next generation networks will lead to shorter transmission distances, resulting in channels which exhibit higher LoS probabilities and stronger LoS components [35]. Therefore, it is clear that advances in tomorrow’s networks which move toward millimetre wave and higher cell densities heavily motivate the study of communication systems under channels with spatial correlation.

As we have already highlighted, the process of channel estimation is a critical part of the baseband processing, and its performance directly effects the reception and transmission of user data. As such, we are motivated in Chapter 4 and 5 to investigate channel estimation techniques that are able to exploit the large antenna array of the massive MIMO BS in the presence of wireless
channels with LoS components.

The related literature on channel estimation techniques often adopt channel models which account for the presence of several LoS paths, and consider the multi-cell setting, often in combination with pilot allocation strategies [18, 19, 56]. We however are first motivated in Chapter 4 to closely examine the fundamental problem of channel estimation in the single user scenario under a pure LoS channel. By doing so, we are able to derive the theoretical optimal estimator and uncover the potential performance gains that can be realised, thus motivating the development of alternate forms of channel estimation, and allowing such schemes to be compared to the optimal and other sub-optimal methods. This is in contrast to results presented in the related literature, which make comparisons of proposed estimation schemes to sub-optimal forms of estimation. Given the promising results of the estimator developed in Chapter 4 we are motivated to study its performance under more realistic Rician channel models, and in the context of multi-cell scenarios in Chapter 5.

1.2 Summary of Chapters and Contributions

In this section we briefly summarise each of the chapters, and their main contributions.

Chapter 2

This chapter introduces important elements of the baseband channel model and network features which are used throughout the thesis. This chapter derives large $M$ SIR expressions for both the maximum receive combining (MRC) and zero forcing (ZF) detectors, under the conventional allocation of pilot sequences where each BS reuses the same pilot set, and a more general scheme where each BS employs a different set. The main contribution of this work is the derivation of the limiting results for the MRC and ZF detectors under a more general form of pilot allocation which uncovers the fact that these detectors have quite different uplink SIRs under such conditions. Further to this, we make the counter-intuitive observation that the large $M$ SIR of the ZF detector is also limited by the intra-cell interference as a result of the pilot contamination under more general pilot allocation schemes.
Chapter 3
This chapter explores user-load capacity under a large number of BS antennas $M$ and presents practical approaches to defining a user admission policy. It makes use of the general pilot allocation scheme and the associated limiting results from Chapter 2 to investigate user-load capacity. A main contribution of this work demonstrates that assigning different orthogonal pilot sets to each BS of the network is superior to reusing the same pilot sets with respect to increasing user-load capacity, where the aim is to guarantee a certain QoS for all users. The main contribution of this chapter is the derivation of the user admission regions for a network employing massive MIMO BSs, with a practically large and finite number of BS antennas. Further to this, it considers the realistic effects of log-normal shadowing and best cell selection. The chapter analyses the uplink SIR under such conditions, and develops an approximation to the SIR which can be used to define the BS-dimensioned user admission region, i.e., the number of users permitted by BSs of the network in order to ensure a certain QoS for all users. Relevant approximations to such regions are made to yield constraints which can be very efficiently evaluated and implemented by cooperative admission control policies in practice.

The work in this chapter was largely presented in conference paper, A. Sivamalai and J. S. Evans, "On Uplink User Capacity for Massive MIMO Cellular Networks," 2016 IEEE Global Communications Conference (GLOBECOM), Washington, DC, 2016, pp. 1-7.

Chapter 4
The main contribution of this chapter is the development of a novel and practical method for estimating the channel vectors of the massive MIMO BS antenna array. The chapter first explores the channel estimation problem for a single user under a pure LoS channel model. A comparison is made amongst popular methods such as the LMMSE estimator to the theoretically optimal MMSE estimator, for which we identify a large gap in performance. We propose a new assisted LMMSE (A-LMMSE) estimator, which delivers performance close to the optimal MMSE estimator with a similar run-time complexity to the efficient LMMSE estimator. Furthermore, we propose and analyse the performance of practical low run-time complexity implementations of the A-LMMSE estimator. We then extend the channel model to a Rician fading channel model, and analyse the
performance of the above mentioned estimators. We rigorously derive the channel estimation performance limits of the A-LMMSE and LMMSE estimators under an infinite number of BS antenna elements. In particular, we derive the limiting performance of a low complexity variant of the A-LMMSE estimator as a simple expression of the SNR and available memory, which can be easily used to evaluate further performance measures at the BS when such an estimator is employed in the practical setting of a memory constrained device.

The work in this chapter was presented under "On Uplink Channel Estimation for Massive MIMO under Line-of-Sight channels" and the recipient of the Best Student Poster Award at the 2018 Australian Communications Theory Workshop (AusCTW), Newcastle, Australia. https://www.ausctw2018.org/news/prize-winners/

Chapter 5

The main contribution of this chapter is the evaluation of the A-LMMSE channel estimation scheme in multi-user and multi-cell scenarios, under Rician fading. Under the multi-cell scenario which introduces the further impediment of pilot contamination, the A-LMMSE and LMMSE estimators are extended as cooperative variants which utilise a minimal amount of additional a-priori knowledge. The chapter analyses the receive beamforming patterns to demonstrate the effectiveness of the A-LMMSE estimator in combating the effects of interference and pilot contamination. We present symbol error rate results for quadrature phase shift keying (QPSK) modulation under various inter-cell Rician K-factor fading scenarios, and demonstrate the performance gains when utilising the A-LMMSE channel estimation scheme.

1.3 List of Publications

Chapter 2
Performance Limits of Linear Detectors

Pilot contamination has been shown to be the dominant impairment in the performance limits of detectors in the Multi-user (MU) massive MIMO scenario. In this chapter we consider two popular detectors, the Maximal Ratio Combiner (MRC) and Zero Forcer (ZF) detector and investigate the effects of pilot contamination on the resulting symbol detection performance at the Base station (BS). Furthermore, we look at the contamination effects under two pilot allocation schemes, the reused set scheme and the different set scheme. We derive large antenna limits of the received signal-to-interference ratio (SIR) for both MRC and ZF detectors, under both pilot allocation schemes. It has been shown in the literature that the SIRs of MRC and ZF detectors converge to the same limit as the number of receive antennas increase. We demonstrate that this convergence is in fact a special case of a more general result, where the ZF and MRC can have significantly different limiting expressions. We show that while the limits converge for the reused set scheme, they are significantly different under the more general different set scheme. Interestingly, we make the counter-intuitive observation in which the large antenna SIR of the ZF detector is in part limited by the intra-cell interference due to pilot contamination under more general forms of pilot allocation. Based on numerical examples, we further analyse the scenarios in which the ZF SIR limit outperforms the MRC SIR limit and vice versa.

2.1 Introduction

The MU massive MIMO scenario modelled in this work results in all network users transmitting pilot and data symbols using the same frequency and time resources. Consequently, the received signal at the BS is a superposition of all user transmissions. The detector at the BS has the task to separate the desired user’s data symbols from the received symbols in the presence of AWGN. A coherent detector relies on having access to estimates of the wireless channel in order to perform this task.

It is widely understood that when applying Shannon’s capacity theorem to channels in the
presence of AWGN, the achievable data rate is proportional to the logarithm of the signal-to-interference-plus-noise ratio (SINR) at the detector. We therefore have a good insight into the data rate performance of the detector from its achievable SINR. In this chapter we define a SINR for the linear processing of the detectors analysed, and study their form when a large number of antenna elements are utilised at the BS.

The use of linear coherent detectors, i.e., where the detected symbol is derived from linear operations on the received data symbols, is preferable in practice due to their low complexity. Interestingly, the performance gain of optimal forms of non-linear detection over linear detection is small when there exists a modest ratio of BS antennas \( M \) to users per BS \( K \). For example, the simulation results in [31] show only a 5% increase in spectral efficiency from the use of optimal detectors under the scenario \( M/K = 4 \). As \( M \) heads to infinity, linear detection is optimal [9], and as such the large \( M \) SINRs of linear detectors have been studied in detail by a large number of works including [7–9, 24, 26].

There exist several approaches to linear detection under finite \( M \). The optimal linear detector which maximises the UL SINR (and consequently the spectral efficiency) is known as the multi-cell minimum mean squared error (M-MMSE) detector [31]. The M-MMSE achieves this optimality by minimising the mean squared error between the desired symbols and the detected symbols. However, it does so by utilising the channel estimates of all users connected to all BSs. Given the hefty overhead of sharing all channel state information in this co-operative scheme, lower complexity non-cooperative detectors are more attractive in practice. One approach would be to simply utilise averages over the inter-cell channel estimates instead of the actual estimates, where such variants are known as the single-cell minimum mean squared detector (S-MMSE) [31, 32]. However, the runtime complexity of such methods are still a concern due to the required costly \( M \times M \) matrix inverses under large finite \( M \).

In this chapter we consider two popular non-cooperative linear coherent detectors, the MRC and ZF, which are widely adopted in the massive MIMO literature due to their significantly lower complexity, when compared to detection schemes such as the M-MMSE and S-MMSE detectors. While their performance under finite \( M \) can be significantly poorer than the MMSE detectors [31], this chapter focuses on the asymptotic performance, where such estimators are optimal [9]. The MRC detector utilises an estimate of the desired user’s channel only. As the name suggests, it
uses the channel estimate of the desired user to try and maximise the received signal power of the
desired user’s transmission over all other users’ transmissions. The more sophisticated ZF utilises
channel estimates of both the desired and interfering users’ channels to detect the desired user’s
transmission. However, unlike the M-MMSE and S-MMSE detectors described above, it only
utilises channel state information of its own users. It uses these estimates to project the received
signal onto the subspace which is orthogonal to the subspace spanned by the intra-cell interfering
users’ channels.

When using coherent detectors, the quality of the channel estimate has a significant effect on
the detectors performance. A large impediment in obtaining the required channel estimates is the
problem of pilot contamination. Pilot contamination occurs when the BS receives indistinguish-
able uplink training pilot signals from the desired user and interfering users. This is depicted
in Figure 2.1, where BS $j$ is receiving pilots transmissions from its user and an interfering pilot
transmission from a user in adjacent cell $l$. Hence the estimated channel for the desired user is
compromised, and the subsequent detection of the desired user’s data transmission by ZF or MRC
detection suffers. Under spatially uncorrelated i.i.d. Rayleigh channels in the large $M$ regime, the
influential works of [7, 8, 33] have shown that detection performance was limited by the effects of
pilot contamination, and has therefore been of significant research interest since. The analytical
tractability that results from using spatially uncorrelated channel models has been a driving factor
for their wide spread adoption by the research community, and as such are also adopted here in this
chapter. Furthermore, the rich scattering environments that are modelled by spatially uncorrelated
channels that result in higher ranking channel matrices have been a well-known requisite for ca-
pacity gains in the single user MIMO setting [6, 34]. (On the other hand, in the multi-user scenario
investigated later in Chapter 5, we show how correlated channels facilitate the decomposition of
the channel and resulting covariance matrices into signal and unwanted interference components,
significantly improving the overall detection performance). However, it should be noted that fully
spatially uncorrelated channels only hold under unrealistic assumptions [15, 35, 36]. Some degree
of spatial correlation will exist intuitively from the fact that stronger signals will be received from
some certain directions in the physical environment, and from the fact that antenna elements of the
array will exhibit non-uniform radiation patterns [31]. Consequently, we also investigate massive
MIMO performances under channel models with spatial correlation in Chapters 4 and 5. Inter-
Interestingly, the recent work of [15] somewhat counters the main body of existing massive MIMO literature by showing that while pilot contamination is a dominant impairment, under channels which result in just a small amount of correlation amongst antenna elements, and with the use of more complex detectors, i.e., the M-MMSE detector, the performance is not limited by pilot contamination and capacity increases with $M$ without bound [15, Theorem 4].

However under all models, the presence of pilot contaminators is a major impediment on the UL detection performance. The adopted pilot allocation scheme will govern which users are considered pilot contaminators, and consequently impact the level of pilot contamination. Therefore, we are motivated to explore the linear detection performance under more general forms of pilot allocation.

The main contributions of the chapter can be summarised as follows,

- We derive large antenna limits of the received SIR for a general pilot allocation scheme where every BS employs a different orthogonal set, and utilises MRC and ZF linear detectors. While in the literature [8, 9, 31] it was shown that both ZF and MRC detectors share the same large antenna SIR, we show this is in fact a result of a special case of the pilot allocation, and under a more general analysis they are significantly different.

- Our analysis leads to a counter-intuitive observation in which the large antenna SIR of the ZF is also limited by the intra-cell interference as a result of pilot contamination under more general forms of pilot allocation.
2.2 System Model

We consider the uplink of a fully loaded cellular network that consists of \( L \) BSs, each with \( K = \tau \) single antenna users. Each BS is equipped with \( M \) co-located antennas. We employ a narrowband model over a coherence time of \( \tau_{coh} \) symbols, divided into \( \tau \) symbols for the transmission of the user pilot sequence, and \( \tau_{data} = 1 \) symbols for user data transmission. For such a network, we can express the baseband symbol matrix \( Y_j \in \mathbb{C}^{M \times \tau} \) received at BS \( j \) during the channel estimation phase as,

\[
Y_j = \sum_{l}^{L} G_{jl} S_l + N_j,
\]

(2.1)

where \( G_{jl} \in \mathbb{C}^{M \times K} \) contains the \( M \)-length channel vectors of the \( K \) users in BS \( l \) to BS \( j \), and \( S_l \in \mathbb{C}^{K \times \tau} \) is the pilot sequence set used by BS \( l \). The additive white Gaussian noise (AWGN) at the BS is represented by \( N_j \in \mathbb{C}^{M \times \tau} \), with all elements of the matrix being (i.i.d.) as \( \mathcal{CN}(0, 1) \).

Similarly, the \( \mathbb{C}^{M \times 1} \) vector received at BS \( j \) during the single symbol data detection phase is given by,

\[
y_j = \sum_{l}^{L} G_{jl} x_l + n_j,
\]

(2.2)

where \( x_l \in \mathbb{C}^{K \times 1} \) is the vector of single symbol transmissions for all users connected to BS \( l \), and \( n_j \in \mathbb{C}^{M \times 1} \) is the vector of complex Gaussian noise elements at BS \( j \). We can further decompose (2.2) as,

\[
y_j = g_{jkj} x_{kj} + \sum_{k' \neq k}^{K} g_{jk'j} x_{k'j} + \sum_{l \neq j}^{L} \sum_{k'}^{K} g_{jkl} x_{k'l} + n_j,
\]

(2.3)

where \( x_{kj} \) is the symbol transmitted by the \( k \)-th user connected to cell \( j \). The \( \mathbb{C}^{M \times 1} \) channel vector \( g_{jk'1} \) contains the channel values from the \( k' \)-th user connected to the \( l \)-th BS, to BS \( j \). The \( m \)-th element of \( g_{jk'1} \) is modelled as \( g_{mjkl} = h_{mjkl} \sqrt{\beta_{jk'l}} \), with \( h_{mjkl} \sim \mathcal{CN}(0, 1) \) and \( \beta_{jk'l} \) representing the small-scale and large-scale fading gain respectively, between the \( k' \)-th user connected to BS \( l \) and the \( m \)-th antenna at BS \( j \). Note that the large-scale fading gain \( \beta_{jk'l} \) is assumed common across all BS antennas.
2.2.1 Pilot allocation

In this chapter we adopt and investigate two practically feasible pilot allocation schemes, namely the reused pilot set scheme and the different pilot set scheme. Under the reused pilot set scheme, which is employed in nearly all massive MIMO literature, [7,8,18,26,31] amongst others, each BS reuses a set of orthogonal pilots. Under a fully loaded network, the pilot reception for each desired user is contaminated by a single interferer per BS of the network. Under the different pilot set scheme, which is also explored in the seminal paper [7], every BS employs its own independent, different set of orthogonal pilot sequences, meaning that every user connected to every other BS contaminates the pilot reception of a desired user. Although there are more pilot contaminators for a desired user, the level of contamination from a single pilot contaminator, is a fraction of the contamination from a single contaminator under the reused pilot set scheme.

The different set scheme employs an orthogonal pilot sequence set for each of the $L$ BSs, i.e. the entire set of pilot sequence sets used by the network $P_d = \{ S_1, S_2, \ldots, S_L \}$, with $S_l$ denoting the pilot sequence set used by BS $l$. As such, the re-used pilot set scheme represents a special case of the different set scheme where $S_n = S$, $\forall n$. In our different set scheme, the set $P_d$ is selected arbitrarily from the full set of orthogonal pilot sequences $P$, where $P_d \subset P$. A more efficient variation of this pilot allocation scheme would select $P_d$ so as to minimise the total interference on the network such as the scheme proposed in [37]. Given a pilot sequence length of $\tau$, a corresponding orthogonal pilot sequence set can contain a maximum of $\tau$ pilots sequences i.e., $S \in \mathbb{C}^{\tau \times \tau}$. Therefore, the number of orthogonal users that can be supported by the set $S$ is $K \leq \tau$.

For analytical convenience, we assume that the pilot sequences, i.e., the $\tau$ row vectors $s \in \mathbb{C}^{1 \times \tau}$ of $S$, are not just orthogonal, but orthonormal, i.e., $SS^H = I$, and $S$ is uniformly distributed on the set of unitary matrices (known as a Haar matrix [38, definition 2.5]), which is also adopted in [7].

2.2.2 The Path Loss Model

As part of our baseband system model, we assume a simplified path loss model from the user terminal transmitter to the BS antenna array, as outlined in [34]. We can express the received
2.2 System Model

multiplicative large-scale fading gain $\beta_{jkl}$ in Section 2.2, as the dimensionless quantity,

$$\sqrt{\beta_{jkl}} = \frac{P_{rx}}{P_{tx}} = z_{jkl} K \left( \frac{r_0}{r_{jkl}} \right)^\gamma,$$  \hspace{1cm} (2.4)

where $P_{rx}, P_{tx}$ are the receive and transmit power in watts, $K$ is a unitless constant that depends on several factors such as average channel attenuation and antenna characteristics. The quantity $r_0$ is the reference distance for the antenna far field (which for urban macrocell deployments would be between 10-100 metres [34]) and $\gamma$ is the path loss exponent. For simplicity we assume that $K r_0^\gamma = 1$.

The large-scale fading gain incorporates the effects of distance based path loss $r_{jkl}^{-\gamma}$ and shadow fading $z_{jkl}$. The users are uniformly distributed over the network coverage area, where the random quantity $r_{jkl}$, is the distance in metres between BS $j$ and the $k$-th user connected to BS $l$. The path loss due to shadowing is modelled by the corresponding log-normal random variable $z_{jkl}$, where $\mathcal{N}(0, \sigma_{shadow}) = 10 \log_{10} z_{jkl}$.

2.2.3 Channel Estimation

Channel estimation is the main focus of Chapters 4 and 5, where we explore sophisticated stochastic methods in various massive MIMO scenarios. In this chapter we are focused on investigating the performance of data detection in the multi-BS massive MIMO scenario, and therefore are motivated to employ a simple channel estimation scheme. Given our orthogonal pilot set $S$ is by nature full rank $^1$, we can apply the well known least squares (LS) projection to the received symbols at the BS during the channel estimation phase in (2.1). The LS projection is one of the simplest channel estimation schemes as it does not require any statistical a-priori information of the random quantities, unlike stochastic schemes such as the minimum mean squared error (MMSE) estimation methods presented in Chapters 4 and 5. The LS estimation only utilises the knowledge of the fixed pilot sequence set employed by the BS of interest. Furthermore, employing the LS estimation method allows the direct comparison of the results throughout this chapter with the seminal work in [7], where LS channel estimation has also been used. The unique LS channel estimates at BS $j$ of all its users given by,

\footnote{$^1$S contains \(\tau\) linearly independent rows (or columns) and hence is non-singular, resulting in the invertible matrix $SS^H$.}
\[ \hat{G}_j = Y_j S_j^H (S_j S_j^H)^{-1} \]  
(2.5)

\[ = Y_j S_j^H, \]  
(2.6)

given \( S_j \) is orthonormal. Consequently, the channel estimation from the received symbols \( Y_j \) in (2.6) is effectively a correlation with the known sequences \( S_j \).

### 2.2.3.1 Reused Pilot Sets

Let us first consider the scheme most widely found in the literature, where all BSs of the network employ the same pilot sequence set. Using the received symbols during channel estimation in (2.1), (2.6) yields the resulting matrix of LS channel estimates,

\[ \hat{G}_j = \sum_l G_{jl} S_j^H + N_j S_j^H. \]  
(2.7)

Therefore, for the single desired \( k \)-th user connected to BS \( j \) using the pilot sequence \( s_k \) we have,

\[ \hat{g}_{kj} = g_{kj} + \sum_{l \neq j} g_{jk} + N_j s_k^H. \]  
(2.8)

The effects of pilot contamination are readily observable from (2.8), where the desired users estimate is contaminated with the channels of \( L - 1 \) users, i.e contaminated by every other user on the network using the same pilot sequence.

### 2.2.3.2 Different Pilot Sets

Under this scheme, each BS of the network uses a different set of orthogonal pilot sequences. Using the received symbols during channel estimation in (2.1), (2.6) yields the resulting matrix of LS channel estimates,

\[ \hat{G}_j = \sum_l G_{jl} S_j^H + N_j S_j^H \]  
(2.9)
The matrix $\Phi_{jl}$ captures the correlation of the pilot sequences between BS $j$ and BS $l$. The channel estimates of the users connected to BS $j$ are now contaminated by all other users not connected to BS $j$. For the single desired $k$-th user connected to BS $j$ using pilot sequence $s_k$ we have,

$$\hat{g}_{jkj} = g_{jkj} + \sum_{l \neq j}^{L} \sum_{k'}^{K} g_{jk'l} \phi_{jkk'} + N_j s_k^H,$$

(2.11)

where $\phi_{jkk'}$ denotes the correlation between the pilot sequence of the $k$-th user of interest connected to BS $j$, with the pilot sequence of the interfering $k'$-th user connected to BS $l$. All sequences within a cell are orthogonal, i.e., $\phi_{jkk'} = 0$ for $j = l$, and otherwise we have in general $0 \leq |\phi_{jkk'}|^2 \leq 1$. This generalisation allows us to express the reused set scheme as a special case of the different set scheme, where under the reused set scheme we have only the two cases $\phi_{jkk'} = 0$ for $k \neq k'$, and $\phi_{jkk'} = 1$ for $k = k'$.

### 2.2.4 Data Detection

Let $G_j$ be the $M \times K$ channel matrix with the $k$-th column denoted by $g_{jkj}$. Given a linear detector $W$, the $K$ transmitted symbols detected at BS $j$ can be written as $\hat{x}_j = W_j^H y_j$, where for MRC detector,

$$W_j = \hat{G}_j,$$

(2.12)

and for the ZF detector,

$$W_j = \hat{G}_j (\hat{G}_j^H \hat{G}_j)^{-1},$$

(2.13)

with $\hat{G}_j$ denoting the LS estimate of $G_j$. Let $w_{kj}$ denote the $k$-th column of matrix $W_j$. The received signal vector $y_j$ in (2.3) is processed with the linear detector to yield an estimate for the
symbol transmitted by the $k$-th user of the $j$-th BS as given below,

$$\hat{x}_{kj} = w^H_{kj} y_j$$

(2.14)

$$= w^H_{kj} g_{jkj} x_{kj} + \sum_{k' \neq k}^K w^H_{kj} g_{jk'j} x_{k'j} + \sum_{l \neq j}^L \sum_{k'}^K w^H_{kj} g_{kj'l} x_{k'j} + w^H_{kj} n_j.$$  

(2.15)

Averaging over the data symbols and the noise terms, the instantaneous SIR used for this thesis follows from the decomposition (2.15),

$$\text{SIR}_{kj} = \frac{|w^H_{kj} g_{jkj}|^2}{\sum_{k' \neq k}^K |w^H_{kj} g_{jk'j}|^2 + \sum_{l \neq j}^L \sum_{k'}^K |w^H_{kj} g_{kj'l}|^2}.$$  

(2.16)

We acknowledge that a SIR (or SINR) in itself is not a real-world performance metric, such as commonly used metrics like capacity and spectral efficiency. In the scenario where the receiver has perfect knowledge of the channel, it is well known that the capacity (in terms of an information rate in bits per second) can be expressed as the base-two logarithm of 1 plus a simple SINR-like expression [6, 9]. Given the detectors make use of imperfect estimates of the channel in our modelled uplink massive MIMO scenario, formulating an expression for channel capacity which is a function of an SINR-like expression requires care, and has been the topic of much discussion [39]. In the literature, lower bounds on the uplink ergodic channel capacity have been derived [31, Theorem 4.4] and [9, Eq. (3.26) and (3.35)]. However, given the focus of this chapter is to make comparisons of the detectors’ performance under the two pilot allocation schemes in an interference limited setting, we choose to make use of an instantaneous SIR expression (2.16) which is indicative of the resulting detector performance, while providing a means to analytically compare the detectors’ behaviour.

### 2.3 The Large $M$ SIR expressions

In this section we study the behaviour of the MRC and ZF detectors as $M \to \infty$ in the MU massive MIMO setting. We derive large $M$ limits of the SIR expressions based on the reused and different pilot schemes described in Section 2.2.1.
2.3 The Large M SIR expressions

2.3.1 Maximal Ratio Combining

Based on (2.12), the linear detector for the \( k \)-th user connected to BS \( j \) is \( w_{kj} = \hat{g}_{kj} \), where \( \hat{g}_{kj} \) denotes the LS estimate of \( g_{kj} \).

2.3.1.1 Reused Pilot Sets

We first present some preliminary results, which look at the inner products of the random channel vectors as \( M \to \infty \). The channel model in Section 2.2 defines a small-scale fading channel vector \( h \) with i.i.d. elements such that for the \( m \)-th element \( E\{h_m\} = 0, E\{|h_m|^2\} = 1 \). It is well known [7, 40] that for the inner products between two such identical channel vectors, with i.i.d. elements \( CN(0, 1) \),

\[
\frac{h^H_{kJ} h_{kJ}}{M} \to 1, \quad \frac{g^H_{kJ} g_{kJ}}{M} \to \beta_{kJ} \quad \text{as } M \to \infty. \tag{2.17}
\]

While for the inner products between two different channel vectors,

\[
\frac{h^H_{kJ} h_{kJ}'}{M} \to 0, \quad \frac{g^H_{kJ} g_{kJ}'}{M} \to 0 \quad \text{as } M \to \infty. \tag{2.18}
\]

The effect described by (2.18) and (2.17) is known as channel hardening [6, 9]. Consequently, all inner products between non identical vectors are insignificant, and only inner products between identical quantities remain significant as \( M \to \infty \).

Under the reused pilot scheme for the MRC detector, \( w_{kj} \) is given by (2.8). When substituted into the (2.15), and applying the results of (2.17) and (2.18) it is evident that the intra-cell interference and noise terms (the 2nd and 4th terms of (2.15) respectively) disappear as \( M \to \infty \), exposing the limiting result [7],

\[
SIR_{kj} \to \frac{\beta_{kJ}^2}{\sum_{l \neq j} \beta_{jkl}^2}, \tag{2.19}
\]

The limiting result (2.19) is the ratio of the slow fading gains of the \( k \)-th user in the \( j \)-th cell of interest, to the sum of slow fading gains from the \( L - 1 \) inter-cell interferers to BS \( j \) who employ the same pilot sequence. Note that the effects of intra-cell interference, estimation error, noise
and fast fading are eliminated and the SIR is limited by the so-called coherent interference [31] from the pilot contaminators. For the MRC and ZF detectors, the coherent interference is the interference that grows linearly with $M$, i.e., at the same rate as the growth of the signal term due to the array gain offered by large antenna array.

The noise power, i.e., the last term in (2.15) is a function of the inner product between the coherent detector $w_{kj}$ containing noise and contamination terms from the channel estimation phase, and the noise vector of the data decoding phase. Given that the additive noise is i.i.d. $\mathcal{CN}(0,1)$ in both phases, as $M \to \infty$, $w_{kj}n_j \to 0$ for all detectors. Therefore under large $M$, we are operating in a interference limited regime, i.e., the effects of noise are negligible, and from here onwards consider only the SIR defined in (2.16) (as opposed to an expression of SINR).

**2.3.1.2 Different Pilot Sets**

Under the different pilot set allocation, $w_{kj}$ is given by (2.11) for the MRC detector. Applying the same results in the derivation of (2.19) we have the large $M$ expression

$$SIR_{kj} \to \frac{p_{jk}^2}{\sum_{l \neq j} \sum_{k'} |\phi_{jk,k'|}^2| \beta_{jk,k'|}^2}. \quad (2.20)$$

Similarly to (2.19), the effects of intra-cell interference do not appear in (2.20), but now every inter-cell user in the network is an interferer, since all these users have used a pilot sequence which is no longer orthogonal to the pilot of the user of interest. This results in an interference which is the sum of $K \times (L - 1)$ terms.

The expression (2.20) models $\phi$ as a random quantity as opposed to [7, Eq. 26], where only the expected value of $\phi$ is used in the limiting SIR expression.

**2.3.2 Zero-Forcing**

Given (2.13), $w_{kj} = [\hat{G}_j(\hat{G}_j^H\hat{G}_j)^{-1}]_k$, with $[.]_k$ denoting the $k$-th row of a matrix. Next, we proceed to regroup $w_{kj}^H$ as

$$w_{kj}^H o = [(\hat{G}_j^H\hat{G}_j)^{-1}]_k \hat{G}_j^H o, \quad (2.21)$$
where the dummy variable \( o \) corresponds to the desired channel vector or the intra/inter-cell interference vectors in (2.16). This allows us to evaluate each term in (2.16) in two parts. In the following, we analyse \((\hat{G}_j^H \hat{G}_j)^{-1}_k\) and \(\hat{G}_j^H o\) in (2.21) separately, as \( M \) goes to infinity, and then combine the results in order to calculate each of the \( w_{kj}^H o \) terms in (2.16).

### 2.3.2.1 Reused Pilot Sets

Under the reused set scheme,

\[
\frac{\hat{G}_j^H \hat{G}_j}{M} \rightarrow \Psi \quad \text{as} \quad M \rightarrow \infty, \tag{2.22}
\]

where \( \Psi = \text{diag}\{\tilde{\beta}_{j1}, \tilde{\beta}_{j2}, \ldots, \tilde{\beta}_{jK}\} \) with \( \tilde{\beta}_{jk} = \beta_{jk} + \sum_{l \neq j}^{L} \beta_{jkl} \). The convergence in (2.22) is achieved based on the strong law of large numbers [41] where, \((\hat{g}_{jk}^H \hat{g}_{jk}/M) \rightarrow \beta_{jk} + \sum_{l}^{L} \beta_{jkl}\) and \((\hat{g}_{jk}^H \hat{g}_{jk'}/M) \rightarrow 0 \) as \( M \rightarrow \infty \). Similarly, we evaluate \(\hat{G}_j^H o\) in (2.21) for large \( M \) with the relevant vector in place to derive

\[
\frac{\hat{G}_j^H g_{jkl}}{M} \rightarrow \nu_{jkl} \quad \text{as} \quad M \rightarrow \infty, \tag{2.23}
\]

where \( \nu_{jkl} \) is a vector of length \( K \), with the \( k \)-th element equal to \( \beta_{jkl} \) and all remaining elements equal to 0. Substituting this along with (2.22) into (2.16), the SIR expression for large \( M \) can be computed as

\[
\text{SIR}_{kj} \rightarrow \frac{|[\Psi^{-1}]_k \nu_{jk}|^2}{\sum_{k' \neq k}^{K} |[\Psi^{-1}]_k \nu_{jkl}|^2 + \sum_{l \neq j}^{L} \sum_{k'}^{K} |[\Psi^{-1}]_k \nu_{jkl}|^2} = \frac{\beta_{jk}^2}{\sum_{l \neq j}^{L} \beta_{jkl}^2} \tag{2.24}
\]

Note that the large \( M \) limit in (2.24) derived for the ZF detector is the same as that for the MRC detector derived in (2.19), and confirms the results in [8].
2.3.2.2 Different pilot sets

Under the different set scheme, we follow the same steps to obtain

\[
\frac{\langle \hat{g}_{jk}^H \hat{g}_{jk} \rangle}{M} \rightarrow \beta_{jkj} + \sum_{l \neq j} \sum_{k'}^K |\phi_{j;k;k'}| \beta_{jk'}^{k'},
\]

\[
\frac{\langle \hat{g}_{jk}^H \hat{g}_{jk} \rangle}{M} \rightarrow \sum_{l \neq j} \sum_{k'}^K \phi_{j;k;k'}^* \phi_{j;k;k'} \beta_{jk'}^{k'}, \tag{2.25}
\]

as \( M \rightarrow \infty \) with \( \phi_{j;k;k'}^* \) denoting the complex conjugate of \( \phi_{j;k;k'} \). It is important to note that in the case of reused set scheme, (2.25) converges to zero making the analysis quite straightforward. However under the different set scheme, the inner products between channel estimates of intra-cell users do not converge to zero as \( M \) goes to infinity because they are correlated through the pilot contamination. As such, the limit in (2.22) can be reexpressed as a non-diagonal matrix given by

\[
\frac{\hat{G}_j^H \hat{G}_j}{M} \rightarrow \mathbf{Y} \quad \text{as} \quad M \rightarrow \infty, \tag{2.26}
\]

where the \((k, \bar{k})\)-th element in \( \mathbf{Y} \) is given by

\[
\mathbf{Y}(k, \bar{k}) = \begin{cases} 
\beta_{jkj} + \sum_{l \neq j}^{L} \sum_{k''}^{K} |\phi_{j;k;k''}|^2 \beta_{jk'}^{k''}, & \text{if } k = \bar{k} \\
\sum_{l \neq j}^{L} \sum_{k''}^{K} \phi_{j;k;k''}^* \phi_{j;k;k''} \beta_{jk'}^{k''}, & \text{if } k \neq \bar{k}.
\end{cases}
\]

Similarly, (2.23) can be reexpressed as

\[
\frac{\hat{G}_j^H g_{jkl}}{M} \rightarrow \begin{cases} 
v_{jkl}, & \text{if } l = j \\
u_{jkl}, & \text{if } l \neq j \tag{2.27}
\end{cases}
\]

where \( u_{jkl} \) is a vector of length \( K \), with the \( k'\)-th element equal to \( \beta_{jkl} \phi_{j;k;k'}^* \). Substituting (2.26) and (2.27) into (2.16), the large \( M \) SIR expression can be computed as

\[
\text{SIR}_{kj} \rightarrow \frac{\beta_{jkj}^2 |Y^{-1}(k, k)|^2}{\sum_{k' \neq k}^{K} |Y^{-1}(k', 1)\beta_{jk'}^{k'}|^2 + \sum_{l \neq j}^{L} \sum_{k''}^{K} \beta_{jk''}^{k''} \sum_{k}^{K} |Y^{-1}(k, k')\phi_{j;k;k'}^*|^2}. \tag{2.28}
\]
It is important to note that (2.28) is significantly different to the large $M$ SIR expression in (2.20) derived for the MRC detector. We observe that the intra-cell interference has completely disappeared in (2.20). Intuitively, one would expect the intra-cell interference to disappear from the SIR limit of ZF, because the ZF actively attempts to suppress the intra-cell interference by projecting the received signals onto a subspace orthogonal to the intra-cell channels. However, when using the ZF detector under the different set scheme (which does not result in pilot contamination from pilot transmissions of fellow intra-cell users), the intra-cell interference contains terms which are correlated through the inter-cell pilot contamination. As a result, the ZF detector cannot completely suppress the intra-cell interference as can be seen from the remaining $\beta_{jk'l}$ terms in (2.28) and, the large $M$ SIR of ZF is limited by both inter and intra-cell interference. That is, the coherent interference contains channel terms from both the intra-cell and inter-cell users, while in the case of the MRC detector, only the inter-cell users are present. This interesting observation points to a performance gap between the ZF and MRC detectors which is somewhat different under the general different set scheme, as compared to the reused scheme. Moreover, it highlights the importance of uplink power control in reducing the effect of intra-cell interference when the ZF detector is used under the different set scheme. As we have seen in this section, the convergence of the SIR limits in the reused set scheme is a special observation that results from the orthogonality between the sequences of the allocated pilot sets.

### 2.4 Comparison of the Large $M$ SIR expressions

In this section, we compare the large $M$ SIR expressions for MRC and ZF detectors under the different set scheme. Under the special case of the reused set scheme, (2.20) and (2.28) both reduce to the same SIR expression given by (2.19) and (2.24). However, this convergence is also true under the broader constraint of only $\phi_{jk:k'} = 0$ for $k \neq k'$. The SIR expression of the ZF detector in (2.28) contain several $(K \times K)$ matrix inversions, which under a general $K$ users scenario, render it too complex to compare with the SIR expression for the MRC detector in (2.20). Given the relatively simple form of the matrix inverse for $K = 2$, for analytical tractability, we choose to consider the dual-user scenario in comparing the asymptotic SIR expressions of the detectors.
Let us consider a cellular network with only two users, i.e., \( K = 2 \) in each of the \( L \) cells. Without loss of generality we focus on the performance of user 1 in cell \( j \). Based on the different set scheme, we examine the regime where the large \( M \) SIR ZF is greater than that for the MRC, i.e.,

\[
\frac{\beta^2_{j1j}}{\sum_{l \neq j} \sum_{k'}^K |\phi_{j1,k'l}|^2 \beta^2_{jkl}} < \frac{\beta^2_{j2j} \psi^2}{|\omega \beta_{j2j}|^2 + \sum_{l \neq j} \sum_{k'}^K \beta^2_{jkl} |\psi \phi_{j1,k'l}^* - \omega \phi_{j2,k'l}^*|^2},
\]

where \( \psi = \beta_{j2j} + \sum_{l \neq j} \sum_{k'}^K |\phi_{j2,k'l}|^2 \beta_{jkl} \) and \( \omega = \sum_{l \neq j} \sum_{k'}^K \phi_{j1,k'l}^* \phi_{j2,k'l} \beta_{jkl} \).

Assuming that the received signal at BS \( j \) is subjected to identical inter-cell interference we set \( \beta_{jkl} = \beta \) for \( l \neq j \), and further simplify (2.29) as

\[
|\omega|^2 \beta^2_{j2j} < \beta^2 \sum_{l \neq j} \sum_{k'}^K |\phi_{j1,k'l}|^2 |\phi_{j2,k'l}^* + \omega \phi_{j1,k'l}^* |\phi_{j2,k'l} - |\omega|^2 \beta_{j2j}^2 |\phi_{j2,k'l}|^2,
\]

for \( \phi_{j1,k'l} \neq 0 \). Next, we substitute the expressions for \( \psi \) and \( \omega \) and perform some mathematical manipulations to obtain

\[
\beta^2_{j2j} - 2 \beta \beta_{j2j} - \sum_{l \neq j} \sum_{k'}^K |\phi_{j2,k'l}|^2 < 0.
\]

Based on (2.31) and the fact that \( \beta_{j2j}/\beta > 0 \), we conclude that the large \( M \) SIR for ZF is greater than that for MRC when

\[
0 < \frac{\beta_{j2j}}{\beta} < 1 + \sqrt{1 + \sum_{l \neq j} \sum_{k'}^K |\phi_{j2,k'l}|^2}.
\]

The condition in (2.32) points to a sensitivity of the large \( M \) SIR performance to the ratio of the intra-cell slow gain to the inter-cell slow gain. Specifically, the SIR limit for the ZF is greater than that for the MRC detector when this ratio is small, i.e., for fixed inter-cell interference, the weaker the intra-cell interference is, the more probable the SIR limit of the ZF is greater than the MRC detector. This counter-intuitive observation in the limiting expressions points to the importance of uplink power control in ZF detector. Outside the region specified by the condition in (2.32) the large \( M \) SIR limit for MRC outperforms that of ZF (with the exception of when they are the same
2.5 Uplink Power Control

In order to improve the uplink SIR of users on the cell edge, uplink power control (ULPC) is utilised by the terminals. We incorporate a modest open loop scheme that does not attempt to compensate for fast fading. We assume perfect path loss estimation at the terminal and under ULPC every terminal transmits with a power which is the inverse of the path loss to its own base station. The terminal transmits the pilots and data symbols of a given slot at the same power.

Under ULPC, the slow gains can be redefined as,

\[ \beta_{jkl} = \begin{cases} 1, & \text{for } l = j \\ \frac{\hat{\beta}_{jl}}{p_{ul}}, & \text{for } l \neq j \end{cases} \]

(2.33)

For clarity, the analytical results in the preceding sections of this chapter have been presented without the use of ULPC. In the following sections of this chapter, and all remaining chapters, ULPC is employed.

2.6 Numerical Results

In this section, we present numerical examples to highlight the performance differences between the ZF and MRC detectors, for both the reused and different set pilot allocation schemes. Based on the LTE uplink simulation settings used in [7], we consider pilot sequences of length 42 symbols, and can therefore support a maximum of \( K = 42 \) orthogonal users within each cell. We consider a grid of hexagonal cells with the BSs placed at the centre of each 1600m radius cell. We assume log-normal shadowing with a variance of 8 dB and a path loss exponent of 4. Users are distributed uniformly at random locations over the network coverage area, and connect to whichever BS offers them the best channel gain. Note in the presence of log-normal shadowing, this is not

---

2. The path loss at the terminal can be estimated using a combination of higher layer signalling of the transmitted power from the BS and power measurements at the terminal, e.g., during the random access procedure in LTE [42].

3. We consider the area within 100m radius from the BS center as an exclusion zone where no users are placed.
necessarily the closest BS. By considering the cells in the two outside tiers we set $L = 19$. All terminals utilise ULPC as defined in Section 2.5.

First, we consider a fully loaded system with the reused set scheme. Figure 2.2(a) plots the cumulative distribution function (CDF) curves of the SIR in dB for the MRC and ZF detectors under this scheme. We consider both the finite $M$ case, i.e., for $M = \{10^2, 10^3, 10^4\}$ where the instantaneous SIR in (2.16) is averaged over $10^4$ iterations and plotted. We also plot the large $M$ SIR expressions as derived in Section 2.3 for the reused set scheme. Large $M$ SIR results for MRC and ZF detectors were generated based on (2.19) and (2.24), respectively. For each simulation iteration, users are placed at random locations uniformly distributed within the cells, and small-scale channel fading and noise components are drawn from an independent complex Gaussian distribution.

For finite $M$, it is well known that the more complex ZF detector generally outperforms the MRC detector in the absence of noise due to its ability to actively suppress the intra cell interference [24, 26], which are naturally the dominant interferers due to their close proximity to the desired BS. Figure 2.2(a) shows this performance improvement of the ZF over the MRC detector under the plotted scenarios. We observe a performance gain of the ZF over the MRC detector within 1dB under both pilot allocation schemes for $M = 100$. In comparing (2.12) and (2.13), the ZF has an increased runtime complexity from the additional matrix inverse and multiplication operations. This relatively small performance gain of the ZF at the cost of the increased complexity motivates our use of the MRC in the further work of the following chapters. The figure also clearly illustrates how both detectors approach the same large $M$ SIR limit as we increase $M$, with the ZF detector converging faster to the limiting SIR.

Figure 2.2(b) plots CDF curves of the finite $M$ SIR and large $M$ SIR expressions for the MRC and ZF detectors under the different set scheme. Under this pilot allocation scheme, we also observe that the ZF detector is able to achieve a higher SIR for finite $M$. However, in contrast to Figure 2.2(a), the detectors approach different large $M$ SIR limits, with the ZF detector outperforming the MRC detector. This highlights the effect of the pilot allocation scheme on the limiting performance of the ZF and MRC detectors.

In Figure 2.3, we illustrate the effects of the intra-cell and inter-cell interference on the limiting SIR of the ZF and MRC detectors. We consider three scenarios and plot the CDF curves of the
2.6 Numerical Results

(a) SIR CDF under the reused pilot set scheme.

(b) SIR CDF under the different pilot set scheme.

Figure 2.2: The finite $M$ and large $M$ SIR CDFs for the MRC and ZF detectors under the reused and different pilot set allocation schemes, under the simulation parameters in Table 2.1.
difference between the SIR limits denoted by $\text{SIR}_{ZF} - \text{SIR}_{MRC}$. In Scenario I, we consider a lightly loaded system with $K = 2$ users in each cell. In Scenario II, we create a non-uniform system load scenario with 42 users in the cell of the desired user, surrounded by lightly loaded interfering cells with 2 users in each cell. This results in an increase in the intra-cell interference while the inter-cell interference remains the same when compared to Scenario I. In Scenario III, we consider a fully loaded system with $K = 42$ users in each cell. It is interesting to observe that, in both Scenarios I and II, the SIR limit of the MRC outperforms that of the ZF for around 60% of the iterations. As suggested by (2.32), the difference in limits is even larger in Scenario II due to the increased intra-cell interference. In Scenario III the SIR limit of the ZF outperforms the MRC, agreeing with the large $M$ limits illustrated in Figure 2.2(b).

Finally, in Figure 2.4 we focus on Scenario II as described above, where we further observe the performance gains of the MRC detector under large but finite $M$. The plots were generated using Monte-Carlo simulations by setting $M = 10^6$. Agreeing with the limiting performance observed under Figure 2.3, Figure 2.4 shows that the MRC outperforms the ZF 60% of the iterations under large but finite $M$ for Scenario II.

While the mean SIR achieved for both detectors in the different set scheme of Figure 2.2(b) is substantially less than in Figure 2.2(a) under the reused set scheme, the following chapter will explore scenarios where the different pilot set scheme is in fact preferable.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of BSs in network</td>
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<td>19</td>
</tr>
<tr>
<td>Number of users per BS</td>
<td>$K$</td>
<td>42</td>
</tr>
<tr>
<td>Pathloss exponent</td>
<td>$\gamma$</td>
<td>4</td>
</tr>
<tr>
<td>Lognormal shadowing standard deviation</td>
<td>$\sigma_{\text{shadow}}$</td>
<td>8 dB</td>
</tr>
<tr>
<td>Distance between BSs</td>
<td></td>
<td>2771 m</td>
</tr>
<tr>
<td>Exclusion radius of BS</td>
<td></td>
<td>100 m</td>
</tr>
<tr>
<td>Frequency reuse factor</td>
<td>$w$</td>
<td>1</td>
</tr>
<tr>
<td>Pilot sequence length</td>
<td>$\tau$</td>
<td>42</td>
</tr>
<tr>
<td>Data sequence length</td>
<td>$\tau_{ul}$</td>
<td>1</td>
</tr>
</tbody>
</table>
Figure 2.3: CDF of the SIR difference under different set scheme, under the simulation parameters in Table 2.1 with scenario dependent values for $K$.

Figure 2.4: SIR CDF for MRC and ZF detectors under Scenario II.
2.7 Conclusions and Future Work

We have presented limiting expressions for the instantaneous received SIR of MRC and ZF detectors under the reused and different pilot schemes. These will form the basis of further analysis in the following chapter. We have shown through simulation that the finite $M$ SIR expressions do approach the derived limiting expressions as $M \to \infty$.

In the available literature [8, 9, 31], it is shown that the uplink SIR of the MRC and ZF detectors converge to the same limiting expression. We have shown that it is in fact a special case of a more general result where the detectors have significantly different limiting expressions. We adopted two pilot allocation schemes, namely the reused and different set schemes. Our analysis illustrates how the SIR limits converge under the reused set scheme and differ under the general case of the different set scheme, highlighting the effect of pilot allocation on the limiting performance of the MRC and ZF detectors. More importantly, under such generalised pilot allocation schemes we have demonstrated the counter-intuitive observation of the large antenna performance of the ZF being limited by both inter-cell and intra-cell interference, as a result of pilot contamination.

Although we have exposed the effects of pilot contamination under a general, different set pilot allocation scheme, such a scheme still ensures in-cell orthogonality. Extensions of this work could include a similar limiting analysis for pilot allocation schemes which do not necessarily ensure in-cell orthogonality, whilst comparing the two popular linear detectors. Such schemes are attractive as they do not limit the number of BS users by the pilot sequence length $\tau$. Furthermore, a lack of in-cell orthogonality in the baseband might also arise from imperfections in other parts of the transmit and receive chains, and therefore understanding its effects on the uplink SIR is of interest. Deriving the limiting SIR under such pilot allocation schemes can enable comparisons of the schemes themselves along with the detector performance, in terms of sum rate and user-load capacity.

While the limiting SIR has been analysed in the uplink, a corresponding analysis of the limiting SIR in the downlink for both detectors under the general different set pilot scheme would bring further insights.
Chapter 3
Admission Control

The explosion in the number of connected devices puts further pressure on next generation networks to maximise and accurately evaluate network user-load capacity. This chapter analyses user-load capacity and derives user admission regions which can be used to form non-cooperative and cooperative admission control policies to ensure a certain quality of service (QoS) for all users of the network. Under conditions where performance in a massive MIMO network is limited by pilot contamination, the uplink signal-to-interference ratio (SIR) exhibits different distributions when employing different pilot allocation schemes. By utilising different sets of orthogonal pilot sequences, as opposed to reused sequences amongst adjacent cells, the resulting SIR distribution is more favourable with respect to maximising the number of users on the network while maintaining a given QoS for all users. Under the different pilot set allocation and realistic operating conditions, that is, by considering log-normal shadowing and a practically feasible number of BS antennas, we analyse the uplink SIR and derive network user admission regions which can be used to form efficient cooperative admission control policies.

3.1 Introduction

GIVEN the imminent leap in the number of connected devices, the next generation of networks will need to find ways to maximise user-load capacity, i.e., maximising the number of users the network can support. In the context of the multi-user Massive MIMO setting, the uplink network user-load capacity is a function of the maximum number of users that the BS can simultaneously receive reliable data transmissions from. In order to properly support its users, the network additionally requires user admission policies to determine if granting access to a new user can be achieved without compromising the QoS of the currently connected users. In this work, we assume a single QoS for all users of the network, which is defined by a minimum receive SIR at the BS for user transmissions, with a given outage probability.
The general increase in BS density along with the deployment of small cells needed to meet network demands mean that such admission policies will be more complex in nature, and have to be evaluated more frequently. Furthermore, the sub millisecond end user latency targets of 5G networks imply the evaluation of such policies will have to be extremely efficient [27]. The network must make admission decisions based on the current “state” of the network, which given our QoS, implies the need to evaluate the levels of interference on the network in a timely manner. One approach to such a policy could evaluate the state of the network from instantaneous interference measurements, however due to the mobility of the users through the network, the time scale of changes in the measured interferences would render it inappropriate for admission control. A more suitable policy would be based around measures which vary under larger time scales, such as the policies developed in this chapter, which are based around the number of users connected to particular BSs. For a user connected to a given BS, under the Rayleigh fading channel model in Section 2.2 and traditional cellular layout, we are in fact limited by the number of users connected to certain sets of BSs which introduce similar interferences. Each set of BSs and their connected users belong to a particular tier of interferers. Additionally, which BSs belong to which tier will also be dictated by the frequency reuse pattern utilised by the network.

A coarse approach to an admission policy could simply be to determine a fixed limit to the number of users that connect to each BS, based on a required QoS. Such a non-cooperative approach is explored in Section 3.3, with the aim of introducing the methods to deriving the user-load capacity while exploring the effects of pilot allocation. It is clear however such an approach would not be able to capitalise on the fact that some of the neighbouring BSs might be very lightly loaded, and in fact allowing other neighbouring BSs to serve additional users over such a fixed limit could still result in acceptable levels of interference. In Section 3.4, we present more refined and practical cooperative user admission policies which dictate the total number of users that can be connected to particular interfering BSs.

Underpinning such practical policies is the application of statistical multiplexing. Statistical multiplexing has been used extensively in deriving effective bandwidths for users of Code Division Multiple Access (CDMA) cellular networks. In [43] several effective bandwidth models are presented, which are then used to develop capacity specifications and consequently call admission procedures for multi-cell CDMA networks with multiple classes. A similar approach is
employed in this chapter to derive an effective interference for multiple tiers of interferers, which then enables our dimensioning of user-load capacity.

While a large amount of research in massive MIMO systems focuses on maximising cell sum rate capacity and spectral efficiency [7, 8, 31, 44], there has been little with a focus on user-load capacity [22, 23]. The recent work in [23] extends the single-cell user-load capacity expressions in [22] by presenting expressions for multi-cell user-load capacity and like [22] examines the downlink user-load capacity while deriving optimal pilot sequences. In contrast to [23], this chapter focuses on uplink user-load capacity and first considers the pilot allocation most commonly used in massive MIMO literature. We then adopt a more general pilot allocation scheme which proves to be better in terms of user-load capacity, and then go on to derive the admission regions of the network.

The chapter is divided into two main sections, Section 3.3 and Section 3.4. In Section 3.3 we compare the two simple pilot allocation schemes introduced in Chapter 2, with respect to uplink user-load capacity. We use the asymptotic results developed in Chapter 2 and derive an analytical approximation for the distribution of the SIR, primarily based on a distance-based path loss model and random user locations. Using our understanding of the SIR, we investigate the gains that can be achieved through statistical multiplexing when admitting users on a network, with the constraint of a minimum SIR and outage probability. We develop a very simple non-cooperative user admission policy in order demonstrate the benefits of the different pilot set allocation in terms of user-load capacity.

Section 3.4 employs the different set pilot allocation and extends the channel model to a much more realistic model which considers the effects of log-normal shadowing and best cell selection. Despite having significant effects on the interference present on the network, such effects are seldom modelled in the massive MIMO literature [9]. Further to this, given the slow convergence of the finite $M$ results to the limiting expressions developed in Chapter 2, in a similar fashion to [26] and [24], we present SIR expressions that can be used for practically finite $M$. The key differences here being the necessity of a particular form of the SIR for use in our methods of determining user-load capacity, the adoption of a more general pilot set allocation and the use of a simple channel estimation method that does not assume knowledge of all slow gains of the network. Using the methods from Section 3.3, we first specify admission regions in terms of the
allowable interferers from particular tiers, from the perspective of a single BS in Section 3.4.4.1. Naturally, the specification of a network-based admission region in terms of the number of users connected to individual BSs follows in Section 3.4.4.2. These regions are the basis of a cooperative user admission policy that can be used to ensure a minimum QoS is experienced by users of the network.

The main contributions of the chapter can be summarised as follows,

- The demonstration of the advantages of a general pilot allocation scheme which uses different orthogonal pilot sets in each BS compared to the popular reused pilot set allocation, when attempting to maximise the number of users in a network for a given QoS.
- The derivation of effective interferences for different tiers of interferers under realistic channel conditions of log-normal shadowing and best cell selection. It is shown that the up to the second tier of inter-cell interferers need to be considered in the typical urban macrocell scenario.
- The derivation of Tier-dimensional admission regions for a given BS, and consequently BS-dimensional admission regions for a massive MIMO network with a practically large finite number of BS antennas. The methodology is able to define admission regions which are simple and sufficiently accurate to form the basis of practical admission policies.

### 3.2 Frequency Reuse

In order to combat uplink interference and meet higher QoS requirements, networks employ a frequency reuse factor, which we denote by \( w \), using traditional cellular frequency reuse patterns to reduce the interference experienced by the desired user. The patterns are shown in Figure 3.1, where the numbers indicate the frequency resource utilised by each BS.

In Figure 3.1, the desired user is connected to the BS which utilises frequency resource 1, and is marked in red. Under a reuse factor of 1, all users of the network utilise frequency resource 1 and are potentially interferers to the desired user. By exclusively utilising different frequency resources (i.e., different bands of the spectrum) in the adjacent BSs, all users connected to the adjacent BSs have orthogonal channels to the desired user and are no longer a source of interference. Therefore under a frequency reuse factor 3, the potential interferers also utilising frequency
3.2 Frequency Reuse

resource 1 (connected to BSs marked in blue) are essentially moved further away, and therefore generate less interference to the desired user. Such cellular frequency reuse patterns have a fixed granularity due to the assignment of the divided frequency resources in a common pattern across the network, and therefore only offer a very coarse means to reduce the interference.

This reduction of interference to the desired user comes at the cost of a reduction to the maximum number of users that can be supported by each BS, since the available frequency resources for all BSs to train its users, i.e., perform channel estimation, has been also reduced by the frequency reuse factor. Given the orthogonality of the pilot sequences, the number of supported users connected to a particular BS is limited by the pilot sequence length $\tau$. If higher frequency reuse factors $w$ are employed by the network to meet a given QoS, this in turn reduces the number of resources available for each BS to estimate the channels of its users by a factor of $w$. Consequently, the number of users all BSs can support is constrained by,

$$K \leq \left\lfloor \frac{\tau}{w} \right\rfloor.$$  \hspace{1cm} (3.1)

Figure 3.1 also depicts the grouping of interferers into tiers, where the tier 1 interferers are those users connected to the 6 surrounding BSs shown in dark blue. Tier 2 interferers are those connected to the 12 surrounding BSs shown in light blue. As shown in Figure 3.1, which BS’s users belong to which tier of interferers is dependent on the value of $w$. Tier 0 interferers are the intra-cell interferers connected to BS of interest as shown in red in Figure 3.1.

Given the cellular structure in Figure 3.1, the constraint (3.1) can also be expressed for a particular tier $t$,

$$n_t \leq \left\lfloor \frac{6t\tau}{w} \right\rfloor, \quad t > 0,$$  \hspace{1cm} (3.2)

where $n_t$ is the number of users tier $t$ can support.
Figure 3.1: Tier 1 interferers (dark blue) and tier 2 interferers (light blue) to the BS of interest, with intra-cell tier 0 interferers (red), under different frequency reuse factors $w \in \{1, 3, 7\}$.

### 3.3 User Load Capacity and Pilot Allocation

#### 3.3.1 System Model and Limiting SIR expressions

Given that the QoS upheld by our admission policy is specified in terms of a minimum SIR, understanding the SIR is fundamental in formulating such policies. For this section, we assume the use of the MRC detector at the BS, as covered in Chapter 2, for the detection of the desired user’s transmission. Under such a detector, as highlighted in Figures 2.2(a) and 2.2(b) of Chapter 2, convergence to the asymptotic behaviour given by the large $M$ SIR expression (2.19) is slow, and requires an impractical large number of antennas. In spite of this, we use the large $M$ SIR expressions as a means to compare the two pilot allocation schemes through analytical methods, where later Monte Carlo simulation results under finite $M$ confirm the differences of the two schemes. Furthermore, later in Section 3.4, we consider more accurate and realistic dimensioning of the user admission region under practically large and finite $M$.

As detailed in Chapter 2, under our system model and MRC detection, the large $M$ SIR is limited by the effects of pilot contamination. Pilot contamination occurs when neighbouring users employ pilot sequences which are not orthogonal to the sequence used by the desired user, and consequently corrupting the channel estimates of the desired user. In the large $M$ limit of the MRC
3.3 User Load Capacity and Pilot Allocation

detector, this reduces to an interference which is proportional to the sum of the squared slow gains of the interfering users as described by (2.19). Here, we employ ULPC\textsuperscript{1} as described in Section 2.5 and explicitly re-express the large $M$ SIR expressions under ULPC, for the reused pilot set scheme (2.19) as,

$$SIR = \frac{\left(\frac{\beta_{jl}}{\beta_{kl}}\right)^2}{\sum_{l \neq j} \left(\frac{\beta_{jl}}{\beta_{kl}}\right)^2} = \frac{1}{\sum_{l \neq j} \left(\frac{\beta_{jl}}{\beta_{kl}}\right)^2}$$  \hspace{1cm} (3.3)$$

and different pilot set schemes (2.20),

$$SIR = \frac{1}{\sum_{l \neq j} \sum_{k=1}^{K} \phi_{kl} \left(\frac{\beta_{jl}}{\beta_{kl}}\right)^2}$$ \hspace{1cm} (3.4)$$

where as defined in Section 2.2, $\beta_{jkl}$ represents the large-scale fading gain between the $k$-th user connected to BS $l$, and BS $j$. In the interests of reducing notational clutter we assume a focus on the $k$-th user of BS $j$ and define $\phi_{kl} \triangleq |\phi_{jk}:kl|^2$, where $\phi_{jk:k'l}$ denotes the correlation between the pilot sequence of the $k$-th user of interest connected to BS $j$, with the pilot sequence of the interfering $k'$-th user connected to BS $l$.

In this section we model the slow fading gain by distance-based path loss alone, where $\beta_{jkl} = r_{jkl}^{-\gamma}$. The users are uniformly distributed over the cell area, where the random quantity $r_{jkl}$ is the distance between the $k$-th user in the $l$-th cell and BS $j$. The constant $\gamma$ is the path loss exponent. Including the effects of log-normal shadowing in the slow fading gain of our analysis of the SIR expression is non-trivial, and handled later in Section 3.4. Therefore, we begin by defining

$$y_{kl} = \phi_{kl} \left(\frac{r_{jkl}}{r_{jkl}}\right)^{2\gamma}$$ \hspace{1cm} (3.5)$$

\textsuperscript{1}Under such a system, the use of ULPC results in a reduced mean SIR and reduced uplink sum rate data capacity, however the variance of the inter-cell interference is reduced significantly, and in the context of statistical multiplexing, results in greater user-load capacity.
so that the SIR in (3.4) can be expressed as

\[
SIR = \frac{1}{\sum_{l \neq j} \sum_{k=1}^{K} y_{kl}}.
\]  

(3.6)

### 3.3.2 Effective Interference

One of the objectives of any user admission policy is to ensure that admitting a new user to the network will still guarantee a certain QoS for the existing users of a network. In the context of our massive MIMO network, whose performance is interference limited, such an admission policy would predict if admitting a new user would add an acceptable level of interference to the users of the network. Specifically, a user admission policy would ensure that users of the network experience a QoS, governed by a minimum SIR, \( S \) and an outage probability \( \alpha \). For (3.6), this condition can be written as,

\[
P \left( \frac{1}{\sum_{l \neq j} \sum_{k=1}^{K} y_{kl}} > S \right) \geq 1 - \alpha.
\]  

(3.7)

As shown in Section 3.3.1, the SIR expression (3.6) is a function of the sum of the \( y_{kl} \) terms, which are functions of several different random variables, and hence not easy to predict.

Fortunately, this problem of determining how a common resource (in our case interference) is loaded by the sum of independent entities, is of a similar nature to an array of well understood problems. For example, in broad-band ISDN networks where we want to evaluate whether the sum of bursty traffic streams exceeds the maximum bit rate the network can provide, or in CDMA networks where we want to determine whether the number of calls that can be handled exceeds our maximum system bandwidth, given a certain (outage) probability.

Just as [43] describes an effective bandwidth to treat the later, we introduce the notion of an effective interference, \( \bar{y} \) for each user. Assigning an effective interference for each user which is equal to the maximum interference results in a very conservative admission policy which does not benefit at all from statistical multiplexing. Assigning an effective interference equal to the mean interference implies the unrealistic scenario of an infinite number of users on the network, where the sample mean is in fact the true mean. Therefore, we derive an effective interference for each user which is somewhere in between the mean interference and the maximum interference.
3.3 User Load Capacity and Pilot Allocation

The random interference $y_{kl}$ is a function of three different random quantities, $r_{kl}$, $r_{jkl}$ and the inner product of the pilot sequences $\phi_{kl}$. Therefore the nature of the interference is still quite complex. The adopted approach to deriving an effective interference $\tilde{y}$ for each of the $n$ interferers, where $\tilde{y}n < 1/S$ for a given outage probability, is to approximate the sum of i.i.d. $y_{kl}$ random variables by a Gaussian random variable by means of the central limit theorem [6, 45]. We note that while the Chernoff bound [46] offers tight bounds on the tail probabilities of the sum of i.i.d. random variables, the complexity of the given interference makes it hard to derive the moment generating functions required by such methods. Under the Gaussian approximation, the condition (3.7) can be expressed as,

$$P \left( \frac{1}{z} > S \right) \geq 1 - \alpha,$$

where $z \sim \mathcal{N}(\mu, \sigma)$, which is equivalent to the condition,

$$\frac{1}{S} - \frac{\mu}{\sigma} \geq Q^{-1}(\alpha),$$

(3.8)

where $\mu = n\mu_{y_{kl}}$, $\sigma^2 = n\sigma_{y_{kl}}^2$, $Q(u) = 1 - \Theta(u)$, and $\Theta(u)$ is the CDF function for $\mathcal{N}(0, 1)$.

3.3.3 Analysing the Interference

In order to make the Gaussian approximation in (3.8), we require the mean and variance of the quantity $y_{kl}$. Under our large $M$ scenario we can easily derive approximations to the first and second moments analytically.

In the interests of readability for the remaining part of this section, we drop the $kl$ subscript by just considering the two BS scenario shown in Figure 3.2, with our BS of interest (BS $j$) and an arbitrary interferer connected to BS $l$ in an adjacent interfering cell (cell $l$). Note that we have approximated the hexagonal cell by a circle as will be explained in more detail shortly.

Due to the statistical independence between the pilot sequences and the remaining random quantities, the mean can be written as $E\{y\} = E\{\phi\}E\{x\}$, where the random variable $x = \left( r_l / r_j \right)^{2\gamma}$. The value of $E\{x\}$ can be determined analytically as follows.

Assuming users which are uniformly distributed over each cell of our cellular network, the
probability distribution functions (PDFs) of the two random variables $r_l$ (the distance of the interferer to their own BS) and $r_j$ (the distance of the interferer to the base station of interest) are dependent on the geometry of the individual cells and the layout of the cells within the network. These two random quantities are clearly not independent and therefore, computing a joint PDF is difficult. An alternative approach, which is also used in [47], is to re-express $x$ in terms of $\theta$ and $r_l$, i.e.,

$$x(r_l, r_j) = \left(\frac{r_l}{r_j}\right)^{2\gamma}$$

becomes,

$$x(r_j, \theta) = \left(\frac{r_j^2}{r_j^2 + a^2 - 2ar_j \cos \theta}\right)^\gamma,$$  \hspace{1cm} (3.9)

where $a$ and $b$ are the known deterministic quantities shown in Figure 3.2.

If we now approximate the hexagon shaped cell with a circular cell, we achieve two important points. The circle cell approximation results in very simple PDFs, where $f_{r_l}(r_l) = 2r_l/b^2$ for $0 \leq r_l < b$ and $f_{\theta}(\theta) = 1/\pi$ for $0 \leq \theta < \pi$. Secondly, we have statistical independence between $r_l$ and $\theta$, and can write the mean and variance of $x$ as,

$$E\{x\} = \int_0^\pi \int_0^b x(r_l, \theta) f_{r_l}(r_l)f_{\theta}(\theta) \, dr_l \, d\theta \hspace{1cm} (3.10)$$

$$\sigma_x^2 = \int_0^\pi \int_0^b (x(r_l, \theta) - E\{x\})^2 f_{r_l}(r_l)f_{\theta}(\theta) \, dr_l \, d\theta \hspace{1cm} (3.11)$$

If we now approximate the hexagon shaped cell with a circular cell, we achieve two important points. The circle cell approximation results in very simple PDFs, where $f_{r_l}(r_l) = 2r_l/b^2$ for $0 \leq r_l < b$ and $f_{\theta}(\theta) = 1/\pi$ for $0 \leq \theta < \pi$. Secondly, we have statistical independence between $r_l$ and $\theta$, and can write the mean and variance of $x$ as,
respectively.

Under the reuse scheme as described in Section 2.2.1, the moments (3.10) and (3.11) readily enable the Gaussian approximation to the denominator of (3.6), since $\phi \in \{0, 1\}$ is a known constant based on which pilot sequences are allocated. If we consider the scheme where each BS uses a different orthogonal set, we will have to derive the mean $\mu_y$ and variance $\sigma_y^2$ where the random nature of $\phi$ is accounted for. As described in Section 2.2.1, the pilot sequences from the orthonormal set $S$ are assigned randomly to the $K$ users of the cell, therefore $\phi$ is statistically independent from $x$, (the user locations). The matrix $S$ is uniformly distributed on the set of unitary matrices such that the random quantity $\phi = |ss^H|^2$, has an expectation of $1/\tau$ and a variance of $1/\tau^2$. Therefore,

$$\mu_y = E\{\phi x\} = E\{\phi\} \mu_x = \mu_x / \tau. \quad (3.12)$$

In order to derive the variance $\sigma_y^2$, a series of simple substitutions can be made, resulting in,

$$\sigma_y^2 = \frac{1}{\tau^2} \left( 2\sigma_x^2 + \mu_x^2 \right). \quad (3.13)$$

The mean $\mu_y$ and variance $\sigma_y^2$ have thus been expressed in terms of $\mu_x$ and $\sigma_x^2$ in (3.12) and (3.13). This shows that the mean and variance of the interference caused by a interfering user will be reduced by a factor $\tau$ and roughly $2/\tau^2$ respectively when different pilot sets are used. If we consider a fully loaded network, where every BS has $K = \tau$ users, under the different pilot set scheme, we have effectively $K$ more interferers. Given (3.12), these additional $K$ interferers do not result in an increase in the average total interference (i.e., the sum in the denominator of (3.6)). However, the reduction of $2/K^2$ in the variance of each interferer in (3.13) means the variance of the total interference under this scheme is reduced significantly by a factor of $2/K$. It is this reduction in variance, in the context of statistical multiplexing, which results in the increased uplink user-load capacity of under the different set scheme.

We can examine these effects closer in Figure 3.3, where the CDFs of the SIR in (3.6) have been plotted from a Monte Carlo simulation for both pilot allocation schemes under typical system parameters. While the CDF of the SIR under the reused pilot scheme shows more users who experience higher SIR levels, these users contribute more to maximising the sum rate but not the
sum user-load capacity. In the context of increasing user-load capacity, our aim is to guarantee only a certain minimum QoS for all users. Since typical QoS values involve outage probabilities between [0, 0.1], we are more interested in the lower tail of the distributions. As shown clearly by the CDF curves of Figure 3.3, the region of outage probabilities from [0, 0.1] offers higher values of SIR when using the different pilot sets scheme.

3.3.4 Non-cooperative User Admission Policy

The central limit theorem allows us to approximate the sum of interference from \( n \) i.i.d interferers\(^2\) by a Gaussian random variable with moments \( \mu = n \mu_y \) and \( \sigma^2 = n \sigma_y^2 \). However, depending on the network topology, the surrounding cells containing the interferers could be at different distances, resulting in interference with different means and variances. The Gaussian approximation can be extended to approximate the sum of these \( T \) different types, or tiers, of interferers. The total interference is now approximated by a sum of \( T \) normal distributions, which itself is another

\(^2\)In this approximation, we consider the weak correlation in the interference terms induced by the unitary property of the pilot sequence matrix negligible.
normal distribution, with \( \mu = \sum_{t=1}^{T} n_t \mu_{y_t} \) and \( \sigma^2 = \sum_{t=1}^{T} n_t \sigma_{y_t}^2 \), resulting in

\[
\frac{1}{S - \sum_{t=1}^{T} n_t \mu_{y_t}} \geq Q^{-1}(\alpha),
\]

where \( n_t \) is the number of interferers belonging to tier \( t \), and \( \mu_{y_t} \) and \( \sigma_{y_t}^2 \) are the mean and variance respectively of the interference \( y_t \) from users belonging to tier \( t \).

If we consider the traditional cellular layouts as shown in Figure 3.1, we only need to consider the tier 1 interferers \( (T = 1) \), i.e., the set of interferers (marked in dark blue for different frequency reuse factors) which connected to the BSs closest to the BS of interest (marked in red). This is because under the current channel model, the interference contribution from users in the next tiers of interfering cells \( (T > 1) \) is negligible. For instance, given a frequency reuse factor of 1 (with \( x_{t=n} \) denoting the random interference at tier \( n \) ), we have \( E\{x_{t=1}\} \simeq 500 \ E\{x_{t=2}\} \), and \( \text{var} \ E\{x_{t=1}\} \simeq 10^6 \text{var} \ E\{x_{t=2}\} \). Under later extensions to the channel model in Section 3.4, more tiers of interferers become relevant and need to be accounted for by the admission policy.

With \( T = 1 \), for a given \( \alpha \) and \( S \), we can solve for the maximum number of interferers \( n_{\text{max}} \) that satisfies the constraint (3.14), where \( n_{\text{max}} = \lfloor n_1 \rfloor \). As a result, each of these interferers contributes an effective interference \( \tilde{y} \), given in [3] as:

\[
\tilde{y} = \frac{\mu_{y_1}}{1 + \frac{2}{z}(1 - \sqrt{1 + z})},
\]

where

\[
z = \frac{4\mu_1}{(Q^{-1}(\alpha))^2 \sigma_1^2 S}.
\]

From (3.15) we can define the constraint on the interference based on the number of connected users in tier 1, \( n_1 \),

\[
n_1 \leq n_{\text{max}},
\]
where the maximum number of allowed interferers $n_{\text{max}}$ across all tier 1 interfering cells is,

$$n_{\text{max}} = \left\lfloor \frac{1}{yS} \right\rfloor.$$  \hfill (3.17)

We wish to define a simple non-cooperative user admission policy, and therefore impose the requirement that each BS can decide to admit a user autonomously, i.e., BSs do not require information from other BSs for user admission. This implies an upper limit to the number of users connected to a BS, which will apply to all BSs of the network. From the constraint (3.16), we define the following constraint of an unregulated limit $k_\mu$,

$$K \leq k_\mu = \left\lfloor \frac{1}{6yS} \right\rfloor,$$  \hfill (3.18)

where $K$ is the number of users connected to every BS of the network. The upper limit to the number of users in each BS of the network to be upheld by a non-cooperative admission policy is where both the constraints (3.18) and (3.1) are satisfied, which can expressed through a defined limit $k_{\text{max}}$,

$$K \leq k_{\text{max}} = \min(k_\mu, \left\lfloor \frac{\tau}{w} \right\rfloor).$$  \hfill (3.19)

### 3.3.5 Numerical Results

#### 3.3.5.1 Scenario

We adopt our system parameters from the urban macrocell LTE uplink modelled in [7]. Consequently, we consider the possible frequency reuse factors $w \in \{1, 3, 7\}$, assume a coherence bandwidth of 14 subcarriers, and a coherence time divided into 7 OFDM symbols, 3 of which are used for uplink training. As a result, our pilot sequence is of length $3 \times 14 = 42$ symbols, and can therefore support a maximum of $K = 42$ users. The users connected to each BS are distributed uniformly over the area of the cell. Each BS is positioned in the centre of a hexagonal cell of radius $a = 1600$ metres and with cell-hole radius $a_h = 100$ metres, where the BSs are placed in a grid forming a hexagonal tessellation. A path loss exponent of 4 is used. ULPC as described in Section 2.5 is employed by all users of the network. The main simulation parameters are summarised in
Table 3.1: Main Simulation parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of BSs in network</td>
<td>(L)</td>
<td>19</td>
</tr>
<tr>
<td>Maximum number of users per BS</td>
<td>(K)</td>
<td>42</td>
</tr>
<tr>
<td>Pathloss exponent</td>
<td>(\gamma)</td>
<td>4</td>
</tr>
<tr>
<td>Lognormal shadowing standard deviation</td>
<td>(\sigma_{\text{shadow}})</td>
<td>0 dB</td>
</tr>
<tr>
<td>Distance between BSs</td>
<td></td>
<td>2771 m</td>
</tr>
<tr>
<td>Exclusion radius of BS</td>
<td></td>
<td>100 m</td>
</tr>
<tr>
<td>Frequency reuse factor</td>
<td>(w)</td>
<td>{1, 3, 7}</td>
</tr>
<tr>
<td>Pilot sequence length</td>
<td>(\tau)</td>
<td>42</td>
</tr>
<tr>
<td>Data sequence length</td>
<td>(\tau_{\text{ul}})</td>
<td>1</td>
</tr>
</tbody>
</table>

3.3.5.2 M in the Limit

In Figure 3.4 we plot the maximum number of users per cell, \(k_{\text{max}}\) as defined in (3.19), across the range of SIR and outage probabilities for relevant QoS targets, under both pilot allocation schemes. Figure 3.4 shows clearly that under the outage probabilities and SIR values of interest, a pilot allocation scheme which employs different pilot sets, as opposed to reusing sets amongst BSs, is superior with regards to \(k_{\text{max}}\). Figure 3.4 also indicates that the outage probability component of the QoS has less of an effect on \(k_{\text{max}}\) compared to changes in the minimum SIR.

In order to examine the differences in detail, Figure 3.5 shows a cross-section of Figure 3.4, with both \(k_u\) and \(k_{\text{max}}\) plotted for a fixed outage probability \(\alpha = 0.05\). Firstly, for both \(w = 1\) and \(w = 3\), as expected from our analysis in Section 3.3.3, \(k_u\) is significantly higher when different pilot sets are used.

Under SIR requirements of 0dB for both schemes, the number of users connected to each BS is restricted by the resources required for uplink training from the constraint (3.1). The limit \(k_u\) shows we would be able to support many more users at this QoS level based on their interference contribution, but can only support 42 based on our pilot sequence length \(\tau\). Therefore, the maximum number of users that can be supported by the BS, \(k_{\text{max}} = 42\), is the same, regardless of the pilot allocation scheme.

However, to meet a SIR requirement which is greater than 1dB under the reused pilot set
Figure 3.4: Maximum number of users per cell, $k_{\text{max}}$ vs. QoS, under the simulation parameters in Table 3.1.

scheme, we are forced to employ $w = 3$, indicated by the switching point circled on Figure 3.5, while under the different pilot sets scheme, we are able to continue using $w = 1$. Moving to $w = 3$ reduces the effective interference $\tilde{y}$ by a factor of 500 since the interferers are now positioned in cells further away. However, it also means that fewer users can be supported due to constraint (3.1), as shown by the first drop in $k_{\text{max}}$ for the reused set scheme in Figure 3.5.

Under the different pilot set scheme, $k_{\text{max}}$ only starts to decline for SIRs greater than 6dB, where we are able to support the increasing SIR requirements by simply admitting fewer users in the cell. This is another significant advantage of the different pilot set scheme which enables finer control on the levels of interference on the network. For SIRs greater than 9dB we would have to admit less than 14 users to meet this requirement, and therefore it makes more sense that we employ $w = 3$ (as indicated by the switching point, where the red $k_u$ curve falls under the $k_{\text{max}}$ line). It is the region between these switching points, indicated by the shaded blue in Figure 3.5, which is the region of gain when using the different pilot sets scheme.

While outside the visible $K$ range plotted in Figure 3.5, but obvious from the steep slope of the $k_u$ curves, for both schemes under $w = 3$ we would be able to support many users for all SIRs up to 26dB. However, frequency resources for uplink training have consequently been reduced by
3.3 User Load Capacity and Pilot Allocation

Figure 3.5: Maximum number of users per cell, $k_{\text{max}}$ and unregulated maximum number of users $k_u$ vs. SIR under fixed outage probability $\alpha = 0.05$ and simulation parameters in Table 3.1.

...a factor of 3, and in practice we can only train a maximum of $42/3 = 14$ users. Therefore, both schemes now support the same number of users up until a SIR of around 30dB, where the reused pilot set scheme must again switch to $w = 7$, while the switching point for the different pilot set is differed to 35dB and hence able to support more users. The shaded light blue regions in Figure 3.5 clearly show for the given QoS, the scale and region where significant gains in user-load capacity can be realised by utilising different orthogonal pilot sets amongst cells.

3.3.5.3 The Gaussian Approximation

We now briefly look at the worst-case accuracy of the Gaussian approximation under large $M$ used to generate the results shown in Figure 3.5. As posed by the central limit theorem, we expect that as the numbers of interferers increases, the distribution of interference closer approaches that of a normal distribution, and the accuracy of our estimation improves. Given our restriction of a simple user admission policy and standard cellular frequency reuse patterns, our worst-case approximation would be under a frequency reuse factor of 7. Under this level of frequency reuse we could support a maximum of 6 users in each of the interfering tier 1 cells. Figure 3.6 shows a comparison between the Gaussian approximation (in combination with our circular cell approximation) and a Monte Carlo simulation, for uplink SIR distributions under both pilot allocation
schemes with frequency reuse factor 7. When specifying meaningful QoS we deal typically with outage probabilities of 0.1 or less, and therefore the main area of interest is in the lower tail of the distributions. As expected, the Gaussian approximation under the different pilot set scheme follows more closely the results from simulation, a consequence of the interference being from the 36 interferers from the 6 cells, which is significantly more interferers than the 6 under the reused pilot set scheme. Importantly, our approximated user-load capacity gain from the different pilot set allocation is conservative, since around this region, the Gaussian approximation yields results significantly more optimistic in the reused pilot sets case when compared to the different pilot set case. We once again emphasise that the intention of this section is compare the user-load capacity of the two pilot allocation schemes, while in the upcoming Section 3.4, we consider more accurate and realistic dimensioning of the user admission region.
Table 3.2: Values of $k_{\text{max}}$ derived from Monte Carlo simulation for finite practical $M = 500$ under varying QoS and simulation parameters in Table 3.1.

<table>
<thead>
<tr>
<th>QoS</th>
<th>Low 0db/0.01</th>
<th>Medium 10db/0.05</th>
<th>High 25db/0.05</th>
<th>Very High 30db/0.005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reused</td>
<td>14</td>
<td>14</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Different</td>
<td>42</td>
<td>14</td>
<td>14</td>
<td>6</td>
</tr>
</tbody>
</table>

3.3.5.4 Finite $M$

As discussed previously in Chapter 2, as $M$ approaches infinity, the effects of intra-cell interference, noise and fast fading are no longer present in the resulting uplink SIR. Consequently, we have fewer remaining terms to consider and are able to derive a suitable approximation for the uplink SIR analytically. However, in the case of finite $M$ all such cross terms are present, and consequently the expression for the uplink SIR at the BS contains several more random quantities.

A proper analysis of the uplink SIR and user-load capacity under practically large $M$ is deferred until Section 3.4, and as such we briefly present values of $k_{\text{max}}$ which result from an exhaustive search, i.e., the largest value of $k_{\text{max}}$ which still satisfies the required QoS under Monte Carlo simulation. Given the use of experimental massive MIMO test beds consisting 128 element arrays [48, 49], we select $M$ of similar order, where $M = 500$. The results in Table 3.2 show that for various QoS, there are user-load capacity gains to be achieved using the different set scheme under practically finite $M$. Note that due to the presence of all intra and inter-cell interference terms, we expect estimates of user-load capacity which are less than our previous analysis with $M$ in the limit. The values in Table 3.2 are indicative of the effects of meeting the required QoS by increasing the frequency reuse factor $w$, where resulting interference levels fall well below the requirement, and we are left with values of $k_{\text{max}}$ which are governed by the constraint (3.1).
3.3.6 Summary

We derived large $M$ uplink SIR expressions for our system model, which was then used as a basis to derive a simple admission policy to dictate the maximum number of users which can be admitted to each cell of a massive MIMO network. In doing so, we have shown for both large $M$ and finite $M$ that using different orthogonal pilot sets in each cell as compared to reusing pilot sets amongst all cells, allows us to admit the same or significantly more users while upholding a given QoS for all users of the network, under a non-cooperative admission policy. This is despite the fact that such a pilot scheme introduces significantly more pilot contaminators. In the context of the user-load capacity derived in this Chapter, we summarise the following significant advantages of using different pilot sets when compared to reusing the same pilot sets amongst BSs,

1. The reduced variability of the interferers in the context of statistical multiplexing under the different pilot scheme makes it superior when trying to maximise the number of users that can be admitted onto a network.
2. The different pilot scheme provides a finer granularity to control the inter-cell interference before having to resort to increasing the level of frequency reuse which is a much coarser adjustment - limiting the number of users will now effectively reduce the inter-cell interference, allowing it to come closer to the required QoS.
3. Under the central limit theorem, which we have used to approximate the distribution of the sum of interferers by a Gaussian distribution, we essentially have significantly more interferers ($K$ more under a fully loaded network) and as a consequence, the Gaussian approximation is significantly more accurate.

The main drawback of using different pilot sets is a reduction in the network sum-rate capacity when compared with the reused pilot set scheme. As discussed in Section 3.3.3, this evident from the resulting SIR distribution from the reused pilot set scheme, which clearly shows a larger proportion of users who experience a significantly higher SIR. Consequently, the use of different pilot sets amongst BSs is well suited for increasing the user load capacity for a network of low rate devices, e.g., in the context of wireless sensor networks.
3.4 User Load Capacity under Large-scale Shadowing

3.4.1 System Model and Limiting SIR expressions

In this section we consider the effects of large-scale shadowing in our system model. The path loss due to shadowing can be modelled by the corresponding log-normal random variable $z_{jkl}$, where $10 \log_{10} z_{jkl} \sim \mathcal{N}(0, \sigma_{\text{shadow}})$. Therefore our slow gains are defined as $\beta_{jkl} = z_{jkl} r_{jkl}^{-\gamma}$, incorporating the effects of distance-based path loss $r_{jkl}^{-\gamma}$ and shadow fading $z_{jkl}$. Users are distributed uniformly at random locations over the network coverage area, and connect to whichever BS offers them the best channel slow gain, where the random quantity $r_{jkl}$ is the distance between BS $j$ and the $k$-th user connected to BS $l$. The constant $\gamma$ is the path loss exponent.

We model the same cellular network as Section 3.3, utilising the same levels of frequency reuse (Section 3.2), and ULPC (Section 2.5).

Motivated by the numerous advantages of the different pilot set scheme in the context of user-load capacity, as summarised in Section 3.3.6, in this section we adopt the different pilot set scheme. In order to expose the effects of the log-normal shadowing, we briefly return to the SIR limiting expressions for the different set scheme under ULPC developed in (3.4). These can be re-expressed as,

$$\text{SIR}_j = \frac{1}{\sum_{l \neq j} \sum_{k=1}^{K} \phi_{kl} \left( \frac{z_{jkl}}{z_{jkl}} \right)^2 \left( \frac{r_{jkl}}{r_{jkl}} \right)^{2\gamma}}.$$  \hspace{1cm} (3.20)

As done in Section 3.3.1, we define the slow gains under ULPC to the $j$-th BS, $x_{kl} \triangleq \left( \frac{p_{kl}}{p_{jkl}} \right)^2$, where the SIR expression for a user connected to the $j$-th BS is given by,

$$\text{SIR}_j = \frac{1}{\sum_{l \neq j} \sum_{k=1}^{K} \phi_{kl} x_{kl}}.$$  \hspace{1cm} (3.21)

3.4.2 BS selection

Due to the large variance of the random quantity $z_{jkl}$, it can be expected that a given user may not necessarily experience the best channel slow gain $\beta_{jkl}$ with the closest BS, or even with the immediately surrounding BSs. We would expect that a realistic base station assignment would
assign the user to the BS to which it experiences the best channel gain. (Of course, this might be not possible if the desired BS is already fully loaded, or network load balancing algorithms result in other selections). As a consequence, for the $k$-th user connected to BS $l$ to be considered as an interferer to the $k$-th user connected to BS $j$, the condition $\beta_{jkl} \leq \beta_{lkl}$ must hold. From this inequality, we then have the constraint on the interference $x_{kl} < 1$, given $x_{kl} \triangleq \left( \frac{\beta_{jkl}}{\beta_{lkl}} \right)^2$. Since users connected to BS $l$ may not be physically located within the cell area of BS $l$, $x_{kl}$ cannot be derived by simply considering uniformly distributed interferers within the cell area as in Section 3.3. Therefore, we highlight here that the last subscript of the slow gain denotes the BS that user $k$ is connected to, and not the cell area where the user is located.

Finding the distribution for $x_{kl}$ analytically is very complex. Even without considering shadowing and assuming the connection to the closest BS, stochastic geometry tools have been widely used to analyse the distribution of the interference, where exact solutions are considered intractable and the yielded approximations are still very complex [24,50]. Previous analytical attempts tackle the problem by restricting the set of BSs a user can connect to, where the user connects to the best BS from a set of the $n$ nearest BSs. Early attempts [51] use such an approach, and present numerical integrals for the total interference, as opposed to an analysis exclusively for users connected to a particular BS $l$, or a particular tier of BSs $t$. More recent attempts, such as [52] show an interference analysis for a user being located in a given cell area. A shortcoming of this work, along with several other works is although the analysis is strictly correct when restricting the user to only connect to the $n$ nearest BSs, it fails to limit the support of the random inter-cell interference ($x_{kl} < 1$), and unrealistic approximations of the moments of $x_{kl}$ result. In any case, the analytical attempts at approximating the interference moments are tedious and involve costly numerical integrations, which require greater effort and complexity, and possibly longer execution time than a more accurate Monte Carlo simulation. Furthermore, larger values of $\sigma_{\text{shadow}}$ and/or smaller values of $\gamma$ increase the set of potential BSs a user may connect to while also increasing the probability that a user will connect to a BS which is further away, which in turn increase the complexity of analytical results such as [51] [52] significantly. Therefore, as carried out in the thorough performance analysis in [9], we turn to using Monte Carlo simulations to properly model and derive the moments of $x_{kl}$, where the user connects to any of the BSs of the network, whichever BS offers the user the best link.
3.4 User Load Capacity under Large-scale Shadowing

3.4.3 Finite M SIR expressions

Further to the complex random SIR given by the expression (3.20), convergence to this limiting result is slow, typically requiring an impractical number of BS antenna elements. For example, \( M = 10^5 \) antenna elements were required in our simulations to come within 15\% of the mean of the limiting SIR. Similar observations are noted in [31]. Therefore, using the limiting expressions would result in effective interferences which are also somewhat unrealistic when a practically finite number of BS antennas are used. In part, this is demonstrated in Section 3.3.5.4, where we have observed a considerably smaller number of admissible users in the finite \( M \) case when compared to the large \( M \) scenario.

We now derive an SIR expression under the same model, but this time explicitly for a practically large finite number of BS antennas. Given the complexity of the uplink SIR expression in (2.16), which is further complicated by the contamination of the channel estimates, we are motivated to derive a simpler expression by means of approximations. The approximated SIR expression is derived by taking the additional expectation over the random fast (small-scale) fading (see Appendix A.1 for the derivation). This particular approximation is motivated by two reasons. (i) In the same vein as the channel hardening presented in Section 2.3.1.1, a consequence of the law of large numbers means that at for a massive MIMO BS which has a large finite value of \( M \), the \( w_{kl}^{H} h_{jk} \) terms of (2.16) are close to their expected value \( E\{w_{kl}^{H} h_{jk}\} \). (ii) As discussed in Section 3.1, a practical admission policy should be based on measurements which vary on a sufficiently slow time scale.

As seen earlier in Section 3.3.2, our method to obtain the effective interference requires that the total interference is expressed as a sum over \( i.i.d \) random quantities, e.g. as in (3.6). Therefore, our derivation also makes approximations where necessary to express the interference in this form. We derive the following uplink SIR expression under finite \( M \), for a user connected to BS \( j \), where the \( l \)-th BS has \( K_l \) connected users,

\[
SIR_j = \frac{1}{\frac{1}{M}(K_j - 1) + \sum_{l\neq j}^{L} \sum_{k}^{K_l} (\varphi_{kl} x_{kl} + \frac{1}{M} \sqrt{x_{kl}} + \frac{1}{M}(K_j - 1) \varphi_{kl} \sqrt{x_{kl}})}.
\]  

(3.22)
By defining $y_{kl} \triangleq \phi_{kl} x_{kl} + \frac{1}{\mathbb{M}} \sqrt{x_{kl}} + \frac{1}{\mathbb{M}} (K_j - 1) \phi_{kl} \sqrt{x_{kl}}$, we can write,

$$SIR_j = \frac{1}{\frac{1}{\mathbb{M}} (K_j - 1) + \sum_{l \neq j}^{L} \sum_{k}^{K_l} y_{kl}}.$$  

(3.23)

In contrast to the form of the large $M$ expression in (3.21), the finite $M$ expression in (3.23) has an additional constant term in the denominator which takes into account the interference from each of the $K$ intra-cell interferers. When using the MRC detector this intra-cell interference is significant, and needs to be accounted for in order to have an accurate expression for the SIR under finite $M$.

Figure 3.7 plots the resulting SIR based on (2.16) from the MRC detector for a fully loaded network under the simulation parameters in Table 3.3, with $M = 500$, and all instances of the involved random quantities generated through Monte Carlo simulation. Figure 3.7 also plots the limiting SIR based on (3.21), an SIR expression averaged over small scale fading without any approximations, and our approximation derived in (3.23). As shown in Figure 3.7, the approximation in (3.23) is much closer to the actual SIR for $M = 500$, when compared to the large $M$ expression in (3.20). This reasonable level of accuracy is still achieved by our approximation despite the fully loaded network being the worst-case, since there are a greater number of cross terms which have been omitted in the derivation of (3.22) (see Appendix A.1). The four interference terms in the denominator of (3.22) are all of the same order of magnitude for $M = 500$, their contributions varying moderately under the level of network loading, and hence are all required for an accurate representation of the total interference. Figure 3.7 shows that our SIR approximation (3.23) in general is slightly larger than the simulated exact result, where under this worst-case we observe a maximum difference of 0.6dB between the CDFs at outage probabilities around 0.1. Intuitively, such differences will result in slightly optimistic user-load capacity results.

A closer look at (3.22) and (3.23) shows clearly the dependence of $y_{kl}$ on the number of intra-cell interferers, $K_j - 1$. In the context of user admission, we consider the intra-cell interferers as tier 0 interferers where $n_0 = K_j - 1$. Given the form of (3.15) in Section 3.3.2, in order to solve for a single effective interference value for each $y_{kl}$, we require a single value of its mean $\mu_{y_{kl}}$ and variance $\sigma^2_{y_{kl}}$. Therefore, the dependency of $y_{kl}$ on $n_0$ in (3.22) is problematic since a complete solution for the effective interference would require values of $\mu_{y_{kl}}$ and $\sigma^2_{y_{kl}}$ for each value of $n_0$. 

Table 3.3: Main Simulation parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of BSs in network</td>
<td>$L$</td>
<td>19</td>
</tr>
<tr>
<td>Maximum number of users per BS</td>
<td>$K$</td>
<td>42</td>
</tr>
<tr>
<td>Pathloss exponent</td>
<td>$\gamma$</td>
<td>4</td>
</tr>
<tr>
<td>Lognormal shadowing standard deviation</td>
<td>$\sigma_{\text{shadow}}$</td>
<td>8 dB</td>
</tr>
<tr>
<td>Distance between BSs</td>
<td></td>
<td>2771 m</td>
</tr>
<tr>
<td>Exclusion radius of BS</td>
<td></td>
<td>100 m</td>
</tr>
<tr>
<td>Frequency reuse factor</td>
<td>$w$</td>
<td>{1, 3, 7}</td>
</tr>
<tr>
<td>Pilot sequence length</td>
<td>$\tau$</td>
<td>42</td>
</tr>
<tr>
<td>Data sequence length</td>
<td>$\tau_{ul}$</td>
<td>1</td>
</tr>
</tbody>
</table>

We can treat this by writing $y_{kl} = u_{kl} + n_0 v_{kl}$, and (3.23) can be expressed as,

$$SIR_j = \frac{1}{\frac{n_0}{M} + \sum_{l \neq j}^{L} \sum_{k}^{K} (u_{kl} + n_0 v_{kl})},$$

(3.24)

where,

$$u_{kl} = \phi_{kl} x_{kl} + \frac{1}{M} \sqrt{x_{kl}},$$

$$v_{kl} = \frac{1}{M} \phi_{kl} \sqrt{x_{kl}}.$$

This decomposition now allows us to readily apply a Gaussian approximation to the sum of the i.i.d. $y_{kl}$ interference terms in (3.23), with the moments of $y_{kl}$ expressed as,

$$\mu_{y_{kl}} = \mu_{u_{kl}} + n_0 \mu_{v_{kl}},$$

(3.25)

$$\sigma_{y_{kl}}^2 = \sigma_{u_{kl}}^2 + n_0^2 \sigma_{v_{kl}}^2 + 2 n_0 \sigma_{uv_{kl}},$$

(3.26)

where $\{\mu_{u_{kl}}, \sigma_{u_{kl}}^2\}$ and $\{\mu_{v_{kl}}, \sigma_{v_{kl}}^2\}$ are the mean and variance of $u_{kl}$ and $v_{kl}$ respectively, and $\sigma_{uv_{kl}}^2$ is the covariance between $u_{kl}$ and $v_{kl}$. With knowledge of the moments $\mu_{u_{kl}}, \mu_{v_{kl}}, \sigma_{u_{kl}}^2, \sigma_{v_{kl}}^2$, and $\sigma_{uv_{kl}}^2$, as previously shown in Section 3.3.2, the constraint (3.8) can be solved for the effective interference. However, unlike the derivation in Section 3.3.2, the effects of log-normal slow fading and best cell selection make the analytical derivation of these moments very difficult (as discussed in Section 3.4.2), and hence we adopt empirical methods.
The effectiveness of the Gaussian approximation was briefly analysed in Section 3.3.5.4, where the approximation was found to be sufficiently tight under the different set scheme. In this section, where the effects of shadowing are accounted for, remarks in [52] suggest that in general, a log-normal approximation to this sum is superior to the Gaussian approximation. However, they also demonstrate that such approximations improve in the presence of ULPC and best cell selection, where in the tail of the SIR distributions, i.e. the region of interest in the context of user-load capacity, the Gaussian approximation improves considerably. Further to this, as pointed out later in Section 3.6, under the different set pilot allocation employed in this work, the Gaussian approximation is sufficiently tight due to the increased number of interferers. The resulting accuracy of our approach is confirmed later by the derived user capacities in the numerical results of Section 3.4.5.
3.4 User Load Capacity under Large-scale Shadowing

3.4.4 User Admission Regions

3.4.4.1 A Tier-dimensioned BS Admission Region

A user admission policy should be able to quickly determine if the interference level experienced by all users, upon potential admission of the new user, is acceptable given a required QoS. Previously in Section (3.3.2), we derived a means to assign a single effective interference value to every interferer, which could then be used by a very simple non-cooperative admission policy which stipulated a maximum number of users per BS, for all BSs, in order to meet the required QoS. In this section we aim to derive more effective cooperative admission policies.

Previously in Section 3.3.2, one tier of interferers was considered when evaluating the constraint (3.14), since under large $M$ the effects of intra-cell interference (tier 0) were no longer present, and the effects from tiers higher than tier 1 were negligible. However, under the extensions to the channel model in this section, we have clearly seen the contribution from the number of tier 0 interferers, $n_0$ in (3.24). Furthermore, (as shown later in Section 3.4.5.2) a consequence of the extended channel model and best cell selection mean that tiers of interferers which are greater than 1 are no longer negligible. Intuitively interferers belonging to different tiers contribute an interference which will be characterised by different first and second moments\(^3\).

We extend the expression (3.14) to include $t = 0$ for the intra-cell interferers and sum over $T > 1$, where the total interference is now approximated by a sum of $T + 1$ normal distributions, which is itself also a normal distribution, with first and second moments given by $\mu = \sum_{t=0}^{T} n_t \mu_{y_t}$ and $\sigma^2 = \sum_{t=0}^{T} n_t \sigma^2_{y_t}$ respectively, and (3.8) becomes:

$$\frac{1}{S} - \frac{\sum_{t=0}^{T} n_t \mu_{y_t}}{\sqrt{\sum_{t=0}^{T} n_t \sigma^2_{y_t}}} \geq Q^{-1}(\alpha). $$

(3.27)

The region described by (3.27) is clearly non-linear with respect to the number of interferers $n_t$. In the same manner as (3.16) and (3.17), we seek a linear approximation to the region (3.27) to enable the assignment of an effective interference $\tilde{y}_t$ for each tier $t$ of interferers, allowing the

\(^3\) It is worth noting that in our definition of the set of BSs which belong to a given tier, we ignore the small differences in distance to the BS of interest amongst the BSs of this set, (and consequently differences in the moments of $y$). Under (3.27), accounting for these small differences and extending $T$ would bring negligible gain, and increase the complexity of admission control considerably.
constraint (3.27) to be approximated by,

\[
\sum_{t=0}^{T} n_t \tilde{y}_t \leq \frac{1}{S}. 
\] (3.28)

This allows for a simpler admission region to evaluate, which is of significant importance, since upon a new user being admitted (i.e., connecting to a BS), we must evaluate this admission region from the point of view of every BS of the network to ensure that the QoS is upheld for all users of the network.\(^4\).

Let us first visualise the region (3.27). In order to do so, we consider a three-dimensional region under \(T = 2\), employing parameters corresponding to a typical use case (minimum SIR = 10dB, \(M = 500\), and outage probability = 0.1) and empirically derive the first and second moments of \(y_t\) using (3.25) and (3.26), denoted by \(\mu_{y_t}\) and \(\sigma_{y_t}^2\). Figure 3.8(a) shows the region (3.27) in blue of the 3 dimensions, given by the number of users in the 3 relevant tiers, i.e., \((n_0, n_1, n_2)\).

\(^4\)In practice, this would be reduced to a set of nearby BSs, and as will be shown later the cardinality of the set being dictated by the propagation conditions.
A simple approximation would be to approximate the region (3.27) by the over bounding planes, formed by the points where the region (3.27) intercepts the \((n_0, ..., n_t)\) axis. Figure 3.8(a) shows this overbounding linear approximation in yellow to the exact region in blue.

The region specified by (3.27) shown in blue in Figure 3.8(a) appears almost linear when dissected along the \(n_0\) axis, and more curved when dissected along the \(n_2\) and \(n_1\) axis. Acknowledging this, a more accurate linear approximation (shown in red in Figure 3.8(b)) follows, defined by the hyperplane formed for a given value of \(n_0\), by the points where the region specified by (3.27) intersects the remaining \(T\) Euclidean axes. This serves as a very good approximation to (3.27) as seen from Figure 3.8(b), where both blue and red regions appear overlapped, and therefore we adopt this approximation to derive the effective interferences.

Such a linear approximation results in effective interferences \(\tilde{y}_t\) which are calculated from the presence of each tier \(t\) individually, together with a given number of \(n_0\) interferers. This corresponds to the intersection points of our chosen approximated region with each \(n_t\) axis for \(t > 0\), for a given value of \(n_0\) i.e., \((n_0, ..., 0, n_1, 0, ..., 0)\). We assign an effective interference of \(1/M\) to the tier 0 interferers, since the SIR expression (3.22) shows this is the actual interference they contribute. Therefore, the effective interference \(\tilde{y}_t\) for each tier \(t\), where \(t > 0\), can be calculated by fixing \(T = t\) and setting the mean and variance of the intra-cell interference as \(\mu_{y_0} = 1/M, \sigma^2_{y_0} = 0\) in (3.27). This yields,

\[
\tilde{y}_t = \begin{cases} 
    \mu_{y_t}/(1 + \frac{2}{z}(1 - \sqrt{1+z})), & \text{if } t > 0 \\
    1/M, & \text{if } t = 0 
\end{cases}
\]  

(3.29)

where,

\[z = \frac{4\mu_{y_t}(1/S - n_0/M)}{(Q^{-1}(\alpha))^2\sigma^2_{y_t}},\]

and \(\mu_{y_t}, \sigma^2_{y_t}\) are defined in (3.25) and (3.26). The result of (3.29) enables the quick evaluation of the interference levels of the network from the perspective of a given BS, by checking the simple condition in (3.28).

It is noteworthy that as a result of the pilot contamination under massive MIMO, the effective
interferences derived in (3.29) are dependent on the number of users connected to the intra BS. Specifically, this is a consequence of the intra-cell interference derived in (A.6) containing inter-cell terms as a result of pilot contamination.

Let $\mathcal{R}_{+}^{(T+1)}$ denote the positive orthant of $(T+1)$-dimensional Euclidean space, then the Tier-dimensioned BS admissible region is the region $(n_0, \ldots, n_T) \in \mathcal{R}_{+}^{(T+1)}$ where (3.28) and (3.2) are satisfied.

### 3.4.4.2 A BS-dimensioned Network Admission Region

Given a BS of interest, the constraint (3.28) specifies the total number of surrounding tier $t$ users $(n_1, \ldots, n_T)$ that can be tolerated for a given QoS, and therefore forms the basis for administering a network user admission policy. A more efficient network user admission policy would consider the number of users which are connected to individual BSs $K_l$, as opposed to only considering the sum of users in the individual tiers $n_t$. That is, the network user admission policy will be governed by a network admission region which is specified by the number of users connected to each BS.

If we at first consider (3.28) for a particular BS $j$, such a condition involves knowledge of $n_t$ for $0 < t < T$, that can only be evaluated under cooperation between the total set of all BSs belonging to all relevant $T+1$ tiers denoted by $V^{(j,T)}$. Consequently, $V^{(j,T)} = \bigcap_{t=0}^{T} V_t^{(j)}$ and $|V^{(j,T)}| = 1 + \sum_{t=1}^{T} 6t$, where $V_t^{(j)}$ denotes the set of BSs that belong to particular tier $t$ of BS $j$. Under a network admission control policy, (3.28) would have to be satisfied on a per BS basis, and for every BS in the network, i.e.,

$$\sum_{n=0}^{T} \tilde{y}_t^{(j)} \sum_{l \in V_t^{(j)}} k_l \leq \frac{1}{S}, \forall j \in V_L,$$  

(3.30)

where $V_L$ denotes the full set of $L$ BSs of the network. The superscript in $\tilde{y}_t^{(j)}$ explicitly indicates that the effective interference of each tier has a dependence on the number of intra-cell users connected to BS $j$ from the presence of $n_0$ in (3.29). Let $\mathcal{R}_+^L$ denote the positive orthant of $L$-dimensional Euclidean space, then the network admissible region is the region $(k_1, \ldots, k_L) \in \mathcal{R}_+^L$ where (3.1) and (3.30) are satisfied.

If we assume that the network meets the required QoS in its current state, we can specify a
much more efficient and practical network admission region to evaluate. Upon evaluating a user’s admission to the network through a connection to BS \( j \), we can specify the region which only involves the BSs affected by such an admission given finite \( T \), i.e.,

\[
\sum_{t=0}^{T} \sum_{l \in \mathcal{V}(j')} \tilde{y}_l^{(j)} k_t \leq \frac{1}{S}, \quad \forall j' \in \mathcal{V}(j,T).
\]  

Let \( w \in \mathcal{V}(j,2T) \) and \( W = |\mathcal{V}(j,2T)| \), then the BS-dimensioned network admission subregion for admitting a user to BS \( j \), is the region \( \{k_w\} \forall w \in \mathcal{R}_+^W \) where (3.1) and (3.31) are satisfied.

In the interests of visualizing a BS-dimensioned region, Figure 3.9 shows a trivial example of a three BS network, \( L = 3 \), with a BS located at each vertex of an equilateral triangle. Therefore, every BS of the network will be affected by two tiers (i.e., \( T = 1 \)), the number of interferers connected to itself, and the sum of interferers connected to the two other BSs. Assuming a pilot sequence length of \( \tau = 42 \), the constraint (3.1) is shown by the yellow shaded cube in Figure 3.9. Given a particular QoS, the BS-dimensioned admission region is the region formed by meeting both constraints (3.1) and (3.30) on the number of interferers allowed in each of the 3 cells, \( (k_0, k_1, k_2) \), and is shown in Figure 3.9 by the region in red.
3.4.5 Numerical Results

3.4.5.1 Scenario

In addition to the setup described in Section 3.3.5, we model the effects of shadowing in the urban macrocell environment using a log-normal shadowing with a variance of 8dB and path loss exponent of 4 [7, 34, 53]. Users are distributed uniformly at random locations over the network coverage area, and connect to whichever BS offers them the best channel gain. Note in the presence of log-normal shadowing, this is not necessarily the closest BS.

In order to evaluate the accuracy of our user admission region defined in Section 3.4.4, we wish to compare it to a somewhat realistic finite $M$ simulation. Given the use of experimental massive MIMO test beds employing 128 elements [48, 49] we select an $M$ of similar order, where $M = 500$.

3.4.5.2 Tiers $T$ to consider

From Section 3.4.3, we have identified that for a finite number of BS antennas, the intra-cell tier 0 interferers are significant and need to be considered. In order to understand how many additional tiers need to be considered, we have computed the moments of the interference from each tier relative to tier 1, (i.e., $\mu_t/\mu_1, \sigma_t^2/\sigma_1^2, t > 1$) from Monte Carlo simulation in Table 3.4. For the typical urban macrocell scenario ($\gamma = 4, \sigma = 8dB$), it can be seen from the moments of the interference that tier 2 interferers are significant while tier 3 are negligible. Therefore for such a scenario, the constraints (3.30) and (3.31) should be met under $T = 2$. Consequently, the corresponding Monte Carlo simulations in this section include only the interference contributions from users connected to the closest 18 BSs surrounding a particular BS.

At the two other extremes of propagation environments, we have the situation where tier 2 and tier 3 are both negligible or both significant.

3.4.5.3 Effective Interferences

In Table 3.5, we take a look at the effective interferences computed from (3.29) under various QoS requirements, where the BS of interest is half loaded, i.e. $n_0 = \tau/2$. As expected, the computed
Table 3.4: Values $\mu_{y_t}$ and $\sigma^2_{y_t}$ for tiers $t \in \{1, 2, 3\}$ relative to tier 1, for $y$ (3.23), under $M = 500$ different propagation conditions. Typical urban macrocell is $\gamma = 4$, $\sigma_{\text{shadow}} = 8$ dB.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma$</th>
<th>$\mu_{y_1}$</th>
<th>$\sigma^2_{y_1}$</th>
<th>$\mu_{y_2}$</th>
<th>$\sigma^2_{y_2}$</th>
<th>$\mu_{y_3}$</th>
<th>$\sigma^2_{y_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>4 dB</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.00289</td>
<td>0.00141</td>
<td>0.00001</td>
<td>0.00000</td>
</tr>
<tr>
<td>4</td>
<td>8 dB</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.14201</td>
<td>0.12500</td>
<td>0.01995</td>
<td>0.01439</td>
</tr>
<tr>
<td>3</td>
<td>12 dB</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.46289</td>
<td>0.42626</td>
<td>0.23574</td>
<td>0.21781</td>
</tr>
</tbody>
</table>

Table 3.5: Effective Interference $\tilde{y}_t$ calculated for different QoS, with $n_0 = \tau/2$, under full frequency reuse $w = 1$, $M = 500$ and $\gamma = 4$, $\sigma_{\text{shadow}} = 8$ dB.

<table>
<thead>
<tr>
<th>SIR</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 0.05$</th>
<th>$\alpha = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{y}_0$</td>
<td>0.00200</td>
<td>0.00200</td>
<td>0.00200</td>
</tr>
<tr>
<td>$\tilde{y}_1$</td>
<td>0.00112</td>
<td>0.00114</td>
<td>0.00145</td>
</tr>
<tr>
<td>$\tilde{y}_2$</td>
<td>0.00015</td>
<td>0.00016</td>
<td>0.00020</td>
</tr>
</tbody>
</table>

Effective interference of the tier 1 and tier 2 interferers increases as the QoS increases. We assign an effective interference of $1/M$ to tier 0 since this is their actual interference contribution, and is therefore unchanged by the required QoS.

3.4.5.4 Tier-dimensioned BS Admission Regions

In this section we take a look at numerical results for the Tier-dimensioned BS admission region $(n_0, \ldots, n_T) \in \mathcal{R}_+^T$, given by (3.28). Figure 3.10 depicts the admission regions from the perspective of a single BS of the network, where $T = 2$, for a range of network QoSs. In order to present the figures in two dimensions, the number of intra-cell users has been fixed, where $n_0 = \tau/2$. As discussed in Section 3.4.4, the admission region is the region which satisfies both constraints, constraint (3.2) indicated by the rectangular region, and constraint (3.28), indicated by the triangular region. Therefore, the admission region is the intersection of these regions, and indicated by the hashed areas in Figure 3.10. The rectangular regions show the constraint (3.2) for the fre-
Figure 3.10: Tier-dimensioned BS admission regions for a particular BS for various QoS requirements, under the simulation parameters in Table 3.3, where $M = 500$. The admission region is the grey hashed area, defined by the number of admissible tier 1 and 2 interferers, $n_1$ and $n_2$, with fixed $n_0 = \tau / 2$. The orange squares are the regions that result from the uplink training constraint (3.2) under $w \in \{1, 3, 7\}$.
frequency reuse factors $w = \{1, 3, 7\}$. The rectangular region which has a solid outline indicates the frequency reuse factor currently employed (with the dashed outline indicating not currently employed) for the particular scenario.

Figure 3.10(a) shows the BS admission region under a QoS where the interference limits are met easily, and we are only constrained by the uplink training resources from (3.2) for $w = 1$. That is, the triangular region is much larger than the rectangular region offered by full frequency reuse, and hence the later constraint defines the applicable admission region. Figure 3.10(b) shows that a higher QoS reduces the admission region, and both (3.2) and (3.28) are bounding constraints. If a further increase in QoS is desired, as depicted in Figure 3.10(c), the admission region is again bound by both constraints. Importantly, the region offered by full frequency reuse is still greater than what would be offered by switching to a frequency reuse factor of 3 (the middle dashed rectangular area). Yet another increase of the required QoS in Figure 3.10(d) results in the small dashed triangle region from the constraint (3.28) under $w = 1$, no longer being larger than the rectangular region from constraint (3.2) under $w = 3$. This increase in the required QoS of scenario (d) would therefore prompt the employment of frequency reuse factor 3 in the network. Naturally, such a change results in smaller effective interferences, illustrated by the region (3.28) under $w = 1$ (shown by the smaller dashed triangle) being now much larger under $w = 3$ (shown by the larger solid triangle). The new admission region for the network for this QoS under $w = 3$ is therefore constrained by (3.2), as shown by the hashed area in Figure 3.10(d).

### 3.4.5.5 Accuracy of the BS Admission Regions

The Tier-dimensioned BS admission regions are shown by the hashed areas of Figure 3.10. In the scenarios of Figure 3.10(b) and Figure 3.10(c), the admission regions are partly constrained by (3.28). Given that the methods employed have made statistical and analytical approximations in deriving the admission region, we wish to look at their accuracy, in particular when constrained by (3.28). We do so in this section by selecting points along the boundary of the specified admission region, and then evaluate the QoS that is actually achieved using Monte Carlo simulation. In Figure 3.10(b) and Figure 3.10(c) we have arbitrarily selected the points along the region boundary shown by the blue dots. The achieved QoS is evaluated twice, firstly using the SIR expression without any averaging or approximations as expressed in (2.16), and secondly by using the approximate
Table 3.6: The target QoS and the QoS achieved through simulation for a given point of the region boundary (indicated by the blue dots in Figure 3.10). The simulation QoS is evaluated for both the target outage probability and SIR (in dB), using an SIR derived with and without approximations.

<table>
<thead>
<tr>
<th>Target QoS</th>
<th>Achieved QoS, SIR (2.16)</th>
<th>Achieved QoS, Approx SIR (3.23)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIR = 6.00, $\alpha = 0.10$</td>
<td>SIR = 5.65, $\alpha = 0.10$</td>
<td>SIR = 6.02, $\alpha = 0.10$</td>
</tr>
<tr>
<td>SIR = 6.00, $\alpha = 0.15$</td>
<td>SIR = 6.00, $\alpha = 0.15$</td>
<td>SIR = 6.00, $\alpha = 0.10$</td>
</tr>
<tr>
<td>SIR = 8.00, $\alpha = 0.10$</td>
<td>SIR = 7.70, $\alpha = 0.10$</td>
<td>SIR = 8.02, $\alpha = 0.10$</td>
</tr>
<tr>
<td>SIR = 8.00, $\alpha = 0.13$</td>
<td>SIR = 8.00, $\alpha = 0.13$</td>
<td>SIR = 8.00, $\alpha = 0.10$</td>
</tr>
</tbody>
</table>

SIR expression which enabled the computation of the admission region as expressed in (3.23). The Monte Carlo simulation results in Figure 3.6 show that the QoS achieved using both evaluations are close to the target QoS for the selected points in Figure 3.10. The close alignment between the target QoS and the simulation result using the SIR approximation in (3.23) demonstrates the accuracy of the methodology, and effectiveness of the applied Gaussian and over-bounding linear approximations (Section 3.4.4.1). Therefore, the slightly lower achieved QoS in simulation from the evaluation of the instantaneous SIR expression (2.16) is the result of the inaccuracies of the SIR approximation in (3.23). Given Figure 3.7 shows that the CDF of the SIR approximation (3.23) follows the same shape as the CDF of the SIR (2.16) with a slightly larger mean, we would naturally expect a slightly optimistic (i.e., larger) admission region, and hence the slightly lower QoS actually achieved at the region boundary.

### 3.5 Conclusions and Future Work

In the first section of this chapter, we have shown that the different pilot set scheme which allocates unique orthogonal pilot sets to each BS is superior in the context of maximising user-load capacity, when compared to reusing pilot sets amongst BSs. This is despite the fact that the different pilot set scheme results in a much larger number of pilot contaminators. In the process of investigating the resulting user-load capacity under these schemes, we developed methods of translating an understanding of the uplink SIR into a simple noncooperative user admission policy.

In the second section of this chapter, we extended those methods to support more realistic channel models and operating conditions. In particular, a reasonably accurate approximation was derived for the uplink SIR under practically finite $M$ to allow the calculation of an effective in-
terference, under log-normal shadowing and realistic BS selection. Under such settings for urban macro cell deployment, we show that in fact only the first two tiers of inter-cell interference need to be considered. We then derived the effective interferences for each tier, which as a consequence of the pilot contamination on the network, had a dependence on the number of intra-cell users. These results were used to specify an efficient cooperative network admission control policy in order to allow the network to guarantee a certain QoS for all users of the network. The resulting BS admission regions were shown to be accurate, with small deviations in SIR of 0.3dB (for a fixed outage probability) between the target and achieved QoS at the region boundaries.

In order to extend the admission policies presented in this chapter into a comprehensive user admission policy, naturally user-load capacity in the terms of the downlink needs to be considered. Such a policy would then be governed by an admission region which is additionally constrained by the admission region in the downlink. This requires an analysis of the downlink interference, which due to the fixed locations of the BSs does not result in i.i.d. interference terms and therefore requires alternative methods to analyse.

The ability to use effective interferences to quickly capture the state of interference on the network also suggests its use in BS selection algorithms. Future work could examine how the effective interferences, in combination with other schemes such as fractional uplink power control, can be used by the BS selection algorithms to minimise the interference and consequently maximise sum-rate capacity on the network.
In light of future networks featuring the use of smaller cells and millimetre wave bands resulting in channels with stronger line-of-sight (LoS) components, we turn our attention to line-of-sight (LoS) channel models to explore the channel estimation problem where spatial correlation exists between the elements of the large antenna array. In this chapter we focus on the scenario of a single user connected to a single massive MIMO base station over a pure LoS channel. We investigate the form and performance of the MMSE and linear MMSE channel estimators, both in terms of estimation error and computational costs. This motivates the development of a new estimator, the assisted LMMSE (A-LMMSE) which offers estimation performance close to that of the optimal MMSE estimator, but with a runtime complexity of the same order as the LMMSE estimator. We then look at efficient implementations of the A-LMMSE estimator and present the channel estimation performance together with the symbol error rate (SER) performance of such implementations. Given our focus on the large antenna array present at the massive MIMO BS, we examine the performance with an increasing number of antenna elements, and analytically derive bounds for the resulting estimation error when the number of antenna elements tends to infinity. We extend the pure LoS channel model to a complex Rician channel model, where the resulting estimators are analysed and similar results are obtained.

4.1 Introduction

COHERENT detectors (i.e., detectors which utilise prior information about the communication channel) offer a very significant performance gain over non-coherent detectors [6] and consequently are widely used. At the BS, they necessitate the acquisition of estimates of the desired user’s channel in order to perform detection of the desired user’s data transmission.

Consider the desired user transmitting symbol $x$ over a channel $g$. The baseband channel can
be modelled by the complex quantity \( g = \sqrt{\beta} h \), where \( \beta \) captures the slow fading effects of the channel. In this chapter, we assume that \( \beta \), i.e., the combined effects of distance-based path loss and shadowing, are known at the receiver. These quantities tend to be easier to acquire because they change relatively slowly and therefore we focus on the problem of estimating the remaining effects of the fast fading, \( h \). At this point we make no assumptions about the distribution of the complex channel \( h \).

We evaluate the quality of the channel estimates under the widely used objective function of mean squared error (MSE), where our ideal channel estimate is

\[
\hat{h} = \arg\min_{\hat{h}} E\{|\hat{h} - h|^2\}.
\]

Typically, the BS attempts to estimate the channel of the desired user \( \hat{h} \) by using deterministic methods such as least squares (LS) or through stochastic methods such as linear minimum mean squared error (LMMSE) estimators. First, let us restrict ourselves to narrow band signals transmitted on a single carrier from a single user to a given BS. Intuitively, a deterministic method would divide the coherence time of the channel into \( \tau \) segments, or symbols, so that it can be sampled and averaged in an attempt to estimate the channel. In a practical setting, we would rarely be motivated to utilise the entire \( \tau \) symbols for the purposes of channel estimation, since we must use symbols of the coherence time for the coherent reception and transmission of user data. Under this scenario, the user transmits \( \tau \) symbols of ”1” over the channel, represented in (4.1) as a column vector of ones \( s \in \mathbb{C}^{\tau \times 1} \), where the BS receives \( \tau \) symbols described by the vector \( y \in \mathbb{C}^{\tau \times 1} \) in the presence of noise, \( n \in \mathbb{C}^{\tau \times 1} \), i.e.,

\[
y = sh + n.
\]  

(4.1)

Given the well-known LS solution [54, Eq. 2.2.6], where \( y \) is projected onto \( sh \), we can apply the estimation matrix \( W \) as,

\[
\hat{h} = Wy = (s^H s)^{-1} s^H y = s^H y / \tau
\]  

(4.2)
4.1 Introduction

For our simple example, the reduction to (4.2) is effectively the intuitive approach to the problem i.e., to simply average $\tau$ received symbols of the $y$ vector.

In a multi-user scenario, the users transmit unique orthogonal pilot sequences so that their transmissions can be separated at the BS. Given a pilot sequence length of $\tau$ symbols, the number of strictly orthogonal pilot sequences within a given pilot set is limited to $\tau$, and can therefore support a maximum of $\tau$ users. The pilot sequences are transmitted by all $\tau$ users within the available coherence time, and represented by the matrix $S \in \mathbb{C}^{\tau \times \tau}$. Additionally the $\tau$ users each have an independent channel, represented by the elements of the column vector $h = [h_1, \ldots, h_\tau]$. We can then describe the received vector as

$$ y = Sh + n. \quad (4.3) $$

Given our orthogonal pilot set $S$ is by nature full rank $^1$, we can apply the LS projection in a similar fashion to the single user case to yield unique estimates of all users’ channels given by

$$ \hat{h}_{LS} = (S^H S)^{-1} S^H y. \quad (4.4) $$

Stochastic methods of solving this estimation problem generally yield better results, but are typically more expensive computationally and require additional knowledge of the random quantities. Of these methods, the widely used LMMSE estimator [54, 55] offers a relatively computationally efficient solution, where only knowledge of the first and second moments of the quantities involved need to be known. Given both $h$ and $y$ are zero mean quantities, revisiting the single user scenario described by (4.3), we apply the estimation matrix $W$ of the LMMSE estimator,

$$ \hat{h} = Wy \quad (4.5) $$

$$ = \gamma_{yy}^{-1} y $$

$$ = \sigma_n^2 s^H (\sigma_n^2 s s^H + \sigma_n^2 I)^{-1} y \quad (4.6) $$

$$ = (\sigma_n^2 / \sigma_n^2 + \tau)^{-1} s^H y. \quad (4.7) $$

$^1$S contains $\tau$ linearly independent rows (or columns) and hence is non-singular, resulting in the invertible matrix $S^H S$.
where the final result in (4.7) has been derived in Appendix B.1. In this single user case, the similarities between the stochastic and deterministic approaches are easily observed, e.g., as the noise power is decreased, the result in (4.7) heads towards (4.2).

In the multi-user case, (4.5) is naturally extended as,

$$\hat{h} = E \{ hh^H S^H \} (E \{ Shh^H S^H \} + I \sigma_n^2)^{-1} y.$$  

(4.8)

Such deterministic and stochastic approaches to channel estimation are employed by the macro-cell BS, where the number of BS antennas elements $M$ can be up to 8 (based on current standards such as Release 10 of LTE-Advanced [2]).

If we now turn to the massive MIMO scenario where $M$ is much larger, we have more observations which we can use to better estimate the channel $h$ from a user to a single element of the BS. Intuitively, one would expect a better estimation given that there are now more observations. However, under the typical Rayleigh fading channel model which is the most common model adopted in wireless communications literature, the channel from the desired user to each BS antenna element is modelled by a statistically independent circularly symmetric complex Gaussian random variable. Therefore under such a model, the additional observations from the extra antenna elements of the large array do not offer any further knowledge on $h$. In reality however, some level of correlation between the channel observations at different antenna elements exists due to LoS channel components. This motivates the use of LoS channel models which capture the correlation between antenna elements.

As mobile carriers are pressured to look outside the currently saturated bands to the abundant millimetre wave spectrum, it has been shown that beamforming techniques using antenna arrays can compensate for the increase in free space path loss due to the increase in carrier frequency [29,30] (as given by Friis’ equation [53]). However, signals at these higher frequencies are severely vulnerable to shadowing, resulting in outages under non-LoS channels [30]. This dependence on LoS channels renders the study of such channels vital for the development of millimetre wave communication systems.

This chapter starts by investigating the problem of channel estimation at a BS equipped with a large antenna array in the single user scenario under pure LoS channels. The optimal MMSE
estimator and popular LMMSE estimators are derived for the estimation problem, and we demonstrate there is in fact a large gap in performance as the number of antenna elements increase. This motivates the development of a new assisted LMMSE (A-LMMSE) estimator which exploits the large antenna array to obtain knowledge of the desired user’s AoA, to further assist a LMMSE based estimator in exploiting the additional observations of the large antenna array. Consequently, it is able to offer channel estimation performance close to that of the optimal MMSE estimator, with a low computational complexity close to that of the LMMSE estimator.

Given most related channel estimation work in the massive MIMO literature is focused on mitigating the effects of pilot contamination, we discuss such works where they relate to single user channel estimation in LoS channels, and defer the multi-user multi-cell discussion to Chapter 5. These mentioned works adopt LoS channel models which account for a varied number of distinct LoS paths, where in this chapter we investigate the channel estimation problem under the special case of a single LoS path. Common to most of the proposed methods is the exploitation of the sparsity, or low-rank property of the single user channel matrix. These properties are a result from the small number of LoS components approaching the large BS linear antenna array [18–20, 56], where the small number of components is a consequence of the assumed limited narrow angular spread of the users AoA in the macrocell scenario. Consequently, the channel estimation problem can reduce to the estimation of a smaller set of parameters under alternative decompositions of the channel or covariance matrices, or an estimation problem over smaller domains.

Reducing the dimensionality of the problem through eigenvalue decompositions (EVD) of the covariance matrices is utilised in several works [18, 57, 58]. The work in [57] goes further to identify the conditions when such decompositions can be made without the use of dedicated pilot transmissions, and therefore provides a means to blind channel estimation. In order to learn about the spatial components of the channel, we make use of highly efficient methods such as the FFT which is also used in [19, 56], but in significantly different ways. The proposed solution in [19] utilises the FFT and inverse FFT to perform a window filtering in the spatial domain around a desired angular region, where this filtering is applied as a post processing stage after a deterministic LS estimation as described by (4.4). The work in [56] employs a more careful use of the FFT by proposing a spatial channel decomposition which also takes into account the imperfections of the
discrete methods of the FFT. While we exploit the efficiency of the FFT in our estimation scheme, in contrast to such proposals, we use it to compliment stochastic methods which deal better with the impairments of additive white gaussian noise under low SNR. The channel estimation performance of stochastic LMMSE estimators which are formed based on assumed perfect a-priori information of the user AoAs is analysed in [18] for the multi-cell environment. In this work we provide a holistic approach to the development of such estimators in the single user environment by further taking into account the acquisition of imperfect AoA information, and its utilisation in the construction of the required covariance matrices. The proposed A-LMMSE estimator is agnostic to the signal structure, resulting in a robust estimator that performs well even with poor a-priori information.

The main contributions of the chapter are summarised as follows,

- An analysis on the effects of improved spatial information on the LMMSE estimator performance as \( M \) grows.
- The derivation and analysis of the novel A-LMMSE estimator, together with the optimal MMSE and LMMSE estimators, for the single user channel estimation problem under LoS and Rician channels. We demonstrate that the A-LMMSE estimator is able to deliver a channel estimation performance which is near to the optimal MMSE.
- An evaluation of the runtime complexities of the above channel estimators which demonstrate that the A-LMMSE estimator has a runtime complexity of the same order as the highly efficient LMMSE, despite an estimation performance which is close to the computationally expensive optimal MMSE. We propose practical and computationally efficient implementations of the A-LMMSE estimator by acquiring the AoA information using highly efficient FFT based methods, and a reduction of the covariance terms to an elegant closed form by a reformulation of the problem in the spatial frequency domain. Furthermore, we propose low complexity variants of the A-LMMSE estimator which utilise banks of precomputed estimators under the practical scenario of memory constrained devices.
- The analytical derivation and validation of simple and useful performance bounds of the LMMSE and low complexity variants of the A-LMMSE estimators under LoS and complex Rician channel models, by utilising the results of Wiener smoothing theory. The analytical limits are confirmed by the results of Monte Carlo simulation for finite \( M \). Additionally,
the analysis demonstrates the desirable property of the limiting performance of the low complexity A-LMMSE variant being agnostic to the user’s AoA distribution.
4.2 The Line-of-sight (LoS) Channel Model

A LoS channel exists when radio waves propagate from transmitter to receiver without, or with negligible effects of multipath propagation. At the receiver we attempt to estimate the channel in order to facilitate the detection of the data symbols transmitted by the desired user, e.g., by using a MRC detector. In the macro-cell massive MIMO context, pure LoS channels in the urban environment are rare but do provide a fundamental framework for the estimation problem of more realistic channel models. For example, the widely used channel models adopted from recent studies of millimeter wave propagation are extensions of the LoS channel model [30], where the channel model consists of a handful of distinct LoS paths. More traditional models, such as the Rician channel model, again feature a LoS channel superimposed on the Rayleigh multipath component.

If we consider the LoS channel where a single electromagnetic wave of frequency $f_c$ propagates from the mobile user to the antenna array without obstruction, over a distance much larger than the array itself, the circular wavefronts can be assumed planar as shown in Fig. 4.1. This wave will require some finite time $t_1$ to reach antenna element 1, where it will arrive with a random phase $\phi$ (which accounts for uncertainties such as synchronisation errors) and impinge with a random angle-of-arrival (AoA) $\theta$, in the horizontal plane (azimuth), based on the user’s location. In our model, these are both assumed to be uniformly distributed random quantities where $\phi \sim \mathcal{U}[0,2\pi]$ and $\theta \sim \mathcal{U}[0,\pi/2]$. (The support for the modelled $\theta$ is explained later in Section 4.3.3.1). The wave travels an additional distance of $d \cos(\theta)$ to reach antenna element 2, and therefore the observations at the $m$-th antenna element are

$$y_m(t) = \cos \left( 2\pi f_c \left( t - t_1 - \frac{(m-1)d}{c} \cos \theta \right) - \phi \right) + n_m(t), \quad (4.9)$$

where $c = 3 \times 10^8$ m/s is the speed of light and $n_m$ is the additive Gaussian noise $\sim \mathcal{N}(0,\sigma_n^2)$. Using a particular time instance, say $t = t_1$, the observations at each antenna element relative to the first are given by,

$$y_m(t_1) = \cos \left( 2\pi f_c \left( \frac{(m-1)d}{c} \cos \theta \right) + \phi \right) + n_m \quad (4.10)$$

$$= \cos \left( (m-1)\pi \cos \theta + \phi \right) + n_m, \quad (4.11)$$
where (4.11) has been derived for an array with elements at the critical spacing of $d = \lambda/2$, and $f_c = c/\lambda$. The effects of the channel are summarised by the cosine term in (4.11). The antenna spacing is favourable since when the separation is greater than $\lambda/2$, there can be several pairs of grating lobes present [6]. In addition to focusing less power in the direction of the user, grating lobes introduce ambiguity when trying to ascertain the direction of the user, which is a key component to channel estimation schemes explored later in this section. Spacing elements much closer than $\lambda/2$ introduces significant mutual coupling between antenna elements, necessitating the use of a more complicated model than (4.9) [59].

The result in (4.11) shows the phase shift of the received cosine at the $m$-th antenna element as $(m - 1)\pi \cos \theta$. This strong correlation between the channels of the antenna elements under the LoS channel makes the estimation problem significantly different to estimating the channel under traditional Rayleigh fading channel models. Under Rayleigh fading channel models where the channel at each element is modelled as the summation of many LoS and non-line-of-sight (nLoS) components using independent Gaussian random variables, there is no correlation between the channels of the antenna elements.

In baseband processing of modern communication systems, both in-phase and quadrature arms
are used in the transmission of symbols, and hence complex quantities are used to represent the baseband equivalent channel. Typically the point-to-point complex Rayleigh fading channel model is described by the complex quantity \( g = \sqrt{\beta} h \), with \( h \sim \mathcal{CN}(0,1) \). Under our LoS channel model, where a single electromagnetic wave arrives at antenna element \( m \), we have the complex quantity \( g_m = \sqrt{\beta} h_m \), where \( h_m = e^{-j\alpha} \). We assume the effects of slow fading \( \beta \) are known through standard methods of path loss estimation, and is common to all paths from the desired user. Consequently, the estimation problem reduces to estimating the complex \( h \) component. For simplicity in this chapter, we assume the known slow gains to be unity.

In order to simplify the analysis and notation, we first consider the estimation problem of the real component of \( h \) from the in-phase or real component of the observations \( y \). A very similar approach would be adopted in practice to make use of both the in-phase and quadrature components of \( y \), and is covered later by the complex channel model adopted in Section 4.5.

Given that the single user case does not require the use of pilots (in time) to separate users, and the correlation amongst the antenna elements is captured well by (4.11), we use (4.11) as a basis for our real-valued, spatial, baseband-equivalent channel model. Re-expressing (4.11) in vector form,

\[
y = h(\theta, \phi) + n,
\]

where \( y, h \) and \( n \) are vectors of length \( M \), and the \( m \)-th element of the channel vector \( h \) is given by \( h_m = \cos((m-1)\pi \cos \theta + \phi) \), \( 1 \leq m \leq M \). The noise vector \( n \) models the presence of zero-mean independent AWGN, with elements \( n_m \sim \mathcal{N}(0, \sigma_n^2) \). The baseband channel model in (4.12) implies the transmission of a real pilot symbol of constant unit amplitude, and we therefore define a corresponding channel SNR at the BS as \( \eta = E\{|h_m|^2\} / \sigma_n^2 = \frac{1}{2\sigma_n^2} \).

### 4.3 Channel Estimation under the LoS Channel Model

In this section we explore a variety of approaches to the single user channel estimation problem of estimating the channel vector \( h \), given the observations \( y \) in (4.12).
4.3 Channel Estimation under the LoS Channel Model

4.3.1 MMSE Estimation

The estimator which produces the minimum mean squared error, i.e., produces the estimate that minimises \( E\{\|h - \hat{h}\|^2\} \), is, as the name suggests, the minimum mean squared error (MMSE) estimator. The estimator is the conditional expectation [55],

\[
\hat{h}_{MMSE} = E\{h|y\}. \tag{4.13}
\]

Although this estimator results in the lowest MSE, it must utilise a great deal of a-priori information in order to find the required distribution of the unknown quantity conditional on the observation vector \( y \). Given that the estimation of the unknown channel vector \( h \) is the equivalent of estimating each element \( h_m \) individually [55, Eq. 11.10],

\[
E\{h|y\} = [E\{h_1|y_1 \ldots y_M\} \ldots E\{h_M|y_1 \ldots y_M\}]^T, \tag{4.14}
\]

for now let us first consider the estimation of just a single \( m \)-th element \( h_m \) of the channel vector \( h \). Given the uniform distributions of \( \theta \) and \( \phi \), this simplifies to, (see Appendix B.2 for the derivation),

\[
\hat{h}_{MMSE} = E\{h_m|y\} = \frac{\int_0^{\pi/2} \int_0^{2\pi} h_m(\theta, \phi)g(\theta, \phi)d\theta d\phi}{\int_0^{\pi/2} \int_0^{2\pi} g(\theta, \phi)d\theta d\phi}, \tag{4.15}
\]

where

\[
h_m(\theta, \phi) = \cos((m - 1)\pi \cos \theta + \phi) \tag{4.16}
\]

\[
g(\theta, \phi) = e^{\frac{1}{2\sigma^2} \sum_{m=1}^{M} (y_m - \cos((m - 1)\pi \cos \theta + \phi))^2}. \tag{4.17}
\]

Using (4.15) for all \( m \) elements to form the estimation vector \( \hat{h}_{MMSE} \), we have an estimation of the channel vector \( h \) resulting in the lowest MSE possible.

The integrals in (4.15) do not have a closed form or an efficient approximation. The computational complexity with numerically evaluating the two double integrals per estimation of \( h_m \) makes it unsuitable for practical use. It is evident that even simply evaluating the functions \( h_m(\theta, \phi) \) and \( g(\theta, \phi) \) involves several operations, including cosines, multiplications, and additions. On inspec-
Figure 4.2: The resulting NMSE for increasing antenna lengths, where $M$ is the number of equally spaced antenna elements, $\eta = -3 \text{dB}$.

The summation over $M$ in (4.17) generates a number of operations of order $O(M^2)$. The numerical integration of such a function requires a grid size which sufficiently captures the changes of the highest frequency components of the function. Looking at (4.16) and (4.17) as functions of $\pi \cos(\theta)$ (instead of $\theta$), the relevant frequency is given by $M$. As a result we have $\cos^3(M)$ terms in the numerator and therefore the highest frequency component of the numerator is of order $3M$. The integration over $\phi$ is not affected by the array length $M$, and therefore integrating over $\theta$ and $\phi$ is of complexity order $O(M)$. Given that we require estimates for all $M$ elements, the overall complexity of the MMSE estimator for the vector $h$ is $O(M^4)$. Under large practical $M$, i.e., $64 \leq M \leq 256$, the resulting complexity from deriving the estimation is closer to $M^5$, given that the multiplicative costs are of similar magnitude to $M$. Such computational complexity inhibits the use of the MMSE estimator for use in practice for large antenna arrays. Further to the runtime complexity, (4.17) is problematic to evaluate using a fixed-point processor since for practically large antenna arrays e.g., $M = 128$, the resulting exponential term is of very small magnitudes (in the order of $10^{-30}$).

The normalised mean squared error (NMSE) is a commonly used measure of estimation per-
4.3 Channel Estimation under the LoS Channel Model

By taking into account the estimates produced for the entire antenna array, we define this measure as

\[
NMSE (dB) = 10 \log_{10} \left( \frac{MSE}{E \{ ||h||^2 \}} \right)
\] (4.18)

\[
= 10 \log_{10} \left( \frac{E \{ ||h - \hat{h}||^2 \}}{E \{ ||h||^2 \}} \right).
\] (4.19)

Figure 4.2 shows the NMSE in dB of the MMSE estimator as \( M \) increases. The figure includes results for values of \( M \), i.e., \( \{4, 32\} \), which are typically too small for practical deployments of a massive MIMO BS. This is especially true in the context of millimetre wave systems where LoS channels are prevalent, and where the array gain from large antenna arrays are required to compensate for the increased path loss at the higher millimetre wave band. However, in the simulation results of Chapter 4 and 5 these lower values of \( M \) are included to provide better insight into the behaviour as \( M \) grows, especially since we also constrained by the potentially long simulation execution times which grow with \( M \). From Figure 4.2 we can see that the NMSE (in dB) of the MMSE estimator decreases rapidly in the order of \( 1/M \).

An important point to note is that while the MMSE estimator of the vector \( h \) is the equivalent of the derived estimators for the individual channels \( h_m \) as given by (4.14), the resulting MSE for all \( h_m \) is not the same. This is shown explicitly in Figure 4.3, where the resulting individual channel NMSE is plotted for each element of the array. Interestingly, from Figure 4.3 channels to the middle of the array are better estimated than channels at the ends of the array, a point which we will revisit in the following sections.

4.3.1.1 Large \( M \) Analysis

In an attempt to look at the asymptotic behaviour of this estimator as \( M \to \infty \), we can express the unconditional MSE (for a particular element \( h_m \)) for the MMSE estimator as (4.20) below. However, again due to the nested \( M + 2 \) integrals which do not have a closed form expression, an asymptotic analysis of this function is not readily made.
Figure 4.3: The resulting NMSE(dB) for the estimation of each individual channel $h_m$ for $M = 64$, where $\text{NMSE}_{h_m} = \frac{E\{|h_m - \hat{h}_m|^2\}}{E\{|h_m|^2\}}$ and $\eta = -3$ dB. For the LMMSE estimator, the resulting NMSE from the analytical expression $\text{MSE}$, i.e., the diagonal elements of the trace argument in (4.27), is also plotted.

\[
\text{MSE}_{\text{MMSE}} = \int_{y_1}^{y_{\text{max}}} \int_{\theta}^{\phi} f(\theta)f(\phi)(h_m(\theta, \phi) - \hat{h}_m^{\text{MMSE}})^2 f(y_1..y_M|\theta, \phi) \, d\phi \, d\theta \, dy_1..dy_M
\]

(4.20)

### 4.3.2 LMMSE Estimation

As discussed earlier, the LMMSE estimator is a commonly used estimation method due to its efficiency and small amount of required a-priori information, i.e., only first and second order statistics are required. As such, the LMMSE estimator has been widely adopted in recent work for channel estimation in LoS based channels, [18, 20]. The form of the LMMSE estimator is readily available [54, Eq 3.2.8]. Given the zero-mean quantities of our channel model, $h$ and $y$ in (4.12),
and the use of zero-mean independent AWGN, the estimator reduces to,

\[ \hat{h}_{LMMSE} = W f_{\theta} y(\theta) \]  

(4.21)

\[ = \Gamma_{hy} \Gamma^{-1}_{yy} y \]  

(4.22)

\[ = \Gamma_{hh} (\sigma^2_n + \Gamma_{hh})^{-1} y, \]  

(4.23)

where \( \Gamma_{hh} \) is the channel covariance matrix given by

\[
\begin{bmatrix}
E\{h^2_m\} & E\{h_1h_2\} & \cdots & E\{h_1h_M\} \\
E\{h_1h_2\} & E\{h^2_m\} & \cdots & E\{h_2h_M\} \\
\vdots & \vdots & \ddots & \vdots \\
E\{h_1h_M\} & E\{h_2h_M\} & \cdots & E\{h^2_M\}
\end{bmatrix},
\]

(4.24)

where under the channel model in (4.12), \( E\{h^2_m\} = 0.5 \) for all \( m \). Given the uniform distributions of both \( \theta \) and \( \phi \), the auto-covariance terms \( E\{h_m h_l\} \) reduce to

\[
E\{h_m h_l\} = \frac{\pi}{2} \int_0^{2\pi} f_{\theta}(\theta) f_{\phi}(\phi) \cos((m - 1)\pi \cos \theta + \phi) \cos((l - 1)\pi \cos \theta + \phi) \, d\phi \, d\theta
\]

(4.25)

\[
= \frac{1}{2} \int_0^{\pi/2} f_{\theta}(\theta) \cos((m - 1)\pi \cos \theta) \, d\theta
\]

(4.26)

Note that the subscript of the \( W \) in (4.21) has been placed to explicitly remind the reader that \( W \) is parameterised by the PDF of \( \theta \), as shown in (4.25). The notation of (4.21) also exposes the underlying random variable of interest \( \theta \) when considering the random quantity \( y \). The explicit nature of this notation will allow us to describe and distinguish the estimation techniques covered in later sections.

The integration in (4.26) does not have a fully closed form solution and must be evaluated numerically (however (4.26) can be expressed in terms of well-known Bessel functions as shown later in (B.25)). Furthermore, approximations using Taylor polynomials are inaccurate, exhibiting
large errors for a practical number of terms. Therefore, computing the LMMSE estimate requires first to numerically evaluate (4.26) for all the components of $W$. However, once $W$ is formed, the generation of the estimates requires only a matrix multiplication\(^3\), and therefore the runtime complexity of the LMMSE estimator is of order $O(M^3)$.

In order to observe the MSE, we can alternatively use an analytical expression for the MSE (instead of evaluating (4.18)) and is given by [54, Theorem 3.2.2],

$$\text{MSE}_{\text{LMMSE}} = \text{tr}(\Gamma_{hh} - \Gamma_{hh} \Gamma_{yy}^{-1} \Gamma_{hh}^T).$$  \hspace{1cm} (4.27)

However, given the lack of a closed form expression for (4.26), and the existence of the matrix inverse in (4.27), it is hard to analytically observe the effect of the additional spatial observations on the resulting MSE. Nevertheless, from the highly non-linear nature of $h_m$ in (4.11), one would speculate that an estimator which is constrained to linear operations on the observations will struggle to perform well in such a setting.

If we examine the NMSE of the LMMSE estimator in Figure 4.2, we see some small improvement with the presence of additional antennas elements from $M = 2$ to $M = 16$, however there is very little gain from further increases in $M$. From Figure 4.2, even for a practically large array of $M = 128$, the NMSE is at -2.1dB and does not improve with the use of larger antenna arrays. These results suggest that while the LMMSE is a popularly used approach due to its low complexity, it is poorly suited to estimating LoS channels using the large antenna arrays of the massive MIMO BS, as it is unable to produce gains from the additional antenna elements.

In Figure 4.3, the $M$ resulting NMSE values from estimates of the user channel to individual antenna elements is plotted. In this figure, the dependency of the NMSE on $m$ is evident for the LMMSE estimator when computing the error from the estimates themselves, as in (4.19). Consequently, the NMSE has also been plotted using the analytical MSE expression in (4.27). From the plot of this expression in Figure 4.3 we can observe the dependency on $m$ for both estimators, which shows that the minimum individual element MSE is achieved in the middle of the antenna array. Intuitively, estimating the individual channel to a particular element, using an observation a large number of elements away is likely to be less useful than using an observation

\(^3\)We assume the use of a naive matrix multiplication algorithm of runtime complexity $O(M^3)$ in order to restrict the discussion of runtime complexities to $O(M^n)$ throughout the chapter, where $n \in \mathbb{Z}^+$ for the sake of simplicity.
from a nearby element. This can be observed more formally for our given model by evaluating
the MSE of the LMMSE estimator when using the scalar observation $y_{m+n}$ to estimate the scalar
channel $h_m$. As one might expect, this is a similar expression to the equivalent vector version in
(4.27), and using (B.17), (B.18) and (B.19) can be expressed as,
\[
MSE_{h_m|y_{m+n}} = r_{h}^{[0]} - \frac{(r_{h}[n])^2}{r_y^{[0]}},
\]
where $r_{h}[n] = E\{h_nh_{m+n}\}$. Given that $r_{h}[n]$ is a Bessel function, the function $(r_{h}[n])^2$ decays
with $n$. Therefore, when estimating a channel by using observations further away, i.e., as $n$ in-}
creases, the MSE given by (4.28) generally increases\(^4\). The channel estimated using the largest
number of closest observations will be the channel to the element in the middle of the array, i.e.,
h\(_{M/2}\), and therefore we expect this estimate to have the lowest MSE, as confirmed in Figure 4.3.
Conversely, the estimates at the end of the array have the smallest number of closest observations,
and therefore result in the largest MSE.

### 4.3.2.1 Large $M$ Analysis

We first state the main result of this section.

**Theorem 4.1.** Given an AoA $\theta \sim U[0, \pi/2]$, in the limit as $M \rightarrow \infty$, the resulting NMSE from
the LMMSE estimator when estimating $h$ from observations $y$ in (4.12), with an SNR $\eta$, is given
by
\[
NMSE_{LMMSE_{M \rightarrow \infty}} = \frac{1}{2\pi \ E\{|h_m|^2\}} \int_{-\pi}^{\pi} \frac{1}{2\eta + \sqrt{\pi^2 - \omega^2}} \ d\omega.
\]

**Proof.** See Appendix B.4

Theorem 4.1 is derived by applying results from Wiener smoothing theory, where the linear
smoother refers to the non-causal filtering process which produces estimations based on all pre-
vious and future observations. In the spatial domain, this translates well to our uniform linear
array, where we estimate the channel from the desired user to the antenna element positioned in

\(^4\)Due to the oscillation of $(r_{h}[n])^2$, near zero dips are observed every 3 samples, i.e., for $n \ mod \ 3 = 0$.  \[\]
the middle of the array, i.e., \( h_m \) where \( m = M/2 \), using an infinite number of observations in either direction. The resulting NMSE for the estimation of the vector \( \mathbf{h} \) is equal to the NMSE when estimating element \( h_{M/2} \), since every \( h_m \) of the vector \( \mathbf{h} \) is effectively estimated using an infinite number of observations in each direction. The results from the applied Wiener theory also require that the infinite observations \( y_m \), and the smoothed process \( h_m \) are jointly wide-sense stationary processes, which we demonstrate in Appendix B.4. The application of Weiner smoothing theory is also demonstrated in [60], with a main distinction being the observations are from neighbouring base stations of a network, as opposed to the neighbouring antenna elements in this work. Using the result in Theorem 4.1, we can easily compute the NMSE of the LMMSE estimator under large \( M \). Under an SNR of \( \eta = 1/2 \), we compute a large \( M \) NMSE of -2.1dB from Theorem 4.1 which agrees well with the simulation result shown in Figure (4.2) as \( M \) grows large.

### 4.3.3 Assisted LMMSE Estimation

As highlighted by the preceding sections, none of the previously presented estimation approaches are immediately compelling options to use for the estimation problem. The MMSE estimator with runtime complexity of \( \mathcal{O}(M^4) \) is computationally expensive but is able to achieve the smallest MSE. The LMMSE, while desirable because it has a runtime complexity of \( \mathcal{O}(M^3) \) and less variation of NMSE across elements, it has demonstrated comparatively poor estimation error performance. Furthermore, given that the resulting NMSE of the LMMSE estimator has already converged to the limiting performance defined by (4.29) by \( M = 16 \), the LMMSE estimator is unable to exploit the additional elements offered by the large antenna array, as shown in Figure 4.2.

Figure 4.2 shows that there is in fact a considerable performance gap between the MMSE estimator and the LMMSE. This motivates us to propose an alternate scheme, achieving estimation performance closer to the MMSE estimator, but with significantly less computational expense. We start by asking - what if the highly computationally efficient LMMSE had additional (or tighter) a-priori knowledge about the unknown quantities?

We introduce a scheme termed the assisted linear minimum mean squared error (A-LMMSE) estimator which exploits tighter a-priori knowledge before performing an LMMSE-like estimation of the channel. Given that AoA estimation has been a well-studied area of work for decades, we
4.3 Channel Estimation under the LoS Channel Model

explore assisting the LMMSE estimator with additional a-priori information obtained through AoA estimation. One example of a highly efficient, low complexity AoA estimation technique is a Maximum Likelihood (ML) approach which will be discussed in some depth and used in our scheme in Section 4.3.4.1.

We first consider an illustrative scenario where the desired user is located at an AoA uniformly distributed within smaller fixed ranges, i.e., \( \theta^{(p)} \sim U[\mu_\theta - R_\theta(p), \mu_\theta + R_\theta(p)] \), where we select arbitrarily \( \mu_\theta = \frac{\pi}{4} \) and \( R_\theta(p) = \frac{\pi}{4p}, p \in \mathbb{Z}^+ \). Additionally, let us assume the existence of a genie-assisted LMMSE estimator which has perfect knowledge of the distributions of \( \theta^{(p)} \). For the genie-assisted LMMSE estimator in this illustrative example, the a-priori information modifies the terms of the covariance matrix in (4.25), which become

\[
E\{h_m h_l\}^{(p)} = \frac{1}{4 R_\theta(p)} \int_{\mu_\theta - R_\theta(p)}^{\mu_\theta + R_\theta(p)} \cos((m - l)\pi \cos \theta^{(p)}) \, d\theta^{(p)}.
\]

Figure 4.4 shows how the resulting NMSE for the genie-assisted LMMSE defined above dramatically decreases as \( R_\theta(p) \) decreases. The substantial performance gap between the standard LMMSE estimator given by (4.21) as shown at \( p = 1 \) in Figure 4.4, and when larger values of \( p \) are used is a result of the tighter \( \theta^{(p)} \) distributions employed by the LMMSE estimator under this scenario.

4.3.3.1 The Spatial Frequency A-LMMSE Estimator

While Figure 4.4 demonstrates that there are significant gains when finer a-priori information on \( \theta \) is employed by the estimator, such gains are expensive to realise given the expression for the covariance terms in (4.30) do not have a closed form. The lack of a closed form expression makes them hard to evaluate for both analytical and practical purposes. As noted before, the numerical integration of (4.30) alone introduces \( O(M) \) complexity as a consequence of the required grid size to properly integrate such functions possessing the high frequency components generated from the \( (m - l) \) term. Therefore, a LMMSE based estimator with \( O(M^3) \) runtime complexity, which additionally uses a-priori information on every estimation to form the required covariance matrices, will have a high \( O(M^4) \) runtime complexity. Given our channel model (4.12) is purely
in the spatial domain, in order to circumvent this, we return to expression (4.10), and re-express the channel model as,

\[ y_m = \cos(2\pi f_s(m - 1) + \phi) + n_m, \]  

(4.31)

where

\[ f_s = \frac{f_c d}{c} \cos \theta = \frac{1}{2} \cos \theta \]  

(4.32)

is the spatial frequency.\(^5\) Given the support of the distribution \( \theta \sim \mathcal{U}(0, \pi/2) \), we have a resulting bound on the spatial frequency \( f_s \in [0, 0.5] \).

\(^5\)An important point to note from this relationship is that the estimation of \( \theta \) in the range \([\pi/2, \pi]\) results in negative values of the spatial frequency \( f_s \) which cannot be distinguished from their positive counterparts. In other words, a user with AoA in \([0, \pi/2]\), and second user with AoA in \([\pi/2, \pi]\), with identical phase \( \phi \), may result in the same observation \( y_m \) (in the absence of noise). The observation of a negative frequency can only be made when both a sine and cosine are observed together, i.e., both the in-phase and quadrature components, and given the reduced scope of this introductory chapter making use of only the real components of the observation vector \( y \), we confine the AoA of all users in the range \([0, \pi/2]\).
4.3 Channel Estimation under the LoS Channel Model

Figure 4.5: The resulting NMSE from a genie-assisted LMMSE estimator with covariance terms (4.35), as the range of $f_s^{(p)}$ is reduced, $M = 64$, $\eta = -3dB$.

timator, using a genie-assisted LMMSE estimator which now has a-priori knowledge of the less intuitive $f_s$ quantity (instead of $\theta$), uniformly distributed\(^6\) within smaller fixed ranges, i.e., $f_s^{(p)} \sim \mathcal{U}[\mu_{f_s} - R_{f_s}(p), \mu_{f_s} + R_{f_s}(p)]$, where we select arbitrarily $\mu_{f_s} = \frac{1}{4}$ and $R_{f_s}(p) = \frac{1}{4p^2}, p \in \mathbb{Z}^+$. The resulting covariance terms are now

\[
E\{h_mh_{m+n}\} = \frac{1}{2} \int_{f_s} f(f_s) \cos(2\pi f_s n) \, df_s, \tag{4.33}
\]

\[
E\{h_mh_{m+n}\}^{(p)} = \frac{1}{4R_{f_s}(p)} \int_{\mu_{f_s} - R_{f_s}(p)}^{\mu_{f_s} + R_{f_s}(p)} \cos(2\pi f_s^{(p)} n) \, df_s^{(p)} \tag{4.34}
\]

\[
= \frac{\cos(2\pi n \mu_{f_s}) \sin(2\pi n R_{f_s}(p))}{4\pi n R_{f_s}(p)}. \tag{4.35}
\]

Figure 4.5 shows that using the $f_s$ based genie-assisted estimator above, the resulting NMSE decreases with the reduction in variance of $f_s^{(p)}$ at a similar rate as the $\theta$ based genie-assisted estimator in Figure 4.4. However, the $f_s$ based estimator is much less computationally expensive to construct given the closed form of (4.35).

\(^6\)The resulting $f_s$ distribution from a uniformly distributed $\theta$ is not uniform. We employ a uniform $f_s$ distribution here for illustrative purposes and analytical tractability.
We now introduce the A-LMMSE estimator, shown in Figure 4.6. The estimator applies the ideas above by assuming that some first pass Assistant provides improved a-priori information about the $f_s$ of the desired user (as performed by the 'genie' in the motivating discussion), in the form of an unbiased estimate $\hat{f}_s$ for every estimation of the channel vector, with some error variance $\sigma^2_{\hat{f}_s}$, where $\bar{f}_s = f_s - \hat{f}_s$. We then construct our estimator $W$ based on this tighter a-priori knowledge. The dramatic reduction in the NMSE of Figure 4.5 shows how having tighter a-priori information about the underlying random quantity $f_s$ can improve the estimation of the channel $h(f_s, \phi)$ significantly, and we therefore expect a significant improvement in the channel estimation result from the A-LMMSE scheme. As demonstrated by the genie-assisted example, we require that $\sigma^2_{\hat{f}_s} < \sigma^2_{f_s}$ to realise performance gains using this scheme. The expression in (4.35) shows that the estimator has both a dependency on the variance of the underlying $f_{s(p)}$ quantity (through $R_{f_s}$ and as shown in Figure 4.5) and the mean of the $f_{s(p)}$ quantity, i.e., $\mu_{f_s}$. Figure 4.7 shows the effect of $\mu_{f_s}$ on the resulting NMSE, using a fixed value of $R_{f_s} = 1/20$. The figure shows that the effect of $\mu_{f_s}$ is barely observable, with the exception of when $\mu_{f_s}$ is close to the upper limit of $f_s$. Even under such cases, it would appear from Figure 4.7 that for all values of $\mu_{f_s}$ the NMSE approaches a common limit as $M$ tends to infinity, which we explore in the following Section 4.3.3.2.

4.3.3.2 Large $M$ Analysis

We can analyse the large $M$ NMSE of the above-mentioned A-LMMSE scheme in a similar way to the analysis presented in Section 4.3.2.1, with the main result summarised by Theorem 4.2.

Theorem 4.2. Given an underlying $f_s \sim \mathcal{U}[\bar{f}_s - \bar{R}, \bar{f}_s + \bar{R}]$, where $\bar{f}_s + \bar{R} < 0.5$ and $\bar{f}_s - \bar{R} > 0$,
as $M \to \infty$, the resulting NMSE from the LMMSE estimator when estimating $h$ from observations $y$ in (4.31), with an SNR $\eta$, is given by

$$ NMSE_{M \to \infty} = \frac{1}{E\{|h_m|^2\} \left( \frac{\eta^2}{2R} + 2 \right)}. $$

(4.36)

**Proof.** See Appendix B.5

Using Theorem 4.2, the large $M$ NMSE of the Assisted-LMMSE scheme is given in (4.36), under the assumption that as $M$ heads to infinity, the Assistant which estimates $f_s$ achieves an unbiased estimate $\hat{f}_s$ with a uniformly distributed error $\tilde{f}_s \sim \mathcal{U}(\hat{f}_s - R, \hat{f}_s + R)$. Under this assumption we have a general case of the previously presented genie-assisted LMMSE estimator, where $\hat{f}_s = \mu_{f_s}$ and $R = R_{f_s}(p)$.

The result (4.36) shows that the large $M$ NMSE decreases with reductions in the variance of the underlying $f_s$ distribution, by means of $R$. As $\eta \to 0$, the noisy observations are unable to contribute to the estimation process, and the LMMSE based estimator reduces to using the mean of zero as an estimate, resulting in the $NMSE = E\{|h - \hat{h}|^2\} / E\{|h|^2\} = 1$. The simple
closed form of (4.36) makes such effects readily observable by reducing to 1 when \( \eta = 0 \) (unlike the result of Theorem 4.1, where the same result is obtained but requires integration). The finite results presented earlier in Figure 4.7 also suggested that the same NMSE is obtained for all mean values of the \( f_s \) distribution, i.e., \( \hat{f}_s \), as \( M \) tends to infinity. The large \( M \) NMSE given by (4.36) confirms this, since it is not dependent on \( \hat{f}_s \).

Figure 4.8 plots the large \( M \) expression in (4.36) for different values of \( R \). In order to look at the convergence to the large \( M \) result, we plot the corresponding NMSE of the genie-assisted LMMSE, for both the estimation of the channel to the middle element and of the complete channel vector, with increasing finite \( M \). The figure clearly shows convergence of the NMSE to the limiting result in (4.36), and importantly the speed of convergence justifies the use of the limiting results for performance bounds for practically large and finite \( M \).

When using an \( f_s \) estimator which is asymptotically optimal, i.e., \( R \to 0 \) as \( M \to \infty \), the expression (4.36) shows that the A-LMMSE estimator is also asymptotically optimal.

### 4.3.4 Realising the A-LMMSE Estimator

This section discusses the issues related to realising a practical implementation of the A-LMMSE estimator presented in the previous section. As this section will detail, an important feature of the A-LMMSE estimator is that it does embody very effective practical implementations that can achieve reductions of up to 6dB in the resulting NMSE, when compared to the standard LMMSE estimator.

#### 4.3.4.1 Spatial Frequency and AoA Estimation

In realising the A-LMMSE estimator, the Assistant block of Figure 4.6 (which was handled by the genie in Section 4.3.3) needs to be realised as an AoA estimator. We remind the reader that both random quantities \( \theta \) and \( f_s \) are directly related via (4.32). As mentioned earlier, AoA estimation is a well-studied problem, and there are several well-known algorithms such as Multiple Signal Classification (MuSIC) and Estimation of Signal Parameters via Rotational Invariance Technique (ESPRIT). Given that the estimation of the AoA is outside the main focus of the work, we choose to adopt an \( f_s \) estimation technique based on the criteria of computational efficiency, as opposed
Figure 4.8: The large $M$ NMSE expression in (4.36) for the genie-assisted LMMSE alongside finite $M$ simulation results with increasing $M$. Several values of $R$ are shown, for a fixed $\hat{f}_s = 0.25$ and $\eta = 0dB$. 
to accuracy or robustness.

The $f_s$ estimation technique we have employed is an approximation to a maximum likelihood (ML) estimation approach, (which is effectively the well-known Bartlett method [61]), which we outline briefly here.

The ML method involves finding the unknown parameters which maximises the likelihood of the observations $y$. Given we do not have information about $\phi$, an analytically tractable approach requires the joint ML estimation over $w = [f_s \phi]$, i.e.,

$$
\hat{f}_s^{\text{ML}} = \arg\max_{f_s, \phi} f(y_1 \ldots y_M | f_s, \phi) \quad \text{(4.37)}
$$

$$
= \arg\max_{f_s, \phi} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} \sum_{m=1}^{M} (y_m - \cos(2\pi f_s (m - 1) + \phi))^2}
$$

$$
= \arg\min_{f_s, \phi} \sum_{m=1}^{M} (y_m - \cos(2\pi f_s (m - 1) + \phi))^2. \quad \text{(4.38)}
$$

An efficient approach to minimising (4.38) is shown in [55], where the objective function is reformulated into a system of linear equations, for which the least squares solution yields a minimising $\phi$ in terms of $f_s$. An approximation to the resulting objective function can then be expressed as $^7$

$$
\hat{f}_s^{\text{ML}} = \arg\max_{f_s} \frac{2}{M} \left| \sum_{m=1}^{M} y_m e^{-j2\pi f_s (m-1)} \right|^2. \quad \text{(4.39)}
$$

Our ML $f_s$ estimate is the argument which maximises an objective function which is defined by the DTFT of the observations $y_m$. We make an approximation to (4.39) by utilising a highly efficient $N_f$-point fast fourier transform (FFT) to implement the discrete fourier transform (DFT) of the observations $y_m$,

$$
\hat{f}_s = \frac{1}{N_f} \arg\max_{k, k < \frac{N_f}{2}} \left| \sum_{m=1}^{M} y_m e^{-j\frac{2\pi k}{N_f} (m-1)} \right|^2, \quad \text{(4.40)}
$$

where $N_f$ is the number of points used for the FFT operation. It is well understood that a greater number of observations $M$, increases the spectral resolution of the periodogram in (4.39). Consequently $\hat{f}_s$ approaches $f_s$ as $M$ increases. Additionally, given our approximation of the DTFT with

$^7$The result in (4.39) is derived in [55] under the assumption that $f_s$ is not near 0 and 0.5, which corresponds to an AoA $\theta$ which is not near 0 and $\pi/2$. 


the FFT in (4.40), larger values of $N_f$ will result in improved estimates of $f_s$ and consequently we use $N_f = 10M$ to ensure sufficient $f_s$ estimation accuracy.

In order to realise the A-LMMSE estimator, in addition to the estimate of the spatial frequency, we require knowledge about the error variance of the $f_s$ estimate. Fortunately there are known results for the error variance behaviour of the ML estimator which apply to our estimation problem. The joint ML estimate of the unknown parameters $\mathbf{w} = [f_s \phi]$, is distributed according to $\hat{\mathbf{w}} \sim \mathcal{N}(\mathbf{w}_0, \mathbf{I}^{-1}(\mathbf{w}_0))$ for large values of $M$, where $\mathbf{I}(\mathbf{w}_0)$ is the Fisher matrix evaluated at the true value of the parameter [55, Theorem 7.3]. By definition, the $ii$-th element of $\mathbf{I}^{-1}(\mathbf{w}_0)$ is the Cramer Rao Lower Bound (CRLB) for the estimation error variance of estimating parameter $w(i)$. Therefore, the ML estimator has the property that it is asymptotically optimal and attains the CRLB for large $M$. Given our scenario of critically spaced antenna elements $d = \lambda/2$, and large $M$ antenna array, the approximation to the CRLB given in [55] becomes,

$$\text{CRLB}_{\hat{f}_s} \approx \frac{3}{\pi^2 \eta M^3}. \quad (4.41)$$

Since we are operating with a large but finite number of antenna elements $M$, it would seem logical to investigate whether the CRLB result in (4.41) can be applied to our A-LMMSE scheme. However, we observe that the convergence of the error variance from our $f_s$ estimator in (4.40), to the CRLB in (4.41) is too slow under the range of practical values of $M$. Furthermore, we observe that the rate of convergence of our estimation error variance to the CRLB is highly sensitive to $\eta$. Consequently, we assume a normal distribution for the error variance under large and finite $M$, but make use of an empirical error variance for $\hat{f}_s$. Therefore, we can describe our implementation of the A-LMMSE estimator (in contrast to the standard LMMSE described by (4.21)) as,

$$\hat{\mathbf{h}}_{A-LMMSE} = \mathbf{W}_{\mathcal{N}(\hat{f}_s, \sigma_{\hat{f}_s}^2)} \mathbf{y}(f_s). \quad (4.42)$$

In practice, a two-dimensional lookup table for the $f_s$ error variance can be employed where the dimensions of the table represent the two variables of the CRLB in (4.41), i.e., $\eta$ and $M$. While the joint estimation also provides an estimate of the phase, $\phi_{ML}$, this estimate suffers from a high error variance as highlighted by the CRLB approximation in [55], where $\text{CRLB}_{\hat{\phi}} \approx 4(\eta M)^{-1}$.

The A-LMMSE estimator described by (4.42) can be realised by computing the covariance
terms in (4.33) as before. Utilising integral results \[62, Section 3.897\], we derive the very elegant closed form covariance terms,

\[ E\{ h_m h_{m+n} \}_{A-LMMSE} \approx \frac{1}{2} e^{-2n^2 \pi^2 \sigma^2_{\tilde{f}_s} \cos(2\tilde{f}_s n \pi)} . \tag{4.43} \]

Strictly speaking, the closed form result of (4.43) is an approximation since it has been derived by integrating over the range \([-\infty, \infty]\), where the actual supports are \( f_s \in [0, \frac{1}{2}] \). However, the resulting approximation of (4.43) is accurate under our operating conditions, since the estimator in (4.40) results in sufficiently small values of \( \sigma^2_{\tilde{f}_s} \) such that negligible tails result from the normal distribution \( \mathcal{N}(\tilde{f}_s, \sigma^2_{\tilde{f}_s}) \) outside the range \([0, \frac{1}{2}]\). Naturally, this approximation becomes tighter as \( M \) or \( \eta \) increases. Of course, if the a-priori information from the \( f_s \) estimation is very poor, i.e., the condition \( \sigma^2_{\tilde{f}_s} < \sigma^2_{f_s} \) is not satisfied, then the A-LMMSE scheme should revert back to using the non-assisted LMMSE estimator in (4.21).

The resulting A-LMMSE estimator is highly efficient. For each estimation of the channel vector, the A-LMMSE estimator needs to generate a particular instance of \( W_{\mathcal{N}(\tilde{f}_s, \sigma^2_{\tilde{f}_s})} \) in (4.43). Employing a modest FFT algorithm to perform the \( f_s \) estimation of the Assistant involves a runtime complexity of \( O(M \log M) \), while more advanced FFT algorithms such as the Sparse-FFT can exploit the sparse nature of our single user LoS channel matrix and yield much lower run-time complexities in the order of \( O(\log M) \) \[63\]. The closed form of the covariance terms in (4.43) mean the runtime complexity to construct \( \Gamma_{hh} \) is limited to \( O(M^2) \). Given that \( \Gamma_{yy} \) in (4.22) is Toeplitz, the recursive Levison-Durbin algorithm can be used to compute \( \Gamma_{yy}^{-1} \) efficiently, with runtime complexity \( O(M^2) \). Therefore, the runtime complexity order of the A-LMMSE estimator is limited by the dominant matrix multiplications of the LMMSE form, where the naive algorithm requires a runtime complexity of \( O(M^3) \).

Figure 4.2 shows that the resulting A-LMMSE estimator is able to achieve a similar channel estimation performance to the optimal MMSE estimator, while requiring a runtime complexity order of \( O(M^3) \), a full order lower than the required \( O(M^4) \) for the MMSE estimator.

\[ \text{For example, in simulation with } \eta = -3dB, \text{ with } M = 128, \text{ the estimator (4.40) results in estimates with error variance } \sigma^2_{\tilde{f}_s} = 3.1 \times 10^{-07}. \]
4.3.4.2 Computationally Efficient Variants

In a practical setting, one might want to optimise the A-LMMSE estimator further in an effort to reduce the computational complexity of the estimation process. We briefly describe an optimisation to the A-LMMSE approach here, which we refer to as the A-LMMSE Static estimator. This scheme involves dividing the full range of $f_s$ (and consequently $\theta$), from 0 to $1/2$ into $Q$ segments, and pre-computing a bank of $Q$ estimators, $\{W^{(1)}, W^{(2)}, \ldots, W^{(Q)}\}$, one for each segment. Given the use of large antenna arrays, where each estimator $W$ requires the storage of $M \times M$ terms, we assume that the value of $Q$ is fixed based on the practical setting of a memory constrained device. Based only on the value of $\hat{f}_s$, the corresponding $W^{(q)}$ estimator is used from the bank of precomputed estimators to form the channel estimate $\hat{h}$.

It is clear from such an approach that utilising a-priori knowledge of a normally distributed $f_s$ centred on a particular $\hat{f}_s$ no longer makes sense, since each of the precomputed $W^{(q)}$ estimators must be used for a range of $\hat{f}_s$ estimates. We present two options. Under the constraint of $Q$ estimators, we construct $W^{(q)}$ assuming that given an estimate $\hat{f}_s$, the remaining randomness is uniformly distributed (A-LMMSE Static Uniform), or follows the distribution of $f_s$ (A-LMMSE Static), over one of the $Q$ fixed intervals. These fixed intervals are of equal size, covering the range of $f_s \in [0, 1/2]$.\(^9\) This yields the estimators,

$$
\hat{h}_{A-LMMSE_{\text{Static}}} = W^{(\tilde{q})} |_{f_s \in [\hat{f}_s^{(\min)}, \hat{f}_s^{(\max)}]} y(f_s), \quad (4.44)
$$
$$
\hat{h}_{A-LMMSE_{\text{StaticUniform}}} = W^{(\tilde{q})} |_{U(\hat{f}_s^{(\min)}, \hat{f}_s^{(\max)})} y(f_s), \quad (4.45)
$$

where,

$$
1 \leq \tilde{q} = [2\hat{f}_s Q] \leq Q. \quad (4.46)
$$

Applying standard random variable transformation techniques to the uniform distribution of $\theta \sim U[\theta_{\min}, \theta_{\max}]$ and its relation to $f_s$ in (4.32), we can derive the PDF $f_{f_s}(f_s)$ of the related random

\(^9\) While having $Q$ differently sized intervals that match the non-uniform distribution of users across $f_s$, i.e., employing smaller intervals where more users are located, (recall our model assumes a uniform distribution of users across $\theta$) does result in a lower channel estimation error in the mean sense, it also results in a higher variance of the channel estimation error, i.e., users in particular locations will naturally have better channel estimation performance. We assume that this is an undesirable quality, and therefore present the A-LMMSE Static scheme using equally spaced intervals. Further to this, the distribution of users may not be uniform across $\theta$ or known accurately.
quantity $f_s \in [f_{s,\min}, f_{s,\max}]$,

$$f_{s,\ell}(f_s) = \frac{1}{\cos^{-1}(2f_{s,\min}) - \cos^{-1}(2f_{s,\max})} \frac{2}{\sqrt{1 - 4f_s^2}} f_{s,\min} \leq f_s \leq f_{s,\max}. \quad (4.47)$$

Using (4.43) to derive the covariance terms of $\Gamma^{(q)}_{hh}$, used to form $W^{(q)}$ for each of the estimators yields,

$$E\{h_mh_{m+n}\}^{(q)}_{\text{Static}} = \frac{1}{\cos^{-1}(2f_{s,\min}) - \cos^{-1}(2f_{s,\max})} \int_{f_{s,\min}}^{f_{s,\max}} \cos(2\pi f_s n) \frac{1}{\sqrt{1 - 4f_s^2}} df_s, \quad (4.48)$$

and

$$E\{h_mh_{m+n}\}^{(q)}_{\text{StaticUniform}} = Q \int_{f_{s,\min}}^{f_{s,\max}} \cos(2\pi f_s n) df_s, \quad (4.49)$$

where

$$f_{s,\min}^{(q)} = q - \frac{1}{2Q}, \quad (4.50)$$

$$f_{s,\max}^{(q)} = \frac{q}{2Q}. \quad (4.51)$$

Since our estimate $\hat{f}_s$ has naturally a non-zero error variance, there exists the possibility that the true value $f_{s,0}$ does not lie within the range of the selected estimator $W^{(q)}$. Under such circumstances, i.e., $q_0 = [2f_{s,0}Q]$ and $q_0 \neq \hat{q}$, we expect that the resulting estimation performance to be particularly poor. In order to reduce the likelihood of this occurring, we extend the scheme by instead forming a bank of overlapping estimators. Given $\tilde{f}_s$ is normally distributed for large $M$, we extend the region of each estimator, $f_{s,\max}^{(q)} - f_{s,\min}^{(q)}$, by $3\sigma_{\tilde{f}_s}$ at either boundary to broadly ensure $P(f_{s,0} \notin [f_{s,\max}^{(q)} - f_{s,\min}^{(q)}]) < 0.03$.

Since the $f_s$ estimation performance $\tilde{f}_s$, is decoupled from the memory constraint $Q$, we also limit the overlapping to half of the segment size, i.e., $\frac{1}{4Q}$. This yields new supports for our a-priori distribution,

$$f_{s,\min}^{(q)} = \max\left(\frac{q - 1}{2Q} - \min(3\sigma_{\tilde{f}_s}, \frac{1}{4Q}), 0\right), \quad (4.52)$$

$$f_{s,\max}^{(q)} = \min\left(\frac{q}{2Q} + \min(3\sigma_{\tilde{f}_s}, \frac{1}{4Q}), \frac{1}{2}\right). \quad (4.53)$$
Under the A-LMMSE Static schemes, there is a significant reduction in computational load, since the $W^{(q)}$ matrices are precalculated and simply used to derive the channel estimate. That is, there is no need to compute the $\Gamma_{hh}$ matrix, or compute the inverse of $\Gamma_{hh}$ on each estimate of the channel vector. Therefore the only additional computation to calculate the channel estimation vector $\hat{h}$, when compared to the traditional LMMSE estimator, is the additional load to compute $\hat{f}_s$ which as described earlier, only requires $O(M \log M)$ under a modest FFT algorithm. Therefore, the ALMMSE Static schemes have the same $O(M^3)$ runtime complexity order of the LMMSE scheme.

Another important feature of the A-LMMSE Static schemes are their robustness to $\tilde{\sigma}_f$, since it only requires that the $\hat{f}_s$ results in the correct estimator selected. Therefore it can be used in cases where $\hat{f}_s$ has limited accuracy, or where $\hat{f}_s$ is being derived out-of-band from potentially slowly updated sources (e.g. GPS).

### 4.3.4.3 Large $M$ Analysis

In a similar way to Figure 4.8 for the genie-assisted scenarios, the NMSE of the A-LMMSE Static schemes plotted in Figure 4.9 appear to approach a limit as $M$ heads to infinity. To look at the behaviour of the A-LMMSE Static schemes for large $M$, we first recall that our ML $f_s$ estimator in (4.40) is asymptotically optimal, and obtains the CRLB for large $M$. As $M \to \infty$, the CRLB approximation of (4.41) heads to zero, and consequently, $\hat{f}_s \to f_s$ and $\sigma_{\tilde{f}_s} \to 0$. For the A-LMMSE Static schemes, this implies we always select the correct estimator in the limit, i.e., $\tilde{q} = q_0$. Furthermore, the regions of the estimators given by (4.52) and (4.53) are not overlapping, and are functions of $Q$ only.

#### 4.3.4.3.1 A-LMMSE Static

The main result of this section is given by Theorem 4.3,

**Theorem 4.3.** For AoA $\theta \sim \mathcal{U}[0, \pi/2]$, the resulting NMSE from the A-LMMSE Static estimator
as $M \to \infty$, when estimating $\mathbf{h}$ from observations $\mathbf{y}$ in (4.31), with an SNR $\eta$, is given by

$$\text{NMSE}_{\text{ALMMSE, Static}_{M \to \infty}} = \sum_q \left[ F_q(f_{s, \text{max}(q)}) - F_q(f_{s, \text{min}(q)}) \right] \cdot \text{NMSE}_{\text{ALMMSE, Static}_{M \to \infty}}^{(q)}.$$  

(4.54)

**Proof.** The large $M$ NMSE when using the $q$-th A-LMMSE Static estimator can be expressed using the result in (B.30), i.e.,

$$\text{NMSE}_{\text{ALMMSE, Static}_{M \to \infty}}^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( S_h^{(q)}(e^{j\omega}) - \frac{|S_h^{(q)}(e^{j\omega})|^2}{|S_h^{(q)}(e^{j\omega}) + \sigma_n^2|} \right) d\omega,$$  

(4.55)

where $S_h^{(q)}(e^{j\omega})$ is the power spectral density function of the $\{h_m\}$ process on selection of the $q$-th estimator. Given that the correct estimator is selected in the limit, i.e., $q = q_0$ as $M \to \infty$, the underlying distribution of $f_s$ is bounded by $f_s \in [f_{s, \text{min}(q)}^{(q)}, f_{s, \text{max}(q)}^{(q)}]$, with non-uniform distribution in (4.47). Therefore we can use Lemma B.1 to write $S_h^{(q)}(e^{j\omega})$ for use in (4.55),

$$S_h^{(q)}(e^{j\omega}) = \frac{1}{\cos^{-1}(2f_{s, \text{min}(q)}) - \cos^{-1}(2f_{s, \text{max}(q)})} \frac{1}{2\sqrt{1 - \frac{\omega^2}{\pi^2}}} \int_{2f_{s, \text{min}(q)}}^{2f_{s, \text{max}(q)}} 2\pi f_s^{(q)} \leq \omega \leq 2\pi f_s^{(q)}$$  

(4.56)

An overall NMSE can then be computed by first considering the probability of selecting the $q$-th estimator and then using the law of total probability to yield,

$$\text{NMSE}_{\text{ALMMSE, Static}_{M \to \infty}} = \sum_q \left[ F_q(f_{s, \text{max}(q)}) - F_q(f_{s, \text{min}(q)}) \right] \cdot \text{NMSE}_{\text{ALMMSE, Static}_{M \to \infty}}^{(q)}.$$  

(4.57)

where CDF of $f_s$ is given by $F_{f_s}(f_s) = \frac{2}{\pi} \sin^{-1}(2f_s)$ for $0 < f_s < 0.5$, computed from the PDF in (4.47).

The large $M$ NMSE in (4.54) can be expressed in closed form, however its long convoluted form does not offer any insight into how it behaves with respect to the underlying parameters. The analytical large $M$ NMSE result in (4.54) is plotted in Figure 4.9 and agrees well with the
simulated finite \( M \) results of the A-LMMSE Static estimator as \( M \) increases.

### 4.3.4.3.2 A-LMMSE Static Uniform

The main result is given by Corollary 4.1 which follows from Theorem 4.4.

**Theorem 4.4.** Given \( f_s \) defined by any probability density function bounded by \([f_{s0} - \bar{R}, f_{s0} + \bar{R}]\), the resulting NMSE as \( M \to \infty \) from an LMMSE estimator derived assuming \( f_{s_{11}} \sim U[f_{s0} - \bar{R}, f_{s0} + \bar{R}] \), where \( f_{s0} + \bar{R} < 0.5 \) and \( f_{s0} - \bar{R} > 0 \), when estimating \( h \) from observations \( y \) in (4.31), with an SNR \( \eta \), is given by

\[
NMSE_{M \to \infty} = \frac{1}{E\{|h_m|^2\} \left( \frac{\eta}{2\bar{R}} + 2 \right)}.
\]

**Proof.** See Appendix B.6

Theorem 4.4 is required to analyse the large \( M \) MSE behaviour of the A-LMMSE Static Uniform estimator since the underlying distribution of \( f_s \) is non-uniform as given by (4.47), but the A-LMMSE Static Uniform estimator assumes a uniformly distributed \( f_s \). Theorem 4.4 describes a more general result for the large \( M \) MSE of a linear smoother assuming a uniform distribution of \( f_s \), when the input process is derived from any bounded \( f_s \) distribution.

**Corollary 4.1.** Given AoA \( \theta \) with any distribution bounded by \([0, \pi/2]\), the resulting NMSE from the A-LMMSE Static Uniform scheme utilising \( Q \) estimators, as \( M \to \infty \), when estimating \( h \) from observations \( y \) in (4.31), with an SNR \( \eta \), is given by

\[
NMSE_{ALMMSE_{StaticUniform}}_{M \to \infty} = \frac{1}{E\{|h_m|^2\} \left( 2(\eta Q + 1) \right)}
\]

**Proof.** Given \( q = q_0 \) as \( M \to \infty \), (i.e., the correct estimator is selected from the \( Q \) estimators spanning the range of \( f_s \in [0, 1/2] \) when an infinite number of BS antennas are used), the underlying distribution of the corresponding \( f_s \) is bounded by \( f_s \in [f_{s_{min}}^{(q)}, f_{s_{max}}^{(q)} + 1/(2Q)] \) (with non-uniform distribution given by (4.47)).

The result of Theorem 4.4 has no dependence on \( f_{s0} \). Consequently, under the A-LMMSE Uniform Static scheme the large \( M \) NMSE has no dependence on \( q \) (or \( f_{s_{min}}^{(q)} \)), and Theorem 4.4
defines the large $M$ NMSE achieved by all $Q$ estimators.

The large $M$ NMSE of the A-LMMSE Static Uniform estimator in (4.59) is plotted in Figure 4.9, and agrees well with the finite $M$ results from Monte Carlo simulation. Figure 4.9 shows that for both the finite $M$ and large $M$ results, the channel estimation performance achieved by both A-LMMSE Static schemes is similar, with the A-LMMSE Static only showing a more noticeable performance improvement compared to the A-LMMSE Static Uniform under low SNR conditions ($\eta \sim -10\text{db}$).

An advantage of the A-LMMSE Static Uniform estimator, which is readily observed from (4.1), is that the limiting performance of the estimator is in fact $f_s$ distribution agnostic. In practice, we may not have accurate knowledge, or any knowledge of the distribution of the AoA. As such, the A-LMMSE Static Uniform estimator is a more desirable choice in practice. From an analysis point of view, the large $M$ MSE expression of (4.59) is extremely simple, clearly showing the effects of SNR ($\eta$) and the memory constraints (number of estimators $Q$) at the BS, and as such can be used in further performance bounds. Furthermore, the convergence of the finite $M$ results is sufficiently fast so that the limiting expression of (4.59) can be used to evaluate the performance of the A-LMMSE Static Uniform estimator with a practical number of antenna elements $M$ ($M \geq 128$) and practical number of precomputed estimators $Q$ ($Q \geq 4$).

4.3.4.4 The A-LMMSE Estimator Performance Compared

In this section we evaluate the resulting performance, both in terms of channel estimation and symbol detection when using the estimators presented in the preceding sections. Figure 4.9 shows the NMSE of the A-LMMSE, the A-LMMSE Static Uniform and the LMMSE estimator from Monte Carlo simulation. It is clear that the LMMSE has poor estimation performance comparatively, as the resulting NMSE does not improve as $M$ increases. The A-LMMSE estimator continues to improve with increases in $M$ and delivers impressive channel estimation performance with the same $O(M^3)$ runtime complexity order of the LMMSE estimator. The low runtime complexity A-LMMSE Static Uniform estimator offers a significant improvement in NMSE over the LMMSE estimator, while only requiring an additional runtime computation of $O(M\log M)$ for the execution of the Assistant, and moderate amounts of additional memory to store the pre-computed
To further investigate the performance of the channel estimators, we evaluate their resulting average uncoded symbol error rates (SER) during the coherent detection of data symbols. For the simplicity of the simulation setup, we consider the data transmission of one symbol per slot. Due to our real-valued baseband channel model and processing, we are restricted to use BPSK data symbols. We assume the same level of additive noise at the BS during channel estimation and data detection, where the receive SNR at the BS is given by \( \eta_d = \frac{a^2}{\sigma^2_n} \).

The received baseband vector \( y_d \in \mathbb{R}^{M \times 1} \) at the BS of the single symbol \( x \) transmitted by the user can be written as,

\[
y_d = \mathbf{h}x + \mathbf{n}, \tag{4.60}
\]

where the vectors \( \mathbf{h}, \mathbf{n} \in \mathbb{R}^{M \times 1} \) are the channel and noise vectors respectively, as defined during the channel estimation process in (4.12). We employ a simple MRC detector at the BS which involves multiplying the received vector with the conjugate of the channel estimation vector for
the desired user. The result is normalised to recover a properly scaled version of desired data symbol, and is given by,

\[
\hat{x} = \frac{\hat{h}^T y_d}{\hat{h}^T \hat{h}} = \frac{1}{|\hat{h}|^2} \left( \hat{h}^T h x + \hat{h}^T n \right) \tag{4.61}
\]

Given the symmetry of the BPSK constellation, the SER is given by the probability

\[
P \left( \left( \hat{h}^T h x + \hat{h}^T n \right) < 0 | x = a \right), \quad \text{where } a > 0. \tag{4.63}
\]

We do not readily have the distributions of the underlying random quantities in (4.63), and consequently evaluate (4.63) by effectively conditioning on \( \hat{h} \) and \( h \) by using randomly generated instances in simulation. The distribution of the detected symbol, conditioned on \( \hat{h}, h \) and \( x = a \), is described by the normal distribution \( \mathcal{N} \left( a \hat{h}^T h, \hat{h}^2 \sigma_n^2 \right) \). After normalising this distribution to the standard normal distribution, the SER is the average of the instantaneous symbol error.
probability and can be expressed as,

$$\text{SER} = E\left\{ Q\left( \frac{\sqrt{\eta_d} \hat{\mathbf{h}}^T \mathbf{h}}{||\mathbf{h}||} \right) \right\},$$  \hspace{1cm} (4.64)$$

where the average is taken over the joint distribution of $\hat{\mathbf{h}}$ and $\mathbf{h}$ and $Q(x)$ is the complementary CDF of a standard normal Gaussian random variable.

We can define an effective SNR for the MRC detector as the argument of the $Q$ function in (4.64). The effect of the channel estimation error on the resulting SER can be seen from the inner product of the estimate and the channel vector, $\hat{\mathbf{h}}^T \mathbf{h}$ in (4.64). When the channel estimation error $(\hat{\mathbf{h}} - \mathbf{h})$ (as captured by our main performance metric (4.19)) is reduced, $\hat{\mathbf{h}}$ approaches $\mathbf{h}$ and the inner product $(\hat{\mathbf{h}}^T \mathbf{h})/||\hat{\mathbf{h}}||$ increases, resulting in a lower SER.

The resulting SER using $a = 1/2$ is shown in Figure 4.10, where the effects of the channel estimation performance on the detection performance are evident. An increase in the number of BS antenna elements $M$ results in an increase in the array gain when using the MRC detector, which benefits all estimators, including the LMMSE estimator. As shown, the decoding performance gains of the A-LMMSE estimator and the A-LMMSE Static Uniform estimator with $Q = 9$, are significant when compared the LMMSE estimator. Extending the A-LMMSE estimator to use both in-phase and quadrature components would allow the use of frequently used QAM schemes (e.g., QPSK, 16-QAM) where similar performance trends are presented in later sections.

### 4.4 Channel Estimation under the Rician Channel model

While the pure LoS channel model in the Section 4.3 allowed us to explore the correlation in the spatial domain, such a channel model is largely unrealistic for macro-cell deployments given that the channel will very likely be subject to some degree of multipath fading. The Rician channel model models both the LoS and multipath components. The Rician channel model between the desired user and $m$-th antenna element ($h_{m,R}$) models the multipath component through a Gaussian random variable $\psi_m \sim \mathcal{N}(0, \sigma^2_\psi)$ added to the normalised LoS channel [6], $h_m$ (as defined in (4.12)). Again, we only consider the real component of the baseband signal for simplicity. The
baseband channel model of the vector of noisy observations under Rician fading is given by,

\[ y_R = h_R(\theta, \phi, \psi) + n. \] (4.65)

The model for the channel to the \(m\)-th element is given by,

\[ y_{m,R} = h_{m,R}(\theta, \phi, \psi) + n_m \] (4.66)

\[ = \rho h_m(\theta, \phi) + \psi_m + n_m \] (4.67)

\[ = \rho \cos((m - 1)\pi \cos \theta + \phi) + \psi_m + n_m. \] (4.68)

The normalisation constant \(\rho\) in (4.67) normalises the channel power to that of the LoS model in (4.12), where \(E\{|h_{m,R}|^2\} = \rho^2 E\{|h_m|^2\} + \sigma_\psi^2 = E\{|h_m|^2\} = 1/2\), where \(0 \leq \rho \leq 1\). Consequently, we have the same channel SNR for our Rician channel as the pure LoS channel in (4.12), i.e., \(\eta = E\{|h_{m,R}|^2\}/\sigma_n^2 = 1/2\).

As shown in (4.66), we again assume the slow gain for the channel \(h_{m,R}(\theta, \phi, \psi)\) is known and unity. An important quantity is the strength of the LoS component relative to the multipath component, known as the \(K\)-factor where \(K_f = A^2/(2\sigma_\psi^2)\) and \(A\) is the peak amplitude of the LoS component [64]. For our model described by (4.67), \(K_f = \rho^2/2\sigma_\psi^2\). When \(K_f = \infty\) (\(\sigma_\psi^2 = 0\)) the channel model in (4.68) reduces to the pure LoS model in (4.12). When \(K_f = 0\) (\(\rho = 0\)), the channel model in (4.68) reduces to the commonly used Rayleigh fading channel model. The process of estimating the \(K\)-factor is well-studied [64–66] and outside the scope of our work, and we assume that the \(K\)-factor is a known quantity.

We now briefly examine how the estimation methods previously presented, behave under the Rician channel model.

### 4.4.1 Minimum Mean Squared Estimation

Evaluating the MMSE estimator for a single element of the channel vector, under the Rician channel model (see Appendix B.3) yields,
\[ \hat{h}_{m,R_{MMSE}} = E\{h_{m,R}|y_R\} \]
\[ = \frac{\int_\theta \int_\phi g(\theta,\phi) \left( \rho h_m(\theta,\phi) + \frac{\sigma_\psi^2}{\sigma_n^2 + \sigma_\psi^2} (y_m - \rho h_m(\theta,\phi)) \right) \, d\theta \, d\phi}{\int_\theta \int_\phi g(\theta,\phi) \, d\theta \, d\phi} \]

where
\[ g(\theta,\phi) = e^{-\frac{1}{2(\sigma_n^2 + \sigma_\psi^2)} \sum_k z_k^2} \]
\[ z_k = y_k - \rho h_m(\theta,\phi), \quad h_m(\theta,\phi) = \cos((k-1)\pi \cos \theta + \phi). \]

The final form of the estimator in (4.70) is similar to that of the pure LoS MMSE estimator in (4.15), and hence has the same \( O(M^4) \) runtime complexity. We can observe that if the power of the multipath component \( \sigma_\psi^2 \) heads to zero, the estimator in Rician MMSE estimator in (4.70) reduces to (4.15).

As per definition, the MMSE estimator in (4.70) has the lowest resulting NMSE from all practical estimators under the Rician channel model, and its NMSE is shown in Figure 4.11.

### 4.4.2 Linear Minimum Mean Square Estimation

Adapting the LMMSE estimator to estimate the Rician channel is relatively simple due to the fact that the multipath component of the channel is statistically independent to the other underlying random quantities of interest. Furthermore, the multipath components across the array \( \{\psi_1, \psi_2, \ldots, \psi_M\} \) are all statistically independent quantities.

Therefore, given the normalised channel power in the model (4.67), the LMMSE estimator for Rician channels is identical to that of the LoS channel estimator in (4.23), with the exception of the non-diagonal elements of the \( \Gamma_{hh} \) matrix in (4.24), where \( E\{h_m \ h_{m+n}\} \) is replaced with \( E\{h_{m,R} \ h_{m+n,R}\} \), which are given by,

\[ r_{h,R}[n] = E\{h_{m,R} \ h_{m+n,R}\} = \begin{cases} r_h[0], & n = 0 \\ \rho^2 r_h[n], & \text{otherwise}, \end{cases} \]

where \( r_h[n] \) is given in (B.20).

Figure 4.11 shows a very similar channel estimation performance of the LMMSE estimator under Rician fading, as compared to the pure LoS scenario performance plotted in Figure 4.9.
This behaviour is expected. Under the LoS channel we only observed a small improvement in the NMSE with increasing $M$ over the range $0 < M < 32$, where no further improvement was observed for larger values outside this range. This highlighted the LMMSE estimators very limited ability to benefit from the additional antenna elements and exploit the spatial correlation amongst the elements. Therefore, the reduction in the spatial correlation between the elements under the Rician channel model (through the reduction of the LoS component in (4.67) by $\rho$), causes only a small increase in the resulting NMSE when compared to the pure LoS performance, since the LMMSE estimator is not significantly affected by this correlation. Similarly to the results in the pure LoS scenario, with increasing $M$ we see little performance improvement of the LMMSE estimator after $M = 32$. This limited effect of the spatial correlation on the LMMSE estimator is shown explicitly in Figure 4.12, where the resulting NMSE of the LMMSE estimator barely changes as the strength of the LoS component increases.
4.4 Channel Estimation under the Rician Channel model

4.4.2.1 Large M Analysis

Due to the complete statistical independence of the multipath components, the large $M$ NMSE of the Rician LMMSE estimator can be derived using the steps in Theorem 4.1 for the LoS channel LMMSE estimator. As per (B.22), the power spectral density function of the $\{h_m\}$ process under the Rician channel can be found by taking the DTFT of the covariance function in (4.71). Expressed in terms of $S_h(e^{j\omega})$ we can write,

$$S_{h,R}(e^{j\omega}) = r_h[0] + 2\rho^2 \sum_{n=1}^{\infty} r_h[n]e^{-j\omega n}$$

$$= r_h[0](1 - \rho^2) + \rho^2 S_h(e^{j\omega})$$

(4.72)

Considering the total signal power $r_h[0] = 1/2$, and substituting the result of (B.29) into (4.72) yields,

$$S_{h,R}(e^{j\omega}) = \frac{1 - \rho^2}{2} + \frac{\rho^2}{\sqrt{\pi^2 - \omega^2}} \quad -\pi < \omega < \pi$$

(4.73)

Replacing $S_h(e^{j\omega})$ with the power spectral density function of the Rician channel $S_{h,R}(e^{j\omega})$ in the large $M$ LMMSE result of (B.30) yields,

$$\text{NMSE}_{\text{LMMSE}_{M \to \infty}} = \frac{1}{2\pi E\{|h_m|^2\}} \int_{-\pi}^{\pi} \frac{(\rho^2 - 1)\sqrt{\pi^2 - \omega^2} - 2\rho^2}{2\eta(\rho^2 - 1) - 2\sqrt{\pi^2 - \omega^2} - 4\eta\rho^2} d\omega.$$  

(4.74)

Using the result in (4.74), we can easily compute the NMSE of the LMMSE estimator under large $M$ in Rician fading. Under a channel SNR of $\eta = -3dB$, using (4.74) results in a calculated limiting NMSE of $-2.0dB$ which agrees well with the Monte Carlo simulation results plotted in Figure 4.11 as $M$ increases.

When the power of the multipath component $\sigma^2_\psi$ is zero and consequently $\rho = 1$, the large $M$ NMSE in (4.74) clearly reduces to the large $M$ NMSE of the LoS channel LMMSE estimator given by (4.29).
4.4.3 Assisted Linear Minimum Mean Squared Estimation

Given the A-LMMSE estimator utilises a-priori information on $f_s$ (as opposed to $\theta$) to estimate the channel vector, for completeness we re-express the channel model in (4.68) as,

$$y_{m,R} = \rho \cos((m-1)2\pi f_s + \phi) + \psi_m + n_m.$$  \hspace{1cm} (4.75)

Identical to the modifications described in Section 4.4.2 for the LMMSE estimator, the addition of the independent multipath component modifies the off-diagonal terms of $\Gamma_{hh}$, (i.e., $r_{h,R}[n]$ for $n > 0$) of the A-LMMSE estimators, as specified by (4.71). Given this is the only modification required, the A-LMMSE estimators under the Rician channel have the same runtime complexity of the A-LMMSE estimators under the LoS channel.

Figure 4.11 shows the performance of the A-LMMSE estimator under Rician fading. As observed under the LoS channel, the resulting NMSE decreases as $M$ increases, demonstrating a channel estimation performance similar to that of the optimal MMSE estimator, but with a significantly lower runtime complexity of $O(M^3)$ instead of $O(M^4)$. As before, the A-LMMSE Static Uniform estimator performs much better than the LMMSE estimator with a small increase in runtime complexity.

When comparing the A-LMMSE estimator under Rician fading in Figure 4.11 to the A-LMMSE under the pure LoS channel in Figure 4.9, there is a marked difference in the channel estimation performance. Under the same channel SNR, a degradation of around 5dB in NMSE at $M = 64$ can be observed under the Rician channel. There are two causes of the degradation.

The first cause is from the multipath component acting only as additional noise during the $f_s$ estimation process. For the $f_s$ estimation process under the Rician channel model in (4.75), this has the effect of reducing the signal power and increasing the effective noise power by $\sigma^2_{\psi}$. Naturally, this will result in a looser a-priori information on $f_s$, i.e., a larger resulting $\sigma^2_{f_s}$, and consequently poorer overall channel estimation performance.

The second cause is due to the reduction in the spatial correlation between the channels of the antenna elements in the presence of the multipath component, i.e., the reduction of $\rho$ in (4.71), consequently making it harder for the A-LMMSE estimator to exploit such correlation. The ef-
fects of spatial correlation on the channel estimation performance of the estimators is illustrated in Figure 4.12. This figure shows that the A-LMMSE estimator performs better as the LoS component is made relatively stronger by increasing $K_f$. For example, when increasing $K_f$ from 1 to 10 we see an improvement of 5dB in the NMSE.

In order to investigate which of the two effects above is the dominant degradation, we compare the Rician A-LMMSE estimator to a genie-assisted Rician A-LMMSE estimator, where the Assistant is capable of providing perfect a-priori knowledge, i.e., $\hat{f}_s = f_s$ and $\sigma^2_{\hat{f}_s} = 0$ in (4.42). When using this knowledge we remove the effects of the first cause of degradation during the $f_s$ estimation. Figure 4.11 shows a difference of around 1dB between the two estimators at $M = 64$, which is a small contribution to the 5dB overall degradation, and exposes the reduced spatial correlation as the dominant source of degradation. As $M$ increases, we see that the degradation from the reduced spatial correlation becomes even more dominant. For example at $M = 256$, we observe an overall 10dB difference between the NMSE of the LoS (Figure 4.9) and Rician (Figure 4.11) A-LMMSE estimators, and only a 0.3dB improvement of the genie-assisted A-LMMSE Rician estimator in Figure 4.11.

Figure 4.11 also plots the NMSE from a LoS-genie-assisted LMMSE estimator in which the LMMSE estimator has perfect knowledge of the LoS component, $\rho h_m(\theta, \phi)$. The estimation problem then becomes one of estimating the remaining complex Gaussian quantity $h_{m,R} \sim \mathcal{N}(\rho h_m, \sigma^2_{\rho h_m})$, from Gaussian observations $y_{m,R} \sim \mathcal{N}(\rho h_m, \sigma^2_{n} + \sigma^2_{\rho h_m})$. Consequently $h_{m,R}$ and $y_{m,R}$ are jointly Gaussian and the resulting LMMSE estimator is in fact the optimal MMSE [54] for this scenario. Therefore, the resulting MSE from such an estimator serves as a lower bound for all estimation schemes under the Rician channel model. Since there is no correlation between the observations under this genie-assisted scenario, the estimation is effectively a scalar estimation problem. Applying the standard scalar MSE expression for the LMMSE [54], the resulting $\text{MSE} = \frac{\sigma^2_{\rho h_m}}{\sigma^2_{n} + \sigma^2_{\rho h_m}}$ is the lowest of all the estimators as shown in Figure 4.11, further supporting the NMSE results presented for the estimation schemes.
Figure 4.12: The NMSE for the Rician Channel model for LMMSE and variations of A-LMMSE as \( K_f \) varies, \( M = 64, \eta = -3dB \).

### 4.4.3.1 Large M Analysis

As discussed previously in Section 4.3.4.3, the A-LMMSE Static Uniform estimator provides a practically desirable solution due to its low complexity and impressive channel estimation performance, which is also agnostic to the distribution of \( f_s \) under large \( M \).

The large \( M \) behaviour of the A-LMMSE Static Uniform estimator under Rician conditions can be analysed using the same steps as presented in Theorem 4.4 and Corollary 4.1. Under the Rician channel, the filter which assumes a uniform distribution of \( f_s \) (described by (B.41)) is now given by,

\[
W(e^{j\omega}) = \frac{S_{h,R}(e^{j\omega})}{S_{h,R}(e^{j\omega}) + \sigma_n^2},
\]

(4.76)

where \( S_{h,R}(e^{j\omega}) \) is defined in (4.72), making use of \( S_h(e^{j\omega}) \) for a uniformly distributed \( f_s \) as derived in Lemma B.1. The filter in (4.76) is then used to filter the input with a power spectral density function given by (4.72), where \( S_h(e^{j\omega}) \) for a non-uniformly distributed \( f_s \) is given by (4.56). Applying the MSE expression (B.40), and the results of Theorem 4.4 and Corollary 4.1 yields,
4.5 The Complex Rician Channel model

\[
\text{NMSE}_{\text{ALMMSE Static Uniform}}_{M \rightarrow \infty} = \frac{\eta(\rho^2 - 1)(1 + \rho^2(Q - 1)) - 1}{2 E\{|h_{m,R}|^2\}(\eta(\rho^2 - 1)(1 + \eta(1 + (Q - 1)\rho^2)))}, \quad Q > 1
\]

(4.77)

The large $M$ NMSE for the A-LMMSE Static Uniform estimator in (4.77) is plotted in Figure 4.11, and agrees well with the finite $M$ results from Monte Carlo simulation as $M$ increases. As the multipath component reduces, i.e., $\rho \rightarrow 1$, the large $M$ MSE expression in (4.77) reduces to the large $M$ A-LMMSE Static Uniform NMSE for the LoS channel in (4.59).

4.5 The Complex Rician Channel model

As discussed in the opening sections, a real-valued channel model was adopted in this chapter to facilitate an intuitive and elementary exploration of the estimation problem. The real-valued Rician channel model of Section 4.4 can readily be extended to a complex-valued channel model, i.e., one which accounts for both in-phase and quadrature arms of the baseband receiver. For completeness, in this section we briefly present an equivalent complex channel model and summarise the key differences from the real-valued LMMSE and A-LMMSE estimators.

Employing an equivalent spatial complex channel model to the Rician fading channel model described by (4.75), we can express the $m$-th element of the observation vector $y_R \in \mathbb{C}^{M \times 1}$ as

\[
y_{m,R} = h_{m,R} + n_m
\]

\[
y_{m,R} = \rho e^{-j((m-1)\pi \cos \theta + \phi)} + \psi_m + n_m
\]

\[
y_{m,R} = \rho e^{-j((m-1)2\pi f_s + \phi)} + \psi_m + n_m
\]

for $\theta \in [0, \pi]$, $f_s \in [-0.5, 0.5]$ and $\phi \in [0, 2\pi]$. Note that the complex valued observations allow the unambiguous detection of the AoA of the desired user over the range $[0, \pi]$ as opposed to the previously modelled smaller range, where $\theta \sim U[0, \pi/2]$ (as discussed in the footnote of Section 4.3.3.1). Consequently, our complex Rician model in (4.78) adopts the full AoA range, where $\theta \sim U[0, \pi]$. The additive multipath component is modelled by the complex quantity $\psi_m \sim \mathcal{CN}(0, \sigma^2_\psi)$ and the additive noise term is modelled by the complex quantity $n_m \sim \mathcal{CN}(0, \sigma^2_n)$. 
Therefore, $h_{m,R}$ and $y_{m,R}$ in (4.78) remain zero mean quantities.

Although the channel model in (4.78) is similar to that of (4.75), some of the terms defined in Section 4.4 are naturally redefined due to the presence of both the real and imaginary components. Namely, the channel power $E\{|h_{m,R}|^2\} = \rho^2 E\{|h_m|^2\} + \sigma_\psi^2 = 1$, where $0 \leq \rho \leq 1$, and consequently $\eta = E\{|h_{m,R}|^2\}/\sigma_n^2 = \frac{1}{\sigma_n^2}$, and the $K$-factor, $K_f = \rho^2 / \sigma_\psi^2$.

### 4.5.1 The LMMSE Estimator

Despite the similarities, the estimation of the random complex vector $h_{m,R}$ in the context of the LMMSE has some additional considerations. The LMMSE estimator structure used in (4.23) and (4.24) is only optimal, i.e., results in the minimum mean square error from all linear estimators, under the conditions where the complex vectors $h$ and $y$ are jointly circular [54, pg. 88], i.e., $E\{hh^T\} = 0$ and $E\{hy^T\} = 0$. In the channel model of (4.78), these conditions are met due to $\phi$ being statistically independent and uniformly distributed over $[0, 2\pi]$. Therefore, the form of the LMMSE estimator given in (4.24) is optimal, and we adopt it for use in the LMMSE and the A-LMMSE estimators.

As before, in order to define the LMMSE estimator in (4.23), we are just left to evaluate the covariance terms under this channel model, which are given by,

$$E\{h_{m,R}^* h_{m+n,R}\}_{\text{LMMSE}} = \frac{\rho^2}{\pi} \int_{\theta=0}^{\pi} e^{-jn\pi \cos \theta} \, d\theta$$

$$= \rho^2 f_0(jn\pi) \quad \text{for } n > 0. \quad (4.79)$$

The covariance terms (4.79) are equal to twice the covariance terms under the equivalent real-valued channel model given by (4.71) (where such a relationship is more readily observed from the form in (B.25)). Consequently, the complex $\{h_m\}$ process results in a power spectral density function which is twice that of the equivalent real-valued process in (4.73),

$$S_{h,R}(e^{j\omega}) = 1 - \rho^2 + \frac{2\rho^2}{\sqrt{\pi^2 - \omega^2}}, \quad -\pi < \omega < \pi. \quad (4.80)$$

\(^{10}\text{Note the presence of the transpose } T, \text{ as opposed to the conjugate transpose } H.\)
4.5 The Complex Rician Channel model

The large $M$ MSE of the LMMSE estimator under the complex Rician model can be derived by following the steps of Theorem 4.1. Given the doubling of the power spectral density function $S_{h,R}(e^{j\omega})$ and channel power $E\{|h_{m,R}|^2\}$, for the channel SNR $\eta$ as defined in Section 4.5, the resulting large $M$ NMSE is identical to that for the equivalent real-valued channel model, given by (4.74).

4.5.2 The A-LMMSE Estimator

Naturally, under the complex-valued model, the Assistant in Figure 4.6 of the A-LMMSE and A-LMMSE Static estimators makes use of the complex observation vector $y_R \in \mathbb{C}^{M \times 1}$ to generate the a-priori information on $\hat{f}_s$ (and consequently $\hat{q}$).

The covariance terms for the A-LMMSE estimator are modified in a similar fashion to the modifications described in Section 4.5.1 for the LMMSE estimator. Under the Rician complex channel model, the covariance terms in (4.43) become,

$$E\{h_m^* h_{m+n} + n\} \approx \rho^2 e^{-2n\pi \eta f_s}$$

for $n > 0$. (4.81)

As highlighted in the derivation of (4.43), the expression (4.81) is strictly an approximation since it has been derived by integrating over the range $[-\infty, \infty]$ (See Section 4.3.4.1).

The covariance terms of the A-LMMSE Static estimator defined in (4.49) are modified in a similar way for the complex Rician channel model, where the $Q$ estimators now span the larger range of $f_s \in [-\frac{1}{2}, \frac{1}{2}]$, yielding,

$$E\{h_m^* h_{m+n}\}^{(q)}_{\text{StaticUniform}} = \rho^2 Q \int_{f_{s,\text{min}}}^{f_{s,\text{max}}} e^{-j2\pi f_{s,n}} \, df_s,$$

for $n > 0$.

Following the steps of Theorem 4.3, the large $M$ MSE of the A-LMMSE Static Uniform estimator under the complex Rician channel model can be readily derived. Given the doubling of
Figure 4.13: The NMSE for the single cell scenario for LMMSE and variations of A-LMMSE under the complex Rician channel model, $K_f = 3$, $\eta = -3dB$.

As demonstrated, the resulting LMMSE and A-LMMSE estimators under the complex-valued Rician channel model are similar to those developed under the real-valued channel model. While the estimation problem under the complex channel model requires the estimation of the additional complex components of the $h$ vector, these complex components are still a function of the underlying random variables $\theta$ and $\phi$. Consequently, the complex observation vector $y$, effectively contains additional observations which can be used to better estimate the channel vector. However this potential improvement has been offset by the additional randomness introduced from extending the range of the AoAs $\theta$ in (4.78). As a result, we observe a similar performance of the LMMSE and A-LMMSE estimators under a given SNR $\eta$, between the complex-valued Rician channel model results in Figure 4.13 and the equivalent real-valued results in Figure 4.11.
4.6 Conclusions and Future Work

Estimation of the wireless channel with LoS components poses its own set of challenges. Under channel conditions where a strong LoS component is present, we have shown that a large gap exists in performance between the LMMSE and optimal MMSE estimator, when estimating the channels of the large BS antenna array. The LMMSE estimator is unable to adequately exploit the spatial correlation that exists amongst the channels of the array, and consequently has an almost constant NMSE as the number of elements of the antenna array increases.

The A-LMMSE estimator presented in this chapter is able to effectively exploit the spatial correlation and additional observations from the large antenna array by utilising tighter a-priori information about the desired user’s AoA. Under a pure LoS channel, the estimator is able to achieve an NMSE which is close to optimal, while introducing only a small amount of additional complexity. Such performance gains of the A-LMMSE estimator are also observable in the resulting SER when using the MRC detector for the reception of user data.

The A-LMMSE estimator lends itself well to even lower runtime complexity variants which can be readily implemented in practice. Despite having the same $O(M^3)$ runtime complexity order as the LMMSE estimator, we show that the low complexity A-LMMSE Static Uniform estimator is able to perform significantly better than the LMMSE estimator under the complex Rician channel model. The A-LMMSE Static Uniform estimator is attractive for practical implementations, not only because of its low runtime complexity but also due to its large $M$ performance being agnostic to the distribution of the user AoAs, and its robustness to poor a-priori information. The NMSE of the LMMSE and A-LMMSE estimators have been studied in detail, where the performance differences observed are confirmed by the analytically derived bounds on the NMSE in the limit of $M$.

While we have derived and analysed the A-LMMSE estimator for the massive MIMO BS under pure LoS and Rician channel models, an analysis of the A-LMMSE estimator under channel models used for millimetre wave, i.e. 30GHz - 300GHz, is also of importance. The use of the millimetre wave spectrum for mobile communications is a recent advancement in wireless technology which is part of the 5G standard, and of current research interest. The millimetre wave channel models adopted in the current literature [30] are in fact extensions of the LoS model used in Section 4.3. The millimetre wave channel is modelled by a small number of clusters, where
each cluster contains several distinct paths over a small angular spread. The application of the A-LMMSE estimator to such a model suggests the estimation of the angular centre of each cluster by the Assistant, accompanied with an uncertainty, or variance which is across all paths of the cluster. This per cluster a-priori information can then be utilised by the LMMSE processing. The extension of such an approach to the multi-user millimetre wave scenario also requires further investigation since the separation and identification of clusters from non-orthogonal users in the angular domain becomes non-trivial.

The A-LMMSE schemes presented in this chapter utilise a uniform linear array at the BS, and exploit a-priori information on the azimuth AoA. Such schemes rely heavily on the array gain achieved through efficient beamforming, and the derivation of unambiguous location information, and as discussed in Section 4.2, implies fixed minimum distances of the array elements to avoid undesirable effects when beamforming. Given the array gain is proportional to the number of antenna elements, two-dimensional antenna arrays allow more practical physical form factors of the array while achieving the required array gains. In the context of millimetre wave communications, the increase in path loss at the higher frequencies can be compensated by an increase in the array gain from the additional closer spaced elements of the array. However, this array gain can only be realised if the beamforming is well aligned to the user, as the increase in elements results in narrower beams. The use of two-dimensional antenna arrays allow the steering of the beam in both the azimuth and elevation angles, allowing more enhanced beamforming to better target the users [67], and therefore their application in millimetre wave systems is of even more importance. This motivates potential extensions of this chapter to include two dimensional planar arrays, where the resulting A-LMMSE scheme exploits a-priori information on both the azimuth and elevation AoAs.
Chapter 5
Channel Estimation in Multi-user Massive MIMO

In this chapter we extend the uplink channel estimation techniques explored in the previous chapter, namely the LMMSE and A-LMMSE estimators, and focus on examining their behaviour in multi-user scenarios. We first examine the performance of the estimators in the single BS scenario, where the other connected users of the BS generate interference to the desired user’s data transmission. We extend the scenario to include several neighbouring interfering BSs which introduces the additional impairment of pilot contamination, and explore the resulting performance under various simulated K-factors of the desired and interfering users’ channels. Under both the single and multi-cell scenarios, the A-LMMSE estimator shows significant NMSE and SER performance gains over the LMMSE estimator. We examine the resulting receive beamforming patterns in order to investigate the performance differences of the estimators under pilot contamination and multi-user interference. We show that under channels with LoS components, interfering channels with higher K-factors result in a lower SER at the BS.

5.1 Introduction

The work of the previous chapter provided the motivation and development of the A-LMMSE estimator. Following on from this work, this chapter focuses on investigating the behaviour and performance of the LMMSE and A-LMMSE estimators in more challenging multi-user scenarios. Again, we focus on more realistic channel models where some degree of spatial correlation exists. While rich scattering environments modelled by fully spatially uncorrelated channel models are key to the capacity gains of single user MIMO [6,34], in the multi-user scenario, correlated channels provide a means to decompose the channel and resulting covariance matrices into signal and interference components. As we will observe in this chapter, this makes
spatial correlation beneficial for multi-user communications where ultimately, under the conditions of non-overlapping AoAs, single user interference free channels are observed in the limit of $M$ [18].

In our TDD multi-user scenario, all users transmit pilot and data symbols in fixed positions, using the *same* time-frequency resource as depicted in Figure 1.2. Therefore, the received baseband symbols at the BS are a sum of all transmissions received from all users connected to BS of interest and neighbouring BSs. Consequently,

- in order to separate users’ pilot transmissions to perform estimation of the desired user’s channel, it is possible to employ pilot sequences that are orthogonal or have low cross correlation properties. However, in the multi-cell scenario, all users connected to nearby BSs who use pilot sequences which are non-orthogonal to the sequence used by the desired user (e.g., by reusing the same pilot sequence) will contaminate the pilot transmission of the desired user. Such contamination can significantly affect the subsequent detection performance and is known as pilot contamination.
- in order to separate users’ data transmissions, the BS utilises detectors, such as the MRC and ZF detector, which require estimates of the channel. For this section we again employ the MRC detector as it is highly efficient under a large number of $M$ BS antennas.

Given that pilot contamination has been identified as the dominant impairment in the context of multi-cell multi-user large antenna systems [7, 8], there is a large amount of literature with proposed methods to mitigate its effect, with a comprehensive survey provided in [68]. We can divide the approaches into those involving protocols or pilot allocation schemes, and those which are focused on signal processing and channel estimation methods, while some involve a combination of both. The contents of this chapter is mainly focused on channel estimation methods, as we aim to evaluate the channel estimation performance of the A-LMMSE estimator derived in Chapter 4 in multi-user multi-cell settings.

Recent works present optimised variants of the reused pilot set allocation scheme (see Section 2.2.1), where pilots are allocated amongst the users of the network in a way that attempts to reduce the effects of pilot contamination in the multi-BS scenario [18–20]. In contrast to these works, this chapter assumes the use of a static reused pilot set scheme where the pilot sequences are assigned arbitrarily. However in doing so, we demonstrate the desirable properties of a pilot
allocation scheme which is to operate effectively with the A-LMMSE estimator in the multi-BS setting. In particular, the benefits of having spatially separated pilot contaminants, which is also the common objective of the above-mentioned works [18–20]. The work in [18] summarises the benefits by providing a theorem stating in the limit of $M$, the effects of pilot contamination disappears when the pilot contaminants have disjoint AoAs to the desired user. In an attempt to ensure this condition, they adopt a greedy static pilot allocation algorithm requiring each cell to compute the covariance matrices of all neighbouring users, where the pilot allocation resulting in the smallest computed MSE is selected.

The work in [20] builds on [18] by proposing an alternate cost function to the MSE which captures the effects of the pilot contaminants under finite $M$, without having to evaluate the covariance matrices. Interestingly, the recent work in [20] shows that even complex jointly optimised pilot allocation schemes are unable to guarantee sufficient spatial separation of users all of the time, resulting in a modest 11% increase in downlink per-cell sum rate when compared to the random reuse pilot set allocation, under $M = 64$ and 10 users per cell. The work in [69] adopts alternate cost functions to quantify the interference under Rician channels, in order to guide the allocation of pilots, and report gains of up to 20% in the uplink per-cell sum rate when compared to a random reused pilot set allocation.

In addition to a pilot allocation that attempts to spatially separate contaminators, the work in [70] proposes the decontamination of the estimates by performing a window filtering in the angular domain over the AoAs of the desired user. Subspace based approaches to decontamination in [18, 58] rely on eigenvalue decompositions of the channel covariance matrices obtained from dedicated pilot transmissions, while the work in [57] demonstrates how under some conditions, such decompositions are possible by only using data transmissions.

Schemes such as [18] are expensive given that the network needs to provision the resources for each BS to learn the channel covariance matrices (inclusive of the slow gains) of all potential interferers, and communicate them to a central place to execute the pilot assignment algorithm. In contrast to the cooperative approaches in [18, 20] we explore the use of minimal forms of cooperation where only the pilot contaminants existence in neighbouring cells is shared amongst BSs, and the use of reduced knowledge of inter-cell interferers’ slow gains in the form of second order statistics.
The main contributions of this chapter can be summarised as follows,

- The evaluation of the extended A-LMMSE and LMMSE estimators under Rician channels, and their NMSE and SER performance, in the multi-user single cell and multi-cell environment.
- The evaluation of an extension to the A-LMMSE estimator which employs minimal forms of BS cooperation, under the reused pilot set scheme and practically large finite $M$. We observe that such minimal forms of cooperation are only advantageous under up to certain values of $M$.
- An investigation into the effects of the intra and inter-cell Rician channel K-factor on the resulting performance of the A-LMMSE and LMMSE estimators. We show that interferers with Rician channels to the BS of interest pose a greater interference than Rayleigh channels under MRC detection due to the severe degradation caused in instances when the LoS component of the interferers channel falls within the main lobe of the desired user.

5.2 Single Cell MU Massive MIMO

5.2.1 Orthogonal Pilots

In this section, each of the $K$ users connected to a BS is arbitrarily assigned a pilot sequence from a set of orthogonal sequences, as described in Section 2.2.1. We momentarily re-introduce the time domain into our narrow band channel model, where $\tau$ symbols are used within the coherence time to transmit the pilot sequence. In this chapter we assume that all users adopt ULPC as described in Section 2.5, and consequently are received at their respective BS with the same gain (unity).

Therefore during the channel estimation phase, the received $M \times \tau$ symbol matrix of observations at the BS can be described by the following complex baseband channel model,

$$Y = H(\theta, \phi)S + N,$$

where $Y, N \in \mathbb{C}^{M \times \tau}, H \in \mathbb{C}^{M \times K}, S \in \mathbb{C}^{K \times \tau}$. The channel vector of the $k$-th user $h_k \in \mathbb{C}^{M \times 1}$, has its $m$-th element given by $h_{mk}^{(k)} = e^{-j(m-1)\pi \cos \theta_k + \phi_k}$. The channel vectors $h_k$ form the $K$ columns of $H$, where each user has an independent random AoA $\theta_k$ and phase $\phi_k$. Note the nota-
5.2 Single Cell MU Massive MIMO

The adoption of massive MIMO in single cell setting has been explored, where the subscript of a vector (bold lowercase) denotes the associated user. When referencing single elements of the vector, this association appears in the superscript, e.g., \( h^{(k)}_{m} \) refers to the \( m \)-th element of the channel vector belonging to the \( k \)-th user. The noise is additive and white, where the i.i.d elements of the matrix \( N \) are distributed according to \( \mathcal{CN}(0, \sigma^2_n) \).

As before, we wish to estimate the channel vector of the desired \( k \)-th user, \( h_k \) from \( Y \). Given the use of the orthogonal pilots by all users in the cell, multiplying the received \( Y \) matrix by the orthonormal pilot sequence vector of the desired user \( s_k \in \mathbb{C}^{1 \times \tau} \), leaves us with the observations \( y_k \) of the desired user’s channel vector in the presence of noise, i.e.,

\[
y_k = Ys_k^H = h_k + n, \tag{5.1}
\]

where the i.i.d elements of the vector \( n \) are distributed according to \( \mathcal{CN}(0, \sigma^2_n) \). Therefore, after correlation with the known pilot sequences, we are left with the complex spatial model in (5.2), where the resulting channel estimation performance (NMSE) of the LMMSE and A-LMMSE estimators is given by the single user scenario presented in Figure (4.13) of Chapter 4.

We now examine the resulting SER when employing the estimators during data detection in the multi-user single BS scenario. The received vector \( y \in \mathbb{C}^{M \times 1} \) at the BS during the data reception of one symbol, is the superposition of symbols from all \( K \) users of the cell, and can be expressed as

\[
y = h_kx_k + \sum_{k' \neq k}^{K} h_{k'}x_{k'} + n, \tag{5.3}
\]

where \( x_{k'} \) is the symbol transmitted by user \( k' \) over the channel \( h_{k'} \), and user \( k \) is our desired user. Using the MRC detector defined in (4.61), we have the detected symbol of the desired user,

\[
\hat{x}_k = x_k + \frac{\hat{h}_k^H}{\hat{h}_k^H\hat{h}_k}x_k + \frac{\sum_{k' \neq k}^{K} \hat{h}_{k'}^H}{\hat{h}_k^H\hat{h}_k}x_{k'} + \frac{\hat{h}_k^Hn}{\hat{h}_k^H\hat{h}_k}, \tag{5.4}
\]

where the desired users channel has been decomposed into the sum of the channel estimate and the estimation error, i.e. \( h_k = \hat{h}_k + \tilde{h}_k \). It can be clearly seen in (5.4), that the effective
noise at the MRC detector is made up from the last three terms, the channel estimation error, the interference from the other users, and the additive noise. As expected, an increase in $K$ results in a larger effective noise and consequently an increase in the SER.

The resulting SER has been plotted in Figure 5.1(a) with cell load $K = 24$. The differences in the channel estimation performance shown in Figure (4.13) are reflected in the resulting SER in Figure 5.1(a), with the use of the A-LMMSE estimator with the MRC detector clearly outperforming the SER when the LMMSE estimator is used in the single cell scenario.

We have thus far used the NMSE and resulting SER to evaluate the effectiveness of channel estimates produced by the estimators. Given we employ MRC detection, we can also readily observe the effectiveness of the channel estimate from the resulting spatial beam pattern when the channel estimate (or beamforming vector) $\hat{h}$ is applied to the uniform linear array. The resulting spatial beam pattern clearly illustrates the AoAs with which we maximise our reception at the BS. We assume each antenna element to be an identical isotropic source, and in our case, a normalised spatial pattern can be easily computed using the transform,

$$p(\theta') = c |\hat{h}^H a(\theta')|, \quad 0 < \theta' < \pi$$  \hspace{1cm} (5.5)

where the $m$-th element of $a(\theta')$ is given by $a_m(\theta') = e^{j(m-1)\pi\cos\theta'}$. We generate and plot a discrete version of the beamforming pattern in (5.5) using $N$ samples, where $\theta'_n = \frac{n\pi}{N}$, $1 < n < N$, which are normalised with factor $c = \left(\sum_{m=0}^{N} |\hat{h}^H a(\theta'_n)|\right)^{-1}$.

It is intuitive that the ideal beamforming vector will concentrate most of its energy through a large main lobe in direction of the desired user. Also of importance is the nature of the side lobes formed as they will potentially receive energy from unwanted sources. The width of the lobes produced by the beam pattern is inversely proportional to $M$. The width of the main lobe, specifically the width between the nulls either side of the main peak known as the first null beamwidth (FNBW), under such an antenna array with critical spacing, can be approximated by $FNBW \approx \frac{2\lambda}{Md} = \frac{4}{M}$ [71].

The beamforming pattern in Figure 5.2 clearly shows how the A-LMMSE estimates have a much larger main lobe in the direction of the desired user, compared to the LMMSE estimates. Consequently, the MRC detector using these estimates is able to receive more energy from the desired user, resulting in a better SER when compared to the use of the LMMSE estimates. As
Figure 5.1: The QPSK SER in the single cell scenario, $K = 24, K_f = 3, \eta = -3dB$. 
Figure 5.2: The beamforming patterns under the single cell scenario with orthogonal pilots for all users $K = 24$, $M = 64$, $K_f = 3$, $\eta = -3dB$.

expected, the beam pattern formed by the perfect estimate, $\hat{h} = h$ has a main lobe which is more correctly focused on the AoA of the desired user. At the desired user’s AoA, the magnitude of the perfect estimate beam pattern is slightly lower than the beam pattern from the A-LMMSE estimate, however it results in a substantially better SER. This is a result of the perfect estimate beam pattern having both a more accurately focused main lobe and smaller side lobes, enabling it to receive less power from the interfering users who have AoAs close to that of the desired user.

Therefore, even in the case of perfect estimates, the natural imperfections of beamforming patterns from such an array results in energy being collected from nearby AoAs where the main lobe and adjacent side lobes have non-zero amplitude. Under our channel model in (5.3), the interfering users are received with the same slow gain as the desired user, and hence the generated interference power is significant. Users which have an AoA close to the AoA of the desired user will fall into the main lobe and generate a large amount of interference, which will likely result in symbol detection errors. As such, when operating under high SNR within practical SER regions, i.e., $\text{SER} < 10^{-1}$, instances of the desired and interfering users having similar AoAs often result in symbol detection errors, and dramatically effect the resulting SER. Intuitively, the
worst case interferer is one which lies directly at the AoA of the desired user $\theta_k$. This can be seen more formally from the magnitude of the channel inner products between the desired user and interfering users which make up the interference for MRC detection, as shown in the third term of the right hand side of (5.4). If we consider a single interferer, $|h_k^H h_{k'}|$ is maximised when $\theta_{k'} = \theta_k$, with the resulting interference power equal to the signal power. The variation of the interference power with respect to the interferer’s AoA is discussed in further detail in Section 5.3.4.

The SER is therefore inversely proportional to the probability of the interferer’s LoS component being close to the main lobe of the desired user. Given that the width of the beam is inversely proportional to $M$, the resulting SER is inversely proportional to $M$, and proportional to the total number of interferers $K$ during data detection$^1$.

The varying SER results between the estimators shown in Figure 5.1(a) are a result of different channel estimation error and interference powers, determined by the magnitude of the second and third terms of the right hand side of (5.4) respectively. The channel estimation error is largely represented by the resulting NMSEs, and the interference power a consequence from the above-mentioned effects of the involved AoAs and resulting beam patterns, both of which to varying degrees are a result of the channel estimates used. Therefore, the SER performance gain of the A-LMMSE estimator shown in Figure 5.1(a) clearly demonstrates its ability to provide decidedly better estimates for MRC detection, in the presence of noise and interference.

Such effects during data detection also highlight the importance of user data scheduling in the multi-user scenario, where the BS should attempt to only schedule users which have spatially separated AoAs. Figure 5.1(b) shows the resulting SER under an ideal fully loaded BS which uses AoA information to only schedule users for data transmission which are spatially separated from the desired user (by the FNBW). Under such a scheduling, we can observe that the SER achieved is significantly lower than without scheduling as plotted in Figure 5.1(a), where all parameters have been kept the same including the number of interfering transmitting users.

$^1$Under high SNR and simplifying assumptions about the AoA distribution and antenna pattern, an approximation of the symbol error probability can be reduced to a combinatoric expression [9].
5.2.2 Discussion: Extensions for Non-orthogonal Pilots

In this section we explore the scenario where an interferer connected to the same BS as the desired user, does not utilise a pilot sequence which is orthogonal to the sequence used by the desired user. To emphasise the effects of such a scenario, we assume that the users are using identical pilot sequences. Let us consider a simple two user case, $K = 2$, where there is a desired and interfering user connected to the same BS, both with random, but different AoA within $[0, \pi]$. Again we will assume the presence of ULPC, and as such they are received with the same gain at the BS. After correlation with the pilot sequence of the desired $k$-th user, we are left with the observation vector

$$y_k = h_k + h_I + n,$$

which in contrast to (5.2), is now contaminated with the channel of the interfering user. More importantly, there is nothing statistically different about the interfering channel $h_I$ and the desired user channel $h_k$. As such, estimating the channel under these particular conditions, with no additional information to exploit, will naturally result in poor estimates.

For example, consider the LMMSE estimator. Under these conditions $\hat{h}_k = \hat{h}_I$, since identical covariance matrices of the LMMSE estimator (as given in (4.22)) are formed when attempting to estimate either $h_I$ or $h_k$. This behaviour is also illustrated by the resulting array beamforming pattern shown in Figure 5.3, which is highly undesirable as it focuses an equal amount of signal from both the desired and interfering user.

The A-LMMSE estimator under such a scenario also has challenges. The presence of the interfering user means that during the detection of the AoA, two peaks of the same magnitude will be present in the argument of (4.40). The Assistant may have additional a-priori information consisting of two spatial frequency estimates, however we are unaware which one corresponds to the desired user. In practice, a simple non-cooperative approach would be to perform the A-LMMSE estimation considering each of the spatial frequency estimates as potentially belonging to the desired user, yielding two channel estimate vectors. Furthermore, the other available spatial frequency estimate could also be used to better construct the $\Gamma_{yy}$ matrix. Each channel estimate can then be used in separate detection attempts on the received data vector (as done in (4.62)),...
yielding two data detection results. In mobile standards such as LTE, an unique identifier for each user called the C-RNTI (Cell Radio Network Temporary Identity) is used along with data bits to create the CRC on the data [72], ultimately allowing the identification of which user the detection result belongs to. Therefore, practical approaches exist which can be employed to harness the performance gains of the A-LMMSE estimator.

Furthermore, even though a user using a non-orthogonal pilot sequence implies they are an interferer connected to a neighbouring BS (under the reuse pilot scheme of Section 2.2.1) and their transmission is not of interest, in the case of cooperative schemes such as LTE Coordinated Multipoint (CoMP), the additional detection results of interfering users can be combined across BSs and offer improved decoding performance.

Using the multi-user non-cooperative approach for the A-LMMSE estimator described above, an example beamforming pattern is shown in Figure 5.3. A significant improvement over the LMMSE beamforming pattern can be observed, since a larger amount of energy is directed towards the desired user. Importantly, under these conditions the A-LMMSE estimator has effectively removed the effect of pilot contamination, as the beam pattern formed is as though the interferer is not present. Under large $M$, this effect has been captured by [18, Theorem 1], which states that the properly formed LMMSE estimator will generate estimates as if the contaminators were not present, provided their AoAs do not overlap with the desired user’s AoA. In order to achieve this decontamination, the A-LMMSE estimator needs to be able to distinguish the different AoAs from all pilot contaminators, where the spectral resolution will be ultimately limited by the number of antenna elements $M$. This highlights the significance of intelligent pilot allocation schemes for massive MIMO networks, e.g., as presented in [18–20, 56], where the network prevents the assignment of non-orthogonal pilots to users which have similar AoAs. Such pilot allocation schemes would then allow the A-LMMSE estimator to offer the performance gains over the LMMSE estimator in the presence of interfering users employing non-orthogonal pilots.

5.3 Multi Cell MU Massive MIMO

In this section we investigate the channel estimation performance of the LMMSE and A-LMMSE estimators in the context of a cellular network. The scenario is shown in Figure 5.4, where the BSs
Figure 5.3: Beamforming pattern, two users with LoS channels, with desired user AoA $\theta_k = \pi/6$, and interfering user AoA $\theta_k' = 5\pi/6$ using non-orthogonal pilots $M = 32$, $\eta = -3\,dB$.

are depicted as solid triangles, with the user of interest (indicated by the red dot) connected to the centre BS, and the interfering users (indicated by the blue dots) connected to the nearby BSs.

### 5.3.1 System Model

We approximate the cellular scenario by modelling intra-cell users distributed uniformly at random locations over the area of the inner annulus, and inter-cell users distributed uniformly at random locations over the area of the surrounding annulus. Given that we wish to model a uniform distribution of users over the full AoA range, the model in Figure 5.4 is preferred over non-tessellated cellular arrangements, such as adjacent circular cells. The small region at the centre of the cell is an exclusion region where users are not physically located due to the space occupied by the macro BS site.

We model the effects of Rician fading, and therefore extend the complex Rician channel model presented in Section 4.5. Note, to reduce notational clutter we drop the subscript "$R$" in this chapter and $h \in \mathbb{C}^{M \times 1}$ refers to the complex Rician baseband channel vector, whose $M$ components
are given by $h_{m,R}$ in (4.78). In reality, the channels of the users which are further away from the BS are more likely to have a smaller K-factor, $K_f$, than the channels of users closer to the BS (which can be clearly seen by the real world $K_f$ CDFs plotted over different distances in [29]). We employ a coarse model for such variations by using a separate K-factor for users connected to the inner BS of interest $K_{f,intra}$, and another K-factor for all other interferers present in the interfering region of the annulus $K_{f,inter}$. Therefore, the fast fading channels of our multi-cell channel model, for the intra and inter-cell users, expressed in terms of AoA $\theta$ is given by,

$$h_m^{(k)} = \rho_{intra} e^{-j((m-1)\pi \cos \theta + \phi)} + \psi_m,$$

$$h_m^{(l)} = \rho_{inter} e^{-j((m-1)\pi \cos \theta + \phi)} + \psi_m,$$

and expressed in terms of spatial frequency $f_s$ is given by,

$$h_m^{(k)} = \rho_{intra} e^{-j((m-1)2\pi f_s + \phi)} + \psi_m,$$

$$h_m^{(l)} = \rho_{inter} e^{-j((m-1)2\pi f_s + \phi)} + \psi_m,$$

where $\rho_{intra}^2 = \sigma_\phi^2 K_{f,intra}$ and $\rho_{inter}^2 = \sigma_\phi^2 K_{f,inter}$. Our channel model also differentiates between

Figure 5.4: The multi-cell scenario which models a desired user (red dot), and 3 neighbouring BSs, each with a single pilot contaminator (blue dots).
the slow gains of the users connected to the BS of the desired user, denoted by $\sqrt{\beta}_k$, and the slow gains of the users connected to neighbouring BSs, denoted by $\sqrt{\beta}_l$. The slow gains are described further in Section 5.3.1.1.

We assume a fully loaded network, with full frequency reuse, where each BS uses the same set of pilot sequences for their users (as specified by the reused pilot set scheme outlined in Section 2.2.1). As a result, all of the $L$ surrounding BSs have a connected user which is a pilot contaminator, i.e., a user which uses the same pilot sequence as the desired user. Just as in the single cell case, after performing a correlation with the known sequences (as described by (5.1)), the resulting observation vector only contains the channels of users using the same pilot sequence as the desired user. Consequently, the observation vector $y \in \mathbb{C}^{M \times 1}$ used for channel estimation is given by,

$$y = \sqrt{\beta}_k h_k + \sum_{l} \sqrt{\beta}_l h_l + n,$$

where $h_k$ is the channel vector of the desired $k$-th user we wish to estimate, and the $m$-th element of $y$ is given by,

$$y_m = \sqrt{\beta}_k h^{(k)}_m + \sum_{l} \sqrt{\beta}_l h^{(l)}_m + n_m,$$

where $n_m \sim \mathcal{CN}(0, \sigma^2_n)$.

As assumed throughout this work, the BS maintains knowledge of its users’ slow gains obtained through other accurate means that involve high layer signalling in combination with UE and/or BS measurements. Therefore, at the BS we assume that we have full knowledge of the slow gain of the desired user, i.e. $\sqrt{\beta}_k$. However, full knowledge of all the relevant inter-cell slow gains $\sqrt{\beta}_l$ is unrealistic given the resources and complexity that would be required to obtain them, especially in the presence of shadowing, where they cannot be derived from shared information and need to be estimated separately. Consequently, in contrast to [18], for this chapter we generally assume that we are constrained to knowledge of only the long-term statistics of the interfering users’ slow gains, namely the second moments of $\sqrt{\beta}_l$. In this chapter we also examine the differences of a minimal co-operative approach where the desired users’ BS is also aware of the number of pilot contaminators present, compared to an estimator which only has knowledge
of its own intra-cell users.

5.3.1.1 A Distance Based Pathloss Model

Using the model for the received symbols during channel estimation in (5.6), we can derive the LMMSE and A-LMMSE estimators, with no assumptions made about the properties of the slow gains $\sqrt{\beta_k}$ and $\sqrt{\beta_l}$. In this section we outline some further assumptions about the slow gains which are modelled later in the simulation in Section 5.3.4, and affect the resulting LMMSE and A-LMMSE estimators presented in Sections 5.3.2 and 5.3.3.

We assume a distance only based path loss model, i.e., the model described by (2.4) where $z_{jkl} = 1$. Using such a path loss model and employing ULPC (as described in Section 2.5) yields the slow gains,

$$
\sqrt{\beta_k} = 1,
\sqrt{\beta_l} = \left(\frac{r_l}{r_{lk}}\right)^\gamma,
$$

where $r_l$ is the distance from the interfering user to its BS, and $r_{lk}$ is the distance from the interfering user to the BS of the desired user, as depicted in Figure 5.4. As mentioned earlier, we assume that the BS has knowledge of the second moments of the gains of the inter-cell interferers in (5.8).

We model a fully loaded network, with full frequency reuse and a underlying cellular structure as shown in Figure 5.4. Given the use of a distance only based path loss model, only the Tier 1 interferers (the neighbouring BSs under full frequency reuse) are significant interferers, as highlighted in the numerical results of Section 3.3.4. For simplicity we consider the projection of the BS antenna array in one direction and consequently model the presence of 3 neighbouring BSs, as depicted in Figure 5.4. Each surrounding BS will contain a pilot contaminator under the reused set scheme, with an AoA to the desired users BS uniformly distributed in one of 3 sextants, over the interval $[0, \pi]$ on which the BS antenna array projects. Furthermore, the slow gains from all three interferers will have the same first and second moments.
5.3.2 LMMSE Estimation

The LMMSE estimator follows the structure presented in (4.22), with the exception that we now need to consider the effects of the slow gains and the interferers. The slow gains are completely statistically independent from each other, and independent of the other random quantities of the fast fading model, i.e., the random AoA $\theta$ and phase $\phi$.

5.3.2.1 LMMSE Estimator

As in the previous sections, the LMMSE estimator presented here only considers the desired user in the presence of additive noise. Since we assume knowledge of the desired users slow gain $\beta_k$, this is directly used, and the LMMSE form in (4.22) yields the estimator,

$$\hat{h}_{\text{LMMSE}} = \sqrt{\beta_k \Gamma_{hh}} \Gamma_{yy}^{-1} y$$

$$= \sqrt{\beta_k \Gamma_{hh}} \left( \sigma^2_n + \beta_k \Gamma_{hh} \right)^{-1} y, \quad (5.10)$$

where the covariance terms of the $\Gamma_{hh}$ matrix are given by (4.79).

5.3.2.2 Cooperative LMMSE Estimator

Under the cooperative scheme, the BSs of the network share information of their allocated pilot sequences. Given we model a fully loaded network, we examine the scenario where the BS of the desired user has knowledge of the pilot contaminator connected to each of the surrounding BSs. This allows the calculation of a more accurate covariance matrix, $\Gamma_{yy}$, of the LMMSE estimator.

Let us examine the covariance terms of the $\Gamma_{yy}$ matrix in the general case,

$$E\{y_m^* y_{m+n}\} = \beta_k E\{h_m^{(k)} h_{m+n}^{(k)}\} + \sum_{l=1}^L E\{\beta_l\} E\{h_m^{(l)} h_{m+n}^{(l)}\} \quad n \neq 0. \quad (5.11)$$

Under the model assumptions discussed in Section 5.3.1.1, we can write,

$$E\{y_m^* y_{m+n}\} = \beta_k E\{h_m^{(k)} h_{m+n}^{(k)}\} + E\{\beta_l\} \sum_{l=1}^L E\{h_m^{(l)} h_{m+n}^{(l)}\}, \quad n \neq 0.$$
Taking into consideration that the AoAs of the pilot contaminators will be uniformly distributed in one of the three non-overlapping sections over \([0, \pi]\), we have,

\[
L = 3 \sum_l \mathbb{E}\{ h_m^{(l)\ast} h_{m+n}^{(l)} \} = \frac{\rho^2_{\text{inter}}}{\rho^2_{\text{intra}}} \left( \int_{\theta = 0}^{\pi/3} e^{-jn\pi \cos \theta} \, d\theta + \int_{\theta = \pi/3}^{2\pi/3} e^{-jn\pi \cos \theta} \, d\theta + \int_{\theta = 2\pi/3}^{\pi} e^{-jn\pi \cos \theta} \, d\theta \right)
\]

where the covariance terms \(\mathbb{E}\{ h_m^{(k)\ast} h_{m+n}^{(k)} \}_{\text{LMMSE}}\) are defined in (4.79). Consequently, we have

\[
\mathbb{E}\{ y_m^* y_{m+n} \} = \begin{cases} 
(\beta_k + E\{ \beta_l \} \frac{\sigma^2_{\text{intra}}}{\rho^2_{\text{intra}}}) \mathbb{E}\{ h_m^{(k)\ast} h_{m+n}^{(k)} \}_{\text{LMMSE}}, & n \neq 0 \\
\beta_k + 3E\{ \beta_l \} + \sigma^2_n, & n = 0,
\end{cases}
\]
given \(\mathbb{E}\{|h_m|^2\} = 1\). Therefore, the estimator can be expressed in terms of the underlying \(\Gamma_{hh}\) matrix,

\[
\hat{h}_{\text{LMMSE}} = \sqrt{\beta_k \Gamma_{hh}} (I_{3\times3} (3 - \frac{\rho^2_{\text{inter}}}{\rho^2_{\text{intra}}}) E\{ \beta_l \} + \sigma^2_n) + (\beta_k + \frac{\rho^2_{\text{inter}}}{\rho^2_{\text{intra}}} E\{ \beta_l \}) \Gamma_{hh} \right)^{-1} y,
\]

where the covariance terms of the \(\Gamma_{hh}\) matrix are given by (4.79).

### 5.3.3 A-LMMSE Estimation

#### 5.3.3.1 A-LMMSE Estimator

As in the previous sections, the A-LMMSE estimator presented here only considers the desired user in the presence of additive noise. The A-LMMSE estimator as described in (4.42) is modified to use the desired users slow gain, \(\beta_k\), yielding the estimator

\[
\hat{h}_{\text{A-LMMSE}} = \sqrt{\beta_k \Gamma_{hh}} (y_{yy}^{-1}) y
\]

\[
= \sqrt{\beta_k \Gamma_{hh}} (I_{3\times3} \sigma^2_n + \beta_k \Gamma_{hh} \right)^{-1} y,
\]
where the covariance terms of the $\Gamma_{hh}$ matrix are given by (4.81).

The Assistant obtains the estimate of the spatial frequency $\hat{f}_s$ in order to form the covariance terms in (4.81) in the same way as the single user case. Using the maximum peak of the FFT output of (4.40) also generally performs well in our multi-cell ULPC scenario since we have the condition $\sqrt{\beta_k} \geq \sqrt{\beta_l}$. Section 5.2.2 outlines strategies which can be employed in the case that the largest peak does not belong to the desired user.

5.3.3.2 Cooperative A-LMMSE Estimator

With knowledge of the pilot sequences allocated in neighbouring BSs, the A-LMMSE estimator can better form the $\Gamma_{yy}$ matrix, in a similar way to the LMMSE estimator described in Section 5.3.2.2. With no assumptions about the slow gains, the covariance terms of $\Gamma_{yy}$ can be expressed as in (5.11), however in the case of the A-LMMSE estimator the covariance terms $E\{h^{(k)*}h^{(k)}\}$ are derived using a-priori knowledge of $f_s$, and are given by (4.81).

As discussed in Section 5.2.2, the Assistant in this scenario can potentially form a-priori information about the spatial frequency of the interfering users, which can used to improve the $\Gamma_{yy}$ matrix. Given our network model results in potentially weak interferers which may not always be easy to identify by the Assistant in the presence of noise, we opt for a simpler and more robust approach that instead exploits the modelled cellular structure by accounting for an interferer within each sextant. Therefore, given the model assumptions outlined in Section 5.3.1.1, we substitute (5.12) into (5.11), (noting that this substitution makes use of the covariance terms of the LMMSE estimator as defined in (4.79) as indicated by the LMMSE subscript in (5.13)), and express the covariance terms of $\Gamma_{yy}$ for the A-LMMSE estimator as

$$E\{y^*_my_{m+n}\} = \begin{cases} 
\beta_k E\{h^{(k)*}h^{(k)}\} + E\{\beta_l\} \frac{\mathbf{P}_{\text{extra}}}{\mathbf{P}_{\text{extra}}} E\{h^{(k)*}h^{(k)}\}_{\text{LMMSE}}, & n \neq 0 \\
\beta_k + 3E\{\beta_l\} + \sigma_n^2, & n = 0,
\end{cases} \quad (5.13)$$

This yields the following A-LMMSE estimator,

$$\hat{h}_{A-LMMSE} = \sqrt{\beta_k} \Gamma_{hh} \Gamma_{yy}^{-1} y$$
5.3 Multi Cell MU Massive MIMO

\[
= \sqrt{\beta_k} \Gamma_{hh} \left( \beta_k \Gamma_{hh} + I \left( (3 - \frac{\rho_{inter}^2}{\rho_{intra}^2}) E\{\beta_i\} + \sigma_n^2 \right) + \frac{\rho_{inter}^2}{\rho_{intra}^2} E\{\beta_i\} \Gamma_{hh,LMMSE} \right)^{-1} y,
\]

(5.14)

where the entries of \( \Gamma_{hh} \) are given by (4.81) and the elements of \( \Gamma_{hh,LMMSE} \) by (4.79).

### 5.3.4 Numerical Results and Discussion

We can evaluate the performance of the LMMSE and A-LMMSE estimators in multi-cell scenarios by observing the resulting NMSE and SER in simulation. We simulate the scenario described by Figure 5.4, where we use radii in metres of \( r_{excl} = 40, r_{intra} = 800, r_{inter} = 1600 \) and a path loss exponent \( \alpha = 4 \). We assume that ULPC is enabled. All cells are loaded, with \( K = 24 \) users connected to each BS. During the data transmission in the multi-user scenario depicted in Figure 5.4, the received data vector at the desired users BS is given by,

\[
y_k = \mathbf{h}_k + \sum_{k' \neq k}^{KL} \sqrt{\beta_{k'}} \mathbf{h}_{k'} + \mathbf{n}.
\]

The MRC detection of the desired users symbol under ULPC with slow gains specified in Section 5.3.1.1, is given by,

\[
\hat{x}_k = \frac{\hat{\mathbf{h}}^T \mathbf{y}_k}{\hat{\mathbf{h}}^T \mathbf{h}} = x_k + \frac{1}{\hat{\mathbf{h}}_k^H \mathbf{h}_k} \left( \hat{\mathbf{h}}_k^H \mathbf{h}_k x_k + \sum_{k' \neq k}^K \hat{\mathbf{h}}_k^H \mathbf{h}_{k'} x_{k'} + \sum_{k' \neq k}^{KL} \sqrt{\beta_{k'}} \hat{\mathbf{h}}_k^H \mathbf{h}_{k'} x_{k'} + \hat{\mathbf{h}}_k^H \mathbf{n} \right),
\]

where \( L = 3 \) for the neighbouring BS.

The simulation results in Figure 5.5 show the resulting NMSE and SER for a varying number of propagation conditions, where \( K_{f,intra} \) and \( K_{f,inter} \) have been varied. As shown, the A-LMMSE estimator outperforms the LMMSE estimator significantly in the simulated cases of the multi-cell scenario.

When the channels of the intra-cell users are Rayleigh, i.e., \( K_{f,intra} = 0 \), the A-LMMSE estimator variants reduce to the LMMSE variants, since \( \Gamma_{hh,LMMSE} = \Gamma_{hh,ALMMSE} = I \), due to the lack of any correlation amongst the channels of the antenna array. This can be observed in
Figure 5.5: The NMSE and SER of the LMMSE and A-LMMSE under various propagation conditions, $K = 24, \eta = -3dB$, under the simulation parameters in Table 5.1.
Figure 5.5(f), (where in fact all estimators converge since the cooperative variants offer negligible differentiation due to $E\{β_l\} \ll E\{β_k\}$). Note the condition where the A-LMMSE and LMMSE estimators are the same also occurs when the interference gain $E\{β_l\}$ and noise power $σ_n^2$ are insignificant in comparison to the desired users channel gain $E\{β_k\}$, since all estimators will reduce to a simple scaling of the results, i.e., $\hat{h} = β_k^{-\frac{1}{2}}y$.

From Figure 5.5 there is no observable difference in performance between the cooperative and non-cooperative variants of the estimators, since the slow gains of the inter-cell interferers, in the mean sense (as utilised by the cooperative estimators), is much lower than the unity gain of the desired user. Likewise, varying the channel types of the $L$ pilot contaminators across Figures 5.5(a), 5.5(c) and 5.5(e) is not observable under the averaging of the NMSE.

5.3.4.1 Worst Case Inter-cell Interference

To examine a worst case inter-cell interference scenario, we consider the model where all users are positioned at the cell boundaries, i.e., $\sqrt{β_k} = \sqrt{β_{l'}} = 1$, $\forall l'$. Consequently, under such a scenario, the cooperative variants effectively have perfect knowledge of the inter-cell slow gains. The resulting NMSE and SER is plotted in Figure 5.6. As expected the cooperative estimators which utilise a-priori knowledge of the pilot contaminators’ presence in neighbouring cells perform better. However, the performance gains of the cooperative A-LMMSE compared to the non-cooperative A-LMMSE estimator diminish with $M$ since the a-priori knowledge of the pilot contaminators in the neighbouring cells, i.e., the $Γ_{hh,LMMSE}$ matrix used in (5.14), does not

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of BSs in network</td>
<td>$L$</td>
<td>4</td>
</tr>
<tr>
<td>Number of users per BS</td>
<td>$K$</td>
<td>24</td>
</tr>
<tr>
<td>Pathloss exponent</td>
<td>$γ$</td>
<td>4</td>
</tr>
<tr>
<td>Lognormal shadowing standard deviation</td>
<td>$σ_{\text{shadow}}$</td>
<td>0 dB</td>
</tr>
<tr>
<td>Cell radius</td>
<td>$r_{\text{intra}}$</td>
<td>800 m</td>
</tr>
<tr>
<td>Cell exclusion radius</td>
<td>$r_{\text{excl}}$</td>
<td>40 m</td>
</tr>
<tr>
<td>Frequency reuse factor</td>
<td>$w$</td>
<td>1</td>
</tr>
<tr>
<td>Pilot sequence length</td>
<td>$τ$</td>
<td>24</td>
</tr>
<tr>
<td>Data sequence length</td>
<td>$τ_{ul}$</td>
<td>1</td>
</tr>
<tr>
<td>Data symbol modulation</td>
<td></td>
<td>QPSK</td>
</tr>
</tbody>
</table>

Table 5.1: Main Simulation parameters
improve with $M$ while the covariance matrix $\Gamma_{hh}$ does. The improvement of $\Gamma_{hh}$ is through the reduction of the error variance of the estimate $\hat{\sigma}^2_{f_s}$ provided by the Assistant, where from (4.41) we have $\hat{\sigma}^2_{f_s} \rightarrow 0$ as $M \rightarrow \infty$.

Therefore, as shown in Figure 5.6 (a), exploiting knowledge of the pilot contaminants presence to form the cooperative variant of the A-LMMSE estimator, even with the knowledge of the inter-cell slow gains, results in performance improvements only up to a certain value of $M$, i.e., $M = 128$ under the scenario in Figure 5.6 (a). This observation suggests that while the cooperative A-LMMSE estimator employing this minimal form of BS cooperation requires less information to be shared and is potentially more robust to user mobility, such variants are only advantageous under certain ranges of $M$.

Under further increases of $M \geq 128$, we observe that this cooperation is insignificant compared to the improvements in $\Gamma_{hh}$, where the A-LMMSE estimator and cooperative variant result in similar NMSE. As we head further into the asymptotic regime, from (4.41) we have $\hat{\sigma}^2_{f_s} \rightarrow 0$, leading to $\Gamma_{hh}$ matrices amongst pilot contaminants now spanning orthogonal subspaces and as described by [18, Theorem 1], will ultimately result in both A-LMMSE variants to produce contamination free estimates.

Utilising more accurate information about the pilot contaminants, i.e., their AoAs, as opposed to the positions of the BSs they are connected to, would result in the A-LMMSE cooperative variant offering benefits up to large values of $M$. However this comes at the cost of higher complexity, and increased information sharing at a rate which is more dependent on user mobility and BS density.

5.3.4.2 K-factors of the Interfering User Channel

As observed in Figure 4.12 of the previous chapter, the A-LMMSE estimator favours propagation conditions where the desired users channel features a strong LoS component, since it is better able to estimate $f_s$ and exploit the resulting larger amounts of spatial correlation. This is also demonstrated in the poor NMSE of the A-LMMSE estimator in Figure 5.5(g), when there is no LoS component to exploit, as compared to Figure 5.5(a).

If we consider the channels from interfering users, we would expect that all estimation schemes would favour channels with a strong LoS component over Rayleigh channels. This intuition stems
Figure 5.6: The NMSE and SER of the LMMSE and A-LMMSE when all users are positioned at cell boundaries, $\sqrt{\beta_k} = \sqrt{\beta_{\nu}} = 1$, $\eta = -3dB$, under the simulation parameters in Table 5.1
Figure 5.7: Two user case, with resulting SER and channel inner product magnitude, \( \frac{1}{M} |h^H_k h_{k'}| \) vs AoA difference \( |\theta_k - \theta_{k'}| \), \( M = 64 \).

from the resulting antenna beam patterns and observations made in Section 5.2, i.e., the LoS component of the Rician interferer would be likely to fall outside the main lobe concentrated at the desired user and therefore generate a smaller amount of interference, when compared to a Rayleigh interferer that is present across the entire AoA range, including the main lobe. However, from the simulation results in Figure 5.5 we observe that nearby interferers who have a stronger LoS component, result in poorer SER results for all estimation methods, including the use of perfect estimates. If we consider the perfect estimates, this is seen in the reduction of the SER in Figure 5.5(f) when compared to Figure 5.5(b) for the inter-cell interferers, and the reduction of the SER in Figure 5.5(h) when compared to Figure 5.5(b) for the intra-cell interferers.

To better understand the effects of the interfering channel type we consider a desired user with a pure LoS channel with AoA \( \theta_k \), and compare two extreme cases, the presence of a pure LoS interferer with AoA \( \theta'_{k} \) and the presence of a Rayleigh interferer. We first examine in detail the effects of \( \theta'_{k} \) on the SER as \( |\theta_{k'} - \theta_k| \) varies, for a particular case of \( \theta_k = \pi/2 \) and \( M = 64 \) in Figure 5.7. Figure 5.7 also plots the corresponding magnitude of the inner product of both user channels \( |h^H_k h_2| \) to show that the trends observed in the SER are a direct consequence of the interference terms in (5.4). We have employed 64-QAM symbols since they are more sensitive to
interference and illustrate such effects better.

The resulting SER in Figure 5.7 follows the shape of the $h_k$ beamforming pattern, generally improving as the interferer is moved away from the desired user. In agreement with our earlier intuition, as seen in Figure 5.7 for the LoS interferer, under most values of AoA separation the SER is significantly less than the Rayleigh interferer which has no LoS component and is naturally constant. However when we approach the main lobe, the SER increases to a level significantly higher than the Rayleigh case. While the particular shape of the beamforming patterns at the BS antenna array are a function of the antenna array geometry and will vary to some degree with $\theta_k$, this behaviour can be observed under all values of $\theta_k$ under the uniform linear array employed. As such, when operating under high SNR and practical SER regions, i.e., $\text{SER} < 10^{-1}$, instances of the desired and interfering users having similar AoAs will very likely result in symbols errors, and dramatically affect the resulting SER. This leads to the higher SER observed in Figure 5.7 when either the intra-cell interferers or inter-cell interferers have Rician channels.

In order to observe the impact of such effects, Figure 5.8 plots the SER of the scenario in Figure 5.5(b) when the interfering intra-cell users are now spatially separated (by the FNBW) from the desired user. Achieving the conditions of spatial separation of the intra-cell users is more easily realised when the BS has more connected users than it is able to schedule simultaneously. Under such scheduling we now observe the large improvement shown in Figure 5.8 when the intra-cell users experience Rician channels as opposed to Rayleigh channels, demonstrating the significant effect of spatial separation amongst the intra-cell users in the Rician case. These observations again highlight that while pilot contamination is dominant in the limit of $M$, under practically large finite $M$ the effects of the intra-cell interference is significant, especially since the probability of spatial overlap is higher due to the fact there are likely more intra-cell interferers than pilot contaminators. Such observations support schemes that additionally aim to restrict the concurrent scheduling of data transmissions to users which are spatially separated [56], and not only the spatial separation of pilot contaminators [18–20].

5.3.5 Conclusions and Future Work

In this chapter we extended the LMMSE and A-LMMSE estimators for use in multi-user and multi-cell scenarios, and examined the resulting performance. Under the scenarios explored, the
Figure 5.8: The SER under network conditions where interfering users are separated spatially from desired user $|\theta_k - \theta_{k'}| > \text{FNBW}$, for intra-cell $k'$, and $K_{f,intra} = 5, K_{f,inter} = 5, \eta = -3dB$, under the simulation parameters in Table 5.1.

A-LMMSE estimator offered significant improvements in NMSE and SER over the LMMSE estimator.

Through observation of the resulting beamforming patterns, we highlight the pilot decontamination effect of spatially separated pilot contaminators under Rician channels when using the A-LMMSE estimator, confirming the main results of [18]. Therefore, under channel models which incorporate LoS components, the area of pilot allocation schemes that consider the spatial signatures of interfering users is of significant importance for future work. Such pilot allocation schemes can be designed to work closely with channel estimation approaches such as the A-LMMSE estimator.

Under the simple reused pilot set scheme, the A-LMMSE estimator employing minimal BS cooperation was able to demonstrate improvements in the NMSE up to certain practically large finite $M$. Future work should investigate the trade-offs between pilot contamination mitigation strategies that employ fully cooperative network pilot allocation algorithms versus channel estimation methods that employ a minimal amount of cooperation. Such work should additionally consider the effects of user mobility in high BS density deployments.

We presented the performance gains of the A-LMMSE estimator, which were observable in the
resulting SER even in the presence of other impediments from general beamforming and detection in LoS environments. By examining the SER performance, we have shown that for a desired user experiencing Rician fading, interfering channels from the randomly positioned users with higher K-factors cause more interference. This highlights the importance of user scheduling algorithms at the BS that utilise the AoA information of the connected users. We demonstrated how BS user scheduling which spatially separates users during data transmissions is able to achieve significantly better SER.

The evaluation of the A-LMMSE estimator under more realistic multi-cell multi-user scenarios is also of importance. Such scenarios should include the effects of shadowing and best cell selection since this will significantly alter the random slow gains of the interferers.
Chapter 6

Conclusion

In this thesis we have investigated several key challenges in the context of massive MIMO networks. The thesis presents insightful proposals to address such challenges, supported by thorough analytical workings and simulation results.

Motivated by the well studied effects of pilot contamination, we derived new limiting results for the ZF and MRC detectors under a more generalised pilot allocation scheme. In doing so, we show that in fact the limiting uplink SIR of the ZF and MRC detectors under massive MIMO networks are different. Furthermore, we make the counter intuitive observation of the ZF uplink SIR performance being in part limited by intra-cell interference, due to the pilot contamination resulting from more general pilot allocation schemes that do not strictly reuse the same orthogonal pilot sets amongst BSs.

Using our derived limiting SIR results, we compare the reused pilot set scheme, to a more general different pilot set scheme in the context of user-load capacity. A methodology is presented to derive the user admission regions which quantify the number of users the network can support while maintaining a certain QoS for all users, providing a basis for simple non-cooperative admission policies. We then demonstrate that under such simple non-cooperative admission policies, assigning different pilot sets to the BSs of the network is superior to the reused pilot set allocation, which is adopted by most of the current massive MIMO literature.

Acknowledging the advantages of the different pilot set allocation scheme, we employ such a scheme in more realistic network scenarios which in-cooperate the effects of log-normal shadowing, best cell selection and a practically large finite number of BS antennas. Using analytical and statistical approximations, we derive methods to determine network user admission regions which
Conclusion

can be simply defined under such scenarios, enabling the development of efficient cooperative user admission policies. We show that the resulting user admission regions from our methods are sufficiently accurate under these scenarios.

Finally, we investigated the channel estimation problem in the massive MIMO setting, and presented a new estimation scheme, the Assisted-LMMSE (A-LMMSE) estimator. We derived the optimal MMSE estimator and popular LMMSE estimator for the single user scenario under pure LoS channels to motivate the development of a new estimation scheme, and to provide a means to compare the A-LMMSE estimators performance and runtime complexity. Unlike the LMMSE estimator, we show how the performance of the A-LMMSE estimator increases with the number of antenna elements, therefore making it well suited for the massive MIMO BS. We demonstrate that the A-LMMSE estimator is able to offer near optimal MMSE performance, but with the runtime complexity order of the highly efficient LMMSE estimator. We also investigate low complexity implementations of the A-LMMSE estimator and rigorously derive their limiting channel estimation performance under an infinite number of BS antennas. We extend the analysis to a complex baseband Rician channel model, where similar performance gains in NMSE and SER are demonstrated both analytically and numerically through simulation.

In order to validate the A-LMMSE estimator further, we extend the estimator and examine its performance in the multi-user cellular environment, where we have the additional impairment of pilot contamination. We demonstrate the mitigation of pilot contamination under the A-LMMSE estimator. Furthermore, we investigate the resulting estimation performance under varying K-factors of the interfering channels, where the A-LMMSE estimator offers significant performance gains in the NMSE and SER under the examined scenarios.

6.1 Future Work

6.1.1 Two-Dimensional Antenna Arrays

As discussed in Section 4.2, the separation of the antenna array elements at the BS by a minimum distance (which is a function of the carrier frequency being used) needs to be considered in order
to avoid undesirable effects when beamforming. As a consequence, the physical size of the array becomes a concern as $M$ grows, and therefore different array geometries need to be envisaged. Arrays which have two dimensions permit more practical form factors, and furthermore, enable three-dimensional beamforming, i.e., beamforming in the horizontal (azimuth) plane, as adopted in this thesis) and in the vertical (elevation) plane. Three-dimensional beamforming is superior to two-dimensional beamforming as it allows more enhanced beamforming and spatial multiplexing, resulting in further improvements in spectral efficiency [67].

The work in Chapter 4 and Chapter 5 of this thesis can be extended to investigate channel estimation and in particular the A-LMMSE scheme exploiting both a-priori knowledge of the azimuth AoA ($\theta$) and the elevation AoA in the context of two dimensional antenna arrays at the BS.

### 6.1.2 Downlink Massive MIMO

The thesis has covered important aspects related to the uplink direction of massive MIMO. The work in Chapter 2 can be extended to include the analysis of the downlink finite and limiting SIR of the MRC and ZF linear detectors, under both pilot allocation schemes. Using these results, the work in Chapter 3 can be extended to define the admission regions from the perspective of the downlink QoS. Finally, the admission region resulting from the intersection of the uplink and downlink admission regions can be used to define a unified admission policy.

### 6.1.3 Low resolution ADCs

As we have observed in the scenarios presented, the advantages of massive MIMO become apparent when there are a sufficiently large number of antenna elements at the BS, i.e., typically when $M \geq 64$. In this thesis, we have assumed that each antenna element of the antenna array is connected to a complete RF chain, each chain utilising an ADC with infinite resolution. The power consumption, hardware costs and signal processing under such an implementation might be prohibitive. As a result, the concept of using low resolution ADCs, (e.g., of 1-bit resolution, one for each in-phase and one for each quadrature arm of the digital baseband processing has received recent interest in the research community. Massive MIMO systems models for such
schemes explicitly model the quantisation operation and associated quantised noise in all inputs of
the baseband processing [14]. Using such models, the chapters of this thesis can be extended to in-
clude an analysis and relevant performance comparisons when low resolution ADCs are employed
at the massive MIMO BS.
Appendix A

A.1 Uplink SIR approximation (3.22)

We wish to approximate the instantaneous SIR defined by (2.16) of Section 2.2.4 for the MRC linear detector given in (2.12). The signal power is expressed in the numerator of (2.16). The intra-cell and inter-cell interference powers are given by the first term and second terms of the denominator of (2.16) respectively.

Under the different pilot set scheme, the channel estimates are given by (2.11). Since we operate in the interference limited regime, we form the expression for the estimates without the additive noise terms, i.e.,

\[ \hat{g}_{kj} = g_{kj} + \sum_{l \neq j} L \sum_{k'} K \sum_{k' \neq k'} g_{k'lj} \phi_{k'lj} \]  

(A.1)

We first present some general results that are used in the following derivations when averaging over small scale fading. Given the channel model presented in Section 2.2, for identical channel vectors we have,

\[ \frac{E\{|h_j^H h_j|\}}{M} = E\{|h_{mj}^2|\} M, \]
\[ \frac{E\{|g_j^H g_j|^2\}}{M} = \beta_j^2 M, \]  

(A.2)

where the expectation operation is over the independent channel vectors \( h \), and the second moment of the \( m \)-th element of such channel vectors under our model is given by \( E\{|h_{mj}^2|\} = 1 \). For the remainder of this section, it is assumed that the expectation operation is over taken over \( h \).
Similarly for mismatched channel vectors we have,

\[
\frac{E\{|h_j^H h_k|\}}{M} = E\{|h_m^2|\}, \\
\frac{E\{|g_j^H g_k|^2\}}{M} = \beta_j \beta_k. 
\]  

(A.3)

Using the result of (A.2), the signal power in (2.16) when averaged over small-scale fading can be written as,

\[
E\{|\hat{g}_j^H g_{kj}|^2\} = E\{|(g_{jkj} + \sum_{l \neq j}^L \sum_{k'}^K g_{jkl}\phi_{jkl})^H g_{jkj}|^2\} \\
= M^2 \beta_{jkj}^2 + M \sum_{l \neq j}^L \sum_{k'}^K |\phi_{jkl}|^2 \beta_{jkl} \beta_{jkj}. 
\]  

(A.4)

Neglecting the insignificant cross terms leaves us with the following approximation,

\[
E\{|\hat{g}_j^H g_{kj}|^2\} \approx M^2 \beta_{jkj}^2. 
\]  

(A.5)

The intra-cell interference power generated by the independent user transmissions of BS \(j\), when averaging over small-scale fading can be written as,

\[
E\{\sum_{k' \neq k}^K |\hat{g}_{k'j}^H g_{jkl}|^2\} = E\{\sum_{k' \neq k}^K |(g_{jkj} + \sum_{l \neq j}^L \sum_{k'}^K g_{jkl}\phi_{jkl})^H g_{jkl}|^2\} \\
= M \sum_{k' \neq k}^K \beta_{jkl} \left( \beta_{jkj} + \sum_{l \neq j}^L \sum_{k'}^K |\phi_{jkl}|^2 \beta_{jkl} \right). 
\]  

(A.6)

where \(K_j\) is the number of users connected to BS \(j\), and (A.6) applies result (A.3).

The inter-cell interference power by the independent user transmissions of all users connected to all other BSs, when averaging over small-scale fading can be written as,

\[
E\{\sum_{l' \neq l}^L \sum_{k'}^K |\hat{g}_{l'j}^H g_{jkl'}|^2\} \\
= E\{\sum_{l' \neq l}^L \sum_{k'}^K |(g_{jkj} + \sum_{l \neq j}^L \sum_{k'}^K g_{jkl}\phi_{jkl})^H g_{jkl'}|^2\} \\
= E\{\sum_{l' \neq l}^L \sum_{k'}^K |(g_{jkj} + \sum_{l \neq j}^L \sum_{k'}^K g_{jkl}\phi_{jkl})^H g_{jkl'}|^2\} \\
= E\{\sum_{l' \neq l}^L \sum_{k'}^K |\phi_{jkl}|^2 \beta_{jkl} \beta_{jkl'} |^2\} \\
= E\{\sum_{l' \neq l}^L \sum_{k'}^K |\phi_{jkl}|^2 \beta_{jkl} \beta_{jkl'} |^2\} \\
= E\{\sum_{l' \neq l}^L \sum_{k'}^K |\phi_{jkl}|^2 \beta_{jkl} \beta_{jkl'} |^2\}. 
\]
Under ULPC as described in Section 2.33, the signal power (A.5), intra-cell interference power (A.6) and inter-cell interference power (A.9) equations become,

\[
E\{ \sum_{l' \neq l} \sum_{k'} |g_{jl}^H g_{jk'}|^2 \} \approx \sum_{l' \neq l} \sum_{k'} \beta_{jl}^2 |\phi_{jk}^T|^2 M^2 |\phi_{lk'}^T|^2 + M |\beta_{jl}^T| \beta_{lk}. \tag{A.9}
\]

Collecting these terms, and dividing through by \( M^2 \), we have an approximate SIR expression under ULPC, averaged over small-scale fading, for the \( k \)-th user of BS \( j \), as

\[
SIR_{kj} \approx \frac{1}{M(K_j - 1) + \sum_{l' \neq j} \sum_{k'} |\phi_{jk}^T|^2 \left( \frac{\beta_{jl}}{P_{jl'}/M} \right)^2 + \frac{1}{M} \left( \frac{\beta_{jl}}{P_{jl'}/M} \right) + \frac{1}{M}(K_j - 1)|\phi_{jk'}^T|^2 \left( \frac{\beta_{jl}}{P_{jl'}/M} \right)}. \]
Appendix B

B.1 Single User LMMSE Estimator

The channel estimate for the single user scenario described by (4.6) is given by,

\[ \hat{h} = \sigma_h^2 s^H (\sigma_h^2 s s^H + I \sigma_n^2)^{-1} y, \tag{B.1} \]

where \( s \) is a column vector of length \( \tau \) of all ones. In order to evaluate the matrix inverse above, we apply the Sherman-Morrison formula [73] given by,

\[ (A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}. \]

Let \( A = I \sigma_n^2, u = \sigma_h^2 s \) and \( v = s \), then the matrix inverse in (B.1) can be expressed as

\[ (\sigma_h^2 s s^H + \sigma_n^2)^{-1} = I \frac{\sigma_h^2 s s^H I \frac{1}{\sigma_n^2}}{1 + \sigma_h^2 s s^H I \frac{1}{\sigma_n^2} s} \]

\[ = I \frac{1}{\sigma_n^2} - \left( \frac{\sigma_h^2}{\sigma_n^2 \left( 1 + \sigma_h^2 \tau \right)} \right) s s^H \]

\[ = P - Q s s^H, \tag{B.2} \]

where in (B.2) we employ substitutions \( P \) and \( Q \) for clarity.

We can then express the estimate in (B.1) as,

\[ \hat{h} = \sigma_h^2 s^H (P - Q) s s^H y \]
\[ = \sigma_h^2 (P - \tau Q)s^H y. \]

Substituting the values in for \( P \) and \( Q \), and performing standard algebraic manipulations yields,

\[ \hat{h} = \frac{1}{\frac{\sigma_h^2}{\sigma_h^2} + \tau} s^H y. \quad (B.3) \]
B.2 MMSE Estimator for $M$ antennas in LoS

To reduce notational clutter, let $f(x)$ and $F(x)$ denote the PDF and CDF of the random variable $x$ respectively. Given the system model in (4.12), for the $m$-th element, Equation (4.13) can be re-expressed as,

\[
\hat{h}_{m,\text{MMSE}} = E\{h_m|y\} = \int_0^{\pi/2} \int_0^{2\pi} h_m(\theta, \phi) f(\theta, \phi|y) \, d\theta \, d\phi. \tag{B.4}
\]

Given than $\theta$ and $\phi$ are independent variables, and using Bayes rule, we can rewrite (B.4) as,

\[
\hat{h}_{m,\text{MMSE}} = \frac{1}{f(y)} \int_0^{\pi/2} \int_0^{2\pi} h_m(\theta, \phi) f(\theta) f(\phi) f(y|\theta, \phi) \, d\theta \, d\phi. \tag{B.5}
\]

Noting that both $f(\theta)$ and $f(\phi)$ are independent uniform distributions, we can express the joint distribution $f(y)$ as,

\[
f(y) = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} f(y, \theta, \phi) \, d\theta \, d\phi
= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} f(y|\theta, \phi) f(\theta) f(\phi) \, d\theta \, d\phi
= f(\theta) f(\phi) \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} f(y|\theta, \phi) \, d\theta \, d\phi. \tag{B.6}
\]

Using (B.6), we can express (B.5) as

\[
\hat{h}_{m,\text{MMSE}} = \frac{\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} h_m(\theta, \phi) f(y|\theta, \phi) \, d\theta \, d\phi}{\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} f(y|\theta, \phi) \, d\theta \, d\phi}. \tag{B.7}
\]

Given the system model in (4.12), the conditional density function in (B.5) is,

\[
f(y|\theta, \phi) = \left( \frac{1}{\sqrt{2\pi}\sigma^2} \right)^M e^{-\frac{1}{2\sigma^2} \left[ \sum_{k=1}^{M} (y_k - \cos((k-1)\pi\cos(\theta)+\phi))^2 \right]} \tag{B.8}
\]
Using (B.8), the expression in (B.7) simplifies to the estimator,

\[ \hat{h}_{MMSE} = E\{h_m|y\} = \frac{\int_0^{\pi/2} \int_0^{2\pi} h_m(\theta, \phi) g(\theta, \phi) d\theta d\phi}{\int_0^{\pi/2} \int_0^{2\pi} g(\theta, \phi) d\theta d\phi}, \]

where,

\[ h_m(\theta, \phi) = \cos((m - 1)\pi \cos \theta + \phi) \]

\[ g(\theta, \phi) = e^{-\frac{1}{2\sigma^2} \sum_{m=1}^{M} (y_m - \cos((m - 1)\pi \cos \theta + \phi))^2}. \]
B.3 MMSE Estimator for M antennas in Rician fading

To reduce notational clutter, let \( f(x) \) and \( F(x) \) denote the PDF and CDF of the random variable \( x \) respectively. Given the system model in (4.66), the estimator in (4.69) can be re-expressed as,

\[
\hat{h}_{m,R}^{\text{MMSE}} = E\{h_{m,R}|y\} = \int_{h_{m,R}} h_{m,R} f(h_{m,R}|y) \, dh_{m,R} = \int_{\theta} \int_{\phi} \int_{\psi_m} h_{m,R}(\theta, \phi, \psi_m) f(\theta, \phi, \psi_m|y) \, d\psi_m \, d\theta \, d\phi.
\]

Applying Bayes rule yields,

\[
\hat{h}_{m,R}^{\text{MMSE}} = \frac{1}{f(y)} \int_{\theta} \int_{\phi} h_{m,R}(\theta, \phi, \psi_m) f(y|\theta, \phi, \psi_m) f(\theta, \phi, \psi_m) \, d\psi_m \, d\theta \, d\phi. \tag{B.9}
\]

Given the independence of the random variables, \( \theta \), \( \phi \), and \( \psi_m \), we have,

\[
\hat{h}_{m,R}^{\text{MMSE}} = \frac{1}{f(y)} \int_{\theta} \int_{\phi} f(\theta) f(\phi) \int_{\psi_m} h_{m,R}(\theta, \phi, \psi_m) f(y|\theta, \phi, \psi_m) f(\psi_m) \, d\psi_m \, d\theta \, d\phi. \tag{B.9}
\]

The conditional distribution in (B.9) can be expressed as the joint PDF of \( M \) independent Gaussian random variables. The resulting distribution is the product between the PDF of a Gaussian random variable with variance of the noise alone, and the remaining \( M-1 \) random variables with variance equal to the sum of the noise and nLoS component, i.e., \( (\sigma_n^2 + \sigma_\psi^2) \). Given both \( n_m \) and \( \psi_m \) are zero mean, the conditional distribution is given by,

\[
f(y|\theta, \phi, \psi_m) = \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{\frac{-1}{2\sigma_n^2}(z_m - \psi_m)^2} \left( \frac{1}{\sqrt{2\pi(\sigma_n^2 + \sigma_\psi^2)}} \right)^{M-1} e^{\frac{-1}{2(\sigma_n^2 + \sigma_\psi^2)} \sum_{k=1}^{M} z_k^2}, \tag{B.10}
\]

where \( z_k = y_k - \rho h_k(\theta, \phi) = y_k - \rho \cos((k-1)\pi \cos \theta + \phi) \).
The second exponential term in (B.10) can be moved outside the inner integral of (B.9) as,

\[
\hat{h}_{m,\text{MMSE}} = \frac{1}{f(y)} \int_\theta \int_\phi f(\theta)f(\phi) \left( \frac{1}{2\pi(\sigma_n^2 + \sigma_\psi^2)} \right)^{M-1} e^{\frac{-1}{2\pi(\sigma_n^2 + \sigma_\psi^2)} \sum_{m=1}^{M} z_m^2} \int_{\psi_m} h_{m,\text{R}}(\theta,\phi,\psi_m) \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{\frac{1}{2\pi\sigma_n^2} (z_m - \psi_m)^2} f(\psi_m) \, d\psi_m \, d\theta \, d\phi \quad (B.11)
\]

Evaluating the inner integral of (B.11),

\[
\int_{\psi_m} h_{m,\text{R}}(\theta,\phi,\psi_m) \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{\frac{1}{2\pi\sigma_n^2} (z_m - \psi_m)^2} f(\psi_m) \, d\psi_m \\
= \int_{\psi_m} \left( \rho \, h_m(\theta,\phi) + \psi_m \right) \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{\frac{1}{2\pi\sigma_n^2} (z_m - \psi_m)^2} \frac{1}{\sqrt{2\pi\sigma_\psi^2}} e^{\frac{-\psi_m^2}{2\sigma_\psi^2}} d\psi_m \\
= \left( \frac{1}{\sqrt{2\pi(\sigma_n^2 + \sigma_\psi^2)}} \right) e^{\frac{-1}{2\pi(\sigma_n^2 + \sigma_\psi^2)} \sum_{m=1}^{M} z_m^2} \left( \rho \, h_m(\theta,\phi) + \frac{\sigma_\psi^2}{\sigma_n^2 + \sigma_\psi^2} (y_m - \rho \, h_m(\theta,\phi)) \right) \quad (B.12)
\]

The (B.11) can now be expressed as,

\[
\hat{h}_{m,\text{MMSE}} = \int_\theta \int_\phi f(\theta)f(\phi) \left( \frac{1}{2\pi(\sigma_n^2 + \sigma_\psi^2)} \right)^{M} g(\theta,\phi) \left( \rho \, h_m(\theta,\phi) + \frac{\sigma_\psi^2}{\sigma_n^2 + \sigma_\psi^2} (y_m - \rho \, h_m(\theta,\phi)) \right) \, d\theta \, d\phi,
\]

where \( g(\theta,\phi) = e^{\frac{-1}{2\pi(\sigma_n^2 + \sigma_\psi^2)} \sum_{m=1}^{M} z_m^2} \).

Evaluating the joint distribution \( f(y) \) in (B.13) using the independence of \( \theta \) and \( \phi \) yields,

\[
f(y) = \int_\theta \int_\phi f(y,\theta,\phi) \, d\theta \, d\phi = \int_\theta \int_\phi f(y,\theta) \, f(\theta) \, d\theta \, d\phi. \quad (B.14)
\]

The conditional distribution in (B.14) is equal to the product of the PDFs of \( M \) independent Gaussian variables with variance equal to the sum of the noise and nLoS component, i.e., \( (\sigma_n^2 + \sigma_\psi^2) \).
B.3 MMSE Estimator for $M$ antennas in Rician fading

Therefore we can write,

$$f(y) = \int_\theta \int_\phi \left( \frac{1}{\sqrt{2\pi(\sigma_n^2 + \sigma_\psi^2)}} \right)^M g(\theta, \phi) f(\theta) f(\phi) d\theta d\phi. \quad (B.15)$$

Given we have uniform distributions of $\theta$ and $\phi$, the estimator in (B.13) can be expressed as,

$$\hat{h}_{m, \text{MMSE}} = \frac{\int_\theta \int_\phi g(\theta, \phi) \left( \rho h_m(\theta, \phi) + \frac{\sigma_n^2}{\sigma_n^2 + \sigma_\psi^2} (y_m - \rho h_m(\theta, \phi)) \right) d\theta d\phi}{\int_\theta \int_\phi g(\theta, \phi) d\theta d\phi}, \quad (B.16)$$

where $g(\theta, \phi) = e^{-\frac{1}{2(\sigma_n^2 + \sigma_\psi^2)} \sum_{k=1}^M z_k^2}$. 

Appendix B

B.4 LMMSE Estimator Large $M$ NMSE

Proof of Theorem 4.1. In order to derive the large $M$ NMSE of the LMMSE estimator when estimating $h$ from observations $y$, we start by examining the autocovariance and cross-covariance functions of the $\{h_m\}$ and $\{y_m\}$ processes from the model in (4.12). These functions can be expressed as,

$$ r_h[n] = E\{h_m h_{m+n}\}, \quad (B.17) $$

$$ r_{hy}[n] = E\{h_m y_{m+n}\} = E\{h_m (h_{m+n} + n_{m+n})\} = r_h[n], \quad (B.18) $$

$$ r_y[n] = E\{y_m y_{m+n}\} = E\{(h_m + n_m)(h_{m+n} + n_{m+n})\} = \begin{cases} r_h[0] + c_n^2, & n = 0 \\ r_h[n], & \text{otherwise} \end{cases} \quad (B.19) $$

where the autocovariance function, $r_h[n]$ is given by, (which is essentially (4.26) rewritten),

$$ r_h[n] = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\pi \cos \theta) \, d\theta. \quad (B.20) $$

Since the autocovariance functions in (B.17) and (B.19) (together with (B.20)) are only dependent on lag $n$, the processes $\{h_m\}$ and $\{y_m\}$ are by definition both wide-sense stationary (WSS) processes. They are also both zero-mean processes, given $E\{h_m\} = 0$ where $h_m$ is defined in (4.12), and given the presence of zero mean AWGN. Furthermore, along with the other observations, the sole dependence of the autocovariance function given by (B.18) on $n$, shows that the processes $\{h_m\}$ and $\{y_m\}$ are jointly WSS processes.

Since $h_m$ and $y_m$ are jointly WSS processes, we are able to apply the general results from Wiener smoothing theory, presented in [54], to the spatial domain. Theorem 7.3.1 from [54] states that given two discrete zero mean, jointly wide-sense stationary (WSS) random processes $\{h_m, y_m\}$ the MMSE of the linear smoother of the process $\{h_m\}$, given infinite observations $\{y_m\}$, is a filter with a MSE

$$ \text{MSE}_{\text{LMMSE}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( S_h(e^{j\omega}) - \frac{|S_{hy}(e^{j\omega})|^2}{S_y(e^{j\omega})} \right) d\omega, \quad (B.21) $$
where $S_h(e^{j\omega})$ is the power spectral density function of the $\{h_m\}$ process, and $S_{hy}(e^{j\omega})$ is the cross-spectral density function of processes $\{h_m, y_m\}$.

The spectral and cross-spectral density functions $S_h(e^{j\omega})$, $S_y(e^{j\omega})$ and $S_{hy}(e^{j\omega})$ in (B.21) are found using the definition in [54, Sec. 7.3], and are functions of angular frequency $\omega$ (rad/s). The (cross) power spectral density function of processes $\{x, y\}$ is defined as the Discrete-Time Fourier Transform (DTFT) of the covariance function of the process, $r_{xy}[n]$, i.e.

$$S_{xy}(e^{j\omega}) = \mathcal{F}_2\pi \{ r_{xy}[n] \} = \sum_{n=-\infty}^{\infty} r_{xy}[n] e^{-j\omega n}, \quad (B.22)$$

where the $e^{j\omega}$ argument of the spectral density function is a notation adopted to highlight the $2\pi$ periodicity of the DTFT.

Applying (B.22) to (B.17) - (B.19), we have

$$S_{hy}(e^{j\omega}) = S_h(e^{j\omega}) = r_h[0] + \sum_{n=-\infty, n\neq 0}^{\infty} r_h[n] e^{-j\omega n}, \quad (B.23)$$

$$S_y(e^{j\omega}) = \sigma_n^2 + r_h[0] + \sum_{n=-\infty, n\neq 0}^{\infty} r_h[n] e^{-j\omega n}$$

$$= S_h(e^{j\omega}) + \sigma_n^2. \quad (B.24)$$

The expressions (B.23) and (B.24) show that the relevant spectral densities are simple functions of the spectral density of the $h$ process, $S_h(e^{j\omega})$, which we are now left to determine.

The definition of the power spectral density in (B.22) is specified as the continuous function resulting from the DTFT of the covariance function $r_h[n], n \in \mathbb{Z}$. Let us first consider a continuous version of the covariance function $r_h(\nu), \nu \in \mathbb{R}$. We can express (B.20) as,

$$r_h(\nu) = \frac{J_0(|\nu|\pi)}{2}, \quad (B.25)$$

where by definition $J_m(n) = 1/\pi \int_0^\pi \cos(m \tau - n \sin(\tau))d\tau$ is the $m$-th order Bessel function of the first kind. The Bessel functions are even aperiodic continuous functions of $n$, and therefore have a corresponding aperiodic continuous time fourier transform (CTFT). The CTFTs of these
functions are well defined, as given by \[74, \text{Table 7.7}\],

\[
\mathcal{F}\{J_0(\nu)\} = \begin{cases} 
\frac{2}{\sqrt{1-\nu^2}}, & -1 < \nu < 1 \\
0, & \text{otherwise}.
\end{cases}
\]  
(B.26)

Using the Time Scaling property of the Fourier Transform on (B.26), the CTFT of the covariance function in (B.20), is given by

\[
\tilde{\mathcal{S}}_h(\omega) = \mathcal{F}\{\frac{|\nu|\pi}{2}\} = \begin{cases} 
\frac{1}{\sqrt{\pi^2-\omega^2}}, & -\pi < \omega < \pi \\
0, & \text{otherwise}.
\end{cases}
\]  
(B.27)

The relationship between the DTFT of the covariance function, \(S_h(e^{j\omega}) = \mathcal{F}_{2\pi}\{r_h[n]\}\), and the CTFT of the covariance function \(\tilde{S}_h(\omega) = \mathcal{F}\{r_h(\nu)\} = \int_{-\infty}^{\infty} r_h(\nu)e^{-j\omega\nu}d\nu\), is given by \[70\]

\[
S_h(e^{j\omega}) = \frac{1}{D} \sum_{k=-\infty}^{k=\infty} \tilde{S}_h\left(\frac{\omega}{D} + \frac{2\pi k}{D}\right),
\]  
(B.28)

where \(D\) is the sampling interval, i.e., the discrete sampled version \(r_h[n] = r_h(nD)\). While the spectrum in (B.28) contains periodic copies across the entire frequency range, the range of interest is only across the integration limits \([-\pi, \pi]\) in (B.21). Given \(D = 1\), and the well defined bandlimited spectrum (B.27), the summation across the periodic copies in (B.28) does not affect the centre copy over \([-\pi, \pi]\) \(^1\) and

\[
S_h(e^{j\omega}) = \frac{1}{\sqrt{\pi^2-\omega^2}}, -\pi < \omega < \pi.
\]  
(B.29)

From the results (B.23), (B.24) and (B.29), the MSE in (B.21) can be expressed as,

\[
\text{NMSE}_{\text{LMMSE}_{M\rightarrow\infty}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(S_h(e^{j\omega}) - \frac{|S_h(e^{j\omega})|^2}{S_h(e^{j\omega}) + \sigma_n^2}\right) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\eta + \sqrt{\pi^2-\omega^2}} d\omega.
\]  
(B.30)

\(^1\)This is also described by the well known sampling theorem which states that the sampling frequency must be greater than or equal to twice the frequency of the highest component of the signal.
B.5 A-LMMSE Estimator Large $M$ NMSE

**Proof of Theorem 4.2.** Using Lemma B.1, the power spectral density function of the $\{h_m\}$ process, with underlying $f_s \sim \mathcal{U}[\hat{f}_s - R, \hat{f}_s + R]$, can be written as,

$$S_h(e^{j\omega}) = \frac{1}{8R} \left( \text{rect} \left( \frac{\omega}{4\pi(f_s + R)} \right) - \text{rect} \left( \frac{\omega}{4\pi(f_s - R)} \right) \right), \quad -\pi < \omega < \pi \quad (B.31)$$

We can use the two-sided spectrum defined by (B.31) in the NMSE expression of (B.30) to express the NMSE as,

$$\text{NMSE}_{M \to \infty} = 2 \times \frac{1}{2\pi} \mathbb{E}\{|h_m|^2\} \int_{2\pi(f_s - R)}^{2\pi(f_s + R)} \left\{ \frac{1}{8R} - \frac{|\frac{1}{8R}|^2}{\frac{1}{8R} + \sigma_n^2} \right\} d\omega$$

$$= \frac{1}{\mathbb{E}\{|h_m|^2\}} \left( \frac{\eta R}{2R} + 2 \right).$$  

□

**Lemma B.1.** The power spectral density function of the process $h_m = \cos(2\pi f_s m + \phi)$, where $m \in \mathbb{Z}$, $\phi \sim \mathcal{U}[0, 2\pi]$ and $f_s$ with PDF $f_{f_s}(f_s)$ defined over $f_s \in [\hat{f}_s - R, \hat{f}_s + R]$, where $\hat{f}_s + R < 0.5$ and $\hat{f}_s - R > 0$, is given by

$$S_h(e^{j\omega}) = \frac{1}{4} f_{f_s} \left( \frac{\omega}{2\pi} \right) \left( \text{rect} \left( \frac{\omega}{4\pi(f_s + R)} \right) - \text{rect} \left( \frac{\omega}{4\pi(f_s - R)} \right) \right), \quad -\pi < \omega < \pi,$$

where $\text{rect}(x) = 1$ for $-0.5 \leq x \leq 0.5$, and 0 otherwise.

**Proof.** Recall from Section 4.3.2.1 that for our channel model the large $M$ MSE can be written in terms of the power spectral density $S_h(e^{j\omega})$, which is related to the CTFT of the covariance function $\tilde{S}_h(w)$ via (B.28). Given the spectral densities are in terms of angular frequency, we re-express the covariance function of $h$ given in (4.33),

$$r_h[n] = \frac{1}{2} \int_{\hat{f}_s - R}^{\hat{f}_s + R} f_{f_s}(f_s) \cos(2\pi f_s n) df_s,$$
in terms of angular frequency $\omega_s = 2\pi f_s$,

$$r_h[n] = \frac{1}{4\pi} \int_{2\pi(f_s-R)}^{2\pi(f_s+R)} f_{f_s} \left( \frac{\omega_s}{2\pi} \right) \cos(\omega_s n) \, d\omega_s,$$  \hspace{1cm} (B.32)

where the PDF for the random quantity $\omega_s$ has been generated using standard change of variable techniques given the monotonic relationship $\omega_s = 2\pi f_s$. Applying the relationship described by (B.28) in the same way as in Theorem 4.1, i.e., $k = 0$, $D = 1$, and using the expression (B.32) we can write,

$$S(\omega) = \tilde{S}(\omega) = \mathcal{F}\{r_h(\nu)\}$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\omega} \int_{2\pi(f_s-R)}^{2\pi(f_s+R)} f_{f_s} \left( \frac{\omega_s}{2\pi} \right) \cos(\omega_s \nu) e^{-j\omega \nu} \, d\omega_s \, d\nu$$

$$= \frac{1}{4\pi} \int_{2\pi(f_s-R)}^{2\pi(f_s+R)} f_{f_s} \left( \frac{\omega_s}{2\pi} \right) \mathcal{F}\{\cos(\omega_s \nu)\} \, d\omega_s$$

$$= \frac{1}{4} \int_{2\pi(f_s-R)}^{2\pi(f_s+R)} f_{f_s} \left( \frac{\omega_s}{2\pi} \right) \left[ \delta(\omega + \omega_s) + \delta(\omega - \omega_s) \right] \, d\omega_s$$

$$= \begin{cases} 
\frac{1}{4} f_{f_s} \left( \frac{\omega}{2\pi} \right), & 2\pi(f_s-R) < |\omega| < 2\pi(f_s+R) \\
0, & \text{otherwise.} 
\end{cases}$$ \hspace{1cm} (B.33)
Proof of Theorem 4.4. We start by noting (as covered earlier in Section 4.3.2.1) that as $M \to \infty$, the MSE of the linear smoother, $E\{|h_m - \hat{h}_m|^2\}$, is the same for all $m$ since there are effectively an infinite number of observations in each direction at every antenna element. Let $e_m = h_m - \hat{h}_m$, and therefore $MSE_{M \to \infty} = E\{|e_m|^2\} = r_e[0]$.

With an infinite number of observations we can compute the relevant power spectral densities as given by the infinite sum of the DTFT in (B.22). Using the definition of the inverse DTFT to calculate the covariance function, (i.e., the dual to (B.22)),

$$r_{xy}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xy}(e^{j\omega})e^{j\omega n} d\omega,$$

we can express the MSE as

$$MSE_{M \to \infty} = r_e[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_e(e^{j\omega})d\omega. \quad (B.35)$$

In order to derive the power spectral density function of the error process, we examine its autocovariance function,

$$r_e[n] = E\{e_m e_{m+n}\}$$

$$= E\{(h_m - \hat{h}_m) (h_{m+n} - \hat{h}_{m+n})\}$$

$$= r_h[n] + r_{\hat{h}}[n] - 2r_{h\hat{h}}[n]. \quad (B.36)$$

Using (B.22) and (B.36), we can rewrite the MSE as

$$MSE_{M \to \infty} = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_h(e^{j\omega}) + S_{\hat{h}}(e^{j\omega}) - 2S_{h\hat{h}}(e^{j\omega}) d\omega. \quad (B.37)$$

Given the processes involved are WSS (as previously shown in Section 4.3.2.1) and filter $W$ is space ($m$) invariant with real-valued frequency response $W(e^{j\omega})$, we can re-express the spectral and cross spectral densities in (B.37), as noted in [54],

$$S_{\hat{h}}(e^{j\omega}) = W(e^{j\omega})^2 S_y(e^{j\omega})$$
\[ W(e^{j\omega})^2(S_h(e^{j\omega}) + \sigma_n^2), \quad (B.38) \]
\[ S_{\hat{h}}(e^{j\omega}) = W(e^{j\omega})S_h(e^{j\omega}). \quad (B.39) \]

Substituting (B.38) and (B.39) into (B.37) we can express the large \( M \) MSE as,

\[ \text{MSE}_{M \to \infty} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ (S_h(e^{j\omega}) \left( W(e^{j\omega})^2 - 2W(e^{j\omega}) + 1 \right) + \sigma_n^2 W(e^{j\omega})^2 \right] d\omega. \quad (B.40) \]

It is trivial to show that the optimum linear filter (smoother) which minimises the above, is given by,

\[ W(e^{j\omega}) = \frac{S_h(e^{j\omega})}{S_h(e^{j\omega}) + \sigma_n^2}, \quad (B.41) \]

under which (B.40) reduces to the MSE expression (B.30) of the LMMSE estimator. This is the resulting MSE given by Theorem 4.1 and Theorem 4.2 where the power spectral density function \( S_h(e^{j\omega}) \) used to construct the filter \( W(e^{j\omega}) \) matches the \( \{h_m\} \) process present in input process \( \{y_m\} \).

Under the A-LMMSE Static Uniform estimator we have a mis-match, since the filter \( W \) has been constructed using a uniformly distributed a-priori \( f_s \), while the input is non-uniformly distributed as given by (4.47). In order to clearly differentiate the power spectral density functions resulting from the non-uniform input \( f_s \) distribution used, \( I \) has been used in the subscript of \( S_{hI}(e^{j\omega}) \) herein where necessary.

Using the double sided spectrum for the uniform \( f_s \) distribution as derived in Lemma B.1, to construct (B.41), yields the filter,

\[ W(e^{j\omega}) = \begin{cases} \frac{1}{8R} / \left( \frac{1}{8R} + \sigma_n^2 \right), & \omega \in [2\pi(f_{s0} - R), 2\pi(f_{s0} + R)] \\ 0, & \text{otherwise}. \end{cases} \]

Therefore, we can rewrite (B.40) as,

\[ \text{MSE}_{M \to \infty} = 2 \times \frac{1}{\pi} \left[ W_u^2 - 2W_u + 1 \right] \int_{2\pi(f_{s0} - R)}^{2\pi(f_{s0} + R)} S_{hI}(e^{j\omega}) d\omega + \sigma_n^2 W_u^2 4\pi R, \quad (B.42) \]
where \( W_u \triangleq \frac{1}{8R} / (\frac{1}{8R} + \sigma_n^2) \).

As shown in Lemma B.1, the positive spectrum of \( S_{hi}(e^{j\omega}) \) is also bandlimited to \( \omega \in [2\pi(f_s0 - R), 2\pi(f_s0 + R)] \) for any \( f_s \) distribution bounded by \( f_s \in [f_s0 - R, f_s0 + R] \), and therefore we can write

\[
\int_{2\pi(f_s0-R)}^{2\pi(f_s0+R)} S_{hi}(e^{j\omega}) \, d\omega = \int_{-\pi}^{\pi} \frac{1}{2} S_{hi}(e^{j\omega}) \, d\omega = \pi r_h[0],
\]

(B.43)

where the equality in (B.43) has made use of the definition in (B.34).

If we re-examine the covariance function for \( r_{h}[0] \), we have

\[
r_h[0] = E\{(\cos(2\pi f_s m) + \phi)^2\}
\]

\[
= \frac{1}{2} \left( E\{\cos(4\pi f_s m + 2\phi)\} + 1 \right)
\]

\[
= \frac{1}{2} \left( E\{\cos(4\pi f_s m)\} E\{\cos(2\phi)\} - E\{\sin(4\pi f_s m)\} E\{\sin(2\phi)\} \right) + \frac{1}{2}
\]

(B.44)

\[
= \frac{1}{2},
\]

(B.45)

where the equality (B.44) is due to the independence between \( \phi \) and \( f_s \), and the final equality is from \( E\{\cos(2\phi)\} = E\{\sin(2\phi)\} = 0 \). Therefore, \( r_h[0] \) evaluates to a fixed constant regardless of the PDF \( f_{f_s}(f_s) \).

Substituting (B.43) and (B.45) into (B.42) yields,

\[
MSE_{M \to \infty} = \frac{1}{(4\sigma^2_R)^{-1} + 2}.
\]

(B.46)

The corresponding NMSE for the estimation of the vector \( \mathbf{h} \) is then given by,

\[
NMSE_{M \to \infty} = \frac{1}{E\{|h_m|^2\} \left( \frac{\eta}{2R} + 2 \right)}.
\]

(B.47)
Bibliography


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