Exact Solutions in Multi-Species Exclusion Processes

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Declaration

This thesis is an account of research undertaken between August 2015 and May 2019 at The School of Mathematics and Statistics, Faculty of Science, The University of Melbourne, Melbourne, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university. The thesis is fewer than 100 000 words in length, exclusive of tables, maps, bibliographies.

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July, 2019
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Abstract

The exclusion process has been the default model for the transportation phenomenon. One fundamental issue is to compute the exact formulae analytically. Such formulae enable us to obtain the limiting distribution through asymptotics analysis, and they also allow us to uncover relationships between different processes, and even between very different systems. Extensive results have been reported for single-species systems, but few for multi-component systems and mixtures. In this thesis, we focus on multi-species exclusion processes, and propose two approaches for exact solutions.

The first one is due to duality, which is defined by a function that co-varies in time with respect to the evolution of two processes. It relates physical quantities, such as the particle flow, in a system with many particles to one with few particles, so that the quantity of interest in the first process can be calculated explicitly via the second one. Historically, published dualities have mostly been found by trial and error. Only very recently have attempts been made to derive these functions algebraically. We propose a new method to derive dualities systematically, by exploiting the mathematical structure provided by the deformed quantum Knizhnik-Zamolodchikov equation. With this method, we not only recover the well-known self-duality in single-species asymmetric simple exclusion processes (ASEPs), and also obtain the duality for two-species ASEPs.

Solving the master equation is an alternative method. We consider an exclusion process with 2 species particles: the AHR (Arndt-Heinzl-Rittenberg) model and give a full derivation of its Green’s function via coordinate Bethe ansatz. Hence using the Green’s function, we obtain an integral formula for its joint current distributions, and then study its limiting distribution with step type initial conditions. We show that the long-time behaviour is governed by a product of the Gaussian and the Gaussian unitary ensemble (GUE) Tracy-Widom distributions, which is related to the random matrix theory. Such result agrees with the prediction made by the nonlinear fluctuating hydrodynamic theory (NLFHD). This is the first analytic verification of the prediction of NLFHD in a multi-species system.
Preface

The work contained in this thesis was carried out under the supervision of Professor Jan de Gier at the School of Mathematics and Statistics, The University of Melbourne. The main original contributions of the author are in Chapters 3 and 4. The Chapter 3, a joint work with Jan de Gier and Michael Wheeler, is published as “Integrable Stochastic Dualities and the Deformed Knizhnik-Zamolodchikov Equation” [165], and Chapter 4, a joint work with Jan de Gier, Iori Hiki, and Tomohiro Sasamoto is published as “Exact Confirmation of 1D Nonlinear Fluctuating Hydrodynamics for a Two-Species Exclusion Process” [35].
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Chapter 1

Introduction

This thesis is devoted to the study of one dimensional exclusion processes, especially with multiple species. Among them, the asymmetric simple exclusion process (ASEP) is the simplest and the most widely studied model for interacting particle systems. It also serves as a paradigm for non-equilibrium statistical mechanics. In spite of their simplicity, these models contain all the necessary complexities for non-trivial and interesting physical phenomena. For example, it exhibits a non-equilibrium steady state [64] and non-Gaussian fluctuations [83]. The ASEP is furthermore of mathematical interest due to its integrability [54,74].

The debut of ASEP was as a biophysical model for protein synthesis on RNA by MacDonald and Gibbs in 1969 [116]. It then was introduced as a purely mathematical tool by Spitzer in 1970 [149] to study interactions in Markov processes, where the name “exclusion process” came forward (see also [78,114,115]). After that, the applications of the ASEP thrived, not only in physics and mathematics, but also biology, social science and industrial engineering. In biology, the ASEP can be used to describe, for example, the accumulation of molecular motors on filaments [102], and sequence alignment [26]. The ASEP also simulates physical phenomena such as the transportation of macromolecules crossing capillary vessels [39] or quantum dots with Coulomb interaction [95]. The ASEP and its variants are also a widely applied interpretations for traffic flow [63,137,139].

In words, the ASEP is a stochastic Markov process with particles sitting on the lattice $\mathbb{Z}^d$, where we take $d = 1$ throughout this thesis. The dynamics of the process is described by the random walks performed by those particles, jumping left at rate $q$ and right at a different rate $p$, where we usually impose the normalisation of $p + q = 1$. The non-equilibrium property is then* realised by a bias in jumping direction, $p \neq q$ (we assume $p < q$ intentionally). Such jumping is subject to the “exclusion” law, i.e., each position can only be occupied by at most one particle. In other words, jumping is suppressed if the target position is already occupied, which can be regarded as a “short range” interaction between particles. The following chapter contains a detailed definition of the ASEP.

The totally asymmetric simple exclusion process (TASEP) is a special case of ASEP with all particles only jumping to the right, i.e. $q = 1$. See Fig.1.1 for

*This can also be achieved in other ways, such as external reservoirs with different driven rate, i.e., boundary driven non equilibrium steady state.
schematic diagrams for events in the ASEP and the TASEP.

Figure 1.1: Diagrams for the ASEP and the TASEP.

1.1 Non equilibrium Systems

Statistical mechanics is the study of physical systems with a large amount of degrees of freedom. It is one of the main fields of modern physics. Within statistical mechanics, the study of classical thermodynamics is called equilibrium statistical mechanics.

A system is said to be in equilibrium when its macrovariables, such as temperature, energy, or density, are time independent and spatially homogeneous, and no macroscopic currents exist. It is well known that for an equilibrium system with thermal reservoir, the probability of its microstates (or a given configuration $C$) is described by a Gibbs-Boltzmann distribution:

$$P_{eq}(C) = \frac{\exp(-\beta H(C))}{Z}, \quad Z = \sum_{C} \exp(-\beta H(C)),$$

where $\beta$ is the inverse of temperature, $H(C)$ is the energy of the configuration $C$ and $Z$ is the partition function (see [140]). This provides enough knowledge to describe macrovariables of the system, as well as fluctuations or phase transition. Particularly, the fluctuation is usually predicted to be Gaussian, for example the Brownian motion.

On the contrary, for a system out of equilibrium little is known about the statistics of its microstates. A system is said to be out of equilibrium when the equilibrium states can never be reached. However at long time limit, a stationary state may be reached in the sense that macrovariables are time independent in such a state. These macrovariables may or may not be spatially homogeneous but generically carry macroscopic currents. For example, a system in contact with two particle reservoirs of different particle densities. At long time the system reaches a stationary state such that the particle density is time independent. But a net flux of particles is expected from high density reservoir to low density reservoir, which indicates that the stationary state is non-equilibrium. The current distribution in a non-equilibrium stationary state is not described by the Gibbs-Boltzmann law. The theory that can predict the probability distribution for non-equilibrium stationary states from first principles is far from complete. The search for universal features and non-equilibrium statistics have attracted considerable attention over the past
few decades, and substantial progress has been made through the fluctuation distributions and exact solutions for specific integrable systems.

The driven lattice gases \([98,138]\) provide stochastic models that are mathematically simple enough and yet exhibit enough physical complexity. Among them, the ASEP (or TASEP) serves as a paradigm model to study non-equilibrium systems. It also provides cross-disciplinary bridges between statistical mechanics, random matrices, combinatorics, probability and quantum group theory.

### 1.2 Integrable models

Another important feature of the ASEP is its integrability \([21,22,74,155]\). The precise definition of integrability varies across different contexts. Two examples are the existence of many conserved quantities and the ability to give explicit solutions \([80]\). The ASEP can be “solved” via various approaches, one of the important methods being the Bethe ansatz (see the next chapter for a detailed introduction). Here we refer “Bethe solvable” to “integrable”.

Bethe ansatz relies on the assumption that any many-body interaction can be reduced to a two-body event. This allows us to solve the system via two-body dynamics. Such an assumption imposes constraints on the underlying mathematical structure of the model which are encoded in the \(R\) matrix (the transfer or scattering matrix). The “Yang-Baxter equations” \([9]\) are a restatement of these constraints in the 1+1 dimensional (one spatial dimension and time dimension) models, and must be satisfied by the Bethe solvable systems.

### 1.3 KPZ universality class

Universality in random systems has attracted more and more attention over the last few decades. A large class of models demonstrate universal statistics in their long time (large scale) behaviour. In 1986, Kardar, Parisi and Zhang predicted a universality class, now called the KPZ universality in the study of growth interfaces \([94]\). It is conjectured that a variety of systems that include randomness, including random matrices, growth interfaces, directed polymers, certain stochastic PDEs and interacting particle systems, share the same scaling exponents and a non-Gaussian distribution in long time limit. Numerous books and articles were devoted to study these models \([1,134–136,150]\).

The ASEP also plays an essential role in the study of Kardar-Parisi-Zhang (KPZ) equation \([5,129]\). Over the past years, numerous integrable systems were found to belong to KPZ universality. Due to the ability to find exact solutions and perform rigorous asymptotic analysis in integrable models, these models introduce numerous novel results into the theory of KPZ universality.
1.3.1 Gaussian distribution

The Gaussian (normal) distribution has a probability density function of the form

$$f(x) = \frac{1}{\sqrt{2\pi}b^2} e^{-\frac{(x-a)^2}{2b^2}},$$

which exhibits a “bell” shaped curve. Thus a Gaussian distribution is sometimes called a “bell curve”. The Gaussian distribution has been extremely important not only in probability theory, but also in most of the natural and social sciences, especially after the establishment of the central limit theorem (CLT) (see [67] for a review).

Informally, the CLT states that observables of interest in certain types of models show random fluctuations with typical scaling exponent $1/2$, and the scaled fluctuations are Gaussian. A classic example is flipping a coin $N$ times. Let $H$ be the number of the heads. Then $H$ tends to $N/2$ with error $\sqrt{N} = N^{1/2}$, as $N$ tends to infinity. Moreover, the probability function $P$ of the scaled fluctuation is given by

$$\lim_{N \to \infty} P\left( H < N/2 + \sqrt{N} x/2 \right) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy := F_G(x).$$

However, a significant amount of systems were found not to obey Gaussian universality, and a large amount of research has been devoted to understand these complex systems in both mathematics and physics.

1.3.2 Simple exclusion processes

The KPZ universality (see [83] for a review) predicted that unlike the scaling exponent $1/2$ in Gaussian universality, the scaling exponents for fluctuations and correlations are $1/3$ and $2/3$, and that these are independent of various model parameters. These exponents were first found in 1977 in stochastic equations [47], then in directed polymers [81] and in a driven lattice gas (the ASEP) [163] in 1985*. The work [94] related these models to interface growth, and conjectured that there is a class of models that share the same scaling exponents. The limiting distributions for fluctuations were identified for the TASEP in [89], and for the ASEP in [161]. These distributions are identified as the Tracy-Widom distributions ($F_{GOE}$ or $F_{GUE}$) of the Gaussian orthogonal ensemble (GUE) or the Gaussian unitary ensemble (GOE).

We generalise the ASEP to the simple exclusion process by allowing all values of the drifting rate

$$\gamma = q - p$$

with $0 < \gamma \leq 1$. The ASEP is identified as $1 > \gamma > 0$, while the TASEP is identified as $\gamma = 1$.

The exclusion process can be mapped to a growth process in the following way. Fix a height at a reference point and map† the particle into a slope of +1, the

*The relation between these two model were then understood in [82]
†Sometimes the mapping is reversed according to the initial state.
hole into a slope of $-1$. The state of the simple exclusion process then uniquely determines a height function $h(x,t)$. This provides a connection between the ASEP and a growth process [76].

The KPZ universality states that

- $h(x,t)$ fluctuates around its mean like $t^{1/3}$
- $h(x,t)$ has nontrivial spatial correlations in a transversal scale of order $t^{2/3}$
- Up to some constants, the long time behavior of $c_1 t^{-1/3}[h(0,c_2 t) - c_3 t]$ is governed by Tracy-Widom distribution.

Specifically, the limiting distribution for fluctuations is given by [89,161]

**Theorem 1.1.** For $\gamma \in (0,1]$ and step initial condition, the long time behavior of $h(x,t)$ is governed by

$$\lim_{t \to \infty} P \left( \frac{h(t/\gamma,0) - t/2}{2^{-1/3}t^{1/3}} \right) \geq -s = F_{\text{GUE}}(s).$$

**Remark 1.2.** Note that when $\gamma = 0$, the simple exclusion process becomes the symmetric process (SSEP). The fluctuation scales like $t^{1/4}$ instead of $t^{1/3}$, and the fluctuations are governed by Gaussian distribution. The parameter $\gamma$ hence provides a connection between Gaussian and KPZ universality.

## 1.4 Notation

Throughout this thesis, we will stick to the following notation unless otherwise specified:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_x[\cdot]$</td>
<td>Expectation value with initial state $x$</td>
</tr>
<tr>
<td>$A^\top$</td>
<td>Transpose of matrix $A$</td>
</tr>
<tr>
<td>$</td>
<td>s</td>
</tr>
<tr>
<td>$S_n$</td>
<td>Symmetric group $S_n$ with its element $s_i$</td>
</tr>
<tr>
<td>$\text{sign} (\pi)$</td>
<td>Signature of the permutation $\pi$</td>
</tr>
<tr>
<td>$\delta_{ij}$</td>
<td>The kronecker delta</td>
</tr>
<tr>
<td>$\text{id}$</td>
<td>The identity element</td>
</tr>
<tr>
<td>$1$</td>
<td>The indicator function</td>
</tr>
<tr>
<td>$C^n; C^\infty$</td>
<td>First $n$ derivatives are continuous ; smooth</td>
</tr>
<tr>
<td>$O(z^n)$</td>
<td>The big O notation</td>
</tr>
<tr>
<td>$o(z^n)$</td>
<td>The small O notation</td>
</tr>
<tr>
<td>$\text{Res}_{z_0}(f(z))$</td>
<td>The residue of $f(z)$ at $z = z_0$</td>
</tr>
</tbody>
</table>
1.5 Outline

This thesis mainly focuses on two multi-species exclusion processes: the multi-species ASEP (see Chapter 2 for an explicit definition and discussions) and the two-species Arndt-Heinzl-Rittenberg (AHR) exclusion process (see Chapter 4 for an explicit definition and discussions). Two approaches are proposed to solve the multi-species exclusion processes:

- A systematic method to construct duality functions in multi-species ASEPs.
- The theoretical underpinning of non-linear fluctuating hydrodynamics (NLFHD) in the AHR model, via the exact formula of the Green’s function.

Exact formulae are a major quest in the study of stochastic non-equilibrium systems. An exact formula can not only enable us to perform asymptotic analysis to uncover limit laws, but also provides a guide point between different models, and even different research areas. Over the past few years many results have been found in single species system, but it is not the case for multi-species system. The two approaches mentioned above serve as two novel techniques to obtain exact formulae of the current distribution for multi-species exclusion processes.

The first method relates a complicated multi-species process with, if possible, a simple process with less particles. We propose a consistent and constructive framework to obtain such a relation, which is called “Duality”. Our framework obtains explicit duality functions. To obtain exact formulae for limiting distributions using duality, such as can be found in [17] for the single species ASEP, remains a research problem for future works.

On the other hand, the explicit exact formulae for the AHR model are given with the help of the Green’s function, which is obtained via Bethe ansatz. This approach for single species ASEP was first done by Tracy and Widom through a series of papers [159–161]. With the formula for the Green’s function we then give a theoretical confirmation for the asymptotic behaviour of a joint distribution function in the AHR model, which can be predicted by the phenomenological NLFHD [35].

The following is the detailed outline of this thesis, and a diagrammatic sketch of its structure (see Fig. 1.2).

To make this thesis more accessible, we will begin with Chapter 2, introducing and reviewing briefly known results for the ASEP. We first give a detailed mathematical definition of the ASEP as a Markov chain, followed by an introduction on the master equation and stationary states. Then the above two approaches (duality and Green’s function) are introduced for the single-species ASEP. Firstly, after introducing the explicit infinitesimal generator of the ASEP, we review the duality function of the single species ASEP [17]. Subsequently, the coordinate Bethe ansatz is applied to the single species ASEP to obtain its Green’s function. Finally this chapter is finished by a short review on another important technique: the Matrix product ansatz.

In Chapter 3, a new systematic method to obtain duality functions in multi-species ASEPs (mASEP) is presented, via the solution to the deformed Knizhnik-
Zamolodchikov (KZ) equations. Apart from a transition matrix (or generator) in the vector space of particle configurations, the mASEP can also be represented by a basis $f_\mu$ in the ring of $n$-variable polynomials. The relation between these two representations is the deformed KZ equation. This construction thus can be interpreted as the duality function of a diagonal observable intertwining the vector space and polynomial representations of the mASEP. We will finish this chapter with an explicit duality function between two-species ASEPs and one between a single and a two-species ASEPs.

The second main topic is presented in Chapter 4, where we consider the two-species AHR model. By using a variant of the coordinate Bethe ansatz, an integral formula of the Green’s function is obtained. Therefore, an integral formula for a joint current distribution is naturally derived using this Green’s function. We then perform an asymptotic analysis on the distribution and show that under step type initial condition, the limiting behaviour is governed by a product of the Gaussian and the GUE Tracy-Widom distribution, as predicted by the NLFHD.

Finally, this thesis will be finished by a summary of the main results in these two topics. Then some possible related future research plans are discussed.
Figure 1.2: Structure and Outline
Chapter 2

Asymmetric Simple Exclusion Process

The aim of this chapter is to review existing background of Markov chains and asymmetric simple exclusion processes (ASEPs). We will begin with explicit definition of the ASEP model and discuss some of its important properties as a Markov chain. Next we will introduce the concept of (self-)duality in ASEP in terms of its infinitesimal generator, and give an example of an explicit duality function by proving the duality equation directly. We then focus on diagonalising the ASEP generator by the approach of Bethe ansatz in the case of periodic boundary conditions as well as on the infinite lattice $\mathbb{Z}$. Finally we finish by a briefly introduction on the matrix ansatz of ASEP.

2.1 Definition of the model: a Markov chain

Recall the verbal description of the ASEP given in Chapter 1: it is a stochastic and Markovian process, which describes indistinguishable particles sitting on the lattice $\Omega \subseteq \mathbb{Z}^d$ performing jumps subject to certain rules and the restriction that no more than one particle is allowed per site. In this thesis we only consider $d = 1$.

Suppose $\Omega = \mathbb{Z}$. Then particles sitting on $\mathbb{Z}$ can jump right or left with different rate $p$ and $q$ respectively (see Fig.2.1), with the additional condition that a jump is not allowed in the direction of an occupied neighbouring site. For the purpose of discussion below, the jumping rates are normalised by $p + q = 1$. An important special case of ASEP is when $p = 0$, which is called totally asymmetric, i.e., TASEP.

![Figure 2.1: ASEP on $\mathbb{Z}$](image)

*Here we let $p = 0$ instead of $q = 0$ since we assume particles are drifting to the left.*
The model exhibits the following properties, which characterises the ASEP as an interacting particle system with non-equilibrium behaviour.

**Asymmetry** The jumping rates to the left and right are different, i.e. \( p \neq q \). For convenience, we let \( p < q \), i.e., particles drift to the left. When one of the jumping rates is zero, then the model is called *totally asymmetric*. This biased jumping induces a current and hence breaks the *detailed balance* condition (see definition in Section 2.3), resulting in an irreversible non-equilibrium steady state (NESS) in ASEP.

**Exclusion** The jumping is performed only when the neighbouring site is empty, i.e., a site can be occupied by at most one particle. Exclusion imposes an interaction between particles in the model, and hence produces a \( N \)-body system (or an infinite many body system).

**Remark 2.1.** The special case \( p = q = 1/2 \) defines the symmetric simple exclusion process (SSEP), which results in an equilibrium condition in the bulk, but may gives a NESS subject to boundary conditions. For example, SSEP on a finite lattice, with open boundaries exhibits a boundary driven NESS.

Other than the infinite system \( \Omega = \mathbb{Z} \), the following three boundary conditions are of great interest in the study of ASEP.

1. **Periodic system:** \( N \) particles sitting on a ring of length \( L \) (see Fig. 2.2(b)). \( N \) is conserved. This system can be considered as infinite with site \( i + L \) identified as site \( i \).

2. **Finite system with open boundaries:** Particles living on a one-dimensional lattice of length \( L \). Both ends of the lattice are attached with reservoirs. Particles can enter or leave the lattice at both ends with different rates (see Fig. 2.2(a)).

3. **Finite system with closed boundaries:** Particles sitting on a one-dimensional lattice of length \( L \), where the total number of particles \( N \) is now conserved.

![Figure 2.2](image-url): (a)ASEP on finite lattice with open boundaries; (b)ASEP on a ring.
2.1 Definition of the model: a Markov chain

2.1.1 Dynamics

First, we need to define how the system evolves with time, namely when and how does the jumping occur? According to the jumping rates \( p \) and \( q \), the particle at site \( i \), during the time interval \((t, t + dt]\), jumps to site \( i + 1 \) with probability \( p \, dt \), given that the site \( i + 1 \) is empty. Specifically, the dynamics is described as below:

- Particles carry a clock that rings according to a Poisson process.
- Jumping happens when the clock rings.
- Jumping is performed only when the target site is empty.

According to these rules, the jumping occurs randomly according to exponential clocks, i.e. time interval between two consecutive jumps of a given particle is exponentially distributed. To be specific, let \( \tau \) be the time interval between two consecutive jumps, then the probability density function of \( \tau \) is given by

\[
f(\tau) = \lambda e^{-\lambda \tau},
\]

where \( \lambda > 0 \). Then the expected waiting time can be calculated as follows

\[
E[\tau] = \int_0^\infty \tau \lambda e^{-\lambda \tau} \, d\tau = \frac{1}{\lambda},
\]

which indicates the jumping rate, i.e., the average number of jumps that occur within a unit time, is give by \( \frac{1}{E[\tau]} = \lambda \). So \( \lambda \) is taken to be \( p \) for right jumps and \( q \) for left.

One important result of exponential distribution is the memoryless property, which expressed as

\[
P[\tau > t + s \mid \tau > s] = P[\tau > t]. \tag{2.1}
\]

To prove this equation, we first need to evaluate \( P[\tau > t] \), then (2.1) can be proved by definition of conditional probability.

\[
P[\tau > t + s \mid \tau > s] = \frac{P[\tau > t + s \cap \tau > s]}{P[\tau > s]} = \frac{P[\tau > t + s]}{P[\tau > s]}
\]

\[
= \int_{t+s}^{\infty} \frac{\lambda e^{-\lambda \tau} \, d\tau}{\int_s^{\infty} \lambda e^{-\lambda \tau} \, d\tau} = e^{-\lambda(t+s)}/e^{-\lambda s} = e^{-\lambda t} = P[\tau > t].
\]

From (2.1), one can see that the probability that the jumping does not occur in the next time \( t \), given that it does not happen for time \( s \), is the same as the probability that it does not happen for time \( t \). This means that a particle does not remember how long it has waited. Such a memoryless waiting time distribution encodes the Markov property (see (2.6) and Remark.2.3) embedded inside the model.
Remark 2.2. This exponentially distributed waiting time is not exclusive to the ASEP system. In fact the exponential distribution is the only continuous time distribution equipped with the memoryless property, which originates from the core property (see (2.6) and Remark 2.3) that defines a Markov system.

2.2 Master equation

Given the dynamics in Section 2.1.1, we are now ready to analyse the time evolution. Before specifying the “equation of motion” of ASEP in \( \mathbb{Z} \), we would like to give a general dynamic equation for Markov systems. For detailed discussion see [92, 131]. Consider a Markov system with state space \( S \), and a state (or configuration) \( C \in S \). The transition rate is denote by \( \Gamma(C, C') \):

The system at \( C' \) evolves into \( C \) during \( (t, t + dt] \) with probability \( \Gamma(C, C')dt \).

It is assumed that \( \Gamma(C, C') \) does not depend on previous status of system\(^*\). We also assumed that \( \Gamma(C, C') \) does not vary with time. The main quantity of interest is the transition probability (a.k.a. transition function)

\[
P(C; t | C_0; 0) := \text{the probability of state } C \text{ at time } t \text{ given the initial state } C_0.
\]

With unspecified initial state, we write \( P(C; t | C_0; 0) \equiv P(C; t) \) for short. In order to give the evolution of \( P(C; t) \), we first study \( P(C; t + dt) \)

\[
P(C; t + dt) = P(C; t) + \sum_{C' \neq C} \Gamma(C, C')dt P(C'; t) - \sum_{C' \neq C} \Gamma(C', C)dt P(C; t).
\]

The interpretation of the terms on the right hand side of (2.2) are:

- The first sum (gain terms): The probability \( P(C; t) \) increases due the probability that the other configurations \( C' \) turn into \( C \): \( \Gamma(C, C')P(C'; t) \). The rate \( \Gamma(C, C') \) is thus called arrival rate (see the right part of Fig.2.3).

- The second sum (loss terms): Similarly, \( P(C; t) \) decreases since the probability that \( C \) turns into other configurations \( C' \), namely \( \Gamma(C', C)P(C; t) \). We call the rate \( \Gamma(C', C) \) exit rate (see the left part of Fig.2.3).

*This is the key hypothesis of Markov chain. As we have seen in the previous section, this assumption can be guaranteed by an exponentially distributed waiting time.
Rearranging (2.2) give us

\[
\frac{P(C; t + dt) - P(C; t)}{dt} = \frac{d}{dt} P(C; t) = \sum_{C' \neq C} \Gamma(C, C') P(C'; t) - \sum_{C' \neq C} \Gamma(C', C) P(C; t). \tag{2.3}
\]

This suggests a compact form of evolution equation by defining a diagonal term of transition rate:

\[
\Gamma(C, C) = -\sum_{C' \neq C} \Gamma(C', C). \tag{2.4}
\]

It follows that the dynamics of the transition probability is then governed by master equation (a.k.a. forward equation):

\[
\frac{d}{dt} P(C; t) = \sum_{C'} \Gamma(C, C') P(C'; t). \tag{2.5}
\]

We can write this in matrix notation in the following way. Let \(|C⟩⟩ denote a basis in the vector space of configurations, and write

\[
\Gamma = \sum_{C, C'} \Gamma(C, C') |C⟩⟨C'|, \quad |P(t)⟩ = \sum_C P(C; t) |C⟩,
\]

where \(\Gamma(C, C')\) and \(P(C; t)\) are understood as matrix and vector elements in \(\Gamma\) and \(|P(t)⟩\), respectively. Then we write (2.5) as

\[
\frac{d}{dt} |P(t)⟩ = \Gamma ⋅ |P(t)⟩ \quad \text{or} \quad \frac{d}{dt} P(t) = \Gamma \cdot P(t). \tag{2.6}
\]

The linear operator \(\Gamma\) is known as generator matrix or transition rate matrix*, and \(|P(t)⟩\) is recognised as state probability vector. For the purpose of discussion, we usually consider \(|C⟩⟩ as the canonical basis vector of \(\mathbb{C}^S\) such that it is orthonormal. Additionally, with different initial states, \(P(t)\) can be regarded as a matrix with elements \(P(C; t | C'; 0)\), i.e., \(P(t) = \sum_{C,C'} P(C; t | C'; 0) |C⟩⟨C'⟩\).

Given the way how the diagonal term \(\Gamma(C, C)\) is defined in (2.4), one can conclude that all the diagonal elements of \(\Gamma\) are negative and more importantly, the sum of each column is identically zero:

\[
\sum_C \Gamma(C, C') = 0.
\]

Assuming that the initial probability is normalized \(\sum_C P(C; 0) = 1\), then this fact

---

*The the Markov matrix (a.k.a. transition matrix) is obtained by adding the identity matrix to \(\Gamma\). Hence the key feature of Markov matrix is each column summing to one, instead of zero.
ensures that the process is stochastic:

\[
\frac{d}{dt} \sum_c P(C; t) = \sum_{C'} P(C'; t) \sum_c \Gamma(C, C') = 0 \Rightarrow \sum_c P(C; t) = \sum_c P(C; 0) = 1.
\]

Namely, the total probability is conserved.

**Remark 2.3.** The master equation (2.6) completely defines a continuous time Markov chain through a generator matrix. One can see that the evolution of a state vector only depends on the current state. Solving the master equation (2.6) gives

\[
P(t) = e^{t\Gamma} P(0) = e^{t\Gamma} = \text{id} + t\Gamma + O(t^2),
\]

where the second equality follows by \( P(C'; 0 | C; 0) = \delta_{C'C} \), i.e. \( P(0) \) is the identity matrix \( \text{id} \). Therefore for each element in \( P(t) \), we have

\[
P(C'; t | C; 0) = \delta_{C'C} + t\Gamma(C', C) + O(t^2).
\]

(2.7)

Or equivalently,

\[
\Gamma = \lim_{t \to 0} \frac{P(t) - \text{id}}{t},
\]

where \( \text{id} \) is the identity matrix. Therefore (2.6) can be written as

\[
\frac{d}{dt} P(t) = \lim_{s \to 0} \frac{P(t + s) - P(t)}{s} = \lim_{s \to 0} \frac{e^{(t+s)\Gamma} - e^{t\Gamma}}{s} = \Gamma \cdot P(t) = P(t) \cdot \Gamma,
\]

where the last equation is known as the *backward equation*.

### 2.3 Stationary states

Given a dynamic system, a fundamental issue is to determine the stationary state (or steady state) of the system, i.e., the gain terms and loss terms in (2.3) cancel with each other leading to a zero time derivative.

\[
\frac{d}{dt} |P_{\text{stat}}\rangle = \Gamma \cdot |P_{\text{stat}}\rangle = 0,
\]

(2.8)

which indicates that the steady state is in fact a right eigenvector of the transition matrix \( \Gamma \) with eigenvalue zero.

We now consider an *irreducible* (or ergodic) Markov chain [59], where such stationary state exists and is unique: the transition probability \( P(C; t) \) tends to a unique stationary measure \( P_{\text{stat}}(C) \) at long time limit.

\[
\lim_{t \to \infty} |P(t)\rangle = |P_{\text{stat}}\rangle.
\]

A Markov chain is *irreducible* if every state can be reached from any other state
with a positive transition rate. In principle, $|P_{\text{stat}}|$ can always be found if there is only finite number of configurations. In this case, (2.8) is a finite system of linear ODEs and hence $|P_{\text{stat}}|$ can be found accordingly. In fact, this is guaranteed by the Perron-Frobenius theorem, which states that the irreducible Markov matrix has a non-degenerate eigenvalue 0, and all the other eigenvalues have negative real part.

To describe the stationary condition in detail, we denote each gain term and loss term by

$$J(C, C'; t) := \Gamma(C, C') P(C'; t) - \Gamma(C', C) P(C; t),$$

which describes the local probability current between states $C$ and $C'$ at time $t$. The condition of stationary state (2.8) is now stated as

$$\sum_{C'} J_{\text{stat}}(C, C') = 0. \quad (2.9)$$

In this case, the steady states of Markov chains are classified into two kinds depending on how (2.9) is satisfied.

**Equilibrium** When each term in (2.9) vanishes independently, $J_{\text{stat}}(C, C') = 0$ for any states $C, C'$, this is called detailed balance condition [92,99]. In such case, the system is called reversible in the sense that the measure of any trajectory through state space equals its time reversal image [131]. A direct consequence of detailed balance (or reversibility) is an equilibrium steady state, indicating that there are no fluxes in the steady state. The microstate of the system is then described by a Gibbs-Boltzmann distribution.

**Nonequilibrium** Contrary to the above condition, when detailed balance is broken, there exit fluxes in the steady state. We called the Markov chain irreversible, and it exhibits a nonequilibrium steady state (NESS). See [166, 167] for a detailed classification on NESS.

### 2.4 Infinitesimal generator

The generator matrix gives an (forward) infinitesimal generator which acts on any function$^*$ of states according to

$$G[f](C) := \sum_{C'} G(C, C') f(C') = \sum_{C'} \Gamma(C', C) f(C'). \quad (2.10)$$

An infinitesimal generator uniquely determines a continuous time Markov chain. In fact, the infinitesimal generator for a general stochastic process $C(t)$ is given by its expectation

$^*$From now on, we only consider functions such that the limit of (2.10) exists. See [60,92] for detailed discussions of the domain of generators.
Definition 2.4. If $\mathcal{C}(t)$ is a continuous times Markov process, then the generator is defined by

$$G[f](\mathcal{C}) = \lim_{t \to 0} \frac{\mathbb{E}_\mathcal{C}[f(\mathcal{C}(t))] - f(\mathcal{C})}{t}, \quad (2.11)$$

where $\mathbb{E}_\mathcal{C}$ denotes the expectation value with respect to initial state $\mathcal{C}$.

We shall show in the following that the above definition (2.11) gives rise to (2.10). By the definition of expectation, we have

$$\mathbb{E}_{\mathcal{C}}[f(\mathcal{C}(t))] = \sum_{\mathcal{C}'} P(\mathcal{C}'; t | \mathcal{C}; 0) f(\mathcal{C}').$$

Substitute this and (2.7) into (2.11), then (2.10) is recovered:

$$G[f](\mathcal{C}) = \lim_{t \to 0} \frac{\sum_{\mathcal{C}'} P(\mathcal{C}'; t | \mathcal{C}; 0) f(\mathcal{C}') - f(\mathcal{C})}{t} = \lim_{t \to 0} \frac{\sum_{\mathcal{C}'} \left( \delta_{\mathcal{C}C} + t \Gamma(\mathcal{C}', \mathcal{C}) + O(t^2) \right) f(\mathcal{C}') - f(\mathcal{C})}{t} = \sum_{\mathcal{C}'} \Gamma(\mathcal{C}', \mathcal{C}) f(\mathcal{C}').$$

In the following, we will give the explicit form of the ASEP generators $L$ and $M$ with respect to occupational states and position states (see Section 2.5). Compared this with the forward and backward equation we have

$$\frac{d}{dt} P(t) = G^\top \cdot P(t) = P(t) \cdot G^\top, \quad (2.12)$$

where $G$ is a matrix with elements $G(\mathcal{C}, \mathcal{C}')$.

2.5 Configuration notation

To explicitly write down the generator for ASEP, one needs to describe the lattice configuration of ASEP mathematically. Each site in the lattice are numbered by an integer $\ldots, -1, 0, 1, 2, \ldots$. In the following two notations that denote configurations are described that will be useful in the future discussion. From now on, the lattice $\Omega$ is taken to be $\mathbb{Z}$, and the total number of particles is fixed to be $N$, unless otherwise stated.

**Occupancy** A state of the system can be represented by an infinite binary string $\eta = \cdots \eta_{-1} \eta_0 \eta_1 \eta_2 \cdots \in \{0, 1\}^\mathbb{Z}$ indicating the occupancy of particles at each site. At site $i$, a particle is present if $\eta_i = 1$, otherwise $\eta_i = 0$. For example the state in Fig.2.4 is written as $\eta = 1100101$.

**Position** The lattice configuration can also be characterized by a length-$N$ vector, which gives the position of the particles $\vec{x} \in \mathbb{W}^N := \{\vec{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \cdots < x_N \}$. Namely, the position of the $i$th particle is
denoted by $x_i$. For example, the corresponding position state of Fig. 2.4 is $\vec{x} = (1, 2, 5, 7)$.

![Figure 2.4: An example configuration of ASEP.](image)

### 2.5.1 ASEP generators

First we will introduce the ASEP generator $L$ with respect to the occupational state space $S = \{0, 1\}^\mathbb{Z}$. Suppose $L$ is a generator that acts on local functions $f : \{0, 1\}^\mathbb{Z} \to \mathbb{C}$. According to (2.10), $L[f](\eta) = \sum_{\eta'} \Gamma(\eta', \eta)f(\eta')$. From the definition of $\Gamma$, the off diagonal term $\Gamma(\eta', \eta)$ is nonzero only when $\eta'$ and $\eta$ differ by one particle jump. For instance, if $\eta'$ is obtained by switching the $i$th and $i+1$th components $\eta_i, \eta_{i+1}$ of $\eta$, then $\Gamma(\eta', \eta)$ is given by

\[
\Gamma(\eta', \eta) = \begin{cases} 
p, & \eta_i > \eta_{i+1}, 
q, & \eta_i < \eta_{i+1}, 
0, & \text{otherwise}
\end{cases}
\]

Specifically, when $\eta_i = 1 > \eta_{i+1} = 0$, then the rate from state 10 to state 01 is $p$, i.e., the right jumping rate is $p$. Similarly when $\eta_i = 0 < \eta_{i+1} = 1$. By (2.4), the diagonal term $\Gamma(\eta, \eta)$ is defined such that the row sums to zero.

In conclusion, we can write down the explicit form of $L$:

\[
L[f](\eta) = \sum_{i \in \mathbb{Z}} L_i[f](\eta) = \sum_{i \in \mathbb{Z}} \sum_{\eta'} \ell_i(\eta, \eta')f(\eta'), \quad (2.13)
\]

where $L_i$ is the local generator of $L$, defined by $L = \sum_i L_i$, and the coefficients $\ell_i(\eta, \eta')$ is given by:

\[
\ell_i(\eta, \eta') = \begin{cases} 
p, & \eta_i > \eta_{i+1}, 
q, & \eta_i < \eta_{i+1}, 
0, & \text{otherwise},
\end{cases}
\]

when $\eta \neq \eta'$, and where the diagonal elements are chosen such that the rows sum to zero:

\[
\ell_i(\eta, \eta) = \begin{cases} 
-p, & \eta_i > \eta_{i+1}, 
-q, & \eta_i < \eta_{i+1}, 
0, & \text{otherwise}.
\end{cases}
\]
Other than using matrix elements, one can write the action of $L$ in a more compact form \cite{17}. From above, we only need consider $\ell_i(\eta, \eta)$ and $\ell_i(\eta, \eta')$ where $\eta'$ obtained by one step jumping at site $i$ (or $i+1$) from $\eta$. Naturally this leads to the permutation operators. Consider the adjacent transpositions (or generators) of the symmetric group $s_i$. Then $s_i$ has the following action on compositions:

$$s_i \eta = \cdots \eta_{i-1} \eta_i \eta_{i+1} \eta_{i+2} \cdots.$$ 

Using the transpositions we can now rewrite the elements in (2.13) as:

$$\ell_i(\eta, s_i \eta) = p\eta_i (1 - \eta_{i+1}) + q\eta_{i+1} (1 - \eta_i),$$

and the corresponding master equation (forward equation) reads as

$$\frac{d}{dt} P(\eta; t) = \sum_i \sum_{\eta'} \ell_i(\eta', \eta) P(\eta'; t) = \sum_i [p\eta_{i+1} (1 - \eta_i) + q\eta_i (1 - \eta_{i+1})] (P(s_i \eta; t) - P(\eta; t)).$$

One can see that $p\eta_{i+1} (1 - \eta_i) + q\eta_i (1 - \eta_{i+1})$ is the transition rate from $s_i \eta$ to $\eta$, i.e. $p$ is the transition rate from the state 10 to 01, and $q$ is that from 01 to 10, as we expected.

Note that the above actions (2.13) and (2.14) can also be expressed in terms of position state space $W^N$. However, these will not be repeated here, since in the following subsection, the generator with respect to position state will be studied in reverse ASEP process.

### 2.5.2 Reverse ASEP generators

Similarly, one can also define a reverse ASEP generator $M$ with respect to position state space $S = W^N$, but with the jumping rates exchanged, i.e., particles jump to right with rate $q$ and left with $p$. With the above knowledge, one can directly write the matrix form of $M$. Suppose $f : W^N \to \mathbb{C}$. Then $M$ is given by

$$M[f](\vec{x}) = \sum_{i \in \mathbb{Z}} M_i[f](\vec{x}) = \sum_{i \in \mathbb{Z}} \sum_{\vec{x}'} m_i(\vec{x}, \vec{x}') f(\vec{x}'),$$

where $M_i$ is the local generator of $M$, defined by $M = \sum_i M_i$, and the coefficients $m_i(\vec{x}, \vec{x}')$ are given by:

$$m_i(\vec{x}, \vec{x}') = \begin{cases} q, & x'_{j-1} + 1 < x'_j = x_j + 1 = i + 1 < x_{j+1}, \ x_k = x'_k \ \forall \ k \neq j \\ p, & x_{j-1} + 1 < i + 1 = x_j = x'_j + 1 < x'_{j+1}, \ x_k = x'_k \ \forall \ k \neq j \\ 0, & \text{otherwise}, \end{cases}$$
when $\vec{x} \neq \vec{x}'$, and where the diagonal elements are chosen such that the columns* sum to zero:

$$m_i(\vec{x}, \vec{x}) = \begin{cases} -p, & x_j + 1 = i + 1 < x_{j+1}, \\ -q, & x_{j-1} + 1 < i + 1 = x_j, \\ 0, & \text{otherwise}. \end{cases}$$

As before, the matrix form can again be rewritten into a compact form [17]. Unfortunately, in this case $\vec{x}'$ cannot be written as $s_i \vec{x}$, since $s_i \vec{x}$ is no longer an element of $\mathbb{W}^N$. However, we notice that, due to exclusion property, $\vec{x}'$ is different from $\vec{x}$ at the end of a cluster of particles. Here a cluster denotes a number of particles sitting next to each other. Let $r(\vec{x})$ be the collection of indices of the coordinates of right-end particles in each cluster and $l(\vec{x})$ the collection of left-end coordinate indices. Then the sum over $\vec{x}'$ becomes the sum over $j$ such that $j \in r(\vec{x}) \cup l(\vec{x})$. When $j \in r(\vec{x})$, then $\vec{x}'$ is obtained by the particle at $x_j$ jumping to the right. Let $\vec{x}_j^\pm := (x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_N)$. It follows that $m_i(\vec{x}, \vec{x}_j^\pm) = q$ only when $i = x_j$, otherwise $m_i(\vec{x}, \vec{x}_j^\pm) = 0$. Whereas when $j \in l(\vec{x})$, the particle can jump to the left, and $m_i(\vec{x}, \vec{x}_j^-) = p$ only when $i = x_j - 1$ otherwise $m_i(\vec{x}, \vec{x}_j^-)$ vanishes.

As one would expect, $\vec{x}_i^- := (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N)$. Consequently, we get

$$M[f](\vec{x}) = \sum_{i \in l(\vec{x})} (pf(\vec{x}_i^-) - qf(\vec{x})) + \sum_{i \in r(\vec{x})}(qf(\vec{x}_i^+) - pf(\vec{x})),$$

and the corresponding backward equation is given by

$$\frac{d}{dt}P(\vec{y}; t|\vec{x}; 0) = \sum_i \sum_{\vec{x}'} P(\vec{y}; t|\vec{x}'; 0)m_i(\vec{x}', \vec{x})$$

$$= \sum_{i \in l(\vec{x})}[qP(\vec{y}; t|\vec{x}_i^-; 0) - pP(\vec{y}; t|\vec{x}; 0)] + \sum_{i \in r(\vec{x})}[pP(\vec{y}; t|\vec{x}_i^+; 0) - qP(\vec{y}; t|\vec{x}; 0)],$$

which defines a reverse process with jumping rates $p$ and $q$ exchanged. Rewriting this into forward equation, we can see that it represents the same process as $L$.

## 2.6 ASEP and duality

Duality is an important approach to the study of stochastic processes, especially interacting particle systems. With the above two generators (2.14) and (2.16), we are now able to demonstrate a self duality for ASEP. In 1997, the ASEP duality was first found by Schütz [143], derived from an underlying $U_q(\mathfrak{sl}_2)$ symmetry of ASEP. This section is devoted to review the self duality found for single species

*Such diagonal term is chosen because $M_i$ acts as a backward generator instead of forward.
ASEP [17, 143], including a brief introduction to the duality function for a general Markov process.

2.6.1 Duality

The following discussion is given in terms of Markov processes and is not limited to ASEP. One can find a detailed introduction in [73, 87].

Definition 2.5. Let \( x(t) \) and \( y(t) \) be independent Markov processes with respective state spaces \( X, Y \). Let furthermore \( \psi : X \times Y \to \mathbb{C} \) be a bounded and measurable function. Then, the processes \( x(t) \) and \( y(t) \) are said to be dual with respect to \( \psi \) if

\[
E_x[\psi(x(t), y)] = E_y[\psi(x, y(t))], \tag{2.17}
\]

holds for all \( x \in X, \ y \in Y \) and \( t \geq 0 \).

To construct a duality function for Markov processes using this definition, one will inevitably encounter the transition probability, and hence is required to solve the master equation. To avoid such trouble, we give an equivalent definition for duality in terms of Markov generators. Using the master equation (2.6) and the relation (2.10) between generators and transition rates, we are able to obtain the following proposition.

Proposition 2.6. Let \( x(t) \) and \( y(t) \) be independent Markov processes with respective state spaces \( X, Y \), and infinitesimal generators \( G \) and \( H \). Let furthermore \( \psi : X \times Y \to \mathbb{C} \) be a bounded and measurable function. Then, \( x(t) \) and \( y(t) \) are dual with respect to \( \psi \) if and only if the two generators give the same result on \( \psi(x, y) \), i.e.,

\[
G[\psi(\cdot, y)](x) = H[\psi(x, \cdot)](y), \tag{2.18}
\]

holds for all \( x \in X, \ y \in Y \).

Proof. Let \( P_1(t), P_2(t) \) be the transition probability (see (2.6)) with respect to \( x(t), y(t) \). Similar to showing the relation between generators and transition rates (2.10), one needs to translate the transition probability into the transition rate, and hence in terms of \( G \) and \( H \). The idea is similar except that we consider the matrix \( P(t) \) instead of the vector \( |P(t)\rangle \). From (2.6) and (2.10), we have

\[
P_1(t) = e^{tG^T}P_1(0) = e^{tG^T}, \quad P_2(t) = e^{tH^T}P_2(0) = e^{tH^T}.
\]

where \( P_1(0) \) and \( P_2(0) \) are in fact the identity matrices since their components are kronecker deltas. In this case, it would be easier to consider \( \Psi \) as a matrix with elements \( \psi(x, y) \), i.e., \( \Psi = \sum_{x,y} \psi(x, y) |x\rangle\langle y| \). It follows that

\[
G[\psi(\cdot, y)](x) = \sum_{x'} G(x, x') \psi(x', y) = \sum_{x'} \langle x|G|x'\rangle \langle x'|\Psi|y \rangle = \langle x|G\Psi|y \rangle, \tag{2.19}
\]

holds for all \( x \in X, \ y \in Y \).
\[ H[\psi(x,\cdot)](y) = \sum_{y'} H(y,y') \psi(x,y') = \sum_{y'} \langle x|\Psi|y' \rangle \langle y'|H^\top|y \rangle = \langle x|\Psi H^\top|y \rangle. \quad (2.20) \]

for all \( x \in X, y \in Y \). Thus if (2.18) holds, then \( G\Psi = \Psi H^\top \). We obtain

\[
E_x[\psi(x(t), y)] = \sum_{x'} P_1(x'; t | x; 0) \psi(x', y) = \sum_{x'} \langle x'|P_1(t)|x \rangle \langle x'|\Psi|y \rangle = \sum_{y'} \langle x|e^{tG}|x' \rangle \langle x'|\Psi|y \rangle = \langle x|\Psi e^{tH^\top}|y \rangle = \sum_{y'} \langle x|\Psi|y \rangle \langle y'|P_2(t)|y \rangle = E_y[\psi(x, y(t))], \quad (2.21)
\]

as required. Conversely, if (2.17) is true, then from (2.21), \( e^{tG}\Psi = \Psi e^{tH^\top} \) for all \( t \geq 0 \):

\[
E_x[\psi(x(t), y)] = \sum_{x'} P_1(x'; t | x; 0) \psi(x', y) = \sum_{x'} \langle x'|P_1(t)|x \rangle \langle x'|\Psi|y \rangle = \sum_{y'} \langle x|e^{tG}|x' \rangle \langle x'|\Psi|y \rangle = \langle x|\Psi e^{tH^\top}|y \rangle = \sum_{y'} \langle x|\Psi|y \rangle \langle y'|e^{tH^\top}|y \rangle = \langle x|\Psi e^{tH^\top}|y \rangle.
\]

Hence one can conclude that \( G\Psi = \Psi H^\top \). Then by (2.19) and (2.20), one can recover (2.18). Thus (2.17) is equivalent to (2.18). \( \square \)

**Remark 2.7.** Recall the ASEP and reverse ASEP generators in terms of (2.13) and (2.15). We notice that the global generators \( L \) and \( M \) are given in terms of a sum over local generators \( L_i \) and \( M_i \). Correspondingly, one can also have a local duality, which is defined by local generators in (2.18), and global duality by global generators. Clearly, a local duality implies a global one. However a global duality does not always imply a local one.

### 2.6.2 Self duality in ASEP

We are now ready to introduce the self duality function in ASEP. Instead of using the spin chain representation of ASEP ([143]), we work on the ASEP as a Markov chain. Following [17], we will show the ASEP duality in terms of infinitesimal generators (2.14) and (2.16). Then we will finish by proving (2.18).

The self duality of ASEP is stated in the following theorem.

**Theorem 2.8.** Consider ASEP defined by (2.14) and a reverse ASEP defined by (2.16). Recall the assumption \( p < q \) that particles are drifting to the left. Then
these two processes are dual with respect to the function

$$\psi(\eta, \vec{x}) = \prod_{i=1}^{N} \prod_{j \leq x_i} \tau^{\eta_j},$$  

(2.22)

where $\tau = p/q < 1$. Namely, the function $\psi(\eta, \vec{x})$ satisfies

$$\mathbb{E}_{\eta}[\psi(\eta(t), \vec{x})] = \mathbb{E}_{\vec{x}}[\psi(\eta, \vec{x}(t))].$$  

(2.23)

**Proof.** By Proposition 2.6, the proof of (2.23) is equivalent to proving

$$L[\psi(\cdot, \vec{x})](\eta) = M[\psi(\eta, \cdot)](\vec{x}),$$  

(2.24)

which makes the proof a straightforward calculation. First, let us consider the case that there is only one cluster in $\vec{x}$, i.e., suppose $\vec{x} = (x + 1, \ldots, x + l)$. Recall that the left hand side of (2.24) is given by (2.14). We observe that $\psi(s_i \eta, y) = \psi(\eta, y)$ if $i \neq y$. Because when $i \neq y$ the number of particles in $s_i \eta$ before position $y + 1$ is the same as the one in $\eta$. Thus by definition of $\psi(\eta, \vec{x})$, one can see that the left hand side of (2.24) reduces to

$$LHS = \sum_{i=1}^{l} [p\eta_{x+i}(1 - \eta_{x+i+1}) + q\eta_{x+i+1}(1 - \eta_{x+i})](\psi(s_{x+i}\eta, \vec{x}) - \psi(\eta, \vec{x})).$$  

(2.25)

Then we need to evaluate $\psi(s_{x+i}\eta, \vec{x})$:

$$\psi(s_{x+i}\eta, \vec{x}) = \prod_{j=1}^{l} \psi(\eta, x + j)\psi(\eta, x + i - 1)\tau^{\eta_{x+i+1}} = \psi(\eta, \vec{x})\frac{\tau^{\eta_{x+i+1}}}{\tau^{\eta_{x+i}}},$$

where $\psi(\eta, x) := \prod_{j \leq x} \tau^{\eta_j}$. Substituting this into (2.25), we obtain

$$LHS = \sum_{i=1}^{l} \psi(\eta, \vec{x})[p\eta_{x+i}(1 - \eta_{x+i+1}) + q\eta_{x+i+1}(1 - \eta_{x+i})](\tau^{\eta_{x+i+1}} - \eta_{x+i} - 1)$$

$$= \sum_{i=1}^{l} \psi(\eta, \vec{x})(p\tau^{-\eta_{x+i}} + q\tau^{\eta_{x+i+1}} - 1),$$

where the last line can be checked for the four possible values of the pair $(\eta_{x+i}, \eta_{x+i+1})$. To further simplify the above expression, one may use the fact that for $\eta \in \{0, 1\}$, $p\tau^{-\eta} + q\tau^{\eta} - 1 = 0$. Then all but the first and last term of the last line cancel with each other, leaving us with only

$$LHS = \psi(\eta, \vec{x})(p\tau^{-\eta_{x+1}} + q\tau^{\eta_{x+i+1}} - 1).$$

Now recall the definition of $M$ given in (2.16). We see that $l(\vec{x}) = 1$ and $r(\vec{x}) = l$. 
therefore, the right hand side of (2.24) is given by
\[
\text{RHS} = p\psi(\eta, \bar{x}_-^1) - q\psi(\eta, \bar{x}) + q\psi(\eta, \bar{x}_+^1) - p\psi(\eta, \bar{x}) \\
= p \prod_{i=2}^l \psi(\eta, x + i)\psi(\eta, x) + q \prod_{i=1}^{l-1} \psi(\eta, x + i)\psi(\eta, x + l + 1) - \psi(\eta, \bar{x}) \\
= \psi(\eta, \bar{x})(p\tau^{-\eta_{x+1}} + q\tau^{\eta_{x+l+1}} - 1),
\]
which agrees with the left hand side as required.

For general \(\bar{x} \in \mathbb{W}^N\), the function \(\psi(\eta, \bar{x})\) can be factorised into clusters. Hence \(M\) and \(L\) act on each cluster separately. Then by applying the result for a single cluster, one can conclude that (2.24) holds for general \(\bar{x}\), and hence complete the proof.

The above theorem gives the global duality for ASEP. A similar result can be stated for the local generator \(L_i\) and \(M_i\) given in (2.13) and (2.15). Namely, one can obtain a local duality for ASEP.

**Theorem 2.9.** Consider ASEP defined by (2.13) and a reverse ASEP defined by (2.15). Recall the assumption \(p < q\) that particles drift to the left. Then these two processes are locally dual with respect to the function
\[
\phi(\eta, \bar{x}) = \prod_{i=1}^N \prod_{j < x_i} \tau^{h_j} \eta_{x_i},
\]
where \(\tau = p/q < 1\). Namely, for \(\forall i \in \mathbb{Z}\), the function \(\psi(\eta, \bar{x})\) satisfies
\[
L_i[\phi(\cdot, \bar{x})](\eta) = M_i[\phi(\eta, \cdot)](\bar{x}).
\]

**Proof.** Similar to the previous proof, we first consider a single cluster \(\bar{x} = (x + 1, \ldots, x + l)\). By the definition of \(M_i\), one can easily see that the right hand side of (2.26) vanishes unless \(i = x, x + l\).

\[
M_x\phi(\eta, \cdot)(\bar{x}) = p\phi(\eta, \bar{x}^-_i) - q\phi(\eta, \bar{x}) = \prod_{i=2}^l \phi(\eta, x + i)\psi(\eta, x - 1)(p\eta_x - q\tau^{\eta_x} \eta_{x+1}),
\]
\[
M_{x+l}\phi(\eta, \cdot)(\bar{x}) = q\phi(\eta, \bar{x}^+_i) - p\phi(\eta, \bar{x}) = \prod_{i=1}^{l-1} \phi(\eta, x + i)\psi(\eta, x + l - 1)
\times (q\tau^{h_{x+l}} \eta_{x+l+1} - p\eta_{x+l}),
\]
where \(\psi(\eta, x)\) is defined in (2.22).

On the other hand, by the fact that \(L_i[\phi(\cdot, \bar{x})](\eta) = \sum_{\eta' \neq \eta} \ell_i(\eta, \eta')[\phi(\eta', \bar{x}) - \phi(\eta, \bar{x})]\) and (2.13), one can observe that \(L_i[\phi(\cdot, \bar{x})](\eta) = 0\) unless \(\eta_i \neq \eta_{i+1}\) and \(\eta_i' = s_i \eta\). In addition, \(\phi(s_i \eta, x) = \phi(\eta, x)\) if \(i \notin \{x - 1, x\}\). Thus one may conclude that the left hand side of (2.26) is nonzero only when \(i \in \{x, x + 1, \ldots, x + l\}\).
Consider the case \( L_{x+i}[\phi(\cdot, \vec{x})](\eta) \) when \( 0 < i < l \)

\[
\phi(s_{x+i}(\eta, \vec{x})) - \phi(\eta, \vec{x}) = \sum_{j=1}^{l} \phi(\eta, x+j)\psi^2(\eta, x+i-1)\eta_{x+i}\eta_{x+i+1}[t_{x+i+1} - t_{x+i}],
\]

Recall that \( L_{x+i}[\phi(\cdot, \vec{x})](\eta) \) vanishes unless \( \eta_{x+i} \neq \eta_{x+i+1} \), i.e., \( \eta_{x+i}\eta_{x+i+1} = 0 \). Thus in this case \( \phi(s_{x+i}(\eta, \vec{x})) - \phi(\eta, \vec{x}) = 0 \). In conclusion, when \( 0 < i < l \), \( L_{x+i}[\phi(\cdot, \vec{x})](\eta) = 0 \), which is consistent with the result of \( M_{x+i}[\phi(\cdot, \vec{x})](\eta) \). Therefore, we only need to check two cases \( L_x \) and \( L_{x+i} \).

\[
L_x \phi(\eta, \cdot)(\vec{x}) = |p\eta_x(1 - \eta_{x+1}) + q\eta_{x+1}(1 - \eta_x)|\phi(s_x\eta, \vec{x}) - \phi(\eta, \vec{x})
\]

\[
= |p\eta_x(1 - \eta_{x+1}) + q\eta_{x+1}(1 - \eta_x)|\prod_{i=2}^{l} \phi(\eta, x+i)\psi(\eta, x - 1)
\]

\[
L_{x+i} \phi(\eta, \cdot)(\vec{x}) = |p\eta_{x+i}(1 - \eta_{x+i+1}) + q\eta_{x+i+1}(1 - \eta_{x+i})|\phi(s_{x+i}\eta, \vec{x}) - \phi(\eta, \vec{x})
\]

\[
= |p\eta_{x+i}(1 - \eta_{x+i+1}) + q\eta_{x+i+1}(1 - \eta_{x+i})|\prod_{i=1}^{l} \phi(\eta, x+i)\times
\]

\[
\psi(\eta, x + l - 1)(\eta_{x+i+1} - \eta_{x+i}).
\]

Now the fact that

\[
(p\eta_x - q\eta_{x+1}) = |p\eta_x(1 - \eta_{x+1}) + q\eta_{x+1}(1 - \eta_x)|(t_{x+1} - t_x),
\]

\[
(q\eta_{x+i} - \eta_{x+i+1}) = |p\eta_{x+i+1}(1 - \eta_{x+i+1}) + q\eta_{x+i+1}(1 - \eta_{x+i})|(\eta_{x+i+1} - \eta_{x+i}),
\]

can be checked by exhausting all the 4 possible values of \((\eta_x, \eta_{x+1})\) and \((\eta_{x+i}, \eta_{x+i+1})\). From this, one can completes the proof for a single cluster.

For general \( \vec{x} \), like before, the function \( \psi(\eta, \vec{x}) \) factors into single clusters and hence the result for a single cluster can be applied separately.

Duality is a powerful tool for in calculating expectation values and correlation functions. As we have seen, duality is a covariant function with respect to two processes. With a self-duality, one can easily relate the dynamics of a many-particle system to that of a few-particles system, or even a single particle system. Therefore, the observable of a many particle process can be analysed and calculated via the dual, simpler process. For example, [84] provides the moments formula for ASEP via its self duality, and [17] uses this result to reproduce the well known Tracy-Widom fluctuation limiting theorem for ASEP.

Dualities were constructed manually by educated guessing in early papers, see e.g., [17, 143]. Recently in [105, 107], the underlying quantum group symmetry was used to derive self duality in ASEP in a natural way. In Chapter 2, we will introduce a new framework to derive the ASEP duality systematically.

In the rest of this chapter, we will focus on other useful tools in the study of
ASEP. For the purpose of discussion, we may consider ASEP with other boundary conditions, apart from $\mathbb{Z}$.

### 2.7 Bethe ansatz

An important technique in the study of many-body models is the Bethe ansatz. This method originates from the paper [13] by Hans Bethe published in 1931 (see reviews e.g. [8, 152]). It suggests an ansatz solution for the spectrum of the Hamiltonian of the Heisenberg spin chain, and has since been applied to many other quantum mechanical and statistical mechanics systems [113, 164]. In fact, the ASEP generator can be mapped into the Hamiltonian of the XXZ spin chain, via a similarity transformation [75, 100], and so the Bethe ansatz can be used to solve the master equation of ASEP. The application of Bethe ansatz to ASEP has been further developed, see e.g. [49–51, 144], to write down the explicit Green’s function for the TASEP, as well as to diagonalise the Markov generator for the ASEP with open boundaries.

From the master equation (2.5) and the ASEP generators (2.16), we can write down the master equation in terms of position state

\[
\frac{d}{dt}P(\vec{x};t) = \sum_{i=1}^{N} pP(\vec{x}^{-}_i; t) + qP(\vec{x}^{+}_i; t) - NP(\vec{x}; t) \tag{2.27}
\]

in the cases where $\vec{x}$ satisfies $x_{i+1} > x_i + 1$ for any $i$. To solve this equation, one can consider the corresponding eigenvalue problem of diagonalising the transition rate matrix. The form of (2.27) suggests a solution in terms of plane waves. This is the key idea behind “coordinate” Bethe ansatz*. In this section, we will first demonstrate the basic steps of coordinate Bethe ansatz in the case of periodic ASEP, which is one the simplest systems for Bethe ansatz. This will be followed by an exact solution of the master equation of ASEP on $\mathbb{Z}$ via Bethe ansatz.

#### 2.7.1 Periodic condition

Consider the ASEP on a ring of length $L$ with $N$ particles. We will basically follow [119] to show how the Bethe ansatz is used in diagonalising the transition rate matrix. We start by solving the corresponding eigenvalue problem for the special cases $N = 1, 2, 3$. We denote the eigenfunction and its corresponding eigenvalue by $f_i$ and $E_i$ respectively.

**Single particle** First we start will $N = 1$, and the eigenvalue problem reads as

\[
Ef(x) = pf(x - 1) + qf(x + 1) - f(x).
\]

*Bethe ansatz can be formalised algebraically to retrieve the eigenfunction, i.e. “Algebraic” bethe ansatz (see Remarks 2.11).
By substitution of the ansatz $f(x) = zw$ we obtain the eigenvalue $E = pz^{-1} + qz - 1$. The periodic boundary condition requires that $x + L$ is identified as $x$, namely $f(x + L) = f(x)$. Again the substitution of the ansatz gives $z^{L} = 1$. It follows that $z$ is root of unity $z = e^{2k\pi i/L}$ where $k = 0, \ldots, L - 1$. Such a solution of $z$ is called Bethe root. In conclusion, the eigenfunctions simply are plane waves with momentum $2k\pi/L$:

$$f(x) = Ae^{\frac{2k\pi}{L}x}$$

and the eigenvalues are

$$E_k = pz^{-1} + qz - 1, \quad z_k = e^{2k\pi i/L}.$$

**Two particles** Suppose $\vec{x} = (x_1, x_2)$. Due to the interaction we need to split the problem into the two cases $x_2 > x_1 + 1$ and $x_2 = x_1 + 1$. In the case $x_2 > x_1 + 1$, there is nothing new in the eigenvalue problem

$$Ef(x_1, x_2) = p[f(x_1 - 1, x_2) + f(x_1, x_2 - 1)] + q[f(x_1 + 1, x_2) + f(x_1, x_2 + 1)] - 2f(x_1, x_2). \quad (2.28)$$

This intuitively suggests the ansatz $f(x_1, x_2) = z_{1}^{x_1}z_{2}^{x_2}$ and substitution gives the eigenvalue $E = p(z_1^{-1} + z_2^{-2}) + q(z_1 + z_2) - 2$. Since $E$ is symmetric, there exists another eigenfunction $f(x_1, x_2) = z_{1}^{x_2}z_{2}^{x_1}$ with the same eigenvalue $E$. So the most general ansatz is of the form

$$f(x_1, x_2) = A_{12}z_{1}^{x_1}z_{2}^{x_2} + A_{21}z_{2}^{x_1}z_{1}^{x_2}.$$  

Now consider the case when $x_2 = x_1 + 1$:

$$Ef(x_1, x_1 + 1) = pf(x_1 - 1, x_1 + 1) + qf(x_1, x_2 + 1) - f(x_1, x_1 + 1). \quad (2.29)$$

We notice that the above ansatz $f(x_1, x_2) = A_{12}z_{1}^{x_1}z_{2}^{x_2} + A_{21}z_{2}^{x_1}z_{1}^{x_2}$ satisfies (2.28) without the restriction $x_2 > x_1 + 1$. Hence this suggests that, instead of solving (2.29) directly, we can investigate the difference between (2.28) and (2.29). Compare this with the general case when $x_2 = x_1 + 1$ and we notice that there are three terms missing from the general case: $pf(x_1, x_1), qf(x_1 + 1, x_1 + 1)$ and $-f(x_1, x_1 + 1)$. Hence if we impose an extra boundary condition:

$$pf(x_1, x_1) + qf(x_1 + 1, x_1 + 1) = f(x_1, x_1 + 1),$$

then the special case (2.29) is automatically satisfied. Substituting the ansatz $f(x_1, x_2) = A_{12}z_{1}^{x_1}z_{2}^{x_2} + A_{21}z_{2}^{x_1}z_{1}^{x_2}$ into this extra boundary condition, we obtain the relation between the amplitude $A_{12}$ and $A_{21}$:

$$A_{12} = \frac{p + qz_1z_2 - z_1}{p + qz_1z_2 - z_2}. \quad (2.30)$$
This ratio between the two amplitudes, is called scattering relation.

Finally, there is only one equation left to be satisfied: the periodic condition. Being aware that \( f(x_1, x_2) \) is always defined on \( \mathbb{W}^2 = \{ \vec{x} = (x_1, x_2) \in \mathbb{Z}^2 : x_1 < x_2 \} \), the periodic condition reads as \( f(x_1, x_2) = f(x_2, x_1 + L) \) for any \( x_1 - x_2 < L \). Substituting the ansatz gives

\[
L^{x_1} z^{x_2} (A_{12} - A_{21} z^L_1) + z^{x_1} L^{x_2} (A_{21} - A_{12} z^L_2) = 0.
\]

We observe that each term above should vanish to ensure the whole equation holds for any \( x_1 - x_2 < L \). The resulting equation is thus

\[
\frac{A_{12}}{A_{21}} = z^L_1 = z^{-L}_2.
\]

With the scattering relation (2.30), we obtain the so called Bethe equation:

\[
z_2^{-L} = z_1^L = \frac{p + qz_1 z_2 - z_1}{p + qz_1 z_2 - z_2}.
\]

We can see that in the two-particles case, diagonalising a matrix of size \( L^2 \) turns into solving a system of two equations of order \( L \). Although the latter one still requires some effort to solve, it is more explicit and straight forward. This is the advantage of the Bethe ansatz.

**Three particles** When \( N = 3 \), the eigenvalue problem is similar to the case \( N = 2 \) in the following sense.

Suppose \( x_1 + 2 < x_2 + 1 < x_3 \), the eigenfunction equation reads as

\[
Ef(\vec{x}) = \sum_{i=1}^{3} \left[ pf(\vec{x}_i^-) + qf(\vec{x}_i^+) \right] - 3f(\vec{x}).
\]

As before, this suggests an ansatz \( f(\vec{x}) = A z_1^{x_1} z_2^{x_2} z_3^{x_3} \) with eigenvalue

\[
E = \sum_{i=1}^{3} (pz_i^{-1} + qz_i - 1).
\] (2.31)

Once more, the eigenvalue is symmetric, which implies that any permutation of \( z_1, z_2, z_3 \) still gives a eigenfunction. Hence the corrected ansatz is the sum of all the permutations:

\[
f(\vec{x}) = \sum_{\pi \in S_3} A_x \prod_{i=1}^{3} z_{\pi_i}^{x_i},
\] (2.32)

with eigenvalue given by (2.31).

Now suppose \( x_1 + 1 = x_2 < x_3 - 1 \) and \( x_1 + 2 < x_2 + 1 = x_3 \), the two corresponding
eigenfunction equations are given by

\[
Ef(x_1, x_1 + 1, x_3) = p[f(x_1 - 1, x_1 + 1, x_3) + f(x_1, x_1 + 1, x_3 - 1)] +
q[f(x_1, x_1 + 2, x_3) + f(x_1, x_1 + 1, x_3)] - 2f(x_1, x_1 + 1, x_3),
\]

\[
Ef(x_1, x_2, x_2 + 1) = p[f(x_1 - 1, x_2, x_2 + 1) + f(x_1, x_2 - 1, x_2 + 1)] +
q[f(x_1 + 1, x_2, x_2 + 1) + f(x_1, x_2, x_2 + 2)] - 2f(x_1, x_2, x_2 + 1),
\]

respectively. We can see that these two equations are very similar to (2.29), since the third particle (which is located at \(x_3\) for the first case and \(x_1\) for the second) behaves as an outsider observer which has no effect on the other two particles. Similarly, these two equations can be solved via including the following two extra boundary conditions

\[
pf(x_1, x_1, x_3) + qf(x_1 + 1, x_1 + 1, x_3) = f(x_1, x_1 + 1, x_3),
\]

\[
pf(x_1, x_2, x_2) + qf(x_1, x_2 + 1, x_2 + 1) = f(x_1, x_2 + 1, x_2 + 1).
\] (2.33)

By observation, we can rewrite the above equations in a more compact form: for \(\forall i \in \{1, 2\}\),

\[
pf(\ldots, x_i, x_i, \ldots) + qf(\ldots, x_i + 1, x_i + 1, \ldots) = f(\ldots, x_i, x_i + 1, \ldots).
\] (2.34)

However, apart from the above equations, \(f(\vec{x})\) still needs to fulfill the third case when \((x_1, x_2, x_3) = (x, x + 1, x + 2)\):

\[
Ef(x, x + 1, x + 2) = pf(x - 1, x + 1, x + 2) + qf(x, x + 1, x + 3) - f(x, x + 1, x + 2),
\]

which can be transformed into

\[
pf(x, x + 2) + pf(x, x + 1, x + 1) + qf(x + 1, x + 1, x + 2) + qf(x, x + 2, x + 2) = 2f(x, x + 1, x + 2).
\] (2.35)

One can observe that (2.35) indeed is of the same form of (2.34). In fact, (2.35) can be obtained by adding the two equations in (2.33), and setting \(x_1 = x, x_2 = x + 1, x_3 = x + 2\). Hence (2.34) is the only extra boundary conditions that \(f(\vec{x})\) needs to satisfy.

**Remark 2.10.** The property that the three-particle exclusion equation (3-body collisions) can be reduced (or factorised) into two-particle equation (2-body collisions) is usually called two-body reducible. It leads to the important: Yang-Baxter equation, which is the defining property of integrability. Hence one can see that ASEP is indeed integrable.

Two-body reducibility is a typical condition for Bethe ansatz. As a result, the models that can be solved via Bethe ansatz are integrable. Hence people usually refer “Bethe solvable” to “integrable”. However, integrable systems are not always Bethe ansatz solvable.
Now we are ready to solve for the amplitudes and momenta in the ansatz. Substituting the ansatz (2.32) into (2.34) gives

\[
\sum_{\pi \in S_3} A_\pi \frac{x_1 - x_2}{x_3} (p + q z_{\pi_1} z_{\pi_2} - z_{\pi_3}) = 0,
\]

\[
\sum_{\pi \in S_3} A_\pi \frac{x_1 - x_2}{x_3} (p + q z_{\pi_2} z_{\pi_3} - z_{\pi_1}) = 0.
\]

Since the above equation holds for \(\forall x_1, x_2, x_3 \in \mathbb{W}^3 = \{ x = (x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1 < x_2 < x_3 \} \), the coefficient for each term \(x_1 x_2 x_3\) and \(x_1 x_2 x_3\) must vanish.

Let’s consider the first equation. Note that \(x_1 x_2 x_3 = x_1 x_2 x_3\) where \(s_i\) is the generator of symmetric group that swap \(i\) and \(i+1\). Hence grouping each term \(x_1 x_2 x_3\) in the first equation, the coefficient of each term should vanish:

\[A_\pi (p + q z_{\pi_1} z_{\pi_2} - z_{\pi_3}) + A_{\pi s_1} (p + q z_{\pi_1} z_{\pi_2} - z_{\pi_1}) = 0.\]

Writing the above equations explicitly, we obtain 3 ratios between the amplitude:

\[\frac{A_{123}}{A_{213}} = \frac{p + q z_1 z_2 - z_1}{p + q z_1 z_2 - z_2}, \quad \frac{A_{132}}{A_{312}} = \frac{p + q z_1 z_3 - z_1}{p + q z_1 z_3 - z_3}, \quad \frac{A_{231}}{A_{321}} = \frac{p + q z_2 z_3 - z_2}{p + q z_2 z_3 - z_3}.\]

Repeat these steps for the second equation and we arrive at 3 more scattering relations,

\[\frac{A_{123}}{A_{132}} = \frac{p + q z_2 z_3 - z_2}{p + q z_2 z_3 - z_3}, \quad \frac{A_{213}}{A_{231}} = \frac{p + q z_1 z_3 - z_1}{p + q z_1 z_3 - z_3}, \quad \frac{A_{312}}{A_{321}} = \frac{p + q z_1 z_2 - z_1}{p + q z_1 z_2 - z_2}.\]

Naively, one expects that all six amplitudes are totally determined. However, the last relation is redundant, since it can be determined through the first five ratios.

\[\frac{A_{312}}{A_{321}} = \frac{A_{231} A_{213} A_{123} A_{132} A_{312}}{A_{321} A_{231} A_{213} A_{123} A_{132}}.\]

Therefore all the six amplitudes are uniquely determined up to an overall constant.

Now we are left with the periodic condition, which is stated as \(f(x_1, x_2, x_3) = f(x_2, x_3, x_1 + L)\). Substituting the ansatz (2.32) into this equation gives

\[
\sum_{\pi \in S_3} A_\pi \frac{x_1 - x_2}{x_3} x_3 + x_1 + L = 0.
\]

As before, \(x_1, x_2, x_3\) is arbitrarily chosen in \(\mathbb{W}^3\), and the coefficients of each monomial \(x_1 x_2 x_3\) must vanish. By grouping each term we arrive at \(A_{123} - A_{231} z_1^L = A_{132} - A_{321} z_1^L = A_{213} - A_{312} z_2^L = A_{231} - A_{321} z_2^L = A_{231} - A_{321} z_3^L = 0.\)

Using the scattering ratios above, and rearranging these six equations, we find

\[z_1^L = \frac{A_{123}}{A_{231}} = \frac{A_{123} A_{213}}{A_{213} A_{231}} = \frac{A_{123}}{A_{231}} = \frac{p + q z_1 z_2 - z_1 p + q z_1 z_3 - z_1}{p + q z_1 z_2 - z_2 p + q z_1 z_3 - z_3}.\]
Asymmetric Simple Exclusion Process

\[ z_2^L = \frac{A_{213}}{A_{132}} = \frac{A_{213}A_{123}}{A_{132}A_{123}} = \frac{p + qz_1z_2 - z_2 (p + qz_2z_3 - z_2)}{p + qz_1z_2 - z_1 (p + qz_2z_3 - z_3)}, \]

\[ z_3^L = \frac{A_{312}}{A_{132}} = \frac{A_{312}A_{123}}{A_{132}A_{123}} = \frac{p + qz_1z_3 - z_3 (p + qz_2z_3 - z_3)}{p + qz_1z_3 - z_1 (p + qz_2z_3 - z_2)}, \]

which are the Bethe equations for three particles case.

**General case** With what we have learned so far, we now can solve the general case when there are \( N \) particles. In fact, this case is essentially the same as the three-particle case.

Suppose \( x_{i+1} < x_{i+1} \) for any \( i \in \{1, \ldots, N\} \), the eigenvalue equation reads as

\[ Ef(x) = \sum_{i=1}^{N} [pf(x_i^-) + qf(x_i^+)] - Nf(x), \]

where \( x_i^\pm := (x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N) \). As before, this suggests an ansatz

\[ f(x) = \sum_{\pi \in S_N} A_{\pi} \prod_{i=1}^{N} z_{x_i}^{x_i}, \tag{2.36} \]

with symmetric eigenvalue

\[ E = \sum_{i=1}^{N} (pz_i^{-1} + qz_i - 1). \]

Now suppose \( x_{i+1} = x_{i+1} \), as we have seen before, the corresponding eigenvalue equation can be modified into an extra boundary condition:

\[ pf(x_1, \ldots, x_{i-1}, x_i, x_i, x_{i+2}, \ldots, x_N) + qf(x_1, \ldots, x_{i-1}, x_i + 1, x_i + 1, x_{i+2}, \ldots, x_N) - f(x_1, \ldots, x_{i-1}, x_i + 1, x_i + 1, x_{i+2}, \ldots, x_N) = 0. \]

Substituting the ansatz (2.36), we obtain

\[ \sum_{\pi \in S_N} A_{\pi} \prod_{k=1}^{N} z_{x_k}^{x_k} z_{x_i}^{x_i} (p + qz_{\pi_i}z_{\pi_{i+1}} - z_{\pi_{i+1}}) = 0. \]

Notice that \( \prod_{k=1}^{N} z_{x_k}^{x_k} z_{x_i}^{x_i} = \prod_{k \neq i+1}^{N} z_{x_k}^{x_k} z_{x_i}^{x_i} \) when \( \pi' = \pi s_i \). Then grouping each monomial of \( z \)'s and equating its coefficient to zero results in the scattering relations

\[ \frac{A_{\pi}}{A_{\pi s_i}} = -\frac{p + qz_{\pi_i}z_{\pi_{i+1}} - z_{\pi_{i+1}}}{p + qz_{\pi_i}z_{\pi_{i+1}} - z_{\pi_{i+1}}}. \tag{2.37} \]

Finally substituting the ansatz (2.36) into the periodic condition:
§2.7 Bethe ansatz

\[ f(x_1, \ldots, x_N) = f(x_2, \ldots, x_N, x_1 + L) \]
gives

\[ \sum_{\pi \in S_N} A_\pi \left( \prod_{k=1}^{N} z_{\pi_k}^{x_k} - \prod_{k=1}^{N-1} z_{\pi_k}^{x_{k+1}} z_{\pi_1}^{x_1 + L} \right) = 0. \]

In this case \( \prod_{k=1}^{N} z_{\pi_k}^{x_k} = \prod_{k=1}^{N-1} z_{\pi_k}^{x_{k+1}} z_{\pi_1}^{x_1} \) when \( \pi' = \pi s_1 s_2 \ldots s_{N-1} \). Therefore extracting the coefficient for each monomial leads to

\[ z_L^{\pi_1} = \frac{A_\pi}{A_{\pi s_1 s_2 \ldots s_{N-1}}} = \frac{A_\pi}{A_{\pi s_1}} \frac{A_{\pi s_1}}{A_{\pi s_1 s_2}} \ldots \frac{A_{\pi s_1 s_{N-2}}}{A_{\pi s_1 s_{N-2} s_{N-1}}} \]

\[ = (-1)^{N-1} \frac{p + q z_{\pi_1} z_{\pi_2} - z_{\pi_1} p + q z_{\pi_1} z_{\pi_3} - z_{\pi_1} p + q z_{\pi_1} z_{\pi_3} - z_{\pi_1} p + q z_{\pi_1} z_{\pi_N} - z_{\pi_1}}{p + q z_{\pi_1} z_{\pi_2} - z_{\pi_2} p + q z_{\pi_1} z_{\pi_3} - z_{\pi_3} p + q z_{\pi_1} z_{\pi_N} - z_{\pi_N}}, \]

where the last equation is obtained by the scattering relations. If we set \( \pi_1 = i \), then the Bethe equations are given as

\[ z_L^{i} = (-1)^{N-1} \prod_{k \neq i} \frac{p + q z_{i} z_{k} - z_{i}}{p + q z_{i} z_{k} - z_{k}}, \quad i = 1, \ldots, N. \]

Therefore, Bethe ansatz transforms the spectrum of an ASEP on a ring of length \( L \), into \( N \) coupled equations of order \( L \). Although these equations are still difficult to solve, it is much more straightforward for large \( N \) and \( L \) than diagonalising the transition rate matrix by brute force.

When \( p = q = 1 \) the Bethe equations are the same as the ones derived by H. Bethe [13], which implies that a SSEP can be identified as the isotropic Heisenberg spin chain.

For an ASEP with open boundary (see Fig.2.2(a)), the Bethe equation have been first derived in [49], based on the works of [30, 126]. It was then further studied in [44, 145] via coordinate Bethe ansatz. The complete spectrum were discussed in [50], and the complete eigenvectors were considered in [45]. These results were then used in analysing the current statistics in [52].

**Remark 2.11.** The Bethe equation can be derived in an algebraic framework *algebraic Bethe ansatz*. The algebraic Bethe ansatz originated from the quantum inverse scattering method, which was proposed by the Leningrad School under the leadership of L. D. Faddeev [61, 146, 154] in the late 1970s. Many quantum “integrable” systems can be represented by a similar type of algebra, but in with different representations, and hence can be solved at the level of algebra, i.e., the algebraic Bethe ansatz. One can refer to [65, 147] for a detailed discussion.

### 2.7.2 Infinite system

Additionally, one can solve the master equation of ASEP on \( \mathbb{Z} \) using Bethe ansatz. Suppose the initial position state is denoted by \( \vec{y} \). Then the solution to the
master equation (2.6) is given by

\[ P(\vec{x}; t \mid \vec{y}; 0) = \langle \vec{x}|e^{t\Gamma}|\vec{y}\rangle = \sum_n e^{tE_n} \langle \vec{x}|f_n\rangle \langle f_n|\vec{y}\rangle, \]

where \( \Gamma \) is the transition rate matrix, and \( |f_n\rangle \) is the eigenfunction of \( \Gamma \) with eigenvalue \( E_n \). However, from above we see that the Bethe ansatz is expected to result in a complicated eigenfunction. Moreover, there may exist eigenfunctions that are not obtained by the Bethe ansatz, i.e., the set of eigenfunctions obtained by the Bethe ansatz may not be complete.

In the following, we will explain how to obtain a formula for \( P(\vec{x}; t \mid \vec{y}; 0) \) in [160] using the idea of Bethe ansatz.

As before, we start with the case of a single particle. We want to solve the following system:

\[
P(x; t \mid y; 0) = pP(x; t \mid y; 0) + qP(x; t \mid y; 0) - P(x; t \mid y; 0),
P(x; 0 \mid y; 0) = \delta_{xy}.
\]

By separation of variables and the results from the ASEP on a ring, we can write down the solution to the first equation:

\[
P(x; t \mid y; 0) = \int e^{E(z)x}g(z)dz,
\]

where \( E(z) = pz^{-1} + qz - 1 \). The initial condition \( P(x; 0 \mid y; 0) = \delta_{xy} \) is secured by setting \( g(z) = z^{-y-1}/(2\pi i) \) and choosing the contour \( C_0 \) only round origin. Then by residue theorem

\[
P(x; 0 \mid y; 0) = \frac{1}{2\pi i} \int_{C_0} z^{x-y-1}dz = \text{Res}_{z=0}(z^{x-y-1}) = \delta_{xy}.
\]

In conclusion, the single particle case is solved by

\[
P(x; t \mid y; 0) = \frac{1}{2\pi i} \int_{C_0} e^{E(z)x}z^{x-y-1}dz,
\]

with \( E(z) = pz^{-1} + qz - 1 \).

Next, let’s consider \( N = 2 \). From the previous discussion of the ASEP on a ring, we are able to write the general solution of the master equation. The contour of integration will be fixed by requiring that it also satisfies the initial condition:

\[
P(\vec{x}; t \mid \vec{y}; 0) = \int \int g(z_1, z_2)e^{(E(z_1)+E(z_2))t}(z_1^{x_1}z_2^{x_2} + S_{21}z_1^{x_2}z_2^{x_1})d_z_1d_z_2,
\]

where \( S_{ij} = \frac{p+qz_i-z_j}{p+qz_j-z_i} \) comes from the scattering relation (2.30). Empirically, one can choose \( g(z_1, z_2) = z_1^{y_1-1}z_2^{y_2-1}/(2\pi i)^2 \), and the contour \( C_0 \) only around the
The transition probability for two particles is given by
\[ P(\vec{x}; 0 \mid \vec{y}; 0) = \frac{1}{(2\pi)^2} \oint_{C_0} \oint_{C_0} \left( z_1^{x_1-y_1-1} z_2^{x_2-y_2-1} + S_{21} z_1^{x_2-y_1-1} z_2^{x_1-y_2-1} \right) dz_1 dz_2 \]
\[ = \frac{1}{(2\pi)^2} \oint_{C_0} \oint_{C_0} z_1^{x_1-y_1-1} z_2^{x_2-y_2-1} dz_1 dz_2 \]
\[ = \delta_{x_1 y_1} \delta_{x_2 y_2}. \]

In the second line, we claim that the integration of the second term \( S_{21} z_1^{x_2-y_1-1} z_2^{x_1-y_2-1} \) vanishes. By the change of variable \( z_2 = z/z_1 \), the second term becomes
\[ S_{21} z_1^{x_2-y_1-1} z_2^{x_1-y_2-1} = \frac{p + q z - z/z_1}{p + q z - z_1} z_1^{x_2-y_1-1} z_2^{x_1-y_2-1}. \]

The fraction \( S_{21} \) only contains a simple pole of \( z_1 = 0 \). But the power of \( z_1 \) is \( x_2 - y_1 - x_1 + y_2 \), which is always positive since we only consider the physical region \( x_1 < x_2, y_1 < y_2 \). Therefore the second term now is analytic inside the \( z_1 \) contour, and hence the integration vanishes in the physical domain. Consequently, the transition probability for two particles is given by
\[ P(\vec{x}; t \mid \vec{y}; 0) = \oint_{C_0} \oint_{C_0} e^{E(z_1) + E(z_2) t} \left( z_1^{x_1-y_1-1} z_2^{x_2-y_2-1} + S_{21} z_1^{x_2-y_1-1} z_2^{x_1-y_2-1} \right) d^2 z, \]
where \( S_{21} \) and \( E(z) \) are given above, and \( d^2 z = \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \).

With the above two cases, we now can guess the integral form of transition probability for \( N \) particles. The result is stated in the following theorem and an explicit proof will be given following \[160, 162\]. To formulate the result we first define an inversion. If \( i < j \) and \( \pi_i > \pi_j \), the pair \((\pi_i, \pi_j)\) is called an inversion of \( \pi \). For example, the inversions of \((24135)\) are \((21)\), \((41)\), \((43)\).

**Theorem 2.12** (Tracy and Widom). Consider ASEP on \( \mathbb{Z} \) with \( N \) particles. The transition probability is given by
\[ P(\vec{x}; t \mid \vec{y}; 0) = G_g(\vec{x}; t) := \oint_{C_0} \cdots \oint_{C_0} d^N z \sum_{\pi \in S_N} A_{\pi} \prod_{i=1}^{N} \frac{z_{\pi_i}^{x_{\pi_i}-y_{\pi_i}-1} e^{E_{\pi_i} t}}{z_{\pi_i}}, \tag{2.38} \]
where \( C_0 \) is the circle centered at the origin small enough to exclude poles in \( A_{\pi} \),
\[ d^N z = \prod_{i=1}^{N} \frac{dz_i}{2\pi i}, \quad E_i = p z_i^{-1} + q z_i - 1, \quad \text{and} \]
\[ A_{\pi} = \prod_{(\alpha, \beta) \text{ inversion in } \pi} S_{\alpha \beta}, \quad S_{\alpha \beta} = \frac{-p + q z_{\alpha} z_{\beta} - z_{\alpha}}{p + q z_{\alpha} z_{\beta} - z_{\beta}}. \tag{2.39} \]

**Proof.** From the knowledge of ASEP on a ring, the time evolution of a transition probability can be written into an ODE with a boundary condition. Hence one needs to prove that (2.38) satisfies the following three conditions:
Asymmetric Simple Exclusion Process

(1) Time evolution

\[
\frac{d}{dt} G_{\bar{g}}(\bar{x}; t) = \sum_{i=1}^{N} \left[ p G_{\bar{g}}(\bar{x}_i^-; t) + q G_{\bar{g}}(\bar{x}_i^+; t) \right] - NG_{\bar{g}}(\bar{x}; t). \tag{2.40}
\]

(2) Boundary condition

\[
G_{\bar{g}}(x_1, \ldots, x_i, x_i + 1, \ldots, x_N; t) = p G_{\bar{g}}(x_1, \ldots, x_i, x_i, \ldots, x_N; t) + q G_{\bar{g}}(x_1, \ldots, x_i + 1, x_i + 1, \ldots, x_N; t) \tag{2.41}
\]

(3) Initial condition

\[
G_{\bar{g}}(\bar{x}; 0) = \delta_{\bar{x}\bar{y}} \tag{2.42}
\]

The proof of (2.40) is straightforward. Since the integrand is $C^1$ continuous in time, we can interchange $\frac{d}{dt}$ and the integral. The evolution equation follows from direct calculation.

The proof of (2.41) follows from the discussion of ASEP on a ring. We have seen that the boundary condition is satisfied once (2.37) holds. Thus we only need to show that if $A_{\pi}$ is given by (2.39), then (2.37) holds. Suppose $\pi_i > \pi_{i+1}$. Then $(\pi_i, \pi_{i+1})$ is an inversion in $\pi$, but not an inversion of $\pi s_i$. However, the rest of inversions in $\pi$ and $\pi s_i$ are exactly the same. It follows that

\[
\frac{A_{\pi}}{A_{\pi s_i}} = S_{\pi_i, \pi_{i+1}} = -\frac{p + q z_{\pi_i}, z_{\pi_{i+1}} - z_{\pi_i}}{p + q z_{\pi_i}, z_{\pi_{i+1}} - z_{\pi_{i+1}}},
\]

which is exactly (2.39) as required.

The proof of (2.42) is the hard part of the entire proof. A detailed and rigorous proof can be found in [162], and the sketch of the proof is stated in the following.

The initial condition (2.42) holds when $\pi = \text{id}$, the identity permutation. In this case $A_{\text{id}} = 1$, and thus the integrand is $\prod_{i=1}^{N} z_{x_i - y_{\pi_i} - 1}$, which has a non-zero residue at origin only when $x_i = y_i$ for $\forall i \in [1, N]$. Hence it is sufficient to show

\[
\sum_{\pi \neq \text{id}} I(\pi) = 0, \tag{2.43}
\]

where

\[
I(\pi) = \oint_{C_0} \cdots \oint_{C_0} d^N z A_{\pi} \prod_{i=1}^{N} z_{x_i - y_{\pi_i} - 1}.
\]

First we want to show (2.43) holds when $\pi_N < N$.

\[
\sum_{\pi_N < N} I(\pi) = 0 \tag{2.44}
\]
We prove this by separate $\{\pi \neq \text{id}\}$ into different groups. For $n \in [1, N)$, fix $n - 1$ distinct numbers $i_1, \ldots, i_n \in [1, N)$, and define $A = \{i_1, \ldots, i_n\}$. Then we group $\{\pi \neq \text{id}\}$ into

$$S_N(A) = \{\pi \in S_N : \pi_1 = i_1, \ldots, \pi_{n-1} = i_{n-1}, \pi_n = N\}.$$  

Namely, permutations $\pi$ are grouped according to the position of $N$ in $\pi$ and the value in front of $N$. We claim that,

**Claim.** For each $A$,

$$\sum_{\pi \in S_N(A)} I(\pi) = 0.$$  

**Proof of Claim.** For each integral $I(\pi)$, we make a change of variable:

$$z_N \rightarrow \frac{\eta}{\prod_{i < N} z_i},$$

where the contour of $\eta$ is of course centered at origin and only contains zero. Thus the integral becomes

$$I(\pi) = \oint_{C_0} \cdots \oint_{C_0} d^{N-1} \eta \prod_{j \in B(A)} S\left(\prod_{i < N} \frac{\eta}{z_i}, z_j\right) \prod_{(i,j) \in B_\pi} S_{ij} \prod_{i < N} \frac{z_{x_i-1}^{x_{i-1}-x_N-1+y_N-1} z_{y_i-1}^{x_N-1-y_N-1}}{\eta^{x_N-1-y_N-1}}.$$

where $B_\pi$ denotes the set of all inversions in $\pi$, $B(A)$ is the complement of $A \cup N$ in $[1, N]$, and $S(x, y) = -(p + qxy - x)/(p + qxy - y)$. For $\forall j \in B(A)$, $(N, j) \in B_\pi$, and $(N, j)$ forms all inversions containing $N$. $B(A)$ is fixed once $A$ is fixed, but $B_\pi$ is not. It depends on specific permutation $\pi$.

In the integrand of $I(\pi)$, we call

$$\prod_{i < N} \frac{z_{x_i-1}^{x_{i-1}-x_N-1+y_N-1} z_{y_i-1}^{x_N-1-y_N-1}}{\eta^{x_N-1-y_N-1}} : \text{product factor},$$

$$S\left(\prod_{i < N} \frac{\eta}{z_i}, z_j\right) : \text{$j$-factor},$$

$$\prod_{(i,j) \in B_\pi} S_{ij} : \text{$S$-factor},$$

If $\pi_{N-1} = N$, i.e., $n = N - 1$, then there only exists one inversion containing $N$, i.e., $|B(A)| = 1$. Suppose $j \in B(A)$. Then $\pi_N = j$. There is only a simple pole of $z_j = 0$ in the $j$-factor. But power of $z_j$ in the product factor is $x_N - x_{N-1} + y_N - y_j - 1$, which is positive by the fact that $x_1 < \cdots < x_N$ and $y_1 < \cdots < y_N$. Therefore the integrand of $I(\pi)$ is analytic inside $z_j$ contour. As a result, $I(\pi) = 0$.

The case $\pi_{N-1} = N$ is rather simple, while the proof for the case $\pi_{N-1}^{-1} < N - 1$
(i.e. \( n < N - 1 \)) is not trivial. In the following, we split the proof for \( \pi_{N - 1} < N - 1 \) into two lemmas.

**Lemma 2.13.** When \( \pi_{N - 1} < N - 1 \), any \( I(\pi) \) can be written into sum of lower order integrals. In each integral, \( \exists \) inversions \( (N, i) \) and \( (N, j) \), such that \( z_i = z_j \). We denote such integral as \( I(\pi; z_i = z_j) \).

**Proof.** We choose the contour \( C_0 \) such that all the pose from the \( S \) factor are excluded. Hence apart from the poles at the origin, the poles only come from those \( j \)-factors.

Suppose \( \max(B(A)) = j \neq k \) and \( k \in B(A) \). Note that there are at least two inversions containing \( N \) in \( \pi \), since \( \pi_{N - 1} < N - 1 \). Then the lemma is proved by integrating with respect to \( z_j \) and then \( z_k \).

We observe that \( z_j = 0 \) is a simple pole in the \( j \)-factor, and analytic in the \( i \)-factor when \( i \neq j \). The power of \( z_j \) in the product factor is positive as before. Hence the residue at \( z_j = 0 \) is zero. Then the rest of poles of \( z_j \) in the integrand come from the \( i \)-factor when \( i \neq j \):

\[
\frac{-p + q \eta \prod_{l \neq i, N} z_l^{-1} - \eta \prod_{l < N} z_l^{-1}}{p + q \eta \prod_{l \neq i, N} z_l^{-1} - z_i} = \frac{-pz_j + q \eta \prod_{l \neq i, j, N} z_l^{-1} - \eta \prod_{l \neq j, N} z_l^{-1}}{pz_j + q \eta \prod_{l \neq i, j, N} z_l^{-1} - z_i z_j}.
\]

Fixed \( k \neq j \) and consider the \( k \)-factor. We see that there is a simple pole at

\[
z_j = \frac{q \eta \prod_{l \neq j, k, N} z_l^{-1}}{z_k - p}.
\]

The residue at this pole is obtained by replacing the \( k \)-factor by

\[
-q \eta \frac{z_k + p z_k^{-1} - 1}{(z_k - p)^2} \prod_{l \neq j, k, N} z_l^{-1},
\]

the \( i \)-factor by

\[
-pq(z_k - z_i) + q z_k z_i - z_k + p
\]

where \( i \neq j, k \), and the \( z_j \) in the \( j \)-factor by (2.45). Hence \( I(\pi) \) can be written into sum of these residues, which are integrals of \( N - 1 \) dimension.

Then we integrate these residues with respect to \( z_k \). First consider the pole at origin. We observe that \( z_k = 0 \) is a simple pole in the \( j \)-factor and (2.46), and analytic in (2.47). The power of \( z_k \) in the product factor is \( x_{\pi_{j-1}} - x_{\pi_{N-1}} + y_N - y_k - 1 \). By the assumption that \( N > j > k \), \( y_N - y_k \geq 2 \). Since \( (N, k) \) is an inversion, then \( \pi_k^{-1} > \pi_N^{-1} \). Thus power of \( z_k \) in the product factor is greater than 2, implying that the integrand is analytic at \( z_k = 0 \). The rest of poles come from (2.47) and the \( j \)-factor.

First consider the \( i \)-factor when \( i \neq j, k \). There is a pole at \( z_i = z_k \). Then in
Then solving (2.45) and (2.48) gives \( z_j = z_k \).

Thus, after integrating these residues with respect to \( z_k \), one obtain integrals of \( N - 2 \) dimension where either \( z_i = z_k \), or \( z_j = z_k \), and hence proves the lemma.

**Lemma 2.14.** For each integral in Lemma 2.13, there is a partition of \( S_N(A) \) into pairs \( \{\pi, \pi'\} \) such that \( I(\pi) + I(\pi') = 0 \).

**Proof.** Consider an integral in Lemma 2.13 with \( z_i = z_j \) and permutation \( \pi \). Pair this integral with the one with \( z_i = z_j \) and \( \pi' \), where \( \pi' \) is obtained by exchanging \( i \) and \( j \) in \( \pi \). Without loss of generality, we assume that \( i < j \) and \( \pi_i^{-1} < \pi_j^{-1} \). Namely, \( \pi = (\pi_1, \ldots, i, \ldots, j, \ldots, \pi_N) \) and \( \pi' = (\pi_1, \ldots, j, \ldots, i, \ldots, \pi_N) \). When \( z_i = z_j \), the product factor is exactly the same for both integrals. We aim to show that \( A_\pi \) and \( A_{\pi'} \) are negative of each other when \( z_i = z_j \).

Obviously \( S(z_j, z_i) \) is a factor in \( A_{\pi'} \) but not a factor in \( A_\pi \). When \( z_i = z_j \), \( S(z_j, z_i) = -1 \). Moreover, by the construction of \( \pi' \), the inversions that do not contain \( i \) or \( j \) are identical for \( \pi \) and \( \pi' \). Thus, it suffices to show that, for any \( k \neq i, j \), the S-factor (2.39) involving \( z_k \) and either \( z_i \) or \( z_j \) is the same for \( \pi \) and \( \pi' \).

Under the assumption \( i < j \) and \( \pi_i^{-1} < \pi_j^{-1} \), there are 3 cases, based on the position of \( k \) in \( \pi \) (or \( \pi' \)). For each case, there are 3 sub-cases, depending on the relation between \( k \) and \( i, j \). The followings list all nine cases.

<table>
<thead>
<tr>
<th>( \pi )</th>
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<td>( \pi_k^{-1} &lt; \pi_i^{-1} &lt; \pi_j^{-1} )</td>
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</tbody>
</table>

When \( z_i = z_j \), \( S(z_j, z_k)S(z_k, z_i) = 1 \). One can observe that in all nine cases, the concerning S-factor are the same for \( \pi \) and \( \pi' \), which proves the Lemma.

Now Lemma 2.13 and Lemma 2.14 together show that when \( \pi_N^{-1} < N - 1 \), \( \sum_{\pi \in S_N(A)} I(\pi) = 0 \) still holds, and hence complete the proof of the claim.

Therefore (2.44) is proved by separating \( \{\pi \neq \text{id}\} \) into the disjoint union of the various \( S_N(A) \).
Now we can use induction on $N$ to prove (2.43). Clearly, (2.43) holds when $N = 1$. Assume that it also holds for $N = M - 1$. Then consider the case $N = M$. When $\pi_M = M$, we can factor the integral $\oint_{C_0} dz_M z_M^{y_M-1}$ out of each term $I(\pi)$. The rest of $I(\pi)$ sums to zero by the induction hypothesis. The case $\pi_M < M$ is already proved by above. Hence we show that (2.43) holds when $N = M$ as required.

**Remark 2.15.** The above theorem and proof follows directly from [160]. In the proof of initial condition, the trouble comes from the pole at (2.45). Then instead of excluding the poles at $A_x$, one also can excludes these poles from the contour.

By redefining the contour to exclude all poles except origin, one can simply the proof of initial condition substantially, especially the proof of Lemma 2.13.

Suppose $C_0$ only contains origin. When $\pi_N^{-1} < N$, let $(N, j)$ be an inversion. The power of $z_j$ is positive as above. Hence the integration around $C_0$ with respect to $z_j$ gives zero, which implies that (2.43) holds when $\pi_N^{-1} < N$. The initial condition can thus be proved by mathematical induction on $N$.

### 2.8 Duality vs. Bethe ansatz

The approaches of Bethe ansatz and duality have been widely used to obtain exact formulas for current distributions. Such formulas reveal long time limit behaviors of the exclusion processes via asymptotic analysis.

The first one, Bethe ansatz, gives rise to the Green’s function, which in principle allows one to compute any current distribution functions of the system. In the case of ASEP, Tracy and Widom, using Bethe ansatz, proved in series of papers [159–162] that the limit law of ASEP is governed by the GUE Tracy-Widom distribution.

The duality function, on the other hand, can be used to compute the moments of the system, via solving the duality ODEs. Then the current distribution is then obtained by some appropriate transformations of the moment formulas. See [17] for applying duality functions to recover the limit law of ASEP.

A review and comparison of these two methods can be found in [40].

### 2.9 Matrix product ansatz

Numerous approaches has been used to study the stationary states properties, e.g. mean-field theories and hydrodynamic approach. However these two methods produce qualitatively correct result instead of explicit quantitative formulas. A useful tool to study the quantitative stationary measure, called Matrix representation method, was first introduced in [55], and then developed in [141]. A more complete review can be found in [14].

The main purpose of this thesis is to discuss two approaches in solving the exclusion processes: duality and Bethe ansatz (see sections before). Therefore we will only mention briefly here about the method “matrix ansatz” used to find the
stationary state and its macroscopic properties, including current distribution and long time fluctuations.

Consider ASEP on a string of length $L$, with boundary exiting and entering rate $\alpha, \beta, \gamma$ and $\delta$ (see Fig. 2.2(a)). Let an occupational state be $\eta = \eta_1 \cdots \eta_L$. The core idea of matrix ansatz is to associate the following matrix elements with each occupational state $\eta$:

$$P(\eta) = \frac{1}{Z_L} \langle W | \prod_{i=1}^{L} (\eta_i D + (1 - \eta_i) E) | V \rangle,$$

(2.49)

with normalisation (or partition function)

$$Z_L = \langle W | \sum_{\eta} \prod_{i=1}^{L} (\eta_i D + (1 - \eta_i) E) | V \rangle = \langle W | \left( \sum_{k=0}^{L} \binom{k}{L} D^k E^{L-k} \right) | V \rangle = \langle W | (D + E)^L | V \rangle = \langle W | (C)^L | V \rangle,$$

(2.50)

where the second equality holds because there are $\binom{k}{L}$ kinds of $\eta$ such that it contains $k$ one’s, and $D + E = C$. Therefore we have

$$\sum_{\eta} P(\eta) = 1,$$

as required. Then [55] claims that (2.49) gives the stationary measure of the occupational state $\eta$, $P_{\text{stat}}(\eta) = P(\eta)$, if the matrices $D, E$ and vectors $\langle W |, | v \rangle$ satisfies the following relations:

$$pDE - qED = D + E,$$

(2.51a)

$$(\beta D - \delta E) | V \rangle = | V \rangle,$$

(2.51b)

$$\langle W | (\alpha E - \gamma D) = \langle W | .$$

(2.51c)

To show this, we need to prove the matrix product form of the stationary probability (2.49) satisfies the stationary state condition (2.8). This can be proved by using the infinitesimal generators. The element of the transition matrix $\Gamma$ is given by generator elements (2.10), and the explicit form of the ASEP generator in terms of occupational states is specified by (2.14). Therefore combining (2.3), (2.10) and (2.14), one should be able to show that (2.49) meets (2.8) given that (2.51) is satisfied.

Plenty of representations were found to meet the reduction relation (2.51). One that is frequently used is matrices of infinite dimension, as discussed in [56, 62, 133].

The matrix product formula can be used to calculate the stationary state properties quantitatively. For example, the stationary current, representing the average number of particles flowing through site $i$ and $i + 1$ under stationary state, in terms
of matrix product is given by

\[
J_{\text{stat}} = p \langle \eta_i (1 - \eta_{i+1}) \rangle - q \langle \eta_{i+1} (1 - \eta_i) \rangle = \frac{1}{Z_L} \langle W | C_{i-1} (pDE - qED) C_{i-1}^{-1} | V \rangle
\]

\[
= \frac{1}{Z_L} \langle W | C_{i-1} (D + E) C_{i-1}^{-1} | V \rangle = \frac{Z_{L-1}}{Z_L}
\]

where \( \langle f(\eta) \rangle := E[f(\eta)] \equiv \sum_{\eta} P(\eta) f(\eta) \). The second lines follows from binomial coefficient as in (2.50). Similarly, one point function \( \langle \eta_i \rangle \) or even \( n \)-point correlation function \( \langle \eta_i \cdots \eta_k \rangle \) can be computed in a similar manner using the matrix product form (2.49).

To obtain \( J_{\text{stat}} \), we need to compute the partition function \( Z_L \). There are various method to obtain \( Z_L \), three of which were given in [14]. For example, the reduction relation (2.51) is satisfied by using the \( q \)-deformed oscillators algebra representation [56,62,133]. The normalisation \( Z_L \) is then followed by diagonalization of the matrix \( C = D + E \). Such \( Z_L \) is given in an explicit integral formula. By evaluating the integral, we can show that there are 3 different values of \( Z_L \), corresponding to 3 steady currents, and 3 phase of ASEP. Summarizing these 3 case, one recovers the phase diagram as shown in [133].
In this chapter, we will present a new method for obtaining duality functions in multi-species asymmetric exclusion processes (mASEP), from solutions of the deformed Knizhnik-Zamolodchikov equations. Our method reproduces, as a special case, duality functions for the self-dual single species ASEP on the integer lattice. Note that this chapter is published as a paper [165].

3.1 Introduction

3.1.1 Background

Duality plays an important role in stochastic Markov processes where the time evolution is described by a linear generator. Early applications appear in [46, 149] for the self-dual symmetric exclusion process, and in [79] for the contact process. Apart from these classical applications, duality is also a valuable tool for proving the limits of particle systems to stochastic partial differential equations; see [42, 43].

A duality functional is an observable that co-varies on the configuration spaces of two stochastic processes; see for example [88, 114]. Duality functionals are most powerful when expectation values and correlation functions of many-particle processes are related to those containing few particles. Models with few particles can be analysed in great detail and therefore expectation values can often be calculated analytically via such dualities. A well-known recent example is that of the duality between the stochastic Kardar–Parisi–Zhang (KPZ) equation for interface growth [43, 58, 94] and the integrable one-dimensional quantum Bose gas [24, 25, 27, 93]. Indeed, much progress has been made in recent years using duality in the setting of integrable stochastic processes such as [16, 17, 31, 32, 41, 72, 73, 85], where several powerful tools are available.

In early attempts duality functionals have been constructed in a more or less ad hoc fashion and only recently attempts have been made to systematically derive dualities in integrable stochastic models using quantum group symmetries [10–12, 33, 34, 105–107, 142]. In this chapter we propose a new approach for method-
ically constructing integrable dualities by exploiting the algebraic structure provided by the $t$-deformed Knizhnik–Zamolodchikov (KZ) equations [70, 103], which are consistency equations expressed in terms of the R-matrix of a quantum group, or alternatively, in terms of the Hecke algebra.

We will work in the context of the integrable (multi-species) asymmetric exclusion simple process (mASEP) with hopping rate $t$. The mASEP can be realized in two ways via representations of the Hecke algebra. The first is a standard description in which each particle configuration $\mu$ is identified with a basis element of a vector space, and where the local Markov generator is a matrix acting on this space. The second realization is on a basis $f_\mu$ of the ring of $n$-variable polynomials, in which the local Markov generator becomes a divided-difference operator (a polynomial representation of a Hecke generator). The $t$-deformed KZ equations connect these two realizations, and can in turn be interpreted as the duality relations of a diagonal observable intertwining the vector space and polynomial representations of the mASEP.

In order to go beyond this tautological diagonal observable, and obtain non-trivial observables on the two processes, our main technical tool will be a family of $n$-variable polynomials $f_\mu$ studied in [29]. These polynomials are a standard basis for the polynomial realization of the mASEP, and are closely related to the theory of symmetric Macdonald polynomials [117, 118, 120] and their non-symmetric versions [37, 38, 127]. The $f_\mu$ polynomials depend on two parameters: the mASEP hopping parameter $t$, and another parameter $q$ which appears when supplementing the $t$-deformed KZ equations by a certain cyclic boundary condition; collectively, these parameters are the $(q, t)$ of Macdonald polynomial theory. The presence of the second parameter $q$ is crucial to our approach, for while it has no direct physical meaning in the mASEP, its value can be tuned. In particular, when the $(q, t)$ parameters satisfy a resonance condition of the form,

$$q^kt^l = 1, \quad k, l \in \mathbb{N}, \quad (3.1)$$

the $f_\mu$ polynomials may become singular and (after appropriately normalizing, to remove poles) degenerate into a sum of the form $\sum_\nu \psi(\nu, \mu; t)f_\nu$, for certain coefficients $\psi(\nu, \mu; t)$. In other words, the condition (3.1) creates linear dependencies between the $f_\mu$ polynomials and thus gives rise to non-trivial intertwining solutions of the $t$-deformed KZ equations. It is these solutions that produce duality relations in the mASEP; the duality functionals end up being nothing but (rescaled versions of) the expansion coefficients $\psi(\nu, \mu; t)$.

In the rest of the introduction, we describe our methodology in greater detail.

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*Our methods work specifically in models described by deformed Knizhnik–Zamolodchikov equations, which are synonymous with the notion of Yang–Baxter integrability. We shall make no comment on dualities in the non-integrable setting. 

†Note that in this chapter, we only consider ASEP on $\mathbb{Z}$. Even though our solutions of the $t$-deformed KZ equations have a certain cyclic symmetry, ASEP on the ring will make no appearance in this work.
3.1.2 Functional definition of duality

The standard definition of a stochastic duality is in terms of a function $\psi$ which takes values on the configuration spaces of two (possibly different) Markov processes. Let us begin by restating this definition in some generality.

Let $\mathbb{A}$ and $\mathbb{B}$ be two discrete (countable) sets, whose elements we denote by $a$ and $b$, respectively. Let $\mathcal{F}$ be the space of all functions $\psi$ of the form

$$\psi : \mathbb{A} \times \mathbb{B} \to \mathbb{C}.$$ 

Consider two linear functionals $L$ and $M$ which act on functions in $\mathcal{F}$ as follows:

$$L[\psi(\cdot, b)](a) := \sum_{a' \in \mathbb{A}} \ell(a, a')\psi(a', b), \quad M[\psi(a, \cdot)](b) := \sum_{b' \in \mathbb{B}} m(b, b')\psi(a, b'),$$

where $\ell : \mathbb{A} \times \mathbb{A} \to \mathbb{C}$ and $m : \mathbb{B} \times \mathbb{B} \to \mathbb{C}$ are some pre-specified functions (in the language of stochastic processes, these will be the matrix entries of the Markov generators $L$ and $M$ of two different processes). Then following [88], $L$ and $M$ are dual with respect to a function $\psi$ if

$$L[\psi(\cdot, b)](a) = M[\psi(a, \cdot)](b), \quad \forall a \in \mathbb{A}, b \in \mathbb{B}. \quad (3.3)$$

Obviously, if the function $\psi$ satisfies (3.3), then $c \cdot \psi$ will also obey the same relation, where $c$ is a constant with respect to the two generators. For this reason, throughout the text we will often consider duality functions modulo an overall multiplicative factor.

3.1.3 Matrix definition of duality

It is useful for our purposes to recast the statement of duality in terms of matrices, rather than functionals. We upgrade the previous sets $\mathbb{A}$ and $\mathbb{B}$ to vector spaces, with basis vectors $|a\rangle$ and $|b\rangle$. Let $\psi$ be a certain function in $\mathcal{F}$ and consider the following vector,

$$|\Psi\rangle := \sum_{a \in \mathbb{A}, b \in \mathbb{B}} \psi(a, b) |a\rangle \otimes |b\rangle.$$ 

(3.4)

Let $L \in \text{End}(\mathbb{A})$ and $M \in \text{End}(\mathbb{B})$ be linear operators given explicitly by

$$L|a\rangle = \sum_{a' \in \mathbb{A}} \ell(a', a) |a'\rangle, \quad M|b\rangle = \sum_{b' \in \mathbb{B}} m(b', b) |b'\rangle,$$

(3.5)

for certain matrix entries $\ell$ and $m$.

**Proposition 3.1.** The duality relation (3.3) is equivalent to the equation

$$L|\Psi\rangle = M|\Psi\rangle.$$ 

(3.6)
Proof. Explicit calculation of the left and right hand sides gives

$$L |\Psi\rangle = \sum_{a,b,a'} \psi(a,b) \ell(a',a) |a'\rangle \otimes |b\rangle = \sum_{a,b} \left( \sum_{a'} \ell(a,a') \psi(a',b) \right) |a\rangle \otimes |b\rangle,$$

$$M |\Psi\rangle = \sum_{a,b,b'} \psi(a,b) m(b',b) |a\rangle \otimes |b'\rangle = \sum_{a,b} \left( \sum_{b'} m(b,b') \psi(a,b') \right) |a\rangle \otimes |b\rangle.$$ 

Requiring that these be equal implies (3.3) for the function $\psi.$

3.1.4 $t$KZ equations as a source of dualities

The local $t$-deformed Knizhnik–Zamolodchikov equations, or $t$KZ equations for short, as introduced by Smirnov in the study of form factors [148], are a system of equations for a polynomial-valued" vector $|\Psi\rangle \in \mathbb{C}[z_1, \ldots, z_n] \otimes \mathcal{V}.$ Here $\mathbb{C}[z_1, \ldots, z_n]$ denotes the ring of polynomials in $n$ variables $(z_1, \ldots, z_n),$ over the field of complex numbers. The vector space $\mathcal{V}$ is obtained by taking an $n$-fold tensor product of local spaces, i.e., $\mathcal{V} := \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n,$ where $\mathcal{V}_i \equiv \mathbb{C}^{r+1}$ for all $1 \leq i \leq n,$ and $r \geq 1$ is some fixed positive integer. The local $t$KZ equations read

$$s_i |\Psi\rangle = \hat{R}(z_i/z_{i+1}) |\Psi\rangle, \quad i \in \{1, \ldots, n-1\}, \quad (3.7)$$

where $s_i$ is a simple transposition acting on $\mathbb{C}[z_1, \ldots, z_n],$ with action

$$s_i g(z_1, \ldots, z_i, z_{i+1}, \ldots, z_n) = g(z_1, \ldots, z_{i+1}, z_i, \ldots, z_n), \quad \forall \ g \in \mathbb{C}[z_1, \ldots, z_n],$$

and $\hat{R}(z_i/z_{i+1})$ denotes the R-matrix associated to quantized affine $\mathfrak{sl}(r+1)$ acting in $\mathcal{V}_i \otimes \mathcal{V}_{i+1}.$ To fix a particular solution of (3.7) these equations are supplemented by a cyclic boundary condition on $|\Psi\rangle,$ which we do not write down at this stage.

It is known (see for example [128,168] and Remark 3.7 in the current text) that the equations (3.7) can be cast in the form

$$L_i |\Psi\rangle = M_i |\Psi\rangle, \quad i \in \{1, \ldots, n-1\}, \quad (3.8)$$

for certain $L_i \in \text{End}(\mathbb{C}[z_1, \ldots, z_n]) \otimes 1$ and $M_i \in 1 \otimes \text{End}(\mathcal{V}).$ This form differs slightly from (3.7), since it separates completely the action on the $\mathbb{C}[z_1, \ldots, z_n]$ part of $|\Psi\rangle$ from that on its $\mathcal{V}$ part. The equation (3.8) is our key to establishing the link between $t$KZ equations and dualities. The connection can be made precise under the following steps:

- We identify the two generic vector spaces appearing in Section 3.1.3 with the

---

*We use the term local to distinguish these equations from the original quantum deformation of the Knizhnik–Zamolodchikov equation introduced by Frenkel and Reshetikhin [70], which involves global scattering matrices. Our use of $t$ rather than $q$ as the deformation parameter stems from the fact that both parameters play a role in this work, as the $(q,t)$ in Macdonald polynomials.

1In many contexts solutions to the $t$KZ equations are in fact in terms of series and elliptic functions.
vector spaces appearing in (3.8), i.e. $A \equiv \mathbb{C}[z_1, \ldots, z_n]$ and $B \equiv \mathbb{V}$.

- We choose suitable bases $\{|a\rangle\}$ and $\{|b\rangle\}$ for $A$ and $B$, and expand both $|\Psi\rangle$ and the linear operators $L_i$ and $M_i$ with respect to these bases, as in (3.4) and (3.5). This yields

$$\sum_{a,a' \in A} \sum_{b \in B} \ell_i(a, a') \psi(a', b) |a\rangle \otimes |b\rangle = \sum_{a \in A} \sum_{b, b' \in B} m_i(b, b') \psi(a, b') |a\rangle \otimes |b\rangle,$$

in the very same way as in the proof of Proposition 3.1.

- The coefficients $\psi(a, b)$ are then duality functions with respect to $n - 1$ pairs of linear functionals $L_i$ and $M_i$, in the same sense as (3.3):

$$\sum_{a' \in A} \ell_i(a, a') \psi(a', b) = \sum_{b' \in B} m_i(b, b') \psi(a, b'), \quad (3.9)$$

where $\ell_i(a, a')$ and $m_i(b, b')$ are the matrix entries of the operators $L_i$ and $M_i$. The $\ell_i(a, a')$ can also be thought of as duality functions with respect to the generators $L := \sum_{i=1}^{n-1} L_i$ and $M := \sum_{i=1}^{n-1} M_i$, simply by summing (3.9) over $1 \leq i \leq n - 1$.

This procedure allows one, in principle, to start from any polynomial solution of the local relations (3.7) and to extract from it duality functions. However, it cannot be applied without due heed to the particulars of the solution that one chooses. For example, finding bases $\{|a\rangle\}$ and $\{|b\rangle\}$ such that the operators $L_i$ and $M_i$ are stochastic matrices may be quite difficult in practice or not even possible. It is also not guaranteed that the functions $\psi(a, b)$ define an interesting statistic on the two configuration spaces $A$ and $B$. In this chapter, we will recover a known interesting statistic from a specific solution of (3.7) which was previously considered in [29,53].

### 3.1.5 Notation and conventions

Let us outline some of the notation to be used in the chapter. A composition $\mu$ is an $n$-tuple of non-negative integers, $(\mu_1, \ldots, \mu_n)$. The elements of $\mu$, $\mu_i \geq 0$, are referred to as parts. The sum of all parts, $|\mu| := \sum_{i=1}^{n} \mu_i$, is referred to as the weight of $\mu$. We say that $\mu$ is a rank-$r$ composition if its largest part is equal to $r$. We define the part-multiplicity function $m_i(\mu)$ as the number of parts in $\mu$ equal to $i$: $m_i(\mu) = \{ k : \mu_k = i \}$. A partition $\lambda$ is a composition with weakly decreasing parts, $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. We also define anti-partitions $\delta$, which are compositions with weakly increasing parts, $(0 \leq \delta_1 \leq \cdots \leq \delta_n)$. Where possible we reserve the letters $\mu, \nu$ for generic compositions, $\lambda$ for partitions, and $\delta$ for anti-partitions. Given a

*In the rest of the chapter we will refer to such coefficients as duality functions rather than functionals. The reason for this is that we only focus on $\psi$ as a function on the underlying configuration spaces, and suppress the fact that configurations $a$ and $b$ are themselves functions of time.
composition \( \mu \), its (anti-)dominant ordering \( (\mu^-) \) \( \mu^+ \) is the unique (anti-)partition obtainable by permuting the parts of \( \mu \).

At times we will consider compositions of infinite length. By this, we shall always mean finitely-supported infinite strings \( \ldots, \mu_{-1}, \mu_0, \mu_1, \ldots \), where \( \mu_i \geq 0 \) for all \( i \in \mathbb{Z} \) and where there exists \( N \) such that \( \mu_i = 0 \) if \( |i| > N \).

Following [97], we define two orders on compositions. The first is the dominance order, denoted by \( \succeq \). Given two compositions \( \mu = (\mu_1, \ldots, \mu_n) \) and \( \nu = (\nu_1, \ldots, \nu_n) \), we define

\[
\mu \succeq \nu \iff \sum_{i=1}^j \mu_i \geq \sum_{i=1}^j \nu_i, \quad \forall \ 1 \leq j \leq n.
\]

The second order is denoted by \( \succ \). Given two compositions \( \mu \) and \( \nu \), we define

\[
\mu \succ \nu \iff (\mu^+ > \nu^+ \text{ or } \mu^+ = \nu^+, \mu > \nu).
\]

This order should not be confused with the interlacing of partitions, which is another standard use of the symbol \( \succ \) in the literature.

We let \( \mathbb{C}_{q,t}[z_1, \ldots, z_n] \) denote the ring of polynomials in \( (z_1, \ldots, z_n) \) with coefficients in \( \mathbb{Q}(q,t) \). We use the shorthand \( z^\mu := z_1^{\mu_1} \cdots z_n^{\mu_n} \) to denote the elements of the monomial basis. Given a polynomial \( g(z_1, \ldots, z_n) \in \mathbb{C}_{q,t}[z_1, \ldots, z_n] \), \( p \in \mathbb{N} \) and \( m \in \mathbb{Q}_{>0} \), we define

\[
\text{Coeff}_p[g, m] := \lim_{q \to t^{-m}} (1 - qt^m)^p g(z_1, \ldots, z_n), \quad (3.10)
\]

where the limit exists. We are mainly interested in simple poles in \( q \), when it is convenient to write \( \text{Coeff}_1[g, m] \equiv \text{Coeff}_1[g, m] \). For two polynomials \( g_1, g_2 \in \mathbb{C}_{q,t}[z_1, \ldots, z_n] \), we write

\[
g_1(z_1, \ldots, z_n) \propto g_2(z_1, \ldots, z_n) \iff \exists \alpha \in \mathbb{Q}(q,t) \text{ such that } g_1 = \alpha g_2. \quad (3.11)
\]

### 3.2 Asymmetric simple exclusion process

The functional definition of duality (3.3), and its matrix version (3.6), are both generic statements that apply for any indexing sets \( A \) and \( B \). In this section we will show how self-duality in the ASEP can be cast within this general framework, forming the foundations of the rest of the chapter.

In the examples of duality in ASEP in [17], duality is exhibited between two different ASEP systems (which contain different numbers of particles, and with their left and right hopping rates interchanged) on the infinite line. This means that we should identify both \( A \) and \( B \) with the set of all infinite binary strings. More precisely, we shall define \( A \) to be the space of all multilinear polynomials in an infinite set of variables \( \{z\} = \{\ldots, z_{-1}, z_0, z_1, \ldots\} \). The basis vectors of this space are \( \prod_{i \in \mathbb{Z}} z_i^{\nu_i} \), where \( \nu \) is an infinite composition with \( \nu_i \in \{0, 1\} \) for all \( i \in \mathbb{Z} \).
The binary string corresponding with a given basis vector is read off simply as the exponents of the variables \( \{ z \} \). On the other hand, we define \( \mathbb{B} \) to be the infinite tensor product \( \bigotimes_{i \in \mathbb{Z}} \mathbb{C}_2^i \) whose basis vectors are \( \bigotimes_{i \in \mathbb{Z}} | \mu_i \rangle_i \), where \( \mu \) is an infinite composition with \( \mu_i \in \{0, 1\} \) for all \( i \in \mathbb{Z} \), and where \( |0\rangle \) and \( |1\rangle \) denote the canonical basis of \( \mathbb{C}^2 \).

### 3.2.1 The ASEP generators \( L_i \) and \( M_i \)

Here we recall the definition of the ASEP generator introduced in Chapter 2. The generator is denoted by \( L \), to match the notation of Section 3.1.2. It is constructed as a sum of local generators, \( L = \sum_{i \in \mathbb{Z}} L_i \). Each local generator \( L_i \) acts on functions \( \psi \) of binary strings \( \nu \). Particles (the ones of the binary string) hop to the left with rate 1 and to the right with rate \( t \):

\[
L_i[\psi](\nu) = \sum_{\nu' \in A} \ell_i(\nu, \nu') \psi(\nu'),
\]

where the coefficients \( \ell_i(\nu, \nu') \), which specify the transition rate from \( \nu \) to \( \nu' \), are given by

\[
\ell_i(\nu, \nu') = \begin{cases} 
  t, & \nu_i > \nu_{i+1}, (\nu_i, \nu_{i+1}) = (\nu'_i, \nu'_i), \nu_k = \nu'_k \forall k \neq i, i + 1, \\
  1, & \nu_i < \nu_{i+1}, (\nu_i, \nu_{i+1}) = (\nu'_i, \nu'_i), \nu_k = \nu'_k \forall k \neq i, i + 1, \\
  0, & \text{otherwise},
\end{cases}
\]

when \( \nu \neq \nu' \), and where the diagonal elements are chosen such that the matrix rows sum to zero:

\[
\ell_i(\nu, \nu) = \begin{cases} 
  -t, & \nu_i > \nu_{i+1}, \\
  -1, & \nu_i < \nu_{i+1}, \\
  0, & \text{otherwise}.
\end{cases}
\]

Similarly, one can define a reverse ASEP generator whose hopping rates have been switched, i.e. particles now hop to the left with rate 1 and to the right with rate \( t \). We shall denote this generator by \( M = \sum_{i \in \mathbb{Z}} M_i \), again in reference to our notation in Section 3.1.2. It acts on functions \( \psi \) of binary strings \( \mu \):

\[
M_i[\psi](\mu) = \sum_{\mu' \in \mathbb{B}} m_i(\mu, \mu') \psi(\mu'),
\]
where the hopping rates are given by

\[
m_i(\mu, \mu') = \begin{cases}
1, & \mu_i > \mu_{i+1}, (\mu_i, \mu_{i+1}) = (\mu'_{i+1}, \mu'_{i}), \mu_k = \mu'_k \forall k \neq i, i + 1, \\
t, & \mu_i < \mu_{i+1}, (\mu_i, \mu_{i+1}) = (\mu'_{i+1}, \mu'_{i}), \mu_k = \mu'_k \forall k \neq i, i + 1, \\
0, & \text{otherwise},
\end{cases}
\]  
(3.16)

when \( \mu \neq \mu' \), and where the diagonal elements are chosen such that the matrix columns sum to zero:

\[
m_i(\mu, \mu) = \begin{cases}
-t, & \mu_i > \mu_{i+1}, \\
-1, & \mu_i < \mu_{i+1}, \\
0, & \text{otherwise},
\end{cases}
\]  
(3.17)

The linear operators \( L_i \) and \( M_i \) with matrix entries \( \ell_i(\nu, \nu') \) and \( m_i(\mu, \mu') \) can be turned into Markov matrices by addition of the identity matrix. Following the standard conventions of the probability literature, \( L_i \) acts to the left, while \( M_i \) acts to the right. However, since we intend to cast \( L_i \) as an operator on the space of polynomials (as explained in the next section), we find that left-action is notationally cumbersome, and instead arrange so that both \( L_i \) and \( M_i \) act to the right.

### 3.2.2 Divided-difference realization of \( L_i \)

Let \( A \) denote the space of multilinear polynomials in \( \{ \ldots, z_{-1}, z_0, z_1, \ldots \} \), and let us seek an operator \( L_i \) whose action on \( A \) faithfully reproduces (3.5) with coefficients given by (3.13)–(3.14). We define a linear operator \( L_i \) on \( A \) by

\[
L_i = \left( \frac{tz_i - z_{i+1}}{z_i - z_{i+1}} \right) (s_i - 1),
\]  
(3.18)

where we recall that \( s_i \) acts on polynomials by the simple transposition \( z_i \leftrightarrow z_{i+1} \).

**Proposition 3.2.** Let \( \nu \) be a binary string and associate to it the monomial \( |\nu\rangle = \prod_{i \in \mathbb{Z}} z_{\nu_i}^i \). Then \( L_i |\nu\rangle = \sum_{\nu' \in \Lambda} \ell_i(\nu', \nu) |\nu'\rangle \), where the expansion coefficients are given by (3.13)–(3.14).

**Proof.** It is easy to check that \( L_i \) has a stable action on the space of multilinear polynomials in \( \{ z \} \), meaning that we can indeed expand \( L_i |\nu\rangle \) on this space. Furthermore it is clear from its definition that \( L_i \) only acts non-trivially on the variables \( (z_i, z_{i+1}) \), meaning that there are only three cases to check:

\[
L_i \left( \prod_{k \in \mathbb{Z}} z_{\nu_k}^k \right) = \prod_{k \in \mathbb{Z}} z_{\nu_k}^k \times \begin{cases}
0, & \nu_i = \nu_{i+1}, \\
(z_{i+1} - tz_i), & \nu_i > \nu_{i+1}, \\
(tz_i - z_{i+1}), & \nu_i < \nu_{i+1},
\end{cases}
\]

(3.19)
where the vanishing of the first case is due to the fact that $L_i$ annihilates any polynomial which is symmetric in $(z_i, z_{i+1})$. The coefficients obtained from (3.19) directly match those in (3.13)–(3.14).

### 3.2.3 Matrix realization of $M_i$

Let $\mathbb{B} = \bigotimes_{i \in \mathbb{Z}} \mathbb{C}_2^i$ and construct basis vectors $|\mu\rangle = \bigotimes_{i \in \mathbb{Z}} |\mu_i\rangle_i$, where each $\mu_i$ takes values in $\{0, 1\}$ and

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Let $M_i$ be the linear operator on $\mathbb{B}$ which acts according to (3.5), with matrix elements given by (3.16)–(3.17). We see that $M_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & +t & 0 \\ 0 & +1 & -t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$ (3.20)

where the subscript indicates that the matrix acts non-trivially only on the spaces $\mathbb{C}_2^i$ and $\mathbb{C}_2^{i+1}$ of the tensor product, acting as the identity on all other spaces.

### 3.2.4 Local duality relation

Now we come to the formulation of duality in the ASEP. We say that $\psi$ is a local ASEP duality function provided that, for all $i \in \mathbb{Z}$,

$$L_i |\Psi\rangle = M_i |\Psi\rangle, \quad \text{where} \quad |\Psi\rangle = \sum_{\nu} \sum_{\mu} \psi(\nu, \mu) \prod_{k \in \mathbb{Z}} z_k^{\nu_k} |\mu\rangle.$$  

(3.21)

As we already showed in Section 3.1.3, this then implies that $\psi$ satisfies the functional version of duality $L_i[\psi(\cdot, \mu)](\nu) = M_i[\psi(\nu, \cdot)](\mu), \quad \forall \ i \in \mathbb{Z}$, with respect to the local ASEP generators (3.12) and (3.15). It is clear that any local duality function $\psi$ will also be a duality function with respect to the global generators $L = \sum_{i \in \mathbb{Z}} L_i$ and $M = \sum_{i \in \mathbb{Z}} M_i$, however the converse is not necessarily true. In the rest of the chapter we will focus on obtaining non-trivial solutions of (3.21) and its higher-rank analogue (3.22), even though we cannot a priori expect to obtain all possible global duality functions in this way.

### 3.2.5 Generalization to multi-species ASEP

All of the notions considered so far admit an extension to the multi-species ASEP. The mASEP is a continuous-time Markov chain of hopping coloured particles, i.e.
it is defined on general strings of non-negative integers, or compositions. In order to study it in our framework, we now identify $A$ and $B$ with the set of infinite compositions. We will assume that the parts of these compositions are bounded by some $r \in \mathbb{N}$, where $r$ denotes the number of particle species present in the mASEP under consideration. The ordinary ASEP is recovered by choosing $r = 1$.

The local mASEP generators $L_i$ and $M_i$ are given by the very same formulae as in Section 3.2.1, i.e. by the equations (3.12)–(3.14) and (3.15)–(3.17). The only difference, compared with the case of ASEP, is that the compositions $\nu$ and $\mu$ are no longer to be understood as binary strings, but rather as strings of non-negative integers taking values in $\{0, 1, \ldots, r\}$.

One might then wonder how to generalize (3.21) to a multi-species setting. To address this question, we begin by elevating $A$ and $B$ to vector spaces, just as we did in the case of the ordinary ASEP. We define $A$ to be the space of all polynomials in an infinite set of variables $\{z_i\}$, whose degree in the individual variable $z_i$ is bounded by $r$, for all $i \in \mathbb{Z}$. $B$ is identified with the vector space $\bigotimes_{i \in \mathbb{Z}} C_r^{i+1}$ with basis vectors $\otimes_{i \in \mathbb{Z}} \|\nu_i\rangle$, where $\nu_i \in \{0, 1, \ldots, r\}$ for all $i \in \mathbb{Z}$ and where $|0\rangle, |1\rangle, \ldots, |r\rangle$ denote the canonical basis vectors of $C_r^{i+1}$. The operators which act on these vector spaces, $L_i$ and $M_i$, are essentially those of Sections 3.2.2 and 3.2.3. $L_i$ is defined as in (3.18), without any modification. $M_i$ is now an $(r+1)^2 \times (r+1)^2$ matrix acting in $C_r^{i+1} \otimes C_r^{i+1}$, with matrix entries given by (3.16)–(3.17).

There is however one point of subtlety compared with the single-species ASEP: how does one choose a basis for $A$, such that $L_i |\nu\rangle = \sum_{\nu' \in A} \ell_i(\nu', \nu) |\nu'\rangle$ for all $\nu$, where the expansion coefficients are given by (3.13)–(3.14)? This motivates the following definition:

**Definition 3.3.** Let $\nu$ denote a composition and fix a basis $\{|\nu\rangle\} = \{f_\nu(z)\}$ of $A$. We say that this basis is admissible if $L_i |\nu\rangle = \sum_{\nu' \in A} \ell_i(\nu', \nu) |\nu'\rangle$ for all $\nu$, where the expansion coefficients are given by (3.13)–(3.14).

**Remark 3.4.** We will say more about one possible construction of an admissible basis in the next section. It is worthwhile pointing out that the simplest basis of $A$, namely $\{|\nu\rangle\} = \{\prod_{i \in \mathbb{Z}} z_i^{\nu_i}\}$, is not admissible for $r \geq 2$.

Given an admissible basis $\{f_\nu(z)\}$ of $A$, we will say that $\psi$ is a local mASEP duality function provided that, for all $i \in \mathbb{Z}$,

$$L_i |\Psi\rangle = M_i |\Psi\rangle, \quad \text{where} \quad |\Psi\rangle = \sum_{\mu \in A} \sum_{\nu \in B} \psi(\nu, \mu) f_\nu(z) |\mu\rangle. \quad (3.22)$$

### 3.3 Connection with the $t$-deformed Knizhnik-Zamolodchikov equations

This section has several aims. First, we establish a connection between the equations (3.22) and the $t$-deformed Knizhnik–Zamolodchikov ($t$KZ) equations (see Section 3.3.1). More precisely, we will show that for $\psi(\nu, \mu) = \delta_{\nu, \mu}$ (trivial duality function), the equations (3.22) are equivalent to the system of $t$KZ equations on the polynomials $\{f_\nu(z)\}$.
Second, in Sections 3.3.2–3.3.4, we discuss how to obtain solutions of the \( t \)KZ equations. For this purpose, it turns out to be convenient to restrict to the space of polynomials in \( n \) variables, when the number of \( t \)KZ equations becomes finite. In particular, we are able to make contact with a family of polynomials \( \{ f_\nu(z_1, \ldots, z_n) \} \) that were considered in \[29, 96, 97\], which have a close connection with the theory of non-symmetric Macdonald polynomials.

Third, we will outline a scheme to obtain non-trivial duality functions \( \psi \) obeying (3.22) in Section 3.3.5, given a solution of the \( t \)KZ equations. It is based on the assumption that the polynomials \( \{ f_\nu(z) \} \) depend on an extra parameter \( q \), and satisfy appropriately nice recursion relations when \( q \) is specialized to certain values. In the case of the polynomials \( \{ f_\nu(z_1, \ldots, z_n) \} \) studied in \[29\], such recursive properties do exist, and are the subject of Sections 3.5 and 3.6.

### 3.3.1 Hecke algebra, ASEP exchange relations and \( t \)KZ equations

Consider a type \( A_{n-1} \) Hecke algebra with generators \( \{ T_i \}_{1 \leq i \leq n-1} \), satisfying the relations
\[
(T_i - t)(T_i + 1) = 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_i, \\
T_i T_j = T_j T_i, \quad \forall i, j \text{ such that } |i - j| > 1.
\]

(3.23)

Both the generator \( T_i \) and its inverse \( T_i^{-1} \) can be realized as operators on the space of polynomials in \( (z_1, \ldots, z_n) \). One can easily show that
\[
T_i = t - \left( \frac{zt_i - z_{i+1}}{z_i - z_{i+1}} \right) (1 - s_i), \quad T_i^{-1} = t^{-1} - t^{-1} \left( \frac{zt_i - z_{i+1}}{z_i - z_{i+1}} \right) (1 - s_i),
\]

compose as the identity, and faithfully represent the relations (3.23).

Let \( \{ f_\nu(z) \} \) be a set of polynomials in the variables \( (z_1, \ldots, z_n) \), indexed by finite compositions \( \nu = (\nu_1, \ldots, \nu_n) \). We say that the family \( \{ f_\nu(z) \} \) is a solution of the mASEP exchange relations provided that, for all \( \nu \) and \( 1 \leq i \leq n-1 \), the following equations hold:
\[
T_i f_{(\nu_1, \ldots, \nu_i, \nu_{i+1}, \ldots, \nu_n)} = \begin{cases} 
\hat{f}_{(\nu_1, \ldots, \nu_i, \nu_{i+1}, \ldots, \nu_n)}, & \nu_i > \nu_{i+1}, \\
t f_{(\nu_1, \ldots, \nu_i, \nu_{i+1}, \ldots, \nu_n)}, & \nu_i = \nu_{i+1}.
\end{cases}
\]

(3.24)

Note that these relations also determine \( T_i f_{(\nu_1, \ldots, \nu_i, \nu_{i+1}, \ldots, \nu_n)} \) when \( \nu_i < \nu_{i+1} \). Indeed, by acting on the top equation in (3.24) with \( T_i \) and using the quadratic relation \( (T_i - t)(T_i + 1) = 0 \), after simplification we obtain for \( \nu_i < \nu_{i+1} \),
\[
T_i f_{(\nu_1, \ldots, \nu_i, \nu_{i+1}, \ldots, \nu_n)} = (t-1) f_{(\nu_1, \ldots, \nu_i, \nu_{i+1}, \ldots, \nu_n)} + t f_{(\nu_1, \ldots, \nu_{i+1}, \nu_i, \ldots, \nu_n)}.
\]

(3.25)

Returning to the local mASEP generator (3.18), we see that \( \text{L}_i = T_i - t \). Defining
\[
\theta_i(\nu) = \begin{cases} 
1, & \nu_i > \nu_{i+1}, \\
0, & \nu_i < \nu_{i+1}, \\
\frac{1}{2}, & \nu_i = \nu_{i+1}.
\end{cases}
\]
so that \( \theta_i(s_i \nu) = 1 - \theta_i(\nu) \), the relations (3.24) and (3.25) can collectively be written as

\[
L_i f_\nu(z) = t^{\theta_i(s_i \nu)} f_{s_i \nu}(z) - t^{\theta_i(\nu)} f_\nu(z) = \sum_{\nu'} \ell(\nu', \nu) f_{\nu'}(z),
\]

(3.26)

where the coefficients in the sum are given by (3.13), (3.14). Therefore, any set of polynomials \( \{ f_\nu(z) \} \) which satisfy the exchange relations (3.24), (3.25) form an admissible polynomial realization of mASEP, in the sense of Definition 3.3.

**Remark 3.5.** Restricting to compositions \( \nu \) such that \( \nu_i \in \{0, 1\} \), one can easily show that \( \{ f_\nu(z) \} = \{ \prod_{i=1}^{n} z_i^{\nu_i} \} \) is a solution of the mASEP exchange relations (indeed, this is just a rewriting of equation (3.19), when it is restricted to finitely many variables).

**Proposition 3.6.** Let \( \{ f_\nu(z) \} \) be a family of polynomials which satisfy the exchange relations (3.24) and (3.25), on which \( L_i \) acts via (3.26). Let \( M_i \) be the matrix with entries (3.16) and (3.17). Then

\[
|I\rangle := \sum_{\mu} \sum_{\nu} \delta_{\nu, \mu} f_\nu(z) |\mu\rangle = \sum_{\mu} f_\mu(z) |\mu\rangle \text{ satisfies } L_i |I\rangle = M_i |I\rangle, \quad \forall 1 \leq i \leq n - 1,
\]
or in other words, the function \( \psi(\nu, \mu) = \delta_{\nu, \mu} \) is a local mASEP duality function.

**Proof.** Writing \( |\Psi\rangle = \sum_{\mu} \sum_{\nu} \psi(\nu, \mu) f_\nu |\mu\rangle \), the polynomial part of the action is calculated using (3.26). For any \( 1 \leq i \leq n - 1 \), we obtain

\[
L_i |\Psi\rangle = \sum_{\mu} \sum_{\nu} \psi(\nu, \mu) \left( t^{\theta_i(s_i \nu)} f_{s_i \nu} - t^{\theta_i(\nu)} f_\nu \right) |\mu\rangle,
\]

\[
= \sum_{\mu} \sum_{\nu} \psi(s_i \nu, \mu) t^{\theta_i(s_i \nu)} f_\nu |\mu\rangle - \sum_{\mu} \sum_{\nu} \psi(\nu, \mu) t^{\theta_i(\nu)} f_\nu |\mu\rangle,
\]

\[
= \sum_{\mu} \sum_{\nu} L_i [\psi(\cdot, \mu)] (\nu) f_\nu |\mu\rangle,
\]

(3.27)

where in the final summation

\[
L_i [\psi(\cdot, \mu)] (\nu) = t^{\theta_i(\nu)} \left( \psi(s_i \nu, \mu) - \psi(\nu, \mu) \right).
\]

(3.28)

In a similar way, the action of \( M_i \) gives

\[
M_i |\Psi\rangle = \sum_{\mu} \sum_{\nu} \psi(\nu, \mu) t^{\theta_i(\nu)} \left( |s_i \mu\rangle - |\mu\rangle \right) f_\nu,
\]

\[
= \sum_{\mu} \sum_{\nu} \psi(\nu, s_i \mu) t^{\theta_i(s_i \mu)} |\mu\rangle f_\nu - \sum_{\mu} \sum_{\nu} \psi(\nu, \mu) t^{\theta_i(\mu)} |\mu\rangle f_\nu,
\]

\[
= \sum_{\mu} \sum_{\nu} M_i [\psi(\nu, \cdot)](\mu) f_\nu |\mu\rangle,
\]

(3.29)
where
\[ M_i [\psi(\nu, \cdot)](\mu) = \left( \theta_i(s_\mu) \psi(\nu, s_i \mu) - t^{\theta_i(\mu)} \psi(\nu, \mu) \right). \] (3.30)

The equality of (3.28) and (3.30) is manifest when \( \psi(\nu, \mu) = \delta_{\nu, \mu} \). We conclude that (3.27) and (3.29) are equal when \( |\Psi\rangle = |I\rangle \).

**Remark 3.7.** The exchange relations (3.24) are also known as the \( t \)-KZ exchange equations. They more commonly appear in the literature in terms of a stochastic higher-rank R-matrix, see e.g. [29]. For example, in the case \( r = 1 \) the exchange relations (3.24), and hence the duality described in Proposition 3.6, are recovered as the components of the equation
\[ s_i |I\rangle = \tilde{R}_i(z_i/z_{i+1}) |I\rangle, \quad \text{for all } i \in \mathbb{Z}, \] (3.31)
where \( \tilde{R}_i(z_i/z_{i+1}) \) is the R-matrix of the stochastic six-vertex model:
\[ \tilde{R}_i(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_-(z) & b_+(z) & 0 \\ 0 & b_-(z) & c_+(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{i,i+1} \] (3.32)
with
\[ b^+(z) = t \left( \frac{1 - z}{1 - tz} \right), \quad b^-(z) = \frac{1 - z}{1 - tz}, \quad c^+(z) = 1 - b^+(z), \quad c^-(z) = 1 - b^-(z). \] (3.33)

It is a simple exercise to show that (3.31) can be cast in the form \( L_i |I\rangle = M_i |I\rangle \), with \( L_i \) given by (3.18) and \( M_i \) by (3.20). This constitutes the two equivalent forms of the \( t \)-KZ equations, as advertised in Section 3.1.4.

In the rest of the chapter we seek to go beyond the diagonal observable in Proposition 3.6, with the aim of finding non-trivial mASEP duality functions. In order to do that, we will make contact with a particular family of polynomials \( f_\nu \) obeying the relations (3.24). This takes us on a brief detour through non-symmetric Macdonald theory.

### 3.3.2 Non-symmetric Macdonald polynomials

Consider polynomials in \( \mathbb{C}_{q,t}[z_1, \ldots, z_n] \) which are indexed by finite compositions \( (\mu_1, \ldots, \mu_n) \), where \( t \) is (as before) related to the hopping rate in ASEP and \( q \) is a new parameter. A well studied basis for \( \mathbb{C}_{q,t}[z_1, \ldots, z_n] \) is the basis of non-symmetric Macdonald polynomials [37,38,127]. Let us recall some facts about them.

Extend the Hecke algebra generated by \{\( T_1, \ldots, T_{n-1} \)\} and their inverses by a generator \( \omega \) which acts cyclically on polynomials in \( \mathbb{C}_{q,t}[z_1, \ldots, z_n] \):
\[ (\omega g)(z_1, \ldots, z_n) := g(qz_n, z_1, \ldots, z_{n-1}). \] (3.34)
The resulting algebraic structure is the affine Hecke algebra of type $A_{n-1}$. It has an Abelian subalgebra generated by the Cherednik–Dunkl operators $Y_i$ [36], where

$$Y_i := T_i \cdots T_{n-1} \omega T_i^{-1} \cdots T_{i-1}^{-1}.$$  \hfill (3.35)

These operators mutually commute and can be jointly diagonalized. The non-symmetric Macdonald polynomials $E_\mu \equiv E_\mu(z_1, \ldots, z_n; q, t)$ are the unique family of polynomials which satisfy

$$E_\mu = z^\mu + \sum_{\nu \prec \mu} e_{\mu,\nu}(q, t) z^\nu, \quad e_{\mu,\nu}(q, t) \in \mathbb{Q}(q, t),$$  \hfill (3.36)

$$Y_i E_\mu = y_i(\mu; q, t) E_\mu, \quad \forall \ 1 \leq i \leq n, \quad \mu \in \mathbb{Z}_{\geq 0}^n,$$  \hfill (3.37)

with eigenvalues given by

$$y_i(\mu; q, t) = q^{\mu_i} t^{\rho(\mu) + n - i + 1}, \quad \rho(\mu) = -w_\mu \cdot (1, 2, \ldots, n),$$  \hfill (3.38)

and $w_\mu \in S_n$ the minimal length permutation such that $\mu = w_\mu \cdot \mu^\dagger$.

**Proposition 3.8.** Let $\mu$ be any composition such that $\mu_i < \mu_{i+1}$. The non-symmetric Macdonald polynomials have the following recursive property:

$$E_{s_i \mu} = t^{-1} \left( T_i + \frac{1 - t}{1 - y_{i+1}(\mu)/y_i(\mu)} \right) E_\mu,$$  \hfill (3.39)

where we abbreviate the eigenvalues (3.38) by $y_i(\mu; q, t) \equiv y_i(\mu)$, and where we use $s_i \mu$ to denote the exchange of the parts $\mu_i$ and $\mu_{i+1}$, i.e. $s_i \mu = (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n)$.

**Proof.** This is a standard fact in the theory, see [104,109,110,132]. \hfill \Box

The non-symmetric Macdonald polynomials are meromorphic functions of the parameter $q$. Their singularities occur at points of the form $q = t^{-m}$, where $m \in \mathbb{Q}_{>0}$. These singularities play a key role, so we give some results which elucidate their structure. The starting point is the following observation from [96]:

**Proposition 3.9.** Define a generating series $Y(w) := \sum_{i=1}^n Y_i w^i$ of the Cherednik–Dunkl operators, and a further generating series $y_\mu(w) := \sum_{i=1}^n y_i(\mu; q, t) w^i$ of their eigenvalues. For any composition $\mu$, we have

$$E_\mu(z; q, t) = \prod_{\nu \prec \mu} \frac{Y(w) - y_\nu(w)}{y_\mu(w) - y_\nu(w)} \cdot z^\nu,$$  \hfill (3.40)

where the product is taken over all compositions $\nu$ which are smaller than $\mu$ with respect to the ordering $\prec$. Note that (3.40) holds for any $w$, even though the left hand side is independent of this parameter.

*More precisely, $E_\mu$ may possess poles at $q = \exp(2\pi ik/\ell) t^{-m/\ell}$ for $\ell, m \in \mathbb{N}$ and $0 \leq k \leq \ell - 1$.* We always focus on singular values of $q$ for which $k = 0$. 
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Proof. By the monicity (3.36) of the non-symmetric Macdonald polynomials, we are able to write

$$z^\mu = E_\mu + \sum_{\nu \prec \mu} d_{\mu,\nu}(q,t) E_\nu,$$

for some coefficients $d_{\mu,\nu}(q,t) \in \mathbb{Q}(q,t)$. We then act on this equation with the product of operators $\prod_{\nu \prec \mu} (Y(\nu(w) - y_\nu(w))(y_\mu(w) - y_\nu(w))^{-1}$. In view of the eigenvalue relations (3.37), all polynomials $E_\nu$ with $\nu \prec \mu$ vanish under this operation, while $E_\mu$ is mapped to itself. Equation (3.40) follows immediately.

3.3.3 Reduction

Although Proposition 3.9 is easy to prove (it can be viewed as Lagrange interpolation), a slight variation of it yields an interesting statement about the structure of the singularities in $E_\mu$:

**Proposition 3.10.** Fix a positive rational number $m$, a natural number $p$ and a composition $\mu$ such that

$$\text{Coeff}_p[E_\mu, m] = \lim_{q \to t^{-m}} (1 - qt^m)^p E_\mu(z; q, t)$$

is well defined and is non-zero. Then one has the expansion

$$\text{Coeff}_p[E_\mu, m] = \lim_{q \to t^{-m}} (1 - qt^m)^p \left( \sum_{\nu \in \mathcal{E}_\mu} c_\nu(q, t) E_\nu(z; q, t) \right)$$

for some family of coefficients $c_\nu(q, t)$, and where the sum is over the set of compositions

$$\mathcal{E}_\mu = \{ \nu : \nu \prec \mu, y_\nu(w) = y_\mu(w) \text{ at } q = t^{-m} \}.$$

Proof. Start from the generic expansion (3.41) and act on it with the product of operators $\prod_{\nu \prec \mu, \nu \notin \mathcal{E}_\mu} (Y(\nu(w) - y_\nu(w))(y_\mu(w) - y_\nu(w))^{-1}$, i.e. the same product as in the proof of Proposition 3.9, excluding compositions in the set $\mathcal{E}_\mu$. Since $(Y(w) - y_\nu(w))$ annihilates $E_\nu$, after acting with the preceding product of operators the only remaining terms in (3.41) will be those for which $\nu \in \mathcal{E}_\mu$. We thus obtain the equation

$$\prod_{\nu \prec \mu, \nu \notin \mathcal{E}_\mu} \frac{Y(\nu(w) - y_\nu(w))}{y_\mu(w) - y_\nu(w)} \cdot z^\mu = E_\mu + \sum_{\nu \in \mathcal{E}_\mu} d_{\mu,\nu}(q,t) E_\nu.$$

Studying the left hand side of the expression (3.44), we see that its singularities occur for compositions $\nu$ such that $y_\mu(w) = y_\nu(w)$; or more explicitly, by (3.38),
compositions such that
\[ q^{\mu_i} t^{\rho(\mu)_i} = q^{\nu_i} t^{\rho(\nu)_i}, \quad \forall \ 1 \leq i \leq n. \]  
(3.45)

For generic \( q \) and \( t \), it is obvious that (3.45) has no solution other than the tautological one, \( \nu = \mu \). On the other hand, for \( q = t^{-m} \) with \( m \in \mathbb{Q}_{>0} \), non-trivial solutions of (3.45) become possible. Since we have demanded that all such compositions \( \nu \) are excluded from the product, the left hand side of (3.44) has a well-defined limit when \( q \to t^{-m} \). Multiplying both sides of (3.44) by \((1 - qt^m)^p\) and sending \( q \to t^{-m} \), the left hand side vanishes. After rearrangement, we recover (3.42). \( \square \)

The following theorem (for a special value of \( p \)) is a stronger version of Proposition 3.10, in which only a single composition in the sum (3.42) is retained. We were unable to locate this result anywhere in the literature.

**Theorem 3.11.** Fix \( m, p, \mu \) as in the statement of Proposition 3.10, and assume in addition that \( p = |E_{\mu}| \), where \( E_{\mu} \) is defined in (3.43). Then there exists a unique composition \( \nu \) for which

\[ E_{\nu}(z; t^{-m}, t) := \lim_{q \to t^{-m}} E_{\nu}(z; q, t) \]

is well defined and such that

\[ \text{Coeff}_p[E_{\mu}, m] \propto E_{\nu}(z; t^{-m}, t), \]  
(3.46)

where we recall the meaning of \( \propto \) given in equation (3.11).

**Proof.** We start from the expression (3.40) for \( E_{\mu} \) and assume there are exactly \( p \) solutions of (3.45), meaning that the cardinality of \( E_{\mu} \) is equal to \( p \). Call these solutions \( \nu[1], \ldots, \nu[p] \) and assume that they have the ordering \( \nu[1] \prec \cdots \prec \nu[p] \). Then by direct calculation on (3.40), we have

\[ \text{Coeff}_p[E_{\mu}, m] \propto \left[ \prod_{\kappa < \mu, \kappa \notin E_{\mu}} Y(w) - y_{\kappa}(w) \prod_{\nu \in E_{\mu}} \prod_{i=1}^p (Y(w) - y_{\nu[i]}(w)) \cdot z^\mu \right]_{q=t^{-m}} \]  
(3.47)

where we suppress the proportionality factors which arise in taking this limit. There cannot be any singularities on the right hand side of (3.47), since \( \nu[1], \ldots, \nu[p] \) are the only compositions for which (3.45) holds, so the specialization \( q = t^{-m} \) can be freely taken.

For generic \( q \), it is an easy consequence of (3.36), (3.37) and (3.41) in combination that

\[ (Y(w) - y_{\mu}(w)) z^\mu = \sum_{\nu < \mu} \tilde{e}_{\mu, \nu}(q, t; w) z^\nu, \]  
(3.48)
where the sum on the right hand side is over compositions $\nu$ which are strictly less than $\mu$ with respect to the ordering $\prec$, for some coefficients $\varepsilon_{\mu,\nu}(q, t; w)$ which are polynomial in $q$. The polynomiality of the coefficients is ensured by (3.34) and (3.35). This equation therefore extends to specializations $q = t^{-m}$. Equation (3.47) can now be further simplified, by the following iterative procedure. Since $y_{\nu[p]}(w) = y_{\mu}(w)$ at $q = t^{-m}$, by repeated use of (3.48) we see that

$$\left[ \prod_{\nu[p] \prec \kappa \prec \mu} \frac{Y(w) - y_{\kappa}(w)}{y_{\mu}(w) - y_{\kappa}(w)} \cdot (Y(w) - y_{\nu[p]}(w)) \cdot z^{\mu} \right]_{q = t^{-m}} \propto \left( z^{\nu[p]} + \sum_{\nu \prec \nu[p]} g_{\nu}(t; w)z^{\nu} \right),$$

for appropriate coefficients $g_{\nu}(t; w)$; i.e. starting from the monomial $z^{\mu}$, it is successively lowered to monomials $z^{\kappa}$ which are smaller in the $\prec$ ordering, until we arrive at $z^{\nu[p]}$. We can then repeat this process, using the fact that $y_{\nu[i]}(w) = y_{\nu[i]}(w)$ at $q = t^{-m}$, for all $1 < i \leq p$. We arrive ultimately at the expression

$$\text{Coeff}_{p}[E_{\mu}, m] \propto \left[ \prod_{\kappa \prec \nu[1]} \frac{Y(w) - y_{\kappa}(w)}{y_{\nu[1]}(w) - y_{\kappa}(w)} \cdot \left( z^{\nu[1]} + \sum_{\nu \prec \nu[1]} h_{\nu}(t; w)z^{\nu} \right) \right]_{q = t^{-m}}$$

for some coefficients $h_{\nu}(t; w)$, and note that all sub-leading terms in the sum vanish under the product of operators, by exactly the same filtering argument used above. We have thus shown that

$$\text{Coeff}_{p}[E_{\mu}, m] \propto \left[ \prod_{\kappa \prec \nu[1]} \frac{Y(w) - y_{\kappa}(w)}{y_{\nu[1]}(w) - y_{\kappa}(w)} \cdot z^{\nu[1]} \right]_{q = t^{-m}} = E_{\nu[1]}(z; t^{-m}, t),$$

establishing both the existence and uniqueness claim. 

Notice that this procedure specifies the $\nu$ appearing in (3.46) as the minimal composition (with respect to $\prec$) which satisfies (3.45) at $q = t^{-m}$. It does not, however, give $\nu$ constructively: one still needs to do the work of finding solutions of (3.45).

Based on experimentation with the non-symmetric Macdonald polynomials we are led to make the following conjecture, generalizing Theorem 3.11 to arbitrary values of $p$, which we were unable to prove in full generality. All of our subsequent results on duality functions can be (and are) proved independently of this conjecture, but it remains an important conceptual cornerstone of this work:

**Conjecture 3.12.** Fix a positive rational number $m$, a natural number $p$ and a composition $\mu$ such that $\text{Coeff}_{p}[E_{\mu}, m]$ is well defined and non-zero. Then there exists a unique composition $\nu$ for which

$$E_{\nu}(z; t^{-m}, t) := \lim_{q \to t^{-m}} E_{\nu}(z; q, t)$$

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is well defined and such that
\[ \text{Coeff}_p[E_\mu, m] \propto E_\nu(z; t^{-m}, t). \] (3.49)

### 3.3.4 Another non-symmetric basis

We will make use of a further set of non-symmetric polynomials, which also comprise a basis of \( \mathbb{C}_{q,t}[z_1, \ldots, z_n] \). We refer to them as ASEP polynomials, and denote them by \( f_\mu = f_\mu(z_1, \ldots, z_n; q, t) \). They are defined as the unique family of polynomials which satisfy
\[
\begin{align*}
  f_\delta(z; q, t) &= E_\delta(z; q, t), \quad \forall \delta = (\delta_1 \leq \cdots \leq \delta_n), \\
  f_{s_i \mu}(z; q, t) &= T_i^{-1} f_\mu(z; q, t), \quad \text{when } \mu_i < \mu_{i+1},
\end{align*}
\] (3.50) (3.51)

where, as before, \( s_i \mu = (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n) \). Clearly by repeated use of (3.51), one is able to construct \( f_\mu \) for any composition, starting from \( f_{\mu^-} = E_{\mu^-} \). Furthermore, because of the Hecke algebra relations (3.23), \( f_\mu \) is independent of the order in which one performs the operations (3.51), making the definition unambiguous.

It can be shown \([29, 97]\) that the ASEP polynomials are equivalently defined as the unique monic polynomials \( f_\mu = z^\mu + \sum_{\nu < \mu} c_{\mu, \nu}(q, t) z^\nu \), for some family of coefficients \( c_{\mu, \nu}(q, t) \), satisfying the \( t \)-KZ relations (3.24) for \( 1 \leq i \leq n - 1 \), and the cyclic boundary condition
\[
\begin{align*}
  f_{\mu_n, \mu_1, \ldots, \mu_{n-1}}(q z_n, z_1, \ldots, z_{n-1}; q, t) &= q^{\mu_n} f_{\mu_1, \ldots, \mu_n}(z_1, \ldots, z_n; q, t).
\end{align*}
\] (3.52)

In view of the discussion in Section 3.3.1, they are therefore fundamental in the study of duality functions for the mASEP. This is not the first time that the family \( \{ f_\mu \} \) has appeared in the context of stochastic processes: in \([29]\) these polynomials also played the role of (inhomogeneous generalizations of) stationary state probabilities in the mASEP on a ring.

We stress that, in general, \( f_\mu \neq E_{\mu^-} \); the non-symmetric Macdonald and ASEP polynomials coincide when their indexing composition is an anti-partition, but are otherwise different, which is readily apparent from their different recursive properties (3.39) and (3.51). One basis can be expanded triangularly in terms of the other, however, as we now show:

**Definition 3.13.** A composition sector is the set of all compositions with a common anti-dominant (or dominant) ordering. If \( \mu \) is a composition, the composition sector \( \sigma(\mu) \) is the following set:
\[
\sigma(\mu) := \{ \nu | \nu^- = \mu^- \}.
\]

**Proposition 3.14.** For any composition \( \mu \), there are unique triangular expand-
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\( E_\mu(z; q, t) = f_\mu(z; q, t) + \sum_{\nu \in \sigma(\mu) \atop \nu < \mu} \tilde{c}_{\mu, \nu}(q, t)f_\nu(z; q, t), \quad (3.53) \)

\( f_\mu(z; q, t) = E_\mu(z; q, t) + \sum_{\nu \in \sigma(\mu) \atop \nu \prec \mu} \tilde{d}_{\mu, \nu}(q, t)E_\nu(z; q, t), \quad (3.54) \)

for some coefficients \( \tilde{c}_{\mu, \nu}(q, t) \) and \( \tilde{d}_{\mu, \nu}(q, t) \), relating the non-symmetric Macdonald and ASEP bases.

**Proof.** The uniqueness claim is immediate, since both families are bases for \( \mathbb{C}_{q,t}[z_1, \ldots, z_n; q, t] \). To prove the form of the expansion (3.53), we note that it holds trivially in the case where \( \mu \) is an anti-partition. Based on this, assume that it holds for some composition \( \mu \) such that \( \mu_i < \mu_{i+1} \), for some \( 1 \leq i \leq n - 1 \). By application of (3.39), we then have

\[
E_{s_i \mu} = t^{-1} \left( T_i + \frac{1 - t}{1 - y_{i+1}(\mu)/y_i(\mu)} \right) \left( f_\mu + \sum_{\nu \in \sigma(\mu) \atop \nu < \mu} \tilde{c}_{\mu, \nu}(q, t)f_\nu \right). \quad (3.55)
\]

We need to act with the Hecke generator \( T_i \) on the sum over ASEP polynomials. The action of \( T_i \) on any given \( f_\nu \) produces some linear combination of \( f_\nu \) and \( f_{s_i \nu} \), as can be seen from (3.24) and (3.25). Both \( f_\nu \) and \( f_{s_i \nu} \) obviously lie in the composition sector \( \sigma(\mu) \equiv \sigma(s_i \mu) \). Now when \( \mu_i < \mu_{i+1} \) and \( \nu \prec \mu \) hold, it is clear that both \( \nu \prec s_i \mu \) and \( s_i \nu \prec s_i \mu \) also hold. Using these observations in (3.55), we can then write

\[
E_{s_i \mu} = f_{s_i \mu} + \sum_{\nu \in \sigma(s_i \mu) \atop \nu < s_i \mu} \tilde{c}_{s_i \mu, \nu}(q, t)f_\nu
\]

for appropriate coefficients \( \tilde{c}_{s_i \mu, \nu}(q, t) \). Note that the coefficient of \( f_{s_i \mu} \) must be 1, using equation (3.25) to calculate \( t^{-1}T_if_\mu \). This proves that (3.53) holds generally, by induction.

Finally, by virtue of (3.53), the matrix \( \tilde{c} \) with entries \( \tilde{c}_{\mu, \nu}(q, t) \) is block-diagonal over composition sectors, with triangular blocks. It can therefore be inverted to yield (3.54), where the transition matrix \( \tilde{d} \) with entries \( \tilde{d}_{\mu, \nu}(q, t) \) is the inverse of \( \tilde{c} \).

Like the non-symmetric Macdonald polynomials, the ASEP polynomials may become singular when \( q = t^{-m}, \ m \in \mathbb{Q}_{>0} \). To clarify the structure of these singularities, we seek a result which directly parallels Conjecture 3.12.

**Theorem 3.15.** Fix a positive rational number \( m \), a natural number \( p \) and an anti-partition \( \delta \) for which Conjecture 3.12 holds. Then there exists a unique
anti-partition $\epsilon$ such that
\[ f_\nu(z; t^{-m}, t) := \lim_{q \to t^{-m}} f_\nu(z; q, t), \]
is well defined for all compositions $\nu \in \sigma(\epsilon)$, and such that
\[ \text{Coeff}_p[f_\mu, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t) f_\nu(z; t^{-m}, t), \tag{3.56} \]
for all $\mu \in \sigma(\delta)$ and suitable coefficients $\psi(\nu, \mu; t)$. Here the definition of \text{Coeff}_p is the usual one; see equation (3.10).

**Proof.** Let us begin by analyzing the case where $\mu = \delta$. In that case, using the direct equivalence of ASEP and non-symmetric Macdonald polynomials and the result of Conjecture 3.12 we have
\[ \text{Coeff}_p[f_\delta, m] \equiv \text{Coeff}_p[E_\delta, m] \propto E_\kappa(z; t^{-m}, t), \]
where $\kappa$ is the minimal composition satisfying the relations $y_i(\delta) = y_i(\kappa)$ at $q = t^{-m}$. Let $\epsilon = \kappa^-$. Using equation (3.53), we know that an expansion of the form
\[ E_\kappa(z; q, t) = f_\kappa(z; q, t) + \sum_{\nu \in \sigma(\epsilon)} \hat{c}_{\kappa, \nu}(q, t) f_\nu(z; q, t) \]
exists, and each $f_\nu$ appearing on the right hand can be obtained by the successive action of inverse Hecke generators $T_i^{-1}$ acting on $f_\epsilon = E_\epsilon$. The action of such generators does not introduce any singular points in $q$, and we know that $\lim_{q \to t^{-m}} E_\epsilon$ is well defined; it follows that one can freely set $q = t^{-m}$ in the above equation, establishing that
\[ \text{Coeff}_p[f_\delta, m] \propto f_\kappa(z; t^{-m}, t) + \sum_{\nu \in \sigma(\epsilon)} \hat{c}_{\kappa, \nu}(t^{-m}, t) f_\nu(z; t^{-m}, t). \tag{3.57} \]
This proves the claim (3.56) for anti-partitions $\mu = \delta$. The general $\mu$ case now follows immediately, by acting on the equation (3.57) with products of inverse Hecke generators. This is permitted, since the action of these generators commutes with the limits being taken, and it allows $f_\delta$ to be converted into an arbitrary ASEP polynomial $f_\mu$. The action of $T_i^{-1}$ on the right hand side of (3.57) also manifestly preserves the sector being summed over. \[\square\]

### 3.3.5 Dualities from reductions of ASEP polynomials

In the previous sections we have outlined some of the theory surrounding the non-symmetric Macdonald and ASEP polynomials, with particular emphasis on their singular points in the parameter $q$. We now apply these results to the construction of non-trivial duality functions in mASEP systems. The following result is the
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central idea of this chapter:

**Theorem 3.16.** Fix a positive rational number $m$, a natural number $p$ and an anti-partition $\delta$ such that for all compositions $\mu \in \sigma(\delta)$ there exists an expansion

$$\text{Coeff}_p[f_{\mu}, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t) f_{\nu}(z; t^{-m}, t),$$

(3.58)

where $\epsilon$ is some other known anti-partition. Then $\psi(\nu, \mu; t) \equiv \psi(\nu, \mu)$ defines a local duality function of the mASEP with generator $L_i$ given by (3.12)–(3.14), and the mASEP with generator $M_i$ given by (3.15)–(3.17). Explicitly, we have

$$L_i[\psi(\cdot, \mu)](\nu) = M_i[\psi(\nu, \cdot)](\mu), \quad \forall 1 \leq i \leq n - 1,$$

(3.59)

where the left hand side of (3.59) is given by (3.28), and the right hand side by (3.30).

**Proof.** From Proposition 3.6, we know that

$$|\mathcal{I} \rangle = \sum_{\mu \in \sigma(\delta)} f_{\mu}(z; q, t) |\mu \rangle$$

satisfies $L_i |\mathcal{I} \rangle = M_i |\mathcal{I} \rangle$ for all $1 \leq i \leq n - 1$. Exploiting the freedom to take limits of $q$, since it does not appear in the local mASEP generators, we see that

$$|\mathcal{I}_{p,m} \rangle := \text{Coeff}_p[|\mathcal{I} \rangle, m] = \sum_{\mu \in \sigma(\delta)} \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t) f_{\nu}(z; t^{-m}, t) |\mu \rangle$$

satisfies $L_i |\mathcal{I}_{p,m} \rangle = M_i |\mathcal{I}_{p,m} \rangle$ for all $1 \leq i \leq n - 1$. Converting this to its functional form, we obtain precisely the relations (3.59).

**Remark 3.17.** The anti-partitions $\delta$ and $\epsilon$ label the particle content of the two mASEP systems appearing in Theorem 3.16. More precisely, Theorem 3.16 presents a duality between one mASEP with $m_i(\delta)$ particles of type $i$ and another mASEP with $m_i(\epsilon)$ particles of type $i$, $0 \leq i \leq r$.

**Remark 3.18.** Theorem 3.16 gives rise to a diverse collection of duality functions. Once the particle content of one mASEP system is fixed by choosing $\delta$, there will in general be multiple choices of $m \in \mathbb{N}$ and $p \in \mathbb{Q}_{>0}$ for which $\text{Coeff}_p[f_{\delta}, m]$ exists and is non-zero. Each such choice will give rise to a different $\epsilon$, labelling the particle content of the second, reduced mASEP system.

It is beyond the scope of the present chapter to explore all possible duality functions arising from Theorem 3.16. One of the obstacles of such a classification

*The expansion (3.58) is guaranteed to be possible if the conditions in Theorem 3.15 are met, namely the validity of Conjecture 3.12. However it is sometimes possible to show that (3.58) holds, independently of Conjecture 3.12, by proceeding via the weaker Proposition 3.10. This is the course of action that we take in Sections 3.5 and 3.6.*
is that one needs a way of calculating the coefficients appearing in (3.58), which is
difficult in full generality. We hope to return to this problem in a future publication.

For the purposes of the current work, we prefer to analyse (3.58) for some
special choices of \( \{ \delta, p, m \} \). Section 3.5 will look at the case \( \{ \delta, p, m \} = \{(0-n^{-m}, r^m), 1, m \} \) for general \( r \geq 1 \). Section 3.6 deals with the case \( \{ \delta, p, m \} = \{(0-n^{-m_1-m_2}, 1^{m_1}, 2^{m_2}), 1, M \} \) for general \( m_1, m_2, M \geq 1 \).

### 3.4 Explicit formulae for the ASEP polynomials

In order to calculate expansions of the form (3.58) explicitly, it naturally helpful
to have explicit expressions for the polynomials \( f_\mu(z; q, t) \) themselves. Such formulae
were obtained in [29, 53], and turn out to be quite expedient for the purposes of
this chapter, since they lay bare the structure of the singularities of \( f_\mu(z; q, t) \) as a function of \( q \).

#### 3.4.1 Matrix product formula for \( f_\mu(z; q, t) \)

Let us recall some of the details of the matrix product Ansatz. Given a compo-
sition \( \mu \) whose largest part is equal to \( r \), one seeks a construction of the form

\[
f_\mu(z_1, \ldots, z_n; q, t) = \Omega_\mu(q,t) \times \text{Tr} \left( A_{\mu_1}(z_1) \ldots A_{\mu_n}(z_n)S \right),
\]

(3.60)

where \( \{ A_i(z) \} \) are a collection of explicit matrices, and \( \Omega_\mu \) is a normaliza-
tion constant (recall that \( f_\mu \) is monic, i.e. it expands as \( f_\mu = z^\mu + \sum_{\nu < \mu} c_{\mu,\nu}(q,t)z^\nu \)).
To proceed with the construction (3.60), two steps are necessary. First, one needs
to translate the exchange relations (3.24) and (3.52), which uniquely characterize
the family \( \{ f_\mu \} \), into algebraic relations between the \( A_i(z) \) and \( S \) operators. The
algebraic structure which arises from this is the Zamolodchikov–Faddeev (ZF) alge-
bra.\(^*\) Second, one needs to seek a suitable representation of this algebra, so that the
trace in (3.60) can be taken.

Following these steps, an explicit matrix product expression (3.60) for \( f_\mu(z; q, t) \)
was obtained in [29]. It involves a family of infinite-dimensional matrices \( \phi, \phi^\dagger, k \)
which satisfy the \( t \)-boson algebra. Their matrix entries are given explicitly by

\[
[\phi]_{i,j} = \delta_{i+1,j}(1-t^i), \quad [\phi^\dagger]_{i,j} = \delta_{i,j+1}, \quad [k]_{i,j} = \delta_{i,j}t^i, \quad \text{for all } i, j \in \mathbb{N}.
\]

It is easy to check that this provides a faithful representation of the \( t \)-boson algebra
\( \bar{B} \), i.e. the matrices obey the relations

\[
\phi\phi^\dagger = 1 - tk, \quad \phi^\dagger \phi = 1 - k, \quad tk\phi = \phi k, \quad k\phi^\dagger = t\phi^\dagger k.
\]

(3.61)

We refer the reader to [29] for the matrix product formula for generic \( f_\mu(z; q, t) \).

\(^*\)In fact the resulting structure is an extended version of the ZF algebra, since it not only
prescribes commutation relations between the operators \( \{ A_i(z) \} \), but also with the “twist” operator
\( S \).
In this chapter we focus on two sub-families of compositions for which the formula (3.60) becomes simple. We detail these below:

**The case \( \mu^- = (0^{n-m}, r^m) \).**

We begin by analyzing the matrix product expression when \( \mu \) is a composition with parts of size 0 and size \( r \), only. Let \( L(z) \) denote the following 2 × 2 matrix, whose entries are \( t \)-bosons:

\[
L(z) = \begin{pmatrix} 1 & \phi \\ z\phi^\dagger & z \end{pmatrix},
\]

i.e. the entries of \( L(z) \) are themselves to be understood as infinite dimensional matrices. From this, construct a two-component vector

\[
\begin{pmatrix} A_0(z) \\ A_r(z) \end{pmatrix} := \underbrace{L(z) \otimes \cdots \otimes L(z)}_{r-1} \begin{pmatrix} 1 \\ z \end{pmatrix}, \tag{3.62}
\]

where \( L(z) \) is composed \( r-1 \) times under the operation \( \otimes \), meaning matrix multiplication combined with taking Kronecker products of matrix entries:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} := \begin{pmatrix} a \otimes e + b \otimes g & a \otimes f + b \otimes h \\ c \otimes e + d \otimes g & c \otimes f + d \otimes h \end{pmatrix}.
\]

The resulting operators \( A_0(z) \) and \( A_r(z) \) are thus polynomial in \( z \), with coefficients in \( \mathcal{B}^{\otimes r-1} \). One can easily calculate the first few examples of these operators:

- \( r = 1 \) : \( A_0(z) = 1 \), \( A_1(z) = z \)
- \( r = 2 \) : \( A_0(z) = 1 + z\phi \), \( A_2(z) = z\phi^\dagger + z^2 \)
- \( r = 3 \) : \( A_0(z) = 1 \otimes 1 + z(1 \otimes \phi + \phi \otimes \phi^\dagger) + z^2(\phi \otimes 1) \), \( A_3(z) = z(\phi^\dagger \otimes 1) + z^2(\phi^\dagger \otimes \phi + 1 \otimes \phi^\dagger) + z^3(1 \otimes 1) \).

**Proposition 3.19.** Let \( \mu \) be a composition with anti-dominant ordering \( \mu^- = (0^{n-m}, r^m) \). Then

\[
f_\mu(z_1, \ldots, z_n; q, t) = \prod_{i=1}^{r-1} (1 - q^i) \times \text{Tr} \left( A_{\mu_1}(z_1)A_{\mu_2}(z_2) \cdots A_{\mu_n}(z_n)(k^u(r-1) \otimes k^u(r-2) \otimes \cdots \otimes k^u) \right), \tag{3.63}
\]

where each operator \( A_i(z) \) is given by (3.62), \( q \) is parametrized through \( u \) via \( q := t^u \), and the trace is taken over \( \mathcal{B}^{\otimes r-1} \) and is to be understood as a formal power series in \( t \).

**Proof.** This follows from the matrix product expression in [29], under some simpli-
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The result in [29] applies to generic compositions \( \mu \), and makes use of \( r \) commuting copies of the \( t \)-boson algebra \( \{ B_i \}_{1 \leq i \leq r} \), where \( r \) is the largest part of \( \mu \). However, whenever \( \mu \) consists of less than \( r \) distinct non-zero parts, the dependence on some of these families drops out. In the case at hand, \( \mu \) consists of only one type of non-zero part (namely, \( r \)), and can therefore be expressed via a matrix product that only uses a single copy of \( B \). It is this simplification of the formula in [29] which gives rise to (3.63); for simplicity we will suppress further details.

Remark 3.20. One can use equation (3.63) to obtain a completely explicit expression for any given polynomial \( f_\mu(z_1, \ldots, z_n; q, t) \), where \( \mu^\ast = (0^{n-m}, r^m) \). The calculation of the trace amounts to taking geometric series, and for that reason \( f_\mu \) acquires denominators of the form \( (1 - qt^{m_1}) \). This is in accordance with the singularities that \( f_\mu \) is expected to have, as a function of \( q \).

The case \( \mu^\ast = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}) \).

An even simpler case is that of compositions whose parts are of size 2, or less. We refer to these as rank-two compositions. In that situation we define directly

\[
A_0(z) = 1 + z\phi, \quad A_1(z) = zk, \quad A_2(z) = z\phi^\dagger + z^2.
\] (3.64)

Proposition 3.21. For any rank-two composition \( \mu \), we have

\[
f_\mu(z_1, \ldots, z_n; q, t) = (1 - qt^{m_1}) \times \text{Tr} \left( A_{\mu_1}(z_1) \cdots A_{\mu_n}(z_n) k^u \right),
\] (3.65)

where \( m_1 = m_1(\mu) \) is the number of parts in \( \mu \) equal to 1, \( q = t^u \), and where the trace is taken over \( B \) and hence is identified as a formal power series in \( t \).

Proof. This is exactly the special case \( r = 2 \) of the matrix product formula in [29]; see Section 3 therein.

3.4.2 Summation formulae

In [53] an alternative formula for \( f_\mu(z; q, t) \) was obtained, in terms of multiple summations over the symmetric group \( S_n \). This expression can be derived from the matrix product formula of [29], by explicitly evaluating all traces which appear. In view of its complexity we do not repeat the general formula here, but again focus on the special cases which are of interest in this chapter.

The case \( \delta = (0^{n-m}, r^m) \).

Let \( \alpha \) and \( \beta \) be rank-one compositions, and for any \( j \geq 1 \) define coefficients

\[
C_j(\alpha, \beta; q, t) := \text{Tr} \left( L(\alpha_1, \beta_1) \cdots L(\alpha_n, \beta_n) k^j u \right),
\]

where \( L(0, 0) = L(1, 1) = 1, L(0, 1) = \phi, L(1, 0) = \phi^\dagger \). These coefficients are rational functions in \( q = t^u \) and \( t \); for given rank-one compositions \( \alpha \) and \( \beta \) they can
be readily evaluated by tracing over the resulting product of infinite-dimensional matrices. We will make use of the following key properties:

**Proposition 3.22.** $C_j(\alpha, \beta; q, t)$ vanishes unless $|\alpha| = |\beta|$, where $|\alpha|$, $|\beta|$ denote the weights of the compositions $\alpha$, $\beta$ (see Section 3.1.5). In the case where $\#\{(\alpha_i, \beta_i) = (0, 1)\} = \#\{(\alpha_i, \beta_i) = (1, 0)\} = m$, one has

$$C_j(\alpha, \beta; q, t) = \frac{p_j(\alpha, \beta; q, t)}{\prod_{i=0}^{m}(1 - q^2t^i)}, \quad (3.66)$$

where $p_j(\alpha, \beta; q, t)$ is a polynomial in $(q, t)$.

**Proof.** Since $\prod_{i=1}^{n} L(\alpha_i, \beta_i)$ contains exactly $m \phi$ operators and $m \phi^\dagger$ operators, by making use of the $t$-boson algebra relations (3.61) one can see that $\prod_{i=1}^{n} L(\alpha_i, \beta_i) = \sum_{i=0}^{m} c_i(t) k^i$, where the coefficients $c_i(t)$ are some appropriate polynomials in $t$. Taking the trace leads to

$$C_j(\alpha, \beta; q, t) = \text{Tr} \left( L(\alpha_1, \beta_1) \ldots L(\alpha_n, \beta_n) k^{ju} \right)$$

$$= \sum_{i=0}^{m} c_i(t) \text{Tr} \left( k^{i+ju} \right) = \sum_{i=0}^{m} c_i(t) \frac{1}{1 - q^2t^i}.$$ 

The statement (3.66) follows immediately, after collecting the terms in the final sum over a common denominator. \qed

**Proposition 3.23.** Fix an anti-partition $\delta = (0^{n-m}, r^m)$ and its rank-one projection, $\delta^* = (0^{n-m}, 1^m)$. The formula

$$f_\delta = \prod_{i=1}^{n} (1 - q^i) \sum_{\mu[1] \in \sigma(\delta^*)} \ldots \sum_{\mu[r-1] \in \sigma(\delta^*)} z^{\delta^*} \left( \prod_{j=1}^{r-1} C_j(\mu[j+1], \mu[j]; q, t) z^{\mu[j]} \right) \quad (3.67)$$

holds, where $\mu[1], \ldots, \mu[r-1]$ are dummy indices, each being summed over all rank-one compositions in the sector $\sigma(\delta^*)$, and $\mu[r] \equiv \delta^*$.

**Proof.** This follows from the matrix product formula (3.63), by decomposing the trace over the $r - 1$ factors in the tensor product, and using the definition (3.62) of the $A_i(z)$ operators. \qed

**The case** $\delta = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2})$.

**Proposition 3.24.** Fix a rank-two anti-partition $\delta = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2})$. The formula

$$f_\delta = \prod_{j=1}^{m_1+m_2} (z_{n-j+1}) \sum_{i=0}^{m_2} t^{m_1} \prod_{j=1}^{i} \left( \frac{1 - t^j}{1 - q t^{m_1 + j}} \right) \times \times \ldots \times \left( \frac{1 - t^j}{1 - q t^{m_1 + j}} \right) \times \ldots \times \left( \frac{1 - t^j}{1 - q t^{m_1 + j}} \right)$$

*Throughout the rest of the chapter, we will mostly use $f_\mu$ as shorthand for $f_\mu(z_1, \ldots, z_n, q, t)$. \*
\( e_i(z_1, \ldots, z_{n-m_1-m_2}) e_{m_2-i}(z_{n-m_2+1}, \ldots, z_n) \)

holds, where \( e_i \) denotes the \( i \)-th elementary symmetric polynomial, given by the generating series expression

\[
\sum_{i=0}^{N} e_i(x_1, \ldots, x_N) y^i = \prod_{j=1}^{N}(1 + x_j y), \quad \text{for any alphabet } (x_1, \ldots, x_N).
\]

**Proof.** Using the matrix product formula (3.65) in the case \( \mu = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}) \), we find that

\[
f_\delta = (1 - qt^{m_1}) \times \text{Tr} \left( \prod_{i=1}^{n-m_1-m_2} (1 + z_i \phi) \cdot \prod_{j=n-m_1-m_2+1}^{n-m_2} (z_j k) \cdot \prod_{l=n-m_2+1}^{n} (z_l \phi^\dagger + z_l^2) \cdot k^u \right)
\]

\[
= (1 - qt^{m_1}) \prod_{j=1}^{n-m_1-m_2} (z_{n-j+1}) \times \text{Tr} \left( \prod_{i=1}^{n-m_1-m_2} (1 + z_i \phi) \cdot \prod_{l=n-m_2+1}^{n} (t^{m_1} \phi^\dagger + z_l) \cdot k^{u+m_1} \right),
\]

where we have used the commutation relation \( k \phi^\dagger = t \phi^\dagger k \) to bring the product \( k^{m_1} \) from the middle to the right of the expression. One can now evaluate the trace directly; the only terms which will have a non-zero trace are those proportional to \( \phi^a \phi^{\dagger a} \), where \( 0 \leq a \leq m_2 \). Summing over all such possibilities, we immediately find that

\[
f_\delta = (1 - qt^{m_1}) \prod_{j=1}^{m_1+m_2} (z_{n-j+1}) \times \sum_{a=0}^{m_2} t^{a \cdot m_1} \text{Tr} \left( \phi^a \phi^{\dagger a} k^{u+m_1} \right) e_a(z_1, \ldots, z_{n-m_1-m_2}) e_{m_2-a}(z_{n-m_2+1}, \ldots, z_n). \tag{3.68}
\]

Finally, the trace in (3.68) can be evaluated explicitly:

\[
\text{Tr} \left( \phi^a \phi^{\dagger a} k^{u+m_1} \right) = \frac{1}{1 - t^{u+m_1}} \prod_{i=1}^{a} \left( \frac{1 - t^i}{1 - t^{u+m_1+i}} \right) = \frac{1}{1 - q t^{m_1}} \prod_{i=1}^{a} \left( \frac{1 - t^i}{1 - q t^{m_1+i}} \right),
\]

under the identification \( t^u \equiv q \). Substituting this into (3.68) yields the desired result.

### 3.5 ASEP dualities

In this section we show how certain self-dualities between asymmetric simple exclusion processes, first found in [143] and later elaborated in terms of ASEP generators in [17], arise within our formalism. This is achieved in three steps: 1. The identification of suitable sectors \( \delta \) and \( \epsilon \) for the use of Theorem 3.16; 2. The calculation of the coefficients \( \psi(\nu, \mu; t) \) in (3.58) for all \( \mu \in \sigma(\delta) \) and \( \nu \in \sigma(\epsilon) \);
3. Checking that the coefficients $\psi(\nu, \mu; t)$ are stable under the transition of the underlying lattice from $[1, \ldots, n]$ to $\mathbb{Z}$, and that they match with the duality functions of [17].

### 3.5 Occupation and position notation

Let us first make contact between our notation and that used in [17]. The ASEP generator in [17] makes particles jump to the left at rate $p$ and to the right at rate $q$, and is expressed in terms of occupation data $\{\eta_i\}_{i \in \mathbb{Z}}$, where $\eta_i \in \{0, 1\}$. In our setting, $p = 1$ and $q = t$, and the generator is also expressed in terms of occupation data $\{\nu_i\}_{i \in \mathbb{Z}}$.

Summing (3.28) over all $i \in \mathbb{Z}$ and manipulating the summand slightly, we see that

$$\sum_{i \in \mathbb{Z}} L_i [\psi(\cdot, \mu)](\nu) = \sum_{i \in \mathbb{Z}} \left( t\nu_i(1 - \nu_{i+1}) + (1 - \nu_i)\nu_{i+1} \right) \left[ \psi(s_i \nu, \mu) - \psi(\nu, \mu) \right],$$

which matches $L_{\text{occ}}$ in [17] under the identifications listed above. The reversed ASEP generator in [17] makes particles jump to the left at rate $q$ and to the right at rate $p$, and is expressed in terms of position data $\vec{x} = \{x_i\}_{1 \leq i \leq m}$, where $x_i \in \mathbb{Z}$ is the position of the $i$-th particle. By abuse of notation, we let $\psi(\nu, \mu) \equiv \psi(\nu, \vec{x})$, where we have translated from occupation to position notation in the second argument of $\psi$. Summing (3.30) over all $i \in \mathbb{Z}$ and converting to the position notation, we find that

$$\sum_{i \in \mathbb{Z}} M_i [\psi(\nu, \cdot)](\vec{x}) = \sum_{k \in \ell(\vec{x})} t \left( \psi(\nu, \vec{x}_{+k}) - \psi(\nu, \vec{x}) \right) + \sum_{k \in r(\vec{x})} \left( \psi(\nu, \vec{x}_{-k}) - \psi(\nu, \vec{x}) \right),$$

where $\ell(\vec{x})$ and $r(\vec{x})$ denote the positions of the leftmost and rightmost particles across all particle clusters, and where $\vec{x}_{\pm k} := (x_1, \ldots, x_{k-1}, x_k \pm 1, x_{k+1}, \ldots, x_m)$. This matches the reversed generator $L_{\text{part}}$ in [17].

**Theorem 3.25** (Schütz [143], Borodin–Corwin–Sasamoto [17]). Let $\nu$ be an infinite composition with parts $\nu_i \in \{0, 1\}$ and fix an ordered $m$-tuple of integers $\vec{x}(\mu) = (x_1 < \cdots < x_m)$, which label the positions of ones in another composition $\mu$. The functions

$$\psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \left( \prod_{i < x} t^{\nu_i} \right) \nu_x$$

---

*A set of inhomogeneous rate parameters $\{a_i\}_{i \in \mathbb{Z}}$ are also employed in [17]; we take all such parameters to be 1.

1 A cluster is a set of $j \geq 1$ particles positioned at coordinate points $(x_i, x_{i+1}, \ldots, x_{i+j-1})$, such that

$$x_i = x_{i+k} - k, \quad \forall 1 \leq k \leq j - 1, \quad x_{i-1} + 1 < x_i, \quad x_{i+j-1} < x_{i+j} - 1.$$
are well defined, since \( \nu_i = 0 \) for sufficiently small \( i \), and satisfy the local duality relation

\[
L_i [\psi(\cdot, \mu)] (\nu) = M_i [\psi(\nu, \cdot)] (\mu), \quad \forall \ i \in \mathbb{Z},
\]

(3.72)

where \( L_i \) and \( M_i \) are given by (3.28) and (3.30), respectively.

The rest of this section is devoted to proving Theorem 3.25 within the framework developed in this chapter.

### 3.5.2 Reduction from rank-\( r \) to rank-one

**Definition 3.26.** Let \( \mu = (\mu_1, \ldots, \mu_n) \) be a composition and \( \rho(\mu) \) be given by (3.38). The \( m \)-staircase of \( \mu \), denoted \( S_m(\mu) \), is an \( n \)-component vector defined as follows:

\[
S_m(\mu) := m\mu - \rho(\mu) = (m\mu_1, \ldots, m\mu_n) + \mu \cdot (1, 2, \ldots, n),
\]

where we recall that \( w_\mu \in S_n \) is the minimal-length permutation such that \( \mu = w_\mu \cdot \mu^+ \).

**Proposition 3.27.** Let \( E_\mu \) and \( E_\nu \) be any two non-symmetric Macdonald polynomials, and let \( y_i(\mu; q, t) \) and \( y_i(\nu; q, t) \) be their eigenvalues under the action of the Cherednik–Dunkl operator \( Y_i \), respectively. Then

\[
y_i(\mu; t^{-m}, t) = y_i(\nu; t^{-m}, t), \quad \forall \ 1 \leq i \leq n \iff S_m(\mu) = S_m(\nu).
\]

**Proof.** The eigenvalues \( y_i(\mu; q, t) \) and \( y_i(\nu; q, t) \) match for all \( 1 \leq i \leq n \) if and only if (3.45) holds. Setting \( q = t^{-m} \) in (3.45) and equating the exponents, it is equivalent to the relation

\[
S_m(\mu) = m\mu - \rho(\mu) = m\nu - \rho(\nu) = S_m(\nu).
\]

\[ \square \]

**Remark 3.28.** Notice that we can also write a weaker version of Proposition 3.27,

\[
y_i(\mu; t^{-m}, t) = y_i(\nu; t^{-m}, t) \quad \forall \ 1 \leq i \leq n \implies S_m(\mu^+) \sim S_m(\nu^+)
\]

where the equivalence relation \( \sim \) is defined as follows:

\[
S_m(\mu) \sim S_m(\nu) \iff \exists \ \sigma \text{ such that } S_m(\mu) = \sigma \cdot S_m(\nu).
\]

In other words, the matching of all eigenvalues is only possible if \( S_m(\mu^+) \) and \( S_m(\nu^+) \) are permutable to each other. For our purposes this is more useful than Proposition 3.27 itself, since the \( m \)-staircase of a partition is just given by

\[
S_m(\mu^+) = (m\mu_1^+, \ldots, m\mu_n^+) + (1, 2, \ldots, n),
\]
obviating the need to calculate \( \rho(\mu) \).

**Theorem 3.29.** Let \( r \) and \( m \) be two positive integers such that \( n - rm \geq 0 \). Consider the anti-partition \( \delta = (0^{n-m}, r^m) \), and let \( f_\delta(z_1, \ldots, z_n; q, t) \) be the associated ASEP polynomial. Then \( \text{Coeff}[f_\delta, m] = \text{Coeff}[f_\delta, m] \) exists, and we have

\[
\text{Coeff}[f_\mu, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t)z^\nu, \quad \forall \, \mu \in \sigma(\delta),
\]

for appropriate coefficients \( \psi(\nu, \mu; t) \equiv \psi(\nu, \mu) \), where \( \epsilon = (0^{n-m}, 1^m) \).

**Proof.** We begin by showing that \( \text{Coeff}[f_\delta, m] \) exists. To establish this, we need to show that the expression \( \frac{1}{1 - qt^m} \) appears at most linearly in \( f_\delta \). Using the summation formula (3.67) together with the results of Proposition 3.22, we see that the coefficient \( C_1(\mu[2], \mu[1]; q, t) \) is the only possible source of the factor \( \frac{1}{1 - qt^m} \) (indeed, another coefficient \( C_j(\mu[j+1], \mu[j]; q, t) \) with \( j \geq 2 \) would need to produce \( \frac{1}{1 - q^j t^m} \) in order to contribute to this factor, which can never happen since the product in the denominator of (3.66) ranges maximally up to \( i = m \)). The existence of \( \text{Coeff}[f_\delta, m] \) is then immediate.

Let us now apply the result of Proposition 3.10, in the case \( \mu = \delta \) and \( p = 1 \). We see that

\[
\text{Coeff}[f_\delta, m] = \text{Coeff}[E_\delta, m] = \lim_{q \to t^{-m}} (1 - qt^m) \left( \sum_{\nu \in E_\delta} c_\nu(q, t) E_\nu(z; q, t) \right)
\]

for some family of coefficients \( c_\nu(q, t) \) and where the sum is over compositions in the set

\[
E_\delta = \{ \nu : \nu \preceq \delta, y_\nu(w) = y_\delta(w) \text{ at } q = t^{-m} \}.
\]

We will show that the only possible compositions \( \nu \) in the set (3.75) are rank-one. By Proposition 3.27 and the remark immediately following it, all compositions in the set (3.75) would need to satisfy the \( m \)-staircase relation

\[
S_m(\delta^+) \sim S_m(\nu^+), \quad |\delta| = |\nu|.
\]

Calculating the \( m \)-staircase of \( \delta^+ \), we find

\[
S_m(\delta^+) = m \cdot (r^m, 0^{n-m}) + (1, \ldots, n) = (rm + 1, \ldots, rm + m, m + 1, \ldots, n),
\]

where we indicate the cardinalities of the two “blocks” in \( S_m(\delta^+) \) underneath, for clarity. On the other hand, in view of the fact that \( |\nu| = rm \), the composition \( \nu \) must have at least \( n - rm \) zeros. We can therefore write the \( m \)-staircase of its
dominant reordering as

\[ S_m(\nu^+) = m \cdot (\nu_1^+, \ldots, \nu_{rm}^+, 0^{n-rm}) + (1, \ldots, n) \]

\[ = (m\nu_1^+ + 1, \ldots, m\nu_{rm}^+ + rm, rm + 1, \ldots, n). \]  

(3.78)

Comparing the final \( n-rm \) parts of the two staircases (3.77) and (3.78), we find that they already agree, without the need to permute their order in any way. Suppressing these parts from both (3.77) and (3.78), the remaining entries of \( S_m(\delta^+) \) are permutable to a “true” staircase (with step-size one). Our problem thus simplifies to finding partitions \( \lambda \) such that

\[ (m + 1, \ldots, rm + m) \sim (m\lambda_1 + 1, \ldots, m\lambda_{rm} + rm), \]

or, after subtracting \( m \) from every component,

\[ (1, \ldots, rm) \sim (m(\lambda_1 - 1) + 1, \ldots, m(\lambda_{rm} - 1) + rm). \]  

(3.79)

A partition solution \( \lambda \) of (3.79) would need to contain two parts \( 0 \leq \lambda_i, \lambda_j \leq r \) such that

\[ m(\lambda_i - 1) + i = 1, \]  

(3.80)

\[ m(\lambda_j - 1) + j = rm, \]  

(3.81)

with \( 1 \leq i, j \leq rm \). Let us examine the possible resolutions of (3.80), (3.81).

(a) If the two parts are equal \( (\lambda_i = \lambda_j) \), subtracting (3.80) from (3.81) we find that \( j - i = rm - 1 \), which implies \( j = rm \) and \( i = 1 \). This identifies \( \lambda_i \) and \( \lambda_j \) as the first and last parts of the partition; all intermediate parts are then forced to assume the same value. All freedom is exhausted, and we find \( \lambda = (1^r) \) as the unique solution in the case \( \lambda_i = \lambda_j \).

(b) Assume a solution exists with \( \lambda_i > \lambda_j \). In that case, subtracting (3.80) from (3.81) leads to the inequality \( rm - 1 < j - i \). There are no values of \( i \) and \( j \) for which this holds.

(c) Finally, assume a solution exists with \( \lambda_i < \lambda_j \). Since \( \lambda \) is a partition, this would imply \( i > j \). Subtracting (3.80) from (3.81), we observe the equation

\[ m(\lambda_j - \lambda_i) = rm - 1 + i - j. \]

The value of \( i - j \) is positive, while \( \lambda_j - \lambda_i \) is bounded by \( r \) (the parts of \( \lambda \) cannot exceed \( r \)), so the only possible resolution in this case is \( \lambda_j = r, \lambda_i = 0, i - j = 1 \). This constrains \( \lambda_k = r \) for all \( k \leq j \) and \( \lambda_k = 0 \) for all \( k \geq j + 1 \), and since \( |\lambda| = rm \), we find that necessarily \( j = m \). We recover the solution \( \lambda = (1^r, 0^{rm-m}) \).

Translating these findings to our original setting, we have shown that (3.76) admits only two types of solutions: compositions \( \nu \) such that \( \nu^+ = (1^r, 0^{n-rm}) \), or \( \nu^+ = (r^m, 0^{n-m}) \). The latter solution is tautological, since it lives in the same
sector as $\delta$; it follows that the set (3.75) consists only of rank-one compositions.\footnote{One can easily check that the composition $\nu = (1^{rm-m}, 0^{n-rm}, 1^m)$ is a particular solution of the equation $S_m(\delta) = S_m(\nu)$, and in fact the minimal one. However for our purposes the precise ordering of parts in $\nu$ is not of interest, since just a statement about the sector of $\nu$ is good enough.} Rank-one non-symmetric Macdonald polynomials are multilinear in $(z_1, \ldots, z_n)$, so the right hand side of (3.74) must also have a multilinear dependence. It follows, by the action of inverse Hecke generators on (3.74), that a general polynomial $f_\mu$ with $\mu \in \sigma(\delta)$ admits the expansion

$$ \text{Coeff}[f_\mu, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu) z^\nu, \quad \epsilon = (0^{n-rm}, 1^{rm}). $$

\[ \square \]

**Theorem 3.30.** The coefficients in equation (3.73) are given by

$$ \psi(\nu, \mu) = d(t) \cdot t^{\Omega(\mu, \nu)} \cdot I(\mu, \nu), \quad \Omega(\mu, \nu) = \sum_{1 \leq i < j \leq n} (1_{\mu_i < \mu_j})(1_{\nu_i = \nu_j = 1}), $$

where $I(\mu, \nu)$ denotes the indicator function

$$ I(\mu, \nu) = \begin{cases} 0, & \exists k : (\mu_k, \nu_k) = (r, 0), \\ 1, & \text{otherwise,} \end{cases} $$

and $d(t)$ is an overall common factor of the coefficients, and need not be specified explicitly, recalling the comment immediately following equation (3.3).

**Proof.** We begin by considering the case $\mu = \delta^+$ of (3.73), namely, the situation when $\mu$ is the unique partition in the sector $\sigma(\delta)$. Using the matrix product formula (3.63) we know that $f_{\delta^+}(z_1, \ldots, z_n; q, t)$ contains the common factor $\prod_{i=1}^m z_i$ (each $A_r(z)$ operator in (3.63) has a common factor of $z$), while being a homogeneous polynomial in $(z_1, \ldots, z_n)$ of total degree $rm$. In addition, this polynomial is symmetric in the subset of variables $(z_{m+1}, \ldots, z_n)$. On the other hand, (3.73) says that $\text{Coeff}[f_{\delta^+}, m]$ admits an expansion on the space of multilinear polynomials in $(z_1, \ldots, z_n)$; the only possible expansion which respects all of these requirements is

$$ \text{Coeff}[f_{\delta^+}, m] = d(t) \cdot \prod_{i=1}^m z_i \cdot e_{(rm-m)}(z_{m+1}, \ldots, z_n) = d(t) \times \sum_{\nu \in \sigma(\epsilon)} I(\delta^+, \nu) f_{\nu} $$

for some constant $d(t)$, where $\epsilon = (0^{n-rm}, 1^{rm})$ and $I(\delta^+, \nu)$ is given by (3.83). This confirms the formula (3.82) for the case $\mu = \delta^+$, since one clearly has $\Omega(\delta^+, \nu) = 0$ for all $\nu$.

We use the preceding special case as the basis for induction. Let us suppose that $\psi(\nu, \mu)$ is given by (3.82) for all $\nu$, where $\mu$ is some composition in the sector $\sigma(\delta)$, which contains (at least) one pair of parts $(\mu_i, \mu_i+1)$ such that $\mu_i > \mu_i+1$. We then
act on \((3.73)\) with \(T_i\), giving

\[
T_i \cdot \text{Coeff}[f_\mu, m] = \text{Coeff}[f_{s_\mu}, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu) (T_i \cdot f_\nu).
\]

For the action of \(T_i\) on \(f_\nu\), we should distinguish the three possibilities (i) \(\nu_i > \nu_{i+1}\), (ii) \(\nu_i = \nu_{i+1}\) and (iii) \(\nu_i < \nu_{i+1}\), as given by equations \((3.24)\) and \((3.25)\). Case (i) means that \((\mu_i, \mu_{i+1}) = (r, 0)\) and \((\nu_i, \nu_{i+1}) = (1, 0)\), and one easily sees that

\[
\psi(\nu, \mu)(T_i \cdot f_\nu) = \psi(\nu, \mu) f_{s_\nu} = \psi(s_i \nu, s_i \mu) f_{s_\nu}.
\]

(3.84)

Case (ii) means that \((\mu_i, \mu_{i+1}) = (r, 0)\) and \((\nu_i, \nu_{i+1}) = (1, 1)\) (we exclude the possibility that \((\nu_i, \nu_{i+1}) = (0, 0)\), since we would then have \((\mu_i, \nu_i) = (r, 0)\), causing the indicator function \((3.83)\) to vanish), and accordingly,

\[
\psi(\nu, \mu)(T_i \cdot f_\nu) = t \psi(\nu, \mu) f_{s_\nu} = \psi(s_i \nu, s_i \mu) f_{s_\nu},
\]

(3.85)

where the final equality exploits the fact that in this case \(\Omega(\mu, \nu) + 1 = \Omega(s_i \mu, s_i \nu)\). Finally, case (iii) means that \((\mu_i, \mu_{i+1}) = (r, 0)\) and \((\nu_i, \nu_{i+1}) = (0, 1)\), which is another situation where the indicator function \((3.83)\) vanishes. We thus have the trivial fact

\[
\psi(\nu, \mu)(T_i \cdot f_\nu) = 0 = \psi(s_i \nu, s_i \mu) f_{s_\nu}.
\]

(3.86)

One finds the same expression for the right hand side in all three cases \((3.84)–(3.86)\); we have thus demonstrated that

\[
\text{Coeff}[f_{s_\mu}, m] = \sum_{\nu \in \sigma(\epsilon)} \psi(s_i \nu, s_i \mu) f_{s_\nu} = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, s_i \mu) f_\nu,
\]

which is the required inductive step. This completes the proof of \((3.82)\). \(\square\)

### 3.5.3 Back to the proof of Theorem 3.25

In the previous subsection we started from an ASEP polynomial \(f_\mu\) such that \(\mu^- = (0^{n-m}, r^m)\), and sent \(q \to t^{-m}\). Quite remarkably, one finds that \(\text{Coeff}[f_\mu, m]\) reduces to a linear combination of ASEP polynomials \(f_\nu\) such that \(\nu^- = (0^{n-rm}, 1^rn)\), where the expansion coefficients are given by \((3.82)\). Applying the result of Theorem 3.16, we now obtain the desired duality statement:

**Corollary 3.31.** In the same notation as Theorem 3.30, the functions

\[
\psi(\nu, \mu) = t^{\Omega(\mu, \nu)} \cdot I(\mu, \nu)
\]

satisfy the local duality relations

\[
L_i[\psi(\cdot, \nu)](\mu) = M_i[\psi(\nu, \cdot)](\mu), \quad \forall \ 1 \leq i \leq n - 1,
\]

(3.88)
where the left hand side is given by (3.28), and the right hand side by (3.30). Note that we have dropped the constant \( d(t) \) from (3.87); we are allowed to do this because it is common to all coefficients \( \psi(\nu, \mu) \) in the sectors we have chosen, and therefore plays no role in (3.88).

**Remark 3.32.** Even though we used a higher-rank ASEP polynomial \( f_\mu \) in the derivation of this duality statement, it is clear that (3.88) itself is a rank-one equation: because of the sector that \( \mu \) belongs to, the \( L_i \) generator sees only particles of type \( r \) and zeros, and so the left hand side of (3.88) describes the evolution of an ordinary (single-species) ASEP.

To complete the proof of Theorem 3.25, one should translate the observable (3.87) into the occupation–position notation employed therein. With \( \vec{x}(\mu) = (x_1(\mu) < \cdots < x_m(\mu)) \) denoting the positions of the \( r \)-particles in the composition \( \mu \), after a simple calculation one finds that

\[
\psi(\nu, \mu) = t^{-m(m-1)/2} \prod_{j=1}^{m} \left( \prod_{1 \leq i < x_j(\mu)} t^{x_i} \right) \nu_{x_j(\mu)},
\]

which matches the form of the right hand side of (3.71) up to the factor \( t^{-m(m-1)/2} \). This factor is spurious; it does not play any role in the equations (3.88) other than as a spectating constant.

Finally, our analysis so far has proceeded on the finite lattice \([1, \ldots, n]\), with closed boundary conditions. It is a trivial matter to transition to the integer lattice. Indeed, the observable (3.87) does not depend on \( n \) in any way (beyond the fact that it is the length of the participating compositions). One can therefore embed the existing observables within the space of functions on \( \mathbb{Z} \times \mathbb{Z} \), simply by padding the finite compositions \( \mu \) and \( \nu \) with zeros on both sides. This reproduces the family of observables (3.71), and finishes our derivation of Theorem 3.25.

### 3.6 Rank-two ASEP dualities

The aim of this section is to produce new types of observables, which generalize those found in [17], being duality functions with respect to two *multi-species* asymmetric simple exclusion processes. We will restrict our attention to dualities between mASEPs with two distinct particle species, in this way finding a natural rank-two extension of Theorem 3.25.

For other recent progress related to higher-rank duality functions, making use of quantum group symmetries, we refer the reader to [10, 33, 34, 105, 107]. It would be interesting to ascertain whether the duality functions obtained in those works are recoverable via the approach in the present chapter.
3.6.1 Reduction relations between a pair of rank-two sectors

Theorem 3.33. Fix three integers \(n, m_1, m_2 \geq 0\) such that \(m_1 + m_2 \leq n\), and an anti-partition

\[
\delta = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}),
\]

Then choosing another integer \(p\) such that \(1 \leq p \leq \min(n - m_1 - m_2, m_2)\), one has the expansion

\[
\text{Coeff}[f_{\mu, p + m_1}] = \sum_{\nu \in \sigma(\epsilon)} \psi(\nu, \mu; t) f_{\nu}(z; t^{-p-m_1}, t), \quad \forall \mu \in \sigma(\delta),
\]

for appropriate coefficients \(\psi(\nu, \mu; t) \equiv \psi(\nu, \mu)\), where

\[
\epsilon = (0^{n-m_1-m_2-p}, 1^{m_1+2p}, 2^{m_2-p}).
\]

Proof. Let us begin by remarking that this theorem is not obvious from the matrix product formula (3.65), for although the latter allows us to manually calculate \(\text{Coeff}[f_{\mu, p + m_1}]\), the resulting expression is not easily re-expressed in the basis of the polynomials \(f_\nu\).

It is therefore best to resort to a similar style of proof as that of Theorem 3.29. In the present situation, given that our starting sector (the sector of \(\delta\)) is rank-two, rather than rank-\(r\), we are able to be a little more explicit. We will show that all members of the set

\[
E_\delta = \{ \nu : \nu \prec \delta, y_\nu(w) = y_\delta(w) \text{ at } q = t^{-p-m_1} \}
\]

live in the composition sector \(\sigma(\epsilon)\), allowing us to conclude that

\[
\text{Coeff}[f_{\delta, p + m_1}] = \lim_{q \to t^{-p-m_1}} (1 - qt^{p+m_1}) \left( \sum_{\nu \in \sigma(\epsilon)} c_\nu(q, t) E_\nu(z; q, t) \right),
\]

for some family of coefficients \(c_\nu(q, t)\). Any compositions in (3.90) would need to satisfy the constraint \(|\nu| = |\delta|\), with parts of at most size two, so it is clearly sufficient to restrict our search to compositions that have the dominant ordering

\[
\nu^+ = (2^{m_2-r}, 1^{m_1+2r}, 0^{n-m_1-m_2-r}),
\]

with \(r \geq 1\) becoming the only degree of freedom. Our aim is to prove that \(r = p\) is the only possible value for \(r\), which we do by exhausting all solutions of the relation \(S_{p+m_1}(\delta^+) \sim S_{p+m_1}(\nu^+)\). With \(\nu^+\) as above and \(\delta^+ = (2^{m_2}, 1^{m_1}, 0^{n-m_1-m_2})\) we see that

\[
\delta^+_i = \nu^+_i, \quad \forall i \in [1, m_2 - r] \cup [m_2 + 1, m_1 + m_2] \cup [m_1 + m_2 + r + 1, n],
\]
and since by assumption \( r > p \) corresponding with the lowest index in \( A \) the smallest element in \( S \) set finds that value matching the right hand side of (3.93). Thus for \( r > p \),

\[
S_1(\delta^+) \cup S_2(\delta^+) \sim S_1(\nu^+) \cup S_2(\nu^+),
\]

(3.92)

where

\[
S_1(\mu) = \{ S_{p+m_1}(\mu)_i | i \in A_1 \}, \quad S_2(\mu) = \{ S_{p+m_1}(\mu)_i | i \in A_2 \},
\]

\[
A_1 = [m_2 - r + 1, m_2], \quad A_2 = [m_1 + m_2 + 1, m_1 + m_2 + r].
\]

Let us first suppose that \( r > p \). Consider the following component of \( S_1(\nu^+) \), corresponding with the lowest index in \( A_1 \):

\[
S_{p+m_1}(\nu^+)_{m_2-r+1} = (p + m_1) \cdot \nu^+_{m_2-r+1} + m_2 - r + 1 = m_1 + m_2 + 1 + p - r.
\]

(3.93)

This element must be reproduced somewhere in \( S_1(\delta^+) \cup S_2(\delta^+) \), or the relation (3.92) does not hold. It is easy to check that the smallest element in \( S_1(\delta^+) \) is given by

\[
S_{p+m_1}(\delta^+)_{m_2-r+1} = 2m_1 + 2p + m_2 - r + 1.
\]

Clearly \( S_{p+m_1}(\delta^+)_{m_2-r+1} > S_{p+m_1}(\nu^+)_{m_2-r+1} \) and hence there is no element in the set \( S_1(\delta^+) \) which reproduces the value on the right hand side of (3.93). Similarly, the smallest element in \( S_2(\delta^+) \) is given by

\[
S_{p+m_1}(\delta^+)_{m_1+m_2+1} = m_1 + m_2 + 1,
\]

(3.94)

and since by assumption \( r > p \), it follows that \( S_{p+m_1}(\nu^+)_{m_2-r+1} < S_{p+m_1}(\delta^+)_{m_1+m_2+1} \). We conclude that there is also no element in \( S_2(\delta^+) \) with value matching the right hand side of (3.93). Thus for \( r > p \), the relation (3.92) has no solutions.

Second, we suppose that \( r < p \). Consider the component of \( S_2(\delta^+) \) corresponding with the lowest index in \( A_2 \), as given by (3.94). This element must be reproduced somewhere in \( S_1(\nu^+) \cup S_2(\nu^+) \). Since \( \nu_i^+ = 1 \) for all \( i \in A_1 \cup A_2 \), the smallest element in \( S_1(\nu^+) \cup S_2(\nu^+) \) is obtained by taking the first index in \( A_1 \). We then find that

\[
S_{p+m_1}(\nu^+)_{m_2-r+1} = m_1 + m_2 + 1 + p - r > m_1 + m_2 + 1 = S_{p+m_1}(\delta^+)_{m_1+m_2+1}.
\]

Hence there is no element in \( S_1(\nu^+) \cup S_2(\nu^+) \) which reproduces the right hand side of (3.94), and accordingly the relation (3.92) has no solutions for \( r < p \).

We have shown that compositions \( \nu \) such that \( \nu^+ = (2^{m_2-p}, 1^{m_1+2p}, 0^{m_1-m_2-p}) \) are the only possible members of the set (3.90).
From here it is quite straightforward to see that
\[ \nu = (1^p, 0^{n-m_1-m_2-p}, 2^{m_2-p}, 1^{m_1+p}) \]
satisfies \( S_{p+m_1}^{\gamma} = S_{p+m_1}^{\nu} \), and is the minimal such composition. The claim (3.91) is proved; one can now follow a similar procedure as in the proof of Theorem 3.15, to transform the right hand side of (3.91) to the basis of ASEP polynomials. This leads to the generic expansion (3.89).

**Theorem 3.34.** The coefficients in equation (3.89) are given by
\[ \psi(\nu, \mu) = d(t) \cdot t^{\Omega(\mu, \nu)} \cdot I(\mu, \nu) = \sum_{1 \leq i < j \leq n} (1_{\mu_i < \mu_j})(1_{\nu_i = \nu_j = 1}), \tag{3.95} \]
where \( I(\mu, \nu) \) denotes the indicator function
\[ I(\mu, \nu) = \begin{cases} 0, & \exists k : \mu_k > \nu_k = 0, \text{ or } \mu_k < \nu_k = 2, \\ 1, & \text{otherwise}, \end{cases} \tag{3.96} \]
and \( d(t) \) is an overall common factor of the coefficients, and need not be specified explicitly, recalling the comment immediately following equation (3.3).

**Proof.** We start from the generic expansion (3.89), as given to us by Theorem 3.33. Since \( z_\nu \) is the leading monomial of the monic polynomial \( f_\nu(z; t^{-p-m_1}, t) \), and unique to that polynomial on the right hand side of (3.89), we can evaluate \( \psi(\nu, \mu) \) by taking the coefficient of \( z_\nu \) in \( \text{Coeff}[f_\mu, p + m_1] \). We then use the matrix product formula (3.65) to perform the calculation:
\[ \psi(\nu, \mu) = \lim_{q \to t^{-p-m_1}} (1 - qt^{p+m_1}) \text{Tr}(A_{\mu_1}(z_1) \ldots A_{\mu_n}(z_n) k^n) \bigg|_{z=\nu} \tag{3.97} \]
and noting the \( z \)-dependence of the operators \( A_i(z) \) in (3.64), we immediately see that \( \psi(\nu, \mu) \) is zero if for some \( 1 \leq k \leq n \) we have \( \mu_k > \nu_k = 0 \) or \( \mu_k < \nu_k = 2 \). This is the reason why the coefficients (3.95) contain the indicator function (3.96); we restrict our attention henceforth to the situation when \( \nu \) is chosen such that \( I(\mu, \nu) \) is non-zero. Using (3.97), we see that
\[ \psi(\nu, \mu) = \lim_{q \to t^{-p-m_1}} (1 - qt^{p+m_1}) \text{Tr}(B_{\mu_1, \nu_1} \ldots B_{\mu_n, \nu_n} k^n) \cdot I(\mu, \nu), \tag{3.98} \]
with \( B_{0,0} = B_{2,2} = 1, B_{1,1} = k, B_{0,1} = \phi \) and \( B_{2,1} = \phi^\dagger \). Since the part-multiplicities of \( \nu \) are already specified by Theorem 3.33, we can assume that \( \#\{B_{0,1}\} = \#\{B_{2,1}\} = p \) and \( \#\{B_{1,1}\} = m_1 \). The product of bosonic operators appearing in (3.98) can then be brought, via repeated use of the relations \( \phi \phi^\dagger = 1 - tk \)}
and $\phi^\dagger \phi = 1 - k$, to a polynomial in $k$:

$$B_{\mu_1, \nu_1} \cdots B_{\mu_n, \nu_n} = \sum_{i=0}^{p} c_{\mu, \nu}(i; t) k^{i+m_1},$$

(3.99)

for suitable coefficients $c_{\mu, \nu}(i; t)$, which for the moment we do not specify. Substituting this into (3.98) and evaluating the resulting traces, we find

$$\psi(\nu, \mu) = \lim_{q \to t^{-p-m_1}} \left(1 - qt^{p+m_1}\right) \cdot \left(\sum_{i=0}^{p} \frac{c_{\mu, \nu}(i; t)}{1 - qt^{i+m_1}}\right) \cdot I(\mu, \nu) = c_{\mu, \nu}(p; t) \cdot I(\mu, \nu).$$

It is straightforward to calculate the top-degree term in (3.99). In the case of a completely ordered string of bosonic operators, i.e.

$$\phi^\dagger \phi \cdots \phi^\dagger \phi \cdots \phi \cdots \phi = d_\mu(p; t) k^{p+m_1} + \text{subleading terms in } k,$$

where $d_\mu(p; t) = (-1)^p (t^{-p})^{m_1 + (p-1)/2}$. As the string becomes disordered, one easily sees that the leading coefficient acquires a factor of $t$ for every pair $\phi \cdots \phi^\dagger$, $k \cdots \phi^\dagger$ or $\phi \cdots k$ that gets created. These pairs are counted by

$$\alpha(\mu, \nu) = \# \{ i < j | (\mu_i = 0, \mu_j = 2), (\nu_i = \nu_j = 1) \},$$

$$\beta(\mu, \nu) = \# \{ i < j | (\mu_i = 1, \mu_j = 2), (\nu_i = \nu_j = 1) \},$$

$$\gamma(\mu, \nu) = \# \{ i < j | (\mu_i = 0, \mu_j = 1), (\nu_i = \nu_j = 1) \},$$

respectively. We conclude that

$$\psi(\nu, \mu) = c_{\mu, \nu}(p; t) \cdot I(\mu, \nu) = d_\mu(p; t) \times t^{\alpha(\mu, \nu) + \beta(\mu, \nu) + \gamma(\mu, \nu)} \cdot I(\mu, \nu),$$

completing the proof of (3.95), with the identification $d(t) \equiv d_\mu(p; t)$.

### 3.6.2 Rank-two duality functions

In the last subsection we studied the reduction of a generic rank-two ASEP polynomial $f_\mu$, such that $\mu^- = (0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2})$, in the limit $q \to t^{-p-m_1}$ with $p$ a positive integer. In Theorem 3.33 we proved that the corresponding expansion over polynomials $f_\nu$ is contained to the sector in which $\nu^- = (0^{n-m_1-m_2-p}, 1^{m_1+2p}, 2^{m_2-p})$, and in Theorem 3.34 we calculated the expansion coefficients. By virtue of Theorem 3.16, we have proved the following duality result:

**Corollary 3.35.** In the same notations as Theorem 3.34, the functions

$$\psi(\nu, \mu) = t^{\Omega(\mu, \nu)} \cdot I(\mu, \nu)$$

(3.100)
satisfy the local duality relations

\[ L_i[\psi(\cdot, \mu)](\nu) = M_i[\psi(\nu, \cdot)](\mu), \quad \forall \ 1 \leq i \leq n - 1, \tag{3.101} \]

where the left hand side is given by (3.28), and the right hand side by (3.30).

Let us now translate the observable (3.100) into an occupation–position notation, similar to that employed in Theorem 3.25. In the rank-two case at hand, the composition \( \mu \) is labelled by two sets of positions: a set \( \vec{x}(\mu) = (x_1 < \cdots < x_{m_1}) \) which labels the positions of 1-particles, and a set \( \vec{y}(\mu) = (y_1 < \cdots < y_{m_2}) \) labelling the positions of 2-particles. The two sets \( \vec{x} \) and \( \vec{y} \) are disjoint, since two particles cannot occupy a single site of the lattice. We introduce a statistic \( \chi(\vec{x}, \vec{y}) \), which counts the number of “crossings” between the two sets \( \vec{x} \) and \( \vec{y} \):

\[ \chi(\vec{x}, \vec{y}) := \# \{(x_i, y_j) \in (\vec{x}, \vec{y}) \mid x_i > y_j\}. \tag{3.102} \]

**Proposition 3.36.** Fix two compositions

\[ \mu \in \sigma(0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}), \quad \nu \in \sigma(0^{n-m_1-m_2-p}, 1^{m_1+2p}, 2^{m_2-p}), \]

chosen such that the inequalities \( \mu_k > \nu_k = 0 \) and \( \mu_k < \nu_k = 2 \) do not occur for any \( 1 \leq k \leq n \). Let \( \Omega(\mu, \nu) \) be given by (3.95). Expressing \( \mu \) in terms of particle-position notation, one has

\[ \Omega(\mu, \nu) + \frac{m_1(m_1-1)}{2} + \frac{p(p-1)}{2} + \chi(\vec{x}, \vec{y}) = \sum_{x \in \vec{x}(\mu)} \sum_{i < x} 1_{\nu_i \geq 1} + \sum_{y \in \vec{y}(\mu)} \sum_{i < y} 1_{\nu_i = 1} 1_{\nu_y = 1}. \tag{3.103} \]

**Proof.** We start from the left hand side of (3.103), and examine what it counts. In the following we always assume that \( i < j \).

- The first term, \( \Omega(\mu, \nu) \), counts all instances such that \( (\mu_i, \mu_j) = (0, 1), (0, 2), (1, 2) \) and \( (\nu_i, \nu_j) = (1, 1) \).

- The second term, \( m_1(m_1-1)/2 \), is equal to the number of times that \( (\mu_i, \mu_j) = (1, 1) \) and \( (\nu_i, \nu_j) = (1, 1) \) (since \( \mu_i = 1 \) forces \( \nu_i = 1 \), by our assumption on the compositions).

- The third term, \( p(p-1)/2 \), is equal to the number of times that \( (\mu_i, \mu_j) = (2, 2) \) and \( (\nu_i, \nu_j) = (1, 1) \). To see this, note that there must be exactly \( p \) pairs \( (\mu_i, \nu_i) = (2, 1) \), by knowledge of the sectors that the two compositions come from.

- The fourth term, \( \chi(\vec{x}, \vec{y}) \), counts the number of times that \( (\mu_i, \mu_j) = (2, 1) \) and \( (\nu_i, \nu_j) = (1, 1), (2, 1) \).
Totaling these possibilities, we find that the left hand side counts 7 different types of pairs \((\mu_i, \mu_j), (\nu_i, \nu_j)\). We proceed to show that the same pairs are recovered on the right hand side of (3.103):

- The first summation, \(\sum_{x \in \vec{x}(\mu)} \sum_{i < x} 1_{\nu_i \geq 1} \nu_i \geq 1\), counts all instances such that \(\nu_i \geq 1, \mu_j = 1\). This can be seen to be equal to
  \[\# \left\{ (\mu_i, \mu_j) = (0, 1), (\nu_i, \nu_j) = (1, 1) \right\} + \# \left\{ (\mu_i, \mu_j) = (1, 1), (\nu_i, \nu_j) = (1, 1) \right\} + \# \left\{ (\mu_i, \mu_j) = (2, 1), (\nu_i, \nu_j) = (1, 1) \right\},\]
  by virtue of the restrictions imposed on \(\mu\) and \(\nu\). This accounts for 4 of the terms on the left hand side of (3.103).

- The second summation, \(\sum_{y \in \vec{y}(\mu)} \sum_{i < y} 1_{\nu_i = 1} \nu_y = 1\), enumerates all the instances such that \(\nu_i = \nu_j = 1, \mu_j = 2\). More explicitly, these instances are given by
  \[\# \left\{ (\mu_i, \mu_j) = (0, 2), (\nu_i, \nu_j) = (1, 1) \right\} + \# \left\{ (\mu_i, \mu_j) = (1, 2), (\nu_i, \nu_j) = (1, 1) \right\} + \# \left\{ (\mu_i, \mu_j) = (2, 2), (\nu_i, \nu_j) = (1, 1) \right\}.
  \]
  This accounts for the remaining 3 types of terms on the left hand side of (3.103).

Using the result of Proposition 3.36 we can now write the observable in Corollary 3.35 as

\[
\psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \prod_{i < x} \left( t^{1_{\nu_i \geq 1}} \right) \prod_{y \in \vec{y}(\mu)} \prod_{i < y} \left( t^{1_{\nu_i = 1} 1_{\nu_y = 1}} \right) \cdot t^{-\chi(\vec{x}, \vec{y})} \cdot I(\mu, \nu), \tag{3.104}
\]

where we have dropped an irrelevant overall factor of \(t^{-(m_1(m_1-1)+p(p-1))/2}\), which comes from (3.103), but plays no role in the duality relations (3.101) (see the comment immediately following equation (3.3)). One can view (3.104) as a duality function on \(\mathbb{Z} \times \mathbb{Z}\), by extending the compositions \(\mu, \nu\) to infinite length in a stable way, as we discussed in Section 3.5.3.

The duality function (3.104) is a generalization of the observable (3.71) to the rank-two setting. It is easily seen to degenerate to the latter when both \(\mu\) and \(\nu\) are chosen to have no 2-particles. Another special case of interest is when \(\mu\) is a generic rank-two composition, while \(\nu\) is purely rank-one. In that case, (3.104) simplifies, and we obtain the following result:

**Corollary 3.37.** Let \(\nu\) be an infinite rank-one composition, with \(\nu_i \in \{0, 1\}\) for all \(i \in \mathbb{Z}\). Let \(\mu\) be an infinite rank-two composition, with \(\vec{x}(\mu) = (x_1 < \cdots < x_{m_1})\) for all \(x_i \in \mathbb{Z}\). Then

\[
\psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \prod_{i < x} \left( t^{1_{\nu_i \geq 1}} \right) \prod_{y \in \vec{y}(\mu)} \prod_{i < y} \left( t^{1_{\nu_i = 1} 1_{\nu_y = 1}} \right) \cdot t^{-\chi(\vec{x}, \vec{y})} \cdot I(\mu, \nu),
\]

where \(\chi(\vec{x}, \vec{y})\) is the difference in the number of 2-particles in \(\vec{x}\) and \(\vec{y}\).
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\( x_{m1} \) and \( \vec{y}(\mu) = (y_1 < \cdots < y_{m2}) \) labelling the positions of its 1 and 2-particles, respectively. Then the function

\[
\psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu) \, i < x} (t^{\nu_x}) \cdot \prod_{y \in \vec{y}(\mu) \, i < y} (t^{\nu_y}) \cdot t^{-\chi(\vec{x}, \vec{y})} \tag{3.105}
\]

satisfies the relations (3.101) for all \( i \in \mathbb{Z} \).

### 3.7 Duality functions without indicators

In Section 3.5 we presented a new derivation of the rank-one observable (3.71), proving that it is a solution of the local duality equations (3.72). A second observable was considered in [17]. This observable differs from (3.71) in two main ways: first, it does not contain any indicator functions, meaning that the observable does not vanish for any values of \( \mu \) and \( \nu \); second, the resulting observable does not satisfy the local relations (3.72), but rather the global relation obtained by summing over all \( i \in \mathbb{Z} \). In Section 3.7.1 we briefly review these facts.

In Section 3.7.2, we will show that the rank-two observable obtained in equation (3.105) also gives rise to a “partner” observable without indicator functions, which satisfies global duality relations. Once this result is written down, it is not hard to see that it in fact generalizes to arbitrary rank: hence in Section 3.7.3 we find a non-vanishing observable valued on a rank-one ASEP and an arbitrary rank mASEP, which is a duality function with respect to the generators of the two processes.

#### 3.7.1 Rank-one duality functions without indicators

**Proposition 3.38** (Borodin–Corwin–Sasamoto [17]). Let \( \nu \) be an infinite composition with parts \( \nu_i \in \{0, 1\} \) and fix an ordered \( m \)-tuple of integers \( \vec{x}(\mu) = (x_1 < \cdots < x_m) \), which label the positions of ones in another composition \( \mu \). Then the observable

\[
H(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \prod_{i \leq x} t^{\nu_i} \tag{3.106}
\]

satisfies the equation

\[
\sum_{i \in \mathbb{Z}} L_i [H(\cdot, \mu)](\nu) = \sum_{i \in \mathbb{Z}} M_i [H(\nu, \cdot)](\mu), \tag{3.107}
\]

where \( L_i \) and \( M_i \) are given by (3.28) and (3.30), respectively.

**Proof.** Once guessed, this result can be proved by direct computation; see [17]. We do not know of a more constructive proof, for example making use of solutions of the \( tKZ \) equations, although it would be very interesting to find one.
3.7.2 Rank-one/rank-two duality functions without indicators

Proposition 3.39. Let $\nu$ be an infinite rank-one composition, with $\nu_i \in \{0, 1\}$ for all $i \in \mathbb{Z}$. Let $\mu$ be an infinite rank-two composition, with $\vec{x}(\mu) = (x_1 < \cdots < x_{m_1})$ and $\vec{y}(\mu) = (y_1 < \cdots < y_{m_2})$ labelling the positions of its 1 and 2-particles, respectively. Recall also the definition of $\chi(\vec{x}, \vec{y})$, as given by (3.102). Then the observable

$$H(\nu, \mu) = \prod_{x \in \vec{x}(\mu), i \in x} (t^{\nu_i}) \cdot \prod_{y \in \vec{y}(\mu), i \in y} (t^{\nu_i}) \cdot t^{-\chi(\vec{x}, \vec{y})}$$

(3.108)

satisfies the equation

$$\sum_{i \in \mathbb{Z}} L_i [H(\cdot, \mu)](\nu) = \sum_{i \in \mathbb{Z}} M_i [H(\nu, \cdot)](\mu),$$

(3.109)

where $L_i$ and $M_i$ are given by (3.28) and (3.30), respectively.

Proof. It is convenient to define the set

$$\vec{z}(\mu) = (z_1 < \cdots < z_{m_1 + m_2}) = \vec{x}(\mu) \cup \vec{y}(\mu),$$

obtained by taking the union of the two sets of coordinates $\vec{x}$ and $\vec{y}$. We begin by considering the case of a single particle cluster in $\mu$. This refers to the situation in which $\vec{z}(\mu) = (z+1, \ldots, z+l)$ for some $z \in \mathbb{Z}$, and where we abbreviate $m_1 + m_2 \equiv \ell$.

To begin, notice that the observable (3.108) can be expressed in the form

$$H(\nu, \mu) = H(\nu, \mu^*) t^{-\chi(\vec{x}, \vec{y})},$$

(3.110)

where $\mu^*$ is the rank-one composition obtained by the following “colour-blind” projection$^*$ of $\mu$:

$$\mu_i^* = \begin{cases} 0, & \mu_i = 0, \\ 1, & \mu_i \geq 1, \end{cases} \quad \forall i \in \mathbb{Z},$$

and $H(\nu, \mu^*)$ denotes a rank-one observable of the form (3.106):

$$H(\nu, \mu^*) = \prod_{x \in \vec{x}(\mu^*), i \in x} (t^{\nu_i}).$$

(3.111)

Studying firstly the right hand side of the proposed identity (3.109), we see from the action (3.30) of $M_i$ that we can localise the summation over $i$ as follows:

$$\sum_{i \in \mathbb{Z}} M_i [H(\nu, \cdot)](\mu) = \sum_{i \in \{z, z+l\} \cup d_1(\vec{x}, \vec{y}) \cup d_2(\vec{x}, \vec{y})} M_i [H(\nu, \cdot)](\mu),$$

(3.112)

$^*$This is also known as Markov projection in the probability literature.
where we have defined the sets
\[ d_1(\vec{x}, \vec{y}) = \{ x_i \in \vec{x} | x_i + 1 \in \vec{y} \}, \quad d_2(\vec{x}, \vec{y}) = \{ y_i \in \vec{y} | y_i + 1 \in \vec{x} \}. \]

Indeed, it is unnecessary to retain any other terms in the summation (3.112), since \( M_i \) has a vanishing action on the observable for all other values of \( i \). Let us simplify (3.112) further. Clearly, when sites \( i \) and \( i + 1 \) of \( \mu \) are occupied by particles, regardless of their types, \( M_i \) has no effect on the set \( \vec{z} \), and hence acts directly on \( t^{-\chi(\vec{x}, \vec{y})} \). Therefore, using (3.110), we find that
\[
\sum_{i \in d_1(\vec{x}, \vec{y})} M_i[H(\nu, \cdot)](\mu) = \sum_{i \in d_1(\vec{x}, \vec{y})} M_i[t^{-\chi(\cdot)}](\vec{x}, \vec{y}). \quad (3.113)
\]

One can now easily show that the right hand side of (3.113) vanishes. To see this, note that when \( i \in d_1(\vec{x}, \vec{y}) \) (namely, when \( \mu_i = 1 \) and \( \mu_{i+1} = 2 \)), we have
\[
M_i[t^{-\chi(\cdot)}](\vec{x}, \vec{y}) = t \cdot t^{-\chi(\vec{x}, \vec{y})} - t^{-\chi(\vec{x}, \vec{y})} = 0. \quad (3.114)
\]

Similarly, when \( i \in d_2(\vec{x}, \vec{y}) \) (namely, when \( \mu_i = 2 \) and \( \mu_{i+1} = 1 \)), we have
\[
M_i[t^{-\chi(\cdot)}](\vec{x}, \vec{y}) = t^{-\chi(\vec{x}, \vec{y})} - t \cdot t^{-\chi(\vec{x}, \vec{y})} = 0. \quad (3.115)
\]

Combining (3.112)–(3.114), we conclude that
\[
\sum_{i \in \mathbb{Z}} M_i[H(\nu, \cdot)](\mu) = \sum_{i \in \{z, z+1\}} M_i[H(\nu, \cdot)](\mu), \quad (3.116)
\]
reducing the action of the generator to just the two sites \( z \) and \( z + l \). By assumption, \( \mu_z = \mu_{z+l+1} = 0 \), meaning that \( \chi(\vec{x}, \vec{y}) \) is invariant under the action of both \( M_z \) and \( M_{z+1} \) (since the number of 1 and 2-particle crossings will be preserved). This allows us to rewrite (3.116) as
\[
\sum_{i \in \mathbb{Z}} M_i[H(\nu, \cdot)](\mu) = t^{-\chi(\vec{x}, \vec{y})} \sum_{i \in \{z, z+l\}} M_i[H(\nu, \cdot)](\mu^*), \quad (3.117)
\]
in which the final expression is a purely rank-one quantity.

Turning to the left hand side of (3.109), we use (3.110) to write
\[
\sum_{i \in \mathbb{Z}} L_i[H(\cdot, \mu)](\nu) = t^{-\chi(\vec{x}, \vec{y})} \sum_{i \in \mathbb{Z}} L_i[H(\cdot, \mu^*)](\nu) = t^{-\chi(\vec{x}, \vec{y})} \sum_{i \in \mathbb{Z}} M_i[H(\nu, \cdot)](\mu^*), \quad (3.118)
\]
where the second equality is deduced from the rank-one duality relation of Proposition 3.38. The final term in (3.118) can be simplified further, since \( M_i \) has a
vanishing action on the rank-one observable for any \( i \neq z, z+l \). Hence,

\[
\sum_{i \in \mathbb{Z}} L_i [H(\cdot, \mu)](\nu) = t^{-\chi(\bar{x}, \bar{y})} \sum_{i \in \{z, z+l\}} M_i [H(\nu, \cdot)](\mu^*). \tag{3.119}
\]

Comparing (3.117) and (3.119) yields the proof of (3.109) in the case of one particle cluster. A generic configuration \( \bar{z}(\mu) = \bar{x}(\mu) \cup \bar{y}(\mu) \) can be written as a union of clusters, which then provides a natural splitting of the generator \( \sum_{i \in \mathbb{Z}} M_i \) into finite disjoint pieces of the form (3.112). One can apply the preceding logic mutatis mutandis to each such piece, leading to the proof of (3.109) in full generality.

### 3.7.3 Generalization to arbitrary rank

Having arrived at the observable (3.108), valued on the configuration spaces of a rank-one and rank-two process, one can immediately see how to generalize the two-species process to arbitrary rank:

**Corollary 3.40.** Let \( \nu \) be an infinite rank-one composition, with \( \nu_i \in \{0, 1\} \) for all \( i \in \mathbb{Z} \). Let \( \mu \) be an infinite rank-\( r \) composition, with

\[
\bar{x}^{(j)}(\mu) = (x_1^{(j)} < \cdots < x_m^{(j)})
\]

labelling the positions of its \( j \)-particles, for all \( 1 \leq j \leq r \). Define an all-rank extension of the crossing statistic (3.102) as follows:

\[
\chi(\bar{x}^{(1)}, \ldots, \bar{x}^{(r)}) := \# \{ x \in \bar{x}^{(i)}, y \in \bar{x}^{(j)} \mid i < j, x > y \}.
\]

Then the observable

\[
H(\nu, \mu) = \prod_{j=1}^{r} \left( \prod_{x \in \bar{x}^{(j)}(\mu)} \prod_{y \in x} (t^{\nu_i}) \right) \cdot t^{-\chi(\bar{x}^{(1)}, \ldots, \bar{x}^{(r)})} \tag{3.120}
\]

satisfies the equation

\[
\sum_{i \in \mathbb{Z}} L_i [H(\cdot, \mu)](\nu) = \sum_{i \in \mathbb{Z}} M_i [H(\nu, \cdot)](\mu),
\]

where \( L_i \) and \( M_i \) are given by (3.28) and (3.30), respectively.

**Proof.** One defines the union of all particle positions,

\[
\bar{z}(\mu) = \bar{x}^{(1)}(\mu) \cup \cdots \cup \bar{x}^{(r)}(\mu),
\]

and proceeds along similar lines as in the proof of Proposition 3.39, considering firstly the case in which \( \bar{z} \) is a single cluster. None of the steps are substantively changed; the sole exception being that the sets \( d_1 \) and \( d_2 \) used in (3.112) should be
replaced by the sets

\[
    d_\prec \left( \bar{x}^{(1)}, \ldots, \bar{x}^{(r)} \right) = \left\{ x \in \bar{x}^{(i)} \bigg| x + 1 \in \bar{x}^{(j)}, \ i < j \right\},
\]

\[
    d_\succ \left( \bar{x}^{(1)}, \ldots, \bar{x}^{(r)} \right) = \left\{ x \in \bar{x}^{(i)} \bigg| x + 1 \in \bar{x}^{(j)}, \ i > j \right\},
\]

respectively.
Chapter 4

The Green’s Function in a Two-Species Exclusion Process

In this chapter, we will study fluctuations of a two-species asymmetric exclusion process, known as the Arndt-Heinzel-Rittenberg model [2]. For a step-Bernoulli initial condition with finite number of particles, we provide an explicit multiple integral expression for a certain joint current probability distribution. By performing an asymptotic analysis we prove that the joint current distribution is given by a product of a Gaussian and a Gaussian unitary ensemble (GUE) Tracy-Widom distribution [157, 158] in the long time limit, as predicted by non-linear fluctuating hydrodynamics.

There are 3 main results in this Chapter, which are given in Theorem 4.1 (the Green’s function of the AHR model), Theorem 4.10 (the joint current distribution under a step type initial condition), and Theorem 4.13 (the long-time limit of the joint current distribution).

4.1 Definition of the model

In 1997, The two species Arndt-Heinzel-Rittenberg (AHR) exclusion process was first studied in [2], where the stationary state was analysed using quadratic algebras, and three phases were found in the stationary state [3, 4].

Similar to ASEP, the AHR model is an exclusion process, and hence a Markov chain, defined on a one-dimensional lattice with two species particles. We consider the one-dimensional lattice $\mathbb{Z}$ (see Fig.4.1). Each site can be occupied by either of the two kinds of particles, which are called “+” (positive) and “-” (negative) particles. The positive particle can hop forward while the negative hops backward. In addition, the particle can swap with its adjacent particle if they are of different kinds*. Obviously, since the AHR model is defined to be a Markov process, the jumps (and swaps) occur according to exponential clocks. Additionally, the jumps (and swaps) are suppressed if neighbouring sites are occupied, in order that the model is an exclusion process. We refer to Chapter 2 for the rigorous definition of the dynamics of a Markov process.

*The swap only occurs in one direction (see the explicit jumping rate)
The explicit jumping rates are:

\begin{align*}
(+, 0) \rightarrow (0, +) & \text{ with rate } p, \\
(0, -) \rightarrow (-, 0) & \text{ with rate } q, \\
(+, -) \rightarrow (-, +) & \text{ with rate } 1.
\end{align*}

Intentionally, we assume that the rates $p$ and $q$ sum up to unity (see Fig. 4.1), in which case the stationary state is factorised [2–4]. In addition, this condition will allow us to construct the eigenfunctions of the time evolution equation in the form of a product of plane waves, and hence also its Green’s function. This property, to a large extent, simplifies the integral formula of the Green’s function and its derivation, and hence the correlation functions.

The Yang-Baxter integrability of the AHR model was proved in [28]. In [2], the authors studied the stationary state of the model on a ring and observed an interesting condensation phenomena [3, 4], which was further studied in [71, 90, 91, 130] by using a connection to the Al-Salam-Chihara polynomials. The dynamic critical exponents and scaling exponents were discussed in [101] and [66].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.1.png}
\caption{The AHR model on $\mathbb{Z}$, with black balls as positive particles and white balls as negative particles.}
\end{figure}

In the following, we first obtain the master equation for the AHR model and give a full derivation of the solution for the Green’s function.

### 4.2 Master equation

We consider the AHR model with $N$ positive particles and $M$ negative. Let $\vec{x} = (x_1, \ldots, x_N)$ and $\vec{y} = (y_1, \ldots, y_M)$ be the position of positive and negative particles, respectively. Suppose the transition probability, a.k.a. the Green’s function, is $P(\vec{x}, \vec{y}; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0)$, where $\vec{x}^{(0)}$ and $\vec{y}^{(0)}$ are the initial positions of the particle. Sometimes we write $P(\vec{x}, \vec{y}; t)$ or $P(\vec{x}; \vec{y})$ for abbreviation.

From the general setup for Markov processes, we write down the master equation for $P(\vec{x}, \vec{y}; t)$ in the AHR model for the case where there are no neighbouring particles:

\[
\frac{d}{dt} P(\vec{x}, \vec{y}; t) = p \sum_{i=1}^{N} P(\vec{x}^+_i, \vec{y}; t) + q \sum_{j=1}^{M} P(\vec{x}, \vec{y}^+_j; t) - (pN + qM)P(\vec{x}, \vec{y}; t), \tag{4.1}
\]

where $\vec{x}^+_i := (x_1, \ldots, x^+_i, \ldots, x_N)$. We emphasise that (4.1) only describes the
time evolution of hopping, it does not include swaps and exclusions. On the right-hand side, the positive terms describe the arrival events of the change in $P(\vec{x}, \vec{y}; t)$. Namely, the state $(\vec{x}^{-}, \vec{y})$ turns into the state $(\vec{x}, \vec{y})$ at rate $p$, while $(\vec{x}, \vec{y}^{+})$ turns into the state $(\vec{x}, \vec{y})$ at rate $q$. On the other hand, the negative term on the right-hand side comes from the exit events of the change in $P(\vec{x}, \vec{y}; t)$. Specifically, the state $(\vec{x}, \vec{y})$ evolves into the state $(\vec{x}^{-}, \vec{y})$ at rate $p$ and the state $(\vec{x}, \vec{y}^{-})$ at rate $q$.

In order to add interactions between particles, i.e., exclusions and swappings, one can include appropriate Kronecker delta functions on the left-hand side of (4.1). Alternatively, one can impose the following boundary conditions. For all eligible $i,j$,

- **Interactions between positive particles**:
  $$P(x_{1}, \ldots, x_{i}, x_{i+1} = x_{i}, \ldots, x_{N}; y) = P(x_{1}, \ldots, x_{i}, x_{i+1} = x_{i} + 1, \ldots, x_{N}; y), \quad (4.2)$$

- **Interactions between negative particles**:
  $$P(\vec{x}; y_{1}, \ldots, y_{i-1} = y_{i}, y_{i}, \ldots, y_{M}) = P(\vec{x}; y_{1}, \ldots, y_{i-1} = y_{i} - 1, y_{i}, \ldots, y_{M}), \quad (4.3)$$

- **Interactions between positive and negative particles**:
  $$P(\vec{x}; y_{1}, \ldots, y_{j} = x_{i} + 1, \ldots, y_{M}) = pP(\vec{x}; y_{1}, \ldots, y_{j} = x_{i}, \ldots, y_{M}) + qP(x_{1}, \ldots, x_{i-1}, x_{i} + 1, x_{i+1}, \ldots, x_{N}; y_{1}, \ldots, y_{j} = x_{i} + 1, \ldots, y_{M}). \quad (4.4)$$

We observe that (4.1)–(4.4) are well defined for $P(\vec{x}; y)$ on $\mathbb{Z}^{N} \times \mathbb{Z}^{M}$. However, we know that the position states are defined on $(\vec{x}, \vec{y}) \in \mathbb{W}^{N} \times \mathbb{W}^{M}$, recalling from Chapter 2 that $\mathbb{W}^{N} := \{ \vec{x} = (x_{1}, \ldots, x_{N}) \in \mathbb{Z}^{N} : x_{1} < x_{2} < \cdots < x_{N} \}$. One can show that $P(\vec{x}; y)$ satisfying (4.1)–(4.4) indeed gives the Green’s function of the AHR model on $\mathbb{W}^{N} \times \mathbb{W}^{M}$, but $P(\vec{x}; y)$ is no longer a probability on $(\mathbb{Z}^{N} \times \mathbb{Z}^{M}) \setminus (\mathbb{W}^{N} \times \mathbb{W}^{M})$, even if it satisfies (4.1)–(4.4). One can refer to Section 4.4 for a detailed explanation of the boundary conditions.

The boundary conditions are much easier to work with than the Kronecker delta functions. In the following section, we give an exact integral formula of the solution to (4.1)–(4.4), i.e., the Green’s function of the AHR model.

### 4.3 Transition probability

We give a formula for the transition probability (or the Green’s function) for the AHR model on $\mathbb{Z}$ in the form of a multiple integral in this section. This formula is a generalisation of the formula for the single species TASEP, first given by Schütz in [144], see also [160]. Our proof will also be similar to the ones in [144, 160], i.e. we show explicitly that the multiple integral satisfies the correct time evolution equation and initial condition.
Theorem 4.1. Consider the AHR model on $\mathbb{Z}$ with $N$ positive and $M$ negative particles. Suppose $r_j$ is the number of positive particles to the right of the $j^{th}$ negative particles, and $r_j^{(0)}$ is $r_j$ at $t = 0$. Namely,

\[ r_j := \# \{ x_i \in \vec{x} \mid x_i \geq y_j \}, \]

\[ r_j^{(0)} := \# \{ x_i^{(0)} \in \vec{x}^{(0)} \mid x_i^{(0)} \geq y_j^{(0)} \}. \]

The transition probability is given by

\[
P(\vec{x}, \vec{y}; t \mid \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \oint_{C_0} \ldots \oint_{C_0} d^N z d^M w e^{A_{N,M} t} \prod_{i=1}^{N} z_i^{-x_i^{(0)} - 1} \prod_{j=1}^{M} p_j^{r_j(y_j^{(0)} - 1)} \sum_{\pi \in S_N} \prod_{i=1}^{N} \left( \frac{1}{1 - z_{\pi_i}} \right) \sum_{\sigma \in S_M} \prod_{j=1}^{M} \left( \frac{1}{1 - w_{\sigma_j}} \right) \times w_{r_j}^{-p} \prod_{k=1}^{r_j} \frac{1}{q z_{\pi_{N-k+1}} + pw_{\sigma_j}}, \tag{4.5}\]

where

\[
d^N z := \prod_{i=1}^{N} \frac{dz_i}{2\pi i}, \quad d^M w := \prod_{j=1}^{M} \frac{dw_j}{2\pi i}, \tag{4.6}\]

\[
\Lambda_{N,M} := p \sum_{i=1}^{N} (z_i^{-1} - 1) + q \sum_{j=1}^{M} (w_j^{-1} - 1). \tag{4.7}\]

The contours of integration are all around the origin and nested such that all $z$-contours exclude the poles at $z_i = -qw_j/p$.

Proof. We need to prove the right-hand side of (4.5) satisfies (4.1)–(4.4) and the initial condition

\[
P(\vec{x}, \vec{y}; t \mid \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \prod_{i=1}^{N} \delta_{x_i, x_i^{(0)}} \prod_{j=1}^{M} \delta_{y_j, y_j^{(0)}}. \tag{4.8}\]

Then it follows from the above section that (4.5) indeed gives the Green’s function.

Finally, we will show that the solution is unique.

Throughout the entire proof, we call the factor $\prod_{j=1}^{M} \prod_{k=1}^{r_j} \frac{1}{q z_{\pi_{N-k+1}} + pw_{\sigma_j}}$ the z-w factor.

Proof of master equation (4.1)

Clearly the integrand is $C^1$ continuous in time. By interchanging $d/dt$ with the integral and making use of the explicit form of $\Lambda_{N,M}$ in (4.7), the evolution equation (4.1) follows from straightforward direct calculation.

Proof of boundary condition (4.2)
On both sides of (4.2), i.e., when $x_{i+1} = x_i$ and $x_{i+1} = x_i + 1$, the z-w factor remains unchanged within the physical regions $\\mathcal{W}^N \times \mathcal{W}^M$. Moreover, the z-w factor in symmetric in $z_{\pi_i}, z_{\pi_i+1}$ since there are no negative particles between the $i^{th}$ and the $i+1^{th}$ positive particles.

From (4.5) we observe that $x_i$ and $x_{i+1}$ only appear as exponents of $z_{\pi_i}$ and $z_{\pi_i+1}$, and there is no change in the z-w factor. Thus, we only need to compare the $z_{\pi_i}$ and $z_{\pi_i+1}$ component

$$(4.9) \quad \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - z_{\pi_i+1}} \right)^{i+1} z_{\pi_i}^{x_{i+1}} z_{\pi_i+1}^{x_i}$$

in the integrand for both sides of (4.2). For the left-hand side of (4.2), i.e., when $x_{i+1} = x_i$, the factor (4.9) reads as

$$LHS = \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - z_{\pi_i+1}} \right)^{i+1} (z_{\pi_i} z_{\pi_i+1})^{x_i}.$$  

When $x_{i+1} = x_i + 1$, (4.9) then is

$$RHS = \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - z_{\pi_i+1}} \right)^{i+1} (z_{\pi_i} z_{\pi_i+1})^{x_i} z_{\pi_i+1}.$$  

It follows that

$$LHS - RHS = \left[ \frac{1}{(1 - z_{\pi_i})(1 - z_{\pi_i+1})} \right]^i (z_{\pi_i} z_{\pi_i+1})^{x_i} \left( \frac{1}{1 - z_{\pi_i+1}} - \frac{z_{\pi_i+1}}{1 - z_{\pi_i+1}} \right)$$

We observe that this factor is symmetric in $z_{\pi_i}, z_{\pi_i+1}$. Moreover, the z-w factor is also symmetric in $z_{\pi_i}, z_{\pi_i+1}$. Hence summing over $\pi \in S_N$ gives a zero integrand, due to the factor sign($\pi$).

**Proof of boundary condition (4.3)**

The proof of this boundary condition is very similar to the one above. First we notice that when $y_{i-1} = y_i$ and $y_{i-1} = y_i - 1$, the z-w factor is unchanged and symmetric in $w_{\sigma_{i-1}}, w_{\sigma_i}$. Hence as before, we only need to consider

$$(4.10) \quad \left( \frac{1}{1 - w_{\sigma_{i-1}}} \right)^{-i+1} \left( \frac{1}{1 - w_{\sigma_i}} \right)^{-i} w_{\sigma_{i-1}}^{y_{i-1}} w_{\sigma_i}^{-y_i}.$$  

The difference of this factor between the left-hand side and right-hand side of (4.3) is given by

$$LHS - RHS = \left[ \frac{1}{(1 - w_{\sigma_{i-1}})(1 - w_{\sigma_i})} \right]^i (w_{\sigma_{i-1}} w_{\sigma_i})^{-y_i} \left( \frac{1}{1 - w_{\sigma_{i-1}}} - \frac{w_{\sigma_{i-1}}}{1 - w_{\sigma_{i-1}}} \right)$$
Now let us consider the right-hand side of (4.4) where \( y \) for both sides of (4.4). For the left-hand side of (4.4), \( y \) that the \( i \)

\[ (4.5) \]

in these two cases, one only needs to check the factor

\[ \pi \]

residues unless \( \sigma \in S_M \) gives a zero integrand as required.

**Proof of initial condition (4.8)**

We consider the two cases when \( y_j = x_1 \) and \( y_j = x_i + 1 \). From the integrand of (4.5), in these two cases, one only needs to check the factor

\[ \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - w_{\pi_j}} \right)^{-j} \]

\[ z_{\pi_i}^{x_1} w_{\pi_j}^{-y_j} \]

\[ \prod_{k=1}^{r_j} \frac{1}{q z_{\pi_{i+k}} + p w_{\pi_j}} \]

\[ x \]

\[ \text{LHS} = \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - w_{\pi_j}} \right)^{-j} \left( \frac{z_{\pi_i}}{w_{\pi_j}} \right)^{x_i} \prod_{k=0}^{N-i} \frac{1}{q z_{\pi_{i+k}} + p w_{\pi_j}} \]

\[ \text{RHS} = p \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - w_{\pi_j}} \right)^{-j} \left( \frac{z_{\pi_i}}{w_{\pi_j}} \right)^{x_i} \prod_{k=0}^{N-i} \frac{1}{q z_{\pi_{i+k}} + p w_{\pi_j}} + q \left( \frac{1}{1 - z_{\pi_i}} \right)^i \left( \frac{1}{1 - w_{\pi_j}} \right)^{-j} \left( \frac{z_{\pi_i}}{w_{\pi_j}} \right)^{x_i+1} \prod_{k=0}^{N-i} \frac{1}{q z_{\pi_{i+k}} + p w_{\pi_j}} \]

which is exactly the same as RHS.

**Proof of boundary condition (4.4)**

Firstly, we show by mathematical induction that (4.5) at \( t = 0 \) has vanishing residues unless \( \pi_i = i \) and \( x_i = x_i^{(0)} \) for all \( i \). To see this, we first prove the case for \( i = 1 \).

(1) **positive particles** \( x_i^{(0)} \) We only consider \( x_1 \geq x_i^{(0)} \) since positive particles only jump to the right. Since all components of \( \vec{x} \) are well ordered, it follows that \( x_j \geq x_1^{(0)} \) for all \( j \). Consider now an arbitrary permutation \( \pi \) such that \( \pi_k = 1 \). The
exponent of $z_1$ in the integrand of (4.5) is given by

$$z_1^{x_k-x_1^{(0)}-1},$$

and the exponent in this expression is always non-negative unless $k = 1$ and $x_1 = x_1^{(0)}$. If these conditions are not met, then at time $t = 0$ the integrand is analytic at $z_1 = 0$ and therefore has a vanishing residue.

For the induction step we assume first that (4.5) vanishes at $t = 0$ unless $x_i = x_i^{(0)}$ and $\pi_i = i$ for all $i \leq \ell - 1$. If $\pi_k = \ell$, the exponent of $z_\ell$ is $x_\ell - x_\ell^{(0)} - 1$, and by the induction hypothesis we have that $k \geq \ell$, and hence the exponent is non-negative when $k > \ell$, and (4.5) is zero unless $\pi_k = \ell$ and $x_\ell = x_\ell^{(0)}$. Therefore, we can conclude that $G(\vec{x} - \vec{x}^{(0)}, \vec{y} - \vec{y}^{(0)}, 0)$ is nonzero only when $x_i = x_i^{(0)}$ for all $i$ and $\pi = \text{id}$.

When $x_i = x_i^{(0)}$ for all $i \in \{1, \ldots, N\}$ and $\pi = \text{id}$, then after integration over the $z$ variables the Green’s function (4.5) at $t = 0$ becomes

$$P(\vec{x}, \vec{y}; 0 | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \int d^M w \prod_{k=1}^{M} w_k^{r_k} \times \sum_{\sigma \in S_M} \text{sign}(\sigma) \prod_{k=1}^{M} \left( \frac{w_k - 1}{w_{\sigma_k} - 1} \right)^{-k} w_{\sigma_k}^{-r_k} w_k^{(0)} - 1,$$

where $r_j = r_j^{(0)}$ for all $j$ since $x_j = x_j^{(0)}$ for all $j$. Note that $p^{r_j^{(0)}} = p^{-r_j}$ cancels with the $p^{-r_j}$ in $(pw_{\sigma_j})^{-r_j}$.

(2) negative particles $y_j^{(0)}$ We now prove that the above function is nonzero only when $y_j = y_j^{(0)}$ for all $j \in \{1, \ldots, M\}$ and $\sigma = \text{id}$. While positive particles hop to the right, the negative particle always hops to the left and so we have that $y_j^{(0)} \geq y_j$ for all $j$. Consider an arbitrary permutation $\sigma$ such that $\sigma_k = j$ with $j > k$. At time $t = 0$, the exponent of $w_j$ in the integrand in (4.5) is given by

$$w_j^{y_j^{(0)} - y_k + r_j - r_k - 1}.$$

By the definition of $r_j$, we must have $y_j - y_k + r_j - r_k > 0$ for any $j > k$. Hence $y_j^{(0)} - y_k + r_j - r_k - 1 \geq y_j - y_k + r_j - r_k - 1 > 0$. Therefore, the exponent of $w_j$ is always positive unless $\sigma_j = j$. When $\sigma = \text{id}$, the exponent of $w_j$ becomes $y_j^{(0)} - y_j - 1$, which results in a vanishing residue unless $y_j^{(0)} = y_j$.

It remains to show that when $x_i = x_i^{(0)}$ for all $i$, $\pi = \text{id}$ and $y_j = y_j^{(0)}$ for all $j \in \{1, \ldots, M\}$, $\sigma = \text{id}$, the Green’s function is normalized as $G(\vec{x} - \vec{x}^{(0)}, \vec{y} - \vec{y}^{(0)}, 0) = 1$. This can be easily seen by the residue theorem.

**Proof of uniqueness**
The Green’s function is a solution of a Markov equation with bounded initial condition, and the number of particle jumps for a given time $t$ in the AHR model is bounded by a Poisson random variable with parameter given be a constant times $t$. The global existence and uniqueness is therefore guaranteed by general considerations as provided in Proposition 4.9 and Appendix C of [17]. This concludes the proof for the Green’s function.

4.4 Bethe wave functions

In this section, we will show how to derive the Green’s function above, using Bethe ansatz. But this is only an idea to compose the Green’s function, not a rigorous proof. Hence this will be very similar to the section “Bethe ansatz” in Chapter 2.

As before, we start with an eigenvalue problem associated with the master equation, so that

$$P(\vec{x}, \vec{y}; t) = e^{\Lambda t} P(\vec{x}; \vec{y}).$$

Positive particles We will start with the case with only two positive particles. The corresponding eigenvalue equation for $x_{1} + 1 < x_{2}$ is given by

$$\Lambda P(x_{1}, x_{2}) = pP(x_{1} - 1, x_{2}) + pP(x_{1}, x_{2} - 1) - 2pP(x_{1}, x_{2}). \quad (4.11)$$

Substituting $P(x_{1}, x_{2}) = A_{12}z_{x_{1}}^{x_{2}} + A_{21}z_{x_{2}}^{x_{1}}$ gives us the eigenvalue $\Lambda = p(z_{1}^{-1} + z_{2}^{-1} - 2)$. Next, we consider the case $x_{1} = x = x_{2} - 1$:

$$\Lambda P(x, x + 1) = pP(x - 1, x + 1) - pP(x, x + 1). \quad (4.12)$$

By comparing these two equations (4.11) and (4.12), one can obtain the boundary condition

$$P(x, x) = P(x, x + 1). \quad (4.13)$$

Namely, (4.12) is automatically satisfied if one imposes (4.13) on (4.11). Hence by substituting $P(x_{1}, x_{2}) = A_{12}z_{x_{1}}^{x_{2}} + A_{21}z_{x_{2}}^{x_{1}}$, one obtains $A_{12}(1 - z_{2}) + A_{21}(1 - z_{1}) = 0$. This suggests one can take $A_{12} = (1 - z_{1})^{-1}(1 - z_{2})^{-2}$, and $A_{21} = -(1 - z_{2})^{-1}(1 - z_{1})^{-2}$.

Now we can generalise the above results to the case with $N$ positive particles such that $P(\vec{x}) = \sum_{\pi} A_{\pi} \prod_{i=1}^{N} z_{x_{i}}^{x_{i}}$.

- The eigenvalue equation for $x_{i} + 1 < x_{i+1}$ indicates that the contribution to the eigenvalue from the positive particles is

  $$\Lambda = p \sum_{i=1}^{N} (z_{i}^{-1} - 1).$$

- Generalising (4.13) gives the boundary condition that characterises the interaction between positive particles, which is exactly the boundary condition
Substituting the Bethe ansatz $P(\vec{x}) = \sum_{\pi} A_{\pi} \prod_{i=1}^{N} z_{\pi_{i}}$ into the above boundary condition gives the scattering relation

$$\frac{A_{\pi}}{A_{\pi s_i}} = -\frac{1 - z_{\pi_{i}}}{1 - z_{\pi_{i+1}}}.$$  

where $s_i$ is the generators of $S_N$. This suggests that $A_\pi$ is of the form

$$A_\pi = \text{sign}(\pi)\left(\frac{1}{1 - z_{\pi_i}}\right)^i.$$  

Negative particles  
Repeat the above analysis for the case of $M$ negative particles only, and one should obtain for $P(\vec{y}) = \sum_{\sigma} B_{\sigma} \prod_{i=1}^{M} w_{\sigma_{i}}^{-y_{i}}$.  

- The eigenvalue equation for $y_{i} + 1 < y_{i+1}$ indicates that the contribution to the eigenvalue from the negative particles is

$$\Lambda = q \frac{M}{\sum_{i=1}^{M} (w_{i}^{-1} - 1)}.$$  

- The boundary condition that describes the interaction between positive particles is given by (4.3).  

- The above boundary condition gives the scattering relation

$$\frac{B_{\sigma}}{B_{\sigma s_i}} = -\frac{1 - w_{\sigma_{i+1}}}{1 - w_{\sigma_{i}}}.$$  

where $s_i$ is the generators of $S_N$. This suggests that $B_\sigma$ is of the form

$$B_\sigma = \text{sign}(\sigma)(1 - w_{\sigma_{i}})^i.$$  

Positive and negative particles  
Now consider the case that two types of particles exist. Likewise, we start with the simplest case with one positive and one negative particles. When $x \neq y \pm 1$, the corresponding eigenvalue equation reads as

$$\Lambda P(x; y) = qP(x; y + 1) + pP(x - 1; y) - (p + q)P(x; y), \quad (4.14)$$

Substituting $P(x; y) = Cz^{x}w^{-y}$ gives the eigenvalue $\lambda = p(z^{-1} - 1) + q(w^{-1} - 1)$.  
Next, we need to consider the case when two particles sit next to each other.  
When $x = y + 1$, i.e., the positive particle is in front of the negative one,

$$\Lambda P(y + 1; y) = P(y; y + 1) - (p + q)P(y + 1; y). \quad (4.15)$$

As before, comparing the two equations (4.14) and (4.15) gives the boundary con-
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\[ P(y; y+1) = qP(y+1; y+1) + pP(y; y). \]  

(4.16)

This suggests an ansatz with a coefficient that depends on the relative position of the negative and positive particles

\[ P(x; y) = \begin{cases} 
  C_{++}z^xw^{-y} & x < y \\
  C_{+-}z^xw^{-y} & x \geq y 
\end{cases}. \]

Substituting this into boundary condition (4.16) gives

\[ C_{++}(qz + pw) = C_{+-}. \]

When \( y = x + 1 \), i.e., the negative particle is in front of the positive one,

\[ \Lambda P(x; x+1) = pP(x-1; x+1) + qP(x+2; x+1) - P(x; x+1). \]  

(4.17)

We observe that (4.17) agrees with (4.14) without any extra boundary conditions. The reason is that \( p + q = 1 \). This condition leads to a factorised stationary state [2–4], and suggests a factorised form of Bethe ansatz. Thus \( P(x; y) = Cz^xw^{-y} \) is an effective Bethe ansatz for the AHR model. In other words, there is no need for another coefficient to satisfy any “fourth” boundary condition.

Finally, one can generalise the two particles case to the one with \( N \) positive particles and \( M \) negative. The corresponding ansatz is

\[ P(\vec{x}, \vec{y}) = \sum_{\pi, \sigma} A_{\pi} B_{\sigma} C_{\vec{x}, \vec{y}}(\pi, \sigma) \prod_{i=1}^{M} w_{\sigma_i}^{-y_i} \prod_{i=1}^{N} z_{\pi_i}^{x_i}, \]

where \( C_{\vec{x}, \vec{y}}(\pi, \sigma) \) is a coefficient that depends on the relative position of \( \vec{x}, \vec{y} \).

- The eigenvalue is given by
  \[ \Lambda = p \sum_{i=1}^{N} (z_{\pi_i}^{-1} - 1) + q \sum_{i=1}^{M} (w_{\sigma_i}^{-1} - 1). \]

- The boundary condition that characterises the interaction between positive particles is given by (4.4), which is obtained by Generalising (4.16).

- Substitute \( P(\vec{x}, \vec{y}) = \sum_{\pi, \sigma} A_{\pi} B_{\sigma} C_{\vec{x}, \vec{y}}(\pi, \sigma) \prod_{i=1}^{M} w_{\sigma_i}^{-y_i} \prod_{i=1}^{N} z_{\pi_i}^{x_i} \) into the above boundary condition, and we have
  \[ C_{y_j > x_i} = C_{y_j < x_i}(qz_{\pi_i} + pw_{\sigma_j}). \]

This relation suggests

\[ C_{\pi, \sigma} = \prod_{j=1}^{M} \prod_{k=1}^{r_j} \frac{1}{qz_{\pi_N - k+1} + pw_{\sigma_j}}. \]

where \( r_j \) is the number of positive particles in front of the \( j \)th negative particle.
General case  From above, we can conclude that the Bethe wave function for the AHR model is given by

\[ \sum_{\pi \in \mathcal{S}_N} \text{sign}(\pi) \prod_{i=1}^{N} \left( \frac{1}{1 - z_{\pi_i}} \right)^i \sum_{\sigma \in \mathcal{S}_M} \text{sign}(\sigma) \times \prod_{j=1}^{M} \left( \frac{1}{1 - w_{\sigma_j}} \right)^{-j} w_{\sigma_j}^{-y_{\sigma_j}} \prod_{k=1}^{r_j} \frac{1}{q z_{\sigma_{N-k+1}} + pw_{\sigma_j}}. \]  

(4.18)

Hence one can construct the Green’s function as below

\[ P(\vec{x}, \vec{y} ; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \oint \ldots \oint \ d^N z \ d^M w \ f(\vec{z}, \vec{w}) \ e^{\Lambda_{N,M} t} \sum_{\pi \in \mathcal{S}_N} \text{sign}(\pi) \times \prod_{i=1}^{N} \left( \frac{1}{1 - z_{\pi_i}} \right)^i \sum_{\sigma \in \mathcal{S}_N} \text{sign}(\sigma) \prod_{j=1}^{M} \left( \frac{1}{1 - w_{\sigma_j}} \right)^{-j} w_{\sigma_j}^{-y_{\sigma_j}} \prod_{k=1}^{r_j} \frac{1}{q z_{\sigma_{N-k+1}} + pw_{\sigma_j}}, \]

where \( f(\vec{z}, \vec{w}) \) is carefully chosen such that \( P(\vec{x}, \vec{y} ; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) \) satisfies the initial condition. Make use of the identity

\[ \oint_0 z^{x-y-1} \frac{dz}{2\pi i} = \delta_{xy}, \]

where the contour is around the origin. It follows that the Green’s function is given by (4.5).

### 4.5 Symmetrisation identities

Once the Green’s function is given, in principle, we should be able to write down any correlation functions. Our next aim is to study a joint current distribution for the AHR model. Before giving the explicit formula, the following lemmas will be useful in deriving the exact expression.

Note that the following identities are special cases of equation (1.6) in [160]

**Lemma 4.2.**

\[ \sum_{\pi \in \mathcal{S}_n} \text{sign}(\pi) \prod_{i=1}^{n} \left( \frac{z_{\pi_i}}{1 - z_{\pi_i}} \right)^{i-1} \frac{1}{1 - \prod_{j=1}^{n} z_{\pi_j}} = \frac{\prod_{i<j}(z_j - z_i)}{\prod_{i=1}^{n}(1 - z_i)^n}. \]  

(4.19)

**Proof.** This can be proved by mathematical induction on \( n \). Denote the left-hand side of (4.19) by \( f_n(z_1, \ldots, z_n) \). Clearly (4.19) holds for \( n = 1 \). Now assume it holds for \( n - 1 \), and we would like to prove it for \( n \). To make use of the induction assumption, we need to find the relation between \( f_n \) and \( f_{n-1} \). The sum over \( \mathcal{S}_n \) can be split into a sum over \( k \in [1, n] \) such that \( \pi_1 = k \) and a remaining sum over \( \mathcal{S}_{n-1} \). We observe that \( \text{sign}(\pi) = (-1)^{k-1} \text{sign}(\sigma) \) where \( \sigma \) is the permutation \( \pi \).
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restricted on \(\{1, 2, \ldots, k - 1, k + 1, \ldots, n\}\), and \((-1)^{k-1}\) is the signature of permutation \((k, 1, \ldots, k - 1, k + 1, \ldots, n)\). We then obtain the relation between \(f_n\) and \(f_{n-1}\):

\[
f_n(z_1, \ldots, z_n) = \sum_{k=1}^{n} (-1)^{k-1} \prod_{i \neq k} \frac{z_i}{1 - z_i} \frac{1}{\prod_{i=1}^{n} z_i} f_{n-1}(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n),
\]

where the factor \(\prod_{j \neq k} \frac{z_j}{1 - z_j}\) is due to the fact \(\sigma_i = \pi_{i+1}\) for \(i \in [1, n - 1]\).

By employing the induction hypothesis on \(f_{n-1}\), and \(\prod_{i<j} (z_j - z_i) = (-1)^{k-1} \prod_{i \neq k} (z_i - z_k) \prod_{i<j} (z_j - z_i)\), we obtain after simplification,

\[
f_n(z_1, \ldots, z_n) = \prod_{i<j} (z_j - z_i) \prod_{i} z_i \sum_{k=1}^{n} \frac{(1 - z_k)^n}{z_k} \frac{1}{\prod_{i \neq k} (z_i - z_k)}. \tag{4.20}
\]

We are left with the sum over \(k\). This can be evaluated by considering the following function,

\[
F(z) := (1 - z)^n \frac{1}{\prod_{i} (z_i - z)}.
\]

One can easily see that

\[
\sum_{k=1}^{n} \frac{(1 - z_k)^n}{z_k} \frac{1}{\prod_{i \neq k} (z_i - z_k)} = \sum_{k} \text{Res}_{z=z_k} F(z).
\]

These residues can be converted to the residues at infinity and at the origin:

\[
\text{Res}_{z=\infty} F(z) = - \text{Res}_{z=0} z^{-2} F(z^{-1}) = - \text{Res}_{z=0} \left[ \frac{(z - 1)^n}{\prod_{i} (z_i - 1) z} \right] = -1,
\]

\[
\text{Res}_{z=0} F(z) = 1 \prod_{i} \frac{1}{z_i}.
\]

Thus by the residue theorem,

\[
\sum_{k=1}^{n} \frac{(1 - z_k)^n}{z_k} \frac{1}{\prod_{i \neq k} (z_i - z_k)} = - \sum_{k} \text{Res}_{z=z_k} F(z) = \text{Res}_{z=0} F(z) + \text{Res}_{z=\infty} F(z) = \frac{1 - \prod_{i} z_i}{\prod_{i} z_i}.
\]

Substitution this into (4.20) gives the required identity

\[
f_n(z_1, \ldots, z_n) = \prod_{1 \leq i < j \leq n-1} (z_j - z_i) \prod_{i=1}^{n-1} (1 - z_i)^{n-1},
\]

and hence completes the proof. \(\square\)

As an immediate result of the above identity, one can easily obtain another
version of symmetrisation identity, which will also be useful in the following sections.

**Corollary 4.3.**

\[ \sum_{\pi \in S_n} \text{sign}(\pi) \prod_{i=1}^{n} \left( \frac{1 - w_{\pi_i}}{w_{\pi_i}} \right)^i \frac{1}{1 - \prod_{j=1}^{n} w_{\pi_j}} = \prod_{i<j} (w_i - w_j) \prod_{i=1}^{n} w_i^{-n} \quad (4.21) \]

The proof the above corollary follows from the mapping between two permutations \( \pi_i = \rho_{n+1-i} \). Then we have \( \prod_{i=1}^{n} (1 - \prod_{j=1}^{n} z_{\pi_i})^{-1} = \prod_{i=1}^{n} (1 - \prod_{j=1}^{n} \rho_{\pi_j})^{-1} \), and \( \prod_{i=1}^{n} \frac{z_{\pi_i}}{(1 - z_{\pi_i})} )^{i-1} = \prod_{i=1}^{n} \frac{\rho_{\pi_i}}{(1 - \rho_{\pi_i})})^{n-i-1} \). Consequently, the above identity follows from (4.19) after some manipulations.

The following identity is another equality needed later in this chapter. The proof of this identity is very similar to the first one.

**Lemma 4.4.**

\[ \sum_{\pi \in S_n} \text{sign}(\pi) \prod_{i=1}^{n} \left( \frac{z_{\pi_i} - 1}{z_{\pi_i}} \right)^i \frac{1}{1 - (1 - \rho) \prod_{j=1}^{n} z_{\pi_j}} = \prod_{i<j} (z_i - z_j) \prod_{i=1}^{n} (z_i - 1) \prod_{i=1}^{n} z_i^{n-i} \quad (4.22) \]

**Proof.** Likewise, we prove this identity by mathematical induction in \( n \). One can easily see that (4.22) holds for \( n = 1 \). We assume that it holds for \( n - 1 \), and want to prove for \( n \). Let us denote the left-hand side of (4.22) by \( f_n(z_1, \ldots, z_n) \). The key step is then to rewrite \( f_n \) using \( f_{n-1} \). The idea is similar as before. One needs to identify one element \( \pi_i = j \) in the permutation \( \pi \), so that the sum over \( \pi \) on \([1, n]\) can be regarded as sum over \( j \in [1, n] \), and a remaining sum over permutations on \([1, n] \setminus \{i\}\).

From (4.22), we observe that the most convenient fixed element is \( \pi_n = k \), since \( z_{\pi_n} \) can be factored out of \( f_n \). We can see that \( \sum_{\pi} \text{sign}(\pi) = \sum_{k=1}^{N} (-1)^{n-k} \sum_{\sigma} \text{sign}(\sigma) \) where \((-1)^{n-k}\) comes from \( \text{sign}(1, \ldots, k-1, k+1, \ldots, n, k) \), and \( \sigma \) is the permutation \( \pi \) restricted on \([1, N] \setminus \{k\}\). Therefore,

\[ f_n(z_1, \ldots, z_n) = \sum_{k=1}^{N} (-1)^{n-k} \left( \frac{2k - 1}{2k} \right)^n \times \frac{1}{1 - (1 - \rho) \prod_{i=1}^{n} z_i} f_{n-1}(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n). \]

Substituting our induction assumption for \( f_{n-1}(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n) \) and after some rearrangements, we obtain

\[ f_n(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j) \prod_{i=1}^{n} z_i^{n-i} \sum_{k=1}^{N} (-1)^{n+1} \frac{(z_k - 1)^{n-1}}{z_k} \times \prod_{i \neq k} \frac{1}{(z_i - z_k)(1 - (1 - \rho) z_i)}. \]
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As before, in order to show the sum over $k$ gives the expected result, we consider the function defined by

$$F(z) = \frac{(z-1)^{n-1}(1 - (1 - \rho)z)}{z \prod_{i=1}^{n}(z_i - z)(1 - (1 - \rho)z_i)}.$$  

By the residue theorem,

$$( -1)^n \sum_{i=1}^{n} \text{Res}_{z=z_i} F(z) = \sum_{k=1}^{n} \frac{(-1)^{n+1} (z_k - 1)^{n-1}}{z_k} \prod_{i \neq k} \frac{1}{(z_i - z_k)(1 - (1 - \rho)z_i)}$$

$$= (-1)^{n+1} \text{Res}_{z=0} F(z) + (-1)^{n+1} \text{Res}_{z=\infty} F(z)$$

$$= \prod_{i=1}^{n} z_i (1 - (1 - \rho)z_i) + \prod_{i=1}^{n} (1 - (1 - \rho)z_i),$$

which gives the required result to complete the proof.  

4.6 Fredholm determinant

In the case of single species models, a useful approach to establish late time results is to rewrite multiple integral expressions into a Fredholm determinant for which asymptotics is easier to perform. For example, for the most standard case of the GUE random matrix eigenvalues, or in fact for general determinantal point processes [15], such a rewriting is well known and is explained in [69, 121, 144, 156]. In our, higher rank, case it has not been known whether the AHR model is associated with a determinantal processes or not, but the following rewrite in terms of auxiliary variables (or Fourier modes) has turned out to be useful in our analysis.

In this section, we describe a general method of converting multiple contour integrals into Fredholm determinants introduced in [86]. The main idea is to transform the integrand into determinants by using Cauchy matrix identity. The integral therefore can be converted into a determinant according to Cauchy-Binet formula, and then into a Fredholm determinant by swapping the order of matrices.

The reason for this transformation is that it makes possible to estimate long time limit probability distribution, in the way that large parameters $N, M, t$ are converted into the exponential power in the integrand, where methods like the steepest descent can be applied.

Let us consider the $\nu$-fold integral of the form,

$$I_{\nu} = \frac{1}{\nu!} \int_{C} \prod_{i=1}^{\nu} \frac{d\zeta_i}{2\pi i} \frac{\prod_{1 \leq i \neq j \leq \nu} (1 - \zeta_i / \zeta_j)}{\prod_{1 \leq i,j \leq \nu} (1 - a_i / \zeta_j)} \prod_{i=1}^{\nu} \frac{g_{\mu,\nu}(\zeta_i, -s)}{g_{\mu,\nu}(a_i, -s)}$$  

(4.23)

where

$$g_{\mu,\nu}(\zeta, x) = \prod_{j=1}^{\nu} \frac{1}{1 - \alpha_j / \zeta} \prod_{k=1}^{\mu} \frac{1}{1 + \beta_k / \zeta} \times \zeta^{\mu - \nu + x} e^{\zeta t}.  \quad (4.24)$$
The possible singularities inside $C$ are at $0, \alpha_i, a_i, -1/\beta_k, \infty$ for $i \in [1, \nu], k \in [1, \mu]$. The factor $\prod_{i \neq j}(1 - \zeta_i/\zeta_j) / \prod_{i,j}(1 - a_i/a_j)$ in the integrand can be written as determinant via the Cauchy determinant identity,

$$\frac{\prod_{i \neq j}(1 - \zeta_i/\zeta_j)}{\prod_i \zeta_i \prod_{i,j}(1 - a_i/a_j)} = \frac{\prod_{i \neq j}(\zeta_i - \zeta_j)}{\prod_{i,j}(\zeta_i - a_j)} = \det \left( \frac{1}{\zeta_i - a_j} \right)_{1 \leq i,j \leq \nu} \det \left( \frac{1}{\zeta_k - a_\ell} \right)_{1 \leq k,l \leq \nu} \prod_{i,j}(\zeta_i - a_j).$$

Recall the Cauchy-Binet identity (or Andreief identity):

$$\frac{1}{\nu!} \int \det (f_i(x_j))_{1 \leq i,j \leq \nu} \det (h_i(x_j))_{1 \leq i,j \leq \nu} \prod_{i=1}^\nu du(x_i) = \det \left( \int f_i(x)g_j(x)du(x) \right)_{1 \leq i,j \leq \nu},$$

from which, the $\nu$-fold integral is written as a determinant

$$I_\nu = \frac{1}{\nu!} \int_C \prod_{j=1}^\nu \left[ \frac{d\zeta_j}{2\pi i a_j^s} g_{\mu,\nu}(\zeta_j, -s) \prod_\ell(\zeta_j - a_\ell) \right] \times \det \left( \frac{1}{\zeta_i - a_j} \right)_{1 \leq i,j \leq \nu} \det \left( \frac{1}{\zeta_k - a_\ell} \right)_{1 \leq k,l \leq \nu} \prod_{i,j}(\zeta_i - a_j).$$

From this explicit form it is clear that the poles at $\zeta = a_j$ and $\zeta = a_k$ are removable unless $j = k$, and hence the after evaluating residues at $\zeta = a_j$ for all $j$, we obtain

$$I_\nu = \det \left( \delta_{jk} + \int_C \frac{d\zeta}{2\pi i (\zeta - a_j) (\zeta - a_k)} g_{\mu,\nu}(\zeta, -s) \prod_\ell(\zeta - a_\ell) \right)_{1 \leq j,k \leq \nu},$$

where contour $C_r$ includes only the poles at $0, \alpha_j$’s and $-1/\beta_k$’s, but not $a_j$’s. To transform the integral around $C_r$ into a product of two operators (or matrices), rewrite the factor $-\zeta/(\zeta - a_j)$ in terms of a geometric series $\sum_{x=1}^\infty (\zeta/a_j)^x$. Therefore, the integral $I_\nu$ becomes,

$$I_\nu = \det \left( \delta_{jk} - \sum_{x=1}^\infty \int_C \frac{d\zeta}{2\pi i \zeta a_j^s} g_{\mu,\nu}(\zeta, x - s) \prod_\ell(\zeta - a_\ell) \right)_{1 \leq j,k \leq \nu}.$$

Note that this result requires that $|\zeta/a_j| \leq \epsilon < 1$, for all $j$, so that the geometric sum converges uniformly. This condition holds if the poles $\alpha_j$’s and $-1/\beta_k$’s lie inside the disk of radius $r := \min_{1 \leq j \leq \nu}(|a_j|)$, i.e., $|\alpha_j| < r$, and $|\beta_k| > r^{-1}$ for all
The sum over $x \geq 1$ is indeed a product of two matrices of dimension $\nu \times \infty$ and $\infty \times \nu$, and hence the integral $I_\nu$ is converted into a Fredholm determinant by swapping the product order of these two matrices. Namely,

$$I_\nu = \det(1 - AB)_{1 \leq j,k \leq \nu} = \det(1 - BA)_{\ell \in \mathbb{N}},$$

where

$$A(k, x) = \int_{C_r} \frac{d\zeta}{2\pi i} g_{\mu,\nu}(\zeta, x - s) \prod_{\ell \neq k} (\zeta - a_\ell),$$

$$B(x, j) = \left( a_j^3 g_{\mu,\nu}(a_j, x - s) \prod_{\ell \neq j} (a_j - a_\ell) \right)^{-1}.$$

For the purpose of later discussion, we will show that the kernel of the Fredholm determinant can be written into a product of two contour integral, so that the method like the steepest descent can be applied for asymptotic analysis. The kernel $K := BA$ is given by

$$K(x, y) = \sum_{j=1}^{\nu} B(x, j) A(j, y)$$

$$= \sum_{j=1}^{\nu} a_j^3 g_{\mu,\nu}(a_j, x - s) \prod_{\ell \neq j} (a_j - a_\ell) \int_{C_r} \frac{d\zeta}{2\pi i} g_{\mu,\nu}(\zeta, y - s) \prod_{\ell \neq j} (\zeta - a_\ell)$$

$$= \int_{C_r} \frac{d\zeta}{2\pi i} \int_{D} \frac{d\xi}{2\pi i} \frac{1}{\xi - \zeta} \left( \frac{\xi^\nu - s}{\zeta^\nu} \prod_{\ell=1}^{\nu} \frac{\zeta - a_\ell}{\xi - a_\ell} - 1 \right) \frac{g(\zeta, y - s)}{g(\xi, x - s)},$$

where

$$g(\zeta, y) = \zeta^\nu g_{\mu,\nu}(\zeta, y),$$

with $g_{\mu,\nu}$ define in (4.24), and the contour $D$ for the $\xi$ integration includes the poles at $a_j$. The term $-1$ inside the parentheses is inserted for convenience of the next step and does not change the value of the integral.

Using a simple identity in [86],

$$\frac{1}{\zeta - \xi} \left( \frac{\xi^\nu}{\zeta^\nu} \prod_{\ell=1}^{\nu} \frac{\zeta - a_\ell}{\xi - a_\ell} - 1 \right) = \sum_{k=0}^{\nu-1} a_{k+1} (\zeta - a_1) \cdots (\zeta - a_k) \xi^{k+1},$$

the kernel can be written in the form

$$K(x, y) = \sum_{k=0}^{\nu-1} \phi_k(x) \psi_k(y),$$
with
\[
\phi_k(x) = \int_D \frac{d\xi}{2\pi i} \frac{\xi^{k-s}}{g(\xi, x-s)(\xi - a_1)\cdots(\xi - a_{k+1})},
\]
\[
\psi_k(x) = a_{k+1} \int_C \frac{d\zeta}{2\pi i} \zeta^{k+2} g(\zeta, x-s)(\zeta - a_1)\cdots(\zeta - a_k).
\]

### 4.7 Joint current distributions

In the rest of this chapter, we will focus on a joint current distribution for the AHR model. In principle, given the above Green’s function, any physical observable should be able to be calculated analytically. Another main result of this chapter, in addition to the Green’s function, is an exact integral formula for a joint current distribution. The asymptotic behaviour of the AHR model can be analysed through this joint current distribution.

#### 4.7.1 Step initial data

To give an explicit formula for the current distribution, one needs to specify the initial condition. We first consider the simplest case, the step initial data. Such initial data is usually the starting point of asymptotic analysis for universal long time behaviour. For example, Johansson [89] showed that for TASEP with step initial condition, the limiting behaviour of the position of the left-most particle is identified with the distribution of the largest eigenvalue of the Gaussian unitary ensemble in random matrix theory, and hence leads to an universal distribution function for the particle current.

The step initial condition is at time \( t = 0 \) consist of positive particles located at the first \( N \) sites on \( x < 0 \) while negative particles sit at the first \( M \) sites on \( x \geq 0 \), as shown in Fig. 4.2. Specifically, \( x_j(0) = -N + j - 1 \) and \( y_k(0) = k - 1 \). Such initial data is called “step” since the initial particle density function is described by a step function.

![Figure 4.2: Step initial condition](image)

Under such initial data, we will show in the following that an exact formula of a joint current distribution: the probability that all positive and negative particles have crossed the origin by time \( t \). We denote this joint current distribution by \( P_{N,M}(t) \).
Remark 4.5. The kernel of the joint current distribution $P_{N,M}(t)$ is totally factorised and hence can be evaluated asymptotically by the method of steepest descent. Such a nice property is due to the fact that the sum of the jump rate of negative particles and positive particles equals to the swap rate.

Recall that the transition probability (4.5) under the initial condition $x_N^{(0)} < y_1^{(0)}$ and final condition $y_M < x_1$ is given by

$$P(\vec{x}, \vec{y}; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \int_{C_0} \ldots \int_{C_0} d^N z \ d^M w$$

$$e^{A_{N,M} t} \prod_{i=1}^N \prod_{j=1}^M \sum_{\pi \in S_N} \operatorname{sign}(\pi) \prod_{i=1}^N \left( \frac{1 - \bar{z}_i}{1 - z_{\pi_i}} \right)^{x_{\pi_i}^N z_{N-i}}$$

$$\times \sum_{\rho \in S_M} \operatorname{sign}(\rho) \prod_{j=1}^M \left( \frac{1 - w_j}{1 - w_{\rho_j}} \right)^{-j - y_j^1 w_j^{j-1}}. \quad (4.28)$$

The total exchange probability is thus given by a sum of the above Green’s function (4.28) over all possible final configurations $0 \leq x_1 < x_2 < \cdots < x_N$ and $y_1 < y_2 \cdots < y_M \leq -1$. The sums are easily evaluated via geometric series as follows:

$$\sum_{0 \leq x_1 < x_2 < \cdots < x_N} \prod_{i=1}^N (z_{\pi_i} - z_j) = \prod_{i=1}^N \frac{z_{i-1}}{1 - \sum_{j=i}^N z_{\pi_j}},$$

$$\sum_{y_1 < y_2 < \cdots < y_M < -1} \prod_{j=1}^M (w_{\rho_j} - y_j) = \prod_{j=1}^M \frac{w_{M-j+1}}{1 - \sum_{k=1}^M w_{\rho_k}}.$$

Therefore, we obtain the following integral formulas for the joint current:

**Proposition 4.6.** Consider a step initial condition such that positive particles starting at negative integers and the negative at non-negative integers, i.e., $x_j^{(0)} = -N + j - 1$ and $y_k^{(0)} = k - 1$. An explicit formula for $P_{N,M}(t)$ is given by

$$P_{N,M}(t)$$

$$= \int_{C_0} d^N z \ d^M w \ e^{A_{N,M} t} \prod_{1 \leq i < j \leq N} (z_i - z_j) \prod_{1 \leq k \leq M} (w_l - w_k) \prod_{j=1}^N \frac{z_j^{N-j} \prod_{k=1}^M w_k^{k-1}}{\prod_{j=1}^N (z_j - 1)^{N+1-j} \prod_{k=1}^M (w_k - 1)^k \prod_{j=1}^N \prod_{k=1}^M \left( q z_j + p w_k \right)},$$

$$= \frac{1}{N!} \int_{C_0} d^N z \ d^M w \ e^{A_{N,M} t} \prod_{i<j} (z_i - z_j)^2 \prod_{k<l} (w_k - w_l) \prod_{j=1}^N \prod_{k=1}^M \frac{w_k^{k-1}}{\prod_{i=1}^N (z_i - 1)^N \prod_{k=1}^M (w_k - 1)^k \prod_{j=1}^N \prod_{k=1}^M \left( q z_i + p w_j \right)}. \quad (4.29)$$
where all integrals are around the origin and
\[
\Lambda_{N,M} = p \sum_{i=1}^{N} (z_i^{-1} - 1) + q \sum_{j=1}^{M} (w_j^{-1} - 1).
\]

**Proof.** The first equation follows from the above discussion and applying identities (4.19) and (4.21). The second one is obtained via a symmetrisation of the first equation. To see this, we first change the variable \( z_i \rightarrow z_{\pi_i} \) in the integrand.

\[
P_{N,M}(t) = \text{sign}(\pi) \oint_{C_0} d^N z \ d^M w \ e^{\Lambda_{N,M} t} \times
\]
\[
\prod_{1 \leq i < j \leq N} (z_i - z_j) \prod_{1 \leq k < l \leq M} (w_k - w_l) \prod_{j=1}^{N} z_{\pi_j}^{N-j} \prod_{k=1}^{M} w_k^{k-1} \prod_{j=1}^{N} z_{\pi_j}^{N-j} (z_{\pi_j} - 1)^{j-1} \prod_{k=1}^{M} w_k^{k-1} \prod_{j=1}^{N} (z_j - 1)^{N-j} \prod_{k=1}^{M} (w_k - 1)^{k} \prod_{j=1}^{N} \prod_{k=1}^{N} \left( q z_j + p w_k \right),
\]

(4.30)

where \( \text{sign}(\pi) \prod_{1 \leq i < j \leq N} (z_i - z_j) = \prod_{1 \leq i < j \leq N} (z_{\pi_i} - z_{\pi_j}) \). By summing (4.30) over \( \pi \in S_N \), we obtain \( N! \) times of that probability, namely,

\[
P_{N,M}(t) = \frac{1}{N!} \oint_{C_0} d^N z \ d^M w \ e^{\Lambda_{N,M} t} \sum_{\pi \in S_N} \text{sign}(\pi) \prod_{j=1}^{N} z_{\pi_j}^{N-j} (z_{\pi_j} - 1)^{j-1} \prod_{k=1}^{M} w_k^{k-1} \prod_{j=1}^{N} (z_j - 1)^{N-j} \prod_{k=1}^{M} (w_k - 1)^{k} \prod_{j=1}^{N} \prod_{k=1}^{N} \left( q z_j + p w_k \right).
\]

The second equality then follows by applying the identity

\[
\sum_{\pi \in S_N} \text{sign}(\pi) \prod_{j=1}^{n} z_{\pi_j}^{n-j} (z_{\pi_j} - 1)^{j-1} = \prod_{j<k} (z_j - z_k).
\]

which can be proved by a variant of the Vandermonde determinant. \( \square \)

In the case where \( N > M - 2 \) we can evaluate the contour integrals over the variables \( w_k \)'s in (4.29). The eigenvalue \( \Lambda_{N,M} \) given by (4.7) introduces an essential singularity at the origin. It is therefore convenient to replace the contours as enclosing all other possible poles except the origin, and including \( \infty \). First let us consider the residue at \( \infty \). The degree of \( w_k \) in the integrand of (4.29) is \( (M - 1 + k - 1) - (k + N) = M - 2 - N \leq 0 \) so that the residue at \( \infty \) is zero. Next consider the simple poles at \( w_k = -pz_j/q \). Such residues will cancel the variable \( z_j \)

*For the case when \( N < M \), a similar argument follows.
in $\Lambda_{N,M}$, making the integrand analytic in $z_j$ and hence give rise to a zero residue at $z_j = 0$.

The only poles with non-zero contribution are therefore at $w_k = 1$. The residue of the simple pole at $w_1 = 1$ can be easily evaluated. The factor $\prod_{l=2}^{M} (w_l - w_1)$ in such residues will decrease the order of each pole at $w_k = 1$ by one. Hence the second order pole at $w_2 = 1$ becomes a simple pole and its residue can therefore be simply evaluated subsequently. The integration of $w_2$ again reduces the order of the pole at $w_3 = 1$ by one, and hence it also becomes a simple pole. Evaluating all poles at $w_k = 1$ sequentially, we arrive at the following result.

$$P_{N,M}(t) = \frac{1}{N!} \oint d^N z \ e^{\Lambda_{N,0} t} \frac{\prod_{1 \leq i < j \leq N} (z_i - z_j)^2}{\prod_{j=1}^{N} (z_j - 1)^N \prod_{j=1}^{N} (qz_j + p)^M}. \quad (4.31)$$

**Remark 4.7.** Note that the contours around $z_j = 0$ in (4.31) can be changed to the ones around $z_j = 1, -p/q, \infty$. One can check that contributions from $z_j = \infty$ are zero but that contributions from $z_j = -p/q$ are non-zero.

**Corollary 4.8.** When $N = M$ and $p = q = 1/2$, the positive and negative particles can be regarded as two identical single-species TASEPs, but hopping towards opposite directions. Hence we retrieve the same distribution as for the single-species TASEP under step initial condition, i.e.

$$P_{N,N}(t) = \frac{1}{N!} \int \prod_{j=1}^{N} \frac{dx_j}{2\pi i} \ e^{\mathcal{E} t} \frac{\prod_{1 \leq i < j \leq N} (x_i - x_j)^2}{\prod_{j=1}^{N} (x_j - 1)^N}, \quad (4.32)$$

where the contours are still around origin and $\mathcal{E} = \sum_{j=1}^{N} (x_j^{-1} - 1)$.

**Proof.** This is made explicit by the simple change of variable $z_j = x_j/(2 - x_j)$. \[\square\]

### 4.7.2 Bernoulli step data

Instead of the step initial data, we consider a more general initial condition, the “step Bernoulli initial condition”, where the position of positive particles is now characterised by Bernoulli random variables. Namely, $N$ particles of + type are distributed according to Bernoulli measure with density $\rho$ on the negative half of the integer line (i.e., $x \leq -1$), whilst the $M$ negative particles are located at the first $M$ sites of the non-negative half line ($x \geq 0$).

![Figure 4.3: step Bernoulli initial condition](image)
Definition 4.9. In general, for $\rho \in [0, 1]$, a Bernoulli random variable $Y$ takes the value 1 with probability $\rho$, and the value zero with probability $1 - \rho$. Suppose $\{Y_x\}_x$ is a collection of i.i.d. Bernoulli random variables. Denote $\eta_x(t)$ by the occupancy of site $x$ at time $t$. Assign a value 2 to positive particles, a value 1 to negative particles and zero to holes. Then the step Bernoulli initial condition (see Fig. 4.3) for the AHR model is letting

$$\eta_x(0) = \begin{cases} 
2Y_x, & \text{if } -\infty < x < 0, \\
1, & \text{if } 0 \leq x \leq M, \\
0, & \text{otherwise}, 
\end{cases}$$

subjecting to the condition that if there exists an integer $L > 0$ such that $\sum_{x<0} \eta_x(0) = 2M$, then $\eta_y(0) = 0$ for $\forall y < -L$. This restriction ensures that there are $0 < M < \infty$ negative particles.

Under such an initial condition, the probability that all positive and negative particles have crossed the origin by time $t$ is again factorised, and then can be evaluated analytically. This event is illustrated in Fig. 4.4. We are interested in the probability that such event happens before time $t$. The transition probability (4.5)

$$P(\vec{x}, \vec{y}; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \int_{C_0} \cdots \int_{C_0} d^{N} z \, d^{M} w \, e^{A_N M t} \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1}{q_{z_i} + p w_j} \sum_{\pi \in S_N} \text{sign}(\pi) \prod_{i=1}^{N} \left( \begin{array}{c} 1 - z_i \\ 1 - z_{\pi_i} 
\end{array} \right) \sum_{\sigma \in S_N} \text{sign}(\sigma) \prod_{j=1}^{M} \left( \begin{array}{c} 1 - w_j \\ 1 - w_{\sigma_j} \end{array} \right) \frac{1 - y_j}{1 - y_{\sigma_j}} y_{\sigma_j}^{(0)} y_j^{(0)} - 1. \quad (4.33)$$

where $y_j^{(0)} = i - 1$. To obtain the probability of total exchange, we need to sum over all the possible final states $\vec{x}, \vec{y}$ and initial states $\vec{x}^{(0)}$. Since at time $t$, the positive
particles lie on the non-negative half line while the negative particles on the negative half line, one can easily see that the range of final states are \(0 \leq x_1 < x_2 < \cdots < x_N\) and \(y_1 < y_2 \cdots < y_M \leq -1\). However, before summing over the initial position \(\vec{x}^{(0)}\), one needs the probability of positive particles locating at \(y\) and \(\vec{x}\) respectively.

Similarly, the sum over \(\vec{y}\) as,

\[
P(\vec{x}, \vec{y}^0; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) P(\vec{x}^{(0)}; 0),
\]

where \(P(\vec{x}, \vec{y}; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) P(\vec{x}^{(0)}; 0)\) is given by (4.33).

First, let’s consider the sum over \(\vec{x}\). In \(P(\vec{x}, \vec{y}; t | \vec{x}^{(0)}, \vec{y}^{(0)}; 0) P(\vec{x}^{(0)}; 0)\), the factor that depends on \(\vec{x}\) is \(z_{\pi_i}^{-x_i}\). Then by geometric series, one computes the sum as

\[
\sum_{0 \leq x_1 < x_2 < \cdots < x_N} N \prod_{i=1}^{N} \frac{z_{\pi_i}^{-x_i}}{1 - z_{\pi_i}} = \prod_{i=1}^{N} \frac{z_{\pi_i}^{-1}}{1 - z_{\pi_i}}.
\]

Similarly, the sum over \(\vec{y}\) and \(\vec{x}^{(0)}\) is calculated as

\[
\sum_{y_1 < y_2 \cdots < y_M \leq -1} M \prod_{i=1}^{M} \frac{w_{\sigma_i}^{-y_i}}{1 - w_{\sigma_i}} = \sum_{y_1 > y_2 \cdots > y_M \geq 1} M \prod_{i=1}^{M} \frac{w_{\sigma_i}^{y_i}}{1 - w_{\sigma_i}}.
\]

With the above calculations and previous identities, we are ready to give our second main result in the following theorem, an exact formula for \(P_{N,M,\rho}(t)\).

**Theorem 4.10.** Consider the AHR model on \(\mathbb{Z}\) with Bernoulli initial data given in Definition 4.9. The joint current distribution such that all positive and negative particles have crossed the origin at time \(t\) is given by

\[
P_{N,M,\rho}(t) = \rho^N \int_{C_0} \cdots \int_{C_0} d^{N} z \ d^{M} w \ e^{A_{x,w} t} \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1}{q z_i + p w_j}.
\]
§4.8 Asymptotic analysis

\[
\prod_{i<j}(z_i - z_j) \prod_{k<l}(w_k - w_l) \prod_{j=1}^N z_j^{N-j} \prod_{k=1}^M w_k^{k-1} \\
\prod_{i=1}^N (z_i - 1)^{N+1-j} (1 - (\rho\prime) z_i) \prod_{k=1}^M (w_k - 1)^k
\]

(4.35)

Proof. Applying the above summations in (4.34), Theorem 4.10 is proved after a symmetrisation of (4.34) using (4.22).

4.8 Asymptotic analysis

We are now in a position to perform an asymptotic analysis of the joint current distribution in Theorem 4.10. Using shorthand, \( \rho' = 1 - \rho \), and \( \oint z \) for the contour integral around the point \( z \), the integral formula of the joint current distribution is given by (4.35):

\[
P_{N,N,\rho}(t) = \oint_0 d^N z \ d^M w \ e^{\Lambda_{N,M} t} \times \\
\frac{\rho^N \prod_{1 \leq i<j \leq N} (z_i - z_j) \prod_{1 \leq k<l \leq M} (w_k - w_l) \prod_{j=1}^N z_j^{N-j} \prod_{k=1}^M w_k^{k-1} }{\prod_{j=1}^N (z_j - 1)^{N+1-j} (1 - \rho' z_j) \prod_{k=1}^M (w_k - 1)^k \prod_{j=1}^N \prod_{k=1}^M (q z_j + p w_k)}
\]

(4.36)

where second line follows from symmetrisation in the \( z \) variables as well as in the \( w \) variables, and we defined the following abbreviation to make the formulas more compact,

\[
\Delta_N(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j).
\]

From now on we set \( p = \frac{1}{2} \) and also define

\[
\Lambda_{N,M} = \frac{1}{2} \sum_{j=1}^N (z_j^{-1} - 1) + \frac{1}{2} \sum_{k=1}^M (w_k^{-1} - 1), \quad S_{N,M}(z,w) = \prod_{j=1}^N \prod_{k=1}^M \left( \frac{1}{2} (z_j + w_k) \right).
\]

Before performing an asymptotic analysis, we first rearrange (4.36), so that we can separate the integrals over \( z_j \)'s and those over \( w_k \)'s. The contours in (4.36) are all around the origin and we can choose the \( z \)-contours to enclose the \( w \) contours. By deforming the \( z \)-contours we can place them around all the other poles, including infinity, in the complex plane. One can see that the integrand has a degree of \( 2(N-1) - N - 1 - M = N - 3 - M \) in the \( z_j \)-variable. Assuming that \( N < M + 4 \),
then there is no pole at \( z_j = \infty \) and the only other poles are located at \( z_j = 1 \) and \( z_j = 1/\rho' \). Therefore, the target distribution becomes

\[
P_{N,M,\rho}(t) = \frac{\rho^N}{N!M!} \int_0^1 d^M w \int_1 d^N z \times \frac{e^{\lambda_{N,M,t}} \Delta_N(z)^2 \Delta_M(w)^2}{\prod_{j=1}^N (1 - \rho' z_j) \prod_{j=1}^N (z_j - 1)^N \prod_{k=1}^M (w_k - 1)^MS_{N,M}(z,w)}.
\]

The poles at \( z_j = 1/\rho' \) are simple poles and hence can be easily evaluated. Due to the symmetry of the integrand, the contribution of all poles at \( z_j = 1 \) is just \( N \) times that of the pole \( z_N = 1/\rho' \). It can be easily seen that after evaluating the pole at \( z_N = 1/\rho' \), the Vandermonde product produces a factor \( \prod_{j \neq N} (z_j - 1/\rho')^2 \), which cancels all other poles at \( z_j = 1/\rho' \). Therefore, we obtain

\[
P_{N,M,\rho}(t) = \frac{\rho^N}{N!M!} \int_0^1 d^M w \int_1 d^N z \times \frac{e^{\lambda_{N,M,t}} \Delta_N(z)^2 \Delta_M(w)^2}{\prod_{j=1}^N (1 - \rho' z_j) \prod_{j=1}^N (z_j - 1)^N \prod_{k=1}^M (w_k - 1)^MS_{N,M}(z,w)}
\]

\[+ \frac{e^{-\rho t/2}}{(\rho')^{N-1}(N-1)!M!} \int_0^1 d^M w \int_1 d^N_{-1} z \frac{e^{\lambda_{N-1,M,t}} \Delta_{N-1}(z)^2 \Delta_M(w)^2}{\prod_{j=1}^{N-1} (1 - \rho' z_j) \prod_{j=1}^N (z_j - 1)^N \prod_{k=1}^M (w_k - 1)^M} \times \frac{S_{N-1,M}(z,w)}{\prod_{k=1}^M (\frac{1}{2}(1/\rho' + w_k))}.
\]

In the following we determine the asymptotic behaviour of (4.37). To that end we first rewrite it in a more suitable form which is the starting formula for this section.

**Lemma 4.11.** The probability \( P_{N,M,\rho}(t) \) can be written as

\[
P_{N,M,\rho}(t) = \frac{\rho^N}{N!M!} \int_0^1 d^M w \int_1 d^N z \times \frac{e^{\lambda_{N,M,t}} \Delta_N(z)^2 \Delta_M(w)^2}{\prod_{j=1}^N (1 - \rho' z_j) \prod_{j=1}^N (z_j - 1)^N \prod_{k=1}^M (w_k - 1)^MS_{N,M}(z,w)}
\]

\[+ \frac{e^{-\rho t/2}}{(\rho')^{N-1}(N-1)!M!} \int_0^1 d^M w \int_1 d^N_{-1} z \frac{e^{\lambda_{N-1,M,t}} \Delta_{N-1}(z)^2 \Delta_M(w)^2}{\prod_{j=1}^{N-1} (1 - \rho' z_j) \prod_{j=1}^N (z_j - 1)^N \prod_{k=1}^M (w_k - 1)^M} \times \frac{S_{N-1,M}(z,w)}{\prod_{k=1}^M (\frac{1}{2}(1/\rho' + w_k))}.
\]

\[
(2(1 - \rho)/2 - \rho)^M \prod_{k=1}^M (w_k - 1)^M S_{N-1,M}(z,1)(1/\rho' + 1)^M = I_1 + I_2.
\]

where we denote the first term by \( I_1 \) and the second term by \( I_2 \).

**Remark 4.12.** For the case when \( M < N \), equation (4.36) can be evaluated by integrating over the \( w_j \) variables first. The residue of \( w_j = 0 \) is converted to \( w_j = \infty, 1, -qz_i/p \). There is no contribution from \( w_j = \infty \) since \( M < N \), while
residue at $z_i = 0$ vanishes after evaluating the residue at $w_j = -qz_i/p$. Hence we only left with residue at $w_j = 1$, which can be evaluated by using the first line of (4.36) and integrating from $w_1 = 1$ in order (see section 4.7.1).

Then the resulting probability is of the same form of $Z_1$ and hence is governed by the same limiting distribution (see (4.65)): the GUE Tracy-Widom distribution. In fact, this can be seen from the condition that there are more positive particles than the negative, and hence the positive particles distributed randomly initially will take much more time to cross the origin than the negative particles, which are located at the first $M$ sites at $t = 0$. In this case therefore, the late time AHR model is equivalent to the TASEP, whose limiting distribution is the GUE Tracy-Widom distribution.

We can perform an asymptotic analysis of the multiple integral formula from Lemma 4.11 using the rewrite in terms of a Fredholm determinant as in Section 4.6. A novel feature compared to the single species case is that one encounters dynamic poles arising from additional integration variables in the kernel of a Fredholm determinant. In our case we will see that, by taking certain poles at the beginning, one can evaluate the effects of the interaction and observe that in fact the two sets of variables decouple asymptotically as the time $t$ tends to infinity. As a consequence, we can study the long time limit of the joint distribution as follows, which is our third main result.

**Theorem 4.13.** Assume the Conjecture 4.25 is true. We will give a rigorous proof of Conjecture 4.25 in a forthcoming paper, and an idea of the proof of 4.25 will be given in Section 4.8.3. For the joint current distribution given in Theorem 4.10, we have

$$\lim_{t \to \infty} P_{N,M,p}(t) = F_2(s_-(N,M;t)) \cdot F_G(s_+(N,M;t)). \quad (4.39)$$

Here $F_2$ and $F_G$ are the distributions functions of the GUE Tracy-Widom and the standard Gaussian distributions respectively, and the variables $s_{\pm}$ are defed as

$$s_-(N,M;t) =: \eta_2 = \frac{1}{c_2 t^{1/3}} \left( (1 + \rho)N - (3 - \rho)M + \frac{1}{2}(1 - \rho)(1 - (1 - \rho)^2/4)t \right),$$

$$s_+(N,M;t) =: \eta_g = \frac{1}{c_g t^{1/2}} \left( - (2 - \rho)N + \rho M + \frac{1}{2}(2 - \rho)(1 - \rho)t \right), \quad (4.40)$$

where the constants $c_2$ and $c_g$ are given by

$$c_2 = (3/32)^{1/3}(1 - \rho)(3 - \rho)^{2/3}(1 + \rho)^{2/3},$$

$$c_g = 3/2(1 - \rho)^{3/2} \sqrt{\rho(2 - \rho)}. \quad (4.41)$$

The proof of this theorem is given in the rest of this chapter, but one may wonder what the meaning is of the variables $s_{\pm}$ and the reason for the appearance of the limiting distribution in (4.39). In fact, these can be understood from nonlinear
fluctuating hydrodynamics (NLFHD) which is a heuristic physics theory for studying the long time behaviour of one dimensional multi-component systems.

4.8.1 NLFHD and scaling limit

Macroscopic evolution of many physical particle systems is described by a hydrodynamic theory. To describe fluctuations and correlations it is customary to add noise to a linearized equation resulting in the theory of fluctuating hydrodynamics (FHD), see e.g. [108]. Despite its successes, FHD is often insufficient especially in low dimensions where anomalous transport is observed [7, 57, 111, 112] and one has to consider a nonlinear theory (NLFHD) [48, 125, 153].

NLFHD was first proposed in [77, 151] to provide concrete predictions for the long time behaviour of one dimensional Hamiltonian dynamics with nonlinear interaction. One first writes down the hydrodynamic equations for three conserved quantities of the original system, stretch, momentum and energy, and takes fluctuation effects into account by adding white noise. Diagonalising the advection term in the hydrodynamic part one switches to normal modes which have intrinsic propagating speeds.

It is natural to study fluctuations of the normal modes. A key idea of NLFHD is that if the speeds of the normal modes are different, then the interaction among them should be irrelevant in the long time limit and thus the fluctuations for each mode would be described by the single species noisy-Burgers equation (aka KPZ equation [6, 94, 136]). Fluctuations in the long time limit would then generically be given by the Tracy-Widom type distributions.

NLFHD has also been formulated for stochastic models and gives concrete predictions for the distribution of currents for multispecies models. A prototypical model is in fact the AHR model, which has two obvious conserved quantities, the number of $+$ and that of $-$ particles. The hydrodynamic equation is given by

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} + \frac{\partial j(\mathbf{u}(x,t))}{\partial x} = 0,$$

where $\mathbf{u}(x,t) = (\rho_+(x,t), \rho_-(x,t))$ is the density vector at position $x$ and time $t$, and $j(\mathbf{u}) = (j_+(\mathbf{u}), j_-(\mathbf{u}))$ denotes the macroscopic current of $\pm$ particles given by

$$j_+(\mathbf{u}) = \rho_+(1 - \rho_+ + \rho_-) + 2\rho_+\rho_-,$$

$$j_-(\mathbf{u}) = -(1 - \rho_+ - \rho_-)\rho_- - 2\rho_+\rho_-.$$  

(4.42)

(4.43)

(4.44)

The two normal modes for the AHR model follow from diagonalisation of the Jacobian matrix $\partial j/\partial \mathbf{u}$. For the step-Bernoulli initial condition, (4.44) now becomes a Riemann problem and hence one can obtain the density and current at the origin $\mathbf{u}_0 := \mathbf{u}(0, t), j_0 := j(\mathbf{u}_0)$.

Let $n_\pm(t)$ be the number of particles that have crossed the origin at time $t$, and let $\mathbf{N}(t) = (n_+(t), -n_-(t))$. For a system with infinitely many particles, NLFHD [122, 123] predicts that $\mathbf{N}(t) - j_0 t$ will have fluctuations that follow the GUE Tracy-
Widom and the standard Gaussian distributions in the directions of two normal modes. It follows that the probability that $N(t) = (n, -m)$ in the long time limit is given by

$$\lim_{t \to \infty} P\left(n_+(t) = n, n_-(t) = m\right) = F_2'(s_-) \cdot F_G'(s_+),$$

where $s_\pm$ are given in (4.40). By connecting the probability of the infinity particles system to the one with finite particles, one can obtain the prediction of $P_{N,M,\rho}(t)$, which is given by (4.39).

We thus focus on the following scaling limit for $N$ and $M$, obtained by solving the modes $s_\pm$ (4.40),

$$N = j_+(\rho)t - \frac{1}{6(1 - \rho)}(3 \rho_2 t^{1/3} + (3 - \rho)c_\eta \eta t^{1/2}),$$

$$M = j_-(\rho)t - \frac{1}{6(1 - \rho)}((2 - \rho)c_\eta t^{1/3} + (1 + \rho)c_\eta \eta t^{1/2}),$$

where the macroscopic currents $j_\pm$ are defined by

$$j_+ = \rho(3 - \rho)^2 \frac{1}{16}, \quad j_- = (1 + \rho)^2(2 - \rho) \frac{1}{16}.$$

In the rest of this chapter, we will show, theoretically, that in this scaling limit, the probability (4.38) tends to a product of the Gaussian distribution and the GUE Tracy-Widom distribution [157,158]: $F_G(s_+)F_2(s_-)$, in the long time limit, i.e. the proof of Theorem 4.39.

4.8.2 Asymptotic analysis of the first term

We separately analyse the two terms in (4.38) of Lemma 4.11, starting with the first one. We observe that the $z$-integral can be easily evaluated, and the result is stated below.

**Lemma 4.14.** The first term in (4.38) is equal to

$$\mathcal{I}_1 := \frac{\rho^N}{N!M!} \int_0^1 d^M w \int_1^1 d^N z \times \frac{e^{\Delta_{N,M}(w)^2} \Delta_N(z)^2 \Delta_M(w)^2}{\prod_{j=1}^N (1 - \rho^2 z_j) \prod_{j=1}^N (z_j - 1)^N \prod_{k=1}^M (w_k - 1)^M S_{N,M}(z,w)}.$$

$$= \frac{1}{M!} \int_0^1 dw \frac{e^{\Delta_{N,M}(w)^2} \Delta_M(w)^2}{\prod_{k=1}^M (w_k - 1)^M S_{N,M}(1,w)}. \quad (4.46)$$

**Proof.** The proof of Lemma 4.14 is easy to see by de-symmetrising the $z$ variables,
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\[
\frac{\rho^N}{M!} \int_0^M d^M w \int_1^N d^N z \frac{e^{\lambda_{N,M} t} \prod_{1 \leq i < j \leq N} (z_i - z_j) \prod_{j=1}^N z_j^{N-j} \Delta_M(w)^2}{\prod_{j=1}^N (z_j - 1)^{N-j+1} \prod_{k=1}^M (w_k - 1)^M (1 - \rho' z_j) S_{N,M}(z, w)}.
\]

On the right-hand side, the pole at \( z_N = 1 \) is first order and its residue can be evaluated in an easy way. In doing so, the Vandermonde product produces a factor \( z_N - 1 \), making the pole at \( z_N = 1 \) first order. This pole can then be subsequently evaluated in an easy way. Proceeding successively all poles at \( z_j = 1 \) an be easily evaluated from \( j = N \) down to \( j = 1 \), giving the final result stated in the lemma.

The asymptotic behaviour of the right-hand side of (4.46) can be obtained in a standard way where we first transform it to a Fredholm determinant according to the procedure outlined in Section 4.6, and then perform a saddle point analysis of the Fredholm kernel. To fit (4.46) into (4.23), we first changing variables \( w_k \to 1/w_k \) so that the contours lie around the poles at \( w_k = \infty \), and then deforming these contours to surround the poles other than the ones at \( w_k = \infty \), resulting in

\[
I_1 = \frac{1}{M!} \int_{0, 1, -1} d^M w \frac{e^{\hat{\lambda}_{N,M} t} \Delta_M(w)^2}{\prod_{k=1}^M (w_k - 1)^M S_{N,M}(1, w)}
\]

where \( \hat{\lambda}_{N,M} = \frac{1}{2} \sum_{j=1}^N (z_j - 1) + \frac{1}{2} \sum_{k=1}^M (w_k - 1) \). This expression can be written as

\[
I_1 := \frac{1}{M!} \int_{0, 1, -1} d^M w \prod_{k=1}^M w_k \prod_{k, \ell=1}^M (1 - a_\ell/w_k) \prod_{k=1}^M \frac{g_{N,M}(w_k, 0)}{g_{N,M}(a_k, 0)},
\]

fitting the standard form (4.23) with,

\[
\nu = M, \quad \mu = N
\]

\[
a_k = 1, \quad 1 \leq k \leq M, \quad a_j = 0, \quad 1 \leq j \leq M,
\]

\[
\beta_j = 1, \quad 1 \leq j \leq N,
\]

and

\[
g(w, x) = w^M g_{N,M}(w, x) = \left( \frac{w}{w+1} \right)^N w^x e^{x/t}.\]

Assuming first that \(|a_k| < 1\) and \(|\beta_j| > 1\), and then analytically continue to \( a_k = 1 \) and \( \beta_j = 1 \), we may apply the same procedure as in Section 4.6. The integral \( I_1 \) defined in (4.46) can thus be written as a Fredholm determinant with the kernel

\[
K(x, y) = \sum_{k=0}^{M-1} \phi_k(x) \psi_k(y), \quad (4.47)
\]
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with

\[
\phi_k(x) = \int_1 \frac{dw}{2\pi i} \frac{1}{w^2(w-1)} \left( \frac{1+w}{w} \right)^N \left( \frac{w}{1-w} \right)^k e^{-w^2/2},
\]

(4.48a)

\[
\psi_k(x) = \int_{0,-1} \frac{dw}{2\pi i} w^{x-2} \left( \frac{w}{1+w} \right)^N \left( \frac{1-w}{w} \right)^k e^{wt/2},
\]

(4.48b)

in which the pole at \( w = 1 \) is separated from the poles at \( w = -1, 0 \).

In the following, we will prove the scaling limit of the kernel (4.47), by computing the scaling limit of functions \( \phi_k(x) \) and \( \psi_k(x) \). Recall that the limit is taken under the scaling limit (4.45), and we scale \( k, x \) as \( k = M - \kappa_1 t^{1/3}, x = \xi_1 t^{1/3} \). The basic idea of the proof is given as follows [18–20]:

- First we prove a uniform convergence of \( \phi_k(x) \) for bounded \( \kappa_1, \xi_1 \) (Proposition 4.16).
- Then we evaluate the bounds of \( \phi_k(x) \) for large \( \kappa_1, \xi_1 \) (Proposition 4.19).
- These two results are repeated for \( \psi_k(x) \) (Proposition 4.18 and 4.20).
- The scaling limit of \( K(x,y) \) (4.47) is then obtained through these two limits in Theorem 4.21.

Before giving the uniform convergence of \( \phi_k(x) \) on bounded sets of \( \kappa_1, \xi_1 \), we first define a steep descent contour of the function \( \phi_k(x) \). Since \( t \to \infty \), we only consider the term in the integrand of (4.48a) that is of the order \( t \), i.e.,

\[
\phi_{M-\kappa_1 t^{1/3}}(\xi_1 t^{1/3}) = \int_1 \frac{dw}{2\pi i} \exp \left( g_1(w)t + O \left( t^{1/2} \right) \right)
\]

where \( g_1(w) = \frac{\rho(3-\rho)^2}{16} \log \left( \frac{1+w}{w} \right) + \frac{(1+\rho)^2(2-\rho)}{16} \log \left( \frac{w}{1-w} \right) - \frac{w}{2} \). Regarding to \( g_1(w) \), we define the following descent contour.

**Lemma 4.15. (Steepest descent contour of \( \phi \))** Suppose for \( 0 < \rho < 1 \)

\[
g_1(w) = \frac{\rho(3-\rho)^2}{16} \log \left( \frac{1+w}{w} \right) + \frac{(1+\rho)^2(2-\rho)}{16} \log \left( \frac{w}{1-w} \right) - \frac{w}{2} \]

then \( g_1'(w) = 0 \) has a double root at \( w_1 = (1-\rho)/2 = \rho'/2 \) and a single root \( w_2 = \rho - 1 = -\rho' \). The path \( \Gamma = \bigcup_{i=1}^3 \Gamma_i \) (see fig. 4.5) given by (4.49) is a steep descent path of \( g_1(w) \) passing through \( w_1 \). Namely, \( w = w_1 \) is the strict global maximum point of \( \text{Re}(g_1) \) along \( \Gamma \), i.e., \( \text{Re}(g_1(w)) < \text{Re}(g_1(w_1)) \) except when \( w = w_1 \). Moreover, \( \text{Re}(g_1) \) is monotone along \( \Gamma \) except two points where it reaches its maximum and minimum.

\[
\Gamma_1 = \left\{ w = \rho'/2 - se^{\pi i/3} : \ s \in [-1-\rho)/2, 0] \right\},
\]

(4.49a)

\[\text{The argument of a steepest path crossing a double root is given by} \ \theta \ \text{such that} \ \cos(3\theta + a) = -1, \ \text{where} \ a \ \text{is the argument of} \ g_1''(w_1)\]
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$$
\Gamma_2 = \left\{ w = \rho'/2 + se^{-\pi i /3} : \ s \in \left[ 0, (1 + \rho)/2 \right] \right\}, \quad (4.49b)
$$

$$
\Gamma_3 = \left\{ w = 1 + \frac{1 + \rho}{2} e^{i\theta} : \ \theta \in \left[ -2\pi/3, 2\pi/3 \right] \right\}. \quad (4.49c)
$$

**Figure 4.5:** Steep descent contour of integration in $\phi_k(x)$, around 1, passing the saddle point at $w_1 = \rho'/2$.

**Proof.** The double root and single root $w_{1,2}$ can be checked easily. We choose the steepest descent contour passing the double root $w_1 = \rho'/2$ instead of $w_2 = -\rho'$, since the one passing $w_2$ needs to cross the pole at origin and hence would change the value of $\phi_k(x)$ (4.48a). To prove $\Gamma$ gives the steep descent contour, we first show that $\text{Re}(g_1)$ is decreasing along $\Gamma_2 \cup \Gamma_3$. On $\Gamma_2$, we find

\[
\frac{d\text{Re}(g_1)(s)}{ds} = 2s^2 \left[ -3(3-\rho)(1-\rho)^2(1+\rho) - 2s(1-\rho)(3+10\rho-5\rho^2) - 2s^2(11-6\rho+3\rho^2) - 12s^3(1-\rho) - 8s^4 \right] / \left[ (3s^2+(1+\rho-s)^2)(3s^2+(1+s-\rho)^2)(3s^2+(3+s-\rho)^2) \right].
\]

One can verify that $d\text{Re}(g_1)(s)/ds < 0$ for $\forall s \in [0,2]$, $\forall \rho \in [0,1]$, implying that $\text{Re}(g_1)$ is decreasing along $\Gamma_2$. It follows by symmetry that, $\text{Re}(g_1)$ is increasing along $\Gamma_1$. In fact, $\text{Re}(\log(w)) = \log(|w|)$, so $g_1(w)$ is symmetric with respect to $y$-axis.

It remains to check the monotonicity of $\text{Re}(g_1)$ along $\Gamma_3$. Again taking the derivative along $\Gamma_3$, we have

\[
\frac{d\text{Re}(g_1)(\theta)}{d\theta} = \frac{2(1+\rho)^2 \sin(w) \cos(w/2)^2 [17 - 5\rho + 3\rho^3 + 8(1+\rho) \cos(w)]}{[5 + 2\rho + \rho^2 + 4(1+\rho) \cos(w)][17 + 2\rho + \rho^2 + 8(1+\rho) \cos(w)]},
\]

which equals zero only at $\theta = 0, \pi$, i.e. $\text{Re}(g_1)$ decreases when $s \in [-2\pi/3, 0]$, and increases when $s \in [0, 2\pi/3]$. In fact $17 - 5\rho + 3\rho^3 + 8(1+\rho) \cos(w) \leq 9 - 13\rho + 3\rho^3 + \rho^3$. The derivative of $9 - 13\rho + 3\rho^3 + \rho^3$ with respect to $\rho$ is $-13 + 6\rho + 3\rho^2 \leq -13 + 6 + 3 < 0$, indicating that $9 - 13\rho + 3\rho^3 + \rho^3 \geq 9 - 13 + 3 + 1 = 0$. 

Similarly, one can see that \( 5+2\rho+\rho^2+4(1+\rho)\cos(w) \) and \( 17+2\rho+\rho^2+8(1+\rho)\cos(w) \) are nonnegative for any \( \theta \) and \( 0 \leq \rho \leq 1 \).

Therefore, we conclude that \( \text{Re}(g_1) \) is strictly monotone along \( \Gamma \) except at its minimum point \( w = (3 + \rho)/2 \) and maximum point \( w_1 = \rho'/2 \). \( \square \)

With the steep descent contour given above, we arrive at the uniform convergence of \( \phi_k(x) \) for bounded \( \kappa_1, \xi_1 \). We first show the contribution from the contour away from the saddle point \( w_c = \rho'/2 \) vanishes as \( t \to \infty \), then the convergence is obtained by a Taylor expansion near \( w_c = \rho'/2 \) and a change a variable.

**Proposition 4.16. (Uniform convergence of \( \phi \) on a bounded set)** Let \( M, N \) be scaled as \( (4.45) \). Then for any fixed \( L > 0 \), the function \( \phi_k(x) \) defined in \( (4.48a) \) converges uniformly on \( \xi, \kappa \in [-L, L] \) to

\[
\lim_{t \to \infty} t^{1/3} c(t) \phi_{M-\lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) = c_0 \text{Ai}(\eta_2 + \xi + \kappa),
\]

where \( c_0 \in \mathbb{R}, \lambda_1 > 0, \lambda_2 > 0, \) and \( c(t) \) is some function of \( t \) that may or may not be bounded as \( t \to \infty \). Specifically,

\[
c_0 = \left( \frac{4(3-\rho)}{3(1+\rho)^2} \right)^{1/3}, \quad \lambda_1 = \frac{c_2}{3-\rho}, \quad \lambda_2 = 2c_2/[(3-\rho)(1+\rho)],
\]

and

\[
c(t) = \exp \left( -g_1(\rho'/2) t - g_2(\rho'/2) t^{1/2} - g_3(\rho'/2) t^{1/3} \right)
\]

where \( c_2, c_3 \) given in (4.41), and \( g_i(w) \) given by

\[
g_1(w) = \frac{\rho(3-\rho)^2}{16} \log \left( \frac{1+w}{w} \right) + \frac{(1+\rho)^2(2-\rho)}{16} \log \left( \frac{w}{1-w} \right) - \frac{w}{2}, \quad (4.52a)
\]

\[
g_2(w) = -\frac{(3-\rho)c_2\eta_2}{6(1-\rho)} \log \left( \frac{1+w}{w} \right) - \frac{(1+\rho)c_2\eta_2}{6(1-\rho)} \log \left( \frac{w}{1-w} \right), \quad (4.52b)
\]

\[
g_3(w) = -\frac{\rho c_2 \eta_2}{6(1-\rho)} \log \left( \frac{1+w}{w} \right) - \frac{(2-\rho)c_2\eta_2}{6(1-\rho)} + \kappa_1 \log \left( \frac{w}{1-w} \right) - \xi_1 \log(w). \quad (4.52c)
\]

Proof. To compute the long time limit of (4.48a), we rewrite its integrand such that the factor with large parameter \( t \) is in the exponential. So that method of steepest descent can be applied. Let \( k = M - \kappa_1 t^{1/3}, \ x = \xi_1 t^{1/3}, \)

\[
\phi_{\kappa}(x) = \oint_{\Gamma} \frac{dw}{2\pi i} e^{f(w,t)},
\]

where

\[
f(w,t) = N \log \left( \frac{1+w}{w} \right) + (M-\kappa_1 t^{1/3}) \log \left( \frac{w}{1-w} \right) - \xi_1 t^{1/3} \log(w) - \frac{w t}{2} \log(w-1).
\]
Scaling $N, M$ according to (4.45) and collecting together terms of same order of $t$, we rewrite $f(w, t)$ as $f(w, t) := g_1(w)t + g_2(w)t^{1/2} + g_3(w)t^{1/3} + g_4(w)$, where $g_i(w)$ are given in (4.52) for $i = 1, 2, 3$, and $g_4(w)$ is given by

$$g_4(w) = -\log(w - 1).$$

(4.53)

In the following, we will consider

$$\lim_{t \to \infty} t^{1/3}e^{-f(w, t)}\phi_{M - \kappa_1 t^{2/3}}(\xi_1 t^{1/3}).$$

From Lemma 4.15, we know $w_c = \rho'/2$ is a double root of $g_1(w) = 0$, and is the strict global maximum point along the steepest descent contour $\Gamma$ defined in Lemma 4.15. Denote a part of the contour $\Gamma$ by $\Gamma_{\delta}$, where $\delta > 0$ is small. Specifically, we can let $\delta \sim t^{-12}$ when $t \to \infty$. Deform the contour of $\phi_k(x)$ to $\Gamma$, and divide the integral into two parts:

$$t^{1/3}e^{-f(w, t)}\phi_{M - t^{1/3}\kappa_1}(t^{1/3}\xi_1) = t^{1/3}\left( \int_{\Gamma_{\delta}} + \int_{\Gamma_{\delta}} \right) \frac{dw}{2\pi i} e^{f(w, t) - f(w_c, t)}.$$

We will first give an estimate on the second integral. On $\Gamma \setminus \Gamma_{\delta}$, $\exists c_1 > 0$, such that $\text{Re}(g_1(w) - g_1(w_c)) < -c_1$. In our case, since $w_c$ is a double root, then $c_1 = c_1(\delta) \sim \delta^3 \sim t^{-1/4}$. This is proved in Lemma 4.15. Hence for $t$ large enough, the integrand is bounded by

$$\left| e^{f(w, t) - f(w_c, t)} \right| \leq e^{-c_1 t + O(t^{1/2})} \Rightarrow \left| \int_{\Gamma \setminus \Gamma_{\delta}} \frac{dw}{2\pi i} e^{f(w, t) - f(w_c, t)} \right| \leq c_3 e^{-c_1 t + O(t^{1/2})},$$

for some constant $c_3 > 0$. Note that this bound is only uniform for bounded $\kappa_1$ and $\xi_1$. Moreover, $e^{O(t^{1/2})}$ and $c_3 t^{1/3}$ are both bounded by $e^{c_1 t/4} \sim e^{t^{3/4}}$ for large enough $t$. Therefore, the integral over $\Gamma \setminus \Gamma_{\delta}$ is bounded by

$$\left| t^{1/3} \int_{\Gamma \setminus \Gamma_{\delta}} \frac{dw}{2\pi i} e^{f(w, t) - f(w_c, t)} \right| \leq e^{-c_1 t/2},$$

(4.54)

uniformly for bounded $\kappa_1, \xi_1$. Note that $e^{-c_1 t} \sim e^{-t^{3/4}}$ goes to zero as $t$ goes to $\infty$.

Next we consider the integral on $\Gamma_{\delta}$, where we can Taylor expand $f(w, t)$ around $w_c$ since $|w - w_c| \ll 1$. Recall that $\Gamma_{\delta}$ is parameterized by $w = w_c + v e^{\pm i\pi/3}$ where $0 \leq v \leq \delta$. Applying Taylor expansion, we obtain

$$g_1(v) - g_1(w_c) = -\frac{2v^3}{(3 - \rho)(1 + \rho)} + O(v^4),$$

(4.55a)

$$g_2(v) - g_2(w_c) = -\frac{8c_2h\kappa_1 e^{\pm 2i\pi/3}}{3(3 - \rho)(1 - \rho)^2(1 + \rho)} v^2 + O(v^3),$$

(4.55b)

$$g_3(v) - g_3(w_c) = -\frac{4c_2\rho^2 + 4(3 - \rho)\kappa_1 + 2(3 - \rho)(1 + \rho)\xi_1}{(3 - \rho)(1 - \rho)(1 + \rho)} e^{\pm i\pi/3} v + O(v^2),$$

(4.55c)
\[ g_3(v) - g_4(w_c) = O(v), \] (4.55d)

where the errors are uniform on bounded sets of \( \kappa_1, \xi_1 \). Denote the function \( g_i(w) \) without the error terms by \( \tilde{g}_i(w) \), and the corresponding \( f(w, t) \) by

\[ \bar{f}(w, t) := \tilde{g}_1(w)t + \tilde{g}_2(w)t^{1/2} + \tilde{g}_3(w)t^{1/3} + \tilde{g}_4(w). \]

Instead of estimating the error arising in \( f(w, t) - \bar{f}(w, t) \), we consider the difference between \( f(w, t) \) and \( \tilde{f}(w, t) := \tilde{f}(w, t) - \tilde{g}_2(w)t^{1/2} + g_2(w_c)t^{1/2} \). Because the contribution of \( e^{g_2(w)t^{1/2}} \) vanishes as \( t \) gets large enough, which will be shown below.

We separate the integral on \( \Gamma_\delta \) into two parts

\[
t^{1/3} \int_{\Gamma_\delta} \frac{dw}{2\pi i} e^{f(w, t) - \bar{f}(w, t)} = t^{1/3} e^{-\bar{f}(w, t)} \left[ \int_{\Gamma_\delta} \frac{dw}{2\pi i} e^{f(w, t)} + \int_{\Gamma_\delta} \frac{dw}{2\pi i} e^{f(w, t)} \left( e^{f(w, t) - \bar{f}(w, t)} - 1 \right) \right].
\]

Let us now estimate the second integral. By the inequality \( |e^x - 1| \leq e^{x^2}|x| \), the second integral is bounded by

\[
t^{1/3} \int_0^\delta \frac{dv}{2\pi i} \left| e^{f(w, v) - \bar{f}(w, v)} \right| e^{a_2v^2t^{1/2}} e^{O(v^4t, v^3t^{1/2}, v^2t^{1/3}, v)} \times \left[ a_2v^2t^{1/2} + O(v^4t, v^3t^{1/2}, v^2t^{1/3}, v) \right] = t^{1/3} \int_0^\delta \frac{dv}{2\pi i} \left| e^{-a_1v^3t - a_3vt^{1/3}} \right| \left| e^{O(v^4t, v^3t^{1/2}, v^2t^{1/3}, v)} \right| O(v^4t, v^2t^{1/2}, v^3t^{1/2}, v),
\]

where in the second line \( O(v^2t^{1/2}) \) is dominated by \( O(v^2t^{1/2}) \), and \( a_i \) are some positive constants given by the Taylor expansion in (4.55). Specifically, \( a_1 = 2/(3 - \rho)(1 + \rho) \) and similarly for \( a_2, a_3 \). By the change of variable \( u = vt^{1/3} \), the bound becomes

\[
\int_0^{\delta t^{1/3}} \frac{du}{2\pi i} \left| e^{-a_1u^3 - a_3u} \right| \left| e^{O(u^4t^{-1/3}, u^2t^{-1/6}, u^3t^{-1/2}, ut^{-1/3})} \right| O(u^4, u^2, u^3, u) \times O(u^4t^{-1/3}, u^2t^{-1/6}, u^3t^{-1/2}, ut^{-1/3}) \leq t^{-1/6} \int_0^{\delta t^{1/3}} \frac{du}{2\pi i} \left| e^{-a_1u^3 - a_3u} \right| \left| e^{O(u^4t^{-1/3}, u^2t^{-1/6}, u^3t^{-1/2}, ut^{-1/3})} \right| O(u^4, u^2, u^3, u),
\]

where \( t^{1/3} dv = du \) and \( O(v^4t, v^2t^{1/2}, v^3t^{1/2}, v) = O(u^4t^{-1/3}, u^2t^{-1/6}, u^3t^{-1/2}, ut^{-1/3}) \). For convenience, in the second line we adopt the term \( O(u^4t^{-1/6}) \) instead of \( O(v^4t) \). Since \( O(u^4t^{-1/3}) \) is dominated by \( O(u^4t^{-1/6}) \) for large enough \( t \), likewise \( O(u^4t^{-1/6}) \) dominates \( O(u^{-1/3}) \), and \( O(u^3) \) dominates \( O(u^3t^{-1/2}) \) when \( t \) is large enough. Namely, \( O(u^4t^{-1/3}, u^2t^{-1/6}, u^3t^{-1/2}, ut^{-1/3}) \ll t^{-1/6} O(u^4, u^2, u^3) \).

On the other hand, we observe that the integral is on \([0, \delta t^{1/3}]\), i.e., \( 0 \leq u \leq \delta t^{1/3} \), where \( \delta < 1 \). It follows that \( O(u^4t^{-1/3}) < u^3/2 \) for small enough \( \delta \), i.e., large
where \( c \) is a term of order \( O(\infty) \). We re-parameterize \( \Gamma_{\delta} \) by

\[
\gamma = \frac{v}{(3/2) \delta^{1/3}} \left( \frac{1 + \rho}{1 + \rho} \right)^{1/3},
\]

where \( 0 \leq v \leq \delta \). The remaining integral then becomes

\[
c_5 \int_{\delta t^{1/3} v \pi/3 / c_5}^{\delta t^{1/3} v \pi/3 / c_5} \frac{dz}{2 \pi i} \exp \left( \frac{1}{3} z^3 - (\eta_2 + \kappa + \xi) z \right)
\]

where \( c_5 = \left( \frac{(1 + \rho)(3 - \rho)}{6} \right)^{1/3} \), \( dw = t^{-1/3} c_5 \, dz \), and \( c_2 \) is given by (4.41), and

\[
\kappa = \kappa_1 (3 - \rho)/c_2 := \kappa_1/\lambda_1, \quad \xi = \xi_1 (3 - \rho)(1 + \rho)/(2c_2) := \xi_1/\lambda_2.
\]

We are allowed to extend the paths to \( t = \infty \), by which we only pick an error term of order \( O(e^{-c' \delta^{3/2}}) \) for some \( c' > 0 \). The boundary of the integral is then \( \infty e^{\pm i \pi/3} \), since \( \delta t^{1/3} \sim t^{1/4} \to \infty \). From the extension and the integral form of an Airy function

\[
\text{Ai}(y) = \frac{1}{2 \pi i} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp \left( \frac{1}{3} x^3 - yx \right) \, dx,
\]
we arrive at the final result
\[
\lim_{t \to \infty} \frac{t^{1/3}c(t)\phi_{M-\lambda_1t^{1/3}}(\lambda_2t^{1/3}\xi)}{r_1} = \frac{2c_5}{\rho + 1} \text{Ai}(\eta_2 + \kappa + \xi),
\]
which agrees with (4.50). The corresponding error terms are given by (4.54) and (4.56).

Similar to the above, we now define the steepest descent contour of \(\phi_k(x)\).

**Lemma 4.17. (Steepest descent contour of \(\psi\))** Suppose for \(0 < \rho < 1\)
\[
g_1(w) = \frac{\rho(3-\rho)^2}{16} \log \left( \frac{1+w}{w} \right) + \frac{(1+\rho)^2(2-\rho)}{16} \log \left( \frac{w}{1-w} \right) - \frac{w}{2},
\]
then \(g_1(w) = 0\) has a double root at \(w_1 = (1-\rho)/2 = \rho'/2\) and a single root \(w_2 = \rho - 1 = -\rho'\). The path \(\Sigma = \bigcup_{i=1}^{4} \Sigma_i\) (see Fig. 4.6) given by (4.57) is a steep descent path of \(-g_1(w)\) passing through \(w_1\).

![Figure 4.6: Steep descent contour of integration in \(\psi_k(x)\) passing the saddle point at \(w_1 = \rho'/2\).](image)

Namely, \(w = w_1\) is the strict global maximum point of \(-\text{Re}(g_1)\) along \(\Sigma\), i.e., \(-\text{Re}(g_1(w)) < -\text{Re}(g_1(w_1))\) except when \(w = w_1\). Moreover, \(-\text{Re}(g_1)\) is monotone along \(\bigcup_{i=1}^{3} \Sigma_i\) and \(\Sigma_4\) except four points where it reaches its local maximum and minimum.

\[
\Sigma_1 = \left\{ w = \rho'/2 + se^{2\pi i/3} : s \in [0, \rho'/4] \right\}, \quad (4.57a)
\]
\[
\Sigma_2 = \left\{ w = \frac{\sqrt{3}}{4} \rho' e^{i\theta} : \theta \in [\pi/6, 11\pi/6] \right\}, \quad (4.57b)
\]
\[
\Sigma_3 = \left\{ w = \rho'/2 - se^{5\pi i/3} : s \in [-\rho'/4, 0] \right\}, \quad (4.57c)
\]
\[
\Sigma_4 = \left\{ w = -1 + \left(1 - \frac{1}{2}\rho'\right) e^{i\theta} : \theta \in [0, 2\pi] \right\}. \quad (4.57d)
\]
Choosing appropriate deformed contour according to coefficient \( - \) point. Combining these two bounds, one can arrive with the bound \( e^{\lambda w} \). Let \( \psi_{k}(x) \) be the parts in \( f(w,t) \) defined in (4.52). Then for large enough \( t \) and \( L \) large enough but independent of \( t \),

\[
\lim_{t \to \infty} t^{1/3} e^{-f(w,t)} \phi_{M-\lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) = c_0 \mathrm{Ai}(\eta_2 + \xi + \kappa), \tag{4.58}
\]

where \( c(t) \) is some function of \( t \) that may or may not be bounded as \( t \to \infty \). Specifically,

\[
c_0 = \frac{4}{(1-\rho)^2} \left( \frac{(1+\rho)(3-\rho)}{6} \right)^{1/3},
\]

and \( \lambda_1, \lambda_2, c(t) \) are given in (4.51).

In the following, we will give an estimate of function \( \phi_M - \lambda_1 t^{1/3} \kappa \) and \( \phi_{M-\lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) \) for unbounded \( \kappa, \xi \).

\textbf{Proposition 4.19. (Estimate of function} \( \phi \)) Consider the function \( \phi_M \) defined in (4.48a), with \( N, M \) scaled as (4.45), and \( f(w,t) = g_1(w)t + g_2(w)t^{1/2} + g_3(w)t^{1/3} + g_4(w) \) defined in (4.52). Then for large enough \( t \) and \( L \) large enough but independent of \( t \),

\[
\left| t^{1/3} e^{-f(w,t)} \phi_{M-\lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) \right| \leq e^{-(\kappa+\xi)},
\]

for \( \kappa, \xi \geq -L \) and \( \kappa + \xi \geq L \).

\textbf{Proof.} Recall that

\[
\phi_{M-\lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) = \oint_{1} \frac{dw}{2\pi i} e^{f(w,t)},
\]

where \( f(w,t) = g_1(w)t + g_2(w)t^{1/2} + g_3(w)t^{1/3} + g_4(w) \) is defined in (4.52). Let \( \tilde{g}_3(w) \) be the parts in \( g_3(w) \) that are independent of \( \kappa \) and \( \xi \):

\[
\tilde{g}_3(w) := g_3(w) + \lambda_1 (\kappa + L) \log[w/(1-w)] + \lambda_2 (\xi + L) \log(w),
\]

and the corresponding \( \tilde{f}(w,t) := g_1(w)t + g_2(w)t^{1/2} + \tilde{g}_3(w)t^{1/3} + g_4(w) \). Here we factoring out the term \( \kappa + L \) (or \( \xi + L \)) instead of \( \kappa \) (or \( \xi \)) is to make sure the coefficient \( (\kappa + L \) or \( \xi + L \)) of the unbounded term is always positive.

Instead of the steep descent path \( \Gamma \) given in (4.49), we consider a deformed contour crossing a point close to the saddle point \( w_c = \rho'/2 \). Along the contour, we first estimate the term \( \exp(\tilde{f}(w,t) - \tilde{f}(w_c,t)) \) via Taylor expansion, then bound the term \( \lambda_1 \kappa \log(w_c(1-w)/|w(1-w_c)|) + \lambda_2 \xi \log(w_c/w) \) by the value at its maximal point. Combining these two bounds, one can arrive with the bound \( e^{-\xi - \kappa} \) by choosing appropriate deformed contour according to \( \xi \) and \( \kappa \).
(i) Estimate of terms independent of $\xi, \kappa$ Deform the contour $\Gamma$ to be $\Gamma'$ by a vertical part (the blue part in Fig. 4.7) $\Gamma_{\text{vert}} = \{w = w_c + w_c\delta(1 + si) : s \in [-\sqrt{3}, \sqrt{3}]\}$, instead of crossing $w_c$. We let $0 < \delta \ll 1$ (see Fig. 4.7).

- First we consider the contribution from the unmodified path $\Gamma'/\Gamma_{\text{vert}}$. We choose $\delta \ll t^{-1/12}$ for large enough $t$. Then we separate $\Gamma'/\Gamma_{\text{vert}}$ into two parts $\Gamma'/\Gamma_{\text{vert}} = \Gamma_1' \cup \Gamma_2'$, where $\Gamma_1' = \{|w - w_c| \geq t^{-1/12}\}$ and $\Gamma_1' = \{2w_c\delta < |w - w_c| < t^{-1/12}\}$. We choose $t$ large enough such that $\Gamma_2'$ is contained inside the red part in Fig. 4.7, i.e., $\Gamma_2' = \{w = w_c + ve^{\pm i/3} : v \in [2w_c\delta, t^{-1/12}]\}$.

When $w \in \Gamma_1'$, we can see from Proposition 4.16 that $t^{1/3}e^{f'(w,t) - f'(w_c,t)}$ is bounded by $e^{-at/2}$ for some positive $a \sim t^{-1/4}$. When $w \in \Gamma_2'$, we apply the Taylor expansion around $w = w_c$.

We know that $w_c = \rho'/2$ is a double root of $g_1(w)$, i.e., $g_1'(w_c) = g_1''(w_c) = 0$. The Taylor expansion of function $g_1(w)$ is then given by

$$g_1(w) - g_1(w_c) = 2a_1(w - w_c)^3 + (w - w_c)^4h_1(w),$$

where $a_1 = \frac{1}{(3\rho)(1+\rho)} > 0$. Suppose $|h_1(w)|$ is bounded by $H_1 > 0$ along $\Gamma'$. Since $\Gamma'$ is finite and $g_1(w)$ is analytic along $\Gamma'$, then $H_1$ is a finite constant independent of $t$. Choose $t$ large enough such that $|w - w_c| < t^{-1/12} < a_1/H_1$. We thus can bound the error term by $|(w - w_c)^4h_1(w)| < a_1|(w - w_c)^3|$. Substituting $w = w_c + ev^{\pm i/3}$ gives us

$$|e^{g_1(w) - g_1(w_c)}t| < e^{-a_1v^2t}.$$

Similarly, for large enough $t$, we have

$$|e^{g_2(w) - g_2(w_c)}t^{1/2}| < e^{a_2v^2t^{1/2}},$$

$$|e^{g_3(w) - g_3(w_c)}t^{1/3}| < e^{a_3vt^{1/3}},$$

$$|e^{g_4(w) - g_4(w_c)}| < e^{a_4v},$$

where $a_i$ are positive constants.

If $v > 2\sqrt{3}/a_1t^{-1/3}$, then $v > t^{-1/2}\max\{4a_2/a_1, 2\sqrt{a_4/a_1}\}$. This can be achieved by assuming $\delta > ct^{-1/3}$ where $c = \sqrt{a_3/a_1/w_c}$. It follows that

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4_7.png}
\caption{An illustration of $\Gamma'$ with $\Gamma_{\text{vert}}$ (the blue part).}
\end{figure}
max\{a_2 v^2 t^{1/2}, a_3 v t^{1/3}, a_4 v\} < a_1 v^3 t/4, i.e.,
\[ \exp(\tilde{f}(w, t) - \tilde{f}(w_c, t)) < \exp(-a_1 v^3 t/4). \]

By change of variable \( vt^{1/3} = u \), we obtain
\[ t^{1/3} \int_0^{t^{1/3}} e^{\tilde{f}(w, t) - \tilde{f}(w_c, t)} dw < 2 t^{1/3} \int_0^{t^{1/3}} e^{-a_1 v^3 t/4} dv < 2 \int_0^{t^{1/3}} e^{-a_1 u^3/4} du. \]

Clearly the last integral can be bounded for some constant \( C \).

Combine the above results we have
\[ t^{1/3} \int_{\Gamma'/\Gamma_{vert}} e^{\tilde{f}(w, t) - \tilde{f}(w_c, t)} dw = e^{-at|\Gamma'_1|} + 2 \int_0^{t^{1/3}} e^{-a_1 u^3/4} du < C. \] (4.59)

where \( |\Gamma'_1| \) is the length of \( \Gamma'_1 \), and for large enough \( t \), \( e^{-at|\Gamma'_1|} \) is bounded by 1, which is absorbed into the constant \( C \).

- Then we need to estimate the contribution from \( \Gamma_{vert} \). Again we apply the Taylor expansion around \( w_c \). Similarly, for large enough \( t \) we have
\[ \exp\left((g_2(w) - g_2(w_c))t^{1/2} + (\bar{g}_3(w) - \bar{g}_3(w_c))t^{1/3} + g_4(w) - g_4(w_c)\right) < \exp(a_1 |w - w_c|^3 t) \leq 8a_1 w_c^3 \delta^3 t, \]
where \( |w - w_c| \leq 2 w_c \delta \) when \( w \in \Gamma_{vert} \). As before, we require \( \delta > c't^{-1/3} \) for some positive constant \( c' \).

Then for \( g_1(w) \) in this case, we have
\[ |g_1(w) - g_1(w_c)| < 3a_1 |w - w_c|^3 t \leq 24a_1 w_c^3 \delta^3 t. \]
The bound is different from the one in \( \Gamma'/\Gamma_{vert} \) since in \( \Gamma_{vert} \) \( w \) can not be parameterised by \( w = w_c + u \delta^{\pm 1/3} \).

In conclusion when \( w \in \Gamma_{vert} \)
\[ |\tilde{f}(w, t) - \tilde{f}(w_c, t)| < 32a_1 w_c^3 \delta^3 t, \]
i.e.,
\[ t^{1/3} \int_{\Gamma_{vert}} e^{\tilde{f}(w, t) - \tilde{f}(w_c, t)} dw < t^{1/3} e^{32a_1 w_c^3 \delta^3 t}|\Gamma_{vert}| = 2\sqrt{3}w_c \delta^{1/3} e^{32a_1 w_c^3 \delta^3 t} \] (4.60)

- Recall that the condition \( \delta \) for the above bounds is \( ct^{-1/3} < \delta \ll t^{-1/12} \) where \( c' \) is absorbed into \( c \).
(ii) Estimate of terms depending on $\xi, \kappa$  Now we want to bound the term depending on $\xi, \kappa$:

$$e^{-\lambda_1(\kappa+L)t^{1/3}\log\left(w(1-w_c)/[w_c(1-w)]\right)} - \lambda_2(\xi+L)t^{1/3}\left(\log(w/w_c)\right).$$

- We first consider the term depends on $\kappa$:

$$\exp\left[-\lambda_1(\kappa + L)t^{1/3}\log\left(w(1-w_c)/[w_c(1-w)]\right)\right].$$

Since $\kappa + L \geq 0$, the maximum of $|1-w|$ along $\Gamma'$ is obtained at $w = w_c$. Namely, $|1-w|/|1-w_c| \leq 1$. Then since $\lambda_1 > 0, \kappa + L \geq 0$, we have

$$|e^{\lambda_1(\kappa+L)t^{1/3}[\log(1-w)-\log(1-w_c)]}| \leq 1$$

We are now left with $|e^{-\lambda_1(\kappa+L)\log(w/w_c)t^{1/3}}|$. Along $\Gamma'$, $|w| \geq |w_c(1+\delta)|$, indicating that $\log |w/w_c| \geq \log [1+\delta]$. When $\delta < 2$, one can verify that $\log [1+\delta] > \delta/2$. It follows that $-\log |w/w_c| < -\delta/2$. Consequently,

$$\left|e^{-\lambda_1(\kappa+L)t^{1/3}\log\left(w(1-w_c)/[w_c(1-w)]\right)}\right| \leq e^{-t^{1/3} \lambda(\kappa+L)\delta} < e^{-t^{1/3} \lambda_1\delta}, \quad (4.61)$$

where $\lambda = \frac{1}{2} \min\{\lambda_1, \lambda_2\}$ is a positive constant.

- Then we consider the term depends on $\xi$:

$$\exp\left[-\lambda_2(\xi + L)t^{1/3}\left(\log(w/w_c)\right)\right].$$

Since $\xi + L > 0$ and $\lambda_2 > 0$, from above we can see that $-\log |w/w_c|$ is bounded by $-\delta/2$. Hence,

$$\left|e^{-\lambda_2(\xi+L)t^{1/3}\left(\log(w/w_c)\right)}\right| \leq e^{-t^{1/3} \lambda(\xi+L)\delta} < e^{-t^{1/3} \lambda_2\xi\delta}, \quad (4.62)$$

where $\lambda = \frac{1}{2} \min\{\lambda_1, \lambda_2\}$.

- Combining (4.61) and (4.62), we have

$$\left|e^{-\lambda_1(\kappa+L)t^{1/3}\log\left(w(1-w_c)/[w_c(1-w)]\right)} - \lambda_2(\xi+L)t^{1/3}\left(\log(w/w_c)\right)\right| \leq e^{-t^{1/3} \lambda(\kappa+\xi)\delta}. \quad (4.63)$$

(iii) Total estimate  It follows from (4.60), (4.59) and (4.63) that

$$\left|\frac{t^{1/3}e^{-f(w_c,t)}\phi_M-\lambda_1t^{1/3}\lambda_2t^{1/3}\xi}{t^{1/3}e^{-f(w_c,t)}\phi_M-\lambda_1t^{1/3}\lambda_2t^{1/3}\xi}\right| \leq e^{-t^{1/3} \lambda(\kappa+\xi)\delta} \times$$

\[= e^{-t^{1/3} \lambda(\kappa+\xi)\delta} \left(\int_{\Gamma'/\Gamma_{\text{vert}}} + \int_{\Gamma_{\text{vert}}}\right) t^{1/3}e^{\tilde{f}(w,t)-f(w_c,t)}dw \]

\[\leq e^{-t^{1/3} \lambda(\kappa+\xi)\delta} \left( C + 2\sqrt{3\omega_c} \delta t^{1/3}e^{3\omega_c^2w_c^2}\right)\]
where $C$ and $2\sqrt{3}w_c e^{2\alpha_1 w_c^3 t}$ are bounded by $e^{a't^3}$ for some appropriate constant $a' > 0$. This can be achieved since we require $ct^{-1/3} < \delta \ll t^{-1/12}$. Note that the constants $c, a'$ are not depending on $t, L$.

- When $\kappa + \xi \leq 2a't^{1/3}/\lambda$, let $\delta = \sqrt{(\kappa + \xi)\lambda/(2a')t^{-1/3}}$. The restriction of $\delta$: $ct^{-1/3} < \delta \ll t^{-1/12}$, is automatically satisfied if $L > 2a'c^2/\lambda$, since $\kappa + \xi > L$. In this case, when $L$ is large enough,

$$1 + \delta t^{1/3} = 1 + \sqrt{(\kappa + \xi)\lambda/(2a')} < e^{12\sqrt{3}c^2/(\kappa + \xi)}.$$ 

As a result,

$$\left| t^{1/3} e^{-f(w_c t)} \phi_{M-\lambda t^{1/3}}(\lambda_2 t^{1/3} \xi) \right| \leq \frac{1}{2\sqrt{3}c} \left( \frac{\lambda}{\xi} \right)^{3/2} e^{12\sqrt{3}c^2/(\kappa + \xi)} e^{-\frac{1}{2\sqrt{3}c} \left( \frac{\lambda}{\xi} \right)^{3/2}} = e^{-\frac{1}{2\sqrt{3}c} \left( \frac{\lambda}{\xi} \right)^{3/2}}.$$

Note that $L$ is still independent of $t$.

- When $\kappa + \xi > 2a't^{1/3}/\lambda$, choose $\delta = t^{-1/6}$. The restriction of $\delta$: $ct^{-1/3} < \delta \ll t^{-1/12}$, is automatically satisfied. In this case, for large enough $t$

$$(1 + \delta t^{1/3}) e^{(1-\lambda t^{1/6}/2)(\kappa + \xi)} \leq (1 + t^{1/6}) e^{(\lambda t^{1/6}/4 - \lambda t^{1/6}/2)(\kappa + \xi)} = (1 + t^{1/6}) e^{-\lambda t^{1/6}/4(\kappa + \xi)}.$$ 

Thus it follows that

$$\left| t^{1/3} e^{-f(w_c t)} \phi_{M-\lambda t^{1/3}}(\lambda_2 t^{1/3} \xi) \right| \leq e^{\lambda t^{1/6} \delta(\kappa + \xi) / 2} e^{-\lambda t^{1/6}(\kappa + \xi)} \left( 1 + \delta t^{1/3} \right) = e^{-\lambda t^{1/6}(\kappa + \xi) / 2} \left( 1 + \delta t^{1/3} \right) = e^{-\kappa(\kappa + \xi)(1 + \delta t^{1/3})} e^{-\lambda t^{1/6}/4(\kappa + \xi)} \leq e^{-(\kappa + \xi)}.$$

Therefore we can conclude that when $\kappa + \xi \geq -L$ and $\kappa + \xi \geq L$,

$$\left| t^{1/3} e^{-f(w_c t)} \phi_{M-\lambda t^{1/3}}(\lambda_2 t^{1/3} \xi) \right| \leq e^{-(\kappa + \xi)}$$

for $t$ large enough and $L$ large enough but independent of $t$.  

Likewise, we also have the following bound for $\psi_k(x)$.

Proposition 4.20. (Estimate of function $\psi$) Consider the function $\psi_k(x)$ defined in (4.48b), with $N, M$ scaled as (4.45), and define $f(w, t) = g_1(w)t + g_2(w)t^{1/2} + g_3(w)t^{1/3} + g_5(w)$ with $g_1(w)$ given in (4.52) and $g_5(w) = -2 \log(w)$.
Then for large enough $t$ and $L$ large enough but independent of $t$,

$$\left| t^{1/3} e^{f(w,t)} \psi_M - \lambda_1 t^{1/3} \bar{K}(\lambda_2 t^{1/3} \xi) \right| \leq e^{-(\kappa + \xi)},$$

for $\kappa, \xi \geq -L$ and $\kappa + \xi > L$.

**Proof.** The proposition follows by the same steps in the proof of Proposition 4.19, except the $\Sigma_4$ contour (4.57d) around $w = -1$. As before let $\bar{f}(w, t)$ be the parts in $f(w, t)$ that are independent of $\kappa$ and $\xi$. But along $\Sigma_4$, the factor $e^{f(w,t) - \bar{f}(w,t)}$ can be bounded by $e^{-ut}$ for some $u > 0$. We observe that the singularity of integrand at $w = 1$ only lies in the factor $e^{f(w,t) - \bar{f}(w,t)}$. Hence the integral along $\Sigma_4$ is bounded by zero as $t \to \infty$. \(\Box\)

We now are ready to conclude the limiting kernel is the *Airy kernel* [157]:

$$\int_0^\infty A\iota(\kappa + \xi) A\iota(\kappa + \zeta) d\kappa := A(\xi, \zeta) \quad (4.64)$$

where $A\iota(x)$ is the Airy function [23,68,124].

**Theorem 4.21.** Consider the kernel defined in (4.47), and define the rescaled kernel

$$K_t(\xi, \zeta) = (\rho'/2)^{\lambda_2(\xi-\zeta)} \lambda_2 t^{1/3} K(\lambda_2 t^{1/3} \xi, \lambda_2 t^{1/3} \zeta),$$

with $\lambda_2$ given in (4.51c). Then we have

(i) $\det[1 - K_t(\xi, \zeta)] = \det[1 - K(x, y)]$, where $(x, y) = (\lambda_2 t^{1/3} \xi, \lambda_2 t^{1/3} \zeta)$.

(ii) For any fixed $L > 0$

$$\lim_{t \to \infty} K_t(\xi, \zeta) = \int_0^\infty \Ai(\eta_2 + \kappa + \xi) \Ai(\eta_2 + \kappa + \zeta) d\kappa,$$

uniformly on $(\xi, \zeta) \in [-L, L]^2$.

(iii) For any fixed $L > 0$ and $t$ large enough,

$$|K_t(\xi, \zeta)| \leq c e^{-\max\{0, \xi\} - \max\{0, \zeta\}}$$

for some constant $c > 0$ and $\xi, \zeta \geq -L$.

**Proof.** Obviously, $K_t(\xi, \zeta) = \lambda_2 t^{1/3} K(\lambda_2 t^{1/3} \xi, \lambda_2 t^{1/3} \zeta)$ gives the same Fredholm determinant as the kernel $K(x, y)$ (4.47) by the change of variable $(x, y) \to (\gamma t^{1/3} \xi, \gamma t^{1/3} \zeta)$. Then if we conjugate the kernel with $(\rho'/2)^{t^{1/3} \lambda_2(\xi-\zeta)}$, we have

$$\det(K_t(\xi, \zeta)) = \det((\rho'/2)^{t^{1/3} \lambda_2(\xi-\zeta)} K_t(\xi, \zeta)),$$

from which, (i) is proved. Next we want to show (ii).

$$K_t(\xi, \zeta) = (\rho'/2)^{t^{1/3} \lambda_2(\xi-\zeta)} \lambda_2 t^{1/3} \sum_{k=0}^{M-1} \phi_k(\lambda_2 t^{1/3} \xi) \psi_k(\lambda_2 t^{1/3} \zeta)$$
\[= \left( \rho' / 2 \right)^{1/3} \lambda_2 (\xi - \zeta) \lambda_1 \lambda_2 t^{2/3} \left( \int_{L'}^{M t^{-1/3} / \lambda_1} + \int_{-1/3 / \lambda_1} \right) \phi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) \psi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \zeta) d\kappa. \]

According to Proposition 4.19 and 4.20, the integrand in the first sum \( \int_{L'}^{M t^{-1/3} / \lambda_1} \) is bounded by

\[\left| \left( \rho' / 2 \right)^{1/3} \lambda_2 (\xi - \zeta) \phi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) \psi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \zeta) \right| \leq c e^{-2\kappa},\]

for some constant \( c \). From this we see that

\[\lim_{t \to \infty} \left( \rho' / 2 \right)^{1/3} \lambda_2 (\xi - \zeta) \lambda_1 \lambda_2 t^{2/3} \int_{L'}^{M t^{-1/3} / \lambda_1} \phi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) \psi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \zeta) d\kappa \leq \lambda_1 c e^{-2L'/2},\]

which vanishes by taking \( L' \to \infty \). For the second sum, we have, as a result of Proposition 4.16 and 4.18,

\[\lim_{t \to \infty} \left( \rho' / 2 \right)^{1/3} \lambda_2 (\xi - \zeta) \lambda_1 \lambda_2 t^{2/3} \int_{-1/3 / \lambda_1}^{L'} \phi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \xi) \psi_{M - \lambda_1 t^{1/3} \kappa}(\lambda_2 t^{1/3} \zeta) d\kappa = \frac{4 \lambda_1 \lambda_2}{(1 - \rho)^2} \left( \frac{1 + \rho}{1 - \rho} \right)^{1/3} \left( \frac{4(3 - \rho)}{3(1 + \rho)^2} \right) \int_{0}^{L'} \text{Ai}(\eta_2 + \kappa + \xi) \text{Ai}(\eta_2 + \kappa + \zeta) d\kappa = \int_{0}^{L'} \text{Ai}(\eta_2 + \kappa + \xi) \text{Ai}(\eta_2 + \kappa + \zeta) d\kappa,\]

uniformly on \((\xi, \zeta) \in [-L, L]^2\) for any fixed \( L > 0 \). Putting the results above together and taking \( L' \to \infty \), the statement (ii) is then proved.

The bound (iii) can be easily seen from Proposition 4.19 and 4.20, along with the fact that the airy function \( \text{Ai}(x) \) tends to \( e^{-x} \) as \( x \to +\infty \), and is bounded by a constant as \( x \to -\infty \).

An general statement on the convergence of a Fredholm determinant is given below.

**Lemma 4.22.** Suppose a kernel \( K_t \) satisfies

(i) For any fixed \( L > 0 \)

\[\lim_{t \to \infty} K_t(\xi, \zeta) = H(\xi, \zeta),\]

uniformly on \((\xi, \zeta) \in [-L, L]^2\).

(ii) For any fixed \( L > 0 \) and \( t \) large enough,

\[|K_t(\xi, \zeta)| \leq c e^{-\max\{0, \xi\} - \max\{0, \zeta\}}\]
Asymptotic analysis

for some constant $c > 0$ and $\xi, \zeta \geq -L$.

Then we have

$$\lim_{t \to \infty} \det(1 - K_t) = \det(1 - H)$$

See Lemma C.2 in [86] for a detailed proof. Consequently, we can conclude that

**Theorem 4.23.** Under the scaling limit (4.45), the kernel $K$ defined in (4.47) satisfies

$$\lim_{t \to \infty} I_1 = \lim_{t \to \infty} \det(1 - K) = \det(1 - A) |_{\rho^2(\eta_2, \infty)} = F_2(\eta_2) = F_2(s_-) \quad (4.65)$$

where $A$ is the Airy kernel defined in (4.64).

### 4.8.3 Asymptotic analysis of the second term

Now we consider the second term in (4.38), which can be written as below.

**Lemma 4.24.** The second term in (4.38) is given by

$$I_2 := I_z \times I_w. \quad (4.66)$$

where

$$I_z \times I_w :=
\frac{e^{-\rho t/2}}{(\rho')^{N-1}(N-1)!} \int_1^{N-1} z \frac{e^{\Lambda_{N-1,M}(z) / 2}}{\prod_{j=1}^{N-1} (z_j - 1)^N} S_{N-1,M}(z, 1) \left( \frac{2(1 - \rho)}{2 \rho'} \right)^M \prod_{k=1}^M \frac{1 + 1/\rho'}{1 + w_k/\rho'} \prod_{j=1}^{N-1} \frac{1 + z_j}{1 + z_j w_k}, \quad (4.67)$$

and $\Lambda_{N,M} = \frac{1}{2} \sum_{j=1}^N (z_j - 1) + \frac{1}{2} \sum_{k=1}^M (w_k - 1)$.

**Proof.** The lemma follows easily from changing the variables $w_k \to 1/w_k$ in the second term of (4.38). The $w$-contours around the origin are therefore deformed to contours around the infinity, and hence can be replaced by contours around $w_k = 0, \pm 1, -\rho'$.

**Evaluation of $I_w$**

We analyse the two integrals $I_z$ and $I_w$ separately. In fact, using Mathematica, one can see that numerically, the long time limit of $I_z \times I_w$ is dominated by setting $z_j = 1$ in $I_w$. Namely, the $z$-dependence of $I_w$ can be replaced by $z_j = 1$.

**Conjecture 4.25.** Under the scale (4.45) and the limit $t \to \infty$, the dominant contribution of $I_w$ (4.66) is given by $z_j = 1$ for all $j$. We have seen this result numerically for large $N, M$ by Matlab [35]. We strongly believe this argument holds under the scale (4.45) and the limit $t \to \infty$. However, due to limited time, we haven’t completed the rigorous proof yet. We will finish the proof in our next paper.
The basic idea of the proof is to rewrite (4.68) into a Fredholm determinant depending on \( z_j \). By expanding the kernel \( k(x, y; z_j) \) of this Fredholm determinant around the point \( z_j = 1 \), we found that only the linear order terms of \( \prod_j (z_j - 1) \) survive after some manipulations. Then we collect all the unwanted term together with the \( z \)-integration (4.67). The conjecture is proved by applying the steepest descent to these terms.

The results in the following are all based on Conjecture 4.25. Letting \( z_j = 1 \), we can see that (4.68) fits into the standard form (4.23) with

\[
\begin{align*}
\nu &= M, \quad \mu = N, \\
a_k &= 1, \quad 1 \leq k \leq M, \\
\alpha_j &= 0, \quad 1 \leq j \leq M, \\
\beta_j &= 1, \quad 1 \leq j \leq N - 1, \quad \beta_N = 1/\rho'.
\end{align*}
\]

and

\[
g(w, x) = w^M g_{N, M}(w, x) = \frac{w^{N+x}}{1 + w/\rho'} e^{wt/2} \left( \frac{1}{1 + w} \right)^{N-1}.
\]

Again we apply the method described in Section 4.6, and the integral \( I_w \) is therefore written as a Fredholm determinant with kernel

\[
K(x, y) = \sum_{k=0}^{M-1} \phi_k(x) \psi_k(y), \tag{4.69}
\]

and

\[
\begin{align*}
\phi_k(x) &= \oint_{1} \frac{dw}{2\pi i} \frac{1 + w/\rho'}{w^x(w - 1)(1 + w)} \left( \frac{1 + w}{w} \right)^N \left( \frac{w}{1 - w} \right)^k e^{-wt'/2}, \\
\psi_k(x) &= \oint_{0, -1, \rho'} \frac{dw}{2\pi i} \frac{w^{x-2}(1 + w)}{1 + w/\rho'} \left( \frac{w}{1 + w} \right)^N \left( \frac{1 - w}{w} \right)^k e^{wt/2}.
\end{align*}
\]

Following by the same analysis in Section 4.8.2, one can obtain the long time limit of \( I_w \). The detailed analysis and proofs follow very similarly, we will only give a brief sketch of the analysis below.

Under the scaling limit (4.45) and setting \( k = M - \kappa_1 t^{1/3}, \quad x = \xi_1 t^{1/3} \), the function \( \phi_k(x) \) and \( \psi_k(x) \) are rewritten as

\[
\begin{align*}
\phi_k(x) &= \oint_{1} \frac{dw}{2\pi i} e^{f(w, t) + g_4(w)}, \\
\psi_k(x) &= \oint_{0, -1, \rho'} \frac{dw}{2\pi i} e^{-f(w, t) + g_5(w)},
\end{align*}
\]

where \( f(w, t) := g_1(w) t + g_2(w) t^{1/2} + g_3(w) t^{1/3} \) with \( g_i(w) \) given in (4.52), and

\[
\begin{align*}
g_4(w) &= \log(1 + w/\rho') - \log(w) - \log(w - 1) - \log(1 + w), \\
g_5(w) &= \log(1 + w) - 2 \log(w) - \log(1 + w/\rho').
\end{align*}
\]

With the same \( g_1(w), g_2(w), g_3(w) \) as in Section 4.8.2, the steepest descent of
\( \phi_k(x) \) and \( \psi_k(x) \) again lies through the saddle point \( w_c = \rho'/2 \). Thus the long time limits of \( \phi_k(x) \) and \( \psi_k(x) \) are given by

\[
\lim_{t \to \infty} \left( \frac{6t}{(1 + \rho)(3 - \rho)} \right)^{1/3} e^{-f'(\rho'/2,t)} \phi_{M - \lambda^{1/3} t} (\gamma_j t^{1/3} \xi) = e^{\gamma t^{1/3} \xi} \phi_{M - \lambda^{1/3} t} (\eta + \kappa + \xi),
\]

\[
\lim_{t \to \infty} \left( \frac{6t}{(1 + \rho)(3 - \rho)} \right)^{1/3} e^{f'(\rho'/2,t)} \psi_{M - \lambda^{1/3} t} (\gamma_j t^{1/3} \xi) = e^{\gamma t^{1/3} \xi} \psi_{M - \lambda^{1/3} t} (\eta + \kappa + \xi),
\]

from which it follows that

\[
\lim_{t \to \infty} I_w = \lim_{t \to \infty} \det (1 - K) = F_2(s). \tag{4.70}
\]

The detailed proof of this is basically the same as the asymptotic analysis of the first term.

**Evaluation of \( I_z \)**

We are now left with the \( z \)-integral:

\[
I_z = \frac{e^{-\rho t/2}}{\rho^{N-1} (N - 1)!} \oint_{1,1/\rho'} d^n z \frac{e^{\lambda^{1/3} t} \prod_{i<j<N} (z_i - z_j)^2 \prod_{i=1}^{N-1} (1 - \rho' z_i) \left( \frac{2\rho'}{1 + \rho} \right)^M}{\prod_{i=1}^{N-1} (z_i - 1)^N [\frac{1}{2} (z_i + 1)]^M}.
\]

It is hard to apply the method introduced in Section 4.6, since the order of the pole at \( z_j = 1 \) is one more than the number of variables \( z_j \)'s. Hence it is reasonable to consider the following \( N \)-fold integral instead of \( I_z \),

\[
J = \frac{\rho^N}{N!} \oint_{1,1/\rho'} d^n z \frac{e^{\lambda^{1/3} t} \prod_{i<j<N} (z_i - z_j)^2 \prod_{i=1}^{N-1} (1 - \rho' z_i) (z_i - 1)^N [\frac{1}{2} (z_i + 1)]^M}{\prod_{i=1}^{N-1} (z_i - 1)^N [\frac{1}{2} (z_i + 1)]^M} := \oint_{1,1/\rho'} d^n z J_z. \tag{4.71}
\]

**Lemma 4.26.** The integral \( I_z \) defined in (4.67) can be written into \( I_z = J - 1 \) with \( J \) given by (4.71).

**Proof.** There is a simple pole at \( z_N = 1/\rho' \), which has residue

\[
\text{Res}_{z_N=1/\rho'} J_z = \frac{\rho^N e^{\lambda^{1/3} t} \prod_{i<j<N} (z_i - z_j)^2 \prod_{i=1}^{M-1} (1 - \rho' z_i)^2 z_N - 1/\rho'}{N! \prod_{i=1}^{N-1} (1 - \rho' z_i) (z_i - 1)^N [\frac{1}{2} (z_i + 1)]^M} \left( \frac{1}{1 + \rho} \right)^M - \frac{\rho^N e^{\gamma t^{1/3} \xi} \prod_{i<j<N} (z_i - z_j)^2 \prod_{i=1}^{N-1} (1 - \rho' z_i)^2}{N! \prod_{i=1}^{N-1} (z_i - 1)^N [\frac{1}{2} (z_i + 1)]^M} \left( \frac{2\rho'}{1 + \rho} \right)^M.
\]

Since \( \text{Res}_{z_N=1/\rho'} J_z \) has no poles at \( z_j = 1/\rho' \), and all the other contributions of
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The poles at \( z_j = 1/\rho' \) is just \( N \) times \( \text{Res}_{z_N=1/\rho' N} J_z \), we therefore obtain

\[
I_z = J - \frac{\rho^N}{N!} \oint_N dNz \frac{e^{N,\alpha t} \prod_{i<j} (z_i - z_j)^2}{\prod_{i=1}^N (1 - \rho' z_i)(z_i - 1)^N \left[ \frac{1}{2}(z_i + 1) \right]^M} = J - \frac{\rho^N}{N!} \oint_N dNz \frac{e^{N,\alpha t} \prod_{i<j} (z_i - z_j) \prod_{i=1}^N z_i^{N-1}}{\prod_{i=1}^N (1 - \rho' z_i)(z_i - 1)^N \left[ \frac{1}{2}(z_i + 1) \right]^N} = J - 1. \tag{4.72}
\]

The second equality follows from anti-symmetrisation. The third line is obtained by evaluating poles at \( z_j = 1 \) sequentially. Starting with the simple pole at \( z_N = 1 \), its residue will decrease the order of other poles at \( z_j = 1 \) by one, and hence all the poles can be evaluated sequentially.

The asymptotic behaviour of \( I_z \) is now given by the asymptotics of \( J \), which now can be analysed via the previous method. Specifically, after rearrangements,

\[
J = \frac{1}{N!} \oint_{1,\rho'} d\rho dz \prod_{i=1}^N \frac{\prod_{i\neq j} (1 - z_i/z_j)}{\prod_{i=1}^N (1 - 1/z_i)N} g_{M,N}(z_i,0) g_{M,N}(1,0),
\]

where

\[
g(z, x) = z^n g_{M,N}(z, x) = \frac{z^x}{1 - \rho'/z} \left( \frac{z}{1+z} \right)^M e^{zt/2}.
\]

According to Section 4.6, \( J \) is thus written as a Fredholm determinant with the kernel,

\[
K = \sum_{k=0}^{N-1} \phi_k(x) \psi_k(y), \tag{4.73}
\]

where

\[
\phi_k(x) = \oint_1 dw w^k \frac{(1 - \rho'/w)}{w^x(w-1)^x+1} \left( w + 1 \right) M e^{-w t/2}
\]

\[
\psi_k(x) = \oint_{\rho'} dw \frac{w^x(w-1)^k}{(1 - \rho'/w)w^{k+2}} \left( w + 1 \right) M e^{w t/2}.
\]

Like before, under the scaling limits (4.45) and setting \( k = N - \kappa t^{1/2}, x = \xi t^{1/2} \), we rewrite the integrand into exponents:

\[
\phi_k(x) = \oint_1 \frac{d\omega}{2\pi i} \left( \omega - \rho' \right) e^{f(\omega, t)+g_4(\omega)},
\]

\[
\psi_k(x) = \oint_{\rho'} \frac{d\omega}{2\pi i} \left( \omega - \rho' \right) e^{-f(\omega, t)+g_5(\omega)} = \frac{1}{\rho'} e^{-f(\rho', t)}, \tag{4.74}
\]

where \( f(\omega, t) = g_1(\omega)t + g_2(\omega)t^{1/2} + g_3(\omega)t^{1/3} \) with \( g_i(\omega) \) given below, and the pole
Proof.

A rigorous proof falls into the same pattern as in Proposition 4.18. To avoid reiterating ourselves, here we only give a basic idea of the proof. Recall that the maximum point of $\text{Re}(g)$ hence vary the estimate of the integral. Through $w = c \rho \eta \xi$ be scaled as (4.45). Then for some constant $c_0 \in \mathbb{R}$ and $\lambda_1 > 0, \lambda_2 > 0$, the function $\phi_k(x)$ defined in (4.48a) converges uniformly in a bounded set of $\xi, \kappa$ to

$$
\lim_{t \to \infty} t c(t) \phi_{N-\lambda_1 t^{1/2}, \lambda_2 t^{1/2}}(\lambda_2 t^{1/2} \xi) = c_0 (\eta + \kappa + \xi) e^{-(\eta + \kappa + \xi)^2},
$$

where $c(t)$ is some function of $t$ that may or may not be bounded as $t \to \infty$. Specifically,

$$
c_0 = \frac{8(2 - \rho)}{9\sqrt{\pi}(1 - \rho)^2}
$$

$$
\lambda_1 = c_g / (2 - \rho)
$$

$$
\lambda_2 = c_g / (\rho(2 - \rho))
$$

$$
c(t) = \exp \left( -g_1(\rho') t - g_2(\rho') t^{1/2} - g_3(\rho') t^{1/3} \right),
$$

with $g_i(w)$ given below in (4.75), and $c_2, c_g$ given in (4.41).

Proof. A rigorous proof falls into the same pattern as in Proposition 4.18. To avoid reiterating ourselves, here we only give a basic idea of the proof. Recall that the proposition. 

$$
\phi_k(x) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{dw}{2\pi i} (w - \rho') e^{f(w,t) + g_4(w)},
$$

where $f(w,t) = g_1(w) t + g_2(w) t^{1/2} + g_3(w) t^{1/3}$ with $g_i(w)$ given in (4.75). Solving $g_1'(w) = 0$ gives us $w_1 = \rho'$ and $w_2 = -\rho'/2$. One can obtain a steepest descent through $w_1 = \rho'$, since the one passing $w_2$ would include extra poles at origin and hence vary the estimate of the integral.

One can see that the following contour $\Gamma = \Gamma_1 \cup \Gamma_2$ (see Fig. 4.8) is a steep descent path of $g_1(w)$ passing through $\rho'$. Namely, $w = \rho'$ is the strict global maximum point of $\text{Re}(g_1)$ along $\Gamma$. This can be proved by calculating $\frac{\text{dRe}(g_1)(s)}{ds}$
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along $\Gamma$.

$$\Gamma_1 = \left\{ w = \rho' - \frac{s}{\sqrt{3}} : s \in [-\rho, \rho] \right\}, \quad (4.77a)$$

$$\Gamma_2 = \left\{ w = 1 + \frac{2\rho}{\sqrt{3}} e^{is} : s \in [-5\pi/6, 5\pi/6] \right\}. \quad (4.77b)$$

\[ \text{Figure 4.8: Steep descent contour of integration in } \phi_k(x) \text{ passing the saddle point at } w_1 = \rho'. \]

As shown in the proof of Proposition 4.18, we can prove that for large enough $t$, only the part $\Gamma_\delta := \{ w \in \Gamma : |w - w_1| \leq \delta \}$ for some small $\delta > 0$ contributes to the integral. Near $w_1 = \rho'$, the Taylor expansion of $g_i(w)$ are given by

$$g_1(w) - g_1(w_1) = \frac{9(1 - \rho)(w - \rho)^2}{16(2 - \rho)\rho} + \mathcal{O}[(w - \rho)^3], \quad (4.78a)$$

$$g_2(w) - g_2(w_1) = -\frac{c_g \eta_g + (2 - \rho)(\kappa_1 + \xi_1 \rho)}{\rho(1 - \rho)(2 - \rho)} (w - \rho) + \mathcal{O}[(w - \rho)^2], \quad (4.78b)$$

$$g_3(w) - g_3(w_1) = \mathcal{O}[(w - \rho)], \quad (4.78c)$$

$$g_4(w) - g_4(w_1) = \mathcal{O}[(w - \rho)]. \quad (4.78d)$$

Denote the function $g_i(w)$ without the error terms by $\bar{g}_i(w)$. As in Proposition 4.18, we can show that for large $t$, only the term $e^{\bar{g}_1(w)t + \bar{g}_2(w)t^{1/2} + \bar{g}_3(w)t^{1/3} + \bar{g}_4(w)}$ contributes to the integral. We re-parameterize $\Gamma_\delta$ by

$$w - \rho' = iv^2 \left( \frac{\rho(2 - \rho)}{t(1 - \rho)} \right)^{1/3},$$

where $-c_\delta t^{1/2} \leq v \leq c_\delta t^{1/2}$, and $c = \frac{3}{2} \sqrt{\frac{(1 - \rho)}{\rho(2 - \rho)}}$. Thus we are left with

$$\lim_{t \to \infty} t \int_{\Gamma_\delta} \frac{d\omega}{2\pi i} (w - \rho') e^{\bar{g}_1(w)t + \bar{g}_2(w)t^{1/2} + \bar{g}_3(w)t^{1/3} + \bar{g}_4(w)} - f(\rho', t)$$
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\[\begin{align*}
\lim_{t \to \infty} & \frac{-t}{\rho(1-\rho)} \int_{\Gamma} \frac{dw}{2\pi i} (w - \rho') \exp \left( \frac{9(1-\rho)w^2}{16(2-\rho)\rho} - \frac{c_g \eta_g + (2-\rho)(\kappa_1 + \xi_1 \rho)}{\rho(1-\rho)(2-\rho)} wt^{1/2} \right) \\
= & \lim_{t \to \infty} \frac{4(2-\rho)}{9(1-\rho)^2} \int_{-c^2t^{1/2}}^{c^2t^{1/2}} \frac{dv}{2\pi i} v e^{-v^2/4 - vi(\eta_g + \kappa + \xi)} \\
= & \frac{4(2-\rho)}{9(1-\rho)^2} \int_{-\infty}^{\infty} \frac{dv}{2\pi i} v e^{-v^2/4 - vi(\eta_g + \kappa + \xi)} \\
= & \frac{8(2-\rho)(\eta_g + \kappa + \xi)}{9\sqrt{\pi}(1-\rho)^2} e^{-(\eta_g + \kappa + \xi)^2}.
\end{align*}\]

where \( \kappa = (2-\rho)\kappa_1/c_g := \kappa_1/\lambda_1, \quad \xi = (2-\rho)\xi_1/c_g := \xi_1/\lambda_2, \)
as required. \( \square \)

Followed by the same fashion used in Proposition 4.19, we have a corresponding estimate of the function \(\phi_k(w)\) defined in (4.74).

**Proposition 4.28. (Estimate of function \(\phi\))** Consider the function \(\phi_k(x)\) defined in (4.74), with \(N,M\) scaled as (4.45), and \(f(w,t) = g_1(w)t + g_2(w)t^{1/2} + g_3(w)t^{1/3}\) defined in (4.75). Then for large enough \(t\) and \(L\) large enough but independent of \(t\),

\[\left| \frac{te^{-f(\rho',t)}}{\rho(1-\rho)} \phi_{N-\lambda t^{1/2}k}(\lambda_2 t^{1/2} \xi) \right| \leq e^{-(\kappa + \xi)},\]

for \(\kappa, \xi \geq -L\) and \(\kappa + \xi \geq L\).

![Figure 4.9: Deformed steepest descent contour of integration in \(\phi_k(x)\).](image)

With a deformed contour given in Fig. 4.9, the proof for this proposition follows exactly the same method as in the proof of Proposition 4.19. Therefore we will not repeat it here.

Consequently, these two Proposition 4.27 and 4.28, together with the fact that

\[e^{f(\rho',t)}\psi_{N-\lambda t^{1/2}k}(\lambda_2 t^{1/2} \xi) = \frac{1}{1-\rho},\]

(4.79)
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give the following theorem.

**Theorem 4.29.** Consider the kernel defined in (4.73), and define the rescaled kernel
\[ K_t(\xi, \zeta) = \rho \lambda_2 t^{1/2} \lambda_2 t^{1/2} K(\lambda_2 t^{1/2} \xi, \lambda_2 t^{1/2} \zeta), \]
with \( \lambda_2 \) given in (4.51c). Then we have

(i) \( \det[1 - K_t(\xi, \zeta)] = \det[1 - K(x, y)]. \)

(ii) For any fixed \( L > 0 \)
\[ \lim_{t \to \infty} K_t(\xi, \zeta) = e^{-\eta_g + \xi} \frac{1}{\sqrt{\pi}}, \]
uniformly on \( (\xi, \zeta) \in [-L, L]^2 \).

(iii) For any fixed \( L > 0 \) and \( t \) large enough,
\[ |K_t(\xi, \zeta)| \leq c e^{-\max\{0, \xi\}} \]
for some constant \( c > 0 \) and \( \xi, \zeta \geq -L \).

**Proof.** Obviously, (i) can be easily proved by the change of variable \( (x, y) \to (\gamma t^{1/3} \xi, \gamma t^{1/3} \zeta) \) and then conjugating the kernel with \( \rho \lambda_2 t^{1/2} (\xi - \zeta) \).

For (ii), we have
\[ K_t(\xi, \zeta) = \rho \lambda_2 t^{1/2} \lambda_2 t^{1/2} \sum_{k=0}^{N-1} \phi_k(\lambda_2 t^{1/2} \xi) \psi_k(\lambda_2 t^{1/2} \zeta) \]
\[ = \rho \lambda_2 t^{1/2} \lambda_2 t^{1/2} \int_{L'}^{L} \phi_{N-1-t^{1/2}/\lambda_1} \psi_{N-1-t^{1/2}/\lambda_1} \lambda_1 t^{1/2} d\kappa. \]

From Proposition 4.28, one can see the first term vanishes by the same argument as in Proposition 4.21. For the second sum, we have, as a result of Proposition 4.27 and (4.79)
\[ \lim_{t \to \infty} \rho \lambda_2 t^{1/2} \lambda_2 t^{1/2} \int_{L'}^{L} \phi_{N-1-t^{1/2}/\lambda_1} \psi_{N-1-t^{1/2}/\lambda_1} \lambda_1 t^{1/2} d\kappa \]
\[ = \frac{8(2 - \rho)}{9 \sqrt{\pi} (1 - \rho)^2} \frac{1}{1 - \rho} \lambda_1 t^{1/2} \int_{0}^{L'} (\eta_g + \kappa + \xi) e^{-\eta_g + \kappa + \xi} d\kappa \]
\[ = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} (\eta_g + \kappa + \xi) e^{-\eta_g + \kappa + \xi} d\kappa, \quad \text{as } L' \to \infty \]
\[ = e^{-\eta_g + \xi} \frac{1}{\sqrt{\pi}}, \]
uniformly on \( (\xi, \zeta) \in [-L, L]^2 \) for any fixed \( L > 0 \).

The bound (iii) can be easily seen from Proposition 4.28 and (ii). \( \square \)
The above results with the Lemma 4.22 allow us to take the limit of the Fredholm determinant into the kernel, i.e.,

\[
\lim_{t \to \infty} \det(1 - K) = \det \left( 1 - \lim_{t \to \infty} K_t(\xi, \zeta) \right) = 1 - \int_0^\infty \frac{e^{-(\eta_g + \xi)^2}}{\sqrt{\pi}} d\xi = \int_{-\infty}^0 \frac{e^{-(\eta_g + \xi)^2}}{\sqrt{\pi}} d\xi = \int_{-\infty}^{\eta_g} \frac{e^{-\xi^2}}{\sqrt{\pi}} d\xi = \frac{\int e^{-\xi^2} d\xi}{\sqrt{\pi}} = F_G(\eta_g),
\]

which implying that

\[
\lim_{t \to \infty} J = F_G(\eta_g) = F_G(s_+).
\] (4.80)

Combining (4.38), (4.65), (4.66), (4.70), (4.72), and (4.80) together, we arrive at the final results of the asymptotic analysis of the joint current distribution

\[
\lim_{t \to \infty} P_{N,M,\rho}(t) = F_2(s_-) + F_2(s_-)(F_G(s_+ - 1) = F_2(s_-)F_G(s_+),
\]

and hence complete the proof of Theorem 4.39. This results agrees with the predictions made by the NLFHD [35], and is the first analytic confirmation of predictions of 1D non-linear fluctuating hydrodynamics for multi-component systems.
The Green’s Function in a Two-Species Exclusion Process
A summary of the thesis followed by several speculations is given in this chapter

\section*{Conclusion and Outlook}

\subsection*{Summary}

\subsubsection*{Duality in mASEP}

The asymmetric simple exclusion process (ASEP) is regarded as the default stochastic model for transport phenomena. It describes particles performing biased hopping in a preferred direction on a one-dimensional lattice. \textit{Duality} is a powerful tool in the study of exclusion processes. A duality function is an observable that co-varies in time with respect to the generators of two processes. It relates physical quantities, such as particle flow, in a system with many particles to another system with few particles, so that the quantity of interest in the first process can be calculated explicitly via the second process.

In Chapter 3, we propose a new method to derive duality functions systematically, by exploiting the algebraic structure provided by the quantum Knizhnik-Zamolodchikov (qKZ) equations \cite{70}, and the Hecke algebra. Using this method, we can not only reproduce the well-known self-duality in the single species ASEP \cite{143}, but also extend this construction to generic multi-species ASEPs.

We consider the multi-species asymmetric exclusion simple process (mASEP) with left hopping rate $t$, and right hopping rate 1. The mASEP can be realised in two ways via representations of the Hecke algebra. The first is a standard description in which each occupational configuration $\mu$ is identified with a basis element of a vector space. By setting 

\begin{align*}
|0\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{align*}

$|\mu\rangle$ is identified as $|\mu\rangle = \bigotimes_{i \in \mathbb{Z}} |\mu_i\rangle$, and the local Markov generator is the standard generator matrix acting on this space.

The second realisation is via a polynomial basis $\{f_\mu(z)\}$ in the variable $(z_1, \ldots, z_n)$ indexed by compositions (i.e. the occupational state) $\mu = (\mu_1, \ldots, \mu_n)$. Let $f_\mu(z)$ be the solution to the $t$KZ equations (see (3.24) in Chapter 3).
dynamics of mASEP is then recovered through a divided-difference operator
\[ L_i = - \left( \frac{t z_i - z_{i+1}}{z_i - z_{i+1}} \right) (1 - s_i). \]

In order to obtain a matrix product formula for the polynomials \( f_\mu \) (see (3.60) in Chapter 3), we impose a cyclic boundary condition with parameter \( q \). We called these polynomials ASEP polynomials. At some specialization \( q^{kt} = 1 \) with \( k, l \in \mathbb{N} \), the ASEP polynomials \( f_\mu \) is singular and hence degenerate into a sum of other polynomials \( \sum_\nu \psi(\nu; \mu; t) f_\nu \). The coefficients \( \psi(\nu; \mu; t) \) in the expansion is shown to be the duality function between mASEP:

**Theorem** (Chen, de Gier, Wheeler). Suppose \( f_\mu \) is a polynomial as described above. Fix a positive rational number \( m \), and a natural number \( p \) such that there exists an expansion
\[
\lim_{q \to t^{-m}} (1 - qt^m)^p f_\mu(z; q, t) = \sum_\nu \psi(\nu; \mu; t) f_\nu(z; t^{-m}, t).
\]

Then \( \psi(\nu; \mu; t) \) is local duality function between the first process with configuration labels \( \mu \), and the second process with configurations labels \( \nu \).

In fact, we make a stronger statement, namely that for any anti-partition \( \mu^- \), and assuming that \( \lim_{q \to t^{-m}} (1 - qt^m)^p f_{\mu^-}(z; q, t) \) exists and is of degree \( \nu^* \), then for any permutation \( \mu \) of \( \mu^+ \) we have
\[
\lim_{q \to t^{-m}} (1 - qt^m)^p f_{\mu^-}(z; q, t) = \sum_{\nu \in S_{\nu^*}} \psi(\nu; \mu; t) f_\nu(z; t^{-m}, t).
\]

Using the matrix product formula of \( f_\mu \), we also obtain the following explicit form of duality functions in two-species ASEP.

**Theorem** (Chen, de Gier, Wheeler). Let \( \mu \) be a rank-two composition with \( m_1 \) ones, \( m_2 \) twos and \( m_0 \) zeros. For any \( 1 \leq p \leq \min(m_0, m_2) \), at \( q = t^{-p-m_1} \),
\[
\mathrm{Res}_{q=t^{-p-m_1}} f_\mu(z; q, t) \propto \sum_\nu C_{\mu,\nu}(t) f_\nu(z; t^{-p-m_1}, t),
\]

The sum is taken over all compositions \( \nu \) with \( m_1 + 2p \) ones, \( m_2 - p \) twos and \( m_0 - p \) zeros, such that
\[
\mu_i = \begin{cases} 
0, & \Rightarrow \nu_i \neq 2, \\
1, & \Rightarrow \nu_i = 1, \\
2, & \Rightarrow \nu_i \neq 0,
\end{cases}
\]

The coefficients are given by \( C_{\mu,\nu}(t) = t^{\sum_{i<j} 1_{\mu_i<\nu_j} 1_{\nu_i=\nu_j=1}} \).

This result agrees with the result on [17], and a higher rank duality is studied in [105–107] by using quantum group symmetries. The duality in [105–107] has a similar structure to ours, but varies in the power of the two hopping rates’ ratio.
5.1.2 The AHR model

Hydrodynamics is the (heuristic) theory of the macroscopic evolution of many-particle systems. When dealing with fluctuations and correlations, one needs to use a non-linear as well as stochastic extension of hydrodynamics, i.e. Non-linear Fluctuating Hydrodynamics (NLFHD) [125]. This heuristic theory has provided several predictions for one-dimensional systems with particles interacting via non-linear potentials and anharmonic chains. However, such predictions lack firm analytic validation.

In Chapter 4, we provide the first confirmation from first principles of the predictions of NLFHD for a two-component stochastic system. Using techniques of solvable lattice models, we derive an exact multiple integral formula for a joint current distribution for the two species exclusion process AHR [2]. An asymptotic formula for this integral formula in the scaling limit is then obtained using a steepest descent analysis. The limiting distribution is found to be a product of a Gaussian and a GUE Tracy-Widom distribution from random matrix theory [157], as predicted by NLFHD.

The AHR model is a Markovian stochastic process consisting of two families of particles, the positive particles and negative particles, that hop in opposite directions on a one dimensional lattice. The positive particle hops to the right at rate $p$ while the negative particle hops to the left at rate $q$. Two different particles swap with each other at a rate $p + q = 1$. The Green’s function of the AHR model is found by constructing the eigenfunctions of its generator using a form of the Bethe ansatz.

**Theorem** (Chen, de Gier, Hiki, Sasamoto). Consider the AHR model on $\mathbb{Z}$ with $N$ positive and $M$ negative particles. Suppose the positions of particles are given by $(x_1, \ldots, x_N), (y_1, \ldots, y_M)$ at general time $t$, and $(x_1^{(0)}, \ldots, x_N^{(0)}), (y_1^{(0)}, \ldots, y_M^{(0)})$ at initial time $t = 0$. When $x_1^{(0)} < \cdots < x_N^{(0)} < y_1^{(0)} < \cdots < y_M^{(0)}$ and $y_1 < \cdots < y_M < x_1 < \cdots < x_N$, the Green’s function is given by

$$G(\vec{x}, \vec{y} \mid \vec{x}^{(0)}, \vec{y}^{(0)}; 0) = \int_C \cdots \int_C \prod_{i=1}^{N} \frac{dz_i}{2\pi i z_i} \prod_{j=1}^{M} \frac{dw_j}{2\pi i w_j} e^{\Lambda} \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1}{q z_i + p w_j} \times \sum_{\pi \in S_N} \text{sign}(\pi) \prod_{i=1}^{N} \left( \frac{1 - z_i}{1 - z_{\pi_i}} \right)^i \frac{x_i}{z_i^{(0)}} \sum_{\sigma \in S_M} \text{sign}(\sigma) \prod_{j=1}^{M} \left( \frac{1 - w_j}{1 - w_{\sigma_j}} \right)^{-j} w_j^{y_j^{(0)}} w_j^{y_j},$$

where $\Lambda = p \sum_{i=1}^{N} (z_i^{-1} - 1) + q \sum_{j=1}^{M} (w_j^{-1} - 1)$, and the contour $C$ is a small circle around the origin.

We then consider a step Bernoulli initial condition, in which initially $N$ positive particles are distributed by the Bernoulli measure with density $\rho$ on $x \leq -1$, while the first $M$ sites on $x \geq 0$ are occupied by negative particles. Through the Green’s function, we found an exact formula for the quantity $P_{N,M,\rho}(t)$, the probability that all positive and negative particles have crossed the origin by time $t$. This probability
is obtained by summing the Green’s function $G(\vec{x}, \vec{y}; t \mid \vec{x}^{(0)}, \vec{y}^{(0)}; \vec{0})$ over all possible final and initial positions:

**Theorem** (Chen, de Gier, Hiki, Sasamoto). Consider the AHR model with Bernoulli initial condition of density $\rho$. The joint current distribution such that all positive and negative particles have crossed the origin at time $t$ is given by

$$P_{N,M,\rho}(t) = \frac{\rho^N}{N!} \oint_{C} \prod_{i=1}^{N} \frac{dz_i}{2\pi i} \prod_{j=1}^{M} \frac{dw_j}{2\pi i} \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1}{qz_i + pw_j} e^{At} \prod_{i<j}(z_i - z_j)^2 \prod_{k<l}(w_k - w_l) \prod_{k=1}^{M} w_k^{k-1} \prod_{i=1}^{N} (z_i - 1)^N (1 - (1 - \rho)z_i) \prod_{k=1}^{M} (w_k - 1)^k. \quad (5.1)$$

where the contour $C$ is a small circle around the origin.

In the case when $N < M$, an asymptotic analysis of $P_{N,M,\rho}(t)$ gives a product of the Gaussian and the GUE Tracy-Widom distribution $F_2(s_-)F_G(s_+)$, where $s_\pm$ are the two normal modes of the AHR model. This results coincides with the conjecture given by NLFHD.

**Theorem.** [Chen, de Gier, Hiki, Sasamoto] The long time limit of the joint current distribution (5.1) is given by,

$$\lim_{t \to \infty} P_{N,M,\rho}(t) = F_2(s_-)F_G(s_+).$$

### 5.2 Outlook

As an extension to the above two approaches, we give here some possible future research directions.

- The first step in possible future research regarding duality functions is to obtain asymptotic formulas in the multi-species ASEP. With the explicit duality function between single species and two-species ASEP [165], one should be able to obtain an explicit integral formula for a certain observable in two-species ASEP. We expect that such an observable gives enough information to describe the distribution of any configuration. Therefore, we can subsequently evaluate the long time limit of fluctuations in two-species ASEP. From this result, a generalisation to the generic $M$ species ASEP setting is expected.

- Another possible research direction following [35] also concerns the asymptotics of multi-species models. The Green’s function of the AHR model is given by an explicit integral formula, from which one should be able to compute joint distributions for other initial conditions. By applying similar asymptotic analyses, we

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*The Theorem is based on Conjecture 4.25. We will give a rigorous proof of Conjecture 4.25 in our next paper.*
can then obtain alternative long time limits of the AHR models. We also believe that the method of Green’s function should be applicable to other integrable multi-species models, which then gives an analytical evaluation of the long time behaviour in multi-species systems.
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