Reduced-order Modeling for Membrane Wings at Low Reynolds Numbers

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Declaration of Authorship

This is to certify that:

- this thesis comprises only my original work except where indicated,
- due acknowledgement has been made in the text to all other material used,
- this thesis is fewer than 100,000 words in length, exclusive of tables, bibliographies and appendices.

Massimiliano Nardini                                      October 2019
Abstract

The present work develops reduced-order models for the fluid-structure interaction of two-dimensional membrane wings at low Reynolds numbers. The fluid-structure interaction models are obtained from the coupling between a linear unsteady aerodynamic model and a structural membrane model. The linear aerodynamic model is derived from high-fidelity simulations at low Reynolds numbers using a system identification procedure. It is able to model the unsteady aerodynamic loads generated by a deflection of the wing, decomposed into its rigid and flexible degrees of freedom. The rigid degrees of freedom are represented by pitching and plunging, while the flexible degrees of freedom are decomposed into Fourier modes. The aerodynamic model is presented in state-space form, hence it is compatible with modern feedback control frameworks. Feedback control techniques are employed to investigate the performance of the individual degrees of freedom of the wing in tracking a reference lift and rejecting external disturbances. A one-dimensional nonlinear equation is adopted to model the structural dynamics of the membrane wing. From the nonlinear model, two lower-order approximations are derived by means of a truncated Taylor expansion: a quasi-linear model and a linear model. From the coupling with the linear aerodynamic model, a nonlinear, a quasi-linear and a linear fluid-structure interaction model are obtained. These models are adopted to investigate the static aeroelastic response and the stability of the membrane wing at different Reynolds numbers. Using Harmonic Balance methods, the forced aeroelastic response to pitching and the autonomous response of linearly unstable cases are also studied. Finally, an adjoint-based optimization procedure is employed to optimize the aerodynamic response of the membrane to an external disturbance in the form of a convective vortex. The results are compared against high-fidelity Direct Numerical Simulation to validate the predictive capabilities of the models and to discuss their range of applicability.
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List of Symbols

The next list describes several symbols that will be later used within the body of the document

Aerodynamic variables and symbols

\( \alpha \) Angle of Attack

\( \bar{x} \) Fluid-dynamic state variables for linear aerodynamic model

\( \Delta C_p \) Sinusoidal Fourier modes for the pressure coefficient

\( \Delta C_p \) Pressure coefficient \( \Delta C_p = 2\Delta p/\rho U^2 \)

\( \Delta p \) Pressure difference between lower and upper surface of the wing

\( (\tilde{x}, \tilde{y}) \) Cartesian axis for the fluid domain

\( (u, v) \) Vortex velocity components in the \( \tilde{x} \) and \( \tilde{y} \) direction

\( \nu \) Kinematic viscosity

\( \omega \) Reduced frequency

\( \rho \) Fluid density

\( c \) Wing chord

\( C_L \) Lift coefficient \( C_L = 2L/\rho U^2c \)

\( c_v \) Vortex intensity

\( L \) Wing’s dimensional lift

\( M \) Mach number

\( p^{low} \) Pressure on the wing’s lower surface

\( p^{up} \) Pressure on the wing’s upper surface

\( r_v \) Vortex viscous radius

\( Re \) Reynolds number \( Re = Uc/\nu \)
Non-dimensional time $t$
Free-stream velocity $U$

Membrane and fluid-structure interaction variables

$\bar{E}_s$ Membrane non-dimensional modulus of elasticity $\bar{E}_s = E_s h / \rho U^2 c$
$C_f$ Sinusoidal Fourier modes for the transverse force coefficient
$\mathcal{W}$ Sinusoidal Fourier modes for the deflection
$\Phi^{m2s}$ Transformation matrix from Fourier space to physical space
$\Phi^{s2m}$ Transformation matrix from physical space to Fourier space
$f^{NL}$ Vector containing the value of nonlinear term of the nonlinear membrane equation evaluated at the structural points
$\delta$ Membrane elongation
$\delta_0$ Membrane pre-stretch
$\mu$ Density ratio $\mu = \rho_s h / \rho U$
$\rho_s$ Membrane density
$C_d$ Membrane non-dimensional damping coefficient $C_d = \rho_s C_{ds} / \rho U$
$C_f$ Transverse force coefficient $C_f = 2 F / \rho U^2$
$c_T$ Membrane tension coefficient $c_T = c_{T0} + 2 \bar{E}_s \delta$
$C_{ds}$ Membrane damping coefficient
$c_{T0}$ Zero-deflection tension coefficient $c_{T0} = 2 \bar{E}_s \delta_0$
$F$ External transverse force
$h$ Membrane thickness
$N$ Total number of deflection modes considered
$N_p$ Number of equispaced structural points along the membrane wing’s chord
$w$ Membrane’s transverse deflection
$x$ Non-dimensional abscissa in the direction of the wing’s chord

State-space variables and symbols

$A_{00}^0$ One of the entries of the state matrix $A$ of a state-space system
$B_{00}^0$ One of the entries of the input matrix $B$ of a state-space system
One of the entries of the output matrix $C$ of a state-space system

One of the entries of the feedthrough matrix $D$ of a state-space system

One of the entries of the matrix containing the nonlinear terms

Input vector of a state-space system

Generic state variables

Output vector of a state-space system

$(A, B, C, D)$ System matrices of a state-space system

**Harmonic Balance formulation variables and symbols**

$\hat{F}$ Matrix containing the frequency-domain harmonic coefficients of $f$

$\hat{Q}$ Matrix containing the frequency-domain harmonic coefficients of $q$

$\hat{R}$ Matrix containing the frequency-domain harmonic coefficients of $r$

$\tilde{F}$ Matrix containing the time-domain harmonic coefficients of $f$

$\tilde{Q}$ Matrix containing the time-domain harmonic coefficients of $q$

$\tilde{R}$ Matrix containing the time-domain harmonic coefficients of $r$

$f$ Forcing term of a general dynamic system discretized in space

$q$ State variables of a general dynamic system discretized in space

$r$ Nonlinear term of a general dynamic system discretized in space

$N_H$ Total number of harmonics

**Adjoint formulation variables and symbols**

$\bar{\alpha}_k$ Step length from line search algorithm for minimization procedure evaluated at iteration step $k$

$\mathcal{J}()$ Derivatives of the cost function with respect to a generic variable

$\phi = [\bar{x}^T \bar{W}^T \dot{W}^T \alpha \dot{\alpha}]^T$ State variables for fluid-structure interaction models

$\xi = [\chi^T \mathcal{U}^T \mathcal{V}^T \beta \gamma]^T$ Adjoint variables for fluid-structure interaction models

$B_k$ Positive definite matrix for optimization procedure evaluated at iteration step $k$

$g$ Control parameters

$p_k$ Descent direction for optimization procedure evaluated at iteration step $k$
$\mathcal{F}(\mu)$ Penalties term on density ratio $\mu$

$\mathcal{J}$ Generic cost function

$\mathcal{J}_{\text{rig}}$ Cost function for rigid wing

$\mathcal{L}$ Lagrange functional

$\mathcal{N}$ State equations for fluid-structure interaction model

$N_g$ Number of control parameters

$N_\phi$ Number of state variables

**Acronyms**

BDIM Boundary Data Immersion Method

BFGS Broyden-Fletcher-Goldfarb-Shanno algorithm

CFD Computational Fluid Dynamics

DNS Direct Numerical Simulation

ERA Eigensystem Realization Algorithm

FDHB Frequency-Domain Harmonic Balance

FDTC AIAA Fluid Dynamics Technical Committee

HB Harmonic Balance

HDHB High Dimensional Harmonic Balance

LCO Limit-cycle Oscillations

LES Large Eddy Simulation

RANS Reynolds Averaged Navier-Stokes

RHP Right Half Plane

TDHB Time-Domain Harmonic Balance


Chapter 1

Introduction

1.1 Motivation

Micro-Air Vehicles (MAVs) are a class of unmanned aerial vehicles characterized by reduced dimensions. In recent years, they have gained a lot of attention for their potential capabilities in the field of surveillance, reconnaissance and remote sensing, both for civil and military applications. Their success is related to their small size. Not only are they non-invasive and hence able to access areas that are impractical for larger vehicles, but they also present advanced flight-dynamic capabilities. MAVs are able to perform agile and strict maneuvers, significantly outperforming large-scale aerial vehicles in terms of maneuverability. This is because the inertia forces are of the same order of magnitude as the aerodynamic loads, and, consequently, the aerodynamic and the flight dynamic time scales of MAVs are comparable [Shyy et al., 1999]. On the other hand, their reduced size makes MAVs extremely sensitive to external aerodynamic disturbances such as wind gusts. Another consequence of the small scale is the reduction of the Reynolds number: when reducing the scale and the flight speed of an aerial vehicle, the chord-based Reynolds number is also reduced. For conventional wings, the low Reynolds number regime is characterized by reduced aerodynamic performance due to phenomena such as vortex shedding, low lift-to-drag ratio and flow separation [Lissaman, 1983].

The inspiration for the design of MAVs with enhanced aerodynamic performance comes from nature, which provides a large variety of examples of extremely efficient flyers. To provide a general insight on the different aerodynamic scales, Fig. 1.1 shows the approximate chord-based Reynolds number range for common flying species. Despite the significant differences between the innumerable species of insects, birds and bats in terms of scale, flight-dynamics and aerodynamic characteristics [Shyy et al., 1999], all natural flyers present two features in common:

- flapping wing motion,
- flexible wings.
Natural flyers use flapping wings to generate unsteady aerodynamic effects that allow them to improve the unfavorable steady aerodynamic performance caused by the low Reynolds regime. The unsteady aerodynamics generated by the interaction of the wing and the airflow also plays a pivotal role in enabling fast maneuvers and responding to disturbances [Shyy et al., 2013]. This is achieved with wings that, in any case, present some degree of flexibility, from the high-aspect ratio flexible wings of dragonflies [Hefler et al., 2018, Thomas, 2004], to the highly-deformable membrane wings of bats [Elangovan et al., 2007, Swartz et al., 2007]. Hence wing flexibility and wing morphing, combined with flapping flight, play a fundamental role in the generation of lift and thrust, as well as in the rejection of external disturbances [Carruthers et al., 2007, Lentink et al., 2007, van Oorschot et al., 2016, Yu & Guan, 2015].

In particular, within the category of flexible wings, membrane wings offer several benefits for low Reynolds number applications. First of all, because of their membrane structure, they present extremely low thickness (often as low as 0.1% of the wing chord). This directly translates into lower weight compared to thicker flexible wings, given the same wing area and the same material. Lightweight solutions are obviously crucial for the design of energy-efficient MAVs. Second, from an aerodynamic perspective, the high compliance of membrane wings offers desirable improvements over less flexible solutions such as delayed stall and higher lift at high angles of attack [Song et al., 2008]. This is also the conclusion of Waszak et al. [2001], who tested the aerodynamic performance of a membrane wing MAV in a wind tunnel. By reducing the number of battens along the wing span—and hence increasing the compliance and the stretching of the membrane wing—larger lift coefficients are achieved at high incidence, without significantly affecting the performance at lower angles of attack. Finally, from a control perspective, the flexibility of membrane wings and their self-adaptation to changes in the aerodynamic field [Gordnier, 2009, Gordnier & Attar, 2014, Rojratsirikul et al., 2009] can also be exploited for disturbance rejection purposes, in order to improve the flight-dynamic performance of MAVs. Unlike active solutions, passive disturbance rejection does not require sensors and actuators, which will need to be accurately designed and optimized for the fast aerodynamic time-scales typical of the low Reynolds regime. The design of sensors and actuators also has to take into account the strict size and weight requirements of MAVs. Hence, given their structural, aerodynamic and control-related benefits, lightweight highly flexible membrane wings present themselves as appealing solutions for high-efficiency flapping-flight at low Reynolds numbers.
If, on the one hand, the strong interaction between the flexible membrane and the fluid offers several benefits for small-scale aerodynamics, on the other hand it also represents a major challenge for scientists and engineers. The membrane exhibits passive shape adaptation and self-cambering under the effect of the aerodynamic forces generated by the free-stream and, at the same time, the membrane’s unsteady deflection strongly changes the aerodynamic field around the wing surface. The high compliance of membrane wings often triggers vortex shedding and flow separation, making fluid-structure interaction of highly deformable surfaces a challenging and multi-disciplinary topic [Gordnier, 2009]. Experiments [Arbós-Torrent et al., 2013, Rojratsirikul et al., 2009, Song et al., 2008] and high-fidelity simulations [De Matteis & De Socio, 2008, Gordnier, 2009, Gordnier & Attar, 2014, Serrano-Galiano et al., 2018, Shyy et al., 2008, Smith & Shyy, 1995, 1996] can provide reliable and accurate solutions and have been successfully adopted for the study of membrane wing aerodynamics, but they are expensive and time consuming, which generally limits their applicability to specific cases. Therefore, they are not suitable for parametric studies. From an engineering perspective, they often fail to provide general relationships between the structural and kinematic parameters and the aerodynamic and flight-dynamic performance of the wing, which are fundamental for the design of artificial flyers. Hence, there exist a need for simplified low-order models to represent the key aspects of the fluid-structure interaction of membrane wings at low Reynolds numbers.

1.2 Fluid-structure interaction: a coupled framework

Small-scale flapping-flight is characterized by the mutual interaction between a deformable surface, the flexible wing, and an unsteady flow that might exhibit strong nonlinear aerodynamic phenomena, such as vortex-shedding and flow separation. For this reason, numerical simulations and reduced-order models generally rely on the coupling between two different physical models: one for the aerodynamics and one for the flexible wing. In particular, in the present work we focus our attention on a specific class of flexible wings: the membrane wing. We now introduce an overview of the main aerodynamic models present in the literature, followed by an overview of structural and fluid-structure interaction models for membrane wings.

1.2.1 Unsteady aerodynamic modeling

A correct representation of the unsteady aerodynamic loads generated by the interaction of an object with a flow is pivotal in the design and the optimization of a variety of modern engineering applications. Small-scale bio-inspired aerodynamics represents one of the most discussed and challenging examples [Shyy et al., 2010, 1999]. From an engineering design-oriented point of view, an investigation of the mechanism of flapping flight does not only have to understand the underlying
physics, but has to obtain general relations that can be used to design flapping-wing MAVs with desired performance. The goal is to link performance-related aerodynamic parameters of interest such as, for example, lift, drag and thrust, to design variables such as flapping frequency, wing kinematics or wing flexibility. Hence, there is a need for reduced-order aerodynamic models that are able, under certain assumptions, to preserve the accuracy provided by high-fidelity methods in predicting the aerodynamic loads generated by the deflection of a flexible wing, significantly reducing the computational cost.

A comprehensive literature review of the fundamental aerodynamic models for flapping-flight can be found in Ansari et al. [2006], Shyy et al. [1999] and Taha et al. [2012]. Ansari et al. [2006] present an overview of the characteristics of flapping-flight with a focus on the modeling techniques for insect flight aerodynamics. They classify the aerodynamic models into steady, quasi-steady, semi-empirical and unsteady. A similar classification is presented by Taha et al. [2012] and Taha et al. [2014], who also provide a discussion on the complexity of the considered models in terms of degrees of freedom and flow characteristics that they are able to capture. Since many models adopted for flapping-flight and fluid-structure interaction of flexible wings are based on the incompressible thin-airfoil theory, it is worth mentioning the overview of Leishman [2002b] on the modeling of airfoil unsteady aerodynamics for helicopter applications. Here we will provide a summary of some of the key aspects of the models present in the literature.

Steady-state aerodynamic models are simplified models based on momentum theory and actuator disk approximations [Weis-Fogh, 1972] or simplified vortical wakes [Rayner, 1979]. They are derived from first principles and they ignore the unsteady variations of the wake and of the aerodynamic loading of the wing. For this reason, they can only provide a low-fidelity representation of limited aspects of flapping-wing aerodynamics. Quasi-steady models are based on the assumption that the aerodynamic loads on the wing statically depend, at every time instant, on parameters such as the instantaneous velocities or the instantaneous angle of attack of each wing section. Most quasi-steady models are based on two-dimensional aerodynamics and are extended to three-dimensional wings using blade element theory (BET) [Osborne, 1951, Weis-Fogh, 1973]. In BET, a three-dimensional blade is considered composed of two-dimensional wing sections that are aerodynamically independent of each other. Therefore each wing section is considered, from an aerodynamic perspective, as a two-dimensional airfoil with a local angle of attack and a local relative velocity [Leishman, 2002a]. Dickinson et al. [1999] developed a quasi-steady model with semi-empirical corrections based on experiments, that was able to capture the effect of the stable leading edge vortex and of the returning wake on the aerodynamic forces of the wing. This model was extended by Pesavento & Wang [2004], Andersen et al. [2005] and Andersen & Pesavento [2005] to include viscous and added-mass effects. Steady and quasi-steady methods cannot fully represent the unsteady nature of flapping wing aerodynamics and their application to fluid-structure interaction of membrane wings is therefore very limited. For small-scale aerodynamics and, in general, for applications that present large accelerations of a structure in the flow, such as aeroelastic problems, agile maneuvers or gusts response, unsteady aerodynamics effects can dominate and thus a steady
or a quasi-steady model is not able to fully represent the unsteady contributions to the lift given by acceleration-related forces [Brunton et al., 2013]. In this context Tiomkin & Raveh [2017] showed that a quasi-steady aerodynamic model fails to predict the stability envelope of a two-dimensional membrane, compared to an unsteady aerodynamic model. In summary, then, steady and quasi-steady models are unsuitable to fully represent the dynamics of membrane wings and models that are truly unsteady are required.

In the category of unsteady analytical models, the models of Wagner [1925] (indicial response to a step change in the angle of attack) and Theodorsen et al. [1935] (lift and pitching moment response of a two-dimensional rigid flat plate to sinusoidal pitching and plunging) have been widely used in aerodynamic and aeroelastic studies. Both models are derived analytically using the small perturbations theory applied to incompressible, potential flow. Their accuracy is thus limited by the hypothesis of inviscid flow and they are not able to represent viscosity-related effects. Although the models of Wagner and Theodorsen remain a fundamental reference for many unsteady applications, they are not sufficiently high-fidelity to be applied to low Reynolds flapping-flight. This is the case for all linear indicial aerodynamic models based on thin-airfoil theory and potential aerodynamics, of which Wagner’s model represents an example. Indicial models are based on indicial functions, which represent the response of an airfoil to a disturbance (e.g. a change in the angle of attack) in the form of a step. Their success in unsteady aerodynamics [Edwards, 2008, Leishman, 2002b, Taha et al., 2014], aeroelasticity [Bergami et al., 2013, Isogai, 1980, Mikkelsen & Jakobsen, 2017] and fluid-structure interaction [Tiomkin & Raveh, 2017] is due to the fact that they can be generalized to obtain the unsteady response of an airfoil to any disturbance by superposition of individual indicial aerodynamic responses using the Duhamel integral [Leishman, 2002a]. These models can also be obtained from experimental data or high-fidelity numerical simulations, allowing for the representation of limited viscous effects [Brunton & Rowley, 2013, Ghoreyshi & Cummings, 2014]. As noted by Brunton & Rowley [2013], the presence of convolution integrals in the mathematical formulation of indicial models represents a limitation when it comes to integrating these models in modern feedback control frameworks.

With the help of high-fidelity simulations and through a system identification procedure, it is possible to obtain an unsteady aerodynamic model, based on the Theodorsen model, that naturally takes into account aerodynamic loads generated by pitching and plunging of a two-dimensional wing. Such a model, introduced in Brunton et al. [2013] and extended in Brunton et al. [2014] to include a parametrization of the pitching axis location and the contribution of surge, outperforms the Theodorsen model because it is also able to capture the effect of viscosity on the aerodynamic loads, including unsteady transients and the presence of the wake. The model is also low order and in state-space form, hence compatible with a feedback control design framework. The model is linear and it is obtained by linearizing the Navier-Stokes equations about an equilibrium condition, and consequently it does not take into account the contribution from nonlinear effects (such as flow separation or leading-edge vortex shedding) on the aerodynamic loads, limiting its predictive capability to flows that present weak nonlinearities.
1.2.2 Membrane models and fluid-structure interaction

The first fluid-structure interaction analytical studies of two-dimensional inextensible constant-tension membranes (sails) were presented by Thwaites [1961] and Nielsen [1963], who derived an analytical relationship between a sail’s shape, tension and the static aerodynamic loads applied. The aerodynamic loads were calculated using thin-airfoil theory for potential flows and thus are a good approximation for membranes that present small deflections, small angles of attack and thin, fully-attached boundary layers. Vanden-Broeck [1982] extended the study to nonlinear constant-tension membranes. A good overview of early models based on potential flow can be found in Newman [1987]. More refined analytical models for static equilibrium based on potential flows were introduced by Sneyd [1984], Song et al. [2008] and Waldman & Breuer [2017]; such models consider nonlinear extensible elastic membranes (with or without pre-stretch), in which the tension is a function of the membrane elongation and increases with the deflection.

Since the fluid-structure interaction of a membrane is essentially a dynamic problem, a static approach only offers limited insights on the membrane behavior. Song et al. [2008] represented the dynamic behavior of a membrane using a one-dimensional wave equation and obtained the analytical structural natural frequencies of the membrane as a function of the Weber number, defined as the ratio between the membrane lift and the product between the membrane’s modulus and thickness. Comparison with experiments showed that the most energetic vibrational modes of the membrane corresponded to multiples of the natural frequencies predicted by the model. Newman & Paidoussis [1991] presented a dynamic model based on the assumption of standing wave solutions of the velocity potential applied to the dynamic linear membrane equation through the pressure term. Despite different values of the static stability limits compared to previous work from Nielsen [1963]—probably due to a representation of the membrane based only on two modes and the use of approximated solution methods—their method was able to qualitatively address the instability mechanisms due to divergence and flutter. Recently, Tiomkin & Raveh [2017] developed an unsteady analytical model by coupling the linear dynamic membrane equation with unsteady potential flow aerodynamics based on thin-airfoil theory, with the inclusion of an unsteady wake. They showed that the stability depends on two parameters, the tension coefficient and the membrane mass ratio, and they analyzed the membrane stability for different combinations of these two parameters.
1.3 Outline of the present work and contributions

The aim of the present work is to investigate the aeroelastic performance of membrane wings at low Reynolds numbers using reduced-order models. The low-order models for fluid-structure interaction are obtained by coupling a linear unsteady aerodynamic model for low Reynolds number flows with a nonlinear one-dimensional membrane model and its quasi-linear and linear approximations. Here we provide an outline of the individual chapters that form this thesis.

Chapter 2: Linear aerodynamic model
A novel linear unsteady aerodynamic model for the aerodynamic loads generated by the deflection of a flexible wing is presented. The model considers both rigid and flexible degrees of freedom of the wing and it represents an extension of the work of Brunton & Rowley [2013]. We present the mathematical formulation and the system identification procedure to generate the model from high-fidelity simulations for different Reynolds numbers. The present aerodynamic model is integrated in a feedback control loop to actuate the wing in order to track a reference lift and to reject external disturbances. Two publications are associated with this chapter:


• Nardini, M., Schlanderer, S. C., Sandberg, R. D. & Illingworth, S. J. 2016 “Reduced Order Modeling for Feedback Control of a Flexible Wing at Low Reynolds Numbers”, 20th Australasian Fluid Mechanics Conference (AFMC)

Chapter 3: Fluid-structure interaction models:
This chapter contains the formulation of the nonlinear membrane model and the derivation of its quasi-linear and linear approximations, followed by the introduction of nonlinear, quasi-linear and linear fluid-structure interaction models. The fluid-structure interaction models are obtained from the coupling of the aerodynamic model with the membrane models. The coupling procedure is presented in the appendices.

Chapter 4: Static aeroelastic response and linear stability of the membrane
The fluid-structure interaction models are adopted to investigate the steady aeroelastic response of membrane wings at different angles of attack and at Reynolds numbers in the range 100–10,000. The performance of the linear, the quasi-linear and the nonlinear fluid-structure interaction models are discussed and compared to Direct Numerical Simulations. The linear stability of the membrane at zero angle of attack is also investigated. The results have been presented in the following conference proceedings:

Chapter 5: Forced and autonomous response of the membrane
Harmonic balance methods are used to investigate the aeroelastic response of a membrane wing pitching about its leading edge. The autonomous limit-cycle oscillations of linearly unstable membranes are also discussed. Results in this chapter are contained in the following publications:


Chapter 6: Optimization of gust response through the adjoint method
In this chapter, we discuss the disturbance-rejection performance of membrane wings at zero angle of attack and compare them with rigid wings. Membranes with an optimized response to a disturbance in the form of a vortex are generated using an adjoint-based optimization procedure. The adjoint equations of the fluid-structure interaction models are presented, together with the optimization framework based on a quasi-Newton iterative method. The membrane is optimized by controlling two structural parameters, density and stiffness. Results from the optimization are compared to the response of the ideal feedback controllers presented in Chapter 2.
Chapter 2

Linear Aerodynamic Model

A state-space linear reduced-order model for the aerodynamic loads generated by the unsteady deflection of a two-dimensional flexible wing is presented. The framework is obtained via a system identification procedure applied to input-output signals from Direct Numerical Simulation (DNS) performed at low Reynolds numbers. The unsteady prescribed motion of the wing is decomposed into rigid motion, pitching and plunging, and flexible degrees of freedom, represented by the Fourier modes of deflection. Both rigid and flexible degrees of freedom represent the input of the system. The aerodynamic outputs are the lift and the pressure distribution at the wing’s surface. In this chapter, first we introduce the mathematical formulation of the unsteady aerodynamic model and the system identification procedure based on pulse responses from DNS. Then we validate and discuss the performance of the models for the case of the lift generated by a prescribed maneuver, comparing the results with DNS.

A novel state-space model is used to understand how the flexible wing can be actuated in order to achieve good aerodynamic performance in terms of reference lift tracking and disturbance rejection. This is achieved by designing closed-loop controllers for the degrees of freedom of the wing using feedback control theory, especially focusing on rigid pitching and on the first two deflection modes. The main discussion will focus on the performance limitations associated with actuating the wing using the first two deflection modes individually. We will see that the present reduced-order models offer insights on how to overcome these limitations by introducing a multi-input single-output framework and actuating the wing using a combination of modes simultaneously. From a computational perspective, the performance of the controllers is tested by comparing the lift step response from reduced-order model and DNS. A test case with a disturbance in the form of a convected vortex is also used to compare the disturbance rejection capabilities of each controller. All the tests are performed at Reynolds 100 (the controller’s design condition), and are repeated for Reynolds 1000, in order to investigate the robustness of the controllers and their applicability to off-design cases.
2.1 Mathematical formulation and system identification of the aerodynamic model

We present now the mathematical formulation of the extended reduced-order model with the inclusion of the wing’s flexibility, as well as the system identification procedure used to obtain the coefficients of the models from lift pulse responses generated via DNS. We begin by summarizing the modeling procedure introduced by Brunton et al. [2013] for rigid pitching, before extending it to include the deflection of the wing. Even though in this chapter we will mainly focus our attention on the aerodynamic lift, we also present a model that includes the aerodynamic pressure at the wing’s surface as an additional output. This is because it is necessary for the coupling with a membrane model in the fluid-structure interaction framework that will be presented Chapter 3.

2.1.1 Mathematical formulation

2.1.1.1 Unsteady lift from rigid pitching and plunging

To develop a reduced-order model, we follow the modeling procedure presented in Brunton et al. [2013] and Brunton et al. [2014] for pitching and plunging. Details of the procedure, briefly summarized here, can be found in those references, as well as a validation of the model and some discussion concerning its predictive capabilities. We will start by looking at the governing equations for pitching and plunging, before extending the model to take into account the wing’s additional degrees of freedom due to flexibility.

A general model of a physical system can be written as:

\[
\begin{align*}
\dot{x} &= f(x, u, \bar{\mu}) \\
y &= g(x, u, \bar{\mu}),
\end{align*}
\] (2.1)

where \( f \) and \( g \) are generic non-linear functions used to model the physics of the system, \( x \) is the state vector, \( u \) the input vector, \( y \) the output vector and \( \bar{\mu} \) a set of parameters related to the physical properties of the system.

For the specific case of the lift generated by a pitching flat plate, Eq. 2.1 can be expressed as

\[
\begin{align*}
\dot{x} &\triangleq \frac{d}{dt} \begin{bmatrix} \bar{\alpha} \\ \dot{\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} f_{NS}(\bar{x}, \alpha, \dot{\alpha}, \ddot{\alpha}) \\ \dot{\bar{\alpha}} \end{bmatrix} \\
y &= g_{lift}(\bar{x}, \alpha, \dot{\alpha}, \ddot{\alpha}).
\end{align*}
\] (2.2)

\( \alpha, \dot{\alpha} \) and \( \ddot{\alpha} \) represent the angle of attack and its first and second time derivatives, respectively, and \( f_{NS} \) and \( g_{lift} \) are functions related to the non-linear Navier-Stokes equations. We will use the superscript dot to indicate a time derivative.
Brunton introduced a state-space reduced-order model based on the linearization of the two-dimensional, unsteady, incompressible Navier-Stokes equations around an equilibrium condition for a rigid flat plate. The equilibrium condition is defined by the base angle of attack \( \alpha_0 \) and by the choice of the parameters \( \bar{\mu} \), which include the Reynolds number and the pitching axis location.

The state-space mathematical form of the model representing the lift generated by the unsteady motion of a pitching wing is presented in Eq. 2.3. The input to the model is the pitching acceleration of the wing \( \ddot{\alpha}(t) \). This choice reflects the need to retain the explicit dependency of the aerodynamic lift on the acceleration (added-mass effect) [Brunton et al., 2014, Theodorsen et al., 1935]. This dependency is here modeled by the two terms \( B^x_\alpha \ddot{\alpha} \) and \( C^L_\alpha \ddot{\alpha} \) present in Eq. 2.3. As a natural consequence of this choice, \( \alpha \) and \( \dot{\alpha} \) appear in the model as state-variables. If \( \ddot{\alpha} \) was a state-variable instead of the input, we would have to express its time derivative \( \dot{\ddot{\alpha}} \) as a function of \( \bar{\mu}, \alpha \) and \( \dot{\alpha} \), which is improper and would require an approximation. This would result in a large number of states required to represent the system dynamics. Brunton et al. [2014] show that a model that has \( \dot{\alpha} \) as the input fails to capture the unsteady acceleration-related aerodynamic effects, which give a non-negligible contribution to the lift during, for example, rapid maneuvers.

The output chosen for the present work is the wing lift, but the modeling framework can be easily generalized to include additional outputs such as drag or pitching moment. We will present an extended formulation of the model to take into account the pressure distribution around the wing in Section 2.1.1.3. In a linear state-space form, Eq. 2.2 becomes

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \alpha \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} A^x_\alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \alpha \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} B^x_\alpha \\ 0 \\ 1 \end{bmatrix} \ddot{\alpha} \\
C_L = \begin{bmatrix} C^L_\alpha & C^L_\dot{\alpha} & C^L_{\ddot{\alpha}} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \alpha \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} C^L_\alpha \ddot{\alpha} \end{bmatrix},
\]

(2.3)

where the state \( \bar{x} \) is related to the velocity of the fluid and the output of the model is the lift coefficient \( C_L \) of the two-dimensional wing section. By definition \( C_L = 2L/\rho U^2c \), where \( L \) is the lift, \( \rho \) the fluid density of the undisturbed flow, \( U \) the free-stream velocity and \( c \) the wing section chord. \( C^L_\alpha \) is defined as \( \partial C_L/\partial \alpha \) and similarly \( C^L_\dot{\alpha} = \partial C_L/\partial \dot{\alpha} \) and \( C^L_{\ddot{\alpha}} = \partial C_L/\partial \ddot{\alpha} \). The following definitions apply: \( A^x_\alpha = \partial f_{NS}/\partial \bar{x}, B^x_\alpha = \partial f_{NS}/\partial \dot{\alpha} \) and \( C^L_\alpha = \partial g_{lift}/\partial \bar{x} \). From Eq. 2.2, a model for plunging can also be obtained by simply changing the input of the system from pitching acceleration to plunging acceleration [Brunton et al., 2013]. Pitching and plunging can also be combined into a multi-input single-output system [Brunton et al., 2014].
2.1.1.2 Contribution to the lift of the flexible degrees of freedom

The procedure introduced by Brunton and summarized in Section 2.1.1.1 is now extended to take into account the deflection of a two-dimensional flexible wing. The deflection of the wing $\bar{w}(\bar{x}, \bar{t}) = w(\bar{x}, \bar{t})/c$, defined as the displacement of the wing mean line in the direction perpendicular to the chord, is normalized with respect to the wing’s chord $c$. $\bar{x} = x/c$ and $\bar{t} = tU/c$ represent the dimensionless abscissa in the direction of the wing’s chord and the dimensionless time (also referred to as convective time), respectively. From now on, the overbar will be omitted and $x$ and $w(x,t)$ are taken to be nondimensional. In order to reduce the degrees of freedom of the wing, the deflection $w(x,t)$ is expressed using a truncated Fourier series as:

$$w(x,t) = \sum_{k=1}^{N} W_k(t) \cdot \sin (k\pi x), \text{ for } x \in [0,1], \quad (2.4)$$

where the subscript $k$ indicates the deflection mode, $W_k(t)$ the amplitude of the $k$-th Fourier mode and $N$ the number of Fourier modes taken into account. A decomposition based on sinusoidal Fourier modes will be justified in Chapter 3, where the aerodynamic model is coupled to a structural model representing a membrane pinned at its two ends. However, at this stage no assumptions are made about the shape of the wing and the decomposition in Fourier modes (Eq. 2.4) is applied to the mean line. Figure 2.1 shows an example of a generic deflection of the mean line of the wing and the first four Fourier modes. In analogy with Eq. 2.3, for each deflection mode $k$, the following coefficients can be defined: $C_{Lk}^{W} = \frac{\partial C_L}{\partial W_k}$, $C_{\dot{L}}^{W} = \frac{\partial C_L}{\partial \dot{W}_k}$ and $C_{\ddot{L}}^{W} = \frac{\partial C_L}{\partial \ddot{W}_k}$. Consequently, the following
Linear Aerodynamic Model

relations hold:

\[
\begin{bmatrix}
\frac{d}{dt} x_k \\
\frac{d}{dt} W_k \\
\end{bmatrix} =
\begin{bmatrix}
A_{x_k}^{z_f} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_k \\
W_k \\
\end{bmatrix}
+ \begin{bmatrix}
B_{W_k}^{z_f} \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{W}_k \\
\ddot{W}_k \\
\end{bmatrix}
\]

(2.5)

\[
C^L_k = [C^L_{x_k} C^L_{\alpha W} C^L_{W} C^L_{\dot{W}}] \begin{bmatrix}
x_k \\
W_k \\
\end{bmatrix}
+ [C^L_{W} C^L_{\dot{W}}] \begin{bmatrix}
\ddot{W}_k \\
\dddot{W}_k \\
\end{bmatrix}
\]

(2.6)

\(C^k_L\) represents the lift generated by a deflection in the \(k\)-th mode, while \(A_{x_k}^{z_f}, B_{W_k}^{z_f}\) and \(C^L_{x_k}\) follow from the analogy with Eq. 2.3. The global lift due to all considered modes is then given by simply adding together the contributions from each mode (i.e. simple superposition). The contributions of the rigid degrees of freedom can also be included. When the only rigid degree of freedom considered is pitching (an extension to include plunging is straightforward), we have

\[
\begin{bmatrix}
\bar{x} \\
\alpha \\
\dot{\alpha} \\
\end{bmatrix}
= \begin{bmatrix}
\alpha & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\alpha \\
\dot{\alpha} \\
\end{bmatrix}
+ \begin{bmatrix}
B_{\alpha}^{z_f} & B_{\dot{W}}^{z_f} \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & I \\
\end{bmatrix}
\begin{bmatrix}
\ddot{\alpha} \\
\dddot{\alpha} \\
\end{bmatrix}
\]

(2.6)

\[
[C^L] = [C^L_{x} C^L_{\alpha} C^L_{\dot{W}} C^L_{\ddot{W}}] \begin{bmatrix}
\bar{x} \\
\alpha \\
\dot{\alpha} \\
\end{bmatrix}
+ [C^L_{W} C^L_{\dot{W}}] \begin{bmatrix}
\ddot{W}_k \\
\dddot{W}_k \\
\end{bmatrix}
\]

The vector containing deflection modes from 1 to \(N\) is indicated with \(W\), while the deflection velocity and the deflection acceleration are expressed as \(\dot{W}\) and \(\ddot{W}\), respectively. Following from Eq. 2.5, matrices \(C^L_{W} \), \(C^L_{\dot{W}} \), and \(C^L_{\ddot{W}} \) contain the derivatives of \(C^L\) with respect to the deflection modes \(W_1\) to \(W_N\) and their corresponding time derivatives. Similarly, \(B_{W_k}^{z_f}\) contains matrices \(B_{\ddot{W}}^{z_f}\) from Eq. 2.5, while the \(A^{z_f}_x\) and \(C^{z_f}_x\) matrices contain the \(A^{z_f}_x\) and \(C^{z_f}_x\) terms introduced in Eq. 2.3 and Eq. 2.5. A model in the form of Eq. 2.6 represents the lift generated by a wing that is actuated in its rigid (pitching) and flexible degrees of freedom (deflection of the mean line), as shown in Fig. 2.2.

2.1.1.3 Additional outputs: pressure distribution on the wing

The versatility of the present formulation allows the introduction of additional outputs to the aerodynamic model. Since the aim of this work is to couple to aerodynamic model to a membrane structural model, we want to include the pressure distribution on the wing surface caused by the rigid and flexible motion of the wing. We introduce now the assumption that the wing’s thickness is small and we define the pressure difference between lower and upper surface as
\( \Delta p(x, t) = p^{\text{low}} - p^{\text{up}} \), as shown in Fig. 2.3. Consequently, we indicate the pressure coefficient as \( \Delta C_p = 2\Delta p / \rho U^2 \). Adopting the same modal decomposition introduced for the deflection (see Eq. 2.4), the pressure modes are determined by the following truncated Fourier series:

\[
\Delta C_p(x, t) = \sum_{k=1}^{N} \Delta C_{pk}(t) \cdot \sin \left( k \pi x \right). \tag{2.7}
\]

From now on, the vector containing the modal pressure coefficients from 1 to \( N \) will be indicated with \( \Delta C_p \).

The multi-input single-output model expressed in Eq. 2.6 can be generalized into a multi-input multi-output model with the acceleration of the degrees of freedom.
of the wing as the input and lift coefficient and modes of pressure as the output:

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \alpha \\ \dot{\alpha} \\ W \\ \dot{W} \end{bmatrix} = \begin{bmatrix} A_x^f & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \alpha \\ \dot{\alpha} \\ W \\ \dot{W} \end{bmatrix} + \begin{bmatrix} B_{\alpha}^f & B_{W}^f \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{W} \end{bmatrix}
\]

This state-space system represents a multi-input multi-output framework with pitching acceleration and \( N \) modal coefficients for the acceleration of the deflection as inputs and \( C_L \) and \( N \) modal coefficients for the pressure distribution as outputs. In analogy with the lift coefficient, \( C_{p}^{p_f} \), \( C_{p}^{p_f} \), \( C_{p}^{p_f} \) represent the derivatives of the pressure modes \( \Delta C_p \) with respect to \( \alpha \), \( \dot{\alpha} \) and \( \ddot{\alpha} \) respectively, while \( C_{W}^{p_f} \), \( C_{W}^{p_f} \) and \( C_{W}^{p_f} \) are matrices containing the derivatives of \( \Delta C_p \) with respect to the deflection modes \( W_1 \) to \( W_N \) and their time derivatives. Finally, \( A_x^f \), \( B_{\alpha}^f \), \( B_{W}^f \), \( C_x^f \) and \( C_{x}^{p_f} \) are system matrices that will be obtained using system identification and model-reduction techniques.

### 2.1.2 System identification procedure

The general state-space reduced-order models introduced in Section 2.1.1 can be applied to a specific wing-flow configuration by identifying the unknown coefficients for the specific model of interest. This is achieved via system identification, a methodology adopted to obtain mathematical models of complex systems, using input-output measurements from experiments or numerical simulations. Due to the complexity of the non-linear Navier-Stokes equations, system identification techniques are used in combination with Direct Numerical Simulations to find an approximation of the lift and the surface pressure distribution generated by the unsteady deflection of a two-dimensional wing, in the form of the state-space models presented in the previous section. The matrices and coefficients of the models depend on parameters such as the wing shape, the Reynolds number and the equilibrium condition chosen to linearize the Navier-Stokes equations.

When lift is the output, the aim of the procedure is to identify the values of the quasi-steady and added-mass force coefficients \( C_{W_k}^{L_f} \), \( C_{W_k}^{L_f} \) and \( C_{W_k}^{L_f} \) (or \( C_{\alpha}^{L_f} \), \( C_{\alpha}^{L_f} \) and \( C_{\alpha}^{L_f} \) when pitching is considered) from input-output data obtained from DNS and to find a low-order approximation of the matrices \( A_{x}^{f} \), \( B_{W_k}^{f} \), \( B_{\alpha}^{f} \) and \( C_{x}^{p_f} \) representing the system’s dynamics. Analogous considerations apply when pressure is taken into account.
The present system identification procedure is based on lift (or pressure) pulse responses simulated with DNS. A pulse represents an input signal aimed to produce an output of the system that is rich in frequency. If the amplitude of the pulse is small enough to ensure that nonlinearities in the response are weak, we can then apply system identification and model-reduction techniques to obtain a linear low-order realization of the system’s frequency response. The pulse response-based system identification procedure adopted in the present work is now introduced.

A deflection velocity pulse for every deflection mode \( k \) and for rigid pitching is applied to the wing and the lift (pressure) responses are obtained from DNS (see Section 2.2). It is important to remark that the input to our model is the acceleration of each deflection mode or the pitching acceleration, hence we should have simulated an acceleration pulse instead of a velocity pulse. However, an acceleration pulse results in a linearly increasing deflection (or angle of attack), that is not suitable for linear system identification because it quickly excites strong nonlinear effects in the response [Brunton et al., 2014]. For this reason, velocity pulses are simulated and the corresponding lift (pressure) pulse responses are later manipulated to obtain stable acceleration-based pulse responses.

When lift is considered, the lift coefficient for the steady-state lift from each case \( (C_{LW}^{f}) \) or \( (C_{L\alpha}^{f}) \) from Eq. 2.8) is identified and subtracted from the velocity-based pulse responses simulated via DNS. As previously discussed, since the input to the model is the acceleration of the deflection (or the pitching acceleration), the lift signals obtained from an impulsive velocity must be integrated in order to obtain the lift from an impulsive acceleration, and hence accurately capture the unsteady acceleration-related features of the flow [Brunton et al., 2014]. Having removed the steady-state lift coefficient from the velocity-based pulse response ensures that the acceleration-based pulse response obtained from the integration will produce a stable realization of the system dynamics. Again, the steady-state lift coefficients \( C_{LW}^{f} \) or \( C_{L\alpha}^{f} \) are identified and their contribution is subtracted from the pulse responses. A similar procedure is applied when pressure is considered: pulse responses for each pressure mode are obtained from DNS and each signal is integrated in time after subtracting the steady-state value of the response.

Once the contribution of the deflection velocity (or pitching velocity) is identified and subtracted from the integrated pulse responses, the remaining signals are used to derive a low order approximation of \( A_{x}^{f} \), \( B_{Wk}^{f} \), \( B_{\alpha}^{f} \) and \( C_{x}^{f} \) (or \( C_{p}^{f} \)). This is achieved using the Eigensystem Realization Algorithm (ERA), introduced in Juang & Pappa [1985].

Considering the vector \( \mathbf{Y} = [Y(0) \ Y(1) \ Y(2) \ldots \ Y(i)] \) containing the unit pulse response samples of a single-input single-output discrete-time system (also called Markov Parameters), and considering a generic system in classical state-space form

\[
\dot{x} = Ax + Bu \\
y = Cx + Du,
\]
then \( Y \) can be expressed as
\[
Y(0) = D \\
Y(1) = CB \\
Y(2) = CAB \\
Y(3) = CA^2B \\
\vdots \\
Y(i) = CA^{i-1}B.
\]
Hence, given the Markov parameters, it is possible to obtain a realization of the state-space system matrices \( A, B, C \) and \( D \) describing the system dynamics. The ERA system identification procedure involves forming a block Hankel matrix of Markov Parameters that can be decomposed using a Singular Value Decomposition (SVD). The SVD expansion can be truncated to retain the \( n \) most energetic singular values (model reduction). From the truncated SVD, after some algebraic manipulation of the resulting matrices (known in the literature as controllability and observability matrices of the system), a balanced low-order realization of the system matrices \( A, B, C \) and \( D \) can be extracted [Juang & Phan, 2001]. The procedure can be easily generalized to take into account multi-input multi-output systems [Juang & Phan, 2001]. The ERA has been extensively used for feedback control in numerous fluid-dynamic applications, as reported in Brunton et al. [2014], Flinois & Morgans [2016], Illingworth et al. [2011] and Illingworth et al. [2012]. The model obtained using the ERA is equivalent to a model obtained by Balanced Proper Orthogonal Decomposition, as demonstrated in Ma et al. [2011]. The entire identification procedure for a pitching plate is explained in detail in Brunton et al. [2013], while a summary of the identification of the extended model introduced in this work is presented in Fig. 2.4.

It is interesting to notice that, since the input to the system identification procedure is the lift (pressure) pulse response, this can be obtained not only with DNS, but also with other computational methods (i.e. RANS, LES or panel methods) or with experiments, making the model applicable to a large variety of unsteady applications, not just limited to small-scale aerodynamics.

### 2.1.3 Effect of the Mach Number

The procedure introduced in Brunton et al. [2013] is based on pulse responses obtained using an incompressible code. In the present work, a compressible code is used, introducing the Mach number \( M \) as an additional parameter included in \( \bar{\mu} \) (see Eq. 2.1). Hence the reduced order model also depends on the Mach number.

Compressibility has an effect on the pulse response (and, consequently, on the identified model), introducing a phase lag between the acceleration (input) and the high-frequency lift response. In a compressible formulation there is no direct
Figure 2.4: Summary of the system identification procedure used to obtain the linear aerodynamic model in state-space form from DNS.

proportionality between acceleration and lift, as is the case in an incompressible formulation [Jaiman et al., 2013]. However, the models presented in Secton 2.1.1 can also be applied to compressible flows: the main difference is that in the compressible formulation the high-frequency added-mass, instead of being captured by the coefficients $C_{\alpha}^{L_f}$ and $C_{\omega}^{L_f}$, is included in the model as a transient. A more detailed discussion about compressibility effects will be presented in Section 2.3.2.
2.2 Direct Numerical Simulation lift pulse Responses for model identification

Simulations to obtain the input-output relationship necessary for system identification are performed using DNS. For the sake of brevity, we will only show results and validations for the lift response in the present chapter, but the same simulation setup has been used to obtain pulse responses for the pressure modes as well. We will provide extensive indirect validations for the full aerodynamic model with pressure (see Eq. 2.8) when coupled to the membrane solver in Chapters 4 and 5.

Direct Numerical Simulations are carried out using HiPSTAR, a well-validated DNS solver for compressible, viscous flow. The geometry of the flexible wing immersed in the flow and the consequent no-slip boundary condition are represented by a Boundary Data Immersion Method (BDIM) originally proposed by Weymouth & Yue [2011] for incompressible flows and recently extended to compressible flows by Schlanderer & Sandberg [2015]. The DNS code and the BDIM method have been rigorously validated on a number of example problems, including the two-dimensional flow around a cylinder at $Re = 100$, aeroacoustic problems such as noise radiation from a transversely oscillating cylinder [Schlanderer et al., 2017] and the flow around a membrane at low and moderate Reynolds numbers [Serrano Galiano & Sandberg, 2015].

2.2.1 Simulation setup

For the present study, a two-dimensional flexible wing of unit length ($c = 1$) and constant 0.75% thickness is immersed in a uniform free-stream at zero angle of attack and $Re = 100$. The Reynolds number is defined as $Re = Uc/\nu$, where $U$ is the free stream velocity, $c$ is the chord length of the wing and $\nu$ the kinematic viscosity of the fluid. For each simulation, the initial condition at time $t = 0$ is the steady-state solution with the wing’s wake and boundary layer fully developed. The Mach number used for the simulations is $M = 0.2$. The effect of the Mach number will be discussed in detail in Section 2.3.2.

A grid convergence study has been performed in order to determine the size of the fluid domain and the grid resolution that guarantees grid-independent results. The grid is an Eulerian Cartesian grid of dimensions $120c \times 120c$, with a resolution of $1,285 \times 1,031$ grid cells. The grid is refined in the vicinity of the wing and the minimum grid spacing in both directions is $\Delta \tilde{x} = \Delta \tilde{y} = 0.75 \times 10^{-3}$. Axis $\tilde{x}$ and $\tilde{y}$ are defined as in Fig. 2.5. In the $\tilde{x}$-direction, the grid is refined towards the leading and trailing edges. The wing is represented by 200 solid-body points equally distributed along the wing chord: 100 on the upper surface and 100 on the lower surface. The leading edge and the trailing edge are modeled as sharp edges. The solid-body points represent the capturing locations for pressure and viscous forces necessary for the evaluation of the global lift. Additional simulations performed with round edges did not show significant differences in the global measured lift. A schematic representation of the computational domain is shown in Fig. 2.5.
2.2.2 Lift pulse response

Since the reduced-order modeling procedure is based on lift pulse responses, a pulse in the deflection velocity of the wing according to one of the deflection modes is prescribed in the DNS and the lift response is obtained. We will mostly focus on responses to deflections in mode 1 and mode 2 (see Eq. 2.4 and Fig. 2.1), because we will see that they qualitatively represent the response of the system to deflections in any odd mode and even mode, respectively. Rigid pitching about the leading edge is considered as well, in order to provide a comparison with the response from the flexible degrees of freedom.

2.2.2.1 Linear ramp maneuver

A pulse in the deflection velocity (or in angular velocity, when pitching is considered) represents a step change in the deflection (angle of attack). In the DNS, following [Brunton & Rowley, 2013], a step change is approximated by the linear ramp maneuver derived from the canonical ramp-up, hold, ramp-down maneuver introduced by the AIAA Fluid Dynamics Technical Committee (FDTC) low Reynolds number discussion group [Ol et al., 2010] (see Appendix A.1 and A.2 for more details). The linear ramp maneuver is represented by the following expression:

\[
W_k(t) = W_{k}^{\text{max}} \frac{G(t)}{\max(G(t))}
\]

\[
G(t) = \log \left[ \frac{\cosh(a(t - t_1)) \cosh(-a(t_2))}{\cosh(a(t - t_2)) \cosh(-a(t_1))} \right].
\]  \hspace{1cm} (2.10)

The maneuver is shown in Fig. 2.6. \(W_k(t)\) represents the temporal component of the deflection according to the considered mode \(k\) (as shown in Eq. 2.4), \(W_k^{\text{max}}\)
is the maximum value of the deflection reached during the maneuver, \( t_1 \) and \( t_2 \) represent the beginning and ending time of the ramp-up maneuver and \( a \) is a parameter that determines the steepness of the velocity gradient of the maneuver. For pitching, \( W_k(t) \) and \( W_k^{max} \) can be simply replaced by the angle of attack \( \alpha(t) \) and the value of the maximum angle of attack \( \alpha^{max} \), respectively.

The duration of the pulse is \( t_2 - t_1 = 0.001 \), which is sufficient to identify the model at high frequencies. The parameter \( a \) is set to 1000 and the maximum deflection is arbitrarily set to \( W_1^{max} = W_2^{max} = 0.0017453 \), where the subscripts 1 and 2 refer to mode 1 and 2 respectively. For pitching, the same settings are used and \( \alpha^{max} = W_k^{max} \); the angle of attack is measured in radians.

### 2.2.2.2 Mode 1 and 2

DNS is used to generate the lift pulse responses necessary to obtain the reduced-order model. The responses of the lift to a deflection velocity pulse in mode 1 and
mode 2 are shown in Fig. 2.7. Different scales are adopted to adequately represent the characteristics of the lift responses. The initial part of the response (corresponding to high frequencies in the frequency domain) is related to the acceleration of the deflection, which generates pressure waves traveling at the speed of sound. The time-scale of the first part of the response is strictly related to the duration of the pulse. The fast acceleration-related response is followed by a slower transient, before the system reaches a steady-state. The pulse responses shown in Fig. 2.7 are used to generate a reduced-order model in the form of Eq. 2.5 for each mode.

The fast dynamics of a deflection in mode 1 produce a significant peak in the lift response: a positive deflection causes higher pressure on the wing upper surface and lower pressure on the lower surface, resulting in a global lift that has opposite sign to the deflection itself. The initial lift response is therefore opposite in sign to the steady-state lift. This inverse response will lead to limitations in the performance of a closed-loop controller, as we will discuss in Section 2.4.

Mode 2 generates an initial pulse response orders of magnitude smaller than that of mode 1. This is because the anti-symmetrical shape of the deflection generates higher pressure and lower pressure regions equally on both upper and lower surfaces, resulting in a global lift close to zero. This behavior is of fundamental importance from a control point of view: the smaller initial response of mode 2 indicates a time delay, which also poses limitations on performance of any feedback controller designed for the system. This is because the actuation has an effect only after waiting a time $\tau$.

### 2.2.2.3 Additional modes

Lift pulse responses have also been obtained for higher Fourier modes. The pulse responses for odd modes from 1 to 9 are compared in Fig. 2.8a-c. All odd modes exhibit the same inverse response behavior discussed for mode 1, due to the shape of the deflection that generates higher pressure values on the upper surface and lower values on the lower surface in the first time instants of the response.

A similar investigation is performed for even modes from 2 to 10; the results are represented in Fig. 2.8d-e. The initial response for all the even modes is negligible compared to the scale of the response of the odd modes, because of the anti-symmetrical nature of the deflection. This behavior, already discussed for mode 2, indicates a time-delay in the lift response.

For both even and odd modes, as the mode number increases, the magnitude of the pulse response decreases. From an engineering point of view, this justifies the approach of using a truncated Fourier series (and thus a limited number of modes) to represent the wing shape, since the contribution of higher modes becomes increasingly negligible. For a feedback control investigation, mode 1 will be taken as representative of odd modes and mode 2 will be taken as representative of even modes. All further discussion about controllers and their performance applied to
Figure 2.8: (a-b-c) Lift pulse responses for odd modes from 1 to 9 (black to grey). All the responses present the same inverse response behavior. The magnitude of the lift decreases as the mode number increases. (a) initial acceleration-related lift. (b) transient. (c) steady-state. (d-e-f) Lift pulse responses for even modes from 2 to 10 (black to grey). (d) the negligible acceleration-related response represents a time delay between input and output. (e) transient. (f) steady-state.

mode 1 can be easily extended to all the other odd modes. Similarly, the discussion to follow concerning control performance for mode 2 can be applied to all other even modes. This will simplify the discussion without losing the generality of the approach.

2.2.2.4 Pitching

It is interesting to analyze the characteristics of the pulse response for rigid pitching, in order to use pitching as a reference to compare the performance of the flexible degrees of freedom of the wing. The lift pulse response for pitching about the leading edge is shown in Fig. 2.9. The characteristics of the pulse response are the same as those observed for mode 1 and 2: an initial acceleration-related response, followed by a transient and a steady-state. Pitching about the leading edge does not suffer from the limitations that affect mode 1 and mode 2 (inverse response and time delay), because the initial response is not negligible with respect to the other parts of the response and because it has the same sign of the steady-state lift. However, this behavior only applies to the considered pitching axis position. Changing the pitching axis location along the wing chord will change the properties of the system. Intuitively, we can imagine that the system will exhibit a time-delay when the wing is pitching about mid-chord (negligible contribution to lift of the added-mass due to symmetry), while it will present an inverse response
if the pitching location is downstream of the mid-chord point. This behavior is discussed in Brunton et al. [2013] observing the Bode plots for Theodorsen’s model. For the present work, only pitching about the leading edge will be considered.

2.3 System identification results and model validation

2.3.1 Validation

The lift pulse responses for mode 1, mode 2 and pitching are used to obtain reduced-order models for the lift generated by the unsteady deflection of the wing. As already mentioned, the models are linear and in state-space form. In order to test their accuracy, the frequency behavior of the reduced-order models is compared to the frequency behavior measured directly using DNS. Numerical simulations were performed with the flexible wing deflecting at prescribed frequencies using a sinusoidal function to represent the acceleration (input). Deflections in mode 1, mode 2 and pitching about the leading edge are considered for the validation. The magnitude of the ratio between lift (output) and acceleration (input) and the phase difference are evaluated and compared against the Bode plot from the reduced-order models. Results for pitching about the leading edge are shown in Fig. 2.10a, while Fig. 2.10b represents the validation for mode 1 and Fig. 2.10c represents mode 2. Frequencies in the Bode plots are normalized with the wing chord and the free-stream velocity. For the three systems, the model and the DNS are in excellent agreement.

A validation in the time domain for the lift generated by rigid actuation in pitching combined with a generic deflection involving modes 1 to 3 at $Re = 100$ is shown in Fig. 2.11. Details on the actuation of the wing, prescribed using the AIAA canonical ramp-up, hold, ramp down function from Appendix A.1, are summarized in Table 2.1.
2.3.2 Compressibility effects on the lift response and on the reduced-order models

For steady and quasi-steady simulations, flows with low Mach numbers ($M << 1$) can be treated as if they were incompressible. From an engineering point of
view, the value $M = 0.3$ is usually considered a reasonable threshold for many applications [Buresti, 2012]. This is not the case for unsteady flows, that might show significant compressibility effects if the frequencies present in the flow (and responsible for the generation of lift) are high, relative to the speed of sound, even if the Mach number is low. Considering an oscillating flat plate with reduced frequency $\bar{\omega} = \omega c / U$, where $\omega$ is the frequency of the oscillations, the assumption of incompressible flow can be justified only if, in addition to $M << 1$, $M \omega r << 1$ also holds [Leishman, 2002a]. From now on, the overbar will be omitted and the reduced frequency will be indicated as $\omega$.

For most practical low Reynolds number flapping-flight applications, the combination of Mach number and reduced frequencies present in the flow is such that the flow can be modeled as fully incompressible. When simulating low Mach number flows with a compressible solver, the Navier-Stokes equations become stiff, resulting in an increase of the computational resources necessary to accurately resolve the flow time scales [Freziger & Peric, 2002]. Reaching the incompressible limit for unsteady simulations using a compressible solver is therefore not computationally feasible.

The aim of the present work is not to target a specific application, but to provide a general reduced-order modeling procedure for the lift generated by a flexible wing that can be applied to different flow regimes. It is therefore important to investigate the effect of the Mach number on the lift response and whether the model is able to capture these effects or not. Pitching about the leading edge will be taken as an example and discussed, but the same considerations can be applied to all the other degrees of freedom applicable in the flexible wing configuration.

Following the results already presented in Section 2.2 for $M = 0.2$, here we use a series of lift pulse responses to generate reduced-order models at different
Mach numbers. The Bode plots for each reduced-order model are represented in Fig. 2.12. The plot also shows the Bode plot for the reduced-order model obtained by Brunton using an incompressible DNS solver [Brunton et al., 2013]. From Fig. 2.12 we notice that compressibility effects change the amplitude and phase of the lift response to an unsteady motion, compared to the incompressible case. In a compressible formulation, pressure waves generated by an unsteady maneuver are traveling at a finite speed (speed of sound); therefore the initial response of the lift to an acceleration of the body strongly depends on the Mach number, because it is due to the pressure waves generated by the acceleration [Leishman, 2002b]. Consequently, compressibility is a high-frequency effect, because it is an acceleration related effect: it affects the mid and high-frequency response, but has little influence on the low frequencies. This can be observed looking at the Bode plots in Fig. 2.12: at low frequencies, the lines for different Mach numbers collapse onto each other, while at mid-frequencies, compressibility effects start to show up, affecting both the magnitude and the phase of the response. As the Mach number increases, the frequency at which the compressibility effects appear decreases. Reducing the Mach number, the frequency behavior of the reduced-order model tends to approximate the incompressible solution.

It is important to notice that although compressibility effects change the magnitude and the phase of the lift response, they do not alter the characteristics of the response, as suggested also by the Bode plots in Fig. 2.13 for mode 1 and mode 2 obtained at different Mach numbers. Hence, all the considerations in terms of
inverse response and time delay discussed for $M = 0.2$, can be generalized to other Mach numbers and are still present in the incompressible limit.

There are no differences in the modeling procedure when using an incompressible or a compressible DNS solver. The main difference in the results is the identification of the high-frequencies. In an incompressible formulation, the high-frequency response is directly proportional to the acceleration; this can also be inferred by the high-frequency phase difference between acceleration and lift in Brunton’s model for the pitching case (Fig. 2.12), that tends to 0 degrees. In a compressible formulation, direct proportionality between acceleration and lift no longer holds because the pressure waves generated by the wing’s acceleration travel at a finite speed [Jaiman et al., 2013]. This is also indicated by the Bode plot in Fig. 2.12: for the compressible cases, the slope of the magnitude curve at high frequency is approximately $-20 \text{ dB/decade}$ and the phase plot is close to $-90$ degrees. Although the high-frequency trend of the Bode plot in Fig. 2.12 resembles the behavior of an integrator, it is unclear if this behavior continues to hold at higher frequencies, or if it is a localized (in frequency) effect. What we can conclude from the plot is that compressibility introduces a frequency dependence in the high-frequency response of the wing. In a compressible formulation, a model in the form of Eq. 2.5 can still be used, with the only difference that the main portion of the high-frequency response will be captured as a transient and included in matrix $C^{L_f}_{x_k}$ and not in the term $C^{L_f}_a$ (or $C^{L_f}_{W_k}$ in case of deflection). For this reason, the duration of the pulse and the modeling procedure pose a limit on the highest frequency that can be accurately identified by the model. It is then important to ensure that the pulse duration is short enough to guarantee that those frequencies, that are not captured by the model, correspond to a small magnitude in the Bode plot, such that their effect on the lift can be neglected.
Hence, the reduced-order modeling procedure has been proven to be applicable, without any alterations in the main formulation, to compressible and incompressible flows. Although it has been shown that compressibility effects cannot be neglected when the product between the Mach number and the reduced frequency of the oscillations responsible of the lift generation is not small, it is also true that they do not affect the qualitative behavior of the models. All the considerations made in the present work for \( M = 0.2 \), in terms of system behavior and controller design (presented in the next sections), can be easily generalized to different Mach numbers and are also applicable to the incompressible case, not investigated in the present work due to computational resource limits.

### 2.4 Lift feedback control using rigid and flexible degrees of freedom of the wing

#### 2.4.1 Introduction

Reduced-order models for pitching and deflection in mode 1 and mode 2 are now used to design robust feedback controllers for the lift generated by the wing. Robust controllers are designed using \( H_\infty \) loop-shaping techniques [Skogestad & Postlethwaite, 2005], based on specifying the shape of the open-loop transfer function in frequency, in order to obtain desired controller performance. In particular, low-frequency tracking of a reference lift (necessary, as an example, to perform a specific maneuver) and rejection of external disturbances (such as, for example, a convecting vortex or a gust of wind), as well as high-frequency noise attenuation (typically from sensors), are the main goals of the design process. We will see that the inverse response of mode 1 and the time delay of mode 2 will pose performance limitations on the controllers, affecting the time scales and the shape of the lift response, and that the limitations present for modes 1 and 2 individually can be overcome by combining them using a multi-input approach.

![Feedback control scheme](image)

**Figure 2.14:** Feedback control scheme. A controller is designed to actuate the wing in order to track a prescribed reference lift.

The feedback control framework is described in Fig. 2.14. The controller actuates the wing by generating an input to the unsteady aerodynamic framework, represented by the reduced-order model or the DNS framework. The actuation
depends on the controller chosen and on the difference between the prescribed lift signal $C_{L}^{ref}$ and the measured lift $C_L$. To test the performance of the controller, the response to a step input in the lift is used as a test case. The results of the controllers applied to the linear reduced-order models and to the DNS are compared and discussed. Additionally, the Navier-Stokes equations present nonlinearities that are not modeled by the reduced-order models; the robustness of the controllers is of fundamental importance to ensure that the performance of the controllers is maintained even in the presence of unmodeled phenomena. We will test the robustness of the controllers by comparing the results of step changes in the lift of different magnitudes from DNS and reduced-order model. Finally, the performance of the controllers is tested adding to the flow a disturbance in the form of a convective vortex simulated through DNS and requiring the controllers to keep the lift constant.

2.4.2 Step response

2.4.2.1 Pitching

Extensive work for the design of a controller applied to pitching has been presented in Brunton et al. [2014]. Following Brunton’s approach, a frequency loop-shape $L_d = \frac{3840(s + 4)}{s^2(s + 80)}$ is chosen to generate a controller for pitching about the leading edge. The desired loop shape presents a double integrator for good low-frequency tracking and a cross-over frequency of around 40 rad/s $c/U$. To increase stability margins, the desired loop is optimized using the Glover-McFarlane $\mathcal{H}_\infty$ loop-shaping algorithm implemented in MATLAB [Glover & McFarlane, 1989, McFarlane & Glover, 1992].

Results from DNS and from the reduced-order model for the closed-loop response to step changes in the lift coefficient $C_L$ of different magnitude are shown in Fig. 2.15a, together with a comparison between the angle of attack predicted by the reduced order model and the one obtained with the DNS in Fig. 2.15b. The angle of attack required to track the smaller lift steps in the plot is sufficiently small to ensure that the flow nonlinearities play a negligible role in the generation of the lift: the lift step response and the angle of attack predicted by the model are very close to the results from DNS. As the angle of attack required to track the desired lift increases, the model is not able to accurately predict the flow behavior due to the increasing contribution of the nonlinear effects to the lift. Especially for a step lift of 0.8, the nonlinear effects decrease the quasi-steady lift slope, requiring a higher angle of attack to achieve the same lift with respect to the angle predicted by the model. Despite the nonlinearities in the flow, the most important result is that the controller is robust enough to be able to track the reference lift without a significant performance decrease, as can be observed in Fig. 2.15a by comparing the lift from the reduced-order model with the lift from DNS. Therefore the same control performance is achieved (Fig. 2.15a), even though the control input required to achieve it is different (Fig. 2.15b).
2.4.2.2 Mode 1

The lift pulse response in mode 1 presented in Section 2.2 showed that the system exhibits an inverse response. An inverse response behavior implies that the response to a positive step input is characterized by an initial high-frequency undershoot, and only after a transient, the output will settle to a steady-state positive value. In a pole-zero diagram, inverse response is indicated by the presence of a zero in the right-half plane (RHP), which is an indicator of competing effects of slow and fast dynamics. A system with one or more RHP zeros is called a non-minimum phase system. A RHP zero poses a limit on the controller bandwidth, consequently limiting the controller’s performance [Skogestad & Postlethwaite, 2005].

Indicating with \( G_{k1} \) the transfer function of the state-space reduced-order model of the lift generated by a deflection in mode 1, the desired loop-shape for the mode...
Linear Aerodynamic Model

The selected gain gives, after the loop-shape is optimized with $H_\infty$ techniques, a cross-over frequency around $0.2 \text{ rad/s } c/U$, which is close to the maximum value allowed by the RHP zero; such frequency is two orders of magnitude smaller than the cross-over frequency achieved for the pitching case. This narrow bandwidth is a result of the limitation posed by the opposite response of fast and slow dynamics and it is a limitation that comes directly from the physics of the system, as already discussed when analyzing the pulse response in Section 2.2.

The results for the lift step response are shown in Fig. 2.16a. As expected, all the lift responses present an undesired initial undershoot. We also see that the time scales of the response are two orders of magnitude larger than the pitching case, because of the smaller bandwidth of the controller in mode 1. A comparison of the deflection of the wing between model and DNS and the pressure field from DNS are presented in Fig. 2.16b and Fig. 2.16c, respectively: in this configuration, the main contribution to the lift comes from the suction area behind the mid-chord point on the upper surface.

As already observed for pitching, as the magnitude of the reference lift that needs to be tracked increases, the wing maximum deflection increases as well, generating a nonlinear flow response that is not captured by the model, as can be observed in

Figure 2.16: (a) Lift step response from a controller in mode 1, obtained with reduced-order model (blue line) and DNS (red line). Results are plotted for different lift steps (black dashed line). The inverse response can be observed in the first time instants of the response. As the deflection increases, nonlinear effects become non negligible, without significantly affect the controller’s performance. (b) Deflection corresponding to each step response (c) Qualitative representation of the pressure field ($p - p_\infty$) around the wing at different time-steps, indicated in the lift plot with black circles.
the deflection comparison between reduced-order model and DNS. To achieve the same lift, the wing simulated with the DNS reaches a higher deflection due to the suction losses caused by nonlinear phenomena. Although there are differences in the lift response as well between model and DNS, the controller is robust enough to guarantee good low-frequency reference tracking even for the largest deflection. As the deflection increases, the controller lift response becomes slightly slower, as shown by the shift in the lift peak, but the qualitative shape of the response does not change.

It is important to notice that for a step in the lift of 0.15, nonlinear effects are already quite significant. This means that to achieve the same lift, the required deflection in mode 1 generates larger perturbations in the flow-field with respect to the required angle of attack corresponding to the same lift. From this perspective, lift generation and control using pitching is more efficient, because pitching is able to generate higher lift with smaller flow perturbations.

2.4.2.3 Mode 2

A deflection according to an even mode is characterized by a time delay between the input and the output. This means that the initial high-frequency response to a step change of the input is zero (or negligible) and the system will start responding with a non-zero output only after a delay \( \bar{\tau} \). Time delays limit the controller performance, limiting the maximum bandwidth achievable by the controller [Skogestad & Postlethwaite, 2005].

The chosen loop-shape for the mode 2 controller is \( Ld_{k2} = W_{k2} G_{k2} \), where \( G_{k2} \) is the reduced-order model for mode 2 and \( W_{k2} = 4(s + 1)/(s + 2) \). Again, the gain is chosen to guarantee the maximum bandwidth achievable in the presence of a time delay, that is a result of the symmetrical deflection of mode 2 that generates a negligible lift in the very first time-steps of the response. After loop-shape optimization, the transfer function cross-over frequency is around 5 rad/s \( c/U \).

From the results from the step response for different lift steps and the corresponding deflection, presented in Fig. 2.17a and Fig. 2.17b, similar conclusions to the previous systems can be drawn. The controller is robust enough to ensure good reference tracking even in the presence of unmodeled nonlinearities, as shown by the comparison of the lift and the deflection plots. Due to the nature of the deflection, nonlinearities tend to increase the slope of the quasi-steady lift, generating higher pressures on the lower surface of the wing with respect to a perfectly linear case. The result is that the deflection necessary to generate a certain lift is smaller than the deflection predicted by the linear model.

Observing the pressure distribution in Fig. 2.17c, we notice that the deflection generates a suction area on the upper surface corresponding to the first hump of the wing, between 0.25\( c \) and 0.5\( c \) from the leading edge, and a corresponding region of high pressure on the lower surface. These are the main contributions to
the lift. The rear part of the wing, close to the trailing edge, is characterized by a smaller suction region on the lower surface, that gives a negative contribution to the global lift.

2.4.2.4 Overcoming limitations: combination of mode 1 and mode 2

Individual feedback controllers using either mode 1 or mode 2 have shown poor performance compared to pitching because of the controller bandwidth limitations posed by the inverse response and the time delay. Instead of considering one single deflection mode, it is possible to combine multiple deflection modes in a multi-input single-output state-space model: the idea is to combine the properties of each individual input to overcome the intrinsic physical limitations that they would have when used separately.

We now consider a MISO reduced-order model combining modes 1 and 2 together. The $1 \times 2$ matrix transfer function is denoted by $G_{k1k2}$. The desired loop-shape is constructed using a $2 \times 2$ pre-compensator $W_1$, a $2 \times 1$ controller $K_{k1k2}$ and a post-compensator $W_2$, resulting in $L_d = G_{k1k2}W_1K_{k1k2}W_2$. $W_1$ is chosen as the identity matrix, to separate the controller matrix entries on the two inputs and $W_2 = 2$ is a pure gain. The controller is chosen as $K = [1 \ 1]^T$, in
Figure 2.18: (a) Lift step response of a multi-input single-output controller that simultaneously uses mode 1 and mode 2. The results from reduced-order model (blue line) and DNS (red line) are compared. The response does not show inverse response behavior or time delays and the time scales are comparable with the ones obtained for a controller in pitching. (b) Deflection corresponding to each step response. Solid lines represent $W_1$ and dashed lines represent $W_2$. (c) Qualitative representation of the pressure field $(p - p_{\infty})$ around the wing at different time-steps (indicated in the lift plot with circles).

order to consider the contribution to mode 1 and mode 2 to the lift with the same weight. More complex loop-shapes can be prescribed for the controller and the compensator, but for the present case a simple approach with no frequency dependency of the controller is sufficient to ensure acceptable performance. The gain is selected in order to have a cross-over frequency, after optimization, around 40 $\text{rad/s } c/U$, similar to the one achieved with the controller for pitching. As shown in Fig. 2.18a, the controller that simultaneously uses mode 1 and mode 2 outperforms the controllers that individually use mode 1 or mode 2, showing a much faster response without time delays and inverse response. Because of the similar bandwidth, the controller time-scales are of the same magnitude as those obtained for pitching. Figure 2.18b shows the maximum deflection for both modes; for small deflections, the model and the results from DNS are in good agreement. As the deflection increases, the model overestimates the deflection necessary to achieve a certain lift. However, the model is robust enough to ensure that the effect of the flow nonlinearities on the controller performance is negligible. The deflection of the wing and the corresponding pressure field are shown in Fig. 2.18c. The pressure distribution is qualitatively similar to the one generated by a deflection in mode 2, given the higher contribution of mode 2 to the global deflection compared to mode 1. Similar to the previous controllers, the deflection necessary to generate
the same lift introduces higher nonlinear perturbations with respect to pitching only.

2.4.2.5 Rejection of disturbance in the form of a vortex

Controllers are designed using models from lift pulse responses obtained in a uniform free-stream. In practical applications, controllers have to deal with a lot of uncertainties not taken into account by the model, such as flow non-uniformities, uncertainties in the operating conditions and other kinds of external disturbances.

We will test the disturbance rejection capabilities of the controllers using a disturbance in the form of a two-dimensional vortex simulated with DNS; the vortex is convected downstream by the free-stream velocity and it impinges on the wing, generating a non-zero global lift. The simulation setup is shown in Fig. 2.19. The simulation parameters are the same as those used for the previous simulations. The grid, composed of \(3,814 \times 1,031\) points, has been obtained from the previous mesh presented for system identification, simply refining the region in front of the wing’s leading edge, in order to accurately capture the convected vortex. Additional simulations performed refining the mesh around the body as well did not show significant differences in the lift of the wing. At time instant \(t = 0\), the distance of the vortex from the leading edge of the wing is \(4.5c\). The components \(u\) and \(v\) of the velocity field induced by the vortex in the \(\tilde{x}\) and \(\tilde{y}\) directions, respectively, are defined by the following equation:

\[
\begin{align*}
    u(\tilde{x}, \tilde{y}) &= + \frac{c_v}{M} \left(0.5 (\tilde{x}^2 + \tilde{y}^2)/r_v^2\right) \tilde{y} \\
    v(\tilde{x}, \tilde{y}) &= - \frac{c_v}{M} \left(0.5 (\tilde{x}^2 + \tilde{y}^2)/r_v^2\right) \tilde{x},
\end{align*}
\]

where \(c_v\) indicates the intensity of the vortex, \(r_v\) its radius and \(M\) the Mach number.

---

**Figure 2.19:** Schematic representation of the simulation setup for the vortex test case. At \(t = 0\), the vortex is placed in front of the leading edge at a distance of \(4.5c\) and it is convected downstream by the free-stream velocity.

Four vortices with different initial radii are simulated. The radius and the intensity of each vortex at \(t = 0\) are summarized in Table 2.2. Both parameters have been chosen in such a way that all the vortices will have the same energy when they reach the wing’s leading edge.
Table 2.2: Initial radius and intensity for each vortex tested. The radius of the vortex determines the frequency content of the lift response.

<table>
<thead>
<tr>
<th>Vortex</th>
<th>Radius ( r_v )</th>
<th>Intensity ( c_v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>V1 - Re=100</td>
<td>0.10c</td>
<td>5.77</td>
</tr>
<tr>
<td>V2 - Re=100</td>
<td>0.25c</td>
<td>0.28</td>
</tr>
<tr>
<td>V3 - Re=100</td>
<td>0.5c</td>
<td>0.053</td>
</tr>
<tr>
<td>V4 - Re=100</td>
<td>1c</td>
<td>0.014</td>
</tr>
</tbody>
</table>

The first set of simulations is performed modeling the wing as rigid and not actuated (open-loop). The lift generated by each vortex is shown in Figure 2.20a. The frequency content of the lift has been obtained using the Fourier transform and it is presented in Fig. 2.20b. The lift frequency content depends on the vortex radius: the smaller the radius, the higher the frequencies present in the lift.

![Lift response and frequency content](image)

Figure 2.20: Lift response of a rigid non-actuated wing (a) and lift frequency content (b) for the vortices listed in Table 2.2 (initial radius from 0.1c to 1c, lines from grey to black).

Simulations are repeated, now in closed-loop, allowing the control framework to actuate the wing in order to keep the lift constant at zero during the passage of the vortex. The optimized controllers for pitching, mode 1, mode 2 and mode 1+2 previously introduced are tested. An efficiency parameter \( \eta \) is introduced for each vortex-controller pair. \( \eta \) is defined as \( \eta = \int C_L^k(t)^2 dt / \int \bar{C}_L(t)^2 dt \), where \( C_L^k \) is the lift coefficient obtained applying generic controller \( k \) to the wing for the considered vortex and \( \bar{C}_L \) is the lift coefficient for the same vortex in the case of a rigid wing with no actuation (open-loop). The smaller \( \eta \), the better the controller will be able to maintain the lift close to zero. The results are summarized in Fig. 2.21, where \( \eta \) is plotted against the main frequency of the lift frequency content for each vortex.

The bandwidth of a controller can be considered as the frequency range in which the controller is effective. A controller will not be able to reject disturbances with frequencies outside its bandwidth. Looking at Fig. 2.21a, it is clear that the
Figure 2.21: (a) Summary of the performance of each controller in rejecting the vortex disturbance for different vortex radii. The lower the value of $\eta$, the better the controller’s performance. Results are shown for reduced-order model (circles) and DNS (star symbol). (b) Integral difference between the deflection squared predicted by the model and deflection squared from DNS in terms of percentage of the integral of the squared model deflection, $\gamma = 100 \cdot \left( \int W_{k_{\text{ROM}}}^2(t) dt - W_{k_{\text{DNS}}}^2(t) dt \right) / \int W_{k_{\text{ROM}}}^2(t) dt$. For the controller $k_1k_2$ the solid line represents the $\gamma$ associated to the deflection in mode 1 and the dashed line represents mode 2. Differences of $\gamma$ smaller than 1% indicate that the flow nonlinearities are small and the flow behaves linearly.

bandwidth of the controller in mode 1 is so small that the main frequency of all vortices lies outside of the bandwidth, resulting in a low efficiency. Compared to the controller bandwidth, all vortices are high-frequency disturbances and they will induce a high-frequency response, that for mode 1 is characterized by an inverse response. For this reason, the vortex disturbance is not rejected but amplified by mode 1. Disturbance rejection improves using mode 2, because of its higher bandwidth. However, as the lift frequency content increases, the performance of mode 2 deteriorates because of the increasing contribution to the lift of vortex frequencies outside of the controller bandwidth. Pitching and mode 1+2 show good disturbance rejection for all the considered vortices, although they also suffer from a performance loss for higher frequency vortices.

The advantage of having a state-space reduced order model lies in the possibility to infer the performance of a controller without having to perform expensive simulations. Qualitative results for the vortex case can be predicted simply by comparing the bandwidth of the controller with the vortex frequency content: the controller will have problems rejecting frequencies outside of its bandwidth, resulting in poorer performance. For quantitative results, the open-loop lift for each vortex from DNS can be fed to the reduced-order model, which can then be simulated in closed-loop. The results in terms of efficiency of the controller are shown in Fig. 2.21a and compared to the DNS results, showing good predictive capabilities for all the considered cases. Accuracy of results is guaranteed if the simulated flow retains the same properties of the model (e.g. linearity) but it has been seen
that robust controllers will maintain their performance, even in the presence of moderate nonlinearities.

For qualitative information about the nonlinearity of the flow, a comparison of deflection from DNS and model must be considered. Fig. 2.21b summarizes capabilities of the model to predict the deflection by introducing the parameter \( \gamma = 100 \cdot \left( \int W_{k_{\text{ROM}}}^2(t) - W_{k_{\text{DNS}}}^2(t) dt \right) / \int W_{k_{\text{ROM}}}^2(t) dt \), that indicates how small is the squared integral difference between deflection predicted by the model \( W_{k_{\text{ROM}}}(t) \) and deflection obtained with DNS \( W_{k_{\text{DNS}}}(t) \), expressed in percentage of the predicted deflection. For all the considered cases, the differences between model and DNS are below 1%, indicating that the flow behavior is linear.

### 2.4.2.6 Discussion

Optimized controllers for pitching about the leading edge, mode 1, mode 2 and a combination of mode 1 and 2 have been presented. Because of the symmetrical nature of the deflection with respect to the mid-chord, controllers in mode 1 have bandwidth limits imposed by the inverse response behavior. Similarly, because of the anti-symmetrical deflection, responses in mode 2 present time delays between input and output, again resulting in limitations on the bandwidth of the controllers. These limitations can be overcome with a controller that actuates the wing using mode 1 and mode 2 simultaneously.

Although the decomposition of the deflection using Fourier modes is arbitrary, it gives useful information on how to best actuate the wing to achieve good control performance. Any deflection that is symmetrical with respect to the mid-chord of the wing, such as Fourier mode 1 and all the odd Fourier modes, will always generate an inverse response of the lift, that poses a limit on the maximum controller bandwidth achievable. This is valid for any arbitrary symmetrical deflection and for symmetrical rigid motion as well: vertical displacement (plunging), also shows inverse response [Brunton et al., 2013]. Instead, any anti-symmetrical deflection with respect to the mid-chord, such as mode 2 and the additional even modes, presents time delays in the lift response, again resulting in controller performance limitations. Similar considerations can be extended to the particular case of pitching about the mid-chord [Brunton et al., 2013]. Hence, in order to overcome the limitations imposed by symmetrical and anti-symmetrical deflections, the symmetry has to be broken using a combination of symmetrical and anti-symmetrical modes, such as mode 1 and mode 2. Similarly, pitching about the leading edge can be seen as a combination of plunging (symmetrical) and pitching about the mid-chord (anti-symmetrical).

Controllers for the lift for pitching about the leading edge or mode 1+2 can be designed with similar bandwidth, and thus similar performance in time, as shown by comparing the step responses from Section 2.4.2. Hence, considering lift only and from the perspective of the linear model, controllers in pitching and mode 1+2 achieve similar performance. On the other hand, the two controllers actuate the wing differently: the controller in pitching changes the angle of attack and the
controller in mode 1+2 changes the shape of the wing according to the first two deflection modes. We also saw that, in order to achieve the same lift, the angle of attack required using a controller in pitching is smaller than the deflection in mode 1 and mode 2. Hence, the flow generated by the two configurations is different. In particular, for the same lift and in the nonlinear case simulated by DNS, the perturbations introduced in the flow by pitching the wing are smaller with respect to the perturbations introduced deflecting it. As a consequence, nonlinear flow effects are larger for the deflection case than for pitching, when tracking the same lift. Linearly and considering lift only, pitching and deflection in mode 1+2 are comparable. On the other hand, from a nonlinear point of view, pitching about the leading edge is more efficient than deflection, because it introduces smaller perturbations in the flow field.

Additional parameters such as, for example, the pitching moment generated (fundamental for the longitudinal flight dynamics of an air vehicle), the total drag, the energy requirements to actuate the wing and the feasibility of the solution (how to implement an actuator for pitching or deflection in a real application), can be taken into account in order to determine whether rigid pitching or deflection offers better performance. Such an investigation partially depends on the specific application to target, but it could benefit from the information obtained by extending the present model to include additional output such as pitching moment or drag coefficient.

2.4.3 Extension to higher Reynolds numbers

The reduced-order modeling procedure introduced in Section 2.1.2 has been tested and discussed for a Reynolds number of 100, but can be easily applied to higher Reynolds number flows. Because the model is linear, good accuracy can be expected provided that the nonlinearities in the flow are weak.

2.4.3.1 Reduced-order model

Fig. 2.22 shows a comparison of the reduced-order model for pitching, mode 1 and mode 2 obtained for $Re = 100$ and $Re = 1000$. The settings used to obtain the pulse response necessary to generate the models are the same as those introduced in Section 2.2. Since the main physical phenomena behind the generation of the aerodynamic forces do not change with the Reynolds number, as long as the deflection of the wing is sufficiently small, the frequency behavior of the models remains qualitatively the same.

2.4.3.2 Step response

A real life application for the controllers, such as a MAV with a flexible wing, could operate at different flow regimes, therefore it is interesting to apply the controllers
Linear Aerodynamic Model

Figure 2.22: Bode plot comparison for reduced-order model for mode 1, mode 2 and pitching (from black to grey) at Re=100 (solid lines) and Re=1000 (dash-dot lines).

designed for $Re = 100$ to the models obtained for $Re = 1000$, in order to test their performance and their robustness in an off-design condition. The results for a step change in the lift of 0.05 are summarized in Fig. 2.23. For each model, the results from the model and DNS for $Re = 1000$ are shown, together with the step response of the models for $Re = 100$. The nominal performance of the closed-loop systems is not affected by the Reynolds number, as observed from a comparison between the reduced-order model step responses for $Re = 100$ and $Re = 1000$. All the systems maintain similar properties in terms of shape of the response and time scales. Controllers applied to $Re = 1000$ for mode 1 and mode 2 are slightly more aggressive, resulting in a smaller rise time and a slightly larger overshoot for mode 1, and in a series of moderate oscillations for mode 2. Hence, the controllers developed and optimized for $Re = 100$ are robust enough to ensure similar nominal performance for Reynolds number 1000. Although the lift step responses are similar, because of the different properties of the flow for $Re = 100$ and $Re = 1000$, the magnitude of the deflection (or angle of attack, in case of pitching) necessary to obtain such lift is different, as observed from the right plots in Fig. 2.23.

The nominal performance discussed so far are the performance predicted by the linear models. The lift step response from DNS for $Re = 1000$ follows the behavior predicted by the model for all cases, with small differences in the lift response for mode 2 and mode 1+2 due to nonlinearities in the flow. This can also be observed in the deflection plots: for mode 1 and pitching, the model predicts the deflection very well, while mode 2 and mode 1+2 show larger differences in the deflection. Because of the higher Reynolds number, the nonlinear effects are larger compared to the case with same lift step and $Re = 100$: for a step in the lift of 0.05, the differences for lift and deflection between model and DNS for $Re = 100$ are smaller.
Linear Aerodynamic Model

2.4.3.3 Vortex response

The vortex disturbance test case is repeated for $Re = 1000$, to test the disturbance rejection capabilities of the controllers at higher Reynolds number. The same with respect to the differences between model and DNS for $Re = 1000$, indicating that nonlinear unmodeled effects are more relevant for the higher Reynolds number case.
simulation setup introduced in Section 2.4 for Re = 100 is used for the Re = 1 000 cases. Because of the different Reynolds number, and thus the different ratio between convective forces and diffusive forces, the vortex decay is different from the Re = 100 cases and, consequently, the resulting lift frequency content is different. Also, in order to have lift responses of the same magnitude of those obtained for Re = 100, the intensity of the vortices is redefined as in Table 2.3.

<table>
<thead>
<tr>
<th>Vortex</th>
<th>Radius $r_v$</th>
<th>Intensity $c_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>V1 - Re=1000</td>
<td>0.10c</td>
<td>0.848</td>
</tr>
<tr>
<td>V2 - Re=1000</td>
<td>0.25c</td>
<td>0.126</td>
</tr>
<tr>
<td>V3 - Re=1000</td>
<td>0.5c</td>
<td>0.040</td>
</tr>
<tr>
<td>V4 - Re=1000</td>
<td>1c</td>
<td>0.014</td>
</tr>
</tbody>
</table>

Table 2.3: Initial radius and intensity for each vortex tested at Re = 1 000.

Results from DNS and reduced-order model are compared in Figure 2.24: Figure 2.24a shows the efficiency of each controller and Fig. 2.24b represents the deflection of the wing. Efficiency from reduced-order model for Re = 100 is also included in Fig 2.24a. As expected, mode 1 has the lowest efficiency because of its limited bandwidth, followed by mode 2. Performance improve using mode 1+2 and pitching gives the best results for low frequency vortices. An interesting result is that mode 2, pitching and mode 1+2 perform better than the same controllers at Re = 100 for the same frequencies. This is probably because their response is a bit more aggressive, as discussed comparing the lift step responses. Vortices at Re = 1 000 have a slower decay due to the smaller diffusivity of the flow, resulting in higher frequencies excited in the wing’s lift for the vortices with smaller radius. The performance of the controllers deteriorate as the frequency of the lift increases, because of the higher frequency content outside of the bandwidth. Mode 1 shows opposite behavior, having an increase in the efficiency at higher frequency; this is probably due to the fact that its bandwidth is so narrow, that most of the vortex frequencies lie outside of it, resulting in the impossibility of qualitatively predict its performance by simply looking at the bandwidth. Finally, it is interesting to notice that mode 1+2 performs better than pitching at high frequency. Comparison between DNS and model shows that the accuracy of the model for mode 2 and mode 1+2 decreases at low frequency, where the vortex has a larger radius. Observing Fig. 2.24b, we can see from the deflection plot that largest nonlinear effects are observed for mode 2 and mode 1+2 for vortices with $r_v = 1c$ and $r_v = 0.5c$.

2.4.3.4 Discussion

We have seen that the same modeling strategy presented in Section 2.1.1 can be used to obtain models at different Reynolds numbers from lift pulse responses, without any change in the procedure. From the comparison between the Bode plots at Re = 100 and Re = 1 000 for pitching, mode 1 and mode 2, we also notice that the Reynolds number does not affect the qualitative behavior of the systems in this frequency range. In particular, mode 1 retains its inverse response and
mode 2 responds with a time delay at higher Reynolds as well. This allows the controllers designed at $Re = 100$ to be applicable to the case with $Re = 1000$.

Controllers have been tested at $Re = 1000$ using the step response and the vortex disturbance test cases introduced for the low Reynolds number discussion. Despite the different properties of the flow due to the different Reynolds number and, consequently, the different deflection necessary to achieve a desired lift, the controllers are robust enough to be applied to $Re = 1000$ without any significant deterioration in their performance. The model is also able to accurately predict the system behavior as long as the nonlinearities in the flow are weak.

### 2.5 Summary

A state-space reduced-order model for the lift generated by the unsteady deflection of a two-dimensional wing at low Reynolds numbers has been presented. The contribution of the deflection is taken into account by decomposing the global deflection into a series of modes using a truncated Fourier series; the total lift is reconstructed using superposition, assuming that the flow behaves linearly. We have seen that the contribution to the lift of a single mode decreases as the mode number increases, justifying the use of a truncated series. The reduced order-model has been validated against high-fidelity DNS and its characteristics have been discussed from a feedback control perspective. Analyzing the lift pulse response of
each mode separately, we saw that mode 1 and odd modes in general display an inverse response and that mode 2 and even modes show an initial delay between input and output. Both phenomena represent physical limitations for feedback control, affecting the bandwidth and the response of the closed-loop systems.

Optimized controllers for Reynolds number 100 have been developed for pitching, mode 1 and mode 2 and have been tested on the lift step response test case, using both the reduced-order model and DNS. An important result of the present work is that the limitations of the individual controllers have been overcome using a combination of mode 1 and mode 2 in a multi-input single-output framework. From the step response case, we observed that the model accurately predicts the lift and the deflection of the wing when the flow is sufficiently linear, but the controllers are robust enough to ensure that their performance is not affected by the presence of moderate nonlinearities. A second test case with a disturbance in the form of a convective vortex has been studied, showing that the controllers are able to reject disturbances provided the frequency of the disturbances lies outside the controller bandwidth, hence demonstrating the importance of a combined mode 1+2 controller to overcome the bandwidth limitations of the individual controllers. Finally, it has been shown that the range of applicability of the controllers is not limited to Reynolds 100. Simulations performed at $Re = 1000$ for step response and vortex disturbance have proven that the controllers maintain a similar behavior, without any significant deterioration in control performance.

The reduced-order model has been proven to be a powerful tool not only to predict the lift generated by the deflection of a two-dimensional wing, but also to design feedback controllers, providing useful information on their performance without having to run an expensive campaign of simulations. The model can be extended to include additional outputs such as pitching moment coefficient, drag coefficient or pressure distribution along the wing.
Chapter 3

Fluid-structure Interaction Models

In Chapter 2 we introduced a linear reduced-order model for the lift generated by a change in configuration of a flexible wing. The motion of the wing, decomposed into rigid and flexible degrees of freedom, represents an external input to the aerodynamic model and it is prescribed. As an example, we discussed how such an aerodynamic model can be coupled to an external controller in order to actuate the wing to track a reference lift and to reject disturbances. We now want to model the fluid-structure interaction of a membrane wing immersed in a free-stream. In this case, the deflection of the wing is not prescribed or defined by a controller, but it is determined by the structural dynamics of the thin elastic membrane. In other words, the deflection of the wing is determined by an equation that models the physical behavior of an elastic membrane. There is a mutual interaction between the structure and the flow: the membrane is deformed by the aerodynamic loads, which are affected by the unsteady deformation on the wing itself. To model this mutual interaction between membrane and fluid, we need to couple the linear aerodynamic solver with an equation for the structure. In this chapter, we first present a nonlinear one-dimensional model for the elastic membrane and we subsequently introduce quasi-linear and linear models based on approximations of the nonlinear membrane. These models are then coupled to the linear aerodynamic model to generate nonlinear, quasi-linear and linear unsteady fluid-structure interaction models in the form of sets of partial differential equations.

3.1 Membrane models

The structural model for the membrane adopted in the present work follows the theory of one-dimensional nonlinear elastic strings introduced by Carrier [1945] and Narasimha [1968]. The equilibrium of a deformed infinitesimal membrane element is shown in Fig. 3.1. From the application of Newton’s Second Law to this
Figure 3.1: Representation of the original and the deformed configuration of an infinitesimal element of the membrane.

For an infinitesimal element, we obtain the following equations:

\[
\mu \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial}{\partial \bar{x}} \left[ c_T \cos \theta \right] + F_x \\
\mu \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + 2C_d \frac{\partial \bar{w}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left[ c_T \sin \theta \right] + F_y.
\] (3.1) (3.2)

\(\bar{x} = x/c\) and \(\bar{y} = y/c\) represent the local coordinate system of the membrane: \(\bar{x}\) is the longitudinal direction, aligned with the direction of the undeflected membrane, while \(\bar{y}\) is the transversal direction. Both are non-dimensionalized with the wing chord \(c\). \(t = tU/c\) is the dimensionless time and \(U\) is the free-stream velocity. \(\bar{u}(x,t) = u(x,t)/c\) and \(\bar{w}(x,t) = w(x,t)/c\) are, respectively, the dimensionless longitudinal and transversal displacement. From now on, the overbar will be omitted and \(t, x, y, u(x,t)\) and \(w(x,t)\) are taken to be nondimensional. The tension of the elastic membrane is expressed using a dimensionless tension coefficient \(c_T\), introduced later in Eq. 3.7. \(\mu = \rho_s h/\rho c\) is the density ratio and represents the ratio between the density of the membrane \(\rho_s\) and the density of the fluid \(\rho\), while \(h\) and \(c\) indicate the wing’s thickness and chord, respectively. \(F_x\) and \(F_y\) are the longitudinal and transversal force per unit length, respectively. \(C_d\) is the dimensionless damping coefficient \(C_d = \rho_s C_{ds}/\rho U\), where \(C_{ds}\) indicates the wing’s structural damping. The angle \(\theta\) can be obtained from simple trigonometric considerations, resulting in:

\[
\sin \theta = \frac{\frac{\partial w}{\partial \bar{x}}}{\sqrt{1 + \left( \frac{\partial u}{\partial \bar{x}} \right)^2 + \left( \frac{\partial w}{\partial \bar{x}} \right)^2}} \\
\cos \theta = \frac{1 + \frac{\partial u}{\partial \bar{x}}}{\sqrt{1 + \left( \frac{\partial u}{\partial \bar{x}} \right)^2 + \left( \frac{\partial w}{\partial \bar{x}} \right)^2}}.
\] (3.3)

In order to simplify the membrane model represented by Eqs. 3.1 and 3.2, the two following approximations are introduced:

1. the longitudinal motion of the membrane is assumed to be small enough—when compared to the transversal motion—to be neglected, resulting in \(u \approx 0\). Consequently, Eq. 3.1 can be discarded and the following approximation
for \( \sin \theta \) is obtained:

\[
\sin \theta \approx \frac{\partial w}{\partial x} \sqrt{1 + \left( \frac{\partial w}{\partial x} \right)^2}.
\]  

(3.4)

It is important to remark that, from a fluid-structure interaction perspective, neglecting the longitudinal deformations implies neglecting aerodynamic viscous shear effects on the membrane. This approach has already been adopted in the works of Gordnier [2009] and Serrano-Galiano et al. [2018] in the context of high-fidelity simulations of membrane wings. For low Reynolds number flows, viscous forces might contribute to elongate or shrink the membrane, hence affecting its tension. However, their effect on the membrane’s dynamics is secondary with respect to the aerodynamic pressure. We will therefore only retain the aerodynamic pressure in our models and in the DNS as the main contributor to the coupled aero-structural dynamics.

2. The membrane is elastic and the tension coefficient \( c_T \) is assumed to be uniform. The stress-strain relationship that relates the tension of the membrane to its elongation can be written as follows:

\[
c_T = c_{T_0} + 2\bar{E}_s\delta, \quad c_{T_0} = 2\bar{E}_s\delta_0, \quad \bar{E}_s = \frac{E_s h}{\rho U^2 c},
\]

\[
\delta = \int_0^1 \sqrt{1 + \left( \frac{\partial w}{\partial x} \right)^2} \, dx - 1.
\]  

(3.5)

\( c_T \) is the tension coefficient and \( c_{T_0} \) is the initial tension due to the pre-stretching of the membrane in its undeflected configuration. \( E_s \) is the modulus of elasticity of the material and it is assumed to be uniform along the wing. \( \delta_0 \) and \( \delta \) are the initial pre-strain and the total length increase of the membrane due to its deflection, respectively.

Equation 3.2, together with the modeling assumptions introduced in this section, represents the starting point for the development of the nonlinear, quasi-linear and linear membrane models.

3.1.1 Nonlinear membrane

The structural dynamics of the membrane wing are represented by the approximated governing equation of the nonlinear one-dimensional extensible membrane subjected to a transverse force introduced in Section 3.1. Considering the approximations explained in Section 3.1, the nonlinear membrane equation in nondimensional form is represented as follows:

\[
2\mu \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + 2C_d \frac{\partial \bar{w}}{\partial \bar{t}} - c_T \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} \left[ 1 + \left( \frac{\partial \bar{w}}{\partial \bar{x}} \right)^2 \right]^{-3/2} = \Delta C_p + C_f
\]  

(3.6)
\[c_T = c_{T_0} + 2\bar{E}_s \delta, \quad \delta = \int_0^1 \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2} \, dx - 1.\] (3.7)

\[\Delta C_p = 2 (p_{\text{low}} - p_{\text{up}})/\rho U^2\] represents the pressure difference between lower and upper surfaces of the wing. Indicating with \(F\) an external transverse force, \(C_f = 2F/\rho U^2\) represents the force coefficient. The membrane is pinned at the two ends, resulting in \(w(x=0,t) = w(x=1,t) = 0\). Equations 3.6 and 3.7 will be referred to as the nonlinear membrane equations. They contain two nonlinear terms: the tension coefficient \(c_T\), which depends on the nonlinear elongation of the membrane \(\delta\), and the term \(\left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]^{-3/2}\), which is related to the curvature of the membrane.

### 3.1.2 Quasi-linear membrane

A weakly nonlinear model, which we will refer to as the quasi-linear model, is derived for small deflections from Eqs. 3.6 and 3.7 to reduce the order of the structural nonlinearity and to make the problem tractable with low order approaches. The mathematical derivation of the model, presented in Appendix B.1, involves representing the nonlinear terms of Eq. 3.7 as a truncated Taylor series, retaining terms up to second order. In the quasi-linear model, the nonlinear term due to the curvature of the membrane is neglected (see Appendix B.1), but the nonlinear contribution of the increasing tension caused by the elongation of the membrane is retained and approximated. To obtain the quasi-linear model, the deflection of the membrane is decomposed into the contribution of \(N\) deflection modes, \(W_k(t)\) with \(k = 1, \ldots, N\), using a truncated Fourier series, as introduced in Section 2.1.1. The quasi-linear membrane equation in physical coordinates can be expressed as

\[2\mu \frac{\partial^2 w}{\partial t^2} + 2C_d \frac{\partial w}{\partial t} - c_T \frac{\partial^2 w}{\partial x^2} = \Delta C_p + C_f\] (3.8)

\[c_T = c_{T_0} + 2\bar{E}_s \delta, \quad \delta = \frac{\pi^2}{4} \sum_{m=1}^{N} m^2 \psi_m^2.\] (3.9)

Equation 3.8 can be decomposed into a system of equations representing the contribution of the individual deflection modes, as shown in Appendix B.1. The equations are coupled by the nonlinear term \(\delta\), which contains contributions from all deflection modes. The quasi-linear model in modal form is

\[2\mu \ddot{\mathbf{W}} + 2C_d \dot{\mathbf{W}} + \pi^2 c_{T_0} \mathbf{\Gamma}^2 \mathbf{W} + \frac{\pi^4}{2} \bar{E}_s \mathbf{\Gamma}^2 (\mathbf{W}^T \mathbf{\Gamma}^2 \mathbf{W}) \mathbf{W} = \Delta C_p + C_f,\] (3.10)

where the coefficients \(\Delta C_{p_k}\) and \(C_{f_k}\) are obtained by expressing the pressure term \(\Delta C_p\) and forcing term \(C_f\) in modal form using a truncated Fourier series, in the same way as we did with \(w\) in Eq. 2.4. Matrix \(\mathbf{\Gamma}\) is a diagonal matrix defined as
Fluid-structure Interaction Models

follows:

\[
\Gamma = \begin{bmatrix}
1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & N
\end{bmatrix}.
\] (3.11)

### 3.1.3 Linear membrane

A linear membrane model is derived from the quasi-linear model by neglecting second order terms (see Appendix B.1). In physical coordinates, the linear membrane takes the following form:

\[
2\mu \frac{\partial^2 w}{\partial t^2} + 2C_d \frac{\partial w}{\partial t} - c_{T_0} \frac{\partial^2 w}{\partial x^2} = \Delta C_p + C_f
\] (3.12)

\[
c_{T_0} = 2 \bar{E_s} \delta_0, \quad \delta \approx 0.
\] (3.13)

From Eq. 3.13 we notice that the elongation of the membrane due to the deflection is neglected and, consequently, the tension is constant and does not depend on the deflection itself. In modal form the linear membrane model becomes

\[
2\mu \ddot{W} + 2C_d \dot{W} + \pi^2 c_{T_0} \Gamma^2 W = \Delta C_p + C_f.
\] (3.14)

### 3.2 Fluid-structure interaction: membrane-fluid coupling

We now present the governing equations of the fluid-structure interaction systems obtained by coupling the membrane models introduced in Section 3.1 with the linear aerodynamic model introduced in Section 2.1.1. The following fluid-structure interaction models are obtained:

- Nonlinear fluid-structure interaction model: nonlinear membrane coupled to linear fluid
- Quasi-linear fluid-structure interaction model: quasi-linear membrane coupled to linear fluid
- Linear fluid-structure interaction model: linear membrane coupled to linear fluid

### 3.2.1 Nonlinear fluid-structure interaction model

The nonlinear membrane equation is coupled to the linear aerodynamic model to obtain a nonlinear fluid-structure interaction model. For the coupling, we use
the aerodynamic model from Eq. 2.8, containing the rigid and flexible degrees of freedom of the wing as inputs and the lift coefficient and the pressure modes as outputs. The coupling terms are the pressure and the acceleration of the deflection, as summarized in Fig. 3.2. A spatial set of discrete $N_p$ points $x_i$, $i = 1, \ldots, N_p$ is introduced to discretize the membrane model in space, obtaining $w_i(t) = w(x_i, t)$, $\Delta C_p(t) = \Delta C_p(x_i, t)$ and $C_f(t) = C_f(x_i, t)$. The discrete nonlinear terms are then defined as follows:

$$f_{NL}^i = c_T_i \left( \frac{\partial^2 w}{\partial x^2} \right)_i \left[ 1 + \left( \frac{\partial w}{\partial x} \right)_i \right]^{-3/2},$$

$$c_T_i = c_{T_0} + 2 \bar{E}_s \left( \int_0^1 \sqrt{1 + \left( \frac{\partial w}{\partial x} \right)_i^2} \, dx - 1 \right).$$

Finally, we indicate with $w$, $\Delta C_p$, $C_f$ and $f^{NL}$ the column vectors containing the corresponding discrete variables. Since the aerodynamic model is expressed in modal form and the membrane equation in physical space, the variables of the two systems require a conversion between physical and modal space. This is achieved by means of conversion matrices, defined as follows:

$$\Phi^{m2s} = \begin{bmatrix}
\sin(\pi x_1) & \sin(2\pi x_1) & \cdots & \sin(N\pi x_1) \\
\sin(\pi x_2) & \sin(2\pi x_2) & \cdots & \sin(N\pi x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\sin(\pi x_{N_p}) & \sin(2\pi x_{N_p}) & \cdots & \sin(N\pi x_{N_p})
\end{bmatrix}_{N_p \times N},$$

$$\Phi^{s2m} = \frac{2}{N_p - 1} \begin{bmatrix}
\sin(\pi x_1) & \sin(\pi x_2) & \cdots & \sin(\pi x_{N_p}) \\
\sin(2\pi x_1) & \sin(2\pi x_2) & \cdots & \sin(2\pi x_{N_p}) \\
\vdots & \vdots & \ddots & \vdots \\
\sin(N\pi x_1) & \sin(N\pi x_2) & \cdots & \sin(N\pi x_{N_p})
\end{bmatrix}_{N \times N_p}. $$

**Figure 3.2:** Schematic representation of the coupling between the nonlinear membrane model and the aerodynamic solver. A variable transformation between physical and modal space is necessary for the coupling.
For the deflection, for example, we have
\[ w = \Phi^m w, \quad \mathcal{W} = \Phi^s w. \] (3.17)

The procedure to couple the nonlinear membrane to the linear aerodynamic model is detailed in Appendix B.2. The system of nonlinear equations representing the nonlinear fluid-structure interaction model in state-space form is the following:

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix}
\dot{x} \\
\dot{\alpha} \\
w \\
\dot{w}
\end{bmatrix} &= \begin{bmatrix}
\dot{A}_{x}^{x} & \dot{A}_{\alpha}^{x} & \dot{A}_{w}^{x} & \dot{A}_{\dot{w}}^{x} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\dot{x} \\
\dot{\alpha} \\
w \\
\dot{w}
\end{bmatrix} + \begin{bmatrix}
\dot{B}_{\alpha}^{x} & \dot{B}_{w}^{x} \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\ddot{\alpha} \\
\dot{w}
\end{bmatrix} + \begin{bmatrix}
\dot{F}_{x}^{x} \\
0
\end{bmatrix} + \begin{bmatrix}
\dot{F}_{\alpha}^{x} \\
0
\end{bmatrix} + \begin{bmatrix}
\dot{F}_{w}^{x} \\
0
\end{bmatrix} [f]_{NL}.
\end{align*}
\] (3.18)

For the sake of clarity, the matrices appearing in Eq. 3.18 are omitted in this section, but they are listed in Appendix B.2. Vector \( \begin{bmatrix} \dot{x} & \dot{\alpha} & \dot{w} \end{bmatrix}^T \) contains the state variables of the system, \( \ddot{\alpha} \) and \( C_f \) are the inputs and \( C_L \) and \( w \) are the outputs. The initial conditions of the system are provided by assigning initial values to the state variables. The time integration is performed by means of a five-step fourth-order Runge-Kutta scheme. The derivatives present in the nonlinear term are carried out using a second-order finite difference discretization.

### 3.2.2 Quasi-linear fluid-structure interaction model

Similarly to what has already been done for the nonlinear membrane in the previous section, the quasi-linear membrane is coupled to the linear aerodynamic model, resulting in a quasi-linear fluid-structure interaction model. Since the quasi-linear membrane can be expressed in modal form (see Eq. 3.10), there is no need for conversion matrices for the coupling. The procedure is summarized in Fig. 3.3. The

![Figure 3.3: Schematic representation of the coupling between the quasi-linear membrane model and the aerodynamic solver.](image)
quasi-linear fluid-structure interaction model in state-space form can be expressed as follows:

\[
\begin{bmatrix}
\dot{x}_x \\
\dot{\alpha}_x \\
\dot{W}_x \\
\dot{W}_y
\end{bmatrix} =
\begin{bmatrix}
A_{xf}^x & A_{xf}^f & A_{xf}^x & A_{xf}^{f-1} & A_{xf}^{f-l} \\
0 & 0 & 1 & 0 & 0 \\
A_{xf}^{f-1} & A_{xf}^{f-l} & A_{xf}^x & A_{xf}^{f-1} & A_{xf}^{f-l} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_x \\
\ddot{\alpha}_x \\
\ddot{W}_x \\
\ddot{W}_y
\end{bmatrix} +
\begin{bmatrix}
B_{xf}^x & B_{xf}^f \\
0 & 0 \\
B_{xf}^{f-1} & B_{xf}^{f-l} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix} =
\begin{bmatrix}
C_{xf}^x & C_{xf}^f & C_{xf}^{f-1} & C_{xf}^{f-l} \\
0 & 0 & 0 & 0 \\
C_{xf}^{f-1} & C_{xf}^{f-l} & C_{xf}^x & C_{xf}^{f-1} & C_{xf}^{f-l} \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix} +
\begin{bmatrix}
D_{xf}^x & D_{xf}^f \\
0 & 0 \\
D_{xf}^{f-1} & D_{xf}^{f-l} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix}
\]

The definition of the system matrices in Eq. 3.19 and the steps of the coupling procedure are presented in Appendix B.3. The state variables are represented by \( \begin{bmatrix} \ddot{x} \quad \dot{\alpha} \quad \ddot{W} \quad \dot{W} \end{bmatrix}^T \), the inputs are \( \dddot{\alpha} \) and \( \dddot{W} \) and the outputs are \( C_L \) and \( W \).

As for the nonlinear model, the time stepping is carried out with a fourth-order Runge-Kutta method.

### 3.2.3 Linear fluid-structure interaction model

Finally, a linear fluid-structure interaction model is presented. The framework is obtained by coupling the linear membrane to the linear fluid model. The linear equations that define the dynamics of the linear fluid-structure interaction system are the following:

\[
\begin{bmatrix}
\ddot{x}_x \\
\ddot{\alpha}_x \\
\ddot{W}_x \\
\ddot{W}_y
\end{bmatrix} =
\begin{bmatrix}
A_{xf}^x & A_{xf}^f & A_{xf}^x & A_{xf}^{f-1} & A_{xf}^{f-l} \\
0 & 0 & 1 & 0 & 0 \\
A_{xf}^{f-1} & A_{xf}^{f-l} & A_{xf}^x & A_{xf}^{f-1} & A_{xf}^{f-l} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\ddot{x}_x \\
\ddot{\alpha}_x \\
\ddot{W}_x \\
\ddot{W}_y
\end{bmatrix} +
\begin{bmatrix}
B_{xf}^x & B_{xf}^f \\
0 & 0 \\
B_{xf}^{f-1} & B_{xf}^{f-l} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix} =
\begin{bmatrix}
C_{xf}^x & C_{xf}^f & C_{xf}^{f-1} & C_{xf}^{f-l} \\
0 & 0 & 0 & 0 \\
C_{xf}^{f-1} & C_{xf}^{f-l} & C_{xf}^x & C_{xf}^{f-1} & C_{xf}^{f-l} \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix} +
\begin{bmatrix}
D_{xf}^x & D_{xf}^f \\
0 & 0 \\
D_{xf}^{f-1} & D_{xf}^{f-l} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dddot{x}_x \\
\dddot{\alpha}_x \\
\dddot{W}_x \\
\dddot{W}_y
\end{bmatrix}
\]

As for the previous models, the mathematical derivation of the equations is presented in Appendix B.3, together with the definition of the system matrices.
3.3 Summary

The mathematical details of three different fluid-structure interaction models have been presented. Such models, indicated as nonlinear, quasi-linear and linear fluid-structure interaction models, couple a linear aerodynamic model to a nonlinear, a quasi-linear and a linear membrane model, respectively. The quasi-linear and the linear membrane models have been derived from the full nonlinear equation by performing a Taylor expansion and retaining only low order terms. All the models are presented in state-space form.
Chapter 4

Static Aeroelastic Response and Linear Stability of the Membrane

Overall, the vast majority of the steady and unsteady reduced-order models present in the literature are based on potential flow theory. Comparison between prediction from potential-based aerodynamic models and experiments have been carried out, often with mixed results [Smith & Shyy, 1995]. Flow visualizations and high-fidelity numerical simulations indicate that discrepancies between inviscid theory and experimental or numerical data are mainly due to the existence of a thick boundary layer at low Reynolds numbers and, in many cases, regions of separated flow towards the membrane trailing edge [Gordnier, 2009, Rojratsirikul et al., 2009]. Hence the importance of fluid-structure interaction models that are able to take into account some of the viscous effects on the aerodynamic loads for small-scale applications.

The purpose of this chapter is to test and validate the predictive capabilities of the fluid-structure interaction reduced-order models previously introduced, both for static equilibrium and for dynamic instability of the membrane at different Reynolds numbers. In particular, we investigate how the range of applicability and the performance of the linear, quasi-linear and nonlinear models are affected by the aerodynamic and the structural parameters. First, the fluid-structure interaction models are used to compare the steady-state deflection and lift of the membrane at different angles of attack and for different values of the tension coefficient over a range of Reynolds numbers. The aim is to demonstrate how the Reynolds number affects the static aeroelastic behavior of the wing. Second, we prove the benefits of a linear model to address the aeroelastic stability of the wing at zero angle of attack as a function of aerodynamic and structural parameters. All the results will be validated by comparison with DNS.
4.1 Static aeroelastic response

The reduced-order models are adopted to investigate the membrane steady-state deflection and lift for different angles of attack and different values of the tension coefficient $c_{Th}$. The angle of attack is defined as the angle between the free-stream velocity vector and the wing chord. The maximum camber and the modal coefficients of the deflection, as well as the global lift of the wing, are discussed. Simulations are repeated for different Reynolds numbers, to investigate the effect of the Reynolds number on the membrane static aeroelastic performance. The models are validated by comparing the results against high-fidelity DNS. The full nonlinear membrane equation (see Eqs. 3.6 and 3.7) is simulated in HiPSTAR by discretizing the spatial derivatives using the same finite difference scheme adopted for the fluid, while the time-stepping is performed using a fourth-order Runge-Kutta method. The membrane equation is solved on a one-dimensional Lagrangian grid made of 100 equispaced points. The points are derived from the projection on the membrane mean line of the 200 symmetrical body points used by the BDIM, as discussed in Section 2.2.1. The grid and the DNS setup are the same introduced and discussed in 2.2.1. More details about the coupled fluid-structure solver can be found in Serrano-Galiano et al. [2018].

4.1.1 Effect of the Reynolds number and of the tension coefficient on the static response

The prediction of the equilibrium condition of the membrane-fluid system for different angles of attack is used to validate the static performance of the reduced-order models. A membrane wing placed in a uniform free-stream at an angle of attack will deflect under the influence of the pressure forces acting on the wing and generated by the fluid. In the absence of instabilities in the system, the membrane will reach an equilibrium state, in which the tension perfectly balances the pressure distribution. In this section we will only test stable membranes. The occurrence of aerodynamic and aeroelastic instabilities will be discussed later in this chapter.

The performance of the models is tested by gradually introducing structural and aerodynamic nonlinearities in the membrane-fluid system by increasing the angle of attack and the Reynolds number. In particular, we want to understand the circumstances under which both the membrane and the fluid can be considered as linear and when, instead, we need a nonlinear model for accurate predictions. Since we are only interested in the steady-state, linear, quasi-linear and nonlinear steady fluid-structure interaction models can be derived from the unsteady models introduced in Chapter 3 by setting all the time derivatives to zero. For the nonlinear model, a steady solution is obtained by solving the following system of
nonlinear equations:

\[
\begin{bmatrix}
\bar{A}_x^{fs} & \bar{A}_x^{fs} & \bar{A}_x^{fs} \\
\bar{A}_w^{fs} & \bar{A}_w^{fs} & \bar{A}_w^{fs}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{\alpha} \\
\bar{w}
\end{bmatrix}
+ \begin{bmatrix}
\bar{B}_C^{fs} \\
\bar{B}_C^{fs}
\end{bmatrix}
\begin{bmatrix}
C_f \\
f_{NL}^{fs}
\end{bmatrix}
= 0
\]
\]
\[C_L = \begin{bmatrix}
C_{Lx}^{fs} & C_{La}^{fs} & C_{Lw}^{fs}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{\alpha} \\
\bar{w}
\end{bmatrix}
+ \begin{bmatrix}
\bar{D}_C^{fs} \\
\bar{D}_C^{fs}
\end{bmatrix}
\begin{bmatrix}
C_f \\
f_{NL}^{fs}
\end{bmatrix}.
\]

Similarly, for the quasi-linear model we have

\[
\begin{bmatrix}
A_x^{fs} & A_\alpha^{fs} & A_w^{fs} + A_w^{fs} \\
A_x^{fs} & A_\alpha^{fs} & A_w^{fs} + A_w^{fs}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{\alpha} \\
\bar{w}
\end{bmatrix}
+ \begin{bmatrix}
B_C^{fs} \\
B_C^{fs}
\end{bmatrix}
\begin{bmatrix}
C_f \\
f_{NL}^{fs}
\end{bmatrix}
= 0
\]
\]
\[C_L = \begin{bmatrix}
C_{Lx}^{fs} & C_{La}^{fs} & C_{Lw}^{fs} + C_{Lw}^{fs} \\
C_{Lx}^{fs} & C_{La}^{fs} & C_{Lw}^{fs} + C_{Lw}^{fs}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{\alpha} \\
\bar{w}
\end{bmatrix}
+ \begin{bmatrix}
D_C^{fs} \\
D_C^{fs}
\end{bmatrix}
\begin{bmatrix}
C_f \\
f_{NL}^{fs}
\end{bmatrix}.
\]

and for the linear model we have

\[
\begin{bmatrix}
A_x^{fs} & A_\alpha^{fs} & A_w^{fs} \\
A_x^{fs} & A_\alpha^{fs} & A_w^{fs}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{\alpha} \\
\bar{w}
\end{bmatrix}
+ \begin{bmatrix}
B_C^{fs} \\
B_C^{fs}
\end{bmatrix}
\begin{bmatrix}
C_f \\
f_{NL}^{fs}
\end{bmatrix}
= 0
\]
\]
\[C_L = \begin{bmatrix}
C_{Lx}^{fs} & C_{La}^{fs} & C_{Lw}^{fs} + C_{Lw}^{fs} \\
C_{Lx}^{fs} & C_{La}^{fs} & C_{Lw}^{fs} + C_{Lw}^{fs}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\bar{\alpha} \\
\bar{w}
\end{bmatrix}
+ \begin{bmatrix}
D_C^{fs} \\
D_C^{fs}
\end{bmatrix}
\begin{bmatrix}
C_f \\
f_{NL}^{fs}
\end{bmatrix}.
\]

Validations of the models are carried out using DNS. In the DNS, the steady-state of the membrane-fluid system is reached by running a simulation long enough to allow all transients to decay. For the present simulations, \(t = 75\) convective time units has been determined to be a temporal interval long enough to reach convergence on the value of the steady-state camber. Differences in the camber achieved for \(t > 75\) are below 1%.

By observing the linear and nonlinear steady membrane equations (see Eqs. 4.1, 4.2 and 4.3), one might think that the steady-state is independent of the density ratio \(\mu\), since \(\mu\) does not appear in the equations. This is only true if we strictly look at the steady equations alone. In real life applications, experiments or high-fidelity simulations such as DNS, we cannot ignore the time-dependent nature of the results. In these cases, a steady-state solution cannot be evaluated a priori, but it can be reached with a time domain simulation (or experiment) that is long enough to allow all the initial transients to decay. For time-domain simulations to converge to a time-independent steady-state (e.g. no limit-cycle oscillations), the dynamic stability of the membrane has to be ensured by accurately selecting a combination of \(\mu\), \(c_{T0}\) and Reynolds number. Hence there exists an implicit dependency of the steady-state on the density ratio \(\mu\). The stability of the membrane about the undeflected zero angle of attack equilibrium condition will be extensively
Static Aeroelastic Response and Linear Stability

Figure 4.1: Steady-state deflection from linear model (blue), quasi-linear model (red), nonlinear model (yellow) and DNS (black) for $c_{T0} = 2$ (a-b-c), $c_{T0} = 3$ (d-e-f) and $c_{T0} = 4$ (g-h-i) at an angle of attack of 1 degree. Lower values of $c_{T0}$ indicate larger membrane compliance, hence larger deflections. Results are shown for Reynolds 100 (a-d-g), 1000 (b-e-h) and 10 000 (c-f-i).

discussed in Section 4.2. For the results shown in the present section, only combinations of $\mu$, $c_{T0}$ and Reynolds number that ensure a linearly stable membrane around the zero angle of attack equilibrium will be chosen. Since membranes are thin structures, we assume that their structural damping is negligible compared to the damping provided by the fluid. For this reason, we will always use a zero damping ratio $C_d = 0$.

Results from the models and the DNS at an angle of attack of 1 degree are represented in Fig. 4.1 for Reynolds numbers of 100, 1000 and 10 000. In order to investigate the effect of the membrane’s compliance on the static aeroelastic behavior of the system, different values of the zero deflection tension coefficient $c_{T0}$ are also tested. For a linear membrane (see Eq. 3.12) the structural compliance is affected by the zero-deflection tension coefficient $c_{T0} = 2\bar{E}_s\delta_0$ only. This means that combinations of $\bar{E}_s$ and $\delta_0$ that produce the same $c_{T0}$ will result in identical
responses of the membrane. This is not the case for the quasi-linear and the non-linear membrane. Because of the presence of $\bar{E}_s$ in the nonlinear term (see Eqs. 3.6 and 3.8), for two different membranes both $E_s$ and $\delta_0$ must be matched in order to obtain the same response. For the present validations, membranes with values of $cT_0$ of 2, 3 and 4 are tested. For the nonlinear cases, this is achieved by selecting $\delta_0 = 0.005$ and $\bar{E}_s = 200$, $E_s = 300$ and $\bar{E}_s = 400$, respectively. The steady-state deflection indicates perfect equilibrium between the pressure distribution and the restoring force given by the membrane tension. As can be observed in Fig. 4.1 for all the Reynolds numbers considered, reducing $cT_0$ has the effect of increasing the compliance of the membrane, resulting in larger deflections. When the deflection is small, both the membrane and the fluid present a linear behavior and the linear fluid-structure interaction model performs as well as the nonlinear models and the DNS. However, as the deflection amplitude increases, a linear fluid-structure interaction model quickly becomes inadequate to represent the behavior of the system. The linear model overestimates both the pressure difference across the wing and the compliance of the membrane, since it does not take into account the increase of the tension due to the deflection. As a consequence, the deflection required to reach an equilibrium condition is higher compared to the deflection predicted by the DNS and the nonlinear models, as evident from Fig. 4.1.

When the deflection amplitude increases, nonlinear effects are introduced both in the fluid and the membrane. However, for the structural parameters considered, at an angle of attack of 1 degree the main source of nonlinearity is the membrane, while the nonlinearities in the flow, although present, are weak. For this reason, the quasi-linear and the nonlinear fluid-structure interaction models, which are based on the assumption of linear aerodynamics and nonlinear structure, not only outperform the linear model, but they also match the results from the DNS well. Since the tension coefficient affects the compliance of the membrane, reducing the tension coefficient has the effect of increasing the deflections. Consequently the linear model performs better with membranes with larger values of $cT_0$, as observed in Fig. 4.1.

Simulations are repeated for an angle of attack of 2.5 degrees and the results are shown in Fig. 4.2. Because of the higher aerodynamic loads generated by the larger angle of attack, the validity of the linear model is restricted to lower values of the compliance and of the Reynolds number. The Reynolds number represents a key parameter for the nonlinear behavior of the flow, as can be inferred from both Fig. 4.1 and Fig. 4.2. As the Reynolds number increases, aerodynamic nonlinearities arise for smaller values of the deflection. Hence, for the same deflection, the prediction performance of the model deteriorates at higher Reynolds numbers. It is interesting to observe that at higher Reynolds numbers, membranes with the same tension coefficient show larger deflections: this implies that the pressure difference across the wing increases with the Reynolds number, as also indicated by Gordnier [2009]. For the same tension coefficient, the membrane would need a larger deflection to balance the higher pressure and to reach an equilibrium condition. A larger deflection also contributes to an additional increase in the pressure difference across the membrane. This is also evident when observing the lift coefficient for the considered cases at different Reynolds numbers, as summarized in
Fig. 4.2: Steady-state deflection from linear model (blue), quasi-linear model (red), nonlinear model (yellow) and DNS (black) for $c_{T0} = 2$ (a-b-c), $c_{T0} = 3$ (d-e-f) and $c_{T0} = 4$ (g-h-i) at an angle of attack of 2.5 degrees. When vortex shedding is present, the time-averaged deflection from DNS is represented with a solid black line, while the black dashed lines represent the instantaneous maximum and minimum deflections. Results are shown for Reynolds 100 (a-d-g), 1 000 (b-e-h) and 10 000 (c-f-i).

Fig. 4.3. For $c_{T0} = 2$, the lift coefficient increases by approximately a factor of 2 from $Re = 100$ to $Re = 10 000$ for both values of the angle of attack. This highlights the importance of a correct representation of Reynolds-number effects in the reduced-order model when targeting low-Reynolds-number applications. In addition, the compliance of the membrane represents the sensitivity of the structure to variations in the aerodynamic loads. Highly deformable membranes are more affected by small variations in the aerodynamic field compared to membranes with a lower compliance. This can be observed in Figs. 4.1, 4.2 and 4.3 by comparing the changes in the deflection and lift with Reynolds number for different values of $c_{T0}$.
Figure 4.3: Steady-state lift coefficient $C_L$ from linear model (blue), quasi-linear model (red), nonlinear model (yellow) and DNS (black). Results are shown for Reynolds 100 (a-d), 1000 (b-e) and 10000 (c-f) and for angles of attack of 1 degree (a-b-c) and 2.5 degrees (d-e-f). When vortex shedding is present, the time-averaged lift from DNS is represented with a filled black circle, while the empty black circles represent the instantaneous maximum and minimum lift.

Figure 4.4: Steady-state deflection modes ratio $W_2/W_1$ and $W_3/W_1$ from linear model (blue), quasi-linear model (red), nonlinear model (yellow) and DNS (black). Results are shown for Reynolds 100 (a-d), 1000 (b-e) and 10000 (c-f) and for angles of attack of 1 degree (a-b-c) and 2.5 degrees (d-e-f). When vortex shedding is present, time-averaged values from DNS are represented with filled black symbols, while the empty symbols represent the instantaneous maximum and minimum values of the modal deflection ratio.
In order to investigate how the shape of the membrane changes with the membrane compliance and with the Reynolds number, the steady-state deflection is decomposed into its leading modes from 1 to 3 (see Eq. 2.4). Contributions of mode $W_2$ and mode $W_3$ to the final shape are plotted as a fraction of the leading mode $W_1$ for different values of the tension coefficient. Results for Reynolds numbers 100, 1000 and 10000 are shown in Fig. 4.4. The steady-state shape is dominated by mode 1, $W_1$ being the largest of the deflection coefficients. The contributions of mode 2 and mode 3 increase with the tension coefficient and are responsible for moving the point of maximum camber towards the trailing edge as the deflection increases. Overall, the Reynolds number does not have a significant effect on the shape of the membrane, except for the case with $Re = 10000$ and $c_T = 4$. In this case, vortex shedding is present. The unsteady loads generated by the shed vortices constantly change the shape of the membrane, affecting the modal deflection ratio. For the same Reynolds number and the same angle of attack, by decreasing the tension coefficient $c_{T_0}$—and hence increasing the compliance of the membrane—vortex shedding is suppressed, as observed from Figs. 4.2, 4.3 and 4.4.

4.1.2 Results at moderate and large angles of attack

Finally, we want to test the performance of the models for moderate and large values of the angle of attack, when large nonlinearities are present in the flow. This is achieved by performing simulations at different angles of attack in the range $0.5 - 20$ degrees for $Re = 100$, $Re = 1000$ and $Re = 10000$. A tension coefficient $c_{T_0} = 3$ and a density ratio $\mu = 0.5$ are chosen for the membrane. Results for the deflection of the wing at an angle of attack of 2.5, 5, 10, 15 and 20 degrees are shown in Fig. 4.5. While models and simulations agree well for small values of the angle of attack, large incidences introduce strong nonlinear effects in the flow, which result in discrepancies between the prediction of the models and the DNS results. In particular, the linear model overestimates the pressure loads, hence overestimating the maximum camber, as already observed for the 1 degree angle of attack case with higher membrane compliance. For $Re = 1000$ and $Re = 10000$, at moderate and large angles of attack, the flow exhibits unsteady nonlinear vortex shedding. As a result, the membrane does not present a steady state, but constantly oscillates under the effect of the vortices. Because of the assumption of linear aerodynamics, this behavior is not taken into account by the models. When vortex shedding is present, the average deflection of a period of 15 convective time units obtained from DNS is shown in Fig. 4.5 with a solid black line. The time instants corresponding to the maximum and the minimum deflection are indicated with black dashed lines. Interestingly, the quasi-linear and the nonlinear models are able to capture reasonably well the average membrane deflection at large angles of attack. This is probably an indicator that the nonlinearities in the mean flow are weak compared to the strong vortices present in the instantaneous flowfield and, most importantly, that their effect on the mean deflection of the membrane is small.
Figure 4.5: Steady-state deflection from linear model (blue), quasi-linear model (red), nonlinear model (yellow) and DNS (black) for $c_{T_0} = 3$. When vortex shedding is present, the time-averaged deflection from DNS is represented with a solid black line, while the black dashed lines represent the instantaneous maximum and minimum deflections. Results are shown for Reynolds numbers 100 (a-d-g-l-o), 1000 (b-e-h-m-p) and 10000 (c-f-i-n-q). The angle of attack of the wing in degrees is the following: 2.5 (a-b-c), 5 (d-e-f), 10 (g-e-h), 15 (i-l-m) and 20 (o-p-q).
Figure 4.6: steadystate lift from linear model (blue), quasi-linear model (red), non-linear model (yellow) and DNS (black) for \(c_T = 3\) as a function of the angle of attack. When vortex shedding is present, the time-averaged lift from DNS is represented with a filled black circle, while the empty black circles represent the instantaneous maximum and minimum lift. Results are shown for Reynolds numbers 100 (a-d), 1 000 (b-e) and 10 000 (c-f).

The lift coefficient \(C_L\) as a function of the angle of attack is plotted in Fig. 4.6. As expected, the linear model predicts a linear relationship between the lift and the angle of attack, while DNS shows that for higher angles of attack, due to nonlinear aerodynamic effects such as vortex shedding and separation, the lift decreases with respect to linear predictions. Although the quasi-linear and the nonlinear model perform reasonably well for the average deflection even in the presence of vortex shedding, their performance for the lift at angles of attack larger than 10 degrees is poor. This means that the aerodynamic field is sensitive to changes in the wing configuration: small changes in the wing deflection cause moderate changes in the aerodynamic loads.


4.2 Membrane dynamic aeroelastic stability

As discussed in the previous section, the predictive capabilities of the linear fluid-structure interaction model are limited to small angles of attack and small deflections. Then one might ask what the advantage of having a linear model is and why not just use the nonlinear models. The reason is that a linear framework, within its range of applicability, can provide useful information on the behavior of the system without having to run a campaign of simulations. This is the case for the membrane’s dynamic stability at zero degree angle of attack. When the angle of attack is zero, the undeformed wing represents an equilibrium condition of the membrane-fluid system. This equilibrium condition can be stable or unstable. If it is stable, the membrane recovers, after a transient, its original undeflected configuration every time that a vanishing perturbation is applied. If the equilibrium is unstable, the nonlinear membrane might present undamped oscillations that can be detrimental for its aerodynamic performance. It is then important to investigate under what conditions the zero-deflection configuration represents a stable equilibrium for the membrane.

4.2.1 Dynamic stability

The linear unsteady model of Eq. 3.20 can be conveniently represented in the classical state-space form

\[
\begin{align*}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}.
\end{align*}
\] (4.4)

The analogy between Eq. 3.20 and Eq. 4.4 is straightforward. The dynamic aeroelastic stability of the membrane can be investigated using the linear model by observing the eigenvalues of the state matrix \( \mathbf{A} \) from Eq. 4.4. Indicating with \( \lambda_n = a_n + ib_n \) the generic complex \( n \)-th eigenvalue (with \( n \) from 1 to \( n_{fs} \)), the aeroelastic stability of the membrane depends on the following conditions:

- if all \( a_n < 0 \): stable,
- if any \( a_n = 0 \) \( \land \) \( b_n \neq 0 \): neutrally-stable flutter,
- if any \( a_n > 0 \) \( \land \) \( b_n \neq 0 \): flutter,
- if any \( a_n > 0 \) \( \land \) \( b_n = 0 \): divergence.

For the sole membrane model from Eq. 3.12 with no structural damping (\( C_d = 0 \)) and no interaction with the fluid (\( \Delta C_p = 0 \)), any vanishing perturbation of the equilibrium condition will result in finite-amplitude undamped oscillations for any values of \( c_{T0} \) and \( \mu \). In fact, Eq. 3.12 with \( C_d = 0 \) and \( \Delta C_p = 0 \) represents an undamped membrane oscillating in a vacuum; if no external forces are applied, there
is a continuous exchange of elastic potential energy and kinetic energy, without any dissipation.

When coupling the membrane with the fluid, there is a mutual interaction between the structure and the free-stream. The flow provides energy to the system through the free-stream velocity and there is a continuous exchange of energy between fluid and structure through the coupling terms involving pressure and deflection. If the real part $a_n$ of any of the eigenvalues $\lambda_n$ is strictly positive, it means that the system is absorbing energy from the free-stream over a cycle, and consequently the amplitude of the response of the membrane increases in time (flutter and/or divergence). If the real part of all the eigenvalues is negative, the system presents positive damping: the system is stable and the perturbed membrane will return to the equilibrium position after the perturbation and all the transients have decayed.

By analyzing the eigenvalues of the matrix $\mathbf{A}$ for different values of $\mu$ and $c_{T_0}$, a stability plot is generated for Reynolds numbers 100, 500, 1 000 and 10 000, as represented in Fig. 4.7. The stability plot for Reynolds 100 only presents instability due to flutter. The instability arises for low values of the tension coefficient.
and it shows a dependence on the density ratio as well; as the density ratio $\mu$ decreases, the tension coefficient required to have a stable system decreases as well. Higher Reynolds numbers show the same trend with respect to flutter, but divergence also appears as an additional form of instability for some combination of $\mu$ and $c_{T_0}$. For Reynolds number 500, divergence shows up for values of the tension coefficient between 0.314 and 0.694. For higher Reynolds numbers, the divergence region covers a broader range of tension coefficients, increasing from $0.317 < c_{T_0} < 0.864$ for Reynolds 1000 to $0.079 < c_{T_0} < 1.328$ for Reynolds 10000. As the Reynolds number increases, the upper limit of the divergence instability approaches the value of $c_{T_0} = 1.72745$, indicated by various authors as the static instability threshold obtained using thin-airfoil potential aerodynamics [Nielsen, 1963, Tiomkin & Raveh, 2017]. For higher Reynolds numbers, the trend of the stability plots seems to qualitatively tend to the results of Tiomkin & Raveh [2017] obtained with potential aerodynamics.

Since the current stability plots are based on models obtained from compressible DNS, they are specific to the Mach number used in the current simulations, $M = 0.2$. A stability analysis based on the Mach number is beyond the purpose of the current work, but it is important to mention that the Mach number affects the dynamic high-frequency response of the system (see Section 2.3.2) and, consequently, compressibility effects might affect the membrane stability diagrams as well. This is especially important when trying to compare the stability plots with results based on incompressible aerodynamics.

It is also important to recall that in the present work we are neglecting the effect of viscous shear forces on the membrane, as discussed in Section 3.1. We can imagine that viscous effects might change the tension coefficient by locally shrinking or stretching the membrane, hence affecting the stability plots. It is beyond the scope of this work to investigate the stability of the membrane in the presence of viscous effects. However, we can speculate that their effect on the linear stability might still be simulated in the present framework by changing the initial pre-stretch of the membrane $\delta_0$. Information from DNS would be required to determine the correct value of $\delta_0$ and extensive validation should be carried out to verify the results.

4.2.2 Validation of stability and transients

The unsteady capabilities of the reduced-order models are validated against DNS for different operating conditions, indicated with dots in the stability plots of Fig. 4.7. For all the operating conditions tested, an external forcing is applied to the membrane at rest and the response in terms of lift coefficient and of the leading deflection modes $W_1$ and $W_2$ is measured. The external forcing is a transverse distributed force in the form of a sine wave in space $C_f(x,t) = C_f(t) \sin(\pi x)$. In time, the force is a ramp-up, ramp-down function (see Appendix A.1) and its
Figure 4.8: Membrane aeroelastic response to an external forcing with $\bar{f}_{\text{max}} = 0.395$ for $E_s = 600$, $\delta_0 = 0.001$ ($cT_0 = 1.2$) and $\mu = 3$ at Reynolds 100. As predicted by the model, the membrane is stable. Results from the linear (blue), the quasi-linear (red) and the nonlinear model (yellow) are compared with DNS (black) for the lift response (a) and the deflection modes from 1 (b) and 2 (c). The membrane’s deflection at different time intervals is represented in figures from (d-i).

A temporal history is defined by the following expression

$$C_{f1}(t) = \bar{f}_{\text{max}} \frac{G_1(t)}{\max G_1(t)},$$

with $G_1$ defined by Eq. A.2 in Appendix A.1. The following values are adopted: $a = 10.5$, $t_1 = 0.95$, $t_2 = 1.05$, $t_3 = 1.45$ and $t_4 = 1.55$. Figure 4.8 shows the responses for $E_s = 600$, $\delta_0 = 0.001$ (resulting in $cT_0 = 1.2$) and $\mu = 3$ at $Re = 100$ to an external forcing with $\bar{f}_{\text{max}} = 0.395$. For the reduced-order models and the DNS the membrane is stable, as predicted by the linear stability theory applied to the linear model. The response from the quasi-linear model and the nonlinear model are in excellent agreement with the DNS, while the linear model is unable to accurately capture the transient dynamics. The deflection of the membrane from the models and the DNS is plotted in Fig. 4.8(d-i) for different
Figure 4.9: Membrane aeroelastic response to an external forcing with $f_{\text{max}} = 0.197$ for $E_s = 50$, $\delta_0 = 0.001$ ($c_{T_0} = 0.1$) and $\mu = 3$ at Reynolds 100. For the considered operating condition the linear membrane presents a flutter instability. Results from the linear (blue), the quasi-linear (red) and the nonlinear model (yellow) are compared with DNS (black) for the lift response (a) and the deflection modes from 1 (b) and 2 (c). The membrane’s deflection at different time intervals is represented in figures from (d-i). Because of the membrane’s instability, results from the linear model are omitted from figures (d) to (i).

time instants using 10 deflection modes. By comparing the results from the quasi-linear and the nonlinear models against DNS, it is clear that 10 deflection modes are sufficient to represent the shape of the membrane for the considered cases. The linearly unstable condition with $E_s = 50$, $\delta_0 = 0.001$ ($c_{T_0} = 0.1$) and $\mu = 3$ with $f_{\text{max}} = 0.197$ are presented in Fig. 4.9. The amplitude of the response of the linear model grows exponentially because of the instability. For the quasi-linear and the nonlinear response, the tension increase due to the deflection results in limit-cycle oscillations in the lift and in the deflection, as also predicted with the DNS. Despite minor differences, the nonlinear models and the DNS results are in excellent agreement.

The membrane’s stability at a Reynolds number of 1 000 is now investigated for different operating conditions. Figure 4.10 shows the stable response of a
Figure 4.10: Membrane aeroelastic response to an external forcing with $\tilde{f}_{\text{max}} = 0.395$ for $\tilde{E}_s = 750$, $\delta_0 = 0.001$ ($cT_0 = 1.5$) and $\mu = 1$ at Reynolds 1000. For the considered operating condition the linear membrane is stable. Results from the linear (blue), the quasi-linear (red) and the nonlinear model (yellow) are compared with DNS (black) for the lift response (a) and the deflection modes from 1 (b) and 2 (c). The membrane’s deflection at different time intervals is represented in figures from (d-i).

membrane with $\tilde{E}_s = 750$, $\delta_0 = 0.001$ (resulting in $cT_0 = 1.5$) and $\mu = 1$. Similar to the case at $Re = 100$, the linear model is unable to represent the transient dynamics, while the predictions from the quasi-linear and nonlinear model agree very well with the DNS. Results for a linearly unstable case with an instability due to divergence are shown in Fig. 4.11, while results from a case with a flutter instability are plotted in Fig. 4.12. The parameters of the membrane-fluid system are the following: $\tilde{E}_s = 300$, $\delta_0 = 0.001$ ($cT_0 = 0.6$), $\mu = 1$ and $\tilde{f}_{\text{max}} = 0.395$ for the case with divergence and $\tilde{E}_s = 100$, $\delta_0 = 0.001$ ($cT_0 = 0.2$), $\mu = 1$ and $\tilde{f}_{\text{max}} = 0.0395$ for the case with flutter. For the case with divergence, due to the linear instability in the vicinity of the equilibrium condition with zero deflection, the responses of the quasi-linear and nonlinear model settle to a new equilibrium condition, as indicated in Fig. 4.11. For the linearly unstable case with flutter, the quasi-linear and the nonlinear membrane present limit-cycle oscillations in the response (see Fig. 4.12). Nonlinear limit-cycle oscillations will be discussed more
Figure 4.11: Membrane aeroelastic response to an external forcing with $\tilde{f}_{\text{max}} = 0.395$ for $E_s = 300, \delta_0 = 0.001 \ (cT_0 = 0.6)$ and $\mu = 1$ at Reynolds 1000. For the considered operating condition the linear membrane has an instability due to divergence. Results from the linear (blue), the quasi-linear (red) and the nonlinear model (yellow) are compared with DNS (black) for the lift response (a) and the deflection modes from 1 (b) and 2 (c). The membrane’s deflection at different time intervals is represented in figures from (d-i). Because of the membrane’s instability, results from the linear model are omitted from figures (d) to (i).

in detail in Chapter 5.

Validations for Reynolds 500 and 10 000 have been omitted for the sake of brevity, but the models show the same accuracy as that seen for Reynolds 100 and 1 000. Based on the results presented, it is possible to conclude that the linear model provides useful insights on the membrane stability, but it is unable to capture the transient dynamics of the membrane due to nonlinearities in the structure. The quasi-linear and the nonlinear reduced-order models outperform the linear model and, as long as the nonlinearities introduced by the membrane deflection in the flow are weak, they are able to accurately represent the dynamic interaction between the membrane and fluid for different Reynolds numbers.
Figure 4.12: Membrane aeroelastic response to an external forcing with $f_{\text{max}} = 0.0395$ for $E_s = 100$, $\delta_0 = 0.001$ ($c_{T0} = 0.2$) and $\mu = 1$ at Reynolds 1000. For the considered operating condition the linear membrane presents a flutter instability. Results from the linear (blue), the quasi-linear (red) and the nonlinear model (yellow) are compared with DNS (black) for the lift response (a) and the deflection modes from 1 (b) and 2 (c). The membrane’s deflection at different time intervals is represented in figures from (d) to (i). Because of the membrane’s instability, results from the linear model are omitted from figures (d) to (i).

4.3 Summary

The unsteady fluid-structure interaction reduced-order models have been employed for the static aeroelastic analysis of a two-dimensional membrane at low and moderate Reynolds numbers. From predictions obtained from different models, we concluded that the Reynolds number has an effect on both static and dynamic aeroelastic performance of the membrane, indicating the importance of models that accurately capture the Reynolds number aerodynamic effects in the analysis of small-scale membrane applications. In particular, the models have been validated for the prediction of steady camber and lift of the membrane for different values of the tension coefficient and the angle of attack. The linear model is able to accurately match the results from DNS only when both the nonlinearities in
the flow and in the structure are small. The quasi-linear and the nonlinear models outperform the linear model when moderate and large structural nonlinearities are present. Simulations were repeated for different Reynolds regimes between 100 and 10,000, showing how the Reynolds number has an effect on both the maximum deflection and the lift of the membrane wing. The performance of the models have been tested in the case of moderate and large angles of attack, when strong aerodynamic nonlinearities such as vortex shedding are present. Although the quasi-linear and the nonlinear models provide good predictions for the time-averaged deflection, they are unable to accurately represent the lift coefficient. This highlights the limitation of the linear aerodynamic assumption and introduces the need for a representation of the nonlinear aerodynamics at large angles of attack.

Despite its limited range of applicability, the linear model presents the advantage of being able to provide useful information on the membrane stability for different values of the tension coefficient and the density ratio, without having to run an expensive campaign of simulations. Stability maps have been introduced and discussed for different Reynolds numbers. The results have been validated against the nonlinear models and the DNS.
Chapter 5

Forced and Autonomous Nonlinear Response of the Membrane

In the present chapter, the quasi-linear and the nonlinear fluid-structure interaction models are solved using Harmonic Balance methods. The two models in combination with Harmonic Balance methods are adopted to investigate the unsteady aeroelastic response of a membrane wing pitching about its leading edge at $Re = 100$, for different values of the pitching angle. Differences between the two Harmonic Balance approaches are discussed and validations of the methods are carried out using high-fidelity Direct Numerical Simulations. The effect of the Reynolds number on the wing’s lift response is taken into account by increasing the Reynolds number to $Re = 1000$. Harmonic Balance methods are also used to analyze the limit-cycle autonomous oscillations of the membrane for linearly unstable cases. The frequency of the oscillations can be predicted by introducing a frequency-searching algorithm in the solution procedure.

5.1 Introduction

So far we have focused our attention on the static aeroelastic response of the membrane wing and on its linear stability. However, small-scale flapping-flight is characterized by the unsteady mutual interaction between a highly deformable surface, the membrane wing, and an unsteady flow. In the present chapter, we investigate the forced unsteady aeroelastic response of a membrane wing subjected to a periodic forcing. The periodic forcing is represented by a rigid pitching motion, which can be seen as a first approximation of a flapping cycle. For this purpose, the linear fluid-structure interaction model in state-space form provides useful insights on the membrane’s aeroelastic response in the frequency domain, without the need for running simulations. However, we have seen in the previous chapter that its range of applicability is limited by the assumption of small deflections.
For natural flyers such as bats, the maximum camber of the wing can easily reach values exceeding 10% of the chord during flight [Song et al., 2008], resulting in strong nonlinear unsteady effects excited in the wing’s structural response. Hence, the importance of analytical models that can represent the membrane’s nonlinear unsteady behavior for moderate and large deflections, while avoiding the direct numerical integration of the governing equations.

A solution is represented by the quasi-linear and the nonlinear fluid-structure interaction models presented in Chapter 3 in combination with Harmonic Balance methods. Classical Harmonic Balance (HB) methods, also referred to as Frequency-Domain Harmonic Balance (FDHB) methods, have been successfully used in many applications to model the periodic response of nonlinear dynamic systems to harmonic inputs. FDHB methods are based on the idea of representing the input and the output of the system as a sum of harmonics in time using a truncated Fourier series and solving for the harmonic coefficients of the output through an iterative procedure. A brief literature review on classical Harmonic Balance methods with references to fluid-structure interaction problems can be found in Liu et al. [2006]. One of the disadvantages of the FDHB is the difficulty to represent systems with strong nonlinearities.

In order to overcome this limitation, Hall et al. [2002] and Thomas et al. [2002] proposed a novel formulation in which the frequency-domain harmonic coefficients are cast into the time domain. This alternative approach, often called High Dimensional Harmonic Balance (HDHB) or Time-Domain Harmonic Balance (TDHB), can be easily applied to highly nonlinear systems and it is suitable for large-scale problems. HDHB has been particularly successful in fluid-dynamic applications due to its adaptability to pre-existing time marching nonlinear solvers commonly used in CFD problems. A good overview on HDHB applied to periodic flows and fluid-structure interaction, including turbomachinery, helicopter rotors and unsteady airfoils, is presented in Hall et al. [2013] and Yao & Marques [2015]. HDHB has also been used for the prediction of limit-cycle oscillations in turbomachinery [Ekici & Hall, 2011] and airfoils in transonic flows [Yao & Marques, 2015]. The work of LaBryer & Attar [2010] applies the HDHB to a plunging membrane wing and to a flapping dragonfly wing.

In many cases, HDHB methods tend to produce aliased non-physical solutions in addition to the physical ones. A comparison with classical HB and an explanation of the origin of the aliased solution is given by Liu et al. [2006]. A de-aliasing technique based on a low-pass filtering of the field variables has been implemented and successfully tested on a Duffing oscillator by LaBryer & Attar [2009]. When tested on a plunging membrane wing, the same de-aliasing procedure failed to eliminate the non-physical solutions for moderate and large deflections [LaBryer & Attar, 2010]. A different approach to reduce the generation of non-physical solutions was proposed by Dai et al. [2013]. They introduced a time collocation method—proving its equivalence to a HDHB method [Dai et al., 2012]—and they extended it by collocating the time-domain harmonic coefficients on a number of time samples larger than the number of equations of the method. The resulting over-determined system, when solved by minimizing the norm of the residual, was
able to significantly reduce the occurrence of non-physical solutions with respect to the original HDHB method.

## 5.2 Harmonic Balance methods for the aerodynamic response of the membrane

The nonlinear response of a membrane wing immersed in a fluid is investigated in the present chapter using Harmonic Balance methods. First, we will introduce the Frequency-Domain Harmonic Balance (FDHB) method, often referred to as classical Harmonic Balance in the literature, and we will discuss the difficulties associated with the representation of strongly nonlinear systems. Finally, we will present an alternative HB method in the time domain, introduced in order to overcome some of the limitations associated with FDHB techniques.

### 5.2.1 Frequency-Domain Harmonic Balance

The dynamics of a general membrane model can be expressed by the following equation

\[ 2\mu \ddot{q} + 2C_d \dot{q} + r = f, \]  

(5.1)

where \( \ddot{q} \) and \( \dot{q} \) are the time derivatives of the variable \( q \), \( f \) represents the forcing vector and \( r \) contains all nonlinear terms. The introduction of a generic variable \( q \) allows us to conveniently represent both the nonlinear and the quasi-linear model with Eq. 5.1: for the nonlinear model, \( q \) represents the vertical displacement of membrane points \( w \), while for the quasi-linear model it represents the deflection modes \( \mathbf{W} \), as listed in Tables C.1 and C.2 in Appendix C.3.

The classical Harmonic Balance method is based on the assumption that the periodic steady-state of the system, resulting from the application of a prescribed periodic forcing, can be approximated by a truncated Fourier series that contains multiples of the fundamental frequency \( \omega \) of the forcing term \( f \):

\[ q(t) \approx \hat{q}_0 + \sum_{h=1}^{N_H} \left( \hat{q}_{2h-1} \cos(h\omega t) + \hat{q}_{2h} \sin(h\omega t) \right), \]  

(5.2)

\[ r(t) \approx \hat{r}_0 + \sum_{h=1}^{N_H} \left( \hat{r}_{2h-1} \cos(h\omega t) + \hat{r}_{2h} \sin(h\omega t) \right), \]  

(5.3)

\[ f(t) = \hat{f}_0 + \sum_{h=1}^{N_H} \left( \hat{f}_{2h-1} \cos(h\omega t) + \hat{f}_{2h} \sin(h\omega t) \right). \]  

(5.4)

\( N_H \) indicates the number of harmonics used to represent the periodic solution of the membrane’s dynamics. Substituting Eqs. 5.2, 5.3 and 5.4 in Eq. 5.1, the system
of equations obtained by splitting Eq. 5.1 into its harmonic components can be written as the following algebraic system of equations:

\begin{align*}
\hat{r}_0 &= \hat{f}_0 \\
-2\mu\omega^2 h^2 \hat{q}_{2h-1} + 2C_d\omega h \hat{q}_{2h} + \hat{r}_{2h-1} &= \hat{f}_{2h-1} \\
-2\mu\omega^2 h^2 \hat{q}_{2h} - 2C_d\omega h \hat{q}_{2h-1} + \hat{r}_{2h} &= \hat{f}_{2h},
\end{align*}

which can be conveniently cast in matrix form, resulting in

\begin{equation}
2\mu\omega^2 \bar{A}^2 \hat{Q} + 2C_d\omega \bar{A} \hat{Q} + \hat{R} = \hat{F}. \tag{5.5}
\end{equation}

The matrices \( \hat{Q}, \hat{R} \) and \( \hat{F} \) are the harmonic coefficients of the system’s response in matrix form, while the matrix \( \bar{A} \) is defined as follows

\[
\bar{A} = \begin{bmatrix}
0 \\
J_1 \\
J_2 \\
\vdots \\
J_{N_H}
\end{bmatrix}, \quad J_h = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix}, \text{ with } h=1,2,\ldots,N_H. \tag{5.6}
\]

Matrix \( \hat{R} \) contains the harmonic coefficients of the nonlinear terms in matrix form. All matrices and equations for the frequency-domain Harmonic Balance applied to the two membrane models are summarized in Tables C.1 and C.2.

Depending on the nature of the nonlinearity, finding a closed expression for \( \hat{R} \) can be a tedious and cumbersome task. This is the case for the nonlinear membrane equation, whose nonlinearity contains the integral of a square root (see Eq. 3.7). To simplify the representation of the system’s dynamics in the frequency domain, the quasi-linear membrane model can be adopted. This model, indicated by Eqs. 3.8 and 3.9, simplifies the nature of the nonlinear terms and allows for a more convenient representation of the membrane’s deflection through Fourier structural modes, while still retaining the nonlinear effect of the elongation of the membrane. Given the difficulties related to representing the nonlinear membrane in a frequency-domain harmonic balance formulation, we will apply the FDHB only to the quasi-linear fluid-structure interaction model (Eq. 3.19), whose variables can
be expressed by the following harmonic decompositions:

\[
\alpha(t) = \alpha_0 + \sum_{h=1}^{N_H} (\alpha_{2h-1} \cos(h\omega t) + \alpha_{2h} \sin(h\omega t)) \tag{5.7}
\]

\[
W_k(t) \approx W_{k,0} + \sum_{h=1}^{N_H} (W_{k,2h-1} \cos(h\omega t) + W_{k,2h} \sin(h\omega t)) \tag{5.8}
\]

\[
\Delta C_{p_k}(t) \approx \Delta C_{p_k,0} + \sum_{h=1}^{N_H} (\Delta C_{p_k,2h-1} \cos(h\omega t) + \Delta C_{p_k,2h} \sin(h\omega t)) \tag{5.9}
\]

\[
C_L(t) \approx C_{L,0} + \sum_{h=1}^{N_H} (C_{L,2h-1} \cos(h\omega t) + C_{L,2h} \sin(h\omega t)) \tag{5.10}
\]

\[
C_f(t) = C_{f,0} + \sum_{h=1}^{N_H} (C_{f,2h-1} \cos(h\omega t) + C_{f,2h} \sin(h\omega t)). \tag{5.11}
\]

Given the harmonic coefficients of the inputs \(\alpha\) and \(C_f\), Eq. 3.10 coupled with the aerodynamic model can be solved for the \((2N_H+1) \times N_H\) coefficients \(W_{k,2h-1}, W_{k,2h}\) (with \(N\) representing the total number of structural modes and \(N_H\) the total number of harmonics considered) using a Newton-Raphson iterative procedure (see Appendix C.4.1). The harmonic coefficients of \(C_L\) are finally obtained from \(\ddot{\alpha}\) and \(W_k\) through the aerodynamic model. The number of harmonics retained in the solution is decided a priori and represents one of the parameters of the method. When large nonlinear effects are excited in the system, the number of harmonics that give a non negligible contribution to the response increases. The full derivation of the nonlinear terms in Eqs. 3.8 and 3.9 can be found in Appendix C.1.

### 5.2.2 Time-Domain Harmonic Balance

Harmonic Balance methods in the time domain offer an alternative approach to FDHB for the determination of time periodic solutions, allowing for an easier representation of strong nonlinear terms. One of the most successful time-domain approaches in the literature is often referred to as the High Dimensional Harmonic Balance (HDHB). It is based on the idea of discretizing Eq. 5.1 by sampling its variables in the time domain at \(2N_H + 1\) equally-spaced time samples over one period of oscillation. The method takes as its unknowns the solution at the \(2N_H + 1\) time samples. These coefficients are related by a matrix transformation to the \(2N_H + 1\) coefficients in the frequency domain introduced in Eqs. 5.2, 5.3 and 5.4. The original mathematical formulation of the HDHB can be found in Hall et al. [2002].

One of the main drawbacks of the HDHB is the generation of non-physical aliased solutions, due to some approximations introduced when representing the nonlinear term, as discussed in Liu et al. [2006]. For polynomial nonlinearities these non-physical solutions can be eliminated with a de-aliasing procedure based on the introduction of a low-pass filter [LaBryer & Attar, 2009], but this method
fails when more complicated nonlinearities are taken into account, such as in the case of a membrane [LaBryer & Attar, 2010]. To reduce, and even eliminate, non-physical solutions in the case of systems with strong nonlinearities, Eq. 5.1 can be discretized by sampling the solution in time on a number of collocation points \( N_c > 2N_H + 1 \) [Dai et al., 2013]. This results in an over-determined system of equations, that can be solved by means of a least-square method by minimizing the norm of the residual [Dai et al., 2013]. Introducing \( t_i = \frac{2\pi}{N_c} (i - 1) \) for \( i = 1, 2, \ldots, N_c \), the \( N_c \) equally-spaced collocation time samples used to discretize Eq. 5.1, we introduce the following matrices:

\[
\tilde{Q} = \begin{bmatrix} q(t_0)^T \\ q(t_1)^T \\ \vdots \\ q(t_{N_c})^T \end{bmatrix}_{N_c \times N_x}, \quad \tilde{R} = \begin{bmatrix} r(t_0)^T \\ r(t_1)^T \\ \vdots \\ r(t_{N_c})^T \end{bmatrix}_{N_c \times N_x}, \quad \tilde{F} = \begin{bmatrix} f(t_0)^T \\ f(t_1)^T \\ \vdots \\ f(t_{N_c})^T \end{bmatrix}_{N_c \times N_x}.
\]

Matrices \( \tilde{Q}, \tilde{R} \) and \( \tilde{F} \) are related to their counterparts in the frequency domain, \( \hat{Q}, \hat{R} \) and \( \hat{F} \), by the following matrix transformations:

\[
\hat{Q} = E^{-1} \tilde{Q}, \quad \hat{R} = E^{-1} \tilde{R}, \quad \hat{F} = E^{-1} \tilde{F};
\]

\[
\hat{Q} = E \hat{Q}, \quad \hat{R} = E \hat{R}, \quad \hat{F} = E \hat{F}.
\]

Expressions for the Discrete Fourier Transform rectangular matrix \( E \) and its corresponding inverse Fourier transform matrix \( E^{-1} \) are provided in Appendix C.2. Using the transformations presented in Eq. 5.13, we can now recast Eq. 5.5 into a new algebraic form that only contains the time-domain coefficients:

\[
2\mu\omega^2 \tilde{D}^2 \tilde{Q} + 2C_{\delta T} \tilde{D} \tilde{Q} + \hat{R} = \hat{F},
\]

with \( \tilde{D} \) defined in Appendix C.2. Equation 5.15 has the same form as Eq. 5.5, but it contains coefficients expressed in the time domain rather than in the frequency domain. The solution can be found through an iterative Newton-Raphson procedure (see Appendix C.4.2 for the details) because, thanks to the time-domain formulation, the residual of Eq. 5.15 is now known. In the present work, the HDHB will be applied to the nonlinear membrane only, since the quasi-linear model can be more conveniently represented in the frequency domain. For the nonlinear membrane, the matrices listed in Table C.1 can be transformed into the time domain by applying the transformations indicated in Appendix C.2.

### 5.2.3 Methods of solution for the membrane’s autonomous response

A linear stability analysis of a membrane wing at zero-degree angle of attack and low Reynolds number with respect to structural parameters such as the zero-deflection tension coefficient \( c_{T0} = 2E_s \delta_0 \) and the density ratio \( \mu \) has been presented in Section 4.2. For specific combinations of the structural parameters, the membrane is linearly unstable. When the instability is due to flutter, the membrane
absorbs energy from the free-stream and its linear aeroelastic response is characterized by unbounded, exponentially growing oscillations. This is because the tension coefficient, in a linear model, is constant and for an unstable membrane it is too low to counteract the destabilizing effect of the aerodynamic forces. For a nonlinear membrane, the restoring force provided by the tension increases with the deflection, hence limiting the amplitude of the oscillations. The resulting autonomous limit-cycle oscillations (LCO) can be investigated using Harmonic Balance methods, as already successfully done by Ekici & Hall [2011] for turbomachinery, Yao & Marques [2015] for an airfoil and a delta wing in transonic flow and McMullen [2003] and McMullen et al. [2006] for cylinder wakes at low Reynolds numbers.

For the autonomous response of a membrane wing, the Harmonic Balance equations contain an additional unknown variable, represented by the frequency of oscillation $\omega_{LCO}$. This is in contrast to a forced membrane, for which the fundamental frequency $\omega$ of the forcing term is given, and the solution of the Harmonic Balance equations (e.g. Eq. 5.5) is a fixed-point iteration problem that can be carried out with a Newton-Raphson method. For the autonomous response, a frequency-searching algorithm therefore has to be added to the solution procedure in order to determine $\omega_{LCO}$.

The procedure adopted in the present work, introduced by Yao & Marques [2015], is based on updating the frequency $\omega$ using a Newton-Raphson iteration in which the Jacobian is obtained by minimizing the $L_2$ norm of the residual $\mathcal{R}$ defined from Eq. 5.5 as follows:

$$L_2^2 = \frac{1}{2} \mathcal{R}^T \mathcal{R} =$$

$$= \frac{1}{2} \left[ 2 \mu \omega^2 \bar{A}^2 \hat{Q} + 2 C_{d\omega} \bar{A} \hat{Q} + \hat{R} - \hat{F} \right]^T \left[ 2 \mu \omega^2 \bar{A}^2 \hat{Q} + 2 C_{d\omega} \bar{A} \hat{Q} + \hat{R} - \hat{F} \right]. \quad (5.16)$$

Formally, indicating with $\omega_n$ the frequency at the iteration step $n$ (the LCO subscript has been dropped for simplicity), the frequency at the next iteration $\omega_{n+1}$ is expressed by the following relationship:

$$\omega_{n+1} = \omega_n - \lambda J_{\omega}^{-1} L_2^2, \quad J_{\omega} = \frac{\partial L_2^2}{\partial \omega}, \quad (5.17)$$

with $\lambda < 1$ serving as a relaxation parameter. The Jacobian $J_{\omega}$ can be evaluated either numerically or analytically. To improve the convergence of the results, when evaluating $J_{\omega}$ it is important to take into account the change of the aerodynamic forces due to a change in $\omega$ [Yao & Marques, 2015].

As already mentioned in Sections 5.2.1 and 5.2.2, Eq. 5.5 coupled with the aerodynamic model can be solved for the forced response of the membrane using an iterative Newton-Raphson procedure, providing an initial guess $\hat{Q}_0$ for matrix $\hat{Q}$. For the autonomous response, an initial guess for the unknown $\omega_{LCO}$ has to be provided in addition to the matrix $\hat{Q}_0$. If the initial guess $\omega_0$ differs from $\omega_{LCO}$, then the iteration for $\hat{Q}$ will not be able to converge or will converge to the trivial solution $\hat{Q} = 0$. For this reason, the additional step of updating $\omega$ (Eq. 5.17) has to be introduced in the iterative procedure. The frequency update is performed
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every \( n_i \) iteration steps, to improve the efficiency of the computation. For the present work, \( n_i \) is taken between 10 and 20. The frequency updating procedure, here introduced for the FDHB, can also be applied to the HDHB by simply using Eq. 5.15 instead of Eq. 5.5.

5.3 Results

The lift frequency response of a pitching membrane wing at a chord-based Reynolds number \( Re = 100 \) is now analyzed by means of Harmonic Balance methods. As introduced in Section 3.2, the membrane aeroelastic behavior is represented by two different reduced-order frameworks: a nonlinear fluid-structure interaction model and an approximated quasi-linear model. A FDHB method is used to obtain the response of the quasi-linear model, while the HDHB is applied to the nonlinear case. The results from the two methods are discussed and compared with time integration of the equations and DNS. The analysis is repeated for \( Re = 1000 \), to investigate the effect of the Reynolds number on the wing’s aerodynamic response. Finally, the autonomous limit-cycle oscillations of the membrane for linearly unstable cases are taken into account and investigated over a range of structural parameters of the membrane.

5.3.1 Membrane’s aeroelastic forced response to pitching at \( Re = 100 \)

A membrane with \( \mu = 0.5, \bar{E}_s = 250, C_d = 0 \) and \( \delta_0 = 0.002 \) (resulting in \( c_{T_0} = 1 \)) pitching about its leading edge at \( Re = 100 \) is considered. First, we analyze the aerodynamic response of the membrane; the characteristics of the frequency response are investigated and discussed using the quasi-linear and the nonlinear model through their corresponding HB formulations. In particular, the quasi-linear model is solved using the classical FDHB method, while for the nonlinear model it is necessary to employ a HDHB, in order to overcome the difficulties associated with the representation of the nonlinear term in the frequency domain (see Section 5.2). The results from the two models are compared and validated against direct numerical integration of the governing equations. The deflection response of the membrane is also discussed, in order to investigate how the nonlinearity of the structure affects the performance of the present models. Finally, the convergence of the HB methods is addressed.

The aerelastic frequency response of the membrane is obtained for a sinusoidal single-harmonic input \( \ddot{\alpha}(t) = -\alpha_2 \omega^2 \sin(\omega t) \), where \( \alpha_2 \) is the coefficient \( \alpha_{2h} \) from Equation 5.7 when \( h = 1 \). Different pitching amplitudes are considered, since, given a set of structural and aerodynamic parameters, the membrane’s response is not only a function of the frequency \( \omega \), as in the linear case, but also depends on the amplitude of the input \( \ddot{\alpha} \). The frequency \( \omega \) represents a dimensionless reduced frequency, formally \( \omega_r = \omega c/U \); from now on, the subscript \( r \) will be
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dropped for simplicity. The solution of the Harmonic Balance equations for both FDHB and HDHB is achieved with a Newton-Raphson method starting from an initial frequency \( \omega_0 \) and marching in frequency with increments of \( \Delta \omega \); the solution at a generic \( \omega_i \) is used as the initial guess for the following iterations at \( \omega_{i+1} = \omega_i + \Delta \omega \). Due to hysteresis, different solution branches can be obtained, depending on whether the equations are solved by marching forward in frequency (positive increments of \( \Delta \omega \)) or backwards (negative increments of \( \Delta \omega \)). For the sake of brevity, the two solution branches will be referred to as the forward solution and backwards solution. The quasi-linear FDHB solution is used as an initial guess for the nonlinear HDHB, in order to reduce the occurrence of non-physical solutions. A discussion concerning non-physical solutions associated with the HDHB method is presented in Section 5.3.1.2.

5.3.1.1 Characteristics of the membrane’s response and comparison between quasi-linear and nonlinear model

The aerodynamic response of a pitching membrane is shown in Fig. 5.1 for four values of the pitching angle: \( \alpha_2 = 1 \) degree, \( \alpha_2 = 2.5 \) degrees, \( \alpha_2 = 5 \) degrees and \( \alpha_2 = 10 \) degrees. The steady-state Harmonic Balance solutions have been obtained with the FDHB method applied to the quasi-linear model and with the HDHB method applied to the nonlinear model. Results from direct numerical time integration of the governing equations of both models are also included: in this case, the discretization of the equations is carried out by means of finite differences for the spatial derivatives and the integration is performed via a five-step fourth-order Runge-Kutta scheme. Figure 5.1 shows the frequency response of the lift magnitude, defined as

\[
Mag C_L = 20 \log_{10} \frac{\max |C_L(t)|}{\max |\hat{\alpha}(t)|},
\]

(5.18)

with \( \hat{\alpha} \) expressed in rad/s\(^2\)c\(^2\)/U\(^2\). For the HB solutions, \( \max |C_L(t)| \) represents the maximum value of the lift coefficient over one period of oscillation, reconstructed from the HB coefficients through Eq. 5.10. For the numerical simulations, \( \max |C_L(t)| \) is the maximum value of the lift coefficient over one period of oscillation when all transients have decayed and the steady-state is reached. Since we are considering a single-frequency input, \( \max |\hat{\alpha}(t)| = |\omega^2 \alpha_2| \). A total number of 7 harmonics (\( N_H = 7 \)) is used to represent the HB solutions. The choice of the number of harmonics is based on a compromise between accuracy over a range of pitching angles and computational cost of the method. Convergence is discussed in Section 5.3.1.4. For the HDHB, \( N_c = 40 \) collocation points have been used to generate the solution.

We start by discussing the aerodynamic lift response of the wing by comparing the two fluid-structure interaction models through their corresponding HB solutions. For a pitching angle of 1 degree (Fig. 5.1a-b), the two models are in excellent agreement. This indicates that any nonlinearities in the response are weak and can be accurately represented by the approximated quasi-linear model. Over a
Figure 5.1: Comparison between the quasi-linear model solved through the FDHB (black line) and the nonlinear model solved through the HDHB (grey line) at $Re = 100$ for $\alpha_2 = 1^\circ$ (a-b), $\alpha_2 = 2.5^\circ$ (c-d), $\alpha_2 = 5^\circ$ (e-f) and $\alpha_2 = 10^\circ$ (g-h). The plots in the right column (b-d-f-h) represent a zoom of the corresponding plots in the left column (a-c-e-g) to highlight the hysteresis regions. Validations from direct integration of the nonlinear model are indicated with symbols: diamonds for simulations with the frequency increasing with time, up to the steady-state value, and squares for simulations with the frequency decreasing in time. Validations for the quasi-linear model are represented with black dots.
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range of frequencies, two different solutions can be found by the HB methods, depending on whether the frequency stepping is performed forward (from low to high frequency), or backwards (from high to low frequency). This is an indicator of an hysteretic behavior of the membrane-fluid interaction, commonly found in many nonlinear systems; an example is represented by the Duffing oscillator [Ghandchi-Tehrani et al., 2015], to which the quasi-linear structural model here presented is analogous. When the pitching angle is increased to 2.5 (Fig. 5.1c-d) and to 5 degrees (Fig. 5.1e-f), the hysteresis regions and the discontinuities in the lift response shift to lower frequencies. The two models are still in good agreement, with minor differences in the hysteresis region. For a pitching angle of 10 degrees (Fig. 5.1d), differences in the response from the two models are present over a range of frequencies, due to the stronger nonlinear response of the membrane. Despite the limitations of the small deflection approximation, the quasi-linear model is still able to capture the overall trend of the response even in the presence of moderate and large nonlinear effects, providing acceptable results from a reduced-order modeling perspective. Further discussion on the validity of the two models is provided in Section 5.3.1.3. It is interesting to notice that for $\alpha_2 = 10$ degrees the membrane does not present hysteresis for the considered frequencies and the forward and backwards solutions coincide. Additional simulations carried out with pitching angles up to 20 degrees have shown that, with the present choice of structural parameters and over the range of frequencies considered, large angles of attack do not present hysteresis in the membrane’s response.

In order to validate the HB solutions for the two models, solutions from direct integration of the quasi-linear and the nonlinear models are included in Fig. 5.1. For large angles of attack the predictions of the two models differ because of the increasing contribution of the nonlinear structural dynamics to the aerodynamic response. The quasi-linear FDHB solution matches the quasi-linear time-integrated results, while the nonlinear HDHB solution matches the nonlinear time integrated results, hence validating both HB approachs. Around the high frequency hysteresis region for pitching angles of 1 and 2.5 degrees, some results from time integration significantly differ from the prediction of the models. The reason is that the time-integrated response presents additional frequencies in its spectrum that differ from the fundamental forcing frequency and its integer multiples. The response also becomes sensitive to initial conditions: small differences in the initial conditions result in large differences in the responses. This chaotic behavior is known in the literature for many nonlinear systems, including simple Duffing oscillators (Barnett et al. [2011], Kovacic & Brennan [2011], Ueda [1980]). Harmonic Balance methods assume periodicity in the response of the system and they are unable to represent non-periodic solutions or solutions that present sub-harmonics of the fundamental frequency that are not known a priori. However, given the limited occurrence of non-periodic and sub-harmonic solutions, in the present work we will only focus on the membrane’s periodic response.

The hysteresis mechanism present in the results shown in Fig. 5.1 hints to the presence of an additional unstable branch [Ghandchi-Tehrani et al., 2015], which cannot generally be obtained by direct integration in time of the membrane equation. The unstable branch can be found with the HB methods by initializing the
Figure 5.2: (a) Hysteretic behavior obtained from the quasi-linear model using the FDHB method for $\alpha_2 = 2.5$ degrees. The two stable branches are represented in grey and the unstable branch with a solid black line.

Figure 5.3: Hysteretic behavior of the structural response for modes 1 (a), 2 (b) and 3 (c) obtained from the quasi-linear model using the FDHB method for $\alpha_2 = 2.5$ degrees. The two stable branches are represented in grey and the unstable branch with a solid black line.

initial guess of the Newton-Raphson iteration with random numbers until a new solution, distinct from the two stable cases, is reached. An example of stable and unstable solution from the quasi-linear model solved with the FDHB for $\alpha_2 = 2.5$ degrees is shown in Fig. 5.2.

The stable and unstable branches for the structural response of the case presented in Fig. 5.2 are shown in Fig. 5.3 for modes 1 to 3. As observed for the corresponding aerodynamic response, the structural response is also continuous when moving along the stable-unstable branches. As we will discuss in Section 5.3.1.3, for this range of frequencies the magnitude of the response decreases as the mode number increases, with mode 1 giving the largest contribution to response.

Finally, a representation in physical space of the shape of the membrane for different harmonics at $\omega = 4.5$ is shown in Fig. 5.4. The frequency $\omega$ has been selected to fall in the hysteresis region, in order to compare the deflection of the stable and unstable branches. For the considered frequency, the first harmonic
Figure 5.4: Shape of the deflection for the two leading harmonics $h = 1$ (a-b) and $h = 3$ (c-d) for $\alpha_2 = 2.5$ degrees and $\omega = 4.5$. The solution from the stable forward branch is represented with a dash-dot grey line, the one from the stable backwards branch with a grey dashed line and the solution for the unstable branch with a solid black line.

$h = 1$ gives the largest contribution to the structural response, presenting amplitudes that are one order of magnitude larger than those obtained for $h = 3$. The structural modes 1 and 2 can be clearly observed in the shape of the membrane for $h = 1$, with mode 1 dominating the response. For $h = 3$, the contribution of mode 3 can also be identified, even though its amplitude is smaller than the amplitude of the two leading modes 1 and 2.

### 5.3.1.2 Non-physical solutions of the HDHB method

The nonlinear model can represent the full nonlinear behavior of the membrane without approximations. The strong nonlinearity of the structural equation makes the problem of finding steady-state solutions via a classical FDHB method non-tractable, due to the difficulty associated with representing the nonlinear term in the frequency domain. For this reason, we introduced a HDHB method. A disadvantage of the HDHB is the occurrence of non-physical solutions. By introducing an alternative formulation based on an over-determined system of equations (see Section 5.2.2) and by using the results from the quasi-linear model via the FDHB as the initial guess for the iterative solution of the HDHB equations, non-physical solutions have been completely eliminated for the considered cases, as indicated by a comparison with time integration in Fig. 5.1.

It is worth considering the case in which the HDHB is used without the results from the FDHB, in order to address the performance of the method when an alias-free preliminary solution is not available. The method is initialized with
random numbers and, when marching in frequency, the solution at $\omega_i$ is used as initial guess at the following step $\omega_{i+1}$. In the presence of discontinuities due to hysteresis, when the previous solution does not guarantee a converged result for the following iteration, the previous solution is perturbed with random numbers until a new solution is found. If a single branch is present, the HDHB solution does not present any aliasing. This is a significant improvement of the over-determined HDHB over the standard HDHB of LaBryer & Attar [2009], that gave rise to aliasing even in the absence of hysteresis. When two solutions are present in the vicinity of the hysteresis jumps, the method tends to converge to a non-physical branch. An example for $\alpha_2 = 2.5$ degrees is shown in Fig. 5.5. The grey lines represent the physical solutions, while the black lines indicate the non-physical branches. Aliasing in the vicinity of the hysteresis branches has been also reported by Dai et al. [2013] for a Duffing oscillator.

5.3.1.3 Structural frequency response

The main difference between the quasi-linear model and the nonlinear model lies in the representation of the elongation $\delta$. The maximum elongation as a percentage of the chord over one period of oscillation as a function of the frequency for different values of the pitching amplitude is presented in Fig. 5.6. Both models give similar results for elongations up to 6% of the wing’s chord. Figure 5.6 also contains the maximum value of the first three deflection modes as a function of the frequency.

The excellent agreement between the solutions from the two membrane models for the set of structural parameters and for the pitching angles considered, indicates that the quasi-linear model represents a good approximation of the nonlinear behavior even for moderate and large deflections (exceeding 15% of the chord). The quasi-linear model can be represented by the FDHB method which does not suffer from aliasing and hence does not require the solution of an over-determined system. The quasi-linear model also allows us to represent the deflection using
deflection modes, hence reducing the number of unknowns in the HB equations by one order of magnitude ($N \approx 10$ for the deflection modes against $N_x \approx 100$ for the deflection on an equispaced grid along the membrane’s chord). For these reasons, the FDHB presents fewer convergence issues and it is computationally less expensive. Given the similarity between quasi-linear and nonlinear models even for large deflections of the membrane, the quasi-linear model offers an efficient and appealing alternative to the nonlinear model to represent the aeroelastic response of a membrane wing, overcoming some of the main issues related to the HDHB approach, without compromising the accuracy of the results.

5.3.1.4 Convergence of the Harmonic Balance methods

Harmonic Balance methods represent the periodic solution of a nonlinear system as a sum of harmonic contributions that are multiples of the fundamental frequency...
of the forcing term. When solving the harmonic equation represented in Eq. 5.5 (or Eq. 5.15, for the corresponding system in the time domain), the number of harmonics (and hence the number of unknowns) is decided a priori and it represents one of the parameters of the solution. The amplitude of the harmonics in the solution generally decreases with the harmonic index $h$, indicating that the lower harmonics give the largest contribution to the response, hence justifying the use of a truncated Fourier series to represent the system’s response. Fig. 5.7 shows the amplitude $A_h = \sqrt{C_{L2h-1}^2 + C_{L2h}^2}$ of the individual harmonics present in the membrane’s lift response for pitching angles from 1 to 10 degrees. For the sake of clarity, only the forward branch of the solution is shown. Because of the nature of the nonlinearity of the membrane’s equation, only odd harmonics give contributions to the response; the amplitude of the even harmonics is zero. When the highest harmonic ($h = 7$ for the case represented in Fig. 5.7) is orders of magnitude smaller than the lowest harmonic ($h = 1$), an increase in the number of harmonics will not improve the solution, since the contribution of higher harmonics will be negligible with respect to the lowest ones. By looking at Fig. 5.7, it is evident that the main contribution to the response is given by the first harmonic, with the contributions of the higher harmonics decreasing with the harmonic index $h$. Exceptions are represented by the cases with $\alpha_2 = 2.5$ and $\alpha_2 = 5$ degrees, in which for a small range of frequencies the third harmonic $h = 3$ gives the largest contribution to the lift response. This phenomenon is known in the literature as super-harmonic solution [Lakshmanan & Murali, 1996].

The ratio between the amplitude of the lift of the highest and the lowest harmonic is also represented in Fig. 5.7 for both HB methods. If the ratio is small, it means that the contribution of the higher harmonic to the lift response is negligible. Because we generally expect the contribution to the response to decrease as the harmonic number increases, a small ratio indicates that the solution is converged. As the pitching angle increases, stronger nonlinear effects are excited in the lift response and the number of harmonics that give a non-negligible contribution to the response increases. By looking at the amplitude ratio, it is often possible to identify those solutions that might benefit from an increase in the number of harmonics.

An example of the effect of the number of harmonics on the quality of the solution is shown in Fig. 5.8 for a pitching angle of 10 degrees. The lift coefficient over one period of oscillation for two different frequencies is reconstructed from the HB coefficients for increasing values of the total number of harmonics, from $N_H = 1$ to $N_H = 7$. Results are compared to the reference obtained with direct time integration. Figures 5.8a, 5.8b, 5.8c and 5.8d represent a case with a nondimensional reduced frequency $\omega = 0.596$, corresponding to a large amplitude ratio in Fig. 5.7d, while Figs. 5.8e, 5.8f, 5.8g and 5.8h correspond to $\omega = 9.05$, a case that presents large deflections, as observed from Fig. 5.6d. For both cases, as the number of harmonics increases, the HB solution yields better approximations of the lift coefficient in time over the entire period.
Figure 5.7: Amplitude ratio $A_7/A_1$ and amplitude of the individual harmonics $h = 1$ (blue), $h = 3$ (red), $h = 5$ (yellow) and $h = 7$ (purple) obtained from the quasi-linear FDHB solution (solid line) and from the nonlinear HDHB solution (dashed line), for different values of the pitching angle, $\alpha_2 = 1^\circ$ (a), $\alpha_2 = 2.5^\circ$ (b), $\alpha_2 = 5^\circ$ (c) and $\alpha_2 = 10^\circ$ (d).

5.3.2 Comparison with the forced response at $Re = 1000$

The effect of the Reynolds number is investigated by obtaining the lift frequency response of a pitching membrane at a higher Reynolds number of 1000. Results from the quasi-linear FDHB solution and from the nonlinear HDHB solution are shown in Fig. 5.9, for pitching angles ranging from 1 to 10 degrees. The structural parameters of the membrane are the same as those introduced in Section 5.3.1. By comparing the results in Fig. 5.9 with the corresponding cases at Reynolds 100 in Fig. 5.1, it is possible to notice that even though the Reynolds number influences the magnitude of the response, the qualitative behavior of the response, including the hysteresis regions, is not significantly affected.

A comparison with the lift response from direct time integration of the governing equations of the nonlinear fluid-structure interaction model at $Re = 1000$ is included in Fig. 5.9. As for the $Re = 100$ case, sub-harmonic responses are present close to the hysteresis regions and in these cases the differences between HB and simulations are significant. For a pitching angle of 10 degrees, as already observed for the same case at $Re = 100$, differences arise between the nonlinear and the quasi-linear model due to the large deflections involved. Although the nonlinear
model is able to capture the full nonlinear structural response, it is important to mention that the aerodynamic reduced-order framework does not take into account nonlinear fluid-dynamic phenomena such as vortex shedding. Hence the validity of the present investigation is limited to small deflections by the assumption of linear aerodynamics. For \( Re = 1000 \), as we will see in Section 5.3.2.1, strong nonlinear aerodynamics effects arise for pitching angles smaller than 10 degrees, making the linear aerodynamics assumption a stronger constraint than the quasi-linear assumption for the structure.

5.3.2.1 Effect of the nonlinear aerodynamics

The effect of the nonlinear aerodynamics on the membrane’s response is examined by means of DNS. The membrane, represented in the high-fidelity framework by the full nonlinear equation (Eqs. 3.6 and 3.7), is coupled to the fluid solver with a Boundary Data Immersion Method (BDIM) (Schlanderer et al. [2017]). Details on the DNS solver can be found in Serrano-Galiano et al. [2018]. The two-dimensional fluid domain for the \( Re = 100 \) case is represented by a \( 1284 \times 1030 \) Cartesian grid, refined in the proximity of the membrane. The minimum grid spacing, necessary for the requirements of the BDIM method, is \( \Delta x = 7.5 \times 10^{-4} \). A refinement downstream of the trailing edge is performed for the \( Re = 1000 \) case, resulting in a \( 1792 \times 1030 \) Cartesian grid, in order to accurately resolve the wake vortices closer to the membrane. The membrane wing is represented by 200 equally spaced structural points: 100 are distributed on the wing upper surface and 100 on the
Figure 5.9: Comparison between the quasi-linear model solved through the FDHB (black line) and the nonlinear model solved through the HDHB (grey line) at Re = 1000 for $\alpha_2 = 1^\circ$ (a), $\alpha_2 = 2.5^\circ$ (b), $\alpha_2 = 5^\circ$ (c) and $\alpha_2 = 10^\circ$ (d). Validations from direct integration of the nonlinear model are indicated with symbols: diamonds for simulations with the frequency increasing with time, up to the steady-state value, and squares for simulations with the frequency decreasing in time.
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Figure 5.10: Validations of the quasi-linear FDHB response (black line) and the nonlinear HDHB response (grey line) with DNS (circles) for different Reynolds numbers. (a) $Re = 100 - \alpha_2 = 2.5$ degrees. (b) $Re = 1000 - \alpha_2 = 2.5$ degrees. (c) $Re = 100 - \alpha_2 = 5$ degrees. (d) $Re = 1000 - \alpha_2 = 5$ degrees.

lower surface. A Mach number of $M = 0.2$ is chosen; a discussion on the choice of the Mach number can be found in Section 2.3.2.

The steady-state lift magnitude obtained from DNS is compared to the predictions from the frequency-domain Harmonic Balance for $\alpha_2 = 2.5^\circ$ and $\alpha_2 = 5^\circ$ at $Re = 100$ and $Re = 1000$. Results are shown in Fig. 5.10. The DNS results are in excellent agreement with the model prediction at $Re = 100$, indicating that for pitching angles up to at least 5 degrees, the flow still behaves linearly. Good agreement is also found for a pitching angle of 2.5 degrees at $Re = 1000$. Predictions at $Re = 1000$ for $\alpha_2 = 5$ are still reasonably accurate for lower frequencies, even though nonlinear vortex shedding is present in the lift response. One possible explanation is that the lift response is dominated by unsteady acceleration-related linear effects that prevail on the aerodynamic loads generated by the nonlinear shed vortices. For higher frequencies, the magnitude of the lift response and the deflection of the membrane decrease and the contribution to the lift of the nonlinear aerodynamics increases, explaining the consequent large discrepancies between model and simulations.

Since the lift response in Fig. 5.10 only takes into account the maximum value of the lift coefficient over one period of oscillation, it is interesting to compare the entire lift response obtained from HB methods and DNS (after we let all the transients decay) over a period. Results for $Re = 1000$ and $\alpha_2 = 5$ degrees are shown in Fig. 5.11 for values of the pitching reduced frequency from $\omega = 4$ to $\omega = 10$. A single snapshot of the vorticity field for each case from DNS is also represented in Fig. 5.11. Although nonlinear vortex shedding is present in all cases, the trend of the lift signal is well approximated by the quasi-linear model, as well
Figure 5.11: Vorticity field from DNS for $\alpha_2 = 5$ degrees and $Re = 1000$ for a membrane pitching with a frequency $\omega = 4$ (a), $\omega = 6$ (b), $\omega = 8$ (c) and $\omega = 10$ (d). Positive vorticity is represented in red, negative vorticity in blue. The lift over one period of oscillation from DNS (dash-dot line), quasi-linear solution through FDHB (black line) and nonlinear solution through HDHB (grey line) is represented in (e) for $\omega = 4$, in (f) for $\omega = 6$, in (g) for $\omega = 8$ and in (h) for $\omega = 10$.

as the value of the maximum lift achieved, with significant differences only present in the case with $\omega = 10$. 
5.3.3 Membrane’s autonomous response

The autonomous limit-cycle oscillations of a linearly unstable membrane wing are now considered. In contrast to the forced response analyzed in Sections 5.3.1 and 5.3.2, for the autonomous case the oscillations of the membrane are not driven by an external periodic forcing. On the contrary, the driving force is internal to the system and it is represented by the mutual interaction between the membrane and the free-stream. Hence, the fundamental frequency of the oscillations is not known a priori as in the forced case, but it depends on the parameters of the membrane-fluid system. For this reason, in order to investigate the limit-cycle oscillations using HB methods, a frequency-searching algorithm has been introduced (see Section 5.2.3). The frequency of the oscillations now represents an additional unknown of the problem.

Because of the smaller number of unknowns of the FDHB and because the HDHB presents an over-determined system of equations, we will only investigate the autonomous response of the membrane using the quasi-linear model solved via the FDHB approach. We will see that the deflections associated with the LCO for the chosen structural parameters are generally small enough to justify the choice of the quasi-linear model. The frequency of the LCO and the membrane’s aeroelastic response are obtained at \( Re = 100 \) for a range of the structural parameters \( \bar{E}_s \) and \( \mu \), with \( \delta_0 = 10^{-5} \). The structural damping coefficient \( C_d \) is assumed to be zero.

The results are shown in Fig. 5.12 for a number of harmonics \( N_H = 3 \). The maximum lift coefficient and the maximum deflection are strongly influenced by the zero-deflection tension coefficient \( c_{T0} \): as the zero-deflection tension coefficient (and hence the membrane’s restoring force) decreases, the maximum deflection reached by the membrane increases, and so does the lift. As observed in Fig. 5.12b, the main contribution to the structural response from the autonomous oscillations comes from the second structural Fourier mode. The LCO frequency \( \omega_{LCO} \) is shown in Fig. 5.12c. The LCO frequency is most affected by the density ratio \( \mu \): the lower the density ratio, the higher the frequency of the oscillations. This is probably related to the fact that when \( \mu \) decreases, the inertia of the membrane becomes smaller with respect to the aerodynamic forces and the aeroelastic response is dominated by the aerodynamic effects.

The HB method applied to the autonomous membrane always admits a trivial solution (zero deflection of the membrane) and the frequency-searching algorithm tends to converge to that trivial solution, unless the initial guess provided is close enough to the non-trivial LCO solution. Convergence has been found to be affected in particular by the initial guess provided for the LCO frequency. For this reason, the results shown in Fig. 5.12 have been obtained by evaluating the HB solution for a specific set of parameters \( \mu \) and \( \bar{E}_s \) and then updating these parameters by small increments, using the previous solution as an initial guess for the following iteration. The membrane autonomous responses obtained with this method present periodic limit-cycle oscillations of the lift and of the individual deflection modes with a zero mean value and with non-trivial contributions coming from the odd harmonics only (the harmonic coefficients for \( h = 0, 2, \ldots \) are zero).
Figure 5.12: Characteristics of the limit-cycle oscillations predicted by the quasi-linear model solved with the FDHB for $Re = 100$ and for different values of the membrane’s parameters. A number of harmonics $N_H = 3$ is chosen to represent the solution. (a) Lift coefficient. (b) Deflection modes $W_1$ (blue), $W_2$ (red) and $W_3$ (yellow). (c) Frequency of the LCO.

The results from the FDHB method have been validated and compared to simulations performed via direct time integration of the quasi-linear and the nonlinear model. It emerged that the autonomous membrane presents additional types of LCO, depending on the structural parameters chosen and on the initial conditions. The autonomous limit-cycle responses observed can be classified into three categories: periodic zero-mean value LCO, discussed in the previous paragraph; periodic non-zero mean value LCO; and aperiodic LCO. The three types of LCO are represented in Fig. 5.13 using phase space diagrams for the structural response of the first deflection mode and corresponding lift oscillations in time. Comparisons with the quasi-linear HB solutions are included in Fig. 5.13. For the zero-mean value LCO (Figs. 5.13a and 5.13d), the quasi-linear HB solution and the time-integrated solutions for both the quasi-linear and the nonlinear model are in excellent agreement. The non-zero mean LCO (Figs. 5.13b and 5.13e) can be identified by an asymmetric phase space diagram and by a non-zero mean value of the lift response. For this case, depending on the initial guess of the solution provided, the quasi-linear HB method can converge to a zero mean value LCO or to a non-zero mean value LCO, as indicated in Figs. 5.13b and 5.13e. Although
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The HB solution predicts two different types of LCO, for the case represented in Figs. 5.13b and 5.13e simulations in the time domain performed with different initial conditions have always converged to the non-zero mean LCO. This could be an indication that the zero mean LCO is unstable for the structural parameters considered. Additional analysis have to be conducted to confirm this hypothesis.

This specific case highlights the limitations associated with the present HB method in predicting the autonomous response of a linearly unstable membrane. The present HB method can potentially determine the membrane’s autonomous response, but the convergence of the method depends on the initial solution provided; in order to ensure that finding non-trivial solutions of the quasi-linear FDHB formulation with the frequency-searching algorithm requires less computational effort than running a simulation in time, some a priori knowledge of the system’s response is required (e.g. a close approximation of the LCO frequency). In addition, the HB method does not provide information on the stability of the solution. Finally, the HB method is based on the hypothesis of having a periodic response, hence it is not able to represent aperiodic responses, as indicated by the case represented in Figs. 5.13c and 5.13f. This case is characterized by aperiodic behavior, characteristic of many nonlinear systems [Thompson et al., 2002].

Validations of the membrane’s autonomous response obtained by comparing the time-integrated solutions of the quasi-linear and the nonlinear model against DNS

Figure 5.13: Comparison between the quasi-linear HB solution with $N_H = 5$ (yellow) and direct time integration of the quasi-linear (blue) and nonlinear model (red) for the different types of LCO observed. The results are shown using phase space diagrams for the first deflection mode (top) and the corresponding lift response in time (bottom). (a-d) $E_s = 50$, $\mu = 1.5$, $\delta_0 = 0.001$, zero mean value LCO (b-e) $E_s = 50$, $\mu = 2.5$, $\delta_0 = 0.001$, non-zero mean value LCO (c-f) $E_s = 50$, $\mu = 4$, $\delta_0 = 0.001$, aperiodic response.
5.4 Summary

Harmonic Balance methods have been adopted to investigate the nonlinear aeroelastic response of a membrane wing at low Reynolds numbers. The governing equations of the fluid-structure interaction problem are given by a linear aerodynamic model coupled with a nonlinear structural model for the membrane. Two approaches to the solution of the governing equations through Harmonic Balance methods are introduced, one in the frequency-domain and the other in the time-domain. Frequency-domain methods are not suitable for strong nonlinear systems and require a simplified version of the nonlinear membrane, here introduced as the quasi-linear membrane equation. On the other hand, time-domain methods can solve the full nonlinear membrane equation, but require an over-determined
system to reduce the generation of non-physical aliased solutions and are generally computationally more expensive.

The two HB methods are used to obtain the aeroelastic response of a membrane wing to forced oscillations in pitching at $Re = 100$. We have seen that nonlinear phenomena in the lift response, such as hysteresis, occur for pitching angles as low as 1 degree for the structural parameters considered, justifying the need for nonlinear structural models of compliant membrane wings. The quasi-linear model represents an excellent approximation of the nonlinear dynamics for small and moderate deflections, providing a more reliable and computationally less expensive solution in combination with the FDHB method, compared to the HDHB method applied to the full nonlinear equation. Only for large pitching amplitudes, when the contribution of the structural nonlinearities to the response of the membrane is larger, does the nonlinear HDHB solution outperform the quasi-linear FDHB method. Although a formulation based on an over-determined system significantly reduces the occurrence of non-physical oscillations associated with the HDHB solution, the method still benefits from having an alias-free approximate solution via the FDHB as an initial guess, in order to completely eliminate the aliased results.

Solutions for the membrane’s forced response are also obtained for a Reynolds number of 1000. In this case, the nonlinear aerodynamic effects associated with the higher Reynolds number limit the validity of the models—which are based on linear aerodynamics—to small pitching amplitudes, as shown by a comparison with DNS.

Finally, the autonomous response of the membrane at $Re = 100$ for linearly unstable cases has been investigated, providing results for a broad range of structural parameters. Due to the small amplitude of the deflection of the autonomous oscillations, the FDHB method has been applied to the quasi-linear case with a frequency-searching procedure. The HB method can accurately evaluate the frequency of the limit-cycle oscillations, as well as the lift coefficient and the deflection of the membrane, as verified by comparison with direct time integration of the quasi-linear model. However, the convergence of the method and the type of LCO predicted depend on the initial guess of the solution provided, making the HB method impractical, unless some a priori information about the autonomous response is provided. Validations against DNS have shown that the autonomous response from the models represents a good approximation of the DNS results.
Chapter 6

Optimization of Vortex-induced Response Using an Adjoint Method

In this chapter, we aim to optimize the aeroelastic response of a membrane wing at zero incidence that is subjected to an aerodynamic disturbance in the form of a small-scale convective vortex. The idea is to understand if the compliance of the membrane can be efficiently exploited for disturbance-rejection purposes. The optimization procedure is based on the solution of the state equations for the linear and quasi-linear models, in conjunction with their corresponding adjoint equations, in order to efficiently provide the gradient of a cost function with respect to some control parameters. For the cases considered in this chapter, the chosen cost functions are related to the integral of the lift response in time and to the peak of the lift. As a preliminary investigation on the performance of the wing with respect to its structural properties, two parameters only are chosen as control parameters: the stiffness $\tilde{E}_s$ and the density ratio $\mu$. The vortex-induced lift responses of the optimal solutions from the linear and the quasi-linear optimization procedures are compared against the full nonlinear solution and against DNS. Finally, the performance of the membrane is compared to the performance of an ideal feedback controller based on an active actuation of the wing.

6.1 Introduction

A key characteristic of Micro Air Vehicles (MAVs) is their reduced size. Their small scale makes MAVs ideal candidates for applications such as surveillance and reconnaissance in environments that are impractical for larger vehicles. In addition, reduced dimensions translate into small inertial forces with respect to the aerodynamic forces [Shyy et al., 1999]. This allows MAVs to perform agile and rapid maneuvers but, on the other hand, it makes them extremely sensitive to changes in the aerodynamic loads caused by external disturbances. Since MAVs
often operate at low altitude, they are subjected to a large variety of aerodynamic
disturbances arising from gusts of wind or generated by the interaction of the
atmospheric boundary layer with buildings, trees and other objects [Geyman et al.,
2014, Lian, 2012]. Another source of disturbance that has gained attention in
recent years is represented by the wake vortices of other small-scale air vehicles
when MAVs swarms are considered. MAVs can potentially fly in close formation
patterns composed of hundreds or even thousands of units [Li et al., 2018, Nægeli
et al., 2014, You & Shim, 2011]; the wake vortices generated by a single vehicle
in maneuver can affect the aerodynamic and flight dynamic performance of other
vehicles [Saban et al., 2009]. Hence, the sensitivity of MAVs to aerodynamic distur-
bances and the variety of disturbances that MAVs can encounter poses challenges
to their design and their controllability.

Active disturbance-rejection solutions have the disadvantage of requiring sensors
and actuators. In addition to the extreme challenges posed by the fast aerodynamic
time-scales on the design of sensors and actuators, another complication is the
strict limitations on size and weight. Hence, the idea of exploiting the flexibility of
the wing as a passive solution for aerodynamic disturbance rejection. Unlike rigid
wings, membrane wings passively deform under the effect of aerodynamic loads.
Consequently, as also discussed in Chapter 4 and Chapter 5, the aerodynamic
response of membrane wings might significantly differ from the response of rigid
wings and it is strongly affected by the structural parameters of the membrane,
such as stiffness and density. This is extensively documented in the literature,
from both experimental and computational studies [Gordnier, 2009, Gordnier &
Attar, 2014, Rojratsirikul et al., 2009, Song et al., 2008].

In the present chapter, we want to investigate whether membrane wings can be
used to effectively reduce the effect of an external disturbance in the
form of a convective vortex. In particular, we want to find values of the membrane
stiffness and density that minimize the lift disturbance caused by a certain vortex.
In order to find this “optimal” membrane, an adjoint-based optimization procedure
is adopted. The adjoint is a method used in many disciplines to find the gradient of
an objective functional that one wants to minimize—often referred to as the cost
function—, with respect to some control parameters. For the present case, the
control parameters are represented by the structural properties of the membrane
and the cost function is related to the lift disturbance. The advantage of an
adjoint formulation lies in the possibility of calculating the gradient of the cost
function by solving a set of direct and adjoint equations whose computational
cost is independent of the number of control parameters [Gunzburger & Wood,
2003]. The adjoint method has been successfully applied to a large variety of
flow control problems, often involving fluid-structure interaction and aerelasticity.
One of the first applications of the adjoint to flow control can be found in Bewley
et al. [2001]. They used wall transpiration to effectively reduce turbulent kinetic
energy and drag of a turbulent plane channel flow. Other applications involve jet
noise reduction [Freund, 2011, Schulze et al., 2011], shape optimization and aero-
structural optimization of winglets [Khosravi & Zingg, 2017], helicopter rotors in
forward flight [Mishra et al., 2016] and full aircraft configurations [Kenway et al.,
2014]. In the context of MAVs, the adjoint method has been adopted to optimize
composite laminated membrane wings [Stanford, 2008] and to optimize the shape
and the kinematic of a three-dimensional hovering wing in unsteady viscous flow
at low Reynolds number [Jones & Yamaleev, 2014].

6.2 Optimization procedure based on adjoint equations applied to the linear and quasi-linear models

Optimizing a system means finding values of some control parameters that min-
imize a certain cost function. Specifically, our objective is to minimize the effect
of an aerodynamic disturbance on the wing’s lift. Details on the choice of the
aerodynamic disturbance are provided in Section 6.3. The optimization of the
disturbance response is achieved by using the flexibility of the membrane wing
and, in particular, by selecting the structural properties of the membrane. As
a preliminary investigation, we will consider two structural parameters only: the
dimensionless stiffness $\bar{E}$ and the density ratio $\mu$.

In this section, we first introduce the general mathematical formulation of the
adjoint optimization procedure by deriving the governing equations. Then we
present the derivation of the adjoint equations for the simplified quasi-linear and
the linear fluid-structure interaction models, leaving the higher-order nonlinear
model as a validation tool only. Finally, we introduce a gradient-based minimiza-
tion method combined with a line search algorithm for the update of the control
parameters and the minimization of the objective function.

6.2.1 General adjoint formulation and optimality iterative procedure

The optimization procedure aims to find values of the structural parameters $\bar{E}$ and
$\mu$, here considered as control parameters and grouped into vector $g$, to minimize a
certain cost function $J$. The definition of the cost function is given in Section 6.3.2.
For now, we just assume that $J$ is a function of the $N_g$ control parameters $g$ and
of the $N_\phi$ state variables of the system $\phi$, formally $J = J(\phi(g), g)$. Note that
the state $\phi$ depends on the control parameters $g$. The optimization of the cost
function with respect to the control parameters is carried out using the gradient-
based quasi-Newton BFGS method, introduced in Section 6.2.4. The BFGS can
not only be efficiently implemented to handle a large number of variables (in case
the problem is extended to consider additional control parameters such as a non-
uniform density distribution), but it also provides a fast iterative estimate of the
Hessian matrix, without the need to calculate it explicitly. Like every gradient-
based optimization method, the BFGS requires information about the gradient of
the cost function with respect to the $N_g$ control parameters:

$$\nabla J = \begin{bmatrix} \frac{D J}{D g_1}, & \ldots, & \frac{D J}{D g_{N_g}} \end{bmatrix}^T,$$

with

$$\frac{D J}{D g_k} = \frac{\partial J}{\partial g_k} + \sum_{i=1}^{N_\phi} \frac{\partial J}{\partial \phi_i} \frac{d \phi_i}{d g_k} \quad \text{for} \ k = 1, \ldots, N_g. \ (6.2)$$

The derivatives $d \phi_i / d g_k$ are called sensitivities of the system. The evaluation of $\nabla J$ using a direct method such as finite differences rapidly becomes computationally impractical as the number of control parameters $N_g$ increases. For this reason, we employ the adjoint method to calculate $\nabla J$. A complete overview of adjoint-based optimization methods, including the mathematical derivation of the equations of the method is presented in Gunzburger & Wood [2003]. Here, we will summarize the key concepts and equations.

The adjoint-based optimization procedure aims to find values of $\phi$, $g$ and of a set of $N_\phi$ variables $\xi$ (called adjoint variables or co-state variables) for which the following Lagrange functional is stationary:

$$L(\phi, g, \xi) = J(\phi, g) - \sum_{i=1}^{N_\phi} \xi_i N_i(\phi, g). \quad (6.3)$$

Equations $N_i(\phi, g) = 0$ with $i = 1, \ldots, N_\phi$ represent the state equations of the system and are, in general, nonlinear. By taking the derivatives of $L$ with respect to the state variables $\phi$ and the adjoint variables $\xi$, and enforcing them to be zero, we obtain the following set of equations:

$$N_i(\phi, g) = 0, \quad \text{for} \ i = 1, \ldots, N_\phi \quad \text{State equations} \quad (6.4)$$

$$\sum_{j=1}^{N_\phi} \left( \xi_j \frac{\partial N_j}{\partial \phi_i} \right) = \frac{\partial J}{\partial \phi_i}, \quad \text{for} \ i = 1, \ldots, N_\phi \quad \text{Adjoint equations} \quad (6.5)$$

By differentiating $N_i(\phi, g) = 0$, the sensitivity equations are obtained:

$$\frac{d \phi_i}{d g_k} = - \left[ \frac{\partial N_i}{\partial \phi_i} \right]^{-1} \frac{\partial N_i}{\partial g_k} \quad \text{for} \ i = 1, \ldots, N_\phi. \quad (6.6)$$

Finally, by combining Eqs. 6.2, 6.5 and 6.6, the following expression for $\nabla J$ is obtained:

$$\frac{D J}{D g_k} = \frac{\partial J}{\partial g_k} - \sum_{i=1}^{N_\phi} \xi_i \frac{\partial N_i}{\partial g_k} \quad \text{for} \ k = 1, \ldots, N_g. \quad (6.7)$$

The optimal solution is found through an iterative procedure that can be summarized as follows:

- Provide an initial guess for $g$
• Solve the state equations (Eq. 6.4) forward in time to find $\phi$
• Solve the adjoint equation (Eq. 6.5) backwards in time to find $\xi$
• Evaluate $\nabla J$ from Eq. 6.7
• Find new values of the control parameters $g$ and repeat the procedure until convergence is achieved.

6.2.2 Adjoint equations for the quasi-linear model

The mathematical derivation of the adjoint equations for the quasi-linear fluid-structure interaction model can be found in Appendix D.1.1. Once the following adjoint variables have been defined

$$\xi = [\chi \quad U \quad V \quad \beta \quad \gamma]^T,$$

the quasi-linear adjoint equations can be written as

$$\frac{d}{dt} \begin{bmatrix} \chi \\ \beta \\ \gamma \\ U \\ V \end{bmatrix} = \begin{bmatrix} \left(A^x_{\chi,ql}\right)^T & 0 & 0 & 0 & \left(A^w_{\chi,ql}\right)^T \\ \left(A^\alpha_{\beta,ql}\right)^T & 0 & 0 & 0 & \left(A^\alpha_{\gamma,ql}\right)^T \\ \left(A^x_{\chi,ql}\right)^T & 0 & 0 & 0 & \left(A^w_{\chi,ql}\right)^T \\ \left(A^x_{\beta,ql}\right)^T & 0 & 0 & 0 & I \\ \left(A^x_{\gamma,ql}\right)^T & 0 & 0 & 0 & \left(A^w_{\gamma,ql}\right)^T \end{bmatrix} \begin{bmatrix} \chi \\ \beta \\ \gamma \\ U \\ V \end{bmatrix} + \begin{bmatrix} J^x \\ J^\alpha \\ J^\gamma \\ J^\delta \\ J^\omega \end{bmatrix},$$

with

$$A_{\chi,ql} = A_{\chi,ql}^{x,fs} + A_{\chi,ql}^{x,fs} (\mathbf{W}^T \Gamma \mathbf{W}) + 2 A_{\chi,ql}^{x,fs} \mathbf{W} \mathbf{W}^T \Gamma^2;$$
$$A_{\gamma,ql} = A_{\gamma,ql}^{w,fs} + A_{\gamma,ql}^{w,fs} (\mathbf{W}^T \Gamma \mathbf{W}) + 2 A_{\gamma,ql}^{w,fs} \mathbf{W} \mathbf{W}^T \Gamma^2.$$

The terms $A_{\chi,ql}^{x,fs}$ and $A_{\chi,ql}^{w,fs}$ are defined in Appendix D.1.1. The adjoint equations are linear with respect to the adjoint variables, but they contain the state variables that must be obtained by solving the direct quasi-linear fluid-structure interaction equations (see Eq. 3.19). Equations 6.9 are solved backwards in time, with zero initial conditions.
6.2.3 Adjoint equations for the linear model

In analogy with what has already been done for the quasi-linear model, the adjoint equations for the linear model are

\[
\frac{d}{dt} \begin{bmatrix} \chi \\ \beta \\ \gamma \\ U \\ V \end{bmatrix} = \begin{bmatrix} (A_x f_{s \alpha})^T & 0 & 0 & 0 & (A_w f_{s \alpha})^T \\ (A_x f_{s \beta})^T & 0 & 0 & 0 & (A_w f_{s \beta})^T \\ (A_x f_{s \gamma})^T & 0 & 0 & 0 & (A_w f_{s \gamma})^T \\ (A_x f_{s \chi})^T & 0 & 0 & 0 & (A_w f_{s \chi})^T \\ (A_x f_{s \psi})^T & 0 & 0 & I & (A_w f_{s \psi})^T \end{bmatrix} \begin{bmatrix} \chi \\ \beta \\ \gamma \\ U \\ V \end{bmatrix} + \begin{bmatrix} J_{\bar{x}} \\ J_{\bar{\alpha}} \\ J_{\bar{\gamma}} \\ J_{\bar{\chi}} \\ J_{\bar{W}} \end{bmatrix}.
\]

(6.10)

The derivation of Eq. 6.10 can be found in Appendix D.2. Because Eq. 6.10 represents the adjoint of a linear system, it does not contain the state variables of the linear fluid-structure interaction model.

6.2.4 BFGS method and line search algorithm

Once the gradient of the cost function with respect to the control parameters has been obtained from the solution of the direct and adjoint equations, a gradient-based optimization procedure can be used to minimize \( J \). We adopt the BFGS algorithm [Nocedal & Wright, 2006], which falls in the category of quasi-Newton methods. Given the gradient of the cost function, the BFGS iteratively constructs an approximation of the Hessian, providing faster convergence than the steepest descent method. Indicating with \( g_k \) and \( J(g_k) \) the control parameters and the cost function at the \( k \)-th step of the iteration procedure, the new iteration is given by

\[
g_{k+1} = g_k + \bar{\alpha}_k p_k. \tag{6.11}
\]

In Eq. 6.11, \( \bar{\alpha}_k \) is a positive scalar representing the length of the \( k \)-th iteration step and \( p_k \) is a descent search direction, defined as

\[
p_k = -B_k^{-1} \nabla J_k, \tag{6.12}
\]

with \( B_k^{-1} \) representing a \( N_g \times N_g \) symmetric positive definite matrix that is updated at every iteration using the following formula:

\[
B_{k+1} = B_k - \frac{B_k s_k r_k^T B_k}{s_k^T B_k s_k} + \frac{r_k r_k^T B_k}{r_k^T s_k}. \tag{6.13}
\]

Equation 6.13 represents an alternative version of the BFGS method and it is referred to as damped BFGS [Nocedal & Wright, 2006]. It guarantees that \( B_{k+1} \)
is always a positive definite matrix and that, consequently, \( p_{k+1} \) represents a descending direction. The terms in Eq. 6.13 are defined as follows:

\[
\begin{align*}
    s_k &= g_{k+1} - g_k \\
r_k &= \theta_k y_k + (1 - \theta_k) B_k s_k,
\end{align*}
\]

with

\[
\begin{align*}
y_k &= \nabla J_{k+1} - \nabla J_k \\
\theta_k &= \begin{cases} 
1 & \text{if } s_k^T y_k \geq 0 \\
2 s_k^T B_k s_k & \text{if } s_k^T y_k < 0.
\end{cases}
\]

The idea of the method is to update the control parameters using Eq. 6.11 in such a way that at every iteration the function \( J \) is reduced, until a minimum is reached. \( g_k \) is updated at every iteration by performing a line search along the direction defined by vector \( p_k \). Ideally we want to obtain the value of \( \bar{\alpha}_k \) that minimizes the cost function \( J \) along \( p_k \). This is often computationally impractical, hence inexact line search algorithms have been introduced. The aim of an inexact line search is to find a value of \( \bar{\alpha}_k \) that will ensure that the following relationship is satisfied:

\[
J(\bar{g}_k + \bar{\alpha}_k p_k) \leq J(\bar{g}_k) + c_1 \bar{\alpha}_k \nabla J^T p_k. \tag{6.14}
\]

Equation 6.14 is known as the first Wolfe condition and ensures that, at every iteration, the cost function is sufficiently reduced. It is often paired to a second inequality, the second Wolfe condition, that prevents the step \( \bar{\alpha}_k \) from being too small [Nocedal & Wright, 2006]. In the present work, we choose a backtracking line search algorithm to avoid enforcing the second Wolfe condition directly. The algorithm is summarized in Fig. 6.1. An initial step length \( \bar{\alpha}_0 \) is chosen. If the new cost function obtained by updating \( g \) using a step length \( \bar{\alpha}_0 \) does not satisfy the Wolfe condition, then \( \bar{\alpha} \) is reduced until this condition is met (step 1 in Fig. 6.1). At this point \( \bar{\alpha} \) is decreased again until the cost function begins to increase (step 2). A small parameter \( \epsilon \) is introduced to stop the iteration when \( \bar{\alpha} \) becomes too small. After step 2, or if the iteration step achieved with \( \alpha_0 \) satisfies the Wolfe condition, \( \bar{\alpha} \) is increased until the cost function starts increasing (step 3) and then decreased again with a smaller step until an \( \bar{\alpha} \) that brings the cost function close to a minimum is reached (step 4). This value of \( \bar{\alpha} \) is selected as the step length \( \bar{\alpha}_k \) for the \( k \)-th iteration step.

### 6.3 Vortex-induced response and definition of the cost function

In the optimization of the response of the membrane to an aerodynamic disturbance, the optimized wing depends on the choice of the disturbance and on the
Optimization of Vortex-induced Response

Figure 6.1: Representation of the backtracking line search algorithm.

cost function. Different disturbances and different optimization objectives result in different solutions. As a preliminary investigation, we choose a disturbance in the form of a convective vortex. It represents a simple approximation of the vortices present in the wake of bluff bodies or produced by the unsteady maneuver of a rigid or flexible wing, as shown in Fig. 6.2. Concerning the cost function, we introduce two objectives: (i) the minimization of the integral of the lift in time; and (ii) the minimization of the sum of the integral lift and the maximum peak lift observed during the response. We will analyze both and compare their optimal responses.
6.3.1 Vortex scaling and response of a rigid wing

The streamwise ($\tilde{x}$ direction) and transverse ($\tilde{y}$ direction) velocity components $u$ and $v$ generated by a convective vortex are represented by the following expressions:

$$u(\tilde{x}, \tilde{y}) = + \frac{c_v}{M} \left( 0.5 (\tilde{x}^2 + \tilde{y}^2)/r_v^2 \right) \tilde{y}$$

$$v(\tilde{x}, \tilde{y}) = - \frac{c_v}{M} \left( 0.5 (\tilde{x}^2 + \tilde{y}^2)/r_v^2 \right) \tilde{x}.$$ 

Once the Mach number $M$ has been defined, the vortex is uniquely determined by two parameters: the viscous radius $r_v$ and the intensity $c_v$. Since the membrane response and, consequently, the optimized solution depend on the vortex considered, we will test vortices with different radii and different intensities. The aim is to obtain insights or to infer general principles concerning the dependency of the membrane optimal response on the vortex characteristics. We focus our attention on small-scale vortices and we test four different cases: $r_v = 0.25c$, $r_v = 0.5c$, $r_v = 0.75c$ and $r_v = 1c$. The intensity of the vortices are denoted with $c_{0.25}$ for the vortex with $r_v = 0.25c$, $c_{0.5}$ for $r_v = 0.5c$, $c_{0.75}$ for $r_v = 0.75c$ and $c_1$ for $r_v = 1c$. For the smallest vortex with $r_v = 0.25c$, we test 30 different values of the intensity $c_{0.25}$, equally distributed between 0.0776 and 3.8824. We will indicate each case by assigning them a scaling index $s_c$ from 1 to 30. Since we want the effect of vortices of different radii on the wing to be comparable, we scale the intensities of all vortices in such a way that the integral of the lift generated on a rigid wing is the same for the same scaling index. The results for $Re = 100$ are summarized in Table 6.1. The maximum velocity $v_{rxx} = \max (\sqrt{\dot{u}^2 + \dot{v}^2})$, where the subscript $rxx$ indicates the vortex radius, of each vortex at time $t = 0$ is normalized with the free stream velocity $V_\infty$ and summarized in Table 6.2. Figure 6.3 shows the lift generated by the vortices on a rigid wing as a function of time. The vortices are placed 4.5 chords upstream of the leading edge from where they convect downstream. The DNS setup is the same as that described in Section 2.4.2.5. The time-dependent vortex-induced pressure distribution obtained from DNS for a rigid wing is applied
Table 6.1: Scaling of the intensity for vortices of different radius.

<table>
<thead>
<tr>
<th>$s_c$</th>
<th>$c_{0.25}$</th>
<th>$c_{0.5}$</th>
<th>$c_{0.75}$</th>
<th>$c_{1.0}$</th>
<th>$\int_0^t |C_{L,rig}|dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0776</td>
<td>0.0088</td>
<td>0.0030</td>
<td>0.0015</td>
<td>0.0318</td>
</tr>
<tr>
<td>5</td>
<td>0.6024</td>
<td>0.0687</td>
<td>0.0233</td>
<td>0.0116</td>
<td>0.2468</td>
</tr>
<tr>
<td>10</td>
<td>1.2584</td>
<td>0.1435</td>
<td>0.0488</td>
<td>0.0242</td>
<td>0.5156</td>
</tr>
<tr>
<td>15</td>
<td>1.9144</td>
<td>0.2184</td>
<td>0.0742</td>
<td>0.0368</td>
<td>0.7843</td>
</tr>
<tr>
<td>20</td>
<td>2.5704</td>
<td>0.2932</td>
<td>0.0997</td>
<td>0.0494</td>
<td>1.0531</td>
</tr>
<tr>
<td>25</td>
<td>3.2264</td>
<td>0.3681</td>
<td>0.1251</td>
<td>0.0620</td>
<td>1.3219</td>
</tr>
<tr>
<td>30</td>
<td>3.8824</td>
<td>0.4429</td>
<td>0.1506</td>
<td>0.0746</td>
<td>1.5906</td>
</tr>
</tbody>
</table>

Table 6.2: Summary of the maximum velocities generated by the different vortices at time $t = 0$.

<table>
<thead>
<tr>
<th>$s_c$</th>
<th>$v_{0.25}/V_\infty$</th>
<th>$v_{0.5}/V_\infty$</th>
<th>$v_{0.75}/V_\infty$</th>
<th>$v_{1.0}/V_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0500</td>
<td>0.0114</td>
<td>0.0058</td>
<td>0.0038</td>
</tr>
<tr>
<td>5</td>
<td>0.3879</td>
<td>0.0885</td>
<td>0.0452</td>
<td>0.0298</td>
</tr>
<tr>
<td>10</td>
<td>0.8103</td>
<td>0.1849</td>
<td>0.0943</td>
<td>0.0623</td>
</tr>
<tr>
<td>15</td>
<td>1.2328</td>
<td>0.2813</td>
<td>0.1435</td>
<td>0.0948</td>
</tr>
<tr>
<td>20</td>
<td>1.6552</td>
<td>0.3777</td>
<td>0.1927</td>
<td>0.1272</td>
</tr>
<tr>
<td>25</td>
<td>2.0776</td>
<td>0.4741</td>
<td>0.2418</td>
<td>0.1597</td>
</tr>
<tr>
<td>30</td>
<td>2.5000</td>
<td>0.5704</td>
<td>0.2910</td>
<td>0.1922</td>
</tr>
</tbody>
</table>

Figure 6.3: Example of the lift generated by 4 vortices with different radii on a rigid flat plate at $Re = 100$. The intensity of the vortices is scaled such that the integral of the lift in time is the same for all vortices. Lines from black to grey indicate increasing radius, from $r_v = 0.25c$ to $r_v = 1c$.

to the fluid-structure interaction models as an external forcing (see Eqs. 3.18, 3.19 and 3.20), in order to simulate the effect of the vortices on the membrane for the optimization procedure.
6.3.2 Cost function

Minimizing the lift disturbance caused by a convected vortex means that we want the lift to be as close as possible to some reference lift, even in the presence of disturbances. Mathematically, we quantify this using the two norm of the difference between the lift response and the reference lift. The resulting cost function is the following:

\[ J_1 = \int_{t=0}^{t=T} (C_L(t) - C_{L,ref}(t))^2 \, dt. \]  

(6.15)

Since we are considering the case of a membrane wing at zero angle of attack, the reference lift in the unperturbed case is \( C_{L,ref} = 0 \). Minimizing \( J_1 \) means minimizing the global effect of the vortex on the aerodynamic response of the wing. For small air vehicles, we also want to prevent large peaks in the lift response, since MAVs respond quickly to sudden changes in the aerodynamic field, due to their small inertia. Large lift peaks have a detrimental impact on flight dynamics performance. For this reason, we introduce a second cost function \( J_2 \) that includes the contribution of the maximum absolute value of the lift:

\[ J_2 = J_1 + \max |C_L(t) - C_{L,ref}|. \]  

(6.16)

When considering a rigid wing, the corresponding cost functions will be indicated as \( J_{1,rig} \) and \( J_{2,rig} \), respectively.

Additional requirements are necessary to ensure a favorable aeroelastic response of the membrane, especially when considering its impact on the flight mechanics of small-scale air vehicles. The first requirement is to enforce a reasonably fast decay of the transients, with particular attention to avoid small amplitude undamped limit-cycle oscillations for linearly unstable cases. This can be enforced by introducing a penalty term \( \bar{\sigma} > 0 \) for \( t > t^* \) on the integral lift in the cost function. In the present case, we adopt \( \bar{\sigma} = 5000J_{1,rig}, T = 100 \) and \( t^* = 50 \). A justification of this choice is given in Appendix D.3.1. We also want to prevent numerical instabilities when the value of the density is too small for practical applications. Hence, a penalty term \( F(\mu) \) in the form of a ramped step is introduced, as described in Appendix D.3.2. The two penalty terms now introduced are gathered in the term \( J_p \), resulting in

\[ J_p = \bar{\sigma} \int_{t=t^*}^{t=T} (C_L(t) - C_{L,ref}(t))^2 \, dt + F(\mu). \]  

(6.17)

The contribution of \( J_p \) to the two cost functions adopted in the optimization procedure can be expressed as follows:

\[ J_{1p} = J_1 + J_p, \]  

(6.18)

\[ J_{2p} = J_2 + J_p. \]  

(6.19)

In the results section, the optimal response and the optimal structural parameters obtained for the two cost functions will be compared and discussed. It is important to mention that for the optimization procedure we adopt the two penalized cost
functions $J_{1p}$ and $J_{2p}$. However, when discussing the results of the optimization and for comparisons with the corresponding rigid wing cases, we always use $J_1$ and $J_2$, without the penalty terms.

### 6.4 Results: optimization at $Re = 100$

The optimization of the membrane is carried out at $Re = 100$ and at an angle of attack of 0 degrees for four vortices with different radii, from $r_v = 0.25c$ to $r_v = 1c$. A steady wing at an incidence of zero degrees represents a critical condition for disturbances, since the flapping motion can reduce the impact of aerodynamic disturbances and wind gusts [Lian, 2009].

First, we use a linear optimization procedure based on the lower order and computationally more efficient linear fluid-structure interaction model to generate optimal values of $\bar{E}_s$ and $\mu$. The linear optimization can be summarized as follows:

- Initial guess of the control parameters $\bar{E}_s$ and $\mu$
- Evaluation of the membrane’s response by directly solving the governing equations of the linear model forward in time to obtain the state variables
- Solution of the adjoint equations for the linear model (see Eq. 6.10) backwards in time to obtain the adjoint variables
- Obtaining $\nabla J$ from Eq. 6.7
- Updating the control parameters and repeating the iteration until convergence.

Second, we repeat the analysis using a quasi-linear optimization procedure. The steps are the same described for the linear procedure, but in this case the linear model and the linear adjoint equations are replaced by the quasi-linear model and the quasi-linear adjoint equations. We will compare the results from the two optimization procedures to understand whether the linear model represents a good approximation of the nonlinear behavior. The optimization results are then validated against direct simulation with the nonlinear model and DNS.

#### 6.4.1 Linear optimization

The results from the linear optimization in terms of the percentage ratio between the optimal cost function and the value of the cost function for a rigid wing are shown in Fig. 6.4. For the integral cost function $J_1$, the results are independent of the vortex intensity. For the smaller vortices $r_v = 0.25c$ and $r_v = 0.5c$ we achieve the largest reduction in the cost function, with values of $J_1$ as low as 17% of the corresponding values for a rigid wing. Ratios of $J_1/J_1^{rig}$ approximately 20% and
22% are achieved for vortices with \( r_v = 0.75c \) and \( r_v = 1c \), respectively. It is important to point out that the two largest vortices display both a local minimum and an absolute minimum. The optimization procedure converges to one of the two solutions, depending on the initial guess provided for the control parameters \( \bar{E_s} \) and \( \mu \). Results for the optimization of \( J_2 \) are shown in Fig. 6.4b. Because \( J_2 \) combines a quadratic function (the integral of the lift squared) with a linear function (the largest peak of the lift response), the results present a dependency on the vortex intensity. Even in this case, the best performance is achieved for vortices with smaller radius. Larger vortices display a smaller reduction of the cost function and the occurrence of local minima.

The values of the control parameters \( \bar{E_s} \) and \( \mu \) that result in an optimal response are shown in Fig. 6.5 for both cost functions. The optimal structural parameters
are strongly affected by the vortex radius and by the choice of the cost function. For $J_1$, it is hard to find a direct correlation between vortex radius and optimal $\mu$ and $\bar{E}_s$: for $r_v = 0.25c$ to $r_v = 0.75c$, as the vortex radius increases, the stiffness and the density also increase. For $r_v = 1c$ the trend is inverted, since the optimal solution is achieved for smaller values of both $\bar{E}_s$ and $\mu$. The results for $J_2$ present a more regular trend: the optimal density ratio increases with the vortex radius, while the optimal stiffness decreases as the radius increases.

### 6.4.2 Quasi-linear optimization

The optimization is repeated employing the quasi-linear fluid-structure interaction model. Results for the two cost functions obtained for different initial estimates of the control parameters are shown in Fig. 6.6. By observing the results, it is evident that both cost functions present several local minima. This represents an issue for gradient-based optimization procedures, which tend to converge to any minimum of the cost function [Nocedal & Wright, 2006]. Since we are using a reduced-order
model, we can afford to test different initial values of the control parameters and then select those that give the lowest values of the cost function. For all the vortices considered, the membrane wing is able to achieve a reduction of the cost function. This reduction becomes less significant as the vortex intensity increases. For $J_1$, the ratio between the optimal cost function and the corresponding value for the rigid wing goes from $15\% - 25\%$ for the less intense vortices, up to $65\% - 75\%$ for the largest scale factors. Similar performance are achieved when $J_2$ is considered. Hence, weaker vortices are easier to reject than stronger vortices.

**Figure 6.6:** Results from the adjoint-based optimization of the quasi-linear membrane response to vortices of different radii. The abscissa represents the scaling index of the vortex intensity. Different initial values of $\bar{E}_s$ and $\mu$ are provided to the optimization procedure because of the presence of local minima. Results for a vortex with $r_v = 0.25c$ are shown in blue, $r_v = 0.5c$ in red, $r_v = 0.75c$ in yellow and $r_v = 1c$ in purple. The minimum values of the cost function are indicated with circles ($\circ$). Figure (a) shows the optimization of cost function $J_1$ and (b) the results for $J_2$.

The optimal values of $\bar{E}_s$ and $\mu$ are represented in Fig. 6.7. As for the linear case, there is no obvious relation between the optimal structural properties and the vortex characteristics. Again, there is a strong dependency of the optimal solution on the cost function. An exception is represented by large vortices ($r_v = 0.75c$ and $r_v = 1c$) with large scale factors, which present very similar solutions when
either $J_1$ or $J_2$ are minimized. This indicates that the solution that minimizes the integral of the lift, also minimizes the lift peak.

What is the effect of the choice of the cost function on the lift response? A comparison between the response of the membrane from the quasi-linear optimization using cost functions $J_1$ and $J_2$ is plotted in Fig. 6.8, together with the response of the rigid wing. Small vortices with moderate and large intensities optimized using $J_1$ display a strong negative lift peak followed by a transient with lift values close to zero. Adopting $J_2$ in the optimization clearly reduces the magnitude of the lift peak, with reductions above 50% for some cases. This reduction has a price: the transient following the peak is significantly larger with respect to the case optimized with $J_1$, resulting in a larger total integral lift. For large vortices, because of the larger radius and, consequently, because of the lower frequency content of the vortex, the peak in the response is less abrupt compared to smaller vortices. For this reason, the optimized membrane obtained with $J_1$ coincides with that obtained with $J_2$.

What is the best solution? The answer depends on the specific application of interest. When optimizing with different objectives (e.g. minimizing both the...
Figure 6.8: Comparison of the lift response obtained from the quasi-linear optimization using $J_1$ (blue line) and $J_2$ (red line). The response of a rigid wing is also shown for comparison (black line). Results are shown for $r_v = 0.25c$ (a-b-c), $r_v = 0.5c$ (d-e-f), $r_v = 0.75c$ (g-h-i) and $r_v = 1c$ (l-m-n). The scaling indices of the considered cases are 1 (a-d-g-l), 15 (b-e-h-m) and 30 (c-f-i-n).

Integral lift and the peak at the same time), there is always a trade-off between the requirements. For the considered cases, minimizing the integral lift often causes strong negative peaks in the lift response and minimizing both the integral lift and the peak results in larger transients. Better performance might be achieved by employing more sophisticated cost functions and by increasing the number of control parameters.
6.4.3 Comparison between linear and quasi-linear optimization

We have discussed the properties of the individual optimized solutions obtained with a linear and a quasi-linear optimization. How do they compare? Is a linear optimization able to provide structural parameters that represent a good approximation of those obtained with a quasi-linear optimization? To answer this question, we simulate by direct integration of the equations the linear vortex response of the membrane obtained with a linear optimization, and the quasi-linear vortex response of the membrane obtained with the quasi-linear optimization. It is important to remark that we are not comparing, in general, the response of the same membrane. We are comparing the linear response of the membrane whose structural parameters have been obtained with a linear optimization, against the membrane obtained with the quasi-linear optimization and then simulated using the quasi-linear model. This comparison is shown in Fig. 6.9. The solutions have been obtained using cost function $J_2$, but similar considerations apply when $J_1$ is adopted. For weak vortices, irrespective of the vortex size, the membrane’s behavior is approximately linear. Hence, a linear approximation is sufficient to model the response of the membrane. As the strength of the vortices increases, the optimal linear solution and the optimal quasi-linear solution significantly differ, indicating that the membrane exhibits strongly nonlinear behavior.

As additional proof of the inadequacy of the linear model for this optimization, we simulate with the quasi-linear model the vortex-induced response of the membrane whose structural parameters have been obtained from the linear optimization (Fig. 6.9). We want to see if the linear optimization generates a membrane that has optimal properties when directly simulated with the quasi-linear model. For moderate and large intensities of the vortices, this solution presents larger peaks compared to the optimal quasi-linear solution. We can conclude that the linear model is inadequate to approximate the behavior of the membrane unless the vortices are weak, resulting in a very limited range of applicability. Consequently, an optimization based on this model will result in a solution that is far from optimal when tested with a higher-order model. Hence, at least a quasi-linear model is necessary to accurately model the vortex-induced response of the membrane. We will validate and compare the performance of the quasi-linear optimization against the nonlinear model and DNS in the next section.

6.4.4 Validation of the quasi-linear optimization results

The validation of the results from the quasi-linear optimization procedure is achieved by first comparing the optimal solutions against direct integration of the quasi-linear model, to prove that the optimization converges to a minimum of the cost function. Then the optimal solutions are compared against the nonlinear model and DNS.
Figure 6.9: Comparison of linear and quasi-linear optimized lift gust responses using cost function $J_2$. The blue line represents the lift from the optimization of the linear response, the red line the quasi-linear lift from the quasi-linear optimization and the yellow line indicates the quasi-linear response generated using the structural parameters from the optimized linear membrane. Results are shown for $r_v = 0.25c$ (a-b-c), $r_v = 0.5c$ (d-e-f), $r_v = 0.75c$ (g-h-i) and $r_v = 1c$ (l-m-n). Three values of the vortex intensity are shown: scaling index 1 (a-d-g-l), scaling index 15 (b-e-h-m) and 30 (c-f-i-n).

6.4.4.1 Validation of the quasi-linear optimization procedure

To validate the quasi-linear optimization procedure, we want to verify that the results from the optimization really represent minimum values of the cost function. To prove this, the values of the cost function are evaluated with direct simulations using the quasi-linear model for different $\bar{E}_s$ and $\mu$. As shown in Fig. 6.10 for $J_2$, the optimal solution generated by the quasi-linear optimization procedure always lies in a region of lower values of the cost function with respect to the neighboring points, hence representing a minimum. As introduced before and as confirmed by
Figure 6.10: Validation of the optimization of the quasi-linear response with cost function $J_2$ for a scaling index of 20 and vortices with $r_v = 0.25c$ (a-b-c), $r_v = 0.5c$ (d-e-f), $r_v = 0.75c$ (f-g-h) and $r_v = 1c$ (i-l-m). In each row of figures are displayed different local minima obtained initializing the optimization algorithm with different initial solutions for the same considered case (same vortex radius and same scaling factor). Values of the cost function $J_2$ are represented with colors from blue (lower values) to yellow (higher values). The red dots indicate the solutions from the adjoint-based optimization procedure. Each box is centered around the optimal solution and has an amplitude of $0.14\mu$ and $0.14E_s$.

In the validation, some cases present local minima, that can be found by initializing the optimization procedure with different initial values of the control parameters.
6.4.4.2 Validation of the quasi-linear optimized solutions against non-linear model and DNS

The quasi-linear model represents an approximation of the nonlinear fluid-structure interaction model, as shown in Chapter 3. The optimal cases generated with the quasi-linear optimization are simulated with the nonlinear model and compared. The results are shown in Fig. 6.11 for both cost functions. For small and moderate vortex intensities, the nonlinear response is well approximated by the quasi-linear model, especially for the cases optimized using $J_2$. Differences arise when the vortex intensity is large, since larger vortices excite a stronger nonlinear response in the membrane. In particular, nonlinear simulations show that the optimal membrane is more “efficient” with respect to the quasi-linear predictions, since the resulting value of the cost function is smaller.

A comparison in the time domain between nonlinear simulations and quasi-linear simulations in terms of lift response and structural response for small and large scaling indices is shown in Fig. 6.12 and Fig. 6.13. We test the cases generated with the quasi-linear optimization using $J_1$, since they present the largest differences when the cost function is obtained from direct simulation, as indicated by the results in Fig. 6.11. For the four cases considered, the quasi-linear response well approximates the nonlinear response, especially for the less intense vortices, with small differences arising when the deflection of the membrane exceeds 10% of the wing’s cord, as indicated by the structural response in modes 1 and 2.

Results from DNS are also presented in Fig. 6.12 and Fig. 6.13. For a scaling index of 1, the DNS results are almost indistinguishable from those from the low order models. For a scaling index of 30, despite the large values of the deflection of the membrane, the models still approximate the DNS response very well. The good agreement between the results from the quasi-linear model and nonlinear model and DNS indicates that for the considered cases, the quasi-linear model approximates the full nonlinear behavior well. However, our observations are limited to the considered cases, which result from the quasi-linear optimization procedure. This does not guarantee that a nonlinear optimization procedure will produce the same results. A full nonlinear optimization, based on the nonlinear model (or on high-fidelity DNS), would be necessary to demonstrate this argument. As a preliminary investigation, we will limit our discussion to the quasi-linear optimization, concluding that we have achieved a reduction in the lift disturbance generated by a convected vortex for different cases and that the results have been validated against the nonlinear model and DNS.

6.5 Comparison with controller

In Chapter 2, a linear aerodynamic model for the lift generated by the first two deflection modes was coupled to a linear controller to prove that it was possible to track a reference lift and to reject disturbances, by using the flexibility of the wing. This ideal controller was based on the assumption that we were able to perfectly
Figure 6.11: Comparison of the values of the cost functions for the optimized membranes resulting from the quasi-linear optimization, simulated using the quasi-linear model and the nonlinear model. Results are shown for $J_1 (\bigcirc)$ and $J_2 (\square)$. Both cost functions are normalized with the corresponding values obtained for a rigid wing. The blue symbols represent solutions simulated with the quasi-linear model. The values of the cost function from the same membrane simulated with the nonlinear model are represented in red. Results are shown for $r_v = 0.25c$ (a), $r_v = 0.5c$ (b), $r_v = 0.75c$ (c) and $r_v = 1c$ (d).

actuate the wing in mode 1 and mode 2, something that would be difficult to achieve in a real application. We then tested the performance of the controller on the same vortex case presented in this chapter.

We now want to compare the performance of this ideal controller to the performance of the optimized membranes. Since the aerodynamic model is based on the assumption of a linear fluid, the performance of the controller only depends on the vortex radius and not on the vortex intensity (see Chapter 2). To compare the response of the controller with the response of the membrane, we use the same
Figure 6.12: Comparison of quasi-linear optimized lift and structural gust responses simulated using the quasi-linear model, the nonlinear model and DNS. The structural response is represented by the first two deflection modes. The cost function used for the optimization is $J_1$. The blue line indicates the response simulated with the quasi-linear model. The response of the same membrane simulated with the nonlinear model is represented in red. DNS results are presented in yellow. Results are shown for $r_v = 0.25c$ (a-b-c), $r_v = 0.5c$ (d-e-f), $r_v = 0.75c$ (g-h-i) and $r_v = 1c$ (l-m-n). The scaling index of the considered cases is 1.

Parameter $\eta$ defined in Chapter 2:

$$\eta = \frac{\int_0^T (C_L - C_L^{ref})^2 \, dt}{\int_0^T (C_L^{ref})^2 \, dt}.$$  

(6.20)
Figure 6.13: Comparison of quasi-linear optimized lift and structural gust responses simulated using the quasi-linear model, the nonlinear model and DNS. The structural response is represented by the first two deflection modes. The cost function used for the optimization is $J_1$. The blue line indicates the response simulated with the quasi-linear model. The response of the same membrane simulated with the nonlinear model is represented in red. DNS results are presented in yellow. Results are shown for $r_v = 0.25c$ (a-b-c), $r_v = 0.5c$ (d-e-f), $r_v = 0.75c$ (g-h-i) and $r_v = 1c$ (l-m-n). The scaling index of the considered cases is 30.

Results are summarized in Table 6.3. Although we can significantly reduce the effect of the vortex by using an optimally designed membrane with respect to a rigid wing, an active controller presents better performance. Values of $\eta$ for the controller are at least an order of magnitude smaller than those of the membrane.

The lift and the modal response for mode 1 and 2 in the time domain are
Optimization of Vortex-induced Response

Optimization method | $r_v = 0.25c$ | $r_v = 0.5c$ | $r_v = 0.75c$ | $r_v = 1c$
--- | --- | --- | --- | ---
Quasi-linear with $J_1$ and $s_c = 1$ | 16.24% | 16.95% | 19.47% | 22.18%
Quasi-linear with $J_1$ and $s_c = 30$ | 67.10% | 70.67% | 70.38% | 72.00%
Quasi-linear with $J_2$ and $s_c = 1$ | 17.29% | 17.07% | 20.40% | 25.86%
Quasi-linear with $J_2$ and $s_c = 30$ | 85.44% | 91.70% | 69.97% | 68.39%
Controller | 1.083% | 0.759% | 0.480% | 0.290%

Table 6.3: Comparison of the performance of the optimized membranes and the controller, measured through the value of parameter $\eta$.

represented in Fig. 6.14 for the membrane and the controller. The scaling index chosen for the membrane cases is 1. The trend of the lift in time looks quite different between the membrane and controller and it is difficult to understand if there are, somehow, some similarities. More interesting is the comparison of the structural response. The structural response in mode 1 and the actuation for the same mode have a very similar trend, although the membrane seems to have a slightly faster response. Different is the case for the response in mode 2. The actuation of the controller is faster than the response of the membrane, that also presents a small initial peak of opposite sign.

Although the results have indicated that the ideal controller outperforms the membrane in rejecting the vortex disturbance, it is important to mention that the membrane has been optimized using two control parameters only. An optimization performed with additional parameters such as the pre-stretch $\delta_0$ or a non-uniform distribution of thickness might improve the membrane’s performance.

6.6 Summary

An adjoint-based optimization framework has been presented. For a given fluid-structure interaction model, the state equations are solved in conjunction with the corresponding adjoint equations in order to find the gradient of a cost function with respect to some control parameters. A gradient-based minimization algorithm is then employed to minimize the cost function. The optimization framework has been used to reduce the effect of an external disturbance on the aerodynamics of a membrane wing by exploiting the flexibility of the membrane. As a preliminary investigation, the optimization has been carried out on a membrane wing at zero angle of attack subjected to a disturbance in the form of a convective vortex. Multiple small-scale vortices, with a viscous radius of the order of magnitude of the wing chord or smaller, have been tested. The control parameters that have been chosen to control the response of the membrane are the membrane stiffness $\bar{E}_s$ and the density ratio $\mu$.

Two optimization frameworks have been employed: a linear optimization, based on the linear fluid-structure interaction model and its corresponding adjoint equation, and a quasi-linear optimization, based on the quasi-linear model and its adjoint. For the cost function, a first approach aimed to minimize the integral of the
Figure 6.14: Comparison of quasi-linear optimized lift and structural gust responses simulated using the quasi-linear model and the nonlinear model and the response of an external controller. The structural response is represented by the first two deflection modes. The cost function used for the optimization is $J_1$. The blue line indicates the lift response simulated with the quasi-linear model. The lift response of the same membrane simulated with the nonlinear model is represented in red. The controller’s response is plotted in black. Results are shown for $r_v = 0.25c$ (a-b-c), $r_v = 0.5c$ (d-e-f), $r_v = 0.75c$ (g-h-i) and $r_v = 1c$ (l-m-n). The scaling index of the considered cases is 1.

Lift generated by the vortex, while a second approach aimed to minimize both the integral and the peak of the lift response. Both cost functions have been compared and discussed. The linear optimization procedure, although more convenient from a mathematical and from a computational point of view, generated solutions that,
when tested with higher order models, were far from the optimum. An exception was represented by those cases where the vortex intensity was smaller. Given its limited range of applicability, the linear optimization was discarded in favor of a quasi-linear optimization. The optimization carried out using the quasi-linear model produced solutions whose vortex-induced response in time was successfully validated with higher order models such as the nonlinear fluid-structure interaction model and DNS. Although this type of validation does not guarantee that the quasi-linear solutions are optimal solutions in the nonlinear range, it proves that the models achieve a significant reduction of the cost function that can be reproduced in the nonlinear framework.

Finally, the aeroelastic vortex-induced response of the optimized membranes has been compared to the response of ideal controllers generated with loop-shaping techniques. Despite the reduction in the disturbance provided by the membrane, the ideal controllers achieve a reduction that is one order of magnitude larger. Future work should aim to develop a full nonlinear optimization procedure based on the nonlinear fluid-structure interaction model and should incorporate additional control parameters such as the membrane’s pre-stretch or non-uniform distributions of density and stiffness. In addition, vortex rejection during unsteady maneuvers should be considered, in order to provide a more realistic approximation of a flapping-wing during operative flight conditions.
Chapter 7

Conclusions and Future Work

7.1 Summary and conclusions

The main contribution of the present work is the investigation of the steady and unsteady aeroelastic performance of membrane wings using novel linear and nonlinear reduced-order models tailored for low Reynolds number applications. These models are adopted in conjunction with mathematical tools such as feedback control theory, harmonic balance methods and adjoint-based optimization methods to provide valuable insights into the performance of membrane wings from an engineering perspective.

The fluid-structure interaction models are obtained from the coupling of a linear unsteady aerodynamic model with one-dimensional membrane models. The linear aerodynamic model represents a low-order representation of the unsteady aerodynamic loads generated by the rigid motion and the deflection of a two-dimensional wing immersed in a free-stream at low Reynolds numbers. The mathematical formulation was presented by Brunton & Rowley [2013] for the rigid degrees of freedom of the wing, and it has been extended in this work to include the flexible degrees of freedom and the pressure distribution around the wing. These inclusions are fundamental for the coupling with structural models. The aerodynamic model is obtained from Direct Numerical Simulations using a system identification procedure and it is presented in state-space form, hence it is suitable for the integration with modern feedback control frameworks. The aerodynamic model outperforms linear models based on thin-airfoil theory for inviscid, potential flows because it is able to capture some of the effects of the low Reynolds number regime, such as the steady-state lift reduction due to the boundary layer.

A one-dimensional nonlinear membrane model successfully adopted in previous works related to membrane wings [Gordnier, 2009, Serrano-Galiano et al., 2018, Smith & Shyy, 1996] is simplified using a modal decomposition of the deflection and a Taylor expansion of the nonlinear terms. This results in two lower-order models: a quasi-linear and a linear model. The nonlinear, quasi-linear and linear membrane equations are coupled to the linear aerodynamic model previously introduced to
obtain nonlinear, quasi-linear and linear fluid-structure interaction models. These models are presented in state-space form. Thanks to the DNS-based aerodynamic model, they are tailored for the low-Reynolds-number regime.

Investigations on the steady-state aeroelastic response at different angles of attack and different Reynolds numbers in the range 100–10000 have shown the importance of a correct representation of the aerodynamic forces in membrane wing studies, especially when considering high compliance membranes. The Reynolds number has an effect on both the lift and the deflection of the membrane. This study has shown that a linear fluid-structure interaction model, although appealing from a mathematical point of view because of its linearity, has a very limited range of applicability. The increase of the tension of the membrane due to its deflection is a nonlinear effect that cannot be neglected a priori. The quasi-linear model has been shown to be able to accurately model the effect of the nonlinear tension increase, with the advantage of retaining the modal decomposition of the deflection, hence presenting a lower number of degrees of freedom with respect to the nonlinear model. Comparison with the nonlinear model has shown excellent agreement even for moderate deflections of the membrane. The linear fluid-structure interaction model is adopted to investigate the linear stability of the membrane at zero angle of attack. The stability envelope is significantly affected by the Reynolds number, further confirmation of the importance of low-Reynolds-number effects on the aeroelastic response of small-scale membrane wings.

The investigation of the aeroelastic performance of the membrane is extended to the unsteady regime. In particular, the aeroelastic response of a membrane undergoing periodic oscillations in pitching (forced response) about its leading edge is considered. Harmonic balance methods in the frequency domain and in the time domain have been applied to the reduced-order models to obtain the periodic steady-state response of the membrane. The pitching membrane represents a first approximation of a wing undergoing a flapping cycle, but the harmonic balance formulation can be easily generalized to include periodic maneuvers obtained from a combination of pitching and plunging. A comparison with DNS has demonstrated the accuracy of the models and of the corresponding harmonic balance solutions over a broad range of pitching frequencies and even in the presence of weak/moderate nonlinear aerodynamic effects in the flow. Harmonic balance method have also been employed to study the autonomous limit-cycle oscillations of linearly unstable membranes. Since in this case the frequency of the oscillations is not known a priori, a frequency-searching algorithm has been presented. The predictions from the harmonic balance method applied to the quasi-linear model are in good agreement with DNS results, but the convergence of the method strongly depends on the initial solution provided to the iterative procedure, hence requiring some a priori knowledge of the system’s response.

Finally, an optimization of the response of the membrane to an external disturbance in the form of a vortex has been carried out. The investigation aims to design optimized membranes that are able to reduce the effect of aerodynamic disturbances with respect to rigid wings, thanks to their compliance. As a preliminary investigation, a membrane at zero angle of attack is optimized with two
control parameters, the stiffness and the density ratio. The optimization procedure is based on the solution of the governing equations of the reduced-order models and their adjoint. Direct and adjoint equations provide a computationally efficient framework to evaluate the gradient of a cost function with respect to the control parameters. Once the gradient has been obtained, the cost function is minimized using a quasi-Newton minimization algorithm. Results have shown that the flexibility of the membrane is able to reduce the lift disturbance caused by a convective vortex with respect to a rigid wing for all the considered cases. Weaker vortices are easier to reject than stronger vortices. Results have also shown that the structural parameters of the optimized membrane are significantly affected by the intensity and the radius of the vortex: a membrane that is optimized for a certain vortex might present poor performance when tested with a different vortex. Although membranes can reduce the effect of an external disturbance without the need for sensors and actuators, a comparison with ideal feedback controllers has shown that actuating the wing is theoretically more efficient in rejecting disturbances. An optimization based on additional control parameters such as the pre-stretch of the membrane or a non-uniform distribution of thickness should be carried out to investigate whether the disturbance-rejection performance of the membrane can be improved.

7.2 Future work

The fluid-structure interaction reduced-order models presented in this work represent a starting point for possible future work. Here we present some of the possible developments.

Extension to 3D
The present fluid-structure interaction models allow, although with some technical complications, for an extension to 3D. A 3D linear unsteady aerodynamic model can be generated from 3D DNS for different wing geometries (e.g. a rectangular wing [Brunton et al., 2008, Moriche et al., 2016, Taira & Colonius, 2009]) by introducing additional degrees of freedom in the spanwise direction. Such a linear model can then be coupled to existing linear and nonlinear two-dimensional membrane models to obtain fully 3D fluid-structure interaction models. Such a contribution would be valuable to investigate the effect of three-dimensional parameters such as the aspect ratio on the aeroelastic performance of membrane wings.

Nonlinear aerodynamic modeling
One of the limitations of the present models is the adoption of a linear aerodynamic model. Most natural flyers exploit the benefits offered by nonlinear aerodynamics, such as the additional lift provided by leading-edge vortex shedding. Hence, nonlinear unsteady aerodynamics should be taken into account in the early design stages of high-performance MAVs. The development of fully nonlinear reduced-order unsteady aerodynamic models for low Reynolds numbers presents several challenges and represents an open problem in aerodynamics, aeroelasticity, fluid-structure interaction and flight-dynamics. Recently Wang & Eldredge [2013] were
able to develop a low order point vortex model for the unsteady aerodynamics of a two-dimensional rigid flat plate. This model was able to capture, although with some limitations, the nonlinear effects of the leading and trailing edge vortices for large amplitude pitching maneuvers. The predictive capabilities of the model were later improved by adjusting the initial vortex strength and its rate of change using information from high-fidelity simulations via an optimization procedure [Hemati et al., 2014]. An alternative to vortex methods is represented by newly developed mathematical frameworks such as the resolvent method [McKeon & Sharma, 2010], which might provide solutions to model the nonlinear dynamics of turbulent and separated flows at low Reynolds numbers.
Appendix A

Functions for Unsteady Maneuvers

A.1 AIAA canonical ramp-up, hold, ramp-down maneuver

The canonical ramp-up, hold, ramp-down maneuver was introduced by the AIAA Fluid Dynamics Technical Committee (FDTC) low Reynolds number discussion group [Ol et al., 2010] as a representative maneuver for numerical and experimental studies in unsteady aerodynamics at low Reynolds numbers [Eldredge et al., 2009]. The maneuver was originally introduced for pitching motions. Here, we present the function in its general form introducing \( \Psi(t) \) and its time derivatives. \( \Psi(t) \) represents a generic rigid (pitching or plunging) or flexible degree of freedom of the wing. The canonical ramp-up, hold, ramp-down maneuver can be represented as follows:

\[
\begin{align*}
\Psi(t) &= \Psi_{\text{max}} \frac{G_1(t)}{\max(G_1(t))} \\
\dot{\Psi}(t) &= \Psi_{\text{max}} \frac{\dot{G}_1(t)}{\max(G_1(t))} \\
\ddot{\Psi}(t) &= \Psi_{\text{max}} \frac{\ddot{G}_1(t)}{\max(G_1(t))},
\end{align*}
\]

(A.1)

with

\[
\begin{align*}
G_1(t) &= \log \left[ \frac{\cosh(a(t - t_1)) \cosh(a(t - t_4))}{\cosh(a(t - t_2)) \cosh(a(t - t_3))} \right] \\
\dot{G}_1(t) &= a \left[ \tanh (a (t - t_1)) + \tanh (a (t - t_4)) + \tanh (a (t - t_2)) - \tanh (a (t - t_3)) \right] \\
\ddot{G}_1(t) &= a^2 \left[ \sech^2 (a (t - t_1)) + \sech^2 (a (t - t_4)) + \sech^2 (a (t - t_2)) + \sech^2 (a (t - t_3)) \right].
\end{align*}
\]

(A.2)
\( \Psi_{\text{max}} \) represents the maximum value of \( \Psi(t) \) reached during the maneuver and \( a \) is a smoothing parameter that determines how gradual the maneuver is. In particular, the parameter \( a \) determines the magnitude of the velocity and of the acceleration. \( \Psi(t), \dot{\Psi}(t) \) and \( \ddot{\Psi}(t) \) for a generic maneuver are represented in Fig. A.1. Time instants \( t_1, t_2, t_3 \) and \( t_4 \) define the intervals of the different phases of the maneuver.

![Figure A.1: Canonical ramp-up, hold, ramp-down maneuver. (a) Maneuver applied to a generic degree of freedom of the wing expressed by \( \Psi(t) \). (b) Velocity \( \dot{\Psi}(t) \). (c) Acceleration \( \ddot{\Psi}(t) \).]

### A.2 Ramped step

A linear ramp-up maneuver is derived from the canonical ramp-up, hold, ramp-down maneuver. It was introduced to represent a gradual step change in pitching and it was adopted by Brunton & Rowley [2013] as to obtain the lift pulse response of the wing for system identification. Indicating with \( \Psi(t) \) a generic degree of freedom of the wing, the linear ramp maneuver is represented by the following expressions:

\[
\begin{align*}
\Psi(t) &= \Psi_{\text{max}} \frac{G_2(t)}{\max(G_2(t))} \\
\dot{\Psi}(t) &= \Psi_{\text{max}} \frac{\dot{G}_2(t)}{\max(G_2(t))} \\
\ddot{\Psi}(t) &= \Psi_{\text{max}} \frac{\ddot{G}_2(t)}{\max(G_2(t))},
\end{align*}
\]

with

\[
\begin{align*}
G_2(t) &= \log \left[ \frac{\cosh(a(t-t_1)) \cosh(-a(t_2))}{\cosh(a(t-t_2)) \cosh(-a(t_1))} \right] \\
\dot{G}_2(t) &= a \left[ \tanh(a(t-t_1)) - \tanh(a(t-t_2)) \right] \\
\ddot{G}_2(t) &= a^2 \left[ \text{sech}^2(a(t-t_1)) - \text{sech}^2(a(t-t_2)) \right].
\end{align*}
\]

The maneuver is represented in Fig. A.2. As already mention for the canonical ramp-up, hold, ramp-down maneuver in the previous section, \( \Psi_{\text{max}} \) represents the
maximum value of the function and the amplitude of the step, while $a$ represents a smoothing parameter.

**Figure A.2:** Linear ramp maneuver. (a) Maneuver applied to a generic degree of freedom of the wing expressed by $\Psi(t)$. (b) Velocity $\dot{\Psi}(t)$. (c) Acceleration $\ddot{\Psi}(t)$. 
Appendix B

Derivation of Fluid-structure Interaction Models

B.1 Derivation of quasi-linear and linear membrane model

The nonlinear terms in Eqs. 3.6 and 3.7 can be expressed using a Taylor series expansion around \( \partial w/\partial x = 0 \), which can be subsequently approximated by the truncated Fourier series introduced in Eq. 2.4, resulting in

\[
\left[ 1 + \left( \frac{\partial w}{\partial x} \right)^2 \right]^{1/2} = 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + O \left( \left( \frac{\partial w}{\partial x} \right)^4 \right) = \\
= 1 + \frac{1}{2} \left( \sum_{k=1}^{N} \pi k \mathcal{W}_k \cos (\pi k x) \right)^2 + O(\mathcal{W}^4)
\]

(B.1)

\[
\left[ 1 + \left( \frac{\partial w}{\partial x} \right)^2 \right]^{-3/2} = 1 - \frac{3}{2} \left( \frac{\partial w}{\partial x} \right)^2 + O \left( \left( \frac{\partial w}{\partial x} \right)^4 \right) = \\
= 1 - \frac{3}{2} \left( \sum_{k=1}^{N} \pi k \mathcal{W}_k \cos (\pi k x) \right)^2 + O(\mathcal{W}^4),
\]

(B.2)

with \( \mathcal{W}_k = O(\mathcal{W}) \). We are assuming that the contributions of the individual modes to the membrane’s structural response are of the same order of magnitude (for the considered number of modes, \( N = 10 \)).
Following Eq. B.1, the elongation $\delta$ can be expressed as

$$
\delta = \frac{1}{2} \int_0^1 \left( \sum_{k=1}^{N} \frac{\pi k W_k \cos (k \pi x)}{2} \right)^2 dx = \frac{1}{2} \int_0^1 \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \pi^2 i j W_i W_j \cos (i \pi x) \cos (j \pi x) \right) dx = \frac{\pi^2}{4} \sum_{k=1}^{N} k^2 W_k^2 + O(W^4),
$$

(B.3)

since

$$
\int_0^1 \sum_{i=1}^{N} \sum_{j=1}^{N} \pi^2 i j W_i W_j \cos (i \pi x) \cos (j \pi x) dx = 0, \quad \text{for } i \neq j. \quad \text{(B.4)}
$$

Assuming that the initial pre-strain $\delta_0$ has the same order of magnitude as the elongation $\delta$ (formally, $\delta_0 = O(\delta) = O(W^2)$), following Eq. B.2 and B.3 and only retaining terms up to the second-order, the quasi-linear approximation of Eq. 3.6 is

$$
2\mu \ddot{w} + 2C_d \dot{w} = \Delta C_p + C_f
$$

(B.5)

Equation B.5, which serves as the quasi-linear model, allows a low order representation of the nonlinearity present in Eq. 3.7, while still retaining the modal decomposition of the membrane’s deflection and without neglecting the nonlinear contribution of the elongation $\delta$. In terms of Fourier deflection coefficients, Eq. B.5 becomes

$$
2\mu \ddot{\mathbf{W}} + 2C_d \dot{\mathbf{W}} + \pi^2 c_{T0} \Gamma^2 \mathbf{W} + \frac{\pi^4}{2} \bar{E}_4 \mathbf{G}^2 (\mathbf{W}^T \mathbf{G} \mathbf{W}) \mathbf{W} = \Delta C_p + C_f.
$$

(B.6)

A linear membrane model is derived from a first-order approximation of Eqs. B.1 and B.2, resulting in the following equation in modal form:

$$
2\mu \ddot{\mathbf{W}} + 2C_d \dot{\mathbf{W}} + \pi^2 c_{T0} \Gamma^2 \mathbf{W} = \Delta C_p + C_f.
$$

(B.7)

From Eq. B.2 it is clear that a fundamental result of the first-order approximation is that $\delta \approx 0$, which means that the elongation of the membrane due to its deflection is neglected.
B.2 Derivation of the nonlinear fluid-structure interaction model

The nonlinear fluid-structure interaction model is determined by the coupling between the linear aerodynamic model and the nonlinear membrane and its dynamics are defined by the following system of equations:

\begin{align*}
2\mu \ddot{w} &= -2C_d \dot{w} - f^{NL} + \Delta C_p + C_f, \quad \text{(B.8)} \\
\dot{x} &= A_\alpha^x x + B_\alpha^x \dot{w} + B_W^x \dot{\dot{w}}, \quad \text{(B.9)} \\
C_L &= C_x^L \dot{x} + C_\alpha^L \dot{\alpha} + C_W^L \dot{W} + C_y^L \dot{Y} + C_y^L \dot{\dot{W}} + C_y^{NL} \dot{W} + C_L^L \dot{\dot{\alpha}} + C_L^{NL} \dot{\dot{W}} \quad \text{(B.10)} \\
\Delta C_p &= C_x^{p/} \dot{x} + C_\alpha^{p/} \dot{\alpha} + C_W^{p/} \dot{W} + C_y^{p/} \dot{Y} + C_y^{p/} \dot{\dot{W}} + C_R^{p/} \dot{\dot{\alpha}} + C_R^{p/} \dot{\dot{W}} \quad \text{(B.11)}
\end{align*}

Since the membrane equation is expressed in physical space and the aerodynamic model in modal space, we introduce the following transformations:

\begin{align*}
w &= \Phi^{m2s} W, \\
\Delta C_p &= \Phi^{m2s} \Delta C_p, \\
C_f &= \Phi^{m2s} C_f,
\end{align*}

(B.12) (B.13) (B.14)

similar considerations also hold for the time derivatives of \(w\) and \(W\). Rearranging Eq. B.9, substituting \(\Delta C_p = \Phi^{m2s} \Delta C_p\) from Eq. B.11 and using transformation matrices \(\Phi^{m2s}\) and \(\Phi^{s2m}\), we get

\begin{align*}
\left(2\mu I - \Phi^{m2s} C_W^{p/} \Phi^{s2m}\right) \ddot{w} = \\
\left(\Phi^{m2s} C_W^{p/} \Phi^{s2m} - 2C_d I\right) \dot{w} - f^{NL} + \Phi^{m2s} C_W^{p/} \Phi^{s2m} \dot{w} + \\
+ \Phi^{m2s} C_x^{p/} \dot{x} + \Phi^{m2s} C_\alpha^{p/} \dot{\alpha} + \Phi^{m2s} C_W^{p/} \dot{\dot{W}} + \Phi^{m2s} C_R^{p/} \dot{\dot{\alpha}} + C_f,
\end{align*}

from which we can obtain an expression for \(\ddot{w}\) by multiplying both sides by \(\bar{M}^{-1} = \left(2\mu I - \Phi^{m2s} C_W^{p/} \Phi^{s2m}\right)^{-1}\), resulting in

\begin{align*}
\ddot{w} &= \bar{M}^{-1} \left(\Phi^{m2s} C_W^{p/} \Phi^{s2m} - 2C_d I\right) \dot{w} - \bar{M}^{-1} f^{NL} + \\
&+ \bar{M}^{-1} \Phi^{m2s} C_W^{p/} \Phi^{s2m} \dot{w} + \bar{M}^{-1} \Phi^{m2s} C_x^{p/} \dot{x} + \bar{M}^{-1} \Phi^{m2s} C_\alpha^{p/} \dot{\alpha} + \\
&+ \bar{M}^{-1} \Phi^{m2s} C_R^{p/} \dot{\dot{\alpha}} + \bar{M}^{-1} \Phi^{m2s} C_R^{p/} \dot{\dot{W}} + \bar{M}^{-1} C_f.
\end{align*}
For the sake of clarity, we introduce now the following matrices

\[
\begin{align*}
\bar{A}_{x}^{wfs} &= \bar{M}^{-1} \Phi m^{2s} C_{x}^{pf} \\
\bar{A}_{\alpha}^{wfs} &= \bar{M}^{-1} \Phi m^{2s} C_{\alpha}^{pf} \\
\bar{A}_{\bar{\alpha}}^{wfs} &= \bar{M}^{-1} \Phi m^{2s} C_{\bar{\alpha}}^{pf} \\
\bar{A}_{w}^{wfs} &= \bar{M}^{-1} \Phi m^{2s} C_{w}^{pf} s^{2m} \\
\bar{A}_{w}^{wfs} &= \bar{M}^{-1} (\Phi m^{2s} C_{w}^{pf} s^{2m} - 2C_{d}I) \\
\bar{B}_{\alpha}^{wfs} &= \bar{M}^{-1} \Phi m^{2s} C_{\alpha}^{pf} \\
\bar{B}_{Cf}^{wfs} &= \bar{F}_{f}^{wfs} = \bar{M}^{-1},
\end{align*}
\]

that can be used to generate a more compact expression for \( \bar{\omega} \):

\[
\begin{align*}
\bar{\omega} &= \bar{A}_{w}^{wfs} \bar{\omega} - \bar{F}_{f}^{wfs} \bar{f}^{NL} + \bar{A}_{w}^{wfs} \bar{\omega} + \\
&+ \bar{A}_{x}^{wfs} \bar{x} + \bar{A}_{\alpha}^{wfs} \bar{\alpha} + \bar{A}_{\bar{\alpha}}^{wfs} \bar{\bar{\alpha}} + \bar{B}_{\alpha}^{wfs} \bar{\beta} + \bar{B}_{Cf}^{wfs} \bar{C}_{f}.
\end{align*}
\]

In order to find an expression for \( \hat{x} \), \( \bar{\mathbf{W}} = \Phi s^{2m} \bar{\omega} \) is obtained from Eq. B.15 and substituted into Eq. B.10, resulting in

\[
\begin{align*}
\hat{x} &= \left( \bar{A}_{x}^{xfs} + \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{A}_{x}^{wfs} \right) \bar{x} + \left( \bar{B}_{\alpha}^{xfs} + \bar{B}_{\beta}^{xfs} \Phi s^{2m} \bar{B}_{\alpha}^{wfs} \right) \bar{\alpha} + \\
&+ \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{A}_{\alpha}^{wfs} \bar{\alpha} - \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{F}_{f}^{xfs} \bar{f}^{NL} + \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{A}_{w}^{wfs} \bar{\omega} + \\
&+ \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{A}_{\alpha}^{wfs} \bar{\alpha} + \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{A}_{\bar{\alpha}}^{wfs} \bar{\bar{\alpha}} + \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{F}_{Cf}^{wfs} \bar{C}_{f}.
\end{align*}
\]

Introducing the following matrices

\[
\begin{align*}
\bar{A}_{x}^{xfs} &= \bar{A}_{x}^{xfs} + \bar{B}_{x}^{xfs} \Phi s^{2m} \bar{A}_{x}^{wfs} \\
\bar{A}_{\alpha}^{xfs} &= \bar{A}_{\alpha}^{xfs} + \bar{B}_{\alpha}^{xfs} \Phi s^{2m} \bar{A}_{\alpha}^{wfs} \\
\bar{A}_{\bar{\alpha}}^{xfs} &= \bar{A}_{\bar{\alpha}}^{xfs} + \bar{B}_{\bar{\alpha}}^{xfs} \Phi s^{2m} \bar{A}_{\bar{\alpha}}^{wfs} \\
\bar{A}_{w}^{xfs} &= \bar{A}_{w}^{xfs} + \bar{B}_{w}^{xfs} \Phi s^{2m} \bar{A}_{w}^{wfs} \\
\bar{A}_{w}^{xfs} &= \bar{A}_{w}^{xfs} + \bar{B}_{w}^{xfs} \Phi s^{2m} \bar{A}_{w}^{wfs} \\
\bar{B}_{\alpha}^{xfs} &= \bar{B}_{\alpha}^{xfs} + \bar{B}_{\beta}^{xfs} \Phi s^{2m} \bar{B}_{\alpha}^{wfs} \\
\bar{B}_{Cf}^{xfs} &= \bar{B}_{Cf}^{xfs} + \bar{B}_{Cf}^{xfs} \Phi s^{2m} \bar{B}_{Cf}^{wfs} \\
\bar{F}_{f}^{xfs} &= -\bar{B}_{x}^{xfs} \Phi s^{2m} \bar{F}_{f}^{wfs},
\end{align*}
\]

we have

\[
\hat{x} = \bar{A}_{x}^{xfs} \bar{x} + \bar{B}_{\alpha}^{xfs} \bar{\alpha} + \bar{B}_{w}^{xfs} \bar{\omega} + \bar{F}_{f}^{xfs} \bar{f}^{NL} + \bar{A}_{w}^{xfs} \bar{\omega} + \\
+ \bar{A}_{\alpha}^{wfs} \bar{x} + \bar{A}_{\bar{\alpha}}^{wfs} \bar{\bar{\alpha}} + \bar{B}_{Cf}^{wfs} \bar{C}_{f}.
\]

A similar procedure can be repeated for \( C_{L} \), resulting in

\[
C_{L} = C_{x}^{Lfs} \bar{x} + C_{\alpha}^{Lfs} \bar{\alpha} + C_{\bar{\alpha}}^{Lfs} \bar{\bar{\alpha}} + C_{w}^{Lfs} \bar{\omega} + C_{w}^{Lfs} \bar{\omega} + D_{\bar{\alpha}}^{Lfs} \bar{\alpha} + \\
+ D_{Cf}^{Lfs} \bar{C}_{f} + F_{f}^{Lfs} \bar{f}^{NL}
\]
with

\[
\begin{align*}
\ddot{C}_x^L &= C_x^L + C_{Wx}^L \Phi x^m \ddot{A}_x^w, \\
\ddot{C}_\alpha^L &= C_\alpha^L + C_{W\alpha}^L \Phi x^m \ddot{A}_\alpha^w, \\
\ddot{C}_\alpha^L &= C_{\dot{\alpha}}^L + C_{W\alpha}^L \Phi x^m \ddot{A}_\alpha^w, \\
\ddot{C}_w^L &= C_{Ww}^L \Phi x^m + C_{Ww}^L \Phi x^m \ddot{A}_w^w, \\
\ddot{C}_w^L &= C_{Ww}^L \Phi x^m + C_{Ww}^L \Phi x^m \ddot{A}_w^w, \\
\ddot{D}_\alpha^L &= C_{\dot{\alpha}}^L + C_{W\alpha}^L \Phi x^m \ddot{B}_\alpha^w, \\
\ddot{D}_C^L &= C_{W\alpha}^L \Phi x^m \ddot{B}_C^w, \\
\ddot{F}_f^L &= -C_{Wf}^L \Phi x^m \ddot{F}_f^w.
\end{align*}
\]

By combining Eqs. B.15, B.16 and B.17, the nonlinear fluid-structure interaction model in state-space form can be expressed as follows:

\[
\frac{d}{dt}\begin{bmatrix} \ddot{x} \\ \ddot{w} \end{bmatrix} = \begin{bmatrix} A_x^L & A_{\alpha}^L & A_w^L & A_{\dot{\alpha}}^L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{w} \end{bmatrix} + \begin{bmatrix} B_{\ddot{x}}^L & B_{\ddot{w}}^L \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\omega} \end{bmatrix} + \begin{bmatrix} C_C^L \\ C_f^L \\ \ddot{f}_{NL} \end{bmatrix} + \begin{bmatrix} \ddot{F}_f^L \\ \ddot{F}_f^L \end{bmatrix}
\]

\[\text{(B.18)}\]

\[
\begin{bmatrix} C_L \\ w \end{bmatrix} = \begin{bmatrix} C_x^L & C_{\alpha}^L & C_w^L & C_{\dot{\alpha}}^L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{w} \end{bmatrix} + \begin{bmatrix} \ddot{D}_{\ddot{\alpha}}^L & \ddot{D}_{\ddot{\omega}}^L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\omega} \end{bmatrix} + \begin{bmatrix} \ddot{C}_C^L \\ \ddot{C}_f^L \\ \ddot{f}_{NL} \end{bmatrix} + \begin{bmatrix} \ddot{F}_f^L \\ \ddot{F}_f^L \end{bmatrix}.
\]

\[\text{(B.19)}\]

\[\text{B.3 \ Derivation of the quasi-linear and linear fluid-structure interaction models} \]

The governing equations of the quasi-linear fluid-structure interaction model are derived from the equations of the quasi-linear membrane coupled to the linear aerodynamic model:

\[
2\mu \ddot{W} = -2C_d \ddot{W} - \pi^2 c_{b_2} \Gamma^2 \ddot{W} - \Delta C_p + C_f,
\]

\[\text{(B.19)}\]

\[
\Delta C_p = C_x^{pL} \ddot{x} + C_{\alpha}^{pL} \ddot{\alpha} + C_w^{pL} \ddot{W} + C_{\dot{\alpha}}^{pL} \ddot{\dot{\alpha}} + C_W^{pL} \ddot{W} + C_{\dot{\alpha}}^{pL} \ddot{\dot{\alpha}} + C_W^{pL} \ddot{W} + C_{\dot{\alpha}}^{pL} \ddot{\dot{\alpha}} + C_W^{pL} \ddot{W}.
\]

\[\text{(B.20)}\]

\[
\ddot{x} = \ddot{A}_x^L \ddot{x} + B_{\ddot{\alpha}}^L \ddot{\dot{x}} + B_{\ddot{W}}^L \ddot{W}
\]

\[\text{(B.21)}\]

\[
C_L = C_x^{L} \ddot{x} + C_{\alpha}^{L} \ddot{\alpha} + C_w^{L} \ddot{W} + C_{\dot{\alpha}}^{L} \ddot{\dot{\alpha}} + C_W^{L} \ddot{W} + C_{\dot{\alpha}}^{L} \ddot{\dot{\alpha}} + C_W^{L} \ddot{W}.
\]

\[\text{(B.22)}\]
By substituting the pressure term from the aerodynamic model into the membrane equation and by introducing $M^{-1} = \left(2\mu I - C_{w}^f\right)^{-1}$, we have

$$\ddot{W} = A_{w_{fs}}^{w_{fs}}\dot{W} + \left(A_{w_{fs}}^{w_{fs}} + A_{w_{ql}}^{w_{fs}}\right) W + A_{x}^{w_{fs}} x + A_{\alpha}^{w_{fs}} \dot{\alpha} + A_{\alpha}^{w_{fs}} \ddot{\alpha} + B_{\alpha}^{w_{fs}} \dot{\alpha} + B_{c_{f}}^{w_{fs}} C_{f},$$  \hspace{1cm} (B.23)

with

$$A_{x}^{w_{fs}} = M^{-1} C_{x}^{w_{fs}}$$
$$A_{\alpha}^{w_{fs}} = M^{-1} C_{\alpha}^{w_{fs}}$$
$$A_{\alpha}^{w_{fs}} = M^{-1} C_{\alpha}^{w_{fs}}$$
$$A_{w_{ql}}^{w_{fs}} = M^{-1} C_{w_{ql}}^{w_{fs}} - \frac{\pi^2}{2} C_{T_0} \Gamma^2$$
$$A_{w_{Ws}}^{w_{fs}} = -M^{-1} \frac{\pi^4}{2} E_s \Gamma^2 (W^T \Gamma^2 W)$$
$$A_{w_{ws}}^{w_{fs}} = M^{-1} \left(C_{w_{ws}}^{w_{fs}} - 2C_{w} \hat{I}\right)$$
$$B_{\alpha}^{w_{fs}} = M^{-1} C_{\alpha}^p$$
$$B_{c_{f}}^{w_{fs}} = M^{-1}.$$

By replacing Eq. B.23 into Eq. B.21 and Eq. B.22 respectively, we get

$$\dot{x} = A_{x_{fs}}^{x_{fs}} x + A_{y_{ws}}^{x_{fs}} \dot{W} + \left(A_{y_{ws}}^{x_{fs}} + A_{y_{ws}}^{x_{fs}}\right) W + A_{\alpha}^{x_{fs}} \dot{\alpha} + A_{\alpha}^{x_{fs}} \ddot{\alpha} + B_{\alpha}^{x_{fs}} \dot{\alpha} + B_{c_{f}}^{x_{fs}} C_{f}$$
$$C_{L} = C_{x}^{L_{fs}} x + C_{\alpha}^{L_{fs}} \dot{\alpha} + C_{\alpha}^{L_{fs}} \ddot{\alpha} + \left(C_{y_{ws}}^{L_{fs}} + C_{y_{ws}}^{L_{fs}}\right) W + C_{w_{ws}}^{L_{fs}} \dot{W} + D_{\alpha}^{L_{fs}} \dot{\alpha} + D_{c_{f}}^{L_{fs}} C_{f}$$
with

\[
\begin{align*}
A^x_{fs} &= A^x_f + B^x_f A^w_{fs} \\
A^x_{fs} &= B^x_f A^w_{fs} \\
A^x_{fs} &= B^x_f A^w_{fs} \\
A^x_{W,l} &= B^x_f A^w_{W,l} \\
A^x_{W,q} &= B^x_f A^w_{W,q} \\
A^x_{fs} &= B^x_f A^w_{fs} \\
B^x_{fs} &= B^x_f + B^x_w A^w_{fs} \\
B^x_{fs} &= B^x_f B^w_{fs} \\
C^L_{fs} &= C^L_f + C^L_w A^w_{fs} \\
C^L_{fs} &= C^L_f + C^L_w A^w_{fs} \\
C^L_{fs} &= C^L_f + C^L_w A^w_{fs} \\
C^L_{W,W} &= C^L_f + C^L_w A^w_{W} \\
C^L_{W,q} &= C^L_w A^w_{W,q} \\
C^L_{W} &= C^L_w A^w_{W} \\
D^L_{fs} &= C^L_f + C^L_w B^w_{fs} \\
D^L_{fs} &= C^L_f B^w_{fs} \\
\end{align*}
\]

The quasi-linear fluid-structure interaction model in state-space form can then be expressed as:

\[
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\alpha} \\
\dot{W} \\
\dot{W}
\end{bmatrix}
= \begin{bmatrix}
A^x_{fs} & A^x_{fs} & A^x_{fs} & A^x_{W,l} & A^x_{W,q} & A^x_{W} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
A^w_{fs} & A^w_{fs} & A^w_{fs} & A^w_{W,l} & A^w_{W,q} & A^w_{W}
\end{bmatrix}
\begin{bmatrix}
\bar{x} \\
\alpha \\
\dot{W} \\
\dot{W}
\end{bmatrix}
+ \begin{bmatrix}
B^x_{fs} \\
B^x_{fs} \\
B^x_{fs} \\
B^x_{fs}
\end{bmatrix}
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\alpha} \\
\dot{W} \\
\dot{W}
\end{bmatrix}
\]

\[
\begin{bmatrix}
C^L_{fs} \\
C^L_{fs} \\
C^L_{fs} \\
C^L_{W,W} + C^L_{W,q} + C^L_{W} \\
C^L_{W} + C^L_{W,q}
\end{bmatrix}
= \begin{bmatrix}
D^L_{fs} \\
D^L_{fs} \\
D^L_{fs} \\
D^L_{fs}
\end{bmatrix}
\begin{bmatrix}
\dot{\bar{x}} \\
\dot{\alpha} \\
\dot{W} \\
\dot{W}
\end{bmatrix}
\]

(B.24)
Similarly, the linear fluid-structure interaction model is obtained from Eq. B.24 by neglecting the nonlinear term, resulting in

$$\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \dot{\alpha} \\ \dot{\bar{W}} \\ \dot{W} \end{bmatrix} = \begin{bmatrix} A_{x}^{fs} & A_{\alpha}^{fs} & A_{\bar{W},l}^{fs} & A_{W}^{fs} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ A_{x}^{wfs} & A_{\alpha}^{wfs} & A_{\bar{W},l}^{wfs} & A_{W}^{wfs} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \dot{\alpha} \\ \dot{\bar{W}} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} B_{\alpha}^{fs} & B_{\bar{C}_{f}}^{fs} \\ 0 & 0 \\ \dot{\alpha} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{C}_{f} \end{bmatrix}$$

(B.25)
Appendix C

Harmonic Balance Methods

C.1 Frequency Domain Harmonic Balance for the quasi-linear membrane

Following the definitions introduced in Section 5.2.1, the variables present in the quasi-linear membrane equation (Eq. 3.10) are expressed in the frequency domain by means of a truncated Fourier series. In order to simplify the expressions presented here, Eqs. 5.8 and 5.9 are expressed by incorporating the $h = 0$ term in the summation:

$$W_k = \sum_{h=0}^{N_H} (W_{k,2h} \cos(h\omega t) + W_{k,2h+1} \sin(h\omega t)) \tag{C.1}$$

$$\Delta C_{p_k} = \sum_{h=0}^{N_H} (\Delta C_{p_k,2h} \cos(h\omega t) + \Delta C_{p_k,2h+1} \sin(h\omega t)) \tag{C.2}$$

$$C_{f_k} = \sum_{h=0}^{N_H} (C_{f_k,2h} \cos(h\omega t) + C_{f_k,2h+1} \sin(h\omega t)) \tag{C.3}$$

Replacing Eqs. C.1, C.2 and C.3 into Eq. 3.10 and assuming $a_{1,k} = \frac{k^2 \pi^2 c_T}{2}$ and $a_{2,k} = \frac{\pi^4 k^2 E_s}{2}$, for the generic structural mode $k$ we have

$$-2\mu \sum_{h=0}^{N_H} (h^2 \omega^2 W_{k,2h} \cos(h\omega t) + h^2 \omega^2 W_{k,2h+1} \sin(h\omega t)) +$$

$$+2C_d \sum_{h=0}^{N_H} (-h\omega W_{k,2h} \sin(h\omega t) + h\omega W_{k,2h+1} \cos(h\omega t)) + k^2 [S_k] \left[ a_{1,k} + a_{2,k} \sum_{m=1}^{N} (m^2 S_k) \right]^2 =$$

$$\sum_{h=0}^{N_H} (\Delta C_{p_k,2h} \cos(h\omega t) + C_{p_k,2h} \sin(h\omega t)) + \sum_{h=0}^{N_H} (C_{f_k,2h} \cos(h\omega t) + \Delta C_{f_k,2h+1} \sin(h\omega t)),$$
with

\[ S_k = \sum_{h=0}^{N_H} (W_{k,2h} \cos(h \omega t) + W_{k,2h+1} \sin(h \omega t)). \]

The nonlinear term expressed by the following relationship

\[ H_{nl} = k^2 |S_k|^2 \sum_{m=1}^{N} \left( m^2 S_k \right)^2 \]

is rearranged in a more convenient form, resulting in

\[
H_{nl} = k^2 a_{2,k} \sum_{m=1}^{N} \left( m^2 S_k \right)^2 \left( \right.
\begin{array}{l}
+ W_{k,2h} W_{m,2p} W_{m,2r} \cos(h \omega t) \cos(p \omega t) \cos(r \omega t) \\
+ W_{k,2h} W_{m,2p+1} W_{m,2r+1} \cos(h \omega t) \sin(p \omega t) \sin(r \omega t) + \\
+ 2 W_{k,2h} W_{m,2p} W_{m,2r+1} \cos(h \omega t) \cos(p \omega t) \sin(r \omega t) + \\
+ W_{k,2h+1} W_{m,2p} W_{m,2r} \sin(h \omega t) \cos(p \omega t) \cos(r \omega t) + \\
+ W_{k,2h+1} W_{m,2p+1} W_{m,2r+1} \sin(h \omega t) \sin(p \omega t) \sin(r \omega t) + \\
+ 2 W_{k,2h+1} W_{m,2p} W_{m,2r+1} \sin(h \omega t) \cos(p \omega t) \sin(r \omega t) \left. \right) \\
\end{array}
\]

Substituting product-sum trigonometric identities (see Barnett et al. [2011]) in Eq. C.4, such as

\[ \sin(h \omega t) \sin(p \omega t) \sin(r \omega t) = \frac{1}{4} \left( - \sin(h \omega t - p \omega t - r \omega t) + \sin(h \omega t + p \omega t - r \omega t) + \sin(h \omega t - p \omega t + r \omega t) - \sin(h \omega t + p \omega t + r \omega t) \right), \]
and after rearranging terms, we arrive at an expression for the quasi-linear model that does not contain products of sinusoids:

\[-2\mu \sum_{h=0}^{N_H} \left(h^2\omega^2W_{k,2h}\cos(h\omega t) + h^2\omega^2W_{k,2h+1}\sin(h\omega t)\right) +
+2C_d\sum_{h=0}^{N_H} (-h\omega W_{k,2h}\sin(h\omega t) + h\omega W_{k,2h+1}\cos(h\omega t)) +
+a_{1,k}k^2\sum_{h=1}^{N_H} (W_{k,2h}\cos(h\omega t) + W_{k,2h+1}\sin(h\omega t)) +
+a_{2,k}k^2\sum_{m=1}^{N} \left( \frac{m^2}{4} \left( \sum_{h=0}^{N_H} \sum_{p=1}^{N_H} \sum_{r=1}^{N_H} \left[ p_{1}^{h,k,m,r} \sin(h\omega t - p\omega t - r\omega t)
+ p_{2}^{h,k,m,r} \cos(h\omega t - p\omega t - r\omega t) + p_{3}^{h,k,m,r} \sin(h\omega t + p\omega t - r\omega t)
+ p_{4}^{h,k,m,r} \cos(h\omega t + p\omega t - r\omega t) + p_{5}^{h,k,m,r} \sin(h\omega t - p\omega t + r\omega t)
+ p_{6}^{h,k,m,r} \cos(h\omega t - p\omega t + r\omega t) + p_{7}^{h,k,m,r} \sin(h\omega t + p\omega t + r\omega t)
+ p_{8}^{h,k,m,r} \cos(h\omega t + p\omega t + r\omega t) \right] \right) \right) =
= \sum_{h=0}^{N_H} (\Delta C_{p_k,2h}\cos(h\omega t) + \Delta C_{p_k,2h+1}\sin(h\omega t)) +
+ \sum_{h=0}^{N_H} (C_{f_k,2h}\cos(h\omega t) + C_{f_k,2h+1}\sin(h\omega t))

with

\[ p_{1}^{h,k,m,r} = -W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} + W_{k,2h+1}W_{m,2p}W_{m,2r} - 2W_{k,2h+1}W_{m,2p}W_{m,2r+1} \]
\[ p_{2}^{h,k,m,r} = +2W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} - W_{k,2h}W_{m,2p+1}W_{m,2r+1} + W_{k,2h}W_{m,2p}W_{m,2r} \]
\[ p_{3}^{h,k,m,r} = +W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} + W_{k,2h+1}W_{m,2p}W_{m,2r} + 2W_{k,2h+1}W_{m,2p}W_{m,2r+1} \]
\[ p_{4}^{h,k,m,r} = -2W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} + W_{k,2h}W_{m,2p+1}W_{m,2r+1} + W_{k,2h}W_{m,2p}W_{m,2r} \]
\[ p_{5}^{h,k,m,r} = +W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} + W_{k,2h+1}W_{m,2p}W_{m,2r} - 2W_{k,2h+1}W_{m,2p}W_{m,2r+1} \]
\[ p_{6}^{h,k,m,r} = +2W_{k,2h+1}W_{m,2p}W_{m,2r+1} + W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} + W_{k,2h+1}W_{m,2p}W_{m,2r} \]
\[ p_{7}^{h,k,m,r} = -W_{k,2h+1}W_{m,2p+1}W_{m,2r+1} + W_{k,2h+1}W_{m,2p}W_{m,2r} + 2W_{k,2h+1}W_{m,2p}W_{m,2r+1} \]
\[ p_{8}^{h,k,m,r} = -2W_{k,2h+1}W_{m,2p}W_{m,2r+1} - W_{k,2h}W_{m,2p+1}W_{m,2r+1} + W_{k,2h}W_{m,2p}W_{m,2r} \]

The full system of Eq. C.5 can then be coupled with the linear aerodynamic model to express the coefficients \( \Delta C_{p_k,2h}, \Delta C_{p_k,2h+1} \) as a function of \( W_{k,2h}, W_{k,2h+1} \) in order to have a system of nonlinear equations in which \( W_{k,2h}, W_{k,2h+1} \) are the only unknowns. This system can be solved using an iterative Newton-Raphson method.
C.2 Transformation from frequency domain to time domain

The matrices $\hat{Q}$, $\hat{R}$ and $\hat{F}$ containing the frequency domain coefficients for the general membrane model from Equation 5.1, and their counterparts in the time domain $\tilde{Q}$, $\tilde{R}$ and $\tilde{F}$ are related by the following matrix transformations:

\begin{align*}
\tilde{Q} &= E^{-1}\hat{Q}, \\
\tilde{R} &= E^{-1}\hat{R}, \\
\tilde{F} &= E^{-1}\hat{F},
\end{align*}

(C.6)

\begin{align*}
\hat{Q} &= E\tilde{Q}, \\
\hat{R} &= E\tilde{R}, \\
\hat{F} &= E\tilde{F},
\end{align*}

(C.7)

where

\[
E = \frac{2}{N_c} \begin{bmatrix}
    1/2 & 1/2 & \ldots & 1/2 \\
    \cos t_0 & \cos t_1 & \ldots & \cos t_{N_c} \\
    \sin t_0 & \sin t_1 & \ldots & \sin t_{N_c} \\
    \cos 2t_0 & \cos 2t_1 & \ldots & \cos 2t_{N_c} \\
    \sin 2t_0 & \sin 2t_1 & \ldots & \sin 2t_{N_c} \\
    \vdots & \vdots & \vdots & \vdots \\
    \cos N_H t_0 & \cos N_H t_1 & \ldots & \cos N_H t_{N_c} \\
    \sin N_H t_0 & \sin N_H t_1 & \ldots & \sin N_H t_{N_c}
\end{bmatrix}_{(2N_H+1) \times N_c}.
\]

(C.8)

\[
E^{-1} = \begin{bmatrix}
    1 & \cos t_0 & \sin t_0 & \ldots & \cos N_H t_0 & \sin N_H t_0 \\
    1 & \cos t_1 & \sin t_1 & \ldots & \cos N_H t_1 & \sin N_H t_1 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    1 & \cos t_{N_c} & \sin t_{N_c} & \ldots & \cos N_H t_{N_c} & \sin N_H t_{N_c}
\end{bmatrix}_{N_c \times (2N_H+1)}.
\]

(C.9)

Substituting Equations C.7 into Equation 5.1 and multiplying both sides by $E^{-1}$, gives an expression that involves only the time domain coefficients:

\[
2\mu\omega^2 \tilde{D}^2 \tilde{Q} + 2C_d\omega \tilde{D} \tilde{Q} + \tilde{R} = \tilde{F},
\]

(C.10)

with $\tilde{D} = E^{-1} \tilde{A} E$. 

C.3 Summary of the Harmonic Balance formulations for the membrane models

### Nonlinear membrane

#### Structural dynamics

\[
q = w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N_x} \end{bmatrix}_{N_x \times 1}, \quad f = \Delta C_p + C_f = \begin{bmatrix} \Delta C_{p_1} + C_{f_1} \\ \Delta C_{p_2} + C_{f_2} \\ \vdots \\ \Delta C_{p_{N_x}} + C_{f_{N_x}} \end{bmatrix}_{N_x \times 1}
\]

\[
q = w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N_x} \end{bmatrix}_{N_x \times 1}, \quad f = \Delta C_p + C_f = \begin{bmatrix} \Delta C_{p_1} + C_{f_1} \\ \Delta C_{p_2} + C_{f_2} \\ \vdots \\ \Delta C_{p_{N_x}} + C_{f_{N_x}} \end{bmatrix}_{N_x \times 1}
\]

\[
r = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{N_x} \end{bmatrix}_{N_x \times 1}, \quad r_i = c_T \left( \frac{\partial^2 w}{\partial x^2} \right)_i \left[ 1 + \left( \frac{\partial w}{\partial x} \right)^2_i \right]^{-3/2}, \quad i = 1, N_x
\]

\[
c_T = 2\bar{E}_s(\delta_0 + \delta), \quad \delta = \int_0^1 \sqrt{1 + \left( \frac{\partial w}{\partial x} \right)^2} \, dx - 1
\]

#### Harmonic Balance matrices

\[
\hat{Q} = \begin{bmatrix} w_{0,1} & w_{0,2} & \cdots & w_{0,N_x} \\ w_{1,1} & w_{1,2} & \cdots & w_{1,N_x} \\ \vdots & \vdots & \ddots & \vdots \\ w_{2N_H-1,1} & w_{2N_H-1,2} & \cdots & w_{2N_H-1,N_x} \\ w_{2N_H,1} & w_{2N_H,2} & \cdots & w_{2N_H,N_x} \end{bmatrix}_{(2N_H+1) \times N_x}
\]

\[
\hat{R} = \begin{bmatrix} r_{0,1} & r_{0,2} & \cdots & r_{0,N_x} \\ r_{1,1} & r_{1,2} & \cdots & r_{1,N_x} \\ \vdots & \vdots & \ddots & \vdots \\ r_{2N_H-1,1} & r_{2N_H-1,2} & \cdots & r_{2N_H-1,N_x} \\ r_{2N_H,1} & r_{2N_H,2} & \cdots & r_{2N_H,N_x} \end{bmatrix}_{(2N_H+1) \times N_x}
\]

\[
\hat{F} = \begin{bmatrix} \Delta C_{p_{0,1}} + C_{f_{0,1}} & \cdots & \Delta C_{p_{0,N_x}} + C_{f_{0,N_x}} \\ \Delta C_{p_{1,1}} + C_{f_{1,1}} & \cdots & \Delta C_{p_{1,N_x}} + C_{f_{1,N_x}} \\ \vdots & \vdots & \vdots \\ \Delta C_{p_{2N_H-1,1}} + C_{f_{2N_H-1,1}} & \cdots & \Delta C_{p_{2N_H-1,N_x}} + C_{f_{2N_H-1,N_x}} \\ \Delta C_{p_{2N_H,1}} + C_{f_{2N_H,1}} & \cdots & \Delta C_{p_{2N_H,N_x}} + C_{f_{2N_H,N_x}} \end{bmatrix}_{(2N_H+1) \times N_x}
\]

**Table C.1:** Frequency Domain Harmonic Balance formulation applied to the nonlinear membrane.
### Quasi-linear membrane

#### Structural dynamics

\[ q = \mathcal{W} = \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{pmatrix}_{N \times 1} \]

\[ f = \Delta C_p + C_f = \begin{pmatrix} \Delta C_{p_1} + C_{f_1} \\ \Delta C_{p_2} + C_{f_2} \\ \vdots \\ \Delta C_{p_N} + C_{f_N} \end{pmatrix}_{N \times 1} \]

\[ r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix}_{N \times 1}, \quad r_i = i^2 \pi^2 C_f W_i, \quad i = 1, \ldots, N \]

\[ c_T = 2E_s(\delta_0 + \delta), \quad \delta = \frac{\pi^2}{4} \sum_{m=1}^{N} m^2 W_m^2. \]

#### Harmonic Balance matrices

\[ \hat{Q} = \begin{bmatrix} \mathcal{W}_{0,1} & \mathcal{W}_{0,2} & \cdots & \mathcal{W}_{0,N} \\ \mathcal{W}_{1,1} & \mathcal{W}_{1,2} & \cdots & \mathcal{W}_{1,N} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{W}_{2N_H-1,1} & \mathcal{W}_{2N_H-1,2} & \cdots & \mathcal{W}_{2N_H-1,N} \\ \mathcal{W}_{2N_H,1} & \mathcal{W}_{2N_H,2} & \cdots & \mathcal{W}_{2N_H,N} \end{bmatrix}_{(2N_H+1) \times N} \]

\[ \hat{R} = \begin{bmatrix} r_{0,1} & r_{0,2} & \cdots & r_{0,N} \\ r_{1,1} & r_{1,2} & \cdots & r_{1,N} \\ \vdots & \vdots & \vdots & \vdots \\ r_{2N_H-1,1} & r_{2N_H-1,2} & \cdots & r_{2N_H-1,N} \\ r_{2N_H,1} & r_{2N_H,2} & \cdots & r_{2N_H,N} \end{bmatrix}_{(2N_H+1) \times N} \]

\[ \hat{F} = \begin{bmatrix} \Delta C_{p_{0,1}} + C_{f_{0,1}} & \cdots & \Delta C_{p_{0,N}} + C_{f_{0,N}} \\ \Delta C_{p_{1,1}} + C_{f_{1,1}} & \cdots & \Delta C_{p_{1,N}} + C_{f_{1,N}} \\ \vdots & \vdots & \vdots \\ \Delta C_{p_{2N_H-1,1}} + C_{f_{2N_H-1,1}} & \cdots & \Delta C_{p_{2N_H-1,N}} + C_{f_{2N_H-1,N}} \\ \Delta C_{p_{2N_H,1}} + C_{f_{2N_H,1}} & \cdots & \Delta C_{p_{2N_H,N}} + C_{f_{2N_H,N}} \end{bmatrix}_{(2N_H+1) \times N} \]

Table C.2: Frequency Domain Harmonic Balance formulation applied to the quasi-linear membrane.
C.4 Newton-Raphson iteration

C.4.1 Newton-Raphson for the FDHB

The residual in matrix form of Eq. 5.5 can be expressed as follows

\[
\hat{\mathbf{R}}(\hat{\mathbf{Q}}) = \begin{bmatrix}
\hat{R}_{0,1} & \hat{R}_{0,2} & \cdots & \hat{R}_{0,N} \\
\hat{R}_{1,1} & \hat{R}_{1,2} & \cdots & \hat{R}_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{R}_{2N_H,1} & \hat{R}_{2N_H,2} & \cdots & \hat{R}_{2N_H,N}
\end{bmatrix}_{(2N_H+1)\times N} = \begin{bmatrix}
\hat{R}_{0,1} \\
\hat{R}_{0,2} \\
\vdots \\
\hat{R}_{2N_H,1}
\end{bmatrix}_{(2N_H+1)\times 1} = 2\mu_\omega^2 \hat{A}^2 \hat{Q} + 2C_d \bar{\omega} \hat{A} \hat{Q} + \hat{R} - \hat{F},
\]

while its vector form is

\[
\hat{\mathbf{R}}_v = \begin{bmatrix}
\hat{R}_{0,1} \\
\hat{R}_{0,2} \\
\vdots \\
\hat{R}_{2N_H,1}
\end{bmatrix}_{(2N_H+1)\times 1} = \begin{bmatrix}
\hat{Q}_{0,1} \\
\hat{Q}_{0,2} \\
\vdots \\
\hat{Q}_{2N_H,1}
\end{bmatrix}_{(2N_H+1)\times 1}.
\]

Finding a solution \(\hat{\mathbf{Q}}_s\) of Eq. 5.5 means to find \(\hat{\mathbf{Q}} = \hat{\mathbf{Q}}_s\) for which each entry of the residual is zero. This condition is mathematically equivalent to state that \(\|\hat{\mathbf{R}}_v(\hat{\mathbf{Q}}_s)\|_2 = 0\), where \(\| \|_2\) indicates the \(l^2\)-norm. After introducing the vector form of the frequency-domain harmonic coefficient matrix \(\hat{\mathbf{Q}}\), whose expression is

\[
\hat{\mathbf{Q}}_v = \begin{bmatrix}
\hat{Q}_{0,1} \\
\hat{Q}_{0,2} \\
\vdots \\
\hat{Q}_{2N_H,1}
\end{bmatrix}_{(2N_H+1)\times 1},
\]

it is possible to define the following Jacobian square matrix

\[
\hat{\mathbf{J}}_{NR,ij} = \frac{\partial \hat{\mathbf{R}}_{v,i}}{\partial \hat{\mathbf{Q}}_{v,j}}, \tag{C.14}
\]

which allows us to obtain a local linear approximation of the function \(\hat{\mathbf{R}}_v(\hat{\mathbf{Q}}_v)\).

Starting from an initial guess of the solution \(\hat{\mathbf{Q}}_v^*\), a linear approximation of the residual function \(\hat{\mathbf{R}}_v(\hat{\mathbf{Q}}_v)\) is obtained about the point \(\hat{\mathbf{R}}_v(\hat{\mathbf{Q}}_v^*)\), thanks to the evaluation of its Jacobian \(\hat{\mathbf{J}}_{NR}\big|_{\hat{\mathbf{Q}}_v^*}\). This is the idea behind the Newton-Raphson method, here adopted to iteratively update the solution vector \(\hat{\mathbf{Q}}_v\), until the norm of the residual becomes smaller than a pre-defined threshold \(\epsilon > 0\). If \(\hat{\mathbf{Q}}_v^*\), corresponding to the \(k\)-th step of the iteration, is known, the solution vector at the following iteration \(k+1\) is as follows

\[
\hat{\mathbf{Q}}_v^{k+1} = \hat{\mathbf{Q}}_v^* - \lambda_{NR} \left[\hat{\mathbf{J}}_{NR}^k\right]^{-1} \hat{\mathbf{R}}_v^k, \tag{C.15}
\]
where \( \lambda_{NR} \) is a relaxation parameter (generally \( 0 < \lambda_{NR} \leq 1 \)) introduced to improve convergence. In the present work, \( \tilde{J}_{NR} \) is evaluated numerically using a second-order central finite difference scheme.

### C.4.2 Newton-Raphson for the HDHB

For the HDHB method, a solution procedure similar to that introduced in Appendix C.4.1 is adopted. The main difference is that we now have an over-determined system of equations, in which the number of equations is larger than the number of unknowns. The proposed Newton-Raphson iterative method, based on the work of Dai et al. [2013], aims to find an approximate solution that minimizes the residual in a least-square sense. As we did for the FDHB, we introduce the matrix residual

\[
\tilde{R} \left( \tilde{Q} \right) = \begin{bmatrix} \tilde{R}_{0,1} & \tilde{R}_{0,2} & \ldots & \tilde{R}_{0,N} \\ \tilde{R}_{1,1} & \tilde{R}_{1,2} & \ldots & \tilde{R}_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{R}_{2NH,1} & \tilde{R}_{2NH,2} & \ldots & \tilde{R}_{2NH,N} \end{bmatrix}_{N_c \times N_x} = 2\mu\omega^2\bar{D}^2\tilde{Q} + 2C_d\omega\bar{D}\tilde{Q} + \tilde{R} - \tilde{F},
\]

(C.16)

and its corresponding vector form

\[
\tilde{R}_v = \left[ \tilde{R}_{0,1} \ldots \tilde{R}_{0,N} \tilde{R}_{1,1} \ldots \tilde{R}_{1,N} \tilde{R}_{2NH,1} \ldots \tilde{R}_{2NH,N} \right]_{N_cN_x \times 1}^T.
\]

(C.17)

Similarly, the vector containing the unknown time-domain coefficients is expressed as

\[
\tilde{Q}_v = \left[ \tilde{Q}_{0,1} \ldots \tilde{Q}_{0,N} \tilde{Q}_{1,1} \ldots \tilde{Q}_{1,N} \tilde{Q}_{2NH,1} \ldots \tilde{Q}_{2NH,N} \right]_{(2NH+1)N_x \times 1}^T.
\]

(C.18)

Unlike the FDHB case described in Appendix C.4.1, with the HDHB the two vectors \( \tilde{Q}_v \) and \( \tilde{R}_v \) do not have the same size. Hence the corresponding Jacobian, defined as

\[
\tilde{J}_{NR,ij} = \frac{\partial \tilde{R}_{v,i}}{\partial Q_{v,j}},
\]

is a rectangular \( N_cN_x \times (2NH + 1)N_x \) matrix. In order to have a well-determined system, following Dai et al. [2013], we introduce a new \( (2NH + 1)N_x \times 1 \) residual, defined as

\[
\tilde{R}_v^* = \left[ \tilde{J}_{NR}^* \right]^T \tilde{R}_v,
\]

(C.20)

and a new \( (2NH + 1)N_x \times (2NH + 1)N_x \) square Jacobian, whose mathematical expression is the following:

\[
\tilde{J}_{NR}^* = \left[ \tilde{J}_{NR} \right]^T \tilde{J}_{NR}.
\]

(C.21)
Hence, in analogy with Eq. C.15, the Newton-Raphson iteration for the HDHB method becomes

\[
\tilde{Q}^{k+1}_v = \tilde{Q}^k_v - \lambda_{NR} \left[ \begin{bmatrix} \tilde{J}^{*k}_{NR} \end{bmatrix}^T \begin{bmatrix} \tilde{J}^{*k}_{NR} \end{bmatrix} \right]^{-1} \tilde{R}^k_v, \tag{C.22}
\]

and a solution is reached when \( \left\| \tilde{R}^*_v \right\|_2 \) falls below a desired threshold \( \epsilon > 0 \).
Appendix D

Adjoint Equations

D.1 Adjoint equations for the fluid-structure interaction models

D.1.1 Adjoint equations for the quasi-linear model

From the quasi-linear fluid-structure interaction model (see Eq. 3.19), the following state variables are identified:

\[
\phi = \begin{bmatrix} \bar{x}^T \ W^T \dot{W}^T \alpha \dot{\alpha} \end{bmatrix}^T. 
\]  

(D.1)

We introduce now the co-state variables or adjoint variables, defined as follows:

\[
\xi = \begin{bmatrix} \chi^T \ U^T \ V^T \beta \gamma \end{bmatrix}^T. 
\]  

(D.2)

By considering a generic cost function \( J \), a Lagrangian functional \( L \) is defined as

\[
L = J - \int_0^T \chi^T \left( \frac{d\bar{x}}{dt} - A_{\bar{x}}^{f,s} \bar{x} - A_{\alpha}^{f,s} \alpha - A_{\dot{\alpha}}^{f,s} \dot{\alpha} - A_{W,l}^{f,s} W \right)
A_{E_s}^{f,s} (W^T \Gamma^2 W) W - A_{W}^{f,s} \dot{W} - B_{\alpha}^{f,s} \dot{\alpha} - B_{C_f}^{f,s} C_f \right) dt + \int_0^T \beta^T \left( \frac{d\alpha}{dt} - \dot{\alpha} \right) dt + 
- \int_0^T \dot{U}^T \left( \frac{d\dot{W}}{dt} - \ddot{W} \right) dt - \int_0^T \gamma^T \left( \frac{d\dot{\alpha}}{dt} - \ddot{\alpha} \right) dt - \int_0^T \dot{V}^T \left( \frac{d\dot{W}}{dt} - A_{\bar{x}}^{f,s} \bar{x} + 
- A_{\alpha}^{f,s} \alpha - A_{\dot{\alpha}}^{f,s} \dot{\alpha} - A_{W,l}^{f,s} W - A_{E_s}^{f,s} (W^T \Gamma^2 W) W - A_{W}^{f,s} \dot{W}
- B_{\alpha}^{f,s} \ddot{\alpha} - B_{C_f}^{f,s} C_f \right) dt + i.c., \quad (D.3)
\]
where the following substitutions have been performed

\[ A_{x,ql}^{fs} = A_{x,fs}^{E} (\mathbf{W}^T \Gamma^2 \mathbf{W}) \]
\[ A_{w,ql}^{fs} = A_{w,fs}^{E} (\mathbf{W}^T \Gamma^2 \mathbf{W}) , \]

with

\[ A_{E}^{w,fs} = - M^{-1} \frac{\pi^4}{2} \bar{E} \Gamma^2 , \]
\[ A_{E}^{x,fs} = - B_{x}^M M^{-1} \frac{\pi^4}{2} \bar{E} \Gamma^2 . \]

The term i.c. indicates the initial conditions. Since we are considering a membrane wing at zero angle of attack starting from rest, we have zero initial conditions for the state variables, formally \( \phi(t = 0) = \phi_0 = 0. \) In order to obtain the adjoint equations, the Fréchet derivatives of \( \mathcal{L} \) with respect to the state variables are evaluated [Gunzburger & Wood, 2003]. For the nonlinear terms, indicating with \( \tilde{\mathbf{W}} \) an arbitrary variation of variable \( \mathbf{W} \) and with \( \epsilon \) a small number, we have

\[
\lim_{\epsilon \to 0} \left( \frac{A_{E}^{w,fs} (\mathbf{W} + \epsilon \tilde{\mathbf{W}})^T \Gamma^2 (\mathbf{W} + \epsilon \tilde{\mathbf{W}}) (\mathbf{W} + \epsilon \tilde{\mathbf{W}}) - A_{E}^{w,fs} (\mathbf{W}^T \Gamma^2 \mathbf{W}) \mathbf{W}}{\epsilon} \right) = A_{E}^{w,fs} (\mathbf{W}^T \Gamma^2 \mathbf{W}) \tilde{\mathbf{W}} + 2 A_{E}^{w,fs} \mathbf{W} \tilde{\mathbf{W}}^T \Gamma^2 \tilde{\mathbf{W}}
\]

By grouping the terms corresponding to the same variables, we end up with the following system of adjoint equations:

\[ \int_{0}^{T} \left( \frac{d}{dt} \chi^T + \chi^T A_{x,fs}^{E} + \nu^T A_{x,fs}^{E} \right) dt = \mathcal{J}_{\dot{x}}^T \quad (D.4) \]
\[ \int_{0}^{T} \left( \frac{d}{dt} \beta + \chi^T A_{\alpha,fs}^{x} + \nu^T A_{\alpha,fs}^{w} \right) dt = \mathcal{J}_{\alpha}^T \quad (D.5) \]
\[ \int_{0}^{T} \left( \frac{d}{dt} \mathbf{U}^T + \chi^T A_{\chi,ql}^{x} + \nu^T A_{\chi,ql}^{w} \right) dt = \mathcal{J}_{\chi}^T \quad (D.6) \]
\[ \int_{0}^{T} \left( \frac{d}{dt} \gamma + \chi^T A_{\alpha,fs}^{x} + \beta^T + \nu^T A_{\alpha,fs}^{w} \right) dt = \mathcal{J}_{\alpha}^T \quad (D.7) \]
\[ \int_{0}^{T} \left( \frac{d}{dt} \mathbf{V}^T + \chi^T A_{\chi,ql}^{w} + \mathbf{U}^T + \nu^T A_{\chi,ql}^{w} \right) dt = \mathcal{J}_{\chi}^T \quad (D.8) \]

with

\[ A_{\chi,ql}^{x} = A_{\chi,ql}^{x} + A_{\chi,ql}^{x} (\mathbf{W}^T \Gamma^2 \mathbf{W}) + 2 A_{E}^{x,fs} \mathbf{W} \tilde{\mathbf{W}}^T \Gamma \]
\[ A_{\chi,ql}^{w} = A_{\chi,ql}^{w} + A_{\chi,ql}^{w} (\mathbf{W}^T \Gamma^2 \mathbf{W}) + 2 A_{E}^{w,fs} \mathbf{W} \tilde{\mathbf{W}}^T \Gamma \quad (D.9) \]

The adjoint equations are linear with respect to the adjoint variables, but because of the nonlinearity of the original system of equations (see Eq. 3.19), they contain the state variables \( \mathbf{W} \) that must be obtained from the solution of the state
equations. The adjoint equations must to be solved backwards in time, with zero “initial” conditions $\xi_T = 0$ at $t = T$. The derivatives of the cost function with respect to the state variables appearing in the adjoint equations are defined as follows

$$
\mathbf{J}_x = \left[ \frac{\partial J}{\partial x_1} \ldots \frac{\partial J}{\partial x_i} \ldots \frac{\partial J}{\partial x_n} \right]^T \\
\mathbf{J}_\alpha = \frac{\partial J}{\partial \alpha} \\
\mathbf{J}_{\dot{\alpha}} = \frac{\partial J}{\partial \dot{\alpha}} \\
\mathbf{J}_w = \left[ \frac{\partial J}{\partial W_1} \ldots \frac{\partial J}{\partial W_i} \ldots \frac{\partial J}{\partial W_N} \right]^T \\
\mathbf{J}_{\dot{w}} = \left[ \frac{\partial J}{\partial W_1} \ldots \frac{\partial J}{\partial W_i} \ldots \frac{\partial J}{\partial W_N} \right]^T.
$$

**D.1.2 Derivative of the cost functions for the quasi-linear model**

The two cost functions $\mathbf{J}_1$ and $\mathbf{J}_2$ have been defined in Section 6.3.2. From the expression of the lift coefficient obtained from the quasi-linear model (see Eq. 3.19), the derivatives of $\mathbf{J}_1$ with respect to the state variables are as follows

$$
\mathbf{J}_{1x} = \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{xfs}^{L} dt + \bar{\sigma} \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{xfs}^{L} dt \\
\mathbf{J}_{1\alpha} = \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{\alpha}^{L} dt + \bar{\sigma} \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{\alpha}^{L} dt \\
\mathbf{J}_{1\dot{\alpha}} = \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{\dot{\alpha}}^{L} dt + \bar{\sigma} \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{\dot{\alpha}}^{L} dt \\
\mathbf{J}_{1w}^{ql} = \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) \left( C_{W,l}^{L} + C_{W,ql}^{L} + 2 C_{E}^{L} \mathbf{WW}^T \mathbf{\Gamma}^2 \right) dt \\
+ \int_{t=t^*}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) \left( C_{W,l}^{L} + C_{W,ql}^{L} + 2 C_{E}^{L} \mathbf{WW}^T \mathbf{\Gamma}^2 \right) dt \\
\mathbf{J}_{1\dot{w}} = \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{\dot{w}}^{L} dt + \bar{\sigma} \int_{t=0}^{t=T} 2 \left( C_L(t) - C_{L_{ref}} \right) C_{\dot{w}}^{L} dt,
$$
with $C_{E}^{Lfs} = C_{W}^{Lfs} A_{E}^{Wfs}$. For $J_2$, we have

$$J_2z = J_1z + s_d C_{Lmax}^{Lfs}$$

$$J_2a = J_1a + s_d C_{Lmax}^{Lfs}$$

$$J_2a = J_1a + s_d C_{Lmax}^{Lfs}$$

$$J_{2W}^{W} = J_{1W}^{W} + s_d C_{Lmax}^{Lfs} \left( C_{W,l}^{Lfs} + C_{W,ql}^{Lfs} + 2C_{E}^{Lfs} \mathcal{W} \mathcal{W}' \Gamma^2 \right)$$

$$J_{2W} = J_{1W} + s_d C_{Lmax}^{Lfs}$$

with $C_{Lmax} = \max |C_L - C_{Lref}|$ and

$$s_d = (C_L - C_{Lref}) / \left( |C_L - C_{Lref}| \right). \quad (D.11)$$

### D.2 Adjoint equations for the linear model

Following the same procedure adopted for the quasi-linear model, the adjoint equations for the linear fluid-structure interaction model are expressed by the following relationships:

$$\int_0^T \left( \frac{d}{dt} \chi^T + \chi^T A_{x}^{xfs} + \mathcal{V}^T A_{x}^{wfs} \right) dt = J_{x}^T$$

$$\int_0^T \left( \frac{d}{dt} \beta + \chi^T A_{\alpha}^{xfs} + \mathcal{V}^T A_{\alpha}^{wfs} \right) dt = J_{\alpha}^T$$

$$\int_0^T \left( \frac{d}{dt} \mathcal{U}^T + \chi^T A_{\chi,l} + \mathcal{V}^T A_{\chi,l} \right) dt = J_{\chi,l}^T$$

$$\int_0^T \left( \frac{d}{dt} \gamma + \chi^T A_{\alpha}^{xfs} + \beta^T + \mathcal{V}^T A_{\alpha}^{wfs} \right) dt = J_{\alpha}^T$$

$$\int_0^T \left( \frac{d}{dt} \mathcal{V}^T + \chi^T A_{W}^{xfs} + \mathcal{U}^T + \mathcal{V}^T A_{W}^{wfs} \right) dt = J_{W}^T,$$

with

$$A_{\chi,l} = A_{\chi,l}^{xfs}$$

$$A_{\chi,l} = A_{\chi,l}^{wfs}.$$

### D.2.1 Derivatives of the cost functions for the linear model

From the expression of the lift coefficient obtained from the linear model (see Eq. 3.20), the derivatives of $J_1$ with respect to the state variables are the same
defined for the quasi-linear case, except for the following term:

\[
J_{lW}^1 = \int_{t=0}^{t=T} 2 (C_L(t) - C_{L_{ref}}) C_{W,l}^{fs} dt + \int_{t=t^*}^{t=T} 2 (C_L(t) - C_{L_{ref}}) C_{W,l}^{fs} dt.
\]

For \( J_2 \), we have

\[
J_{2W}^l = J_{1W}^l + s_d C_{L_{max}} C_{W,l}^{fs},
\]

with \( C_{L_{max}} = \max \left| C_L - C_{L_{ref}} \right| \) and

\[
s_d = \left( C_L - C_{L_{ref}} \right) / \left( \left| C_L - C_{L_{ref}} \right| \right) .
\]

(D.19)

**D.3 Penalty terms for the cost functions**

**D.3.1 Penalty term for the integral lift**

The effect of a vortex disturbance on a rigid wing quickly decays as the vortex is convected downstream. For the vortices considered in Chapter 6, results from DNS show that the vortex-induced response of a rigid wing becomes negligible after approximately 20 convective time units. This corresponds to the vortex being carried 20 chord lengths downstream of the wing. For a membrane wing, any contribution to the vortex-induced response for \( t > 20 \) is the result of the transient dynamics of the membrane-fluid system. This transient, although caused by the passage of the vortex, is not due to a direct forcing of the wing from the vortex and it mostly depends on the structural properties of the membrane and on the characteristics of the free-stream. In order to ensure favorable flight dynamic performance of small-scale flyers, we want these transients to decay rapidly. Hence, in order to reduce the computational cost of the optimization procedure, the length of the simulations is selected as \( T = 100 \) for the evaluation of the cost functions \( J_1 \) and \( J_2 \) (see Eqs. 6.15 and 6.16). If the transients decay rapidly, their contribution to the response for \( t > 100 \) is negligible and choosing a value of \( T > 100 \) will only increase the computational cost of the simulation without affecting the solution.

Since the largest contribution to the aerodynamic response of the membrane wing comes from the time intervals corresponding to the vortex impinging on the wing, an optimization carried out using \( T = 100 \) sometimes results in an optimized response that presents small amplitude limit-cycle oscillations (see Fig. D.1). Although their contribution to the cost function is negligible, they are not desirable from a flight dynamic perspective. To reduce (or to eliminate) the occurrence of these limit-cycle small amplitude oscillations without increasing the computational cost (hence without increasing the length of the simulations \( T \)), a penalty term \( J_p \) on the integral lift for \( t > t^* \) is introduced (see Eq. 6.17). For the present cases, we want all the undesired oscillations to decay after 50 convective time units, hence we adopt \( t^* = 50 \).
The penalty term $J_p$ also includes the parameter $\bar{\sigma}$ that has to be selected. A sensitivity analysis on the dependence of the optimized solution on $\bar{\sigma}$ has been carried out. Results for a vortex scaling factor of 1 are shown in Fig. D.2, while Fig. D.3 contains the same results obtained for a scaling factor of 30. For the considered cases, values of $\bar{\sigma}$ such that $500 J_1^{rig} < \bar{\sigma} < 10000 J_1^{rig}$ ensure that the residual oscillations $\Delta C_L$ for $t > t^*$ are negligible (we consider $10^{-3}$ as a threshold) and that the optimization procedure converges to the same solution. Similar results, not reported here for the sake of brevity, have been found for other values of the scaling factor. From the results of the sensitivity analysis, we select $\bar{\sigma} = 5000 J_1^{rig}$ for our optimization campaign.

**Figure D.1**: Optimized response for a vortex with $r_v = 0.25c$ and for a scaling factor of 30, obtained using cost function $J_{1p}$ with $\bar{\sigma} = 0$. 
Figure D.2: Dependence of the optimized solution on the choice of $\bar{\sigma}$ for a scaling factor of 1 and for vortices with a radius $r_v$ of $0.25c$ (a-b-c), $0.5c$ (d-e-f), $0.75c$ (g-h-i) and $1c$ (l-m-n). The dashed line indicates $\bar{\sigma} = 10000 \mathcal{F}^{rig}_1$. 
Figure D.3: Dependence of the optimized solution on the choice of $\tilde{\sigma}$ for a scaling factor of 30 and for vortices with a radius $r_v$ of 0.25$c$ (a-b-c), 0.5$c$ (d-e-f), 0.75$c$ (g-h-i) and 1$c$ (l-m-n). The dashed lines indicate $\tilde{\sigma} = 500J_1^{rig}$ and $\tilde{\sigma} = 10000J_1^{rig}$. 
D.3.2 Penalty term for the density ratio $\mu$

The penalty term $J_p$ from Eq. 6.17 contains a penalization of the density ratio $\mu$, indicated as $F(\mu)$. Function $F(\mu)$ is represented by the following ramped step:

$$F(\mu) = \sigma_\mu \left( 1 - \frac{G(\mu)}{\max(G(\mu))} \right)$$

$$G(\mu) = \log \left[ \frac{\cosh(a(\mu - \mu_1)) \cosh(-a(\mu_2))}{\cosh(a(\mu - \mu_2)) \cosh(-a(\mu_1))} \right].$$

with $\sigma_\mu = 50$ (selected as an arbitrary large value), $\mu_1 = 0.007$, $\mu_2 = 0.015$ and $a = 600$. The penalty function is plotted in Fig. D.4. This penalty term on the density ratio has been introduced to prevent $\mu$ to becoming too small during the optimization procedure. Small values of $\mu$ require small timesteps in order to avoid instabilities when integrating the governing equations of the quasi-linear model in time (see Eq. 3.19). This results in an increase of the computational time. In order to prevent instabilities in the solution without increasing the computational cost, the values of the cost function are increased when $\mu$ decreases below a threshold $\bar{\mu} \approx 0.02$. Although this threshold has been chosen for computational reasons, it is well below typical values of the density ratio encountered in real-life applications (see Table D.1). Hence, it does not affect the validity of the optimization.

**Figure D.4:** Penalty term on the density ratio $\mu$ in the form of a ramped step

<table>
<thead>
<tr>
<th>Membrane type</th>
<th>$h$ [mm]</th>
<th>$c$ [mm]</th>
<th>$\rho_s$ [kg/m$^3$]</th>
<th>$\rho_{\text{air}}$ [kg/m$^3$]</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Large bat wing</td>
<td>$\approx 0.2^1$</td>
<td>$\approx 151^1$</td>
<td>1100$^2$</td>
<td>1.225</td>
<td>$\approx 1$</td>
</tr>
<tr>
<td>Small bat wing</td>
<td>$\approx 0.04^2$</td>
<td>$\approx 30$</td>
<td>1100$^2$</td>
<td>1.225</td>
<td>$\approx 1$</td>
</tr>
<tr>
<td>Latex membrane</td>
<td>0.1 – 0.2$^3$</td>
<td>50 – 150</td>
<td>1000$^3$</td>
<td>1.225</td>
<td>$\approx 0.5 – 3$</td>
</tr>
</tbody>
</table>

**Table D.1:** Typical values of the density ratio $\mu$ for bat wings and for a latex membrane.

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$^1$Values for a grey-headed flying fox (*Pteropus poliocephalus*) taken from Watts *et al.* [2001].

$^2$Values for a small bat with a mass $m < 10g$ taken from Swartz *et al.* [1996].

$^3$Values taken from the experiments of Rojratsirikul *et al.* [2009].
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