Published for SISSA by Springer

Received: May 13, 2020
Accepted: June 1, 2020
Published: June 17, 2020

n-th parafermion $\mathcal{W}_N$ characters from $\mathrm{U}(N)$ instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$

Masahide Manabe

School of Mathematics and Statistics, University of Melbourne, Royal Parade, Parkville, Victoria 3010, Australia

E-mail: masahidemanabe@gmail.com

Abstract: We propose, following the AGT correspondence, how the $\mathcal{W}^{\text{para}}_{N,n}$ (n-th parafermion $\mathcal{W}_N$) minimal model characters are obtained from the $\mathrm{U}(N)$ instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation by imposing specific conditions which remove the minimal model null states.

Keywords: Supersymmetric Gauge Theory, Conformal and W Symmetry, Conformal Field Theory

ArXiv ePrint: 2004.13960

Dedicated to the memory of Professor Omar Foda

https://doi.org/10.1007/JHEP06(2020)112

JHEP06(2020)112

Contents

1 Introduction

1.1 AGT correspondence

The AGT correspondence [1] with various generalizations makes the connection between 4D supersymmetric gauge theory with Ω-deformation [2] and 2D conformal field theory (CFT) with a generic central charge. In this paper we will focus on the correspondence between a 4D $\mathcal{N}=2$ U(N) supersymmetric gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ and a 2D CFT with the symmetry algebra

$$\mathcal{A}(N,n;p)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_N\oplus\frac{\widehat{\mathfrak{sl}}(N)_n\oplus\widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{n+p-N}},$$

which acts on the equivariant cohomology of instanton moduli space [3–5] (see also [6]). Here $\mathcal{H}$ is the affine Heisenberg algebra, and p, which parametrizes the central charge in the 2D CFT, is related to the ratio ϵ_1/ϵ_2 of the Ω-deformation parameters ϵ_1, ϵ_2 on $\mathbb{C}^2/\mathbb{Z}_n$ (see eqs. (2.7) and (2.8)). The 2D CFT, in particular, has the $\mathcal{W}^{\rm para}_{N,n}$ (n-th parafermion $\mathcal{W}_N$) symmetry [7, 8] described by the third (coset) factor [9–11] in the algebra $\mathcal{A}(N,n;p)$. When $n=1$, it gives the $\mathcal{W}_N$ algebra in [12–14] which contains higher spin currents.

For the gauge theory with an adjoint hypermultiplet, the AGT-corresponding CFT lives on a torus T^2. In the case of $\epsilon_1+\epsilon_2=0$ (corresponding to $p\to\infty$), the U(N) instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ for the massless adjoint hypermultiplet yields the partition function of an $\mathcal{N}=4$ twisted Yang-Mills theory enumerating torus fixed points on the moduli space of instantons, which is labelled by N-tuples of n-coloured Young diagrams $(Y_1^{\sigma_1},\dots,Y_N^{\sigma_N})$ with $\mathbb{Z}_n$ charges $\sigma_I\in\{0,1,\dots,n-1\}$, $1\le I\le N$ (see e.g. [15, 16]). The twisted partition function is well-known to give a character of the 2D CFT [17–19], where a string theory interpretation is given in [20].

1.2 AGT correspondence for minimal models

The AGT correspondence for minimal models was proposed in [21–23] (see also [24, 25] for early works) when $n=1$, and it was generalized to $n\ge 2$ in [26]. When p in the algebra $\mathcal{A}(N,n;p)$ is an integer with $p\ge N$, one finds that the U(N) instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ has *non-physical poles* which need to be removed and supposed to correspond to $\mathcal{W}^{\rm para}_{N,n}$ minimal model null states. The poles are parametrized by positive integers r_I and s_I, $0\le I<N$, with $\sum_{I=0}^{N-1}r_I=p$ and $\sum_{I=0}^{N-1}s_I=p+n$, and shown to be removed by imposing *Burge conditions*

$$Y^{\sigma_I}_{I,i}\ge Y^{\sigma_{I+1}}_{I+1,i+r_I-1}-s_I+1\qquad\text{for}\quad i\ge 1,\ 0\le I<N,$$

on N-tuples of n-coloured Young diagrams $(Y_1^{\sigma_1},\dots,Y_N^{\sigma_N})$, where $Y_0^{\sigma_0}=Y_N^{\sigma_N}$, and the $\mathbb{Z}_n$ charges σ_I satisfy the $\mathbb{Z}_n$ *charge conditions* $\sigma_I-\sigma_{I+1}\equiv -r_I+s_I \pmod n$, $0\le I<N$, with $\sigma_0=\sigma_N$.

Following the algebra $\mathcal{A}(N,n;p)$, the generating functions of the coloured Young diagrams with the Burge conditions and the $\mathbb{Z}_n$ charge conditions, that we will refer as *Burge-reduced generating functions*, are expected to be decomposed into $\widehat{\mathfrak{sl}}(n)_N$ WZW

JHEP06(2020)112

(Wess-Zumino-Witten model) characters [27] and $\mathcal{W}^{\rm para}_{N,n}$ $(p, p+n)$-minimal model characters (branching functions of the coset factor in $\mathcal{A}(N, n; p)$ [10, 28]) up to a Heisenberg factor. In [26] we discussed the special case $p = N$ in which the coset factor in $\mathcal{A}(N, n; p)$ is trivialized and, using the results in the crystal graph theory of [29], showed that the Burge-reduced generating functions indeed give the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters. The aim of this paper is to generalize it to integral $p \geq N$ and propose how the $\mathcal{W}^{\rm para}_{N,n}$ $(p, p+n)$-minimal model characters are obtained from the Burge-reduced generating functions.

1.3 Plan of the paper

In section 2, we summarize the minimal ingredients about the AGT correspondence for minimal models and introduce SU(N) Burge-reduced generating functions of n-coloured Young diagrams by subtracting the overall U(1) factor corresponding to $\mathcal{H}$. We then recall that the Burge-reduced generating functions in the special case $p = N$ agree with the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters. In section 3 we generalize it to $p \geq N$ and propose Conjecture 3.5 which states a decomposition of the Burge-reduced generating functions into the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters and the $\mathcal{W}^{\rm para}_{N,n}$ $(p, p+n)$-minimal model characters. The conjectural decomposition formula is considered to be a generalization of a character decomposition formula in [30, 31] for $p \to \infty$ established in the context of the level-rank duality [32–34]. We check the conjecture, by extracting the $\mathcal{W}^{\rm para}_{N,n}$ $(p, p+n)$-minimal model characters from the Burge-reduced generating functions, for $(N, n, p) = (2, 2, 4), (3, 3, 4)$ in section 4 and for $(N, n, p) = (2, 3, 3), (2, 4, 4), (3, 2, 4), (4, 2, 5)$ in appendix C. Section 5 is devoted to summary and outlook. In appendix A we summarize some string functions, and in appendix B we give some examples of the dominant integral weights of $\widehat{\mathfrak{sl}}(n)_N$ which are dual to the dominant integral weights of $\widehat{\mathfrak{sl}}(N)_n$ defined in section 3.2.

1.4 Notation

We use the following notation of affine Lie algebras (see [26, appendix A]).

Consider the affine Lie algebra $\widehat{\mathfrak{sl}}(M)$, and define the index sets $\mathcal{I}_M = \{0, 1, \ldots, M-1\}$ and $\overline{\mathcal{I}}_M = \{1, 2, \ldots, M-1\}$. Let α_i and Λ_i for $i \in \mathcal{I}_M$ be the simple roots and fundamental weights of $\widehat{\mathfrak{sl}}(M)$. For the standard inner product $\langle\cdot,\cdot\rangle$ they satisfy

$$\langle \alpha_i, \alpha_j \rangle = A_{ij}, \quad \langle \alpha_i, \Lambda_j \rangle = \delta_{ij}, \quad \langle \Lambda_i, \Lambda_j \rangle = \min\{i, j\} - \frac{i\, j}{M} \tag{1.1}$$

for $i, j \in \mathcal{I}_M$, where A is the Cartan matrix of $\widehat{\mathfrak{sl}}(M)$. For the inner product we use a notation $|\Lambda|^2 = \langle \Lambda, \Lambda \rangle$. The Weyl vector ρ is defined by $\rho = \sum_{i \in \mathcal{I}_M} \Lambda_i$. The level-$m$ weight lattice $P_{M,m}$, the level-m dominant weight lattice $P^+_{M,m}$, the level-m regular dominant weight lattice $P^{++}_{M,m}$ and the root lattice $\overline{Q}_M$ are defined by

$$\begin{aligned} P_{M,m} &= \left\{ \Lambda \in \bigoplus_{i \in \mathcal{I}_M} \mathbb{Z}\, \Lambda_i \,\middle|\, \Lambda = \sum_{i \in \mathcal{I}_M} d_i\, \Lambda_i, \quad \sum_{i \in \mathcal{I}_M} d_i = m \right\}, \\ P^+_{M,m} &= P_{M,m} \cap \bigoplus_{i \in \mathcal{I}_M} \mathbb{Z}_{\geq 0}\, \Lambda_i\,, \qquad P^{++}_{M,m} = P_{M,m} \cap \bigoplus_{i \in \mathcal{I}_M} \mathbb{Z}_{>0}\, \Lambda_i\,, \\ \overline{Q}_M &= \bigoplus_{i \in \overline{\mathcal{I}}_M} \mathbb{Z}\, \alpha_i\,. \end{aligned} \tag{1.2}$$

JHEP06(2020)112

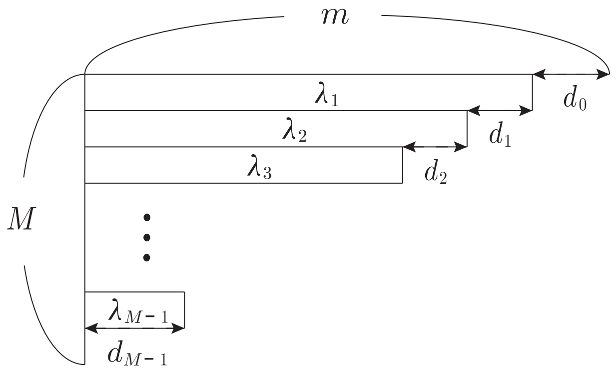

Figure 1. The partition par(Λ) for a dominant weight $\Lambda = [d_0, d_1, \ldots, d_{M-1}] \in P^+_{M,m}$.

We often use the notation $[d_0, d_1, \ldots, d_{M-1}]$ of Dynkin labels to denote $\Lambda = \sum_{i \in \mathcal{I}_M} d_i \Lambda_i$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ for $\Lambda = [d_0, d_1, \ldots, d_{M-1}] \in P^+_{M,m}$ is introduced by

$$\lambda_i = \begin{cases} \sum_{j=i}^{M-1} d_j & \text{if } \ 1 \leq i < M, \\ 0 & \text{if } \ i \geq M, \end{cases} \tag{1.3}$$

and denoted by par(Λ) (see figure 1). The transposed partition of par(Λ) is denoted by par$(\Lambda)^T = (\lambda_1^T, \lambda_2^T, \ldots)$, and one can write $\Lambda = \text{par}^{-1}(\lambda)$ when m is specified.

2 AGT correspondence for U(N) instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$

In this section, we recall some contents in sections 2, 3, 4 and 5 of [26] about the U(N) instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$, the AGT correspondence for minimal models and the Burge conditions.

2.1 U(N) instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$

The instanton moduli space $\mathcal{M}_{N,n}$ of U(N) instantons on $\mathbb{C}^2/\mathbb{Z}_n$ is characterized by the fixed point set of $\mathrm{U}(1)^2 \times \mathrm{U}(1)^N$ torus action on $\mathcal{M}_{N,n}$, where the $\mathrm{U}(1)^2$ torus is generated by the Ω-deformation parameters ϵ_1, ϵ_2, through $(z_1, z_2) \in \mathbb{C}^2 \ \rightarrow \ (\mathrm{e}^{\epsilon_1} z_1, \mathrm{e}^{\epsilon_2} z_2)$, and the $\mathrm{U}(1)^N$ torus is generated by the Coulomb parameters a_I, $I = 1, 2, \ldots, N$, which parametrize the Cartan subalgebra of U(N). The fixed point set has the colour coding induced by the $\mathbb{Z}_n$ orbifold of $\mathbb{C}^2$ as $(z_1, z_2) \ \rightarrow \ (\mathrm{e}^{\frac{2\pi i}{n}\sigma} z_1, \mathrm{e}^{-\frac{2\pi i}{n}\sigma} z_2)$, $\sigma = 0, 1, \ldots, n-1$, and is described by N-tuples of n-coloured Young diagrams $\boldsymbol{Y^\sigma} = (Y_1^{\sigma_1}, \ldots, Y_N^{\sigma_N})$ as follows [35, 36].

A coloured Young diagram Y^σ, with $\mathbb{Z}_n$ charge $\sigma \in \{0, 1, \ldots, n-1\}$, is a Young diagram whose box at position $(i, j) \in Y^\sigma$ has a colour $\sigma - i + j \pmod{n}$. The length of the i-row in Y^σ is denoted by Y_i^σ, and the total number of boxes in Y^σ is $|Y^\sigma| = \sum_i Y_i^\sigma$.

Let k_i, $0 \leq i < n$, be the total number of boxes with colour i in $\boldsymbol{Y^\sigma}$, and $\mathcal{P}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}$ be the set of N-tuples of n-coloured Young diagrams $\boldsymbol{Y^\sigma}$ labelled by the $\mathbb{Z}_n$ charges $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ and $\boldsymbol{\delta k} = (\delta k_1, \ldots, \delta k_{n-1})$, where $\delta k_i = k_i - k_0$. The charges $\boldsymbol{\sigma}$ define the non-negative

JHEP06(2020)112

integers N_i as the number of coloured Young diagrams with charge i, and we have

$$|\boldsymbol{Y}^{\boldsymbol{\sigma}}| := \sum_{I=1}^{N} \left|Y_I^{\sigma_I}\right| = \sum_{i=0}^{n-1} k_i, \qquad N = \sum_{i=0}^{n-1} N_i. \tag{2.1}$$

As a characterization of the $\mathrm{U}(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$, consider the first Chern class $c_1 = \sum_{i=0}^{n-1} \mathfrak{c}_i\, c_1(\mathcal{T}_i)$ of the gauge bundle. Here $c_1(\mathcal{T}_i)$ is the first Chern class of an individual vector bundle $\mathcal{T}_i$ associated with the $\mathbb{Z}_n$ orbifold, where $c_1(\mathcal{T}_0) = 0$, and

$$\mathfrak{c}_i = N_i + \delta k_{i-1} - 2\delta k_i + \delta k_{i+1} = N_i - \sum_{j=0}^{n-1} A_{ij}\, \delta k_j, \tag{2.2}$$

for $0 \le i < n$, where $k_n = k_0$, $k_{-1} = k_{n-1}$, and A denotes the Cartan matrix of $\widehat{\mathfrak{sl}}(n)$.

Now, it is useful to identify the non-negative integers N_i, $0 \le i < n$, with the Dynkin labels of $\widehat{\mathfrak{sl}}(n)$ in the level-N dominant weight lattice as $\boldsymbol{N} = [N_0, N_1, \dots, N_{n-1}] \in P^+_{n,N}$. We then introduce a generating function which enumerates the fixed points of $\mathrm{U}(1)^2 \times \mathrm{U}(1)^N$ torus action on the instanton moduli space $\mathcal{M}_{N,n}$.

Definition 2.1. *For* $\boldsymbol{N} = [N_0, N_1, \dots, N_{n-1}] \in P^+_{n,N}$*, the* $\mathrm{SU}(N)$ $\mathfrak{t}$*-refined generating function of* n*-coloured Young diagrams is defined by*

$$\widehat{X}_{\boldsymbol{N}}(\mathfrak{q}, \mathfrak{t}) = \sum_{\boldsymbol{\delta k} \in \mathbb{Z}^{n-1}} \widehat{X}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) \prod_{i=1}^{n-1} \mathfrak{t}_i^{\mathfrak{c}_i(\boldsymbol{\delta k})}, \tag{2.3}$$

where $\mathfrak{c}_i(\boldsymbol{\delta k}) = N_i + \delta k_{i-1} - 2\delta k_i + \delta k_{i+1}$ *are the Chern classes* (2.2)*, and*

$$\widehat{X}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) = (\mathfrak{q};\mathfrak{q})_\infty \sum_{\boldsymbol{Y}^{\boldsymbol{\sigma}} \in \mathcal{P}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}} \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}^{\boldsymbol{\sigma}}|}. \tag{2.4}$$

Here $\widehat{X}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q})$ *does not depend on the ordering of the* $\mathbb{Z}_n$ *charges* $\boldsymbol{\sigma}$ *and* (2.3) *is well-defined,*[1] *and the prefactor* $(\mathfrak{q};\mathfrak{q})_\infty = \prod_{n=1}^{\infty}(1-\mathfrak{q}^n)$ *subtracts the* $\mathrm{U}(1)$ *factor in* $\mathrm{U}(N)$ *gauge theory.*

As mentioned in the introduction, the generating function (2.4) originates with the $\mathrm{U}(N)$ instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ with a massless adjoint hypermultiplet in the case of $\epsilon_1 + \epsilon_2 = 0$ and pertains to the partition function of an $\mathcal{N} = 4$ twisted Yang-Mills theory on $\mathbb{C}^2/\mathbb{Z}_n$ [19] (see [20] for a string theory interpretation).

2.2 Algebra $\mathcal{A}(N, n; p)$

For a $\mathrm{U}(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ with Ω-deformation, the relevant AGT-corresponding CFT possesses symmetry algebra

$$\mathcal{A}(N, n; p) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{n+p-N}}, \tag{2.5}$$

[1] If the ordering $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_N$ is assumed, the $\mathbb{Z}_n$ charges are described by the partition (1.3) as $(\sigma_1, \sigma_2, \dots, \sigma_N, 0, 0, \dots) = \mathrm{par}(\boldsymbol{N})^T$.

which acts on the equivariant cohomology of $\mathcal{M}_{N,n}$ [3–5] (see also [17, 18] for the early notable works by Nakajima), where $\mathcal{H}$ is the affine Heisenberg algebra. This implies that the AGT-corresponding CFT is a combined system of $\widehat{\mathfrak{sl}}(n)_N$ WZW model with the additional $\mathcal{H}$ symmetry and a 2D CFT with the $\mathcal{W}^{\text{para}}_{N,n}$ (n-th parafermion $\mathcal{W}_N$) symmetry described by the coset [9–11]

$$\frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{n+p-N}}. \tag{2.6}$$

The parameter p is related to the Ω-deformation parameters ϵ_1, ϵ_2 by

$$\frac{\epsilon_1}{\epsilon_2} = -1 - \frac{n}{p}, \tag{2.7}$$

and controls the central charge of the 2D CFT with $\mathcal{W}^{\text{para}}_{N,n}$ symmetry by

$$c\left(\mathcal{W}^{\text{para}}_{N,n}\right) = \frac{n\,(N^2-1)}{N+n}\left(1 - \frac{N\,(N+n)}{p\,(p+n)}\right). \tag{2.8}$$

Here, if we take the limit $p \to \infty$ corresponding to $\epsilon_1 + \epsilon_2 = 0$, the algebra $\mathcal{A}(N,n;p)$ is formally reduced to $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \widehat{\mathfrak{sl}}(N)_n$. Since the central charge of $\widehat{\mathfrak{sl}}(n)_N$ WZW model is

$$c(\widehat{\mathfrak{sl}}(n)_N) = \frac{N(n^2-1)}{N+n}, \tag{2.9}$$

the AGT-corresponding CFT with this symmetry algebra has the central charge

$$1 + \frac{N(n^2-1)}{N+n} + \frac{n(N^2-1)}{N+n} = Nn, \tag{2.10}$$

and is considered to be described by Nn free fermions (see below (3.22) and e.g. [20]).

2.3 Burge conditions

When

$$p \in \mathbb{N} \quad \text{with} \quad p \geq N, \tag{2.11}$$

the ratio of the Ω-deformation parameters (2.7) becomes rational, and then the instanton partition function in 4D $\mathcal{N}=2$ U(N) Yang-Mills theory on $\mathbb{C}^2/\mathbb{Z}_n$ has non-physical poles [26] (see also [21–23] for early works in the case of $n=1$). By the AGT correspondence, these poles should correspond to the null states in $\mathcal{W}^{\text{para}}_{N,n}$ $(p, p+n)$-minimal models, which are described by the coset (2.6), and are parametrized by positive integers r_I and s_I, $0 \leq I < N$, with

$$\sum_{I=0}^{N-1} r_I = p, \qquad \sum_{I=0}^{N-1} s_I = p+n. \tag{2.12}$$

Similarly to $\boldsymbol{N} = [N_0, N_1, \ldots, N_{n-1}] \in P^+_{n,N}$, in what follows, we identify the positive integers r_I and s_I, $0 \leq I < N$, with the Dynkin labels of $\widehat{\mathfrak{sl}}(N)$ in the level-n regular dominant weight lattices as $\boldsymbol{r} = [r_0, r_1, \dots, r_{N-1}] \in P^{++}_{N,p}$ and $\boldsymbol{s} = [s_0, s_1, \dots, s_{N-1}] \in P^{++}_{N,p+n}$.

One finds that the poles can be removed by imposing the Burge conditions [21–23, 26] (see also [37–41] for Burge conditions),

$$Y^{\sigma_I}_{I,i} \geq Y^{\sigma_{I+1}}_{I+1,i+r_I-1} - s_I + 1 \quad \text{for } i \geq 1,\ 0 \leq I < N, \tag{2.13}$$

on N-tuples of n-coloured Young diagrams $\boldsymbol{Y}^{\boldsymbol{\sigma}} = (Y^{\sigma_1}_1, \dots, Y^{\sigma_N}_N)$, where $Y^{\sigma_0}_0 = Y^{\sigma_N}_N$. The $\mathbb{Z}_n$ charges σ_I are related to $\boldsymbol{r}$ and $\boldsymbol{s}$ by the $\mathbb{Z}_n$ charge conditions [26]

$$\sigma_I - \sigma_{I+1} \equiv -r_I + s_I \pmod{n}, \quad 0 \leq I < N, \tag{2.14}$$

where we set $\sigma_0 = \sigma_N$.

2.4 Burge-reduced generating functions

Let $\mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}$ be the subset of $\mathcal{P}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}$,

$$\mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}} \subset \mathcal{P}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}, \tag{2.15}$$

whose elements satisfy the Burge conditions (2.13) and the $\mathbb{Z}_n$ charge conditions (2.14). We now introduce Burge-reduced generating functions of coloured Young diagrams by subtracting the overall U(1) factor corresponding to $\mathcal{H}$.

Definition 2.2. *For $\boldsymbol{N} = [N_0, N_1, \dots, N_{n-1}] \in P^{+}_{n,N}$, the $\mathrm{SU}(N)$ $\mathfrak{t}$-refined Burge-reduced generating function of n-coloured Young diagrams, which is reduced by the Burge conditions (2.13) for $\boldsymbol{r} = [r_0, r_1, \dots, r_{N-1}] \in P^{++}_{N,p}$ and $\boldsymbol{s} = [s_0, s_1, \dots, s_{N-1}] \in P^{++}_{N,p+n}$, is defined by*

$$\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{N}}(\mathfrak{q}, \mathfrak{t}) = \sum_{\boldsymbol{\delta k} \in \mathbb{Z}^{n-1}} \widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) \prod_{i=1}^{n-1} \mathfrak{t}_i^{\mathfrak{c}_i(\boldsymbol{\delta k})}, \tag{2.16}$$

where $\mathfrak{c}_i(\boldsymbol{\delta k}) = N_i + \delta k_{i-1} - 2\delta k_i + \delta k_{i+1}$ are the Chern classes (2.2), and

$$\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) = (\mathfrak{q};\mathfrak{q})_\infty \sum_{\boldsymbol{Y}^{\boldsymbol{\sigma}} \in \mathcal{C}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}} \mathfrak{q}^{\frac{1}{n}|\boldsymbol{Y}^{\boldsymbol{\sigma}}|}. \tag{2.17}$$

Here, for fixed $\boldsymbol{r}$ and $\boldsymbol{s}$ the $\mathbb{Z}_n$ charge conditions (2.14) fix the charges $\boldsymbol{\sigma}$ up to the shifts $\sigma_I \to \sigma_I - k$ modulo n by $k \in \mathbb{Z}_n$ and the cyclic permutations $\sigma_I \to \sigma_{I-\theta}$ by $\theta \in \mathbb{Z}_N$, where $\sigma_{I+N} = \sigma_I$ and the latter ambiguities exist only if $s_0 - r_0 \equiv s_1 - r_1 \equiv \ldots \equiv s_{N-1} - r_{N-1} \pmod{n}$. Once we fix $\boldsymbol{N}$, the former ambiguities are fixed. As seen from the Burge conditions (2.13), $\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q})$ is invariant under the cyclic permutations $\sigma_I \to \sigma_{I-\theta}$, $r_I \to r_{I-\theta}$ and $s_I \to s_{I-\theta}$, where $r_{I+N} = r_I$ and $s_{I+N} = s_I$, and so (2.16) is well-defined. This also implies

$$\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{N}}(\mathfrak{q}, \mathfrak{t}) = \widehat{X}^{\boldsymbol{r}^{(\theta)},\boldsymbol{s}^{(\theta)}}_{\boldsymbol{N}}(\mathfrak{q}, \mathfrak{t}), \quad \theta \in \mathbb{Z}_N, \tag{2.18}$$

where $\boldsymbol{r}^{(\theta)} = [r^{(\theta)}_0, \dots, r^{(\theta)}_{N-1}]$ with $r^{(\theta)}_I = r_{I-\theta}$ and $\boldsymbol{s}^{(\theta)} = [s^{(\theta)}_0, \dots, s^{(\theta)}_{N-1}]$ with $s^{(\theta)}_I = s_{I-\theta}$.

JHEP06(2020)112

Consider the special case $p = N$ in which the algebra $\mathcal{A}(N,n;p)$ is reduced to $\mathcal{A}(N,n;N) = \mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_N$, and then $\boldsymbol{r} = \boldsymbol{1} = \rho$ is fixed by (2.12). In [26, Corollary 5.5], using the results of [29], it was shown that the $\mathfrak{t}$-refined Burge-reduced generating function (2.16) for $\boldsymbol{N} = [N_0, N_1, \ldots, N_{n-1}] \in P^+_{n,N}$ agrees with the $\widehat{\mathfrak{sl}}(n)_N$ WZW character as[2]

$$\widehat{X}^{\boldsymbol{1},\boldsymbol{s}}_{\boldsymbol{N}}(\mathfrak{q},\mathfrak{t}) = \mathfrak{q}^{\,w_{\boldsymbol{N}}-h_{\boldsymbol{N}}}\,\chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}}(\mathfrak{q},\hat{\mathfrak{t}}) \quad \textit{if} \;\; p = N, \tag{2.19}$$

where

$$\hat{\mathfrak{t}}_i = \mathfrak{q}^{-\frac{i\,(n-i)}{2n}}\,\mathfrak{t}_i, \qquad w_{\boldsymbol{N}} = \frac{\langle \boldsymbol{N},\rho\rangle}{n} = \sum_{i=1}^{n-1}\frac{i\,(n-i)}{2n}\,N_i, \qquad h_{\boldsymbol{N}} = \frac{\langle \boldsymbol{N},\boldsymbol{N}+2\rho\rangle}{2(n+N)}. \tag{2.20}$$

Here the $\widehat{\mathfrak{sl}}(n)_N$ WZW character is defined by (an overall normalization factor $\mathfrak{q}^{\,-\frac{1}{24}\,c(\widehat{\mathfrak{sl}}(n)_N)}$ is further introduced in the literature),

$$\chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}}(\mathfrak{q},\hat{\mathfrak{t}}) = \mathrm{Tr}_{L(\boldsymbol{N})}\,\mathfrak{q}^{\,L_0}\prod_{i=1}^{n-1}\hat{\mathfrak{t}}_i^{\,H_i}, \tag{2.21}$$

where $L(\boldsymbol{N})$ is the level-N irreducible highest-weight module of $\widehat{\mathfrak{sl}}(n)$, and the Virasoro generator L_0 and the Chevalley elements H_i in the Cartan subalgebra of $\widehat{\mathfrak{sl}}(n)$ act on the modules in the representation of a highest-weight state with the eigenvalues $h_{\boldsymbol{N}}$ and N_i, respectively. As in (2.16), we now expand the $\widehat{\mathfrak{sl}}(n)_N$ WZW character (2.21) as

$$\chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}}(\mathfrak{q},\hat{\mathfrak{t}}) = \mathfrak{q}^{\,h_{\boldsymbol{N}}}\sum_{\boldsymbol{\delta k}\in\mathbb{Z}^{n-1}} a^{\boldsymbol{N}}_{\mathfrak{c}(\boldsymbol{\delta k})}(\mathfrak{q})\prod_{i=1}^{n-1}\hat{\mathfrak{t}}_i^{\,\mathfrak{c}_i(\boldsymbol{\delta k})} = \sum_{\boldsymbol{\delta k}\in\mathbb{Z}^{n-1}} \hat{a}^{\boldsymbol{N}}_{\mathfrak{c}(\boldsymbol{\delta k})}(\mathfrak{q})\prod_{i=1}^{n-1}\mathfrak{t}_i^{\,\mathfrak{c}_i(\boldsymbol{\delta k})}, \tag{2.22}$$

where $a^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q})$ is known as a (normalized) $\widehat{\mathfrak{sl}}(n)$ string function of level-N (see also (3.1)) and

$$\hat{a}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q}) = \mathfrak{q}^{\,h_{\boldsymbol{N}}-w_{\mathfrak{c}}}\,a^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q}), \tag{2.23}$$

is also introduced, where $\mathfrak{c} = \mathfrak{c}(\boldsymbol{\delta k})$. From (2.19), by comparing (2.16) with (2.22) one obtains [26]

$$\widehat{X}^{\boldsymbol{1},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) = \mathfrak{q}^{\,\frac{1}{n}\sum_{i=1}^{n-1}\delta k_i}\,a^{\boldsymbol{N}}_{\mathfrak{c}(\boldsymbol{\delta k})}(\mathfrak{q})\,, \tag{2.24}$$

where $w_{\boldsymbol{N}} - w_{\mathfrak{c}(\boldsymbol{\delta k})} = \frac{1}{n}\sum_{i=1}^{n-1}\delta k_i$ was used.

3 $\mathcal{W}^{\mathrm{para}}_{N,n}$ minimal model characters from the instanton counting

In this section, we first recall a formula of $\mathcal{W}^{\mathrm{para}}_{N,n}$ $(p,p+n)$-minimal model characters (branching functions), and then propose Conjecture 3.5 about how the $\mathfrak{t}$-refined Burge-reduced generating functions of coloured Young diagrams are decomposed into the characters following the algebra $\mathcal{A}(N,n;p)$.

[2] When $\boldsymbol{N}$ is fixed, by the $\mathbb{Z}_n$ charge conditions $\sigma_I - \sigma_{I+1} \equiv s_I - 1 \pmod{n}$ with $\sum_{I=0}^{N-1}(s_I - 1) = n$ and $s_I - 1 \geq 0$, the generating function $\widehat{X}^{\boldsymbol{1},\boldsymbol{s}}_{\boldsymbol{N}}(\mathfrak{q},\mathfrak{t})$ is ambiguous only for the cyclic permutations $s_I \to s_{I-\theta}$ by $\theta \in \mathbb{Z}_N$. By (2.18), this is not the actual ambiguity of $\widehat{X}^{\boldsymbol{1},\boldsymbol{s}}_{\boldsymbol{N}}(\mathfrak{q},\mathfrak{t})$, and one can assume $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N$ and $s_I = \sigma_I - \sigma_{I+1} + 1 + n\,\delta_{I,0}$.

3.1 $\mathcal{W}^{\rm para}_{N,n}$ minimal model characters

We introduce a normalized $\widehat{\mathfrak{sl}}(N)$ string function $\hat{c}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$ of level-n for a dominant highest-weight $\boldsymbol{\ell}=[\ell_0,\ell_1,\dots,\ell_{N-1}]\in P^+_{N,n}$ and a maximal-weight $\boldsymbol{m}=[m_0,m_1,\dots,m_{N-1}]\in P_{N,n}$ by normalizing the $\widehat{\mathfrak{sl}}(N)$ string function $a^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$ of level-n in (2.22) with the exchange $N\leftrightarrow n$ as[3]

$$\hat{c}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})=\mathfrak{q}^{\,h_{\boldsymbol{\ell}}-\frac{1}{2n}|\boldsymbol{m}|^2}\,a^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q}). \tag{3.1}$$

Here $\hat{c}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$ is related to the string function $c^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$ in [27, 42] by $c^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})=\mathfrak{q}^{-\frac{1}{24}c(\widehat{\mathfrak{sl}}(N)_n)}\,\hat{c}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$, where $c(\widehat{\mathfrak{sl}}(N)_n)=\frac{n(N^2-1)}{n+N}$ is the central charge of $\widehat{\mathfrak{sl}}(N)_n$ WZW model. Note that, for non-zero string functions, the highest-weight $\boldsymbol{\ell}$ and the maximal-weight $\boldsymbol{m}$ should satisfy

$$\sum_{I=1}^{N-1}(\ell_I-m_I)\,\Lambda_I\in\overline{Q}_N,\quad i.e.\quad \sum_{I=1}^{N-1}I\,(\ell_I-m_I)\equiv 0\pmod N, \tag{3.2}$$

where $\overline{Q}_N$ is the root lattice in (1.2). Note also that, under the outer automorphisms of $\widehat{\mathfrak{sl}}(N)$ which cyclically permutes the Dynkin labels as $\ell_I\to\ell_{I-\theta}$ and $m_I\to m_{I-\theta}$ for all $I=0,1,\dots,N-1$ by $\theta\in\mathbb{Z}_N$, the string functions (3.1) are invariant, where we set $\ell_{I+N}=\ell_I$ and $m_{I+N}=m_I$. Here, by $\hat{a}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})=\mathfrak{q}^{\frac{1}{2n}|\boldsymbol{m}|^2-w_{\boldsymbol{m}}}\,\hat{c}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$, and $\frac{1}{2n}|\boldsymbol{m}|^2-w_{\boldsymbol{m}}=\frac{1}{2nN}\sum_{0\le I<J\le N-1}(I-J)(N+I-J)m_Im_J$, the normalized string functions $\hat{a}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})$ in (2.23) with the exchange $N\leftrightarrow n$ are also invariant under the outer automorphisms. Some string functions are summarized in appendix A.

Let us now recall the branching functions of the coset (2.6) that we refer as the $\mathcal{W}^{\rm para}_{N,n}$ characters when p is taken to be infinity (or a generic value) and the $\mathcal{W}^{\rm para}_{N,n}$ $(p,p{+}n)$-minimal model characters when p is an integer with $p\ge N$. Up to a normalization factor, the $\mathcal{W}^{\rm para}_{N,n}$ characters are given by the $\widehat{\mathfrak{sl}}(N)$ string functions (3.1) of level-n (see [43] for $N=2$), and the $\mathcal{W}^{\rm para}_{N,n}$ $(p,p{+}n)$-minimal model characters, labelled by $\boldsymbol{\ell}=[\ell_0,\ell_1,\dots,\ell_{N-1}]\in P^+_{N,n}$, $\boldsymbol{r}=[r_0,r_1,\dots,r_{N-1}]\in P^{++}_{N,p}$ and $\boldsymbol{s}=[s_0,s_1,\dots,s_{N-1}]\in P^{++}_{N,p+n}$ with the non-zero condition

$$\sum_{I=1}^{N-1}(\ell_I+r_I-s_I)\,\Lambda_I\in\overline{Q}_N,\quad i.e.\quad \sum_{I=1}^{N-1}I\,\ell_I\equiv\sum_{I=1}^{N-1}I\,(s_I-r_I)\pmod N, \tag{3.3}$$

are given by [10, 28],

$$C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}}(\mathfrak{q})=\sum_{\substack{\boldsymbol{m}\in P^+_{N,n}\\ \sum_{I=1}^{N-1}I\,(m_I-\ell_I)\equiv 0\,(\mathrm{mod}\,N)}}\hat{c}^{\boldsymbol{\ell}}_{\boldsymbol{m}}(\mathfrak{q})\sum_{w\in\overline{W}}\sum_{\boldsymbol{k}\in K^{\boldsymbol{r},\boldsymbol{s}}_w(\boldsymbol{m})}(-1)^{|w|}\,\mathfrak{q}^{\,B_{p\boldsymbol{k}+\boldsymbol{r},w(\boldsymbol{s})}-B_{\boldsymbol{r},\boldsymbol{s}}}. \tag{3.4}$$

Here

$$K^{\boldsymbol{r},\boldsymbol{s}}_w(\boldsymbol{m})=\bigcup_{w'\in\overline{W}}\left\{\boldsymbol{k}\in\overline{Q}_N\,\middle|\,p\boldsymbol{k}+\overline{\boldsymbol{r}}-w(\overline{\boldsymbol{s}})+w'(\overline{\boldsymbol{m}})\equiv 0\ (\mathrm{mod}\ n\,\overline{Q}_N)\right\} \tag{3.5}$$

[3] A maximal-weight $\boldsymbol{m}$ in $P_{N,n}$ is obtained from a dominant maximal-weight in $P^+_{N,n}$ by an action of the affine Weyl group of $\widehat{\mathfrak{sl}}(N)$, and the string function is invariant under the action (see Proposition 2.12 (a) and eq. (2.17) in [42]).

JHEP06(2020)112

with $\overline{\boldsymbol{m}} = \sum_{I=1}^{N-1} m_I \Lambda_I$, $\overline{\boldsymbol{r}} = \sum_{I=1}^{N-1} r_I \Lambda_I$, $\overline{\boldsymbol{s}} = \sum_{I=1}^{N-1} s_I \Lambda_I$, and $\overline{W}$ is the finite part of the affine Weyl group of $\widehat{\mathfrak{sl}}(N)$,[4] $|w|$ is the length of w, and

$$B_{\boldsymbol{r},\boldsymbol{s}} = \frac{|(p+n)\,\boldsymbol{r} - p\boldsymbol{s}|^2}{2np\,(p+n)}\,. \tag{3.6}$$

Note that the formula in [10, 28] corresponding to (3.4) has the summation over $\boldsymbol{m} \in P_{N,n}/n\overline{Q}_N$ instead of $\boldsymbol{m} \in P^{+}_{N,n}$ and the set corresponding to (3.5) does have the union over $w' \in \overline{W}$. Here to rewrite it we used the invariance of the string functions in footnote 3. We also used the fact that the simple affine Weyl reflection s_0 on $\Lambda = \sum_{I=0}^{N-1} d_I \Lambda_I \in P_{N,n}$ given by s_0: $d_I \mapsto d_I - A_{0I} d_0 \equiv d_I - \sum_{J,K=1}^{N-1} A_{IJ} d_K \pmod{n\overline{Q}_N}$ is also written as $\mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_{N-1}\mathsf{s}_{N-2}\cdots\mathsf{s}_2\mathsf{s}_1 \in \overline{W}$, i.e. $\mathsf{s}_0 \equiv \mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_{N-1}\mathsf{s}_{N-2}\cdots\mathsf{s}_2\mathsf{s}_1$ on Λ modulo $n\overline{Q}_N$.

Remark 3.1. Up to a normalization factor, the branching function $C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}}(\mathfrak{q})$ in (3.4) is defined by

$$\chi^{\widehat{\mathfrak{sl}}(N)_n}_{\boldsymbol{\ell}}(\mathfrak{q},\hat{\mathfrak{t}})\,\chi^{\widehat{\mathfrak{sl}}(N)_{p-N}}_{\boldsymbol{r}-\boldsymbol{1}}(\mathfrak{q},\hat{\mathfrak{t}}) \sim \sum_{\boldsymbol{s}\in P^{++}_{N,p+n}} C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}}(\mathfrak{q})\,\chi^{\widehat{\mathfrak{sl}}(N)_{n+p-N}}_{\boldsymbol{s}-\boldsymbol{1}}(\mathfrak{q},\hat{\mathfrak{t}}), \tag{3.7}$$

where $\boldsymbol{1} = \rho$.

Example 3.2. When $n=1$, $\mathcal{W}^{\text{para}}_{N,1} = \mathcal{W}_N$ [12–14]. The string functions (3.1) for $n=1$ (i.e. $\mathcal{W}_N$ characters) do not depend on the dominant highest-weight $\boldsymbol{\ell} \in P^{+}_{N,1}$ and are given by

$$\hat{c}(\mathfrak{q}) = \frac{1}{(\mathfrak{q};\mathfrak{q})^{N-1}_\infty}\,. \tag{3.8}$$

Similarly, the $\mathcal{W}_N$ $(p,p+1)$-minimal model characters (3.4) for $n=1$ do not depend on the dominant highest-weight $\boldsymbol{\ell} \in P^{+}_{N,1}$ and are given by [44, 45],

$$C^{\boldsymbol{r},\boldsymbol{s}}(\mathfrak{q}) = \frac{1}{(\mathfrak{q};\mathfrak{q})^{N-1}_\infty} \sum_{w\in\overline{W}} \sum_{\boldsymbol{k}\in\overline{Q}_N} (-1)^{|w|}\,\mathfrak{q}^{\,B_{p\boldsymbol{k}+\boldsymbol{r},w(\boldsymbol{s})} - B_{\boldsymbol{r},\boldsymbol{s}}}\,. \tag{3.9}$$

Example 3.3. When $N=2$, the $\mathcal{W}^{\text{para}}_{2,n}$ $(p,p+n)$-minimal model characters, labelled by $\boldsymbol{\ell} = [n-\ell,\ell] \in P^{+}_{2,n}$, $\boldsymbol{r} = [p-r,r] \in P^{++}_{2,p}$ and $\boldsymbol{s} = [p+n-s,s] \in P^{++}_{2,p+n}$ with $\ell+r-s \in 2\mathbb{Z}$, are computed by [46–48],

$$C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}}(\mathfrak{q}) = \mathfrak{q}^{-B_{r,s}} \sum_{\substack{m=0\\ m\equiv\ell(\text{mod}2)}}^{n} \hat{c}^{[n-\ell,\ell]}_{[n-m,m]}(\mathfrak{q}) \left(\sum_{\substack{k\in\mathbb{Z}\\ pk-\frac{r-s}{2}\equiv\pm\frac{m}{2}(\text{mod}n)}} \mathfrak{q}^{B_{2pk+r,s}} - \sum_{\substack{k\in\mathbb{Z}\\ pk-\frac{r+s}{2}\equiv\pm\frac{m}{2}(\text{mod}n)}} \mathfrak{q}^{B_{2pk+r,-s}} \right), \tag{3.10}$$

where $B_{r,s} = ((p+n)r - ps)^2/(4np(p+n))$ and the string functions $\hat{c}^{[n-\ell,\ell]}_{[n-m,m]}(\mathfrak{q})$ are given in (A.5).

[4]The Weyl group $\overline{W}$ is generated by the simple Weyl reflections s_I, $1 \le I < N$, acting on $\overline{\Lambda} = \sum_{I=1}^{N-1} d_I \Lambda_I$ as $\mathsf{s}_I(\overline{\Lambda}) = \overline{\Lambda} - \langle \alpha_I, \overline{\Lambda}\rangle\,\alpha_I$, i.e. s_I: $d_J \mapsto d_J - \overline{A}_{IJ} d_I$, where the simple Weyl reflections have the relations $\mathsf{s}_I^2 = 1$ for $1 \le I < N$, $(\mathsf{s}_I\mathsf{s}_{I+1})^3 = 1$ for $1 \le I < N-1$, $\mathsf{s}_I\mathsf{s}_J = \mathsf{s}_J\mathsf{s}_I$ for $|I-J| \ge 2$, and $\overline{A}$ is the Cartan matrix of $\mathfrak{sl}(N)$.

JHEP06(2020)112

3.2 Dual dominant integral weights

For proposing our conjecture, let us define a dominant integral weight

$$\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)} = [N_0, N_1, \dots, N_{n-1}] \in P_{n,N}^+ \tag{3.11}$$

of $\widehat{\mathfrak{sl}}(n)_N$ which is dual to or associated with the dominant integral weight $\boldsymbol{\ell} = [\ell_0, \ell_1, \dots, \ell_{N-1}] \in P_{N,n}^+$ of $\widehat{\mathfrak{sl}}(N)_n$. Here a non-negative integer $f < \max\{N, n\}$, which classifies the dominant weights in $P_{N,n}^+/\overline{Q}_N$ and $P_{n,N}^+/\overline{Q}_n$, respectively, as the $\mathbb{Z}_N$ orbits and the $\mathbb{Z}_n$ orbits, is introduced by[5]

$$\sum_{I=1}^{N-1} I\,\ell_I \equiv f \pmod{N}, \qquad \sum_{i=1}^{n-1} i\,N_i \equiv f \pmod{n}. \tag{3.12}$$

Let $\mathrm{par}(\boldsymbol{\ell}) = (\lambda_1, \lambda_2, \dots)$ be the partition for $\boldsymbol{\ell}$ in (1.3), and then the first relation in (3.12) is written as $\sum_{I=1}^{N-1} \lambda_I \equiv f \pmod{N}$. We define the Dynkin labels N_i in (3.11) as the multiplicity of $i = \sigma_I^*$ in $\{\sigma_1^*, \dots, \sigma_N^*\}$, where $\sigma_I^* \in \{0, 1, \dots, n-1\}$, $1 \le I \le N$, correspond to the $\mathbb{Z}_n$ charges (on the gauge side) defined by

$$\sigma_I^* \equiv \lambda_I + \sigma_N^* \pmod{n}, \quad 1 \le I < N, \quad \sigma_N^* \equiv -\frac{1}{N}\left(\sum_{I=1}^{N-1} \lambda_I - f\right) \pmod{n}. \tag{3.13}$$

Here the shifted transposed partition $\widetilde{\mathrm{par}}(\boldsymbol{\ell})^T = (\lambda_1^T - \lambda_n^T, \lambda_2^T - \lambda_n^T, \dots)$ by λ_n^T naturally defines a *'transposed'* (dual) dominant integral weight $\boldsymbol{\ell}^T = [\ell_0^T, \ell_1^T, \dots, \ell_{n-1}^T] \in P_{n,N}^+$ by inverting (1.3). Then the first relations in (3.13) imply that the dual dominant integral weight $\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)} = [N_0, N_1, \dots, N_{n-1}]$ is defined by

$$N_i = \ell_{i-\sigma_N^*}^T, \qquad 0 \le i < n, \tag{3.14}$$

where we set $\ell_{i+n}^T = \ell_i^T$. Note that, by (3.13) we see that the $\mathbb{Z}_n$ charges σ_I^* have the relations

$$\sigma_I^* - \sigma_{I+1}^* = \ell_I - n\,\delta_{I,N-g}, \quad 1 \le I \le N, \qquad \sum_{I=1}^{N} \sigma_I^* \equiv f \pmod{n}, \tag{3.15}$$

and are ordered as $\sigma_{1-g}^* \ge \sigma_{2-g}^* \ge \dots \ge \sigma_{N-g}^*$, where $\sigma_{I+N}^* = \sigma_I^*$, $\ell_{I+N} = \ell_I$, and $g \in \{0, 1, \dots, N-1\}$ is[6]

$$g \equiv \frac{1}{n}\left(\sum_{I=1}^{N} \sigma_I^* - f\right) \equiv \frac{1}{n}\left(\sum_{i=1}^{n-1} i\,N_i - f\right) \pmod{N}. \tag{3.16}$$

Here the second relation in (3.15) gives the second relation in (3.12) by $\sum_{I=1}^{N} \sigma_I^* = \sum_{i=1}^{n-1} i\,N_i$. Some examples of $\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}$ are provided in appendix B.

[5] The numbers of the dominant integral weights $|P_{N,n}^+| = \frac{(n+N-1)!}{(N-1)!\,n!}$ in $\widehat{\mathfrak{sl}}(N)_n$ and $|P_{n,N}^+| = \frac{(n+N-1)!}{(n-1)!\,N!}$ in $\widehat{\mathfrak{sl}}(n)_N$ are related by $|P_{N,n}^+|/N = |P_{n,N}^+|/n$.

[6] In terms of the $\mathbb{Z}_n$ charges $\sigma_I = \sigma_{I-g}^*$ with the ordering $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_N$, the first relations in (3.15) are written as $\sigma_I - \sigma_{I+1} = \ell_{I-g} - n\,\delta_{I,0}$, $0 \le I < N$.

JHEP06(2020)112

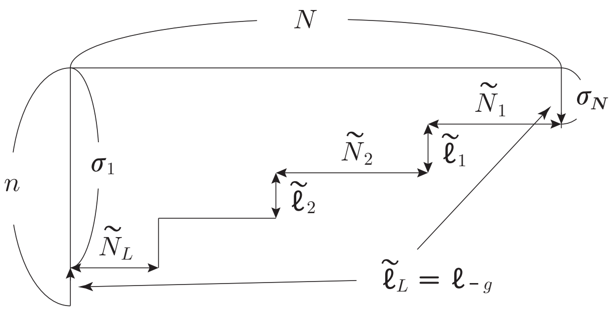

Figure 2. The finite sequence (3.20), where we follow the notation in footnote 6.

Remark 3.4. Consider the dual dominant integral weight $\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}$. Then, the normalization factors of string functions in (2.23) for $\mathfrak{c} = \boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}$ and in (3.1) for $\boldsymbol{m} = \boldsymbol{\ell}$ are related by

$$w_{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}} - h_{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}} = h_{\boldsymbol{\ell}} - \frac{1}{2n}|\boldsymbol{\ell}|^2. \tag{3.17}$$

Proof. The left and right hand sides are, respectively, obtained as

$$w_{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}} - h_{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}} = \frac{1}{2n(n+N)} \sum_{0\le i<j<n} (j-i)\,(n-j+i)\,N_i\,N_j\,, \tag{3.18}$$

and

$$h_{\boldsymbol{\ell}} - \frac{1}{2n}|\boldsymbol{\ell}|^2 = \frac{1}{2n(n+N)} \sum_{0\le I<J<N} (J-I)\,(N-J+I)\,\ell_I\,\ell_J\,. \tag{3.19}$$

We now take all the non-zero components $(\widetilde{N}_1, \widetilde{N}_2, \ldots, \widetilde{N}_L) = (N_{i_1}, N_{i_2}, \ldots, N_{i_L})$ with $i_k < i_{k+1}$ from $\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}$, and $(\widetilde{\ell}_1, \widetilde{\ell}_2, \ldots, \widetilde{\ell}_L) = (\ell'_{I_1}, \ell'_{I_2}, \ldots, \ell'_{I_L})$ with $I_k > I_{k+1}$ from $\boldsymbol{\ell}$, where $\ell'_I = \ell_{I-g}$, $0 \le I < N$, in footnote 6. Then consider the finite sequence

$$\widetilde{N}_1\,,\ \widetilde{\ell}_1\,,\ \widetilde{N}_2\,,\ \widetilde{\ell}_2\,,\ \ldots,\ \widetilde{N}_L\,,\ \widetilde{\ell}_L\,, \tag{3.20}$$

which is described as in figure 2. For $\widetilde{N}_a = N_{i_a}$, $\widetilde{N}_b = N_{i_b}$ with $a < b$, we see that $i_b - i_a = \sum_{a\le A<b} \widetilde{\ell}_A$ and $n - i_b + i_a = \sum_{A<a} \widetilde{\ell}_A + \sum_{A\ge b} \widetilde{\ell}_A$. This shows that (3.18) is equal to

$$\frac{1}{2n(n+N)} \left(\sum_{1\le A<a\le B<b\le L} + \sum_{1\le a\le A<b\le B\le L} \right) \widetilde{\ell}_A\,\widetilde{\ell}_B\,\widetilde{N}_a\,\widetilde{N}_b\,. \tag{3.21}$$

Similarly, (3.19) is also shown to be equal to (3.21), and thus (3.17) is proved. □

When we consider the special case $\epsilon_1 + \epsilon_2 = 0$ ($p \to \infty$), the central charge (2.10) of the AGT-corresponding CFT is reminiscent of a conformal embedding

$$\mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \widehat{\mathfrak{sl}}(N)_n \subset \widehat{\mathfrak{gl}}(Nn)_1, \tag{3.22}$$

which preserves the central charge Nn and is utilized to explain the level-rank duality between $\widehat{\mathfrak{sl}}(n)_N$ and $\widehat{\mathfrak{sl}}(N)_n$, where $\widehat{\mathfrak{gl}}(Nn)_1 \cong \mathcal{H}^{\oplus N} \oplus \widehat{\mathfrak{sl}}(n)_1^{\oplus N}$ is described by Nn free fermions [30–34] (see also [20] for an elegant string theory interpretation by intersecting D4 and D6-branes). Actually, the generating function (2.4) of coloured Young diagrams for general N and n is obtained by [49, 50]

$$\widehat{X}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q}) = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty^{N-1}} \sum_{\boldsymbol{\delta k}_1+\cdots+\boldsymbol{\delta k}_N=\boldsymbol{\delta k}} \prod_{I=1}^{N} \widehat{X}_{(\sigma_I);\boldsymbol{\delta k}_I}(\mathfrak{q}) \,, \tag{3.23}$$

where

$$\widehat{X}_{(\sigma);\boldsymbol{\delta k}}(\mathfrak{q}) = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty^{n-1}} \mathfrak{q}^{\sum_{i=1}^{n-1}\left(\delta k_i^2+\frac{\delta k_i}{n}-\delta k_{i-1}\,\delta k_i-\delta_{\sigma i}\,\delta k_i\right)}, \tag{3.24}$$

is the generating function for $N=1$ which gives the $\widehat{\mathfrak{sl}}(n)_1$ WZW character. Let $\boldsymbol{\sigma}_{\boldsymbol{m}}^{(f)} = (\sigma_1,\ldots,\sigma_N)$ be the $\mathbb{Z}_n$ charges with the ordering $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N$ which follow from the dual dominant integral weight $\boldsymbol{N}_{\boldsymbol{m}}^{(f)}$ in (3.11). Following the algebra $\mathcal{A}(N,n;p)$ for $p\to\infty$, we find that the generating function (3.23) is decomposed into the $\widehat{\mathfrak{sl}}(N)$ string functions (3.1) of level-n and the $\widehat{\mathfrak{sl}}(n)$ string functions (2.23) of level-N as

$$\begin{aligned}\widehat{X}_{\boldsymbol{\sigma}_{\boldsymbol{m}}^{(f)};\boldsymbol{\delta k}}(\mathfrak{q}) &= \sum_{\substack{\boldsymbol{\ell}\in P_{N,n}^+ \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f\,(\mathrm{mod}\,N)}} \hat{c}_{\boldsymbol{m}}^{\boldsymbol{\ell}}(\mathfrak{q}) \times \hat{a}_{\mathfrak{c}(\boldsymbol{\delta k})}^{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}}(\mathfrak{q}) \\ &= \sum_{\substack{\boldsymbol{\ell}\in P_{N,n}^+ \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f\,(\mathrm{mod}\,N)}} \mathfrak{q}^{\frac{1}{2n}(|\boldsymbol{\ell}|^2-|\boldsymbol{m}|^2)+\frac{1}{n}\sum_{i=1}^{n-1}\delta k_i}\, a_{\boldsymbol{m}}^{\boldsymbol{\ell}}(\mathfrak{q}) \times a_{\mathfrak{c}(\boldsymbol{\delta k})}^{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}}(\mathfrak{q}) \,,\end{aligned} \tag{3.25}$$

where, in the second equality the relation (3.17) was used. The same decomposition was shown for the conformal embedding (3.22) in [31] (see also [20, appendix A]). In terms of the SU(N) $\mathfrak{t}$-refined generating functions (2.3) of n-coloured Young diagrams, the above decomposition boils down to the decomposition into the $\widehat{\mathfrak{sl}}(N)$ string functions (3.1) of level-n ($\mathcal{W}_{N,n}^{\mathrm{para}}$ characters) and the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters (2.21) as

$$\widehat{X}_{\boldsymbol{N}_{\boldsymbol{m}}^{(f)}}(\mathfrak{q},\mathfrak{t}) = \sum_{\substack{\boldsymbol{\ell}\in P_{N,n}^+ \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f\,(\mathrm{mod}\,N)}} \hat{c}_{\boldsymbol{m}}^{\boldsymbol{\ell}}(\mathfrak{q}) \times \chi_{\boldsymbol{N}_{\boldsymbol{\ell}}^{(f)}}^{\widehat{\mathfrak{sl}}(n)_N}(\mathfrak{q},\hat{\mathfrak{t}}) \,, \tag{3.26}$$

where $\hat{\mathfrak{t}}_i = \mathfrak{q}^{-\frac{i\,(n-i)}{2n}}\,\mathfrak{t}_i$.

3.3 Conjecture

Based on the symmetry algebra $\mathcal{A}(N,n;p)$ in (2.5), we now propose the following conjecture for integers $p \geq N$ that generalizes the decomposition formula (3.26) for $p\to\infty$.

JHEP06(2020)112

Conjecture 3.5. *The* SU(N) $\mathfrak{t}$*-refined Burge-reduced generating functions* (2.16) *of* n*-coloured Young diagrams can be decomposed into the* $\mathcal{W}^{\rm para}_{N,n}$ $(p,p+n)$*-minimal model characters* (3.4) *and the* $\widehat{\mathfrak{sl}}(n)_N$ *WZW characters* (2.21) *as*

$$\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{N}^{(f)}_{\boldsymbol{m}}}(\mathfrak{q},\mathfrak{t}) = \sum_{\substack{\boldsymbol{\ell}\in P^+_{N,n} \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f\,(\mathrm{mod}\,N)}} C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}^{(\omega)}}(\mathfrak{q}) \times \chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}^{(f)}_{\boldsymbol{\ell}}}(\mathfrak{q},\hat{\mathfrak{t}})\,, \tag{3.27}$$

where $\hat{\mathfrak{t}}_i = \mathfrak{q}^{-\frac{i\,(n-i)}{2n}}\,\mathfrak{t}_i$. *The dominant weight* $\boldsymbol{\ell}^{(\omega)} = [\ell^{(\omega)}_0, \ell^{(\omega)}_1, \ldots, \ell^{(\omega)}_{N-1}] \in P^+_{N,n}$ *with* $\ell^{(\omega)}_I = \ell_{I-\omega}$, $\ell_{I+N} = \ell_I$, $0 \le I < N$, *is shifted by*

$$\omega \equiv \frac{1}{n}\left(\sum_{i=1}^{n-1} i\,N_i - f\right) + \frac{1}{n}\sum_{I=1}^{N-1} I\,\Big(s_I - r_I - \sigma_I + \sigma_{I+1}\Big) \pmod N, \tag{3.28}$$

where $\boldsymbol{\sigma}^{(f)}_{\boldsymbol{m}} = (\sigma_1, \ldots, \sigma_N)$ *are the* $\mathbb{Z}_n$ *charges associated with* $\boldsymbol{N}^{(f)}_{\boldsymbol{m}} = [N_0, N_1, \ldots, N_{n-1}] \in P^+_{n,N}$, *and the ordering of* $\boldsymbol{\sigma}^{(f)}_{\boldsymbol{m}}$ *depends on* $\boldsymbol{r}$ *and* $\boldsymbol{s}$ *by the* $\mathbb{Z}_n$ *charge conditions* (2.14). *Here the non-zero condition* (3.3) *for the characters* $C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}^{(\omega)}}(\mathfrak{q})$ *is shown to be satisfied as*

$$\sum_{I=1}^{N-1} I\,\ell^{(\omega)}_I \equiv \sum_{I=1}^{N-1} I\,\ell_I + \omega\,n \equiv \sum_{I=1}^{N-1} I\,(s_I - r_I) \pmod N, \tag{3.29}$$

where in the second equality we used $\sum_{I=1}^{N-1} I\ell_I \equiv f$ *and* $\sum_{I=1}^{N-1} I(\sigma_I - \sigma_{I+1}) \equiv \sum_{i=1}^{n-1} i\,N_i$ $\pmod N$. *By the expansions* (2.16) *and* (2.22), *the conjectural formula* (3.27) *is equivalent to*

$$\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma}^{(f)}_{\boldsymbol{m}};\boldsymbol{\delta k}}(\mathfrak{q}) = \sum_{\substack{\boldsymbol{\ell}\in P^+_{N,n} \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f\,(\mathrm{mod}\,N)}} C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}^{(\omega)}}(\mathfrak{q}) \times \hat{a}^{\boldsymbol{N}^{(f)}_{\boldsymbol{\ell}}}_{\mathfrak{c}(\boldsymbol{\delta k})}(\mathfrak{q})\,. \tag{3.30}$$

We make some remarks to support Conjecture 3.5.

Remark 3.6. From the invariance (2.18) of the $\mathfrak{t}$-refined Burge-reduced generating functions under the cyclic permutations $\sigma_I \to \sigma_{I-\theta}$, $r_I \to r^{(\theta)} = r_{I-\theta}$ and $s_I \to s^{(\theta)} = s_{I-\theta}$, $\theta \in \mathbb{Z}_N$, one finds that the conjectural formula (3.27) gives a relation

$$\sum_{\substack{\boldsymbol{\ell}\in P^+_{N,n} \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f\,(\mathrm{mod}\,N)}} \Big(C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}^{(\omega)}}(\mathfrak{q}) - C^{\boldsymbol{r}^{(\theta)},\boldsymbol{s}^{(\theta)}}_{\boldsymbol{\ell}^{(\omega+\theta)}}(\mathfrak{q})\Big)\, \chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}^{(f)}_{\boldsymbol{\ell}}}(\mathfrak{q},\hat{\mathfrak{t}}) = 0. \tag{3.31}$$

Here ω is defined by (3.28) and

$$\frac{1}{n}\sum_{I=1}^{N-1} I\,\Big(s^{(\theta)}_I - r^{(\theta)}_I - \sigma_{I-\theta} + \sigma_{I-\theta+1}\Big) \equiv \frac{1}{n}\sum_{I=1}^{N-1} I\,\Big(s_I - r_I - \sigma_I + \sigma_{I+1}\Big) + \theta \pmod N, \tag{3.32}$$

is used. The relation (3.31) then implies the invariance

$$C^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\ell}}(\mathfrak{q}) = C^{\boldsymbol{r}^{(\theta)},\boldsymbol{s}^{(\theta)}}_{\boldsymbol{\ell}^{(\theta)}}(\mathfrak{q}), \quad \theta \in \mathbb{Z}_N, \tag{3.33}$$

of the minimal model characters (branching functions).

Remark 3.7. In the special case $p = N$, let us show that the conjectural formula (3.27) yields the formula (2.19). In this special case, by

$$C_{\boldsymbol{\ell}}^{\mathbf{1},\boldsymbol{s}}(\mathfrak{q}) = \begin{cases} \mathfrak{q}^{\,h_{\boldsymbol{\ell}}-\frac{1}{2n}|\boldsymbol{\ell}|^2} & \text{if } \ s_I = \ell_I + 1 \ \text{ for } \ 0 \le I < N, \\ 0 & \text{otherwise}, \end{cases} \tag{3.34}$$

which follows from the definition (3.7) with taking into account of the normalization factor, the conjectural formula (3.27) is

$$\widehat{X}^{\,\mathbf{1},\boldsymbol{s}}_{\boldsymbol{N}^{(f)}_{\boldsymbol{m}}}(\mathfrak{q},\mathbf{t}) = \sum_{\substack{\boldsymbol{\ell}\in P^{+}_{N,n} \\ \sum_{I=1}^{N-1} I\,\ell_I \equiv f \,(\mathrm{mod}\, N)}} \mathfrak{q}^{\,h_{\boldsymbol{\ell}^{(\omega)}}-\frac{1}{2n}|\boldsymbol{\ell}^{(\omega)}|^2}\, \chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}^{(f)}_{\boldsymbol{\ell}}}(\mathfrak{q},\hat{\mathbf{t}}) \prod_{I=0}^{N-1} \delta_{\ell^{(\omega)}_I, s_I - 1}\,. \tag{3.35}$$

Following footnote 2, by taking $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_N$ and $\sigma_I - \sigma_{I+1} = s_I - 1 - n\,\delta_{I,0}$, the shift parameter ω is now given by $\omega = g$ in (3.16) and then $m^{(g)}_I = s_I - 1$ by footnote 6, where note that the $\mathbb{Z}_n$ charges σ_I are associated with $\boldsymbol{N}^{(f)}_{\boldsymbol{m}}$. As a result, (3.35) yields

$$\widehat{X}^{\,\mathbf{1},\boldsymbol{s}}_{\boldsymbol{N}^{(f)}_{\boldsymbol{m}}}(\mathfrak{q},\mathbf{t}) = \mathfrak{q}^{\,h_{\boldsymbol{m}^{(g)}}-\frac{1}{2n}|\boldsymbol{m}^{(g)}|^2}\, \chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}^{(f)}_{\boldsymbol{m}}}(\mathfrak{q},\hat{\mathbf{t}})\,. \tag{3.36}$$

Therefore, by $h_{\boldsymbol{m}^{(g)}} - \frac{1}{2n}|\boldsymbol{m}^{(g)}|^2 = h_{\boldsymbol{m}} - \frac{1}{2n}|\boldsymbol{m}|^2$ following from (3.19), and by the relation (3.17) we obtain the formula (2.19).

Remark 3.8. When $n = 1$, the conjectural formula (3.27) yields

$$\widehat{X}^{\,\boldsymbol{r},\boldsymbol{s}}_{[N]}(\mathfrak{q}) = C^{\,\boldsymbol{r},\boldsymbol{s}}(\mathfrak{q}), \tag{3.37}$$

which gives the $\mathcal{W}_N$ $(p, p+1)$-minimal model characters in Example 3.2.[7]

4 Examples of Burge-reduced generating functions

In this section, we test Conjecture 3.5 by extracting the $\mathcal{W}^{\mathrm{para}}_{N,n}$ $(p, p+n)$-minimal model characters from the SU(N) Burge-reduced generating functions of n-coloured Young diagrams in the cases of $(N, n, p) = (2, 2, 4)$ and $(3, 3, 4)$. By assuming the formula (3.30) with the use of the $\widehat{\mathfrak{sl}}(n)$ string functions in appendix A we will check that the minimal model characters in (3.4) are obtained.

4.1 $(N, n) = (2, 2)$ and minimal super-Virasoro characters

When $(N, n) = (2, 2)$, the $\mathcal{W}^{\mathrm{para}}_{2,2}$ algebra is the super-Virasoro algebra [52] and studied in the context of the AGT correspondence in [3, 53–57]. Here we consider the $(4, 6)$-minimal model $(p = 4)$ which has central charge $c\left(\mathcal{W}^{\mathrm{para}}_{2,2}\right) = 1$ by (2.8). The SU(2) Burge-reduced generating functions $\widehat{X}^{\,\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};(\delta k)}(\mathfrak{q})$ in (2.17) of 2-coloured Young diagrams are labelled by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ with $0 \le \sigma_1, \sigma_2 \le 1$, $\delta k \in \mathbb{Z}$, and $\boldsymbol{r} = [r_0, r_1] \in P^{++}_{2,p}$, $\boldsymbol{s} = [s_0, s_1] \in P^{++}_{2,p+2}$ with

[7] See [51, section 3.4], where note our normalization of string functions as below (3.1) by $\frac{1}{24}\,c(\widehat{\mathfrak{sl}}(N)_1) = \frac{1}{24}\,(N-1)$.

$s_1 - r_1 \equiv \sigma_1 - \sigma_2 \pmod 2$, where $\boldsymbol{\sigma}$ and δk define $\mathfrak{c} = [\mathfrak{c}_0, \mathfrak{c}_1] = [N_0 + 2\delta k, N_1 - 2\delta k] \in P_{2,2}$ in (2.2).

The Burge-reduced generating functions for $\boldsymbol{N} = [2,0]$ and $\mathfrak{c} = [2,0]$, $[0,2]$ are obtained as

$$\begin{aligned}
\widehat{X}^{[3,1],[5,1]}_{(0,0);(0)}(\mathfrak{q}) &= 1+\mathfrak{q}+5\mathfrak{q}^2+10\mathfrak{q}^3+25\mathfrak{q}^4+48\mathfrak{q}^5+101\mathfrak{q}^6+185\mathfrak{q}^7+350\mathfrak{q}^8+615\mathfrak{q}^9+\cdots, \\
\widehat{X}^{[3,1],[5,1]}_{(0,0);(-1)}(\mathfrak{q}) &= \mathfrak{q}^{\frac12}+3\mathfrak{q}^{\frac32}+7\mathfrak{q}^{\frac52}+16\mathfrak{q}^{\frac72}+35\mathfrak{q}^{\frac92}+70\mathfrak{q}^{\frac{11}{2}}+137\mathfrak{q}^{\frac{13}{2}}+256\mathfrak{q}^{\frac{15}{2}}+465\mathfrak{q}^{\frac{17}{2}}+\cdots, \\
\widehat{X}^{[2,2],[4,2]}_{(0,0);(0)}(\mathfrak{q}) &= 1+3\mathfrak{q}+10\mathfrak{q}^2+25\mathfrak{q}^3+57\mathfrak{q}^4+121\mathfrak{q}^5+243\mathfrak{q}^6+465\mathfrak{q}^7+862\mathfrak{q}^8+\cdots, \\
\widehat{X}^{[2,2],[4,2]}_{(0,0);(-1)}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac12}+6\mathfrak{q}^{\frac32}+16\mathfrak{q}^{\frac52}+38\mathfrak{q}^{\frac72}+84\mathfrak{q}^{\frac92}+172\mathfrak{q}^{\frac{11}{2}}+338\mathfrak{q}^{\frac{13}{2}}+636\mathfrak{q}^{\frac{15}{2}}+\cdots,
\end{aligned} \tag{4.1}$$

and using the $\widehat{\mathfrak{sl}}(2)$ string functions (A.5) of level-2 with $\hat{a}^{[N_0,N_1]}_{[\mathfrak{c}_0,\mathfrak{c}_1]}(\mathfrak{q}) = \mathfrak{q}^{\frac18 \mathfrak{c}_1(\mathfrak{c}_1-2)}\, \hat{c}^{[N_0,N_1]}_{[\mathfrak{c}_0,\mathfrak{c}_1]}(\mathfrak{q})$, from the formula (3.30) we obtain

$$\begin{aligned}
C^{[3,1],[5,1]}_{[2,0]}(\mathfrak{q}) &= 1+\mathfrak{q}^2+\mathfrak{q}^3+3\mathfrak{q}^4+3\mathfrak{q}^5+7\mathfrak{q}^6+8\mathfrak{q}^7+14\mathfrak{q}^8+17\mathfrak{q}^9+27\mathfrak{q}^{10}+\cdots, \\
C^{[3,1],[5,1]}_{[0,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac32}+\mathfrak{q}^{\frac52}+2\mathfrak{q}^{\frac72}+3\mathfrak{q}^{\frac92}+5\mathfrak{q}^{\frac{11}{2}}+7\mathfrak{q}^{\frac{13}{2}}+11\mathfrak{q}^{\frac{15}{2}}+15\mathfrak{q}^{\frac{17}{2}}+22\mathfrak{q}^{\frac{19}{2}}+\cdots, \\
C^{[2,2],[4,2]}_{[2,0]}(\mathfrak{q}) &= 1+\mathfrak{q}+2\mathfrak{q}^2+4\mathfrak{q}^3+6\mathfrak{q}^4+10\mathfrak{q}^5+15\mathfrak{q}^6+22\mathfrak{q}^7+32\mathfrak{q}^8+46\mathfrak{q}^9+\cdots, \\
C^{[2,2],[4,2]}_{[0,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac12}+2\mathfrak{q}^{\frac32}+3\mathfrak{q}^{\frac52}+5\mathfrak{q}^{\frac72}+8\mathfrak{q}^{\frac92}+12\mathfrak{q}^{\frac{11}{2}}+18\mathfrak{q}^{\frac{13}{2}}+27\mathfrak{q}^{\frac{15}{2}}+38\mathfrak{q}^{\frac{17}{2}}+\cdots.
\end{aligned} \tag{4.2}$$

We see that they agree with the $\mathcal{W}^{\text{para}}_{2,2}$ (4,6)-minimal model characters in (3.10). Similarly, from a Burge-reduced generating function for $\boldsymbol{N} = [1,1]$ and $\mathfrak{c} = [1,1]$,

$$\widehat{X}^{[3,1],[4,2]}_{(1,0);(0)}(\mathfrak{q}) = 1+3\mathfrak{q}+8\mathfrak{q}^2+20\mathfrak{q}^3+44\mathfrak{q}^4+92\mathfrak{q}^5+183\mathfrak{q}^6+348\mathfrak{q}^7+640\mathfrak{q}^8+1144\mathfrak{q}^9+\cdots, \tag{4.3}$$

we obtain

$$C^{[3,1],[4,2]}_{[1,1]}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{16}}\left(1+\mathfrak{q}+2\mathfrak{q}^2+4\mathfrak{q}^3+6\mathfrak{q}^4+10\mathfrak{q}^5+15\mathfrak{q}^6+22\mathfrak{q}^7+32\mathfrak{q}^8+46\mathfrak{q}^9+\cdots\right). \tag{4.4}$$

4.2 $(N, n) = (3, 3)$ and minimal super-$\mathcal{W}_3$ characters

When $(N, n) = (3, 3)$, the $\mathcal{W}^{\text{para}}_{3,3}$ algebra is supposed to be the super-$\mathcal{W}_3$ algebra, and here we consider the (4,7)-minimal model ($p = 4$) with the central charge $c\left(\mathcal{W}^{\text{para}}_{3,3}\right) = 10/7$ by (2.8) which ensures the associativity of the $\mathcal{W}^{\text{para}}_{3,3}$ algebra and has a unitary representation [58–61].[8] The SU(3) Burge-reduced generating functions $\widehat{X}^{\boldsymbol{r},\boldsymbol{s}}_{\boldsymbol{\sigma};\boldsymbol{\delta k}}(\mathfrak{q})$ of 3-coloured Young diagrams are labelled by $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with $0 \le \sigma_1, \sigma_2, \sigma_3 \le 2$, $\boldsymbol{\delta k} = (\delta k_1, \delta k_2) \in \mathbb{Z}^2$, and $\boldsymbol{r} = [r_0, r_1, r_2] \in P^{++}_{3,4}$, $\boldsymbol{s} = [s_0, s_1, s_3] \in P^{++}_{3,7}$ with $s_1 - r_1 \equiv \sigma_1 - \sigma_2$, $s_2 - r_2 \equiv \sigma_2 - \sigma_3 \pmod 3$, where $\boldsymbol{\sigma}$ and $\boldsymbol{\delta k}$ define $\mathfrak{c} = [\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{c}_2] \in P_{3,3}$ in (2.2).

[8] See [62, 63] for the generalization to the minimal super-$\mathcal{W}_N$ algebra corresponding to $(N, n, p) = (N, N, N+1)$.

The Burge-reduced generating functions for $\boldsymbol{N} = [3,0,0]$ and $\mathfrak{c} = [3,0,0]$, $[1,1,1]$, $[0,3,0]$, $[0,0,3]$ are obtained as

$$\begin{aligned}
\widehat{X}^{[2,1,1],[5,1,1]}_{(0,0,0);(0,0)}(\mathfrak{q}) &= 1+2\mathfrak{q}+11\mathfrak{q}^2+42\mathfrak{q}^3+144\mathfrak{q}^4+448\mathfrak{q}^5+1303\mathfrak{q}^6+3510\mathfrak{q}^7+\cdots, \\
\widehat{X}^{[2,1,1],[5,1,1]}_{(0,0,0);(-1,-1)}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{3}}+5\mathfrak{q}^{\frac{4}{3}}+24\mathfrak{q}^{\frac{7}{3}}+89\mathfrak{q}^{\frac{10}{3}}+299\mathfrak{q}^{\frac{13}{3}}+896\mathfrak{q}^{\frac{16}{3}}+2503\mathfrak{q}^{\frac{19}{3}}+\cdots, \\
\widehat{X}^{[2,1,1],[5,1,1]}_{(0,0,0);(-2,-1)}(\mathfrak{q}) &= \mathfrak{q}+8\mathfrak{q}^2+35\mathfrak{q}^3+132\mathfrak{q}^4+426\mathfrak{q}^5+1261\mathfrak{q}^6+3443\mathfrak{q}^7+\cdots, \\
\widehat{X}^{[2,1,1],[5,1,1]}_{(0,0,0);(-1,-2)}(\mathfrak{q}) &= \mathfrak{q}+8\mathfrak{q}^2+35\mathfrak{q}^3+132\mathfrak{q}^4+426\mathfrak{q}^5+1261\mathfrak{q}^6+3443\mathfrak{q}^7+\cdots,
\end{aligned} \tag{4.5}$$

and using the $\widehat{\mathfrak{sl}}(3)$ string functions (A.7) of level-3 with $\hat{a}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{9}(\mathfrak{c}_1^2+\mathfrak{c}_2^2+\mathfrak{c}_1\mathfrak{c}_2)-\frac{1}{3}(\mathfrak{c}_1+\mathfrak{c}_2)}\,\hat{c}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q})$, from the formula (3.30) we find the $\mathcal{W}^{\text{para}}_{3,3}$ $(4,7)$-minimal model characters

$$\begin{aligned}
C^{[2,1,1],[5,1,1]}_{[3,0,0]}(\mathfrak{q}) &= 1+\mathfrak{q}^2+2\mathfrak{q}^3+3\mathfrak{q}^4+4\mathfrak{q}^5+8\mathfrak{q}^6+10\mathfrak{q}^7+\cdots, \\
C^{[2,1,1],[5,1,1]}_{[1,1,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{3}{2}}+2\mathfrak{q}^{\frac{5}{2}}+3\mathfrak{q}^{\frac{7}{2}}+6\mathfrak{q}^{\frac{9}{2}}+10\mathfrak{q}^{\frac{11}{2}}+16\mathfrak{q}^{\frac{13}{2}}+\cdots, \\
C^{[2,1,1],[5,1,1]}_{[0,0,3]}(\mathfrak{q}) &= \mathfrak{q}^4+\mathfrak{q}^5+3\mathfrak{q}^6+5\mathfrak{q}^7+\cdots, \\
C^{[2,1,1],[5,1,1]}_{[0,3,0]}(\mathfrak{q}) &= \mathfrak{q}^4+\mathfrak{q}^5+3\mathfrak{q}^6+5\mathfrak{q}^7+\cdots.
\end{aligned} \tag{4.6}$$

Similarly, the Burge-reduced generating functions for $\boldsymbol{N} = [1,1,1]$ and $\mathfrak{c} = [3,0,0]$, $[1,1,1]$, $[0,3,0]$, $[0,0,3]$,[9]

$$\begin{aligned}
\widehat{X}^{[2,1,1],[1,3,3]}_{(0,1,2);(1,1)}(\mathfrak{q}) &= 3\mathfrak{q}^{\frac{2}{3}}+18\mathfrak{q}^{\frac{5}{3}}+84\mathfrak{q}^{\frac{8}{3}}+312\mathfrak{q}^{\frac{11}{3}}+1028\mathfrak{q}^{\frac{14}{3}}+3052\mathfrak{q}^{\frac{17}{3}}+8425\mathfrak{q}^{\frac{20}{3}}+\cdots, \\
\widehat{X}^{[2,1,1],[1,3,3]}_{(0,1,2);(0,0)}(\mathfrak{q}) &= 1+10\mathfrak{q}+50\mathfrak{q}^2+203\mathfrak{q}^3+693\mathfrak{q}^4+2136\mathfrak{q}^5+6031\mathfrak{q}^6+15967\mathfrak{q}^7+\cdots, \\
\widehat{X}^{[2,1,1],[1,3,3]}_{(0,1,2);(-1,0)}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac{2}{3}}+16\mathfrak{q}^{\frac{5}{3}}+79\mathfrak{q}^{\frac{8}{3}}+302\mathfrak{q}^{\frac{11}{3}}+1009\mathfrak{q}^{\frac{14}{3}}+3018\mathfrak{q}^{\frac{17}{3}}+8364\mathfrak{q}^{\frac{20}{3}}+\cdots, \\
\widehat{X}^{[2,1,1],[1,3,3]}_{(0,1,2);(0,-1)}(\mathfrak{q}) &= 3\mathfrak{q}^{\frac{2}{3}}+18\mathfrak{q}^{\frac{5}{3}}+84\mathfrak{q}^{\frac{8}{3}}+312\mathfrak{q}^{\frac{11}{3}}+1028\mathfrak{q}^{\frac{14}{3}}+3052\mathfrak{q}^{\frac{17}{3}}+8425\mathfrak{q}^{\frac{20}{3}}+\cdots,
\end{aligned} \tag{4.7}$$

give

$$\begin{aligned}
C^{[2,1,1],[1,3,3]}_{[3,0,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{8}{3}}+2\mathfrak{q}^{\frac{11}{3}}+5\mathfrak{q}^{\frac{14}{3}}+8\mathfrak{q}^{\frac{17}{3}}+15\mathfrak{q}^{\frac{20}{3}}+\cdots, \\
C^{[2,1,1],[1,3,3]}_{[1,1,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{6}}+2\mathfrak{q}^{\frac{7}{6}}+4\mathfrak{q}^{\frac{13}{6}}+8\mathfrak{q}^{\frac{19}{6}}+15\mathfrak{q}^{\frac{25}{6}}+26\mathfrak{q}^{\frac{31}{6}}+43\mathfrak{q}^{\frac{37}{6}}+\cdots, \\
C^{[2,1,1],[1,3,3]}_{[0,0,3]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{2}{3}}+\mathfrak{q}^{\frac{5}{3}}+3\mathfrak{q}^{\frac{8}{3}}+4\mathfrak{q}^{\frac{11}{3}}+8\mathfrak{q}^{\frac{14}{3}}+12\mathfrak{q}^{\frac{17}{3}}+21\mathfrak{q}^{\frac{20}{3}}+\cdots, \\
C^{[2,1,1],[1,3,3]}_{[0,3,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{2}{3}}+\mathfrak{q}^{\frac{5}{3}}+3\mathfrak{q}^{\frac{8}{3}}+4\mathfrak{q}^{\frac{11}{3}}+8\mathfrak{q}^{\frac{14}{3}}+12\mathfrak{q}^{\frac{17}{3}}+21\mathfrak{q}^{\frac{20}{3}}+\cdots.
\end{aligned} \tag{4.8}$$

The Burge-reduced generating functions for $\boldsymbol{N} = [2,1,0]$ and $\mathfrak{c} = [2,1,0]$, $[0,2,1]$, $[1,0,2]$,

$$\begin{aligned}
\widehat{X}^{[2,1,1],[4,2,1]}_{(1,0,0);(0,0)}(\mathfrak{q}) &= 1+5\mathfrak{q}+26\mathfrak{q}^2+104\mathfrak{q}^3+367\mathfrak{q}^4+1151\mathfrak{q}^5+3329\mathfrak{q}^6+8969\mathfrak{q}^7+\cdots, \\
\widehat{X}^{[2,1,1],[4,2,1]}_{(1,0,0);(-1,-1)}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{3}}+8\mathfrak{q}^{\frac{4}{3}}+39\mathfrak{q}^{\frac{7}{3}}+156\mathfrak{q}^{\frac{10}{3}}+532\mathfrak{q}^{\frac{13}{3}}+1638\mathfrak{q}^{\frac{16}{3}}+4631\mathfrak{q}^{\frac{19}{3}}+\cdots, \\
\widehat{X}^{[2,1,1],[4,2,1]}_{(1,0,0);(0,-1)}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac{2}{3}}+13\mathfrak{q}^{\frac{5}{3}}+62\mathfrak{q}^{\frac{8}{3}}+234\mathfrak{q}^{\frac{11}{3}}+777\mathfrak{q}^{\frac{14}{3}}+2322\mathfrak{q}^{\frac{17}{3}}+6435\mathfrak{q}^{\frac{20}{3}}+\cdots,
\end{aligned} \tag{4.9}$$

[9] Note the ordering $\sigma_1 \leq \sigma_2 \leq \sigma_3$ for $(\sigma_1,\sigma_2,\sigma_3) = (0,1,2)$.

JHEP06(2020)112

give

$$\begin{aligned} C^{[2,1,1],[4,2,1]}_{[2,1,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{9}}\,(1+\mathfrak{q}+3\mathfrak{q}^2+5\mathfrak{q}^3+9\mathfrak{q}^4+14\mathfrak{q}^5+24\mathfrak{q}^6+37\mathfrak{q}^7+\cdots)\,,\\ C^{[2,1,1],[4,2,1]}_{[1,0,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{9}}\,(\mathfrak{q}^{\frac{4}{3}}+2\mathfrak{q}^{\frac{7}{3}}+4\mathfrak{q}^{\frac{10}{3}}+7\mathfrak{q}^{\frac{13}{3}}+13\mathfrak{q}^{\frac{16}{3}}+21\mathfrak{q}^{\frac{19}{3}}+\cdots)\,,\\ C^{[2,1,1],[4,2,1]}_{[0,2,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{9}}\,(\mathfrak{q}^{\frac{5}{3}}+2\mathfrak{q}^{\frac{8}{3}}+4\mathfrak{q}^{\frac{11}{3}}+8\mathfrak{q}^{\frac{14}{3}}+14\mathfrak{q}^{\frac{17}{3}}+24\mathfrak{q}^{\frac{20}{3}}+\cdots)\,. \end{aligned} \tag{4.10}$$

5 Summary and outlook

Following the AGT correspondence for $\mathrm{U}(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$, we conjectured the decomposition formula (3.27) of the Burge-reduced generating functions of N-tuples of n-coloured Young diagrams with the Burge conditions and the $\mathbb{Z}_n$ charge conditions for integral $p \geq N$. This conjectural decomposition generalizes the decomposition formula (3.26) of the generating functions of N-tuples of n-coloured Young diagrams for $p \to \infty$ (or for a generic central charge), and gives the $\mathcal{W}^{\text{para}}_{N,n}$ $(p, p+n)$-minimal model characters (branching functions of the coset factor in $\mathcal{A}(N,n;p)$). When $p = N$, the central charge of the $\mathcal{W}^{\text{para}}_{N,n}$ $(N, N+n)$-minimal model is vanished, and in Remark 3.7 the conjectural formula is indeed shown to yield the formula (2.19) which gives the $\widehat{\mathfrak{sl}}(n)_N$ WZW characters.

In [26] we also introduced the $\mathrm{SU}(N)$ *Burge-reduced instanton partition functions* on $\mathbb{C}^2/\mathbb{Z}_n$ with $2N$ (anti-)fundamental hypermultiplets, where the Burge conditions and the $\mathbb{Z}_n$ charge conditions for $p = N$ were imposed. We then conjectured that they give the specific integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks in [64]. Similarly to the conjectural decomposition (3.27) of the Burge-reduced generating functions, the Burge-reduced instanton partition functions for integral $p \geq N$ are also expected to be decomposed into $\mathcal{W}^{\text{para}}_{N,n}$ $(p, p+n)$-minimal model conformal blocks and $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks (see [57] in the case of $(N,n) = (2,2)$ with a generic central charge). It would be interesting to pursue this direction as was discussed in [23] when $n = 1$.

Acknowledgments

The author would like to thank O Foda, N Macleod and T Welsh for the stimulating collaboration on [26] and useful discussions, and the Australian Research Council for support of this work.

A Some string functions

In this appendix we summarize some normalized $\widehat{\mathfrak{sl}}(M)$ string functions of level-m.

The normalized string function $\hat{c}^{\Lambda}_{\gamma(\boldsymbol{\ell})}(\mathfrak{q})$, for a dominant highest-weight $\Lambda = [d_0, d_1, \ldots, d_{M-1}] \in P^{+}_{M,m}$ and a maximal-weight $\gamma(\boldsymbol{\ell}) = [\gamma_0, \gamma_1, \ldots, \gamma_{M-1}] \in P_{M,m}$, is obtained from the $\widehat{\mathfrak{sl}}(M)_m$ WZW character in (2.21) as (see eqs. (2.22) and (3.1) with the normalization by the central charge),

$$\chi^{\widehat{\mathfrak{sl}}(M)_m}_{\Lambda}(\mathfrak{q}, \hat{\mathfrak{t}}) = \mathfrak{q}^{\frac{1}{2m}|\gamma(\boldsymbol{\ell})|^2} \sum_{\boldsymbol{\ell} \in \mathbb{Z}^{M-1}} \hat{c}^{\Lambda}_{\gamma(\boldsymbol{\ell})}(\mathfrak{q}) \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{\gamma_i(\boldsymbol{\ell})}, \tag{A.1}$$

where $\gamma_i = \gamma_i(\boldsymbol{\ell}) = d_i + \ell_{i-1} - 2\ell_i + \ell_{i+1}$ with $\ell_M = \ell_0 = 0$, $\ell_{-1} = \ell_{M-1}$. The WZW characters can be computed by the Weyl-Kac character formula [27] (see also [51, appendix B.2] and [26, appendix A.8]),

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(M)_m}(\mathfrak{q}, \hat{\mathfrak{t}}) = \frac{\mathcal{N}_\Lambda\left(\mathfrak{q}, \hat{\mathfrak{t}}\right) \mathfrak{q}^{h_\Lambda}}{(\mathfrak{q};\mathfrak{q})_\infty^{M-1} \prod_{1\le i<j\le M}\left(\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_j/\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{j-1};\mathfrak{q}\right)_\infty \left(\mathfrak{q}\,\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{j-1}/\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_j;\mathfrak{q}\right)_\infty} \prod_{i=1}^{M-1} \hat{\mathfrak{t}}_i^{d_i}, \tag{A.2}$$

with

$$\mathcal{N}_\Lambda(\mathfrak{q}, \hat{\mathfrak{t}}) = \sum_{\substack{(k_1,\dots,k_M)\in\mathbb{Z}^M \\ k_1+\cdots+k_M=0}} \det_{1\le i,j\le M}\left(\left(\hat{\mathfrak{t}}_i/\hat{\mathfrak{t}}_{i-1}\right)^{\left(M+m\right)k_i-\lambda_i+i+\lambda_j-j} \mathfrak{q}^{\frac{1}{2}\left(M+m\right)k_i^2+\left(\lambda_j-j\right)k_i}\right), \tag{A.3}$$

where $(\mathfrak{q};\mathfrak{q})_\infty = \prod_{n=1}^\infty(1-\mathfrak{q}^n)$, $\hat{\mathfrak{t}}_0 = \hat{\mathfrak{t}}_M = 1$ and $(\lambda_1, \lambda_2, \dots) = \mathrm{par}(\Lambda)$ in (1.3). Note that the string functions are invariant under the outer automorphisms of $\widehat{\mathfrak{sl}}(M)$ as

$$\hat{c}^{\Lambda}_{\gamma(\boldsymbol{\ell})}(\mathfrak{q}) = \hat{c}^{\Lambda^{(\theta)}}_{\gamma(\boldsymbol{\ell})^{(\theta)}}(\mathfrak{q}), \qquad \theta \in \mathbb{Z}_M, \tag{A.4}$$

where $\Lambda^{(\theta)} = [d_0^{(\theta)}, d_1^{(\theta)}, \dots, d_{M-1}^{(\theta)}]$ with $d_i^{(\theta)} = d_{i-\theta}$, $d_{i+M} = d_i$, and $\gamma(\boldsymbol{\ell})^{(\theta)} = [\gamma_0^{(\theta)}, \gamma_1^{(\theta)}, \dots, \gamma_{M-1}^{(\theta)}]$ with $\gamma_i^{(\theta)} = \gamma_{i-\theta}$, $\gamma_{i+M} = \gamma_i$.

A.1 $\widehat{\mathfrak{sl}}(2)$

When $M = 2$, the $\widehat{\mathfrak{sl}}(2)$ string functions $\hat{c}^{[m-d,d]}_{[m-\gamma,\gamma]}(\mathfrak{q})$ of level-m, with $d-\gamma \in 2\mathbb{Z}$, for $[m-d,d] \in P^+_{2,m}$ and $[m-\gamma,\gamma] \in P_{2,m}$ are given by [65],

$$\begin{aligned}\hat{c}^{[m-d,d]}_{[m-\gamma,\gamma]}(\mathfrak{q}) = \frac{\mathfrak{q}^{\frac{d(d+2)}{4(m+2)}-\frac{\gamma^2}{4m}}}{(\mathfrak{q};\mathfrak{q})^3_\infty} \sum_{k_1,k_2=0}^\infty &(-1)^{k_1+k_2}\, \mathfrak{q}^{\frac{1}{2}k_1(k_1+1)+\frac{1}{2}k_2(k_2+1)+(m+1)k_1k_2} \\ &\times\left(\mathfrak{q}^{\frac{1}{2}(d-\gamma)k_1+\frac{1}{2}(d+\gamma)k_2} - \mathfrak{q}^{m+1-d+\frac{1}{2}(2m+2-d+\gamma)k_1+\frac{1}{2}(2m+2-d-\gamma)k_2}\right),\end{aligned} \tag{A.5}$$

and satisfy $\hat{c}^{[m-d,d]}_{[m-\gamma,\gamma]}(\mathfrak{q}) = \hat{c}^{[d,m-d]}_{[\gamma,m-\gamma]}(\mathfrak{q})$.

A.2 $\widehat{\mathfrak{sl}}(3)$

Here we summarize the $\widehat{\mathfrak{sl}}(3)$ string functions $\hat{c}^\Lambda_\gamma(\mathfrak{q})$ of level-2 and 3 given in [42].

The $\widehat{\mathfrak{sl}}(3)$ string functions of level-2 are

$$\begin{aligned}
\hat{c}^{[2,0,0]}_{[2,0,0]}(\mathfrak{q}) - \hat{c}^{[2,0,0]}_{[0,1,1]}(\mathfrak{q}) &= \frac{\left(\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}^{\frac{1}{2}}\right)_\infty \left(\mathfrak{q},\mathfrak{q}^{\frac{3}{2}},\mathfrak{q}^{\frac{5}{2}};\mathfrak{q}^{\frac{5}{2}}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^4_\infty}, \\
\hat{c}^{[2,0,0]}_{[0,1,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}} \frac{(\mathfrak{q}^2;\mathfrak{q}^2)_\infty (\mathfrak{q}^2,\mathfrak{q}^8,\mathfrak{q}^{10};\mathfrak{q}^{10})_\infty}{(\mathfrak{q};\mathfrak{q})^4_\infty}, \\
\hat{c}^{[0,1,1]}_{[0,1,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{10}} \frac{(\mathfrak{q}^2;\mathfrak{q}^2)_\infty (\mathfrak{q}^4,\mathfrak{q}^6,\mathfrak{q}^{10};\mathfrak{q}^{10})_\infty}{(\mathfrak{q};\mathfrak{q})^4_\infty}, \\
\hat{c}^{[0,1,1]}_{[0,1,1]}(\mathfrak{q}) - \hat{c}^{[0,1,1]}_{[2,0,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{10}} \frac{\left(\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}^{\frac{1}{2}}\right)_\infty \left(\mathfrak{q}^{\frac{1}{2}},\mathfrak{q}^2,\mathfrak{q}^{\frac{5}{2}};\mathfrak{q}^{\frac{5}{2}}\right)_\infty}{(\mathfrak{q};\mathfrak{q})^4_\infty},
\end{aligned} \tag{A.6}$$

where $(a_1, a_2, \ldots, a_k; \mathfrak{q})_\infty = \prod_{i=1}^k (a_i; \mathfrak{q})_\infty = \prod_{i=1}^k \prod_{n=1}^\infty (1 - a_i \mathfrak{q}^{n-1})$.

The $\widehat{\mathfrak{sl}}(3)$ string functions of level-3 are

$$\begin{aligned}
&\hat{c}\,^{[3,0,0]}_{[3,0,0]}(\mathfrak{q}) - \hat{c}\,^{[3,0,0]}_{[0,3,0]}(\mathfrak{q}) = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty (\mathfrak{q}^3;\mathfrak{q}^3)_\infty}, \qquad \hat{c}\,^{[1,1,1]}_{[1,1,1]}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{6}} \frac{(\mathfrak{q}^2;\mathfrak{q}^2)^3_\infty (\mathfrak{q}^3;\mathfrak{q}^3)^2_\infty}{(\mathfrak{q};\mathfrak{q})^6_\infty (\mathfrak{q}^6;\mathfrak{q}^6)_\infty}, \\
&\hat{c}\,^{[2,1,0]}_{[2,1,0]}(\mathfrak{q}) + \hat{c}\,^{[2,1,0]}_{[0,2,1]}(\mathfrak{q}) + \hat{c}\,^{[2,1,0]}_{[1,0,2]}(\mathfrak{q}) = \frac{\mathfrak{q}^{\frac{1}{9}}}{(\mathfrak{q};\mathfrak{q})_\infty \left(\mathfrak{q}^{\frac{1}{3}};\mathfrak{q}^{\frac{1}{3}}\right)_\infty}, \\
&\hat{c}\,^{[3,0,0]}_{[3,0,0]}(\mathfrak{q}) - 3\,\hat{c}\,^{[3,0,0]}_{[1,1,1]}(\mathfrak{q}) + 2\,\hat{c}\,^{[3,0,0]}_{[0,3,0]}(\mathfrak{q}) + \hat{c}\,^{[1,1,1]}_{[1,1,1]}(\mathfrak{q}) - \hat{c}\,^{[1,1,1]}_{[3,0,0]}(\mathfrak{q}) = \frac{\left(\mathfrak{q}^{\frac{1}{2}};\mathfrak{q}^{\frac{1}{2}}\right)^3_\infty \left(\mathfrak{q}^{\frac{1}{3}};\mathfrak{q}^{\frac{1}{3}}\right)^2_\infty}{(\mathfrak{q};\mathfrak{q})^6_\infty \left(\mathfrak{q}^{\frac{1}{6}};\mathfrak{q}^{\frac{1}{6}}\right)_\infty},
\end{aligned} \tag{A.7}$$

where $\hat{c}\,^{[2,1,0]}_{[2,1,0]}(\mathfrak{q}) \in \mathfrak{q}^{\frac{1}{9}}\,\mathbb{Z}[[\mathfrak{q}]]$, $\hat{c}\,^{[2,1,0]}_{[0,2,1]}(\mathfrak{q}) \in \mathfrak{q}^{\frac{4}{9}}\,\mathbb{Z}[[\mathfrak{q}]]$, $\hat{c}\,^{[2,1,0]}_{[1,0,2]}(\mathfrak{q}) \in \mathfrak{q}^{\frac{7}{9}}\,\mathbb{Z}[[\mathfrak{q}]]$, $\hat{c}\,^{[3,0,0]}_{[1,1,1]}(\mathfrak{q}) \in \mathfrak{q}^{\frac{2}{3}}\,\mathbb{Z}[[\mathfrak{q}]]$ and $\hat{c}\,^{[1,1,1]}_{[3,0,0]}(\mathfrak{q}) \in \mathfrak{q}^{\frac{1}{2}}\,\mathbb{Z}[[\mathfrak{q}]]$.

B Examples of dual dominant integral weights

Here we provide some examples of the dominant integral weights $\boldsymbol{N}^{(f)}_{\boldsymbol{\ell}} = [N_0, N_1, \ldots, N_{n-1}] \in P^+_{n,N}$ of $\widehat{\mathfrak{sl}}(n)_N$ in (3.11), which are labelled by a non-negative integer $f < \max\{N, n\}$ and dominant integral weights $\boldsymbol{\ell} = [\ell_0, \ell_1, \ldots, \ell_{N-1}] \in P^+_{N,n}$ of $\widehat{\mathfrak{sl}}(N)_n$.

For $(N, n) = (2, 2)$,

$$\boldsymbol{N}^{(0)}_{[2,0]} = [2,0], \quad \boldsymbol{N}^{(0)}_{[0,2]} = [0,2], \quad \boldsymbol{N}^{(1)}_{[1,1]} = [1,1]. \tag{B.1}$$

For $(N, n) = (2, 3)$,

$$\begin{aligned}
&\boldsymbol{N}^{(0)}_{[3,0]} = [2,0,0],\ \boldsymbol{N}^{(0)}_{[1,2]} = [0,1,1],\ \boldsymbol{N}^{(2)}_{[3,0]} = [0,2,0],\ \boldsymbol{N}^{(2)}_{[1,2]} = [1,0,1], \\
&\boldsymbol{N}^{(1)}_{[2,1]} = [1,1,0],\ \boldsymbol{N}^{(1)}_{[0,3]} = [0,0,2].
\end{aligned} \tag{B.2}$$

For $(N, n) = (2, 4)$,

$$\begin{aligned}
&\boldsymbol{N}^{(0)}_{[4,0]} = [2,0,0,0],\ \boldsymbol{N}^{(0)}_{[2,2]} = [0,1,0,1],\ \boldsymbol{N}^{(0)}_{[0,4]} = [0,0,2,0], \\
&\boldsymbol{N}^{(2)}_{[4,0]} = [0,2,0,0],\ \boldsymbol{N}^{(2)}_{[2,2]} = [1,0,1,0],\ \boldsymbol{N}^{(2)}_{[0,4]} = [0,0,0,2], \\
&\boldsymbol{N}^{(1)}_{[3,1]} = [1,1,0,0],\ \boldsymbol{N}^{(1)}_{[1,3]} = [0,0,1,1],\ \boldsymbol{N}^{(3)}_{[3,1]} = [0,1,1,0],\ \boldsymbol{N}^{(3)}_{[1,3]} = [1,0,0,1].
\end{aligned} \tag{B.3}$$

For $(N, n) = (3, 2)$,

$$\boldsymbol{N}^{(0)}_{[2,0,0]} = \boldsymbol{N}^{(2)}_{[0,2,0]} = [3,0], \quad \boldsymbol{N}^{(0)}_{[0,1,1]} = \boldsymbol{N}^{(2)}_{[1,0,1]} = [1,2], \quad \boldsymbol{N}^{(1)}_{[1,1,0]} = [2,1], \quad \boldsymbol{N}^{(1)}_{[0,0,2]} = [0,3]. \tag{B.4}$$

For $(N, n) = (3, 3)$,

$$\begin{aligned}
&\boldsymbol{N}^{(0)}_{[3,0,0]} = [3,0,0], \quad \boldsymbol{N}^{(0)}_{[1,1,1]} = [1,1,1], \quad \boldsymbol{N}^{(0)}_{[0,0,3]} = [0,3,0], \quad \boldsymbol{N}^{(0)}_{[0,3,0]} = [0,0,3], \\
&\boldsymbol{N}^{(1)}_{[2,1,0]} = [2,1,0], \quad \boldsymbol{N}^{(1)}_{[1,0,2]} = [0,2,1], \quad \boldsymbol{N}^{(1)}_{[0,2,1]} = [1,0,2], \\
&\boldsymbol{N}^{(2)}_{[1,2,0]} = [2,0,1], \quad \boldsymbol{N}^{(2)}_{[2,0,1]} = [1,2,0], \quad \boldsymbol{N}^{(2)}_{[0,1,2]} = [0,1,2].
\end{aligned} \tag{B.5}$$

JHEP06(2020)112

For $(N,n)=(4,2)$,

$$\boldsymbol{N}^{(0)}_{[2,0,0,0]}=\boldsymbol{N}^{(2)}_{[0,2,0,0]}=[4,0],\quad \boldsymbol{N}^{(0)}_{[0,1,0,1]}=\boldsymbol{N}^{(2)}_{[1,0,1,0]}=[2,2],\quad \boldsymbol{N}^{(0)}_{[0,0,2,0]}=\boldsymbol{N}^{(2)}_{[0,0,0,2]}=[0,4],$$
$$\boldsymbol{N}^{(1)}_{[1,1,0,0]}=\boldsymbol{N}^{(3)}_{[0,1,1,0]}=[3,1],\quad \boldsymbol{N}^{(1)}_{[0,0,1,1]}=\boldsymbol{N}^{(3)}_{[1,0,0,1]}=[1,3]. \tag{B.6}$$

C More examples of Burge-reduced generating functions

In this appendix, in addition to the examples in section 4, we give some more examples of the SU(N) Burge-reduced generating functions of n-coloured Young diagrams in the cases of $(N,n,p)=(2,3,3),(2,4,4),(3,2,4)$ and $(4,2,5)$ and check Conjecture 3.5.

C.1 $(N,n,p)=(2,3,3)$

Consider the case of $(N,n)=(2,3)$ and $p=3$. The $\mathcal{W}^{\text{para}}_{2,3}$ $(3,6)$-minimal model has central charge $c\left(\mathcal{W}^{\text{para}}_{2,3}\right)=4/5$. The Burge-reduced generating functions for $\boldsymbol{N}=[2,0,0]$ and $\mathfrak{c}=[2,0,0]$, $[0,1,1]$ are obtained as

$$\widehat{X}^{[2,1],[5,1]}_{(0,0);(0,0)}(\mathfrak{q})=1+2\mathfrak{q}+11\mathfrak{q}^2+32\mathfrak{q}^3+97\mathfrak{q}^4+246\mathfrak{q}^5+610\mathfrak{q}^6+1388\mathfrak{q}^7+3067\mathfrak{q}^8+\cdots,$$
$$\widehat{X}^{[2,1],[5,1]}_{(0,0);(-1,-1)}(\mathfrak{q})=\mathfrak{q}^{\frac{1}{3}}+5\mathfrak{q}^{\frac{4}{3}}+18\mathfrak{q}^{\frac{7}{3}}+56\mathfrak{q}^{\frac{10}{3}}+154\mathfrak{q}^{\frac{13}{3}}+389\mathfrak{q}^{\frac{16}{3}}+922\mathfrak{q}^{\frac{19}{3}}+2072\mathfrak{q}^{\frac{22}{3}}+\cdots, \tag{C.1}$$

and using the $\widehat{\mathfrak{sl}}(3)$ string functions (A.6) of level-2 with $\hat{a}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q})=\mathfrak{q}^{\frac{1}{6}(\mathfrak{c}_1^2+\mathfrak{c}_2^2+\mathfrak{c}_1\mathfrak{c}_2)-\frac{1}{3}(\mathfrak{c}_1+\mathfrak{c}_2)}\,\hat{c}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q})$, from the formula (3.30) we obtain the $\mathcal{W}^{\text{para}}_{2,3}$ $(3,6)$-minimal model characters

$$C^{[2,1],[5,1]}_{[3,0]}(\mathfrak{q})=1+\mathfrak{q}^2+\mathfrak{q}^3+2\mathfrak{q}^4+2\mathfrak{q}^5+4\mathfrak{q}^6+4\mathfrak{q}^7+7\mathfrak{q}^8+\cdots,$$
$$C^{[2,1],[5,1]}_{[1,2]}(\mathfrak{q})=\mathfrak{q}^{\frac{7}{5}}+\mathfrak{q}^{\frac{12}{5}}+2\mathfrak{q}^{\frac{17}{5}}+2\mathfrak{q}^{\frac{22}{5}}+4\mathfrak{q}^{\frac{27}{5}}+5\mathfrak{q}^{\frac{32}{5}}+8\mathfrak{q}^{\frac{37}{5}}+\cdots. \tag{C.2}$$

Similarly, the Burge-reduced generating functions for $\boldsymbol{N}=[0,1,1]$ and $\mathfrak{c}=[2,0,0]$, $[0,1,1]$,

$$\widehat{X}^{[2,1],[4,2]}_{(2,1);(1,1)}(\mathfrak{q})=2\mathfrak{q}^{\frac{2}{3}}+10\mathfrak{q}^{\frac{5}{3}}+36\mathfrak{q}^{\frac{8}{3}}+110\mathfrak{q}^{\frac{11}{3}}+300\mathfrak{q}^{\frac{14}{3}}+752\mathfrak{q}^{\frac{17}{3}}+1770\mathfrak{q}^{\frac{20}{3}}+3956\mathfrak{q}^{\frac{23}{3}}+\cdots,$$
$$\widehat{X}^{[2,1],[4,2]}_{(2,1);(0,0)}(\mathfrak{q})=1+5\mathfrak{q}+20\mathfrak{q}^2+65\mathfrak{q}^3+185\mathfrak{q}^4+481\mathfrak{q}^5+1165\mathfrak{q}^6+2665\mathfrak{q}^7+5822\mathfrak{q}^8+\cdots, \tag{C.3}$$

give

$$C^{[2,1],[4,2]}_{[2,1]}(\mathfrak{q})=\mathfrak{q}^{\frac{1}{15}}\left(1+\mathfrak{q}+2\mathfrak{q}^2+3\mathfrak{q}^3+4\mathfrak{q}^4+6\mathfrak{q}^5+9\mathfrak{q}^6+12\mathfrak{q}^7+17\mathfrak{q}^8+\cdots\right),$$
$$C^{[2,1],[4,2]}_{[0,3]}(\mathfrak{q})=\mathfrak{q}^{\frac{5}{3}}+\mathfrak{q}^{\frac{8}{3}}+2\mathfrak{q}^{\frac{11}{3}}+3\mathfrak{q}^{\frac{14}{3}}+4\mathfrak{q}^{\frac{17}{3}}+6\mathfrak{q}^{\frac{20}{3}}+9\mathfrak{q}^{\frac{23}{3}}+\cdots. \tag{C.4}$$

C.2 $(N,n,p)=(2,4,4)$

When $(N,n)=(2,4)$, the $\mathcal{W}^{\text{para}}_{2,4}$ is known as the S_3 parafermion algebra [66] and also discussed in the context of the AGT correspondence in [67, 68]. Here we consider the case of $p=4$, and the $\mathcal{W}^{\rm para}_{2,4}$ $(4,8)$-minimal model has central charge $c\left(\mathcal{W}^{\rm para}_{2,4}\right)=5/4$. The Burge-reduced generating functions for $\boldsymbol{N}=[2,0,0,0]$ and $\mathfrak{c}=[2,0,0,0],[0,1,0,1],[0,0,2,0]$ are obtained as

$$\begin{aligned}\widehat{X}^{[3,1],[7,1]}_{(0,0);(0,0,0)}(\mathfrak{q})&=1+3\mathfrak{q}+19\mathfrak{q}^2+72\mathfrak{q}^3+272\mathfrak{q}^4+877\mathfrak{q}^5+2680\mathfrak{q}^6+7546\mathfrak{q}^7+\cdots,\\ \widehat{X}^{[3,1],[7,1]}_{(0,0);(-1,-1,-1)}(\mathfrak{q})&=\mathfrak{q}^{\frac14}+7\mathfrak{q}^{\frac54}+34\mathfrak{q}^{\frac94}+137\mathfrak{q}^{\frac{13}{4}}+481\mathfrak{q}^{\frac{17}{4}}+1528\mathfrak{q}^{\frac{21}{4}}+4490\mathfrak{q}^{\frac{25}{4}}+\cdots,\\ \widehat{X}^{[3,1],[7,1]}_{(0,0);(-1,-2,-1)}(\mathfrak{q})&=2\mathfrak{q}+14\mathfrak{q}^2+66\mathfrak{q}^3+252\mathfrak{q}^4+852\mathfrak{q}^5+2614\mathfrak{q}^6+7460\mathfrak{q}^7+\cdots,\end{aligned}\tag{C.5}$$

and by the formula (3.30) with $\hat{a}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q})=\mathfrak{q}^{\frac14|\mathfrak{c}|^2-w_{\mathfrak{c}}}\,\hat{c}^{\boldsymbol{N}}_{\mathfrak{c}}(\mathfrak{q})$ we obtain the $\mathcal{W}^{\rm para}_{2,4}$ $(4,8)$-minimal model characters

$$\begin{aligned}C^{[3,1],[7,1]}_{[4,0]}(\mathfrak{q})&=1+\mathfrak{q}^2+\mathfrak{q}^3+3\mathfrak{q}^4+3\mathfrak{q}^5+7\mathfrak{q}^6+8\mathfrak{q}^7+\cdots,\\ C^{[3,1],[7,1]}_{[2,2]}(\mathfrak{q})&=\mathfrak{q}^{\frac43}+\mathfrak{q}^{\frac73}+3\mathfrak{q}^{\frac{10}{3}}+4\mathfrak{q}^{\frac{13}{3}}+8\mathfrak{q}^{\frac{16}{3}}+11\mathfrak{q}^{\frac{19}{3}}+\cdots,\\ C^{[3,1],[7,1]}_{[0,4]}(\mathfrak{q})&=\mathfrak{q}^3+\mathfrak{q}^4+3\mathfrak{q}^5+4\mathfrak{q}^6+7\mathfrak{q}^7+\cdots.\end{aligned}\tag{C.6}$$

The Burge-reduced generating functions for $\boldsymbol{N}=[0,1,0,1]$ and $\mathfrak{c}=[2,0,0,0],[0,1,0,1],[0,0,2,0]$,

$$\begin{aligned}\widehat{X}^{[3,1],[5,3]}_{(3,1);(1,1,1)}(\mathfrak{q})&=3\mathfrak{q}^{\frac34}+21\mathfrak{q}^{\frac74}+105\mathfrak{q}^{\frac{11}{4}}+419\mathfrak{q}^{\frac{15}{4}}+1469\mathfrak{q}^{\frac{19}{4}}+4636\mathfrak{q}^{\frac{23}{4}}+13544\mathfrak{q}^{\frac{27}{4}}+\cdots,\\ \widehat{X}^{[3,1],[5,3]}_{(3,1);(0,0,0)}(\mathfrak{q})&=1+9\mathfrak{q}+50\mathfrak{q}^2+217\mathfrak{q}^3+803\mathfrak{q}^4+2651\mathfrak{q}^5+8019\mathfrak{q}^6+22618\mathfrak{q}^7+\cdots,\\ \widehat{X}^{[3,1],[5,3]}_{(3,1);(0,-1,0)}(\mathfrak{q})&=4\mathfrak{q}^{\frac34}+22\mathfrak{q}^{\frac74}+110\mathfrak{q}^{\frac{11}{4}}+426\mathfrak{q}^{\frac{15}{4}}+1490\mathfrak{q}^{\frac{19}{4}}+4666\mathfrak{q}^{\frac{23}{4}}+13616\mathfrak{q}^{\frac{27}{4}}+\cdots,\end{aligned}\tag{C.7}$$

give

$$\begin{aligned}C^{[3,1],[5,3]}_{[4,0]}(\mathfrak{q})&=\mathfrak{q}^{\frac34}+\mathfrak{q}^{\frac74}+3\mathfrak{q}^{\frac{11}{4}}+4\mathfrak{q}^{\frac{15}{4}}+8\mathfrak{q}^{\frac{19}{4}}+11\mathfrak{q}^{\frac{23}{4}}+19\mathfrak{q}^{\frac{27}{4}}+\cdots,\\ C^{[3,1],[5,3]}_{[2,2]}(\mathfrak{q})&=\mathfrak{q}^{\frac{1}{12}}\left(1+\mathfrak{q}+3\mathfrak{q}^2+5\mathfrak{q}^3+10\mathfrak{q}^4+15\mathfrak{q}^5+26\mathfrak{q}^6+\cdots\right),\\ C^{[3,1],[5,3]}_{[0,4]}(\mathfrak{q})&=\mathfrak{q}^{\frac74}+2\mathfrak{q}^{\frac{11}{4}}+3\mathfrak{q}^{\frac{15}{4}}+6\mathfrak{q}^{\frac{19}{4}}+10\mathfrak{q}^{\frac{23}{4}}+16\mathfrak{q}^{\frac{27}{4}}+\cdots.\end{aligned}\tag{C.8}$$

The Burge-reduced generating functions for $\boldsymbol{N}=[1,1,0,0]$ and $\mathfrak{c}=[1,1,0,0],[0,0,1,1]$,

$$\begin{aligned}\widehat{X}^{[3,1],[6,2]}_{(1,0);(0,0,0)}(\mathfrak{q})&=1+7\mathfrak{q}+37\mathfrak{q}^2+157\mathfrak{q}^3+575\mathfrak{q}^4+1889\mathfrak{q}^5+5704\mathfrak{q}^6+16081\mathfrak{q}^7+\cdots,\\ \widehat{X}^{[3,1],[6,2]}_{(1,0);(0,-1,-1)}(\mathfrak{q})&=2\mathfrak{q}^{\frac12}+15\mathfrak{q}^{\frac32}+74\mathfrak{q}^{\frac52}+297\mathfrak{q}^{\frac72}+1039\mathfrak{q}^{\frac92}+3284\mathfrak{q}^{\frac{11}{2}}+9598\mathfrak{q}^{\frac{13}{2}}+\cdots,\end{aligned}\tag{C.9}$$

give

$$\begin{aligned}C^{[3,1],[6,2]}_{[3,1]}(\mathfrak{q})&=\mathfrak{q}^{\frac{1}{16}}\left(1+\mathfrak{q}+2\mathfrak{q}^2+4\mathfrak{q}^3+7\mathfrak{q}^4+11\mathfrak{q}^5+18\mathfrak{q}^6+\cdots\right),\\ C^{[3,1],[6,2]}_{[1,3]}(\mathfrak{q})&=\mathfrak{q}^{\frac{25}{16}}+2\mathfrak{q}^{\frac{41}{16}}+4\mathfrak{q}^{\frac{57}{16}}+7\mathfrak{q}^{\frac{73}{16}}+12\mathfrak{q}^{\frac{89}{16}}+19\mathfrak{q}^{\frac{105}{16}}+\cdots.\end{aligned}\tag{C.10}$$

JHEP06(2020)112

C.3 $(N, n, p) = (3, 2, 4)$

Consider the case of $(N, n) = (3, 2)$ and $p = 4$. The $\mathcal{W}^{\text{para}}_{3,2}$ $(4, 6)$-minimal model for $p = 4$ has central charge $c\left(\mathcal{W}^{\text{para}}_{3,2}\right) = 6/5$. The Burge-reduced generating functions for $\boldsymbol{N} = [3, 0]$ and $\mathfrak{c} = [3, 0], [1, 2]$ are obtained as

$$\begin{aligned}
\widehat{X}^{[2,1,1],[2,3,1]}_{(0,0,0);(0)}(\mathfrak{q}) &= 1 + 3\mathfrak{q} + 11\mathfrak{q}^2 + 30\mathfrak{q}^3 + 77\mathfrak{q}^4 + 176\mathfrak{q}^5 + 385\mathfrak{q}^6 + 792\mathfrak{q}^7 + 1575\mathfrak{q}^8 + \cdots, \\
\widehat{X}^{[2,1,1],[2,3,1]}_{(0,0,0);(-1)}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac{1}{2}} + 7\mathfrak{q}^{\frac{3}{2}} + 22\mathfrak{q}^{\frac{5}{2}} + 56\mathfrak{q}^{\frac{7}{2}} + 135\mathfrak{q}^{\frac{9}{2}} + 297\mathfrak{q}^{\frac{11}{2}} + 627\mathfrak{q}^{\frac{13}{2}} + 1255\mathfrak{q}^{\frac{15}{2}} + \cdots,
\end{aligned} \tag{C.11}$$

and using the $\widehat{\mathfrak{sl}}(2)$ string functions (A.5) of level-3 with $\hat{a}^{[N_0,N_1]}_{[\mathfrak{c}_0,\mathfrak{c}_1]}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{12}\mathfrak{c}_1(\mathfrak{c}_1-3)}\,\hat{c}^{[N_0,N_1]}_{[\mathfrak{c}_0,\mathfrak{c}_1]}(\mathfrak{q})$, from the formula (3.30) we obtain the $\mathcal{W}^{\text{para}}_{3,2}$ $(4, 6)$-minimal model characters

$$\begin{aligned}
C^{[2,1,1],[2,3,1]}_{[0,2,0]}(\mathfrak{q}) &= 1 + \mathfrak{q} + 2\mathfrak{q}^2 + 3\mathfrak{q}^3 + 6\mathfrak{q}^4 + 9\mathfrak{q}^5 + 15\mathfrak{q}^6 + 22\mathfrak{q}^7 + 35\mathfrak{q}^8 + \cdots, \\
C^{[2,1,1],[2,3,1]}_{[1,0,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{3}{5}} + 2\mathfrak{q}^{\frac{8}{5}} + 4\mathfrak{q}^{\frac{13}{5}} + 7\mathfrak{q}^{\frac{18}{5}} + 12\mathfrak{q}^{\frac{23}{5}} + 19\mathfrak{q}^{\frac{28}{5}} + 31\mathfrak{q}^{\frac{33}{5}} + 46\mathfrak{q}^{\frac{38}{5}} + \cdots.
\end{aligned} \tag{C.12}$$

The Burge-reduced generating functions for $\boldsymbol{N} = [1, 2]$ and $\mathfrak{c} = [3, 0], [1, 2]$,

$$\begin{aligned}
\widehat{X}^{[2,1,1],[3,1,2]}_{(1,1,0);(1)}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}} + 5\mathfrak{q}^{\frac{3}{2}} + 15\mathfrak{q}^{\frac{5}{2}} + 42\mathfrak{q}^{\frac{7}{2}} + 101\mathfrak{q}^{\frac{9}{2}} + 231\mathfrak{q}^{\frac{11}{2}} + 490\mathfrak{q}^{\frac{13}{2}} + 1002\mathfrak{q}^{\frac{15}{2}} + \cdots, \\
\widehat{X}^{[2,1,1],[3,1,2]}_{(1,1,0);(0)}(\mathfrak{q}) &= 1 + 3\mathfrak{q} + 11\mathfrak{q}^2 + 30\mathfrak{q}^3 + 77\mathfrak{q}^4 + 176\mathfrak{q}^5 + 385\mathfrak{q}^6 + 792\mathfrak{q}^7 + 1575\mathfrak{q}^8 + \cdots,
\end{aligned} \tag{C.13}$$

give

$$\begin{aligned}
C^{[2,1,1],[3,1,2]}_{[0,2,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{3}{2}} + 2\mathfrak{q}^{\frac{5}{2}} + 4\mathfrak{q}^{\frac{7}{2}} + 6\mathfrak{q}^{\frac{9}{2}} + 11\mathfrak{q}^{\frac{11}{2}} + 16\mathfrak{q}^{\frac{13}{2}} + 26\mathfrak{q}^{\frac{15}{2}} + \cdots, \\
C^{[2,1,1],[3,1,2]}_{[1,0,1]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{10}}\left(1 + \mathfrak{q} + 3\mathfrak{q}^2 + 5\mathfrak{q}^3 + 9\mathfrak{q}^4 + 14\mathfrak{q}^5 + 23\mathfrak{q}^6 + 35\mathfrak{q}^7 + \cdots\right).
\end{aligned} \tag{C.14}$$

The Burge-reduced generating functions for $\boldsymbol{N} = [2, 1]$ and $\mathfrak{c} = [2, 1], [0, 3]$,

$$\begin{aligned}
\widehat{X}^{[1,1,2],[2,2,2]}_{(1,0,0);(0)}(\mathfrak{q}) &= 1 + 5\mathfrak{q} + 17\mathfrak{q}^2 + 48\mathfrak{q}^3 + 120\mathfrak{q}^4 + 277\mathfrak{q}^5 + 600\mathfrak{q}^6 + 1237\mathfrak{q}^7 + 2448\mathfrak{q}^8 + \cdots, \\
\widehat{X}^{[1,1,2],[2,2,2]}_{(1,0,0);(-1)}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac{1}{2}} + 8\mathfrak{q}^{\frac{3}{2}} + 24\mathfrak{q}^{\frac{5}{2}} + 66\mathfrak{q}^{\frac{7}{2}} + 160\mathfrak{q}^{\frac{9}{2}} + 360\mathfrak{q}^{\frac{11}{2}} + 768\mathfrak{q}^{\frac{13}{2}} + 1560\mathfrak{q}^{\frac{15}{2}} + \cdots,
\end{aligned} \tag{C.15}$$

give

$$\begin{aligned}
C^{[1,1,2],[2,2,2]}_{[1,1,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{10}}\left(1 + 2\mathfrak{q} + 4\mathfrak{q}^2 + 8\mathfrak{q}^3 + 13\mathfrak{q}^4 + 22\mathfrak{q}^5 + 35\mathfrak{q}^6 + 54\mathfrak{q}^7 + \cdots\right), \\
C^{[1,1,2],[2,2,2]}_{[0,0,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}} + 2\mathfrak{q}^{\frac{3}{2}} + 3\mathfrak{q}^{\frac{5}{2}} + 6\mathfrak{q}^{\frac{7}{2}} + 10\mathfrak{q}^{\frac{9}{2}} + 16\mathfrak{q}^{\frac{11}{2}} + 26\mathfrak{q}^{\frac{13}{2}} + 40\mathfrak{q}^{\frac{15}{2}} + \cdots.
\end{aligned} \tag{C.16}$$

C.4 $(N, n) = (4, 2, 5)$

Consider the case of $(N, n) = (4, 2)$ for $p = 5$. The $\mathcal{W}^{\text{para}}_{4,2}$ $(5, 7)$-minimal model for $p = 5$ has central charge $c\left(\mathcal{W}^{\text{para}}_{4,2}\right) = 11/7$. The Burge-reduced generating functions for $\boldsymbol{N} = [4, 0]$

JHEP06(2020)112

and $\mathfrak{c} = [4,0], [2,2], [0,4]$ are obtained as

$$\begin{aligned}
\widehat{X}^{[2,1,1,1],[2,3,1,1]}_{(0,0,0,0);(0)}(\mathfrak{q}) &= 1 + 3\mathfrak{q} + 11\mathfrak{q}^2 + 34\mathfrak{q}^3 + 93\mathfrak{q}^4 + 234\mathfrak{q}^5 + 552\mathfrak{q}^6 + \cdots , \\
\widehat{X}^{[2,1,1,1],[2,3,1,1]}_{(0,0,0,0);(-1)}(\mathfrak{q}) &= 2\mathfrak{q}^{\frac{1}{2}} + 7\mathfrak{q}^{\frac{3}{2}} + 25\mathfrak{q}^{\frac{5}{2}} + 70\mathfrak{q}^{\frac{7}{2}} + 185\mathfrak{q}^{\frac{9}{2}} + 441\mathfrak{q}^{\frac{11}{2}} + \cdots , \\
\widehat{X}^{[2,1,1,1],[2,3,1,1]}_{(0,0,0,0);(-2)}(\mathfrak{q}) &= 2\mathfrak{q} + 9\mathfrak{q}^2 + 31\mathfrak{q}^3 + 88\mathfrak{q}^4 + 227\mathfrak{q}^5 + 541\mathfrak{q}^6 + \cdots ,
\end{aligned} \tag{C.17}$$

and using the $\widehat{\mathfrak{sl}}(2)$ string functions (A.5) of level-4 with $\hat{a}^{[N_0,N_1]}_{[\mathfrak{c}_0,\mathfrak{c}_1]}(\mathfrak{q}) = \mathfrak{q}^{\frac{1}{16}\mathfrak{c}_1(\mathfrak{c}_1-4)}\,\hat{c}^{[N_0,N_1]}_{[\mathfrak{c}_0,\mathfrak{c}_1]}(\mathfrak{q})$, from the formula (3.30) we obtain the $\mathcal{W}^{\rm para}_{4,2}$ (5,7)-minimal model characters

$$\begin{aligned}
C^{[2,1,1,1],[2,3,1,1]}_{[0,2,0,0]}(\mathfrak{q}) &= 1 + \mathfrak{q} + 2\mathfrak{q}^2 + 4\mathfrak{q}^3 + 7\mathfrak{q}^4 + 12\mathfrak{q}^5 + 21\mathfrak{q}^6 + \cdots , \\
C^{[2,1,1,1],[2,3,1,1]}_{[1,0,1,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{2}{3}} + 2\mathfrak{q}^{\frac{5}{3}} + 5\mathfrak{q}^{\frac{8}{3}} + 9\mathfrak{q}^{\frac{11}{3}} + 18\mathfrak{q}^{\frac{14}{3}} + 30\mathfrak{q}^{\frac{17}{3}} + \cdots , \\
C^{[2,1,1,1],[2,3,1,1]}_{[0,0,0,2]}(\mathfrak{q}) &= \mathfrak{q}^2 + 2\mathfrak{q}^3 + 5\mathfrak{q}^4 + 9\mathfrak{q}^5 + 17\mathfrak{q}^6 + \cdots .
\end{aligned} \tag{C.18}$$

The Burge-reduced generating functions for $\boldsymbol{N} = [2,2]$ and $\mathfrak{c} = [4,0], [2,2], [0,4]$,

$$\begin{aligned}
\widehat{X}^{[2,1,1,1],[3,1,2,1]}_{(1,1,0,0);(1)}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}} + 5\mathfrak{q}^{\frac{3}{2}} + 18\mathfrak{q}^{\frac{5}{2}} + 55\mathfrak{q}^{\frac{7}{2}} + 149\mathfrak{q}^{\frac{9}{2}} + 371\mathfrak{q}^{\frac{11}{2}} + \cdots , \\
\widehat{X}^{[2,1,1,1],[3,1,2,1]}_{(1,1,0,0);(0)}(\mathfrak{q}) &= 1 + 3\mathfrak{q} + 14\mathfrak{q}^2 + 41\mathfrak{q}^3 + 119\mathfrak{q}^4 + 295\mathfrak{q}^5 + 706\mathfrak{q}^6 + \cdots , \\
\widehat{X}^{[2,1,1,1],[3,1,2,1]}_{(1,1,0,0);(-1)}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{2}} + 5\mathfrak{q}^{\frac{3}{2}} + 18\mathfrak{q}^{\frac{5}{2}} + 55\mathfrak{q}^{\frac{7}{2}} + 149\mathfrak{q}^{\frac{9}{2}} + 371\mathfrak{q}^{\frac{11}{2}} + \cdots ,
\end{aligned} \tag{C.19}$$

give

$$\begin{aligned}
C^{[2,1,1,1],[3,1,2,1]}_{[0,2,0,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{3}{2}} + 2\mathfrak{q}^{\frac{5}{2}} + 5\mathfrak{q}^{\frac{7}{2}} + 8\mathfrak{q}^{\frac{9}{2}} + 16\mathfrak{q}^{\frac{11}{2}} + \cdots , \\
C^{[2,1,1,1],[3,1,2,1]}_{[1,0,1,0]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{1}{6}}\left(1 + \mathfrak{q} + 4\mathfrak{q}^2 + 7\mathfrak{q}^3 + 15\mathfrak{q}^4 + 25\mathfrak{q}^5 + \cdots\right), \\
C^{[2,1,1,1],[3,1,2,1]}_{[0,0,0,2]}(\mathfrak{q}) &= \mathfrak{q}^{\frac{3}{2}} + 2\mathfrak{q}^{\frac{5}{2}} + 5\mathfrak{q}^{\frac{7}{2}} + 8\mathfrak{q}^{\frac{9}{2}} + 16\mathfrak{q}^{\frac{11}{2}} + \cdots .
\end{aligned} \tag{C.20}$$

References

[1] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. **91** (2010) 167 [arXiv:0906.3219] [INSPIRE].

[2] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003) 831 [hep-th/0206161] [INSPIRE].

[3] V. Belavin and B. Feigin, *Super Liouville conformal blocks from $\mathcal{N} = 2$ SU(2) quiver gauge theories*, JHEP **07** (2011) 079 [arXiv:1105.5800] [INSPIRE].

[4] T. Nishioka and Y. Tachikawa, *Central charges of para-Liouville and Toda theories from M5-branes*, Phys. Rev. **D 84** (2011) 046009 [arXiv:1106.1172] [INSPIRE].

[5] A.A. Belavin, M.A. Bershtein, B.L. Feigin, A.V. Litvinov and G.M. Tarnopolsky, *Instanton moduli spaces and bases in coset conformal field theory*, Commun. Math. Phys. **319** (2013) 269 [arXiv:1111.2803] [INSPIRE].

[6] G. Bonelli, K. Maruyoshi, A. Tanzini and F. Yagi, $\mathcal{N}=2$ *gauge theories on toric singularities, blow-up formulae and W-algebrae*, JHEP **01** (2013) 014 [arXiv:1208.0790] [INSPIRE].

[7] V.A. Fateev and A.B. Zamolodchikov, *Parafermionic Currents in the Two-Dimensional Conformal Quantum Field Theory and Selfdual Critical Points in Z(n) Invariant Statistical Systems*, *Sov. Phys. JETP* **62** (1985) 215 [INSPIRE].

[8] D. Gepner, *New Conformal Field Theories Associated with Lie Algebras and their Partition Functions*, *Nucl. Phys.* **B 290** (1987) 10 [INSPIRE].

[9] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, *Coset Construction for Extended Virasoro Algebras*, *Nucl. Phys.* **B 304** (1988) 371 [INSPIRE].

[10] P. Christe and F. Ravanini, $G_N \otimes G_L/G_{N+L}$ *Conformal Field Theories and Their Modular Invariant Partition Functions*, *Int. J. Mod. Phys.* **A 4** (1989) 897 [INSPIRE].

[11] P. Bowcock and P. Goddard, *Coset Constructions and Extended Conformal Algebras*, *Nucl. Phys.* **B 305** (1988) 685 [INSPIRE].

[12] A.B. Zamolodchikov, *Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory*, *Theor. Math. Phys.* **65** (1985) 1205 [INSPIRE].

[13] V.A. Fateev and A.B. Zamolodchikov, *Conformal Quantum Field Theory Models in Two-Dimensions Having* Z_3 *Symmetry*, *Nucl. Phys.* **B 280** (1987) 644 [INSPIRE].

[14] V.A. Fateev and S.L. Lukyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with* Z_n *Symmetry*, *Int. J. Mod. Phys.* **A 3** (1988) 507 [INSPIRE].

[15] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[16] T. Okuda and V. Pestun, *On the instantons and the hypermultiplet mass of* $N=2^*$ *super Yang-Mills on* S^4, JHEP **03** (2012) 017 [arXiv:1004.1222] [INSPIRE].

[17] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, *Duke Math. J.* **76** (1994) 365 [INSPIRE].

[18] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, *Duke Math. J.* **91** (1998) 515.

[19] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074] [INSPIRE].

[20] R. Dijkgraaf, L. Hollands, P. Sulkowski and C. Vafa, *Supersymmetric gauge theories, intersecting branes and free fermions*, JHEP **02** (2008) 106 [arXiv:0709.4446] [INSPIRE].

[21] M. Bershtein and O. Foda, *AGT, Burge pairs and minimal models*, JHEP **06** (2014) 177 [arXiv:1404.7075] [INSPIRE].

[22] K.B. Alkalaev and V.A. Belavin, *Conformal blocks of* $\mathcal{W}_N$ *minimal models and AGT correspondence*, JHEP **07** (2014) 024 [arXiv:1404.7094] [INSPIRE].

[23] V. Belavin, O. Foda and R. Santachiara, *AGT, N-Burge partitions and* $\mathcal{W}_N$ *minimal models*, JHEP **10** (2015) 073 [arXiv:1507.03540] [INSPIRE].

[24] R. Santachiara and A. Tanzini, *Moore-Read Fractional Quantum Hall wavefunctions and* SU(2) *quiver gauge theories*, *Phys. Rev.* **D 82** (2010) 126006 [arXiv:1002.5017] [INSPIRE].

[25] B. Estienne, V. Pasquier, R. Santachiara and D. Serban, *Conformal blocks in Virasoro and W theories: Duality and the Calogero-Sutherland model*, *Nucl. Phys.* **B 860** (2012) 377 [arXiv:1110.1101] [INSPIRE].

[26] O. Foda, N. Macleod, M. Manabe and T. Welsh, $\widehat{\mathfrak{sl}}(n)_N$ *WZW conformal blocks from* $\mathrm{SU}(N)$ *instanton partition functions on* $\mathbb{C}^2/\mathbb{Z}_n$, *Nucl. Phys.* **B 956** (2020) 115038 [arXiv:1912.04407] [INSPIRE].

[27] V. Kac, *Infinite Dimensional Lie algebras*, 3rd edition, Cambridge University Press, Cambridge, U.K., (1990).

[28] P. Bouwknegt, J.G. McCarthy and K. Pilch, *On the freefield resolutions for coset conformal field theories*, *Nucl. Phys.* **B 352** (1991) 139 [INSPIRE].

[29] E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Paths, Maya Diagrams and representations of* $\widehat{\mathfrak{sl}}(r,\mathrm{C})$, *Adv. Stud. Pure Math.* **19** (1989) 149.

[30] M. Jimbo and T. Miwa, *On a Duality of Branching Rules for Affine Lie Algebras*, Algebraic Groups and Related Topics, 17–65, Mathematical Society of Japan, Tokyo, Japan, (1985).

[31] K. Hasegawa, *Spin module versions of Wely's reciprocity theorem for classical Kac-Moody Lie algebras — An application to branching rule duality*, *Publ. Res. Inst. Math. Sci.* **25** (1989) 741.

[32] I.B. Frenkel, *Representations of affine lie algebras, hecke modular forms and Korteweg-De Vries type equations*, In D. Winter ed., *Lie Algebras and Related Topics. Lecture Notes in Mathematics*, vol. 933, Springer, Berlin Heidelberg, (1982).

[33] S.G. Naculich and H.J. Schnitzer, *Duality Between* $\mathrm{SU}(N)_k$ *and* $\mathrm{SU}(k)_N$ *WZW Models*, *Nucl. Phys.* **B 347** (1990) 687 [INSPIRE].

[34] T. Nakanishi and A. Tsuchiya, *Level rank duality of WZW models in conformal field theory*, *Commun. Math. Phys.* **144** (1992) 351 [INSPIRE].

[35] P.B. Kronheimer and H. Nakajima, *Yang-Mills instantons on ALE gravitational instantons*, *Math. Ann.* **288** (1990) 263.

[36] F. Fucito, J.F. Morales and R. Poghossian, *Multi instanton calculus on ALE spaces*, *Nucl. Phys.* **B 703** (2004) 518 [hep-th/0406243] [INSPIRE].

[37] W.H. Burge, *Restricted partition pairs*, *J. Combin. Theor.* **A 63** (1993) 210.

[38] O. Foda, K.S.M. Lee and T.A. Welsh, *A Burge tree of Virasoro type polynomial identities*, *Int. J. Mod. Phys.* **A 13** (1998) 4967 [q-alg/9710025] [INSPIRE].

[39] I.M. Gessel and C. Krattenthaler, *Cylindric Partitions*, *Trans. Am. Math. Soc.* **349** (1997) 429.

[40] B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum continuous* gl_∞*: Semi-infinite construction of representations*, *Kyoto J. Math.* **51** (2011) 337 [arXiv:1002.3100].

[41] B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum continuous* gl_∞*: Tensor products of Fock modules and* W_n *characters*, arXiv:1002.3113 [INSPIRE].

[42] V.G. Kac and D.H. Peterson, *Infinite dimensional Lie algebras, theta functions and modular forms*, *Adv. Math.* **53** (1984) 125 [INSPIRE].

[43] Z. Kakushadze and S.H.H. Tye, *Kac and new determinants for fractional superconformal algebras*, *Phys. Rev.* **D 49** (1994) 4122 [hep-th/9310160] [INSPIRE].

[44] S. Mizoguchi, *The Structure of Representation of the* $W_{(3)}$ *Algebra*, *Int. J. Mod. Phys.* **A 6** (1991) 133 [INSPIRE].

[45] E. Frenkel, V. Kac and M. Wakimoto, *Characters and fusion rules for W algebras via quantized Drinfeld-Sokolov reductions*, Commun. Math. Phys. **147** (1992) 295 [INSPIRE].

[46] D. Kastor, E.J. Martinec and Z.-a. Qiu, *Current Algebra and Conformal Discrete Series*, Phys. Lett. **B 200** (1988) 434 [INSPIRE].

[47] J. Bagger, D. Nemeschansky and S. Yankielowicz, *Virasoro Algebras with Central Charge $c > 1$*, Phys. Rev. Lett. **60** (1988) 389 [INSPIRE].

[48] F. Ravanini, *An Infinite Class of New Conformal Field Theories With Extended Algebras*, Mod. Phys. Lett. **3A** (1988) 397 [INSPIRE].

[49] S Fujii and S Minabe, *A combinatorial study on quiver varieties*, SIGMA **13** (2017) 052 [math/0510455].

[50] M.N. Alfimov, A.A. Belavin and G.M. Tarnopolsky, *Coset conformal field theory and instanton counting on C^2/Z_p*, JHEP **08** (2013) 134 [arXiv:1306.3938] [INSPIRE].

[51] O. Foda and T.A. Welsh, *Cylindric partitions, $\mathcal{W}_r$ characters and the Andrews-Gordon-Bressoud identities*, J. Phys. **A 49** (2016) 164004 [arXiv:1510.02213] [INSPIRE].

[52] P. Goddard, A. Kent and D.I. Olive, *Unitary Representations of the Virasoro and Supervirasoro Algebras*, Commun. Math. Phys. **103** (1986) 105 [INSPIRE].

[53] G. Bonelli, K. Maruyoshi and A. Tanzini, *Instantons on ALE spaces and Super Liouville Conformal Field Theories*, JHEP **08** (2011) 056 [arXiv:1106.2505] [INSPIRE].

[54] A. Belavin, V. Belavin and M. Bershtein, *Instantons and 2d Superconformal field theory*, JHEP **09** (2011) 117 [arXiv:1106.4001] [INSPIRE].

[55] G. Bonelli, K. Maruyoshi and A. Tanzini, *Gauge Theories on ALE Space and Super Liouville Correlation Functions*, Lett. Math. Phys. **101** (2012) 103 [arXiv:1107.4609] [INSPIRE].

[56] Y. Ito, *Ramond sector of super Liouville theory from instantons on an ALE space*, Nucl. Phys. **B 861** (2012) 387 [arXiv:1110.2176] [INSPIRE].

[57] A. Belavin and B. Mukhametzhanov, *$N = 1$ superconformal blocks with Ramond fields from AGT correspondence*, JHEP **01** (2013) 178 [arXiv:1210.7454] [INSPIRE].

[58] T. Inami, Y. Matsuo and I. Yamanaka, *Extended Conformal Algebras With $N = 1$ Supersymmetry*, Phys. Lett. **B 215** (1988) 701 [INSPIRE].

[59] A. Bilal, *A note on super W-algebras*, Phys. Lett. **B 238** (1990) 239 [INSPIRE].

[60] K. Hornfeck and É. Ragoucy, *A Coset Construction for the Super W_3 Algebra*, Nucl. Phys. **B 340** (1990) 225 [INSPIRE].

[61] C.-h. Ahn, K. Schoutens and A. Sevrin, *The full structure of the super W_3 algebra*, Int. J. Mod. Phys. **A 6** (1991) 3467 [INSPIRE].

[62] K. Schoutens and A. Sevrin, *Minimal super-W_N algebras in coset conformal field theories*, Phys. Lett. **B 258** (1991) 134 [INSPIRE].

[63] K. Hornfeck, *The Minimal supersymmetric extension of WA_{n-1}*, Phys. Lett. **B 275** (1992) 355 [INSPIRE].

[64] V.G. Knizhnik and A.B. Zamolodchikov, *Current Algebra and Wess-Zumino Model in Two-Dimensions*, Nucl. Phys. **B 247** (1984) 83 [INSPIRE].

[65] J. Distler and Z.-a. Qiu, *BRS Cohomology and a Feigin-fuchs Representation of Kac-Moody and Parafermionic Theories*, Nucl. Phys. **B 336** (1990) 533 [INSPIRE].

JHEP06(2020)112

[66] V.A. Fateev and A.B. Zamolodchikov, *Representations of the Algebra of 'Parafermion Currents' of Spin 4/3 in Two-dimensional Conformal Field Theory. Minimal Models and the Tricritical Potts Z_3 Model*, *Theor. Math. Phys.* **71** (1987) 451 [INSPIRE].

[67] N. Wyllard, *Coset conformal blocks and $\mathcal{N} = 2$ gauge theories*, arXiv:1109.4264 [INSPIRE].

[68] M.N. Alfimov and G.M. Tarnopolsky, *Parafermionic Liouville field theory and instantons on ALE spaces*, *JHEP* **02** (2012) 036 [arXiv:1110.5628] [INSPIRE].

JHEP06(2020)112