On Discrete-Time Risk Models with Premiums Adjusted According to Claims

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Abstract

This thesis studies the discrete-time risk models with premiums adjusted according to claims experience. These proposed models are inspired by the well-known principle in the non-life insurance industry, the so-called ‘bonus-malus system’. The main goal of this thesis is to evaluate the risk of ruin for the insurers who implement the bonus-malus system with two different premiums correction strategies: by aggregate claims or by claim frequency. In addition, this thesis is contributing to the literature of risk models with claim correlated premiums on three aspects. The first contribution lies in an extension of existing bonus-malus risk models by introducing an external Markovian environment. To be more specific, the distribution of periodic claims are assumed to be governed by the external Markov process and the premium adjustment rules depend on the external environment as well. We find that in this context the initial premium level has a high impact on the insolvency risk when the initial surplus level is low, and the initial external environment state also influences the ruin probabilities under the proposed premiums adjustment rules. The second contribution is grounded on incorporating the delayed settlements of certain insurance claims into the bonus-malus risk models. Assuming two types of claims, the so-called main claims and by-claims, the by-claims are induced by the main claims. The settlements of by-claims are assumed to be either at the end of the reporting period or delayed to next policy period with a certain probability. According to our main findings, both the probability of delaying by-claim settlements and the correlation between the main claims and their associated by-claims differentiate the ruin probabilities. Moreover, in general, the ruin probabilities under the principle of adjusting premiums by settled claims are higher than those under the reported claims principle. Within the last part of this thesis, a surplus-related premium correction framework is proposed. In other words, the premium adjustment rules are assumed to vary according to the current surplus level of the insurance company. Further, the Parisian type of ruin is also considered, where the premium adjustment rules are different between the positive and negative surplus levels. We found that when the initial premium level is low, the ruin probabilities under the surplus-related premium adjustment rules are lower than those with fixed premium adjustment rules. On the contrary, for high initial premium levels, the ruin probabilities under these two premium correction scenarios are comparable.
Preface

This thesis was completed under the supervision of Associate Professor Xueyuan Wu and Professor Shuanming Li in the Centre for Actuarial Studies at the University of Melbourne and supported by the financial support from the Faculty of Commerce and Accountancy, Chulalongkorn University and a Faculty of Business and Economics Doctoral Program Scholarship. Chapter 2 to 4 contain the original research of this thesis, except as otherwise noted.

Chapter 2 is based on the paper ‘Discrete-Time Risk Models with Claim Correlated Premiums in a Markovian Environment’, co-authored by Xueyuan Wu. The research and writing was done by Dhiti Osatakul, with supervision, proofreading and editing by Xueyuan Wu.

Chapter 3 is based on the working paper ‘Discrete-time risk models with time-delayed claims and varying premiums’, co-authored by Xueyuan Wu and Shuanming Li. The research and writing was done by Dhiti Osatakul, with supervision, proofreading and editing by Xueyuan Wu and Shuanming Li.

None of the work appearing here has been submitted for any other qualifications, nor was it carried out prior to Ph.D. candidature enrollment.
Declaration

This is to certify that:

1. The thesis comprises only my original work towards the Ph.D. except where indicated in the Preface,

2. Due acknowledgement has been made in the text to all other material used,

3. The thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Dhiti Osatakul
June 2021
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Chapter 1

Introduction

1.1 Motivations and Contributions

The commonly adopted bonus-malus system in the general insurance industry is based on a principal that insurance premiums can be adjusted based on the historical claims record of individual policyholders. To be specific, the policyholders, who make no claim or small claims in the latest policy year, will be offered a premium discount (also called ‘bonus’) on renewal. On the other hand, policyholders who make more claims than the given thresholds in the current policy year may need to pay higher premiums (also called ‘malus’) if they decide to renew their policies. The bonus-malus system plays an important role in the insurance industry, in particular in motor vehicle insurance sector, because this system defines risk specific premium levels that help to sustain the total premium pool in covering all motor insurance claims of the given insurance portfolio. The discount in premium acts as an incentive to retain low-risk policyholders and to attract new customers; on the other hand, the malus component prevents high-risk policyholders from taking advantage of the low-risk policyholders by receiving disproportionate insurance benefits. The purpose of the bonus-malus system is to create heterogeneity by separating low-risk policyholders and high-risk policyholders. In other words, it acts as a posteriori classifier that differentiates the policyholders according to their real claim experience besides the priori rating factors. In addition, in terms of commercial purposes, the bonus-malus system is widely adopted in pricing by motor vehicle insurance companies to enhance their competitiveness in the insurance
market. Some references regarding the bonus-malus system can be found in Lemaire (1995) and Denuit et al. (2007).

The bonus-malus system has been studied by many researchers on a variety of topics, including ultimate-time ruin probabilities and finite-time ruin probabilities under a bonus-malus system, which is in the context of risk models with claim-dependent premiums (more details see Section 1.2.2). However, all of the current literature regarding the risk models under a bonus-malus system do not effectively address some challenges from real-world insurance practice that insurers, who apply the bonus-malus system in their normal business, are confronted with constantly.

The first type of real-life challenges arise from changes in an external environment which could affect the insurance business on several aspects, such as claim frequency and claim severity, investment returns, reserving and more. The changes in external environments often account for unsuitable transition rule for premium adjustments. The term ‘external environment’ in this context means the factors that the insurers are not able to control but have an impact on the performance of their insurance business. It can include economic conditions, weather conditions, government policies and regulations, competitiveness of competitors and so on. A good example for the change in the external environment can be referred to the situation that the claim costs increase due to inflation or decrease resulting from deflation. Another example is the higher claim frequency observed in wet seasons vs lower claim frequency resulting from the drought. In the case of significant changes in external environment status, if the insurers keep using the same transition rules for premium adjustments, then they might face significant changes in basis assumptions that are likely to cause serious overestimation or underestimation in long-term expected premium income. Moreover, the consequential changes in the claims distribution under varying external environments can also distort the premium adjustment rules under the bonus-malus system. Specifically, in an extreme environment, higher claims costs and/or claim frequencies would increase the chance of the policyholders to be penalised by having their premiums moving towards the malus end that would upset the policyholders and would lead to high drop-outs. On the contrary, in a non-severe environment, the policyholders may receive the bonuses easily due to the lower claims costs and/or claim frequencies, which could bring extra risks to insurers. To address the issues, the renewal premiums should be determined not
only by individual policyholder’s claims experience but also by the current external environment status.

The second type of challenges are related to the nature of general insurance business, where general insurers often need to deal with the issue of delayed claim settlements. There are many factors that prevent the insurers from settling claims promptly after the claims are lodged. One of the main causes of settlement delay is the investigation time insurers spend on verifying and assessing the reported claims. A typical example is casualty insurance. According to the usual claiming process of casualty insurance policies, after the policyholders notify the insurance company of the incident that causes losses or damages to their property, the surveyors/loss assessors will assess the damages caused by the reported incident to evaluate the repair/replacement costs. This process may also involve the police department and some third parties so may require a lot of time, which can result in delays in claim settlements. Another cause of delayed claim settlements is delayed claim reporting. This issue occurs when the policyholders report previously incurred insurable losses to their insurer after their insurance policies have expired. In an insurance terminology, this type of claims is known as incurred-but-not-reported claims or simply IBNR claims. As the name says, these claims are not reported in a timely manner which certainly delay the whole process of claims management. In terms of the solvency risk, the delayed claims have significant impact on the loss modeling by the actuaries, since the timing of settled claims are inconsistent with the incident occurrence times. It may lead to underestimation/overestimation of claim experience in the time period under consideration which will reduce the effectiveness of the insolvency measures developed by usual loss models.

Lastly, government policies and regulations are another challenge that the insurers might encounter with when they apply the bonus-malus system in their insurance business. In some circumstances, the insurers are not able to continue implementing the same transition rules for premium adjustments. Especially when an insurance company has dangerously low level of capital and assets, the regulators are likely to grant a sanction to the insurance company and require the insurer to apply very conservative premium rules to decrease the insolvency risk. Moreover, the competitions in the insurance markets may also impact the implementation of the bonus-malus premium adjustment rules and lead to modifications in the rules. To be more specific, when the insurer’s surplus level is sufficiently high, the insurer can
take more aggressive premium policies to enhance the company’s competitiveness in the insurance market.

Because the challenges mentioned above are not considered in the current literature, the main goal of this thesis is to extend the risk models with claim-dependent premiums by considering these issues. Although the continuous-time risk models gain more interests from the researchers, compared with the discrete-time setting, this thesis only focuses on discrete-time risk models because of its practical advantages. Specifically, the results of the discrete-time setting are more programmable which makes it easier to deliver numerical studies compared with the continuous-time setting. Additionally, the discrete-time risk models are also more realistic on certain aspects due to the fact that an insurance company’s financial status is normally reviewed periodically, e.g. weekly, monthly or annually according to certain accounting procedure.

The structure of this thesis with the contributions is shown as follows. Chapter 2 is an extension of the models of Wu et al. (2015). As for the models of Wu et al. (2015), there are some strong assumptions regarding claim amounts and premiums. For example, the claims were assumed to take only three integer values and the differences between claims amount and premiums must be multiples of the lowest premium level. In addition, there are only three premium levels in their models. Taking into account these limitations, this chapter aims to relax the strict assumptions such that the individual claims can take any non-negative integer values and the premiums vary according to claims record in a broader bonus-malus context. Recursive equations are derived to calculate the finite-time ruin probabilities in the discrete-time setting. A similar approach was adopted in (Cai and Dickson (2004), sect. 2). In addition to considering premium adjustments according to aggregate claim amounts, this chapter also considers the option of adjusting premiums based on recorded claim numbers, whereas Wu et al. (2015) only focused on the former case. Moreover, this chapter combines inhomogeneity in claims experience with the bonus-malus system by introducing an external Markovian environment, which could see more applications in the real world. This is inspired by the fact that the external environment can affect the implementation of the bonus-malus system and should be taken into account by the insurers. As being said in Niemiec (2007), the external environment can affect the claim frequency and the bonus-malus system should be evaluated according to this factor. In this
chapter, we only focus on external environments that affect the implementation of the bonus-malus system in terms of varying the aggregate claims distribution such as economic environment and weather conditions. Regarding our proposed premium adjustment rules, we assume that the external environment is governed by an external Markov process. When the policies are renewed, the insurers will apply certain premium correction rules that correspond to the current external environment condition to determine the renewal premiums.

Chapter 3 studies the risk models with claim correlated premiums and time-delayed by-claims. The proposed models can be used to evaluate the risk of ruin for insurers who have to face both delayed claim settlements and varying premiums in their everyday business, such as the automobile insurance companies. To obtain the models, the study in Yuen and Guo (2001) is extended by assuming that periodic premiums are adjustable by previous claims experience. Regarding the principle of delayed claims, the term ‘main-claim’ is defined, which refers to the initial claims that induces another type of claims, so-called by-claims, with different severity distribution and occurrence feature. This principle is similar to that of Yuen and Guo (2001) but the assumption for a correlation between main-claims and by-claims in this chapter is different. To be specific, in Yuen and Guo (2001), the main-claims and by-claims are assumed to be independent to each other which is a restrictive assumption, while this chapter inherits the assumptions from Wu and Li (2012) which weakened the assumptions in Yuen and Guo (2001) by allowing the dependence between main-claims and by-claims. In the term of claims settlements, the main-claims are assumed to be settled at the end of the time period of occurrence, whereas the settlement of by-claims may be delayed to the end of the next time period with a certain probability $q$. This chapter derives recursive formulae for the finite-time ruin probabilities by a similar technique to the one used in chapter 2. Then the impact of the correlation between main-claims and by-claims, the probability of delayed claim settlement $q$ as well as the premium adjustment rule on the finite-time ruin probabilities is further examined through several numerical examples. Four transition rules are proposed for premium adjustment: premiums adjusted according to aggregate reported claims; premiums adjusted according to aggregate settled claims; premiums adjusted according to reported claims number; and premiums adjusted according to settled claims number.

Chapter 4 studies discrete-time risk models with premium adjusted according to
claims and surplus level. In this chapter, the financial status of insurers is assumed to be measured by their surplus levels. If the current surplus level is greater than a threshold \( d \), the insurers’ financial status are defined as ‘healthy’ state or ‘unhealthy’ state otherwise. In the healthy state, it is assumed that the less conservative transition rules for premium adjustment should be applied, while in the unhealthy state, the insurers will be sanctioned by the regulator and required to use more conservative transition rules to adjust premiums aiming to lower the insolvency risk. This chapter derives recursive formulae for the finite-time ruin probabilities and examines the impact of the threshold \( d \) on the finite-time ruin probabilities through some numerical studies with two propose rules: premium adjusted according to aggregate claims and premium adjusted according to claim frequency. Additionally, this chapter also considers one special type of ruin, the so-called ‘Parisian ruin’, which occurs when the insurers operate their business with negative surplus level for a certain time period. The principle of Parisian ruin was introduced by Dassios and Wu (2009) and the term ‘Parisian’ came from the Parisian option which is triggered when the underlying asset price is continuously above or below the barrier for a given time period. In the term of risk measures for ruin, the Parisian ruin probability may be more realistic than normal ruin probability in a real life situation because in general, the insurers do not get bankrupt and wind up their business immediately after the surplus level becomes negative. On the contrary, insurers are usually able to continue running their business for a certain time period before the true bankruptcy. In respect of the transition rules for premium adjustments under the negative surplus scenario, a very conservative set of transition rules for premium corrections should be applied, whereas less conservative rules can be implemented when the surplus level is positive. For real life applications, regulators could use the main results obtained in this chapter as a general guideline helping to determine the suitable threshold \( d \) which could lower the insolvency risk of insurers. Besides, the insurers can also adopt these models to help enhancing their premium policies when regulatory ruin policies are in place.

Chapter 5 provides overall concluding remarks and discussions of the three topics covered in this thesis together with possible topics of research in the future. Because this thesis extends the risk models with claim-correlated premiums by considering existing premium correction principles in the literature, the Markov modulated risk models, the risk models with delayed claims, the risk models with surplus-dependent
premises as well as Parisian ruin probability, it is worth reviewing the relevant literature in the following section.

1.2  A literature review

In this section, a review of the relevant literature is provided. We shall start with the reviews of the compound binomial model and discrete-time Sparre Andersen model because these two models are fundamental of other discrete-time risk models, including the proposed models in this thesis. Next, we shall review the relevant literature regarding the risk models with varying premiums, the Markov modulated risk models, the risk models with delayed claims, the risk models with surplus-level-correlated premiums and the Parisian ruin probability in some risk models.

1.2.1  The compound binomial model and discrete-time Sparre Andersen model

The compound binomial risk model was introduced by Gerber (1988). Its surplus process is defined as

$$U_k = U_0 + k - \sum_{t=1}^{N(k)} X_t, \quad \text{for } k \in \mathbb{N}^+, \quad (1.2.1)$$

where \( \mathbb{N} = \{0, 1, 2, \ldots\} \), \( \mathbb{N}^+ = \{1, 2, \ldots\} \), \( U_k \) is the surplus level at time \( k \), and \( U_0 \) is the initial surplus level. \( \{X_t\}_{t \in \mathbb{N}^+} \) are individual claims amounts at time \( t \in \mathbb{N} \) which are independent and identically distributed (i.i.d.) with common probability mass function (P.M.F.) \( f_X(x), x \in \mathbb{N}^+ \), and mean \( \mu < 1 \). \( N(k) \) is the number of claims at time \( k \) which is governed by a binomial process with the probability of having a claim \( q \) and no claim \( p = 1 - q \). The premium per time unit is 1. We assume that the occurrences of a claim in different time periods are independent events and the individual claim amounts are independent of the claims number process. We define the ultimate ruin probability with an initial surplus level \( u \) as

$$\psi(u) = \mathbb{P}\left\{ \bigcup_{k=1}^{\infty} (U_k \leq 0) \big| U_0 = u \right\}.$$
Gerber (1988) obtained the following explicit results for the ultimate ruin probability for the compound binomial model defined in (1.2.1), for \( u \in \mathbb{N}^+ \),

\[
\psi(u) = \sum_{m=1}^{\infty} \frac{1}{m!} \left( \frac{q}{p} \right)^m \mathbb{E} \left( \left( S_m - u \right)_{+} \right) \frac{p^{S_m - u}}{1 - q \mu},
\]

with \( \psi(0) = q \mu \), where \( S_0 = 0 \), \( S_m = X_1 + X_2 + \ldots + X_m \) and \( a^{(m)} = m! \binom{a}{m} \).

Moreover, the explicit formulae for the corresponding survival probability for the model in (1.2.1), denoted by \( \phi(u) = 1 - \psi(u) \), were given in Shiu (1989):

\[
\phi(0) = 1 - q \mu \quad \text{and} \quad \phi(u) = \phi(0) \sum_{m=0}^{\infty} \left( \frac{-q}{1 - q} \right)^m \mathbb{E} \left( \frac{u + m - S_m}{m} \right) \left( 1 - q \right)^{S_m - u} \left[ 1 + (u - S_m) \right],
\]

for \( u \in \mathbb{N}^+ \), where \( 1_+(a) = 1 \) for \( a \in \mathbb{N} \) and 0 otherwise. However, Shiu (1989) defined the ruin probability in a different way from Gerber (1988). Shiu (1989) assumed that ruin occurs when the surplus level is strictly negative, whereas according to Gerber (1988), ruin occurs when the surplus level is negative or zero.

In addition to the ultimate ruin probability, finite-time ruin probabilities for the model (1.2.1) were studied by Willmot (1993). Let the \( n \)-period finite-time ruin probability with an initial surplus level \( u \) be defined as

\[
\psi(u, n) = \mathbb{P} \left\{ \bigcup_{k=1}^{n} (U_k < 0) \mid U_0 = u \right\}
\]

with the corresponding \( n \)-period finite-time survival probability \( \phi(u, n) = 1 - \psi(u, n) \). Willmot (1993) obtained the explicit formulae of \( n \)-period finite-time survival probability for the model (1.2.1):

\[
\phi(0, n) = \frac{\sum_{m=0}^{n} (n - m + 1) g_m(n + 1)}{(1 - q)(n + 1)}, \quad n \in \mathbb{N},
\]

\[
\phi(u, n) = G_{u+n}(n) - (1 - q) \sum_{m=0}^{n-1} \phi(0, n - 1 - m) g_{u+m+1}(m), \quad u, n \in \mathbb{N}^+,
\]

where \( g_a(m) = \mathbb{P}(S_m = a) \) and \( G_a(m) = \sum_{i=0}^{a} g_i(m) \).
Furthermore, Cheng et al. (2000) studied the joint distribution of the surplus level just before ruin and deficit at ruin given the certain initial surplus level (‘discounted’ probability of ruin); Li and Garrido (2002) derived a recursive formula for the expected discounted penalty function; Landriault (2008) studied the expected discounted penalty function considering a general positive integer-valued premium rate.

Now we redefine the claims number process $N(k)$ in the model (1.2.1) as

$$ N(k) = \max\{m : W_1 + W_2 + \ldots + W_m \leq k\}, $$

where $\{W_i\}_{i \in \mathbb{N}^+}$ are i.i.d. claim inter-arrival time random variables with common P.M.F. $f_{W}(w)$, $w \in \mathbb{N}^+$. We assume that the claim amounts $\{X_i\}_{i \in \mathbb{N}^+}$ and the claim waiting times $\{W_i\}_{i \in \mathbb{N}^+}$ are independent of each other. Other assumptions are the same as the ones for the model (1.2.1). This revised model is known as the discrete-time Sparre Andersen model which is more general than the compound binomial model, because it allows the claims number to be governed by any stochastic processes besides the binomial process. Let the probability generating function (p.g.f.) of $f_{W}(w)$ be defined as $\hat{f}_{W}(s) = \sum_{w=1}^{\infty} s^w f_{W}(w)$, for $s \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers. If $\hat{f}_{W}(s)$ can be expressed in terms of a ratio between two polynomials of order $m \in \mathbb{N}^+$, then the claim waiting time distribution is classified as a discrete $K_m$ distribution.

For the discrete-time Sparre Andersen model with $K_m$ inter-claim times, Pavlova and Willmot (2004) derived the expected discounted penalty functions in terms of the relationship between the ordinary and the stationary discrete renewal risk models. Li (2005a) obtained the recursive formula for the expected discounted penalty function. Li (2005b) derived the explicit formula for the expected discounted penalty function and the distributions of the surplus before ruin, the deficit at ruin, and the claim causing ruin in terms of a compound geometric distribution function.

Regarding general inter-claim times, Cossette et al. (2006) derived an upper bound and an asymptotic expression for the ruin probabilities and proposed the method to approximate the corresponding results for the continuous-time risk model (Sparre Andersen model). Wu and Li (2008) studied the ruin probability and the distribution of the deficit at ruin when the distribution of individual claim amounts is a discrete phase-type distribution. Wu and Li (2009) derived an explicit
expression for the expected discounted penalty function when claim amounts follow a zero-truncated geometric distribution, a constant size of 2, and mixed geometric distribution.

In addition, studies of the discrete-time Sparre Andersen models with dependence between claim sizes and their associated inter-claim times can be found in Marceau (2009) and Woo (2012). For Marceau (2009), the discounted penalty function is considered, whereas Woo (2012) considered the generalised discounted penalty function which included two more variables in the penalty function; the minimum surplus level before ruin occurs and the surplus immediately after the second last claim before ruin occurs. Woo and Liu (2018) studied the discounted aggregate claim costs until ruin and the discounted penalty function with this variable.

1.2.2 Risk models with varying premiums

In this section, we shall provide a review on the current literature regarding the risk models with varying premiums. Unlike the classical risk model which assumes a constant premium rate, these models allow varying premiums which can be adjusted according to certain criteria. These models can be used to study the risk of ruin under a bonus-malus system that is the main focus of this thesis.

As for the risk models with varying premiums in a continuous-time setting, Afonso et al. (2009) studied the finite-time ruin probabilities in a continuous-time risk model that allows the premiums to be adjusted according to historical surplus levels. Li et al. (2015a) studied the ruin probabilities under a continuous-time risk model with premium rates adjusted according to the historical number of claims. Specifically, the Bayesian credibility theory was applied to find the posteriori expected number of claims and then the premium rate was adjusted according to the posteriori estimator of claim numbers. The authors studied the impact of the two risks classified by ‘defectiveness’, i.e. the ‘historical’ stream and the ‘unforeseeable’ stream, on ruin probabilities. Meanwhile, Constantinescu et al. (2016) studied ruin probabilities in a regenerative risk process. The authors assumed serial correlation between inter-arrival times of the risk process. To be more specific, the distribution of the waiting time for next claim depends on the waiting time between the current claim and the previous one. This assumption enables the premium rate to be
adjusted following the change of the inter-arrival time distribution. For the purpose of simplicity, only two levels of premium rates were evaluated in the paper: the bonus (discount) state and the base state.

Moreover, Kucerovský and Najafabadi (2017) studied a continuous-time risk model for a long term bonus-malus system in a steady state. Closed-form expressions of the ruin probability were obtained by solving an integral equation using complex analysis. Li et al. (2015c) studied a risk model with premium adjusted according to the surplus increments between successive random review times and derived a matrix-form defective renewal equation for the discounted penalty function. Besides, Afonso et al. (2010) studied the ruin probabilities for a portfolio with credibility adjusted premiums. Afonso et al. (2017) analyzed the impact of different well-known bonus-malus scales and transition rules on the finite-time ruin probabilities for a continuous-time risk model. The posteriori premiums in their model were modified according to the historical claim record of each individual policyholder. In the most recent literature, Afonso et al. (2020) studied ruin probabilities and capital requirements for open automobile portfolios with a bonus-malus system based on claim counts.

Under the discrete-time setting, Dufresne (1988) proposed the recursive algorithm to compute the ruin probabilities by using the stationary distributions of the bonus-malus system. Wagner (2002) studied the joint distribution of ruin and severity of ruin and the expected time to surplus zero from a given negative surplus for the two-state Markov chain risk model. Trufin and Loisel (2013) studied the ruin probabilities with premiums adjusted to the claims by Bühlmann credibility theory. They derived the asymptotic formulae for the ultimate ruin probabilities and the Lundberg coefficients for the light-tailed claims. In addition, they also derived the asymptotic formulae for the ultimate ruin probabilities for heavy-tailed claims. Wagner (2001) considered a two-state Markov chain risk model and derived recursive formulae for ruin probabilities. For the models of Wagner (2001), let \( \{J_t\}_{t \in \mathbb{N}} \) be a homogeneous Markov chain with state space \( \{1, 2\} \) and transition matrix \( P = [p_{ij}]_{2 \times 2} \), for \( i, j \in \{1, 2\} \), where \( 0 < p_{12} < 1 \) and \( 0 < p_{21} < 1 \). We assume that for \( t \geq 1 \), if \( J_t = 1 \), the insurer receives a premium \( c > 0 \) and makes a payment of 1 if \( J_t = 2 \). For an initial state \( J_0 = i \in \{1, 2\} \), the surplus process can
be expressed as
\[ U_{k,i} = U_0 + ck - (1 + c) \sum_{t=1}^{k} (J_t - 1), \quad \text{for } k \in \mathbb{N}^+ \text{ and } U_0 \geq 0, \quad (1.2.2) \]

where \( U_{k,i} \) is the surplus level at time \( k \) with an initial state \( i \) and \( U_0 \) is the initial surplus level. The ultimate ruin probability with initial state \( i \) and initial surplus \( u \) can be expressed as
\[ \psi_i(u) = \mathbb{P} \left\{ \bigcup_{k=1}^{\infty} (U_{k,i} < 0) \bigg| U_0 = u, J_0 = i \right\}. \]

According to the model in (1.2.2), Wagner (2001) obtained the following recursive formulae for the ultimate ruin probability:
\[
\begin{align*}
\psi_1(u) &= p_{11} \psi_1 \left( u + \frac{1}{N} \right) + p_{12} \psi_2(u - 1), \\
\psi_2(u) &= p_{21} \psi_1 \left( u + \frac{1}{N} \right) + p_{22} \psi_2(u - 1),
\end{align*}
\]
with initial values
\[
\psi_1(0) = N \left( \frac{p_{12}}{p_{21}} \right), \quad \psi_2(0) = \frac{p_{12}(N - 1) + p_{22}}{p_{11}},
\]
where \( c = \frac{1}{N} \), for \( N = 2, 3, 4, \ldots \).

Further, Wu et al. (2015) derived recursive formulae and explicit formulae for the ultimate ruin probabilities in the case where premiums are correlated to claim amounts by using the two-state Markov Chain model. In Wu et al. (2015), the surplus process is defined as
\[ U_k = U_0 + \sum_{t=1}^{k} C_t - \sum_{t=1}^{k} S_t, \quad k, U_0 \in \mathbb{N}, \quad (1.2.3) \]

where \( U_k \) is the surplus level at time \( k \), \( U_0 \) is the initial surplus level, \( C_t \) is the amount of premium the insurer receives at the beginning of period \( t \) and \( \{S_t\}_{t \in \mathbb{N}^+} \) are total claim amounts in each period which are assumed to be i.i.d. random variables. We assume that \( S_t \) can be only three values: 0, \( M \) and \( N \), for \( M < N \in \mathbb{N}^+ \) with probabilities \( q = 1 - p_1 - p_2, p_1 \) and \( p_2 \), respectively, for \( p_1 > 0 \) and \( p_2 < 1 \) and \( C_t \) satisfies the following condition: \( \mathbb{P}(C_{t+1} = k_2 | S_t = 0) = 1, \mathbb{P}(C_{t+1} = K_2 | S_t = M) = 1 \) and \( \mathbb{P}(C_{t+1} = K_1 | S_t = N) = 1 \), for \( K_1, K_2, k_2 \in \mathbb{N}^+ \). The positive safety loading condition for the model is \( k_2 + p_1(K_2 - k_2 - M) + p_2(K_1 - k_2 - N) > 0 \). We denote the ultimate ruin probability with initial premium \( K_2 \) and initial surplus \( u \)
as \( \psi_2(u) = \mathbb{P}\left\{ \bigcup_{k=1}^{\infty} \{U_k < 0\} \bigg| U_0 = u, C_1 = K_2 \right\} \). For the initial premium \( K_1 \) and \( k_2 \) case, the notation for the ultimate ruin probability is \( \psi(u) \) and \( \psi_1(u) \) respectively and the definition for the ultimate ruin probability is same as \( \psi_2(u) \) with \( C_1 = K_1 \) and \( C_1 = k_2 \) respectively. For the model in (1.2.3), we can compute the ultimate ruin probability recursively by the formulae obtained by Wu et al. (2015):

\[
\psi_2(u) = \begin{cases} 
\frac{1}{q}[\psi_2(u - k_2) - p_1 - p_2], & k_2 \leq u < M + k_2 - K_2 \\
\frac{1}{q}[\psi_2(u - k_2) - p_1 \psi_2(u + K_2 - k_2 - M) - p_2], & M \leq u - k_2 + K_2 < N \\
\frac{1}{q}[\psi_2(u - k_2) - p_1 \psi_2(u + K_2 - k_2 - M)] - p_2 \psi_2(u + K_1 - k_2 - N), & u \geq N - K_2 + k_2
\end{cases}
\]

with the initial values

\[
\psi_2(i) = \begin{cases} 
\frac{p_1 J_1 + p_2 (J_1 + 1)}{1 - p_1}, & i = 0, 1, \ldots, I - 1 \\
\frac{p_1 J_1 + p_2 J_1}{q}, & i = I, I + 1, \ldots, k_2 - 1,
\end{cases}
\]

where \( N - K_1 = J_1 k_2, M - K_2 = J_2 k_2 \), for \( J_1, J_2 \in \mathbb{N}^+, I = K_1 - K_2 \) and \( \psi(u) \), \( \psi_1(u) \) can be computed by the following relationship:

\[
\psi_1(u + K_1) = \psi_2(u + K_1 + k_2 - K_2) \quad \text{and} \quad \psi_2(u + K_1) = \psi(u + K_2).
\]

Besides, Wu et al. (2015) also find that when model (1.2.3) is simplified and only two premium levels are considered instead of three by allowing \( p_1 = 0 \) and \( K_2 = k_2 \), an explicit formula for the ultimate ruin probability can be expresses as

\[
\psi(u) = 1 - \frac{q - p J_1}{q^{j_u + 1}} \left[ 1 + \sum_{n=1}^{\beta} (-pq^{J_1})^n \sum_{k=1}^{j_u - n J_1 - n + 1} a_n(k) \right], \quad \text{for } u \geq K_2,
\]

where \( p = p_2, j_u \) is quotient from dividing \( u \) by \( K_2 \), for \( j_u \in \mathbb{N}^+ \), \( \beta \) is the quotient from dividing \( j_u \) by \( J_1 + 1 \) and \( a_n(k) = \sum_{j=1}^{k} a_{n-1}(j), \quad k = 1, \ldots, j_u - n J_1 - n + 1 \) with \( a_1(i) = 1, \quad i = 1, \ldots, j_u - J_1. \)
1.2.3 Markov-modulated risk models

As mentioned previously, chapter 2 extends some risk models with varying premiums by considering the Markovian environment. Therefore, this section provides a review of the literatures concerning risk models with Markovian environment which are known as Markov-modulated risk models. The main principle of Markov-modulated risk models is that their parameters at the certain time point such as claim amounts distribution, inter-claim waiting times distribution, premium rate, etc. are assumed to be governed by a Markov process.

Regarding the study of the ruin probability for Markov-modulated risk models, Asmussen (1989) proposed some approximations, simulations and numerical method to compute the ruin probability. Asmussen and Rolski (1994) obtained the Cramér-Lundberg approximation for the ruin probability. Cossette et al. (2004) studied the ruin probability for the compound binomial model in a Markovian environment. They assumed that claim occurrences and claim amounts at time $t$ are both governed by a homogeneous and irreducible discrete-time Markov environment process $\{J_t\}_{t \in \mathbb{N}}$ with finite state space $\{1, 2, ..., m\}$ and transition probability matrix $\Gamma = [\gamma_{i,j}]_{m \times m}$, where $\gamma_{i,j} = \mathbb{P}(J_{t+1} = j | J_t = i)$, for $i, j \in \{1, 2, ..., m\}$ with stationary probabilities $\lambda = [\lambda_1, \lambda_2, ..., \lambda_m]$. For $J_t = i$, the probability of having a claim is $p_i \in (0, 1)$, where $p_i \in \mathbf{p} = \{p_1, p_2, ..., p_m\}$, claim amount at period $t$ is $\{S_t\}_{t \in \mathbb{N}}$ with P.M.F. $f_i(s)$ and mean $\mu_i$. Given $J_t = i$, $\{S_t\}_{t \in \mathbb{N}}$ are mutually independent and are also independent of the claim occurrence process (Bernoulli process) defined in a Markovian environment. The positive safety loading condition follows $\sum_{i=1}^{m} \lambda_i p_i \mu_i < 1$ (the premium per time unit is one). Regarding the surplus process, it is defined in (1.2.1) and assumed to be governed by the Markov process $\{J_t\}_{t \in \mathbb{N}}$ mentioned above. The ultimate ruin probability with initial surplus level $u$ and initial state $i$ is denoted as $\psi_i(u) = \mathbb{P}\left\{ \bigcup_{k=1}^{\infty} (U_k < 0) \big| U_0 = u, J_0 = i \right\}$ with the survival probability $\phi_i(u) = 1 - \psi_i(u)$ and the corresponding finite-time ruin probability is defined as $\psi_i(u, n) = \mathbb{P}\left\{ \bigcup_{k=1}^{n} (U_k < 0) \big| U_0 = u, J_0 = i \right\}$ with finite-time survival probability $\phi_i(u, n) = 1 - \psi_i(u, n)$. For finite-time survival probabilities, Cossette et al. (2004) obtained the following recursive formula:
For $u \in \mathbb{N}^+$ and $n \in \mathbb{N}$,
\[
\begin{bmatrix}
\phi_1(u - 1, n + 1) \\
\vdots \\
\phi_m(u - 1, n + 1)
\end{bmatrix}
= \Gamma \times (I - \text{diag}(p)) \times
\begin{bmatrix}
\phi_1(u, n) \\
\vdots \\
\phi_m(u, n)
\end{bmatrix}
+ \Gamma \times \text{diag}(p) \times
\begin{bmatrix}
\sum_{s=1}^{u} \phi_1(u - s, n)f_1(s) \\
\vdots \\
\sum_{s=1}^{u} \phi_m(u - s, n)f_m(s)
\end{bmatrix},
\]
where $\text{diag}(p)$ holds for $\text{diag}[p_1, p_2, ..., p_m]$, $I$ is the $m \times m$ identity matrix and $\phi_i(u, 0) = 1$, for $i \in \{1, 2, ..., m\}$ and $u \in \mathbb{N}$. Regarding the ultimate time survival probability computation, it is not easy to obtain the explicit formula or even the recursive formulae. Therefore, Cossette et al. (2004) proposed the following algorithm to approximate to ultimate time survival probability:

- Fix $\phi_i(u) = 1$ for $u = n, n + 1, ...$ and $i \in \{1, 2, ..., m\}$.
- Find $\phi_i(u)$ for $u = 0, 1, ..., n - 1$ and $i \in \{1, 2, ..., m\}$ by solving the following system of $m \times n$ equations for $m \times n$ unknown parameters

\[
\phi_i(u) = \sum_{j=1}^{m} \gamma_{i,j} \left( (1 - p_j)\phi_j(u + 1) + p_j \sum_{s=1}^{u+1} \phi_j(u + 1 - s)f_j(s) \right).
\]

The above method is more accurate when $n$ is large. However, the recursive formulae to compute the ultimate survival probability can be derived with some certain boundary conditions. For example, for the two-state Markov process $\{J_t\}_{t \in \mathbb{N}}$ defined previously, if we assume $p_1 = 0$ for $J_t = 1$, $p_2 = 1$ for $J_t = 2$, and the claim amount distribution $f(s)$ for $J_t = 2$, Cossette et al. (2003) obtained the following recursive formulae to compute the ultimate time survival probability:

\[
\begin{align*}
\phi_1(u) &= \frac{\phi_1(u - 1) - \gamma_{1,2} \sum_{s=1}^{u} \phi_2(u - s)f(s)}{\gamma_{1,1}}, \\
\phi_2(u) &= \frac{\gamma_{2,1}\phi_1(u) - (\gamma_{1,1}\gamma_{2,2} - \gamma_{1,2}\gamma_{2,1}) \sum_{s=2}^{u+1} \phi_2(u + 1 - s)f(s)}{\gamma_{1,1} - (\gamma_{1,1}\gamma_{2,2} - \gamma_{1,2}\gamma_{2,1})f(1)}, \quad \text{for } u \in \mathbb{N}^+.
\end{align*}
\]
with the initial values
\[
\begin{align*}
\phi_1(0) & = \frac{1 - \lambda_2 \mathbb{E}(S)}{1 - \lambda_2} , \\
\phi_2(0) & = \frac{\gamma_{2,1}}{\gamma_{1,1} - (\gamma_{1,1}\gamma_{2,2} - \gamma_{1,2}\gamma_{2,1})f(1)} \phi_1(0).
\end{align*}
\]

Besides, a recursive formula or explicit expression for the ultimate survival probability in the two-stage Markov model can be also found in Chen et al. (2014b), Dickson and Qazvini (2018) and Lu and Li (2005).

In addition to the ruin probability, the problems concerning the ruin-related quantities in the Markov-modulated risk models have been also studied in various literatures. Bäuerle (1996) studied the expected ruin time and derived explicit formulae for the expected ruin time for the two-state model. Ren (2008) derived an explicit formula for distribution of the aggregate discounted claims when inter-claim waiting times follow a Markovian arrival process by using the Laplace transform method. Li and Li (2020) studied some state-specific one-sided exit probabilities. Reinhard and Snoussi (2002) derived recursive formulae for the distribution of the severity of ruin in the discrete-time setting, whereas Lu (2006) and Li and Ren (2013) studied the severity of ruin in the continuous time setting. Ng and Yang (2006) obtained the expression for the joint distribution of surplus before and after ruin in certain cases. For the study of ruin-related quantities in the Markov-modulated risk models, see Reinhard (1984), Li et al. (2015b) and Li et al. (2016).

Lastly, the studies of the penalty function in the Markov-modulated risk model can be found in Albrecher and Boxma (2005), Li and Lu (2008), Cheung and Feng (2013) and Yuen et al. (2017). Literatures regarding Markov-modulated risk models with dividend payment can be found in Badescu et al. (2007), Li and Lu (2007), Zhu and Yang (2008), Cheung and Landriault (2009), Chen et al. (2014a) and Nie et al. (2020).

1.2.4 Risk models with delayed claims

In this section, we shall review the current literature regarding risk models with delayed claims. Unlike the classical models which assume simultaneous claims
payment when claims occur, risk models with delayed claims allow the delayed
claims payment which are more realistic in some situations in the real world. In
chapter 3 of this thesis, we shall extend risk models with claim correlated premiums
by considering delayed claims.

Regarding the literatures on the ruin probability and ruin-related quantities
in risk models with delayed claims, Waters and Papatriandafylou (1985) derived
the upper bounds for the ruin probability of a risk process with delayed claims
settlement. Yuen and Guo (2001) studied ruin probabilities for time-correlated
claims in the compound binomial risk model. They introduced two types of claims;
the main claims and the by-claims, where the by-claims are assumed to be induced
by main claims. The probability of a main claim occurrence in any time period
is \( q \), for \( 0 < q < 1 \) and the probability of no main claims is \( p = 1 - q \). The
occurrences of main claims in different time periods are assumed to be independent.
The probability that a main claim and its by-claim occur simultaneously is \( \theta \) and
the by-claim occurs in the following period with probability \( 1 - \theta \). The i.i.d. main
claim amounts at time \( t \) are denoted as \( \{X_t\}_{t \in \mathbb{N}^+} \) with P.M.F. \( f_X(x) \) and mean
\( \mu_X \), for \( x \in \mathbb{N}^+ \). Similarly, the i.i.d. by-claims amount at time \( t \) are denoted as
\( \{Y_t\}_{t \in \mathbb{N}^+} \) with P.M.F. \( f_Y(y) \) and mean \( \mu_Y \). The P.G.F. for \( \{X_t\}_{t \in \mathbb{N}^+} \) and \( \{Y_t\}_{t \in \mathbb{N}^+} \)
are denoted as \( \hat{f}_X(z) = \sum_{x=1}^{\infty} f_X(x) z^x \) and \( \hat{f}_Y(z) = \sum_{y=1}^{\infty} f_Y(y) z^y \), respectively,
where \( z \in \mathbb{C} \). The aggregate claims amount in the first \( n \) period is denoted as
\( S_n = \sum_{t=1}^{n}(X_t + Y_t) \), for \( n \in \mathbb{N}^+ \) and \( \hat{h}(z,n) = (p + q\hat{f}_X(z)\hat{f}_Y(z))^n \), \( h(s,n) \) and
\( H(s,n) \) are denoted for the P.G.F., P.M.F. and cumulative distribution function of
\( S_n \), respectively. The surplus process is defined as

\[
U_k = U_0 + k - \sum_{t=1}^{k}(X_t + Y_t),
\]

where the premium per time period is one and \( U_0 \) is the initial premium level. The
definition for the ultimate ruin probability \( \psi(u) \) and the \( n \)-period finite-time ruin
probability \( \psi(u,n) \) is given as \( \mathbb{P}\{ \bigcup_{k=1}^{\infty} (U_k < 0) \Big| U_0 = u \} \) and

\[
\mathbb{P}\{ \bigcup_{k=1}^{n} (U_k < 0) \Big| U_0 = u \},
\]

respectively. For the \( n \)-period finite-time survival probability
\( \phi(u, n) = 1 - \psi(u, n), \) Yuen and Guo (2001) obtained:

\[
\begin{align*}
\phi(i - n, n) &= \theta H(i, n) + (1 - \theta) H_1(i, n) - p\theta \sum_{j=0}^{n-1} \phi(0, n - 1 - j) h(i + j + 1 - n, j) \\
&\quad - p^2(1 - \theta) \sum_{j=0}^{n-2} \phi(0, n - 2 - j) h(i + j + 2 - n, j),
\end{align*}
\]

for \( 1 \leq n \leq i \) and \( n \in \mathbb{N}^+ \), where \( H_1(i, n) \) is the corresponding cumulative distribution function of the P.G.F. \( \hat{h}(z, n) = \hat{h}(z, n - 1) \left( p + q \hat{f}_X(z) \right) \) and \( \phi(0, n) \) is computed recursively by

\[
\begin{align*}
\phi(0, n) &= \theta H(n, n) + (1 - \theta) H_1(n, n) - p\theta \sum_{j=0}^{n-1} \phi(0, n - 1 - j) h(j + 1, j) \\
&\quad - p^2(1 - \theta) \sum_{j=0}^{n-2} \phi(0, n - 2 - j) h(j + 2 - j).
\end{align*}
\]

Regarding the survival probability \( \phi(u) = 1 - \psi(u) \) for \( \theta = 1 \), Yuen and Guo (2001) found that it can be obtained by considering the claim amount \( X + Y \) instead of \( X \) in the study of Willmot (1993). For \( \theta = 0 \) and \( f_X(1) = f_Y(1) = 1 \), Yuen and Guo (2001) expressed the survival probability as:

\[
\phi(u) = 1 - \left( \frac{q}{p} \right)^{u+2}, \text{ for } u \in \mathbb{N}.
\]

Additionally, some similar models can be found in Wu and Yuen (2004) which is an extension of Yuen and Guo (2001) by considering the interaction between the dependent classes of business in the models. Yuen et al. (2005) used the martingale theory to obtain an expression for the ultimate ruin probability with the corresponding Lundberg exponent of its non-delayed risk model. Zou and Xie (2010) considered the case when the claims number process follows a renewal process with Erlang(2) inter-arrivals and derived an explicit expression for the survival probability when both main claims amounts and by-claims amounts are exponentially distributed. Trufin et al. (2011) and Ahn et al. (2018) studied the ruin probability by defining delayed claims in terms of IBNR claims. Dassios and Zhao (2013) obtained an asymptotic expression for the ruin probability by exploiting a non-homogenous Poisson model. Besides, the studies of an approximation of ruin
probability can be also found in Gao et al. (2019) and Yang and Li (2019). Xiao and Guo (2007) studied the joint distribution of the surplus immediately prior to ruin and the deficit at ruin in the compound binomial risk model with time-correlated claims and its relationship with the classical compound binomial risk model.

Regarding the dividend problem in risk models with time-delayed claims, Wu and Li (2012) studied the expected present value of dividend payments up to the time of ruin by considering a constant dividend barrier, whereas Zhou et al. (2013) studied the same problem and assumed that the premium income is governed by a binomial process. Liu and Zhang (2015) considered a randomized dividend strategy for the study of the expected present value of dividend payments up to the time of ruin. Further, the literatures concerning the penalty function in risk models with time-delayed claims can be found in Yuen et al. (2013), Zhu et al. (2014), Liu and Bao (2015), Xie and Zou (2017), Wat et al. (2018), Deng et al. (2018), S and Upadhye (2019), Zou and Xie (2019) and Liu et al. (2020).

1.2.5 Risk models with surplus-dependent premiums and Parisian ruin probability in some risk models

In this subsection, we shall review the literatures regarding some risk models with surplus-dependent premiums and risk models in which Parisian ruin probability is studied.

For the risk model with surplus-dependent premiums, Albrecher et al. (2013) obtained an expression for the ruin probabilities and the discounted penalty function, whereas Marciniak and Palmowski (2016) studied an optimal dividend problem. Marciniak and Palmowski (2018) also studied an optimal dividend problem but considered the dual risk model in which premiums are treated as costs and the claims are referred to profits. Besides, the study of the optimal dividend problems for risk models with surplus-dependent premiums can be also found in Li et al. (2020).

Regarding the studies of the Parisian ruin probability in some risk models, Dassios and Wu (2009) derived the Laplace transform for the time of Parisian ruin and the Parisian ruin probability. Czarna and Palmowski (2011), Loeffen et al. (2013), Landriault et al. (2014), Czarna (2016) and Li et al. (2018) studied
the Parisian ruin probability for the spectrally negative Lévy risk process. Debicki et al. (2015) obtained an asymptotic expression for the Parisian ruin probability for the self-similar Gaussian risk processes. Zhao and Dong (2018) derived a compact formula for Parisian ruin probability for the spectrally negative Markov additive risk process. Bai (2018) obtained the asymptotics of the Parisian ruin probability for a Brownian motion risk model. Bladt et al. (2019) considered the Parisian ruin probability for a risk model with dependent phase-type distributed claim sizes and inter-arrivals times. Czarna et al. (2017b) and Palmowski et al. (2018) studied the Parisian ruin probability in a risk model under the discrete-time setting.

Additionally, Wong and Cheung (2015) obtained an expression for the Laplace transform of the Parisian ruin time and studied occupation time problems for some dual renewal risk models. Czarna et al. (2017a) proposed an algorithm to obtain the joint distribution of the Parisian ruin time and the number of claims until Parisian ruin for the classical risk model. Wang and Zhou (2020) studied the draw-down Parisian ruin time for a spectrally negative Lévy risk process. Surya et al. (2020) studied the Parisian ruin problem for a spectrally negative Lévy risk process with capital rejection. Liang and Young (2020) studied the problem of minimizing the discounted probability of exponential Parisian ruin via reinsurance. Lastly, literatures concerning the penalty function at Parisian ruin time for some risk models can be referred to Baurdoux et al. (2016), Yang et al. (2017) and Loeffen et al. (2018). The studies of the dividend problems for the Parisian ruin probability can be found in Czarna and Palmowski (2014), Czarna et al. (2016), Yang et al. (2020) and Czarna et al. (2020).
Chapter 2

Discrete-time risk models with claim correlated premiums in a Markovian environment

2.1 Introduction

This chapter introduces some discrete-time risk models with claim correlated premiums in a Markovian environment. We assume that the claims distributions vary due to the external Markov process. Regarding the proposed premium adjustment rules, the renewal premiums are determined according to the claims experience and also the external environment state and there are two premium adjustments strategies; premiums adjusted by aggregate claims and by claim frequency. As for a real-world application, it is not an issue in practice due to the short-term nature of most non-life insurance contracts. It means that the insurers are able to frequently modify the premium adjustment rules when the external environment changes. By applying this principle, the insurers can better address the systematic risk when implementing the bonus-malus system in pricing. However, it can be tricky when insurers explain the environment-dependent premium changing rules to their policyholders since the rules are likely to change over time.

For the calculations of ruin probabilities in this chapter, recursive formulae are derived to compute the finite-time ruin probabilities and Lundberg-type upper
bounds are derived to evaluate the ultimate-time ruin probabilities. In addition, the joint distribution of premium level and environment state at ruin given the ruin occurs are also studied and obtained by the recursive formulae. At the end of this chapter, two numerical examples are provided to numerically evaluate the impact of bonus-malus system on the ruin probabilities of the proposed risk models within an external Markovian environment. To be specified, we aim to explore the following questions that may arise when implementing the bonus-malus system in practice through some numerical studies based on hypothetical assumptions:

- What should the initial premium level be for new policyholders?
- Which premium adjustment criterion is better in the proposed risk models: premiums adjusted by aggregate claims amount or premiums adjusted by claim frequency only?
- What is the likely impact of the initial external environment condition on the risk of ruin when the proposed premium adjustment rules are implemented?

## 2.2 Models and Assumptions

Let \( c \) denote a premium level set where \( c = \{c_i\}_{i \in \mathcal{L}}, \mathcal{L} = \{1, 2, \ldots, l\}, l \in \mathbb{N}^+, c_i \in \mathbb{R}^+ \) (\( \mathbb{R}^+ \) is the set of positive real numbers). Here \( c_i, i = 1, \ldots, l \), are premium levels per unit volume of risk. Let the external economic environmental status in time period \([t-1, t)\), \( t \in \mathbb{N}^+ \) be represented by a homogeneous and irreducible discrete-time Markov chain \( \{J_t\}_{t \in \mathbb{N}} \) with a finite state space \( \mathcal{R} = \{1, 2, \ldots, r\} \) and a transition probability matrix \( P_J = [p_J(g,h)]_{g,h \in \mathcal{R}}, \) where \( p_J(g,h) = \mathbb{P}(J_t = h|J_{t-1} = g) \). The stationary probability distribution of the Markov process is denoted by \( \lambda = [\lambda_1, \ldots, \lambda_r] \) where \( 0 \leq \lambda_i \leq 1, i = 1, 2, \ldots, r \), and \( \sum_{i=1}^r \lambda_i = 1 \). Let \( \{L_t\}_{t \in \mathbb{N}^+} \) be a stochastic process monitoring the premium levels that the insurance company charges over time. Here \( L_t \in c \) for any \( t \in \mathbb{N}^+ \) and this premium level applies in the time period \([t-1, t)\).
Consider a general insurance surplus process of which the level of surplus at time $k$, $k = 0, 1, \ldots$, is defined by

$$U_k = U_0 + \sum_{t=1}^{k} (C_t - S_t), \quad \text{for} \quad k \in \mathbb{N}^+,$$

where $S_t$ is the aggregate claims amount for time period $[t-1, t)$ payable at time $t$, $t \in \mathbb{N}^+$; $U_0 = u \geq 0$, $u \in \mathbb{N}$ is the initial surplus level; $C_t$ is the total premium for time period $[t-1, t)$ received at time $t-1$ and $C_t = L_t E[S_t]$, $t \in \mathbb{N}^+$. Further, $S_t$ has P.M.F. $f_{S,J_t}(s)$ and mean $\mu_{S,J_t}$, for $J_t \in \mathbb{R}$ and $s \in \mathbb{N}$.

The timing of all cash flows involved in the above insurance surplus process is illustrated through the following timeline given in Figure 2.1 where year $t$ denotes the period of $[t-1, t)$, $t \in \mathbb{N}^+$.

- $S_{t-1}$ is paid at end of year $t-1$.
- $J_{t-1}$ is fixed for year $t-1$.
- $C_t$ is received at the beginning of year $t$.
- $S_t$ is paid at the end of year $t$.
- $J_t$ is fixed for year $t$.
- $C_{t+1}$ is received at the beginning of year $t+1$.

**Figure 2.1:** The timeline of all cash flows.

Given $L_1 = c_i$ and $J_1 = g$, $i \in \mathcal{L}$, $g \in \mathbb{R}$, the first premium amount $C_1$ can be determined as follows

$$C_1 = c_i E[S_1] = c_i \mu_{S_i,g} := \alpha_{i,g}.$$  \hfill (2.2.2)

According to (2.2.2), theoretically $\alpha_{i,g}$ can be a non-integer value. However, an integer-valued $\alpha_{i,g}$ is required for the computation of numerical results. There are
two feasible ways to convert non-integer $\alpha_{i,g}$ to an integer. Firstly, certain scaling-up can be applied to make sure that $\alpha_{i,g}$ for all $i \in L$ and $g \in R$ are integer-valued. However, this method may heavily increase the required computational time if the multiplier is a large number. Another way is to round each of $\alpha_{i,g}$ to its nearest integer, but this method potentially reduces the level of accuracy in the computations. This negative impact might be minimal when considering large insurance portfolios. We remark that in this chapter, we always assume that the parameters are integer-valued when necessary. In addition, all variables in our models are assumed to be integers to suit the derivations of the main recursive formulae in this chapter. Non-integer valued parameters are not feasible in the recursive computational framework.

The $n$-period finite-time ruin probability with initial surplus $u$, initial environment state $g$ and initial premium level $c_i$ is defined by, $n \in \mathbb{N}^+$,

$$
\psi_{i,g}(u,n) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} (U_k < 0) \big| L_1 = c_i, J_1 = g \right\},
$$

where $U_k$ is defined by (2.2.1) and the subscript $u$ represents the condition $U_0 = u$. By convention, for any $i \in L$ and $g \in R$,

$$
\psi_{i,g}(u,n) = \begin{cases} 
0 & \text{if } u \geq 0, n \leq 0, \\
1 & \text{if } u < 0, n \geq 0.
\end{cases}
$$

Further, we assume that the aggregate claim amounts $\{S_t\}_{t \in \mathbb{N}^+}$, are non-negative integer-valued random variables that follow a collective risk model structure:

$$
S_t = \sum_{i=1}^{M_t} W_{it},
$$

where $M_t$ is the total number of claims recorded in time period $[t-1,t)$, $t \in \mathbb{N}^+$, with P.M.F. $f_{M,J_t}(m)$ for $J_t \in R$ and $m \in \mathbb{N}$; $\{W_{it}\}_{i \in \mathbb{N}^+}$, are individual claim sizes settled in time period $[t-1,t)$. For the purpose of simplification, conditional on $J_t$, we assume that $\{W_{it}\}_{i \in \mathbb{N}^+}$ are i.i.d. with common P.M.F. $f_{W,J_t}(w)$, $w \in \mathbb{N}^+$ and $J_t \in R$. The claims number $\{M_t\}$ and individual claim sizes $\{W_{it}\}$ are assumed to be independent of each other given $J_t$. 


One can show that, for $s \in \mathbb{N}$,

$$f_{S,g}(s) = \sum_{m=0}^{s} f_{M,g}(m) f_{W,g}^{*m}(s)$$

with $f_{W,g}^{*m}(s)$ being the $m$-fold convolution of $f_{W,g}(w)$ and $f_{W,g}^{*0}(0) = 1$.

**Remark.** Quite often it is not easy to obtain the explicit expression for $f_{W,g}^{*m}(s)$. Instead, $f_{W,g}^{*m}(s)$ can be calculated recursively using the formula in Dickson (2016).

### 2.2.1 Premiums Adjusted by Aggregate Claims

As introduced previously, this chapter aims to consider a general bonus-malus premium system in a discrete-time setting. Let $\{t_{ij}(s,g)\}_{i,j \in \mathcal{L}, s \in \mathbb{N}, g \in \mathbb{R}}$ denote a general set of time-homogeneous rules for premium variations, where $t_{ij}(s,g) = 1$ if the aggregate claim $S_t = s$ and environment state $J_t = g$ lead to the transition from the premium level $L_t = i$ to $L_{t+1} = j$ and $t_{ij}(s,g) = 0$ otherwise. For example, given that the current environment state is 1, if the transition rule requires the premium level for the next year to move to the next higher level if the aggregate claims is more than 10, then $t_{i(i+1)}(s,1) = 1$, for $s > 10$ and $t_{i(i+1)}(s,1) = 0$, for $s \leq 10$. According to the definition, we have $\sum_{j=1}^{\infty} t_{ij}(s,g) = 1$ for any $i \in \mathcal{L}, g \in \mathbb{R}, s \in \mathbb{N}$.

Let $p_{C,g,h}(i,j)$ denote the probability that the premium level moves from level $i$ in initial environment state $g$ to level $j$ in environment state $h$, which can be expressed as, for any $t \in \mathbb{N}^+$,

$$p_{C,g,h}(i,j) = \mathbb{P}\{L_{t+1} = j, J_{t+1} = h | L_t = i, J_t = g\}$$

$$= p_{J}(g,h) \sum_{s=0}^{\infty} t_{ij}(s,g) f_{S,g}(s), \text{ for } i, j \in \mathcal{L}, g, h \in \mathbb{R}, \quad (2.2.5)$$

where $f_{S,g}(s), s \in \mathbb{N}$, is the P.M.F. of aggregate claims in the environment state $h$ with mean $\mu_{S,h}$. The function $t_{ij}(s,g)$ is determined according to the given transition rule. From the definition of $p_{C,g,h}(i,j)$, considering a state space of $\{(i,g)\}_{i \in \mathcal{L}, g \in \mathbb{R}}$, its one-step transition probability matrix has the form...
Discrete-time risk models with claim correlated premiums in a Markovian environment

\[ P_C = \begin{bmatrix}
  p_{C,1,1}(1,1) & \cdots & p_{C,1,1}(1,l) & \cdots & p_{C,1,1}(1,l) \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{C,1,1}(l,1) & \cdots & p_{C,1,1}(l,l) & \cdots & p_{C,1,1}(l,l) \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{C,r,1}(1,1) & \cdots & p_{C,r,1}(1,l) & \cdots & p_{C,r,1}(1,l) \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{C,r,1}(l,1) & \cdots & p_{C,r,1}(l,l) & \cdots & p_{C,r,1}(l,l)
\end{bmatrix}_{(l \times r) \times (l \times r)} \tag{2.2.6}
\]

Note that the above matrix is constructed based on the combined statuses of pairs of premium level and environment state, which is a generalised version of the usual transition matrix for individual premium levels or transition matrix for individual environment states. The associated stationary probability distribution of \( P_{C,g,h} \) is denoted by \( \pi = [\pi_{ig}]_{l \times r} \), for \( i \in \mathcal{L} \) and \( g \in \mathcal{R} \). From (2.2.5) we have, for \( j \in \mathcal{L} \), \( h \in \mathcal{R} \),

\[ \pi_{jh} = \sum_{i=1}^{l} \sum_{g=1}^{r} \pi_{ig} p_{C,g,h}(i,j) = \sum_{i=1}^{l} \sum_{g=1}^{r} \pi_{ig} p_{J(g,h)} \sum_{s=0}^{\infty} t_{ij}(s,g) f_{s,g}(s). \]

Following a common practice, we would also assume that the surplus process (2.2.1) satisfies the positive safety loading conditions, for any \( i \in \mathcal{L}, g \in \mathcal{R} \),

\[ \alpha_{ig} > \mu_{S,g}. \]

**Remark.** The above positive safety loading condition assumption is more conservative than it usually appears in the Markov modulated risk model literature, because it needs to be in such form to enable the generalised Lundberg inequalities discussed in Section 2.4.
2.2.2 Premiums Adjusted by Claim Frequency

When periodic premiums adjust according to the claim frequency experience rather than the aggregate claim amounts, the transition probability \( p_{C,g,h(i,j)} \) in (2.2.5) needs to be modified as

\[
p_{C,g,h}(i,j) = p_{f}(g,h) \sum_{m=0}^{\infty} t_{ij}(m,g) f_{M,g}(m), \quad \text{for } i, j \in \mathcal{L} \quad \text{and} \quad g, h \in \mathcal{R},
\]

(2.2.7)

where \( f_{M,g}(m) \), for \( m \in \mathbb{N} \), is the P.M.F. of the claims number in the environment state \( g \).

2.3 Finite-Time Ruin Probabilities

In this section, we shall derive recursive formulae to compute the finite-time ruin probabilities that are defined by (2.2.3). We consider the two options of varying premiums separately.

2.3.1 Premiums Adjusted by Aggregate Claims

According to the assumptions in Section 2.2.1, the \( n \)-period finite-time ruin probability \( \psi_{i,g}(u,n) \), satisfies the following recursive formula.

**Theorem 1.** For \( u \geq 0, n \in \mathbb{N}^+, i \in \mathcal{L} \) and \( g \in \mathcal{R} \),

\[
\psi_{i,g}(u,n + 1) = \sum_{h=1}^{r} p_{f}(g,h) \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_{i,g}} t_{ij}(s,g) \psi_{j,h}(u+\alpha_{i,g} - s,n) f_{S,g}(s) + \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s) \tag{2.3.8}
\]

with \( \psi_{i,g}(u,1) = \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s) \).
Proof. From (2.2.3), we have

$$
\psi_{i,g}(u,n+1) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid L_1 = c_i, J_1 = g \right\}
$$

$$
= \sum_{s=0}^{u+\alpha_{i,g}} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \mid L_1 = c_i, J_1 = g, S_1 = s \right\} f_{S,g}(s) + \sum_{s=u+c_i \mu_{S,h+1}}^{\infty} f_{S,g}(s)
$$

$$
= \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_{i,g}} t_{ij}(s,g) \mathbb{P}_u^{u+\alpha_{i,g}-s} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \mid L_2 = c_j, J_1 = g \right\} f_{S,g}(s) + \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s)
$$

$$
= \sum_{h=1}^{r} p_{ij}(g,h) \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_{i,g}} t_{ij}(s,g) \mathbb{P}_u^{u+\alpha_{i,g}-s} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \mid L_2 = c_j, J_2 = h \right\} f_{S,g}(s) + \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s)
$$

Since $\psi_{i,g}(u,1)$ only measures the probability of ruin of the business within one time period, the verification of the given boundary condition is trivial. \(\square\)

2.3.2 Premiums Adjusted by Claim Frequency

For this section, we change the transition rule for premium adjustment from considering the aggregate claims to number of claims. According to the assumptions in Section 2.2.2, the $n$-period-finite-time ruin probability with premiums adjusted according to claims numbers, initial premium level $i$ and initial environment state $g$ satisfies the following the recursive formulae.
Theorem 2. For $u \geq 0$, $n \in \mathbb{N}^+$, $i \in \mathcal{L}$ and $g \in \mathbb{R}$,
\begin{align}
\psi_{i,g}(u, n + 1) &= \sum_{h=1}^{r} p_{J}(g, h) \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_{i,g}} f_{M,g}(m) t_{ij}(m, g) \sum_{s=m}^{u+\alpha_{i,g}} f_{W,g}^{m}(s) \psi_{j,h}(u + \alpha_{i,g} - s, n) \\
&\quad + \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s)
\end{align}
(2.3.9)
with $\psi_{i,g}(u, 1) = \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s)$.

Proof. From (2.2.3), we have, for $u \geq 0$, $n \in \mathbb{N}^+$, $i \in \mathcal{L}$ and $g \in \mathbb{R}$,
\begin{align}
\psi_{i,g}(u, n + 1) &= \mathbb{P}_u \{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, J_1 = g \} \\
&= \sum_{m=0}^{\infty} f_{M,g}(m) \sum_{s=0}^{\infty} f_{W,g}^{m}(s) \mathbb{P}_u \{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, J_1 = g, S_1 = s \} \\
&= \sum_{m=0}^{\infty} f_{M,g}(m) \left( \sum_{s=\alpha_{i,g}+u+1}^{\infty} f_{W,g}^{m}(s) \\
&\quad + \sum_{h=1}^{r} p_{J}(g, h) \sum_{s=0}^{\infty} f_{W,g}^{m}(s) \mathbb{P}_u \{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, J_1 = g, S_1 = s \} \right) \\
&= \sum_{s=\alpha_{i,g}+u+1}^{\infty} f_{S,g}(s) + \sum_{h=1}^{r} p_{J}(g, h) \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_{i,g}} f_{M,g}(m) t_{ij}(m, g) \sum_{s=m}^{u+\alpha_{i,g}} f_{W,g}^{m}(s) \\
&\quad \times \mathbb{P}_u \{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| L_2 = c_j, J_2 = h \} \\
&= \sum_{h=1}^{r} p_{J}(g, h) \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_{i,g}} f_{M,g}(m) t_{ij}(m, g) \sum_{s=m}^{u+\alpha_{i,g}} f_{W,g}^{m}(s) \psi_{j,h}(u + \alpha_{i,g} - s, n) \\
&\quad + \sum_{s=\alpha_{i,g}+u+1}^{\infty} f_{S,g}(s).
\end{align}
Again, since $\psi_{i,g}(u, 1)$ only measures the probability of ruin of the business within one time period, the verification of the given boundary condition is trivial. 

2.4 Lundberg Inequalities for Ruin Probabilities

In previous sections, we studied the finite-time ruin probabilities, which reflect the risk of ruin for insurers within finite terms. However, in practice, insurers and insurance regulators also concern about the risk of ruin in the long term, where ultimate ruin probabilities are the appropriate measurement instead of the finite-time ones. The ultimate ruin probabilities can be defined by letting \( n = \infty \) in (2.2.3), which is denoted by \( \psi_{i,g}(u) \), \( i \in \mathcal{L} \) and \( g \in \mathbb{R} \). Due to the general settings of premium changing rules and aggregate claim distributions, neither explicit results nor recursive formulae for the ultimate ruin probabilities can be obtained easily. How to extend our finite-time ruin probability recursive formulae to the infinite-time context remains an open problem for the future.

Instead of calculating the ultimate ruin probabilities directly, we would derive some upper bounds for them in this section. It is inspired by the Lundberg inequality result in the classical risk model. Some Lundberg-type upper bounds, named as generalised Lundberg inequalities, are obtained with the induction method in both cases of premium variations. A similar approach was used in Cai and Dickson (2004). To calculate the generalised Lundberg inequalities, the stationary distribution \( \pi \) defined in Section 2.2.1 is needed, which represents the long-term probabilities of premium levels and environment states.

We consider the case of premiums adjusted by aggregate claims first. Following the classical risk model Lundberg inequality approach, to derive an upper bound for the ultimate-time ruin probabilities, we need to find the corresponding adjustment coefficient for our generalised risk model first. Let \( \gamma_{i,g} > 0 \), \( i \in \mathcal{L}, g \in \mathbb{R} \), be constants satisfying the following equation,

\[
e^{-\gamma_{i,g} \alpha_{i,g}} \mathbb{E}[e^{\gamma_{i,g} S_1} | J_1 = g] = 1.
\]

Then \( \gamma = \inf_{i \in \mathcal{L}, g \in \mathbb{R}} \{\gamma_{i,g}\} \) is a generalised adjustment coefficient. It can be shown that, by the log-convexity property of the moment generating functions, for any \( i \in \mathcal{L}, g \in \mathbb{R} \),

\[
e^{-\gamma \alpha_{i,g}} \mathbb{E}[e^{\gamma S_1} | J_1 = g] \leq 1.
\]

We can obtain the following main result:
Theorem 3. Let $\gamma > 0$ be the generalised adjustment coefficient defined above, then for any $i \in \mathcal{L}, g \in \mathbb{R}$,

$$\psi_{i,g}(u) \leq \beta e^{-\gamma u}, \quad (2.4.12)$$

where $\beta = \sup_{t \geq 0; g \in \mathbb{R}} e^{\gamma t} \sum_{s=t+1}^{\infty} e^{\gamma s} f_{S,g}(s)$.

Proof. Firstly, we need to prove by induction that for any $n > 0$, $\psi_{i,g}(u,n) \leq \beta e^{-\gamma u}$.

When $n = 1$, we have, for $i \in \mathcal{L}, g \in \mathbb{R}$,

$$\psi_{i,g}(u,1) = \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s)$$

$$= e^{-\gamma(u+\alpha_{i,g})} \sum_{s=u+\alpha_{i,g}+1}^{\infty} e^{\gamma s} f_{S,g}(s) \left[ e^{\gamma(u+\alpha_{i,g})} \sum_{t=u+\alpha_{i,g}+1}^{\infty} e^{\gamma t} f_{S,g}(t) \right]$$

$$\leq \beta e^{-\gamma(u+\alpha_{i,g})} \sum_{s=u+\alpha_{i,g}+1}^{\infty} e^{\gamma s} f_{S,g}(s)$$

$$\leq \beta e^{-\gamma u} e^{-\gamma \alpha_{i,g}} \mathbb{E} \left[ e^{\gamma S_1} | J_1 = g \right]$$

$$\leq \beta e^{-\gamma u}. \quad (2.4.13)$$

Assume the result holds true for the case of $n \geq 1$, i.e.,

$$\psi_{i,g}(u,n) \leq \beta e^{-\gamma u}, \quad i \in \mathcal{L}, g \in \mathbb{R}.$$
We only need to show that it also holds for the case of \( n + 1 \). From (2.3.8) and (2.4.13), we have

\[
\psi_{i,g}(u, n + 1) = \sum_{h=1}^{r} \sum_{j=1}^{l} p_{j}(g, h) t_{ij}(s, g) \psi_{j,h}(u + \alpha_{i,g} - s, n) f_{S,g}(s) \\
+ \sum_{s=u+\alpha_{i,g}+1}^{\infty} f_{S,g}(s) \\
\leq \beta \sum_{s=0}^{u+\alpha_{i,g}} e^{-\gamma(u+\alpha_{i,g} - s)} \sum_{h=1}^{r} p_{j}(g, h) \sum_{j=1}^{l} t_{ij}(s, g) f_{S,g}(s) \\
+ \beta \sum_{s=u+\alpha_{i,g}+1}^{\infty} e^{-\gamma(u+\alpha_{i,g} - s)} f_{S,g}(s) \\
= \beta \sum_{s=0}^{\infty} e^{-\gamma(u+\alpha_{i,g} - s)} f_{S,g}(s) \\
= \beta e^{-\gamma u} e^{-\gamma \alpha_{i,g}} E[e^{\gamma S_1} | J_1 = g] \\
\leq \beta e^{-\gamma u}.
\]

By induction, we conclude that for any \( n > 0 \),

\[
\psi_{i,g}(u, n) \leq \beta e^{-\gamma u}.
\]

Since \( \psi_{i,g}(u) = \lim_{n \to \infty} \psi_{i,g}(u, n) \), this upper bound also holds for \( \psi_{i,g}(u) \).

**Remark.** From the similarity between the recursive formulae given in Theorems 1 and 2, one can see that the Lundberg inequality (2.4.12) also applies to the ultimate ruin probabilities in the case of premiums adjusted by claims frequency.

### 2.5 The Joint Distribution of Premium Level and Environment State at Ruin

In this section, given that ruin occurs, the joint distribution of the premium level and environment state at ruin is studied.
2.5.1 Premiums Adjusted by Aggregate Claims

Let $T_u = \min \{ k : U_k < 0 \mid U_0 = u \}$ be the time of ruin with initial surplus $u$. Define

$$\chi_{i,g}(u,n,j,h) = \mathbb{P}_u\{ T_u \leq n, L_{T_u} = c_j, J_{T_u} = h \mid L_1 = c_i, J_1 = g \}$$

to be the probability that ruin occurs within the first $n$ time periods with the premium level $c_j$ and the environment state $h$ at ruin given the initial surplus $u$, initial premium level $i$ and initial environment state $g$, where $u \geq 0$, $n \in \mathbb{N}^+$, $i, j \in \mathcal{L}$ and $g, h \in \mathbb{R}$. Then, the joint probability of premium level $c_j$ and environment state $h$ at ruin given that ruin occurs within the next $n$ periods with initial surplus $u$, initial premium level $c_i$ and initial environment state $g$, denoted by $\chi'_{i,g}(u,n,j,h)$, takes the form

$$\chi'_{i,g}(u,n,j,h) = \frac{\chi_{i,g}(u,n,j,h)}{\psi_{i,g}(u,n)},$$

where $\psi_{i,g}(u,n)$ can be computed using (2.3.8).

A trivial relationship between $\chi_{i,g}(u,n,j,h)$ and $\psi_{i,g}(u,n)$ is that, for $u \geq 0$, $n \in \mathbb{N}^+$, $i, j \in \mathcal{L}$ and $g, h \in \mathbb{R}$,

$$\sum_{j=1}^l \sum_{h=1}^r \chi_{i,g}(u,n,j,h) = \psi_{i,g}(u,n).$$

Parallel to Theorem 1, we can show that $\chi_{i,g}(u,n,j,h)$ satisfies the following recursive formula.

**Theorem 4.** For $u \geq 0$, $n \in \mathbb{N}^+$, $i, j \in \mathcal{L}$ and $g, h \in \mathbb{R}$,

$$\chi_{i,g}(u,n+1,j,h) = \sum_{h'=1}^r p_{J}(g,h') \sum_{s=0}^{u+c_i} f_{S,g}(s) \sum_{j'=1}^l t_{ij'}(s,g) \chi_{j',h'}(u+c_i-s,n,j,h)$$

$$+ \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s), \quad (2.5.14)$$

where $\chi_{i,g}(u,1,j,h) = \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s)$. 
Proof. Following the definition of $\chi_{i,g}(u, n, j, h)$, we have, for $u \geq 0$, $n \in \mathbb{N}^+$, $i, j \in L$ and $g, h \in R$,

$$
\chi_{i,g}(u, n + 1, j, h)
= \mathbb{P}_u \{ T_u \leq n + 1, L_{T_u} = c_j, J_{T_u} = h \mid L_1 = c_i, J_1 = g \}
= \sum_{h' = 1}^{r} p_J(g, h') \left[ \sum_{s = u + c_i}^{u + c_i} f_{S,g}(s) + \sum_{s = u + c_i + 1}^{\infty} f_{S,g}(s) \right]
\times \mathbb{P}_u \{ T_u \leq n + 1, L_{T_u} = c_j, J_{T_u} = h \mid L_1 = c_i, J_1 = h', S_1 = s \}
= \mathbb{1}_{\{i = j\}} \mathbb{1}_{\{g = h\}} \sum_{s = u + c_i + 1}^{\infty} f_{S,g}(s) + \sum_{h' = 1}^{r} \sum_{s = u + c_i}^{u + c_i} f_{S,g}(s) \sum_{j' = 1}^{l} t_{ij'}(s, g)
\times \mathbb{P}_u \{ T_u \leq n + 1, L_{T_u} = c_j, J_{T_u} = h \mid L_2 = c_j', J_2 = h', S_1 = s \}
= \mathbb{1}_{\{i = j\}} \mathbb{1}_{\{g = h\}} \sum_{s = u + c_i + 1}^{\infty} f_{S,g}(s) + \sum_{h' = 1}^{r} \sum_{s = u + c_i}^{u + c_i} f_{S,g}(s) \sum_{j' = 1}^{l} t_{ij'}(s, g)
\times \mathbb{P}_u + c_i \leq n, L_{T_u + c_i - s} = c_j, J_{T_u + c_i - s} = h \mid L_2 = c_j', J_2 = h'
= \sum_{h' = 1}^{r} \sum_{s = 0}^{u + c_i} f_{S,g}(s) \sum_{j' = 1}^{l} t_{ij'}(s, g) \chi_{j', h'}(u + c_i - s, n, j, h)
+ \mathbb{1}_{\{i = j\}} \mathbb{1}_{\{g = h\}} \sum_{s = u + c_i + 1}^{\infty} f_{S,g}(s).
$$

The verification of the boundary condition is trivial. \qed

### 2.5.2 Premiums Adjusted by Claim Frequency

The notations and definitions are same as the case of premiums adjusted according to aggregate claims. The corresponding $\chi'_{i,g}(u, n, j, h)$ is also computed by $\chi_{i,g}(u, n, j, h)$, where $\psi_{i,g}(u, n)$ is calculated by (2.3.9) and $\chi_{i,g}(u, n, j, h)$ satisfied the following recursive formulae.
Theorem 5. For $u \geq 0$, $n \in \mathbb{N}^+$, $i, j \in \mathcal{L}$ and $g, h \in \mathbb{R}$,

$$\chi_{i,g}(u,n+1,j,h) = \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s) + \sum_{h'=1}^{r} p_j(g,h') \sum_{m=0}^{u+c_i} f_{M,g}(m) \times \sum_{s=m}^{u+c_i} f_{W,g}^m(s) \sum_{j'=1}^{l} t_{ij'}(m,g) \chi_{j',h'}(u+c_i-s,n,j,h), \quad (2.5.15)$$

where $\chi_{i,g}(u,1,j,h) = \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s)$.

**Proof.** Similar to the proof of Theorem 4, we have, for $u \geq 0$, $n \in \mathbb{N}^+$, $i, j \in \mathcal{L}$ and $g, h \in \mathbb{R}$,

$$\chi_{i,g}(u,n+1,j,h)$$

$$= \mathbb{P}_u \left\{ T_u \leq n+1, L_{T_u} = c_j, J_{T_u} = h \mid L_1 = c_i, J_1 = g \right\}$$

$$= \sum_{h'=1}^{r} p_j(g,h') \sum_{m=0}^{\infty} f_{M,g}(m) \sum_{s=0}^{\infty} f_{W,g}^m(s) \times \mathbb{P}_u \left\{ T_u \leq n+1, L_{T_u} = c_j, J_{T_u} = h \mid L_1 = c_i, J_2 = h', S_1 = s \right\}$$

$$= \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s) + \sum_{h'=1}^{r} p_j(g,h') \sum_{m=0}^{u+c_i} f_{M,g}(m) \sum_{s=m}^{u+c_i} f_{W,g}^m(s) \times \mathbb{P}_u \left\{ T_u \leq n+1, L_{T_u} = c_j, J_{T_u} = h \mid L_1 = c_i, J_2 = h', S_1 = s \right\}$$

$$= \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s) + \sum_{h'=1}^{r} p_j(g,h') \sum_{m=0}^{u+c_i} f_{M,g}(m) \sum_{s=m}^{u+c_i} f_{W,g}^m(s) \sum_{j'=1}^{l} t_{ij'}(m,g) \chi_{j',h'}(u+c_i-s,n,j,h)$$

$$\times \mathbb{P}_u \left\{ T_u \leq n+1, L_{T_u} = c_j, J_{T_u} = h \mid L_2 = c_{j'}, J_2 = h' \right\}$$

$$= \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s) + \sum_{h'=1}^{r} p_j(g,h') \sum_{m=0}^{u+c_i} f_{M,g}(m) \sum_{s=m}^{u+c_i} f_{W,g}^m(s) \sum_{j'=1}^{l} t_{ij'}(m,g) \chi_{j',h'}(u+c_i-s,n,j,h)$$

$$\times \mathbb{P}_u \left\{ T_{u+c_i-s} \leq n, L_{T_{u+c_i-s}} = c_j, J_{T_{u+c_i-s}} = h \mid L_2 = c_{j'}, J_2 = h' \right\}$$

$$= \sum_{h'=1}^{r} p_j(g,h') \sum_{m=0}^{u+c_i} f_{M,g}(m) \sum_{s=m}^{u+c_i} f_{W,g}^m(s) \sum_{j'=1}^{l} t_{ij'}(m,g) \chi_{j',h'}(u+c_i-s,n,j,h)$$

$$+ \mathbb{1}_{\{i=j\}} \mathbb{1}_{\{g=h\}} \sum_{s=u+c_i+1}^{\infty} f_{S,g}(s).$$

The verification of the boundary condition is trivial. □
2.6 Some Numerical Results

In this section we shall provide two numerical examples that represent the two varying premium cases discussed previously in this chapter and the numerical results regarding the ruin probabilities are given with some concluding remarks.

2.6.1 An Example for Premiums Adjusted by Aggregate Claims

This example is designed for the case that premiums are adjusted according to aggregate claims. We assume that the economic state is fixed over a year from the beginning, and the economic state affects the aggregate claims distribution in the year. It implies that the yearly aggregate claims distribution changes whenever the economic state changes. The aggregate claims are assumed to be negative binomial distributed as follows:

- Economic state 1 (normal): mean = 10, variance = 101.743;
- Economic state 2 (deflation): mean = 5, variance = 54.664;

The one-step transition probability matrix of the economic state is

\[
P_J = \begin{bmatrix} p_J(g,h) \end{bmatrix}_{g,h=1,2,3} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.65 & 0.05 \\ 0.3 & 0.05 & 0.65 \end{bmatrix},
\]

with stationary distribution \( \lambda = [\lambda_1 = 0.6, \lambda_2 = 0.2, \lambda_3 = 0.2] \). The expected long-term aggregate claim amount is: \((0.6 \times 10) + (0.2 \times 5) + (0.2 \times 15) = 10\).

Suppose the set of premium loading is \( c = \{120\%, 140\%, 160\%, 180\%, 200\%\} \). The rules for adjusting the premiums are given as follows.

1. If the recorded aggregate claims in the current period is no more than the 30th percentile of the aggregate claim distribution, then the premium level for the next period will move to the lower premium level or stay in the lowest one;
2. If the recorded aggregate claims in the current period is more than the 30th percentile but no more than the 70th percentile of the aggregate claim distribution, then the premium level for the next period will remain in the current premium level;

3. If the recorded aggregate claims in the current period is more than the 70th percentile of the aggregate claim distribution, then the premium level for the next period will move to the higher premium level or stay in the highest one.

Remark.

- The transition matrix of economic state given above is a hypothetical one. Different matrices will generate different sequences of premiums in the future. How to obtain a reliable estimate of such a transition matrix in real-life is beyond the scope of this study. Econometric studies could possibly provide answers to this question.

- The above set of premium rules is again a hypothetical one and is a much simplified version of the real-life bonus-malus rules. This helps to simplify the computational process and can sufficiently showcase our key results obtained in the main text before.

- The premium loadings given in \( c \) do not indicate the bonus or malus cases directly. Only when the initial premium level (or base level) is chosen, then we can tell whether a given premium level is a bonus (lower than base level) or a malus case (higher than base level).

According to the above transition rule, we can calculate the transition matrix among premium levels, i.e., \( P_C = [p_{C,g,h}(i,j)]_{(l \times r) \times (l \times r)} \), defined in (2.2.6) in Appendix A. Using \( P_C \), we can find the long-term stationary joint distribution of the premium levels and economics conditions:

\[
\pi = [\pi_{ig}] \in \mathcal{L} \times \mathbb{R} = \\
\begin{bmatrix}
0.1270, & 0.1234, & 0.1199, & 0.1165, & 0.1132 \\
0.0421, & 0.0411, & 0.0400, & 0.0389, & 0.0379 \\
0.0424, & 0.0411, & 0.0400, & 0.0388, & 0.0377
\end{bmatrix}
\]

The expected long-term premium income calculated from \( \pi \) is 15.89 per time unit, which is roughly 60% greater than the expected long-term aggregate claim
amount per time unit. Assuming the initial economic state is 1 (normal condition), using (2.3.8) we calculate the finite-time ruin probabilities $\psi_{i1}(u, 40), i = 1, \ldots, 5,$ and the associated Lundberg upper bounds for the ultimate ruin probabilities $\psi_{i}\gamma(u)$ can be found using (2.4.12). These results are summarised in Table 2.1 and Figure 2.2 below.

As shown in Table 2.1 and Figure 2.2, one can see that $\psi_{i1}(u, 40), i = 1, \ldots, 5$ are ordered by their initial premium levels, the lower the initial premium level is, the higher the finite-time ruin probabilities. Secondly, the initial surplus level $u$ has a significant impact on differentiating the five finite-time ruin probabilities: these probabilities differ more from each other when $u$ is small, but this impact tends to wear off when $u$ becomes larger. For example, for $u = 0, \psi_{11}(0, 40) = 0.581516,$ whereas $\psi_{51}(0, 40) = 0.220787$ (around 0.36 in difference). On the other hand, for $u = 50, \psi_{11}(50, 40) = 0.039369$ and $\psi_{51}(50, 40) = 0.007212$ (only 0.03 in difference). Its implications coincide with the practical concerns when choosing the initial premium level with different $u$ levels: a low initial premium level tends to be very risky when $u$ is small. However, when the initial surplus $u$ is sufficiently large, the insurer has more flexibility to lower the initial premium level that can attract more new policyholders and help with boosting the insurance business without significantly increasing the risk of insolvency.

One obvious observation in Figure 2.2 regarding the upper bound is that it is very loose for all five versions of finite-time ruin probabilities. An argument for this is that the generalised adjustment coefficient $\gamma$ adopted in (2.4.12) is quite conservative, since it is determined by the scenario that has the highest ruin probabilities by its definition. Also, the upper bound given in (2.4.12) is for the ultimate time ruin probabilities, so tend to be fairly loose for finite-time ruin probabilities with $n = 40$. In practice, the insurer can use these upper bounds to evaluate the worst-scenario risk of ruin in the long run.
Table 2.1: $\psi_{i,1}(u, 40)$ values with upper bounds (UB) under the aggregate claims principle

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_{1,1}(u, 40)$</th>
<th>$\psi_{2,1}(u, 40)$</th>
<th>$\psi_{3,1}(u, 40)$</th>
<th>$\psi_{4,1}(u, 40)$</th>
<th>$\psi_{5,1}(u, 40)$</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.581516</td>
<td>0.485600</td>
<td>0.370290</td>
<td>0.278787</td>
<td>0.220787</td>
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</tr>
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<td>10</td>
<td>0.346148</td>
<td>0.268051</td>
<td>0.189482</td>
<td>0.135426</td>
<td>0.106381</td>
<td>0.823486</td>
</tr>
<tr>
<td>20</td>
<td>0.202262</td>
<td>0.147489</td>
<td>0.097952</td>
<td>0.067067</td>
<td>0.052281</td>
<td>0.690207</td>
</tr>
<tr>
<td>30</td>
<td>0.117224</td>
<td>0.081516</td>
<td>0.051458</td>
<td>0.034011</td>
<td>0.026317</td>
<td>0.578500</td>
</tr>
<tr>
<td>40</td>
<td>0.067836</td>
<td>0.045466</td>
<td>0.027558</td>
<td>0.017698</td>
<td>0.013597</td>
<td>0.484872</td>
</tr>
<tr>
<td>50</td>
<td>0.039369</td>
<td>0.025658</td>
<td>0.015062</td>
<td>0.009450</td>
<td>0.007212</td>
<td>0.406397</td>
</tr>
<tr>
<td>70</td>
<td>0.013508</td>
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<td>0.200560</td>
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<td>120</td>
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<td>0.000340</td>
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<td>0.118091</td>
</tr>
<tr>
<td>150</td>
<td>0.000240</td>
<td>0.000144</td>
<td>0.000075</td>
<td>0.000043</td>
<td>0.000031</td>
<td>0.069532</td>
</tr>
<tr>
<td>200</td>
<td>0.000021</td>
<td>0.000012</td>
<td>0.000006</td>
<td>0.000004</td>
<td>0.000003</td>
<td>0.028761</td>
</tr>
</tbody>
</table>

Figure 2.2: $\psi_{i,1}(u, 40)$ with the upper bounds under the aggregate claims principle

First of all, given each initial economic state, we make similar observations in the trending in finite-time ruin probabilities when $u$ and $i$ change. Following that, comparing the three cases with different initial economic states, we have some
interesting findings. Firstly, when \( u = 0 \) and three \( i \) cases of \( u = 10 \), the deflation economic state (state 2) leads to the highest finite-time ruin probabilities whilst the inflation economic state (state 3) has the lowest finite-time ruin probabilities. Secondly, when the initial premium level is at the lowest one \( (i = 1) \) and \( u \geq 10 \), the inflation economic state leads to the highest ruin probabilities whilst the deflation state has the lowest ones. For remaining cases, the inflation state usually ranks the highest in finite-time ruin probabilities and the normal economic state ranks the lowest. We struggle to find intuitive reasons behind these observations. One contributing factor is the initial premium amount. The ranking of ruin probabilities when \( u = 0 \) coincides with the ranking of the initial premium amounts correspond to the three initial economic states, which again confirms the significant role of the early premium income in keeping solvency when there is no capital buffer in the first place. Based on these observations, it is fair to say that both initial economic state and initial premium level play important roles in determining the finite-time ruin probabilities of the insurance business. Therefore, the insurers must keep a close eye on this matter when optimising their premium-changing strategies.

For the initial economics state 2 (deflation condition) and state 3 (inflation condition), the finite-time ruin probability results are given in Tables 2.2 and 2.3, respectively.

**Table 2.2:** \( \psi_{i,2}(u, 40) \) values with upper bounds (UB) under the aggregate claims principle

<table>
<thead>
<tr>
<th>( u )</th>
<th>( \psi_{1,2}(u, 40) )</th>
<th>( \psi_{2,2}(u, 40) )</th>
<th>( \psi_{3,2}(u, 40) )</th>
<th>( \psi_{4,2}(u, 40) )</th>
<th>( \psi_{5,2}(u, 40) )</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.602651</td>
<td>0.530232</td>
<td>0.432010</td>
<td>0.346695</td>
<td>0.290467</td>
<td>0.982500</td>
</tr>
<tr>
<td>10</td>
<td>0.340618</td>
<td>0.280003</td>
<td>0.210953</td>
<td>0.159843</td>
<td>0.132489</td>
<td>0.823486</td>
</tr>
<tr>
<td>20</td>
<td>0.194130</td>
<td>0.151662</td>
<td>0.107550</td>
<td>0.077895</td>
<td>0.063776</td>
<td>0.690207</td>
</tr>
<tr>
<td>30</td>
<td>0.110690</td>
<td>0.083187</td>
<td>0.056257</td>
<td>0.039292</td>
<td>0.031786</td>
<td>0.578500</td>
</tr>
<tr>
<td>40</td>
<td>0.063296</td>
<td>0.046186</td>
<td>0.030090</td>
<td>0.020401</td>
<td>0.016316</td>
<td>0.484872</td>
</tr>
<tr>
<td>50</td>
<td>0.036402</td>
<td>0.025979</td>
<td>0.016437</td>
<td>0.010875</td>
<td>0.008605</td>
<td>0.406397</td>
</tr>
<tr>
<td>70</td>
<td>0.012333</td>
<td>0.008554</td>
<td>0.005196</td>
<td>0.003313</td>
<td>0.002573</td>
<td>0.285494</td>
</tr>
<tr>
<td>90</td>
<td>0.004325</td>
<td>0.002954</td>
<td>0.001750</td>
<td>0.001087</td>
<td>0.000832</td>
<td>0.200560</td>
</tr>
<tr>
<td>120</td>
<td>0.000946</td>
<td>0.000638</td>
<td>0.000369</td>
<td>0.000223</td>
<td>0.000168</td>
<td>0.118091</td>
</tr>
<tr>
<td>150</td>
<td>0.000215</td>
<td>0.000143</td>
<td>0.000082</td>
<td>0.000049</td>
<td>0.000036</td>
<td>0.069532</td>
</tr>
<tr>
<td>200</td>
<td>0.000019</td>
<td>0.000012</td>
<td>0.000007</td>
<td>0.000004</td>
<td>0.000003</td>
<td>0.028761</td>
</tr>
</tbody>
</table>
Table 2.3: $\psi_{i,3}(u, 40)$ values with upper bounds (UB) under the aggregate claims principle

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_{1,3}(u, 40)$</th>
<th>$\psi_{2,3}(u, 40)$</th>
<th>$\psi_{3,3}(u, 40)$</th>
<th>$\psi_{4,3}(u, 40)$</th>
<th>$\psi_{5,3}(u, 40)$</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.536216</td>
<td>0.441881</td>
<td>0.338071</td>
<td>0.259681</td>
<td>0.209647</td>
<td>0.982500</td>
</tr>
<tr>
<td>10</td>
<td>0.362565</td>
<td>0.284586</td>
<td>0.209476</td>
<td>0.157582</td>
<td>0.127362</td>
<td>0.823486</td>
</tr>
<tr>
<td>20</td>
<td>0.240562</td>
<td>0.181306</td>
<td>0.129259</td>
<td>0.095593</td>
<td>0.077312</td>
<td>0.690207</td>
</tr>
<tr>
<td>30</td>
<td>0.157427</td>
<td>0.114621</td>
<td>0.079529</td>
<td>0.057972</td>
<td>0.046900</td>
<td>0.578500</td>
</tr>
<tr>
<td>40</td>
<td>0.101979</td>
<td>0.072065</td>
<td>0.048833</td>
<td>0.035150</td>
<td>0.028439</td>
<td>0.484872</td>
</tr>
<tr>
<td>50</td>
<td>0.065557</td>
<td>0.045126</td>
<td>0.029942</td>
<td>0.021312</td>
<td>0.017240</td>
<td>0.406397</td>
</tr>
<tr>
<td>70</td>
<td>0.026650</td>
<td>0.017546</td>
<td>0.011225</td>
<td>0.007835</td>
<td>0.006334</td>
<td>0.285494</td>
</tr>
<tr>
<td>90</td>
<td>0.010669</td>
<td>0.006769</td>
<td>0.004198</td>
<td>0.002881</td>
<td>0.002327</td>
<td>0.200560</td>
</tr>
<tr>
<td>120</td>
<td>0.002651</td>
<td>0.001606</td>
<td>0.000957</td>
<td>0.000643</td>
<td>0.000519</td>
<td>0.118091</td>
</tr>
<tr>
<td>150</td>
<td>0.000647</td>
<td>0.000377</td>
<td>0.000217</td>
<td>0.000144</td>
<td>0.000116</td>
<td>0.069532</td>
</tr>
<tr>
<td>200</td>
<td>0.000000</td>
<td>0.000033</td>
<td>0.000018</td>
<td>0.000012</td>
<td>0.000009</td>
<td>0.028761</td>
</tr>
</tbody>
</table>

Next, we shall dig further into the relationship between $u$, the premium level and economic state at ruin given ruin occurs. As an example, we calculate the joint distribution of the premium level and economic state at ruin given the ruin occurs within 10 periods, i.e., $\chi'_{i,g}(0, 10, j, h)$, making use of the result (2.5.14).

**Scenario 1.** Let $u = 0$, $i = 1$ and $g = 1$. Values of $\chi'_{1,1}(0, 10, j, h)$ are summarised in Table 2.4.

Table 2.4: Results for $\chi'_{1,1}(0, 10, j, h)$ under the aggregate claims principle

<table>
<thead>
<tr>
<th>$j$ = 1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1$</td>
<td>0.758260</td>
<td>0.066721</td>
<td>0.017387</td>
<td>0.004019</td>
</tr>
<tr>
<td>$h = 2$</td>
<td>0.031033</td>
<td>0.015421</td>
<td>0.005128</td>
<td>0.001372</td>
</tr>
<tr>
<td>$h = 3$</td>
<td>0.062983</td>
<td>0.026394</td>
<td>0.007770</td>
<td>0.001865</td>
</tr>
</tbody>
</table>

In Table 2.4, $\chi'_{1,1}(0, 10, 1, 1)$ is the highest one among all 15 cases, much larger than all the other cases. It means that, given that $u = 0$ and ruin occurs by time 10, the most likely premium level and economic state combination at ruin is the same as the initial combination ($i = 1, g = 1$). The second and third most likely cases are ($j = 2, h = 1$) and ($j = 1, h = 3$) that are adjacent combinations of ($i = 1, g = 1$). This implies that without any capital buffer at the beginning and charging the lowest level of premium, if ruin occurs early, then it will occur within
the first few time units before making many transitions in either premium levels or economic states. Of course, the initial premium level and initial economic state as well as \( u \) all play an important roles in this. We shall continue this investigation in our next scenario.

**Scenario 2.** Keep \( u \) and \( g \) unchanged, but let \( i = 5 \), the highest level of premium. The corresponding results are given in Table 2.5.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( h = 1 )</th>
<th>( h = 2 )</th>
<th>( h = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 1 )</td>
<td>0.000113</td>
<td>0.000726</td>
<td>0.004665</td>
</tr>
<tr>
<td>( j = 2 )</td>
<td>0.000037</td>
<td>0.000231</td>
<td>0.001394</td>
</tr>
<tr>
<td>( j = 3 )</td>
<td>0.000408</td>
<td>0.001594</td>
<td>0.005976</td>
</tr>
</tbody>
</table>

In Table 2.5, \( \chi'_{5,1}(0, 10, 5, 1) \) is the highest one among all 15 cases, much larger than all the others. Again, given that \( u = 0 \) and ruin occurs by time 10, the most likely premium level and economic state combination at ruin is the same as the initial combination. The second and third most likely cases are \( (j = 5, h = 3) \) and \( (j = 4, h = 1) \), the adjacent combinations of \( (i = 5, g = 1) \). It looks like when \( u = 0 \), even charging the given highest level of premium, if we know ruin occurs early, then it will still occur within the first few time units. This is a similar observation to Scenario 1. This implies that, under our model assumptions, the initial surplus plays a more important role in the insolvency risk than the initial premium level. For the purpose of comparison, we shall consider one more scenario of \( u = 100 \) to see any different observations from the \( u = 0 \) scenarios.

**Scenario 3.** Let \( u = 100 \), \( i = 1 \) and \( g = 1 \). The corresponding results are given in Table 2.6.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( h = 1 )</th>
<th>( h = 2 )</th>
<th>( h = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = 1 )</td>
<td>0.016169</td>
<td>0.053910</td>
<td>0.071874</td>
</tr>
<tr>
<td>( j = 2 )</td>
<td>0.002011</td>
<td>0.008437</td>
<td>0.013755</td>
</tr>
<tr>
<td>( j = 3 )</td>
<td>0.098078</td>
<td>0.218514</td>
<td>0.204083</td>
</tr>
</tbody>
</table>

In contrast to the previous two scenarios, in Table 2.6, the top four most likely
combinations of premium level and economic state at ruin are \((j = 2, h = 3)\), \((j = 3, h = 3)\), \((j = 4, h = 3)\), and \((j = 1, h = 3)\). It implies that when the initial surplus is large, given ruin occurs, then ruin would most likely occur under the worst economic state, i.e., \(h = 3\), which by assumption has the highest expected aggregate claim amount per time unit. Again, the premium level at ruin seems relatively less influential than the economic state regarding insolvency. Moreover, the values of \(x'_{1,1}(100, 10, j, h)\) in Table 2.6 are more evenly spread out than the results in Tables 2.4 and 2.5. This is because with \(u = 100\), there are more chances that the insurance business could stay solvent in the first few periods and ruin would occur later. The longer the surplus process runs, then the less predictable it is, thus the premium level and economic state combination at ruin.

2.6.2 An Example for Premiums Adjusted by Claim Frequency

In this example, we shall consider automobile insurance business and we replace the external economic environment by weather conditions. We assume that the premium levels are adjusted according to the claim frequency and the weather condition has a significant impact on the claim frequency of automobile insurance policyholders.

- The claim frequency is modelled by Poisson distribution with mean 1.57, 0.785 and 2.355 for weather state 1 (normal condition), 2 (less severe condition) and 3 (severe weather condition) respectively.

- The one-step transition probability matrix of weather states (environment state) with the corresponding stationary probability distribution are the same as the ones for the external economic states in the previous example. Similar to the previous example, the weather state is assumed to be fixed over a year from the beginning and the premiums depend on the current weather state.

- The individual claim size distribution under each weather state is assumed to be geometric with P.M.F. \(f_W(w) = \left(\frac{1.57}{10}\right)(1 - \frac{1.57}{10})^{w-1}\) for \(w \geq 1\) and mean \(\frac{10}{1.57}\). Then the expected aggregate claim amount under weather state 1, 2, and 3 is 10, 5 and 15, respectively. The expected long-term aggregate claim amount is 10, which is the same as the one in the previous example as well.
We continue to use the same set of premium loading as the one in the previous example, i.e., \( c = \{120\%, 140\%, 160\%, 180\%, 200\%\} \). The rules for adjusting premiums are given as follows:

1. If the number of claims in the current period is 0, then the premium level for the next period will move to the lower premium level or stay in the lowest one;
2. If the number of claims in the current period is greater than 0 but no more than 2, then the premium level for the next period will remain in the current premium level;
3. If the number of claims in the current period is more than 2, then the premium level for the next period will move to the higher premium level or stay in the highest one.

According to the above transition rules for premium adjustments, we can also find its associated transition matrix among the premium levels in the Appendix A, and the following long-term stationary joint distribution of the premium levels and weather states:

\[
\pi = [\pi_{ig}]_{i \in L, g \in R} = \begin{bmatrix}
0.1429 & 0.1214 & 0.1119 & 0.1089 & 0.1150 \\
0.0702 & 0.0394 & 0.0350 & 0.0314 & 0.0241 \\
0.0328 & 0.0374 & 0.0373 & 0.0380 & 0.0545
\end{bmatrix}.
\]

Using \( \pi \), one can see that the expected long-term premium is around 15.9 per time unit, which is about 60% greater than the expected long-term aggregate claims per time unit. Remarkably, the expected long-term premium loading in this example is comparable to the one in the previous example. We make the expected long-term premium income and expected long-term aggregate claims of these two examples comparable on purpose such that a comparison is feasible between the two different types of premium transition rules regarding their impact on the ruin probabilities. Similarly, in the following we use the result (2.3.9) to calculate \( \psi_{i,1}(u, 40) \) and use (2.4.12) to calculate the upper bounds for the ultimate time ruin probabilities \( \psi_{i,1}(u) \). The results are summarised in Table 2.7 and Figure 2.3 below.
Table 2.7: $\psi_{i,1}(u, 40)$ values with upper bounds (UB) under the claim frequency principle

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_{1,1}(u, 40)$</th>
<th>$\psi_{2,1}(u, 40)$</th>
<th>$\psi_{3,1}(u, 40)$</th>
<th>$\psi_{4,1}(u, 40)$</th>
<th>$\psi_{5,1}(u, 40)$</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.605971</td>
<td>0.509785</td>
<td>0.394719</td>
<td>0.299570</td>
<td>0.235311</td>
<td>0.971992</td>
</tr>
<tr>
<td>10</td>
<td>0.388786</td>
<td>0.299805</td>
<td>0.209603</td>
<td>0.146053</td>
<td>0.110407</td>
<td>0.731630</td>
</tr>
<tr>
<td>20</td>
<td>0.236054</td>
<td>0.167432</td>
<td>0.106238</td>
<td>0.068367</td>
<td>0.050195</td>
<td>0.550706</td>
</tr>
<tr>
<td>30</td>
<td>0.137875</td>
<td>0.090424</td>
<td>0.052377</td>
<td>0.031307</td>
<td>0.022445</td>
<td>0.414523</td>
</tr>
<tr>
<td>40</td>
<td>0.078166</td>
<td>0.047692</td>
<td>0.025389</td>
<td>0.014180</td>
<td>0.009959</td>
<td>0.312016</td>
</tr>
<tr>
<td>50</td>
<td>0.043249</td>
<td>0.024708</td>
<td>0.012176</td>
<td>0.006393</td>
<td>0.004407</td>
<td>0.234858</td>
</tr>
<tr>
<td>70</td>
<td>0.012487</td>
<td>0.006372</td>
<td>0.002750</td>
<td>0.001299</td>
<td>0.000865</td>
<td>0.133065</td>
</tr>
<tr>
<td>90</td>
<td>0.003391</td>
<td>0.001581</td>
<td>0.000614</td>
<td>0.000266</td>
<td>0.000172</td>
<td>0.075391</td>
</tr>
<tr>
<td>120</td>
<td>0.000441</td>
<td>0.000186</td>
<td>0.000064</td>
<td>0.000025</td>
<td>0.000015</td>
<td>0.032152</td>
</tr>
<tr>
<td>150</td>
<td>0.000053</td>
<td>0.000021</td>
<td>0.000007</td>
<td>0.000002</td>
<td>0.000001</td>
<td>0.013712</td>
</tr>
<tr>
<td>200</td>
<td>0.000001</td>
<td>0.000001</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.003313</td>
</tr>
</tbody>
</table>

Figure 2.3: $\psi_{i,1}(u, 40)$ with the upper bounds under the claim frequency principle

Firstly, our main findings in Table 2.7 and Figure 2.3 are consistent with those in Table 2.1 and Figure 2.2 in terms of the impact of the initial premium level and initial surplus level on ruin probabilities. Secondly, given the same initial
Discrete-time risk models with claim correlated premiums in a Markovian environment

external environment state, when \( u \) is small the premium adjustment rules based on claim frequency have a negative impact on the finite-time ruin probabilities comparing with the rules by aggregate claim amounts. Also, this effect becomes more significant when the initial premium level is lower. For instance when \( i = 1 \) this effect applies to \( u \leq 50 \) but it only takes effect for \( u \leq 20 \) when \( i = 5 \). This is reasonable since the transition rules that adjust the premiums according to the claim frequency can not fully reflect the historical claims experience, which causes higher risk of ruin when \( u \) is small. On the contrary, when \( u \) is large enough, the premium rules by claim frequency seems to lower the insolvency risk of the insurer comparing with the other type of rules. This implies that, in practice, the insurers should carefully design their bonus-malus rules by taking into account factors like the capital adequacy level since the proposed rules could have significant impact on insolvency risk. Next, we shall investigate the finite-time ruin probabilities with initial weather state 2 and 3, which are shown in Tables 2.8 and 2.9.

**Table 2.8:** \( \psi_{i,2}(u, 40) \) values with upper bounds (UB) under the claim frequency principle

<table>
<thead>
<tr>
<th>( u )</th>
<th>( \psi_{1,2}(u, 40) )</th>
<th>( \psi_{2,2}(u, 40) )</th>
<th>( \psi_{3,2}(u, 40) )</th>
<th>( \psi_{4,2}(u, 40) )</th>
<th>( \psi_{5,2}(u, 40) )</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.647608</td>
<td>0.600217</td>
<td>0.511647</td>
<td>0.414121</td>
<td>0.332302</td>
<td>0.971992</td>
</tr>
<tr>
<td>10</td>
<td>0.410970</td>
<td>0.362287</td>
<td>0.281517</td>
<td>0.204813</td>
<td>0.150624</td>
<td>0.731630</td>
</tr>
<tr>
<td>20</td>
<td>0.251122</td>
<td>0.211238</td>
<td>0.150889</td>
<td>0.099501</td>
<td>0.067739</td>
<td>0.550706</td>
</tr>
<tr>
<td>30</td>
<td>0.148774</td>
<td>0.119900</td>
<td>0.079446</td>
<td>0.047947</td>
<td>0.030482</td>
<td>0.414523</td>
</tr>
<tr>
<td>40</td>
<td>0.085828</td>
<td>0.066553</td>
<td>0.041270</td>
<td>0.023024</td>
<td>0.013769</td>
<td>0.312016</td>
</tr>
<tr>
<td>50</td>
<td>0.048375</td>
<td>0.036238</td>
<td>0.021204</td>
<td>0.011040</td>
<td>0.006249</td>
<td>0.234858</td>
</tr>
<tr>
<td>70</td>
<td>0.014483</td>
<td>0.010237</td>
<td>0.005448</td>
<td>0.002533</td>
<td>0.001305</td>
<td>0.133065</td>
</tr>
<tr>
<td>90</td>
<td>0.004064</td>
<td>0.002744</td>
<td>0.001359</td>
<td>0.000580</td>
<td>0.000276</td>
<td>0.075391</td>
</tr>
<tr>
<td>120</td>
<td>0.000550</td>
<td>0.000353</td>
<td>0.000162</td>
<td>0.000063</td>
<td>0.000027</td>
<td>0.032152</td>
</tr>
<tr>
<td>150</td>
<td>0.000069</td>
<td>0.000042</td>
<td>0.000018</td>
<td>0.000007</td>
<td>0.000003</td>
<td>0.013712</td>
</tr>
<tr>
<td>200</td>
<td>0.000002</td>
<td>0.000001</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.003313</td>
</tr>
</tbody>
</table>
Table 2.9: $\psi_{i,3}(u, 40)$ values with upper bounds (UB) under the claim frequency principle

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_{1,3}(u, 40)$</th>
<th>$\psi_{2,3}(u, 40)$</th>
<th>$\psi_{3,3}(u, 40)$</th>
<th>$\psi_{4,3}(u, 40)$</th>
<th>$\psi_{5,3}(u, 40)$</th>
<th>UB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.555437</td>
<td>0.430304</td>
<td>0.315517</td>
<td>0.231635</td>
<td>0.179284</td>
<td>0.971992</td>
</tr>
<tr>
<td>10</td>
<td>0.354335</td>
<td>0.249205</td>
<td>0.167191</td>
<td>0.115193</td>
<td>0.087273</td>
<td>0.731630</td>
</tr>
<tr>
<td>20</td>
<td>0.212928</td>
<td>0.136511</td>
<td>0.084156</td>
<td>0.054739</td>
<td>0.040872</td>
<td>0.550706</td>
</tr>
<tr>
<td>30</td>
<td>0.122699</td>
<td>0.072029</td>
<td>0.040953</td>
<td>0.025268</td>
<td>0.018672</td>
<td>0.414523</td>
</tr>
<tr>
<td>40</td>
<td>0.068509</td>
<td>0.037011</td>
<td>0.019486</td>
<td>0.011455</td>
<td>0.008399</td>
<td>0.312016</td>
</tr>
<tr>
<td>50</td>
<td>0.037306</td>
<td>0.018650</td>
<td>0.009134</td>
<td>0.005138</td>
<td>0.003744</td>
<td>0.234858</td>
</tr>
<tr>
<td>70</td>
<td>0.010434</td>
<td>0.004543</td>
<td>0.001955</td>
<td>0.001019</td>
<td>0.000736</td>
<td>0.133065</td>
</tr>
<tr>
<td>90</td>
<td>0.002749</td>
<td>0.001066</td>
<td>0.000411</td>
<td>0.000202</td>
<td>0.000144</td>
<td>0.075391</td>
</tr>
<tr>
<td>120</td>
<td>0.000343</td>
<td>0.000116</td>
<td>0.000039</td>
<td>0.000018</td>
<td>0.000013</td>
<td>0.032152</td>
</tr>
<tr>
<td>150</td>
<td>0.000040</td>
<td>0.000012</td>
<td>0.000004</td>
<td>0.000002</td>
<td>0.000001</td>
<td>0.013712</td>
</tr>
<tr>
<td>200</td>
<td>0.000001</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.003313</td>
</tr>
</tbody>
</table>

According to the results in Tables 2.8 and 2.9, the trending of finite-time ruin probabilities when $u$ and $i$ change is still consistent with Table 2.7. In contrast to the previous premium-changing category (by aggregate claims), when the premiums change according to claim frequency experience, there is an overall consistent ranking in the finite-time ruin probabilities among the three initial weather conditions, i.e., less severe initial weather condition (state 2) leads to the highest ruin probabilities whilst the severe initial weather condition (state 3) has the lowest ruin probabilities. This ranking can be explained by the level of right-skewness of the Compound Poisson distributions corresponding to the aggregate claims under each weather condition. Apparently, state 2 has the highest right-skewness whilst state 3 has the lowest, which relates to the risk of insolvency.

At last, we shall use (2.5.15) to compute the joint distribution of the premium level and weather condition at ruin given ruin occurs within 10 time periods. The results are given in Tables 2.10–2.12, where the scenarios considered are the same as Scenario 1–3 in the previous example.
Table 2.10: Results for $\chi'_{1,1}(0, 10, j, h)$ under the claim frequency principle

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1$</td>
<td>0.788065</td>
<td>0.069503</td>
<td>0.012257</td>
<td>0.002256</td>
<td>0.000437</td>
</tr>
<tr>
<td>$h = 2$</td>
<td>0.041473</td>
<td>0.010975</td>
<td>0.002007</td>
<td>0.000384</td>
<td>0.000077</td>
</tr>
<tr>
<td>$h = 3$</td>
<td>0.045738</td>
<td>0.020304</td>
<td>0.005216</td>
<td>0.001088</td>
<td>0.000220</td>
</tr>
</tbody>
</table>

Table 2.11: Results for $\chi'_{5,1}(0, 10, j, h)$ under the claim frequency principle

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1$</td>
<td>0.000410</td>
<td>0.001394</td>
<td>0.005025</td>
<td>0.039455</td>
<td>0.863448</td>
</tr>
<tr>
<td>$h = 2$</td>
<td>0.000204</td>
<td>0.000681</td>
<td>0.002484</td>
<td>0.010320</td>
<td>0.034591</td>
</tr>
<tr>
<td>$h = 3$</td>
<td>0.000116</td>
<td>0.000389</td>
<td>0.001216</td>
<td>0.006851</td>
<td>0.033414</td>
</tr>
</tbody>
</table>

Table 2.12: Results for $\chi'_{1,1}(100, 10, j, h)$ under the claim frequency principle

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 1$</td>
<td>0.066714</td>
<td>0.193377</td>
<td>0.205458</td>
<td>0.115465</td>
<td>0.057055</td>
</tr>
<tr>
<td>$h = 2$</td>
<td>0.017125</td>
<td>0.033472</td>
<td>0.034556</td>
<td>0.020260</td>
<td>0.010142</td>
</tr>
<tr>
<td>$h = 3$</td>
<td>0.020572</td>
<td>0.071252</td>
<td>0.082757</td>
<td>0.047871</td>
<td>0.023924</td>
</tr>
</tbody>
</table>

From Tables 2.10 and 2.11, one can see that under Scenario 1 and Scenario 2 (both with $u = 0$) we get similar main findings to those in the previous example. It is worth noting that in Scenario 3 where $u = 100$, we get opposite findings comparing with our previous example. The top four most likely combinations of premium level and weather conditions at ruin are $(j = 3, h = 1)$, $(j = 2, h = 1)$, $(j = 4, h = 1)$ and $(j = 3, h = 3)$, mostly associated with the normal weather conditions under which the claim experience should be the middle one. This is totally different from the previous example where in Scenario 3 the most likely combinations come from the worst economic state under which the claim experience is the worst. The most likely explanation lies in the way premiums are being adjusted as well as the claim experience assumptions under different weather conditions. To be more specific, we assume that individual claim amounts are not affected by weather conditions, but claim frequency does. Under the normal weather condition ($i = 1$), the current premium level is likely to stay unchanged. However, under severe weather condition ($i = 3$), average claim frequency is high, which leads to moving the next premium level up. To a certain extent, the worsened claim experience is
off-set by the increased premium amount, which leads to a lower overall insolvency risk than the normal weather condition. We remark that this observation might not hold when the parameter assumptions are changed, which result in a different trade-off between the claim experience and premium adjustment.

Last but not least, we make a further comparison between the above two numerical examples, as the risk models constructed in those two examples are in general comparable, for example, equal average aggregate claim amounts as well as the same premium levels. We can see that the case of adjusting premiums according to claim frequency is riskier than the case of adjusting premiums according to aggregate claims under our assumptions. This finding also has some material implication for the insurance companies on how to choose an appropriate premium adjustment strategy.

2.7 Concluding remarks

In this chapter we considered a discrete-time risk model, which allows the premium to be adjusted according to claims experience. The premium correction was based on the well-known bonus-malus system and the claims experience was assumed to depend on an external Markovian environment (economic and/or natural environment). As a result, the evaluation of this unusual bonus-malus framework, which has non-homogeneous premium transition rules, became the main objective of this chapter. To have a better coverage, two types of premium changing criteria were examined throughout the chapter: aggregate claims criterion vs. claim frequency criterion. The basis of our evaluation is the risk of ruin for the proposed risk model with the given set of initial parameters, i.e., initial surplus, initial premium level and initial environment state. On one hand, recursive formulae were obtained to calculate the finite-time ruin probabilities, and the Lundberg-type upper bounds were derived to evaluate the ultimate ruin probabilities in both cases. On the other hand, the joint distribution of premium level and environment state at ruin was also studied. Through our numerical studies, we find that both the initial premium level and the initial environment state have a significant impact on the insolvency risk. We observed that there is no straightforward ordering on the insolvency risk among cases with different initial environmental state, which can be
seen as the consequence of a combined effect of non-homogeneous loss distributions, premium rule assumptions as well as the initial surplus level. This shed a light on the importance of determining the proper base premium level in a given external environment when insurers implement the bonus-malus system in premium corrections.
Chapter 3

Discrete-time risk models with time-delayed claims and varying premiums

3.1 Introduction

This chapter extends the risk models with claim dependent premiums by considering time-delayed claims. Regarding the principle of delayed claims, two types of individual claims are defined: main claims and by-claims. This definition was first introduced by Yuen and Guo (2001). We assume that one main claim is dependent with its associated by-claim and the settlement of the by-claim may be delayed for one time period under a certain probability $q$. There are four proposed transition rules for premium adjustment in this chapter: by aggregate reported claims, by aggregate settled claims, by reported claims number and by settled claims number. In addition, it is worth mentioning that if premiums are to be adjusted by the settled claim experience, then the underlying premium status process would display an in-homogeneous nature, because the transition probability between any two premium levels vary from time to time due to the uncertainty in the settled claims. This property differs from the homogeneity property of the premium status process should the premiums be adjusted by the reported claim experience. This interesting contrast makes our discussions in this chapter more realistic. This chapter aims to answer to following questions:
• What is the impact of the probability of claims settlement delays on the ruin probabilities?
• What is the impact of the correlation between the main claims and by-claims on the ruin probabilities?
• Which of the premium adjustment strategies should be implemented by the insurers?

3.2 Models and Assumptions

We first define a surplus process of discrete-times, denoted by $U_k$, as

$$U_k = U_0 + \sum_{t=1}^{k} (C_t - S_t), \quad k = 0, 1, \ldots, (3.2.1)$$

where $U_0 \in \mathbb{N}$ is the initial surplus, $S_t$ is the total amount of settled claims during the $t^{th}$ unit time period payable at time $t$, and $C_t$ is the premium of the $t^{th}$ period received at the beginning of the period. In this chapter we aim to study varying premiums. Let $c := \{c_1, c_2, \ldots, c_l\}$ be the set of premium levels and $\mathcal{L} = \{1, 2, \ldots, l\}$. Without losing generality, we let $c_1 < \ldots < c_l \in \mathbb{N}^+$.

As we mentioned previously, there are two types of reported individual claims, i.e. main claims and the associated by-claims. They are denoted by $X_t$ and $Y_t$ respectively for $t \in \mathbb{N}^+$. In this chapter, we only consider a very simple case where there is at most one main claim in any time period, and one main claim generates at most one by-claim. Both $\{X_t\}_{t \in \mathbb{N}^+}$ and $\{Y_t\}_{t \in \mathbb{N}^+}$ are i.i.d. sequences of random variables with common P.M.F. $f_X(x), x \in \mathbb{N}$, and $f_Y(y), y \in \mathbb{N}$, respectively. On the other hand, $X_t$ and $Y_t$ are assumed to be correlated with common joint P.M.F. $f_{XY}(x, y), x, y \in \mathbb{N}$. Not surprisingly, one can see that $f_{XY}(0, y) = 0$ for $y \neq 0$.

Assume that main claims are always settled at the end of the reporting time period, which is not the case for by-claims. When a by-claim $Y_k$ occurs, there is a probability $0 \leq q \leq 1$ that its settlement will be delayed to the end of the $(k+1)^{th}$
Thus, the aggregate claim amount settled in time period $t$ is

$$ S_t = \begin{cases} 
X_t & \text{if } Y_t \text{ is delayed; no delayed by-claim from } t-1, \\
X_t + Y_{t-1} & \text{if } Y_t \text{ is delayed; a delayed by-claim from } t-1, \\
X_t + Y_t & \text{if } Y_t \text{ is not delayed; no delayed by-claim from } t-1, \\
X_t + Y_t + Y_{t-1} & \text{if } Y_t \text{ is not delayed; a delayed by-claim from } t-1.
\end{cases} \tag{3.2.2} $$

For a given time horizon $n \in \mathbb{N}^+$, the finite-time ruin probability of $U_k$ with initial premium level $c_i$, for $i \in \mathcal{L}$, is defined as

$$ \psi_i(u, n) = P_u \left\{ \bigcup_{k=1}^n (U_k < 0) \Big| C_1 = c_i \right\}, \tag{3.2.3} $$

where the subscript $u$ represents the condition $U_0 = u$. We have $\psi_i(u, n) = 1$ for $u < 0, n \geq 0$ and $\psi_i(u, 0) = 0$ for $u \geq 0$ by convention.

**Remark.** We assume that there is no delayed by-claim from the time period before the initial time 0. Then, $S_1$ can only be $X_1$ or $X_1 + Y_1$.

Next we shall develop some recursive algorithm to compute the finite-time ruin probabilities under the above proposed risk framework. To enable our derivations, we define the following auxiliary surplus process with an up-front delayed by-claim

$$ U'_k = U_0 + \sum_{i=1}^k (C_i - S_i) - Y_0, \tag{3.2.4} $$

where $Y_0 > 0$ is the up-front delayed by-claim and other notations are exactly the same as those in model (3.2.1). Assume that $Y_0$ is independent of all other random components in model (3.2.4) and follows the P.M.F. $f_Y(y)$. The corresponding $n$-period finite-time ruin probability with initial premium level $c_i$, $i \in \mathcal{L}$, is defined as

$$ \psi'_i(u; z, n) = P_u \left\{ \bigcup_{k=1}^n (U'_k < 0) \Big| C_1 = c_i, Y_0 = z \right\}. \tag{3.2.5} $$

Again, $\psi'_i(u; z, n) = 1$ for $u < 0, n \geq 0$ and $\psi'_i(u; z, 0) = 0$ for $u \geq 0$ by convention.
In the following sections, we shall consider four different premium changing principles, i.e. premiums adjusted according to aggregate reported claims, premiums adjusted according to aggregate settled claims, premiums adjusted according to the reported claim frequency, and premiums adjusted according to the settled claim frequency, respectively.

\section{3.3 Premiums adjusted according to aggregate reported claims}

The premium changing rule considered in this section allows the next periodic premium to be determined based on the current premium level as well as the total reported claims in the current time period. In our previous model setting, we can see that the total reported claims in time period $k$ is $X_k + Y_k$. Whether the settlement of $Y_k$ is delayed or not does not have impact on the next periodic premium level.

We define a bonus-malus system $\Delta = (T, c, i)$, where $i \in \mathcal{L}$ is the state of initial premium level; $T = \{t_{ij}(s)\}_{i,j \in \mathcal{C}, s \in \mathbb{N}}$ denotes a general set of time-homogeneous rules for premium variations. For any $s \in \mathbb{N}$ and $k \in \mathbb{N}^+$, $t_{ij}(s) = 1$ if the total reported claim amount $s$ in time period $k$ leads to the transition from premium level $C_k = c_i$ to $C_{k+1} = c_j$ and $t_{ij}(s) = 0$ otherwise.

For any $k \in \mathbb{N}^+$, the probability that the premium level moves from level $c_i$ in time period $k$ to level $c_j$ in time period $k + 1$ is defined by

$$p_T(i, j) = \sum_{x=0}^{\infty} \sum_{y=0}^{\infty} t_{ij}(x+y) f_{XY}(x,y), \quad \text{for} \quad i, j \in \mathcal{L}. \quad (3.3.6)$$

Using (3.3.6), one can obtain a one-step transition probability matrix for the premium level Markov process

$$P_T = [p_T(i, j)]_{i \times l} = \begin{bmatrix} p_T(1, 1) & \cdots & p_T(1, l) \\ \vdots & \ddots & \vdots \\ p_T(l, 1) & \cdots & p_T(l, l) \end{bmatrix}.$$
Before we present our first main result, we would like to show a simple relationship between the two finite-time ruin probabilities defined before, which will benefit our following discussions.

**Lemma 1.** When premiums are adjusted according to aggregate reported claims, the finite-time ruin probabilities $\psi$ and $\psi'$ satisfy the following relationship, for $n \in \mathbb{N}^+$,

$$
\psi'_i(u; z, n) =
\begin{cases} 
\psi_i(u - z, n) & 0 < z \leq u, \\
\psi'_i(0; z - u, n) & u < z \leq u + c_i, \\
1 & z > u + c_i.
\end{cases}
$$

(3.3.7)

**Proof.**

Because the premiums are adjusted according to the total reported claims, the up-front delayed claim $Y_0$ has no impact on how the next premium is going to change.

When $0 < z \leq u$, it can be seen from (3.2.3) that $U_k'$ with initial surplus $u$ is equivalent to $U_k$ with initial surplus $u - z \geq 0$. So the first case of (3.3.7) holds.

When $z > u + c_i$, the delayed by-claim is large enough to cause ruin, no matter whether there is any new claim in time period 1.

Before we present our first main result, we introduce an auxiliary function that is used to simplify our main results given within the rest of this chapter:

$$
\xi_y(n + y) := \sum_{x=1}^{n} f_{XY}(x, y + n - x).
$$

Our first main result is given below.

**Theorem 6.** Given initial surplus $u \geq 0$ and initial premium level $c_i$, $i \in \mathcal{L}$, the finite-time ruin probability with premiums adjusted according to aggregate reported claims without the up-front delayed by-claim satisfies the following recursive formula,
for \( n \in \mathbb{N}^+ \),

\[
\psi_1(u, n + 1) = \sum_{j=1}^{l} \sum_{x=0}^{u+c_i} \sum_{y=0}^{u+c_i-x} t_{ij}(x+y) \psi_j(u+c_i-x-y, n) f_{X,Y}(x, y) \\
+ q \sum_{j=1}^{l} \sum_{y=1}^{c_i} t_{ij}(u+c_i+y) \psi_j'(0; y, n) \xi_y(u+c_i+y) \\
+ q \sum_{j=1}^{l} \sum_{y=c_i+1}^{\infty} t_{ij}(u+c_i+y) \xi_y(u+c_i+y) \\
+ (1-q) \sum_{y=1}^{\infty} \xi_y(u+c_i+y) + \sum_{x=u+c_i+1}^{\infty} f_X(x), 
\]

(3.3.8)

where \( \psi_1(u, 1) = (1-q) \sum_{y=1}^{\infty} \xi_y(u+c_i+y) + \sum_{x=u+c_i+1}^{\infty} f_X(x) \).

**Proof.** From (3.2.3), we have

\[
\psi_1(u, n + 1) \\
= \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \big| C_1 = c_i \right\} \\
= \sum_{x=u+c_i+1}^{\infty} \sum_{y=0}^{u+c_i-x} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \big| C_1 = c_i, X_1 = x \right\} f_X(x) \\
+ \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} \left[ (1-q) \mathbb{P}_u \left\{ U_1 < 0 \big| C_1 = c_i, X_1 = x, Y_1 = y \right\} \\
+ q \mathbb{P}_u \left\{ \bigcup_{k=2}^{n+1} (U_k' < 0) \big| C_1 = c_i, X_1 = x, Y_1 = y \right\} \right] f_{X,Y}(x, y) \\
= \sum_{x=u+c_i+1}^{\infty} f_X(x) + (1-q) \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} f_{X,Y}(x, y) \\
+ \sum_{j=1}^{l} \sum_{x=0}^{u+c_i} \sum_{y=0}^{u+c_i-x} t_{ij}(x+y) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \big| C_2 = c_j, X_1 = x, Y_1 = y \right\} f_{X,Y}(x, y) \\
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} t_{ij}(x+y) \mathbb{P}_u \left\{ \bigcup_{k=2}^{n+1} (U_k' < 0) \big| C_2 = c_j, X_1 = x, Y_1 = y \right\} f_{X,Y}(x, y) 
\]
Discrete-time risk models with time-delayed claims and varying premiums

\[
= \sum_{x=u+c_i+1}^{\infty} f_X(x) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y) \\
+ \sum_{j=1}^{l} \sum_{x=0}^{u+c_i-1} \sum_{y=0}^{u+c_i-x} t_{ij}(x+y) \mathbb{P}_{u+c_i-x-y}\left\{ \bigcup_{k=2}^{n+1} (U'_k < 0) \right\} f_{XY}(x,y) \\
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=0}^{u+c_i-x+1} t_{ij}(x+y) \mathbb{P}_{u+c_i-x} \left\{ \bigcup_{k=2}^{n+1} (U'_k < 0) \right\} f_{XY}(x,y) \\
= \sum_{x=u+c_i+1}^{\infty} f_X(x) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y) \\
+ \sum_{j=1}^{l} \sum_{x=0}^{u+c_i-1} \sum_{y=0}^{u+c_i-x} t_{ij}(x+y) \psi_j(u+c_i-x-y,n) f_{XY}(x,y) \\
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} t_{ij}(x+y) f_{XY}(x,y) \\
= \sum_{j=1}^{l} \sum_{x=0}^{u+c_i-1} \sum_{y=0}^{u+c_i-x} t_{ij}(x+y) \psi_j(u+c_i-x-y,n) f_{XY}(x,y) \\
+ q \sum_{j=1}^{l} \sum_{y=0}^{c_j} t_{ij}(u+c_i+y) \psi_j'(0; y, n) \xi_y(u+c_i+y) + \sum_{x=u+c_i+1}^{\infty} f_X(x) \\
+ q \sum_{j=1}^{l} \sum_{y=c_j+1}^{\infty} t_{ij}(u+c_i+y) \xi_y(u+c_i+y) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y).
One can also verify that

\[
\psi_i(u, 1) = \mathbb{P}_u \left\{ U_1 < 0 \mid C_1 = c_i \right\} = \sum_{x=u+c_i+1}^{\infty} \mathbb{P}_u \left\{ U_1 < 0 \mid C_1 = c_i, X_1 = x \right\} f_X(x) \\
+ \sum_{x=0}^{u+c_i} \mathbb{P}_u \left\{ U_1 < 0 \mid C_1 = c_i, X_1 = x, Y_1 = y \right\} f_{XY}(x, y) \\
+ \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} \left[ (1-q) \mathbb{P}_u \left\{ U_1 < 0 \mid C_1 = c_i, X_1 = x, Y_1 = y \right\} \\
+ q \mathbb{P}_u \left\{ U_1 < 0 \mid C_1 = c_i, X_1 = x, Y_1 = y \right\} \right] f_{XY}(x, y) \\
= (1-q) \sum_{y=1}^{\infty} \xi_y(u+c_i+y) + \sum_{x=u+c_i+1}^{\infty} f_X(x).
\]

**Remark.** From the definition of \( \xi_y(n+y) \), one can show that

\[
\sum_{y=1}^{\infty} \xi_y(n+y) = \sum_{x=1}^{n} \sum_{y=1}^{\infty} f_{XY}(x, n-x+y) \\
= \sum_{x=1}^{n} \left[ \sum_{y=0}^{\infty} f_{XY}(x, y) - \sum_{y=0}^{n-x} f_{XY}(x, y) \right] \\
= \sum_{x=1}^{n} \left[ f_X(x) - \sum_{y=0}^{n-x} f_{XY}(x, y) \right].
\]

Also, \( \sum_{x=u+c_i+1}^{\infty} f_X(x) = 1 - \sum_{x=0}^{u+c_i} f_X(x) \). Therefore, in the recursive formula given in Theorem 6, there is only one infinite summation left which requires extra attention when use it for computational purpose.

To use the recursive formula obtained in Theorem 6, we need to find a way to determine \( \psi_i'(0; z, n), 0 < z \leq u + c_i, n \in \mathbb{N}^+ \).

**Corollary 1.** The finite-time ruin probability with premiums adjusted according to aggregate reported claims and an up-front delayed by-claim \( z \) satisfies the following recursive formula, for \( 0 < z \leq u + c_i \) and \( n \in \mathbb{N}^+ \),
\[
\psi_i'(0; z, n + 1) = \sum_{j=1}^{l} \sum_{x=0}^{c_i - z} \sum_{y=0}^{c_i - z - x} t_{ij}(x + y) \psi_j(c_i - z - x - y, n) f_{XY}(x, y) \\
+ q \sum_{j=1}^{l} \sum_{y=1}^{c_i} t_{ij}(c_i - z + y) \psi_j'(0; y, n) \xi_y(c_i - z + y) \\
q \sum_{j=1}^{l} \sum_{y=c_j+1}^{\infty} t_{ij}(c_i - z + y) \xi_y(c_i - z + y) \\
+(1 - q) \sum_{y=1}^{\infty} \xi_y(c_i - z + y) + \sum_{x=c_i - z + 1}^{\infty} f_X(x), \quad (3.3.9)
\]

where \( \psi_i'(0; z, 1) = (1 - q) \sum_{y=1}^{\infty} \xi_y(c_i - z + y) + \sum_{x=c_i - z + 1}^{\infty} f_X(x). \)

**Proof.** For \( u < z \leq u + c_i \), the same method in the proof of Theorem 6 can be used to derive a recursive formula for \( \psi_i'(u; z, n + 1) \). Using (3.3.7), (3.3.9) can be obtained by replacing \( \psi_i'(u; z, n) \) with \( \psi_i'(0; z - u, n) \) in the formula. \( \square \)

### 3.4 Premiums adjusted according to aggregate settled claims

Previously, we have discussed the first case of varying premiums based on the total reported claims. In contrast, we shall consider another case where for \( k \in \mathbb{N}^+ \), the premium \( C_{k+1} \) is determined by \( C_k \) and the total settled claims in time period \( k \), i.e. \( S_k \). Other model assumptions are the same as the previous case.

It is worth noting that in this case of premium correction, the underlying Markov process governing the periodic premium levels is not time-homogeneous anymore since the distribution of aggregate settled claims \( S_t \) takes different forms over time, see (3.2.2) for details. Further, Lemma 1 does not hold in this case either as having a by-claim delayed from previous time period or not does matter when determining future premiums. However, we can still follow the main idea in previous section to obtain the following main result.
Theorem 7. Given initial surplus $u \geq 0$ and initial premium level $c_i$, $i \in \mathcal{L}$, the finite-time ruin probability with premiums adjusted according to aggregate settled claims without the up-front delayed by-claim satisfies the following recursive formula, for $n \in \mathbb{N}^+$,

$$
\psi_i(u, n + 1) = \sum_{j=1}^{l} \psi_j(u, n) f_{X,Y}(x, 0) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y)
$$

$$
+ (1 - q) \sum_{j=1}^{l} \sum_{x=1}^{u+c_i-1} \sum_{y=1}^{u+c_i-x} t_{ij}(x) \psi_j(u + c_i - x - y, n) f_{X,Y}(x, y)
$$

$$
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=1}^{u+c_i-x-c_j} t_{ij}(x) \psi_j'(u + c_i - x; y, n) f_{X,Y}(x, y)
$$

$$
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x-c_j}^{\infty} t_{ij}(x) f_{X,Y}(x, y) + \sum_{x=u+c_i+1}^{\infty} f_X(x), \ (3.4.10)
$$

where $\psi_i(u, 1) = \sum_{x=u+c_i+1}^{\infty} f_X(x) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y)$.

Proof. From (3.2.3), we have

$$
\psi_i(u, n + 1) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{u+c_i} \left( U_k < 0 \right) \mid C_1 = c_i \right\}
$$

$$
= \sum_{x=0}^{u+c_i} \sum_{y=0}^{\infty} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} \left( U_k < 0 \right) \mid C_1 = c_i, X_1 = x, Y_1 = y \right\} f_{X,Y}(x, y)
$$

$$
+ \sum_{x=u+c_i+1}^{\infty} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} \left( U_k < 0 \right) \mid C_1 = c_i, X_1 = x \right\} f_X(x)
$$

$$
= \sum_{x=0}^{u+c_i} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} \left( U_k < 0 \right) \mid C_1 = c_i, X_1 = x, Y_1 = 0 \right\} f_{X,Y}(x, 0)
$$

$$
+ \sum_{x=1}^{u+c_i} \sum_{y=1}^{\infty} \left[ (1 - q) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} \left( U_k < 0 \right) \mid C_1 = c_i, X_1 = x, Y_1 = y \right\}
$$

$$
+ q \mathbb{P}_u \left\{ \bigcup_{k=2}^{n+1} \left( U_k < 0 \right) \mid C_1 = c_i, X_1 = x, Y_1 = y \right\} \right] f_{X,Y}(x, y)
$$

$$
+ \sum_{x=u+c_i+1}^{\infty} f_X(x)
$$
\[
= \sum_{x=0}^{u+c_i} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| C_1 = c_i, X_1 = x, Y_1 = 0 \right\} f_{XY}(x, 0) \\
+ (1-q) \left[ \sum_{x=1}^{u+c_i-1} \sum_{y=1}^{u+c_i-x} \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| C_1 = c_i, X_1 = x, Y_1 = y \right\} f_{XY}(x, y) \\
+ \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} f_{XY}(x, y) \right] + \sum_{x=u+c_i+1}^{\infty} f_X(x) \\
+ q \sum_{x=1}^{u+c_i} \mathbb{P}_u \left\{ \bigcup_{k=2}^{n+1} (U'_k < 0) \bigg| C_1 = c_i, X_1 = x, Y_1 = y \right\} f_{XY}(x, y) \\
= \sum_{j=1}^{l} \sum_{x=0}^{u+c_i} t_{ij}(x) \mathbb{P}_{u+c_i-x} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| C_2 = c_j \right\} f_{XY}(x, 0) \\
+ (1-q) \sum_{j=1}^{l} \sum_{x=1}^{u+c_i-1} \sum_{y=1}^{u+c_i-x} t_{ij}(x+y) \mathbb{P}_{u+c_i-x-y} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| C_2 = c_j \right\} f_{XY}(x, y) \\
+ (1-q) \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+1}^{\infty} f_{XY}(x, y) + \sum_{x=u+c_i+1}^{\infty} f_X(x) \\
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=1}^{\infty} t_{ij}(x) \mathbb{P}_{u+c_i-x} \left\{ \bigcup_{k=2}^{n+1} (U'_k < 0) \bigg| Y_1 = y, C_2 = c_j \right\} f_{XY}(x, y) \\
= \sum_{j=1}^{l} \sum_{x=0}^{u+c_i} t_{ij}(x) \psi_j(u+c_i-x, n) f_{XY}(x, 0) + (1-q) \sum_{y=1}^{\infty} \xi_y(u+c_i+y) \\
+ (1-q) \sum_{j=1}^{l} \sum_{x=1}^{u+c_i-1} \sum_{y=1}^{u+c_i-x} t_{ij}(x+y) \psi_j(u+c_i-x-y, n) f_{XY}(x, y) \\
+ \sum_{x=1}^{u+c_i} \sum_{y=1}^{\infty} t_{ij}(x) \psi_j(u+c_i-x, y, n) f_{XY}(x, y) \\
+ q \sum_{j=1}^{l} \sum_{x=1}^{u+c_i} \sum_{y=u+c_i-x+c_j}^{\infty} t_{ij}(x) f_{XY}(x, y) + \sum_{x=u+c_i+1}^{\infty} f_X(x).
\]

Similar to Theorem 6, one can verify that the result for \( \psi(u, 1) \) is just a special case of \( n = 1 \). \( \square \)

To use the recursive formula obtained in Theorem 7, we need to find a way to determine \( \psi_j(u; z, n), 0 < z \leq c_i, n \in \mathbb{N}^+ \).

**Corollary 2.** The finite-time ruin probability with premiums adjusted according to
aggregate settled claims and an up-front delayed by-claim \( z \) satisfies the following recursive formula, for \( 0 < z \leq u + c_i \) and \( n \in \mathbb{N}^+ \),

\[
\psi_j(u; z, n + 1) = \sum_{j=1}^{l} \sum_{x=0}^{l} t_{ij}(x + z) \psi_j(u - z + c_i - x, n) f_{XY}(x, 0)
\]

\[
+ (1 - q) \sum_{j=1}^{l} \sum_{x=1}^{l} \sum_{y=1}^{l} t_{ij}(x + y + z) \times \psi_j(u - z + c_i - x - y, n) f_{XY}(x, y)
\]

\[
+ q \sum_{j=1}^{l} \sum_{x=1}^{l} \sum_{y=1}^{\infty} t_{ij}(x + z) \psi_j(u - z + c_i - y, n) f_{XY}(x, y) \]

\[
+ q \sum_{j=1}^{l} \sum_{x=1}^{l} \sum_{y=u-z+c_i-x+c_j+1}^{\infty} t_{ij}(x + z) f_{XY}(x, y)
\]

\[
+ (1 - q) \sum_{y=1}^{\infty} \xi_y(u - z + c_i + y) + \sum_{x=u-z+c_i+1}^{\infty} f_X(x),
\]  \( (3.4.11) \)

where \( \psi_j(u; z, 1) = \sum_{x=u-z+c_i+1}^{\infty} f_X(x) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u - z + c_i + y). \)

**Proof.** Using \((3.4.10)\), \((3.4.11)\) can be obtained by adding \( z \) into the premium rule function \( t_{ij} \) and replacing \( u \) by \( u - z \) in \((3.4.10)\). \( \square \)

### 3.5 Premiums adjusted according to reported number of claims

In this section, we shall switch the premium correction trigger from aggregate claim experience to claim frequency experience. We still denote the bonus-malus system by \( \Delta = (T, c, i) \), where \( i \in \mathcal{L} \); \( T = \{ t_{ij}(k) \}_{i,j \in \mathcal{L}, k \in \mathbb{N}} \) denotes a general set of time-homogeneous rules with input \( k \) being the number of claims. For any \( k \in \mathbb{N} \) and \( n \in \mathbb{N}^+ \), \( t_{ij}(k) = 1 \) if the total number of claims in time period \( n \) leads to the transition from premium level \( C_n = c_i \) to \( C_{n+1} = c_j \) and \( t_{ij}(k) = 0 \) otherwise.

Now we consider the first type of claim frequency, i.e. the total number of reported claims. Let \( N_t^X \) denote the number of main claims in \( t \)-th time period.
According to the assumption in section 3.2, we have $\mathbb{P}(N^X_t = 0) = f_X(0)$ and $\mathbb{P}(N^X_t = 1) = 1 - f_X(0)$. Similarly, the number of by-claims in time period $t$ is denoted by $N^Y_t$, where $\mathbb{P}(N^Y_t = 0) = f_Y(0)$ and $\mathbb{P}(N^Y_t = 1) = 1 - f_Y(0)$. We assume that $N^Y_t$ is observable at time $t$ no matter if the settlement of $Y_t$ will be delayed or not, so the total number of reported claims in time period $t$ is $N^X_t + N^Y_t$. For any time period $t$, $t \in \mathbb{N}^+$, there are only three cases of reported number of claims:

1) $N^X_t = 0$ and $N^Y_t = 0$;
2) $N^X_t = 1$ and $N^Y_t = 0$;
3) $N^X_t = 1$ and $N^Y_t = 1$.

Using a similar method as the one used in Section 3.3, we obtain the following main result:

**Theorem 8.** Given initial surplus $u \geq 0$ and initial premium level $c_i$, $i \in \mathbf{L}$, the finite-time ruin probability with premiums adjusted according to reported number of claims without the up-front delayed by-claim satisfies the following recursive formula, for $n \in \mathbb{N}^+$,

$$
\psi_i(u, n + 1) = \sum_{j=1}^{l} t_{ij}(0) \psi_j(u + c_i, n) f_{XY}(0, 0) \\
+ \sum_{j=1}^{l} t_{ij}(1) \sum_{x=1}^{u+c_i} \psi_j(u + c_i - x, n) f_{XY}(x, 0) \\
+ \sum_{j=1}^{l} t_{ij}(2) \left( \sum_{x=1}^{u+c_i-1} \sum_{y=1}^{u+c_i-x} \psi_j(u + c_i - x - y, n) f_{XY}(x, y) \\
+ q \sum_{y=1}^{c_j} \psi_j'(0; y, n) \xi_y(u + c_i + y) + q \sum_{y=c_j+1}^{\infty} \xi_y(u + c_i + y) \right) \\
+ (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y) + \sum_{x=u+c_i+1}^{\infty} f_X(x),
$$

(3.5.12)

where $\psi_i(u, 1) = (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y) + \sum_{x=u+c_i+1}^{\infty} f_X(x)$.

**Proof.** It is not hard to see that the formula (3.5.12) can be obtained by replacing the premium rule $t_{ij}$ in (3.3.8) with the new version defined at the beginning of this
section. Since there are only three cases of total number of reported claims in each time period, one can get (3.5.12) straightforwardly. □

Corollary 3. The finite-time ruin probability with premiums adjusted according to reported claims number and an up-front delayed by-claim $z$ satisfies the following recursive formula, for $0 < z \leq c_i$ and $n \in \mathbb{N}^+$,

$$
\psi'_i(0; z, n + 1) = \sum_{j=1}^{l} t_{ij}(0) \psi_j(c_i - z, n) f_{XY}(0, 0) \\
+ \sum_{j=1}^{l} t_{ij}(1) \sum_{x=1}^{c_i - z} \psi_j(c_i - z - x, n) f_{XY}(x, 0) \\
+ \sum_{j=1}^{l} t_{ij}(2) \left( \sum_{x=1}^{c_i - z} \sum_{y=1}^{c_i - z - x} \psi_j(c_i - z - x - y, n) f_{XY}(x, y) \\
+ q \sum_{y=1}^{c_j} \psi'_j(0; y, n) \xi_y(c_i - z + y) + \sum_{y=c_j+1}^{\infty} \xi_y(c_i - z + y) \right) \\
+ (1 - q) \sum_{y=1}^{\infty} \xi_y(c_i - z + y) + \sum_{x=c_i - z+1}^{\infty} f_X(x),
$$

(3.5.13)

where $\psi'_i(0; z, 1) = (1 - q) \sum_{y=1}^{\infty} \xi_y(c_i - z + y) + \sum_{x=c_i - z+1}^{\infty} f_X(x)$.

Proof. Again, the formula (3.5.13) can be obtained by plugging in the reported claims number in the new premium rule function $t_{ij}$ that replaces the one in (3.3.9). □

3.6 Premiums adjusted according to settled claims number

In this section, we consider the second type of claim frequency, i.e. the total number of settled claims in a given time period. For time period $t \in \mathbb{N}^+$, let $M^X_t$ denote the number of main claims in this time period; let $M^Y_t$ denote the number of settled by-claims incurred in the current period; let $M^Z_t$ denote the number of settled by-claims incurred in previous time period. Based on the assumptions in Section
3.2, we know that all of these count random variables can only take a value either 0 or 1. The values of \( M^Z_t \) and \( M^X_t \) have unique interpretations, but the value 0 for \( M^Y_t \) leads to multiple possibilities. To be more specific, \( M^Y_t = 0 \) means either no by-claim incurred in time period \( t \) or the settlement of the incurred by-claim is delayed to next time period. This implies that \( M^Z_{t+1} = 1 \) gives \( M^Y_t = 0 \), but not vice versa.

Here we assume that the premium \( C_{t+1}, t \in \mathbb{N}^+ \), is determined according to the total number of settled claims in time period \( t \), i.e. \( M^X_t + M^Y_t + M^Z_t \), which can take an integer value from 0 to 3:

1) \( M^Z_t = 0, M^X_t = 0, M^Y_t = 0 \Rightarrow M^X_t + M^Y_t + M^Z_t = 0; \)
2) \( M^Z_t = 0, M^X_t = 1, M^Y_t = 0 \Rightarrow M^X_t + M^Y_t + M^Z_t = 1; \)
3) \( M^Z_t = 0, M^X_t = 1, M^Y_t = 1 \Rightarrow M^X_t + M^Y_t + M^Z_t = 2; \)
4) \( M^Z_t = 1, M^X_t = 0, M^Y_t = 0 \Rightarrow M^X_t + M^Y_t + M^Z_t = 1; \)
5) \( M^Z_t = 1, M^X_t = 1, M^Y_t = 0 \Rightarrow M^X_t + M^Y_t + M^Z_t = 2; \)
6) \( M^Z_t = 1, M^X_t = 1, M^Y_t = 1 \Rightarrow M^X_t + M^Y_t + M^Z_t = 3. \)

Similar to Section 3.4, there is lack of time-homogeneity in the underlying Markov process for premiums. Taking into account the complications illustrated above on the total number of settled claims, we obtain the following result for the finite-time ruin probabilities.

**Theorem 9.** Given initial surplus \( u \geq 0 \) and initial premium level \( c_i, i \in \mathcal{L} \), the finite-time ruin probability with premiums adjusted according to settled claims number without the up-front delayed by-claim satisfies the following recursive formula,
for $n \in \mathbb{N}^+$,

$$
\psi_i(u, n + 1) = \sum_{j=1}^l t_{ij}(0)\psi_j(u + c_i, n) f_{XY}(0, 0) \\
+ \sum_{j=1}^l \sum_{x=1}^{u+c_i} t_{ij}(1)\psi_j(u + c_i - x, n) f_{XY}(x, 0) \\
+ (1 - q) \sum_{j=1}^l \sum_{x=1}^{u+c_i-1} \sum_{y=1}^{u+c_i-x} t_{ij}(2)\psi_j(u + c_i - x - y, n) f_{XY}(x, y) \\
+ q \sum_{j=1}^l \sum_{x=1}^{u+c_i} \sum_{y=1}^{u+c_i-x+c_j} t_{ij}(1)\psi'_j(u + c_i - x; y, n) f_{XY}(x, y) \\
+ q \sum_{j=1}^l \sum_{x=1}^{u+c_i} \sum_{y=u+c_j-x+c_j+1}^{\infty} t_{ij}(1) f_{XY}(x, y) \\
+(1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y) + \sum_{x=u+c_i+1}^{\infty} f_X(x), \tag{3.6.14}
$$

where $\psi_i(u, 1) = \sum_{x=u+c_i+1}^{\infty} f_X(x) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u + c_i + y)$.

**Proof.** Similar to the proof of Theorem 8, (3.6.14) can be obtained by replacing the function $t_{ij}$ in (3.4.10) with the new one defined at the beginning of Section 3.6. Then we plug in the total number of settled claims in $t_{ij}$ according to the values of $X$ and $Y$. \hfill \square

**Corollary 4.** The finite-time ruin probability with premiums adjusted according to settled claims number and an up-front delayed by-claim $z$ satisfies the following recursive formula, for $0 < z \leq u + c_i$ and $n \in \mathbb{N}^+$,
\[ \psi'_i(u; z, n+1) = \sum_{j=1}^{l} t_{ij}(1) \psi_j(u - z + c_i, n)f_{XY}(0,0) \]
\[ + \sum_{j=1}^{l} \sum_{x=1}^{u-z+c_i} t_{ij}(2) \psi_j(u - z + c_i - x, n)f_{XY}(x,0) \]
\[ + (1 - q) \sum_{j=1}^{l} \sum_{x=1}^{u-z+c_i-1} \sum_{y=1}^{u-z+c_i-x} t_{ij}(3) \psi_j(u - z + c_i - x - y, n)f_{XY}(x,y) \]
\[ + q \sum_{j=1}^{l} \sum_{x=1}^{u-z+c_i} \sum_{y=1}^{u-z+c_i-x+c_j} t_{ij}(2) \psi'_j(u - z + c_i - x; y, n)f_{XY}(x,y) \]
\[ + q \sum_{j=1}^{l} \sum_{x=1}^{u-z+c_i} \sum_{y=u-z+c_i-x+c_j+1}^{\infty} t_{ij}(2)f_{XY}(x,y) \]
\[ + (1 - q) \sum_{y=1}^{\infty} \xi_y(u - z + c_i + y) + \sum_{x=u-z+c_i+1}^{\infty} f_X(x), \quad (3.6.15) \]

where \( \psi'_i(u; z, 1) = \sum_{x=u-z+c_i+1}^{\infty} f_X(x) + (1 - q) \sum_{y=1}^{\infty} \xi_y(u - z + c_i + y). \)

**Proof.** Using (3.6.14), (3.6.15) can be obtained by adding 1 into the premium rule function \( t_{ij} \) and replacing \( u \) by \( u - z \) in (3.6.14). \( \square \)

### 3.7 Numerical results

In this section we shall provide some numerical examples to illustrate the theoretical results obtained under the previously discussed four premium adjustment principles and to further study the commonality and dissimilarity of the four principles. Since we have been focusing on the finite-time ruin probabilities in this chapter, we shall adopt the finite-time ruin probabilities with a fixed term (say 20) as the proxy to achieve the aforementioned goals. The possible behaviours of finite-time ruin probabilities under each principle when the term changes are not covered here, mainly due to the significantly increased computational costs involved in the completion of the task.
3.7.1 Premiums adjusted according to aggregate reported claims

The first numerical example we give in this section applies the premium correction principle allowing premiums to be adjusted according to aggregate reported claims. As mentioned previously in section 3.3, the aggregate claims are assumed to be reported at the end of each policy period even when the settlement of by-claims is delayed. We shall examine three hypothetical scenarios for the degree of correlation between the main claim $X$ and the by-claim $Y$ in this example: low correlation, moderate correlation and high correlation. For each scenario, two cases of claim settlement delay are considered: $q = 0.2$ or $q = 0.8$. We propose the following joint distributions of $X$ and $Y$:

- high correlation case:

  $$f_{XY}^H(x,y) = \begin{cases} 
  \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^x & x = y = 0, \\
  \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^x \left( \frac{6}{7} \right)^y & x = y > 0, \\
  0 & \text{otherwise},
  \end{cases}$$

  where $E(X)=5$, $E(Y)=5$ and the correlation coefficient $\rho_{XY}=1$;

- low correlation case:

  $$f_{XY}^L(x,y) = \begin{cases} 
  \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^x \left( \frac{1}{7} \right)^y & x = y = 0, \\
  \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^x \left( \frac{6}{7} \right)^y & x > 0, y \geq 0, \\
  0 & \text{otherwise},
  \end{cases}$$

  where $E(X)=5$, $E(Y)=5$ and $\rho_{XY}=0.1443$;

- moderate correlation case: we let

  $$f_{XY}^M(x,y) = 0.5 f_{XY}^H(x,y) + 0.5 f_{XY}^L(x,y),$$

  where $E(X)=5$, $E(Y)=5$ and so $\rho_{XY}=0.5401$.

According to the above assumptions, we can see that $X$ follows the same marginal geometric distribution in all three cases, i.e. $f_X(x) = \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^x$, $x \geq 0$. However,
the three marginal distributions of $Y$ differ from each other, which are listed below, for $y \geq 0$,

\begin{align*}
    f_Y^H(y) &= \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^y; \\
    f_Y^L(y) &= \frac{1}{6} \times I_{\{y=0\}} + \frac{5}{6} \times \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^y; \\
    f_Y^M(y) &= \frac{1}{2} f_Y^H(y) + \frac{1}{2} f_Y^L(y),
\end{align*}

where $I_{\{y=0\}}$ is an indicator function taking 1 when $y = 0$ and 0 otherwise.

The set of premium levels is assumed to be $c = \{c_1, \ldots, c_5\} = \{11, 12, 14, 16, 18\}$ and the initial premium of new policyholders $C_1$ is $c_3$ that is 140% of the expected aggregate reported claims $E(X + Y)$ (i.e. a safety loading factor of 40%). Under our assumption, the premium levels range from 110% to 180% of the expected aggregate reported claims. We propose the following rules of premium adjustment:

1) If the reported aggregate claims in the current period is no more than 3, then the premium level for the next period will move to the lower premium level or stay in the lowest one;

2) If the reported aggregate claims in the current period is between 3 and 14, then the premium level for the next period will remain in the current level;

3) If the reported aggregate claims in the current period is more than 14, then the premium level for the next period will move to the higher premium level or stay in the highest one.

According to the above transition rules, we can calculate the transition probabilities among the premium levels based on (3.3.6). Let $P_T^H$, $P_T^M$ and $P_T^L$ denote the
transition matrix in each of the above correlation cases respectively, then we have

\[
P^H_T = \begin{bmatrix}
0.76743 & 0.23257 & 0 & 0 & 0 \\
0.30556 & 0.46188 & 0.23257 & 0 & 0 \\
0 & 0.30556 & 0.46188 & 0.23257 & 0 \\
0 & 0 & 0.30556 & 0.46188 & 0.23257 \\
0 & 0 & 0 & 0.30556 & 0.69444
\end{bmatrix},
\]

\[
P^M_T = \begin{bmatrix}
0.75712 & 0.24288 & 0 & 0 & 0 \\
0.28407 & 0.47305 & 0.24288 & 0 & 0 \\
0 & 0.28407 & 0.47305 & 0.24288 & 0 \\
0 & 0 & 0.28407 & 0.47305 & 0.24288 \\
0 & 0 & 0 & 0.28407 & 0.71593
\end{bmatrix},
\]

\[
P^L_T = \begin{bmatrix}
0.74681 & 0.25319 & 0 & 0 & 0 \\
0.26258 & 0.48423 & 0.25319 & 0 & 0 \\
0 & 0.26258 & 0.48423 & 0.25319 & 0 \\
0 & 0 & 0.26258 & 0.48423 & 0.25319 \\
0 & 0 & 0 & 0.26258 & 0.73742
\end{bmatrix}.
\]

The corresponding long-term stationary distribution of the premium levels are:

\[
\pi^H = [\pi^H_i]_{i \in \mathcal{L}} = \begin{bmatrix}
0.32082 & 0.24419 & 0.18586 & 0.14146 & 0.10767
\end{bmatrix},
\]

\[
\pi^M = [\pi^M_i]_{i \in \mathcal{L}} = \begin{bmatrix}
0.26699 & 0.22828 & 0.19518 & 0.16688 & 0.14268
\end{bmatrix},
\]

\[
\pi^L = [\pi^L_i]_{i \in \mathcal{L}} = \begin{bmatrix}
0.21482 & 0.20714 & 0.19974 & 0.19259 & 0.18571
\end{bmatrix}.
\]

The long-term expected premium per time period is 13.26, 13.65 and 14.07 in the high, moderate, and low correlation scenario, respectively. Using (3.3.8) and (3.3.9), we calculate \( \psi_3(u, 20) \), \( 0 \leq u \leq 100 \), with the initial premium \( c_3 \) and the results are summarised in Table 3.1 and Figure 3.1. Note that the notations \( H1, M1 \) and \( L1 \) denote the scenarios of high, moderate and low correlation between \( X \) and \( Y \) when \( q = 0.2 \), and the notations \( H2, M2 \) and \( L2 \) correspond to the scenarios when \( q = 0.8 \).
Table 3.1: $\psi_3(u, 20)$ under the aggregate reported claims principle

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_3^{H1}(u, 20)$</th>
<th>$\psi_3^{H2}(u, 20)$</th>
<th>$\psi_3^{M1}(u, 20)$</th>
<th>$\psi_3^{M2}(u, 20)$</th>
<th>$\psi_3^{L1}(u, 20)$</th>
<th>$\psi_3^{L2}(u, 20)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.48789</td>
<td>0.34433</td>
<td>0.46301</td>
<td>0.32119</td>
<td>0.43201</td>
<td>0.29416</td>
</tr>
<tr>
<td>10</td>
<td>0.28527</td>
<td>0.19639</td>
<td>0.23543</td>
<td>0.15643</td>
<td>0.17866</td>
<td>0.11266</td>
</tr>
<tr>
<td>20</td>
<td>0.16386</td>
<td>0.11085</td>
<td>0.11795</td>
<td>0.07688</td>
<td>0.06897</td>
<td>0.04179</td>
</tr>
<tr>
<td>30</td>
<td>0.09279</td>
<td>0.06188</td>
<td>0.05892</td>
<td>0.03797</td>
<td>0.02564</td>
<td>0.01516</td>
</tr>
<tr>
<td>40</td>
<td>0.05194</td>
<td>0.03423</td>
<td>0.02940</td>
<td>0.01878</td>
<td>0.00931</td>
<td>0.00541</td>
</tr>
<tr>
<td>50</td>
<td>0.02880</td>
<td>0.01878</td>
<td>0.01464</td>
<td>0.00929</td>
<td>0.00333</td>
<td>0.00191</td>
</tr>
<tr>
<td>60</td>
<td>0.01583</td>
<td>0.01024</td>
<td>0.00728</td>
<td>0.00459</td>
<td>0.00117</td>
<td>0.00067</td>
</tr>
<tr>
<td>70</td>
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<td>0.00554</td>
<td>0.00361</td>
<td>0.00226</td>
<td>0.00041</td>
<td>0.00023</td>
</tr>
<tr>
<td>80</td>
<td>0.00469</td>
<td>0.00298</td>
<td>0.00178</td>
<td>0.00111</td>
<td>0.00014</td>
<td>0.00008</td>
</tr>
<tr>
<td>90</td>
<td>0.00253</td>
<td>0.00160</td>
<td>0.00088</td>
<td>0.00054</td>
<td>0.00005</td>
<td>0.00003</td>
</tr>
<tr>
<td>100</td>
<td>0.00136</td>
<td>0.00085</td>
<td>0.00043</td>
<td>0.00027</td>
<td>0.00002</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

Figure 3.1: $\psi_3(u, 20)$ under the aggregate reported claims principle

The first observation, a trivial one, from Table 3.1 and Figure 3.1 is that $\psi_3(u, 20)$ decreases when $u$ increases. Moreover, we notice that the correlation level between main claim $X$ and by-claim $Y$ does affect the finite-time ruin probability. Under our previous assumptions, after fixing $u$ and $q$, the higher is the correlation, the higher is the risk of ruin. Although the same premium adjustment rules are applicable for all three correlation scenarios, the joint distribution of $X$ and
Y differentiates the transition probabilities among premium levels as well as the stationary distribution of individual premium levels. The previously calculated $\pi^H$, $\pi^M$ and $\pi^L$ show that the high correlation case has the highest long-term probability to reach low premium levels and the lowest long-term probability for high premium levels. It implies that in long-run, in scenario $H$, the insurer is expected to receive less total premium income than the other two scenarios, which results in the highest finite-time ruin probabilities among the three scenarios. Similar arguments can be made to explain the ordering between cases $M$ and $L$.

In addition, the differences, in terms of percentages, among the finite-time ruin probabilities under the three scenarios increase when the initial surplus $u$ increases. For example, $\psi^H_3(0,20)$ is only about 5.4% higher than $\psi^M_3(0,20)$ and around 12.9% higher than $\psi^L_3(0,20)$. But $\psi^H_3(100,20)$ is about three times $\psi^M_3(100,20)$ and around 68 times $\psi^L_3(100,20)$. This makes sense because when $u$ is small, if ruin occurs then it is more likely to occur within the first few periods. As a result, there is only limited time for the main factors, which vary the finite-time ruin probabilities among these scenarios, to take effect. The same initial premium assumption under all three scenarios also contributed to the small differences in percentage among the finite-time ruin probabilities when $u$ is small. On the contrary, when $u$ is large, if ruin occurs then ruin is more likely to occur in the long run. The dissimilar premium evolving patterns under the three scenarios have plenty of time to drive the underlying surplus processes to different directions, which lead to divergent finite-time ruin probabilities.

Last but not least, it is evident from Figure 3.1 that with all other factors being the same, an increase in $q$ from 0.2 to 0.8 shifted the finite-time ruin probabilities downwards. This is reasonable since when the settlement of by-claims is more likely to be delayed, the insurers can receive more premium income that helps to settle the claims. However, this effect reduces when $u$ is larger, because delaying by-claims for one time unit would not make a big difference for the worst cases (i.e. getting bankrupted with a large initial capital).

### 3.7.2 Premiums adjusted according to aggregate settled claims

This example examines the premium adjustment principle that was discussed in Section 3.4. This principle is worth exploring because that, in certain circumstances,
the total settled claim amounts might better reflect the claims experiences of policyholders than the total reported claim amounts in a given time window due to the fact that in real practice reported claims come with uncertainties in the scale and timing of the real settlements. Therefore, the reported claims are only initial guesses and may not provide accurate information to represent the policyholders’ historical claim experience. In this chapter, for the purpose of simplification, we assumed that the reported and settled by-claims amounts are always equal and the length of delay is always 1. Although these restrictive assumptions are not entirely realistic, they serve as good starting points that could motivate more realistic models in future studies.

We assume the same claim distributions and premium levels as those in previous example, whilst the transition rules of premium levels are modified as:

1) If the settled aggregate claims in the current period is no more than 3, the premium level for the next period will move to the lower premium level or stay in the lowest one;
2) If the settled aggregate claims in the current period is more than 3 but no more than 14, the premium level for the next period will remain in the current premium level;
3) If the settled aggregate claims in the current period is more than 14, the premium level for the next period will move to the higher premium level or stay in the highest one.

By the non-homogeneity nature exhibited under the new rules, there is no constant one-step transition matrix among the premium levels anymore. On the contrary, the one-step transition matrix varies over time and depends on the number of by-claims settled in each given time period. However, we can still study 20-period finite-time ruin probabilities using the recursive formulae (3.4.10) and (3.4.11). The results are given in Table 3.2 and Figure 3.2. We adopt the same notation to denote the scenarios under consideration.
Table 3.2: $\psi_3(u, 20)$ under the aggregate settled claims principle

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_3^{H1}(u, 20)$</th>
<th>$\psi_3^{H2}(u, 20)$</th>
<th>$\psi_3^{M1}(u, 20)$</th>
<th>$\psi_3^{M2}(u, 20)$</th>
<th>$\psi_3^{L1}(u, 20)$</th>
<th>$\psi_3^{L2}(u, 20)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.49739</td>
<td>0.36760</td>
<td>0.47738</td>
<td>0.36262</td>
<td>0.45114</td>
<td>0.35399</td>
</tr>
<tr>
<td>10</td>
<td>0.29196</td>
<td>0.20393</td>
<td>0.24635</td>
<td>0.17862</td>
<td>0.19275</td>
<td>0.14766</td>
</tr>
<tr>
<td>20</td>
<td>0.16826</td>
<td>0.11276</td>
<td>0.12495</td>
<td>0.08811</td>
<td>0.07701</td>
<td>0.05910</td>
</tr>
<tr>
<td>30</td>
<td>0.09555</td>
<td>0.06178</td>
<td>0.06303</td>
<td>0.04346</td>
<td>0.02963</td>
<td>0.02294</td>
</tr>
<tr>
<td>40</td>
<td>0.05361</td>
<td>0.03358</td>
<td>0.03170</td>
<td>0.02143</td>
<td>0.01112</td>
<td>0.00869</td>
</tr>
<tr>
<td>50</td>
<td>0.02978</td>
<td>0.01813</td>
<td>0.01590</td>
<td>0.01056</td>
<td>0.00410</td>
<td>0.00323</td>
</tr>
<tr>
<td>60</td>
<td>0.01640</td>
<td>0.00974</td>
<td>0.00795</td>
<td>0.00519</td>
<td>0.00149</td>
<td>0.00118</td>
</tr>
<tr>
<td>70</td>
<td>0.00896</td>
<td>0.00520</td>
<td>0.00396</td>
<td>0.00254</td>
<td>0.00053</td>
<td>0.00043</td>
</tr>
<tr>
<td>80</td>
<td>0.00487</td>
<td>0.00277</td>
<td>0.00196</td>
<td>0.00125</td>
<td>0.00019</td>
<td>0.00015</td>
</tr>
<tr>
<td>90</td>
<td>0.00263</td>
<td>0.00147</td>
<td>0.00097</td>
<td>0.00061</td>
<td>0.00007</td>
<td>0.00005</td>
</tr>
<tr>
<td>100</td>
<td>0.00141</td>
<td>0.00077</td>
<td>0.00048</td>
<td>0.00030</td>
<td>0.00002</td>
<td>0.00002</td>
</tr>
</tbody>
</table>

As shown in Table 3.2 and Figure 3.2, consistent observations are evident in this aggregate settled claims principle comparing with the aggregate reported claims case. Further, the differences between the two $q$ cases in each correlation scenario also behave interestingly differently. In the high correlation scenario, there is a big gap between the two ruin probability curves showing that a high chance of delaying the highly correlated by-claims results in a big reduction in the risk of
ruin comparing from the case of low chance of delay. On the contrary, when the correlation between main claims and by-claims is low and \( u \) is not small, whether delaying the by-claims or not seem not having a significant impact on the finite-time ruin probabilities. A reasonable interpretation is that when the correlation is low, the main difference between the two cases of \( q \) is that the by-claims settled in each time period are likely to be delayed ones or freshly incurred ones. Since the correlation between the main claims and by-claims is low, the distributions of aggregate settled claims in each period are similar in both cases. Therefore, except the first time period, the surplus process should behave similarly within all remaining time periods in both \( q \) cases that lead to similar finite-time ruin probabilities.

Moreover, we generate comparison results, shown in Figure 3.3 and Figure 3.4, regarding \( \psi_3(u, 20) \) in this and the previous numerical examples. The superscripts \( R \) and \( S \) denote the premium adjustment principle by reported aggregate claims and by settled aggregate claims respectively.

![Figure 3.3: Comparison between \( \psi_3(u, 20) \) in 7.1 and 7.2 when \( q = 0.2 \).](image-url)
As seen in Figure 3.3, the two premium correction principles lead to marginal differences in the finite-time ruin probabilities in the case of \( q = 0.2 \), because when \( q \) is small, the aggregate reported claims in each period are likely to be the same as the aggregate settled claims. Therefore, the periodic premiums are highly likely to follow the same pattern in both cases, which result in similar finite-time ruin probabilities.

On the other hand, according to Figure 3.4, when \( q = 0.8 \) the trends of finite-time ruin probabilities in the two cases differ significantly from one another. However, the differences increase when the correlation between the main claims and by-claims becomes weaker, and they tend to diminish when \( u \) increases. Moreover, when \( q = 0.8 \) the differences among the three correlation scenarios under the aggregate settled claims principle are generally smaller than those in the aggregate reported claims case. A possible interpretation is that when \( q \) is high, after the first couple of time periods, the aggregate settled claims in each period is highly likely to be the summation of a main claim \( X \) of the current period and a by-claim \( Y \) delayed from the previous period (if any), whilst the aggregate reported claims in each period is a current main claim plus a current by-claim (if any). Due to the independence assumption between main claims and by-claims in different time periods, the within-period correlation between main and by-claims becomes
between-period correlation in the aggregate settled claims case, which likely contributes to the above observation.

A consistent finding in both $q$ cases is that the finite-time ruin probabilities under the aggregated settled claims principle are generally higher than the corresponding ones in the aggregate reported claims case. It implies that if the information regarding reported claims is accurate, then the insurers better adopt the aggregate reported claims principle to adjust their periodic premiums, or they will face a higher insolvency risk otherwise.

In the following sections, we shall provide two examples designed to examine the finite-time ruin probabilities with premiums adjustment principles that focus on the claim frequency information.

### 3.7.3 Premiums adjusted according to reported claims number

In this example, we assume that the claim distributions and the set of premium levels are the same as the previous examples. The rules of premium corrections are:

1) If the number of reported claims in the current period is 0, then the premium level for the next period will move to the lower premium level or stay in the lowest one;
2) If the number of reported claims in the current period is 1, then the premium level for the next period will remain in the current premium level;
3) If the number of reported claims in the current period is more than 1, then the premium level for the next period will move to the higher premium level or stay in the highest one.

Next, we shall explore the impact of the correlation between main claims and by-claims as well as the impact of $q$ on the finite-time ruin probabilities. Under the new premium adjustment rules given above, the correlation between the number of main claims $N_t^X$ and by-claims $N_t^Y$ are calculated instead of the correlation between $X$ and $Y$. We find that the correlation between $N_t^X$ and $N_t^Y$ generated by the claims distribution $f_{XY}^H(x, y)$, $f_{XY}^M(x, y)$ and $f_{XY}^L(x, y)$ is $\rho_{N_t^X,N_t^Y} = 1$, $\rho_{N_t^X,N_t^Y} = 0.8272$. 

and $\rho_{N,X,N,Y} = 0.7071$, respectively. These surprisingly high correlations between number of claims are rooted in the model assumptions made in Section 3.2, i.e. one main claim generates at most one by-claim and no main claim means no by-claim.

Similar to the example in Section 3.7.1, we can calculate the transition matrix among premium levels as follows:

$$
P^H_T = \begin{bmatrix}
1/6 & 5/6 & 0 & 0 & 0 \\
1/6 & 0 & 5/6 & 0 & 0 \\
0 & 1/6 & 0 & 5/6 & 0 \\
0 & 0 & 1/6 & 0 & 5/6 \\
0 & 0 & 0 & 1/6 & 5/6
\end{bmatrix},
$$

$$
P^M_T = \begin{bmatrix}
0.22619 & 0.77381 & 0 & 0 & 0 \\
0.16667 & 0.05952 & 0.77381 & 0 & 0 \\
0 & 0.16667 & 0.05952 & 0.77381 & 0 \\
0 & 0 & 0.16667 & 0.05952 & 0.77381 \\
0 & 0 & 0 & 0.16667 & 0.83333
\end{bmatrix},
$$

$$
P^L_T = \begin{bmatrix}
0.28571 & 0.71429 & 0 & 0 & 0 \\
0.16667 & 0.11905 & 0.71429 & 0 & 0 \\
0 & 0.16667 & 0.11905 & 0.71429 & 0 \\
0 & 0 & 0.16667 & 0.11905 & 0.71429 \\
0 & 0 & 0 & 0.16667 & 0.83333
\end{bmatrix}.
$$

The corresponding long-term stationary distribution of the premium levels are:

$$
\pi^H = \begin{bmatrix}
0.00128 & 0.00640 & 0.03201 & 0.16005 & 0.80026
\end{bmatrix},
$$

$$
\pi^M = \begin{bmatrix}
0.00169 & 0.00784 & 0.03642 & 0.16907 & 0.78498
\end{bmatrix},
$$

$$
\pi^L = \begin{bmatrix}
0.00227 & 0.00975 & 0.04177 & 0.17901 & 0.76720
\end{bmatrix}.
$$

The long-term expected premiums in each correlation scenario is 17.50, 17.46 and 17.40 in the $H, M$ and $L$ scenario, respectively. It is worth noting that given the
very different joint distributions of $X$ and $Y$ in the three correlation scenarios, the corresponding long-term expected premiums are very similar under the current premium correction principle. By (3.5.12) and (3.5.13), we obtain results for $\psi_3(u, 20)$ that are summarized in Table 3.3 and Figure 3.5. The notations in Table 3.3 are defined in the same way as in Table 3.1 and 3.2.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\psi_3^{H1}(u, 20)$</th>
<th>$\psi_3^{H2}(u, 20)$</th>
<th>$\psi_3^{M1}(u, 20)$</th>
<th>$\psi_3^{M2}(u, 20)$</th>
<th>$\psi_3^{L1}(u, 20)$</th>
<th>$\psi_3^{L2}(u, 20)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.36310</td>
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<td>0.35810</td>
<td>0.23559</td>
<td>0.34799</td>
<td>0.22890</td>
</tr>
<tr>
<td>10</td>
<td>0.19645</td>
<td>0.12700</td>
<td>0.16968</td>
<td>0.10723</td>
<td>0.13642</td>
<td>0.08316</td>
</tr>
<tr>
<td>20</td>
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<td>0.01038</td>
</tr>
<tr>
<td>40</td>
<td>0.03020</td>
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<td>0.00634</td>
<td>0.00361</td>
</tr>
<tr>
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<td>0.00885</td>
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<td>0.00221</td>
<td>0.00125</td>
</tr>
<tr>
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<td>0.00428</td>
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<td>0.00076</td>
<td>0.00043</td>
</tr>
<tr>
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<td>0.00282</td>
<td>0.00208</td>
<td>0.00127</td>
<td>0.00026</td>
<td>0.00015</td>
</tr>
<tr>
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<td>0.00101</td>
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<td>0.00009</td>
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<tr>
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<tr>
<td>100</td>
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<td>0.00041</td>
<td>0.00024</td>
<td>0.00014</td>
<td>0.00001</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

Figure 3.5: $\psi_3(u, 20)$ under the reported claims number principle
Again, Table 3.3 and Figure 3.5 show us some similar trends to those shown in Table 3.1 & 3.2 and Figure 3.1 & 3.2. First, $\psi_3(u, 20)$ decreases when $u$ increases and the correlation level between $N^X$ and $N^Y$ is positively related to the ruin probabilities. When fixing $u$ and $q$, the higher the correlation, the higher is the ruin probability. Additionally, the decrease in $q$ from 0.8 to 0.2 also causes a lift in the finite-time ruin probabilities in all correlation scenarios. There are two inconsistencies between this example and the previous ones:

- Firstly, the scales of difference in $\rho_{XY}$ and $\psi_3(u, 20)$ among all correlation scenarios in Section 3.7.1 and 3.7.2 are larger than the corresponding differences in this example. An interpretation is that, as given at the beginning of this section, the differences among the three $\rho_{N^X, N^Y}$ values are much smaller than the differences among the three $\rho_{XY}$ values, which makes the three correlation scenarios less distinct from one another.

- Secondly, the relationship between $\rho_{n^x, n^y}$ and the long-term expected premium in this example is opposite to that in Section 3.7.1. To be more specific, in Section 3.7.1, lower $\rho_{XY}$ leads to higher long-term expected premiums, whereas in this example, lower $\rho_{N^X, N^Y}$ gives lower long-term expected premiums. A likely justification of this difference is the change of premium correction objectives from aggregate claim experience to claim frequencies.

### 3.7.4 Premiums adjusted according to settled claims number

In our last numerical example, we shall duplicate the model assumptions but change the premiums adjustment rules following the settled claims number premium principle. The transition rules of premiums are:

1) If the number of settled claims in the current period is 0, then the premium level for the next period will move to the lower premium level or stay in the lowest one;

2) If the number of settled claims in the current period is 1, then the premium level for the next period will remain in the current premium level;
3) If the number of settled claims in the current period is more than 1, the premium level for the next period will move to the higher premium level or stay in the highest one.

We use (3.6.14) and (3.6.15) to calculate \( \psi_3(u, 20) \) and the results are summarised in Table 3.4 and Figure 3.6, adopting the same notations.

**Table 3.4: \( \psi_3(u, 20) \) under the settled claims number principle**

<table>
<thead>
<tr>
<th>( u )</th>
<th>( \psi_3^{H1}(u, 20) )</th>
<th>( \psi_3^{H2}(u, 20) )</th>
<th>( \psi_3^{M1}(u, 20) )</th>
<th>( \psi_3^{M2}(u, 20) )</th>
<th>( \psi_3^{L1}(u, 20) )</th>
<th>( \psi_3^{L2}(u, 20) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.37559</td>
<td>0.27392</td>
<td>0.37074</td>
<td>0.27144</td>
<td>0.36068</td>
<td>0.26506</td>
</tr>
<tr>
<td>10</td>
<td>0.20550</td>
<td>0.15024</td>
<td>0.17838</td>
<td>0.12923</td>
<td>0.14449</td>
<td>0.10328</td>
</tr>
<tr>
<td>20</td>
<td>0.11160</td>
<td>0.08175</td>
<td>0.08534</td>
<td>0.06204</td>
<td>0.05439</td>
<td>0.03884</td>
</tr>
<tr>
<td>30</td>
<td>0.06024</td>
<td>0.04420</td>
<td>0.04106</td>
<td>0.02999</td>
<td>0.01984</td>
<td>0.01424</td>
</tr>
<tr>
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</table>

**Figure 3.6: \( \psi_3(u, 20) \) under the settled claims number principle**
Table 3.4 and Figure 3.6 show very similar trends to our findings from Table 3.3 and Figure 3.5. Again, we generate two comparison graphs, Figure 3.7 and Figure 3.8, between the two claim frequency premium principles for $q = 0.2$ and $q = 0.8$ respectively. The superscript $R$ denotes the reported claims number principle and $S$ denotes the settled claims number one.

**Figure 3.7:** Comparison between $\psi_3(u, 20)$ in 7.3 and 7.4 when $q = 0.2$.

**Figure 3.8:** Comparison between $\psi_3(u, 20)$ in 7.3 and 7.4 when $q = 0.8$. 
From Figure 3.7 and Figure 3.8, we can see that when $q = 0.2$, the finite-time ruin probabilities in this example is slightly higher than the results in section 3.7.3 for fixed $u$ and the correlation level. On the other hand, when $q = 0.8$, the gaps between the finite-time ruin probabilities of this example and those of 3.7.3 are larger. This is because when $q = 0.2$, the by-claim settlements are unlikely to be delayed. Therefore, it is more likely that both main claims and their associated by-claims to be settled in the same time periods, which makes the reported claims number principle and the settled claims number one to work similarly. On the contrary, when $q = 0.8$, the settled claims number in the first time period is likely to be one since there is no up-front delayed by-claim, while the by-claim’s settlement (if any) is likely to be delayed. Therefore, the settled claims number principle would determine the second premium according to the number of main claims in period one, whereas both the number of main claims and by-claims in period one will be used by the reported claims number principle. As a result, it is likely that the second premium under the reported claims number principle will be higher than the one under the settled claims number principle, which varies the whole sequence of future premiums and results in lower ruin probabilities in the former case.

3.8 Concluding remarks

In this chapter we studied a discrete-time risk model with claim-dependent premiums and time-delayed by-claims. Our main goal is to evaluate the impact of the correlation between the main claims and by-claims and the probability of delaying by-claim settlements on the finite-time ruin probabilities under the proposed premium adjustment principles: the aggregate reported claims principle, the aggregate settled claims principle, the reported claims number principle and the settled claims number principle. Under certain assumptions, we found in our numerical studies that higher probability of delaying the by-claim settlements would result in lower finite-time ruin probabilities. Moreover, higher correlation between the main claims and by-claims also leads to higher finite-time ruin probabilities. Lastly, the premium adjustment principles based on settled claims experience (aggregate settled claims or settled claims number) account for higher finite-time ruin probabilities, compared with the principles based on the reported claims experience given all other factors are the same. This difference is more remarkable when the probability of by-claim
delays is high. According to these main findings in our study, the insurers should remain on high alert if a high correlation between the main claims and their associated by-claims is evident or the chance of getting delays in claim settlements is low, because both situations could lead to increased insolvency risk. Further, the premium adjustment principles based on the reported claims experience could be a safer choice than the principles based on settled claims experience, especially in the high probability of delayed by-claim settlement case.
Chapter 4

Discrete-time risk models with premium adjusted according to claims and surplus level

4.1 Introduction

This chapter extends the discrete-time risk models with premium adjusted according to claim experience by taking into account the current surplus level. In other words, the premium adjustment rules are assumed to vary according to the current surplus level. This idea is inspired by the fact that, in practice, the insurers may need to consider their current capital position when they determine the appropriate premium correction rules. For example, when the insurer’s current surplus level is higher than a certain threshold, the financial status is defined as ‘Healthy state’, which has low insolvency risk. In this state, the insurer can apply more aggressive premium adjustment rules which aim at retaining existing policyholders with better claim experience, i.e. offering higher discounts to qualified policyholders. On the contrary, when the insurer’s current surplus level is lower than a certain threshold, we say the insurer is in an ‘Unhealthy state’ and has a high risk of ruin. When in the unhealthy state, the insurer may be sanctioned by insurance regulators and is required to apply more conservative premium standards to recover their financial strength from the unhealthy state. In addition, this chapter also considers a special type of ruin, so-called ‘Parisian Ruin’. The Parisian ruin is the event that the
insurers are in a negative surplus position for a continuous period of time that exceeds a given time frame. This concept is realistic in real life in situations where an insurer may not be bankrupted immediately when the surplus level falls below zero and is allowed to keep the business running with a negative surplus within a regulated period of time to recover from the insolvency. In this chapter, we assume that during the period of negative surpluses, extremely conservative premium adjustment rules are in place to increase the chance of bringing the surplus back to a positive level.

In this chapter we study the finite-time ruin probabilities by using recursive formulae and we shall investigate the impact of threshold level, denoted by $d$, on the finite-time ruin probabilities by numerical examples. Additionally, the Parisian ruin probability as a risk measure is also examined numerically.

### 4.2 Models and Assumptions

As usual, let $c$ denote a premium level set where $c = \{c_i\}_{i \in \mathcal{L}}$, $\mathcal{L} = \{1, 2, \ldots, l\}$, $l \in \mathbb{N}^+$, $c_i \in \mathbb{R}^+$. Here $c_i$, $i = 1, \ldots, l$, are premium levels per unit volume of risk. Let $\{L_i\}_{i \in \mathbb{N}^+}$ be a stochastic process monitoring the premium levels that the insurance company charges over time. Here $L_t \in c$ for any $t \in \mathbb{N}^+$ and this premium level applies in the time period $[t-1,t)$.

Consider a general insurance surplus process of which the level of surplus at time $k$, $k \in \mathbb{N}$, is defined by

$$U_k = U_0 + \sum_{t=1}^{k} (C_t - S_t), \quad \text{for} \quad k \in \mathbb{N}^+,$$

(4.2.1)

where $S_t$ is the aggregate claim amount for time period $[t-1,t)$ payable at time $t$, $t \in \mathbb{N}^+$; $U_0 \geq 0$ is the initial surplus level; $C_t$ is the total premium for time period $[t-1,t)$ receivable at time $t-1$ and $C_t = L_t E[S_t]$, $t \in \mathbb{N}^+$. Let $d \in \mathbb{N}^+$ denote a predetermined surplus threshold level. From (4.2.1), if $0 < d \leq U_k$, then the insurer is classified as in a ‘Healthy State’ at time $k$; or in an ‘Unhealthy State’ at time $k$ if $0 \leq U_k < d$. Further, $\{S_t\}_{t \in \mathbb{N}^+}$ are assumed to be i.i.d. with a P.M.F. $f_S(s)$ and mean $\mu_S$, $s \in \mathbb{N}$. 
The timing of all cash flows involved in the above insurance surplus process is illustrated through the following timeline where year $t$ denotes the time period $[t-1, t)$, $t \in \mathbb{N}^+$. 

- $S_{t-1}$ is paid at end of year $t - 1$.
- The surplus state for year $t - 1$ is determined by $U_{t-1}$.
- $C_t$ is received at the beginning of year $t$.
- $S_t$ is paid at end of year $t$.
- The surplus state for year $t$ is determined by $U_t$.
- $C_{t+1}$ is received at the beginning of year $t + 1$.

Given $L_1 = c_i$, $i \in \mathcal{L}$, the first premium amount $C_1$ can be determined as follows

$$C_1 = c_i E[S_1] = c_i \mu_S := \alpha_i. \quad (4.2.2)$$

The $n$-period finite-time ruin probability with initial surplus $u$, initial premium level $c_i$ is defined by, for $n \in \mathbb{N}^+$,

$$\psi_i(u, n) = \mathbb{P}_u \left\{ \bigcup_{k=1}^n (U_k < 0) \bigg| L_1 = c_i, U_0 = u \right\}, \quad (4.2.3)$$

where $U_k$ is defined by (4.2.1) and the subscript $u$ represents the condition $U_0 = u$. To distinguish between the two surplus states, healthy and unhealthy, we shall retain $\psi_i(u, n)$ if $u \geq d$ and use $\tilde{\psi}_i(u, n)$ instead if $u < d$. By convention, for any $i \in \mathcal{L}$, $\psi_i(u, n) = 0$ if $n \leq 0$, and

$$\tilde{\psi}_i(u, n) = \begin{cases} 
0 & \text{if } 0 \leq u, \ n \leq 0, \\
1 & \text{if } u < 0, \ n \geq 0.
\end{cases}$$
Further, we assume that the aggregate claim amount \( S_t, t \in \mathbb{N}^+ \), is non-negative integer-valued that follows a collective risk structure:

\[
S_t = \sum_{i=1}^{M_t} W_{it},
\]

(4.2.4)

where \( M_t \) is the total number of claims recorded in time period \([t-1, t), t \in \mathbb{N}^+\), with P.M.F. \( f_M(m), m \in \mathbb{N} \); \( \{W_{it}\}_{i \in \mathbb{N}^+} \) are individual claim sizes settled in time period \([t-1, t)\). The claims number \( \{M_t\} \) and individual claim sizes \( \{W_{it}\} \) are assumed to be independent of each other.

For simplicity, we will use the symbol \( f'_m \) to denote \( f_M(m) \) within the rest of this chapter. One can show that, for \( s \in \mathbb{N} \),

\[
f_S(s) = \sum_{m=0}^{s} f'_m f^*_m W(s)
\]

with \( f^*_m W(s) \) being the \( m \)-fold convolution of \( f_W(w) \) and \( f_W^0(0) = 1 \).

**4.2.1 Premiums adjusted according to aggregate claims**

Let \( \{t_{ij}(s)\}_{i,j \in \mathcal{L}, s \in \mathbb{N}} \) denote a general set of rules for premium corrections if the insurance company’s surplus is in the healthy state, where \( t_{ij}(s) = 1 \) if the aggregate claim \( S_t = s \) causes the transition from the premium level \( L_t = i \) to \( L_{t+1} = j \) and \( t_{ij}(s) = 0 \) otherwise. According to the definition, we have \( \sum_{j=1}^{L} t_{ij}(s) = 1 \). Let \( p_C(i, j) \) denote the probability that the premium level moves from level \( i \) to level \( j \) under the healthy state, which can be expressed as, for any \( t \in \mathbb{N}^+ \),

\[
p_C(i, j) = \mathbb{P}\{L_{t+1} = j | L_t = i, \text{healthy}\}
\]

\[
= \sum_{s=0}^{\infty} t_{ij}(s) f_S(s), \quad \text{for } i, j \in \mathcal{L}.
\]

(4.2.5)

The function \( t_{ij}(s) \) is determined according to the detailed premium correction rules.

**Remark.** The general set of rules for premium variations and the transition probability between two premium levels under the unhealthy state is denoted by
Discrete-time risk models with premium adjusted according to claims and surplus level

\[ t'_{ij}(s) \text{ and } p'_C(i, j) \text{ respectively. They also satisfy a similar relationship to (4.2.5) by replacing } p_C(i, j) \text{ and } t_{ij}(s) \text{ with } p'_C(i, j) \text{ and } t'_{ij}(s). \]

4.2.2 Premiums adjusted according to claim frequency

When periodic premiums adjust according to the claim frequency rather than the aggregate claim amounts, the transition probability \( p_C(i, j) \) has a modified expression, for \( i, j \in L \),

\[ p_C(i, j) = \sum_{m=0}^{\infty} t_{ij}(m) f'_m. \] (4.2.6)

Again, under an unhealthy state, we have a similar version involving \( t'_{ij}(s) \) and \( p'_C(i, j) \).

4.3 Finite-time ruin probabilities

In this section, we shall derive recursive formulae to compute the finite-time ruin probabilities defined in (4.2.3). We shall consider the two premium correction options separately.

4.3.1 Premiums adjusted according to aggregate claims

According to the assumptions in section 4.2.1, the \( n \)-period finite-time ruin probability with an initial healthy state \( \psi_i(u, n) \), satisfies the following the recursive formula.

**Theorem 10.** For \( u \geq d, n \in \mathbb{N}^+, i \in \mathcal{L} \),

\[
\psi_i(u, n+1) = \sum_{j=1}^{L} \left( \sum_{s=0}^{u+\alpha_i-d} f_S(s) t_{ij}(s) \psi_j(u+\alpha_i-s, n) + \sum_{s=u+\alpha_i-d+1}^{u+\alpha_i} f_S(s) t'_{ij}(s) \tilde{\psi}_j(u+\alpha_i-s, n) \right)
+ \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \] (4.3.7)

with \( \psi_i(u, 1) = \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \).

**Proof.** From (4.2.3), for \( u \geq d \), we have

\[
\psi_i(u, n+1) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, U_0 \geq d \right\}
\]

\[
= \sum_{s=0}^{u+\alpha_i-d} f_S(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, U_0 \geq d, S_1 = s \right\}
\]

\[
+ \sum_{s=u+\alpha_i-d+1}^{u+\alpha_i} f_S(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, U_0 \geq d, S_1 = s \right\}
\]

\[
= \sum_{j=1}^{l} \left( \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \mathbb{P}_{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| L_2 = c_j, U_1 \geq d \right\} \right)
\]

\[
+ \sum_{s=u+\alpha_i-d+1}^{u+\alpha_i} f_S(s) \mathbb{P}_{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| L_2 = c_j, 0 \leq U_1 < d \right\}
\]

\[
+ \sum_{s=u+\alpha_i+1}^{\infty} f_S(s)
\]

Since \( \psi_i(u, 1) \) only measures the probability of ruin of the business within one time period, the verification of the given boundary condition is trivial. \( \square \)

For the \( n \)-period finite-time ruin probability with an initial unhealthy state \( \tilde{\psi}_i(u, n) \), it satisfies the following the recursive formula.
Corollary 5. For $0 \leq u < d$, $n \in \mathbb{N}^+$, $i \in \mathcal{L}$,

$$
\tilde{\psi}_i(u, n + 1) = \sum_{s = u + \alpha_i + 1}^{\infty} f_S(s) + \sum_{j=1}^{l} \left( \sum_{s=0}^{u+\alpha_i-d} f_S(s) t_{ij}(s) \tilde{\psi}_j(u + \alpha_i - s, n) \right) \\
+ \sum_{s=\max(u+\alpha_i-d+1,0)}^{u+\alpha_i} f_S(s) t_{ij}'(s) \tilde{\psi}_j(u + \alpha_i - s, n),
$$
(4.3.8)

where $\tilde{\psi}_i(u, 1) = \sum_{s = u + \alpha_i + 1}^{\infty} f_S(s)$.

Proof. The proof of (4.3.8) is similar to the proof of (4.3.7) with a changed condition $0 \leq u < d$. \(\square\)

Remark. The recursive formulae (4.3.7) and (4.3.8) are for the case that the premiums are adjusted according to aggregate claims by taking into account the current surplus status. However, the recursive formulae (4.3.7) will reduce to the following version if the current surplus level does not affect the premium adjustments, i.e. letting $d = 0$.

Corollary 6. For $n \in \mathbb{N}^+$ and $i \in \mathcal{L}$,

$$
\psi_i(u, n + 1) = \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \psi_j(u + \alpha_i - s, n) + \sum_{s = u + \alpha_i + 1}^{\infty} f_S(s)
$$
(4.3.9)

with $\psi_i(u, 1) = \sum_{s = u + \alpha_i + 1}^{\infty} f_S(s)$.

Proof. The proof of (4.3.9) is trivial. \(\square\)

4.3.2 Premiums adjusted according to claim frequency

In this section, we change the premium adjustment criterion from aggregate claim amounts to number of claims. According to the assumptions in section 4.2.2, the $n$-period-finite-time ruin probability with premiums adjusted according to claims numbers, initial premium level $i$ and the initial healthy state satisfies the following the recursive formulae.
Theorem 11. For \( u \geq d, n \in \mathbb{N}^+, i \in \mathcal{L}, \)

\[
\psi_i(u, n + 1) = \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_i} f_m'(t_{ij}(m) \sum_{s=m}^{u+\alpha_i-d} f_W^m(s) \psi_j(u+\alpha_i-s, n) + t_{ij}'(m) \sum_{s=\max(u+\alpha_i-d+1,m)}^{u+\alpha_i} f_W^m(s) \psi_j(u+\alpha_i-s, n)) + \sum_{s=\max(u+\alpha_i-d+1,m)}^{\infty} f_S(s) \]

(4.3.10)

with \( \psi_i(u, 1) = \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \) and \( \sum_{a}^{b} = 0 \) for \( b < a. \)

Proof. From (4.2.3), we have, for \( u \geq d, n \in \mathbb{N}^+, i \in \mathcal{L}, \)

\[
\psi_i(u, n + 1) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, U_0 \geq d \right\} 
\]

\[
= \sum_{m=0}^{\infty} f_m \left( \sum_{s=0}^{u+\alpha_i-d} f_W^m(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, U_0 \geq d, S_1 = s \right\} 
\]

\[
= \sum_{m=0}^{\infty} f_m \left( \sum_{s=0}^{u+\alpha_i} f_W^m(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n+1} (U_k < 0) \bigg| L_1 = c_i, U_0 \geq d, S_1 = s \right\} + \sum_{s=u+\alpha_i-d+1}^{\infty} f_W^m(s) \right) 
\]

\[
= \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) + \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_i} f_m'(t_{ij}(m) \sum_{s=m}^{u+\alpha_i-d} f_W^m(s) \times \mathbb{P}_{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| L_2 = c_j, U_1 \geq d \right\} 
\]

\[
+t_{ij}'(m) \sum_{s=\max(u+\alpha_i-d+1,m)}^{u+\alpha_i} f_W^m(s) \mathbb{P}_{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n+1} (U_k < 0) \bigg| L_2 = c_j, 0 \leq U_1 < d \right\} 
\]

\[
(4.3.10)
\]
\begin{align*}
&= \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_i} f'_m(t_{ij}(m)) \sum_{s=m}^{u+\alpha_i-d} f^m_W(s) \psi_j(u+\alpha_i-s,n) \\
&\quad + t'_{ij}(m) \sum_{s=\max(u+\alpha_i-d+1,m)}^{u+\alpha_i} f^m_W(s) \tilde{\psi}_j(u+\alpha_i-s,n) + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s).
\end{align*}

Again, since \( \psi_i(u,1) \) only measures the probability of ruin of the business within one time period, the verification of the given boundary condition is trivial. \( \square \)

Meanwhile, for the \( n \)-period finite-time ruin probability under an initial unhealthy state \( \tilde{\psi}_i(u,n) \), we have

**Corollary 7.** For \( 0 \leq u < d, \ n \in \mathbb{N}^+, \ i \in \mathcal{L} \),

\begin{align*}
\tilde{\psi}_i(u,n+1) &= \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_i} f'_m(t_{ij}(m)) \sum_{s=m}^{u+\alpha_i-d} f^m_W(s) \psi_j(u+\alpha_i-s,n) \\
&\quad + t'_{ij}(m) \sum_{s=\max(u+\alpha_i-d+1,m)}^{u+\alpha_i} f^m_W(s) \tilde{\psi}_j(u+\alpha_i-s,n) + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \tag{4.3.11}
\end{align*}

with \( \tilde{\psi}_i(u,1) = \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \).

**Proof.** The proof of (4.3.11) is similar to the proof of (4.3.10) with the condition \( 0 \leq u < d \). \( \square \)

**Remark.** Similarly to (4.3.7), the recursive formula (4.3.10) can be simplified to the following version if the insurer’s surplus level does not affect its premium adjustment decisions, i.e. \( d = 0 \).

**Corollary 8.** For \( n \in \mathbb{N}^+, \ i \in \mathcal{L} \),

\begin{align*}
\psi_i(u,n+1) &= \sum_{j=1}^{l} \sum_{m=0}^{u+\alpha_i} f'_m t_{ij}(m) \sum_{s=m}^{u+\alpha_i} f^m_W(s) \psi_j(u+\alpha_i-s,n) \\
&\quad + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \tag{4.3.12}
\end{align*}
Discrete-time risk models with premium adjusted according to claims and surplus level

\[ \psi_{i}(u, 1) = \sum_{s=u+\alpha_i+1}^{\infty} f_{S}(s). \]

**Proof.** The proof of (4.3.12) is trivial.

### 4.4 Some numerical results

#### 4.4.1 Example for premiums adjusted according to aggregate claims.

This example shows the numerical results for the case that the premiums are adjusted according to aggregate claims. The aggregate claims are assumed to follow a negative binomial distribution with mean 10. Let the set of safety loading factors be \{110\%, 120\%, 140\%, 160\%, 170\%\}, and there are three threshold scenarios; \( d = 0 \), \( d = 10 \) and \( d = 20 \). The transition rules for premium adjustments are given as follows.

When the current surplus level is greater than or equal to the threshold \( d \) (i.e. under the healthy state),

- if the aggregate claims in the current period is no more than 6, the premium level for the next period will move to the lower premium level or stay in the lowest one;
- if the aggregate claims in the current period is more than 6 but no more than 12, the premium level for the next period will remain in the current premium level;
- if the aggregate claims in the current period is more 12, the premium level for the next period will move to the higher premium level or stay in the highest one.

Otherwise, when the current surplus level is less than the threshold \( d \) (i.e. under the unhealthy state),

- if the aggregate claims in the current period is no more than 6, the premium level for the next period will remain in the current premium level;
Discrete-time risk models with premium adjusted according to claims and surplus level

- if the aggregate claims in the current period is more than 6 but no more than 12, the premium level for the next period will move to the higher premium level or stay in the highest one;
- if the aggregate claims in the current period is more 12, the premium level for the next period will move to the second higher premium level or stay in the highest one.

We apply the results (4.3.7), (4.3.8) and (4.3.9) to compute the 50-period finite-time ruin probabilities $\psi(u, 50)$ that are summarised in Table 4.1 and Figure 4.1 and 4.2 below.

**Table 4.1:** $\psi(u, 50)$ values in the case of adjusted premiums according to aggregate claims

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<th>$u$</th>
<th>$\psi_1^A(u, 50)$</th>
<th>$\psi_2^A(u, 50)$</th>
<th>$\psi_1^B(u, 50)$</th>
<th>$\psi_2^B(u, 50)$</th>
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<td>0.002630</td>
<td>0.033306</td>
<td>0.002630</td>
<td>0.020793</td>
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<td>0.009829</td>
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<tr>
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</tr>
<tr>
<td>35</td>
<td>0.003759</td>
<td>0.000108</td>
<td>0.002983</td>
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<td>0.002059</td>
<td>0.000107</td>
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<td>0.000013</td>
</tr>
<tr>
<td>50</td>
<td>0.000231</td>
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<td>0.000199</td>
<td>0.000004</td>
<td>0.000156</td>
<td>0.000004</td>
</tr>
</tbody>
</table>

**Remark.** The superscript $A, B$ and $C$ represents the scenario $d = 0$, $d = 10$ and $d = 20$ respectively.
Discrete-time risk models with premium adjusted according to claims and surplus level

Figure 4.1: $\psi_1(u, 50)$ against $u$ with premiums adjusted by aggregate claims

Figure 4.2: $\psi_5(u, 50)$ against $u$ with premiums adjusted by aggregate claims
An obvious finding is that given \( u \) and the initial premium level, \( \psi(u, 50) \) decreases when \( d \) increases. However, the speed of decreasing in the ruin probabilities depends on \( u \) and the initial premium level. According to Table 4.1, we see the following dynamics between \( d \) and \( u \) with respect to the finite-time ruin probabilities.

Firstly, assuming the lowest initial premium level \((i = 1)\),

- \( d = 0 \text{ vs } d = 10 \): \( \psi_1^A(0, 50) \) is roughly 20% higher than \( \psi_1^B(0, 50) \), \( \psi_1^A(10, 50) \) is roughly 50% higher than \( \psi_1^B(10, 50) \), whilst \( \psi_1^A(50, 50) \) is roughly 16% higher than \( \psi_1^B(50, 50) \). It shows that having a threshold level of 10 helps to bring \( \psi_1^B(u, 50) \) down significantly comparing with the zero threshold case. This effect peaks at \( u = 10 \).

- \( d = 10 \text{ vs } d = 20 \): \( \psi_1^B(0, 50) \) is roughly 3% higher than \( \psi_1^C(0, 50) \), \( \psi_1^B(10, 50) \) is roughly 27% higher than \( \psi_1^C(10, 50) \), \( \psi_1^B(20, 50) \) is roughly 60% higher than \( \psi_1^C(20, 50) \), whilst \( \psi_1^B(50, 50) \) is roughly 28% higher than \( \psi_1^C(50, 50) \). It shows that increasing the threshold level from 10 to 20 has relatively low impact on \( \psi_1^C(u, 50) \) when \( u = 0 \). This downwards impact slowly increases when \( u \) gets to 10, and increases more quickly after 10. This effect peaks at \( u = 20 \).

Secondly, assuming the highest initial premium level \((i = 5)\),

- \( d = 0 \text{ vs } d = 10 \): \( \psi_5^A(u, 50) \) is roughly the same as \( \psi_5^B(u, 50) \) for all \( u \) levels.

- \( d = 10 \text{ vs } d = 20 \): \( \psi_5^B(0, 50) \) is only 1% higher than \( \psi_5^C(0, 50) \), whilst \( \psi_5^B(50, 50) \) is roughly the same as \( \psi_5^C(50, 50) \).

Figure 4.1 and 4.2 give a more comprehensive illustration of the dynamics between \( u \) and \( d \) described above in respect of \( \psi_i(u, 50) \). To have an even better view of their relationship, we also draw two 3D surface plots, Figure 4.3 and 4.4. Those contour lines confirm our previous findings.
Figure 4.3: $\psi_1(u, 50)$ against $u$ and $d$ with premiums adjusted by aggregate claims

Figure 4.4: $\psi_5(u, 50)$ against $u$ and $d$ with premiums adjusted by aggregate claims
We remark that under our assumptions, having sufficiently high premium income, proposing a small change in the threshold level will not bring any significant impact on the finite-time ruin probabilities. Also, when the insurers have low initial premium income (e.g. imposing aggressive pricing strategies to increase market share), the risk of ruin can be significantly lowered by applying surplus-dependent premium adjustment rules. This impact varies at different $u$ level and is peaked around $d = u$. On the contrary, implementing surplus-dependent premium adjustment rules may not be an effective approach of lowering the insolvency risk when the insurers charge high initial premiums (e.g. imposing conservative pricing strategies).

### 4.4.2 Example for premiums adjusted according to claim frequency.

In comparison with the previous example, this example shows the numerical results for the case that the premiums are adjusted according to the claim frequency. The number of claims is assumed to be Poisson distributed with mean 3.555 and the individual claim amounts are assumed to be geometric distributed with P.M.F. $f_W(w) = \left(\frac{3.555}{10}\right)\left(1 - \frac{3.555}{10}\right)^{w-1}$ for $w \geq 1$ and mean $\frac{10}{3.555}$. We use the same set of safety loading factors and $d$ values as assumed in the previous example. The transition rules for the premium adjustments are given as follows.

When the current surplus level is greater than or equal to the threshold $d$ (i.e. under the healthy state),

- If the claims number in the current period is no more than 2, the premium level for the next period will move to the lower premium level or stay in the lowest one;
- If the claims number in the current period is more than 2 but no more than 4, the premium level for the next period will remain in the current premium level;
- If the claims number in the current period is more than 4, the premium level for the next period will move to the higher premium level or stay in the highest one.

Otherwise, when the current surplus level is less than the threshold $d$ (i.e. under the unhealthy state),
Discrete-time risk models with premium adjusted according to claims and surplus level

- If the claims number in the current period is no more than 2, the premium level for the next period will remain in the current premium level;
- If the claims number in the current period is more than 2 but no more than 4, the premium level for the next period will move to the higher premium level or stay in the highest one;
- If the claims number in the current period is more than 4, the premium level for the next period will move to the second higher premium level or stay in the highest one.

We applied (4.3.10), (4.3.11) and (4.3.12) to compute $\psi_i(u, 50)$ and the results are summarised in Table 4.2 and Figure 4.5 & 4.6 below. We also draw the 3D surface plots for $\psi_1(u, 50)$ and $\psi_5(u, 50)$ for $0 \leq d, u \leq 30$.

**Table 4.2:** $\psi_i(u, 50)$ values in the case of adjusted premiums according to claims frequency

<table>
<thead>
<tr>
<th>u</th>
<th>$\psi_1^A(u, 50)$</th>
<th>$\psi_5^A(u, 50)$</th>
<th>$\psi_1^B(u, 50)$</th>
<th>$\psi_5^B(u, 50)$</th>
<th>$\psi_1^C(u, 50)$</th>
<th>$\psi_5^C(u, 50)$</th>
</tr>
</thead>
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<tr>
<td>0</td>
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<td>0.302684</td>
<td>0.087542</td>
</tr>
<tr>
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</tr>
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<td>0.084063</td>
<td>0.016660</td>
</tr>
<tr>
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<td>0.103743</td>
<td>0.007102</td>
<td>0.072406</td>
<td>0.007067</td>
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</tr>
<tr>
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<td>0.001820</td>
<td>0.000035</td>
<td>0.001261</td>
<td>0.000034</td>
</tr>
</tbody>
</table>
Figure 4.5: $\psi_1(u, 50)$ against $u$ with premiums adjusted by claim frequency

Figure 4.6: $\psi_5(u, 50)$ against $u$ with premiums adjusted by claim frequency
Figure 4.7: $\psi_1(u, 50)$ against $u$ and $d$ with premiums adjusted by claim frequency

Figure 4.8: $\psi_5(u, 50)$ against $u$ and $d$ with premiums adjusted by claim frequency
According to Table 4.2, the finite-time ruin probabilities under the given surplus-dependent premium rules by claim frequency are generally higher than the counterparts in the previous example. A possible interpretation is that the claim frequency records do not fully reflect the claim experience, which imposes additional risk when using the claim frequency records to determine the future premiums. Other than this, we have very similar findings from the numerical results in this example comparing with the previous one. The dynamics between $u$ and $d$ for different initial premium levels are similar in these two examples.

4.5 Parisian ruin probabilities

In this section, we will extend our models to the Parisian type of ruin. As mentioned at the beginning of this chapter, the Parisian ruin is the event that an insurer is in a negative surplus position for a continuous period of time that exceeds a given time frame. It is different from the ‘classical ruin’ in the way that the classical ruin occurs immediately when the insurers surplus level drops below zero. This concept of insolvency is more realistic in the real world than the classical ruin since in many circumstances the insurers may keep their business in operation with negative surpluses within a certain time period before they eventually get bankrupted. The bankruptcy can be avoided if the insurers can fight their way out of the deficit before it is too late. In this section, we assume that the Parisian ruin occurs when insurer’s surplus is negative in two consecutive periods. In addition, whenever the insurer’s surplus drops below zero, it immediately triggers a sanction by the regulators, under which very conservative premium adjustment rules are in force to prevent further deterioration in the following periods. This sanction will be lifted as long as the surplus level recovers from negative given that Parisian ruin does not occur.

For the purpose of simplification, we would not propose an additional surplus threshold level $d > 0$ on top of the Parisian ruin, i.e. only one set of premium correction rules when the surplus level is non-negative.
4.5.1 Models and Assumptions

All assumptions in this section are the same as those in Section 4.2 and the \( n \)-period finite-time Parisian ruin probability with initial surplus \( u \geq 0 \) and initial premium level \( c_i \) is defined by, \( n \in \mathbb{N}^+ \) and \( n \geq 2 \),

\[
\psi^P_i(u, n) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n-1} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \bigg| L_1 = c_i \right\}, \tag{4.5.13}
\]

where \( U_k \) is defined by (4.2.1) and the subscript \( u \) represents the condition \( U_0 = u \). It is trivial that \( \psi^P_i(u, n) = 0 \) for \( u \geq 0, n < 2 \). The corresponding \( n \)-period finite-time Parisian ruin probability with initial surplus \( v < 0 \) and initial premium level \( c_i \) is defined by, for \( n \in \mathbb{N}^+ \) and \( n \geq 3 \),

\[
\tilde{\psi}^P_i(v, n) = \mathbb{P}_v \{ U_1 < 0 \big| L_1 = c_i \} \\
+ \mathbb{P}_v \left\{ (U_1 \geq 0) \cap \left( \bigcup_{k=2}^{n-1} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \right) \bigg| L_1 = c_i \right\}, \tag{4.5.14}
\]

and \( \tilde{\psi}^P_i(v, 1) = \tilde{\psi}^P_i(v, 2) = \mathbb{P}_v \{ U_1 < 0 \big| L_1 = c_i \} \).

**Remark.** Let \( \alpha_i \) denote the first premium amount conditional on \( L_1 = c_i \) (see definition 4.2.2), then \( \tilde{\psi}^P_i(v, n) = 1 \) for \( v + \alpha_i < 0 \) and \( n \in \mathbb{N}^+ \). It is because if the summation of the initial surplus and initial premium is negative, the surplus at the end of the first period will definitely be negative which triggers the Parisian ruin.

Further, we shall use \( \{ t_{ij}(s) \}_{i,j \in \mathcal{L}, s \in \mathbb{N}} \) to denote the set of rules for premium corrections in non-negative surplus situation, and use \( \{ t'_{ij}(s) \}_{i,j \in \mathcal{L}, s \in \mathbb{N}} \) to denote the set of premium adjustment rules in the negative surplus case. The detailed definition of these rules are the same as those given in Section 4.2.1.

4.5.2 Finite-time Parisian ruin probabilities

Similarly to section 4.3, we shall derive the recursive formulæ to compute the finite-time Parisian ruin probabilities with the same two premium adjustment rules.
4.5.2.1 Premiums adjusted according to aggregate claims

According to the assumptions in section 4.5.1, the \( n \)-period finite-time Parisian ruin probability \( \psi_i^P(u, n) \), satisfies the following recursive formula.

**Theorem 12.** For \( u \geq 0, n \in \mathbb{N}^+ \), \( i \in \mathcal{L} \),

\[
\psi_i^P(u, n + 1) = \sum_{j=1}^{l} \left[ \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \psi_j^P(u + \alpha_i - s, n) + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) t_{ij}'(s) \tilde{\psi}_j^P(u + \alpha_i - s, n) \right] \quad (4.5.15)
\]

with \( \psi_i^P(u, 1) = 0 \) and \( \sum_a^b f_S(s) = 0 \) for \( b < a \).

**Proof.** From (4.5.13), for \( u \geq 0 \), we have

\[
\psi_i(u, n + 1) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} \left\{ (U_k < 0) \cap (U_{k+1} < 0) \right\} \big| L_1 = c_i, U_0 = 0 \right\}
\]

\[
= \sum_{s=0}^{u+\alpha_i} f_S(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} \left\{ (U_k < 0) \cap (U_{k+1} < 0) \right\} \big| L_1 = c_i, U_0 = 0, S_1 = s \right\}
\]

\[
+ \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} \left\{ (U_k < 0) \cap (U_{k+1} < 0) \right\} \big| L_1 = c_i, U_0 = 0, S_1 = s \right\}
\]

\[
= \sum_{j=1}^{l} \left( \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \mathbb{P}_{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n} \left\{ (U_k < 0) \cap (U_{k+1} < 0) \right\} \big| L_2 = c_j, U_1 = 0 \right\}
\]

\[
+ \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) t_{ij}'(s) \mathbb{P}_{u+\alpha_i-s} \left\{ (U_2 < 0) \big| L_2 = c_j, U_1 < 0 \right\}
\]

\[
+ \mathbb{P}_{u+\alpha_i-s} \left\{ (U_2 \geq 0) \cap \left( \bigcup_{k=3}^{n} \left\{ (U_k < 0) \cap (U_{k+1} < 0) \right\} \big| L_2 = c_j, U_1 < 0 \right\} \right\}
\]

\[
= \sum_{j=1}^{l} \left( \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \psi_j^P(u + \alpha_i - s, n) + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) t_{ij}'(s) \tilde{\psi}_j^P(u + \alpha_i - s, n) \right)
\]

The term \( \psi_i^P(u, 1) \) measures the probability of Parisian ruin of the business within one time period which is equal to zero because it is impossible that the surplus is negative in consequent two periods within one time period. \( \square \)
Further, the $n$-period finite-time Parisian ruin probability $\tilde{\psi}_i^P(u, n)$ satisfies the following the recursive formula.

**Corollary 9.** For $0 > u \geq -\alpha_i$, $n \in \mathbb{N}^+$, $n \geq 2$, $i \in \mathcal{L}$,

$$\tilde{\psi}_i^P(u, n + 1) = \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \psi_j^P(u + \alpha_i - s, n) + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) \quad (4.5.16)$$

with $\tilde{\psi}_i^P(u, 1) = \tilde{\psi}_i^P(u, 2) = \sum_{s=u+\alpha_i+1}^{\infty} f_S(s)$.

**Proof.** From (4.5.14), for $u < 0$ and $n \geq 2$ we have

$$\tilde{\psi}_i^P(u, n + 1) = \mathbb{P}_u\{U_1 < 0 \mid L_1 = c_i, U_0 < 0\} + \mathbb{P}_u\{(U_1 \geq 0) \cap \left( \bigcup_{k=2}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \right) \mid L_1 = c_i, U_0 < 0\}$$

$$= \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) + \sum_{s=0}^{u+\alpha_i} f_S(s) \mathbb{P}_u^{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \mid U_1 \geq 0 \right\}$$

$$= \sum_{s=u+\alpha_i+1}^{\infty} f_S(s) + \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s)$$

$$\times \mathbb{P}_u^{u+\alpha_i-s} \left\{ \bigcup_{k=2}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \mid U_1 \geq 0, L_2 = c_j \right\}$$

$$= \sum_{j=1}^{l} \sum_{s=0}^{u+\alpha_i} f_S(s) t_{ij}(s) \psi_j^P(u + \alpha_i - s, n) + \sum_{s=u+\alpha_i+1}^{\infty} f_S(s).$$

For $n = 1, 2$, the proof is trivial. $\square$

### 4.5.2.2 Premiums adjusted according to claim frequency

According to the assumptions in section 4.5.1, the $n$-period finite-time Parisian ruin probability $\psi_i^P(u, n)$, satisfies the following the recursive formula.
Theorem 13. For $u \geq 0$, $n \in \mathbb{N}^+$, $i \in \mathcal{L}$,

$$
\psi_i^P(u, n + 1) = \sum_{j=1}^{l} \left( \sum_{m=0}^{u+\alpha_i} f_m t_{ij}(m) \sum_{s=0}^{u+\alpha_i} f_W^m(s) \psi_j^P(u + \alpha_i - s, n) \right. \\
+ \sum_{m=1}^{\infty} f_m t_{ij}'(m) \sum_{s=u+\alpha_i+1}^{\infty} f_W^m(s) \tilde{\psi}_j^P(u + \alpha_i - s, n) \right) (4.5.17)
$$

with $\psi_i^P(u, 1) = 0$ and $\sum_{a} f_W^m(s) = 0$ for $b < a$.

Proof. From (4.5.13), for $u \geq 0$, we have

$$
\psi_i(u, n + 1) = \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \bigg| L_1 = c_i, U_0 \geq 0 \right\} \\
= \sum_{m=0}^{\infty} f_m' \left( \sum_{s=0}^{u+\alpha_i} f_W^m(s) \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \bigg| L_1 = c_i, U_0 \geq 0, S_1 = s \right\} \\
+ \mathbb{P}_u \left\{ \bigcup_{k=1}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \bigg| L_1 = c_i, U_0 \geq 0, S_1 = s \right\} \right) \\
= \sum_{j=1}^{l} \left( \sum_{m=0}^{u+\alpha_i} f_m' t_{ij}(m) \sum_{s=0}^{u+\alpha_i} f_W^m(s) \mathbb{P}_u \left\{ \bigcup_{k=2}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \bigg| L_2 = c_j, U_1 \geq 0 \right\} \\
+ \sum_{m=1}^{\infty} f_m t_{ij}'(m) \sum_{s=u+\alpha_i+1}^{\infty} f_W^m(s) \left( \mathbb{P}_{u+\alpha_i-s} \left\{ (U_2 < 0) \bigg| L_2 = c_j, U_1 < 0 \right\} \\
+ \mathbb{P}_{u+\alpha_i-s} \left\{ (U_2 \geq 0) \cap \left( \bigcup_{k=3}^{n} \{ (U_k < 0) \cap (U_{k+1} < 0) \} \bigg| L_2 = c_j, U_1 < 0 \right\} \right) \right) \\
= \sum_{j=1}^{l} \left( \sum_{m=0}^{u+\alpha_i} f_m' t_{ij}(m) \sum_{s=0}^{u+\alpha_i} f_W^m(s) \psi_j^P(u + \alpha_i - s, n) \\
+ \sum_{m=1}^{\infty} f_m t_{ij}'(m) \sum_{s=u+\alpha_i+1}^{\infty} f_W^m(s) \tilde{\psi}_j^P(u + \alpha_i - s, n) \right) \\
= \sum_{j=1}^{l} \left( \sum_{m=0}^{u+\alpha_i} f_m' t_{ij}(m) \sum_{s=0}^{u+\alpha_i} f_W^m(s) \psi_j^P(u + \alpha_i - s, n) \\
+ \sum_{m=1}^{\infty} f_m t_{ij}'(m) \sum_{s=u+\alpha_i+1}^{\infty} f_W^m(s) \tilde{\psi}_j^P(u + \alpha_i - s, n) \right)
$$

Again, the term $\psi_i^P(u, 1)$ is equal to zero because it is impossible that the surplus is negative in consequent two periods within one time period. \(\square\)

Further, the $n$-period finite-time Parisian ruin probability, $\tilde{\psi}_i^P(u, n)$, satisfies the following the recursive formula.
Corollary 10. For \( u + \alpha_i \geq 0, n \in \mathbb{N}^+, n \geq 2, i \in \mathcal{L} \),

\[
\tilde{\psi}_i^P(u, n + 1) = \sum_{j=1}^l \sum_{m=0}^{u+\alpha_i} f_{m,j}^*(m) \sum_{s=0}^{u+\alpha_i} f_s^w(s) \psi_j^P(u + \alpha_i - s, n) \\
+ \sum_{s=u+\alpha_i+1}^{\infty} f_s(s) \tag{4.5.18}
\]

with \( \tilde{\psi}_i^P(u, n) = \sum_{s=u+\alpha_i+1}^{\infty} f_s(s) \) for \( n = 1, 2 \).

Proof. From (4.5.14), for \( u < 0 \) and \( n \geq 2 \) we have

\[
\tilde{\psi}_i^P(u, n + 1) = \mathbb{P}_u \{ U_1 < 0 | L_1 = c_i, U_0 < 0 \} \\
+ \mathbb{P}_u \{ (U_1 \geq 0) \cap \left( \bigcup_{k=2}^n \{ (U_k < 0) \cap (U_{k+1} < 0) \} \right) | L_1 = c_i, U_0 < 0 \} \\
= \sum_{s=u+\alpha_i+1}^{\infty} f_s(s) + \sum_{m=0}^{u+\alpha_i} f_{m}^* \sum_{s=0}^{u+\alpha_i} f_s^w(s) \mathbb{P}_{u+\alpha_i-s} \left( \bigcup_{k=2}^n \{ (U_k < 0) \cap (U_{k+1} < 0) \} | U_1 \geq 0 \right) \\
= \sum_{s=u+\alpha_i+1}^{\infty} f_s(s) + \sum_{j=1}^l \sum_{m=0}^{u+\alpha_i} f_{m,j}^*(m) \sum_{s=0}^{u+\alpha_i} f_s^w(s) \\
\times \mathbb{P}_{u+\alpha_i-s} \left( \bigcup_{k=2}^n \{ (U_k < 0) \cap (U_{k+1} < 0) \} | U_1 \geq 0, L_2 = c_j \right) \\
= \sum_{j=1}^l \sum_{m=0}^{u+\alpha_i} f_{m,j}^*(m) \sum_{s=0}^{u+\alpha_i} f_s^w(s) \psi_j^P(u + \alpha_i - s, n) + \sum_{s=u+\alpha_i+1}^{\infty} f_s(s). 
\]

For \( n = 1, 2 \), the proof is trivial.

Remark. Again, \( \tilde{\psi}_i^P(u, n) = 1 \) for \( u + \alpha_i < 0 \) and \( n \in \mathbb{N}^+ \) because the surplus at the end of the first period will be always negative under this condition which causes the Parisian ruin.

4.5.3 Some numerical results

In this numerical examples, we shall duplicate the assumptions in 4.4 but consider \( \psi_i^P(u, 50) \) instead of \( \psi_i(u, 50) \).
4.5.3.1 Example for premiums adjusted according to aggregate claims

In this example, we assume that the claim distribution and set of premium levels are the same as the ones given in the example 4.4.1. For comparison purpose, we shall consider three scenarios of premium adjustments under the Parisian ruin:

- Scenario A: the premium adjustment rules under the healthy state given in the example 4.4.1 will be implemented all the time;
- Scenario B: the premium adjustment rules under the healthy state will be implemented when the current surplus is non-negative; the premium adjustment rules under the unhealthy state will be in place whenever the surplus level falls below zero;
- Scenario C: the premium adjustment rules under the unhealthy state will be applied all the time.

Clearly, Scenario A is most risky among the three scenarios in respect of the risk of Parisian ruin, whilst Scenario C is the least risky one. Using formulae (4.5.15) and (4.5.16), we compute the finite-time Parisian ruin probabilities and summarise our results in Table 4.3 and Figure 4.9 below.
Discrete-time risk models with premium adjusted according to claims and surplus level

Table 4.3: $\psi^P_i(u, 50)$ for $u \geq 0$ and $\tilde{\psi}^P_i(u, 50)$ for $u < 0$ in the case of adjusted premiums according to aggregate claims

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\tilde{\psi}^P_{A1}(u, 50)$</th>
<th>$\tilde{\psi}^P_{B1}(u, 50)$</th>
<th>$\tilde{\psi}^P_{C1}(u, 50)$</th>
<th>$\psi^P_{A1}(u, 50)$</th>
<th>$\psi^P_{B1}(u, 50)$</th>
<th>$\psi^P_{C1}(u, 50)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-17</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.99374</td>
<td>0.99374</td>
<td>0.99365</td>
<td></td>
</tr>
<tr>
<td>-14</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.89745</td>
<td>0.89741</td>
<td>0.89609</td>
<td></td>
</tr>
<tr>
<td>-11</td>
<td>0.99610</td>
<td>0.99547</td>
<td>0.70111</td>
<td>0.70101</td>
<td>0.69781</td>
<td></td>
</tr>
<tr>
<td>-8</td>
<td>0.93281</td>
<td>0.92316</td>
<td>0.48241</td>
<td>0.48236</td>
<td>0.48041</td>
<td></td>
</tr>
<tr>
<td>-5</td>
<td>0.79260</td>
<td>0.76743</td>
<td>0.30279</td>
<td>0.30277</td>
<td>0.30158</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>0.67843</td>
<td>0.64434</td>
<td>0.21350</td>
<td>0.21348</td>
<td>0.21262</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>0.62051</td>
<td>0.58316</td>
<td>0.17757</td>
<td>0.17755</td>
<td>0.17683</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>0.56384</td>
<td>0.52416</td>
<td>0.14685</td>
<td>0.14685</td>
<td>0.14623</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.41538</td>
<td>0.32094</td>
<td>0.24240</td>
<td>0.24240</td>
<td>0.24240</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.22508</td>
<td>0.16927</td>
<td>0.10264</td>
<td>0.10264</td>
<td>0.10264</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.11057</td>
<td>0.08348</td>
<td>0.04050</td>
<td>0.04050</td>
<td>0.04050</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.05085</td>
<td>0.03890</td>
<td>0.01521</td>
<td>0.01521</td>
<td>0.01521</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.02224</td>
<td>0.01733</td>
<td>0.00554</td>
<td>0.00554</td>
<td>0.00554</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.00935</td>
<td>0.00743</td>
<td>0.00198</td>
<td>0.00198</td>
<td>0.00198</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.00381</td>
<td>0.00309</td>
<td>0.00070</td>
<td>0.00070</td>
<td>0.00070</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.00151</td>
<td>0.00125</td>
<td>0.00024</td>
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<td>0.00024</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.00059</td>
<td>0.00050</td>
<td>0.00008</td>
<td>0.00008</td>
<td>0.00008</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>0.00022</td>
<td>0.00019</td>
<td>0.00003</td>
<td>0.00003</td>
<td>0.00003</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.00008</td>
<td>0.00007</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00000</td>
<td></td>
</tr>
</tbody>
</table>
From the table 4.3 and figure 4.9, one can see that except the case of $i = 5$, the finite-time Parisian ruin probabilities in other two initial premium cases all display an obvious ordering from A down to C that confirms our initial judgement on the risk levels of these three premium scenarios. In case 5, the differences among the three scenarios are negligible. This is consistent with our findings in Example 4.4.1 that the surplus-dependent premium adjustment rules are not effective in lowering the ruin probabilities when the initial premium is at a high level. Further, the initial premium levels appear to have a significant impact on the ruin probabilities and the gaps between the curves with different initial premium levels are quite substantial. Also, the numerical results imply that the Parisian ruin is just a special case of the classical ruin when we consider the surplus-dependent premium corrections.

4.5.3.2 Example for premiums adjusted according to claim frequency

Again, in this example, we assume that the claims distribution, set of premium levels are the same as in Example 4.4.2. We shall explore three similar premium adjustment scenarios making use of the premium adjustment rules given in Example
4.4.2. Using formulae (4.5.17) and (4.5.18) we compute the finite-time Parisian ruin probabilities and present the results in Table 4.4 and Figure 4.10 below.

Table 4.4: $\psi_P(u, 50)$ for $u \geq 0$ and $\tilde{\psi}_P(u, 50)$ for $u < 0$ in the case of adjusted premiums according to claims frequency

<table>
<thead>
<tr>
<th>$u$</th>
<th>$\tilde{\psi}_1^P,A(u, 50)$</th>
<th>$\psi_1^P,B(u, 50)$</th>
<th>$\tilde{\psi}_1^P,C(u, 50)$</th>
<th>$\psi_5^P,A(u, 50)$</th>
<th>$\psi_5^P,B(u, 50)$</th>
<th>$\psi_5^P,C(u, 50)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-17</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.97418</td>
<td>0.97406</td>
<td>0.97341</td>
</tr>
<tr>
<td>-14</td>
<td>1.00000</td>
<td>1.00000</td>
<td>1.00000</td>
<td>0.84824</td>
<td>0.84775</td>
<td>0.84474</td>
</tr>
<tr>
<td>-11</td>
<td>0.98460</td>
<td>0.98236</td>
<td>0.97965</td>
<td>0.66928</td>
<td>0.66870</td>
<td>0.66470</td>
</tr>
<tr>
<td>-8</td>
<td>0.90403</td>
<td>0.89181</td>
<td>0.87505</td>
<td>0.48654</td>
<td>0.48607</td>
<td>0.48237</td>
</tr>
<tr>
<td>-5</td>
<td>0.78017</td>
<td>0.75585</td>
<td>0.71336</td>
<td>0.33119</td>
<td>0.33086</td>
<td>0.32789</td>
</tr>
<tr>
<td>-3</td>
<td>0.68622</td>
<td>0.65512</td>
<td>0.59481</td>
<td>0.24873</td>
<td>0.24847</td>
<td>0.24600</td>
</tr>
<tr>
<td>-2</td>
<td>0.63841</td>
<td>0.60466</td>
<td>0.53666</td>
<td>0.21389</td>
<td>0.21367</td>
<td>0.21143</td>
</tr>
<tr>
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<td>0.59101</td>
<td>0.55519</td>
<td>0.48067</td>
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<td>0.18087</td>
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<td>0</td>
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<td>0.07221</td>
<td>0.07194</td>
<td>0.06976</td>
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<td>5</td>
<td>0.29861</td>
<td>0.23987</td>
<td>0.14656</td>
<td>0.03244</td>
<td>0.03233</td>
<td>0.03118</td>
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<tr>
<td>10</td>
<td>0.18189</td>
<td>0.14405</td>
<td>0.06940</td>
<td>0.01402</td>
<td>0.01397</td>
<td>0.01339</td>
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<tr>
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<td>0.08346</td>
<td>0.03131</td>
<td>0.00592</td>
<td>0.00590</td>
<td>0.00561</td>
</tr>
<tr>
<td>20</td>
<td>0.05912</td>
<td>0.04686</td>
<td>0.01367</td>
<td>0.00247</td>
<td>0.00246</td>
<td>0.00232</td>
</tr>
<tr>
<td>25</td>
<td>0.03205</td>
<td>0.02559</td>
<td>0.00584</td>
<td>0.00102</td>
<td>0.00102</td>
<td>0.00095</td>
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<tr>
<td>30</td>
<td>0.01691</td>
<td>0.01363</td>
<td>0.00246</td>
<td>0.00042</td>
<td>0.00042</td>
<td>0.00039</td>
</tr>
<tr>
<td>35</td>
<td>0.00872</td>
<td>0.00711</td>
<td>0.00103</td>
<td>0.00017</td>
<td>0.00017</td>
<td>0.00016</td>
</tr>
<tr>
<td>40</td>
<td>0.00441</td>
<td>0.00363</td>
<td>0.00043</td>
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<td>0.00007</td>
<td>0.00006</td>
</tr>
<tr>
<td>45</td>
<td>0.00219</td>
<td>0.00183</td>
<td>0.00018</td>
<td>0.00003</td>
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</tr>
<tr>
<td>50</td>
<td>0.00108</td>
<td>0.00090</td>
<td>0.00007</td>
<td>0.00001</td>
<td>0.00001</td>
<td>0.00001</td>
</tr>
</tbody>
</table>

Remark. The superscript $A,B$ and $C$ are defined in the same way as in the table 4.3.
Figure 4.10: $\psi_i(u, 50)$ and $\tilde{\psi}_i(u, 50)$ against $u$ with premiums adjusted by claim frequency

According to Table 4.4 and Figure 4.10, similar trends to those in Table 4.3 and Figure 4.9 are observed and there is no significant difference in the results between these two examples except the slightly higher values of the finite-time ruin probabilities in this example than the previous example with a reason that was already explained in Section 4.4.1 and 4.4.2.

4.6 Concluding remarks

In this chapter, we proposed some discrete-time risk models with claims-correlated premiums and modified premium adjustment rules that depend on the surplus level. In terms of the implementation in practice, an insurer may apply less conservative premium adjustment rules when the insurance business is in a good shape, i.e. having a high surplus level. On the other hand, when the company’s surplus level is critically low, the insurer may choose to put in place more conservative premium adjustment rules (or being sanctioned by the insurance regulators) to reduce the insolvency risk. Using the derived recursive formulae, we discussed four
numerical examples in terms of the calculated finite-time ruin probabilities under each proposed model, including the model under Parisian ruin. These numerical examples demonstrated the impact of the surplus-dependent premium adjustment rules, using the threshold level as a proxy, on the ruin probabilities. According to our observations, the surplus-dependent premium adjustment rules do show a high impact on the finite-time ruin probabilities when the initial premium level is low, whilst the impact in a high initial premium situation is low. These findings are consistent under both classical ruin and Parisian ruin. The two different premium adjustment principles, i.e. being adjusted by aggregate claims or by claim frequency, do not show significant differences in the results. Moreover, under the classical ruin, we also found that when the initial surplus level is high, all threshold levels have similar impact on decreasing the ruin probabilities, whereas with a low initial surplus level, high threshold levels account for larger decreases in ruin probabilities than having low threshold levels.
At present, the bonus-malus system is widely adopted in the non-life insurance industry for many reasons. First, it acts as the posteriori variable which classifies the policyholders within the same tariff cell and accounts for the heterogeneity within homogeneity groups. It helps the insurers to deal with the adverse selection risk so that they can customise appropriate premium levels for different policyholders, as the bonus-malus system allows the insurers to obtain useful information from the policyholders claim history besides the priori rating variables. Additionally, the bonus-malus system also balances the insurer’s portfolio in terms of maintaining good and bad policyholders with varying premiums. If everyone in the portfolio needs to pay a portfolio-based average premium, then the good policyholders will feel being overcharged and will very likely discontinue their policies. The low premium rate will also attract more bad policyholders into the portfolio. Consequently, the disproportionate bad policyholders in the portfolio will invalidate the original portfolio-based premium calculation and lead to high risk of insolvency. Lastly, the bonus-malus system is implemented by the insurers due to some commercial purposes. Specifically, when the good policyholders receive discounts in their renewal premiums, they tend to feel being fairly treated so more likely to stay with the company. It may also help to attract new good policyholders to join the portfolio which increases the premium growth and enhances the profit of the company.

On the contrary to the aforementioned benefits in adopting the bonus-malus system in non-life insurance pricing, the real-life implementation of the system
always faces some challenges including the change in external natural and/or social-economic environment, the complexity in claim experience records such as delayed claim settlements, and regulatory restrictions on insurance pricing. These challenges affect the performance of the adopted bonus-malus system directly, so should be taken into consideration when designing the bonus-malus system in the first place. This thesis addresses these challenges through evaluating the performance of the bonus-malus mechanism in three customised discrete-time risk models with built-in features accommodating the aforementioned factors. Valuable insights are gained into the ways of setting more effective bonus-malus systems while managing the insolvency risk.

The first discrete-time risk model studied in this thesis assumed that the claims distribution are governed by an external Markov process and the premium correction rules vary when the environment state changes. Two premium adjustment principles were considered in this case: by aggregate claims or by claim frequency. Our numerical studies found that the choice of initial premium level for new policyholders is a critical factor in the finite-time ruin probabilities. Having a low initial premium level tends to be very risky when the company’s initial capital amount is low. On the contrary, when the insurer has a sufficiently large initial capital amount, the insurer has a higher flexibility in deciding the initial premium levels. Competitive premiums, which help to attract more new policyholders and help with boosting the insurance business, can be utilised without significantly increasing the risk of insolvency. In respect of the impact of the external environment, the initial external environment condition does have a significant impact on the risk of ruin under the proposed premium adjustment rules, but the impact was a little different from our first guess. We observed that there is no straightforward ordering on the insolvency risk among cases with different initial environmental state, which can be seen as the consequence of a combined effect of non-homogeneous loss distributions, premium rule assumptions as well as the initial surplus level. Lastly, we found that adjusting premiums according to claim frequency can be riskier than the case of adjusting premiums by aggregate claims. However, how to implement the premium corrections in real life based on the main findings in this model remains a very challenging task, as there will be no fixed premium changing rules when the external environment changes. A possible solution is that the insurer designs a special scoring system and uses it to evaluate its policyholders’ claim experience. For instance, at the end of each policy year a policyholder receives a certain score
based on the ordering of his/her claim experience in the whole insurance portfolio, and then a bonus or malus can be offered in the renewal premium based on the scores. This type of score-based premium rules can effectively address the impact of external environment on premium corrections. This non-standard premium correction practice needs to be carefully communicated with the policyholders in practice.

The second discrete-time risk model with varying premiums allowed delays in by-claim settlements. When determining the renewal premium levels, the claim experience involving delayed claim settlements was taken into consideration. Four premiums correction rules were considered in this model: by aggregate reported claims, by aggregate settled claims, by reported claims number, or by settled claims number. We found that the probability of claim delays differentiates the ruin probabilities. To be specific, a high probability of delayed claims settlement decreases the ruin probabilities, because it is more likely to give the insurers more time to cover the claims. More importantly, the insurers would have received more premium income by the time the claims are due. Moreover, higher correlation between the main claims and by-claims also leads to higher finite-time ruin probabilities. Lastly, the premium adjustment principles based on settled claims experience (aggregate settled claims or settled claims number) account for higher finite-time ruin probabilities, compared with the principles based on the reported claims experience given all other factors are the same. This difference is more evident when the probability of by-claim delays is high. This suggests that the insurers should consider the rule of premiums adjusted by reported claims in such cases because the renewal premiums are determined by the incurred claims rather than settled claims. Otherwise, the renewal premiums determined by the settled claims could be underestimated and results in an increase in the ruin probabilities. However, these findings are based on the provided numerical examples with certain assumptions and may vary when the set of assumptions change.

Our last discrete-time risk model studied surplus-dependent premium adjustments. A pre-determined threshold surplus level guides the way in which the premiums are adjusted. Specifically, when the current surplus level is below the threshold, more conservative rules are applied. Meanwhile, less conservative rules can be applied when the current surplus level is above the line. We also studied Parisian type of ruin, as a special case of the threshold level, i.e. level zero. Our numerical studies in
this part show similar conclusions to the previous two models, i.e. when the initial surplus level is low, the surplus-dependent premium adjustment rules show high impact on the finite-time ruin probabilities, whilst when the initial surplus level is high, the threshold level have similar impact on decreasing the ruin probabilities. These findings are consistent under both classical ruin and Parisian ruin.

Overall, when the insurance company is in a low surplus situation, it is more worthwhile for the actuaries to investigate modified versions of the bonus-malus system, which are not bounded by the three types studied in this thesis. Only when the premium correction rules appropriately reflect the underlying insurance business, they can play the role in enhancing the insurance business as expected. Badly designed bonus-malus rules may bring more harm to the business than good.

Last but not least, we shall talk about some limitations of the studies conducted in this thesis as well as some potential future research problems. The first limitation of this thesis is that it only focused on the discrete-time setting. No cash flow was allowed throughout a given time period other than the beginning and the end. This assumption is far from being realistic. Allowing continuous-time cash flows within each time period could be a great extension, so it is worth studying in the future. Secondly, this thesis employed the finite-time ruin probability as the main objective when assessing the performance of the proposed risk models with various bonus-malus features. We are aware of the limitation of the finite-time ruin probabilities on measuring the long-term insolvency risk. In the future, studying the ultimate-time ruin probability would be a valuable extension. If a theoretical approach is not achievable, then approximations or numerical methods can be considered alternatively. The recursive approach with initial values obtained by simulation is also possible way to study the ultimate-time ruin probabilities. In addition, some other ruin-related quantities, such as the finite-time Gerber-Shiu functions, form another possible extension of this thesis. Thirdly, some assumptions made in this thesis can be replaced with more practical ones. For instance, in chapter 3, we could allow multi-period delays for the settlement of by-claims. Also, we could consider general counting distributions for the number of main claims and by-claims instead of Bernoulli distributions. In chapter 4, when considering the Parisian ruin, we could allow the length of the period of negative surplus to be longer than one time unit. Fourthly, the proposed premium setting strategies rooted on ruin probability minimisation in this thesis are more of a theoretical
nature. Therefore, the findings in this thesis may not find direct applications in a real-life situation since the real insurance operations are far more complicated than the ruin probabilities. For example, in chapter 4, when the insurer has a low initial surplus and minimises the ruin probabilities by using more conservative premium adjustment rules, the policyholders are likely to switch to other companies due to the increased premiums. Consequently, the insurer will receive lower aggregate earned premium which results in an increase in both fixed costs per written policy and the resultant ruin probabilities. Lastly, this thesis studies the bonus-malus system from the insurance portfolio point of view, which is a theoretically feasible angle in terms of the ruin probabilities. However, in practice the bonus-malus framework is implemented at the level of individual policyholders. Allowing every policyholder to have premiums adjusted periodically would undoubtedly make it impossible to build a workable collective risk model and to study the corresponding ruin probabilities. Although this limitation may prevent the results in this thesis from being directly useful in real practice, we still can get some interesting information and insights from the studies. In the future, we can facilitate real commercial bonus-malus scales and transition rules and implement them at the individual level. The risk of ruin for the whole portfolio can be evaluated by putting all individual risks together through simulation studies.
Appendix A

The transition matrix among premium levels

The transition matrix among premium levels in Section 2.6.1

\[ P_C = [p_{C,g,h}(i,j)]_{(l \times r) \times (l \times r)} \]

\[
\begin{bmatrix}
0.5668 & 0.2332 & 0 & 0 & 0 & 0.0709 & 0.0291 & 0 & 0 & 0 & 0.0709 & 0.0291 & 0 & 0 & 0 \\
0.2405 & 0.3263 & 0.2332 & 0 & 0 & 0.0301 & 0.0408 & 0.0291 & 0 & 0 & 0.0301 & 0.0408 & 0.0291 & 0 & 0 \\
0 & 0.2405 & 0.3263 & 0.2332 & 0 & 0 & 0.0301 & 0.0408 & 0.0291 & 0 & 0 & 0.0301 & 0.0408 & 0.0291 & 0 \\
0 & 0 & 0.2405 & 0.3263 & 0.2332 & 0 & 0 & 0.0301 & 0.0408 & 0.0291 & 0 & 0 & 0.0301 & 0.0408 & 0.0291 \\
0 & 0 & 0 & 0.2405 & 0.3263 & 0.5595 & 0 & 0 & 0.0301 & 0.0699 & 0 & 0 & 0 & 0.0301 & 0.0699 \\
0.2114 & 0.0886 & 0 & 0 & 0 & 0.4584 & 0.1919 & 0 & 0 & 0 & 0.0352 & 0.1480 & 0 & 0 & 0 \\
0 & 0 & 0.2114 & 0.0886 & 0 & 0 & 0.4584 & 0.2631 & 0.1919 & 0 & 0 & 0.0150 & 0.0202 & 0.0148 & 0 & 0 \\
0 & 0 & 0 & 0.2114 & 0.0886 & 0 & 0 & 0.4584 & 0.2631 & 0.1919 & 0 & 0 & 0.0150 & 0.0202 & 0.0148 & 0 \\
0 & 0 & 0 & 0 & 0.2114 & 0.0886 & 0 & 0 & 0.4584 & 0.2631 & 0.1919 & 0 & 0 & 0.0150 & 0.0202 & 0.0148 \\
0 & 0 & 0 & 0 & 0 & 0.2114 & 0.2100 & 0 & 0 & 0.4584 & 0.4550 & 0 & 0 & 0 & 0.0150 & 0.0350 \\
0.2129 & 0.0871 & 0 & 0 & 0 & 0 & 0.0355 & 0.0143 & 0 & 0 & 0.4613 & 0.1887 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0355 & 0.0143 & 0 & 0 & 0.4613 & 0.1887 & 0 & 0 & 0 \\
0.0900 & 0.1229 & 0.0871 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0355 & 0.0143 & 0 & 0 & 0.4613 & 0.1887 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0355 & 0.0143 & 0 & 0 & 0.4613 & 0.1887 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0355 & 0.0143 & 0 & 0 & 0.4613 & 0.1887 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0355 & 0.0143 & 0 & 0 & 0.4613 & 0.1887 & 0 \\
\end{bmatrix}
\]
The transition matrix among premium levels in Section 2.6.2

\[ P_C = \begin{bmatrix}
0.6329 & 0.1671 & 0 & 0 & 0 & 0.0791 & 0.0209 & 0 & 0 & 0 & 0.0791 & 0.0209 & 0 & 0 & 0 \\
0.1664 & 0.4664 & 0.1671 & 0 & 0 & 0.0208 & 0.0583 & 0.0209 & 0 & 0 & 0.0208 & 0.0583 & 0.0209 & 0 & 0 \\
0 & 0.1664 & 0.4664 & 0.1671 & 0 & 0 & 0.0208 & 0.0583 & 0.0209 & 0 & 0 & 0.0208 & 0.0583 & 0.0209 & 0 \\
0 & 0 & 0.1664 & 0.4664 & 0.1671 & 0 & 0 & 0.0208 & 0.0583 & 0.0209 & 0 & 0 & 0.0208 & 0.0583 & 0.0209 \\
0 & 0 & 0 & 0.1664 & 0.6336 & 0 & 0 & 0 & 0 & 0.0208 & 0.0792 & 0 & 0 & 0 & 0.0208 & 0.0792 \\
0.2864 & 0.0136 & 0 & 0 & 0 & 0 & 0.6206 & 0.0294 & 0 & 0 & 0 & 0.0477 & 0.0230 & 0 & 0 & 0 \\
0.1368 & 0.1496 & 0.0136 & 0 & 0 & 0 & 0 & 0.3241 & 0.0294 & 0 & 0 & 0 & 0.0228 & 0.0249 & 0.0923 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.2965 & 0.3241 & 0.0294 & 0 & 0 & 0 & 0.0228 & 0.0249 & 0.0023 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2965 & 0.3241 & 0.0294 & 0 & 0 & 0 & 0.0228 & 0.0249 & 0.0023 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2965 & 0.3535 & 0 & 0 & 0 & 0.0228 & 0.0272 \\
0.1745 & 0.1255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2720 & 0 & 0 & 0 \\
0.0285 & 0.1460 & 0.1255 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0617 & 0.1363 & 0.2720 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0617 & 0.1363 & 0.2720 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0617 & 0.5883 \\
\end{bmatrix}_{(l \times r) \times (l \times r)}
Bibliography


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