4. MATHEMATICS FOR SECONDARY TEACHING

Four Components of Discipline Knowledge for a Changing Teacher Workforce

This chapter addresses the mathematics required for teaching in secondary schools, from early adolescence to preparation for university. The chapter works from a vision of good mathematics learning which values working from reasons not just rules, and being able to use whatever mathematics has been learned for solving problems within and beyond mathematics. Four components of mathematical knowledge are needed for teaching: (i) knowing mathematics in a way that has special qualities for teaching; (ii) having experienced mathematics in action solving problems, conducting investigations and modelling the real world; (iii) knowing about mathematics including its history and current developments; and (iv) knowing how to learn mathematics. The chapter includes a short survey of teacher certification requirements in some western countries, and also reviews some reports that highlight shortages of well-qualified mathematics teachers. The policy responses to this situation relate to certification requirements, as well to the adequate provision for practising teachers of experiences that address all four components of discipline knowledge for teaching mathematics.

INTRODUCTION

This chapter addresses the mathematics required for teaching in secondary schools. In the chapter, I consider secondary school as being for students aged approximately 11 to 18 and primary school for students aged approximately 5 to 11. This chapter is concerned with preparation of teachers who will teach a broad spectrum of secondary mathematics through the middle and upper years of secondary school. Secondary school encompasses a very wide range of mathematics learning, from the early adolescent years when some students are still struggling with ideas of whole number place value up to the highest levels of school achievement when some students are prepared to enter the most demanding university mathematics courses. This wide range of mathematical content and student achievement necessitates a breadth of teaching tasks which in turn requires a broad range of mathematical knowledge for teaching. Given that other chapters in this volume deal with mathematics knowledge for teaching from an Asian perspective, this chapter is concerned with ‘western’ countries although a comprehensive review cannot be attempted because of the large variety of

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arrangements for teacher education and of school systems. I hope that the examples chosen serve to illustrate challenges and opportunities in many western countries. The term 'mathematics' is generally used to apply broadly to the mathematical sciences, including pure and applied mathematics, statistics, operations research, and parts of computer science.

In the past in many countries, mastery of the subject matter has been regarded as the only requirement to be a teacher. Aldrich (1990) notes that for much of English history the accepted method of preparation for teaching in a grammar school or public school or university was holding a master's degree from Oxford or Cambridge. Learning to teach was by an informal apprenticeship and teaching was often a family trade. Shulman (1986) notes that no distinction was made between content and pedagogy: those who knew the subject matter well were assumed to be able to teach it. Later, when university education departments were established, they generally prepared primary teachers since content knowledge was still regarded as sufficient for secondary teachers. Aldrich (1990) supports this by noting that even in 1925 there were 4602 students in university education departments in England preparing for primary teaching, but only 917 students preparing for secondary teaching.

Today, teacher education is seen to address several types of knowledge and skills and it is common to follow Shulman (1986) to describe these as content knowledge, general pedagogical knowledge, pedagogical content knowledge and knowledge of various aspects of the education setting (including knowledge of the learners, curriculum, educational contexts, purposes and values). This chapter is principally concerned with content knowledge, but since the purpose of this content knowledge is to teach, there is inevitably a blending with pedagogical content knowledge. Indeed in her study of pedagogical content knowledge (PCK), Chick (2007) found strict separation was unhelpful and so developed a framework for PCK with three components: (i) 'clearly PCK'; (ii) general pedagogical knowledge specifically applied in a content context (e.g., knowledge of what area of applications might be most motivating for students); and (iii) content knowledge in a pedagogical context (e.g., deep understanding of content, ability to deconstruct or connect content). This chapter considers the content knowledge that supports all three of these parts of PCK, with a specific emphasis on the development of content knowledge and content knowledge in a pedagogical context.

Those who use mathematics professionally learn to apply mathematical thinking to fields of endeavour such as engineering, finance or meteorology, and their work is characterised by a productive interplay between their knowledge of mathematics and their knowledge of the field of application. It is the same with teaching; we can think of mathematical knowledge being applied to the task of teaching with a productive interplay between what the teacher knows about mathematics and what the teacher knows about students and curriculum. Mathematical knowledge is applied to solve the real problems of teaching when teachers analyse a task to identify sources of difficulty, select an example with certain properties, identify the important pre-requisite knowledge for a topic and make connections between topics and engage in many other aspects of the work of teaching. These are the
special skills, at the interface between mathematics and teaching, which teachers uniquely need. In this chapter I address the content knowledge needed to support these skills.

Anyone who reflects on what mathematics teachers should know does so in relation to two things – the mathematics that students should learn at school, and the most important features of mathematics as an activity. I believe that a good mathematics education engages all students at every level in age-appropriate activities that develop:
- knowledge of facts
- fluency and accuracy in routine procedural skills;
- deep conceptual understanding;
- understanding of the major applications of mathematics;
- ability to communicate using clear and precise mathematical language;
- ability to tackle non-routine problems systematically;
- ability to apply what has been learned to solve problems in real world contexts;
- ability to conduct investigations using mathematics;
- logical reasoning and a conception of the nature of proof;
- practical ability for measuring, estimating, drawing and constructing;
- sensible use of calculators and computers;
- appreciation of the dynamic role of mathematics in society and the processes by which mathematics grows;
- confidence and a productive disposition, which inclines one to see mathematical activity as useful and worthwhile.

This list does not specify the content of mathematics learning, but it express an orientation to the processes and outcomes of learning any mathematics, which values working from reasons not just rules and being able to use whatever mathematics has been learned for solving problems within and beyond mathematics. The list is in broad agreement the U.S. report Adding It Up (Kilpatrick, Swafford & Findell, 2001) which lists five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition) although with a stronger emphasis on applications and problem solving outside mathematics. I take these orientations and values to apply to all mathematics education – in school and in the preparation of teachers.

This chapter begins with a short survey of the mathematical knowledge required to be a fully qualified mathematics teacher in several western countries. I then discuss in turn several different aspects of the content knowledge needed for teaching, which are widely recognised in the literature as important. These are:
- Knowing mathematics
- Experiencing mathematics in action
- Knowing about mathematics
- Knowing how to learn mathematics

In the final section, I will discuss how the recommendations given in earlier sections, which outline an ideal answer to the question ‘what mathematics should mathematics teachers know?’ are affected by the reality in many countries of a
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decline in the availability of well-qualified teachers. The chapter concludes with some questions for the future.

TEACHER CERTIFICATION REQUIREMENTS AROUND THE WORLD

For certification as a secondary mathematics teacher, educational jurisdictions require studies in mathematics and in education and usually specify the amount of university-level study of content knowledge rather than its nature and characteristics. INCA, the internet archive reviewing curriculum and assessment frameworks that is sponsored by the United Kingdom’s Qualifications and Curriculum Authority (HREF1) provides summary data on the professional education typically required of secondary teachers, including mathematics teachers, in a range of countries. INCA divides teacher training into ‘concurrent’ where teacher education is combined with a degree which results in the award of a Bachelor of Education or similar; ‘combined’ where there is a joint degree in education and a specific subject; ‘consecutive’, where a programme of professional education training is undertaken after an undergraduate degree and ‘on-the-job training’. Almost without exception, prospective teachers are required to gain a background in their discipline area(s) and to study education but there are differences in the balance between discipline content knowledge and pedagogical knowledge. It is the case that specifications for teacher certification apply to all secondary teachers; there are no special requirements listed for mathematics teachers. However, it is sometimes the case that upper secondary teachers (those of students about 15-18 years old) are required to undertake more discipline study than lower secondary teachers.

It is not possible to give a thorough review of the requirements for certification as a teacher of secondary mathematics, because of the substantial variation within and between countries. For example, while the United Kingdom, the Netherlands, Australia, Canada, New Zealand and the United States of America all endorse both the concurrent model and the consecutive model, the total time required to complete the qualification varies between 3 and 6 years. Germany, Hungary, Sweden, and Switzerland, Japan and Korea endorse the concurrent model. Italy and Spain have consecutive models. Stephens (2003) discusses the variety of courses offered in USA, Australia, the Netherlands and Japan in detail.

There is also a variety of institutions where teacher education is conducted. For example, the German Education server (HREF2) states that teacher training is divided into two parts, first studying at a university, and then practical pedagogical training. Teacher training courses are offered at universities, technical universities, pedagogical universities and art and music colleges while the pedagogical training in terms of a preparatory service takes place in school practical seminars and training schools.

Most countries specify only the amount of mathematics (and other subjects) to be studied for qualification as a teacher. In the UK, according to Goulding, Hatch, and Rodd (2003), most secondary mathematics teachers complete a one year Post Graduate Certificate of Education which follows a degree. These candidates are
required to have at least 50% mathematics or content strongly related to mathematics within their undergraduate degree. This is spelt out in broad terms in the UK “Qualifications to teach” (HREF3) as “Teachers should have a secure knowledge and understanding of the subject(s) they are trained to teach. For those qualifying to teach secondary pupils, this knowledge and understanding should be at a standard equivalent to degree level”. Spain provides something of a contrast because while their teachers are required to undertake general then professional studies the particular discipline which may be the focus of their undergraduate general degree may not be linked to the subject they intend to teach. To become a secondary mathematics teacher, it is not necessary for the candidate's general degree qualification to have been in mathematics (HREF1). The requirements for initial teacher training in the UK are further described by McNamara, Jaworski, Rowland, Hodgen, and Prestage (2002), including the introduction of the "Numeracy Skills Test" for all primary and secondary teachers (not just mathematics teachers) by the central Teacher Training Agency as part of a wide-reaching standards agenda. They also conduct audits of the mathematics content of teacher training courses.

France is an example of a system that has stronger national agreement on the content of mathematics that prospective teachers should study. In writing about the French requirements for mathematics teachers, Robert and Hache (2000) draw attention to the strong emphasis placed there on the teachers’ personal knowledge of mathematics. To qualify as a mathematics teacher in France students undertake five years of training beginning at University and with final two years at IUFM (Institut Universitaire de Formation des Maîtres). In the first three years, prospective mathematics teachers are expected to study a standard undergraduate mathematics syllabus including classical linear algebra, real analysis, topology, differential and integral calculus and options such as probability, complex analysis or numerical analysis. The first year at IUFM is again devoted to mathematics, preparing for a highly competitive theoretical examination with written and oral components. The written paper covers problems from the mathematics topics that students are expected to have covered during their university studies (analysis, topology and functional analysis, integration and differential calculus, groups, rings, and linear algebra, geometry) while the oral examination focuses on topics from the later part of the high school curriculum. Henry (2000) notes that this year helps students consolidate their knowledge and reorganise it for teaching. This theme of knowledge being reorganised for teaching is also evident in initiatives that are described in later sections. Clearly France expects a high level of uniformity in the students’ undergraduate mathematics programmes. Only in the fifth year is there any consideration of what the profession will involve. Robert and Hache (2000) observe that in the French system it is accepted that a thorough knowledge of mathematics is the most important ingredient for teaching. They observe that there is a need for teachers to be able to do mathematics themselves but also to take account of the students. Despite their years of study in mathematics, it is not uncommon for teachers to have difficulty in presenting
mathematics in such a way that links are formed and students in their class are enabled to appreciate the structure of mathematics.

In the United States there is no uniform specification by teacher certification agencies of the mathematics which should be studied by prospective secondary teachers. In some institutions the academic home for studies in both mathematics discipline and methods is in the mathematics department. In others they are separated, although the effects of this are not well studied (Graham, Li, & Curran Buck, 2000). The USA endorses both the concurrent and consecutive models but in either case the range and depth of mathematics studied may vary. To provide guidance, the Conference Board of the Mathematical Sciences (CBMS) (2001), which is associated with both the Mathematical Association of America and the American Mathematical Society, put forward the following general recommendations regarding the mathematics curriculum and instruction for prospective teachers. They state that an aim is “to convince faculty that there is more intellectual content in school mathematics instruction than most realize, content that teachers need to understand well” (p. 3).

- Recommendation 1. Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach.
- Recommendation 2. Although the quality of mathematical preparation is more important than the quantity, the following amount of mathematics coursework for prospective teachers is recommended. Prospective high school teachers of mathematics should be required to complete the equivalent of an undergraduate major in mathematics, that includes a 6-hour capstone course connecting their college mathematics courses with high school mathematics.
- Recommendation 3. Courses on fundamental ideas of school mathematics should focus on a thorough development of basic mathematics ideas. All courses designed for prospective teachers should develop careful reasoning and mathematical ‘common sense’ in analysing conceptual relationships and in solving problems.
- Recommendation 4. Along with building mathematical knowledge, mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching.

The CBMS report also lists mathematics topics that they believe should be studied by prospective secondary mathematics teachers, which are discussed below. In parallel with their assertion, strongly backed by extensive research of the mathematics requirements embedded within teaching tasks, they recommend a stronger focus on topics related to school mathematics than might usually be expected in university studies and attention, in Recommendation 4, to the atmosphere within which this content is delivered. This is in contrast to the regulations for certification in nearly all the countries discussed, where the requirements for being a teacher specify the amount of mathematics to be studied but are prescriptive about neither the content of the courses nor the teaching methods and attitudes developed by these courses.
KNOWING MATHEMATICS

In this section and the three following, I discuss the mathematical discipline studies for teaching. This first section discusses the content; the next section discusses how it is essential for mathematics teachers to have experienced doing mathematics through open investigations and modelling the real world. In this regards it is often the case that good mathematics learning for prospective teachers is not different from good mathematics learning for other undergraduates. In the next two sections, I discuss what teachers should know about mathematics as a discipline and how they need to be able to continue to learn mathematics independently.

There is no question that teachers need a sound understanding of mathematics for teaching in secondary schools, but there are important questions of how much, what topics, and what should be the qualities of their mathematical knowledge. As noted elsewhere, most jurisdictions specify only how much mathematics (or any other discipline area) a secondary teacher requires for certification. In this section, we discuss what topics and what qualities this knowledge should have. Before leaving the question of quantity of mathematics teaching however, I note that the specification of some study of university level mathematics indicates several values – that teachers should have strengthened their mathematical knowledge and skills beyond what is needed at the school level, that they should know more than their students, and that they should have some perspective on where school mathematics leads. It is also worthwhile noting that there is an assumption that those who have completed school mathematics know its content well. This assumption is reassessed below: students learning mathematics as mathematically immature adolescents are unlikely to have experienced it with the richness and perspective required for teaching.

Commentaries and opinions on the mathematics that teachers should know reflect ideals of what constitutes good school mathematics and what constitutes good mathematical practice in a wider sense. Regardless of one’s philosophical stance on the nature of mathematics, it is abundantly clear that the existing and ideal school and university curricula and the values about mathematics itself that people hold are socially constructed, with the result that recommendations for teacher education vary from time to time and place to place, always expressing what is seen as the ideal. In the era of the ‘new mathematics’ in school curricula and Bourbaki formalism amongst mathematicians, for example, people valued a highly logical approach to mathematics where even school children drew on set theory and definitions of fundamental objects to deduce their properties. As another example, from Cyprus, Stylianides, Stylianides, and Philippou (2007) document prospective teachers’ knowledge of proof by mathematical induction. The article assumes that all mathematics teachers should have mastered various forms of proof and should be able to give their students rich experiences related to proof. Whilst I would like this to be true in my own country, many fully qualified mathematics teachers in Australia have little understanding of proof. It is also the case that proof and verification currently play a very small part in Australian school mathematics. The 1999 TIMSS video study (Hiebert et al., 2003), for
example, found practically no instances of proof or deduction in large random samples of Grade 8 lessons from Australia, USA and the Netherlands, although instances of proof were prominent in Japan.

The most prominent recent recommendations of mathematical knowledge for teaching have been given by the Conference Board for the Mathematical Sciences (CBMS, 2001) in the USA. Here I discuss only their recommendations for high school teachers (grade levels 9–12). The CBMS report begins by reviewing research that shows that beyond a threshold, having taken additional subject matter courses has only a small effect on teachers' effectiveness. They take this as a challenge to rethink teacher education so that what prospective teachers learn will indeed increase their effectiveness in the classroom in the future. As a consequence they recommend emphasising the underlying nature of the subject matter, a deep understanding of the subject in a way that is organised for teaching, and awareness of historical, cultural and scientific roots to mathematical ideas and techniques. They consider that recent changes in areas of application of mathematics caused a broadening of school mathematics to include more statistics and discrete mathematics and the use of new technologies opened up new opportunities for both teaching strategies and content to be studied. Consequently, they conclude that future teachers need to know more mathematics than before, somewhat different mathematics than before, and to experience learning it in a new way for themselves.

The CBMS report outlines the content of a mathematics major for prospective teachers. In doing this, it is mindful that mathematics departments may not have the student numbers to warrant special programmes for teachers. As a consequence they have to consider whether the mathematical knowledge required by prospective teachers is quite different to that required by students pursuing other mathematics-related professions. Although they initially state that it is quite different, they later propose that the recommendations for prospective teachers, which outline a major study broader than before and with stronger connections to school mathematics and with use of modern technology, may well serve today's US undergraduates better than traditional majors. They note that mathematics courses which traditionally aimed at preparing students for graduate school work in mathematics must now serve a much wider constituency including mathematics majors not planning graduate work, and undergraduates with other major studies, such as engineering or science.

The broad content of the CBMS recommended university major is organised around five themes of the high school curriculum: algebra and number theory, geometry and trigonometry, functions and analysis, statistics and probability, and discrete mathematics and computer science. The report gives an explanation of the special role and emphasis of each of these areas in the education of teachers. They recommend use of new technology, and note that prospective teachers need to understand the differences between electronic calculation to advance learning, human computation to advance learning, and electronic computation as a practical expedient, as well as learning to use a wide range of mathematical software (e.g., CAS, school-level technology such as graphics calculators and dynamic geometry)
and learning some computer-related mathematics and the basics of computer science. In setting out proposed content for each of the courses, the principle that university study of mathematics should illuminate high school mathematics is strong. This is in agreement with Cooney and Wiegels's (2003) Principle 2 that prospective teachers should explicitly study and reflect on school mathematics. For example, in algebra and number theory suggested exercises are to explain how operations of school algebra link with formal axiomatic principles and to justify each step of a common procedure (such as solving a quadratic equation) with field or ring properties. The geometry and trigonometry course is used to demonstrate the nature of axiomatic reasoning, and uses computer graphics and robotics as applications to strengthen students' understanding of modern areas of application and use of technology. Looked at from the British mathematics tradition, the proposed major seems to be lacking an emphasis on the major applications of mathematics, especially relating to partial differential equations. There are also some personal favourite topics of mine that are omitted. For example, I would like to include the study of complex analysis because it gives a strong sense of how the real number line is embedded in the complex plane and because it explains the difficulties with the definitions of log and power functions for negative and fractional numbers; difficulties that impinge on high school mathematics. These examples serve to show the increasing pressure on curriculum time in the mathematics major. The enormous growth in mathematical sciences over the last century is necessarily changing the face of both undergraduate and school mathematics and the place of every component needs to be regularly reassessed.

Although the CBMS report claims that their mathematics major designed for teachers may serve other undergraduates better than a traditional major in mathematics, they also recognize that teachers need some knowledge of mathematics that is unique to teaching. The main need is to assist prospective teachers to make insightful connections between the advanced mathematics they are learning and school mathematics. They recommend that this can be done within each course, but it can also be done by offering a capstone course, taught jointly with mathematics educators. Extensive suggestions for the contribution of each of the five themes to the capstone course are given. For example, linking to the functions and analytic theme, the report suggests that the capstone course looks at the concept of function as a unifying theme in mathematics, examines the role of computers as tools for graphing and computation, examines relations between exploration and proof, and offers some experience of mathematical modeling. Other suggestions include historical perspectives on the development of the idea of function from a 'formula' to a 'mapping' and the cognitive difficulties that modern students experience in making this same transition, examining the use of graphing technology in teaching calculus, and drawing connections between the functions used in different branches of mathematics including probability distributions and log-log plots. How all of this material, with more from the other four themes, could fit within the one capstone course is not addressed. The squeeze on time in the mathematics major comes not only from the expansion of mathematics, but also from the growing appreciation of the need for connections to school mathematics.
Other moves to reform undergraduate education have also found good alignment between the needs of prospective teachers and the needs of the new population of undergraduates now in universities. Pierce, Turville, and Giri (2003) report on the process of review of mathematics courses in an Australian mathematics department where a significant proportion of mathematics majors are prospective teachers. They report that the review was a challenging and reinvigorating process, requiring analysis of what mathematicians valued, but also coming to see their teaching from the students' viewpoints. They argued “mathematical thinking, key skills and conceptual understandings were valued, so too was the exposure of students to various branches of mathematics and their applications. What we needed was to approach this learning from the students’ perspective” (p. 155). Their review set goals for knowledge of mathematics, experience of the process of doing mathematics (“reason mathematically, communicate and solve problems”), and for knowledge about mathematics (“understand and appreciate the role of mathematics and its applications in the real world”). They also specified goals related to career development (“Education students should form a positive view of their potential careers as mathematics teachers”) and goals related to improving their experience of learning mathematics (“incorporate up-to-date teaching technology and utilise methods that enhance student learning”). Pierce et al. (2003) adopted a thematic approach to their new curriculum, and planned for the use of realistic problems to introduce the need for theory. They gave courses non-traditional and enticing names such as “Logic and Imagination”. Practical activities and current technologies were used to enhance the process, and the assessments were chosen to cater for different learning styles and to encourage a range of different skills of communication and analysis. In accordance with the results of many other investigations, surveys revealed that their earlier students often had a narrow perception of mathematics, focussing on routine processing, and so their reinvigorated curriculum for mathematics teachers (and other students) also aimed to heighten enthusiasm for mathematics by using engaging topics in both learning and assessment. The use of new technologies was one tool they used to change undergraduate students' perceptions. In particular they wanted to use new technologies to emphasise a view of mathematics as description and explanation, rather than mathematics as rules for symbol manipulation. Pierce et al. (2003) found that these changes resulted in an increase in both initial enrolments and retention rates, increasing awareness of the relevance of mathematics for other disciplines and every day life, reduction in mathematics anxiety, increasing interest in mathematical thinking and an improved understanding of mathematics. Initiatives such as this may produce teachers with better understanding—although of a possibly more limited curriculum—and with more enthusiasm for mathematics. In turn, by increasing mathematics enrolments, it may produce more mathematics teachers.
Attitudes to and Beliefs about Mathematics

Initiatives such as that of Pierce et al. (2003) and the capstone course of CBMS intend to create more a more productive disposition towards mathematics in future teachers. There is a large literature (reviewed, for example, by Cooney and Wiegels, 2003) which shows that teachers’ attitudes to, and beliefs about, mathematics influence their teaching. Some of these attitudes and beliefs are inconsistent with the reform vision of mathematics set out above, especially as they are often dominated by the view of teaching and learning mathematics by ‘rules without reasons’. As Lachance and Confrey (2003) observe, teacher reform efforts are often “attempting to get teachers to think about and to teach mathematics in ways which they have never experienced as learners” (p. 110). It is clear that all mathematics education, at school and within teacher education, contributes to the attitudes and beliefs that teachers bring to with them to their work and hence to those that they impart to students. Teacher education that supports the implementation of a better form of mathematics teaching will itself need to demonstrate the characteristics of that mathematics teaching.

It is important, though, to remember that there is not only one good way to teach mathematics and that there are different valid views of what is most important about mathematics. Kendal and Stacey (2001) studied two experienced teachers in one school who were incorporating computer algebra systems into their introductory calculus teaching. They incorporated this new technology into their pedagogy in different ways, consistent with their different beliefs and understandings about mathematics. The different conceptions of mathematics influenced their particular choices while using technology, their emphasis, and how to incorporate the graphical and symbolic algebra capabilities of the calculator into their lessons. In turn, these choices affected what their students learned. Although both classes achieved almost identical overall achievement, they showed quite different strengths. One teacher enjoyed the exactness of mathematics and especially liked the capability of the computer algebra system to give exact answers (e.g., fractions, square roots). He privileged the teaching of mathematical procedures, to which he added new technology procedures. His class was better at recognising how differentiation could be applied to solve a problem. The other teacher privileged conceptual understanding of mathematical ideas supported by extensive use of technology graphing and consequently his students were superior at interpretation of mathematics.

It is also important to recall that teachers will move beyond their own preferences in the interest of students. Teachers’ own beliefs about mathematics can be mediated by beliefs about students and their needs, especially as their experience of students and teaching grows. Good teachers aim to give the best education for every individual student, regardless of their mathematical talent and approach to school work, and this can override the teacher’s personal preferences. An excellent example of this was provided in a recent project (Stacey, Stillman & Pierce, no date) where we worked with teachers in six schools to enhance students’ achievement and engagement through using real world problems and new
technology. At the end of an interview Meryl, one teacher in the project, explained that she used the project activities to teach in a way different to her own preferences because she judged doing so was in the interests of her students:

Personally I like algebra. If I could choose to teach children where I don’t need to have activities, where I don’t need do a lot of modelling or real life problems, I would. I would just do the ‘boring’ algebra. I enjoy that most. But the percentage of students in a normal cohort who would gain from that is just a minor part. The majority gain more from this approach.

The Quality of Knowledge

The above discussion has focussed mostly on what teachers should know. However, it is well established that for this content knowledge to be effective for teaching, it requires certain characteristics. Shulman (1986) listed aspects such as the amount and organisation of knowledge, understanding the structures of the subject matter and how truth and falsehood are established, and being able to explain why a proposition is regarded as true, why it is worth knowing and how it relates to other propositions within and beyond the discipline. Many subsequent studies have shown that these characteristics are often lacking even in prospective teachers who have strong undergraduate mathematics backgrounds (see, for example, Ball, 1990; Goulding, Hatch, & Rodd, 2003). Concerns to ensure content knowledge has such characteristics lie behind proposals such as the CBMS capstone course.

There is a growing body of research which aims to measure some of these qualities of teachers’ subject matter knowledge. Connectedness is one that has received considerable prominence in recent literature. Chinnappan and Lawson (2005) present a framework which enabled them to characterise teachers’ content knowledge and content knowledge for teaching. They demonstrated how to map teachers’ content knowledge and their knowledge for teaching one topic (in this case in geometry) in a way which revealed and began to quantify the connectedness of that knowledge. This draws on research which shows that knowledge structures which are more comprehensive and more internally and externally connected are more likely to be useful in problem solving. In this case the ‘problem solving’ is the act of teaching itself. Their maps revealed qualitative and quantitative differences in the connectedness of knowledge of two well-qualified and experienced secondary mathematics teachers. Further developments in research on measuring the qualities of teachers’ knowledge bases will help us understand which differences and which magnitudes of differences impact upon teaching effectiveness.

EXPERIENCING MATHEMATICS IN ACTION

The most influential argument for teachers to experience the process of doing mathematics was put forward by Polya (1962) in his books on how to solve mathematical problems and his exposition of problem solving heuristics. Polya’s
Work was inspired by his “concrete, urgent, practical aim: to improve the preparation of high school mathematics teachers” (p. vii) and so along with case studies of interesting problems and their solutions, he provided a section on “hints for teachers and teachers of teachers” (p. 209) drawn from his own teaching experiences. Writing in the era of the ‘new math’, Polya found the issue of what content should be offered to prospective high school teachers and to their students too controversial for agreement, but he proposed that knowledge of the process of doing mathematics was something upon which experts would agree.

Our knowledge about any subject consists of information and of know how. If you have genuine bona fide experience of mathematical work on any level, elementary or advanced, there will be no doubt in your mind that, in mathematics, know-how is more important than mere possession of information. Therefore, in the high school, as on any other level, we should impart, along with a certain amount of information, a certain degree of know-how to the student. What is know-how in mathematics? The ability to solve problems – not merely routine problems but problems requiring some degree of independence, judgment, originality, creativity. […] The teacher should know what he is supposed to teach. He should show his students how to solve problems – but if he does not know, how can he show them? The teacher should develop his students’ know-how, their ability to reason; he should recognise and encourage creative thinking – but the curriculum he went through paid insufficient attention to his mastery of the subject matter and no attention at all to his know-how, to his ability to reason, to his ability to solve problems, to his creative thinking. Here is, in my opinion, the worst gap in the present preparation of high school mathematics teachers. To fill this gap, the teachers’ curriculum should make room for creative work at an appropriate level (p. vii).

Polya’s recommendations for teaching teachers about mathematical problem solving and investigation have been the basis of much subsequent work. He recommended problems that did not require much knowledge beyond high school mathematics, but did require concentration and judgement. His book was organised around strategies for problem solving, illustrated by problems chosen to highlight the strategies. He also advised that prospective teachers should reflect on the classroom use of such problems. He therefore supplemented the ‘look back’ phase of doing mathematics (where problem solvers reflect on the mathematical solution) by an additional didactically-oriented phase. Polya’s basic ingredients for teaching problem solving of experience, strategies and reflection have formed the basis of many subsequent endeavours, including those of Schoenfeld (1985), and Mason, Burton, and Stacey (1982).

Since the time of Polya, explicit attention to the process aspects of mathematics has been evident as a goal in teacher preparation courses. This is evident, for example, in two of the 3 principles that Cooney and Wieg (2003, p. 806) recommend for mathematics for prospective teachers:
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- Principle 1: Preservice teachers should experience mathematics as a pluralistic subject.
- Principle 2: Preservice teachers should explicitly study and reflect on school mathematics.
- Principle 3: Preservice teachers should experience mathematics in ways that support the development of process-oriented teaching styles.

Experiencing mathematics as a pluralistic subject (Principle 1) includes (among other things) experiencing mathematical investigation and problem solving, and Principle 3 calls for prospective teachers to be supported in making such experiences a reality for their own students. Recently, Ryve (2007) found that 6 of 28 teacher education institutions in Sweden had a specific course directed to problem solving, although he noted different foci. Some courses emphasised the mathematical aspects of the tasks with which the prospective teachers engage. Others dealt with problem solving tasks in relation to the secondary school students’ learning and behaviour. Others emphasised aspects of the problem solving process, such as interpretations of answers, multiple solutions etc. Other important differences are whether a course sets out to examine only problems challenging to school students or aims to extend prospective teachers’ own problem solving, giving them a taste of the genuine problem solving experience which they aim to provide for school students. In my opinion, some attention to the latter (teachers’ own experience of doing mathematics) is essential in order to equip them to provide rich experiences for school students. Stacey (2005), using examples from Singapore and three western countries, documents how the goal of teaching children to be better problem solvers (teaching for problem solving) has now generally been supplanted by the intention of ‘teaching through problem solving’ where students encounter new material about standard topics through investigations. They are expected to acquire problem solving and investigative approaches implicitly. It is my opinion that this approach is unlikely to give student teachers the explicit knowledge of the process of doing mathematics that they require for teaching it well, and that there is a need to provide prospective teachers with supported experience of doing mathematics at their own level, with discussion of strategies and reflection on successful elements.

Ryve (2006) also notes the importance of studies which examine the different character of courses that are provided in teacher education. It is hoped that such studies might extend the primarily quantitative discussions of how much mathematics prospective teachers should learn, to examining qualitatively the nature of their participation in those courses and consequently how their views of mathematics could be extended. The literature provides many examples of how prospective teachers’ views and approaches to solving mathematical problems are limited. Van Dooren, Verschaffel, and Onghena (2003) examined secondary prospective teachers solving algebraic and arithmetic word problems. The secondary teachers had good content knowledge, but their habits and attitudes whilst problem solving were stereotyped and some were not open to look for the alternative methods which their future pupils may use. They characterised the
prospective teachers, at both the beginning and end of their teacher preparation, as tending to have routine expertise rather than adaptive expertise. They concluded that prospective teacher education needed to promote experiences of doing mathematics that encouraged flexible thinking and created a well-organised body of professional knowledge.

Despite the agreement that all mathematics students, and especially teachers, should have direct experience of the processes of mathematical discovery, investigation and application, the history of its implementation has been anything but smooth, in teacher education as well as in schools and universities. Burkhardt and Bell (2007), for example, describe the development of problem solving in UK school mathematics over more than a century. They highlight the many different interpretations of problem solving, from mathematical research by professionals; to the solution of ‘riders’ where knowledge of theorems and proofs is adapted to a novel problem situation; to conducting open investigations such as finding which numbers are not sums of consecutive numbers; to modelling the real world; to teaching for functional numeracy and mathematical literacy (as defined by OECD, 2003). They highlight the gap between the goals for having students’ experience any of the above forms of problem solving and the understanding by school systems of the nature of the change required and the conditions under which it could really be achieved. It is likely that the situation is similar in teacher education, with a mismatch of goals and reality.

Burkhardt and Bell’s article also draws attention to the fact that there is a lot more to the ‘experience of doing mathematics’ than engaging in the pure mathematics investigations in the spirit of Polya and successors (e.g., which numbers are not sums of consecutive numbers), or the investigations that reform-oriented teachers offer to develop students’ understandings of particular concepts (e.g., Chick’s (2007) finding the dimensions of a rectangle of given perimeter with maximum area or Teacher B in Kendal et al. (2001) setting his class the problem of finding a general rule for the derivative of $x^2$, $x^3$, $x^4$ and then $x^n$ by guessing rules from numerical values of the slopes of tangents). Mathematics is studied for its interest and its beauty and for its place in our cultural heritage, but its central role in the school curriculum is due to its usefulness. Teachers therefore need to understand deeply the way in which mathematics is applied. The presence of teachers coming into teaching as a second or subsequent career from a diversity of backgrounds is a strength here.

Mathematical modelling, using mathematics to answer questions about the real world, has a distinctly different flavour to investigation within the real world. It can be understood as consisting of four steps: formulating a mathematical problem from the real world problem, solving the mathematical problem (using all the techniques developed within mathematics itself), interpreting the mathematical solution in real world terms, and evaluating the solution to see if this solution is adequate for the task. The intention of mathematical modelling is similar to that of mathematical literacy as defined by the OECD (2003) for its PISA study of 15 year olds, although the possibilities of actually assessing modelling are severely curtailed by the written timed test format in PISA. Teachers need experience of
mathematical modelling, in addition to pure mathematics investigations, because the process is distinctly different. Only in the simplest of word problems does a mathematical model capture all that is relevant to the real situation (e.g., 6 identical apples shared equally amongst 3 boys). The skill of the mathematical modeller is to identify what are likely to be the most critical variables, but the test for whether they are adequate choices can only come after a full modelling cycle when results are compared with reality. For example, to answer a question about how the times should be set for traffic lights to clear traffic at a busy intersection is it enough to assume that cars pass through the lights every 2 seconds in each lane (this is the time difference that is recommended to learner drivers), or should it be assumed that the first car takes longer and then others pass through at a constant rate, or is it necessary to go for a probabilistic model where cars pass through according to a selected probability distribution? Even the choice to include the rate at which cars pass through the lights as a variable in the model is part of the formulation stage, subject to later verification as to its usefulness. As with pure mathematics investigations, providing prospective teachers with experience of mathematical modelling is a high priority that it often not achieved. When it is achieved, a study by Nicol (2002) shows there may be further work to do to translate this into lively teaching. Prospective teachers were able to see how mathematics was being used in workplaces, but when they created lessons from the experience, the mathematics remained decontextualised. I noticed the same phenomena with prospective teachers from an engineering background. They were expert in applying mathematics in their previous work, but do not see how they could use these experiences to motivate students and illustrate applications.

KNOWING ABOUT MATHEMATICS

Beyond knowledge of mathematics and experience of doing mathematics, teachers need to know about mathematics. This is an area where the preparation of teachers is readily seen to require something different to preparation for other professions. Teachers who know about mathematics – its history in both the East and the West, its ways of working, its major events, and so on – can enliven their teaching and assist students to understand how mathematics works, where it comes from and its role in society. Some knowledge about mathematics comes incidentally as we learn mathematics (assisted by teaching), but some such as the history, epistemology or philosophy of mathematics can be studied separately from mathematics. The CBMS (2001) report recognises this need in both its recommended capstone course as well as in the attention it pays to ‘habits-of-mind goals’ (p. 141) and mathematical thinking.

There is long standing interest in courses on the history of mathematics. Since 1976 there has been, for example, an International Study Group on the relations between the History and Pedagogy of Mathematics (HREF4) affiliated to the International Commission for Mathematical Instruction. One of their aims is to assist mathematics teachers to gain insights on how the history of mathematics may be integrated into teaching and may help students to learn mathematics. Materials
to support courses for prospective teachers are now becoming readily available on
the Internet. For example, Mills (2007) describes an elective course intended for
prospective teachers (although not exclusively), that is based on historical
documents rather than a modern textbook. Mills’ course aims to show how
“mathematics is created by human beings and hence is connected with the culture,
the times and the place where this creative activity takes place” (p. 195). Students
study ancient Egypt (Rhind Papyrus), ancient Greece (Euclid) and medieval
Europe (Fibonacci). In justifying his selection of elementary mathematics topics
for the course, Mills (2007) notes that many more students can enjoy studying the
history of elementary, rather than advanced mathematics. Whilst this is certainly
the case, it is often not the case that the history of simple mathematics is itself
simple. Matrices, for example, can be introduced in a straightforward way to junior
secondary students as storage arrays for data or for coefficients of equations, but
their important place in mathematics derives from advanced work by Cauchy,
Lagrange and others (HREFS) on a variety of topics such as the use of differential
equations to solve problems of celestial mechanics. It is not feasible to motivate the
eyrly study of matrices with this. A serious treatment of history can demand both
difficult mathematical content and difficult historical material, both from the point
of view of the prospective teacher and the instructor. As a consequence, the use of
historical anecdotes to enrich the standard teaching of customary mathematical
topics is a more widespread approach to increasing teachers’ understanding of the
history of mathematics than offering complete courses.

Whereas it is common to use history as a source of enrichment when teaching
many topics, the philosophy of mathematics seems more difficult to encompass in
teacher education. It seems reasonable that prospective teachers should be able to
supply good answers to questions such as: what is a mathematical object? what is
the nature of mathematical truth? is mathematics created or discovered? and why
does mathematics model the real world so well? However, there are no simple
answers here; these are difficult questions at the interface of philosophy, logic and
mathematics, which require serious study. In addition, they are questions which
rarely trouble the working mathematicians who generally provide education in
mathematics for prospective teachers. Davis and Hersh (1981) observed that most
mathematicians act as though they are Platonists, acting on a naïve view that
mathematical objects have an uncomplicated status, although when pressed they
often retreat to a formalist view, where mathematics is viewed as a game played
according to certain rules and where it is not required to specify any further
meaning. In contrast to the concern with mathematical foundations that is attacked
with the tools of logic and mathematics, many mathematics educators are keen to
stress mathematics as a human endeavour and view it from a social perspective.
For example, in their extensive article on mathematics for teacher education,
Cooney and Wiegert (2003) promote the view that a fallibilist view of mathematics
is the most productive to guide courses for prospective teachers – a view where
truth in mathematics is seen as grounded not in pure reason, or on correspondence
with data from the senses, but is a result of a social process. In their view, prevalent
teachers’ beliefs that mathematics is abstract, rigid, unchanging and not based in
human experience have arisen from their mathematical training. They propose that these beliefs present a major obstacle to reform of mathematics in schools, and so should be countered by the fallibilist (social) view. My belief is that the view of mathematics as abstract (only), unchanging, rigid, and unrelated to human experience is a consequence of an inadequate mathematics education, not attending to the four components outlined in this chapter, rather than a consequence of a non-fallibilist philosophy. In a recommendation that I strongly endorse, Cooney and Wieglin also recommend that courses for prospective teachers should present mathematics as a pluralistic subject that includes elements of discovery and investigation alongside appropriate formalism, rather than being dominated by one view.

Beyond history and philosophy of mathematics, what else should teachers know about mathematics? For teachers of science, it is important that they ‘keep up to date’ with their subject, and there is a great deal of information in the press to help them and other concerned citizens, but this is harder for mathematics teachers since advances in mathematics are generally highly technical, and only slowly, if ever, become part of the school mathematics curriculum. Moreover, teachers tend to view mathematics as uncontested, often only as what their own educational jurisdiction sets down for them to teach (Wilson, Cooney, & Stinson, 2005). This view is too limited. At a minimum, teachers need some personal experience that new mathematics and new applications of mathematics continue to be invented and/or discovered. Mathematics ‘general knowledge’ does not have an easy place in formal mathematical training. The Black-Scholes equation is used for pricing options in financial markets by hedging against losing bets. It is now said to be the world’s most used formula and so one might expect that most mathematics teachers would know about it. (Bulmer (2002) provides a suitable introduction for teachers.) However, since pricing options belongs to the mathematics of finance, the Black-Scholes formula is not a central part of a mathematics major today. Moreover it has become important only in recent years, after many teachers have graduated. As a consequence, it is probably the case that most of the world’s mathematics teachers have not heard of the world’s most used formula.

Infinity is a concept that fascinates people of all ages, including young students, and so one might expect that most mathematics teachers would have a good grasp of the infinite cardinals and how to do arithmetic with them. A teacher who understands why a mathematician says there is the same number of fractions as of all integers but more decimals than either, for example, can link into these widespread fascinations. Similarly, the fourth dimension is a fascinating idea and again one might expect that mathematics teachers could explain how the idea of a 2-dimensional net of a 3-dimensional cube can be generalised to the 3-dimensional net of a 4-dimensional cube. Mathematicians have also shown in recent years that it can be useful to calculate non-integer dimensions, for example the fractal dimension of a coast line. These, and many other ideas which are similarly prominent in the popular imagination, rarely have a firm place in a mainstream mathematics major study, because they are not central to the development of knowledge to participate in advanced mathematics. None of these illustrations
above are of themselves essential knowledge for teaching (I expect that there will be individual readers disagreeing with my choice of each of these examples) but there is a cumulative effect of having teachers who can tap into the natural human interest in things mathematical and appreciate the ever increasing role of mathematics in society. This courses back to my point: mathematics teachers need to ‘keep up to date’ with mathematics and, over time, build up a wide general knowledge of mathematics. Consequently, there is a need for materials on new mathematical developments and on perennial favorites to be prepared for the teacher audience, and presented through teachers’ journals and other media.

KNOWING HOW TO LEARN MATHEMATICS

Equally importantly, the need to keep up-to-date throughout a career points to the need for mathematics training at every level to develop skills in learning to learn mathematics. The CBMS report (2001) makes the same point: “Thus, college mathematics courses should be designed to prepare prospective teachers for the lifelong learning of mathematics, rather than to teach them all they will need to know in order to teach mathematics well” (p. 6). An initial preparation in any subject is not sufficient for an on-going career. Being an independent learner of mathematics, both to master new technical skills as required to deal with new topics in the curriculum, and to keep abreast of the concepts behind new developments, is as important for teachers as for any other professional. New demands from the workforce and national economies, new mathematical ideas and applications and new technologies for doing mathematics all contribute to making it a dynamic subject, with consequent needs for continual independent learning.

What do we know about courses that develop good skills for independent learning of mathematics? Seaman and Szydlik (2007) developed a concept of “mathematical sophistication” to encompass a set of values and avenues for doing mathematics that is required to create fundamental understandings. They observed prospective primary teachers attempting to learn from an on-line resource and noted that many of them experienced difficulties that were due to characteristics such as not attempting to make sense of the relevant definitions provided, not focussing on giving meaning to the problem and attending precisely to language, and not using the explanations supplied. Although this was only a small study and it studied primary rather than secondary teachers, this study is relevant to this discussion because it points to the way in which independent learning of mathematics depends on an enculturation into mathematical practices and values. A detailed example of how understanding of the key role of mathematical definitions affects the mathematical work of teaching is given by Chick (2007). She reported episodes of teaching where teachers’ knowledge of mathematics strongly influenced the outcome. In a classroom investigation, students who were finding rectangles with a certain property discarded a square as a solution, but the teacher skillfully drew their attention to how the square met the requirements (definition) of being a rectangle even though it had other properties as well. The point being made was that a definition provides minimal criteria, not a full
Mathematical practice has many other instances, large and small, in which teachers who are not encurtated into its language and practices may find an obstacle in extending their own knowledge. For example, even simple words such as 'some' and 'either' have mathematical meanings that contrast to their everyday meanings. It is true in mathematics that some women are called Kylie Minogue (even if there is only one person with this name, mathematicians can still say some) and it is true that a red square fulfils the criterion of being “either red or square” (being both is not excluded by a mathematical either-or).

The need for teachers to be able to learn mathematics independently is most critical for those teachers teaching ‘out-of-field’. Unfortunately, these are the teachers who are likely to have the more difficulty with learning mathematics and also had less practice. With colleagues, I recently conducted professional development sessions for such teachers of junior secondary mathematics. In one instance, we wanted to focus on pedagogical content knowledge, to demonstrate the different degrees of difficulty of the three basic types of percentage problems. Given similar numbers, type 1 problems (given whole, given part and missing percentage) and type 2 problems (given whole, given percentage and missing part) are easier for students than type 3 problems (given percentage, given part and missing whole). In the professional development session, the demonstration was much more effective than we had intended. In every group, the type 3 problems could not be solved by a small but significant proportion of teachers currently teaching those grade levels that percentage is taught. It is likely that these teachers will avoid setting problems of this type for students, believing they are too difficult, and hence in effect setting expectations of students that are too low.

Wilson and Berne (1999) comment on the dilemmas posed when voluntary professional development exposes teachers’ lack of knowledge, and they also comment on the difficulties of research into teacher knowledge when it too can spotlight gaps in individual teachers’ knowledge base. Even in professional development programmes with a strong focus on content knowledge, Wilson and Berne reported that opportunities to discuss mathematical content were repeatedly missed because instructors and fellow participants did not want to embarrass individuals. These sensitivities make it difficult for teachers to improve their content knowledge. Lachance and Confrey (2003) report similar observations, but note that teaching content in the context of new technology, new to all participants, eased the situation and fostered stronger helping relationships among teachers.

Another instance from the same professional development series highlights how teachers’ ‘big picture’ of mathematics affects their flexibility in adapting teaching ideas. We introduced these ‘out-of-field’ teachers to the dual number line to solve percentage and ratio and proportion problems. The dual number line is marked on one side with the absolute amount and on the other with the percentage. It is used widely in Singapore but not in Australia where the professional development was held. By marking the data on a dual number line, a student can organise the information and hence see how to solve proportional reasoning problems, including problems related to speed, density, prices given cost per gram, and so on. From the viewpoint of more advanced mathematics, all of these problems are problems of
direct linear proportion, using the same mathematical techniques. After I commented quickly that the dual number line was also useful for these problems, I was surprised when teachers asked me how this could work. These teachers, learning mathematics independently and building only on their own school experience, have learned it topic by topic, without seeing the big picture of direct proportion and the multiplicative structure that links them all. After my short explanation of the similarity between the problems, it is likely that they overgeneralised. They may have not seen that the dual number line needs serious modification to be used with other than direct proportion (e.g., conversion between Celsius and Fahrenheit).

GOOD POLICY IN AN ERA OF TEACHER SHORTAGE

The above sections focussed on the mathematics education that is desirable for mathematics teachers. In this section, I confront this discussion with reality. Le Métuis (2002) notes that the trend of mathematics to decline in popularity in secondary schools across a number of western countries, has reduced the pool of specialist mathematics teachers. As a result, there are many non-specialists teaching mathematics, who may not have the expertise or confidence necessary to prepare and motivate students to pursue higher-level studies in mathematics. Stephens (2003) reiterates these concerns, and concludes that traditional pathways will not be able to provide sufficient mathematics teachers in the future, especially since the recent growth in occupations requiring a strong grounding in mathematics sciences exceeds supply.

The qualifications of mathematics teachers in Australia is reported in detail by Harris and Jensz (2006) in a study conducted on behalf of the Deans of Science of Australian universities, as a result of their concern about teacher supply and quality. Harris and Jensz report that 75% of teachers of senior (grade levels 11-12) mathematics had studied some mathematics to third year at university. This may not necessarily be sufficient for a major, as they may have taken mathematics as a small component alongside other major studies. Harris and Jensz express deep concern that about 8% of all secondary mathematics teachers (grade levels 7-12) studied no mathematics at university and 20% of all mathematics teachers studied no mathematics beyond first year (p. iv). The report also notes that many teachers, including a third of those teaching only junior secondary mathematics (grade levels 7-8) had studied no mathematics-specific education. Stephens (2003) cites data that over a quarter of high school students in the USA are taught by ‘out-of-field’ teachers.

It is also the case that the decline in the number of students studying mathematics, the increasing availability of well-paid careers for those qualified in the mathematical sciences, and a decline in the popularity of teaching as a career, leads to prospective teachers with a wide range of undergraduate qualifications (not a traditional mathematics major) entering consecutive courses of teacher education. For example, the University of Melbourne submission to the Australian Review of Teachers and Teacher Education (University of Melbourne, 2002) reported on the
discipline qualifications of entrants to the post-graduate diploma in education from 1998 to 2002. The submission noted that “demographics of current teacher education students do not conform to traditional expectations. They are older, better qualified and have substantial prior work experience” (p. 14). The report noted an increasing diversity of academic backgrounds, reflecting increasingly specialised undergraduate science training. Prospective mathematics and physics specialist teachers had an average age of 31 and about three quarters had previously worked in another profession, most commonly engineering. Only about a quarter of students had done their mathematics study within a Bachelor of Science degree, which indicates that they have mostly been trained in the applications of mathematics (e.g., engineering, applied science, information technology, commerce) rather than studying mathematics for its own sake. The decline in teachers with ‘normal’ qualifications is also indicated in a survey carried out in the Australian state of Victoria by the Mathematical Association of Victoria. The results of this survey indicated that a mathematics major in the undergraduate degree was held by 83% of teachers who had been teaching for more than 20 years but by only 61% of teachers who had been teaching less than 10 years (University of Melbourne, 2002). They have all met the requirements for amount of mathematics in the undergraduate degree as specified by state legislation (a sub-major), but the nature of this training is very different to what it might have been 20 years ago.

These trends in teacher supply raise important issues for answering the question of what mathematics teachers ‘should’ know. On the one hand, it is likely that most countries will have significant numbers of teachers who have studied little or no mathematics at university, and educational systems should plan to meet their needs for professional development in mathematics, as well as in other educational issues. These people will be an important part of the teacher workforce. On the other hand, a growing proportion of ‘fully qualified’ mathematics teachers will have studied mathematics in the service of another profession or discipline, rather than for its own sake. In answering the question of what mathematics teachers ‘should’ know, their needs must also be considered. Their expertise in another discipline is an advantage. The presence of many teachers who have used mathematical sciences in the workplace, in many different occupations, provides a considerable resource for enriching school students’ experience of the applications of mathematics and for understanding the careers in which mathematical expertise is useful. Teacher education must empower these teachers to use these prior experiences in the classroom. On the other hand, having their expertise outside of mathematics means that there may be large gaps in their mathematical knowledge as exemplified in the article on prospective teachers’ knowledge of proof by Stylianides et al. (2007) mentioned previously. It is extremely unlikely that prospective teachers who come to mathematics teaching from engineering or commerce will have experience of proof.

Reports such as that of the Deans of Science (Harris & Jensz, 2006) arise from concern about an inadequate supply of teachers well qualified in mathematics. They advocate that there should be stronger minimum standards of discipline study
and of pedagogy for those teaching mathematics. This is one of the common policy responses to the widely recognised need to improve the quality of the teacher workforce. Boyd, Goldhaber, Lankford, and Wyckoff (2007) have compared the effects of this policy response (to tighten teacher certification requirements) with the opposite response of easing requirements and introducing alternative ways of being certified as a teacher, as implemented in the states of the USA. They conclude that tightening requirements is effective only if the tighter requirements focus on characteristics that lead to better outcomes for students and if they do not deter potential applicants who may become excellent teachers. In other circumstances, the opposite approach of easing requirements becomes the more attractive policy. Whilst calling for further research related to such policies, they note that teachers’ content knowledge in mathematics is known to improve students’ mathematics learning (hence can be seen as a characteristic that leads to better outcomes for students), but they also claim that there is some evidence that teacher certification requirements shrink the pool of people pursuing teaching careers. As a consequence, whilst it is appealing for teacher educators to identify substantial lists of the knowledge that teachers ought to have, it may be that better outcomes for students overall may arise from looser initial requirements and good opportunities for growth within the profession. Stephens (2003) makes the point that even with full teacher education, the powerful blend of content knowledge and pedagogical content knowledge that is envisioned by the Conference Board CBMS recommendations is unlikely to be able to be attained in initial teacher education, due to time constraints, lack of clarity of responsibility between discipline and education components, and lack of experience of the prospective teachers themselves with school student’s mathematical thinking. As a consequence, the responsibility for deep understanding of content knowledge in a pedagogical context has to be taken up by education for practising teachers. These issues need careful consideration at the local level for the formation of good policy.

**CONCLUSION**

In the first part of this chapter, I outlined what a good education in mathematics for prospective teachers would be like. Working from an ideal of mathematics as a subject that is taught for both its interest and its applications, and from a basis of reasons rather than rules, a vision of what teachers should know and how they might know it emerges. The knowledge of teachers was classified into four: knowledge of the content of mathematics; experience of doing mathematics; knowledge about mathematics as a discipline; and knowing how to learn mathematics. In planning the content of mathematics courses for teachers in the past, it seems it was assumed that teachers had an adequate knowledge of school mathematics by having been successful students. However, led by research on the minute-by-minute mathematical requirements of teaching and by research which shows only weak links between university studies of mathematics and effectiveness as a teacher, there has been a movement (encapsulated in the CBMS report) to make school mathematics a central concern of university studies for
teachers. The need to give prospective teachers an understanding of newly important mathematics (notably but certainly not restricted to statistics and computer science) has also put pressure on the traditional mathematics major, leading to a reassessment of the priorities for study. An important research question for the future will be to investigate whether courses along the lines envisaged by CBMS (2001) or initiatives as exemplified by Pierce et al. (2003) really do provide a major in mathematics that well-equiips graduates for their chosen careers. In particular, it is important to investigate whether these initiatives do succeed in providing teachers with knowledge that really does enhance their teaching. Boyd et al. (2007) note that some knowledge of mathematics does improve student outcomes. Can we show that a theoretically excellent education for mathematics teaching actually makes a further measurable difference to student outcomes, and hence is worth requiring?

The later section examined the reality of teacher qualifications for mathematics and considers projections that in the future, shortages of well qualified mathematics teachers are likely to persist. Whereas the first section addressed an ideal with a teacher undertaking a mathematical major, the reality is that many teachers who are fully qualified to teach mathematics in the eyes of regulating bodies will in fact have been trained as users of mathematics in the service of other professions. More seriously, many will not be educated as mathematics teachers at all, but will do their best, as they understand it, from their own experience of being a high school student or undertaking a small component of university mathematics. This situation raises a substantial series of research questions related to how gaps in teacher knowledge can be overcome. There are some concerning reports in the literature (e.g., Wilson & Berne, 1999) that professional development that is intended to focus on content knowledge often avoids facing up to the challenge. In the literature there are many examples of teachers or prospective teachers not knowing the mathematics that will teach. How serious a problem is this in the long-term? What sort of knowledge is likely to be gathered during a teaching career, by teaching topics and engaging with students’ thinking and what support does this require? To what extent do teachers (with differing backgrounds) learn mathematics by teaching it, and what support materials would encourage this? The CBMS report identifies that there is a special knowledge required for teaching, principally an overview of school mathematics as in the capstone course and the type of content that I have outlined above in the ‘knowing about mathematics’ section. How can this best be gained by teachers without the expected content background? I have also noted that many mathematics teachers now have substantial experience of using mathematics in a previous career. How can the educational system harness this?

In recent years, research has demonstrated how mathematical knowledge affects the minute-by-minute decisions that a teacher makes. Most of this research has been conducted with primary teachers, and most of the initiatives to address the identified deficiencies have also been aimed at primary teachers. Possibly this is because usually more of their education is within schools of education, and hence in the hands of those who conduct education research. Secondary teaching has been
neglected on both these accounts. There is less research and the avenues for change are administratively more complex within institutions for secondary majors. Secondary mathematics is extremely important for individual life chances and for national prosperity yet it is often taught as rules without reasons, turning students away. Attending to the mathematics knowledge of the secondary teacher workforce, consisting, as it does, of people with mixed mathematical backgrounds, should be a high priority for researchers, teacher educators, mathematicians and educational systems.

REFERENCES


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