APPLICATION OF THE WAVELET TRANSFORM IN TURBULENCE

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ABSTRACT

Traditionally, Fourier transforms have been used to elicit the scale-based behaviour of the turbulent motion and one speaks synonymously of its wavenumber components with scales (large scales are associated with small wavenumbers and vice-versa). Although, this approach is theoretically correct, many workers have questioned its appropriateness on the grounds that a Fourier mode represents a wave like disturbance which is global in the physical domain, whereas an eddy is a disturbance with finite spatial extent. Consequently, a more appropriate scheme should involve a local decomposition of the velocity field which is more reminiscent of eddy like phenomena. In this paper we have explored the feasibility of the wavelet transform as an analysing tool in deducing the turbulence spectrum.

1 INTRODUCTION

The appropriateness of using Fourier transforms in turbulence has been questioned for a long time, as the analyzing functions used in the Fourier transform are trigonometric functions which would have been appropriate if turbulence were a superposition of waves; and under this circumstance the wavenumbers are well defined and the Fourier spectrum would have been meaningful. On the contrary, the general understanding of the turbulence structure is as being composed of eddies of different scales; though the precise meaning of the term eddy may not be found in the literature, it is certain that an eddy is not a Fourier mode. Tennekes & Lumley (1972) pointed out that the Fourier transform of a velocity field is a decomposition into waves of different wavelengths and each wave is associated with a single Fourier coefficient; whereas an eddy, however, is associated with many Fourier coefficients and the phase relations between them. Thus a Fourier description of the turbulence field may not be sufficient to understand the scale based nature of turbulence and as such a more sophisticated transform will be needed if one wishes to decompose the velocity field into eddies instead of waves.

Some further issues, related to Fourier transforms, which have drawn criticism as regards to its applicability in turbulence are: firstly, the eddies which are the basic turbulence producing structures, are of finite extent in the physical space unlike the sine or cosine waves which are global (continue to infinity). Thus the Fourier transform will not be able to reveal the local information contained in eddies of each scale. Secondly, a description of sharp gradients by Fourier modes is inefficient. This artefact of Fourier transforms arises because of the fact that Fourier transforms tend to extend the basis function over the entire range. It can be shown that the Fourier spectra becomes very hairy and difficult to interpret if a phase-shift is introduced in the signal or if the signal window is not an integral multiple of the period.

The compactness of the wavelet transform has made it an attractive alternative to consider. In the literature, there exists a number of research papers on the applicability of wavelet transforms in turbulence, but as Farge (1992) pointed out, most of them are exploratory in nature. However,
WAVELET TRANSFORM

Wavelet transform is a relatively new mathematical tool originally developed by Morlet in 1981 for the analysis of seismic data. The later developments like the geometrical alism of the continuous wavelet transforms were the joint efforts of Morlet and Grossmann and Saam et al. (1987). Interest in using the wavelet transform for extracting scale dependent mation stems from the success of the wavelet transform in image analysis, acoustic pattern gnition and signal analysis. A comprehensive review on the chronological development of the let transform can be found in Parge (1992).

Wavelet transform is essentially a convolution integral, where the data signal is convolved with a wavelet termed an analyzing wavelet. In physical space, the one dimensional wavelet transform of a data signal \( s(t) \) with a re-scaleable analyzing wavelet \( g(t) \):

\[
S[b, a] = u[a] \int_{-\infty}^{\infty} s(t) g^* \left[ \frac{t - b}{a} \right] dt
\]

\( a \) is the scale factor for rescaling the mother wavelet \( g(t) \), \( b \) is the time-shift parameter, \( u[a] \) is the normalization function and the asterisk (*) stands for the complex conjugate. It may be noted here that the mother wavelet \( g(t) \) is, in general, a complex function that satisfies the admissibility condition: it means that it has to be square integrable and it must have a zero mean. Equation (1) may be expressed in terms of the Fourier transforms as

\[
\tilde{S}[f, a] = u[a] \tilde{g}^*[a] \tilde{s}(f)
\]

It can be shown that, given the forward wavelet transform as in (1), the original signal can be reconstructed from its wavelet transforms using the relation

\[
s(t) = \frac{1}{C_g} \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2 u[a]} S[b, a] g^* \left[ \frac{t - b}{a} \right] db da
\]

The constant factor \( C_g \) depends upon the wavelet function and is given by

\[
C_g = \int_{-\infty}^{\infty} \frac{\left| \tilde{g}[f] \right|^2}{|f|} df
\]

It is obvious from (3) that for the reconstruction formula to work \( C_g \) has to be finite, i.e., \( |\tilde{g}[f]|^2/|f| \) must be equal to zero for \( f = 0 \), meaning that \( g(t) \) must have a zero mean.

A (3), the total energy of the signal can be computed from its wavelet transform using

\[
E_s[a] = 1/2 \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} S[b, a]^* S[b, a] db da
\]

Equation 7 can be used to compute the energy spectrum of the signal \( s(t) \) in terms of the wavelet scales \( a \). Note that in deriving the relations given by (6) and (7) it has been implicitly assumed that both of the functions \( g(t) \) and \( s(t) \) are square-integrable.

It is also worth noting here that \( C_g \) is constant for a given wavelet function and as such appears outside the integrals in (4), (5) and (7). Thus, in principle, a function which does not satisfy the admissibility condition may still be used as a “wavelet” with the consequence that the wavelet transform at each scale will carry information about the mean value of the signal. Therefore, an attempt to reconstruct the final signal from the wavelet transform will produce an invalid result. However, if one wishes to analyze a turbulence signal which has its mean removed, with an inadmissible wavelet, the admissibility requirement may not be a constraint, at least as far as the forward transform is concerned.

The choice of the weighting function is open and varies from application to application. A very common choice of the weighting function is

\[
u[a] = \frac{1}{\sqrt{a}}
\]

This form of the weighting function ensures that the energy of the wavelets at each scale \( a \) is conserved. Other weighting function may be chosen to emphasize different scales; for example, if one is interested in only small scale features, then a choice like \( u[a]=1/a \) will amplify the contributions from the smaller scales. To overcome the effect of the ambiguity associated with \( u[a] \), the wavelet transformed signals were properly normalized in this work.

3 COMMON WAVELETS

The primary reason for using the wavelet transform is its local compactness. To take this advantage it is desirable to choose a function that provides maximum possible localization both in physical and spectral space. According to Papoulis (1982), it can be shown that, within the limits of the uncertainty principle, functions of the form \( g(t)=C e^{i f t} \) are the only class of function which attain this maximum localization. A Gaussian function given by \( g(t)=\sigma e^{-t^2} \) belongs to this group which can be obtained by choosing \( C=1 \) and \( a=m=1 \). Obviously \( g(t) \) is square integrable, but it is not an admissible wavelet as it does not have zero mean (\( \mu=\sigma \sqrt{\pi} \)). However, changing the sign of the derivative of \( g(t) \), we get

\[
g_2(t) = (1 - t^2) e^{-t^2}
\]

which obviously is an admissible wavelet and is commonly referred to as a Mexican Hat wavelet. Apart from this Mexican Hat another widely used wavelet is the so called Morlet “wavelet” given by

\[
g_m(t) = e^{i \omega t} e^{-t^2}
\]

This wavelet is a complex valued function and the reason of placing quotes around the Morlet “wavelet” is that the real part of this analyzing function is not strictly admissible, although for \( f_0=0 \) the peak to mean ratio is \( C(1-\omega^2) \) (of the order of \( 10^{-5} \)) and thus negligible for most practical purposes.

4 WAVELET SCALE AND WAVE NUMBER

The wavelet transform is capable of analyzing a signal locally at different scales because of the local nature of the wavelet function in both physical space and frequency space. By contrast, the Fourier modes, being trigonometric functions (sine or cosine waves), are global (i.e., infinite) in the physical space though discrete in the frequency space. Turbulent flows are known to be intermittent (local) and the practical turbulent flows are generally inhomogeneous (composed of events of different scales) and as such the distinctive property of the wavelet transform is conjectured to be of great advantage. Equation (2) shows the wavelet transform, in the frequency space, as a band-pass filter.
no remarkable success had been achieved on the use of wavelet transforms in experimental uence data.

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Wavelet transform is a relatively new mathematical tool originally developed by Morlet in (Morlet 1981) for the analysis of seismic data. The later developments like the geometrical formalism of the continuous wavelet transforms were the joint efforts of Morlet and Grossmann (Sammartino et al. 1987). Interest in using the wavelet transform for extracting scale dependent information stems from the success of the wavelet transform in image analysis, acoustic pattern recognition and signal analysis. A comprehensive review on the chronological development of the wavelet transform can be found in Pargave (1992).

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$$S[b, a] = w[a] \int_{-\infty}^{\infty} s[t] g^* \left[ \frac{t-b}{a} \right] dt$$

where $a$ is the scale factor for rescaling the mother wavelet $g[t]$, $b$ is the time-shift parameter, $w[a]$ is the windowing function and the asterisk (*) stands for the complex conjugate. It may be noted here that mother wavelet $g[t]$ is, in general, a complex function that satisfies the admissibility condition by which it has to be square integrable and it must have a zero mean. Equation (1) may be expressed in terms of the Fourier transforms as

$$\hat{S}[f,a] = a w[a] \hat{g}^* \hat{s}[f] \hat{g}[f]$$

It is to be shown that, given the forward wavelet transform as in (1), the original signal can be reconstructed from its wavelet transforms using the relation

$$s[t] = \frac{1}{C_p} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\alpha^2 w[a]} S[b, a] \hat{g} \left[ \frac{t-b}{\alpha} \right] db da$$

where the constant factor $C_p$ depends upon the wavelet function $g[t]$ and is given by

$$C_p = \int_{-\infty}^{\infty} \left| \hat{g} \right|^2 d f$$

Equation 7 can be used to compute the energy spectrum of the signal $s[t]$ in terms of the wavelet scales $a$. Note that in deriving the relations given by (6) and (7) it has been implicitly assumed that both of the functions $g[t]$ and $s[t]$ are square-integrable.

It is also worth noting here that $C_p$ is constant for a given wavelet function and as such appears outside the integrals in (6), (7) and (8). Thus, in principle, a function which does not satisfy the admissibility condition may still be used as a "wavelet" with the consequence that the wavelet transform at each scale will carry information about the mean value of the signal. Therefore, an attempt to reconstruct the original signal from the wavelet transform will produce an invalid result. However, if one wishes to analyze a turbulence signal which has its mean removed, with an inadmissible wavelet, the admissibility requirement may not be a constraint, at least as far as the forward transform is concerned.

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$$w[a] = \frac{1}{\sqrt{a}}.$$  

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$$g_1[t] = (1-t^2) e^{-t^2}$$

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$$s[t] = e^{it} e^{-t^2}$$

This wavelet is a complex valued function and the reason of placing quotes around the Morlet "wavelet" is that the real part of this analyzing function is not strictly admissible, although for $s_0=0$ the peak to mean ratio is $O(10^{-5})$ (of the order of $10^{-5}$) and thus negligible for most practical purposes.

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here the filter is given by \( a w[a]^2 g[a] \), i.e., the Fourier transform of the wavelet function. A change in the scale \( a \) will change the bandwidth of the filter and in that case the selection criterion for a wavelet would be its shape in Fourier space.

It is worth noting here that while the Fourier spectrum is a decomposition in terms of wave numbers, a wavelet transform will provide the spectrum in terms of space-scale. However, it is possible to infer a wave number (or frequency \( f \)) corresponding to a wavelet of particular scale \( a \), at least if the purpose of comparing this two types of spectrum is to be shown that when the wavelet transform is used as a band-pass filter, the center frequency \( f_s \) is given by

\[
f_s = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{for Mexican Hat wavelet} \\ \frac{1}{2} \frac{1}{\sqrt{2\pi}} & \text{for Morlet wavelet} \end{cases}
\]

Equation (7) can be used to calculate the energy \( E_s[a] \) of the signal \( s[t] \) at scale \( a \), where \( a \) can be related to frequency (and subsequently to wave number \( k \)) using the above relations. Note that in deducing the correspondence between the scale \( a \) and the wave number \( k \) (or frequency \( f \)), like Tennekes & Lumley (1972), it is assumed that an eddy is contributing a narrow spike to the spectrum as this nd of spectrum produces slowly damped oscillations in the correlation which is not an eddy-like characteristic as one would expect an eddy to loose its identity within a few wavelengths due to interactions with others (see Tennekes & Lumley (1972)). It is likely that contributions of an eddy to the spectrum, thus, will have a fairly broad spike, wide enough to dampen the ringing behaviour of the correlation. Though both the Mexican Hat and the Morlet wavelet provide spatial localization, they do not depict the widely excepted intrinsic structure of the turbulence producing events, at least so far; the streamwise component is concerned. From potential flow theory, it can be shown that the streamwise component \( u_{mz}[x] \) of the velocity field associated with a point vortex along a streamwise flow it has the form

\[
u_m[x] \sim \frac{1}{1 + x^2} \quad (12)
\]

here the constant of proportionality depends on the circulation, the distance and the vortex distance from the vortex center to the streamwise cut (i.e., parallel to \( z \)-axis). As such an aligning wavelet of the shape given by

\[
g_B[t] = \frac{1}{1 + t^2} \quad (13)
\]

could have been an attractive basis function for pattern recognition. Unfortunately as the mean of its function is zero, it does not fall within the rigours of so called admissibility. It was earlier mentioned that this admissibility condition becomes crucial when it comes to the reconstruction of the original signal from its wavelet transforms. Also when one is interested only in turbulent energy, from (2), it can be seen that for a signal \( s[t] \) with zero mean \( \langle S[a] \rangle = 0 \), the non-zero mean sallowing wavelets does not have an impact as far as the forward transform is concerned. It is possible to use the function \( g_B[t] \) as a wavelet to transform the signal \( s[t] \) into a scale dependent ginal \( S[b,a] \), where as before, \( b \) is the time shift parameter and from \( S[b,a] \) the energy associated with a particular scale \( a \) can be computed.

Equation (11) shows scale \( a \), in terms of the frequency \( f \) when the wavelet transform is viewed as a band-pass filter. Recall that the signal \( s[t] \) is filtered with a filter given by \( w[a] g_B[t] \). Now the Fourier transform of \( g_B[t] \) is given by

\[
g_B[f] = \pi e^{-2\pi if} \quad (14)
\]

Clearly \( g_B[f] \) has a peak at the origin and as such, unlike the Mexican Hat or Morlet wavelets, it cannot be used as a band-pass filter function by itself; subsequently no straightforward correspondence between \( f \) and \( a \) can be established. However, this can be overcome by an appropriate choice of the weighting function \( w[a] \). Let the energy \( E_{SB}[s[t]] \) of the signal \( s[t] \) at scale \( a \) be given by

\[
E_{SB}[s[t]] = \int_{-\infty}^{\infty} S[b,a] S^*[b,a] \, db
\]

\[
= \int_{-\infty}^{\infty} \mathbb{S}[f,a] \mathbb{S}^*[f,a] \, df \quad (15)
\]

Considering the conservation of energy between the signal and its wavelet transform

\[
\int_{-\infty}^{\infty} s[t] s^*[t] \, dt = \int_{-\infty}^{\infty} E_{SB}[s[t]] \, da \quad (16)
\]

Equation (16) leads to

\[
\int_{-\infty}^{\infty} s[t] s^*[t] \, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{S}[f,a] \mathbb{S}^*[f,a] \, df \, da
\]

\[
= \int_{-\infty}^{\infty} g_B(f) \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} a^2 w[a] g_B[a] \mathbb{S}[a,f] \, da \right\} \, df
\]

\[
= \int_{-\infty}^{\infty} \frac{a^2}{1 + \frac{a^2}{\pi}} \int_{-\infty}^{\infty} a^2 \left( \frac{a}{\pi} \right)^2 g[B][a] \mathbb{S}[a,f] \, da \, df
\]

\[
\int_{-\infty}^{\infty} a^2 \left( \frac{a}{\pi} \right)^2 g[B][a] \mathbb{S}[a,f] \, da = 1 \quad (17)
\]

An appropriate integral to consider here is

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = 1
\]

Thus rearranging the terms of LHS of (17) and equating with (19)

\[
\int_{0}^{\infty} \frac{a}{16\pi^2} \left( \frac{a}{\pi} \right)^2 g[B][a] \mathbb{S}[a,f] \, da \, df = \int_{0}^{\infty} 4\pi a e^{-4\pi a} \, da
\]

\[
\int_{0}^{\infty} \frac{a}{16\pi^2} \left( \frac{a}{\pi} \right)^2 g[B][a] \mathbb{S}[a,f] \, da \, df = \int_{0}^{\infty} 4\pi a e^{-4\pi a} \, da
\]

This implies that

\[
w[a] = \frac{4\pi f}{\sqrt{\pi}} e^{-\frac{\pi f^2}{\sqrt{\pi}}}
\]

Using (14) in (21), for a bell-shaped wavelet

\[
w_B[a] = \frac{4f}{\sqrt{\pi}}
\]

Thus it can be shown that a band-pass filter function given by \( a w[a] g_B[a] \) will have the center-frequency \( f_s \) as

\[
f_s = \frac{1}{2\pi a}
\]

Please note that the need to redefine the weighting function as given by (8) is not only for finding a relation between the scale parameter \( a \) and the frequency \( f \), but also due to that fact that, in the case of a bell-shaped wavelet, \( C_B \) is not finite (because of non-zero mean of \( g_B[f] \)) and therefore, the use of (7) for calculating energy would be meaningless. As such, in the case of a bell-shaped wavelet, the scale-dependent energy was calculated in a similar manner as shown in (15). However, for the Mexican Hat and the Morlet wavelet (7) in conjunction with the weighting function as given by (8) will be used for the purpose of calculating scale dependent energy.
of the weighting function \( w[a] \). Let the energy \( E_{A}[s][t] \) of the signal \( s[t] \) at scale \( a \) be given by
\[
E_{A}[s][t] = \int_{-\infty}^{\infty} S[b, a] S^*[b, a] \, db
\]
\[
= \int_{-\infty}^{\infty} \tilde{S}[f, a] \tilde{S}^*[f, a] \, df
\]  
(15)

Considering the conservation of energy between the signal and its wavelet transform
\[
\int_{-\infty}^{\infty} s[t] s^*[t] \, dt = \int_{0}^{\infty} E_{A}[s][t] \, da
\]  
(16)

Equation (16) leads to
\[
\int_{-\infty}^{\infty} s[t] s^*[t] \, dt = \int_{0}^{\infty} \int_{0}^{\infty} \tilde{S}[f, a] \tilde{S}^*[f, a] \, df \, da
\]
\[
= \int_{0}^{\infty} \int_{0}^{\infty} a^2 \{ \int_{-\infty}^{\infty} (w[a])^2 \tilde{g}[a f] \tilde{g}^*[a f] \tilde{g}[f] \tilde{g}^*[f] \, df \, da \}
\]
\[
= \int_{0}^{\infty} \tilde{g}[f] \tilde{g}^*[f] \left\{ \int_{-\infty}^{\infty} \frac{a^2}{1 + a^2} (w[a])^2 \, da \right\} \, df
\]  
(17)

Equation (17) will satisfy Parseval's identity if
\[
\int_{0}^{\infty} a^2 (w[a])^2 \tilde{g}[a f] \tilde{g}^*[a f] \, da = 1
\]  
(18)

An appropriate integral to consider here is
\[
\int_{0}^{\infty} e^{-x^2} \, dx = 1
\]  
(19)

Thus rearranging the terms of LHS of (17) and equating with (19)
\[
\int_{0}^{\infty} \frac{4 \pi a f}{(4 \pi a f)^2} \left\{ \int_{0}^{\infty} \frac{a^2}{1 + a^2} (w[a])^2 \tilde{g}[a f] \tilde{g}^*[a f] \, d(4 \pi a f) \right\} = \int_{0}^{\infty} 4 \pi a f e^{-4 \pi a f} \, d(4 \pi a f)
\]  
(20)

This implies that
\[
w[a] = \frac{4 \pi f}{\sqrt{a}} e^{-2 \pi a f}
\]  
(21)

Using (14) in (21), for a bell-shaped wavelet
\[
w_B[a] = \frac{4 f}{\sqrt{a}}
\]  
(22)

Thus it can be shown that a band-pass filter function given by \( w[a] \tilde{g}[a f] \) will have the centre-frequency \( f_c \) at
\[
f_c = \frac{1}{2 \pi a}
\]  
(23)

Please note that the need to redefine the weighting function as given by (8) is not only for finding a relation between the scale parameter \( a \) and the frequency \( f \), but also due to that fact that, in the case of a bell-shaped wavelet \( C_B \) is not finite (because of non-zero mean of \( \tilde{g}_B[f] \)) and therefore, the use of (7) for calculating energy would be meaningless. As such, in the case of a bell-shaped wavelet, the scale-dependent energy was calculated in a similar manner as shown in (15). However, for the Mexican Hat and the Morlet wavelet (2) in conjunction with the weighting function as given by (8) will be used for the purpose of calculating scale dependent energy.
5 COMPARISON BETWEEN FOURIER AND WAVELET SPECTRA

It can be shown that the power spectrum \( \Phi(\ell) \) of a continuing random signal \( s(\ell) \) can be written as

\[
\Phi(\ell) = \lim_{T \to \infty} \frac{1}{2T} |\tilde{s}(\ell)|^2
\]

But strictly speaking this is not true, as it can be shown (e.g. see Goldman (1953)) that \( \lim_{\ell \to \infty} \frac{1}{\ell} |\tilde{s}(\ell)|^2 \) does not exist, at least not in the normal sense and that a generalized function must be used. For \( T \) finite, and for a bandwidth \( \Delta \ell \) where \( \Delta \ell < 1/T \), it can be shown that \( |\tilde{s}(\ell)|^2/2T \) has approximately \( \Delta \ell \) fluctuations; and as \( T \to \infty \), these fluctuations become more numerous and more violent. In fact one ends up with an infinite number of delta functions of finite height and hence of infinitesimal area. Thus the power spectrum (by Fourier transform) will appear noisy. To overcome this, Goldman (1953) suggested the use of so called smoothed power spectrum which in essence is the same as an ensemble average power spectrum in the limit \( T \to \infty \). The smoothed power spectrum \( \Phi_s(\ell) \) of a continuing signal \( s(\ell) \), where the subscript \( s \) denotes smoothed spectrum, is given by

\[
\Phi_s(\ell) = \lim_{\Delta \ell \to \infty} \lim_{T \to \infty} \frac{1}{\Delta \ell} \int_{-\Delta \ell/2}^{\Delta \ell/2} |\tilde{s}(\ell)|^2 \, d\ell
\]

However, this smoothing is a built-in feature of the wavelet transformed spectrum. This is illustrated in figure 1 where a random signal \( s(\ell) \) having a Gaussian distribution with unit variance and zero mean is considered. The spectrum is deduced using a Fourier transform and a wavelet transform with the Morlet wavelet. It can be easily seen that the unsmoothed Fourier power spectrum is very noisy in comparison with the wavelet power spectrum. Figure 2 shows the same plot for an actual turbulent signal in a boundary layer. This smoothing feature of the wavelet transform gives it an advantage over the Fourier transform, since it avoids the introduction of any problem associated with the smoothing routines.

6 EXPERIMENTAL RESULTS

Experimental data presented here corresponds to a turbulent flow with Reynolds number based on the momentum thickness, \( Re = 4140 \). In figure 3 the wavelet transformed spectra at various
5 COMPARISON BETWEEN FOURIER AND WAVELET SPECTRA

It can be shown that the power spectrum $\Phi(f)$ of a continuing random signal $s(t)$ can be written as

$$
\Phi(f) = \lim_{T \to \infty} \frac{1}{2T} \left[ \overline{\left| \tilde{s}(f) \right|^2} \right]
$$

But strictly speaking this is not true, as it can be shown (e.g. see Goldman (1953)) that $\lim_{T \to \infty} \frac{1}{\Delta f} \left| \tilde{s}(f) \right|^2$ does not exist, at least not in the normal sense and that a generalized function must be used. For $T$ finite, and for a bandwidth $\Delta f$ where $\Delta f < 1/T$, it can be shown that $\left| \tilde{s}(f) \right|^2 / 2T$ has approximately $(T \Delta f)$ fluctuations; as $T \to \infty$, these fluctuations become more numerous and more violent. In fact one ends up with an infinite number of delta functions of finite height and hence of infinitesimal area. Thus the power spectrum (by Fourier transform) will appear noisy. To overcome this, Goldman (1953) suggested the use of so called smoothed power spectrum which in essence is the same as an ensemble average power spectrum in the limit $T \to \infty$. The smoothed power spectrum $\Phi_s(f)$ of a continuing signal $s(t)$, where the subscript $s$ denotes smoothed spectrum, is given by

$$
\Phi_s(f) = \lim_{\Delta f \to 0} \lim_{T \to \infty} \frac{1}{\Delta f} \int_{f - \Delta f / 2}^{f + \Delta f / 2} \left| \tilde{s}(f') \right|^2 df'
$$

However, this smoothing is a built-in feature of the wavelet transformed spectrum. This is illustrated in figure 1 where a random signal $s(t)$ having a Gaussian distribution with unit variance and zero mean is considered. The spectrum is deduced using a Fourier transform and a wavelet transform with the Morlet wavelet. It can be easily seen that the unsmoothed Fourier power spectrum is very hairy in comparison with the wavelet power spectrum. Figure 2 shows the same plot for an actual turbulent signal in a boundary layer. This smoothing feature of the wavelet transform gives it an advantage over the Fourier transform, since it avoids the introduction of any problem associated with the smoothing routines.

6 EXPERIMENTAL RESULTS

Experimental data presented here corresponds to a turbulent flow with Reynolds number based on the momentum thickness, $Re = 4140$. In figure 3 the wavelet transformed spectra at various $z/\delta_H$, where $\delta_H$ and $z$ are the boundary layer thickness and the wall-normal distance respectively, is presented. Please note that here, and for all subsequent discussions spectra imply nondimensional streamwise power spectral density $\Phi_{11}[k_1 z]/U^2$, where $U$ is the wall-shear velocity. As the wavenumber space has more physical significance than the frequency space, the argument in power spectral density is changed from frequency $f$ to streamwise wavenumber $k_1$ using Taylor's hypothesis of frozen turbulence. As mentioned earlier, this is done with three types of analyzing wavelets, namely the Mexican Hat, the Morlet and the bell-shaped wavelet. For the purpose of comparison, the smoothed Fourier transformed spectra are also shown on the same plot. It can be seen from these plots that at any $z/\delta_H$, at low wavenumbers the spectra corresponding to the bell-shaped wavelets have the best resemblance with the Fourier spectra; whereas at moderate wavenumbers profiles corresponding to all the three analyzing wavelets virtually collapse with the Fourier spectra; and at higher wavenumbers only those corresponding to the Morlet wavelet match with the Fourier spectra. Also, near the wall, all of the three analyzing wavelets produce identical spectra at low wavenumber; and near the edge of the layer at high wavenumbers the spectral profiles corresponding to those three wavelets asymptote to the Fourier spectra. It is important to note here that the Fourier spectra are smooth ensemble averaged profiles, whereas the wavelet spectra are simply ensemble averaged profiles. Also it may be mentioned that it was observed that at any $z/\delta_H$ the wavelet spectral profiles corresponding to different realizations differ very little implying that less number of samples are required for convergence. From the plots in figure 3, it appears that the Morlet wavelets spectra has the best overall match with the Fourier spectrum; in fact, these two spectra are almost identical except for a few profiles at very low wavenumbers. This may have been expected as Morlet wavelets act like a band pass filter with a band-width symmetrical about and proportional to the center frequency. The spectral profiles corresponding to the Mexican Hat wavelet match with those corresponding to the Morlet wavelet (i.e. the Fourier spectra as well) up to moderately high wavenumber; and at very high wavenumbers they seem to slightly overestimate the energy in comparison with the Fourier or the Morlet wavelet spectra, though the extent in $k_1 z$ to which they remain matched increases with
7 CONCLUSION

The wavelet transform was found to be useful in inferring the turbulence spectra. It shows some promising advantages over the traditional Fourier transform spectra, at least, in terms of its temporal localization and the elimination of the "harvesting" in the spectra without the need of any additional multi-point smoothing. However, the issue which still remains to be answered is: what is the most suitable mother wavelet that resembles the eddy velocity signature?
Figure 4. Comparison of the wavelet spectrum of a Gaussian noise signal as deduced using three different analyzing functions. Note that the power spectrum and the frequency $f$ are in arbitrary units.

$z/6H$. However, this over-estimate of the energy by the Mexican Hat wavelet is not unexpected as a Mexican Hat is asymmetric in the Fourier space with a longer high frequency tail which also increases with the centre frequency. A bell-shaped wavelet is found to have an even higher over-estimate of the energy at high wavenumber.

It is interesting to observe that amongst all, the bell-shaped wavelet spectra shows the most prominent $-5/3$ region across the whole layer even in the buffer zone i.e. $z$, as low as 55, which, even according to less restrictive definition, is within the turbulent wall region. To check whether this trend is intrinsic to a bell-shaped wavelet or not, the power spectrum for the random noise signal $s(t)$ as shown in figure 1(a) were deduced using all three types of wavelets in question and are shown in figure 4. It can seen that no such trend in the spectrum is visible, indicating that the emergence of this $-5/3$ envelope is not intrinsic to a bell-shaped wavelet. At this stage this aspect remains not understood and is being further investigated.

7 CONCLUSION

The wavelet transform was found to be useful in inferring the turbulence spectra. It shows some promising advantages over the traditional Fourier transform spectra, at least, in terms of its temporal localization and the elimination of the "lairness" in the spectra without the need of any additional multi-point smoothing. However, the issue which still remains to be answered is: what is the most suitable mother wavelet that resembles the eddy velocity signature?
SLOW STEADY MOTION OF VISCOUS FLUID PAST A PLANE BORDERLINE

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ABSTRACT

We present a theorem for non-axisymmetric and slow viscous fluid motion past a stationary no-slip plane boundary with the aid of two scalar functions $A_s$ and $B_s$, which are respectively biharmonic and harmonic, and which represent the velocity and pressure fields of the primary motion, due to fundamental singularities in an unbounded incompressible viscous fluid. We further show that Collin’s theorem for axisymmetric flow about a plane boundary follows as a special case of our theorem. A number of illustrative examples are given.

1. INTRODUCTION

The velocity and pressure fields of the fundamental singularities, such as Stokeslet, rotlet, streslet, source and source-doublet, etc., play an important role in the study of the hydromechanics of Stokes flow (slow viscous fluid motion) for different body shapes (including micro-organism) in the neighbourhood of a plane wall. That the need for a knowledge of the singularities in the presence of a wall is required to account for the interactions which both flagella and cilia have with the wall is referred to in [1]. Blake [2] has obtained the image system for the Stokes flow due to a Stokeslet in a no-slip plane boundary by using Fourier transforms. Blake and Cheung [1] have also considered three more Stokes flow problems, the first, second and third ones being respectively concerned with a rotlet, source and source-doublet in a viscous fluid in the vicinity of a stationary no-slip plane boundary, and obtained in each case the image system by using Fourier transforms similar to that of Blake [2]. Happel and Brenner [3, section 7.4] have given a general review on the hydromechanics of Stokes flow involving bodies in the vicinity of walls.
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