Problems of Teaching Mathematics in a Reform-oriented Singapore Classroom

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Abstract

What is the state of mathematics education in Singapore? Singapore students topped the scoring in the second and third Trends in International Mathematics and Science Study (TIMSS). This achievement has drawn attention (and admirers) externally. Within Singapore, there is a mixed response: a sense that the current skill-oriented emphasis is fundamentally sound but reforms are needed to address areas of weakness. Recently, reform efforts are aimed at “teaching thinking” and “integrating technology”. As with most education reforms, the key challenge lies in translating reform-oriented intents into workable forms in the classroom.

This study examines actual problems encountered when these reform goals are incorporated into classroom teaching. The research project was based on a case of teaching a geometry module to an intact Year 7 Singapore class in a naturalistic setting in 2003. The Geometers’ Sketchpad was used as an integral tool to help students learn the required geometrical content. The module covered eleven lessons and each lesson was seventy minutes in duration. I taught the module as the teacher-researcher. My actions and interactions with students in class were captured by video-recording and my thoughts during teaching were recorded in same day audio reflections. A goal-based methodology was employed to analyse the data. The assumption behind this approach is that every teaching action is motivated by one or more instructional goals. The transcribed video data were coded according to the goals in operation across different grain-sizes ranging from an utterance to an entire lesson. Problems of teaching were then identified naturally as hindrances to the attainment of teaching goals. Extending the theme of aeronautical navigation as a metaphor of teaching, I used the construct of turbulence in navigation to frame problems and coping strategies. Turbulent regions refer to those parts of my teaching journey where I experienced interaction between anticipated and emergent goals. Taking into account the multiple complexities in teaching, the task of analysis was carried out in three phases: Phase I was a detailed analysis of one lesson for preliminary findings; Phase II examined and refined these preliminary findings by
broadening the scope of analysis to the entire module; Phase III focused on those parts of teaching where technology was used.

Viewed through the goal-based methodology, reform objectives translate into goals of teaching. As such, in actual practice, reform goals function similarly to other goals of teaching: depending on the instructional situation and due to constraints, they can sometimes be carried out, suppressed, or compromised when weighed against other simultaneously-pressing goals. Coping with the challenge of fulfilling multiple goals and limited resources (such as time) involve constantly improvising ways to prioritise, merge, or postpone worthy instructional goals. This study contributes primarily to the knowledge of goal complexities underlying the work of teaching—an important first step to realising reform ideals.
Declaration

This is to certify that

(i) the thesis comprises only my original work towards the PhD,
(ii) due acknowledgment has been made in the text to all the material used,
(iii) the thesis is less than 100,000 words in length, exclusive of tables, maps bibliographies and appendices

____________________
Leong Yew Hoong
Acknowledgements

To me, doing a PhD is like going on a journey along multiple tracks all at the same time.

Along the academic track, I owe much to my supervisor, Dr Helen Chick, who provided much guidance in both the macro and micro aspects of the project. I also thank my associate supervisor, Dr Eric Wood, who taught me the need to constantly focus by half-jesting, “How many PhDs are you doing?”

There were ups and downs along the spiritual track. I constantly battle against the extremes of over-confidence and despondency. I learn humility by accepting inadequacies and the need to undertake major revisions; I learn perseverance through chipping away a bit at a time. I thank God for these precious lessons.

The turbulences were quite strong along the emotional track too. I am glad I didn’t have to cope with them alone. I am blessed with a supportive family, colleagues, and friends to ride through the storms with me. I specially thank my dear wife for making many sacrifices for me. She is God’s greatest gift for me on this side of eternity. The children were understanding and never once complained about my lack of time for them. Thank you. And newborn Seth also brought so much joy that it often made me forget the heavy-heartedness. I remember, too, the ‘bosses’ for the cuppas and the constant encouragement throughout the journey.

Looking back, the journey has been meaningful and worthwhile. I thank all who have shared memorable moments with me along the way.
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INTRODUCTION

He who can, does. He who cannot, teaches.

George Bernard Shaw

Those who can, do. Those who understand, teach.

Lee Shulman

The famous quote from George Bernard Shaw (1856 – 1950) has rung through the corridors of time since the beginning of the twentieth century and sadly continues to misrepresent the work of teaching to many both outside and inside the education profession. It belittles teachers and the nature of their work. Teachers find the work of teaching challenging and nowhere near the picture of triviality that is implied in the maxim. It takes nothing less than a thinking mind and a “can do” spirit to navigate through the problems that classroom teaching poses.

Happily, the actual experiences of practitioners are joined by the voices of others who have taken an interest in studying the work of teaching. It is perhaps best represented by Shulman’s (1986) counter-slogan. The author shares this view of teaching as one that requires deep and extensive knowledge about the instructional situation. Quite unlike the simplistic work portrayed by Shaw’s quote, teaching is a complex activity that deserves more intense research.

1.1 Motivation for the Study

1.1.1 Teaching as a Complex and Cultural Activity

Over the last decade, an increasing number of classroom researchers have found the work of teaching to be more complex than previously conceived (Ball, 2000; Fleischer, 1995). Teachers juggle multiple and sometimes conflicting goals when carrying out instruction. They attend to many issues—curricular objectives, diverse student competencies, subject content, social conduct of students, consciousness of time, keeping the class focused on productive work—often all at the same time. There are thus complexities in the teacher’s work along social, historical, temporal, and intellectual planes (Lampert, 2001).
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Research into the interaction between these complexities has grown in the last decade, and this study is in line with the knowledge-generation enterprise in this area.

Research into the complexity of teaching should also consider national and cultural factors. International lesson studies have revealed noticeable variations of classroom instructional modes across countries (Stigler & Hiebert, 1999; Clarke, 2006). The diversity reflects the different political and cultural forces that influence classroom practices and teaching behaviour. As such, a Singapore classroom is a worthwhile object of study as it may manifest characteristics both similar to and unique from classrooms in other parts of the world.

1.1.2 Teaching in a Singapore Classroom and Educational Reform

One important reason for a better understanding of classroom teaching practice is its relevance to education policy making. In Singapore, the government made policy changes to the education system to meet the changing needs of society. The following excerpts from speeches by government ministers Goh and Teo illustrate the ongoing review of the education curriculum in the country.

We cannot assume that what worked well in the past will work for the future … . Our Ministry of Education is undertaking a fundamental review of its curriculum … (Goh, 1997, emphasis added).

To prepare our people, the Ministry of Education has been making changes … in the last five years (Teo, 2002, emphasis added).

However, policy changes at the ‘top level’ do not necessarily bring about the intended change in classrooms. Indeed, studies have shown that most ‘top-down’ reform efforts rarely change fundamentally what happens inside the classroom (Cuban, 1993; Griffin, 1995; Tyack & Cuban, 1995). There is often no direct transference of top-level vision into classroom realization, producing a ‘gap’ between reform policies at the ‘top’ and classroom implementation by teachers at the ‘bottom’. This gap is also acknowledged by
the country’s ex-Prime Minister: “Indeed, the best educational policies and programmes can only come alive in the hands of a good teacher” (Goh, 2001).

Ball (2000) argued that the way to address the gap should not begin by analyzing the new curriculum to prescribe what teachers should know, because “little is known about how ‘knowing’ the topics on these [curricula] lists affects teachers’ capabilities” (p. 6). Rather, “our understanding of the content knowledge needed in teaching must start with practice” (p. 6, emphasis added). Ball’s argument suggests that there is a fundamental flaw in most of current education reforms. Current curriculum changes may be informed by political considerations, economic expediencies, achievement scores, or even anecdotes from parents and teachers, but are rarely informed by actual teaching practice. The current education reform undertaken by Singapore is no exception. At the time of writing, the author is not aware of any current or past studies that examined the actual work of teaching mathematics—in all its complexities—in Singapore classrooms. The proposed study here seeks to contribute to this limited knowledge base in the hope that it can help inform the implementation of policy changes.

1.1.3 Teaching Mathematics in a Singapore Classroom
Mathematics is deemed a core subject within the Singapore school curriculum. It is compulsory from the Primary school (Years 1 to 6) through the Secondary levels (Years 7 to 10 or 11). For students who aspire to experience university education, a pass in mathematics in the pre-university Year 12 examinations is a requirement (MOE, 2006). A number of university courses, such as those offered by Science, Medical, and Engineering faculties, also list good grades in mathematics as an entrance requirement. For students who prefer an alternative non-university technical track via Polytechnics, minimum standard requirements in mathematics are similarly imposed. It is thus natural that students’ performance in mathematics, as well as the way it is taught, is a matter of immense scrutiny by educators and parents in Singapore.

The interest in ‘Teaching Mathematics in Singapore’ is, however, one that is not restricted to the local community. In the Third Trends in International Mathematics and
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Science Study (TIMSS), that compared mathematics and science achievement across some forty countries, Singapore students emerged top in the rankings for mathematics (MOE, 2005). The TIMSS results have brought the small island state of Singapore into the focus of many mathematics educators and policy makers in the international community. This study may provide some added perspectives for this international inquiry.

1.1.4 Reform-oriented Teaching in Singapore

As this study seeks to inform education changes, it is therefore appropriate to focus on teaching that is in line with the reforms in the local scene. As in many other parts of the world, the Singapore government is actively urging “review [in] our method of instruction . . . [towards] innovative teaching methods” (Goh, 2001). There is, however, no consensus among scholars concerning what constitutes this new and desirable way of instruction. Terms such as “conceptually-oriented mathematics teaching” (Thompson, Philipp, Thompson, & Boyd, 1994) and teaching by “inquiry” (Cobb, Perlwitz, & Underwood-Gregg, 1998) that are often used in the literature highlight the different facets of what desirable instructional methods could be like. The author prefers the term “reform-oriented”\(^1\) teaching used by Boaler (2002) as it captures the intent of this study—to examine teaching that takes into account the current education initiatives—and allows a variety of instructional approaches.

In Singapore, the ongoing education changes that have a direct impact on mathematics instruction can be traced to these two broad initiatives dating back to 1998:

---

\(^1\) The use of “reform-oriented teaching” language is to alert the reader that the intention of the author in this study is in teaching that seeks to support reform goals. He does not, however, use the term to define or prescribe a narrow or particular way of teaching. He agrees with Kilpatrick, Swafford and Findell (2001) that “much debate centers on forms and approaches to teaching: ‘direct instruction’ versus ‘inquiry’, ‘teacher centred’ versus ‘student centred’, ‘traditional’ versus ‘reform’. These labels make rhetorical distinctions that often miss the point regarding the quality of instruction. Our review of the research makes plain that the effectiveness of mathematics teaching and learning does not rest in simple labels . . . Moreover, effective teaching—teaching that fosters the development of mathematical proficiency over time—can take a variety of forms” (p. 315).
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1. The infusing of thinking skills;
2. Integrating the use of information technology (IT) (MOE, 1998)

These were the primary thrusts of education reform efforts directed by the Ministry of Education. Thus, in examining reform-oriented teaching, this study also incorporates these twin objectives of “teaching thinking” and “integrating IT”.

1.2 Towards the Research Questions

The above paragraphs explain the author’s intent to study mathematics teaching in an actual classroom setting in Singapore. In addition, he proposes to examine reform-oriented mathematics teaching. In the Singapore context, this refers to instructional methods that seek to achieve the goals of “teaching thinking” and “integrating IT”. Question 0 below is a first-draft translation of these research purposes into question form.

Question 0: How does a teacher teach mathematics in a way that supports the reform agenda in Singapore?

This question provides the motivation for this study. However, there are four broad ‘fields’ inherent in the question that, when expressed so broadly, are not suited for a focused inquiry. The scope of each ‘field’ is explicated below.

Field 1: “How does a teacher . . . teach . . .”—this field examines the complex work of teaching in the classroom. There are many aspects to a teacher’s work of teaching. Assessment, supervision, seatwork monitoring, giving instructions, management, organization, leading discussions, etc., all constitute the ‘work of teaching’.
Field 2: “... supports the reform agenda ...”—this field reflects the diverse forms of reform efforts over the years. Each of these reform initiatives carry with it extensive implications for teaching in the classroom.

Field 3: “... teach mathematics ...”—this field addresses what is traditionally known as the ‘content’ of teaching. The ‘content’ of mathematics is undoubtedly wide and rich.

Field 4: “... teach ... in Singapore”—this field indicates that Singapore is the wider context in which this study is located, and that the variety of schooling experiences in Singapore, such as in the levels of schooling and the different performance streams, may influence the nature of teaching.

1.3 Sharpening the Research Foci

For the purpose of research, each of the above fields will be suitably re-interpreted or narrowed where necessary in a way that makes the research achievable without losing its significance to the original question.

Field 1: The Work of Teaching
As suggested earlier, teaching is a complex activity. One way to study the complexities of teaching is by looking at the problems of practice. When we look at the problems that teachers experience when teaching, we are looking at the interaction of the multiple complexities of the teaching task. By “problems of practice”, the author has in mind all problems a teacher encounters that interfere with his goals of teaching. Lampert (2001) indicated what some of these problems might look like:

The problems in teaching are many. Teachers face some students who do not want to learn what they want to teach, some who already know it, or think they do, and some who are poorly prepared to study what is taught. They must figure out how to teach each student, while working with a class of students who are different from one another. They must respond to the many authorities who tell them what to teach. They have a limited amount of time to teach what needs to be taught, and they are interrupted often . . . (p. 1).
The framework that Lampert calls the “basic model of practice” (2001, p. 30) and which she first used to identify the problems of her teaching practice is shown in Figure 1. The various arrows in Figure 1 indicate the interactions that take place in an instructional situation. To her, problems of teaching practice occur along a number of these practice arrows. The teacher encounters problems as she interacts with students (as indicated in the teacher-student arrow) and when she interacts with the content (as indicated in the teacher-content arrow) of what she intends to teach. Lampert also introduced an interaction arrow linking the teacher to the student-content interaction, which reflects her position that the teacher’s problem space is not limited to just teacher-student or teacher-content interactions, but also includes attending to students’ interaction with content. Moreover, because teacher’s actions with relation to students, content, and connection between students and content are simultaneous, the arrows that proceed from the “teacher” in the model are linked by an overlapping arrow in a “single, three-pronged problem space” (p.33). Lampert's basic model of practice above will be used in this study as a starting point to identify the problems of practice in the Singaporean context.

However, research efforts should go beyond merely understanding problems. There should be a study of teacher actions that accommodate these problem contexts and adapt them to carry out the reform agenda. Lampert (2001) stated that “[i]t is possible to teach students and subjects in the classroom, and even to teach well and elegantly [despite the problems of teaching]. How a teacher manages the complexity is what we need to understand” (p. 2, emphasis added). Documentation of these feasible practices can serve
as a guide to future reform efforts. The author refers to such practices as “coping strategies”, a term borrowed from Lampert (1985). She indicated her preference for the term “coping” rather than “solving” to describe teachers’ responses to problems of practice, based on her view that the problems of the teaching practitioner are conceptually “unsolvable” (p. 181) because of inherent conflicting goals in teaching.

The identification of “coping strategies” is closely linked to the identification of “problems of practice”. For the purpose of this research, “coping strategies” are defined as the strategies the teacher uses to cope with the “problems of practice” of which he is aware. This definition necessarily limits the coping strategies under study to those problems that are seen through the teacher’s eyes. This limitation is, however, not unrealistic in actual practice, as teachers in day-to-day practice manage problems seen through their own perspectives, and not through those of others. Since the purpose of this proposed research is to study practice in an actual setting, it is not expected that “coping strategies” as defined here should include those for situations which the teacher has not identified as problems in her classroom.

The field of the “work of teaching” can now be redefined as comprising two inquiries: (1) the “problems of practice” (interferences to goals of teaching); and (2) the “coping strategies” (ways to cope with problems of practice).

Field 2: Reform agenda

*Integrating IT*

Particular types of software have emerged over the last few years that have become popular for mathematics instruction. A survey of the common textbooks used in Singapore schools for lower secondary (Years 7 and 8) mathematics (e.g., Sin, 2001/2002; Teh & Looi, 2001/2002) indicated that Dynamic Geometry Software (DGS) is the most frequently recommended software by secondary mathematics textbook writers.
Internationally, there has also been interest in the advent of DGS on the geometry education scene. Several studies highlight its potential in aiding the teaching of geometry. Clements and Battista (1994) claimed that students who explore geometric concepts using DGS used “the objects on the computer screen as manipulable representations—a ‘mirror’—of [their own] thinking.” Thus, “the students can make conjectures, evaluate visual manifestations of those conjectures, and reformulate their thought” (p. 87). Scher (1999) further argued that the dynamic visuals in the software “can provide not only data to feed a conjecture, but tools to jumpstart ideas and feed a proof” (p. 24, emphases in the original). Using various instruments to measure students’ achievement, several comparative studies have yielded empirical data showing benefits associated with work in DGS environment (Dixon, 1997; Leong, 2002; McCoy, 1991; Vincent, 1998).

Teaching Thinking
The author uses two prevailing views from the literature to refine the idea of “thinking” in a way that is suitable for classroom research. From the Vygotskian perspective, learning is sociocultural in nature. It is an action that internalizes outward forms of social discourse into internal forms of discourses between intramental states. Thought is therefore internal language (Vygotsky, 1986). Seen through this perspective, the social interactions and classroom discourses become important, meaning that the teacher can structure them to promote students’ thinking, as “forms of discourse becomes forms of thinking” (O’Connor, 1998, p. 22).

Another prevalent idea about the implementation of a thinking curriculum is that “thinking skills” should be taught in the context of content-specific domains (Marzano, et al., 1988). This is where the different ‘fields’ can be seen as non-independent: “teaching mathematics” and “teaching thinking” are inextricably linked. The teacher organises instruction that encourages students to think within the mathematical terrain. Indeed, “mathematical thinking” is a subset of “thinking”, and to focus on teaching mathematical thinking will reveal insights about teaching thinking. Lampert (1988) advocated a pedagogy that mirrors how knowledge is constructed within the mathematics discipline.
To her, how mathematics is done in the discipline informs how mathematics should be done in the classroom. Since mathematical knowledge is constructed via reasoning among the practitioners within the community, mathematical thinking within the classroom should be about reasoning among students. By “reasoning” the author draws on Polya’s (1954) broad categorical distinction of mathematical reasoning as comprising “plausible reasoning” and “deductive reasoning”. The former includes more intuitive approaches to mathematics problem-solving such as conjecturing, hunches, guesses and counter-examples; the latter involves the formal mode of logical presentation portrayed in most written forms of mathematics. While deductive reasoning is often the “front” (Hersh, 1997)\(^2\) of mathematics, plausible reasoning, as the “back”, is also essential in the creative process of mathematics.

Thus, in “teaching thinking”, the author has in mind “thinking viewed through the social-cultural and discipline-specific lenses. In other words, the “thinking” to be encouraged in students by the teacher in this study is not the exchange between intra-mental states. These internal interactions within each student may be present but are unobservable by the teacher and cannot be directly supported by the teacher. Neither is the framing of “thinking” in this study rooted outside the domain of mathematics. The “thinking” to be encouraged by the teacher, as conceived in this study, is classroom discourse that approximates the kind of reasoning—both plausible and deductive—among practitioners in the mathematics community. This kind of thinking can then be rightly encouraged, seen, and directly supported by the teacher.

The two initiatives of “integrating IT” and “teaching thinking” within the field of “reform agenda” can now be correspondingly refined as “using Dynamic Geometry Software” and “supports students’ participation in mathematical reasoning”.

\(^2\) Hersh’s “front/back” view of mathematics is an analogy taken from the ‘front’ and ‘back’ of a restaurant. Elaborations of this are given in section 2.3.1 of Chapter 2.
Field 3: Mathematics Content
The choice of content here again highlights the fact that the fields being discussed are certainly not independent of one another. The choice of DGS in Field 2 restricts the field of “mathematics content” to “geometry”. However, this confinement of the mathematics subject-matter in no way limits the potential scope of applicability of this study, as geometry is one of four major components in the school mathematics syllabus in Singapore (MOE, 2007).

In addition, school geometry continues to be valued for its grounding in deductive reasoning—a feature which remains relevant in today’s mathematics educational goals internationally, although its emphasis varies from country to country. Thus, the choice of geometry as “mathematics content” provides the classroom environment with an ingredient that can support students’ deductive reasoning, a form of mathematical thinking as elaborated under Field 2.

Field 4: In Singapore
In Singapore, students go through the first six years of formal schooling in the Primary schools and the next four or five years in the Secondary schools. At the end of year 6 in the Primary schools, all students sit a national examination. Their performances in this examination determine whether they will be channeled into the “Express stream” (four years) or “Normal stream” (five years) in their Secondary-level education. The highest-achieving students can enroll in Independent schools. These schools have some liberty to craft curricula that suit the needs of their students. All the other Secondary schools are known as government schools and follow curricula that are centrally designed and monitored by the Ministry of Education. Most government schools offer courses for students in both the Express and the Normal streams.

The author proposes to focus the study of teaching “in Singapore” to “Secondary classroom in Singapore”. This choice is related to the decisions made in Fields 2 and 3. The emphasis in the Primary mathematics curriculum is towards more concrete and real-life manifestations of mathematics. There is thus a heavier leaning towards arithmetic.
While geometry is introduced in the Primary curriculum, the experience is largely restricted to shape recognition and basic computation of angles. In contrast, Secondary geometry covers a greater proportion of the mathematics syllabus and is singled out as the component that can help students develop mathematical reasoning abilities (MOE, 2007). Moreover, DGS is far more commonly used in the Secondary classrooms than in the Primary levels. Insofar as Secondary geometry increases the potential of realising the ideals of integrating DGS and supporting mathematical reasoning, the choice of “Secondary classroom” is appropriate for this study.

Since there are different types of Secondary schools, there is also a need to further narrow the selection. As the combination of “government” schools and “Express” stream represents the majority of Secondary classrooms in Singapore, the field of “in Singapore” will be refined as “in an Express class of a government school in Singapore”.

1.4 Research Questions

The above process of sharpening the foci of inquiry refines Question 0 above into the following two research questions for this study.

**Question 1:** What is the nature of problems that a Secondary teacher experiences when using dynamic geometry software to teach geometry in a way that supports students’ participation in mathematical reasoning in an Express class of a government school in Singapore?

**Question 2:** How does a teacher cope with the problems of practice?

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3 The Ministry of Education negotiated a discounted price for installing DGS with the vendor in the early 2000s. The offer to buy this discounted site licence was only extended to Secondary schools. This perhaps reflects the Ministry’s view that Secondary geometry is better suited to realise the power of DGS in teaching. The situation now is such that almost all Secondary schools own the software while almost no Primary schools do.
The above questions directly address the goals of the study as elaborated in the earlier paragraphs. The questions are targeted at understanding the actual work of teaching. In particular, they examine the work of teaching involved to bring on board the reform objectives of teaching thinking and integrating IT. While the two research questions limit the scope of the research to one that allows a focused inquiry, it nonetheless has relevance to the broader goal embodied in Question 0.

1.5 Overview of the thesis

This chapter has focused on the motivation of the study as represented by Question 0 and how the various ‘fields’ within the question lead to the research questions. Chapter 2 surveys the literature that sheds light on the ‘fields’ identified in the first chapter. The literature review will provide a theoretical basis on which to base the general pedagogical and methodological approaches discussed in Chapter 3. Chapter 4 is devoted to a specific discussion and setup of the goal-based methodology that is heavily-used in this study. The next three chapters provide detailed analyses of the data collected: Chapter 5 reports some preliminary findings; Chapter 6 tests and refines these findings by broadening the data-base; and Chapter 7 looks more specifically at problems of teaching and coping strategies involving the use of DGS. Chapter 8 provides a summary of findings and relates them to the original Question 0.
The purpose of this chapter is to explore the literature that relates to the main elements of this study. Research papers and scholarly contributions are reviewed with the intention of harnessing them to form a theoretical basis for both the research design as well as the teaching approach to be adopted in this study. The arrangement of the sections in this chapter corresponds roughly to the fields identified in the research questions in the previous chapter.

### 2.1 Teaching Mathematics

In reviewing the literature associated with “teaching mathematics”, the motivations are to understand the different theoretical perspectives on mathematics teaching and how they can be used to inform the research about “teaching mathematics”.

#### 2.1.1 Approaches to Mathematics Teaching

Calls for education reforms are often expressed using “new” versus “traditional” rhetoric. The “new versus traditional” presentation by policy makers highlights contrasts between the two approaches to teaching, with the agenda of alerting and persuading the education community to imbibe the new and reject the traditional.

Current literature on reforms in mathematics education use similar contrastive pairs to present the traditional-new distinction. Kirshner (2002) observed that, in the United States of America, “the Learning Principle propounded in Principles and Standards for School Mathematics (NCTM, 2000) rehearses the familiar distinction between facts/procedures and understanding as a central guiding principle of teaching reform” (p. 46). Boaler (2002) presented the distinction as one between “skill-oriented” and “reform-oriented” teaching approaches. Other researchers, who avoided association with prescriptive methods, sought rather to describe methods that teachers use in their classroom practices. Some of them have also used contrasting dualistic descriptions, as in “calculationally-oriented mathematics teacher” versus “conceptually-oriented mathematics teacher” (Thompson, Philipp, Thompson, & Boyd, 1994), and teaching by “procedural instruction” versus teaching by “inquiry” (Cobb, Perlwitz, & Underwood-Gregg, 1998).
A brief consideration of the contrasting pairs in the paragraph above indicates a commonality in the depiction of traditional teaching in the literature. “Skills”, “facts”, “calculation”, and “procedures” suggest an underlying view of school mathematics as following rules, and of teaching mathematics as getting students to practice towards conformity to these rules. It is interesting to note that what was seen as traditional teaching has not changed much in substance over the past decades despite many calls for reforms littered over the same period. The traditional teaching method as seen by Bruner (1960) focused on “techniques and methods”. Skemp (1986) saw it as “rote memorization”. These terms continue to be used in today’s literature as characteristics of traditional teaching (see the examples mentioned previously).

There is, however, no consensus among scholars concerning what constitutes the ‘new’ approach to teaching. By looking at the terms used above to describe the new teaching approaches—such as “teaching for understanding”, “teaching by inquiry”, and “conceptually-oriented teaching”—one can see a variety of views of what reform-oriented teaching looks like. One way to account for this diversity is by considering the diversity of theoretical positions that researchers and teachers hold. The main traditions underpinning teaching approaches for mathematics are behaviourism, constructivism, and sociocultural theories. Kirshner (2002) reviewed the contributions of these theoretical streams to mathematics pedagogical practice and a summary is given in Table 1.

<table>
<thead>
<tr>
<th>Learning theories</th>
<th>Behaviorism; Information processing</th>
<th>Psychological constructivism</th>
<th>Sociocultural theory (appropriation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>What is gained</td>
<td>Skills</td>
<td>Concepts</td>
<td>Dispositions</td>
</tr>
<tr>
<td>Progression</td>
<td>Incremental</td>
<td>Transformative</td>
<td>Discontinuous</td>
</tr>
<tr>
<td>Pedagogical focus</td>
<td>Repetitive practice</td>
<td>Hypothetical learning trajectory</td>
<td>Classroom microculture</td>
</tr>
<tr>
<td>Pedagogical objective</td>
<td>Proficiency with routine exercises</td>
<td>Conceptual restructuring</td>
<td>Culturally appropriate participation</td>
</tr>
</tbody>
</table>

Table 1: Adapted from Kirshner’s framework for application of learning theory to pedagogical practice (2002, p. 51)
2.1.2 Complexity of Teaching

Relating theories to actual practice is, however, not a straightforward matter. The main difficulty is in the complexity of teaching practice. Lampert (2001) described the work of a teacher as a “complex activity” (p. 1). The complexity is not unidimensional but on a number of ‘planes’. On the ‘social plane’, a teacher must be aware that she interacts not only with one student but with a group of students, nested within bigger groups, extending to the whole class. While engaging with these different pockets of individuals, she has to be aware of the focus and direction of the whole class. On the ‘temporal plane’, the work of teaching is not merely confined to a point in time, but is related to what goes on before and after, across units of time and across lessons extending to the whole year. On the ‘mathematics content plane’, content is not comprised of disjoint pieces. They are related to each other logically and mathematically. Also, the complexity in the content often has to do with students accessing different content in a single problem context—for example, one student may be working on understanding the technical language of the problem, while another is solving the algebraic equations derived from the problem.

The complexity of the instructional task is accentuated by the need of the teacher to ‘simultaneously’ weigh many factors in each decision during ongoing practice. She also does not have the luxury of a researcher—with time to analyse choices carefully—she has at most a few seconds. A teacher also has to juggle different and sometimes conflicting goals. She has to teach mathematics that matches the curriculum goals while inculcating a classroom environment where students do not merely take the teacher’s word as the authority, but rather are able to reason using agreed-upon assumptions. Wood, Cobb, and Yackel (1995) described a teacher’s experience of conflicting priorities as “walking the pedagogical tightrope” (p. 421).

In the TIMSS video study, Stigler and Hiebert (1999) examined lessons conducted in Germany (100 lessons), Japan (50 lessons), and the United States of America (81 lessons) from the perspective of outside observers. They noted that there were distinct differences in the way teaching was conducted between the countries studied but warned against simplistic transference of what appeared to work within a particular jurisdiction to another. They argued that teaching is a complex cultural
activity where the whole practice is not merely an aggregation of different elements of teaching: “Teaching systems, like other complex systems, are composed of elements that interact and reinforce one another; the whole is greater than the sum of the parts” (p. 97).

Since teaching is such a complex activity, it is no wonder that existing unifocal theoretical perspectives cannot account for all the complexities. Lampert (1985) cautioned that “efforts to build generalized theories of instruction, curriculum, or classroom management based on careful empirical research have much to contribute to the improvement of teaching, but they do not sufficiently describe the work of teaching” (p. 179).

2.1.3 Towards Cross-disciplinarity

One way to approach this difficulty of linking theory to practice is to let go of the strict adherence to the one-theory-fits-all conception and adopt a “cross-disciplinary” (Kirshner, 2002) approach. Studies that adopt the use of two or more traditionally competing theories in a unified way for analysis are increasingly common. Jones and Tanner (2002) saw mathematical sense making (which they viewed as derived from constructivism) and developing a mathematical viewpoint (which they see as more akin to enculturation into a community) as important tasks for learners of mathematics in the classroom. They noted that these “two viewpoints need not be mutually exclusive … . Pupils learn by participating in a ‘culture of mathematising’ which is characterized by subjective, personal construction of knowledge through the negotiation of meaning in social interaction” (p.267). Yackel and Cobb (1996), in their study of how students develop mathematical beliefs and values amidst the complexity of classroom life, recorded that

We began the project intending to focus on learning primarily from a cognitive perspective, with constructivism as a guiding framework. However, as we attempted to make sense of our experiences in the classroom, it was apparent that we needed to broaden our interpretative stance by developing a sociological perspective on mathematical activity. (p. 459)
In addition to the use of multiple perspectives in research studies, there are also moves towards negotiation for complementarity at the theory-reconstruction level, as seen in emerging integrative theories such as social constructivism and situated cognition.

Cobb (1994) also argued for a pragmatic stance with regard to applying theoretical ideas into classroom practice. He grounded his arguments for such a position from Rorty’s (1983) view of theoretical perspectives as instruments for coping with things rather than ways of representing intrinsic nature. To Cobb, “it is social reality [as it plays out in practice] that dictates the correct theoretical perspective” (p. 19). Seen from this pragmatic standpoint, the way to evaluate the appropriateness of a theoretical perspective in interpreting practice lies in whether “it works better than another for a given purpose” (Rorty, 1983, p.157, emphasis added).

This pragmatic stance can be extended to the use of multiple instructional approaches in teaching. Like the underlying theoretical motivations, teaching approaches can similarly be complemented to suit the different purposes of teaching. Thus, “effective teaching” may not involve a single teaching approach but a variety of instructional methods suited for different contexts and purposes. This argument runs against the traditional simplistic classification of teaching methods as ‘traditional’ or ‘new’. Quality teaching can be a complex mix-and-match of different instructional forms whose choice is dependent on various factors and competing priorities. The traditional-vs-new rhetoric, instead of accounting for diversity and complexity, tends to add fuel to the ongoing “math wars” (Becker & Jacob, 1998) between conservatives and reformers.

The futility of the dualistic casting of teaching methods in advancing knowledge of quality practice was also attested by the Mathematics Learning Study Committee, a multidisciplinary study group formed by the US National Research Council to review and recommend approaches to improve pre-K to grade eight mathematics education in US schools. Kilpatrick, Swafford and Findell (2001), editors of the resulting report, wrote that
Much debate centers on forms and approaches to teaching: ‘direct instruction’ versus ‘inquiry,’ ‘teacher-centred’ versus ‘student-centred,’ ‘traditional’ versus ‘reform’. These labels make rhetorical distinctions that often miss the point regarding the quality of instruction. Our review of the research makes plain that the effectiveness of mathematics teaching and learning does not rest in simple labels …. Moreover, effective teaching—teaching that fosters the development of mathematical proficiency over time—can take a variety of forms. (p. 315)

2.1.4 A Cross-disciplinary Model of Teaching Practice

Cohen, Raudenbush, and Ball (2002), taking a practice-focused view of analysing the work of teaching, proposed that the practice of teaching should not be restricted to the acts of the teacher, but should rather be viewed as interactions among the teacher-student-content triad. Moreover, these interactions interact with the environment in that the triadic interactions and the classroom-school cultural milieu affect one another bidirectionally. This conception of the work of teaching is clearly consistent with both constructivist and sociocultural views of learning and can thus be seen to be cross-disciplinary in nature. The “instructional triangle” model (p. 88) was used as a diagrammatic representation of the practice of teaching, and is reproduced in Figure 2. This model serves as a starting point in building a language that describes the multi-planar complexities of teaching. A comparison with Lampert’s (2001) “basic model of practice” (Figure 1 in Chapter 1) reveals identical elements at the vertices of the triangle. The alternative model in Figure 2 highlights other sociocultural factors—such as environment and interactions among students—not explicitly described in Lampert’s model. Henceforth, the term “instructional triangle” (or its equivalent) will refer to a merger of features from both of these models.

A growing number of researchers (e.g., Kilpatrick, Swafford & Findell, 2001; Boaler, 2002) have also referred to similar versions of the instructional triangle as having the potential to advance knowledge of instructional capacity and have used the model in their work of research and analyses. The instructional triangle highlights the key elements and relationships that researchers from different theoretical traditions can hold jointly to be crucial in the work of teaching. There is thus potential for more exchanges across traditional disciplinary lines in sharing and building knowledge about the teaching enterprise.
2.1.5 Metaphors as Ways to Link Theory and Practice

As a way to link theory and practice, Kirshner (2002) advocated seeing theoretical conceptions as metaphors to help practitioners access theories relevant to their own practice. Since there “is no [one] magic ‘reform method’ that addresses the multiple forms of learning that teachers may aspire to for their students” (p. 54), the use of metaphors relating theory with practice affords educators the diversity of instructional forms that are part and parcel of daily classroom practice.

Metaphors have a long history of use in linguistics and social sciences. Metaphors are powerful tools that provide information about unfamiliar objects by making corresponding references to familiar representations. Recently, the use of metaphors has featured in the education research literature. The basis of using them to describe and analyse practice is that metaphors “cross the borders between the spontaneous and the scientific, between the intuitive and the formal … [T]hey enable osmosis between everyday and scientific discourses” (Sfard, 1998, p. 4), thus giving them the key position of linking theory and practice. In the current climate of increasing frustration with the inadequacy of theories in directly reforming teaching practice (Christiansen, 1999; Bishop, 1998), the role of metaphors that root theories into forms familiar to practitioners holds promise. Metaphors “permit us to reason about [the target domain] using the knowledge we use to reason about [the source domain]” (Lakoff, 1994. p. 210).
There is, however, no single metaphor that can comprehensively capture all the complexities of teaching. In the following paragraphs, the author explores some metaphors of teaching from the literature, in order to elaborate the model of teaching practice shown in Figure 2. As there are many such metaphors, the author limits this review to those extended metaphors that have surfaced from studies of practice that have sought to capture the complexity inherent in the work of teaching mathematics.

**Teaching mathematics as Jazz Performance**

Viewing the work of teaching as analogous to the work of performing artistes, such as orchestrating or choreographing (Thompson, 1985) is not new in the literature. King (2001) elaborated on this by extending the metaphor and emphasizing one unique feature of jazz performance—improvisation—as being distinctly characteristic of teaching practice. That improvisation is an integral part of teaching is supported by other classroom researchers (e.g., Chazan & Ball, 1999; Lampert, 1985).

In Coker’s (1964) discussion of the improvisational character of jazz performances, he said that

> Improvisation, like composition, is the product of everything heard in past experiences, plus the originality of the moment. The contents of even a very accomplished improviser’s solos are not all fresh and original, but are a collection of clichés, established patterns, and products of memory, rearranged in new sequences, along with a few ideas. (p. 36)

A jazz musician has a source “song”\(^1\), which may be a written music score or stored in the memory of the musician; but unlike most other forms of music performances, the jazz soloist does not adhere strictly to the song, but improvises on it during performance, guided by responses to other instruments in the band and the sensibilities of the audience. The improvisations are not entirely random, but are built upon the structure of the song.

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\(^1\) In jazz language, there is a distinction between “music” and “song”. Music is the actual spontaneous performance and improvisation of the reference song.
King (2001) drew correspondences between “jazz soloist” and “the teacher”, “soloist’s instrument” and “teacher’s discourse”, and “other instruments in the band” and “students”. Like the soloist, the teacher is the lead performer and sets the tone and direction of the classroom instruction. She does so through her “instrument”, that is, her discourse. By discourse, King included also “gestures, non-verbal communication and all written work” (p. 10) that proceeded from the teacher. But how the teacher plays the instrument (conducts her discourse), is not independent of the students’ contributions; rather, the teacher’s work on her instrument is a result of an interaction with other instruments in the band (the students). Seen through this metaphor, students are not passive receivers nor merely an audience, but are active participants of the improvisational process which is at the heart of jazz performance (teaching).

For the model in Figure 2, the metaphor of jazz performance elaborates the teacher-student part of the instructional triangle. The bi-directional arrowhead linking ‘teacher’ and ‘students’ is not symmetrical in the equal-impact sense (as highlighted in Figure 3.1). The teacher is the leader but the students are not. The teacher’s intellectual authority comes from the responsibilities of teachers within the historical and social setup of schooling. Students may bring innovations into the instructional setting by way of novel questions and alternative viewpoints, but the teacher ultimately decides the course of action in the classroom. Teachers’ priorities often differ markedly from the students’ as the former are based on multiple considerations which include (but are not restricted to) subject-matter, beliefs, knowledge of pupils (Ball, 1991), culture, and curriculum coverage (Lampert, 2001). These considerations arise out of their role as teachers, and are certainly not students’ priorities as they do not share this same role. However, while the students are not leaders, neither are they merely followers. They are co-participants in the performance in that they contribute to the music-making and also they affect directly the improvisational moves made by the teacher. The teacher, while leading, does not merely pursue her agenda but must react to students’ participation in the music-making. In other words, the metaphor is consistent with constructivistic views of learning.
What then is “music” in the context of the classroom? According to King (2001), it is “the taken-as-shared knowledge, exhibited through discourse” (p. 11). The term “taken-as-shared” reflects the sociocultural view of knowledge as residing not only in the cognitive domain, but also on the social plane, at the interactions among students and between students and the teacher. Thus Figure 3.1 is modified into Figure 3.2 to take into account the importance of student-student interactions in the production of this taken-as-shared knowledge.

The “song” in the classroom is the “hypothetical learning trajectory (HLT)” (Simon, 1995). “It includes the teacher’s mathematical learning goal for her students, her hypothesis about the process of learning and a plan for learning tasks consistent with this hypothesis” (King, 2001, p. 11). Just as the song provides a guiding structure but not the actual jazz music, the HLT is hypothetical and is “not intended to be played as written, but to be improvised upon” (p. 11). King further clarified that the HLT “may be embodied in a lesson plan, but is more than simply curriculum materials and
teacher’s notes. It includes the teacher’s thoughts by which students are to move toward the intended learning goals …” (p. 11). King clearly distinguished the two types of knowledge that the teacher requires to generate the HLT—knowledge of the subject matter and knowledge of engaging students to learn the subject matter. She writes, “[A] teacher needs to have a deep knowledge of the mathematics she teaches … [and] needs to be able to elicit from her students both the willingness to participate in the discourse and productive discursive interactions” (p. 12). Ball (1991) gave a similar distinction on the kinds of knowledge required in the teachers of mathematics. She posited “subject-matter knowledge” and “other kinds of knowledge” (p. 23). By the former, she meant a combination of “knowledge of mathematics” and “knowledge about mathematics” (p. 20, emphases in the original), which approximately corresponds to the usual constructs of “mathematics content” and “beliefs about mathematics” respectively. By the latter, she meant “ideas about teaching and learning mathematics and ideas about pupils, teachers, and the context of classrooms” (p. 21). In addition, Ball asserted that “subject-matter knowledge” and “other kinds of knowledge” interact with one another in the teacher’s “performance” (improvisational enactment of the HLT). Linking all these additional ideas into Figure 3.2 and the instructional triangle model brings about Figure 3.3.

Figure 3.3: Teacher-content interactions viewed through the jazz metaphor

In carrying out a lesson, the teacher draws upon the mathematics subject-matter and other knowledge of teaching necessary for her classroom setting. Ball (2000) warned against fragmenting these strands of knowledge. For this reason, they are represented
as interacting with one another and subsumed as part of “content” in Figure 3.3.

Materials relevant to the goals of the lesson are selected and put together into a hypothetical plan (song). In playing the song during actual instruction, the song transforms into the actual music through the improvisations of the teacher as she takes into account the students’ responses.

In jazz, the performance is almost always for an audience; the music does not play alone. This leads to the final question pertaining to this metaphor, “who is the audience?” King points out that there is one: “The audience is outside observers of the classroom environment (principal, mathematics supervisor, researcher, fellow teacher, parents)” (King, 2001, p. 12). This acknowledgement of an audience beyond the boundaries of the classroom is consistent with sociocultural views of learning. However, while King recognised that the classroom is not a closed system, she did not build on this wider audience as contributing positively to mathematics education. Instead, she accused some audiences who do not appreciate jazz performances as “unknowledgeable audiences”. She claimed that mathematics teachers who teach in a way that is unfamiliar to audiences schooled in more traditional instructional methods risk being unappreciated. The author here does not share this denial of the need to engage the wider audience. While the world of jazz may be content to work in a closed system among those who can appreciate jazz music, the same cannot be said of mathematics teachers. The success of the teacher’s work lies in the “improvement interventions [that] must attend to the complex relationships that exist among intervention agents, schools, and their social environments” (Cohen & Ball, 1999, p. 1).

Figure 4 represents the final elaboration of the instructional triangle as a summary of what the jazz performance metaphor contributes to the understanding of the work of teaching.

This metaphor does not, however, comprehensively deal with all the possible interactions within the teacher-student-content triad. In particular, the “content” vertex of the triangle is given the least consideration. King tended to see “content” as not directly related to the happenings in classroom interactions, as can be clearly seen in her acknowledgement that “this formulation … places the activity in the classroom
as central, *as opposed to* the mathematical content …” (p. 11, emphasis added). This conception of “content” as being non-integral in classroom activity is not shared by other researchers. In fact, the criticality of content in affecting instructional situations is implicitly attested by the inclusion of “content” as one of the vertices of the instructional triangle model. This point about the important role of “content” will be clearly illustrated when other metaphors are examined.

Figure 4: Instructional triangle viewed through the jazz metaphor

**Teaching mathematics as Managing Fermentation**

An alternative metaphor was presented by Chazan and Ball (1999), who used the process of biological fermentation as a source domain to map correspondences onto the work of teaching as the target domain. To them, teaching involves managing intellectual fermentation during classroom discourse. “Mere sharing of ideas does not necessarily generate learning. For a discussion to be productive of learning, different ideas need to be in play, the air filled with a kind of *intellectual ferment* in which ideas bubble and effervesce” (p. 7, emphasis added). Thus, this metaphor puts a correspondence between “fermentation” and “engagement of ideas” during classroom discourse. Moreover, it is not so much the voicing of ideas as the *interacting* with others’ ideas that is seen as the key to intellectual effervescence. Also, clearly rooted in the constructivistic tenet that teachers cannot directly construct learning for the students, Chazan and Ball stated that “this intellectual process cannot be controlled
directly” (p. 7). However, they suggested that “it can be accelerated by the presence of ‘catalysts’. [As an example], [d]isagreement—the awareness of the presence of alternative ideas—can be an important catalyst” (p. 7). This idea of the teacher’s role being one that introduces catalysts to kickstart and sustain intellectual processes is a helpful one. The traditional view of teaching is that of direct instruction, whereas some modern views tend to overstress the passivity of the teacher’s role by downplaying teachers’ ability to interact with students’ learning. This metaphor of managing fermentation via introducing catalysts avoids either extreme. It recognises the indirectness of the process (via catalysts) while affirming the activeness of the teacher’s role (in introducing catalysts).

Chazan and Ball (1999) further clarified the use of disagreement as catalyst to promote ferment. Chazan described an episode where the students in his class were divided on the answer to a problem he posed. The problem listed the bonus earned by 10 employees and required the students to find the average bonus. What led to the disagreement was that one of the employees was listed with a bonus of $0. The students could not agree on whether that employee should be included in the computation for average. They drew on their prior knowledge of adding followed by dividing as a way to compute average, but some took the divisor as 9 while others used 10. As the classroom discussion went on, each side dug into their positions and students began to talk unproductively instead of engaging in an intellectual discourse. Sensing that the classroom discourse was moving away from reflecting on the subject matter, Chazan intervened and re-directed the class back to the concept of average. He wanted to help them see average as not just a result of a procedure of adding and division, but also the idea of what each employee will get if everyone was to get the same amount. This, however, only re-directed the disagreement to whether an employee having no bonus should be given a share, and the students soon took sides again on either side of the ‘is/isn’t’ divide. While the arguments grew more intense, Chazan began to challenge his own assumptions and to evaluate the validity of the argument that the person who has $0 should not be included in the averaging. “It started to seem silly to say that the one person got a bonus of zero dollars, instead of saying that the person didn’t get a bonus and therefore should not be considered in
computing the average bonus” (Chazan & Ball, p. 4). The account of Chazan’s lesson ended at that point.

In Ball’s lesson, the experience was different. Instead of disagreements, Ball’s students are agreed. The problem was that they were agreeing over something that was wrong. Ball introduced the number line partitioned into eighths from zero to one and asked the students to label each ‘cut’ on the number line. The first student she called upon struggled and then reverted to a more familiar method of drawing a rectangle and dividing it into equal pieces. What surprised Ball was that the student divided the rectangle into 7 pieces. Ball stepped in to elicit from the student the reason for the number of divisions she has chosen. But sensing that the student could not respond, Ball posed the question to the class. What was surprising to her was that the class subsequently agreed that 7 pieces was right! At this point, Ball needed to draw the students’ attention to their error. She re-directed them to previously discussed material and revised a similar problem on a number line divided into fourths. She then showed that by utilizing a similar procedure on the current ‘eighths’ problem, the error was exposed. Despite her attempts to introduce her voice into the discussion to provoke disagreement, Ball “was unsure of what sense [the students] were making of [her] intervention” (Chazan & Ball, 1999, p. 6).

The two episodes described provided elaborations on the metaphor of “fermentation”. The teacher’s role was not merely one of just introducing a catalyst, he/she was also required to monitor the process, sometimes (in the case of Ball’s class) adding a further catalyst by inserting an alternative voice to stimulate disagreements; sometimes (in the case of Chazan’s class) it involved slowing down a catalytic process to avert unproductive ‘over-reaction’; and sometimes (in both classes) there is a need to re-direct the flow of effervescence to a correct focus on appropriate content. “Teachers must have a repertoire of ways to add, stir, slow, [and] redirect the class’s work” (Chazan & Ball, 1999, p. 9).

Unlike King’s jazz performance metaphor, Chazan and Ball (1999) placed strong emphasis on “content” being an integral part of teaching. They were, as teachers, concerned about leading the class towards learning correct mathematics. Students were directed to tap into prior knowledge and also challenged to reorganise their
assumptions. In this way, students were led to focus on the mathematics and to interact with it. However, by “content”, it was not just mathematics as the subject matter of discussion that was in consideration. Chazan and Ball recognised that for students to engage in disagreements without degenerating into a shouting match, they needed to build on one another’s ideas in a civil way. Thus, “content” also referred to other kinds of knowledge that the students need for the social process of fermentation to take place. Moreover, the interaction with content was not restricted to the student-content part of the instructional triangle. Chazan’s rethinking of his own position about computing the average shows that the teacher also interacts with content. The teacher actively acts on the content to make sense of it and to reformulate it for use in the work of teaching.

Figure 5 summarises the contribution of the metaphor of managing fermentation in the context of the teaching triad model.

A further word on “content” is necessary here. Obviously, the “content” in the teacher-content part of the triad is different from the “content” in the student-content part. The teacher interacts with the content primarily for her work of teaching. She taps into her knowledge of teaching mathematics (which, as mentioned before, includes both subject-matter knowledge and other kinds of knowledge) and constantly interacts with that knowledge by way of reflection on classroom experiences and on curriculum materials. The students, however, do not share the teacher’s purposes when interacting with content. Students build on their prior knowledge and construct
their own content consistent with their conception and goals of learning mathematics. Despite the distinction in “content” made here, “content” remains the common vertex in keeping with the diagrammatic simplicity of the instructional triangle model.

Teaching mathematics as *Navigating an Aircraft*

Lampert (2001) used the metaphor of “navigation” (p. 446) to depict the work of teaching. Just as navigators use gadgets and personal knowledge as resources to weather storms and get them to their destination, teachers, too, harness all the resources at their disposal to chart a course to fulfil their teaching goals. However, Lampert does not view “resources” as merely the physical classroom environment, curriculum documents, and teaching aids. She observes,

> The problem space in which the teacher works is full of ideals to be realised and full of worthy destinations. To make it possible for the teacher to realise these ideals and to get to these destinations, students and their relationships with one another must be employed as *resources* to complement the *resources* the teacher brings and the objects that are physically available in the classroom. (p. 447, emphasis added)

As a navigator, a teacher uses her knowledge of content, students, and how the students interact with content to anticipate and control the course of instruction. Anticipation is about “developing the foreknowledge that [the teacher] would need to take advantage of the opportunities that would arise to make mathematical connections” (p. 184). In this sense, anticipation is akin to the navigator’s work of being familiar with maps and charts and rehearsing operating procedures so as to exploit favourable conditions as well as to brave through contingencies. Lampert’s idea of anticipation is therefore similar to Simon’s (1995) conception of the hypothetical learning trajectory (HLT) reviewed earlier. Figure 6.1 summarises this metaphor’s elaboration of the instructional triangle model.

Lampert also introduced an interaction arrow linking the teacher to the student-content interaction, which reflects her position that the teacher’s problem space is not limited to just teacher-student or teacher-content interactions, but should also include attending to students’ interaction with content. Her classroom experiences in a teacher-researcher role have shown that a teacher needs “to take action to make
studying happen, and to make it happen in ways that are likely to result in learning” (p. 32, emphasis added). Interestingly, Lampert used the term “study” to label the student-content part of the triad. Her use of the term is in the old-fashioned sense where people “apply their minds purposefully to the acquisition of knowledge or understanding of something” (p. 32), rather than revising or memorising. The author here shares Lampert’s revived use of “study”.

Figure 6.1: Instructional triangle viewed through navigation metaphor (preliminary)

Lampert’s choice of navigation as a metaphor for teaching also has to do with the work of a navigator in dealing with constantly changing situations, as in teaching. In the classroom, teacher-student, teacher-content, and teacher-study pairs of interactions do not happen sequentially or independently of each other. They happen all at once. The teacher’s decisions at various junctures of the lesson hinges on the intersections of these interacting pairs. To more accurately represent the simultaneity of the teacher’s action or reaction, the arrows that proceed from the “teacher” in the model should reflect this overlap in a “single, three-pronged problem space” (p. 33), as shown in Figure 6.2.

Figure 6.2: Overlap of three arrows in Lampert’s (2001) model
Even this model of the metaphor (Figure 6.2) does not adequately represent the complexity that teachers experience. Teachers do not just interact with students at a point in time or even over one lesson. Teachers build a culture of learning and studying over an extended period of time. These acts (such as the act of culture-building) have little meaning when seen at a particular instant in a lesson but form a part of the teacher’s practice when seen over a protracted period. Also, the “content” that teachers teach should not be seen in isolation from the wider curriculum. Often, a teacher includes (or excludes) particular content in class in the light of its relevance and connection with the mathematics that had been done or needs to be done later. So, in describing a teacher’s work on content, the connections in content over time must also be featured. Thus, a temporal dimension must be included in the model. This is the reason why Lampert chooses the metaphor of navigation on an aircraft, rather than on a ship. The aircraft movement in three-dimensional space (in contrast to a ship’s navigation in two-dimensional plane) emphasises the multi-dimensional complexity in teaching. The additional dimension—time—is therefore included in Figure 7 to highlight the temporal element in the work of teaching.

![Figure 7: Instructional triangle viewed through the navigation metaphor (revised)](image)

A final word about metaphors

At this point, there is a temptation to merge Figures 4, 5, and 7 to provide a ‘comprehensive’ picture of the work of teaching, but the author chooses to reject this temptation. Instead of merging, the author views the different metaphors as affording
different languages to describe teaching and different lenses to view practice. Keeping the model simple also allows each analytic frame to be manageable while keeping the realistic complexity in view through the use of multiple frames instead of one complicated frame. Lampert shares this caution about over-complexifying the model:

If the elaborations of the teacher-student-content relationship I have represented in each part of this chapter could be integrated, we could see the many practice arrows along which the teacher works. And I have surely not identified all of them. Even as I know that the model must be further revised and elaborated, I recognise that this endeavour is dangerous because it forces us back on the question of whether teaching … is an impossible task. (p. 448)

2.1.6 Summary of Section 2.1 and Relevance to the Study
The literature reviewed above points to teaching as a very complex activity. While existing theoretical perspectives cast useful light onto the various aspects of teaching, studies that are based on unifocal views of the work of teaching may miss some vital interacting complexities. For this reason, this study takes a cross-disciplinary approach that avoids locking into any one traditional theoretical stream or falling into the narrow binary view of choosing one particular teaching approach over another. In developing a methodological framework to investigate the research questions, the author takes an approach that incorporates multiple theoretical lenses—such as constructivism and sociocultural perspectives—within its fold. Similarly, in considering teaching methods, there will be a movement away from simplistic casting of contrastive teaching modes—such as calculationally-oriented vs conceptually-oriented. Rather, the conception of quality teaching is one that can contain elements of these various instructional types.

The instructional triangle model presents a cross-disciplinary first step for studying the complexity of teaching. The different metaphors, when presented as elaborations of the triangle model, provide different language and ways to highlight the underlying complexities. Insofar as this study is about problems of teaching and coping strategies, the metaphor-enriched triangle model provides a useful template to see where some problems of teaching can arise and where coping is needed—such as
along the respective practice arrows. The task in this study is to build on the basic elements depicted in the model to uncover deeper complexities-at-work underlying problems and coping strategies in the classroom.

2.2 Teaching geometry

As identified in the research questions, this study zooms in from the view of “teaching mathematics” to a focus on “teaching geometry”. There is thus a need to look at the literature specifically about geometry education. Because studies and reports in this field are varied and numerous, the author limits his survey here to those findings that attract widespread support and that relate directly to the theoretical basis used in this study.

2.2.1 The van Hiele Levels Theory

In the 1950s, Dutch researchers Dina van Hiele-Geldolf and her husband Pierre M. van Hiele proposed a theory of students’ development of geometric thinking based partly on their experiences as high school geometry teachers. The van Hieles’ central motivation was to analyse students’ failure to understand the deductive proofs to theorems, noting that they often resorted to rote-memory of written proof steps. The van Hieles felt that the mode of geometry teaching prevalent in their days was pegged at a higher level than the one in which most students were operating. The theory highlighted the need for teaching to take into serious consideration the entry level of students.

The van Hieles (1986) conjectured that students begin learning geometry with a gestalt and “real-world” approach where geometric figures are viewed holistically, without paying explicit attention to analytic features. This type of geometric thinking develops, with the help of appropriate tasks constructed by the teachers, to higher levels of abstraction and to realising geometry in the context of a mathematical and deductive system. The original work of the van Hieles suggested that learners proceed through five levels of development in geometric thinking. As these levels are now widely known in the literature, only a brief outline is given here (further details are in Clements & Battista, 1992; de Villiers; 2004, Gutierrez & Jaime, 1998; Hoffer, 1983; van Hiele, 1986). Students who recognise geometrical objects only by shape
and gestalt features are operating at the “Visual” or “Recognition” level. When students can view a geometric object as a holder of properties, they enter the “Analysis” stage. Students who can make deductive relationships among properties are said to have reached the higher “Ordering” or “Relational” level. At the “Deduction” level, students construct formal mathematical proofs. The van Hieles also proposed a further “Rigour” level but it is beyond the scope of secondary geometry.

A theory so general in nature and capable of “wide applicability” (Usiskin, 1982, p. 6) is bound to attract intense research. Among the more prominent projects in the USA are the Chicago University project (Usiskin, 1982), the Oregon State University project (Burger & Shaughnessy, 1986), and the Brooklyn College project (Fuys, Geddes & Tischler, 1988). Empirical research has generally borne out that the van Hiele level theory is a good model to describe the development of geometric thinking in students from elementary to college levels (Clements & Battista, 1992). In Singapore, Hang (1993) did a study with students ranging from years 7 to 12. Applying a Rasch model to his data supported the hierachical nature of the van Hiele levels of geometric thought. That the van Hieles’ levels are viewed as having continuing relevance in research and teaching can be seen from recent studies that were based on its theoretical framework (e.g., Halat, 2006; Hartweg, 2005; Groth, 2005; Wu & Ma, 2005).

2.2.2 van Hiele Theory and the Geometry Curriculum

The influence of the van Hieles’ theory on the Dutch Mathematics curriculum was significant. According to the theory, geometry education should take into account students’ natural and informal approaches to geometrical objects—such as rectangles as doors and alternate angles as saws (Fuys, et al., 1988). This implies that teaching should begin with what is experientially ‘real’ to the students. Gravemeijer (1998) attributed the recent Realistic Mathematics Education (RME) movement in the Netherlands partially to this aspect—starting with what is experientially real² to students—of the van Hiele theory.

² The “real” in RME, as clarified by Freudenthal (1991), is not restricted to what the physical senses experience in the space-time world. Freudenthal’s idea of what is reality to an individual encompasses notions that the individual “uses unreflectingly” (p. 17). It is through this definition of ‘reality’ that he
In Singapore, the geometry curriculum reflects heavy influence from van Hiele levels theory (MOE, 2006). Geometry is taught at every grade level throughout Primary and Secondary education (that is, from Year 1 to Year 10). At the earlier grades (Primary 1 and 2), the students are introduced to geometry via shape-naming and shape-differentiation exercises. The focus is on gestalt (shape recognition) and topological (curved vs straight boundaries) impressions. Formal geometric language and properties enter into the syllabus from Primary 3 and proceeds with greater complexity through Primary 6. The focus during these middle grades is on the attributes (angle, perpendicularity, parallelism) of triangles and quadrilaterals. In addition, the use of a compass as a geometrical instrument for construction is introduced at Primary 5.

In the Secondary grades, the coverage of the geometrical domain is wider and there is gradual shift away from merely knowing geometrical attributes towards reasoning about the relationship among these attributes. An example of this movement can be seen from the treatment of “special quadrilaterals”. Properties of special quadrilaterals are taught in the Primary levels. In the Secondary syllabus, the emphasis gradually moves to relationship between the special quadrilaterals. To understand the hierarchical relationship between special quadrilaterals, students need to know not merely the properties of each quadrilateral but also the relationship among their properties. In addition the Secondary geometry syllabus also includes other approaches not previously introduced in the Primary geometry syllabus—such as the synthetic approach (as in simple proofs for congruency and similarity), the analytic approach (as in co-ordinate geometry), and the vector approach (as in addition and subtraction of vectors) taught in the later Secondary grade levels (from Secondary 2 to 4).

Overall, the progression in the Singapore geometry curriculum from shape recognition (Primary 1-2) to identification of attributes (from Primary 3-6), and then spoke of an “expanding reality”, where mental objects (such as mathematical ones) that are previously unfamiliar can, through repeated acquaintances, be brought under its fold. The author shares this definition of ‘real’.
to relationships and diverse approaches (Secondary 1-4) corresponds roughly to the first three levels of the van Hiele theory of geometric development.

2.2.3 The Role of Language in Geometry Teaching
The importance of language acquisition in geometry learning is borne out within the van Hiele theory. Characterizations of each level are based on students’ geometric “objects of focus” (Hoffer, 1983) and this is in turn revealed by the language students use. It is well-documented that language structure is a critical factor in the movement through the van Hiele levels (e.g., Burger & Shaughnessy, 1986; Fuys, et al., 1988; Mayberry, 1983). Students’ reasoning at the beginning level is marked by imprecise language in describing geometric figures and their properties. Examples include the use of “straight lines” to refer to parallel lines and “diamond” for rhombus. Teachers may begin by using everyday language as a point of reference, then proceed to “carefully draw distinctions between common usage and mathematical usage” (Clements & Battista, 1992, p. 433), and later focus their attention on appropriate language to describe parts of figures and their properties (Fuys et al., 1988). This acquisition of appropriate language will help the students abstract and reason with higher level mathematical content.

The need in geometry teaching to move progressively from everyday informal language to increasingly formal geometric terms leads one to question the traditional axiomatic approach to the teaching of geometry. In this approach, popularized by Euclid in his Elements, instruction proceeds directly at the formal level from statements of postulates and definitions and from these proofs of theorems are demonstrated via deductive reasoning. In light of the van Hiele theory, instruction that begins with formal definitions is unlikely to result in student success. A number of studies (e.g., Hershkowitz, 1989; Mitchelmore & White, 2000) indicated that the mere presentation of definition did not make any significant impact on the learning of the targeted concepts.

2.2.4 Geometrical Figures and Concept-Images
The term “concept” is now commonly used in the literature by researchers of diverse traditions and there is no general agreement on what it means. The author will use Fischbein’s (1993) characterization of concept as “an idea, a general, ideal
representation of a class of objects, based on their common features” (p. 139). Based on this definition, geometric figures are mathematical concepts (Hershkowitz et al., 1991). Unlike other mathematical concepts, it is hard to conceive of geometrical concepts (such as triangles, quadrilaterals) in purely verbal modes, as there is a close association between school geometry and spatial objects (Clements & Battista, 1992). There is a long tradition of using figural representations in geometry teaching. Concept formation of geometrical figures involves mental images related to the concept. Fischbein (1993) defined “image” as “a sensorial representation of an object or phenomenon” (p. 139). Thus, the “mathematical concept” and “mental image” duality with which students come to conceptualise and view geometric figures has led Hershkowitz et al. (1991) and Tall (1989) to propose the idea of “concept-image” to depict one’s mental representation of a geometric figure. This construct of concept-image that highlights the strong link between the visual representation and the underlying concept is now well-used in the literature (Clement, 2001; Ouvrier-Buffet, 2006; Schwarz & Hershkowitz, 1999).

Vinner and Hershkowitz (1983) conducted a number of studies involving elementary and middle school students on the development of concept-images of geometric figures. They consistently found that most students have a predominant image strongly associated with each geometric figure. For example, the right-angled triangle was almost always pictured with the ‘legs’ forming the right angle in horizontal-vertical positions. They termed this the “prototype phenomenon”.

The prototype, however, because of its overly-constrained nature, does cause difficulties for students in the development of their concept-image. From the prototype, the students often perceive as critical some non-critical attributes (such as orientation) not inherent in the concept. As such, it was also found that wrong judgments of examples and non-examples of concepts were based on students’ adherence to the prototype, despite teachers having provided them with a verbal definition of the concepts. In order that students form concepts correctly, there would be a need to change the frame of reference from that of the prototype to that of critical attributes of the concept (a movement from the basic van Hiele level to higher levels). The implication for instruction is that teachers should provide a learning environment
where students “construct a meaningful synthesis of this definition [the concept] with a range of exemplars” (Clements & Battista, 1992, p. 447).

### 2.2.5 External Visual Representation in Geometry Teaching

The role of visualisation in students’ development of concept-images is not a straightforward matter. On the one hand, we cannot expect students to form an image of a concept and its examples without visualizing its constituent elements; on the other hand, these visual elements may narrow the concept-image and encourage the prototypical phenomenon (Hershkowitz, 1989). Despite these difficulties associated with the use of visual representations in geometry teaching, it is inconceivable to think of the teaching of geometry completely devoid of representative diagrams. The challenge is in harnessing the useful functions that external representation can perform in teaching geometry.

Mesquita (1998) proposed two main functions for the use of external representations in geometry problems. The first is descriptive and the second is heuristical. To her, “an external representation is descriptive when its sole function is to give a synoptical apprehension of the properties mentioned in the problem statement” (p. 191). She gave an example of a task that required a comparison of the lengths of the semicircles in the context of a problem. Figure 8 is his illustration of an external representation that performs a purely descriptive role as it does not in itself point to solution approaches.

An external representation goes beyond being descriptive and becomes *heuristical* when it “acts as a support for intuition, suggesting transformation that leads to solution” (Mesquita, 1998, p. 191). She gave the example where the problem (as shown in Figure 8.1) required students to show that rectangles 1 and 2 have the same area. Mesquita posited that the diagram in Figure 9.1 serves a heuristical function because it lends itself to the solution: as illustrated in Figure 9.2, the big rectangle is composed of two right angled-triangles of equal area; within each of these right-angled triangles there are three component parts; since two of these corresponding parts can be seen to be of the same area, the remaining corresponding parts (rectangles 1 and 2) are also of the same area. Mesquita maintained that in whichever
of the above roles external representations play, they are useful tools for students in problem-solving.

![Diagram of external representations serving a descriptive function](image)

**Figure 8: External representation serving a descriptive function**

![Diagram of external representation serving a heuristical function](image)

**Figure 9.1: External representation serving a heuristical function**

![Diagram showing the decomposition of a rectangle](image)

**Figure 9.2: Decomposing the rectangle in Figure 9.1**

Although few people question the need to use visual representations to help students build concept-images and to solve problems, some researchers have cautioned against their indiscriminate use in geometry teaching. Mesquita (1998) saw the use of diagrams as supporting intuitive approaches to geometry. However, she cautioned that over-reliance on diagrams can stunt students’ progression beyond figure-based reasoning to more formal geometric reasoning. Re-interpreted through the van Hiele levels theory, Mesquita’s concern was that students’ purely visual-based learning could prevent their progression beyond the first level of geometric thinking. In reporting the Soviet researcher Kabanova-Meller’s work in the use of diagrams in
geometry learning, Clements and Battista (1992) also cautioned that when learning theorems in geometry, students often incorporated information contained in a specific diagram as part of the properties or givens for the proof. They cite the example where students thought an exterior angle was necessarily obtuse because the diagram in which the theorem (of an exterior angle being the sum of the interior opposite angles) was presented showed an obtuse exterior angle. For students to understand and correctly apply the theorem, they would need to alter the mental image (corresponding to the one illustrated in the theorem) to abstract the condition set forth in the theorem.

For these reasons, Hershkowitz (1989) urged that in geometry teaching, students should be provided with “an as-complete-as possible ‘example world’” (p. 75). A suitable range of exemplars of a concept can reinforce the critical attributes of the concept while highlighting the non-essential nature of non-critical ones. Figure 10.1 illustrates what might be appropriate examples to teach the idea of “cyclic quadrilateral”.

![Figure 10.1: Examples of cyclic quadrilaterals](image)

Hershkowitz et al. (1991) also found that non-examples can also be useful. They encouraged students to generate positive examples of concepts and use their errors as non-examples. Continuing the illustration of “cyclic quadrilateral”, non-examples may look like the ones presented in Figure 10.2.
2.2.6 Summary of Section 2.2 and Relevance to the Study

As indicated in the research questions, this study is about teaching in a geometry classroom. Since the van Hiele theory was originally formulated (and still widely used today) to address issues specific to geometry teaching, it is of direct relevance to this study. The theory is applied here in a number of ways: the progression of the stages guides the design of the geometry curriculum that is used in this project; the need to take students’ van Hiele level into consideration when carrying out instruction is taken as a given; language development is seen as a vital part of geometry instruction; helping students attain higher levels of geometric thinking is a pedagogical goal.

The literature about geometrical figures as concept-images and the occurrence of prototypical phenomena indicate that the use of external visual representation in teaching geometry is an issue that teachers need to consider carefully in their classroom instruction. As such, pedagogical applications (such as the use of a suitable range of examples to illustrate a geometrical concept) are incorporated into the preparation and planning of the teaching materials used in this study. In particular, the design of the DGS files takes into consideration issues about external representation discussed earlier. This will be re-examined further under the heading on DGS in a later part of this chapter.

2.3 Teaching Reasoning in a Geometry Classroom

In addition to teaching geometry, this study also focuses on teaching that supports students’ participation in mathematical reasoning. Since the focus is on teaching
geometry, the scope of the literature on mathematical reasoning will be narrowed to the domain of teaching reasoning in a geometry classroom.

### 2.3.1 Mathematical reasoning

Because formal mathematics is commonly presented in written discourse as a set of sequential logical steps, mathematical reasoning is often associated with deductive rigour. Each step of the argument requires justification; warrants for justification include well-established logic rules, definitions, and theorems. The most ancient and well-known publication that illustrates the use of such rigour in mathematical reasoning is the *Elements* whose authorship is traced to Euclid who lived around 300 B.C. Since then, the abstract geometrical world that Euclid constructed (now widely known as Euclidean geometry) continues to be taught in schools today (Clements & Battista, 1992). This points to the importance that deductive reasoning still holds in mathematics teaching and learning.

Polya (1954) was one who extolled the value of deductive reasoning. However, he argued that deductive reasoning does not tell the whole story about the labours of mathematicians. Behind the formal mathematics—presented in all its completeness, certainty, and deductive clarity—real mathematicians experience many false starts, dead ends, and wrong destinations. Polya asserted that the making of a mathematically-literate person should include such experiences of what he called “plausible reasoning”. It includes hunches, observations, and conscious guessing. Lampert (1990) argued that this side of the mathematician’s work—plausible reasoning—should be an important part of learning within the discipline of mathematics but is noticeably lacking in mathematics classrooms today.

Lakatos (1976) continued the Polyan spirit of upholding the importance of plausible reasoning. He imaginatively re-created the fictional discourses surrounding Euler’s formula relating faces, edges, and vertices on polyhedra. Along the way, he demonstrated that producing mathematical propositions goes through a process of conjecturing, refutation, testing, re-conjecturing, and refining before arriving at theorems-to-prove. He labeled the process as “inductive reasoning” and the teaching
approach that encourages students to participate in the joint-conjecturing as the “inductive approach of teaching”.

The two components of mathematical reasoning—deductive and inductive—were given an interesting metaphorical illustration by Hersh (1997). Just like the front of a restaurant that shows the presentable and orderly side of the business, deductive reasoning is the formal ‘side’ of mathematics, and, like the restaurant whose ‘back’ is a flurry of activities and disorderliness, inductive processes are the essential ‘back’ of mathematics.

The Singapore Ministry of Education views both deductive reasoning and inductive reasoning as important in geometry teaching. In the curriculum document elaborating the secondary geometry syllabus, it was advocated that “… concepts are introduced visually, experientially, inductively, and then deductively” (MOE, 2006, p. 20, emphases added).

There is perhaps a need to make an important clarification at this juncture. While the use of the term “inductive reasoning” is useful in highlighting an important aspect of mathematical reasoning that is easily missed out when the ‘front’ of mathematics—the deductive rigour—takes centre stage, there is no claim that the inductive and deductive elements of reasoning are identifiably separate processes in the work of doing mathematics. The work of mathematicians involves both these processes and they are often harnessed simultaneously and inseparably when attacking a mathematical problem. Herbst (2006) pointed out that “conjecturing and proving are concurrent in the actual work of mathematicians” (p. 317, emphasis added). The author’s continued use of the labels of “inductive reasoning” and “deductive reasoning” is not to imply a strict dichotomy of these processes. Rather, the purpose is to highlight the emphasis of each of these processes in the work of teaching mathematical reasoning.

2.3.2 Classroom conditions that support mathematical reasoning
Lampert (1990), inspired by Lakatos’ (1976) quasi-empirical approach to mathematics instruction, advocated classroom pedagogies that mirror the type of
mathematical *inductive* reasoning used by practitioners in the professional mathematics community. She noted that, too often, mathematics in the classroom is seen by students as a set of unchallenged facts and rules handed down by an authority to be remembered and applied. Such a representation of mathematics is at variance with the way mathematicians make sense of mathematics. She argued against the prevalent existing dysjunction in the conception of mathematics and urged teachers to conduct classroom instruction in a way that encourages students to practice the inductive reasoning processes that mathematicians experience—experimenting, observing, making generalizations, conjecturing, refuting, and re-conjecturing.

Other researchers have focused on the use of students’ discourse as a way to encourage argumentation, use of warrants, and justification—reasoning processes that carry a strong *deductive* flavour. Yackel and Cobb (1996) studied the development of “sociomathematical norms” as taken-as-shared negotiated meanings in the classroom. The “social” part of “sociomathematical” is a recognition of learning activity as a social activity, that teaching is partly an acculturation of students into social norms that fulfil the goals of schooling. The “mathematical” part of “sociomathematical” is to distinguish this activity from other classroom acts of socialization. The authors argued that there are norms that are unique in mathematical discourse. Students are to participate in classroom mathematical discourse that approximates the norms that are closer to that of deductive reasoning within the mathematics community. They urged that “teachers initiate the development of social norms that sustain classroom microcultures characterized by explanation, justification, and argumentation” (p. 460). However, in Reid’s (2002) research into the reasoning patterns of grade 5 students in a mathematics class, he found only partial manifestation of deductive reasoning present in their discourses. This is a reminder that, while deductive reasoning is indeed highly valued in mathematics, “[i]t is perhaps not reasonable to expect children of this [young] age to be operating from a completely mathematical … orientation” (p. 26).

While the inductive/deductive labels are useful to highlight the differences in the work of mathematical reasoning, Chazan (1995) cautioned against the emphasis of one to the exclusion of the other in the classroom. He argued that to expect students
to engage in conjecturing without also helping them determine why those conjectures were true or false can lead to students depending on the sole authority of the teacher to decide on whether conjectures are correct or wrong. Herbst (2006) also questioned the treating of inductive and deductive processes as separate activities in the classroom. As such, students should be encouraged to form “reasoned conjectures”, and to engage with “problems that summon the ideas that would allow them to formulate and conceive the sources for the reasonableness of the conjecture—that is, the arguments that make it true—not just to perceive the conjecture as a fact” (p. 318, emphasis in the original).

While it is crucial that students be actively involved in the classroom reasoning process, the role of the teacher is also very important. According to Martino and Maher (1999), teachers contribute to the reasoning process by modeling it so that students can gradually accept that careful listening and exchange of ideas is the accepted mode of communication within practitioners in the mathematics community. Teachers can also use questioning techniques to “promote justification and generalisation on the part of students” (p. 55). They even proposed a sequence for using questioning. Students should first work in groups or pairs to share ideas and solutions “without any teacher intervention” (p. 56, emphasis in the original). Since students do not naturally seek justification or validity of their solutions, teachers can then prompt for students’ justifications by way of questions “after students working alone or together have taken their ideas as far as they can” (p. 56, emphasis in the original). Further “timely questioning” in the form of asking students to explain their solutions or inviting them to consider the justifications provided by other students can help advance students’ reasoning.

Apart from questioning, teachers also play a central role in guiding whole-class mathematical discourses. McNair (2000) noted that teachers’ directions and utterances “provide purpose, structure, and coherence for a discussion”. Executive guidance from teachers is necessary because the plurality of students’ contributions can take whole-class discussions into diverse and conflicting pathways. While alternatives are useful for reasoning, teachers also need to steer the discourse to match the curriculum agenda (Lampert, 1988). This role of teachers in facilitating
mathematical discussions by “legitimis[ing] certain aspects of [students’] mathematical activity and implicitly sanction[ing] others” (p. 466) is also acknowledged by Yackel and Cobb (1996).

Apart from questioning and charting the course of discussion, teachers can also be co-participants in the reasoning process. Yackel (2002) pointed out that there is a “misconception … that if teachers are to support students in constructing their own mathematical understandings, the teachers must refrain from contributing mathematical ideas to the discussion(" (p. 424). Rather, she advocated a proactive role for teachers in guiding mathematical discourses and even in explicating the argumentative basis for claims that students make. This view of teachers assuming an essential role in classroom mathematical reasoning resonates with the metaphor of teaching as managing intellectual fermentation used by Ball and Chazan (1999). Although teachers cannot directly control students’ learning, they can be catalytic in stirring and directing the fermentation process in classroom mathematical reasoning.

2.3.3 Visualisation in Geometrical reasoning
The use of external visual representations is an indispensable part of geometry teaching. When used appropriately, they can help students strengthen their concept-images of underlying geometrical ideas. Visual representations can also help students better ‘see’ solutions and explanations. It is not immediately clear, however, whether the use of visual representations in aid of the reasoning process is more aligned to deductive or inductive reasoning. The author’s position is that visual approaches can overlap both strands. The reason for this stance will be made clear through the literature reviewed in this section.

Visual diagrams are useful in facilitating solution approaches to geometrical problems. Duval (1998) gave the example of the role of using visuals in the classic square-within-a-square proof of the Pythagoras theorem (see Figure 11), where reorganisation of the pieces in the diagram is integral to the proof. He claimed this process “gives to vision its heuristical power in problem-solving” (p. 43).

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3 This understanding is likely drawn from a simplistic interpretation of constructivism.
4 Many definitions of “visualization” are offered in the literature. The author uses the term in a general sense to refer to the use of visual images (whether external or mental) in problem-solving or reasoning processes.
While visual representations like Figure 11 are credited with “heuristical” (Mesquita, 1998) functions, there is generally a distinction between such visual approaches to problem-solving and the formal deductive process which is seen as more mathematically conventional. Duval (1998) made a clear distinction between such visual processes (what he termed “purely configural processes”) and formal deductive processes (which he termed “discursive processes”). This view implies that when it comes to judging formal mathematical acceptability, visual approaches are placed at a level of intuition, pre-logic, and uncertainty; whereas written/verbal reasoning is placed at the level of formality, rigour, and certainty. This separation of visual processes and discursive processes into two categories appears to be borne out by teachers as they carry out classroom instruction. Laborde (2002) noticed in her research that teachers who had been using visual images for students to come up with conjectures of geometrical properties switched to conventional methods when asked by students to justify the conjectures. It appeared that these teachers believed that visuals played a role of providing an intuitive initiatory route to solving problems before the “main game” of conventional deductive justification. Using the inductive-deductive pair, visual processes seem more closely associated with inductive reasoning.

That visual processes are implicitly excluded from deductive reasoning seems also to be borne out by the van Hiele levels. Students who are visually-based in their recognition of geometrical objects operate at the first (Visual) level. Deductive
reasoning begins at the third (Relational) level. At that stage, students’ objects of focus are no longer concrete geometrical shapes but relations between geometrical properties. Thus, visualisation does not appear to be acknowledged in higher level geometrical (deductive) reasoning.

Some researchers, however, reject the visual process/deductive reasoning separation. Hershkowitz (1998) disagreed with Duval (1998) concerning the separation of visual processes from deductive reasoning processes. She also disagreed with the traditional view of visualisation as playing a supportive role that provides merely an initial route to a problem but not necessarily the solution process. To her, visualisation is much more than that:

It includes many, if not most of the aspects attributed to other kinds of reasoning, including analytical aspects, even proving. … [It] can function by itself in order to complete a rigorous mathematical argument or [be] blended with other kinds of reasoning, not necessarily as a preliminary to them. (p. 34, emphases added)

Hershkowitz, Arcavi, and Bruckheimer (2001) demonstrated such roles of visualisation in solving a ‘matchstick problem’. The authors presented in-service teachers with the problem, “how many matches are needed to build a n×n square” and classified the solutions obtained. The solutions were broadly categorized under “numerically driven solutions” (p. 256) and “visually driven solutions” (p.257). The latter were further subdivided into visual strategies used. The main approach was via decomposition and recomposition, although the decomposed unit varied. Some took a unit to be a 1×1 square, others one matchstick and still others into ‘U’’s (3 matchsticks) and ‘L’’s (two matchsticks). The authors claimed that the students’ visual strategies played a prominent and central role in helping the search for patterns. They further asserted that “visualisation … becomes a generalisation which includes its justification, and thus becomes a ‘proof that explains’ (Hanna, 1990)” (p. 264).

5 A “proof that explains” refers to visuals that sufficiently show the proof strategy without the use of additional textual elaborations.
Using the strategies for the matchstick problem examples, Hershkowitz (1998) stated that the visual processes provided (p. 36)

1. A new way of looking at the situation in order to suggest a generalisation,
2. Proof and verification in one process, and
3. An explanation of ‘why’ the generalisation holds.

For this reason, Hershkowitz (1998) raised the status of visualisation to that of “visual reasoning” and thus alerted mathematics educators to the danger of belittling students’ visual approaches as non-mathematical. Also, it reminded teachers of the need to see these approaches as not merely initial forays but as inseparable from or leading to deductive reasoning.

Weighing the contributions on this subject, the author takes the moderate view that “visual reasoning” contains features of both inductive and deductive reasoning. Visualisation involves some exploratory uncertainty (at least initially) that associates it more with inductive processes; but when visualisation reveals the underlying relationships, then all the explanations and justifications become immediately apparent, thus giving it a deductive flavour. This duality potentially places “visual reasoning” in a unique position of linking inductive and deductive approaches in teaching.

2.3.4 Summary of Section 2.3 and Relevance to the Study

One of the foci of this study is to examine teaching in a classroom that encourages mathematical reasoning. As highlighted in the literature surveyed above, both strands of mathematical reasoning—inductive and deductive—are important in mathematical work and should be valued in the mathematics classroom. Apart from teaching deductive rigour, such as providing formal justifications and warrants to statements, instructional work should also focus on encouraging inductive approaches, such as encouraging students to articulate preliminary observations, conjectures, and refutations. Teachers play a crucial role in modeling reasoning in classroom discourses, use questions to ‘stir the intellectual ferment’, and guide the reasoning

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6 A clarification in the order of the dates: Hershkowitz (1998) made a brief reference to the matchstick problem. The details of the problem and how in-service teachers solve the problem were reported in Hershkowitz, Arcavi, and Bruckheimer (2001).
process as co-participants. These pedagogical principles are employed in the teaching project of this study.

The author takes the view that “visual reasoning” can contribute to both the inductive and deductive aspects of reasoning. This in-between position enables visual processes to play an important role in encouraging both strands of reasoning in the classroom. This point will become clearer when visual reasoning is revisited in Section 2.4 on the use of DGS.

For some time, the main emphasis in mathematics classrooms in Singapore has been on skill and technique acquisition to meet the demands of testing. [More discussions on this are found in Section 2.5]. Impelled by recent reform initiatives, the shift towards teaching reasoning is in its beginning stages. As such, there is little research in this area. Thus, a study on the problems and coping strategies involved in teaching geometric reasoning is of particular relevance to the Singapore mathematics community.

2.4 Teaching Geometry using Dynamic Geometry Software

Another focus of this study relates to the use of Dynamic Geometry Software (DGS) in teaching geometry in a way that supports students’ mathematical reasoning. The term “Dynamic Geometry” was originally coined by Jackiw (1991) to refer to the dynamic visuals generated by a particular geometry programme he designed. Since then, DGS has been used to describe a class of software that presents dynamic images and allows users to manipulate objects on screen while retaining geometrical properties in-built within the objects. Members of this DGS family include Cabri™, Geometer’s Sketchpad™, Geometry Inventor™ and Geometric Supersusposer™ (Goldenberg & Cuoco, 1998).

Although these geometry construction programmes share common features of dynamic visuals, each of them is different and was designed for different purposes (Chazan & Yerushalmy, 1998). It is beyond the scope of this study to examine the specific didactic relevance of unique features of each type of DGS. Since the
**Geometer’s Sketchpad™** (henceforth referred to as Sketchpad) is the only DGS used in Singapore schools\(^7\), the focus of the literature review will be on features of DGS within Sketchpad.

### 2.4.1 Features of Dynamic Geometry Software Present in Sketchpad

One feature of DGS is that they come with a toolset of geometric primitives that the user can draw upon to build more elaborate on-screen objects. In the case of Sketchpad (and also Cabri™), the primitives appear in the form of a menu of buttons where composite objects made up of the primitives can be formed. Unlike static diagrams or drawings from software like MacDraw™, visual primitives can be controlled by the computer mouse and manipulated directly. In Figure 12, which is a snapshot of a Sketchpad screen (from version 4), the triangle ABC (formed by using the “line segment” primitive shown on the toolbar on the left) is seen, through the trace lines, to be easily manipulable by clicking-and-dragging the mouse on vertex C. This movement of screen objects by the click-and-drag of the mouse is known as “dragging”.

The direct manipulability of the software contributes to its ease of use. Of greater significance, Chazan and Yerushalmy (1998) also proposed that this feature provided the “tangibility of geometrical abstractions, … because, like the objects in the world from which [geometrical objects] are abstracted, the diagrams [in DGS environment] can be acted upon” (p. 81). The ease in manipulating the onscreen objects renders the underlying abstract geometric figures more tangible and real to the user.

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\(^7\) Via active introduction by the Ministry of Education in the late 1990s, Sketchpad is the first entrant into the Singapore mathematics education scene and has since become the only DGS known to most mathematics teachers. Frequent references to Sketchpad use in geometry are found in most commonly-used secondary mathematics textbooks (e.g., Teh, K.S. & Looi, C.K., 2001/2002; Sin, K.M, 2001/2002). In these textbooks, Sketchpad is taken to be the only DGS and no other software types are mentioned.
Another feature of DGS programs that distinguishes them from other drawing software is the ability to build into the objects geometrical properties that stay invariant upon dragging. These are known as “drag-resistant” figures (Hoyle & Noss, 1994) or “Cabri” figures (Laborde, 1995). Since the latter terminology is associated to a particular type of DGS, the author prefers the former. A drag-resistant square, for example, is one that retains all the critical attributes of a square—such as perpendicularity of adjacent sides and congruence of all sides—while other non-critical attributes such as size and orientation may vary upon dragging. To build such an onscreen object, the software provides options which include construction of parallel and perpendicular lines, and transformations such as translation, reflection, rotation, and dilation. The procedure of building a drag-resistant figure is known as “constructing” (Hoyle & Noss, 1994), to distinguish it from “by-eye drawing” (Laborde, 1995) which will not yield objects that withstand the “drag test”. Figures 13 and 14 illustrate the difference between a constructed figure and a by-eye drawing.
What is “dynamic” in DGS is not restricted to the movements of points, lines, and other constructed objects. Some DGS programs, including Sketchpad, also allow dynamic numerical feedback (Chazan & Yerushalmi, 1998). In the case of Sketchpad, numerical measures of objects can be displayed on-screen. Measures can be taken on angles, lengths, and areas. Simple algebra can also be performed on these measures. Figure 15 illustrates the measures of the interior angles of a triangle and the sum of these measures. These numerical displays are dynamic in the sense that, as one drags any part of the triangle and moves it, the measures on screen are updated, according to the relational changes of the screen objects.
Other features of DGS include the high degree of precision in the construction tools and the numerical feedback compared with conventional construction and measurement tools. It is common to see inaccuracies in construction with conventional compass and straight edge even when procedures are correctly followed. The imprecision is inherent in the limitations of the geometrical tools used. In the DGS environment, however, figural and numerical outputs are of a consistently high level of precision. DGS tools not only produce precise outputs, they also demand procedural precision from the user to produce the constructions intended (Clements & Battista, 1994). To illustrate this requirement of precision by the software, the example of constructing the circumcircle of a triangle is considered. Using paper and pencil, inaccuracies in the constructions steps can be masked so long as the final circle appears to almost pass through the vertices of the triangle; but in the DGS setting, any imprecise move in the construction process cannot escape the scrutiny of the drag-test. Thus, precision has two aspects: in the output displays and in the requirement on the user.

Another obvious feature of the DGS is the speed with which it carries out the required work on screen, compared to conventional paper-and-pencil work. Unlike the laborious work of controlling a compass, aligning a straight edge, and erasing errors, these tasks can be accomplished with a few mouse-clicks in the DGS setting.

2.4.2 DGS in the Teaching of Geometry
The use of dragging has been being harnessed for geometry instruction (Hoyles & Noss, 1994; Laborde, 1995; Leung & Lopez-Real, 2001). For example, when a drag-
resistant parallelogram is dragged, students can be asked by the teacher to explore properties of the parallelogram they find invariant. Laborde (2002) called these activities “tasks for interpreting visual phenomena” and stated that in “this kind of task, children learn to recognise a geometrical property from its various spatial-graphical representations” (p. 18, emphasis added). In other words, such tasks potentially help students to shift their attention away from visually-driven images to the properties of a parallelogram. Re-interpreted in van Hielean terms, these tasks can bridge students’ lower “Visual” thinking to the next level of viewing figures as holders of properties. In addition, Laborde recommended “tasks for producing visual phenomena” (p. 20, emphasis added), where students go beyond interpreting pre-constructed figures to producing or reproducing figures and then checking their drag-resistance. To perform such tasks, Hoyles and Noss (1994) proposed that students would need to “develop the idea that one object is dependent on another” and thus “gain insight into geometric relations” (p. 717, emphasis in the original). Again, seen in the van Hieles’ scheme, such tasks to construct figures can help students move from recognition of properties to the even higher level of seeing relationship between properties.

A study by Vincent (1998) supports the above notions of DGS being instrumental in helping students progress through the van Hiele levels. She studied the development in geometric thinking of a group of twelve Grade 7 and Grade 8 girls as they were guided through a module in angle properties in a Cabri environment. The girls did not have prior exposure to Cabri. Data from van Hiele pre- and posttest scores indicated not only that learning within the Cabri environment facilitated the advancement of students’ van Hiele levels, but that the progress was rapid. Vincent reported that

The increases, which occurred after six Cabri lessons, contrast with the reported 20 lessons taken by Dina van Hiele-Geldof to bring her 12 year-old pupils from Level 1 to Level 2 … [and also the] increases, which occurred after 7 or 8 Cabri lessons, contrast with the 50 lessons taken by Dina van Hiele-Geldof to bring her 12 year-old pupils from Level 2 to Level 3. (p. 647)
The increase in van Hiele levels was further complemented by the observed development of geometric language characteristic of the levels as students communicated with each other and with the teacher. Bobango (as cited in Clements & Battista, 1992) also found that instructions based on the Geometric Supposer significantly raised students’ van Hiele levels of thinking, particularly from Level 1 to Level 2.

Apart from using DGS as tools to interpret and produce visual phenomena, Laborde (1995) observed that students also used the drag-mode “as a validity criterion of their constructions” (p. 63). By dragging, a student can know for himself if his construction was successful, without the teacher peering over his shoulder to evaluate by saying “right” or “wrong”. This encouraged the student’s ownership of the problem and the metacognitive process of re-examining his construction procedure if he failed the drag test. Instead of constantly requiring the teacher to evaluate work, DGS provided the feedback tool—the drag-mode—to motivate students’ ownership and to prompt re-attempts at tasks. Hoyles and Noss (1994) introduced a similar idea of “messing up” in their class where students check their work on Sketchpad by dragging after they consider themselves to have completed the required task. By messing up, “students would be forced to think about the construction of their picture: how the parts fitted together, the relationship that needed to be specified and those that could be left to vary” (p. 717).

2.4.3 External Representations of Geometry in DGS Setting
In the literature on external representations of geometry, there is a distinction made between abstract geometry and the external representations of geometry. The geometry that is formalised in definitions, postulates, and theorems belongs to abstract geometry. Geometrical diagrams seen through the figurative register belong to the domain of external representations. Poincaré (1902) used the terms “geometrical space (GS)” and “representative space (RS)” to describe the two respective domains. Laborde (2002) also saw a distinction between what she called “physical space” (of the world we experience) and “geometry as a theory”. In acknowledging the distinction between the geometrical ideal and the external
diagrammatic representation, Parzysz (1988) proposed to distinguish by the use of the terms “figure” for the theoretical referent and “diagram” for the material entity.

This understanding of the difference between GS and RS naturally led to the notion of the “double status of external representation in geometry” (Mesquita, 1998, p. 185). A drawing of a triangle on paper can represent “either an abstract geometrical object, or a particular concretization” (p. 186). More specifically, a triangle on paper can refer to a general triangle or a particular triangle with the length of sides as indicated in the diagram. In traditional geometry teaching, it is common that diagrams, which are really objects offering spatial-graphical features in physical space, are nevertheless used to represent general or typical geometrical objects, as if the two domains are similar. This can cause confusion to students. Laborde (2002) noted that in using a diagram-on-paper of a parallelogram to represent the theoretical parallelogram, the “graphical-spatial properties of the diagram are incidental … [but] some others are necessary, like the parallelism properties” (p. 16). She questioned the students’ ability to discern the “incidental” features from the “necessary” geometric properties by looking at drawings on paper. Mathematicians work in the GS and are able to control the information coming from the RS, but beginning students of geometry may be confused as they primarily work in the RS.

Kaput (1998), however, noted that the difficulties with external geometrical representations described above were associated with doing mathematics with “static inert media” (p. 271). To him, the subtle connection between seeing geometrical objects as particular objects (in the RS) and seeing them as representing a general figure (as in the GS) “is addressed by computer-based geometry” (p. 271). He subscribed to the potential of dynamic geometry for seeing the general figure (in the GS) in the particular DGS on-screen object (in the RS). Chazan and Yerushalmy (1998) shared this view of the general-in-the-particular potential of DGS:

The diagrams created with geometry construction programmes seemed poised between the particular and the general. They appear in front of us in all of their particularity, but, at the same time, they can be manipulated in ways that indicate the generalities lurking behind the particular. (p. 82)
In Parzysz’s (1988) terms, DGS offers a way for students to see the “figure” while working with the “drawing”.

Laborde (1995) also stated that in the DGS environment there is a close association between the visual and the geometrical aspects of the objects shown on screen. Because the objects on screen are constructed from geometrical primitives, they follow the geometrical ‘laws’ that are in-built by the constructor. So, as the user manipulates the objects on the screen, the figure lurking behind the object constrains the object to move in a certain way. For the user to interpret the movements as he manipulates them, geometrical knowledge is required. Because of this visual-geometric link, she hypothesized that “by designing specific tasks in a software environment, … it is possible to promote the learning of geometrical knowledge as a tool for interpreting some visual phenomena, explaining, producing and predicting them” (p. 44). Thus, it appears that DGS, as compared to traditional media, provides a greater promise in connecting external representations to the intended geometrical ideas.

2.4.4 DGS in Supporting Reasoning
Another much-discussed potential of DGS is its suitability as a tool for encouraging students to learn by exploration and experimenting (Chazan & Yerushalmy, 1998; Clements & Battista, 1994; Lampert, 1988; Laborde, 1995; Leung & Lopez-Real, 2001; Olive, 1998; Vincent, 2002), and thus useful in supporting inductive reasoning. A conducive experimental environment is one where students can make observations leading to conjectures, test conjectures, modify conditions quickly, and retest conjectures. All these are supported by DGS features. Accurate observations of precise data—both graphical and numerical—are possible; testing of conjectures can be done using the drag-mode; changes on screen can be done quickly by retracing the steps using editing functions; and modifications can be performed by clicks on the onscreen menu. This cycle of “observe-conjecture-test-observe” can be performed tirelessly by the computer and provides support for students’ experimental approaches.
Leong’s (2002) study of a class of Singapore students who spent a considerable time learning school geometry with Sketchpad provided evidence for the potential of DGS in aiding students’ experimental inquiry. Selected students from the class were interviewed and asked for their thoughts on the exploratory approach of learning with the software. All of them liked the new experience, with one student providing an interesting insight into the way the software had changed his approach to learning:

With paper-and-pencil constructions, I’m more careful – I think carefully first before doing, otherwise [I] need to erase again which is very troublesome …, [but] exploration [on the computer] is better because I can learn from failures and know what not to do next time. (pp. 103-104)

In other words, while the traditional paper-and-pencil mode requires a high degree of certainty prior to proceeding, the software provides students with the tools to test things they are yet uncertain about, thus encouraging the enterprise of experimenting and conjecturing.

DGS can also be used as a springboard for deductive reasoning. Not only does DGS provide a plethora of flexible visual tools, the behaviour of the onscreen objects upon dragging can prompt users to find reasons why the screen objects behave the way they do. This seeking after justification can lead to deductive reasoning to explain the visual phenomena. Thus, DGS not only provides the animated visuals, it can be harnessed to link the visuals to the intended geometric reasoning (i.e., deductive reasoning about the underlying geometrical ideas involved).

One way where the visuals on the DGS screen can be linked to geometric reasoning is when the drag-mode reveals ‘failures’. Laborde (1995) conducted an experiment with grade 8 and grade 9 pupils who were given a specific construction task in Cabri. She noted that the students used the drag-mode to validate the constructions that they had done. If the visuals indicated that the construction failed the drag-test, students would be led to re-consider the geometrical reasoning behind their constructions. This drag-check-reason-amend-drag reasoning cycle took place throughout the entire process. Hoyles and Noss (1994) similarly got students to apply the drag-test to their
constructions and saw how they were forced to apply geometrical reasoning to review their constructions.

de Villiers (1998) made use of another feature of the drag-mode—not when ‘failures’ occur but when students successfully observed consistent patterns from DGS-use. He found that when students worked on Sketchpad and found visual phenomena that pointed to some geometrical property, such as that of medians of a triangle being consistently concurrent, it provided a good motivation to lead to geometric reasoning: “Why does it work? Prove it.” In such situations, students would have already been convinced from dragging that the property they observed is true. The emphasis, therefore, in the reasoning process is not so much for verification of truth as it is for an explanation of truth. To him, visual stimulus on Sketchpad provides a good motivation for the question, “why do such a visual phenomena occur?” and thus serves to drive the need for deductive reasoning to explain the visuals. Vincent (2002) reported a similar experience of using DGS visuals to lead to geometric reasoning. A pair of students she observed was convinced from the Cabri screen that the mid-point quadrilateral is a parallelogram. The “prove it” call from the teacher prompted the students to consider the geometric reasons for the phenomenon they observed.

DGS, with its rich offerings of dynamic visuals, could perhaps support visual reasoning as well. Visual reasoning shares characteristics of inductive and deductive reasoning, and is thus potentially useful in linking both. Visual reasoning is juxtaposed between the visual-intuitive and the logical-analytical (Hershkowitz, 1998). The precision and manipulability of DGS objects clearly qualifies for the former. The dynamic but property-preserving feature of DGS may also meet the requirements of geometric reasoning. When one drags a drag-resistant square, the non-critical features (such as size, orientation) vary but the critical geometrical attributes (such as equality of sides and perpendicularity of adjacent sides) stay invariant. If the observer of the phenomenon is asked a question intended to prompt geometric reasoning, such as “how do you know this is a square?” a reasonable answer could be, “it looks like one upon dragging”. On the surface, the answer seems to point merely to the visual-intuitive (pointing to inductive reasoning), but the reply
may also hide features of actual deductive reasoning: the equality of sides and perpendicularity of adjacent sides are drag-resistant, so it must be a square.

An even clearer support of the use of visual reasoning in DGS environment is in students’ *construction* of drag-resistant figures. To successfully construct a drag-resistant square, for example, students need to use some sequence of *Sketchpad* tools such as “construct perpendicular sides”, “rotate 90°”, or “produce circle with a given radius”. Each of these tools relate to some underlying attributes of a square. Thus, when students use these tools to build drag-resistant squares, they are, in effect, also harnessing (perhaps without reflection) these geometrical properties analytically and purposefully to solve the construction problem.

### 2.4.5 Summary of Section 2.4 and Relevance to the Study

The literature reviewed above points to DGS being a potentially useful tool to advance geometrical reasoning in the classroom. DGS features that are particularly relevant in encouraging students’ reasoning, such as the drag-mode and the flexibility for experimentation, are incorporated in the teaching project during the course of the study. DGS tasks requiring students to *interpret* visual phenomena (by dragging and observing) and *produce* phenomena (by constructing and dragging to check) are used.

However, to paint the picture of DGS-use as being all-good and problem-free would be inaccurate. In Laborde’s (2002) work of integrating DGS into the French mathematics curriculum, she reported that

> Really integrating technology into teaching takes *time* for teachers to accept that learning might occur in computer-based situations without reference to paper-and-pencil and to be able to create appropriate learning situations. But it also takes *time* for them to accept that they may lose part of their control over what students do. (p. 32, emphasis added)

Laborde seems to warn that integrating DGS into actual schooling is not a straightforward task. Although the software offers potential benefits, *time* is needed for teachers to adjust to a different mode of teaching. Interestingly, Assude (2005) also highlighted “time” as a constraint in the integration of DGS. Her focus, however, was not on the time the teacher needed to adjust to the new DGS environment; rather,
it was on the limited time teachers actually experienced when carrying out DGS tasks while having to fulfil other teaching responsibilities such as syllabus coverage.

Surprisingly, apart from these rare reports on problems, the last decade of intense research interest about the use of DGS has yielded very few deep discussions on actual problems that teachers face in using DGS in geometry classrooms. In focusing on problems and coping strategies involved in DGS-use, this study seeks to contribute to the research in this area.

2.5 Teaching Mathematics in Singapore

The final “piece of the puzzle” in the literature review relates to the field of teaching mathematics in Singapore. Due to obvious limitations of space, only a brief overview of the mathematics education system in Singapore will be attempted. The main purpose of this review is to examine the broader cultural and political context in which this study is based.

2.5.1 Meritocracy and Education in Singapore

The Singapore government has, since the early days of independence some 40 years ago, upheld meritocracy as an ideological pillar in the governance of the island state. That meritocracy still drives the thinking and policy-making of current leaders is clearly explained by Senior Minister\(^8\) Goh (2005): “meritocracy helps build the ethic of hard work. A person’s advancement depends on his ability, performance and contributions”.

The link between meritocracy and education in Singapore is most evident in the streaming policy. In 1980, streaming as a means to differentiate the curriculum to “suit [the] different pace of learning among students” (Goh, 1979) became a key tenet in the country’s education policy. Since then, the streaming methodology has undergone several revisions. The current system performs major streaming at two points—Primary 4 (year 4) and Primary 6 (year 6). At both junctures, the stream that a student is subsequently assigned to is solely determined by achievement scores

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\(^8\) The post of Senior Minister is held by officers who have served as the Prime Minister of the country. Senior Ministers remain in the cabinet and play an advisory role within the Prime Minister’s office.
attained at a common examination at the end of the school year. After Primary 4, students are channeled into two streams—EM1\(^9\) and EM3. After Primary 6, the students enter Secondary 1 in three main paths—Express, Normal (Academic) and Normal (Technical). Each stream has different curriculum tracks with different goals. The EM1/Express track aims for greater rigour in the languages, sciences, and mathematics to prepare students for further academic pursuits. The EM3/Normal (Technical) track aims to provide students with sufficient base knowledge to learn technical trade and general skills later on. The middle Normal (Academic) track allows room for students to develop in either direction.

The social implications of streaming mean that examinations are viewed as high-stakes assessment in Singapore. The pressure faced by parents and students are not only experienced in Year 4 and Year 6 where the streaming examinations take place. Examinations are also conducted at least bi-yearly at each grade levels as milestone preparations for the streaming examinations. It is therefore not surprising that there is a pervasive examination-oriented culture within classroom practices in Singapore schools (Kaur & Yap, 1998).

2.5.2 Mathematics Curriculum and Achievement in Singapore

In Singapore, the mathematics curriculum in the schools is centrally determined by the Ministry of Education—the education authority of the country. All grade levels in the Primary and Secondary schools adhere to this common curriculum. The curriculum spells out the content coverage at each grade level and the assessment standards required at the end of each grade level.

The conceptualization of the Singapore mathematics curriculum is most succinctly represented by the following “pentagonal framework” (MOE, 2000a, p. 10).

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\(^9\) There were originally three streams—EM1, EM2, and EM3. In early 2006, the EM1 and EM2 streams were merged into a single EM1 stream.
The framework guides both the Primary and the Secondary mathematics syllabuses. As depicted in Figure 16, “problem solving” is intended to be at the heart of mathematics education in the schools. The work of studying problem solving is supported by developing the five sides of the pentagon—attitude, skills, concepts, processes, and metacognition, elaborated in Figure 16.

If the effectiveness of mathematics education can be evaluated by paper-and-pencil test outcomes of the students, then the recent Trends in International Mathematics and Science Study (TIMSS) is by far the most extensive study of its kind. Singapore students participated in the first TIMSS (1TIMSS) test in 1994 and the second TIMSS (2TIMSS) test in 1998. In 1TIMSS, more than forty countries participated and students from Grades 3, 4, 7 and 8 took part. In 2TIMSS, about five thousand Grade 8 students were sampled and thirty-eight countries participated. It is significant to note that the student cohort in Grade 4 in 1TIMSS formed the Grade 8 cohort of 2TIMSS. The results showed that “The Singapore Primary 4 cohort (1994) had topped 1TIMSS for mathematics, and maintained their lead in mathematics by the time they reached secondary two (1998) in 2TIMSS” (MOE, 2000b, p. 3). The top ranking of Singapore students’ scores in TIMMS were interpreted by the country’s
Senior Minister himself as a main supporting evidence of a fundamentally sound education structure.

We have a good and high quality education system today. This is shown in the performance of our students in international studies and competitions … . In the TIMSS (1998), our secondary two students were ranked first in Maths … out of 38 countries including developed countries. (Goh, 2001)

2.5.3 Recent Education Reforms in Singapore

Despite renewed affirmation of the robustness of the existing education system, the country’s leaders recognized inadequacies in the current system and proposed a major revamp. Reform efforts began in the late 1990s and the work of monitoring and reviewing implementation continues to the present time:

We cannot assume that what worked well in the past will work for the future … Our Ministry of Education is undertaking a fundamental review of its curriculum and assessment system. (Goh, 1997, emphasis added)

To prepare our people, the Ministry of Education has been making changes … in the last five years … . However such changes have been carried out within the current structure of our current education system … . When MOE embarked on this review, the aim was to make a departure from the current structure. (Teo¹⁰, 2002, emphasis added)

Some might find the avowal of a current “good system” and the need for a “fundamental review” contradictory. However, education officials prefer to view it as a way to balance the retention of what has worked with the introduction of what is necessary for the future. This challenge translates into a working formula in curriculum restructuring efforts: change to prepare students for the changing needs of the future, while keeping the emphasis on mastery of procedural knowledge for excellence in test achievements.

¹⁰ Teo was then the Minister of Education.
Whichever way we cut back and redefine our curriculum, we will ensure our students retain the mastery over the core knowledge, … retain the high standards needed to stretch all our pupils and keep them striving for excellence. (Goh, 1997)

The goal of the education reform that is a response to future challenges is outlined in three broad “initiatives” for schools as part of the broader reform vision known as “Thinking Schools, Learning Nation” (TSLN):

1. The infusing of thinking skills;
2. Integrating the use of information technology; and

While all the initiatives point to serving the needs of the future as a collective goal, upon closer examination, one can see that they are meant for different anticipations of the future. The first initiative addresses the uncertainty of the future. Senior Minster Goh said: “We do not even know what these [future] problems will be … but we must ensure that our young can think for themselves” (Goh, 1997, emphasis added). It seems that the argument can be rephrased in this way: in the face of uncertainty, where no traditional knowledge domain forges prominently ahead of the rest, the safest bet is to go for generic thinking skills that will be useful and transferable to all circumstances. The second initiative directly tackles a specific prediction that the future is one where the usage of technology will be increasingly prevalent. The third initiative reinforces the traditional and continuing role of schooling as performing a function for national cohesion.

The third initiative is not incorporated into this study. One reason is that national education issues are unique to the local context of Singapore and are unlikely to be of general interest to a wider international audience, which this study seeks to address. Another reason is that the author considers the “national education messages” to be carrying a political agenda that interferes with, and is not necessarily complementary to, the work of mathematics education. The third reason is that the third initiative

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11 This study began in 2002 where the three initiatives embodied in TSLN were then the most current reform thrusts. Since that time, other reform catchphrases—such as “Teach Less, Learn More (TLLM)” —have become more prominent, but TLLM is nevertheless said to be built on TSLN: “TLLM builds on the groundwork laid in place by the systemic and structural improvements under TSLN” (MOE, 2007, emphases in the original).
appears to be of least importance with regards to the mathematics curriculum. In the secondary mathematics syllabus revision that follows the incorporation of the new initiatives (UCLES/MOE, 2000), there are no traces of the third initiative in the document, whereas the first two initiatives were clearly stated as “learning aims” (p. 4) of the new mathematics curriculum. For these reasons, the incorporation of the third initiative in the teaching of mathematics is considered to be outside the scope of this study.

2.5.4 Summary of Section 2.5 and Relevance to the Study
This study takes into account the broad cultural and political environment in which the mathematics classroom is situated. In particular, meritocracy and its effects on education—such as examination-orientedness—need to be taken into consideration when planning classroom instruction. In addition, the twin objectives of “integrating technology” and “infusing thinking”, as part of the national reform agenda, are incorporated as teaching goals during the period of this study.

The unique blend of high-achieving, conservative, and yet reform-oriented mathematics education in Singapore is not only of local interest, but is increasingly attracting attention and comparison from researchers and policy-crafters outside Singapore (e.g., Garelick, 2006; Ginsburg, Leinwand, Anstrom, & Pollock, 2005). However, these reports generally focus on broad-grained features of the Singapore mathematics education system such as textbooks, teacher qualifications, and assessment modes. This study aims to contribute to this international discussion by looking beyond the macro elements into the complexities involved when the combined goals associated with high examination performance and reform initiatives are actually brought into the classroom.

2.6 Overall Summary of Chapter

This chapter reviews broadly the research surrounding the various fields captured in the research questions: teaching, geometry, reasoning, dynamic geometry, and Singapore mathematics education. From the survey of the literature, there are theoretical frameworks and pedagogical principles that can be applied into both the
research design and the teaching project of this study. The next two chapters describe how these applications are incorporated into the methodological approaches used in this study.
This chapter relates directly to the motivations and areas of research identified in Chapter 1. In particular, the focus here is to develop the methodology to study research Questions 1 and 2 spelt out on p. 12. The literature surveyed in the previous chapter also contributes to this chapter by supplying a theoretical basis for the methodological approaches.

The author follows the well-accepted distinction between “methodology” and “method”—the former referring to the theoretical orientation which underpins the research approach and the latter referring to the procedural details involved in gathering and processing the research data. The methodology stance is first discussed while the proposed methods that are congruous to the methodology will be discussed later.

The methodological position taken by the author is pragmatic, in that it selects research methods that fit the purpose of the project. The reasons for the pragmatic stance are illustrated in the discussion in the next section involving the now-familiar rhetoric of the quantitative-qualitative divide.

3.1 Methodological Stance

The purpose-driven approach towards research design is increasingly supported by the education research community. Cobb (1994) argued for a pragmatic stance with regard to applying theoretical ideas—and thus methodologies—to the study of classroom practice. He grounded his arguments for such a position in Rorty’s (1983) view of theoretical perspectives as instruments for coping with things rather than ways of representing the intrinsic nature of the subject of study. Seen from this pragmatic standpoint, the way to evaluate the appropriateness of a methodology lies in whether “it works better than another for a given purpose” (Rorty, 1983, p. 157, emphasis added). Similarly, Doyle (1997) argued that the methodological choice boils down to the purpose of research: “much of the debate about … research on teaching hinges on issues of purpose” (p. 93, emphasis added).
The purpose of this study is embedded in the research questions listed in Chapter 1. From the research questions and their justification, it is clear that this study centres on understanding the work of classroom mathematics teaching in actual practice. The focus is thus on documenting and understanding practice within a naturalistic setting. This focus invariably aligns the research closer to the qualitative research methodology, as “descriptive”, “meaning”, and “naturalistic” are key defining characteristics of qualitative research (Bogdan & Biklen, 1998, pp. 4-7).

Also, in considering problems of practice and coping strategies as the central foci of this study, it is not sufficient to look at teaching problems and strategies by breaking down every problem or strategy into variables for aggregation—as is characteristic of the quantitative approach. The actual work of teaching lies in the interaction between problems (whether observable or within the teacher’s thought) and the adaptive strategies, as highlighted in Figure 1 at p. 5. The complexity of the teaching task is accentuated by the need of the teacher to simultaneously weigh all these factors in her moment-by-moment decisions as she carries out instruction (Lampert, 2001). Thus Lampert (1985) cautioned that “efforts to build generalised theories of instruction, curriculum, or classroom management based on careful empirical research have much to contribute to the improvement of teaching, but they do not sufficiently describe the work of teaching” (p. 179). In understanding how the teacher manages the interaction of problems of teaching, one needs to carefully describe the activity of the teaching and the thoughts of the teacher—in all their richness and complexity. This naturally leads to the adoption of a qualitative methodology.

A further point must be made with reference to the quantitative-qualitative rhetoric. The reader will note that the author has thus far avoided “paradigm talk”—that presumes each side of the qualitative-quantitative divide are “paradigms” that are irreconcilable—when elaborating methodological preferences. This avoidance is due to the awareness that “paradigm” tends to pitch researchers into different methodological ‘camps’ and limits the sharing of knowledge across paradigms. Donmoyer (2001) noted this tendency when he pointed out that “implicit in … paradigm talk is the notion of radically different epistemologies. That notion, in turn, implies at least some sense of incommensurability, some sense that we must still, of
necessity, choose up sides” (p. 179). Instead of speaking about paradigms, Donmoyer (2001) argued that paradigms can be complementary and urged for cross-paradigm perspectives when designing research programmes. His proposed conciliatory framework of classifying research methodology is one that is not based on paradigm, but on purpose. Given a particular purpose for a research inquiry, a variety of methods characteristic of different traditional paradigms can be applied.

Donmoyer is not alone in the call to work across paradigmatic lines. Pirie (1998) also recommended that “quantitative and qualitative methods are not alternative paradigms for the same research activities. Each has much to offer, but what is offered and what constitutes the goals of any project must together guide the choice of methodology” (p. 79). The author agrees with this view and the goal-driven choice of methodology. The continued use here of ‘qualitative versus quantitative’ talk is therefore not paradigmatic talk, but merely a way to carry on the familiar historical academic distinction between different methodological traditions.

In line with this pragmatic stance in research design, the author adopts a mixed-method approach that addresses the research questions and are harmonious with the theoretical orientations of this study.

3.1.1 A Case Study of Teaching in a Singapore classroom
The review of the different metaphors of teaching in the previous chapter—King’s (2001) Jazz performance, Chazan and Ball’s (1999) fermentation, and Lampert’s (2001) navigation—shows teaching as a complex activity where teachers constantly improvise and operate simultaneously along different planes of interaction (see the interaction arrows in Figures 4-7 in Chapter 2). An in-depth case study of teaching is therefore a suitable mechanism to investigate the richness of the interactions among different elements within the teaching enterprise.

Moreover, one of the motivations for this study (as elaborated in Chapter 1) arises from recognising the ‘gap’ between reform efforts and the implementation in the classroom, and that steps to bridge the gap should start from understanding actual practice (Ball, 2000). The research here thus focuses on a case study of teaching in an
actual classroom conducted within a naturalistic setting. By “naturalistic”, the author has in mind the conditions and constraints inherent in classroom teaching in Singapore—such as the type of classrooms, the duration of lessons, and the syllabus to be taught.

As identified in the research questions, the site of study is an Express stream secondary geometry class in a government school that possesses the facilities to support the use of DGS and mathematical reasoning in teaching. The research to be conducted is therefore a case study of an attempt to teach an intact class the geometry syllabus using DGS and in a way that encourages students’ reasoning.

3.1.2 Classroom Research Using a Teacher-researcher

Over the last decade, there has been a noticeable increase in reports about classroom research projects that involve persons assuming dual roles of researcher and teacher (e.g., Ainley, 1999; Chazan & Ball, 1999; Fleischer, 1995; Lampert, 2001). These teacher-researchers enter the classroom with both the intention to teach and to conduct research. The effort to research “from the inside” (Mason, 1994) while performing the actual work of teaching has a long history. Zeichner and Noffke (2001) traced the roots of the current teacher-researcher movement from a few traditions. The contributions that were identified include the teacher-as-researcher movement in the United Kingdom (Stenhouse, 1968, 1975; Elliot, 1976-77, 1991, 1997) which began in the 1960s; the participatory action research movement in Australia (Carr & Kemmis, 1986; Grundy & Kemmis, 1988; Kemmis & McTaggart, 1988a, 1988b) which has close intellectual links with the pioneers in the United Kingdom; and the North American teacher research movement (Schön, 1983; Cochran-Smith & Lytle, 1993) which started in the 1980s. That the teacher-researcher mode is gaining increasing use and acceptance is attested by an inclusion of a chapter on “Practitioner Research” (Zeichner & Noffke, 2001) in the fourth edition of the Handbook of Research on Teaching.

There are many reasons for the continued interest in the role and function of the teacher-researcher mode in classroom research projects. One source of motivation is the frustration with traditional theories and their perceived inadequacy in directly
informing and reforming teaching practice (Bishop, 1998; Christiansen, 1999).

Hiebert, Gallimore, and Stigler (2002) argued that “traditional research knowledge” is different from the “craft knowledge” that teachers use. Unlike the knowledge produced by educational researchers, teachers’ craft knowledge “is characterized more by its concreteness and contextual richness than its generalisability and contextual independence” (p. 3). The authors then call for the need to “bridge the gap” between traditional research knowledge and teachers’ practice. What is refreshing in the authors’ argument is that they acknowledge both of these branches of knowledge as valuable. Practitioner knowledge can complement traditional research knowledge in that it is linked with practice: it is detailed, concrete, and specific. Moreover, unlike traditional research that organises knowledge according to types, “in practitioner knowledge, all these types of knowledge [referring to content knowledge, pedagogical knowledge, and pedagogical content knowledge] are intertwined, organised … according to the problem the knowledge is intended to address” (p. 6).

Studying teaching through the teacher-researcher position can potentially tap into this alternative knowledge base of practitioner knowledge in understanding actual teaching practice.

One can, of course, argue that researchers can access practitioner’s knowledge without themselves being the practitioner. That is, they could collect data from teaching practice via conventional methods as an outsider, not as an insider. It is likely that an outside researcher looking at the same data might view instructional interactions differently and can therefore offer an outsider perspective of classroom phenomena. The author shares Kemmis’ (1995) stance in valuing studies of practice both “from the ‘outside’ and from ‘within’ the individual and social relations of the group” (p. 24).

However, it is doubtful if the kind of information gathered from the outside would be as “experience-near” (Geertz, 1983) as those obtained from insider’s accounts. Many of the problems of teaching practice reside not only in externally observable teaching acts, but also in the teacher’s thought-world. The internally unobservable problems include managing dilemmas (Lampert, 1985) and coping with conflicting goals of

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1 The language of “outside/inside” is credited to Cochran-Smith and Lytle (1993).
teaching—what a teacher called “walking the pedagogical tightrope” (Wood, Cobb, & Yackel, 1995). These thought reflections of the teacher, which contribute substantially to understanding teaching practice, would unlikely surface if a similar inquiry is done by an outside ethnographer. Anderson (2002) agreed that “for practitioners, who act daily in the setting …, [their] knowledge [of the practice] is deeper, more nuanced, and more visceral” (p. 23).

Another compelling reason for teacher-researcher work is found in the answer to the question, “Who are the ‘consumers’ of educational research?” It is the author’s persuasion that apart from adding to the knowledge base of researchers and informing policy decisions, classroom research’s primary goal ought to be to add to teachers’ understanding and improvement of their practice. If teachers are the intended main ‘consumers’ of classroom research, then the issue of the kinds of research that is most likely to attract their interest becomes of central importance. Anderson (2002) observed that “many practitioners currently do not find that academic research—formal or applied—is very useful” (p. 24).

Cochran-Smith and Lytle (1993) similarly commented that

Even when educational researchers have addressed problems that are of interest to teachers, their findings have frequently been reported in ways that are inaccessible, seemingly unrelated to the everyday realities of teaching, and counterintuitive to lessons learned from experience. … To many teachers, research is more or less by definition something that is distant, uninteresting, and impenetrable (p. 89).

In contrast, practitioner inquiry taps into practical knowledge that arises from concrete and situated experiences in the classroom which are the source of realities to which teachers can relate. Thus, if the key purpose of classroom research is to benefit practice, then “practitioner research … seems a natural methodological fit for problems of educational practice” (Anderson, 2002, p. 24).

In seeking to explain the lack of penetration of research knowledge into actual practice, Clandinin and Connelly (1995) introduced a metaphor to describe the professional knowledge of teachers—that of a “landscape”. “Landscape”, to them,
appropriately provides the picture of the vast expanse of teacher’s knowledge and its rich and varied nature at different parts of the landscape. They argued that current theoretical knowledge enters the landscape via a “conduit”:

The language of the conduit permeates the out-of-classroom landscape . . . [I]t is a language of abstraction. The language of abstraction, a rhetoric of conclusions, is prepositional, relational among concepts, impersonal, situation-independent, objective, nontemporal, ahistorical, and generic. In contrast, teachers’ language [is that] of [a] story, which is prototypical, relational among people, personal, contextual, subjective, temporal, historical, and specific (p. 14).

There are, however, lingering concerns about the validity of such teacher-researcher studies. The degree of doubt ranges from whether to consider such studies to be counted as research (Huberman, 1996) to the level of status attached to the knowledge generated from teacher-researcher inquiries (Kilpatrick, 2000). There is thus a real need to make the self-study work of teacher-researchers more valid in the sense that it is as accurate a portrayal of the situation as one can render, taking care to be self-critical instead of being self-congratulatory. For this reason, Feldman (2003) proposed the use of multiple forms of data representation and interpretation as ways to improve validity. This recommendation will be adopted in this study and the detailed method of gathering and interpreting different data streams will be reported later in this chapter.

Another concern is about the potential conflicting relationship between the two roles contained in the teacher-researcher’s work. Wong (1995) cautioned that the teacher-researcher enterprise can create tensions between the respective roles. His premise rested on the Aristotelian distinction between theoretical sciences (such as research) and practical sciences (such as teaching). “The primary purpose of research is to learn through investigation . . . . The primary purpose of teaching is to bring others to understand” (p. 22). In carrying out teaching and research simultaneously as teacher-researcher in the classroom, he “felt a distinct tension between trying to be systematic and thorough [as a researcher] and trying to be responsive and compassionate [as a teacher]” (p. 25).
Wilson (1995), however, objected to the “conflict-full” picture that Wong painted of the work of a teacher-researcher. She explicated her experiences engaging in teacher-researcher work in the classroom, where she viewed the two motivations not as being in tension but rather as intentions that she mobilised to improve both the work of teaching and research.

In carrying out the research reported here, the author (henceforth referred to in the first person pronoun as a natural reflection of the dual roles of teacher and research I assumed) took the warnings of Wong seriously. I built within the research design an administrative apportionment of teacher-researcher work across the boundary of the classroom. Outside the classroom, I consciously focused on the research goals in my reflections and in curriculum decision-making. Once I stepped into the classroom, I focused solely on the work of teaching and did not consciously think about my role as a researcher. For me, the overall picture with respect to the teaching and research agendas in the classroom was one where the teaching goals were more visibly portrayed, with the research agenda evident at the background. In almost all aspects of instructional work, the teaching and research purposes were carried out without any conscious compromising of one or the other. This interaction between the teaching and research agendas may perhaps be best illustrated by a picture, as is attempted in Figure 17.

I proposed a shift away from the Wong-Wilson ‘conflicting versus complementary talk’ to the use of a background/foreground metaphor in describing the interactions between the research and the teaching roles in the teacher-researcher enterprise:
METHODOLOGICAL APPROACHES AND METHOD

When I enter the classroom, the teacher-at-work is at the foreground and teaching goals become the main driving force for instructional choices and action; however, the research intents are not altogether absent, but are rather actively working at the background to influence the thoughts and actions of the teacher. In my study, apart from one point of conflict when the research agenda sought to surface to the foreground to challenge the teaching agenda, this teacher-at-the-foreground and researcher-at-the-background situation enabled me to carry out both the goals of teaching and research in a largely productive way (Leong, Chick, & Moss, 2007, p. 23).

There is perhaps a need to conclude the discussion in this section by making a comment about the term “teacher-researcher”. While teacher-researcher studies are conventionally associated with practitioner research, in which teachers take on the work of researching their own practice, the term “teacher-researcher” apparently now also includes someone in the inverse situation—university researchers becoming teachers. This distinction between “practitioner researchers” and the inverse situation of “researchers as practitioners” are often not made clear in most essays which use the term “teacher-researcher” (or its equivalent). Ainley (1999) makes this distinction explicit.

For my case, prior to the project reported here, I spent the first seven years of my professional career teaching mathematics in school settings to students ranging from Year 7 to Year 12. The next stage of my education career was spent in the university, where I have begun learning the ropes of doing research for the last seven years. I find it important to clarify my professional history at this juncture insofar as it helps to clarify my expertise and awareness in teaching and in research. Because of the almost balanced exposure to both of these activities, my work can perhaps fit the label of both “practitioner research” as well as “researcher as practitioner” that Ainley used. For simplicity though, I shall continue the use of “teacher-researcher” to represent the dual roles I play in the study reported here.

3.1.3 Goal-based Approach to Analysing Teaching

The Teacher Model Group at Berkeley, led by Schoenfeld, developed a goal-based methodology to analyse teaching. A significant number of projects and reports from this Teacher Model Group were based on this approach to analysing teaching (Aguirre

There were a number of motivations behind the analytical model they developed. First, their model aimed to take into consideration the various interacting complexities inherent in the work of teaching. Schoenfeld (2000) noted that different branches of educational research traditions tend to focus on different factors that influence teaching—e.g., some on “content”, others on “human factors”. It was asserted that the model, “via the creation of a goal-driven architecture … did equal justice to both the content-related and the human factors aspects of [teaching] interactions” (p. 246).

Also, a key assumption behind the goal-based approach is that every teaching action is motivated by one or more teaching goals. This allows an analysis of teaching not merely at the level of observable actions but also at the underlying intentions of the teacher when those actions were carried out. In other words, in analyzing the work of teaching, the goal-based method places implicit importance on both the observable acts of the teacher as well as the unobservable ones—such as the knowledge employed by the teacher in classroom decision-making.

The methodology also acknowledges that teaching goals can vary depending on the level of ‘zoom’ at which one looks at the teaching enterprise. Schoenfeld (2000) explained that “[g]oals occur at different grain sizes: ‘overarching’ goals for students over the course of weeks, months, or the year; unit goals; lesson goals; goals for particular parts of a lesson, and ‘local’ goals for particular interactions with students” (p. 250). Entire teaching sequences over time can be mapped into an architecture consisting of goals—with finer-grained goals nested into broader-grained goals.

Much of the theoretical basis of this study—such as the view of teaching as a complex activity involving multi-planar interactions and the need to take into consideration teacher’s decision-making processes—is in common with the theoretical assumptions of the goal-based methodology. As such, this approach to analyzing teaching is suited to the purposes of this research project. The work of analysis therefore entails a study into the interaction between my instructional goals as a teacher. Through this goal-
based method, problems of teaching (the focus of the first research question) can be identified as occasions where there are hindrances to the attainment of teaching goals.

The way the goals of teaching are described and viewed in this report is influenced by the “goal-driven architecture” used by the Berkeley group. This method of identifying the goals of my teaching and how these goals interact is a fundamental approach to data analysis in this study. As such, more details will be provided in Chapter 4 to explicate how the method was adopted and modified to fulfill the purposes of this project.

3.1.4 Analysis by Progressive Widening of Focus
As reviewed in the previous chapter, teaching practice is a complex activity (Lampert, 2001; Ball, 2000; Stigler & Hiebert, 1999). The complexity involves a broad inclusion of many elements within the teaching activity, such as mathematical content, curriculum requirements, time factors, and the needs of students, among many others. Another layer of complexity is found when zooming in to each component reveals, nested within the component, sub-components which by themselves carry with them complexities yet at a different level. The metaphor of “picture” may therefore be suitable in illustrating the different parts of complexity in instructional practice and the different levels of grain-sizes that one can zoom into the picture to view complexities at varying degrees of zoom-scale.

But the complexity of the picture is not just in the quantity of the components and sub-components. It is also found in the complex dynamic interactions between constituents within the picture. It is therefore not a static camera snapshot picture, but a moving video strip of component elements dynamically exerting influence on one another. It is therefore a picture that is clearly too complex to analyse in one sweeping attempt.

I therefore propose to adopt a method of analysis which I term “progressive widening of focus”. I start with certain objects within the picture—defined as a “region”—which to me, as the teacher, is of greatest interest in the work of teaching and is central to the entire teaching enterprise. This first region that is selected should
be of *moderate size* and connectivity. By “size”, I refer to the number of elements found in the region; and by “connectivity” is meant the degree of influence and relationship these elements have on one another within the region. The proposal for moderation is to avoid two undesirable situations. On one hand, starting off with too large a region renders the practical work of analysis almost impossible and can result in the temptation to over-simplify the complexities involved. On the other hand, looking at too small a region leads to the danger of atomising the work of teaching. The result of either approach is a distorted portrayal of teaching practice (Fleischer, 1995).

Starting with a moderately-sized, appropriately rich, and core region of the picture allows a feasible work of analysis without compromising its relevance and representative nature. After zooming in on this first part of the picture, I will ‘watch the entire video’ (that is, look at the full duration of the teaching project) with my attention trained solely on the elements in this identified region to flesh out details from it and to provide a more thorough analysis of the constituents. This will complete the first stage of the analysis.

The next stage involves a step taken towards widening the focus. Other objects of the picture which were not included in the first region can then be brought into the wider focus to reveal further complexity. With the understanding of the first region well-grounded after the initial analysis, the introduction into the focus of other parts of the picture should be less daunting compared to the attempt to see this picture all at once. Subsequent stages can then include more parts of the picture that are increasingly on the ‘fringes’ of the picture—those that are less integral in the work of teaching.

The objective of this report, however, is not to paint the picture exhaustively; but to tackle those components of practice that contribute most substantially to problems of teaching. As such, parts of the picture which are on the outermost fringes of the frame may not be included at all in this presentation.
3.2 Method of Study

In the previous section, I mapped out the broad methodological approaches that I employed in this study. I have taken a pragmatic stance towards methodological design in the sense that the selection of approaches was primarily determined by their suitability to the purposes and nature of the study. In summary, this project is a case study of my attempt—as a teacher-researcher—to teach a geometry class using DGS within a naturalistic setting in a way that supports students’ mathematical reasoning. The goal-based framework was used to link teaching actions to underlying instructional goals and thus served as a methodology to study the problems and coping strategies used in teaching through investigating goals interactions. To deal with the multi-complexities in teaching, the analyses took place in phases through progressive widening of focus on the ‘picture’ of actual teaching practice. In the following sections, I detail the methods used to carry out the research design that were characterised by these methodological approaches.

3.2.1 Setting

The class chosen for the study was a Singapore Secondary 1 (Year 7, mostly thirteen-year olds) Express intact class of a government school. The particular school was approached because of my professional associations with some staff members in that school. The Vice-Principal and the Head of Department ( Discipline) were my colleagues when we taught in another secondary school in the earlier part of our career. They were therefore in a position to judge my competencies as a teacher in the classroom. The Principal read my research proposal and gave her approval in writing for the research to take place in the school. The choice of a Secondary 1 class was a follow-up joint decision made by a number of teaching staff together with me. Their considerations included the reservation about involving later grade levels as those classes were nearer important examinations, as well as a concern about the lack of experienced mathematics teachers at Secondary 1, seeing me as one who could provide some experience.

The school was located in a low-middle-class suburb of Singapore and a majority of the student population lived around the area. It was a mixed-gender school with about
equal numbers of male and female students. The academic performance of the school was consistently ‘unranked’. In Singapore, the Ministry of Education publishes annual ‘league tables’ of school results. Based on the performance of the end-of-secondary centralised examinations, school results are listed in the table. For example, the table for the Express students who took the 2005 examination listed 54 schools (out of a total of about 160 secondary schools) ranked into performance bands (MOE, 2006). ‘Unranked’ schools refer to those schools that were not included in the ranked list published because of poorer academic results. These schools were generally referred to as ‘neighbourhood’ schools; the school in which the study was conducted was such a ‘neighbourhood’ school.

During the period of the project conducted in the second semester of 2003, I replaced the resident teacher to assume the role of the class’s mathematics teacher. It was a class of forty students—twenty girls and twenty boys. I taught over a period of four weeks. Three lessons were scheduled for each week, and each lesson had a duration of seventy minutes. I taught in a naturalistic classroom context, where the usual constraints of teaching, such as syllabus coverage within stipulated time and limited resources, were taken as givens. I also included in the lessons the twin components of DGS usage and students’ active participation in reasoning as reflected in the research questions. During the module, students were expected to participate normally in classroom discourse, seatwork, group work, interacting with computer outputs, etc. They were taught the geometry curriculum requirements of other same-level classes in the school.

At the point when I took over the class to teach, the resident teacher had taught them for about eight months. His total teaching experience then was about one year. He acknowledged his inexperience in teaching and mentioned that the school leaders urged him to “learn from me”. As such, he was present in the class with me throughout the entire module when I taught. His role was purely observatory. Instructional planning and teaching work was done by me.

The choice of the duration of twelve scheduled lessons for this study was a decision I took jointly with the resident mathematics teacher of the class and the school leaders.
Factors taken into consideration were the coherence of a teaching module, the duration of support the school could provide, and the amount of data that was appropriate for a study of this scale. According to the curriculum plan followed by the school, twelve lessons were allocated for a self-contained geometry module and that was the period the school recommended for the duration of the study. I was agreeable to that duration as it seemed that the 12-lesson period would provide a substantial amount of data to address the research questions.

I was, however, aware of the obvious limitations of a 12-lesson study. With respect to problems of practice and the coping strategies, which are the key goals to address in this study, the 12-lesson module will not capture those problematic issues which a teacher may experience outside of that teaching time frame. Some of these problems, such as the problem of “teaching to cover the [entire year’s] curriculum” and the problem of “teaching closure [at the end of the school year]”, have been identified and discussed at length by Lampert (2001). These problems and others that are yet to be identified which are not bound by a relatively short teaching span are therefore not within the scope of this study. Nevertheless, there remains a large domain of problems that can pose serious challenges to the teacher within the 12-lesson duration. This problem domain is enlarged with the inclusion of the use of computers and the desire to support students’ reasoning in the teaching practice.

Due to the need for each of the students to regularly access DGS during the course of the teaching duration, the lessons took place in one of the school’s computer labs. In the Singapore setting, computer labs are rooms in schools that are specially equipped with computers and facilities such as air-conditioning. In the school in which the study was conducted, there was a computer lab designated as the “math room”. The room was suitably equipped with mathematics manipulatives and software, including DGS. The school administration thus naturally assigned that math room to me for the lessons I was to conduct as the teacher-researcher. The sketch plan of the room is given in Figure 18 (p. 96). In keeping with the intent to conduct the research within a naturalistic setting, I did not make any changes to the furniture layout of the room. The original arrangement—including the hexagonal tables and the computers—was adopted as given.
As there were only twenty computers in the lab while there were forty students in the class, the students were randomly paired\(^2\) and assigned to the computers. Pairing of students for computer work was meant to provide opportunities for discussions among students that could aid in their mathematical reasoning. During lesson segments when students were not required to access the computers, they were clustered around the hexagonal table for whole-class discussions. Thus, their seating arrangement for non-computer work was such that each pair was seated near to their assigned computer and they could access it easily when required. The names of the students written in this report (and also indicated in Appendix 1) are pseudonyms. The pseudonyms preserved the gender and ethnicity\(^3\) of the students.

### 3.2.2 Module Plan

One of the main professional responsibilities of a mathematics teacher is to complete the teaching of the curriculum content within a given time period. In Singapore, the implementation mechanism of this goal of syllabus coverage takes the form of teaching schedules—known as schemes of work (SOW)—handed down by school authorities which prescribe mathematical topics to be taught within specified time frames. The relevant sections of the school’s SOW for the module I taught are shown in Table 2.

\(^2\) Mindful that random pairings can result in pairs that are too distracting to each other that may render teaching difficult, the paired list was shown to the resident teacher for his views. A few pairs were identified as problematic and changes were made accordingly.

\(^3\) Students in this class, as reflective of the racial-mix in the Singapore population, are from one of these three ethnic origins: Chinese, Malay, and Indian.
Table 2: Extract of the school’s scheme of work for the module

<table>
<thead>
<tr>
<th>Week(^4)</th>
<th>Units of Work</th>
<th>Textbook(^5) Exercises</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>7 – 8</strong></td>
<td><strong>Textbook chapter 13:</strong> Basic geometrical concepts &amp; properties</td>
<td>Ex 13a – 13c. Review questions.</td>
</tr>
<tr>
<td></td>
<td>• Points</td>
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<td></td>
<td>• Lines</td>
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<tr>
<td></td>
<td>• Planes</td>
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<td></td>
<td>• Solids</td>
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<td></td>
<td>• Curved surfaces</td>
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<td></td>
<td>• Intersecting lines</td>
<td></td>
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<tr>
<td></td>
<td>• Angles</td>
<td></td>
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<tr>
<td></td>
<td>• The protractor &amp; angle measure</td>
<td></td>
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<tr>
<td></td>
<td>• Different kinds of angles</td>
<td></td>
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<tr>
<td></td>
<td>• Complementary angles</td>
<td></td>
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<tr>
<td></td>
<td>• Supplementary angles</td>
<td></td>
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<tr>
<td></td>
<td>• Adjacent angles on a line</td>
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<tr>
<td></td>
<td>• To construct an angle using a protractor</td>
<td></td>
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<tr>
<td></td>
<td>• Vertically opposite angles</td>
<td></td>
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<tr>
<td></td>
<td>• Parallel lines</td>
<td></td>
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<tr>
<td></td>
<td>• Drawing parallel lines using a setsquare and a ruler</td>
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<tr>
<td></td>
<td>• Perpendicular lines</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Drawing perpendicular lines using a setsquare and a ruler</td>
<td></td>
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<tr>
<td></td>
<td>• Use of compass</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Perpendicular bisector and angle bisector</td>
<td></td>
</tr>
<tr>
<td><strong>8</strong></td>
<td><strong>Common test</strong></td>
<td></td>
</tr>
<tr>
<td><strong>9 – 10</strong></td>
<td><strong>Textbook chapter 14:</strong> Angle properties of triangles &amp; quadrilaterals</td>
<td>Ex 14a – 14c. Review questions.</td>
</tr>
<tr>
<td></td>
<td>• Polygons</td>
<td></td>
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<td></td>
<td>• Triangles</td>
<td></td>
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<tr>
<td></td>
<td>• Construction of triangles</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Angle properties of triangles</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Exterior &amp; interior opposite angles</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Quadrilaterals</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Tessellations of regular polygons</td>
<td></td>
</tr>
</tbody>
</table>

The SOW listed the topics to be covered but did not provide enough detail to conduct my instructional work for the module. Moreover, I also needed to re-organise the instructional sequence in a way that fulfilled my other goals such as the use of DGS in

\(^4\) “Week” refers to the week number within the term. The study was conducted in the third term of the school year. The entire school year stretches across four terms of ten weeks each.

\(^5\) Textbook exercises were taken from the textbook that the class was using since the beginning of the year. The title of the textbook is *New Syllabus Mathematics 1* (Teh & Looi, 2001).
teaching, which the SOW did not incorporate. I needed to devise a more detailed plan and one also that supported my teaching and research agenda. I shall refer to this as the module plan. Table 3 shows the module plan. As it turned out, the construction of the module plan was by no means a straightforward task. I had to accommodate multiple instructional goals while still keeping to the constraints of the 4-week duration. The actual problems I faced will be more appropriately discussed later in the analyses chapters of this report where I discussed problems of teaching and coping strategies, since they are, in fact, relevant for data analysis. For the current purpose, I shall describe briefly the various components of the module plan and the rationale behind their inclusion.

With respect to the number of lessons in the module plan, the scheduled number was twelve (based on three lessons over a four-week period). However, I noted that one of the scheduled lessons fell on Teacher’s Day—which was a holiday for all schools—and another two lessons coincided with the common test period in Week 8. Since the exact common test slots were not ready at the point when I designed the plan, I made an assumption that at least one of the two lessons would be affected by the common test schedule and would be cancelled. The curriculum design was thus based on the assumption of a maximum of ten lessons. Instead of sticking to the original twelve lessons and thus extending beyond the allocated four weeks, I took the view that such lesson cancellations are normal in school life and since this research sought to study teaching in a naturalistic setting, the problem of cramped teaching schedule (due to ad hoc reduction of lessons) should be taken as a given.

The principles used in guiding the work of partitioning the content into ten lessons were that (1) each lesson be a coherent natural unit of closely-related components; and (2) if the contents in a particular lesson were revisionary in nature (e.g., revisiting topics from the primary curriculum), then more would be packed into that lesson. The column on “math language” was included in the module plan as a reminder to the teacher of the mathematics vocabulary that was needed to be introduced in the course of the module. It also indicates the importance of language in the learning of geometry, as highlighted in the literature review. Consistent with theoretical knowledge on the development of language, it would be introduced in teaching not in
a ‘by-definition’ way, but by tapping into the students’ natural language and gradually rephrasing this to increasingly more formal mathematical terminology.

Since the integration of DGS is an important component of the research design, there was careful consideration of how Sketchpad should be utilised in the module plan. The Ministry of Education envisions the “integration of information technology (IT) to enhance the mathematical experience” (UCLES/MOE, 2000, emphasis added). In other words, it was intended by the Ministry that (1) IT should feature prominently in Mathematics classrooms; (2) IT should be woven tightly into other components of teaching to form a well-coordinated whole; and (3) students are expected to use technological tools actively in order to “enhance the mathematical experience”. Thus, I wanted to integrate Sketchpad as a prominent and essential tool for students’ learning in the lessons I conducted. I aimed to use Sketchpad not merely as a fringe accessory but as a regular tool in the instructional environment blending in with the other components of my instructional practice. The column on “Sketchpad files” in the module plan reflected this attempt at IT-integration. The actual design of the Sketchpad templates was informed by current literature on the educational features of the software. In particular, the features of dynamic visuals, ‘drag-mode’, and ease in testing conjectures were intended to be exploited. The Sketchpad templates were designed by me. My familiarity with the software stemmed from the many hours invested into it for the purposes of teaching (especially in my former work as a professional teacher) and research (Leong, 2003; Leong & Lim, 2003).

Under the column of “Other Materials” in the module plan, I have included the use of ‘worksheets’ as a standard practice. Worksheet-use is a common feature in Singapore mathematics classrooms. The most compelling reason for the use of worksheets was that the textbook exercises were not adequate in supporting my module plan. The worksheets that I crafted complemented the instructional work in the classroom. The tasks included summary charts for computer work, constructions using geometrical instruments, and additional practice questions. They served as templates for students to record their observations and conjectures in class, as well as slates for them to practise the skills and concepts that they learnt. The details of how the worksheets were used in the classroom are provided in the later chapters when the work of teaching is analysed.
<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Content Coverage/progression</th>
<th>Math language</th>
<th>Sketchpad files</th>
<th>Textbook exercises</th>
<th>Other materials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Tessellations of regular polygons (cf tilings in ‘real-world’)</td>
<td>Equilateral triangle</td>
<td>Tessellations</td>
<td>13A, 3-12</td>
<td>Protractor Polynomials</td>
</tr>
<tr>
<td></td>
<td>Introduce concept of polygon (prevent prototypical phenomena of thinking about ‘polygons’ as always regular)</td>
<td>Square</td>
<td>gsp</td>
<td></td>
<td>worksheet</td>
</tr>
<tr>
<td></td>
<td>Specialize to regular polygons</td>
<td>Pentagon</td>
<td></td>
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<tr>
<td></td>
<td>Use of protractor to measure interior angles of polygons</td>
<td>Hexagon</td>
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<tr>
<td></td>
<td>Connect to “angle at a point = 360°” – to explain why some regular polygons doesn’t tessellate the plane.</td>
<td>Octagon</td>
<td></td>
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<tr>
<td></td>
<td>Connect “angle at a point = 360°” to “angle on a line = 180°” and focus on reasoning from one to the other Connect to “vertically opposite angles have the same measure”.</td>
<td>Decagon</td>
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<td></td>
<td></td>
<td>Octagon</td>
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<tr>
<td></td>
<td>Explore properties of parallel lines and a transversal:</td>
<td>Vertically opposite</td>
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<tr>
<td></td>
<td>▪ Interior angles are supplementary (cf this is equivalent to euclid’s parallel postulate)</td>
<td>Angles on a line</td>
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<tr>
<td></td>
<td>▪ Alternate angles are equal (i.e. ‘congruent’ but I use ‘equal’ to be consistent with textbook and thus avoid confusion)</td>
<td>Vertically opposite</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>▪ Corresponding angles are equal</td>
<td>Angles at a point</td>
<td></td>
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<tr>
<td></td>
<td>Show how the above 3 theorems are logically related.</td>
<td>Angles on a line</td>
<td></td>
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<tr>
<td></td>
<td>Lead to perpendicular lines</td>
<td>Vertically opposite</td>
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<td></td>
<td></td>
<td>Angles at a point</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>Using setsquare and ruler to construct</td>
<td>Corresponding angles</td>
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</tr>
<tr>
<td></td>
<td>▪ Parallel lines; and</td>
<td>Interior angles</td>
<td></td>
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<tr>
<td></td>
<td>▪ Perpendicular lines</td>
<td>Alternate angles</td>
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<td></td>
<td></td>
<td>Adjacent angles</td>
<td></td>
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<tr>
<td></td>
<td>Drag-resistant</td>
<td>Parallel and perpendicul</td>
<td>13B</td>
<td></td>
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<td></td>
<td></td>
<td>gsp</td>
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<tr>
<td></td>
<td>Worksheet-transversal</td>
<td>gsp</td>
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</tbody>
</table>
### METHODOLOGICAL APPROACHES AND METHOD

<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Content Coverage/progression</th>
<th>Math language</th>
<th>Sketchpad files</th>
<th>Textbook exercises</th>
<th>Other materials</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Relate construction process to the theorems learnt earlier as a way to explain “why it works”.</td>
<td>Interior angles of a triangle</td>
<td>ar.gsp</td>
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<tr>
<td></td>
<td>Explain what ‘drag-resistant’ means by contrasting with non-drag-resistant objects.</td>
<td>Exterior angles of a triangle</td>
<td></td>
<td>14A – 9-12</td>
<td>Mini-catalogogue of some Sketchpad functions</td>
</tr>
<tr>
<td>4</td>
<td>Use Sketchpad to Construct parallel lines using ‘construct’ and ‘transform’ menu</td>
<td></td>
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<td></td>
<td>Construct perpendicular lines using ‘construct’ and ‘transform’ menu</td>
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<tr>
<td></td>
<td>Relate construction process to the theorems learnt earlier as a way to explain “why it works”.</td>
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<td></td>
<td>Explain what ‘drag-resistant’ means by contrasting with non-drag-resistant objects.</td>
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<td></td>
<td>Use Sketchpad to Construct parallel lines using ‘construct’ and ‘transform’ menu</td>
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<td></td>
<td>Construct perpendicular lines using ‘construct’ and ‘transform’ menu</td>
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<tr>
<td></td>
<td>Lead to sum of interior angles of triangle is 180° and the justification (revise alternate angle theorem.).</td>
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<tr>
<td></td>
<td>Exterior angles of a triangle add up to 360°.</td>
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<td></td>
<td>An exterior angle of a triangle is the sum of the interior opposite angles.</td>
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<td></td>
<td>Lead to explain the results.</td>
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</tr>
<tr>
<td>5</td>
<td>Use the ‘Menu’ function to measure angles – interior angles of a triangle.</td>
<td>Acute, right, obtuse, and reflex angles.</td>
<td>Types of triangles.gsp</td>
<td></td>
<td>Protractor Setsquare Worksheet-types of triangles</td>
</tr>
<tr>
<td></td>
<td>Lead to sum of interior angles of triangle is 180° and the justification (revise alternate angle theorem.).</td>
<td>Acute-angled triangle</td>
<td></td>
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<tr>
<td></td>
<td>Exterior angles of a triangle add up to 360°.</td>
<td>Obtuse-angled triangle</td>
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<tr>
<td></td>
<td>An exterior angle of a triangle is the sum of the interior opposite angles.</td>
<td>Right-angled triangle</td>
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<tr>
<td></td>
<td>Lead to explain the results.</td>
<td>Equilateral triangle</td>
<td></td>
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<tr>
<td></td>
<td>Explore the definition of “acute angle”, “right angle”, “obtuse angle” and “reflex angle”.</td>
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<tr>
<td></td>
<td>Study (by exploring) properties of Acute-angled triangle, Obtuse-angled triangle, and Right-angled triangle.</td>
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<td></td>
<td>Types of triangles.gsp</td>
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<td>Protractor Setsquare Worksheet-types of triangles.</td>
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<tr>
<td>Lesson number</td>
<td>Content Coverage/progression</td>
<td>Math language</td>
<td>Sketchpad files</td>
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<td>Other materials</td>
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<tr>
<td></td>
<td>• Equilateral triangle</td>
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<td>Isosceles triangle</td>
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<td>• Isosceles triangle</td>
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<td>Scalene triangle</td>
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<td>• Scalene triangle</td>
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<td>Triangle classification on Sketchpad:</td>
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<tr>
<td></td>
<td>• Acute-angled triangle</td>
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<tr>
<td></td>
<td>• Obtuse-angled triangle</td>
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<td></td>
<td>• Right-angled triangle</td>
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<td></td>
<td>• Equilateral triangle</td>
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<td></td>
<td>• Isosceles triangle</td>
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<td></td>
<td>• Scalene triangle</td>
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<tr>
<td></td>
<td>Draw an exemplar of each of the above using setsquare, protractor and ruler</td>
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<tr>
<td></td>
<td>Construct using Sketchpad drag-resistant</td>
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<tr>
<td></td>
<td>• Equilateral triangle</td>
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<td></td>
<td>• Isosceles triangle</td>
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<td>• Right-angled triangle</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Classification of special quads:</td>
<td>Square</td>
<td>Classify special quads.gsp</td>
<td></td>
<td>Setsquare special quads</td>
</tr>
<tr>
<td></td>
<td>• Square</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Rectangle</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>• Parallelogram</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Rhombus</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Trapezium</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>• Kite</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Draw an exemplar of each of the above using setsquare, protractor and ruler</td>
<td></td>
<td></td>
<td></td>
<td>Setsquare special quads</td>
</tr>
<tr>
<td>Lesson number</td>
<td>Content Coverage/progression</td>
<td>Math language</td>
<td>Sketchpad files</td>
<td>Textbook exercises</td>
<td>Other materials</td>
</tr>
<tr>
<td>---------------</td>
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<td>----------------</td>
</tr>
<tr>
<td>7</td>
<td>By exploring, study the properties of each of the special quads: ♦ Square ♦ Rectangle ♦ Parallelogram ♦ Rhombus Lead to definition of each of the above (Optional) Lead to hierarchical relationship between the special quads.</td>
<td>Diagonal Definition Bisect</td>
<td>Properties of special quads.gsp</td>
<td>14B</td>
<td>Worksheet—properties of special quads</td>
</tr>
<tr>
<td>8</td>
<td>Construct using Sketchpad ♦ Square ♦ Rectangle ♦ Parallelogram ♦ Rhombus Construct using compass and/or setsquare on paper ♦ Square ♦ Rectangle ♦ Parallelogram ♦ Rhombus</td>
<td>Construct special quads.gsp</td>
<td></td>
<td>14C</td>
<td>Worksheet—construction of quads</td>
</tr>
<tr>
<td>9</td>
<td>Use of compass to construct ♦ Perpendicular bisector ♦ Angle bisector Relate the construction process to the properties of rhombus to show why the method “works”</td>
<td>Perpendicular bisector Angle bisector</td>
<td></td>
<td>13C, 3-6</td>
<td>Worksheet—compass constructions</td>
</tr>
<tr>
<td>10</td>
<td>Identify the following</td>
<td>Cube</td>
<td></td>
<td></td>
<td>Solids set.</td>
</tr>
</tbody>
</table>
**METHODOLOGICAL APPROACHES AND METHOD**

<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Content Coverage/progression</th>
<th>Math language</th>
<th>Sketchpad files</th>
<th>Textbook exercises</th>
<th>Other materials</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cube</td>
<td>Cuboid</td>
<td></td>
<td></td>
<td>Polydrons™</td>
</tr>
<tr>
<td></td>
<td>Cuboid</td>
<td>Prism</td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Prism</td>
<td>Cylinder</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>Cylinder</td>
<td>Pyramid</td>
<td></td>
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<tr>
<td></td>
<td>Pyramid</td>
<td>Cone</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cone</td>
<td>Sphere</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sphere</td>
<td>Build using <em>Polydrons™</em></td>
<td></td>
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</tr>
<tr>
<td></td>
<td>Cube</td>
<td>Cuboid</td>
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<td>Cuboid</td>
<td>Prism</td>
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<td></td>
<td>Prism</td>
<td>Cylinder</td>
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<td>Cylinder</td>
<td>Pyramid</td>
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<tr>
<td></td>
<td>Pyramid</td>
<td>Cone</td>
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<tr>
<td></td>
<td>Cone</td>
<td>Sphere</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sphere</td>
<td>Highlight the difference between prism and pyramid</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sketch the following</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Cube</td>
<td>Cuboid</td>
<td></td>
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<td></td>
<td>Cuboid</td>
<td>Prism</td>
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<td></td>
<td>Prism</td>
<td>Cylinder</td>
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<td></td>
<td>Cylinder</td>
<td>Pyramid</td>
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<tr>
<td></td>
<td>Pyramid</td>
<td>Cone</td>
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<tr>
<td></td>
<td>Cone</td>
<td>Sphere</td>
<td></td>
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<tr>
<td></td>
<td>Sphere</td>
<td>Lateral face</td>
<td></td>
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</tr>
</tbody>
</table>

Table 3: Module plan
METHODOLOGICAL APPROACHES AND METHOD

3.2.3 Video-recording of Lessons

My teaching actions and interactions with students were captured by video-recording. As the study requires in-depth analyses of in-class teaching behaviour and discussions, videos were appropriate media to document such actual instructional details.

The first trial video-recording took place two months before the actual start of the geometry module. The purposes of the trial were to check the adequacy of the video-recording method and for me to try out the facilities in the math room in the context of teaching. I conducted a lesson with a random secondary one class, teaching for seventy minutes on converting word problems to algebraic equations. The choice of the topic was decided jointly with the resident teacher. One unattended video camera was placed at the back of the classroom (at position Ca1 shown in Figure 18) to capture my teaching actions at the front of the class. The camera’s zoom lens was set to cover the entire whiteboard so that my movements along the whole breadth of the board could be recorded. My reflections on the first trial are given below.

The current video data is inadequate. Despite my physically readjusting the camera from time to time [at various junctures of the lesson], the camera cannot capture all the necessary details. It either takes me or the students, but not both. If I zoom out, it will take more but the images are too small to make good sense of facial and bodily features. Also what is written on the board cannot be seen. There is a need to have another camera. One to capture me in front of the class and the actions I take on the board; another capturing the students’ responses and also when I walk around the class to supervise work.

I decided to conduct a second trial with two video cameras—one camera (Ca1) to capture teaching actions in the front of the class and another camera (Ca2) placed at the front with a wide-screen to record students’ participation. Positions of Ca1 and Ca2 are shown in the classroom plan (see Figure 18). Moreover, I planted audio receivers on every hexagonal table to pick up students’ voices during classroom discussions. The second trial took place about 3 weeks before the start of the geometry module and was conducted with the actual class chosen for the project. In that trial, as I had to manage and monitor the additional equipment, I requested the resident teacher to conduct his lesson as per his normal schedule.
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While I checked the effectiveness of the audio and video equipment. My reflections after the second trial are given below.

I use two video cameras this time—one to focus only the teacher at the front and the other a wide-angle lens to focus the students when they are in whole-class discussion mode. It is a vast improvement insofar as more are taken—image of the teacher is big enough to see the facial features and the writings on the board are bigger though still blurr. … [Moreover] when the students move from the whole-class discussion mode to computer-work mode, some of them will be out of the scope of the wide lens of Ca2.

The main problem was that because both Ca1 and Ca2 were fixed and unmanned cameras, closer zoom-in details cannot be captured, especially when the teacher was on the peripheral of the room where the computers were located. This problem would be exacerbated in my module as I planned to spend a significant portion of class time for students’ hands-on work with the computer, where I would walk from computer-to-computer to guide their explorations.

For the video-recording in the actual geometry module, I decided to engage a videographer to man Ca1. In so doing, Ca1 could be trained on me as I moved around in class. Furthermore, its zoom could be adjusted accordingly to capture the relevant details—the camera would zoom-in closer when I engaged in a conversation with a pair at their computer terminal; and when capturing my teaching actions in front of the class, the camera would zoom-out to the appropriate frame-size to capture both my gestures as well as the details on the board. I also carried a portable audio recorder to better capture the verbal exchanges as I engaged in discourse with small groups of students. This improved arrangement served the purposes of the research in capturing sufficiently-rich data on my teaching actions and discourse. No further changes were made to the video-recording setup throughout the course of the module.
METHODOLOGICAL APPROACHES AND METHOD

Key:  - Video camera. Smallest angle indicate direction of focus.  * - Table-top flat microphones.  Wn – Workstation n for pair na and nb

Figure 18: Plan of the Math Lab

Whiteboard and screen
3.2.4 Preparations for Computer Work

Two weeks before the start of the module, I went to the math room to load the Sketchpad files into the students’ computers so that students could easily access the computer task when the module began. I also took the opportunity to do a final check that the hardware environment was ready for my lessons. I noted some minor hardware problems—such as a diskette reader not working for a computer and the remote control for the projector not functioning well. The deficiencies I spotted were compiled and sent to the Head of Technology in the school for his information and follow-up actions. Upon my request, the school also provided two sets of laptops to be stored in the math room. They would serve as back-ups if some of the desktops in the lab malfunctioned during the module.

However, I was informed a few days later that the Sketchpad files that I earlier loaded in the hard disk were automatically erased as soon as the computers were shut down. Only an authorized ‘admin user’ could save files permanently to the hard disks. Thus, I changed the mode of operation by using diskettes instead of loading the files directly into the computers’ hard disks. I duplicated twenty sets of files into diskettes—one for each pair of students. I passed them the diskettes before each lesson. After students completed and saved their work, I collected their diskettes at the end of each lesson.

3.2.5 Relevant Data

There are three primary categories of raw data that were gathered in the attempt to capture the complexities of teaching throughout the period of the study:

1. Documentation of acts of teaching in the classroom
2. Documentation of the work of students
3. Teacher reflections of classroom experience and planning

The method to collect the first category of data was primarily video- and audio-taping, as detailed above. To supplement these sources of data, written aids that I used in class, such as writings on the board, overhead transparencies, and screen projections from the computer were also collected.

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6 The school’s lab maintenance and policy issues are such that I cannot be granted ‘admin user’ status.
The second category of data relates to the tasks that students engaged in both within and outside the classroom. These data were relevant to the project as they revealed students’ responses to the teaching that took place in class. The data collected under this category were artefacts that teachers normally expect students to produce. These included copies of students’ notebooks, homework submissions, and computer-recorded logs of students’ interactions with the Sketchpad tasks.

The third category of data was the reflection on my classroom experience and plans. Data on my view as the teacher were obtained through my journal writings and reflections before and after each lesson. The before-lesson writings were lesson memos that focused on lesson preparations which took into account all the goals of teaching that were brought on board in this study. It also factored in the reflections of previous lessons in which there might be issues for the lesson-in-preparation that I might need to tackle. The after-lesson reflections took place immediately after each lesson on the same day. The reflections were made while reviewing the video-recording, noting, post hoc, my account of the thoughts and decision-processes I had undertaken at various stages of the lessons. Reflections of my thoughts behind the planning of the module was also collected.

While it was part of my research agenda to collect relevant data for the study, I was also very conscious of my responsibilities of teaching within the project. In planning to carry out successfully both the research and teaching agendas in class, I took heed of Wong’s (1995) post hoc suggestion that he “might have concentrated primarily on teaching during class and then later reflected back on the class as a researcher” (p. 25) as a way to lessen the purported conflict between the teaching and research functions in his study. I similarly considered that some form of separation of roles of researcher and teacher across the boundary of the classroom may be helpful in reducing the tension in carrying out the respective agendas in class. I therefore made a conscious decision for the teaching agenda to be my main focus during class time, and when I was outside the classroom I reflected and scrutinised my teaching with research goals primarily in mind. As for carrying out the research agenda in class, I relied on the research consciousness that I brought with me into class and the teaching materials used which were designed with the intent to carry out the import of the research requirements. The “separation” that I refer to here is a deliberate administrative decision on my part as the teacher-researcher to serve the dual purposes of my teaching and research.
work better. It was not intended nor perhaps even possible to actually separate the two functions altogether at any point in the entire teaching/research enterprise.

### 3.2.6 Preparing Data for Analysis
The data collection described above provided multiple sources of data. All the video data of my instructional work in class were converted into text through transcription. The transcribed text was arranged in chronological order according to the lesson sequence in the module. The transcribed video lessons formed the primary source of analysis. The textual form of the video data was then coded based on the goals architecture mentioned earlier. The detailed description of the codes used will be given in the next chapter.

My pre-lesson plans and post-lesson reflections (transcribed from audio form into text) were also inserted into the appropriate temporal junctures of this primary video data source to enrich my *post hoc* interpretation of my instructional work in class. These materials served sometimes as supporting evidence of my initial analysis of the video data but at other times contradict my preliminary conjectures, thus necessitating deeper examination of the surrounding data. This process of finding incoherent evidence which prompted deeper probing was often repeated until an overarching theme emerged that could satisfactorily account for what may be observed in the multiple data sources.

Whenever relevant, additional data features in the form of students’ work—computer work, homework, or written notes—were also checked to provide a richer account or background of classroom work. This referencing is especially useful in the context of analysing teacher’s interactions with particular students or groups, where the subject of such discussions usually centered on the computer work or paper-and-pencil work that these students were producing. In such situations, having access to the actual work students were doing provided the basis of analysing the instructional goals carried out at that juncture of the module.

### 3.2.7 Unit of Analysis
In the study of teaching, which is a complex activity, it is generally agreed among researchers that it must be parsed into units of analyses so that teaching can be studied. The literature on classroom research indicates that the size of the unit of analysis varies greatly. It can be as “small” as a few words or as “big” as a whole-year teaching programme. Schoenfeld,
Minstrell, and van Zee (2000) chose the predominant unit of analysis as an utterance—“[t]he analysis, often carried out on a line-by-line level …” (p. 281)—in their study of a classroom lesson. Hiebert, Gallimore, and Stigler (2002) suggested that to accumulate a knowledge base of teachers’ craft knowledge, “one possible unit of analysis is a natural one for teachers—daily lessons” (p. 8).

Lampert (2001) proposed that units of analysis can be of flexible sizes to suit the different purposes of inquiry. She suggested that the decision on the “frame size” for analysis should be based on “workable representations” that appropriately captures the complexity of the teaching practice. By that she meant that the unit of analysis should be one that “could achieve the purpose of representing the complexities of teaching for productive communication about the problems of practice” (p. 43).

I share Lampert’s view of the use of flexible adjustment of the “frame size” to capture the complexities of teaching that addresses the research purpose. The method of analysis used in this study incorporated this use of flexible frame size to examine teaching practice depending on the details required. It can be an at-the-moment frame to zoom-in on a specific problem faced by a particular student or a wider lens frame to zoom-out over a temporal range of a few lessons to look at the students’ class work over a period of time. This use of a variable “unit of analysis” will be more clearly exemplified when goals of teaching are discussed in the next chapter.

3.2.8 Phases in Data Analysis
Consistent with the method of “progressive widening of focus” explained earlier, I proceeded with the analysis in three phases as indicated in Table 4.

The detailed analysis of phases I, II, and III will be reported in the respective Chapters 5, 6, and 7 of this dissertation. The task of the next chapter is to set up the goals framework upon which the analysis reported in the subsequent chapters was built. Chapter 4 is thus somewhat a hybrid of a conventional “methodology” chapter and an “analysis” chapter. It explicated the goal-based methodology by describing my goals of teaching at various grain sizes. In addition, it performs the work of analysis in the form of coding teaching actions using the underlying goals-in-operation. It is perhaps useful to distinguish the emphasis of “analysis” in Chapter 4 and that which is performed in Chapters 5-7. The former is concerned with
illustrating the process of mapping teaching actions into instructional goals; the latter focuses on how these goals and their interactions can address the main questions of this study—about problems and coping strategies. Figure 19 shows the different foci of analysis diagrammatically.

<table>
<thead>
<tr>
<th>Phase</th>
<th>Activity of analysis</th>
<th>Purposes</th>
</tr>
</thead>
</table>
| I     | Detailed analysis of one lesson                                                     | Develop a workable framework to look at problems of teaching and coping strategies.  
Conjecture some responses to the research questions based on the analysis |
| II    | Use the framework developed in Phase I to analyse the other lessons (not including regions where Sketchpad was used) | Extend the framework to analyse broader regions beyond one lesson  
Test, refute, and refine the conjectures proposed at Phase I in the light of more evidence  
Report new findings about problems and coping strategies not conspicuous in the earlier one-lesson analysis during Phase I |
| III   | Focus the analysis on those regions of the module where Sketchpad was used          | Investigate how Sketchpad-use may alter the nature of problems and coping strategies discussed in Phase II |

Table 4: Analysis of the data in three phases

<table>
<thead>
<tr>
<th>Chapter 4 analysis</th>
<th>Chapter 5-7 analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demonstrate the relationship between Teaching actions</td>
<td>Study the link between Goals (interactions)</td>
</tr>
<tr>
<td>Goals</td>
<td>Problems and coping strategies</td>
</tr>
</tbody>
</table>

Figure 19: Different foci of analysis between Chapter 4 and the subsequent chapters
The research questions of this study are directed at the problems of teaching and the coping strategies used. In Chapter 1, I defined a “problem” as a hindrance against the successful carrying out of one or more goals of teaching. “Coping strategies” are the mechanisms used by the teacher to advance, perhaps partially, some of the goals of teaching despite the constraints posed by the problems. Thus problems and coping strategies are identified in relation to goals of teaching. This necessarily leads to a need to carefully discuss my goals of teaching. A key methodological approach I adopted in this study is thus one that views teaching behaviour as attempts to carry out some goals of teaching.

The method I used to examine my teaching is similar to the goal-based methodology adopted by the Berkeley researchers (Aguirre & Speer, 2000; Schoenfeld, 2000; Schoenfeld, et al., 2000; Sherin, et al., 2000; Zimmerlin & Nelson, 2000) reviewed in the previous chapter. The task in this chapter is to demonstrate the method I used to relate my teaching actions to my goals of teaching. I begin by first stating my overarching goals of teaching—arising from my beliefs and historical experience in teaching and research. I distinguish a second set of goals that are more fine-grained and that emerge from the improvisational work of teaching in the classroom. These goals are then used to code my teaching actions in one lesson I taught within the module. The chapter ends with a section that relates my two sets of instructional goals.

4.1 My Goals of Teaching

I view my goals of teaching as consisting of two kinds. The first set of goals was determined even before I stepped into the classroom. These goals were shaped by my previous experience as a secondary mathematics teacher as well as my interest as a researcher. Their strong influence over me as the teacher was manifested in my classroom instruction throughout the course of the eleven lessons. These goals were thus “overarching goals” (Schoenfeld, 2000) of the whole teaching project and were influenced by both the teaching and the research experiences in me as teacher-researcher.
The second set of goals was more local and immediate to the execution of lessons or even part of a lesson. These goals were subordinate to the overarching goals in the sense that they supported the accomplishment of those bigger goals. While some of these goals might be pre-planned in the sense that they are factored into the pre-lesson plans, others may emerge during the process of teaching (Saxe, 1991). That goals may emerge and evolve is not surprising, especially when we take into account the highly improvisational nature of teaching, as reviewed in Chapter 2 (e.g., Chazan & Ball, 1999; Lampert, 2001, King, 2001). This second set of goals thus allows for “the existence of goals that arise in response to planned and unplanned events” (Aguirre & Speer, 2000). This separation of goals of varying grain size into the longer-term goals of the module and the shorter term (even moment-by-moment) goals of the immediate teaching context is similarly employed in the projects undertaken by the Berkeley researchers.

The two sets of goals thus differ in the level of grain size from which one looks at the teaching enterprise. The overarching goals are broad ‘zoom-out’ goals that underpin my overall teaching agenda, and particularly cover the whole geometry module; the other goals are narrower ‘zoom-in’ goals that motivate how I plan a lesson or part of a lesson and account also for emergent goals that arise in the classroom. It is also important to note that these classes of goals differ in their purposes in this study and in the way that they were identified. My overarching goals of teaching were identified through introspection by considering my beliefs of teaching and my historical experience both as a researcher and as a teacher; the other more fine-grained goals were identified through the analysis of the multiple forms of data about my teaching\(^1\). As much of the analysis in Chapters 4-6 requires frame sizes that cut across different levels of detail, the second set of goals were used predominantly. The overarching goals will be more useful when I look at more broad-grained issues in the closing chapters of this thesis.

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\(^1\) This process will be explained in the later sections of this chapter.
4.2 Focus: Overarching Goals of Teaching Geometry (Excluding the Use of Sketchpad)

For this chapter, I will limit the description of my overarching goals to those I consider most essential to the work of “teaching geometry”. For this purpose, I have initially excluded the consideration of overarching goals that are more specific to the use of Sketchpad in teaching. Chapter 7 will deal with goals, problems, and coping strategies when Sketchpad-use in the teaching of the geometry module is brought into more prominent focus. This gradual multi-phase analytical process is in line with the method of “progressive widening of focus” reviewed earlier in Section 3.1.4. A brief review of the method here: in examining my instructional practice, my first foray will start with a region of the complex ‘picture’ of teaching that is of moderate size and connectivity. The first part of the picture on which I will train my focus is the work of teaching geometry\(^2\) (excluding Sketchpad use). The initial exclusion of Sketchpad use is to keep the analysis manageable and yet of adequate complexity that is representational of actual geometry teaching.

Yet, even with excluding goals directly related to Sketchpad usage at this stage, examining the overarching goals of “teaching geometry” alone covers much of the overall teaching agenda in the module. From viewing the actions of my teaching through the videos, it is clear that the overarching goals of teaching geometry were pervasive in each of the lessons in the module. Moreover, these goals were not merely present; they were often conspicuous and even dominant\(^3\). “Teaching geometry” is thus an appropriate place to start the progressive widening of focus in gaining an insight into the complexities of classroom teaching.

4.2.1 Influence of Beliefs, Knowledge, and Research Experiences to My Overarching Goals
My prior teaching experiences shaped my beliefs and knowledge of teaching, students, and mathematics. Intensive research has focused on the nature and evolution of beliefs and how they affect teaching behaviour (e.g., Aguirre & Speer, 2000). For

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\(^2\) For simplicity, I will, for the rest of this dissertation, use the shortened form of merely “geometry teaching” (or its equivalent) to refer to that portion of teaching geometry which does not include the prominent involvement of Sketchpad.

\(^3\) Evidences of these assertions will be given in the course of this chapter.
the purpose of this study, it suffices to note that researchers agree that such beliefs play a major part in guiding teachers’ goals and actions (Ball, 1991; Cuban, 1993; Malara & Zan, 2002).

My more recent involvement in research also shaped my teaching goals. The influence of my research experience was in a mixing of theoretical knowledge of mathematics teaching—to which I had easier access in my new capacity as a researcher—with the existing knowledge and beliefs acquired in practical teaching experiences. A specific example of how knowledge of van Hiele theory affected the conception of my teaching goals is explored in the next section. This amalgam of knowledge and beliefs derived from both teaching and research experiences helped determine my goals for the geometry module.

4.2.2 My Overarching Goals for Teaching Geometry

My beliefs are both personally-owned as well as built up through perceptions of my social role as a mathematics teacher in Singapore. The socially-influenced character of my belief system therefore takes into consideration the wider schooling culture and the mathematics curriculum in Singapore. In Singapore, the mathematics curriculum in the schools is centrally determined by the education authority of the country. The secondary schools, except for a small minority that cater for high-achievers, adhere to this common curriculum. The scope and objectives of the mathematics curriculum for all grade levels are spelt out in curriculum documents from the education authority. Individual schools in turn elaborate on these curriculum requirements and list what a teacher needs to teach on a week-by-week basis for each semester. Nested within these weekly targets are day-by-day curriculum components. This translates into school-level teaching schedules: school administrators give teachers lists of mathematical topics with time allocations for each topic to be taught in class. Although schools vary in the actual topics covered at each year-level, the practice of teachers following a sequence of topics governed by strict time frames is the modus operandi across schools. One implication of the need to adhere to time-bound topic coverage is that mathematics teachers must be mindful of the utilisation of class time, to cover the curriculum within the time given. When I took over the class from the

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4 For more discussion on the Singapore mathematics curriculum, refer to Section 2.5.2.
responsible teacher for the geometry lessons, it was natural for me to take it as my professional responsibility—as the teacher—to complete teaching what the school had originally expected the resident teacher to cover. Thus, one of my goals for teaching the geometry module was to cover the school’s stipulated syllabus content within the given time period.

Another influential factor in determining what and how mathematics is taught in Singapore schools is the end result of the schooling process—assessment. In Singapore, students sit for a centralised examination, designed by the education authority, at the end of their secondary schooling. The students are evaluated not only at the end of their schooling but are also required to sit bi-yearly for school-based examinations. This leads to a pervasive examination-oriented culture in the schooling environment in Singapore. The school and the society place heavy emphasis on students’ outcomes in examinations. This is, in part, due to the placement implications for future schooling and the perception that an individual’s self-esteem is related to examination performance. The high value placed on assessment grades can be traced to a pervasive meritocratic system of government which has become entrenched over the last forty years. I therefore see it as my social responsibility, as a teacher, to teach students mathematics in a way that will prepare them for the impending examinations. To me, this responsibility includes the teaching of mathematical skills, procedures, and techniques that are directly relevant to tackling exam-type questions.

To me, however, teaching geometry is not merely about covering syllabus and preparing students for subsequent examinations. My own passion for the beauty of the internal structure, consistency, and logic of mathematics requires that I expect the students I teach to be able to glimpse something of this beauty. This appreciation of mathematics was kindled when I read mathematics as a major in my undergraduate days and remains a glowing flame throughout all my years of teaching.

For me, mathematics teaching should aim to develop students’ ability to make deep meaning of what they are able to do on the surface. In the context of geometry

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5 More details and literature about the influence that meritocracy exerts on the Singapore education system is found in Section 2.5.1.
teaching, the endeavour in this study, it means aiming not only at students’ proficiency at answering questions or standard textbook problems, but also at students’ ability to provide geometrical explanations for their answers. An example might illustrate this point. I may show a picture of a rectangle and ask the students what the diagram shows. They may be able to answer, “rectangle.” But I expect students to go deeper than that. They should also be able to provide a geometrical reason to a follow-up question: “How do you know that this is a rectangle?” While the surface justification would possibly be visually-driven, such as “it looks like one”, a deeper geometrical consideration would require students to conceive the rectangle in terms of its geometrical properties, such as perpendicularity of adjacent sides, and equality of opposite sides. My teaching is intended to move the students’ focus gradually towards the latter conception of geometry, with increasing degrees of abstraction.

The above example can be recast in the language of van Hielean literature. Interpreted through the van Hiele stage theory, students who recognise geometrical objects only by their shape and gestalt features are operating at the “Visual” or “Recognition” level. Teachers, through structuring suitable activities, can help students advance towards higher levels of geometry competency. When students can view a geometric object not only in its visual form but also as a holder of properties, they enter into the next stage of “Analysis”. Students who can make deductive relationships among properties have reached the higher “Ordering” or “Relational” level. Then, at the “Deduction” level, students can harness geometrical definitions and theorems to construct mathematical proofs. Seen through the van Hiele framework, one of my goals in this module was to help students make progress towards higher levels of geometric thinking.

Lampert (1988) advocated a mathematics classroom pedagogy that mirrors how knowledge is constructed within the mathematics discipline. For her, how mathematics is done in the discipline informs how mathematics should be done in the classroom. Since mathematical knowledge is constructed via reasoning among

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6 As the van Hiele levels are discussed in greater detail in Section 2.2.1, only a brief illustration is given here.

7 More details of Lampert’s discipline-based pedagogy were given in Section 2.3.2.
practitioners within the mathematics community, learning mathematics within the classroom should involve mathematical reasoning among students. In other words, seen through Lampert’s discipline-based pedagogy, teaching geometric reasoning would mean structuring an instructional setting that allows students to reason—conjecture, refute, evaluate, substantiate, justify, and explain—about geometrical ideas in a way governed by the logic rules within the discipline.

I share this goal of bringing mathematical reasoning into the classroom. As highlighted in Chapter 2, mathematical reasoning in the context of a geometry module is as much about the exploration of geometrical ideas and tools (inductive reasoning) as it is about harnessing them to present an argument (deductive reasoning). Therefore, in teaching geometry, I value students’ observations, their preliminary attack routes to problems, and their agreements/disagreements. All these contribute to the inductive process, as well as their attempts to substantiate and structure their explanations. This goal of encouraging inductive and deductive reasoning is not merely mine. As reviewed in Chapter 2, it is also a goal stated in the Singapore mathematics curriculum in keeping with the reform vision of TSLN (Thinking Schools, Learning Nation).

Quite apart from social, political, and content-related influences, I also have a personal belief about student capacity that sustains my interest and motivation in teaching. This belief is that all students in my class are able to do mathematics, in the sense that each can attain the basic level required by the curriculum. When expressed as a goal of my teaching, it would be that I strive to teach every student according to his/her abilities, especially helping the less proficient ones to achieve the prescribed basics.
A summary of my goals for teaching geometry can now be presented.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>To cover the geometry content allocated within the time frame given by the school</td>
</tr>
<tr>
<td>G2</td>
<td>To prepare students to tackle exam-type questions from the topics within the geometry module</td>
</tr>
<tr>
<td>G3</td>
<td>To help students progress to higher van Hiele levels of geometric competence</td>
</tr>
<tr>
<td>G4</td>
<td>To encourage students to reason within the geometrical domain</td>
</tr>
<tr>
<td>G5</td>
<td>To help every student meet the curriculum objectives</td>
</tr>
</tbody>
</table>

Table 5: Summary of my overarching goals of teaching geometry

The presentation of these teaching goals in a list is not to suggest any order of priority. More significantly, I do not suggest that they are easily separable when observing actual teaching behaviour. In practice, classroom events may fulfil one or more of the goals. Also, the goals may not be independent of one another, since the fulfilment of one may support or hinder the fulfilment of others, with the latter creating a “problem” situation. The interaction between goals during practice is a complex one, and the delineation of my teaching goals in point-form is partly for ease of reference.

There is also no claim of comprehensiveness in the list of goals presented; the stated goals are those that I perceive to be directly related to “teaching geometry”. Other goals of teaching—such as teaching students to follow school rules and to respect one another—are not included in the list of goals here as I do not view them as being central to the work of teaching geometry. Moreover, this study does not claim to examine all the constituents in the complex ‘picture’ of teaching. I aim rather to tackle those components of practice that contribute most substantially to the problems of teaching geometry. This implies that those parts that are at the fringes of the ‘picture’ are beyond the scope of this study.

At this juncture, a reader may rightly question the inclusion of “mathematical reasoning” here as a goal when “integrating technology” has been excluded, since both reflect the twin reform initiatives stated in the research design. It is first useful to restate that the focus in this chapter is on the core supportive goals involved in “teaching geometry”. As geometric reasoning is so tightly linked to the discipline of
geometry, it would artificially distort of the representation of teaching if the former is left out from the main objectives of “teaching geometry”. The other reform initiative of “Integrating technology”, however, is not as integral to the work of geometry teaching. While one can contemplate teaching geometry without technology, to teach geometry devoid of reasoning is not to teach geometry at all. Studying the work of teaching geometry without looking at the associated reasoning would be to examine a region of the complex ‘picture’ of teaching that is not well-connected. This goes against my earlier argument in Section 3.1.4—when discussing the method of “progressive widening of focus”—of the need to select an initial region that is of moderate connectivity. For this reason, while “teaching reasoning” is integral to “teaching geometry” and should thus be considered at this stage of the analysis, Sketchpad usage is not essential to “teaching geometry” and can be analysed at a later stage. It will be discussed separately in Chapter 7.

It is useful to clarify that the goals of geometry teaching described above are mine, influenced by how I view my social and professional roles as a teacher in relation to my interpretation of the social and cultural emphasis of mathematics education in Singapore. They are also shaped by my stance towards how relevant research literature on geometry and thinking can be incorporated into my teaching. When I enter the classroom, these teaching goals accompany me and influence my classroom practice because they have become intertwined and inseparable from the belief and knowledge system within me. They are rightly termed as my teaching agenda because they show my perspective of the teaching task, and this perspective may not all be shared by other teachers.

Nevertheless, it is heartening to know that some of my teaching goals are similarly shared by other teachers, even though the cultural setting in which we operate may be different. In Wiske’s (1995) project, which involved collaborative work with teachers, he reported that the teachers viewed their primary resource for their instructional tasks to be curriculum materials. In particular, “teachers defined the [situation] in terms of the types of problems, taken from their texts, tests, and workbooks….” (p. 193). This reliance on curriculum documents to guide instruction seems to strike a close match to the curriculum-driven and exam-oriented focus of my
teaching agenda as reflected in G1 and G2 above. As to wanting all students to meet curricular objectives [G5], Newman, Griffin, and Cole (1989) encountered teachers who see students’ achievement as the major motivation of their teaching endeavours. They found that “for the teacher, it is important to find ways in which children can succeed as well as possible in their academic work” (p. 145). Regarding G4 and G5, the abundance of literature on van Hiele theory and the teaching of reasoning reflects an ongoing interest in the wider research community on these aspects of teaching. Thus, although the teaching agenda that I bring into this study is indeed my own in the sense that it is conceived and implemented by me as the teacher, there is an added sense that the teaching goals are shared with and can be appreciated by a wider group of teachers and researchers who may be familiar with these objectives.

4.3 The Subordinate Goals

As discussed earlier, I distinguish a second set of goals that are more fine-grained compared to the overarching goals and which emerge within practice. I looked for these goals by examining the transcriptions of the video recordings of classroom instruction and the post-lesson comments which were reflections of my thought processes during teaching. These goals were not necessarily spelt out in advance of instructional events; rather, I studied my teaching plans, actions, and reflections to determine the underlying goals that drove my instructional behaviour. I did not arrive at all these goals in one attempt at examining the data. I started by coding the first lesson of the module with broad goal categories. As I proceeded in the coding process from lesson to lesson, I modified the goals gradually through a process of repeated refinement—by adding new goals to account for practices as I viewed more data, and by collapsing and rephrasing some goals so that they could capture a wider spectrum of actions. The final list of goals derived from the study of the data is shown in Table 6. The first-letter labelling of the goals start with a small letter “g” to reflect their subordination to the capital-lettered “G” overarching goals discussed earlier. The 3-letter labelling of the g-goals in Table 6 gives a summary of what the goal is about for ease of reference.
There is no claim that this is the only way to represent my goals of teaching geometry. Nevertheless, these seventeen goals sufficiently account for all my instructional behaviour exhibited in the module in relation to “teaching geometry”. This point about sufficient coverage will receive partial substantiation in this chapter but will become gradually clearer as data from more parts of the module are examined in subsequent chapters.

There could be an understandable concern that someone else looking at the data may identify my goals differently or may even be able to reduce the set of goals to a smaller number. However, it is important to realise that in trying to look for my goals from the data (which is what this study is about), the outside observer cannot do so directly. By looking at what a teacher does, an observer in class may only make guesses of what the teacher was trying to achieve, by interpreting the teacher’s behaviour. My actual goals of teaching can only be accessed via my thought-world while I was carrying out actions in class. This is where my assuming of roles of both teacher and researcher was advantageous. Through the simple activity of
“introspection”, the teacher-in-me can answer all the questions that the researcher-in-me wants to ask, such as “What were you intending to achieve at that time when you were doing that …?”

Nevertheless, it is important to provide evidence of the occurrences of these goals in my actual work of teaching. One way to provide evidence that all these g-goals were indeed conspicuous in my instructional behaviour during the teaching of the geometry module is to proceed goal by goal—to draw out the relevant supporting data to match each goal. However, to proceed this way may present a rather fragmented picture of teaching, with teaching episodes and reflections disjointed from one another. In keeping with the spirit of presenting teaching with all its inherent interacting complexities, I choose to describe in narrative form how all the g-goals came into play in one lesson that I taught. Moreover, narrating in continuous teaching sequences is in keeping with another overriding purpose of this chapter: to illustrate how I relate actual teaching behaviour in class to my underlying goals.

That a lesson is a suitable unit of consideration here can be seen in two ways. Firstly, a lesson is commonly viewed as a natural “frame size” for both teachers and researchers to study the essential instructional practices of a teacher. Secondly, and as it turned out in this case, there is almost⁸ the right amount of data in the lesson selected to provide evidence of all the g-goals—less data may not account for all the goals and more data may be overwhelming.

The choice of which lesson of the eleven to examine is reduced to a choice between two lessons: Lesson 8 and Lesson 11. This restriction is due to the present focus of teaching geometry without the involvement of Sketchpad. All the other lessons had a prominent Sketchpad component during lesson time and are therefore less suitable for the current purpose. Lesson 11 was primarily about teaching students how to construct perpendicular bisectors and angle bisectors using a straightedge and a pair of compasses. Lesson 8 is richer in terms of the variety of geometrical tasks presented to the students. Besides construction tasks, students were also required to solve other types of geometrical problems. This greater variety of tasks also led to the

⁸ The reason for the use of “almost” will become clearer in section 4.3.6.
greater variety of g-goals displayed in instruction. For this reason, Lesson 8 is the most appropriate candidate for the current focus.

The way I will proceed with the description of the lesson is by first highlighting the macro-components of the lesson and subsequently zooming in to more and more fine-grained components nested within each component or sub-component. As a tool to help me analyse, as well as to provide a language to denote the degree of grain-size of the lesson I am looking at, I use the analogy of ‘tiers’ within a hierarchy. The ‘first tier’ refers to the major components of the lesson; the ‘second tier’ refers to the next-level sub-components of the components identified in the first tier; subsequent ‘tiers’ are further breakdowns of sections nested within the previous ‘tier’. The beginning of a new component or sub-component is identified through an observed shift in the nature of teaching. Motivations for these shifts included changes in instructional content, changes in teaching goals, or movements to different levels of interaction (such as from whole-class instruction to teacher-student supervision of seatwork). To exemplify all seventeen goals, there is no need to study all the fine-grained details of the lesson, so zooming in to the most fine-grained descriptions of teaching actions will only be done selectively.

4.3.1 Overview of Lesson 8 – a First-tier View

Before each lesson, I wrote a lesson memo that sketched my overall plan to guide my lesson sequences. The written lesson memo for Lesson 8 had two headings describing the planned main components of the lesson:

I. Textbook exercises 14b
II. Construct quadrilaterals

The lesson memo elaborates under on “Textbook exercises”:

I begin here because this is what I intended to do in the last lesson but did not quite get to it. This is important insofar as the goal of teaching students how to attempt textbook exercises (in preparation for tests) is concerned.

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9 Both the lesson memos and the post-lesson reflections are part of the data collected for this study. Refer to Section 3.2.5 for a description of the relevant data for this project.
10 14b refers to the second set of exercises in Chapter 14 of the textbook used
Similarly, in my post-lesson reflections\textsuperscript{9} of Lesson 8, I mentioned that

Today I went through with them mainly two broad categories of ideas. One is to revise the properties of quadrilaterals, mapping it more directly to the textbook exercises which require the knowledge of the fact that “interior angles of the square adds up to 90°”, “opposite angles of parallelograms are equal” and others. This is to check that they can apply their knowledge of all these properties into the given exercises . . .

A later section of my reflections clearly identifies the target of the second component of the lesson:

The second part of the lesson is essentially to help them [the students] perform the construction [of quadrilaterals].

Thus, the first part of the lesson on “Textbook exercises” was planned and conducted with teaching students to solve textbook exercises primarily in mind [gsol], while the subsequent section of the lesson on “construct quadrilaterals” was essentially meant to help students perform geometrical constructions [gins].

I earlier asserted that more than one teaching goal can be at work at any part of a lesson. Thus, when I associate a section of a lesson to a particular goal of teaching—as in the above paragraph—I am not contradicting this earlier stance, but merely stating what was to me the most prominent motivation for carrying out that particular component of the lesson. Taking the “Textbook Exercises” section of this lesson as an example, other goals such as “revising geometrical properties of quadrilaterals” [gmrn] were certainly present, but what was at the foreground of my efforts then was producing geometrical constructions [gins].

At the “overview-of-lesson” level only the most prominent goals are uncovered, but other goals will become evident as I delve into increasingly finer-grained details of the lesson. In the language of the ‘tiers’, it can be said that the first-tier analysis of the lesson revealed two main sections, “textbook exercises” followed by “construction”.
A second-tier breakdown of the lesson can proceed along the same line by examining each of these sections of the lesson to find out the component parts.

### 4.3.2 Zooming in to the Second Tier

I started the first section of the lesson by writing a selection of textbook question numbers on the board and asking the students to attempt to solve them individually. To me, this was an appropriate activity as the geometrical properties involved these problems were discussed in the previous lesson just a day before. In requiring students to try the problems on their own without first seeing a teacher solution, I was hoping to encourage them to test their own effort and persevere with the difficulties they may face. Thus, for this sub-section of the lesson, there was an additional second-tier goal of urging students to give good tries at their work [gtry] nested within the gsol goal associated with the first-tier “textbook exercises” component of the lesson.

After a few minutes, there were conspicuous signs of frustration from the students when they could not make significant progress. I drew two diagrams—corresponding to two of the problems from the textbook—on the board as a signal to the class that I was about to discuss the solutions to these problems with them at a whole-class level. The diagrams, as they appeared on the whiteboard in the front of the classroom at that stage of the lesson, are reproduced in Figure 20.

I then proceeded to teach the students the solutions to item 2c, followed by item 4c. Unlike the earlier subsection where there was a more fine-grained gtry goal embedded within the higher order gsol goal, these subsequent parts of the lesson which dealt with solving specific textbook exercises maintained the first-tier gsol flavour. In these cases, the first-tier and second-tier gsol\(^{11}\) goals shared the same underlying purpose, but they perhaps differ in the specificity of the goal. While gsol at the first tier was about teaching solutions to textbook exercises more generally, gsol at the second tier was targeted more specifically at these particular exercise problems.

\(^{11}\)That g-goals (in this case, gsol) can occur at different tier-level analysis will be discussed in greater detail later in Section 4.3.7
Moving on to the construction part of the lesson, I started by distributing a worksheet comprising different construction tasks. I used the worksheet as a blueprint to sequence the activities for the rest of the lesson. All the subsequent construction work, whether undertaken by the teacher with the whole class or carried out individually by the students, were taken from the worksheet tasks. Due to the centrality of these worksheet items in relation to the work of my teaching, the first four items are reproduced in Figure 21.

Based on previous construction activities with the class I believed that the students were still relatively unfamiliar with constructions involving geometrical instruments. Thus, I began, as planned, by demonstrating to the whole class the task required in the first item of the worksheet—constructing a square. I used a whiteboard-sized setsquare, compass, and ruler to perform a ‘gigantic’ version of what they were expected to do on their worksheets. By surveying the students’ construction efforts as they copied my steps, I paused at suitable junctures of the demonstration—and repeated certain steps when necessary—to allow students to follow the construction process closely as they tried to replicate on their worksheets what I did on the whiteboard. The primary purpose of this part of the lesson was thus to help students perform the construction of the required square correctly [gins].

Figure 20: Writings and drawings on the whiteboard at the beginning of lesson
Complete each of the construction tasks below.

<table>
<thead>
<tr>
<th>Construction task</th>
<th>Tools to use</th>
<th>Constructions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construct a square ABCD. The side AB is already drawn for you.</td>
<td>Setsquare, Compass, Ruler</td>
<td><img src="image" alt="A" /> B</td>
</tr>
<tr>
<td>Construct a rectangle PQRS. The sides PQ and RS are already constructed for you.</td>
<td>Compass, Ruler</td>
<td><img src="image" alt="P" /> Q R</td>
</tr>
<tr>
<td>Construct a rhombus JKLM with $\angle J = 30^\circ$. The side JK is already drawn for you.</td>
<td>Compass, Protractor, Ruler</td>
<td><img src="image" alt="J" /> K</td>
</tr>
<tr>
<td>Construct a parallelogram ABCD with AB = 7 cm, AD = 5 cm, and $\angle BAD = 42^\circ$</td>
<td>Protractor, Compass, Ruler</td>
<td>![image]</td>
</tr>
</tbody>
</table>

Figure 21: Sample worksheet items
According to my memo prior to the lesson, I meant to “demonstrate the first two tasks [the constructions of the square and the rectangle] before letting them try the rest [of the worksheet on their own]”, but when I was performing on the whiteboard construction of the square, my post-lesson reflection revealed that “I realise that most of the students are not quite following me”, and at that point, “I thought a better way was to go table by table to see how I can help each student.” I thus abandoned my earlier plan to demonstrate the construction of the rectangle. Instead, I paused and asked the students to proceed with the worksheet tasks on their own. The main purpose of this change of plan was to enable me to walk quickly from student to student to survey their difficulties with the first task (the square) and to highlight and correct their individual errors [gerr] before those same errors got in their way for subsequent tasks. Data from the video showed that this is, indeed, what I did: walking around the class and teaching them to use the instruments to perform constructions correctly. The following conversation illustrates one such quick interaction with a student who was constructing the square:

34.22 Hassan: This one come down is it? [Using his finger to show an arbitrary ‘by-eye’ drawing of a line meant to be perpendicular to AD which he has already drawn with a setsquare]
T: Not like that. You must make sure that it is 90 [pointing at point D] right? Put your ruler here . . . [demonstrates with setsquare and ruler; the positions of the instruments on Hassan’s worksheet is given in Figure 22.1]

![Figure 22.1: Illustration of a correct positioning of setsquare prior to establishing the next right angle in the construction of the square](image-url)
T: … then you slide … [Figure 22.2 shows the new position of the setsquare after sliding]. Then you get 90 you see? Then draw long long [line] lah. Hor\textsuperscript{12}?

![Diagram of sliding a setsquare](image)

**Figure 22.2**: Illustration of a correct way to slide the setsquare to achieve the second right angle in the square

After pointing out errors in construction to a few students like Hassan and showing them quickly one right way, I went back to the board to proceed with the construction demonstration of the rectangle, adopting a very similar purpose [gins] and mode of instruction as with the square construction earlier.

For the remaining parts of the lesson, I let students continue with the rest of the construction tasks in the worksheet [gins] while I went around the class to supervise their individual efforts. As I looked at the students’ work, I noticed difficulties that some of the weaker students had in their constructions and that their struggles often led to frustrations at being ‘stuck’, unable to progress along with others. The thoughts and the decision that I made in response to that observation are shown in my reflections after the lesson:

Here I was again seeing certain students having difficulties and I thought I should just show [that] one time to those who are keen. Here I do not require all to be looking, but I am just demonstrating so that those who have real difficulties will want to see and follow the

\textsuperscript{12}The ‘lahs’ and the ‘hors’ are local, culturally-influenced components of conversational English. They carry subtle contextually-dependent semantics. In this case, “Lah” is roughly equivalent to “thus” or “like this”, and “hor” equates approximately with “okay?” or “do you understand?”
construction procedure. So I want to focus and let them know that we are trying to make use of JK, KL, JM and ML [referring to the rhombus task] using the scratching of arcs so that the students have something to refer to when they have difficulties in their construction.

I therefore directed my attention primarily to those who had difficulties with the rhombus task [gwkk] and demonstrated the construction procedure step-by-step on the board. I also took pains to emphasise critical things—such as not changing the compass arms when marking the sides of the rhombus—that students need to produce a correct construction.

The class proceeded with their individual attempts on the worksheet [gins] after my demonstration on the rhombus task. As I moved around to continue supervising their work, I noticed yet again that some weaker students remained stuck at almost every question:

Here again as I observed around, there are still some people yet struggling with the fourth question about the parallelogram. I thought there is again a need to demonstrate on board as a reference so that they can refer to the parallelogram later.

Towards the end of the lesson, I drew the attention of those who needed help [gwkk] to watch and follow the steps as I again demonstrated the construction process for the parallelogram for their benefit.

For the ease of the reader’s reference, the summary of both the first- and second-tier components and their respective prominent goals is given in Table 7.

As the second-tier components are still too large to see more fine-grained goals nested within, further zooming in will help reveal these underlying goals. However, due to the sheer volume of data involved as the analysis becomes more fine-grained, only a selection of ‘snapshots’ of these details can be undertaken here. To fulfil the purpose of presenting a variety of teaching modes, grain sizes, and underlying goals, more detailed descriptions taken from representative sections I.1 and I.2 will be discussed.
4.3.3 Zooming in Further into Section I.1

The primary mode of instruction in this section of the lesson was teacher-student interaction. I walked from student to student and engaged in didactic conversations with each of them, with the intention of carrying out instruction in a way suited to their individual needs. An extract showing a few such interactions with students illustrates the variety of instructional needs of different students.
The first conspicuous point to note from the above discourses is the difference in the way I carried out my teaching work with different students. An example is the difference between the teacher-Chong and teacher-Yan San exchanges. Although both these conversations were about alerting students of the need to make clear their geometrical reasons to a reader of their work, I provided more guidance to Yan San compared to Chong. My reasons for this difference in the degree of explicit instruction afforded to different students can be seen through my reflections:
And I realized that from student to student I will use different technique[s]. Some who are known to be weaker [like Yan San] I point to them, give them direction to get them started so that they can move on to the next part of the solution. For some who I know to be more advanced [like Chong], I can afford to slow them down, give them some ideas, pointers, redirect their questions back at them, assure them that they are on the right direction, and then set them going. … And to other students like Syarah, perhaps, she gets discouraged easily, I need to constantly help her, and give her directions. … It is almost subconscious of me that I am giving different treatment to different students based on different goals or different purposes that arose to me at that point in time. And based also on a subconscious assessment … of him or her so far in my interactions based on my observations.

Thus, the underlying goals that accounted for the ‘customised’ approach of teaching were that I wanted to teach both the stronger and the weaker students in ways that suited their needs, thus reflecting goals gstg and gwkk. In addition, for each student, there were specific goals targeted for their unique difficulties. For Chong and Kai, as well as for Yan San, the main purpose was to teach them an overall strategy—use a diagram—to solve the problem of not being able to present their solutions in a way that was intelligible to a reader of their work [gpsv]. In the case of Mel, it was to make explicit to her that her solutions were not adequate [gerr]. For Syarah, there was a need to revise with her some geometrical concepts related to the diagrams that she was seeking to make sense of [gmrn]. With these underlying differing needs and purposes of teaching identified, the same discourses with these students can be represented with the goals-hierarchy shown as in the vignette below.

<table>
<thead>
<tr>
<th>Time</th>
<th>Dialogue</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.55</td>
<td>T: [Looks at Chong's work] Would you be able to write it clearer [than] this way? Because without a diagram, anyone looking at your working will find it difficult, right? Chong: [Nods lightly in acknowledgement] T: Would it be better to draw this diagram [pointing to the diagram in the textbook] Kai: [Joining in the conversation] So we do work must draw diagram? T: Draw a diagram to help people follow your reasoning, yeah? Kai: [Nods in agreement]</td>
</tr>
<tr>
<td></td>
<td>gpsv</td>
</tr>
<tr>
<td>7.26</td>
<td>T: [Looks at Mel's work] Would you like to write some reasonings down instead of just answers? Mel: [Inaudible] T: Provide some working steps also.</td>
</tr>
<tr>
<td></td>
<td>gerr</td>
</tr>
<tr>
<td>8.52</td>
<td>T: [Looks at San's work] This reasoning that you give [pointing to his notebook entry] San, that interior angles add up to 90. You are referring to which angle[s]?</td>
</tr>
</tbody>
</table>
Yan San: [pause]
T: These [pointing to the angles in his textbook diagram], isn't it? Now anyone who looks at your working without the diagram will not be able to follow. So it'll be good to put up a diagram so that you'll be able to show people your working clearly.
Yan San: [Nods in agreement]

10.00 Syarah: [Raises hand to catch teacher's attention]
T: [Walks over] Yeah?
Syarah: Is it isosceles? [pointing to a certain portion of a diagram in her textbook]
T: Which one?
Syarah: This one [pointing to an angle within a triangle in her textbook]
T: Which triangle? You point to me.
Syarah: [Points to the same angle]
T: This one? [pointing at the same diagram but using his finger to point at the triangle instead of pointing at an angle in the triangle] Is it isosceles? You mean this triangle [repeats the pointing] - is it isosceles? What do you think? Did they say that it is? What is an isosceles triangle?
Syarah: Base angles the same
T: Base angles the same. Did they say they are the same? No. Then you cannot assume.

4.3.4 Zooming in Even Further into Section I.1

While the purpose of the episode teaching Syarah was primarily that of revising geometrical ideas related to isosceles triangle, contained within this discourse were teacher-Syarah exchanges that revealed other nested goals. These moment-by-moment goals that emerged during the interaction were spontaneous in nature but nevertheless reflected larger teaching goals.

In teaching Syarah, apart from getting her to know the critical attributes of an isosceles triangle—that “base angles are the same”—I was concerned to give her the opportunity to verbalise what she knew about isosceles triangles. Instead of beginning by telling her directly what an isosceles triangle was, the discourse showed instead that I provided an outlet to articulate her conjecture—when she pointed that “this one” was isosceles—and her explanations that base angles are equal for isosceles triangles [gart]. The single goal, gart, however, does not fully capture my intentions. In refusing to answer her question directly, and by ‘bouncing back’ her questions, there was an effort to shift the responsibility for answering questions from the teacher to the student [grel].
There were other goals in operation too. When I confirmed that Syarah was pointing at an angle of the triangle and asked, “Is it isosceles?” it seemed to me that, apart from other misconceptions, she was using the wrong language to convey her doubts. While it was correct to refer to “isosceles triangle”, it was incorrect to talk about “isosceles angle”. Thus, without explicitly saying that she was using incorrect language, I re-phrased the question from “which one?” to “which triangle?” and later “you mean this triangle” with the intention of signalling to her, in an unobtrusive way, the correct language to be used. The other error apparent in her understanding was her assumption that the triangle was isosceles. I therefore drew her attention to this false assumption that she made, and to the need to go beyond a visual-based view of the triangle to one determined by the attributes stated in the question.

Taking this further zoomed-in view of the teacher-Syarah discourse, the episode of “teaching Syarah” can be seen with the included nested goals, as shown in the reproduced vignette below.

10.00 Syarah: [Raises hand to catch teacher's attention]
T: [Walks over] Yea?
Syarah: Is it isosceles? [pointing to a certain portion of a diagram in her textbook]
T: Which one?
Syarah: This one [pointing to an angle within a triangle in her textbook]
T: Which triangle? You point to me.
Syarah: [Points to the same angle]
T: This one? [pointing at the same diagram but using his finger to point at the triangle instead of pointing at an angle in the triangle]
Is it isosceles? You mean this triangle [repeats the pointing] - is it isosceles? What do you think? Did they say that it is? What is an isosceles triangle?
Syarah: Base angles the same
T: Base angles the same. Did they say they are the same? No. Then you cannot assume.

Post hoc analysis of this segment may reveal other causes for Syarah’s use of “isosceles angles” that I did not pick up in class then. However, this study is about analysing actual teaching goals that are carried out in instructional practice. Alternative interpretations of classroom interactions make for interesting study but are beyond the scope of this research.
4.3.5 Zooming in Further into Section I.2
Having looked closely at the teacher-student interactions in Section I.1 of the lesson, I now turn my attention to other types of goals involved in a whole-class instructional segment in Section I.2. This section of the lesson, involving finding angles in the parallelogram shown in problem 2c of Figure 20, can be further subdivided into three natural third-tier parts—I.2i, I.2ii, and I.2iii. The first part was a very brief one, essentially serving as a transition from their earlier individual attempts in Section I.1 to what I was intending to do in Section I.2 of the lesson. Despite its brevity, some underlying goals can be discerned in my teaching comments:

Line 1  Ok we look at this one [pointing to the diagram of 2a on board] first.  
2  Again I see very interesting tries as I walk around.  
3  When Mr Leong walks around it's not just to check your work but to learn from you, remember?  
4  Sometimes I only think of it one way, but as I look around I realise "Wah! You also did it in other ways".  
5  You also did it in other ways”.  
6  So I learn to pretend [to present the solution] to be [like] you.  
7  And then you help me check, can? To see whether it's right.  

These words were meant to encourage the students that their good attempts were noticed by the teacher, thus giving them a sense that good tries on their part were recognised [Line 2] so as to build up their confidence in their ability [gcon]. To further boost this confidence, I told them I would harness some of their ideas [Line 6] in my later teacher-presentation of the solutions to the whole class. Apart from attempting to raise their self-belief in their ability to do mathematics, there were also recognitions that solutions differing from the teacher’s were acceptable [Lines 4 & 5] and indeed encouraged [galt]. Moreover, in acknowledging that “I learn from you [students]” [Line 3] and that they were to check my presentation of solutions instead of just accepting it [Line 7], I was hoping to reduce students’ reliance on my answers but to depend on their own judgement [grel].

Part I.2ii began to deal with the solution to item 2c. A complete solution to the item, which would be acceptable in an exam situation is shown below.
\[ y = 65^\circ \text{ (opposite angles of parallelogram are equal)} \]
\[ x = 125^\circ - 65^\circ \text{ (exterior angle = sum of interior opposite angles)} \]
\[ x = 60^\circ \]

For exam solutions—indeed for mathematically acceptable solutions—the steps of reasoning should be presented in logical sequential order. There is also a requirement to write clearly the geometrical properties whenever they were invoked in the working steps. I recognised that each of these expectations was not trivial. Thus, instead of intertwining them in a one-go presentation of the solution, I adopted a heuristical approach of first focusing on the logical sequence of obtaining the solution, followed by filling in the relevant textual substantiations later. The former was undertaken in this part of the lesson while the latter work of written reasoning belonged to Part I.2iii. This two-part approach to presenting the solution was best captured by the opening words I made to the students before I embarked formally on solving item 2c:

"Ok, again Mr Leong is trying to model [here] how you [should] do it. I don’t want to write down the reasoning first. I want to do on this one [pointing to the diagram of 2c on board] first."

The dominant purpose of Parts I.2ii and I.2iii was therefore to teach students a strategy for attacking the problem [gsvp]: getting the solution steps first (in Part I.2ii), before writing the reasons later (in Part I.2iii). The first prong of the attack strategy for getting the solution sequence used yet another embedded heuristic of relying on the visual stimuli of the diagram to link the steps together [gsvp]. In Figures 23.1, 23.2 and 23.3, I try to present the attack strategy in primarily visual form rather than merely textual form.
As can be seen from the three frames presented in Figures 23.1, 23.2, and 23.3, apart from the intention of showing students the use of diagrams as a heuristic tool to solve the problem, there were other built-in goals. The teaching actions recorded in Frames 1 and 2 showed the purpose of revising specific geometrical theorems.
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[gmrn]—specifically, “equal opposite angles of a parallelogram” for Frame 1, and “exterior angle theorem” for Frame 2. In addition, these first two frames showed clearly my attempts to attribute my solution strategy to the work of students—as seen through the repeated use of “someone says …” —and the subjecting of my working to students’ agreement instead of imposing my answers as the final authority. That provided more evidence of my goal of avoiding full dependence upon teacher’s answers [grel]. Frame 3 dealt with straightforward arithmetic calculation to obtain the required answers and therefore falls directly under the domain of gsol.

After using the diagram to show the sequence of steps that led to the required answers, Part I.2iii was devoted to the written presentation of the reasoning process [gwrs]. I started by writing “\( y = 65^\circ \)” beside the diagram and asked, “What is the reasoning?” A few students mentioned “opposite angles”. I acknowledged the answer and repeated it, with greater accuracy, “opposite angles of a parallelogram are equal” and wrote the abbreviated form “opp. \( \angle \)s of //gram are =” in parentheses beside the “\( y = 65^\circ \)”. The sequence may appear clearer in list form:

I wrote “\( y = 65^\circ \)” on the board;
I asked for a reason from the class;
A few students mentioned “opposite angles”;
I re-phrased students’ responses into the more complete form;
I wrote the abbreviated form in paranthesis as “opp. \( \angle \)s of //gram are =”.

I then proceeded with a similar cycle of “write equation”-“ask for reason”-“rephrase reason”-“write abbreviated reason” for the subsequent part of the written solution in obtaining “\( x \)”:

I wrote “\( x = 125^\circ - 65^\circ \)” on the board;
I asked for a reason from the class;
I got some students giving “exterior angle . . .”;
I rephrased students’ responses into the technically more accurate form; and
I wrote the abbreviated form in parentheses as “ext. \( \angle \) = sum of int. opp. \( \angle \)s”.

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In terms of goals, the first step of each of the cycles was purely calculating answers [gsol]; the second step was primarily to teach the requirements of deductive logic [grsd]; the third step was to encourage students to voice their answers [gart]; the fourth step was to teach the correct technical rendering of the geometric theorems [glan]; and the last step was to introduce abbreviations commonly used in the textbook as part of exercise solutions [gsol].

The goal-analysis at various tier-levels of Parts I.2ii and I.2iii of the lesson can be summarised more succinctly in Table 8.

<table>
<thead>
<tr>
<th>Parts I.2ii &amp; I.2iii</th>
<th>Component Parts</th>
<th>Sub-sec</th>
<th>Activity</th>
<th>Goals at different tier-levels</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.2ii</td>
<td>Teach the use of a diagram as a heuristic tool for solving the problem</td>
<td>Frame 1</td>
<td>Use “opposite angles of parallelogram are equal”</td>
<td>gmrn, grel, gmrn, grel, gmrn, grel, gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Frame 2</td>
<td>Use “exterior angle equals sum of interior opposite angles”</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Frame 3</td>
<td>Calculate to obtain answer for x</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td>I.2iii</td>
<td>Teach students how to write the reasoning steps</td>
<td>Cycle 1</td>
<td>Write “y = 65°”</td>
<td>gmrn, grel, gmrn, grel, gmrn, grel, gmrn, grel, gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ask for reasons</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Opposite angles”</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Rephrase reason</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Write abbreviated form</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Cycle 2</td>
<td>Write “x = 125° − 65°”</td>
<td>gmrn, grel, gmrn, grel, gmrn, grel, gmrn, grel, gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ask for reasons</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>“Exterior angles . . .”</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Rephrase reason</td>
<td>gmrn, grel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Write abbreviated form</td>
<td>gmrn, grel</td>
</tr>
</tbody>
</table>

Table 8: Summary of Parts I.2ii and I.2iii and the underlying goals

4.3.6 Zooming-out to a Broad View of Lesson 8

The close examination of Lesson 8 so far has uncovered nearly all of the g-goals identified in Table 6 lurking beneath teaching actions across a range of tiers and
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throughout the entire lesson. There are, however, two g-goals that were not directly
evident from this close study of Lesson 8. The ‘missing’ g-goals are gcvr and gcwt,
both concerning intentions to complete the coverage of geometry topics within the
time frame allotted for the lesson. That these goals were indeed prominent in other
lessons will be clearer when the discussion shifts in the subsequent chapter into a
study of problems of teaching. It suffices to mention at this point that though gcvr
and gcwt were not manifest directly in Lesson 8, their influence on the lesson as a
whole could still be seen. A board-view comparison between the planned coverage of
the lesson and the actual content coverage that occurred in class shows the influence
of these goals in actual teaching practice. The lesson memo—reflecting the planned
coverage—and the lesson sequence—what actually occurred—are placed side-by-
side in Table 9 to highlight correspondences and contrasts between what was planned
and what was actually carried out in class.

From the component-by-component matching (as shown by arrows in the table
above), it is clear that when I conducted the lesson, I intended to follow my plan.
Insofar as covering the schedule for the day was part of the bigger goal to cover the
geometry topics within the entire module, gcvr is evident in the lesson. In addition,
the careful apportionment of time in the lesson among components of the
lesson—divided between sections I (about 25 minutes) and II (about 35 minutes)
—reflected a consciousness of appropriate time allocations for particular parts of the
lesson [gcwt].
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### Extracts from lesson memo

<table>
<thead>
<tr>
<th>I</th>
<th>Textbook exercises Ex 14b</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.1</td>
<td>Students attempt selected textbook exercises</td>
</tr>
<tr>
<td>I.2</td>
<td>Whole-class instruction on the solution of Item 2(c)</td>
</tr>
<tr>
<td>I.3</td>
<td>Whole-class instruction on the solution of Item 4(c)</td>
</tr>
</tbody>
</table>

### Actual lesson sequence\(^\text{15}\)

<table>
<thead>
<tr>
<th>II</th>
<th>Construct quadrilaterals</th>
</tr>
</thead>
<tbody>
<tr>
<td>II.1</td>
<td>Whole-class instruction on the construction of the required square</td>
</tr>
<tr>
<td>II.2</td>
<td>Teacher checks that students’ construction inadequacies of the square are corrected</td>
</tr>
<tr>
<td>II.3</td>
<td>Whole-class instruction on the construction of the required rectangle</td>
</tr>
<tr>
<td>II.4</td>
<td>Students attempt the worksheet tasks at their own pace</td>
</tr>
<tr>
<td>II.5</td>
<td>Whole-class instruction on the rhombus task, directed to students who had difficulties with the task</td>
</tr>
<tr>
<td>II.6</td>
<td>Students continue to attempt the worksheet tasks at their own pace</td>
</tr>
<tr>
<td>II.7</td>
<td>Whole-class instruction on the parallelogram task, directed to students who had difficulties with the task</td>
</tr>
</tbody>
</table>

Table 9: Comparisons between planned components and actual implementation

### 4.3.7 More about g-goals

From the goal-based analysis of lesson 8, it is clear that the respective g-goals were not tied to a particular tier-level of analysis. A certain g-goal may be evident during teaching practice in a moment-by-moment utterance, while teaching a particular student, in the midst of explaining working steps in front of the whole class, for the period of a major component of the lesson, or even as the overall intent of the entire lesson. An example to illustrate this expression across a range of tiers is gsol. Its

\(^{14}\) The sequence of the lesson presented in this table is identical with the order presented in Table 5 earlier.

\(^{15}\) The […] parts of the memo were comments that did not directly deal with the teaching of geometry or the coverage of geometrical content.
influence can be seen at the first-tier level at section I, at the second-tier level at Sections I.2 and I.3, at a yet more fine-grained level at Frame 3 of Part I.2ii, and also at utterance-level in I.2iii. The method I used in examining teaching thus allows a freedom of occurrence of g-goals at different levels of analysis.

An alternative approach (as used by the Berkeley group of researchers) is to label goals that are identified at different tiers of analysis as different goals altogether. In other words, using the above example of gsol, they would treat gsol when occurring in different levels as separate goals. In the method I used, I focus rather on the similarity in the underlying motivations—that is, the intent to teach the correct solution steps. The difference in the approach could perhaps be traced to the different purposes behind our research. For the Berkeley researchers, their main purpose is to use the goals architecture to model and predict the instructional behaviour of a particular teacher:

When the modelling process is done, the model of a particular teacher will contain representations of the goals, beliefs, and knowledge attributed to the teacher, and a decision-making mechanism that suggests how, in any set of circumstances, those goals, beliefs, and knowledge will shape the teacher’s decision regarding what to do “next” (Schoenfeld, 2000, p. 249).

Their use of goal-distinctions at every tier-level suited their purpose of modelling and differentiating the teaching behaviours of different teachers. The purpose of my research is decidedly different. Rather, my purpose is to study the interaction between my underlying teaching goals. That these goals appear at different tiers of my lesson is not the focus of my study; my focus is where the goals meet each other problematically and in ways that require coping. Moreover, on a pragmatic note, my interest in this research is to keep the number of goals minimal (and yet adequate to account for my goals in teaching geometry). A larger-than-sufficient set of g-goals renders the work of analysing interactions unnecessarily complex and potentially distracting from the central focus.
4.4 Relationship between g-goals and G-goals

My remaining concern in this chapter is to show the relation of these g-goals with my overarching goals I stated earlier. Table 8 gives an overview of the relationship between the g-goals and the G-goals. When the former are placed under the latter in the columns within the table, it indicates a relation of the smaller goal providing support for the larger goal in the sense that the fulfilment of the g-goal directly contributes to the fulfilment of the G-goal.

<table>
<thead>
<tr>
<th>Overarching goals</th>
<th>G1</th>
<th>G2</th>
<th>G3</th>
<th>G4</th>
<th>G5</th>
</tr>
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<tbody>
<tr>
<td>Subordinate goals</td>
<td>'glan</td>
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<td>'gtrv</td>
<td>'gwkk</td>
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<td></td>
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<td>'gart</td>
<td>'gcon</td>
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<td></td>
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<td>'galt</td>
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</tbody>
</table>

Table 10: Relation between g-goals and G-goals

As suggested by the horizontal line in Table 10, the g-goals can be categorised into two types. The first set of g-goals—above the horizontal line—are those that contributed to the fulfilment of more than one of the G-goals. For example, glan is placed under G1, G2, G3, and G4, indicating that glan supports all these four G-goals. The second set of goals are those below the horizontal line. Each of these g-goals is subordinate to exactly one G-goal. To provide ease of reference between the g-G links in the table and the elaboration in the following paragraphs, I have inserted markers (a–f) against goals in Table 10 to link them with the respective blocks of texts below. For the purpose of simplifying the terminology, the first set of g-goals will be known as “plural” g-goals; the second set will then take the label of “singular” g-goals.

Some of the g-G subordinations indicated in the table above are fairly straightforward to establish. An example is the gcwr-G1 relation. Even from the phrasing of the goal-
statements, it is clear that gcvr was a sub-goal of G1 in the sense that achieving gcvr would directly fulfil a part of G1. Other examples of these straightforward g-G relations are found in the gins-G1, gcwt-G1, ghrs-G2, gins-G2, gerr-G2, gpsv-G2, gsol-G2, gstg-G5, and gwkk-G5 subordinations. I shall therefore devote the space to the task of explicating the other g-G relations that are less straightforward.

Some of the g-goals are attitude-targeted. By this I mean that the desired end products of these goals are mental dispositions rather than knowledge in the cognitive sphere. That attitudes play an important part in students’ success in learning is well-known. Middleton and Spanias (1999), in their summary of existing literature about the role of motivation in mathematics achievement, reported that there was agreement that “the effort a person is willing to expend on a task is determined by the expectation that participation in the task will result in successful outcomes” (p. 79). Thus, teaching work directed towards building students’ confidence for success has a potential of raising their level of commitment to meet the requirements of the mathematical tasks presented. This provides evidence for the gcon-G5 link proposed in the table above.

Apart from looking at mathematical achievement, my agenda in teaching aims for other goals, such as the ability of students to reason about geometry via their own explorations (which is part of G4). For these higher mathematical functions, the correct frame of mind is also important. In looking at helping students to engage in plausible reasoning, Polya (1954) stressed the need for students to possess “the inductive attitude” which consisted, among others things, the components of “intellectual courage” and “intellectual honesty”. By “intellectual courage” he meant the mental tenacity to make initially uncertain attempts at arriving at attack routes to problems and the willingness to change tack when unsuccessful; and he defined “intellectual honesty” as the willingness to change one’s approach when there are good alternatives provided by other sources. Lakatos (1976) similarly upheld courage and honesty as important affective qualities of inductive thinkers. Lampert (1990) not only valued these qualities of the inductive attitude but also attempted to introduce them into her mathematics classroom. Her instructional method involved building a “participation structure” where she as the teacher, together with the students, engaged in classroom discourses to arrive at mathematical knowledge. As one of the items on
her teaching agenda was to work towards “the goal of students’ acquiring technical skills and dispositions necessary to participate in disciplinary discourse” (p. 44), she consciously taught the students not to rely on her to give answers, but rather, to rely on the pooling together of their combined knowledge via discourse to arrive at conjectures and generalisations to the problem posed by her. To Lampert, “what is important is developing and defending strategies, making hypothesis, or what Lakatos calls ‘conscious guessing’, and rising to the challenge of articulating and defending the assumptions that led up to a guess” (p. 40). Insofar as I agree with Polya, Lakatos, and Lampert, I see grel, gtry, gart, and galt as contributing to a correct mental attitude as required to fulfil G4.

While the previous paragraph is concerned with encouraging “plausible reasoning” as part of fulfilling G4, “reasoning” also includes, as Polya (1954) indicated, the other component of “deductive logic”. This involves understanding the logic rules in establishing mathematical statements and what counts as valid mathematical justifications. Deductive reasoning also involves the presentation of the process of arriving at solutions in ways that are considered sufficiently rigorous to the wider mathematics community. I see the role of teaching students to reason deductively as comprising both discourses about geometry involving deductive logic as well as direct efforts to express these underlying thought processes in the written form. This explains the attachment of gwrs and grsd to G4.

Deductive reasoning is also linked to the fourth “Deduction” level of the van Hiele theory. In the van Hielean scheme, students reaching this stage of geometric competence are able to marshal mathematical propositions as justifications in the context of a proof. This ability is very similar to the Polyan idea of utility with deductive reasoning as discussed above. Thus, gwrs and grsd can also be seen as working towards students’ progression to a higher van Hiele level of geometric thought. The gwrs and grsd subordinations to G3 are derived from this similarity between the van Hielean and Polyan notions of deductive reasoning.

I turn my attention now to glan, which is about the teaching of the language of geometry. The glan-G1 relation is clear because the language of geometry forms part of the geometrical content. Curriculum documents stipulate the teaching of specific
geometrical terms as part of the task of teaching geometry (MOE, 2000). The glan-G2 connection is also an obvious one. Without knowing the language associated with common geometrical objects and properties, students would not be able to make sense of questions in test papers, let alone attempt to provide solutions to the underlying problems. Teaching the language of geometry is thus seen as directly helpful to preparing students for exam-type questions. The glan-G3 relation can be accounted for by the critical role that the van Hieles placed on language in students’ advancement through the levels (see Burger & Shaughnessy, 1986; Fuys, et al., 1988; Mayberry, 1983). The literature characterises the language used by students operating at the lower levels to be imprecise and non-technical. For these students, van Hielean researchers proposed that teachers should begin by using everyday language as a point of reference, and then proceed to “carefully draw distinctions between common usage and mathematical usage” (Clements & Battista, 1992, p. 433), and later focus their attention on appropriate language to describe parts of figures and their properties (Fuys, et al., 1988). This acquisition of appropriate language will help the students abstract and reason with higher-level geometrical content. As to the glan-G4 link, it is based on the centrality of language to the very enterprise of thinking itself (Vygotsky, 1986; Skemp, 1986; Dewey, 1933/1998), which of course includes geometric reasoning. Although the quoted authors see the ontological relation between thought and language differently, their common emphasis on the importance of language in thinking is taken as a well-accepted proposition.

Goal gmrn is about aiming to teach geometric meanings and relationships underpinning correct facts or procedures. The emphasis on geometric ideas underlying surface solutions strike a close match to the transitions between van Hiele stages where each movement to a different level represents a deeper level of geometric abstraction. An instance from the data might make this association clearer. In Lesson 11, there was a segment in the lesson where I taught students how to construct a perpendicular bisector of a line segment AB using only a pair of compass and an unmarked ruler. After completing the construction procedure, I attempted to lead the students to see the underlying geometrical meaning of the arcs that were constructed—that (a) they were equidistant from both points A and B, (b) their equidistance ensured a rhombus construction, and (c) joining the intersections of arcs would yield the result by using the diagonal properties of rhombus. That attempt to
direct students’ attention from the visual and action-based stimuli of drawings and movements of instruments, to the geometrical properties of those constructions indicated a shift of content from Level 1 to Level 2 of the van Hiele stages. But I did not stop there; I went on subsequently to indicate that a kite construction would also yield the same result, thus bringing the consideration of this diagonal property beyond the rhombus to other special quadrilaterals. This kind of work is more characteristic of the higher van Hiele “Relational” level mode of operation. From this episode, it becomes clearer how the uncovering of geometric ideas beneath surface solutions supports the progression of students’ level of geometric abstraction, which is another way of stating the gmrn-G3 connection.

A final clarification about g-G relations is necessary. The reader may perhaps wonder if other g-G links exist which were not explicated above. One possible consideration is the subordination of grsd to G2. The argument might follow this vein: when we teach students to reason deductively within the geometrical domain (grsd), we are helping students to internalise the deductive processes and thus helping also to apply them in paper-and-pencil settings (G2). While this may be true in certain testing settings and cultural contexts, there is no evidence from my data or from the research literature to conclusively confirm this hunch that “teaching students to reason” will contribute directly to “preparing them for exam performance”. This explains the absence of a grsd-G2 relation in above table. It also explains the absence of other g-G links which a reader may think to exist. In asserting g-G subordinations, I have relied solely on substantiations from clear reason, from the literature, and from the data, as evidenced in the supporting paragraphs above.

4.5 Concluding Comments

In this chapter, I have demonstrated the use of the goal-based methodology to examine my teaching actions through my goals of teaching in one lesson. I maintain two sets of goals (G- and g-) to account for goal-analysis at different grain sizes and also established the g-G subordinations. In explicating the g-goals, I have also illustrated the method of analysis of my teaching by using the hierarchical tier-levels. These goals and the multi-tier method of examining teaching that are described in this chapter will be used in subsequent chapters. The focus of analysis in the next chapter
will shift to the identification and analysis of my problems of teaching and the related coping strategies in the same lesson of study.
In this chapter, I focus on Phase I of the analysis process (refer to Table 4 in Chapter 3 for the stages in analysis). I use the goals and the methodology described in the previous chapter to study the goal interactions in the same lesson, Lesson 8, of the module, and now, consistent with the foci of this study, I examine the problems and coping strategies that arose in my teaching practice. As defined in the introductory chapter, problems are regarded as *interferences to my attempt to carry out one or more goals of teaching*; coping strategies are seen as *the marshalling of resources as a teacher to advance some goals of teaching when encountering problems*.

### 5.1 Turbulence as an Extension of the Navigation Metaphor

In viewing the work of teaching geometry as fulfilment or hindrances to fulfilment of instructional goals, two natural regions emerge—parts of teaching that advance the intended goals and parts that did not. I shall refer to the latter as *turbulent regions* of the lesson. The idea of turbulence is inspired by the metaphor of navigation in an aircraft used by Lampert (2001), which was reviewed in greater detail in Chapter 2. When teaching goals were carried out as intended and when I did not sense any need to alter the planned course of instructional activity, it is very similar to the *smooth* parts of an aircraft journey where it glides along in a predetermined direction and where the navigator’s role is to maintain routine operations, keep to course, and to monitor the controls. However, during *turbulent* parts of the journey, the picture is very different. A number of warning signals may light up and the attention level of the navigator is heightened by the prospect of danger. He has to make improvisational decisions to steady the aircraft and to minimize the effects of the turbulence. This latter picture of turbulence approximates the regions of practice where interferences to the goals of teaching were experienced. As this study is on the problems of teaching, the focus of discussion here is thus on the turbulences rather than the smooth parts of the journey.

Since the metaphor of turbulence will pervade the language of discussion in this chapter, there is perhaps a need to elaborate on what aircraft turbulence entails. Such a turbulence usually begins with a perturbation in the form of rough weather, failure of mechanical parts, or other unexpected causes. These perturbations *trigger* a state
of tension in the navigator. He assesses the situation, weighs the options, and then decides on certain improvisational moves. The purpose of these reactive measures is to minimize the impact of the turbulence so that the aircraft can proceed with the journey and reach the destination. The outcomes of his improvisation, however, may not always be as intended. Sometimes, he succeeds in fulfilling this purpose; but at other times, the turbulence may persist or be exacerbated. In some cases, the travel plan may have to be changed drastically to accommodate the effects of the turbulence. Thus, turbulence involves “trigger”, “tension”, “improvisation”, and “outcomes”.

This metaphorical language will be used to depict the turbulences in my work of teaching the geometry module. How the components surface in my teaching practice will be examined in the following discussions about an example of turbulence experienced in Lesson 8. I build on the goal-analyses in Chapter 4 to analyse this example. I will then review the descriptions of turbulences to focus more directly on the problems of teaching and the coping strategies.

5.2 Turbulence Situation 1: An Example of Turbulence with the ‘Rhombus Problem’

Recall that there were two planned components of Lesson 8. The first component was meant to deal with solving some textbook problems [gsol] and the second component was to teach constructions using geometrical instruments [gins]. The first component can be further divided into three sections—I.1 on letting students attempt textbook exercises [grel], I.2 on discussing the solution of item 2c [gsol], and I.3 on discussing the solution of item 4c [gsol].

The turbulence to be discussed here involved mainly Section I.3 of the lesson. The trigger of the turbulence was, however, not in that section of the lesson but could be traced to Section I.1 of the lesson when I was moving from table to table to observe the students’ attempts at the textbook problem. Towards the end of that section of the lesson at around the 14-minute mark, I noticed that students were beginning to attempt item 4c, which I refer to as the ‘rhombus problem’. The portion of the
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whiteboard showing the ‘rhombus problem’ is reproduced below for ease of reference.

![Rhombus Diagram]

Figure 24: ‘Rhombus problem’ represented on the board

I observed that a number of students were wrongly assuming perpendicular adjacent sides—properties of squares—with the rhombus. An extract of my post-lesson reflection below summarised my observations of their attempts:

In the question the diagram given was that of a rhombus. But the rhombus looked very much like a square. And a number of them wrongly assumed that the interior angle of the rhombus was 90 degrees. … So a number of them could not yet see but are just drawn by the appearance of the diagram rather than focusing on the properties.

My original plan was to focus on teaching the solutions of some typical textbook problems [gsol] during the remaining parts of Section I of the lesson. From looking at the students’ work, however, I realised that I could not merely have gsol in mind when discussing the problem with them at the later part of the lesson. I also had to address their rootedness to the visual form and the need to view the rhombus as holders of properties. In other words, I needed to point out their error in assuming interior right angles in a rhombus [gerr] as well as to help them move beyond their visual-based mode to focusing on geometric attributes of the rhombus. To teach with that geometric focus in mind is to fulfil the goal of teaching geometrical meaning and relationships [gmrn]. That I was conscious of a trigger and the need to make adjustments to cater to an unplanned-for situation is clearly illustrated in another reflection excerpt.
Here I walked around and I notice that especially the ‘rhombus problem’ is very interesting . . . so I decided to bring up the . . . rhombus [as in drawing it on board to prepare for discussion later]. The rhombus one was interesting because I noticed that some of them were assuming that all the interior angles are right angles. And I know in my mind that as I draw it I can see myself thinking through how I should present the solutions later . . .

To see the improvisations I actually made to address the additional goals of teaching the rhombus problem, I skip an account of the intervening sections of the lesson to consider that part of Section I.3 that dealt with the error. That one of my purposes was to help students discover their error [gerr] can be detected easily from my introductory words:

18.56  T: … I notice some of you do this - can Mr Leong put on the board for you to consider? Things that we do not know we use black colour, ha. Someone suggested that this was 90 degrees [marks one interior angle of the rhombus with the right angle square symbol using black marker, then pauses to see reaction]

What followed from this challenge to comment on my conjecture was an equivocal chorus of Yes/No from the students, signaling to me that the error was persistent and instructional work was needed along the gmrn and gerr direction.

It is perhaps worthwhile to pause from and re-examine the situation thus far using the turbulence metaphor. I was gliding along the first section of the lesson with the goal of helping students apply what they have learnt in the module so far to some typical textbook problems [gsol]. Towards the end of Section I.1 of the lesson, there was a trigger that indicated a need to adjust the plan to accommodate the changing circumstance [of gmrn and gerr]. I continued flying while thinking about ways to handle the turbulence. At Section I.3 of the lesson, I took the opportunity, while discussing the rhombus problem, to confirm the need to address these emergent goals. The response from students showed that moving along the original path [of gsol] without also dealing with gmrn and gerr would be inadequate. However, the assessment of the situation at that point in time brought about certain tensions. For the reader to appreciate my internal struggles, it is necessary to go beyond the confines of Lesson 8 to view my broader teaching plan for special quadrilaterals—and, in particular, the rhombus—in the context of preceding lessons.
5.2.1 Zooming out to View the Context Surrounding the Tension

I introduced special quadrilaterals, including rhombuses, in Lesson 6. In the first section of the lesson, students worked on a given Sketchpad template to classify the different special quadrilaterals shown on screen. Upon opening the sketch, the screen appears as in Figure 25.1.

**Figure 25.1: Screenshot of the sketch when it is initialised**

Although all the quadrilaterals appear as squares when the file is opened, they are each in-built with geometrical properties that uniquely match the respective special quadrilaterals listed. As the students had done a similar classifying exercise on Sketchpad in an earlier Lesson 5 on special types of triangles, they were familiar with the need to explore each figure by click-and-drag to look for drag-resistant properties. Upon dragging, the figures reveal their intrinsic properties. Figure 25.2 shows how the screen may appear after drag-mode is applied.
After working on dynamic figures of the respective special quadrilaterals, the next section of the lesson required students to draw representations of each of these quadrilaterals. The extract below shows an example of the drawing task. The drawing activity was intended to reinforce students’ visual familiarity with each type of quadrilaterals. At this stage, the drawings were done using a setsquare (for drawing perpendicular and parallel sides) and a marked ruler. The inclusion of a set-square in the drawing process was my attempt to help students focus on the parallel and perpendicular properties of special quadrilaterals. The more demanding type of Euclidean constructions using straightedge and compass would require greater familiarity with geometrical properties and they were not introduced in this lesson but at a later point in the module.

For each type of special quadrilaterals listed below, draw a representative example. You may make use of a setsquare, a ruler and a pencil to do the drawing.

<table>
<thead>
<tr>
<th>Special Quadrilateral</th>
<th>Draw an example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td></td>
</tr>
</tbody>
</table>

Figure 25.2: A screenshot of the sketch upon dragging
Lesson 7 continued with the study of special quadrilaterals by shifting the focus from the more gestalt approach of looking at the quadrilaterals to an emphasis on their geometrical properties. In the first section of the lesson students explored, in turn, Sketchpad templates featuring a square, a rectangle, a parallelogram, and a rhombus. They were instructed to use the ‘Measure’ option and drag-mode to observe and write down conjectures of properties of each of these quadrilaterals. The rhombus Sketchpad template is reproduced in Figure 26.

![Figure 26: Screenshot of the rhombus sketch for students’ exploration](image)

In the second section of Lesson 7, I conducted a whole-class discussion based on the students’ observations during the Sketchpad activity. The commonly-agreed conjectures were then recorded on a summary sheet (shown in Figure 27) in the form of an overhead transparency projected on the screen in the front of the classroom for students’ reference.

Re-examining the concept development track of “rhombus” in lessons 6 and 7, the reader can see my deliberate attempt to begin from gestalt recognition (the Sketchpad quadrilateral classification activity) to a consciousness of parallel properties (in the setsquare drawing task) to a conjecturing of properties by observing the figure (in the Sketchpad activity utilising the ‘Measure’ option) to a direct consideration of properties (in the discussion over the summary sheet). That gradual increase of complexity in the instructional activities corresponded to the goal of helping the students progress to higher levels of geometric abstraction [gmrn]. Viewed from the
framework of overarching goals, the progression in the instructional activities corresponded to the goal of helping students advance to a higher van Hiele level of geometric competence [G3].

![Figure 27: Summary of properties of some special quadrilaterals](image)

### 5.2.2 Zooming Back in to the Point of Tension

Having described the developmental track of the ‘rhombus’ concept, I return to that juncture in Lesson 8 where I experienced the tension. The rhombus problem required an understanding of rhombus properties, that is, that students could operate at the van Hiele “Analysis” stage with respect to knowledge about rhombus. However, my observations during Lesson 8 were that most of them were still at the lower visual-based mode of operation. My original plan was to teach the solution of the problem [gsol], with the assumption that the previous lessons had prepared them sufficiently to understand the solution. With this assumption in doubt, the struggle was between proceeding with the demonstration of the solution [gsol] as planned or to take time first to address their yet visually-driven ideas of quadrilaterals [gmrn].

To do the former would be to teach with the knowledge that a significant portion of the class would be unprepared to appreciate the solution strategy. That would be a direct violation of my belief in teaching every student, including those weaker in mathematics [gwkk]. To do the latter would mean that a substantial amount of time would be taken to engage the students to re-consider rhombus beyond merely its gestalt features. That would mean postponing some components planned for Lesson 8 to subsequent lessons, which would go against the desire to complete the geometry
content within the stipulated time period [gcwt]. Figure 28 represents graphically a simplified version of the dilemma I faced.

![Figure 28: Illustration of goals interaction I](image)

Both paths were problematic, as each hindered the fulfilment of some goal. There was no clean solution in the sense of fulfilling all the goals satisfactorily. I had to make a decision based on what I was aware of at that time to continue the classroom instruction. My improvisation was to proceed with the original plan of discussing the solutions and in the process do as much as I could—without taking more than a few minutes—to bring as many students as I could to operate beyond the visual-based mode. My post-lesson reflection also indicated this reluctance to devote too much time to tackling these emergent goals:

I did it [i.e., addressed the error] by presenting a *quick* draw [of a rhombus and a square] … (emphasis added)

The improvisation that I made was not a solution to the problem; it was an attempt to fulfil at least one goal—in this case gsol—while minimising the violation of the other goals of teaching.

It is perhaps appropriate at this juncture to acknowledge that *post hoc* analysis of the situation can possibly yield other possible improvisation solutions. A teacher,
however, does not have the luxury of time to consider other options. As a teacher, I usually only had a few moments to make my instructional decisions. This study is not an investigation of hypothetical scenarios arising from careful post-lesson analysis. It is a study of actual thoughts and decisions of a teacher while carrying out the work of teaching with multiple goals in mind.

5.2.3 Improvisation and the Subsequent Outcome

I proceeded by beginning with drawings of a rhombus and a square and hoped to help those students who were still only visually-driven to shift their focus instead on inherent properties of figures [gmrn]:

T: [Draws a free-hand square and rhombus on another side of the board] Mr Leong has drawn two diagrams on the board. Can you guess - can you give a name to each of them?
Chorus: [Mixture of “square” and “rhombus”]
T: This [pointing] is a rhombus because – why? - all the sides are equal [marks the equality of the sides on the diagram] …. And this is a square [pointing]. What is the difference between them?
Hassan: The shape
Farin: The angle
T: [Nods] This [pointing to the square] is a rhombus-like figure but it has an additional feature, that is, each of the angles are 90 degrees [marks the right angle symbols in each of the interior angles]. A square must have right angles, huh?
Dickvan: Must?
T: Must [emphasis, with repeated nodding of head]. For a rhombus must have right angles or not [pointing at the interior angles of the rhombus]?
Chorus: No
T: No need.

By comparing the square and the rhombus, my purpose was to point out that while having perpendicular adjacent sides is a critical attribute of a square, it is non-critical for a rhombus. Having made that observation, I returned to the rhombus problem:

20.20 T: [Walks over back to the diagram for 4c] You see - though [pointing to the diagram] it looks like a square, but nobody says it is. It says it is a rhombus [pointing to the label “rhombus” written above the diagram]. A few students: Yeah.
T: So can we assume that this is [pointing to earlier black-marked right-angle square symbol] 90 or not?
Chorus: [A mixture of yes and no, with “no” louder and trying to shout the “yes” group down]

I sensed at this stage that despite my explanation, there were still some students who still could not operate at the “Analysis” level. However, this ‘detour’ to deal with this
problem was meant to be a short treatment. Even though I knew that I did not manage to help all students understand me then [undermining gwkk and gmrn], I needed to move on with the solution of the rhombus problem [gsol], as planned. I therefore wrapped up the turbulence control measure by again emphasising the error:

20.40 T: Now [erases the black-marked right angle square symbol away] unless the question says this is a square, even though it looks like one, we cannot assume right? Remember the diagram can deceive. So we cannot assume this is a square. So this [pointing to the angle] is not 90°. Careful, ha. So if you have 90 here it means it is wrong. Nobody says so, even though it looks like. This is a very important point …

Having done a quick explanation of the error [gerr] and the geometrical differences between square and rhombus [gmrn], I returned to the solution of the rhombus problem [gsol].

5.3 Turbulence Situation 2: More Turbulence on the Rhombus Problem

Following the earlier discussion on the error of assuming the interior angles as 90 degrees, I was about to embark on the path with the students towards the correct solution for “y”. As I turned to look at the rhombus problem again on the board, an innovative thought came to me. I had not realised before that there was an alternative solution to the problem from the one that I had planned to use just before the lesson began. Prior to the lesson, my solution for the problem was along the line of approach shown in Figure 29.1 (henceforth referred to as Method 1). The alternative approach which occurred to me at that moment, however, followed the different sequence of thought in Figure 29.2 (henceforth referred to as Method 2).³

¹ To be precise, I should have added, “although it COULD be 90 degrees”.
² The solution for “x” was done in the earlier part of the lesson. It was a straightforward solution using the “alternate angle theorem” and it belonged to the smooth region of the lesson.
³ Again, I am aware there are other methods upon post hoc analysis. The two methods shown here were those I was aware of at that point in time when I was teaching in class. Other methods involve a re-sequencing of similar geometrical knowledge.
Figure 29.1: Illustration of Method 1 approach to solving for “y”

Figure 29.2: Illustration of Method 2 approach to solving for “y”
The geometrical knowledge required to perform the respective methods follows:

**Method 1**
Step 1: Diagonals of a rhombus are perpendicular to each other
Step 2: Interior angles of a triangle are supplementary
Step 3: Base angles of an isosceles triangle are equal

**Method 2**
Step 1: Each diagonal of a rhombus bisects the interior angles incident to it
Step 2: Interior angles of a triangle are supplementary
Step 3: Diagonals of a rhombus are perpendicular to each other

While the requisite knowledge in Method 1 was purportedly familiar knowledge for the students—in the sense that it had either been taught in previous lessons or was covered in the Primary school syllabus—Method 2 requires knowledge that was yet to be explicitly taught in class. In particular, Method 2 Step 1 (henceforth termed as Property 2.1 for ease of reference) had not been discussed so far in my lessons and was thus a novel idea for the students:

Property 2.1: Each diagonal of a rhombus bisects the interior angles incident to it.

There was an important factor to me at that point in time in the comparison of methods. To see how this factor fitted into my internal mental discourse of alternative methods, I again need to go beyond the confines of Lesson 8 to the previous lesson. Towards the end of Lesson 7, I was trying to collate all the students’ conjectures of properties of each of the special quadrilaterals. The final summary is shown in Figure 27. The order in which I discussed the quadrilateral properties with the class was column-wise followed by row-wise. In other words, I started with the properties of a shape’s sides (first column), beginning with the square and going down to the rhombus, and ending with all the diagonal properties (last column), from square down to the rhombus (last row). By the time I reached that last entry of the table, it was very close to the end of the lesson. Realising that I had little time to complete the
remaining planned component of the lesson, the last part of the discussion on diagonal
properties of the rhombus was, as a result of the time pressure, given a relatively brief
treatment. It was only after Lesson 7 that I noticed I had left out Property 2.1 from
the table; I had only included that diagonals bisect each other. Since Property 2.1 is
included in the geometry syllabus list (MOE, 2000a, p. 36), I had since been thinking
about finding a suitable opportunity to teach it [gcvr].

The rhombus problem appeared to me an opportune moment to slip in the teaching of
Property 2.1. In addition to teaching students the solution to the rhombus problem
[gsol], I had the bonus of being able to bring in the coverage of Property 2.1 in the
process [gcvr]. With the prospect of attaining these goals together, I found Method 2
at that point in time clearly preferable.

However, I did not anticipate then that the introduction of an additional goal [gcvr]
also represented an adjustment from the original overall intention of solving the
rhombus problem [gsol]. It was a trigger for another wave of turbulence. The sense
of turbulence build-up was captured by in my post-

However I was caught in this bind where I have to make use of the property that the
diagonals of a rhombus bisect all the interior angles but we have not talked about it before
[emphases added].

For the reader to appreciate how “we have not talked about it before” contributed to
“this bind” that I was in, there is again a need to look beyond the confines of Lesson 8
to view the principles I employed in teaching geometrical ideas not previously taught
before. The starting point for the developmental track of teaching about the properties
of special quadrilaterals was in Lesson 6. The primary goal in that lesson was to help
students identify the various types of special quadrilaterals—including the rhombus—
by their gestalt-visual features. In Lesson 7, I wanted to take a step further to teach
students the geometrical properties of each of the special quadrilaterals in relation to
their sides and angles. The manner in which I intended to proceed with this task is
illustrated by an extract of the lesson notes I wrote in preparation for the lesson:
II Properties of special quads – square, rectangle, parallelogram, rhombus

Continue exploration using Sketchpad\(^4\).

[Students] fill in the worksheet on “properties of special quadrilaterals”\(^5\).

Have a discussion leading to properties …

The sequence I employed in teaching the novel properties of the rhombus (and the special quadrilaterals, in general) can be summarised by the key stages of explore, record (“fill-in”), and discuss, as highlighted by italics in the lesson plan excerpt above. Instead of direct telling, that choice of lesson structure represented a deliberate effort to get students to observe and make conjectures towards what I intended them to learn. In teaching geometrical ideas I considered novel to the students, my usual instructional mode was to provide opportunities for students to make observations, articulate their conjectures, and engage in a discussion towards a conclusion they found to be jointly reasonable [gart]. My role then was not to give quick answers but to guide the discussion towards the properties I wanted them to agree upon. In so doing, I hoped to reduce students’ reliance on the teacher for direct answers [grel].

5.3.1 Assessment of the Situation Following the Trigger

Returning to that juncture of Lesson 8 where I was in a “bind”, I was about to introduce Property 2.1 in the solution of the rhombus problem. However, students had not yet been taught this property. That presented a situation where my modus operandi was to invite students to offer suggestions of observations or ways to proceed with the task. My role then was to use their ideas and lead them towards an agreement based on mathematical reasoning, in keeping with goals gart and grel explicated above.

The “bind” was, therefore, a state of tension between a number of goals of teaching. I had to consider how to teach the new Property 2.1 to the students. If I did it in the usual way of giving students time to re-explore the rhombus and to engage in a whole-class discussion towards a joint agreement of the property, it would be in keeping with the spirit of grel and gart; however, to do so would not only take time

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\(^4\) The Sketchpad template referred to here is the same as that illustrated in Figure 26.

\(^5\) A copy of this worksheet was shown in Figure 27.
away from subsequent components of the lesson [against gcwt], it would also be taking a step further away from solving the rhombus problem with the risk of losing students’ focus on the problem altogether [against gsol]. If, however, I chose to skip the discussion with the class and resort to direct telling, I would have covered the teaching of Property 2.1 [gcvr] but I would have undermined the grel and gart way of building up shared knowledge in class which I had so far tried to uphold.

The third alternative was to avoid mentioning Property 2.1 altogether and revert to method 1 of solving the problem. This choice, however, would have ‘wasted’ the opportunity to slip in the teaching of this attribute of the rhombus [gcvr]. Thus, similar to the situation in the previous turbulence, assessment of the circumstances revealed competing goals but no obvious solutions that did not prevent one or other goal being achieved. Figure 30 provides a simplified representation of the goal considerations.

![Figure 30: Illustration of goals interaction II](image)

Having weighed the options and being in a situation where each pathway would yield no wholly satisfactory solution, my improvisational move was to proceed with the “usual process of joint discovery” towards Property 2.1. While doing so, I was conscious of the need to continually monitor the gcwt and gsol indicator lights closely while weathering the turbulence.
5.3.2 Improvisation and Subsequent Effects

I proceeded by inviting students to offer their ways of approaching the problem, with a mind to connect them to Property 2.1 in the process. Mark started the class going:

20.50 T: So how to solve this [pointing to y]? 
Mark: The middle is 90. 
T: Ah . . . Mark suggests this is 90 [mark the right angle symbol at the angle where the diagonals intersect], do you agree? 
Chorus: Yes [loud]. 
T: We learnt it yesterday right? That the diagonals [using two index fingers to show the intersection] of a rhombus are perpendicular . . . . So this is 90 [pointing to the right angle symbol]. 
T: But we still cannot solve you know? [pause, looking at the diagram]

I was glad that Mark set the discussion on a productive track. Both Methods 1 and 2 required knowledge of the property that the diagonals of a rhombus are perpendicular. My purpose was to build on this good beginning prompted by Mark to head towards the use of Method 2. By saying that “we still cannot solve the problem”, I meant to signal the need to have other knowledge about the rhombus, thus providing me the opportunity to slip in Property 2.1. Yuxin joined in the discussion and I was hoping that he could help the class connect to Property 2.1:

21.05 Yuxin: y is also 42 
T: y is also 42? How do you know that? Can you listen to Yuxin? Remember we must think together? Yuxin, please. 
Yuxin: because the side angle to the triangle is 43 
T: Which one is 43? This one [pointing and looking at Yuxin for confirmation]? This one [point to y and then following Yuxin's pointing, wrote 43 to the angle adjacent to the original 42-degree angle]. Yuxin say that angle is 43. We use black colour for things not sure [labels the angle 43 degrees] 
Kai: Huh? 
Xiao: Why? 
[some undiscernible mumbles from students in the background] 
T: Can you tell us why, Yuxin? 
Yuxin: [long pause]

It turned out that Yuxin made a wrong conjecture that “y is 42”. My usual practice in such situations was not to give an immediate judgment [grel] but instead to hear out the justification so that the class can be drawn in to either support or refute the conjecture, thus encouraging joint reasoning [gart]. In this case, however, Yuxin took yet another step of error by asserting that the angle adjacent to the 42-degree angle was 43 degrees. Despite my patience to allow him to provide a rationale for his
conjectures, I was aware that the suggestions by Yuxin were heading further and further away from my intended objective of connecting it to Property 2.1. Sensing the unproductiveness of pursuing that path of discussion, I decided to steer the discourse closer to my goal of bringing in Property 2.1 by asking a more direct question:

22.09 T: Start from this one [erase the 43 degrees written earlier]. What do you think is this angle [finger pointing the just-erased angle]? [Then waited for answers]
Yan San: Don’t know
[Amidst undiscernible mumbles]

I realised at this point, based on their responses to my question, that if I continued this whole-class reasoning with the students, it would be unlikely to lead to their discovery of Property 2.1. In fact, it was straying further and further away from that objective. Meanwhile, two minutes of lesson time had gone by [against gcwt] and we were not any nearer to the solution of the problem [against gsol]. Internally, the gcwt and gsol indicator lights were flashing wildly for attention. They were sending out another turbulence signal.

5.3.3 Turbulence Situation 3: Turbulence within Turbulence
The need to reserve time for other planned components of the lesson [gcwt] and the focus on solving the rhombus problem [gsol] were important considerations in assessing the situation anew. It seemed that the longer I continued with the current track of discussion, the further I was moving from fulfilling the goals of gcwt and gsol. Faced with this consequence of the previous improvisation, I ‘backtracked’ to the situation as depicted in Figure 30 to consider the circumstance of the turbulence. Once again, in such circumstances, the decision to be taken would not solve the problem of conflicting goals. I could abandon the thought of introducing Property 2.1 altogether and stick to the ‘safe path’ (as it was the original intention) of proceeding with solving the rhombus problem with method 1. Another option was to pursue the reasoning path with the students further in hope of finally helping them connect to the property, although the earlier foray has greatly reduced my confidence in that direction. My decision at that moment was to resort to the direct-telling of Property 2.1. To alleviate partially the problem involved in this departure from classroom-reasoning-towards-agreement mode to that of direct-teaching, I brought out the overhead transparency (see Figure 27) used in Lesson 7 to highlight the context in
which Property 2.1 should have been brought up. In so doing, I hoped to relate Property 2.1 to other diagonal properties of a rhombus. The teaching episode reproduced below illustrates my approach.

```
T: [went to desk to pick up the overhead transparency to show] Anyway remind you of this. We are looking at rhombus - last row [project the overhead on the whiteboard] . . . . We say the diagonals bisect each other, but we forgot to say something about the rhombus. Not only does the diagonals bisect each other [pointed to the diagram], the diagonals of a rhombus . . . [pause]
Edward: 90 degrees
T: bisect the interior angle as well [using the flat palm to simulate the diagonal bisection of the angle]. Say one more time [slower]: the diagonals bisect the interior angles as well.
T: Take too long for Mr Leong to set up the Sketchpad to show you. But at the moment - I don’t usually do this - can you take it from me that this is true? Later we will find out whether it is true. The diagonals will bisect this angle in a rhombus. [pause] You think about this more carefully. We will revisit this, but this is a good time to bring out this.
T: Diagonal bisect this angle [pointing]. This is 42 [pointing] so how much will this be?
A few students: 42
T: [writes 42 degrees] bisect means into two equal parts right?
T: so this [pointing] is 42 . . .
```

From the almost apologetic manner in which I proceeded with the task of teaching this part of the lesson, it is clear that I experienced a deep internal struggle to juggle the unresolved conflicting goals of teaching at that time. The tension involved the violation of my usual practice of engaging students in discussion towards a joint-agreement of a new geometrical idea in order to fulfil other goals of teaching. Based on my reflections after the lesson, it was yet another “bind” I experienced:

Here I was again caught in a bind because I realize I couldn’t think of a better way to bring out this idea of putting up the transparency (of what we discussed yesterday) to summarise all the properties and I thought that was a good opportunity for me to remind them of the properties of a rhombus and for me to indicate at that juncture what is the property [2.1] of a rhombus. And I told them in a very awkward way that you take it from me but you will later find out whether it is true . . . (emphases added)
5.4 Return to the Focus of this Study – Viewing the Problems of Teaching and Coping Strategies in Relation to Turbulences

After zooming in to the complex details of the turbulences involved in the teaching of the rhombus problem, it is appropriate to pause and re-consider how the descriptions of these turbulent experiences relate to the objectives of this study, namely in the investigation of the problems of teaching and the coping strategies amidst these complexities.

The first point to clarify is that turbulences are not to be equated to problems of teaching. Turbulence, as explained earlier, is marked by a trigger of emergent goals of teaching that are not previously accounted for in the anticipation process. These new goals interact with planned ones, giving rise to a need for a fresh appreciation of the situation and the weighing of instructional options. This consideration of choices will then lead to an improvisational decision to cope with the juggling of these multiple goals while proceeding with teaching. I therefore take “turbulence” in teaching to refer to the part of my teaching journey where I encounter a swirl of interaction between anticipated and emergent goals. Seen as such, turbulences by themselves need not be problematic. When the navigator detects the need to adjust to new goals in his journey, he may be able to incorporate these new goals into his existing ones in his flight plan in a way that does not violate any of these goals. When that happens, not only is the turbulence unproblematic, it can potentially help him acquire new experiences and knowledge to deal with other turbulences in future.

However, the example with the rhombus problem shows that there are occasions when the goal-interactions lead to a situation where the teacher cannot devise an instructional path that fulfils all the goals at that point in time. When that happens, some goals of teaching that are deemed important at that juncture of the lesson are hindered from being carried out, rendering the situation problematic. Thus, while turbulence involves the whole instructional segment whereby the goals-mix is in a state of flux, problems of teaching refer to those points of tension within the turbulence where there are no solutions resulting in the complete fulfilment of the goals in consideration.
As for “coping strategies”, I define them as the harnessing of tools to deal with the situation of multiple goals amidst the encountering of problems. As seen in the turbulence involved in the teaching of the rhombus problem, the tools used can be both mental—as in the mental tossing up of different instructional paths to take—or taken in the more usual sense of external resources—as in the use of diagrams on the board. This view of coping strategy broadens the conventional use of the word “strategy” as a chain of teaching behaviour to one that involves the marshalling of both mental and external tools to tackle the challenge of teaching with multiple goals in mind. The coping strategy therefore does not necessarily begin with external teacher actions; it can also include the mental activity of assessing the goals and weighing the options beforehand.

5.4.1 An Overview the Turbulences involved in the Teaching of the Rhombus Problem

I now take a step back for a zoomed-out view of the whole duration of turbulences involved in the teaching of the rhombus problem. From the description above, the turbulences started from the first trigger of students’ error in seeing the rhombus as having the squarish property of perpendicular adjacent sides. From that moment, there was a series of interlocking activities—both mental and external—throughout the period of turbulences, terminating at the point when I resorted to direct telling of Property 2.1 to help me move into the solution of “y”. I shall use the language of “region of turbulence” to depict that entire instructional period in discussion. I have two ideas in mind when I choose the term “region”. Firstly, it is appropriate in depicting the close associations between the three turbulences described in the earlier paragraphs. Although each of these turbulences had their unique features and involved the interplay of different goals, yet they were held together by the common context of the subject matter—that of solving the rhombus problem. Instead of merely seeing these turbulences as isolated and separate, tying them together as part of a “region of turbulence” correctly reflects their interconnectedness in the wider context of my attempt to teach the solution of the rhombus problem. The time line in Figure 31.1 marks the boundaries of the region of turbulence.
“Region” also has analogical links to the landscape metaphor. Just like a region in a land area has a variety of geographic forms, a region of turbulence may show a rich diversity of experiences. Within the region the turbulence may not be constant. Rather, among the turbulent situations are different types of terrain. I therefore think of “region of turbulence” to encapsulate the rich mix of instructional experiences surrounding the turbulences. With respect to the context of the rhombus problem, the first, second and third turbulences correspond to the three account of turbulences described in the text above. These turbulent periods are added into the timeline to add details to the region of turbulence (see Figure 31.2).

<table>
<thead>
<tr>
<th>13.55</th>
<th>20.20</th>
<th>24.02</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; trigger</td>
<td>2&lt;sup&gt;nd&lt;/sup&gt; trigger</td>
<td>Tell Prop 2.1</td>
</tr>
</tbody>
</table>

**Figure 31.2: Turbulences within the turbulent region**

Problems of teaching, involving unresolvable tensions of conflicting goals, can be represented in the ‘landscape’ as peaks within the respective turbulences. These peaks suitably depict the heightened mental states of struggling with options to deal with competing goals of teaching. Figure 31.3 shows these problematic peaks, together with the goals in consideration<sup>6</sup>, along the timeline within the respective turbulent ranges.

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<sup>6</sup> The diagram shows only the broad overview of the problems in the context of the overall region of turbulence. For the details on how the goal-interactions bring about the conflicts, the reader may like to review Figures 28 and 30.
5.4.2 Problems of Teaching

Through using the navigation metaphor and the concomitant notion of turbulence in the midst of the journey, I have attempted to describe the complexities and problems involved in carrying out various goals of teaching throughout my attempt to teach the solution of the rhombus problem. Using the region of turbulence involved in the teaching of the rhombus problem in Lesson 8 as a ‘prototype’, I now advance a few conjectures about the problems I faced in the teaching of geometry.

P1. Every problem of teaching was traceable to a trigger.

Based on Figure 31.3, there is a one-to-one correspondence between problems of teaching and the triggers that set off the turbulence in the first place. The first trigger was the observation that students were wrongly ascribing the property of perpendicular adjacent sides—a critical attribute of a square—to the rhombus. Although there was an intervening period of about 8 minutes between this trigger and the associated problem peak, during which I was primarily teaching the solution of a parallelogram problem, the subsequent tension among conflicting goals experienced in the first problem peak was traceable to the first trigger. The trigger sparked off the turbulence, of which the problem peak was a part. Without the trigger, there would not be the problem of the inability to meet the differing goals of teaching.
This link between the trigger and the problem peak was similarly observed during the second and third turbulences. In the second trigger, my realisation of a second method of solving the rhombus problem threw up the possibility of including other goals into the instructional process which then led to the goals-in-conflict situation in the second problem peak. In the third turbulence, it was again an awareness of the potential futility of proceeding with students’ reasoning that triggered the third problem peak. The criticality of the triggers in the build up to each of the problem peaks in the rhombus problem strengthens the conjecture that every problem of teaching is traceable to a trigger.

Note that these triggers share common features. All of them were unexpected and therefore not anticipated in the pre-lesson planning process. In navigational language, they were ‘off the radar screen’. Conventional navigational charts and equipment did not detect their existence in advance and so navigators were unable to make preparations before they caused navigational perturbations. The suddenness of their occurrences heightened their problem-causing potential. The unexpected nature of these triggers was perhaps one reason that accounted for the turbulences they caused.

P2. Triggers need not be easily detectable by an external observer.

One common feature about the triggers in the rhombus problem was their seeming insignificance relative to the total teaching time. These triggers were very brief and it is easy to overlook these parts of the lesson as unimportant moments. The first trigger of observing students’ errors happened over a mere few minutes as I walked around to look at students’ work; the second trigger spanned only a few seconds in my thought world as I considered an alternative solution; the third trigger was based on a momentary observation that students’ discussions were not linking to the idea I intended. Although these triggers occupied short amounts of instructional time, their influence were disproportionately great, in terms of how they affected subsequent courses of teaching choices and actions.

There are also, however, important differences between these triggers. The first and third turbulences were sparked by events external to the teacher. The perturbation for the first turbulence, for example, began with a phenomenon—the failure of some
students to see the rhombus as being holders of properties—that was directly observable, even to a third party observer, assuming one was present in the classroom while I taught. The trigger for the second turbulence—my being aware of an alternative method—however, belonged entirely in the teacher’s thought world, invisible to external observation. Despite the subjective and unobservable nature of this internal trigger, the analysis above shows that it contributed significantly to the complexity of teaching.

Another contrasting feature between the triggers is the difference in trigger-problem peak time lapse. Figure 31.3 shows that for the first turbulence there was a time difference of about 8 minutes between the trigger and the conflicting-goals problem. The trigger-problem peak time difference shortened to about 20 seconds in the second turbulence. In the case of the third turbulence, the time lapse was almost imperceptible. These show that the triggers can be more or less near—in terms of time—the occurrence of the actual problem.

Thus, significant triggers to turbulences can appear remote in at least three ways. They can be remote with respect to direct observability—as when hidden in the thought world of the teacher and therefore not observable externally; they can be remote with respect to duration—as in a momentary occurrence; and they can be remote with respect to direct causality—as when there is time lag between trigger and turbulence. For these reasons, I use the term “trigger” not merely to refer to immediately and directly observable phenomena, but also all other activities—including mental ones—that cause an awareness of the need to adjust from the prevailing goals of teaching.

P3. One of the ingredients for problems to occur was the attempt to improvise as a response to triggers.

Triggers by themselves do not bring about the problems of teaching. In the first turbulence, the trigger brought to my awareness the need to consider gmrm. It was my desire to infuse gmrm into the existing gsol purpose that caused the tension when the instructional paths were weighed one against the other. In the second turbulence, the trigger also brought about the opportunity for me to slip in gcvr, grel, and gart. The
subsequent dilemma of not being able to carry out all the goals was a consequence of the conscious act of wanting to bring these additional goals on board in teaching. The third trigger sparked a similar concern with other goals (gcwt and gsol) that led to the struggle between instructional paths. In each of these situations, talk of turbulence would be irrelevant if there were no awareness of triggers nor the effort to improvise to the situation. The problems arose out of a conscious desire on the part of the teacher to improvise his instructional moves according to newfound goals emerging in the classroom context.

This conjecture may appear startling at first glance. An implication of this conjecture seems to be that the more a teacher is aware of and wants to react to triggers in the classroom—such as students’ difficulties—the more he creates problematic situations! If true, this seems to suggest that, to avoid problems, teachers should choose to be unaware of the changing needs of the classroom and not attempt to improvise accordingly—which is an absurd instructional stance. Thankfully, my rhetoric so far need not necessarily lead to this bind. By claiming that a teacher’s attempt to incorporate emergent goals is an ingredient for problems of teaching, I am not asserting that such improvisational attempts by teachers necessarily result in problems of teaching. There may be improvisations that successfully carry out all the goals—both prevailing and emergent—in other regions of my teaching practice within the eleven lessons. In such cases, a teacher’s conscious adaptations to bring in these new goals would then not result in problematic situations.

Moreover, although my consciousness of emerging goals contributed to the problem build-up, it can be argued that a complete disregard for these goals at critical junctures may merely postpone bigger instructional problems for a later time. Taking the example of the first turbulence, not noticing or wanting to react to students’ errors may avoid the situation of the goal tensions. However, it would mean that students’ misconceptions remained uncorrected. Ultimately, this would result in teaching that is not responsive to the instructional needs of the students. Such a scenario can hardly qualify as productive teaching at all. Seen through the navigation theme, one can ignore warning signs along the journey, but it is a sure recipe for impending disaster!
P4. Problems of teaching always involved multiple overarching (G) goals.

I earlier categorized two types of goals of teaching. They are the overarching goals—the G-goals—which I bring along with me as I enter the class to carry out instructional work, and the g-goals which were actually played out in the classroom setting. I have attempted to analyse the problematic situations in the teaching of the rhombus problem via the interaction of g-goals. Since each of the g-goals is subordinate to some G-goals, the goal interactions at each of the problem peaks in the region of turbulence can also be viewed at another level as conflict among (and within) the G-goals of teaching.

Looking at the first turbulence, the g-goals involved were gmrn, gcwt, gsol, and gwkk. The gmrn-G3, gcwt-G1, gsol-G2, and gwkk-G5 subordinations were established earlier. There was thus a number of overarching goals—G1, G2, G3, and G5—involved in the first turbulence. In the second and third turbulences, a different set of g-goals were in operation. They were gcvr, gcwt, gsol, grel, and gart. Similarly, the subordinations to the G-goals can be seen from the gcvr-G1, gcwt-G1, gsol-G2, grel-G4, and gart-G4 links. Viewed from the level of overarching goals, the latter two turbulences involved conflict among and within G1, G2, and G4 goals.

In all three problem peaks in the episode of the rhombus problem, a number of G-goals were involved in the tension struggles. This seems to suggest that the tensions were not merely at the level of emergent goals arising from the dynamic circumstances of teaching. The conflict appeared to strike at the ‘fundamental’ level of my overarching goals of teaching. By ‘fundamental’, I mean the essential and core agenda of what teaching geometry is about to me. The G-goals are indeed fundamental as they represent my beliefs, experiences, and purposes of teaching mathematics. Conflicts among G-goals thus reflected a tension at the very heart of what I believe and know teaching mathematics is all about. The fundamental nature of the conflicts perhaps explains why the struggles to make instructional decisions at these problem peaks were so intensely felt.

7 The reader may like to refer to the Table 10 in Chapter 4 for the summary of g-G subordinations.
P5. Studying a particular problem of teaching without considering the wider context of the region of turbulence can result in a distortion of the complex work of teaching.

This conjecture questions the approach of studying problems of teaching in isolation from the neighbouring instructional context. Taking the example of the third turbulence, the turbulence associated with this section of the lesson was better appreciated by viewing it as within the second turbulence. Suppose, to the contrary, that I extract the third problem peak from the context of the wider region of turbulence and consider it separately by itself. If that is done, the problem under scrutiny reduces only to that part of the lesson where I gave up on student talk about their suggested solutions and changed to teacher-show about how Property 2.1 could be introduced into the solution. However, such a limited view of the problem omits some crucial aspects of the work of teaching. It misses out, for example, on how the turbulence built up to that particular juncture to render it a problem. Without an appreciation of the wider context of the goals consideration, the prior teaching actions carried out and the attempts to juggle the multiple goals, focusing on the problem peak or the decision point itself is inadequate in understanding the complexities underlying the problem.

Taking an overly and inappropriately narrow view of the problem also limits its potential in explaining my relatively short experimentation with students’ reasoning before I shifted to the teacher-tell mode. Seen, however, through the wider perspective of the second turbulence, the picture becomes clearer. Even at the point when I decided to go ahead with letting students articulate their solution paths, I was already feeling the tension of potentially drawing further away from the solution of the rhombus (against gsol) and the time pressure (against gcwt). Thus I was monitoring these gsol and gcwt signals from the very moment I started on the student-talk segment. The constant and increasingly loud signals from gsol and gcwt explain why I decided to switch to teacher-talk in the middle of the earlier improvisation when I sensed that the discussion was unlikely to lead to Property 2.1. All these tensions involved among the goals would be missed out if one looks solely at the point when the problem of teaching was most acutely experienced.
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In cautioning against viewing problems of teaching in a separatist way, I am, however, not arguing for a blanket fixed-size region of lesson to conduct analysis of teaching. My argument is that the restricting of the region of analysis to merely the problem alone can sometimes distort the view of the work of teaching, with all its inherent connections to the wider context where the problem is situated. Rather, problems need to be framed within a turbulent range whose size is suitable to bring out the true complexities as faced by the teacher.

P6. Using one lesson alone as the ‘unit of analysis’ of problems of teaching may be inadequate for understanding the internal tension involved when dealing with competing goals of teaching.

The argument in support of this conjecture is similar to the one offered in the discussion of the fifth conjecture, but at a different ‘scale’ level. Not only were there occasions where the problems of teaching were better appreciated in the wider context of the entire region of turbulence, there may also be a need to see the problems beyond the confines of the lesson itself.

Looking again at the description of the teaching of the rhombus problem, I ‘broke away’ from the Lesson 8 itself at a few junctures to provide a history of the development of the related ideas. I did so by referring the reader to the earlier Lessons 6 and 7, such as when discussing the role of Property 2.1 in the solution of the rhombus problem. Property 2.1 came to mind primarily because it was left out in the earlier treatment of the rhombuses in Lesson 7 and it naturally occurred to me then in Lesson 8 to be a suitable juncture to include it.

Another example of this referencing beyond Lesson 8 was when I discussed the development of special quadrilaterals in the light of the error I noticed in students’ attributing the property of perpendicular adjacent sides to the rhombus. It is only through an appreciation of a careful development of the concept of rhombus—from viewing its gestalt features to recognition of angle properties to it being a holder of properties—that the reader can glimpse the teacher’s experience of tension when confronted with students being ‘stuck’ at the visual level.
These examples illustrate the relevance of varying the ‘grain-size’ of analysis sometimes to a level that is even beyond a single lesson. Although one lesson is often seen as a natural ‘unit of analysis’ in the literature (e.g., Hiebert, Gallimore, & Stigler, 2002), the examination of the goals of teaching of the rhombus problem seems to show that a teacher may take into consideration an instructional history that is beyond the boundaries of a single lesson in weighing and deciding upon the goals of teaching.

5.4.3 Coping Strategies Used Within the Turbulent Region

My account of the teaching of the rhombus problem highlighted not only the tensions involved with conflicting goals, but also my efforts at managing the turbulences surrounding those problematic regions of practice. Those coping strategies employed were not solutions to the problems—if “solution” means a way to carry out all the desired teaching goals. Rather, they were ways to keep the turbulences under some measure of control to a point that, despite the need to abandon or compromise other worthy goals of instruction, the overall teaching activity in the classroom could still proceed with a large degree of success in achieving some other teaching goals. In relation to the navigation metaphor, “coping strategies” can be compared to the actions taken to ‘steady the aircraft’ when undergoing turbulent patches. Like “coping”, such efforts to manage the aircraft during perturbations do not guarantee zero damage, but rather strive for overall stability and air-worthiness despite the possibility of suffering some relatively minor losses. Just as conjectures about the problems of teaching can be drawn from my experience with teaching the rhombus problem, I will similarly propose some conjectures about the coping strategies that I employed in the lesson.

C1. Coping involved an ongoing monitoring of changes in the instructional situation.

That there were changes in the instructional situation throughout the region of turbulence is easily seen in Figure 31.3. The changes were reflected in the presence of triggers as well as the different sets of goals-in-consideration as a result of the introduction of these triggers. The fact that these changes were noted and acted upon
showed that I was first of all aware of the existence and influence of these elements in the classroom setting. Awareness precedes action or decision. Whether it was the alertness to the first trigger of students’ error with assuming perpendicular adjacent sides of the rhombus, or the consciousness of an alternative method of solution to find “y” that resulted in the second trigger, or the sensitivity to the third trigger of time pressure and the futility of pursuing a discursive track, these examples showed my awareness of the changes as part of my way to cope with the related problems of teaching.

In addition, it should be noted that monitoring changes in the instructional situation is not of a one-off kind. Rather, as seen from the various changes within the relatively short 10-minute duration of the turbulent region, there was an ongoing ‘update’ of the changing situations in response to the dynamic goal-interaction situations throughout that period. Returning to the analogy of navigating an aircraft, the constant monitoring of the instructional situation can be likened to the heightened vigilance with which the navigator watches the control panels, signal lights, data dials, surrounding terrain, etc., as he weathers a turbulent portion of the aircraft’s flight path.

As a clarification of terms, my use of the phrase “instructional situation” is not restricted to only the physical conditions of the classroom or merely the behaviour of the students. It includes also all the non-physical elements during instruction that contribute directly to the teaching decisions in the classroom. It certainly includes all weighing of instructional goals and alternatives within the mind of the teacher. That these internal activities of the teacher can influence teaching behaviour and should rightly be considered as constituents of the “instructional situation” is clearly evidenced by their roles in altering my instructional paths throughout the turbulent region described above.

C2. Many of the resources I harnessed during coping were invisible to others.

Some of the resources I used to cope with the turbulences were clearly visible ones. They included the whiteboard and writing materials that I used to quickly present the difference between a square and a rhombus. Also included was the overhead
transparency with which I introduced Property 2.1 in the context of the other properties of a rhombus. These easily observable tools I utilised as improvisations to my instructional activity were “resources” in the way the term is often used in the literature on “teaching resources”.

However, my use of the term “resources” here extends beyond the conventional references to outward equipment and aids to include all other kinds of resources I drew upon during coping. Quite clearly, even prior to the use of the whiteboard and the transparency, I must first possess the knowledge of how to use these materials to fulfil my intentions of harnessing these tools appropriately. This knowledge, and not just the actual hardware, is therefore part of the resources that I used to ride out the turbulence.

But the resource of knowledge is not necessarily linked to the use of external resources. In the process of weighing possible instructional tracks at different problematic points of the turbulence, I was also demonstrating the use of my knowledge of possible teaching options in relation to the ongoing changes in the instructional situation. This knowledge resource of possible instructional choices is best illustrated by Figures 28 and 30, and certainly featured prominently as a major part of my coping strategies.

This conjecture about invisible resources will appear more convincing when considered alongside P2. Just as triggers to problems of teaching occurred largely within my thought world and were inaccessible to outside observers, many of the resources that I used in coping were also tapped from my internal knowledge base and were thus not easily observable to others.

C3. While coping, I was influenced by phenomena outside the immediate context of the instructional situation.

In conjecture P6, I mentioned that a suitable frame-size to view a problem of teaching may extend beyond the confines of a single lesson. Similarly, when looking at coping strategies, the elements that shape the way I deal with problems need not be contained within the immediate temporal vicinity of the problems themselves.
Within the account of teaching the rhombus problem, when I was about to introduce Property 2.1 to the class, there was indeed a phenomenon external to the immediate context—that “we have not talked about it before” (post-lesson reflections)—which had a substantial impact on how I conducted the subsequent instruction. The realisation of what I had done (or rather, not done) in the past lessons with the class meant that I was aware that Property 2.1 would be unfamiliar to the students, which prompted the need to devise a coping strategy. Had I not invoked that memory of no prior mention of Property 2.1, I would have proceeded with the lesson in a very different way. The problems and the coping mechanisms that followed would have been different too.

That there are actual influences from happenings and events outside the neighbouring vicinity of the turbulence should not be taken to mean that the immediate elements of the instructional context are unimportant. Rather, the point that is significant is that, while the work of coping was primarily in finding ways to deal with the flux of triggers, goals, and problems within the immediate context, other phenomena within the instructional history with the class can also influence the process in tangible ways.

Interestingly, historical data and experience can also be seen as important elements when seen through the navigation metaphor. When deciding coping actions in a rough patch of the journey, the navigator relies not only on the live information fed to him via the dials, controls, and monitors, but also on knowledge of the aircraft and the terrain based on previous flying experiences.

C4. Coping involved prioritisation of goals, and the priorities needed not remain the same throughout the turbulent region.

In the turbulent region under consideration, problems occurred when there was no practical way to resolve the conflict between different goals of teaching. Under such circumstances, coping must necessarily involve a choice of one set of goals over another. In other words, in all the problem peaks experienced in the teaching of the rhombus problem, I proceeded with subsequent instructional decisions based on my
priority of certain goals at each of those junctures. In the first problem peak (see Figure 31.3), my priority was gsol over gmrn; in the next problem peak, my preference was grel and gart against the other choices of gsol and gcvr; and in the final problem situation, gcvr was seen as more important than the other goals. Thus, at every situation of competing agendas, the goals under consideration were weighed and some were given priority over the others.

What is also noteworthy is that the act of prioritisation of goals was not a once-and-for-all decision in the sense that a particular goal or set of goals then remained the top choice throughout the entire turbulent region. Goal gsol, for example, was the priority in the first turbulence but was not the preferred choice in the second problem peak. Similarly, gcvr was not taken up in the second problem peak but was given the priority in the third turbulence. This shows that goal priority was not a static state but fluctuated along with the dynamic interaction with other goals under focus.

In fact, upon closer examination of problem peaks two and three, the goals-mix in the two situations was exactly the same and yet different goals emerged to be the priority under each case. In the second problem peak, I was favouring grel and gart because I wanted students to actively think about diagonal properties of a rhombus instead of relying on my direct-telling, even though I had already felt the time pressure then (to pursue gcvr); but in the third problem peak, the time pressure element became much stronger because of the students’ unproductive responses, resulting in gcvr overtaking grel and gart in prominence. This switch of goal dominance seemed to indicate that the choice of which goals to prioritise was not tied to a fixed hierarchical relationship with other goals, but was influenced by the complex and ongoing changes within the instructional situation during turbulence.

C5. Actions were taken to alleviate the potential ill effects of unfulfilled goals.

Although decision-making during problems resulted in the sacrifice of some goals in order to fulfil other goals in preference, there were efforts to minimise the negative consequences. Such attempts can be seen in all the three problem situations.
In the first problem peak, there was a purposeful attempt at achieving gmrn even though it was by a presentation of a “quick draw” of a rhombus and a square so as to bring as many students as possible to operate at a higher level within the short time. As for the second problem, I was aware of the limited time I had and so tried to steer the students’ conjectures into productive directions as soon as I detected a veering away from my intended destination. In the third turbulence, my attempt at downplaying my departure from using students’ discourse to arrive at jointly-agreed facts was by being apologetic and by using time limitations to justify a different instructional approach.

Although the methods used were different in all the three problem situations, there were clear attempts in each case at alleviating the potential ill effects of goals that were left out. Again, turning to the navigation metaphor, such efforts can be likened to the work of ‘damage control’ when steering through troubled weather. There are times when the pilot cannot avoid minor damage sustained through turbulent regions, but the main focus is always to stay on the main flight course while keeping the damages to the bare minimum.

5.5 From One lesson to the Whole Module

This chapter corresponded to the first phase of the analysis by looking at the problems and coping strategies encountered in a single lesson. A number of provisional conjectures have been advanced in the light of the examination of a turbulent region within the lesson. In the next chapter, I will proceed to the second phase of analysis by considering other turbulent regions experienced throughout the entire module. The earlier conjectures will be subjected to scrutiny against the data afforded in other lessons of the module.
The conjectures in the previous chapter will be scrutinised and refined in this chapter as I zoom out from the region of turbulence in Lesson 8 to consider the whole geometry module. The main geometrical development track from lesson to lesson is presented in Table 11. Within the topics, there were components which involved Sketchpad use. The use of computer tools will be the direct focus of analysis in the next chapter. Here I consider only those portions of the module that used traditional media.

<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Main geometrical ideas/skills taught</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Polygons</td>
</tr>
<tr>
<td>2</td>
<td>Angles related to point, line, right angle, intersecting lines, parallel lines</td>
</tr>
<tr>
<td>3</td>
<td>Construct parallel and perpendicular lines using ruler and setsquare</td>
</tr>
<tr>
<td>4</td>
<td>Angle properties of triangles</td>
</tr>
<tr>
<td>5</td>
<td>Apply angle properties of triangles in textbook exercise items</td>
</tr>
<tr>
<td>6</td>
<td>Draw special quadrilaterals using ruler and setsquare</td>
</tr>
<tr>
<td>7</td>
<td>Side, angle, and diagonal properties of special quadrilaterals</td>
</tr>
<tr>
<td>8</td>
<td>Apply properties of special quadrilaterals in textbook exercise items.</td>
</tr>
<tr>
<td>9</td>
<td>Hierarchical relationships between special quadrilaterals</td>
</tr>
<tr>
<td>10</td>
<td>Construction of quadrilaterals using a marked ruler and a pair of compasses</td>
</tr>
<tr>
<td>11</td>
<td>Construction of perpendicular bisectors and angle bisectors using a straightedge and a pair of compasses</td>
</tr>
</tbody>
</table>

Table 11: Development of main geometrical ideas and skills within the module

Due to space constraints, I cannot describe all teaching segments throughout all the lessons in the same way as for the rhombus problem in Chapter 5. In surveying the module, I organise the presentation chronologically by turbulent regions. As established in the previous chapter, turbulent regions are suitable frames of analysis to view problems of teaching and the coping strategies. In selecting the turbulent regions for discussion in this chapter, I considered examples (when taken as a whole) that allow the reader to follow the overall geometrical development and only those whose problems I felt most intensely during teaching. When taken together, these selected turbulent regions presented problems and coping strategies involving all the g-goals, across different time ranges, and dealt with different geometrical ideas in the module. Table 12 shows the turbulences and the parameters along g-goals, time duration, and geometrical content covered.
PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Turbulent region</th>
<th>Lesson No. involved</th>
<th>g-goals involved</th>
<th>Approximate duration of turbulent region</th>
<th>Main geometrical content involved</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>grel, gtry, gsol, gart, gcon</td>
<td>11 minutes</td>
<td>Polygons</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>gmrn, gcvr, gcbt, gret, gart, gwkk, gstg</td>
<td>20 seconds</td>
<td>Angles at a point</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>gins</td>
<td>2 minutes</td>
<td>Setsquare construction of parallel lines</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>gerr, gart, gmrn, gsol, gcbt, galt, grsd</td>
<td>Beyond 1 lesson</td>
<td>Angle properties of triangles</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>glan</td>
<td>12 seconds</td>
<td>Apply alternate angle theorem</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>gmrn, gins</td>
<td>10 minutes</td>
<td>Recognition of quadrilaterals</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>gsol, gpsv, gwws, gmrn, gcbt, galt</td>
<td>Before and during first 15 min of lesson</td>
<td>Apply angle properties involving parallel lines</td>
</tr>
<tr>
<td>8</td>
<td>7-9</td>
<td>gwkk, gstg, gcon, gmrn</td>
<td>3 lessons</td>
<td>Hierarchical relations between special quadrilaterals</td>
</tr>
<tr>
<td>9</td>
<td>7-10</td>
<td>gerr, gsol, gins, gmrn, gcvr</td>
<td>4 lessons</td>
<td>Construction of quadrilaterals</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>grel, gwkk</td>
<td>20 minutes</td>
<td>Construction of perpendicular and angle bisectors</td>
</tr>
</tbody>
</table>

Table 12: Selected turbulent regions

Since this chapter focuses on reviewing the conjectures proposed in the last chapter, these conjectures are reproduced for ease of reference.

P1. Every problem of teaching was traceable to a trigger.

P2. Triggers need not be easily detectable by an external observer.

P3. One of the ingredients for problems to occur was the attempt to improvise as a response to triggers.

P4. Problems of teaching always involved multiple overarching (G) goals.
PHASE II ANALYSIS

P5. Studying a particular problem of teaching without considering the wider context of the region of turbulence can result in a distortion of the complex work of teaching.

P6. Using one lesson alone as the ‘unit of analysis’ of problems of teaching may be inadequate for understanding the internal tension involved when dealing with competing goals of teaching.

C1. Coping involved an ongoing monitoring of changes in the instructional situation.

C2. Many of the resources I harnessed during coping were invisible to others.

C3. While coping, I was influenced by phenomena outside the immediate context of the instructional situation.

C4. Coping involved prioritisation of goals, and the priorities need not remain the same throughout the turbulent region.

C5. Actions were taken to alleviate the potential ill effects of unfulfilled goals.

6.1 Turbulent Region 1: Teaching about Polygons in Lesson 1

In the Component I of Lesson 1, I listed my expectations for students’ learning behaviour since this was my first meeting with the class. We then moved to the Math lab (where all subsequent lessons were held), and Component II of the lesson was devoted to explaining lab rules and routines. I then moved into the day’s geometrical topic—polygons. The main goals for teaching about polygon are evident in my lesson memo:

<table>
<thead>
<tr>
<th>Lesson component III. POLYGONS.</th>
</tr>
</thead>
<tbody>
<tr>
<td>III.1 Write “Polygons” on board and get them to talk about polygons …</td>
</tr>
<tr>
<td>III.2 Do the worksheet Question 1 and 2.</td>
</tr>
<tr>
<td>III.3 Discuss worksheet and show (after looking from table to table) alternative answers.</td>
</tr>
<tr>
<td>III.4 Lead to ‘definition’ of polygon and mention that regular polygons are special polygons.</td>
</tr>
</tbody>
</table>
The first two questions of the worksheet for Section III.2 are reproduced in Figure 32.

1. For each of the figures below, write a name of the figure by filling in the blank provided. If it is not a polygon, fill-in ‘non-polygon’.

<table>
<thead>
<tr>
<th>a.</th>
<th>b.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image_url" alt="Figure a" /></td>
<td><img src="image_url" alt="Figure b" /></td>
</tr>
<tr>
<td>________________</td>
<td>________________</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c.</th>
<th>d.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image_url" alt="Figure c" /></td>
<td><img src="image_url" alt="Figure d" /></td>
</tr>
<tr>
<td>________________</td>
<td>________________</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>e.</th>
<th>f.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image_url" alt="Figure e" /></td>
<td><img src="image_url" alt="Figure f" /></td>
</tr>
<tr>
<td>________________</td>
<td>________________</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>g.</th>
<th>h.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image_url" alt="Figure g" /></td>
<td><img src="image_url" alt="Figure h" /></td>
</tr>
<tr>
<td>________________</td>
<td>________________</td>
</tr>
</tbody>
</table>
2. Based on the answers to the exercise 1 above, complete the table below.

<table>
<thead>
<tr>
<th>POLYGONS</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Polygons must . .</td>
<td>Polygons need not . .</td>
</tr>
<tr>
<td>(eg.) have more than 2 sides</td>
<td>(eg.) have equal sides</td>
</tr>
</tbody>
</table>

Figure 32: Question 1 and 2 of the worksheet

My overall intent in Component III of the lesson, apart from teaching the topic on polygons [gcvr], was to encourage students to take an active role in learning about polygons instead of relying upon the teacher for information.

The Primary Mathematics syllabus includes “triangles and quadrilaterals” as geometry topics. Students at year seven would thus be familiar with triangles and quadrilaterals as special polygons, even though the formal term “polygon” might not have been introduced. My intention was to invoke students’ earlier knowledge to help them generalise properties of triangles and quadrilaterals to beyond three or four sides. I expected them initially to guess what polygons were in Section III.1 in a whole-class setting. I planned to follow-up in Section III.2 by having students check the plausibility of these guesses using Question 1 of the worksheet, and to form preliminary conjectures about attributes of polygons in Question Two. In Section III.3, I would re-discuss students’ various ideas of polygons as a class, leading them in Section III.4 to a conventional definition of polygon. My primary goals during Component III were to imbue in students a habit of learning mathematics via their own attempts at making sense of the situation [gtry], and to move away from relying on the teacher for simple answers [grel].
As planned, I started Section III.1 by asking students to comment on what they thought polygons were. The first trigger of turbulence occurred shortly after that, in the form of non-response from the class:

A minute has passed and I noticed that students were not very open or used to this kind of whole-class [discussion] setting. I thought it’s not worthwhile to continue this track. What I had originally intended is for them to write down a few conjectures of what polygons are. I thought the better way is to give them the worksheet and let them try and perhaps they will figure out more things through that and then we can discuss afterwards. [Post-lesson reflections]

On sensing that students were perhaps uncomfortable with classroom discourse that required them to voice their guesses, I improvised on the original plan by moving to Section III.2 of the lesson sooner than intended. I hoped that after students had attempted the worksheet, the whole-class discussion following that would be more productive:

6.18  T: OK. Instead of going on with the discussion, I am going to give you a worksheet. I am going to give you ten minutes to complete the worksheet on your own. After that we'll discuss and we'll know more about polygons …

6.36  T: I am distributing the worksheet now. Try on your own Question 1 and 2. You've only got ten minutes so you have to work very quickly. [Distributes worksheet]

Despite shortening section III.1, the emphasis in section III.2 on “trying on your own” reflected my attempt to still achieve gtry. I wanted to maintain the message that they needed to cultivate the disposition to make guesses based on existing knowledge even when they were uncertain.

The second trigger occurred as I walked around looking at what students were doing on their worksheets. I noticed that some students were not fulfilling my gtry purpose of having “good tries” at the worksheet. They were simply leaving Question 1 blank because they were unsure of the ‘correct answer’. My table-side prompts to May, Elsie, and Shi Hui illustrate my awareness of this trigger:
My reflections on the lesson reveal the reasons for my attempts at persisting with gtry and grel despite these repeated signs of turbulence:

I was trying to encourage this girl May—“give it a try”. And the words I seem to be using all the time was “give it a good try!” In a sense I wanted them to switch their culture of doing math from one where you move every step with great certainty, following the teacher’s way, to one where they can conjecture, can try and re-order their thinking later on. So I was not just teaching content, I was teaching the necessary moral qualities of “giving a good try” to learn mathematics.

I continued with the gtry and grel motivations, but the signals issued by the two triggers indicating turbulence made me increasingly aware of internal resistance from the students. I attributed their struggles with my goals of gtry and grel to a drastic shift away from their previous mathematics learning experiences.

As I moved away from May’s table to other tables, I noticed similar cases of leaving the first page blank. Each observation added to my tension, building until there emerged a conscious sense of a problem of conflicting goals:

At this juncture I have been seeing more and more students leaving the first page blank. I suspect they reached a frustration level where they just gave up trying because teacher was not teaching in the traditional way. Here, [I was] going against cultural norm. Yet I was reluctant to give up this attempt to change the cultural setting [into] one where they can boldly make attempts and give credit and regard to their own answers and their own thinking. I can do either one of these things now—to stop their work and say “come, let’s listen to me, I’ll teach you how to do it, … [to] release you from the struggle and provide answers for you.”

Observations of the students seemed to show that they were unready to make that “cultural shift” from the traditional “teacher-show” way of teaching to the one that I
intended then. An alternative to “releasing” the students from this difficulty was for me to switch into a mode they would be more familiar with—just telling them what a polygon was—and in so doing fulfil gsol as well. However, this would have undermined the disposition of gtry and grel which I was trying to build and which underpinned my overarching G4 agenda throughout the module. However, if I persisted in asking them to keep trying, I risked accentuating their frustration. As a result, students might lose confidence in their own ability to learn the required geometrical idea [against gcon]. I was in this problematic situation of having no solution no matter which path I chose. My reflections describe the decision I made at that first problem peak at 11.55:

I was reluctant to do that [referring to giving them the answers], because if I do that, I will fail, in my opinion, in getting the students into the groove of thinking for themselves, coming up with answers on their own. So I wanted to give it a good shot, and I wanted to persist on [in trying on their own]. So I gave them a small pep talk—telling them that in my class it is important to “give it a good try”.

However, I was conscious also of the build-up in their frustration level so the way I managed their growing unrest while pursuing the gtry course was to adopt a damage-control strategy:

[I] choose the time [to stop the Section III.2 segment] to discuss [their conjectures at the next Section III.3]. . . by looking at their body language and the work that is done on paper.

When I judged that most students had made some attempts at the worksheet items and could go no further on their own, I moved on to Section III.3 where I gathered their answers together and led them to discuss the critical and non-critical attributes of “Polygon”. I started by getting students to build upon their earlier forays into what polygons were via the worksheet items. Based on what I observed in Section III.2 earlier, I wrote some of their answers on the transparency version of the worksheet. I projected the overhead worksheet on the screen and invited students to articulate their views [gart]:
The equivocal responses provided me with what I thought to be the right opportunity to connect to the geometrical concept I wanted them to learn—what polygons were—and I therefore asked a follow-up question, “So what makes a polygon a polygon?” A mixture of responses from a small portion of the class greeted my question. When Faizul put up his hand, I invited him to start the discussion going.

I made use of Faizul’s attempted ‘definition’ of polygon to test whether diagram 1(a) was a polygon. However, the agreement of only a few students with this determination of “polygon” triggered another turbulence. I had earlier anticipated that most students would agree that 1(a) was a polygon based on some implicit guesses of what polygons were (although I did not expect them at that stage to be able to verbalise the attributes formally). I intended to proceed in a similar way in determining polygon/non-polygon for each of the 1(b) – 1(h) items, so that the minority who had wrong conjectures could modify their mental concept of “polygons” along the way. I was thus surprised when only a handful could ‘see’ that the hexagon in 1(a) was a polygon.

1 This was an error. The statement in p. 2 of the worksheet states “more than 2 sides”. I was not aware of the error at that point in time but corrected it when the lesson later progressed to Question 2 of the worksheet discussion.
In order to move on to Question 2 where students articulate the critical and non-critical attributes of a polygon [gart], they needed to base their conjectures on correct classification of polygons/non-polygons in Question 1. However, most students seemed unable to provide the correct answers for Question 1, and I was not prepared to just give them the correct answers [counter gret]. That trigger of students’ uncertainty brought about another irresolvable problem of opposing goals. The path of continual reliance on students to come up with answers [grel] led to a logical dead end—the impossibility of conjecturing polygonal properties in Question 2 without first knowing which objects were polygons in Question 1 [against gart]; the path of teacher’s direct telling of answers [gsol] would go headlong against the entire G4 motivation intended for that part of the lesson.

As a form of damage control, I told them to accept my authority of judgment for the time being and wait later for a fuller understanding of what a polygon is:

17.04 T: Umm … Anyway we say it's a polygon first, OK? For those who say it's not, just wait for a while.

Nevertheless the struggle was clearly experienced and again captured in the post-lesson reflection for that part of the lesson:

Here this is probably one of the most difficult parts of the lesson because … I don’t want them to just say “teacher says this is correct, you are wrong”. But I want them to move towards a realisation why polygon is called a polygon and I wanted them to participate [towards that answer-seeking process]. I don’t think I can avoid at this point in time and say, “now fine, just accept this as a polygon”. The one who say “non-polygon” I told him to just “hold it”. And I told him that we’ll come back to you later but “just hold it and find out what is a polygon, what is not a polygon”.

Subsequently, I tried to go over the polygon/non-polygon classification for each of the remaining diagrams quickly so that I could reach the more meaningful discussion on the critical and non-critical attributes of polygons in Question 2 sooner. In the process, I alleviated the undermining of G4 by also getting students to voice their agreement too instead of a solely direct-answer-telling mode by the teacher. The region of turbulence discussed thus far is summarised in Figure 33.
6.1.1 Review of Turbulent Region 1

With reference to P1 that “every problem is traceable to a trigger”, this segment adds evidence for the significance of triggers in kick starting turbulences which led to problems. However, the possibility of a plurality of triggers rather than a single trigger corresponding to a problem is revealed. In Lesson 1, the first trigger of students’ non-response alone did not directly result in the problematic situation later. It sensitised me to students’ discomfort with my emphasis of trying without being certain of answers. My awareness of their uneasiness grew when I observed their reticence during seatwork in the second trigger, which directly led to the conflict of whether to prioritise grel and gtry or to pay more attention to gsol and gcon. Neither is it helpful to focus only on the second trigger alone without viewing it in the wider context of the influence by the first trigger. Forcing a one-trigger-to-one-problem correspondence distorts the complexity inherent in the region of turbulence under investigation. Thus, by saying that “every problem is traceable to a trigger”, it should be clarified that “a trigger” need not mean a unique trigger.

The vignette also supports conjecture C3, that my way of coping with problems was influenced by phenomena outside the immediate temporal context. When I observed the students’ response to the worksheet assignment, I did so knowing what they had learnt in Primary school. That knowledge served as a background for me to formulate
reasonable expectations$^2$ of what they knew, thus influencing my instructional choices.

In addition to the retrospective view, there is evidence of anticipating future events. My knowledge of what I planned to do in Section III.2 allowed me to present it earlier than intended. So, the phenomena “outside the immediate context of the instructional situation” in conjecture C3 can refer to temporal perspectives that are both before and after the situation under consideration.

The other conjectures P1 – P6 and C1 – C5 receive general support from the description of Turbulent Region 1. Comparing Figure 31.3 in Chapter 5 about the rhombus problem and Figure 33 reveals similarity in the ‘build-up’ of the problems via triggers and the presence of multiple linked turbulences within the turbulent regions. Table 13 provides a summary of the strength of each of the conjectures advanced earlier when evaluated against the problem and coping experiences encountered in Lesson 1.

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$^2$ Post hoc analysis made me wonder if some expectations I had—such as students being able to make sensible guesses of “polygons” without first being taught what it was—was reasonable. If not, then the turbulence reported here may have been a consequence of this unreasonable expectation on my part as the teacher. I was initially tempted not to discuss this turbulent region here so as to avoid any potential ‘negative’ views of me being presented. I choose to resist this temptation in favour of keeping to the principles of selection of turbulent regions. I also readily admit that some of the problems of teaching reported may be traceable to my mistakes or wrong anticipations. I take comfort that I am learning to moderate my expectations through such reflections of each lesson and that the best teachers can also make similar miscalculations.
6.2 Turbulent Region 2: Teaching Angles Related to Point, Line, Right Angle, Intersecting lines, Parallel Lines in Lesson 2

Lesson 1 ended with students working on a Sketchpad activity to tessellate the plane with different regular polygons—equilateral triangle, square, regular pentagon,
regular hexagon, regular octagon, and regular decagon. As an example of one of the tiling tasks, Figure 34 shows how the initial layout of the regular hexagonal tiles can be re-arranged—by dragging the points at the vertex of each hexagon to rotate and dragging the interior of the tiles to translate—to close up the gaps in between the tiles to present a tessellation.

![Figure 34: Sketchpad activity on tessellating regular hexagons](image)

In Lesson 2, one objective was to build upon the Sketchpad tiling activity to lead them to the fact that “angles at a point add up to 360 degrees” was the underlying reason why certain regular polygons tile the plane while others do not. How I intended to proceed with the fulfilment of this objective can be seen from an excerpt of my lesson memo, written prior to conducting the lesson:

<table>
<thead>
<tr>
<th>Component I. Revise POLYGONS.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.1   Didn’t manage to do what I intended in the previous lesson. Will start by quickly recalling the regular polygon tiling activity by using Sketchpad (teacher show).</td>
</tr>
<tr>
<td>I.2   Get them to conjecture what determines some to tessellate but not others.</td>
</tr>
<tr>
<td>I.3   Lead (using manipulatives to give a few examples) that angles at a point totals 360 degrees.</td>
</tr>
</tbody>
</table>

It is clear that apart from the goal of teaching “that angles at a point total 360 degrees” as a way to cover the required geometry syllabus (gcvr), there were also other goals, evident in phrases like “get them (students) to conjecture . . .” (gart) and “lead [not tell] . . .” (grel). In addition, there was a gmrn goal involved. Not only did I want to help students know which regular polygons tessellate the plane, I also wanted them to know the underlying geometrical reasons behind those observations. Time pressure was felt in this lesson since what I planned to do in Lesson 1—to link their Sketchpad
explorations to the “angles at a point” property—had to be carried over to Lesson 2. There was thus a gcwt at the back of my mind as I taught.

The turbulent region in this lesson was associated with Section I.2:

This is what I consider to be the most important point of the lesson. And that is to let them see the connection between what they have observed in the tessellation Sketchpad exercise and the fact that angles at a point must add up to 360 degrees. I do that by asking the question why certain regular polygons will tessellate the plane while others don’t … [Post-lesson reflections]

I started that segment of the lesson by showing the overhead copy (OHT) of their worksheet (Table 14) which summarised the information obtained so far concerning which regular polygons tessellate the plane.

<table>
<thead>
<tr>
<th>Regular Polygon</th>
<th>Drawing</th>
<th>Measure of one interior angle of the regular polygon (an approximate)</th>
<th>Is this regular polygon able to tile the floor? (Yes/No)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Square</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Regular pentagon</td>
<td></td>
<td></td>
<td>✗</td>
</tr>
<tr>
<td>Regular hexagon</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>
Using this information, I started the discussion on the reasons why some regular polygons tessellate the plane.

Even though Edward’s contribution did not link towards the geometrical idea I intended, it was not a wrong answer and so I acknowledged his input as a way to encourage others to join the discussion. In so doing, I was trying to advance my agenda of getting students to participate in classroom discourse as a way to learn geometrical content. The class seemed to take the cue from my openness to Edward’s answer, as seen from the next conjecture:
4.35 Rashid: Too many sides.
T: [Stretched out left hand invitingly and leaning forward]: Say again?
Rashid: Too many sides [louder].
T: Too many sides? [pause, looking tentative]
A few others: Yes.

At this point Rashid’s answer could have been used to open up a productive discussion by tossing his conjecture back to the class for substantiation or refutation (gart). I could have taken a backseat and allowed students to use the data in the OHT to come up with their own agreed conclusions instead of my telling the answers (grel). At that juncture, however, I was beginning to feel some tension signaling turbulence amidst conflicting goals. The time pressure weighed heavily in my consideration of instructional choices. This section was meant to be complete unfinished work from the previous lesson. I thought I needed to finish this part of the lesson quickly so that I could proceed with what was originally planned for Lesson 2 itself (gcwt). It did not help the gcwt cause when Rashid gave a ‘wrong’ answer. To steer the class away from considering sides to looking at angles (the gmrn intention) via the gart and grel route of asking further questions, waiting upon students to observe for themselves, allowing them to voice refutations and counter conjectures seemed too time-consuming at that instance:

But there was one occasion where someone said whether the polygon tessellates the plane is determined by the number of sides … . I was pressured by time. It [time] is also a very significant issue. When I find something important to tease out the reasoning, it often goes against time limitations. This is yet another problem in teaching. [Post-lesson reflections]

Rashid’s wrong answer was a trigger to a situation involving competing priorities. The gart motivation would have led me to encourage discussions about Rashid’s comment, but was seen as taking too long a route (clashing against gcwt) to address the “angles at a point” property (gcvr). On the other hand, if I intervened immediately to redirect the discussion to “angles” to fulfil gcwt, I would have undermined my attempts to reduce students’ reliance on me for simple answers to questions (against grel). The tension I experienced between conflicting sets of goals was yet another problem in teaching. I had to make a decision which cannot avoid the problem of
violating some goals of teaching. My improvisation, in reaction to Rashid’s conjecture, was influenced more strongly by the gcwt motivation:

4.45  T: No. You see - hexagon has more sides than pentagon right? But this one [pointing to the regular hexagon on the OHT] can tile but this one [pointing to the regular pentagon on the OHT] cannot tile.

The outcome was a flurry of other conjectures from a number of students. I was glad that my block of Rashid’s talk of “sides” did not discourage other students in the class from trying conjecturing. Amidst the chorus of responses, I could discern “the sides are different”, “different shapes”, and a voice talking about “angles”. That reference to “angles” provided an opportunity to use a student’s contribution to connect to my agenda of teaching the “angles at a point” property (gcvr). I therefore quietened the class and specifically invited that particular student to ‘lead the way’:

4.50  T: Shhh … [finger on lips] Remember we have to listen to one another, so one by one.
[Hand extended to Kai’s direction in an inviting way, smiling] Yes?
Kai: The angles are different
T: [Pause]

My pause following Kai’s answer was actually yet another trigger to a follow-up turbulence. As I paused, I wondered, “Does the rest of the class understand what Kai was pointing towards?” If I assumed that the class followed Kai’s way of thinking, then we were close to my target of seeing “angles at a point sum up to 360 degrees” as an explanation of why certain regular polygons tessellate. All I had to do was to highlight the fact that for those polygons that do tile the floor, the degree measures of each of their interior angles are factors of 360. That would have quickly (in the spirit of gcwt) fulfilled the gmrn intention of teaching the underlying geometric meanings behind the tessellations phenomena. However, I was unsure if many others in the class were at the “level of reasoning”:

Kai gave a suggestion but I think the rest of the class is not at his level of reasoning yet.
I am aware that I am not just teaching Kai but also the rest of the class. If I follow Kai’s line of argument, I may lose the class altogether. [Post-lesson reflections]

My concern was that if I proceeded with the reasoning process from the point which Kai stated, then I might leave some students behind (and hence violating gwkk). I
was not sure if many in the class understood Kai’s statement—“the angles are different”—in the first place. On the other hand, if I did not follow up on Kai’s statement, I would miss a good opportunity to teach students who were ready to handle the next level of the reasoning process. The resulting instructional sequence would then not cater to such more able students (against gstg). Kai’s statement was therefore a trigger to a tension between teaching the stronger students (gstg) and addressing the needs of the weaker students (gwkk). Whichever decision I made, I would not be able to teach satisfactorily those two sets of students simultaneously (fulfilling gstg and gwkk). This is again a problem of conflicting goals.

My improvisation then was skewed to the needs of the weaker students. I wanted the assurance that most of the students saw that, indeed, “the angles are different”. In the overhead (Table 11), the data field in the third column from the right labeled as “Measure of one interior angle” was still not entered. Without the numerical data in that field, it reduced students’ ability to verify the statement that “the angles are different”. I proceeded with asking students to help me fill up the approximate measures of one interior angle of the respective regular polygons in the third column of the transparency. With the numerical data presented before them, I hoped they could more easily affirm the differences in angle measures as a way to explain the tessellating characteristics.

Thus, in Lesson 2, the region of turbulence began from the first trigger of Rashid’s “too many sides” comment to my decision to cope with Kai’s “the angles are different” answer. The peaks and duration of the turbulences are summarised in Figure 35. In the remaining parts of Component I of the lesson, I attempted to help students relate the numerical measures of the interior angles to the respective tessellation behaviours. I led students to the conclusion that for a regular polygon to tile the plane, the measure of its interior angle must be a factor of 360, as intended.
6.2.1 Review of Turbulent Region 2

In the first problem peak, the goals involved were grmn, gcvr, gcbt, grel, and gart. Due to the grmn-G3, gcvr-G1, gcbt-G1, grel-G4, and gart-G4 connections, the conflict within the first turbulence was of the inter-G kind, further supporting conjecture P4. However, the second problem peak only involved gstg and gwkk, and both of these goals are subordinated to the common overarching goal G5, and thus of an intra-G type. There is therefore a need to amend the conjecture to take into account this other type of teaching problems which involves only one overarching goal:

P4a. Problems of teaching can involve one or more overarching (G) goals.

The experience of turbulence presented in Lesson 2 supports the other conjectures. The summary of the evaluation of each conjecture is presented in Table 15 for ease of reference.
<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 2</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>Each problem was traceable to a trigger</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>Triggers were remote to external observer that they occurred within a narrow time frame of a few seconds</td>
</tr>
<tr>
<td>P3 Problems Improvising</td>
<td>Supported</td>
<td>Problems occurred as a consequence of my intention to stop Rashid’s focus on “sides” (in the first peak) and my reluctance to immediately build on Kai’s statement that “the angles are not the same” (in the second peak)</td>
</tr>
<tr>
<td>P4 Inter-G interactions</td>
<td>Requires amendment</td>
<td>More elaborations above.</td>
</tr>
<tr>
<td>P5 Wider context</td>
<td>Supported</td>
<td>Close relation between the two turbulences within the region of turbulence</td>
</tr>
<tr>
<td>P6 One lesson</td>
<td>Supported</td>
<td>Time pressure trigger felt in the first turbulence was contributed by unfinished segment of the tiling activity in lesson one moved over into lesson 2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 2</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Monitored time, the discussion direction, and students’ level of readiness to build upon Kai’s conjecture</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>Tapped on students’ conjectures (visible) as well as my knowledge (invisible) of the productiveness of these conjectures in linking to my intended agenda.</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>Aware that students did not fill up the angle measures column of the worksheet in the previous lesson. I then devoted some time to complete those entries before returning to the discussion</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Supported</td>
<td>gmrn, gcvr, and gcbt seen as more important than grel and gart (in the first peak) and gwkk was given the priority over gsgt (in the second peak)</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>Seized upon Kai’s conjecture to salvage the spirit of grel and gart in the first problem. In the second peak, I implicitly sought the patience of stronger students implicitly by asking, “Shall we fill up all the angles first?”</td>
</tr>
</tbody>
</table>

Table 12: Review of conjectures based on analysis of Turbulent Region 2
6.3 Turbulent Region 3: Teaching Construction of Parallel and Perpendicular Lines using Setsquare and Ruler in Lesson 3

I started the lesson by showing the students how to construct parallel lines using a setsquare and a ruler. I drew a line on a transparency and proceeded to show the class how the construction can be carried out, as shown in Figure 36. The students were instructed to follow along in their notebooks with their own setsquares and rulers. I similarly demonstrated the construction of perpendicular lines.

![Figure 36: Visual overview of using a setsquare to construct parallel lines](image)

Following the construction demonstrations students had to try out individual constructions from a worksheet. The worksheet items covered a variety of drawing contexts involving parallel and perpendicular lines. While they worked, I walked from student to student inspecting their constructions to detect errors and direct them to correct methods of drawing. I anticipated that students would make errors in the drawing tasks—as that was their first experience at such tasks—and so I had set aside some time for managing students’ errors. Thus, when I noticed that students had erroneous construction steps, I was not surprised and it did not trigger the need for goal adaptations, since I had already planned to devote time to dealing with this. The last drawing task in the worksheet required the students to draw a rectangle, given one of its sides. The “rectangle problem”—as presented in the worksheet—is reproduced in Figure 37.
My post-lesson reflection indicates why I chose the rectangle problem as a context for addressing students’ errors:

[I] noticed the rectangle problem is the one that exposes glaringly their errors in construction. Either their product [the construction] is non-rectangle [parallelogram-like] or they were using merely their eye to check for parallelism and perpendicularity of sides. I chose the rectangle problem because I thought it was a good problem requiring them to synthesise the abilities to construct both parallel and perpendicular lines.

In focusing on the rectangle problem, I had in mind the fulfilment of the goal of correcting their construction errors [gerr] as well as the goal of reiterating the construction of parallel and perpendicular lines using the required instruments [gins].

The trigger to turbulence came, however, when I contemplated how to discuss the rectangle problem. I would have preferred to have a transparency version of the rectangle problem—identical to the worksheet—so that my visual demonstration could be easily followed by the students, but I had not prepared one. I readily admit here that I contributed to the resulting build-up of turbulence. If I had prepared an OHT version of the worksheet, the subsequent problem of teaching might not have occurred. As there was no facility in the classroom to make transparencies quickly, I had to think of other ways of fulfilling gerr. I decided to use the wooden ruler and a whiteboard-size setsquare that were available to demonstrate the construction procedures on the whiteboard.

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3 However, the study here is about actual problems of teaching which thus necessarily include problems as a result of a teacher’s lapses of anticipation. The last clause is not meant to excuse my oversight in planning but to highlight what can normally take place in the classroom, including improvisations arising from unanticipated triggers.
Since my instruction on the rectangle problem combined visual and verbal elements, the instructional segment is presented here in a form that preserves the interaction between these modes of communication. I was trying to present to the students what some of them did wrongly. I saw a number of them constructing non-rectangular parallelograms instead of rectangles. My purpose was to highlight to them that such a construction would not result in the required rectangle. I started by reproducing the line segment AB on the whiteboard with the wooden ruler, as in Figure 38.1:

![Figure 38.1: Reproducing the rectangle problem on the board](image)

The next move I made was to construct a line that was parallel to AB [gins], as shown in Figure 38.2.
So far, I was enacting what some students might have done, including the language they might use. My next step was intended to involve sliding the setsquare along the ruler. To effect the sliding movement, I used my right hand (shown as \( \text{} \) in Figure 38.3) to hold on to the handle of the setsquare while using my left hand (shown as \( \text{} \) in Figure 38.3) to grab hold of the left end of the ruler.

**Figure 38.2:** Setting up the ruler and setsquare to construct a line parallel to AB

**Figure 38.3:** Position of my hands before sliding
The problem occurred when I started to slide the setsquare. To maintain constant contact between the ruler and the setsquare meant that I needed to exert a force from the ruler to the setsquare (controlled by my left hand) and also a force from the setsquare to the ruler (controlled by my right hand). The two forces needed to be balanced carefully so that the instruments did not slip. As the setsquare got further along the ruler, the balance of the forces became more difficult to control. Eventually, the moment exerted by my left hand was greater than the counteracting moment by my right hand and a ‘slip’ occurred, as shown in Figure 38.4.

This difficulty partly arose from the unsuitable equipment. The contact between the smooth wooden ruler and the glossy whiteboard surface provided insufficient friction for me to control the sliding movement of the setsquare. Also, the ruler was narrow and did not have a handle, thus aggravating the lack of control. That I was unfamiliar with this combination of ruler and setsquare did not help either; however, I had to ‘make do’ with what was available.

I tried to show the sliding again but this time I shifted my left hand closer to the centre of the ruler. Despite the adaptation, the smooth surfaces again disrupted my efforts, with a slip in the opposite direction. My post-lesson reflection highlights my frustration:
I am obviously having difficulties with the gigantic ruler and setsquare. One problem is the ruler is not the correct one. It is very slim and it’s not easy for me to get a hold on it.

At that point, I just wanted to get the embarrassing sliding motion over and done with. On my third attempt, to better coordinate the forces exerted from both hands, I adjusted my hand positions again and finally achieved a ‘successful’ sliding action. The ‘success’ came with a price, however. I was unaware that as my left hand moved towards the centre of the ruler, my body blocked a substantial part of the segment AB, thus rendering students’ view of the construction process more difficult. This blocking was noticed in the post-lesson viewing of the video.

The entire struggle with the white-board instruments constituted a problem of teaching to me. In the turbulent region described above, I was trying to carry out gins—to revise the construction of parallel lines in the context of the rectangle problem. The repeated failed attempts at demonstrating the procedures on the board were hindrances to my effort to fulfil gins. Thus, the inappropriateness of the instruments and my lack of familiarity with them caused problems of teaching insofar as they interfered with my intended objective of advancing gins in teaching. The region of turbulence relating to the teaching of the rectangle problem is summarised in Figure 39.

![Figure 39: Turbulent Region 3](image)

Following this setsquare construction of a line parallel to AB, I continued with the intended wrong demonstration of rectangle construction: I drew a line segment that joined B to an arbitrary point on the constructed line, and then completed the fourth
side by using the setsquare to construct a side parallel to this line segment. Some students could see that my construction resulted in a non-rectangular parallelogram. I then proceeded with a correct construction of a rectangle.

6.3.1 Review of Turbulent Region 3
This region of turbulence involved a teaching goal (gins) different from those under examination in the previous lessons. Not only was the type of goal different, the number of goals also differed. While the problems in Lessons 1 and 2 involved a plurality of g-goals, the problems here centred around a single g-goal.

In comparing the turbulences in this region with those reported about Lesson 8’s rhombus problem in Chapter 5, it is noteworthy that apparently identical phenomena did not invoke the same response in me. I refer to the phenomena of “spotting students’ errors” which occurred in both Lesson 8 and in this Lesson 3. In the Lesson 8 segment, I noticed that some students wrongly attributed the property of perpendicular adjacent sides to the rhombus. That observation became a trigger and which subsequently build up to a problematic peak. In contrast, in this Lesson 3 episode, although I saw students performing erroneous constructions of the rectangle, it did not become a trigger to a turbulence. The difference in reaction has to do with my level of anticipation of students’ errors. This further highlights the necessity of an internal response for turbulence to start, and adds strength to conjecture P3.

Comparing this region of turbulence to the others examined so far brings about another obvious difference. While the problems encountered in the earlier lessons were results of multiple goals-in-interaction, the obstacles experienced here involved only one goal—gins. The problem in this case was not a consequence of conflict of goals. It was caused by lack of suitable resources and my inexperience in the use of existing resources. The ‘blame’ cannot be apportioned to a single factor alone. If a magnetic ruler had been available, then the issue of my inexperience would not have been relevant as I am familiar with its use. The lack of such suitable resources is a teaching problem related to “external” resources. On the other hand, problems related to my lack of expertise with existing materials belong to the “internal” resources
domain, as it has to do with my knowledge and skills. Both the internal and external resources were lacking in tackling the problem of trying to carry out gins.

However, not all problems of teaching had to do with resource deficiencies in both the internal and external domains. To illustrate this point, a look at another problem situation in Lesson 3 may be helpful. It has to do with a fault in the projection screen’s mechanism:

22.41 [Rolls down the screen . . .]
[Attempts to keep the screen at a certain draw-down position, but screen does not stay in position – it tends to roll up]
[Tries to hold screen in position longer, but still it moves up upon releasing his hand from screen handle] Not working very well.
[Squats down to look for something below the whiteboard to hold down the screen] See if this works.
[ Finds a nail below the whiteboard. Ties the screen to the nail with a cord extending from the handle]
23.02 [Manages to secure the cord to the nail] OK.

It took me about twenty seconds to sort out the problem. The struggle with the screen shows that the interference of external resources to the goals of teaching was not a one-off happening. In this case, however, it was purely the external factor that was affecting the progress of instruction. As for internal resources, not only were they not deficient, they contributed to minimising the effects of the interference, as seen from my on-the-spot improvisation of using the cord to secure the screen.

Table 16 summarises the review of the conjectures in the light of the analysis and discussion about Turbulent Region 3.
## PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 3</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>Each problem was traceable to at least one trigger</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>While there were triggers closer temporally to the problems (like the slips of the ruler) that were visible externally, the earlier triggers (such as the first and second triggers) leading to these subsequent triggers would not be easily noticed as they resided within the mind of the teacher. See Figure 32.</td>
</tr>
<tr>
<td>P3 Problems</td>
<td>Supported, with more insight</td>
<td>Elaborated above</td>
</tr>
<tr>
<td></td>
<td><strong>Improvising</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Supported, with more insight</td>
<td>Problems need not involve more than one g-goals. Thus, it need not involve conflict of goals. In this case, it was caused by deficiency in resources available. More elaborations above.</td>
</tr>
<tr>
<td></td>
<td><strong>Inter- and intra-G interactions</strong></td>
<td></td>
</tr>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Monitored the use of equipment used for whiteboard constructions and its effectiveness in fulfilling gins</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>Ongoing ‘learning’ of how to deal with the slippery wooden ruler—as demonstrated in changing positions of my hands to cope with different imbalance of forces exerted</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>Considered the options of using the OHT version of the worksheet and the magnetic ruler although these resources were not available within the immediate context</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Not applicable</td>
<td>Only one g-goal involved so “prioritisation” of goals was not applicable in this case. In other words, C4 is only relevant in situations when the problems involve more than one goals of teaching</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>When I realised my difficulty with the board equipment, I tried adjusting with what knowledge I had then to complete a ‘successful’ construction</td>
</tr>
</tbody>
</table>

Table 16: Review of conjectures based on analysis of Turbulent Region 3
6.4 Turbulent Region 4: Teaching Angle Properties of Triangles in Lesson 4

In each of the previous lessons, the turbulent regions were temporally confined within a lesson segment. The turbulent region for this lesson, however, stretches beyond the lesson itself.

To achieve syllabus coverage, the overall module plan partitioned the geometry content into lesson-size bits. This allowed me, as the teacher, to compare my actual coverage with the module plan and moderate my pace accordingly. Up to Lesson 3, my in-class instructional work closely matched the coverage in the module plan. The adjustment for tessellations from Lessons 1 and 2 had been managed in a timely fashion.

At the onset of Lesson 4, however, I needed to add to what was in the module plan. Between Lessons 3 and 4 I collected students’ homework assignments for marking. Here, I detected some common errors. In the spirit of giving timely feedback, I intended to include a discussion of these errors in Lesson 4. I thus modified the goals-mix in Lesson 4 by adding the goal of correcting students’ errors [gerr]. The main goals, as expressed in my original module plan, were to help students observe and conjecture angle properties related to triangles [gart], to teach the underlying geometrical explanations of these properties [gmrn], and deal with solutions to some related textbook-type problems [gsol]. The inclusion of gerr was reflected in Component I of the lesson memo I wrote just before Lesson 4 (Table 17).

| I.  | Revision of homework problems |
| II. | Sketchpad work on angles of triangles |
| III. | Conjecturing of properties in the whole-class discussion |
| IV.  | Textbook exercises |

Table 17: Notes on Lesson 4 just before the lesson

The introduction of gerr, however, changed the goals-combination of the lesson in a way that was problematic to me. Since the duration of a lesson is fixed, adding another component to the lesson—in this case Component I—naturally reduces the
time for other components (II-IV). As I saw it, there were only two courses of action. The first was to reduce the depth of coverage of the original components to make time for Component I. That course would advance goal gbct—to complete the teaching of the planned components within the required time. However, it was problematic as it would lead to compromises for the intended goals in the other lesson components. In this sense, introducing gerr into the pool of goals dilutes the prominence of each of the goals in the lesson. The alternative course was to retain the pace of coverage but ‘push’ one of the components to the next lesson. In my mind, though, I knew this might simply postpone the problem, with accumulating un-covered content. That, however, would violate gcvr—the goal of covering all geometry topics as given in the schedule. A diagrammatic summary of the problematic situation is in Figure 40.

![Figure 40: Consideration of instructional courses before Lesson 4](image)

My decision was to take the ‘incorporate all’ route, and the details of my lesson memo reveal some anticipated dilution in the coverage of the other components of the lesson. The most obvious reduction was in component III of the lesson. I had originally wanted to cover three theorems about angles in triangles, but my plan revealed that I “will select one or two [angle properties of triangle] to explain”. Despite that improvisation to cut down on the number of properties to discuss, I was still concerned about having enough time.
As planned, I started the lesson with a whole-class discussion of the homework assignments. I selected two homework items for discussion as they were the ones most poorly done. For the first exercise, I presented a sequence of incorrect solution strategies to highlight the kind of errors they made in their homework submissions. I invited students to evaluate those approaches. After I explained the errors, I moved on to discuss the alternative correct solutions. I completed the discussion of the first exercise by writing out the detailed solution steps with the help of students voicing the steps.

I then asked for suggestions on ways to tackle the next problem. At that point I looked up at the clock and realized that more than twenty minutes of the lesson time had gone by. My post-lesson reflections reveal the problem associated with time pressure:

I was moving along the second question and I glanced at the time and immediately felt the time pressure . . . [B]ecause of the time pressure, I chose to go directly at presenting one of the correct solutions given by one of the students.

Normally, I would have asked the students to evaluate the errors [gerr], to articulate their methods [gart], and possibly to explore alternatives [galt], as was done for the first question. However, due to the “time pressure”, I chose to directly use one student’s correct approach and presented the working steps that followed from that solution path. I had to ‘sacrifice’ the ideals of gerr, gart, and galt in the discussion process to the more immediately goal of gcbt—to complete the teaching of all the planned components within the lesson time.

The second awareness of time pressure occurred during Component II when I was demonstrating some features of Sketchpad that they would be using. I was midway through a demonstration when the first bell rang, indicating the halfway mark of the lesson. However, because of the need to carry out the remaining planned components of the lesson, I had to ‘cut short’ the time on the computer exploration of angles by limiting it to only twenty minutes:
Twenty-one minutes later, the students were back in their seats for the next section of the lesson. As planned, Component III was about getting students to conjecture angle properties of triangles based on what they noticed in the computer-exploration. I invited students to state their observations, in keeping with gart. To encourage their contributions, I wrote their conjectures on the left of the board. The students were forthcoming and soon listed relevant observations that could be extended to properties like “exterior angles add up to 360 degrees”, “exterior angle equals sum of interior opposite angles”, and “interior angles add up to 180 degrees”. In fact, student Xiao was keen to continue the conjecturing process but I stopped him to get on with the next part of the lesson. The desire to keep to time and to complete the planned teaching [gcbt] was again stronger than the motivation to encourage students to voice their conjectures [gart].

According to the lesson memo, I was to “select one or two” of the conjectures on the board to discuss the underlying geometrical reasons for why they were true [gmrm] as well as to show the proofs of those stated results—as a way to teach deductive reasoning [grsd]. The three ‘theorems’ were:

(i) Exterior angles of the triangle add up to 360 degrees;
(ii) An exterior angle is the sum of interior opposite angles;
(iii) Interior angles of the triangle add up to 180 degrees.

As mentioned earlier, the “one or two” was already a reduction from the “three” targeted theorems in the module plan. With very limited time remaining in the lesson, I had to make a further reduction, to only one theorem. This was problematic in that it was a further dilution of the original goals of gmrm and grsd.

My choice at that point was theorem (ii) above. The preference for (ii) over (i) was motivated by curriculum coverage [gcvr]. Theorem (i) would be more formally

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4 To be more mathematically precise, it should be “angle measures” instead of “angles”. I use the ‘looser’ version to match the language used in the textbook.
introduced in the Year 8 syllabus whereas (ii) was required immediately. Similarly, (ii) was also preferred over (iii) as aspects of the latter were already taught at the Primary levels.

The final trigger of time pressure came when the second bell rang—signalling the end of lesson time. I was still only halfway into explaining the overall proof strategy of theorem (ii) using a diagram on the board, reproduced in Figure 41.

As I was reluctant to stop at this unnatural juncture, I pressed on to complete the proof explanation by writing down the steps of the proof:

\[
\begin{align*}
    a + b &= 180 \\
    b + c + d &= 180 \\
    a + b &= b + c + d \\
    a &= c + d
\end{align*}
\]

![Figure 41: Diagram on the board when the second bell rang](image)

However, with time already up, I had to present this in an unusually quick manner, wondering later in my reflections if they “got it”. This hasty proof explanation was another interference to the fulfilment of grsd.
Ironically, despite sensing the need to speed up at many points in the lesson—and doing something in response—I had to end the lesson with a large chunk—Component IV—not done. This component comprised textbook exercises. The primary intent behind Componet IV was gsol, but the premature end of the lesson meant that the gsol goal was completely abandoned.

Looking back at the turbulence region, the various triggers were largely associated with time pressures and they seemed to form a linked chain. Each consecutive awareness of the time limitations seemed to heighten the sense of the importance of keeping to time and thus ‘keep things short’. That tendency culminated at the end of the lesson where the last bell just set me rushing to finish off. Figure 42 illustrates the linkages between the triggers and the increasing time pressure as the lesson drew closer to the end.

6.4.1 Review of Turbulent Region 4
The problems of teaching involved in this lesson need a wider ‘zoomed out’ lens to view the entire region of turbulence. Indeed, the turbulent region extends beyond the traditional boundaries of a lesson into the planning stages. If, hypothetically, the unit of problem analysis chosen was a rather fine-grained one, such as zooming in only at the close vicinity of one problem peak, then the actual links and the culminating effect of each problem peak to another would not have surfaced so clearly as was discussed.
earlier. The data in this turbulent region contributed further to the ongoing examination of the conjectures on problems and coping strategies (see Table 18).

The study of the goals interaction in this lesson brings out a different way in which goals interact problematically. In previous lessons, problems occurred mainly via the conflict of goals as they competed with one another. Taking a path that favoured a particular set of goals necessarily meant a violation of another set. In this turbulent region, however, the problems were not due to goal-conflict. Re-examining the first problem peak, the introduction of gerr into the then-existing gmrn-gmrn-gsol goals-pool did not contradict any of those goals. Rather, the introduction of this additional goal heightens the possibility for one or more goals to be unfulfilled in the lesson. While the emergent goal did not directly conflict with the existing goals, its introduction diluted the goals-mix in such a way as to render one or more goals to be compromised. This idea of dilution adds another dimension in my consideration of how goals interact in a problematic way.
### PHASE II ANALYSIS

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<th>Some details of the data</th>
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<tbody>
<tr>
<td>P1 Problem ↑ triggers</td>
<td>Supported</td>
<td>Each problem was traceable to one or more triggers</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>Although some of the triggers occur by way of audible prompts—e.g., the bell rings—most of the other triggers arose internally</td>
</tr>
<tr>
<td>P3 Problems ↑ Improvising</td>
<td>Supported</td>
<td>Each problem came about as a result of the injection of a new goal—gcbt—into the goal-mix, prompting improvisation</td>
</tr>
<tr>
<td>P4 Inter-G interactions</td>
<td>Supported</td>
<td>In this case, the problems all involved inter-G goals</td>
</tr>
<tr>
<td>P5 Wider context</td>
<td>Supported</td>
<td>Clearly shown by the interlinking chain of triggers</td>
</tr>
<tr>
<td>P6 One lesson ×</td>
<td>Supported</td>
<td>The 1st trigger occurred during the pre-lesson planning stages and that was outside the temporal confines of the lesson time. The inclusion of homework revision has to do homework assignment given out in a previous lesson.</td>
</tr>
</tbody>
</table>

### Table 18: Review of conjectures based on analysis of Turbulent Region 4

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 4</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Constant time-monitoring</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>The tossing up of which theorems out of (i), (ii), or (iii) to explain was an example of digging into mental resources</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>In considering the theorems (i)-(iii), I tapped on what the students had learnt in the Primary school (looking back in time) and also the geometry content to be covered in Year 8 (looking forward in time)</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Supported</td>
<td>Ongoing priorisation between goals. In this lesson, because of the strong time pressure right from the beginning, gcbt was given prominence in each prioritisation</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>Throughout, there were attempts to ‘salvage’ the fulfilment of goals although in a compromised form. An example was the reduction in the number of theorems explained from three to finally one.</td>
</tr>
</tbody>
</table>
6.5 Turbulent Region 5: Teaching the Application of Theorems to Textbook Problems in Lesson 5

Having ‘zoomed out’ to look at the region of turbulence stretching beyond the confines of a single lesson, I now ‘zoom in’ to a very small unit of analysis—a few utterances—in this lesson. The wider context of the lesson was that students were doing solving assigned problems from the textbook. The textbook items were chosen to allow students to apply the theorems they had learnt in previous lessons. As the students worked, I walked from table to table examining their attempts provided guidance. The vignette arose in one of my table-side conversation with Shi when she was working on item 9(k) from the textbook, reproduced in Figure 43.1.

I noticed that Shi wrote “$b = 71^\circ + 37^\circ$” on her notebook without elaboration. I wanted to know her underlying thoughts and lead her towards providing written reasons for her statement:

7.16  T:  [Looking over the shoulders of Shi ] Umm … How did you get b as 71 plus 37 [pointing at the line she wrote in her notebook]?
Shi:  Because this one and this one [pointing at her textbook diagram the 71-degree angle and the angle adjacent to it (the angles she referred to are indicated in Figure 36.2)]
T:  Aha, aha, very good.
Shi’s answer surprised me as I was expecting most students to think of the “37°” to come from the angle adjacent to angle $a$, as an application of the “exterior angle equals the sum of interior opposite angles” theorem.

Most students I thought usually assume that alternate angles are acute [as Shi did in obtaining the angle adjacent to 71° to be 37°]. But Shi managed to [also] see the equality of two obtuse alternate angles that are bounded between parallel lines (emphasis added). [Post-lesson reflections]

I was persuaded by Shi’s answer that she had a sound geometrical basis to make the “$b = 71° + 37°$” statement. I then shifted my attention to helping her use the right language of geometry to describe the justifications she drew upon [glan]. This, however, triggered turbulence as I debated what language to use. To clarify this, I need to discuss a part of Lesson 2.

In that lesson, the students explored Sketchpad templates featuring angle properties related to parallel lines. After the computer work, I led the students in a whole-class setting to summarise the observations they made. Figure 44 shows the diagram of parallel lines presented on the board to help students describe their observations.
During the discussion, the relationship of “$a = c$” surfaced. I asked them what “they [the pair of equal angles] are called”. It was the students who brought up the idea of visualising a ‘Z’ to depict the relationship between the two angles:

I found the ‘Z’ useful as a way to bridge their visual-based and intuitive language to that of the technical language of “alternate angles”. That it came from the students’ was an added incentive to use it, reflecting my goal of taking into account their viable suggestions [gart]. Thus, instead of disregarding their ‘Z’ language and presenting
only ‘proper’ geometry language, I chose to accept the Z language a way to remember
the relationships, but also reminded them of the importance of knowing the formal
written form of “alternate angles”. In addition, I highlighted the critical requirement
of parallel lines for \( a \) and \( c \) to be equal.

That I found the transition from the ‘Z’ angle terminology to the “alternate angles”
language useful in my teaching was evident from my re-use of it in Lesson 4. It was
used in a whole-class instructional segment as part of the solution to a textbook
problem.

Returning to my table-side conversation with Shi in Lesson 5, I was aware then that
the ‘Z’ imagery did not quite fit the ‘\(~\)’ shape of the obtuse alternate angles in
Figure 43.2 and wondered if the use of the ‘Z’ language in that context would be
advantageous or confusing. My post-lesson reflection revealed the concern I had:

Shi managed to see the equality of two obtuse alternate angles that are bounded between
parallel lines. But she didn’t quite assign the “alternate angles” language to explain why
they are equal. So here I sounded to her … [but I thought about the] limitation of the ‘Z’
angle prototypical language …

By “prototypical”, I was referring to the dangers of the “prototypical phenomenon”
(Hershkowitz, 1989) where students view non-critical attributes of representative
diagrams as critical. In this case, the concern was that the repeated use of the ‘Z’
picture might suggest “acute” is critical for the concept of “alternate angles”. It did
not occur to me before this exchange with Shi that the use of ‘Z’ could have restricted
the students’ concept image of “alternate angles” to only acute ones. That Shi saw the
equality of obtuse alternate angles made me re-consider the suitability of continuing
with the ‘Z’ language. The problem I faced then was whether to bring in the ‘Z’
language when I tried to help Shi learn the use of “alternate angles” to justify her
written steps.

I chose to go with the ‘Z’ to “alternate angles” transition as the class had become
familiar with the language:
7.28 T: And this one - b [pointing at the angle on her textbook] is making use of what property? So you must put down [pointing to the space after her statement “b = 71° + 37°” on her notebook]. What are you making use of?
T: It's a Z angle, right? You using the Z angle?
Shi: Aha [nods].
T: So that's alternate angles comma parallel line

Happily for me, Shi’s verbal response—“aha”—and her physical reaction—[nods]—seemed to suggest that she could make sense of my use of the ‘Z’ term. She appeared to have taken the ‘Z’ form to include the obtuse alternate angles case. Or, she was able to mentally stretch the ‘Z’ to take it also to include the ‘\(\sim\)’. To Shi, some kind of generalisation of the ‘Z’ term to its more flexible use to mean “alternate angles” might have taken place. Figure 45 shows the turbulence associated to this short teaching episode.

Figure 45: Turbulent Region 5

6.5.1 Review of Turbulent Region 5

In weighing whether or not to use the ‘Z’ language, the problem I encountered was not about differing goals. Regardless of whether the ‘Z’ was used, my goal was to teach Shi the language of “alternate angles” [glan]. Rather, the problem was in the path to take in fulfilling the goal. One path was the ‘tested’ one (used successfully in previous lessons)—using ‘Z’ as a familiar starting point to link to the less-familiar “alternate angles” language; the other path was to abandon the ‘Z’ use as I became aware of the risk of promoting the ‘prototypical phenomenon’. What caused internal turbulence was my knowledge of the theoretical dangers of prototypical phenomenon.
The source of that knowledge is the research literature that I have encountered through my work as a researcher. This kind of knowledge may be classified as “theoretical knowledge” of teaching. In contrast, the path I eventually chose was based on personal experiences of using the same instructional strategy with the class in earlier encounters. The decision was thus based on “craft knowledge”\(^5\).

It is important to clarify that in distinguishing between the two categories of knowledge, I am not suggesting that they are always distinct when teaching. However, there may be regions of my practice of teaching where these two domains of knowledge do not cohere and can become a source of tension in my attempt to carry out the goals of teaching. In such cases, theoretical knowledge may not apply directly during teaching. It interacts with craft knowledge—sometimes in a problematic way—actively during classroom instruction. Nevertheless, reflections of how both types of knowledge interact contribute towards the building up of knowledge in teaching.

The discussion above further strengthens the earlier conjecture that a sufficiently flexible time frame may be necessary to understand the context of a problem. While the problem peak occurred at the point of decision over whether to use the ‘Z’ language, the ‘lead-up’ to the problem—including the evolution of the ‘Z’ language—can only be appreciated when seen over a longer time frame. The historical context of how I earlier talked about alternate angles using the ‘Z’ language has a strong relation to the considerations I made and the problems of teaching I faced during the actual turbulence in Lesson 5.

Even though the nature of the problem of teaching highlighted above in Lesson 5 is again quite different from those described in the earlier lessons, the evidence here gives support to the conjectures, as shown in Table 19.

\(^5\) For the discussion on craft knowledge and theoretical knowledge of teaching, refer to Section 3.1.2.
Table 19: Review of conjectures based on analysis of Turbulent Region 5

<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 5</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>The problem peak was traceable to triggers</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>Although the 1st trigger was visible, the 2nd trigger was an internal motivation</td>
</tr>
<tr>
<td>P3 Problems</td>
<td>Supported</td>
<td>The response of wanting to fulfil glan led to the tension experienced</td>
</tr>
<tr>
<td>Improvising</td>
<td>Supported</td>
<td>Only one g-goal was involved. More elaborations above</td>
</tr>
<tr>
<td>P4 Inter-G interactions</td>
<td>Supported</td>
<td>The region of turbulence, though short temporally, provided the context to understand the problem encountered</td>
</tr>
<tr>
<td>P5 Wider context</td>
<td>Supported</td>
<td>Elaborated above</td>
</tr>
<tr>
<td>P6 One lesson ✗</td>
<td>Supported</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 5</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Monitored Shi’s responses to my questions as well as my reactions to her novel way of looking at ‘Z’ angle</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>The consideration of different instructional pathways and knowledge of the problems each of them posed were internal resources</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>Elaborated above, together with the elaborations for P6</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Not applicable</td>
<td>Only one g-goal involved so “prioritisation” of goals was not applicable here, like as was the case in Lesson 3. There was, however, a prioritisation between “craft knowledge” and “theoretical knowledge”, as elaborated above</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Not applicable</td>
<td>Shi’s “aha” was interpreted by me as successful goal-fulfilment and so no need for ‘damage control’</td>
</tr>
</tbody>
</table>

6.6 Turbulent Region 6: Teaching Special Quadrilaterals in Lesson 6

Lesson 6 had five components, with each component marked by a change in the nature of class activity, as shown in Table 20.
### Table 20: Components of Lesson 6

<table>
<thead>
<tr>
<th>Component</th>
<th>Duration (correct to the nearest minute)</th>
<th>Description of main activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>33</td>
<td>Students work with the Sketchpad to classify special quadrilaterals</td>
</tr>
<tr>
<td>II</td>
<td>8</td>
<td>Teacher shows how to draw a rhombus using a setsquare and a marked ruler on whiteboard to the whole class</td>
</tr>
<tr>
<td>III</td>
<td>9</td>
<td>Students draw other quadrilaterals using a setsquare and a marked ruler on a distributed worksheet.</td>
</tr>
<tr>
<td>IV</td>
<td>3</td>
<td>Teacher shows how to draw a kite using a setsquare and a marked ruler on whiteboard to the whole class</td>
</tr>
<tr>
<td>V</td>
<td>8</td>
<td>Students continue with the worksheet</td>
</tr>
<tr>
<td>Total</td>
<td>61</td>
<td></td>
</tr>
</tbody>
</table>

The problems of interest occurred in Component III of the lesson, but the context leading to the region of turbulence is necessary. The students had encountered square, rectangle, parallelogram, rhombus, and trapezium in Primary school. In this particular lesson, the first on quadrilaterals at the secondary level, the focus was on revisiting the different special quadrilaterals mainly through visual processes. Efforts to develop students’ awareness of properties of special quadrilaterals began with the intuitive approaches of observing figures and drawing representative diagrams rather than explicitly discussing “properties”. The latter would be done in Lesson 7. In van Hielean terms, Lesson 6 targeted the lowest “Recognition” level of geometric competence, making gmrn a goal throughout the lesson.

I wanted to incorporate the use of geometrical instruments [gins], as seen in Components II-IV of the lessons. I saw this drawing activity as a bridge from the “Recognition” tasks done during Component I to the “Analysis” tasks planned for the next lesson. By getting students to draw the quadrilaterals using setsquare and ruler, I was expecting students to reinforce their visual familiarity of the objects and develop implicit awareness of properties such as perpendicularity and parallelism, thus moving towards the second van Hiele level.

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6 Although each lesson is nominally 70 minutes, the time taken for students to move between rooms can take up to 10 minutes, which accounts for the total duration here being 61 minutes.
PHASE II ANALYSIS

With the twin goals of gins and gmrn, I started Component II of the lesson with a demonstration of the use of the setsquare and ruler in drawing a rhombus on the board. I set the stage by drawing two equal adjacent sides of the rhombus (as shown in Figure 46.1). I proceeded to arbitrarily draw a third side without the use of the instruments and asked students to evaluate my move. The purpose was to point out that a ‘by-eye’ drawing was inaccurate as well as to emphasise the critical attributes of “parallel opposite sides” and “equal adjacent sides” in rhombus.

I paused after that to invite students to go to the board to construct a third side of the rhombus. Wanxia and Mark took turns to try unsuccessfully. Their attempts are shown in Figures 46.2 and 46.3 respectively. After thanking the students for their attempts, I proceeded to show the class a possible way of obtaining a third side by constructing parallel opposite sides, as shown in Figure 46.4. I checked that the length of the third side was equal to the other two sides and completed the task by joining up the fourth side of the rhombus.

Figure 46.1: Two sides of a rhombus

Figure 46.2: Attempt by Wanxia

---

7 I used a broad magnetic ruler in Lesson 6 which was different from the slim wooden ruler used in Lesson 3. The change reflected a lesson learnt by the teacher-in-me.
The turbulence began in Component III as the students attempted the drawing tasks on the worksheet that I gave out. A sample item of the worksheet is reproduced below.

For each type of special quadrilaterals listed below, draw a representative example. You may make use of a setsquare, a ruler and a pencil to do the drawing.

<table>
<thead>
<tr>
<th>Special Quadrilateral</th>
<th>Draw an example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td></td>
</tr>
</tbody>
</table>

I was surprised that, despite the discussion devoted to methods of construction, some students were either not using the instruments correctly or not even using the instruments at all:
Through walking around I realize that a number of them were … not using the setsquare for the parallelism and perpendicularity. Although they use setsquare they are not aware that they are not constructing them correctly, but most of them are still ‘by-eye’ …

[Post-lesson reflections]

That observation triggered a re-consideration of whether the students—despite doing setsquare construction of parallel and perpendicular lines in Lesson 3 and having just been given an additional demonstration—were prepared for the setsquare construction of special quadrilaterals. I wondered if my twin goals of teaching them recognition of special quadrilaterals [gmrn] and using instruments to draw them [gins] were feasible. My interpretation of their difficulties was that “they perhaps need to know many properties [of the special quadrilaterals] before they can even perform this task [of setsquare construction]” [post-lesson reflections]. Some students appeared unready to use the knowledge of parallel sides, perpendicular sides, and equal sides properties of special quadrilaterals—requiring competence at the “Analysis” level—to draw the quadrilaterals using a setsquare. In this case, my insistence on pursuing gins would deny such students this content, thus violating gwkk. On the other hand, abandoning the drawing activity would have gone against the gins purpose of this part of the lesson. Either instructional course would result in a conflict with one of my teaching goals. My way of coping with the problem then was to prioritise one goal over the other:

However, knowing at the back of my mind that the main purpose of this is to get them to draw special quadrilaterals as a way that will help them remember [the types of special quadrilaterals], and the construction using a setsquare … being a secondary purpose, I have chosen . . . to help them focus on whether they get the visual rhombus and the visual square and so on. (Italics added, bold in the original) [Post-lesson reflections]

I chose to set gmrn as my primary goal and gins to be a secondary goal. My main focus was on helping all students to be able to identify the different types of special quadrilaterals [gmrn]; as a secondary goal, I would also help students who were ready to perform the setsquare constructions [partial gins]. Although I would be unable to fulfil gins for all students in the class, I took comfort that some would do so. Selected
vignettes of my one-to-one interactions with students reveal the partial fulfilment of gins in this component of the lesson:

44.09  T:  [To Tsan] Do you ensure they [pointing at adjacent sides of the ‘rectangle’ she draws on her worksheet] are perpendicular also to each other? Not just being parallel to each other [pointing to opposite sides], these two lines [pointing to adjacent sides] must be perpendicular as well, isn't it? You got to check for perpendicularity [as he walks to the next pair].

45.39  T:  [To Derrick] … the ruler [demonstrating to Derrick the construction of parallel lines using a setsquare] … like that … like that … slide … . Then you can see a parallel line, can you see that?
Derrick:  [Nods his head]
T:  You need a longer ruler than that. Borrow from your friend.

46.48  T:  [As he places a hand on Hassan’s shoulder] … Your rhombus doesn't look rhombusic to me. Remember rhombus must have all sides equal? … You see [using a ruler to measure adjacent sides of the ‘rhombus’] … 3cm right? And this one? Aiyo! Close to 4cm. Not equal. OK? So you must - try the parallelogram first then maybe it can help you do the [rhombus]. This one looks more like a parallelogram than a rhombus.

Hassan was having difficulties even recognising the types of quadrilaterals, confusing parallelogram and rhombus. For students like Hassan with fundamental difficulties, I abandoned setsquare construction [counter gins] and focused instead on helping them identify critical differences between the quadrilaterals [gmrn]. This same gmrn emphasis was employed with students like Tsan who may have had a gestalt sense of the different quadrilaterals but who did not represent the critical attributes correctly in their diagrams. I made explicit these properties for their consideration. Finally, for students like Derrick who were ready to perform setsquare constructions that correctly represented the inherent properties of the special quadrilaterals, I helped them revise the construction procedures [gins].

Throughout the remainder of Component III, a majority of the students were like Hassan and Tsan—struggling either to recognise features of special quadrilaterals or to represent them correctly in their diagrams. I focused on the gmrn goal for these students and postponed gins until they were more prepared for it. For the remaining students, I could include both the goals of gmrn and gins as originally planned. Thus, a large part of that section of the lesson was problematic in the sense that I could not fulfil the gins goal with most of the students. Figure 47 represents the picture of turbulence.
The difference between the problem experience here and that in earlier descriptions is that the tension of conflicting goals was not confined to ‘peaks’; rather it was felt continuously throughout the region. Except for a few students, where I could slip in the teaching of constructions [gins], I had to keep gmrn as my priority, helping students identify special quadrilaterals and visually verify perpendicular and parallel properties. For this reason, I have represented the region of problem as a grey area—neither black nor white—neither complete abandonment nor complete implementation of gins.

6.6.1 Review of Turbulent Region 6

The problem of teaching considered here showed yet another way in which teaching goals can experience interference. The problematic situation may not result simply in a straightforward non-fulfilment of goals; it can be a partial fulfilment of a goal (in this case, gins). The goal was carried out throughout the region of turbulence with some (not all) students.

It is interesting to note the difference in my reaction to “students’ construction errors” between the episode in Lesson 3 and that described here. In Lesson 3, where I introduced construction with geometrical instruments to the class, I anticipated the difficulties students would have and even included the goal of correcting the errors (gerr) for that part of the lesson. The students’ errors did not come as a surprise to me and so did not constitute a trigger to turbulence. Here, however, the students’
struggles with the worksheet tasks triggered the problematic region described above. Clearly, I did not anticipate such difficulties; my surprise when they arose contributed to the power of the trigger. Thus the same phenomenon of “students’ construction errors” can spark starkly different responses. This shows that my internal reaction to phenomena is a key ingredient for turbulences to occur, reinforcing Conjecture P3.

One obvious reason for the surprise is my differing expectation of students’ ability in relation to their preparation. In Lesson 3, it was the students’ first attempt at setsquare construction. Naturally, I considered their inexperience and anticipated errors at the start. In contrast, in Lesson 6 I expected the experience gained in Lesson 3 and my revision of the procedures in Lesson 6 to help students become more proficient the instruments. That the students performed below expectations despite my careful prior work seems a likely cause for its trigger effect.

This connection between response to students’ errors, expectations, and instructional history is also supported by the turbulent experience in Lesson 8 (Chapter 5). There, the students erred in imposing the “perpendicular sides” property on the rhombus. My surprise then with their error had also to do with my expectations due to a background of teaching them rhombus properties since Lesson 6.

Despite the apparently different picture of turbulence shown in Figure 47, the conjectures forwarded earlier continue to receive general support from the problem and coping experience in this lesson, as summarised in Table 21.
## PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 6</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>P1</strong> Problem triggers</td>
<td><strong>Supported</strong></td>
<td>The problem region was traceable to a trigger</td>
</tr>
<tr>
<td><strong>P2</strong> Invisible triggers</td>
<td><strong>Not applicable</strong></td>
<td>In this case, the trigger of students’ non-/wrong use of the setsquare can be seen by an external observer</td>
</tr>
<tr>
<td><strong>P3</strong> Problems</td>
<td><strong>Supported</strong></td>
<td>Problem arose because I pared down gins, as I felt it not realistic to persist with the originally-planned level of gins fulfilment. More elaborations above.</td>
</tr>
<tr>
<td><strong>P4</strong> Improvising</td>
<td><strong>Supported</strong></td>
<td>In this case, the problems involved inter-G goals</td>
</tr>
<tr>
<td><strong>P5</strong> Wider context</td>
<td><strong>Supported</strong></td>
<td>The entire problematic region (not isolated junctures) served as the appropriate context to view the gins vs gmrn conflict. Moreover, why the students’ errors constituted a trigger can only be appreciated against the wider context of instruction prior to the turbulent region</td>
</tr>
<tr>
<td><strong>P6</strong> One lesson</td>
<td><strong>Supported</strong></td>
<td>The comparison of my response to students’ errors between segments in lessons 3 and 6 provides a clearer understanding of the problematic situation. See elaborations below.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 6</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>C1</strong> Monitoring</td>
<td><strong>Supported</strong></td>
<td>Ongoing monitoring of students’ constructions as I moved from table to table</td>
</tr>
<tr>
<td><strong>C2</strong> Invisible resources</td>
<td><strong>Supported</strong></td>
<td>The choices I made between instructional pathways and with each student used data from my mental resources</td>
</tr>
<tr>
<td><strong>C3</strong> Outside context</td>
<td><strong>Supported</strong></td>
<td>I drew on instructional experience on setsquare construction from Lesson 3 and Component II of this lesson</td>
</tr>
<tr>
<td><strong>C4</strong> Prioritisation of goals</td>
<td><strong>Supported</strong></td>
<td>Set gmrn as primary goal and gins as secondary goal throughout the turbulent region</td>
</tr>
<tr>
<td><strong>C5</strong> Address ill-effects</td>
<td><strong>Supported</strong></td>
<td>gmrn was not abandoned, but partially fulfilled</td>
</tr>
</tbody>
</table>

Table 21: Review of conjectures based on analysis of Turbulent Region 6
6.7 Turbulent Region 7: Revising Homework Problems in Lesson 7

At the end of Lesson 6, I collected the students’ homework submissions. As there was a four-day (including weekend) separation between Lessons 6 and 7, I was able to mark and return their submissions at the beginning of Lesson 7. While marking, I noticed that two exercises were problematic to many students. I therefore started Lesson 7 by directing the students’ attention to these problems on the board, as shown in Figure 48.

![Figure 48: Exercises 3(g) and 3(i)](image)

Some students had left blanks in their assignment for these items; some attempted but did not successfully devise a solution strategy; some provided workable solution strategies but did not provide any geometric reasons; and some gave geometric reasons for some steps but did not communicate their reasoning clearly. Very few provided well-argued correct solutions. The distribution of the different solution types is given in Table 22.
PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Types of students’ solutions</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3(g)</td>
</tr>
<tr>
<td>I</td>
<td>No (feasible) solution</td>
</tr>
<tr>
<td>II</td>
<td>Feasible solution without (complete) geometric reasons</td>
</tr>
<tr>
<td>III</td>
<td>Well-reasoned feasible solutions</td>
</tr>
<tr>
<td></td>
<td>Total</td>
</tr>
</tbody>
</table>

Table 22: Classification of student solutions of items 3(g) and 3(i)

Samples of each type of solutions are given in Figures 49.1 to 49.4.

Figure 49.1: Type I solution—Shamu’s attempt of 3(g)

Figure 49.2: Type I solution—Lyn’s attempt of 3(i)

There were 40 students in the class, but one student did not submit the worksheets, homework assignments, and note books at the end of the module as part of data collection.
As so many students were unable to devise feasible solutions to the two problems, one of my goals in Lesson 7 was to teach the solutions to these textbook problems [gsol]. However, the fact that students were still unable to solve such problems despite exposures to them in Lessons 2, 4, and 5 indicated that I should also teach the overall strategy for solving this type of problems [gpsv]. In other words, I should not just show the correct solution, I should also explicitly teach general attack routes for those kinds of problems.

Type II solutions also revealed students’ inability to write geometric reasons in support of their working steps. Shi Hui’s solution in Figure 49.3 shows only sporadic geometric reasoning. I needed to include the goal of teaching students how to present geometric reasons in written answers [gwr]. Thus, the inclusion of a “homework
revision” component in this lesson concomitantly introduced goals gsol, gpsv, and gwrs into the planning of the lesson.

However, these goals were *additional* to the original teaching agenda. The original module plan intended that Lesson 7 focus on properties of special quadrilaterals, to form a van Hielean progression from the visual approaches of Lesson 6. That corresponded to a gmrn goal. Moreover, there was also the gsol goal of covering exercises 14B that required applications of quadrilateral properties.

Including homework revision into the lesson was a trigger to turbulence, similar to the goal-dilution situation experienced before Lesson 4. The introduction of gsol-gpsv-grsd did not directly conflict with the original gmrn-gsol, but without an increase in class time the goals were diluted in the sense that less time could be allocated to the fulfilment of each. I was reluctant to move the original components to make way for the homework revision, as this would delay coverage of the required geometry content for the module (against gcvr). I decided to include the new goals while keeping the incumbent ones, noting that time would then be very limited for each component. The heads of my lesson memo are given below; the turbulence described here is from the first component.

| I. Returning homework     | II. Properties of special quads – square, rectangle, parallelogram, rhombus | III. Demonstrate doing Ex 14B |

For Component I of the lesson, my general approach to showing the solutions to homework items 3(g) and 3(i) comprised two steps: first, I gave a broad overview of how I ‘attacked’ the problem [gpsv]; then I revisited the solution in greater detail by writing down reasons in support of each critical step [gwrs].

The problem-solving strategies I advocated in the first step [gpsv] were:

1. When dealing with a geometry problem, develop the overall solution first before revisiting it to write out the detailed solutions;
2. Use auxiliary lines not originally in the diagram; and
3. Use diagrams to support your thinking about the ‘attack route’.
Figures 50.1-50.3 provided snapshots of how these heuristics were explained and demonstrated to the students when I presented the solution of “x” in item 3(g).

**Figure 50.1: Explaining heuristics in attacking problem 3(g) – first snapshot**

**Figure 50.2: Explaining heuristics in attacking problem 3(g) – second snapshot**

**Figure 50.3: Explaining heuristics in attacking problem 3(g) – third snapshot**
After the overall strategy was presented, I proceeded to the second step of writing down the solution together with the geometrical reasons [gwrs] beside the diagram on the board. Students were invited to provide the reasons before I wrote them on the board. The completed solution for “x” as it appeared on the board is shown in Figure 51.

Figure 51: Completed solution for x of 3(g) on the board

Another trigger of turbulence occurred at the point when I completed writing the solution of “x”, as reflected in what I told the students:

6.56 T: … [pointing to the written solutions on the board] This is the kind of working I am looking for. This doesn’t mean … Mr Leong is not meaning that this is the only way. That's not what I meant. I am demonstrating only one of the ways. But because of time, I may not be able to show all the possible ways.

I wanted to avoid giving students the impression that what I had presented was the only possible solution path. During the module, I had been trying to encourage students to find alternative solutions to problems [galt]. In fact, at the beginning of this lesson, before I even started discussion 3(g), I reminded students that there might be more than one way of attacking the problem:

00.56 T: There are many ways - many interesting ways but let’s discuss one possible way …

Thus, at that point when I completed writing the solution for “x”, I contemplated presenting an alternative solution as a way to fulfil the emerging galt goal. This trigger of wanting to fulfil galt brought about a problematic situation because, as
discussed, introducing the homework revision meant I was running on a very tight schedule. Devoting time to show an alternative solution (to fulfil galt) would jeopardize the implementation of later lesson components, thus conflicting with gcrt. On the other hand, failure to discuss actual alternative solutions would undermine galt. Once again, I was in a conflicting goals situation where pursuing one goal would lead to non-fulfilment of the other. As the extract of the lesson transcript above shows, my decision was to put aside galt at that juncture of the lesson as the time element felt the more pressing factor.

I went on to show students the solution for “y” in question 3(g), again showing the overall strategy before proceeding with the written reasoning. The finished solution on the whiteboard was as shown in Figure 52.

![Figure 52: Completed solution for x and y of problem 3(g)](image)

When I wrote the line “\( (y + 15) + 122 = 360 \) (angles at a pt = \( \angle = 360^\circ \)”, I glanced at the classroom clock at the front of the classroom, which prompted yet another trigger. I saw that nearly ten minutes of lesson time had passed:

Here I am progressively aware – though I haven’t finish showing [the full written solution of “y”] yet – of the time pressure now that there is a next thing in my plan. So I have to make the decision of whether to continue showing the solution or to leave it...
“[vertical] dot dot dot” because the procedures are almost the same as the solution of \( x \). I want to leave it to them to solve for \( y \). [Post-lesson reflections].

Again, because of time pressure, I had to choose between completing the solution of 3(g) [gsol] and moving to the “next thing in my plan” [gcbt]. Having already taken ten minutes for 3(g) alone, I felt even further behind than I was at the first trigger juncture. This sense of “going too slowly” meant that I elected to move on to 3(i) instead of completing the written solution of 3(g).

As before, I began 3(i) by explaining the overall strategy for solving the problem. The diagram on the board after this overview is shown in Figure 53.

![Figure 53: Auxiliary lines drawn for problem 3(i)](image)

At that juncture, I again felt aware of time pressure. Simultaneously, I was aware that my discussion focused on only one solution instead of encouraging a variety of solution tracks [galt]. The thought of introducing alternative attack routes to problems had occurred during the earlier second trigger but it was suppressed then. This urge to fulfil galt re-surfaced for problem 3(i). These calls from within to heed other goals of teaching was captured in my post-lesson reflections for this part of the lesson:

So here I am also not just showing the immaculate solution but I also am modeling the process of attacking the problem. But the weakness of this method is that my first attack is a successful attack. I should also attack this problem in different ways and help them see what are the possible different ways in which the attack can be made …

However, similar to the tension at the second trigger, time pressure [gcbt] interfered with galt:
But yet I am also aware that the purpose of this portion of the lesson is to model how it is presented – that is the main focus … Also here I am aware that I must be careful to make sure of my time … [Post-lesson reflections, emphases added]

Given the strong sense of time limitations, I had to choose between writing the reasons to complete the solution [gwrs] or offering other alternative solutions strategies [galt]. As indicated in the reflections above, my priority was to proceed complete the written solution for 3(i).

The overall turbulence experienced in the first component of the lesson is presented in Figure 54.

**Figure 54: Turbulent Region 7**

6.7.1 Review of Turbulent Region 7

The pattern of turbulence highlighted for this portion of Lesson 7 is very similar to that in Lesson 4. Both lessons featured turbulence associated with inserting new goals into the lesson. This diluted the goals-mix in that the same instructional time has to accommodate a larger set of goals. The resulting continual consciousness of time pressure limited the instructional options when emergent goals of teaching were detected, producing the problematic situations of choosing one goal at the expense of the other(s). Another similarity between Lessons 4 and 7 is that despite being aware of time and trying to cover all components, ironically, I failed to do so. In Lesson 7, although I intended to solve items from exercise 14B of the textbook at the end of the
lesson, the computer work that followed the homework revision filled the remaining lesson time. Because of the strong resemblance in the structure of turbulence between the episodes highlighted in Lessons 4 and 7, it is not surprising that Table 23 on the evaluation of conjectures based on the data in Lesson 7 closely resembles Table 18.

The similar turbulent experiences in both Lessons also showed the dominant force of time pressure in creating triggers that ultimately led to problematic situations. In both cases, I started the lessons already anticipating time shortage because of the additional goals. That burden built up further when I sensed that I was falling behind and formed a strong barrier against the fulfilment of other worthwhile emergent goals. Time limitations played a vital role in my teaching decisions in other lesson too. In Lesson 8, for example, there was no time pressure at the beginning of the lesson, but consciousness of the need to keep to time was felt midway in the lesson when one instructional component took too long to complete. The influence of time pressure cannot be underestimated in my analysis of problems of teaching.

Despite the similarities between the turbulent regions of Lessons 4 and 7, there are important differences as well. Firstly, although both lessons started off with a re-visit of homework problems, the goal structures underlying these seemingly identical activities were different. In Lesson 4, homework questions were presented to elicit and highlight students’ common error. The main purpose was thus to correct students’ errors [gerr]. In contrast, the goal for the homework problems in Lesson 7, the goal was to teach students (a) a feasible solution [gsol], (b) a way to approach such problems [gpsv], and (c) how proper reasoning steps should be written[gwrs]. Because of the different underlying goals, it is not surprising that there were also differences in the triggers, the reactions, and the decisions made in each lesson. There is one particularly clear difference. In Lesson 4, whenever the time pressure was felt, the decision was to cut short the activity in order to move on to the next section [gcbt]. While such decisions to ‘move on’ were also made in Lesson 7, it was not the only kind of response to time pressures. In the last trigger, when the homework revision was taking too long, I nevertheless completed writing the solutions and reasoning on the board. The decision reflected the value and priority I placed on teaching written reasoning [gwrs].
## PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 7</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>Each problem was traceable to a trigger</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>All the triggers of insertion of new goals and time consciousness were invoked internally and therefore not easily detectable by an external observer</td>
</tr>
<tr>
<td>P3 Problems</td>
<td>Supported</td>
<td>Each problem was a result of the inclusion of one or more goals which prompted improvisation</td>
</tr>
<tr>
<td>P4 Improvising</td>
<td>Supported</td>
<td>All the problems here involved inter-G goals</td>
</tr>
<tr>
<td>P5 Inter-G interactions</td>
<td>Supported</td>
<td>There was a chain of triggers linked together by the increasing concern of time pressure</td>
</tr>
<tr>
<td>P6 One lesson</td>
<td>Supported</td>
<td>The 1st trigger occurred before the lesson and that was outside lesson time. Moreover, the nature of the goal-mix took into account prior instructional experience in Lessons 2, 4, and 5</td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Constant time-monitoring and the awareness of the need to follow the galt spirit of giving alternative solution paths</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>The weighing of the goals-at-play and the consequences of decisions made at each trigger points were not visible to others</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>In deciding gcbt over galt in the 2nd &amp; 4th triggers and gcbt over gsol in the 3rd trigger, I was motivated by the need to cover the later components of the lesson.</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Supported</td>
<td>Ongoing prioritisation between goals. In this lesson, because of the strong time pressure right from the beginning, gcbt was given prominence in each prioritisation. More elaborations on the strong influence of time pressure above</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>There were explicit attempts at ‘damage control’. One example was the explanation to the students why I could not go into alternative solutions (to fulfil galt) due to the lack of time</td>
</tr>
</tbody>
</table>

Table 21: Review of conjectures based on analysis of Turbulent Region 7
6.8 Turbulent Region 8: Teaching Hierarchical Relationships Among Special Quadrilaterals

There are three main components in Lesson 9, as shown in my lesson memo:

I. Show different ways of constructing a square using Sketchpad.
II. Students work on Sketchpad to construct selected special quadrilaterals.
III. (Optional) Hierarchical relations between special quadrilaterals.

The focus of analysis is the last component. The inclusion of hierarchical relations between special quadrilaterals was the culmination of earlier work. Lesson 6 had focused on identification and attempts to draw special quadrilaterals. During Lesson 7, the focus shifted from visual forms to geometric properties. Students worked with each quadrilateral in Sketchpad and listed their angle, side, and diagonal properties. In Lesson 8, students applied those properties to solve textbook problems and to construct each of the special quadrilaterals using only a pair of compasses and a ruler.

The instructional development of special quadrilaterals from Lesson 6 to Lesson 9 in corresponded to a progression in the van Hiele levels of geometric competencies. Lesson 6 required involved visual and gestalt features, emphasising the first van Hiele level of “Recognition”. Lessons 7 and 8 corresponded to the “Analysis” level. Lesson 9 was intended to move towards the “Ordering” level, in accordance with my goal of teaching to help students progress through the van Hiele levels [gmrn].

While I wanted to move on to the higher van Hiele inclusion relations, I was apprehensive about doing so, evidenced by my tagging of that part of the lesson as “optional” in the memo. To appreciate my apprehension, some background of related struggles in previous lessons is necessary.

The subject of inclusion relations was actually first given cursory coverage in Lesson 7, although unplanned. It arose from a student’s response to my invitation to the class to define a square:
57.26 Syarah: Square is a rhombus …
T: Square is a rhombus - wow! [exclamation]. Somebody just said something - wow [jerk body to show surprise] - high-level thinking question. Syarah just asked me a question and I just got shocked.

My reaction to Syarah’s unexpected contribution is captured in my post-lesson reflections:

Syarah suddenly says “a square is a rhombus” (but actually she clarified later that she took it from the textbook – “square is a rhombus with all interior angles 90 degrees”). But I took from that moment and I said “wow”. I thought she understood what she was saying and that the class was ready to have a discussion on that issue. So I pick up from that and use it to ease into the hierarchical relationship of the special quadrilaterals. But I only realised later that the class was simply not prepared for that.

I assumed that the class was ready for such a Level 3 type of discussion and seized upon the opportunity that Syarah provided to launch into a discourse with the students on the statement that “a square is a rhombus”. I showed the list of critical properties of rhombuses and asked the students to check if the square fulfils all those attributes of a rhombus to determine the verity of the statement. However, before long I sensed, through their body language and the increasing noise, that they were not ready for this. That hunch about a mismatch between the discussion and their level of operation seemed to be confirmed by their reactions after I finished my explanation that the square has all the attributes of a rhombus:

61.28 T: OK. Look here. [taps on OHP with fingers] That’s the final thing we want to talk about before we go. Rhombus has equal sides, square has equal sides with interior angles 90 degrees. Syarah said something just now, ” a square is a rhombus”
A few: Wrong
Many others: [Undiscernible voices]
T: [Sensing confusion] You are not ready for this yet …

This short teaching episode in Lesson 7 made me realise that students were not prepared for geometric reasoning at Level 3. To avoid a similar mismatch between my instruction and the students’ level of operation, I was mindful that future attempts at introducing inclusion relations should take into account better judgment of the students’ geometric competence. I therefore proceeded to Lesson 8 with a greater awareness that I needed to assess continually the students’ level of geometric thinking.
Lesson 8 revealed that some students were still operating at the lower van Hiele levels despite the instructional efforts in Lessons 6 and 7. That these students wrongly assumed the perpendicularity of adjacent sides in a rhombus showed that they were still visual-based in their view of the special quadrilaterals (see Chapter 5 for details). However, in the second part of the Lesson 8 on construction tasks, it was apparent that other students could do the more difficult construction tasks, demonstrating performance at the “Analysis” level. Such a task is given in Figure 55. One possible way to complete the construction is to (1) measure angle 30 degrees at J using a protractor, (2) mark out M using a pair of compasses by taking JK as the radius, and (3) use the intersection of arcs with centres at K and M (and the same radii as (2)) to obtain L. The ability to do steps (2) and (3) above demonstrates the knowledge of equality of all sides as an attribute of a rhombus. Other construction procedures are possible, but all require some applications of properties of rhombus. Students who could perform these constructions tasks successfully were showing signs of proficiency at the “Analysis” level.

<table>
<thead>
<tr>
<th>Construction task</th>
<th>Tools to use</th>
<th>Constructions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construct a rhombusJKLM</td>
<td>Compass</td>
<td>J</td>
</tr>
<tr>
<td>with $\angle J = 30^\circ$</td>
<td>Protractor</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Ruler</td>
<td>K</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 55: Sample of a construction task in Lesson 8

At the end of Lesson 8, therefore, I was aware of students’ diverse competencies, as shown in my post-lesson reflections:

I could see that there was a wide range. Some of them were very advanced, they were moving on; some of them were still struggling.

These experiences through Lessons 6 to 8 provide the backdrop for my apprehension about introducing hierarchical relationships among special quadrilaterals in Lesson 9. The internal struggle that I had before lesson nine, then, was between two goals of teaching: the goal of teaching to meet the needs of students who were ready for higher-level content [gstg] and the goal of catering to the other students who were still...
struggling with more basic geometrical properties of some special quadrilaterals [gwkk]. Both goals are part of the overarching goal of teaching all students [G5].

I therefore started Lesson 9 without having decided whether I should proceed with that last component of the lesson on inclusion relations. In fact, I was even apprehensive about the second component, where students were required to construct drag-resistant special quadrilaterals using Sketchpad. To perform such constructions, students would need to know properties of various special quadrilaterals and apply them appropriately. For example, to construct a square, students would need to apply the attributes of perpendicular adjacent sides, equal sides, etc; that is, they would need to operate at the “Analysis” level—and I was yet unsure of their readiness. As it turned out, however, a number of students were proficient with the constructions:

… when they go into the Sketchpad activity—I was two ways about it when I started this lesson whether they are prepared about this, “they have not done any serious constructions before. Will they be able to handle that?”—But it turned out that a number of students were really very confident, whereas there will be the other extreme where they are completely struggling, like Hassan I would remember. Lyn and Shi were also having their own struggles … . But that few that I saw who were able to do the constructions I was impressed by. And that again teach me as a teacher never to cap the abilities of the students. [Post-lesson reflections].

The proficiency tilted my decision towards carrying out the component on inclusion relations. While I was aware that some students (like Hassan, Lyn, and Shi) were still having difficulties with “Analysis” level material, I felt the greater urge to cater to the needs of the students ready to progress to the next van Hiele level [in line with gmrn and gstg].

I started component III of the lesson with a re-visit of the statement, “A square is a rhombus.” I showed it on an overhead transparency and invited students to decide whether the statement was true or false and to provide explanations for their stand. What followed was a lively discussion with a number of students contributing their views on the issue:
I just want to ask this question, "Is a square a rhombus?"

Class: [almost equally] Yes/No.

T: This is most interesting to Mr Leong. Whenever you don’t agree that is most interesting. You know why? Because it shows you are thinking. Now “yes” and “no” is not as important as "explain" - why? [showing dramatically the desire to know “why” with opens arms and clap of hands] …

T: I remember who says "yes". I remember Rashid says "yes". OK can you tell us why? Explain why you think a square is a rhombus.

Rashid: Don’t know [Meanwhile Farin raises his hand]

T: Farin wants to help. Yes Farin?

Farin: It has two pairs of parallel lines. Rhombus also parallel lines

T: [Broadcast loudly to whole class] Farin says … Is it a square has two pairs of - I think he meant opposite parallel sides, am I right? [writes down this statement on the board]

T: So what? Farin is thinking … You see … Farin is thinking of something but you must follow what he is thinking. Farin says a square is a rhombus because a square has two pairs of opposite parallel sides. So what? Yes Shi? You put up your hands and want to say something. Yes Shi [walking towards her, inviting her to speak up]?

Shi: A square - all sides must be equal. [softly] And rhombus also.

T: [Broadcast loudly to class] She said "squares has all sides equal" and she said something softly - I also hear - and rhombus also. [writes down on the board]. That's what she said. Square has all sides equal …

T: But there are some people who say a square is not a rhombus. Why? [saw a few hands raised, including Ian’s] Yes Ian? [walking towards him to invite him to speak up]

Evan: Rhombus: Not all angles the same.

T: Oh . . . [simulating disappointment] Rhombus not all angles the same.

Mark: Square must be 90 degrees

Dickvan: Yeah.

T: Mark also said no. Dickvan also said no. Some things square has that rhombus do not have.

This discussion continued for some time, comparing and contrasting properties of squares and rhombuses and I sensed a good level of active student participation:

I was again surprised that everyone seemed—based on what I saw—they … were all focusing on the statements. And asking that kind of questions, giving that kind of feedback which I wrote on the board. [Post-lesson reflections]

There was more productive discussion this time over the “the square is a rhombus” questions compared to Lesson 7. That encouraged me to proceed with the next level of geometric reasoning. I was, however, aware that while the knowledge of similar and different properties between the two quadrilaterals is a pre-requisite to relating them in a hierarchical way, if the discussion stayed only on that level of reasoning, it would not be sufficient for students to understand the inclusion of squares within the set of rhombuses. I therefore made a deliberate move to switch the students’ attention directly to whether the square would satisfy all the critical attributes of a rhombus. I
did so firstly by projecting the teacher’s computer screen on the board and dragging a vertex of the *Sketchpad* rhombus slowly until it looked like a square (as shown in the Figure 56). This demonstration was meant to help students see the squares as being included in rhombuses.

![Figure 56: Illustration of dragging a rhombus to the point that it looks like a square](image)

In addition, I drew a diagram of a square on the board and asked if all the properties of a rhombus would be fulfilled by the square:

49.51  T:  What are the properties of a rhombus?
Yuxin:  All sides equal
T:  All sides equal.  Look at the square [pointing to the square].  All sides equal or not?
A few:  Yes
T:  [nods head] What other properties of rhombus?  Opposite angles equal?  [points to square] opposite angles equal or not [pointing to a pair of opposite angles]
A few:  Yes
T:  Equal or not [pointing to another pair of opposite angles]
A few:  Yes
T:  Diagonals bisect each other?  [Draws in the diagonals] Diagonals bisect the angles?
A few:  Yes
T:  Everything the rhombus needs to have – needs [emphasis] to have - the square also has.
Yuxin:  So it’s a rhombus
T:  So a square … [pointing to statement on OHT] is a rhombus

However, in the middle of this discourse, I could already sense, from the limited participation of only a small group of students, that despite all the positive signs I thought I saw earlier, where students seemed ready for a discussion at the “Ordering” level, few students in the class could actually understand the reasoning behind the inclusion relation:
Am not sure actually how many of these students follow these logic statements. Perhaps a few voices here and there. But I was trying to let them see by drawing a diagram of a square and ask the class whether they can see what were the required properties of a rhombus in a square. If so, then the square is a rhombus. … I can sense that some of the students were sort of slowly losing [the reasoning] … and I’m not sure how many students will get this kind of discussion. So this is a big problem too in teaching … of them not being able to reach the level of geometric thinking. [Post-lesson reflections]

I was clearly disappointed that in my effort to help some students attain to a higher level of geometric reasoning as a goal of my teaching [gmrn and gstg], I was unable, at the same time, to cater sufficiently to the needs of other students who were still struggling at the lower levels of competence [against gwkk and gcon]. This tension between goals that led to a non-fulfilment of some goals of teaching created a problematic situation in teaching practice.

Reviewing this problem, the region of turbulence began in the unsuccessful attempt at teaching the “square is a rhombus” at the end of lesson seven. This alerted me to the conflict between my goal of teaching the more competent students [gstg] and the goal of catering to the students who were grappling with lower-level content [gwkk]. Starting from that point, there was a series of triggers that increased my apprehension about advancing to “Ordering”-level tasks. However, nearing and within Lesson 9, there were indications that students were becoming more comfortable with “Analysis”-type activities, suggesting they might be ready for discussion on inclusion relations among quadrilaterals. When I actually went ahead with explaining why “a square is a rhombus”, I realised that only a few students could follow the logic of reasoning, whereas most other students seemed still not ready in their development of geometric reasoning. A summary of the whole region of turbulence described above is illustrated in Figure 57.
6.8.1 Review of Turbulent Region 8

The region of turbulence stretched across three lessons and is the most broad-grained of all the lesson analyses reported so far. This provides further evidence that teaching problems can indeed occur at a wide range of grain-sizes—from frames of a few seconds (e.g., in Lesson 5) to a larger scope of a few lessons (as shown in Figure 57).

The presence of several triggers in the turbulent region discussed above suggest chain-like links between the triggers, as was also the case in Lessons 4 and 7 (see Turbulent Regions 4 and 7). However, closer scrutiny reveals a difference between the trigger-chain in Lesson 9 and that found in the earlier lessons. In Lessons 4 and 7, each subsequent trigger increased the tension with respect to the urgency of carrying out the same emergent goal, gbc italiane—the need to keep the teaching schedule to time.

With each successive trigger, the awareness gbc italiane grew, creating numerous problems along the way. Using the navigation metaphor, the relationship of the triggers in those lessons was like a signal beeping with increasing volume over time, urging the navigator to attend to the perturbations, sometimes to the neglect of other worthwhile goals.

The pattern of triggers associated with Lesson 9, however, presented a different picture of trigger-links. Although the signals of gwkk began to sound
initially—arising from concern that students with weaker mathematical abilities were unable to access higher van Hiele levels—their intensity varied rather than steadily increased. At the third trigger, the gwkk signals decreased when I noticed that some students were comfortable with the mathematical tasks given. This downward trend in the gwkk signal was sustained through the fourth and fifth triggers, before bursting into prominence again at the second problem peak. Thus, unlike the always-upward trend of gcbt intensity in Lessons 4 and 7, the situation in Lesson 9 reflected an undulating pattern between more and less urgency of attending to gwkk demands. This oscillation may explain why there were no major changes in the instructional course in Lesson 9, as compared to the improvisations made in Lessons 4 and 7 when the urgency was more greatly felt.

The goals interactions in the turbulence here also bear a strong resemblance to the gwkk versus gstg intra-G5 conflict experienced in Lesson 2. In both cases, I was torn between catering to the learning needs of the weaker students and those who were ready to move ahead. In Lesson 2, the decision favoured the weaker students: I slowed down deliberately, and instead of continuing the discussion initiated by Kai, I took a step back to fill in the angle measures on the worksheet as a way to prepare the weaker students for the discussion. The decision taken in the similar intra-G5 conflict in Lesson 9, however, was different: I instead leaned towards helping students move on to more complex content. Even though the same set of goals-in-conflict was involved, the contextual differences in the instructional situations resulted in different priorities being placed on different goals. In the case of Lesson 9, my interest in the gmrn goal—of wanting to see stronger students progress to higher van Hiele levels—tilted my preference for gstg over gwkk.

Table 24 below shows how the conjectures are evaluated against the data described in this large-sized turbulent region.
PHASE II ANALYSIS

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<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>In this turbulent region, each problem peak was linked to a plurality of triggers. The pattern of trigger-links in this region is interesting and is elaborated above</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>Observer of Lesson 9 alone would not be able to detect most of the triggers that occurred outside the temporal limits of the lesson</td>
</tr>
<tr>
<td>P3 Problems Improvising</td>
<td>Supported</td>
<td>Both problems arose out of my response to include gmrn and gstg as goals of teaching</td>
</tr>
<tr>
<td>P4 Inter-G interactions</td>
<td>Supported</td>
<td>Most of the goals involved—gstg, gwkk, and gcon—were subordinate to G5, although gmrn was the exception. The problems in the region of turbulence had a strong intra-G5 flavour</td>
</tr>
<tr>
<td>P5 Wider context</td>
<td>Supported</td>
<td>The triggers were so closely linked that examining a small part of the turbulent region in isolation misses out on vital aspects of the work of teaching</td>
</tr>
<tr>
<td>P6 One lesson ✗</td>
<td>Supported</td>
<td>That the turbulent region spans three lessons provides evidence in support of the conjecture</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 8</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Ongoing monitoring of students’ abilities to access the mathematics that I intended to teach</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>The choices I weighed at different junctures of the turbulent period required internal mental resources</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>The decisions I made took into account the instructional history, especially about how some students could and could attain to various van Hiele levels of thinking</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Supported</td>
<td>Both problems involved prioritisation between goals. The choices made between the g-goals within G5 presented an interesting contrast against a similar situation in Lesson 2. More elaborations above</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>The efforts to show how the Sketchpad rhombus can be dragged to form a square, and the drawing of a square on the board to check for rhombusic properties were ways at ‘damage control’ to help weaker students for proceeding with gstg and gmrn goals</td>
</tr>
</tbody>
</table>

Table 24: Review of conjectures based on analysis of Turbulent Region 8
6.9 Turbulent Region 9: Teaching the Construction of Quadrilaterals—as Revision of Homework Problems

A few minutes before the start of Lesson 10, a rather significant disruption occurred. It was a trigger that brought upon a strong sense of time pressure and the need to make drastic adjustments to Lessons 10 and 11. To fully appreciate the tensions I felt, there is a need to take into account triggers leading up to Lesson 10.

After Lesson 7
The first trigger in this zoomed-out period of turbulence began after Lesson 7 when I was unable to complete teaching all the components listed in the module plan. In particular, I was supposed to discuss some textbook questions from Exercise 14B. However, due to the insertion of new goals into the lesson and the turbulence generated as a result of the changes, I could not complete that section of the lesson (see Turbulent Region 7). I decided to cover Exercise 14B items in the first part of Lesson 8 as it met my goal of teaching students how to attempt textbook exercises (in preparation for tests) [gsol].

According to the original module plan for Lesson 8, I was supposed to teach (1) construction of special quadrilaterals using compass and/or setsquare; and (2) construction of special quadrilaterals using Sketchpad. I was reluctant at that time to simply insert a new section on Exercise 14B into Lesson 8 in addition to the two other existing components listed in the module plan. From my teaching experience with the students during Lesson 7, I sensed that many were still unfamiliar with the geometrical properties of the quadrilaterals and needed more time on related activities. Consequently, I was not inclined to shorten the time for either of activities (1) and (2) above by squeezing in Exercise 14B.

Therefore, to allow more time for each set of activities, I chose to move Component (2) on construction with Sketchpad to Lesson 9. The component on Exercise 14B therefore displaced the original Component (2) in Lesson 8. The movements of components in Lessons 7 and 8 are presented in Table 25.
### PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Intended coverage according to module plan</th>
<th>Actual coverage in the lesson</th>
<th>Reasons for retention/change</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>(A) Explore properties of special quadrilaterals on <em>Sketchpad</em></td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(B) Lead to definition of each of the special quadrilaterals</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(C) Exercise 14B</td>
<td>Moved to Lesson 8</td>
<td>No time to complete in lesson 7</td>
</tr>
<tr>
<td>8</td>
<td>Exercise 14 B</td>
<td></td>
<td>Important for gsol</td>
</tr>
<tr>
<td></td>
<td>(1) Construct quadrilaterals using compass and/or setsquare</td>
<td>✓</td>
<td>Students need more time with properties. So (2) needs to move out to retain sufficient time for (1)</td>
</tr>
<tr>
<td></td>
<td>(2) Construct quadrilaterals using <em>Sketchpad</em></td>
<td>Moved to Lesson 9</td>
<td></td>
</tr>
</tbody>
</table>

Table 25: Plan after lesson 7—Ex 14B displaces existing component in Lesson 8

Clearly, the implication of such displacements—if continued into subsequent lessons—would be a failure to cover the required content at the end of the module [against gcvr]. I was prepared to make the displacement at that juncture because it seemed I would have time for a Lesson 11. The original module plan was based on ten lessons to allow for unanticipated lesson cancellations (see Section 3.2.2). In the earlier lessons, I was conservative in the use of time as I had a ‘ten-lesson module’ in mind; nearing the end of the module, and having no lesson cancellations so far, I began to envisage ‘eleven-lesson module’ timeframe in mind and could consider component-displacements across lessons.

**Before Lesson 9**

At the beginning of Lesson 9, the increased confidence of a spare lesson triggered my decision to spend the whole of Lesson 9 on students’ construction of quadrilaterals
with Sketchpad, shifting the original contents of Lesson 9 and 10 to Lessons 10 and 11 respectively:

According to [the module] plan, today’s lesson is scheduled to be for compass construction of perpendicular and angle bisectors. But since there was that ‘extra lesson’ … I reckon I can afford to let them work longer [that is, the entire lesson] on the Sketchpad on the construction of quads. [Extracted from memo for Lesson 9].

Table 26 illustrates the plan I had in mind before Lesson 9.
<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Intended coverage according to module plan</th>
<th>My new plan of coverage, as revised at start of lesson 9</th>
<th>Reasons for retention/change</th>
</tr>
</thead>
</table>
| 9             | (I) Use of compass to construct perpendicular and angle bisectors  
               (II) Relate construction process to the properties of rhombus | (2) Construct quadrilaterals using *Sketchpad*  
               (I) and (II) moves to lesson 10 | Displaced from Lesson 8  
               Students need more time with *Sketchpad* constructions. Can afford to ‘space out’ since I have ‘extra’ Lesson 11 |
| 10            | (I) Use of compass to construct perpendicular and angle bisectors  
               (II) Relate construction process to the properties of rhombus | (a), (b), and (c) moves to Lesson 11 | Originally-planned Lesson 9 components can shift to Lesson 10  
               Originally-planned Lesson 10 components can shift into the ‘extra’ Lesson 11 |
| 11            | *Lesson 11 was not part of the original 10-lesson module plan* | (a) Identify different 3-D solids  
               (b) Build, using *Polydrons*, different 3-D objects  
               (c) Highlight differences between prism and pyramid |  |

Table 26: Plan before Lesson 9—displacing components into Lessons 10 and 11
After Lesson 9

At the end of Lesson 9, I collected the homework assignments from the students. My lesson memo for Lesson 10 reveals a trigger to further changes for Lessons 10 and 11:

I glanced through the students’ homework submission. Noticed that the construction exercises [in Exercise 14C of the textbook] were badly done, especially those [types of questions] that we did not have time to do in class. … I will therefore present the construction procedures for Q7 and 9 … . It also looks like I need to revise Exercise 14B p. 317 questions 4d, 5a first [before the Exercise 14C items].

Seeing the need to present correct solutions to homework problems to address the students’ errors and difficulties [g sol, gerr], I decided to include this additional “homework revision” component into Lesson 10 on top of the existing instructional Components (I) and (II) (shown in Table 26). I felt distinctly the time pressure of inserting additional components to the lesson, as evidenced by my post-Lesson 10 reflections:

So I need to address these errors [in the students’ homework submissions] they made in this lesson. At the same time, I wanted to press on with today’s lesson on perpendicular bisector and angles bisectors. At the same time I needed to be aware that in the next lesson, I must deal with solids. So I feel a distinct pressure with a number of competing goals.

Seeing that the planned components on “solids” for Lesson 11 would likely take up the time for a full lesson, I was hesitant to move components from Lesson 10 into Lesson 11 to make way for “homework revisions”. Instead, my decision then was to add “homework revisions” to the other two components in Lesson 10, as shown in the corresponding heads of my lesson memo:

9 (Ia) Review of Homework submitted earlier
(II) Construction of perpendicular bisectors and angles bisectors
(II) Discuss why ‘it works’ [geometrically].

9 The actual labels in the lesson plan are (I), (II), and (III). I have re-labelled them as (Ia), (I), and (II) to correspond to the earlier labels used for the same lesson components.
Before Lesson 10 – the critical trigger

Against the background of actual and anticipated displacement and insertion of lesson components spanning Lessons 7 to 11, I return to the point a few minutes before the start of Lesson 10. The school authority announced a last-minute decision to have all Secondary 1 students gather at the school hall to view a dance performance. This “special assembly” was expected to last for more than half an hour, reducing my lesson time with students and thus necessitating adjustments to my goals for that lesson. Within an already tight schedule, the disruption to lesson time created a problematic situation for me. The tensions that I felt and the decision that I made are revealed in my reflections of Lesson 10:

When the lesson started at about 0935, there were a number of decisions I had to make. I re-looked at the goals [of the lesson components] to see if I could pare them down in terms of expectations and still attempt to carry them out. I had to reorganise and reprioritise my goals. I had already planned to go through the homework errors and I decided to carry on with that, as I feared that the mistakes they made, if not addressed, can result in further problems in learning down the line. … With these thoughts in mind, I decided to cut off the whole section of curriculum on “solids” altogether. In making this decision, I bore in mind that the class had previously done something on volume of cuboids and cones etc with their usual teacher under the topic of “mensuration”. So it is less critical for me to go through solids again. In its place in the last Lesson [11], I would then cover angle and perpendicular bisectors. I made this decision also with the knowledge now that they are yet to be ‘steady’ in compass constructions and more time allocated for that would help. Also, the curriculum I know requires construction as a major part of the requirement and thus justifies my spending more time to help them to be more proficient in it.

So this lesson [only about 30 minutes left] I focused on the homework questions that were problematic to the students … .

I hope to do construction of perpendicular bisectors and angle bisectors with them in the next lesson [eleven] and be able the complete the worksheet that accompanies the topic. I also hope to help them see why the construction method works – due to the angle properties of rhombuses.
When I stood at the beginning of Lesson 10, frustratedly aware that over half the original lesson time was lost, I had to decide how to make use of the remaining one and a half lessons. I weighed the instructional choices: Squeeze the contents of two lessons into one and a half lessons and risk goal-dilution situation; or, maintain usual lesson pace but leave some components uncovered at the end of Lesson 11 [against gcvr]. My decision was to delete the 3-D solids material from the original Lesson 10—labeled as (a), (b), and (c) in Table 26. Knowing that 3-D solids appeared elsewhere in the curriculum made me feel less guilty about omitting this\textsuperscript{10}. The contrast between the teaching plan I devised at the start of Lesson 9 and the actual content coverage is illustrated in Table 27.

<table>
<thead>
<tr>
<th>Lesson number</th>
<th>My plan of coverage, as revised at start of lesson 9</th>
<th>Actual plan, as revised just before lesson 10</th>
<th>Reasons and goals consideration for retention/change</th>
</tr>
</thead>
</table>
| 10            | (I) Use of compass to construct perpendicular and angle bisectors  
                (II) Relate construction process to the properties of rhombus | Revision of homework on Ex 14B and 14C  
                (I) and (II) moved to lesson 11 | Students had errors and difficulties. [gerr, gsol]  
                Lesson time cut short by half because of “special assembly” |
| 11            | (a) Identify different 3-D solids  
                (b) Build, using Polydrons, different 3-D objects  
                (c) Highlight differences between prism and pyramid | (I) Use of compass to construct perpendicular and angle bisectors  
                (II) Relate construction process to the properties of rhombus  
                (a), (b), and (c) moved out of the 11-lesson time period | (I) and (II) kept in module because students need more time with these. [gins, gmnn]  
                Chose to put priority on gosol, gerr, gins, and gmnn. |

\textsuperscript{10}I informed and apologised to the resident teacher—who resumed as the students’ mathematics teacher once my eleven-lesson sequence was over—about my inability to finish the 3-D section of the plan.
The entire period of turbulence leading to the problem of discarding the section on 3-D objects thus started way back in Lesson 7 when there was movement of one component of that lesson to the next. There followed a series of movements, displacements, insertions, and disruptions that acted as falling dominoes linking one trigger to the next. The picture of related triggers is summarised and illustrated in Figure 58.

<table>
<thead>
<tr>
<th>Trigger</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>Cannot cover Ex 14B</td>
</tr>
<tr>
<td>2nd</td>
<td>Ex 14 B displaces Sketchpad constructions to lesson 9</td>
</tr>
<tr>
<td>3rd</td>
<td>Assumed more time with ‘extra lesson 11’. Devote whole lesson to Sketchpad constructions. Move lessons 9 &amp; 10 to 10 &amp;11 respectively</td>
</tr>
<tr>
<td>4th</td>
<td>Insert Homework revision. Time pressure</td>
</tr>
<tr>
<td>5th</td>
<td>Announce “special assembly”</td>
</tr>
</tbody>
</table>

**Figure 58: Turbulent Region 9**

6.9.1 Review of Turbulent Region 9

The turbulence discussed here spread across a temporal region that is even broader than Turbulent Region 8. Here, the turbulence spans four lessons, adding further support to the premise that the problems of teaching can surface at different analytical grain-sizes.

In addition, the experiences highlight the realities of classroom teaching concerning the relationship between teaching plan and actual implementation. Teaching is not just a straightforward enactment of what is previously planned—whether at the modular level or at the lesson level. The dynamic situation of classroom teaching (such as the insertion of homework revision) and school life (such as a last minute
“special assembly”) can influence instructional decisions, causing substantial changes in planning and classroom practice.

As on previous occasions (e.g., Lessons 4 and 7), the insertion of the homework revision component before Lesson 10 caused a now-familiar goal-dilution situation. This was problematic as it compromised the fulfilment of all the objectives within the goal-mix by injecting more teaching goals without correspondingly increasing the instructional time. Although occurrences of goals inclusion need not always involve homework revisions, the importance I attached to homework as part of students’ learning and my responsibility to give feedback made this a recurring event.

A closer look at the underlying g-goals for each occasion of including homework revision into the lesson reveals a common theme. In Lesson 4, the purpose for bringing in the homework component was gerr; in Lesson 7, the reason was gsol, gpsv, and grsd; and in Lesson 10, there were gerr and gsol intentions. But all these g-goals—grsd, gerr, gpsv, and gsol—are subordinate to the overarching G2 goal of preparing students to tackle exam-type questions. The common motivation underlying my urge to address unsatisfactory homework submissions in my classroom teaching despite the tight time schedule was a strong sense of responsibility to prepare students for forthcoming examinations. Thus, the influence of G2 in my work of teaching cannot be underestimated when considering problems of practice. The importance of G2 is traceable to the broader cultural environment of placing great emphasis on students’ excelling in examinations (see Section 4.2.2).

The frequent occurrences of goal-dilution situations indicate that a common way I coped with emergent goals was to take the risk of bringing all goals on board in the classroom. However, the second trigger in Turbulent Region 9 showed that ‘incorporate all goals’ was not the only way I dealt with the problem. After Lesson 7, when I realized Exercise 14B items were not covered in the lesson, instead of adding of this component into Lesson 8, I moved some pre-planned sections of Lesson 8 into the next lesson. That I chose “displacing components” rather than “incorporate all” had to do with my prior knowledge of students’ need to spend more time with Exercise 14B, as well as the awareness of more instructional time afforded by an
‘extra’ Lesson 11. The differences in my approach to emergent goals provide further evidence that the work of coping involved my knowledge of the instructional situation and its wider temporal context.

The pattern of trigger relations leading to Lesson 10 is closer to the increasing-decreasing picture described in Lesson 9 rather than the always-ascending trend experienced in Lessons 4 and 7. The first trigger kick-started the awareness of gcyr being in danger. At the second and third triggers, however, the gcyr pressure eased because of my increased assurance of the added instructional time afforded by Lesson 11. The gcyr warning light started beeping again at the fourth trigger when I planned to insert the additional homework revision component into Lesson 10, but became most intense just before Lesson 10 when the “special assembly” was announced.

The conjectures are again examined in the light of Turbulent Region 9, as shown in Table 28.
## PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 9</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>Like in lesson 9, the trend of trigger-links in this region is ‘undulating’. More elaborations below</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>Except for the 5th trigger, the other triggers were not open to public observation as they resided in my thought world and occurred before and after lesson time</td>
</tr>
<tr>
<td>P3 Problems Improvising</td>
<td>Supported</td>
<td>Problems were direct consequences of my insertion and removal of lesson components</td>
</tr>
<tr>
<td>P4 Inter-G interactions</td>
<td>Supported</td>
<td>In this case, Inter-G goals were involved</td>
</tr>
<tr>
<td>P5 Wider context</td>
<td>Supported</td>
<td>Without understanding background of relations among the chain of triggers, the difficulty of fulfilling gcvr cannot be adequately appreciated</td>
</tr>
<tr>
<td>P6 One lesson</td>
<td>Supported</td>
<td>Turbulent region stretched across four lessons</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 9</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Ongoing monitoring of time and the feasibility of fitting lesson components into current and future lessons</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>Tapped heavily on my knowledge of the module plan, the ‘extra’ lesson, and my estimate of how much content can fit into given time limits</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>In choosing to leave out “solids” from the 11 lessons, I took into consideration the curriculum coverage of solids in the earlier part of the students’ school year. Also, there was an ongoing ‘looking forward’ of the potential time contained in Lesson 11. More elaborations above</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Supported</td>
<td>Prioritisation was particularly evident in the second problem peak where I chose the gerr, gsol, gins, and gmrn goals over gcvr</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>In the first problem peak, I tried to incorporate all the goals in Lesson 10; in the second problem peak, when I knew that gcvr would be compromised, I apologised to the resident teacher and left it to him to decide if he would teach “solids” some other time</td>
</tr>
</tbody>
</table>

Table 28: Review of conjectures based on analysis of Turbulent Region 9
6.10 Turbulent Region 10: Teaching Dickvan the Construction of Perpendicular Bisectors and Angle bisectors in Lesson 11

The main focus of the final lesson was to help students learn the construction of perpendicular bisectors of line segments and bisectors of angles using only a pair of compasses and an unmarked ruler. These constructions are clear requirements of the school’s curriculum. From my experience in previous lessons and from the homework submissions, I realized that the students were still unfamiliar with the use of compasses in construction. Thus, I went into Lesson 11 with a deliberate intention of going slower with my demonstrations as well as giving more time for students to practise the constructions. The two main components of the lesson were (I) teacher’s demonstration of the construction procedures, followed by (II) students’ practice of constructions using a worksheet.

I started the first component of the lesson by explaining what “perpendicular bisector” meant and demonstrating on the whiteboard steps involved in constructing it. I did so with the whiteboard-sized compass and the magnetic ruler. As I performed the procedure one step at a time on the board, I paused to check that students were replicating the procedures in their notebooks.

I went through … the procedure of the compass construction. I can see as I view it now that I went through it really slowly. I made each step of the instruction explicit, pausing regularly to look up to see that they were following. I reminded students a few times of the need not to change the arms of the compasses when it was required for them [to lift the compass up]. I also asked them what I should do next after getting the two arcs’ intersection. They suggested I joined them and I did accordingly. The advantage of doing it on the whiteboard this way is that they can see the actions and results correspondingly as if they are on paper … . I walked around to ensure that students were able to construct the perpendicular bisector before I proceeded to do the angle bisector construction. I know that if they do not follow this part they may be constantly frustrated later by always getting their working wrong. I corrected some of their errors individually and even had to show them on a one-to-one basis how it can be rectified. [Post-lesson reflections, emphasis in the original].
After demonstrating the perpendicular bisector, and being reasonably satisfied that students could replicate the procedure, I moved on with angle bisector construction in the same slow, step-by-step, stop-to-check-students way:

Went through also a very slow and similar procedure as the last [one on perpendicular bisector] to show the construction of angle bisector. At various junctures I stopped to check that they were following by sounding some reminders of critical things they need to watch out for. I was also visually maintaining a contact with students that they were looking up when each small step was completed and they were following me. … I also remembered when I was constructing that in marking students’ homework, they were not careful in proper labeling. So I took the opportunity here to explicitly mention the importance of labeling the points and the final bisectors and demonstrated to them how that can be done. [Post-lesson reflections]

When I completed the constructions, the texts and visuals on the whiteboard were as shown in Figure 60.

![Diagram of constructions](image)

**Figure 59: Visuals on board after the construction demonstration**

I ended the first component by relating the construction procedures to the properties of a rhombus. A worksheet was given to students to facilitate their practice of construction procedures. For the rest of the lesson, the primary activity of the classroom centred around students’ working on the worksheet while I moved from table to table to provide guidance along the way. The first page of the worksheet is shown in Figure 60.
Thus far, I have depicted the work of teaching mainly in whole-class situations. To highlight another aspect, I focus my attention now on those parts of Component II of the lesson where I attempted to teach Dickvan how to perform the constructions presented in the worksheet. When teaching Dickvan, a series of my interactions with him formed a turbulent region.

*Complete each of the construction tasks below. You are to use only a pair of compasses and a ruler. Do not erase the construction traces you create in the process of doing the tasks.*

<table>
<thead>
<tr>
<th>Construction task</th>
<th>Constructions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construct a perpendicular bisector of the line segment AB shown on the right.</td>
<td><img src="image1" alt="AB construction" /></td>
</tr>
<tr>
<td>Construct a perpendicular bisector for each of the three sides of the triangle PQR shown on the right.</td>
<td><img src="image2" alt="PQR construction" /></td>
</tr>
<tr>
<td>Given the line segment JK on the right, construct a line that passes through J and is perpendicular to JK.</td>
<td><img src="image3" alt="JK construction" /></td>
</tr>
<tr>
<td>Construct an angle bisector of the given ( \angle DEF ).</td>
<td><img src="image4" alt="DEF construction" /></td>
</tr>
</tbody>
</table>

*Figure 60: First page of the worksheet on construction tasks*
First interaction with Dickvan

Soon after I gave out the worksheet, I heard Dickvan calling for me, “cher\textsuperscript{11}, cher …”. I was a few tables away from him but I walked over immediately to see how I could help him. He was working on the first question of the worksheet to bisect the segment AB when our discourse began.

20.24 Dickvan: Can extend the line?
T: What do you mean? [as he is walking over]
Dickvan: [Pointing to segment AB] Can extend this line?
T: Can. Why cannot?
T: [Looks at what Dickvan was doing as he ‘extends’ segment AB longer] Are you sure you want to extend it? What is the reason for extending it?
Dickvan: Because make it more simpler [almost inaudible].
T: Make it what?
Dickvan: More—4cm. Easier.
T: [Sensing what Dickvan wants to do] No, you are not supposed to do that. You are supposed [as he places his hand on Dickvan’s shoulder] to use the compass [uses his two fingers to simulate the compass arms movement] to construct. You can extend but in this case you must think why you want to extend it. [Pause]
T: I want to bisect AB you know? I don’t want to bisect your longer one [pointing at Dickvan’s worksheet]. Not by measurement ah.

I initially took his question about extending AB at face value and thought that he wanted to employ another method of bisecting AB. As I stood beside him a while longer and probed further to investigate his intentions, I sensed through what he said and his placing of the ruler along AB that he was actually trying to “make AB to be 4cm long” so that he can read off the midpoint of AB at the 2cm mark:

Dickvan asked if he could extend the line segment AB. He missed the whole idea behind the construction altogether. He wanted to extend the segment AB to be 4cm so that it’s easier for him to put down to 2 cm, as I supposed that’s what he wanted to do. He was way out of the correct solution. … What the question required was to bisect AB, and not the longer extended segment. This is yet another problem—the students’ range of abilities. While I have to worry about other students at other tables moving too fast and not sufficiently challenged, here I have with students like Dickvan who was way out, and having no sense it seemed of the whole purpose of the task. [Post-lesson reflections]

The impression I formed of Dickvan then was that he was very weak in his grasp of the content at hand. His weakness in the subject matter can be traced beyond the confines of this lesson to my previous interactions with him in the module. I brought

\textsuperscript{11} Singaporean students commonly address teachers in the shortened from of “‘cher”.

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this awareness with me when I interacted with Dickvan here. His difficulty was all the more conspicuous when seen in the context of my careful demonstrations in the earlier part of the lesson. He did not seem to ‘get it’ despite my deliberate examples on the board. My subsequent interactions with him were therefore mainly motivated by the goal of trying to help him cope with the basic requirements of the curriculum [gwkk]: that of being able to do the basic compass and straightedge constructions of perpendicular and angle bisectors.

**Second interaction with Dickvan**

The second time that Dickvan called for my help in the lesson was over his confusion between perpendicular bisector and angle bisector, as shown in the vignette below.

---

24.30 Dickvan: Cher [quite loudly]
T: Yes? Who’s calling me? [Looks up from talking to Wanyan and looks around]
Dickvan: [With one hand raised] Mr Leong.
T: Yes? [As he walks over]
Dickvan: Cher this one [pointing the second question in his worksheet about constructing perpendicular bisectors in a triangle] - this one is angle or it bisect the [his pencil moving in a direction that shows the act of bisecting a side of triangle PQR]?
T: What do you think? What does the question [pointing to the question text] say?
[Pause]
T: Construct a perpendicular bisector for each of the 3 sides [reading off from the worksheet question] right? So you don’t worry about so many sides. Just look at side PQ [pointing] first. Construct a perpendicular bisector [flattens his palm vertically to simulate the action of bisecting the side PQ]. Then look at QR [pointing] then construct another one [uses the palm-simulation again].
Dickvan: [Pause, then nods] Aor [colloquial for “yes”].

Although the definitions of “perpendicular bisector” and “angle bisector” were carefully clarified at the earlier part of the lesson and were still on the board in front of the class (as shown in Figure 52), Dickvan did not seem to be accessing those instructional elements at all. In fact, he appeared neither to check the definitions visually from the board nor do a careful reading of the question. My uneasiness about his unhealthy reliance on me to provide him with immediate answers can be detected by my attempt to re-direct him to read the requirements of the question:

For Dickvan, who asked me “is it right?” I decided to asked him back, “what do you think?” to get him less reliant on me to evaluate his work for him. I asked him to read the instructions of the question himself. [Post-lesson reflections, emphasis added]
I wanted to teach him the basic responsibility of not being dependent on the teacher for basic information he can access on his own. However, the awareness of Dickvan’s overall weakness in the subject made me conscious that he might perhaps be so poor in the grasp of the ideas that I should further lower my expectations of him and help revise with him the meaning of “perpendicular bisector”. The motivation in me towards Dickvan was strong and made me more inclined to helping him with his difficulties rather than leaving him to sort it out on his own.

Third interaction with Dickvan

The third time Dickvan called for my attention further confirmed that he was struggling even with the mechanical procedure of constructing the perpendicular bisector on one of the sides of the triangle.

```
29.44 Dickvan: Cher cher cher. Like that, correct or not?
T: [Walks over to check] Yes … [looks closer] but how come your construction - no no no. Your construction is not correct. Your construction [picks up a pair of compasses] tends to be a bit too short [points at Dickvan’s constructed perpendicular bisector of side QR of triangle PQR]
Dickvan: Oh
T: You see - you only got one point [pointing at only one intersection of arcs]. One point is not enough to draw the line Dickvan.
T: OK I show you an example ah [as he erases Dickvan’s line and arcs and proceeds to demonstrate to him a correct construction procedure]
```

The ‘perpendicular bisector’ of QR that Dickvan constructed used only one intersection of arcs. He drew a line joining this intersection point to a point which is roughly the midpoint of QR—perhaps measured by the scale on the ruler, resulting in an incorrect construction, as shown in Figure 61.

![Diagram of perpendicular bisector]

**Figure 61: Illustration of Dickvan’s construction**
Most of Dickvan’s classmates had already completed this question and progressed to later questions. This led me to think that he needed extra help from me to get him performing at the most basic level [gwkk]. I patiently rehearsed the entire procedure of constructing perpendicular bisector with him by demonstrating the process on his worksheet itself. I emphasized that he needed to obtain two intersections of arcs instead of one in order to obtain the correct bisector as required.

Fourth interaction with Dickvan

The fourth time Dickvan asked for my attention, I saw improvement in his ability to construct perpendicular bisectors. He used the correct procedure for the perpendicular bisectors of all the three sides of triangle PQR. The transcript below shows my satisfaction at his progress and my attempt to boost his confidence.

32.48  Dickvan: Cher
T: [Walks over] Yes?
T: [Looks at Question 2 of Dickvan’s worksheet] Ah … better, better [pats his shoulder, and smiling]. … Ah this one [pointing to the intersection of all the three perpendicular bisectors] you manage to join it ah? Good.

Before I continue to describe my later interactions with Dickvan, it is helpful to examine these four exchanges with Dickvan with respect to my goals of teaching him. In the first instance when Dickvan called for my help, I detected that he had weak knowledge of the underlying geometry. It heightened in me a gwkk goal as my main emphasis for Dickvan. When Dickvan called for me the second time over his confusion between perpendicular bisector and angle bisector, I felt uncomfortable at his over-reliance on me for answers accessible from the board and worksheet instructions [against grel]. Yet because of my perception that he was so weak, the gwkk motivation in me dominated my teaching reactions. The third interaction with Dickvan, over re-demonstrating the construction process on his worksheet, confirmed the need to put gwkk as my priority. On the fourth occasion, however, I sensed that things might be turning around. Dickvan’s success with the perpendicular bisectors of the three sides of the triangle provided evidence of progress that could be built upon by trying other problems on his own. My stance towards him could now allow gwkk to diminish while making grel more prominent.
Before the fifth encounter with Dickvan, I showed the students on the board the construction process for the third question, requiring students to construct a perpendicular at a particular point, as I sensed that a number of them were unable to solve it and were frustrated. My explanation was very similar to those in the earlier part of the lesson. I proceeded slowly, checking visually that students were following the procedure at each step of the way. The diagram on the board at the end of that demonstration is given in Figure 62.

![Figure 62: The construction arcs I demonstrated on the board for the third question](image)

Fifth interaction with Dickvan

Shortly after I finished the construction, Dickvan called for me for the fifth time. When I went over to Dickvan, I saw that the line that he drew passing through J was visibly not perpendicular to JK. Despite my demonstration on the board, I was perturbed by the fact that Dickvan did not attempt to follow the procedures. It seemed to me that he was dependent solely on my going to him to teach him in a one-to-one setting. My growing disappointment with his over-reliance on me is apparent in the vignette below.

Dickvan: [Shouting from the other side of class] Teacher, Mr Leong, like that ah?
T: [While walking over] Check against what I've done on the board
T: [Looks at Question 3 of his worksheet, and shows a frown] Doesn’t look accurate.
You look at this line [pointing to the line passing J that Dickvan drew] - doesn’t look [shakes head] perpendicular to this line [pointing to JK]. What did you do?
Dickvan: [blank look]
Evan: [sitting next to Dickvan, interrupts and shows his worksheet Question 3] Looks OK?
T: [Glances] Looks OK.
T: [Returning to Dickvan] You check - you must understand the meaning of- [notices Dickvan looking away, he places both his hands on Dickvan’s shoulders to redraw his attention] Dickvan - perpendicular. So this line [flattens his palm vertically to simulate drawing a line] must be perpendicular to this [flattened palm aligns to JK] - does it look like 90 degrees or not?
Dickvan: 90 degrees ah?
T: Ah you try, you try on your own.
[ended 43.03]

The overall sentiment in this fifth discourse with Dickvan reflected a rising discomfort with Dickvan’s over-dependence on me for one-one teaching [against grel], especially when I needed to attend to thirty-nine other students in the class, and a seeming lack of effort on his part to make sense of the question as well as to access my earlier whole-class demonstration. That he could perform the constructions in Question 2 correctly on his own earlier indicated to me that he could do better in Question 3 if he tried harder. In other words, with respect to teaching Dickvan, my grel intention was strengthening in comparison to gwkk. I was questioning my readiness to help him on a one-to-one setting because that would accentuate his over-reliance. The tension was building up.

Sixth interaction with Dickvan
The tension reached a problematic point in the sixth encounter with Dickvan about a minute later. I was at the same table where he was seated, but teaching another student, when Dickvan interrupted again:

44.02 Dickvan: [Interrupting T’s demonstration to Yan San] Teacher, like that correct or not [pointing to his worksheet]?
T: [complete explanation to Yan San]
Yan San: [Nods]
T: [Looks at Dickvan and glances at his worksheet] I am more interested in how you do it. Can you erase it and then show it and then I come and take a look at what you have done?

What I saw on Dickvan’s worksheet was an attempt by him to thicken the line drawn previously—presumably by drawing multiple lines over it—in hope that the broader line would look visually more perpendicular to JK. What he did seemed to confirm in me his lack of motivation at trying the problem on his own. He even seemed to have taken my earlier patience with him as a license to be sloppy in his work. I was
concerned at his attitude and realized the problematic situation of conflicting goals in me. If I continued teaching him individually at his ‘beck and call’, I would not encourage him to be self-motivated in seeking for answers and reasoning [against grel]. On the other hand, if I left him to work it out on his own [grel], I was afraid that he could not handle the worksheet questions [against gwkk]:

But some of them I could see they were way out. I shouldn’t just leave them wandering into unproductive territories. Thus I gave them some idea of what could be a productive way to go. Where there were mistakes I detected, I also pointed it out to them [gwkk] …

I was not too happy with … [those who] seemed to want to get confirmation from me from the frequent requests of “am I getting it right?” This shows still the lack of confidence in themselves in checking for themselves the correctness of the working [grel]. [Post-lesson reflections]

For Dickvan, I took a stand with grel and sent a strong signal to him that he had to learn to be interested in the work for himself (as shown in the vignette above). He had to demonstrate some attempts at finding the answers before I would come in to help him in areas he could not handle for himself. One contributing factor that swayed my preference towards grel was the knowledge that Evan, who sat beside Dickvan and was on good terms with him, was progressing well with the construction work and could be a resource for Dickvan when he needed help. This thought alleviated my sense of uneasiness with leaving Dickvan to struggle it out by himself. As it turned out, that was the last time Dickvan called for my help in the lesson. I devoted the rest of the class time on helping other students in the class. The turbulence experienced with teaching Dickvan over the six interactions is summarized in Figure 63.
6.10.1 Review of Turbulent Regions 10

After looking at zoomed-out frames which stretch across a few lessons in the last two turbulent regions, the frame-size here covers a portion within a lesson. However, instead of involving my interactions with lesson components, external perturbations, or whole-class responses, my problematic experience in Lesson 11 concerned my one-to-one instructional sequences with a particular student. This reveals that problems of teaching exist at such socially—not just temporally—fine-grained levels.

The way I interacted with Dickvan at each of the encounters revealed my tapping into a ‘historical log’ that I perhaps keep for every student in the class. For Dickvan, at each conversation with him, I was digging into a mental data bank of past experiences with Dickvan to help me make sense of his behaviour so that I could make appropriate judgments in relation to my goals of teaching for him. This historical log that I keep of students’ behaviours and performances may explain the strong existence of mental and temporal elements in conjectures C2 and C3 while I cope with problems of teaching.

The review of the conjectures in the light of my attempt to teach Dickvan is given in Table 29.
# PHASE II ANALYSIS

<table>
<thead>
<tr>
<th>Conjecture about problems</th>
<th>Evaluation of conjecture based on Turbulent Region 10</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 Problem triggers</td>
<td>Supported</td>
<td>Similar to Lessons 7 and 9, the triggers-links reflected flip-flips in prominence between goals gwkk and grel.</td>
</tr>
<tr>
<td>P2 Invisible triggers</td>
<td>Supported</td>
<td>Although Dickvan’s written work can be easily seen by another observer, the goal considerations, especially of grel, which contributed to the triggers, were internal activities</td>
</tr>
<tr>
<td>P3 Problems Improvising</td>
<td>Supported</td>
<td>From an initial concern for gwkk, the 2nd trigger prompted a response within to also consider grel, which resulted in a problematic situation of conflict between the two goals</td>
</tr>
<tr>
<td>P4 Inter-G interactions</td>
<td>Supported</td>
<td>Due to the grel-G4 and gwkk-G5 subordinations, the conflict was of a G4 vs G5 kind</td>
</tr>
<tr>
<td>P5 Wider context</td>
<td>Supported</td>
<td>The chain-like relationship between the triggers provide the broader context to understand the grel vs gwkk tension</td>
</tr>
<tr>
<td>P6 One lesson Not refuted</td>
<td>Not refuted</td>
<td>No relevant data in the episode presented</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Conjecture about coping</th>
<th>Evaluation of conjecture based on Turbulent Region 10</th>
<th>Some details of the data</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1 Monitoring</td>
<td>Supported</td>
<td>Ongoing monitoring of Dickvan’s attempts with the worksheet tasks as well as his level of dependence on me</td>
</tr>
<tr>
<td>C2 Invisible resources</td>
<td>Supported</td>
<td>I used my knowledge of Dickvan and the subject matter to help him fulfil my grel and gwkk goals for him. The weighing of the instructional options for Dickvan was also drawn from internal resources</td>
</tr>
<tr>
<td>C3 Outside context</td>
<td>Supported</td>
<td>In my consideration of how I should react to Dickvan’s request for help, I was influenced by my recollections of our earlier encounters, and also what I have taught the class in the previous section of the lesson. More elaborations below</td>
</tr>
<tr>
<td>C4 Prioritisation of goals</td>
<td>Supported</td>
<td>I had to prioritise between grel and gwkk goals</td>
</tr>
<tr>
<td>C5 Address ill-effects</td>
<td>Supported</td>
<td>My knowledge of Evan’s ability to help Dickvan out alleviated my guilt of compromising on gwkk for Dickvan</td>
</tr>
</tbody>
</table>

Table 29: Review of conjectures based on analysis of Turbulent Region 10
6.11 Summary of Phase II Analysis

Having surveyed the turbulences experienced across the entire geometry module, it is appropriate to perform a brief summative evaluation of the conjectures on problems of teaching and coping strategies.

P1. Every problem of teaching was traceable to a trigger

Status: Strongly supported.

Elaborations: The conjecture was true for every problem studied in all the eleven lessons. The emphasis in this conjecture is that each problem can be traceable to at least one trigger, but not necessarily only one trigger.

P2. Triggers need not be easily detectable by an external observer

Status: Strongly supported.

Elaborations: Most of the triggers were not easily detectable by an external observer. Some occurred in a split moment and so were hardly noticeable; some others were temporally distant from the actual occurrence of related problems and hence remote to an observer; yet other triggers existed purely in the mind of the teacher.

P3. One of the ingredients for problems to occur was the attempt to improvise as a response to triggers

Status: Strongly supported.

Elaborations: All the problems were linked to my active attempt to improvise, mainly by way of including or considering emergent goals of teaching.
P4a. Problems of teaching can involve one or more overarching goals

Status: Supported, but only very weakly for the case of problems involving only one G-goal. Only one problematic situation—during Lesson 2—involved solely one G-goal.

Elaborations: The picture of G-goal interactions across the eleven lessons are as follows:
Lesson 1: G2-G5 vs G4; G2 vs G4;
Lesson 2: G1-G3 vs G4; *intra*-G5 (gwkk vs gstg);
Lesson 3: G1&G2 (only gins)
Lesson 4: dilutes G2-G3-G4; G1 vs G2-G3-G4; G1 vs G4; dilutes G1-G3-G4; G1 vs G3-G4
Lesson 5: G1,G2,G3&G4 (only glan)
Lesson 6: G3 vs G1-G2
Lesson 7: dilutes G2-G3-G4; G1 vs G4; G1 vs G2; G2-G3-G4(gwrs) vs G4(galt);
Lesson 8: G2 vs G3; G2 vs G1 vs G4;
Lesson 9: G3-G5(gstg) vs G5(gwkk, gcon); G3-G5(gwkk) vs G5(gwkk,gcon);
Lesson 10: dilutes G1-G2-G3; G1(g4)-G2-G3 vs G1(gcvr);
Lesson 11: G5 vs G4; G5 vs G4.

P5. Studying a particular problem of teaching without considering the wider context of the region of turbulence can result in a distortion of the complex work of teaching.

Status: Strongly supported.

Elaborations: Problems can only be adequately studied in the context of the turbulent region surrounding the problems. Linked triggers can in some cases be remote from the immediate vicinity of the problem peaks.
P6. Using one lesson alone as the ‘unit of analysis’ of problems of teaching may be inadequate for understanding the internal tension involved when dealing with competing goals of teaching

Status: Moderately supported.

Elaborations: Except for the problems discussed in Turbulent Regions 1, 3, and 10, all the other problems had clear links with teaching events and experiences outside the temporal bounds of individual lessons.

C1. Coping involved an ongoing monitoring of changes in the instructional situation

Status: Strongly supported.

Elaborations: In all the problem situations, I constantly monitored time, equipment, students, as well as my own instructional motivations.

C2. Much of the resources I harnessed during coping were invisible to others

Status: Strongly supported.

Elaborations: While there were sometimes external resources harnessed during coping, such as whiteboard construction equipment and students’ responses, internal resources—such as knowledge of students and instructional choices—were always utilised in each of the turbulent regions.

C3. While coping, I was influenced by phenomena outside the immediate context of the instructional situation

Status: Strongly supported.

Elaborations: When I coped with problems, I was not ‘locked’ in the time-space zone within which the particular problem occurred. I tapped into my knowledge of the
students’ learning history as well as looked forward in time to what was yet to take place in their instructional experience.

**C4. Coping involved prioritization of goals, and the priorities needed not remain the same throughout the turbulent region**

Status: Strongly supported, but only in situations where two or more g-goals were involved.

Elaborations: When coping in situations with multiple g-goals, I concentrated on at least one goal as main focus of teaching. In Turbulent Regions 4, 6, and 7, the goal priority remained the same throughout the turbulent regions; but in Lesson 8, and Turbulent Regions 9 and 10, the priorities changed considerably over the respective turbulent regions.

**C5. Actions were taken to alleviate the potential ill effects of unfulfilled goals**

Status: Supported.

Elaborations: Although some goal(s) were compromised in almost all the problematic situations, there were always efforts at ‘damage control’.

Apart from providing data to evaluate the conjectures, the lesson-by-lesson analyses also revealed greater insight into how the problems of teaching and coping strategies interacted with other significant instructional elements surrounding and within the turbulent regions. Some elements—such as triggers, goals, external resources, internal resources, and time—were conspicuous throughout the teaching of the entire module. This chapter concludes by reviewing how problems and coping strategies in teaching were related to these key elements of instructional practice.

**6.11.1 Problems and Coping Strategies in Relation to Triggers**

Triggers are *necessary* conditions for problems of teaching to occur [P1]. In all the turbulent regions analysed over the whole module, each of the problems of teaching
was traceable to at least one trigger, often more. The temporal separation between the trigger(s) and the related problem also varied, from virtually coinciding to several lessons. Despite these differing patterns of triggers, it is clear that the problems of teaching are greatly influenced by the type and occurrences of triggers.

However, triggers alone are not sufficient conditions for problems to take place. The ingredient that transformed triggers to actual problems was my improvisational responses towards the triggers [P3]. Triggers merely bring about an awareness of a novel instructional situation. Reactions to the triggers include the teacher’s deliberate attempts to change or preserve the instructional situation in such a way that some goals of teaching can be achieved. Such improvisations bring about problematic situations as one or more goals of teaching become compromised.

Also, triggers do not just lead to problems in an unmediated way. Triggers, once detected, were monitored closely with regards to how much ‘damage’ they could wreak [C1]. Such constant monitoring of trigger effects was most clearly illustrated by the ongoing awareness of changing goal-prominence across a trigger-chain—as seen in Turbulent Regions 4, 7, 9, and 10.

6.11.2 Problems and Coping Strategies in Relation to Goals Interaction
Problems of teaching come about because one or more goals of teaching are prevented from being carried out as intended. In most situations, as seen in the analyses, the interferences to goal-fulfilment were actually other goals of teaching. Goals that emerged as the teaching work was carried out interacted with pre-existing goals to bring about problematic regions in which at least one of the goals of teaching was compromised.

There are different ways in which the goals interacted problematically. One type of problematic goal interaction is when sets of teaching goals directly conflict with each other. This phenomenon takes place when the teacher confronts multiple instructional options, but choosing any particular one abandons or downplays the goals associated with the other instructional pathways. The final outcome of such conflicting situations is invariably a partial fulfilment of goals. This inability to completely fulfil
all the teaching goals results in deficiency either in terms of the number of goals achieved or the number of students taught with those goals in mind. In most cases, partial fulfilment means only a subset of all the goals-in-consideration are being carried out. In Turbulent Region 6, however, partial fulfilment took the form of goals being implemented with some (but not all) of the students in class.

Goals can also interact problematically by way of dilution. When new goals are added into an existing pool of instructional goals, they need not be in direct conflict with each other. Rather, the insertion of additional goals without a corresponding increase in time or resources to carry out the bigger goal-set results in a dilution of goals. When this happens, time dedicated to the fulfilment of each goal is reduced, thus decreasing the likelihood of successful implementation of all the goals. The Turbulent Regions 4, 7, and 9 provided ample illustration of problems associated with goals dilution.

To cope with the flux of multiple goals under consideration, I repeatedly used the strategy of prioritization of goals [C4] to focus on the most significant goal(s) for that particular instructional situation and to ensure that at least one goal of teaching was kept in mind and carried out during the turbulent regions. In addition, actions were always taken to alleviate the potential ill effects of compromising other worthy goals [C5].

Except in one case in Lesson 2 where the goals involved were of the intra-G type, all the other problems encountered had to do with inter-G interactions. The number of instances that involved more than one G-goal may indicate that the problems discussed involved a wide scope of the teaching enterprise—since all the five G-goals represent the main agenda for teaching mathematics. In other words, the problems of teaching encountered were not trivial or peripheral ones that had little effect on my main work of teaching. Rather, they created upheavals over a sizeable landscape of the teaching enterprise.

Although most of the problems analysed arose from goals interaction, there were a few where the turbulences came from other sources. In Turbulent Region 3, the
interference was due to a lack of suitable construction instruments and my inadequate proficiency with the available equipment. Towards the end of the same lesson, the faulty retractable mechanism of the screen also created frustration with my intention to carry out the gmrn goal. In Turbulent Region 5, the conflict was not between teaching goals. Rather, it was a tension between following theoretical knowledge of avoiding prototypical phenomena and capitalizing on the craft knowledge of using the language of ‘Z’ to describe obtuse alternate angles. The presence of these other difficulties highlighted the importance of other less obvious issues—such as availability of suitable resources, proficiency with resources, dealing with the interaction between theoretical and craft knowledge—when considering problems of teaching and coping strategies.

6.11.3 Problems and Coping Strategies in Relation to Time

In Lessons 2, 4, 7, 8, and 10, the issue of time pressure featured prominently in the respective turbulent regions, when there was too little time to deal with too many instructional goals. Problems occurred when I could not fit all the intended goals into the available time. Time is seen as a teacher’s resource; time pressure is felt when there are overwhelming demands—goals—vying for this resource. When that happens, the lack of time acts as a constraint on the fulfilment of all the intended goals of teaching, thus resulting in problematic situations.

The problem of time limitations was most clearly seen in my attempts to “cover the syllabus” [G1]. I set time targets for content coverage for periods as short as a part of lesson to as long as a few lessons. Time-related stress set in when I was unable to finish a section of a lesson within the time allocated for it (as in Lesson 8), or when an entire component of a lesson had to be postponed to a subsequent lesson (as in Lesson 2), or when I sensed that such postponements might eventually lead to a subsequent failure to complete the teaching of the module within the eleven lessons (as in Turbulent Region 9). Time constraints become accentuated when other worthy, but time-demanding, instructional goals emerged along the way. Problems of teaching occur when some of the goals of teaching are—of necessity—trimmed to fit within given time constraints.
But time, as a resource, should not be seen merely as a constraint. Time, as an accessible region within a continual temporal landscape, acts as an affordance for the purpose of coping with the problems of teaching. There were occasions where I harnessed the resource of time outside the immediate instructional context to cope with the problematic situations [C3]. The case of teaching Dickvan in Lesson 11 illustrates how time beyond the actual teaching moment can provide an affordance. When interacting with Dickvan, my thoughts and decisions were based not only on current content of discourse; rather I also tapped into a ‘historical log’ of past exchanges and impressions about his attitude and abilities towards learning mathematics. Similarly, in other turbulent regions, I frequently drew on knowledge of previous experiences in the class as a resource for determining what instructional courses to take [C2]. In other words, while the physical and observable elements of teaching are locked in the ‘here and now’, the resources I bring to the ongoing instructional work in the classroom need not be so confined to the present. They can come from outside the temporal boundaries of the lesson.

Furthermore, not only can coping strategies be based on retrospective reflections, they can also be influenced by anticipations about what is to come. There were a number of junctures in the module where I mentally moved forward in time to anticipate the instructional content and used that information to adjust current teaching choices. Successful anticipations, such as the knowledge of students’ difficulties with construction tasks and the impact on time pressure from introducing additional components to a lesson, help the teacher prepare for the challenges ahead. Time, when seen as temporal space where I can move backward and forward mentally, can therefore be an affordance to teaching and form part of my coping strategies.

Another issue related to time is the suitable timeframe needed for the study of teaching problems and coping strategies. The strong and moderate support for P5 and P6 respectively question the traditional convention of using fixed temporal frames for the analysis of classroom teaching (see Section 3.2.7). Using a rigid unit of time to view instructional behaviour misses out on the contextual variations in the size of the turbulent regions and thus distorts the complexity inherent within and across the boundaries of these fixed time intervals. The evidence presented here supports the
use of turbulent regions of flexible framesizes that suitably capture the complexity of the teaching phenomenon.

6.11.4 Problems and Coping strategies in Relation to Diversity in Students’ Mathematical Abilities

One prominent feature in the survey of the lessons is the challenge of teaching in a way that will meet the needs of students of varying mathematical abilities. The analyses of Turbulent Regions 2 and 9 showed the tensions I experienced in attempting to teach both the more proficient students [gstg] as well as those who struggled with the mathematical content [gwkk]. While it was a strong motivation for me to teach every student regardless of their abilities [G5]—a belief I considered to be fundamental to the work of teaching—there were indeed some occasions where I could not fulfil the learning needs of both sets of students at the same time.

Nevertheless, there were attempts to cope with the gstg versus gwkk conflict as well. Apart from prioritizing between the two [C4] according to the contextual influences surrounding the problem, there were also efforts each time to alleviate the problem by partially addressing the needs of the neglected group of students [C5]. In Lesson 2, I pleaded for patience as I deliberately slowed down the pace of teaching to cater for the slower students; in Lesson 9, I used the Sketchpad demonstration as a way to help the weaker students gain something from my instruction.

6.11.5 Problems and Coping Strategies in Relation to Observability

Many of the problems and coping mechanisms of teaching reside in the “thought-world” of the teacher and as such are unobservable to an external observer [P2, C2]. Even when the turbulences are manifest in observable phenomena, they are at best incomplete but at worst misleading depictions of deeper underlying complexities when the internal counterpart of the turbulence is not considered as well. To adequately study the complexity of teaching there is, therefore, a need to look at both teaching actions as well as the underlying motivations behind those actions.
The importance of the unobservable parts of teaching suggests that conventional ways to observe and study teaching may be inadequate for understanding the underlying complexity of the task. Research on teaching tends to focus predominantly on teaching actions. In my case, however, much of the work of teaching—such as weighing the alternatives, toying with ideas and available resources, making decisions and being aware of the consequences—were hidden from outside observers, but nevertheless contribute directly to my teaching behaviours. Without access to this thought-world of the teacher, the observer of teaching misses a vital component of the work of teaching. Moreover, to understand why a teacher makes certain instructional choices over others, there is a need to view his teaching actions in light of his goals of teaching and how these goals interact while he attempts to fulfil them in the classroom.

6.11.6 Problems and Coping Strategies in Relation to Teaching Mathematical Reasoning

I argued earlier that part of the work of teaching students to reason mathematically involves helping them acquire attitudes and mental dispositions that support plausible reasoning. These include the willingness to rely less on the teacher for answers and direct instructions, to try things out themselves, to work in environments where formative ideas are as yet uncertain, and to have the courage to make conscious guesses. However, problems occurred in my instructional work when I needed to go against these very virtues that I advocated in order to accommodate other goals of teaching. In Lesson 1, I imposed my authority to resolve uncertainty so that the subsequent part of the lesson could proceed without confusion; in Lesson 8, I felt that the encouragement for students to voice their guesses freely resulted in the classroom discourse diverging so I intervened directly to steer the discussion towards the intended curricular objectives; in Lesson 11, I struggled between helping Dickvan who was weak in mathematics and allowing him to make independent attempts on his own without being over-reliant on me. These accounts showed that encouraging dispositions conducive for reasoning may conflict with other instructional needs in the classroom. Other goals such as maintaining lesson flow, connecting to curricula agendas, and helping particular weaker students are also important priorities competing with the goal of engendering good habits that support reasoning.
Apart from encouraging suitable attitudes and mindsets as ways to teach reasoning, I also demonstrated and modeled what the reasoning process was like during the lessons. These in-class enactments of reasoning, like all other instructional activities, use up lesson time. A number of the situations described involved the desire to teach reasoning pitted against the sensitivity of time constraints—engaging in conjecturing and explanation in a whole-class setting while watching time in Lessons 2 and 7, seeking alternatives but mindful of available time in Lesson 8, and diluting the goals of discussing and proving due to limited time in Lesson 4. Thus, helping students to develop their reasoning abilities clearly requires substantial amounts of lesson time.

The instructional work involved in modeling the reasoning process is particularly time-intensive. In an environment where time is a critical resource, it is no wonder that my attempts at enacting reasoning were so prone to problems associated with time pressures.

Despite these problems, I nevertheless tried to fulfil the goal of teaching mathematical reasoning. One common coping strategy that I used was to attribute productive ideas during the reasoning process to students’ contribution whenever feasible. In so doing, I could efficiently advance the discussion while showing that their voices were included in the process. I often “seized upon” a particular student’s contribution midway through a whole-class discussion to direct the discourse to a track that helped me connect to curricular requirements (as illustrated in Lessons 2 and 7). But in situations where no such positive helps were given by the students, I then tried to slip in my voice to keep the reasoning process going and to avoid further time delay. To blunt the authoritative nature of ‘my voice’, I made attempts to subject my contributions to joint agreement from the class (as illustrated in Lessons 1, 3, and 7) or to the drag-test verification of the Sketchpad (in Lesson 7) as ways to reduce the students’ reliance on my words as the final verdict.

### 6.12 Conclusion

This chapter corresponded to Phase II of the data analysis process (see Section 3.2.8 for the summary of phases). The conjectures derived from the first phase of analysis (reported in Chapter 5) were scrutinized and refined by studying a number of
turbulent regions spread over the entire module. The findings here added further overall support to the conjectures and revealed how problems and coping strategies interacted with other instructional elements in a complex way.

Thus far, the focus was primarily on regions of practice where Sketchpad use did not feature prominently. In the next chapter, the attention is turned specifically to problems and coping strategies involved when integrating Sketchpad into the geometry module.
In Chapters 4 to 6, the inquiry focused mainly on the teaching practice without the use of Sketchpad. In this chapter, the final phase of a “progressive widening of focus”, I examine specifically those regions of practice where Sketchpad was used.

The analytical framework of “progressive widening of focus” has been employed as follows in this study: Using the metaphor of viewing video footage of a landscape, the work of Chapter 5 [detailed analysis of the rhombus problem in Lesson 8] was like zooming in on a short video segment of prominent terrain in the landscape [“teaching geometry”]. That in-depth exploration of Lesson 8 allowed me to develop tools in the method of study—the G/g-goals and the use of turbulent regions—as well as posit initial conjectures on problems and coping strategies. In Chapter 6, I viewed the entire video segment [the whole module] of the same terrain [“teaching geometry”] using the same tools of analysis to test the conjectures and refine them. In this stage of the analysis, the view of problems and coping strategies of teaching was based solely on looking at that prominent terrain [“teaching geometry”]. In this chapter, I look at some of the additional features of the landscape [Sketchpad use in teaching] in parts of the module where technology was used, to examine how these features affect the view developed so far.

Before I discuss the problems and coping strategies associated with Sketchpad use in Sections 7.4 to 7.7, I begin by clarifying the enlarged goal-based framework that takes into account instructional work involving Sketchpad. Section 7.1 describes the overarching goals; Section 7.2 examines the emerging goals as evidenced from actual work of teaching; and Section 7.3 relates these two sets of goals.

### 7.1 My Overarching Goals Associated with Sketchpad Use

In Sections 4.1 and 4.2 I described my goals of teaching geometry as being of two types—overarching and emerging—with the latter subordinate to the former. I use a similar distinction here when considering my teaching goals associated with Sketchpad use. Like my other G-goals of teaching, the motivations behind the use of Sketchpad were influenced by other elements in the classroom milieu such as my...

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1. Recall that I use “teaching geometry” for geometry teaching that excludes the use of Sketchpad.
beliefs, my teaching and research experience, and the educational expectations of the wider community.

I was aware that Sketchpad was likely to be new to most Secondary 1 students as very few Primary schools in Singapore own the software\(^2\). As with the introduction of any software, students need time to become familiar with its features. Moreover, constructions in Sketchpad are very different from other drawing tools (such as the commonly-used drawing tools in Microsoft Office\(^{TM}\)). It is no wonder that some students find the initial experience with Sketchpad frustrating (Pokay & Tayeh, 1997), a problem I had previously encountered myself (Leong, 2002). Successful integration of Sketchpad into geometry teaching needs time investment to help students become familiar with the functions and features of the software.

Although helping students become proficient with Sketchpad was necessary, the focus was still on students’ learning of geometry. My intent in using Sketchpad was that students could make connections between work in Sketchpad and the underlying geometrical relationships. An example may illustrate this point better. One way to construct a drag-resistant square is to use the “construct perpendicular” and “circle” tools in Sketchpad. My focus was not merely that students could use these tools to construct a square; I also wanted to highlight the correspondences between these tools and the critical attributes of squares, namely perpendicular adjacent sides and equal adjacent sides.

My overarching teaching goals associated with the use of Sketchpad in the geometry module were as follows:

T1. To help students become familiar with working in the Sketchpad environment.

T2. To use Sketchpad to develop geometrical ideas and reasoning.

Figure 64 illustrates the T-goals and my perception of their roles.

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\(^2\) This has largely to do with uneven efforts to encourage Sketchpad use. The Ministry of Education and local vendors of the software combined to offer site-licence discounts to Secondary schools, but the same offer was not made to Primary schools.
Goal T2 is about helping students to connect what they do and observe in the Sketchpad environment to the wider geometrical world. I do not use the terms “Sketchpad environment” and the “geometry world” to imply that they are separate. I recognise the possibility that when students work with Sketchpad, they can make non-deliberate connections to related ideas in the “geometry world”; hence the perforated boundary around Sketchpad in Figure 64. As an example, students may absorb geometrical language while clicking on menu tabs like “perpendicular line”, “parallel line”, or “midpoint” when performing onscreen constructions. However, the work of teaching should be more deliberate than just relying on students to make such implicit associations on their own. I believe there is a need to help students know the direct links between Sketchpad phenomena and the related geometrical ideas and reasoning.

While goals T1 and T2 are shaped by my beliefs about the usefulness of Sketchpad in teaching, they are also in line with the policy of the Singapore education authorities for technology use in teaching. Helping students become proficient with the software and connecting its use to geometry learning is in keeping with the reform initiative of “integrat[ing] information technology to enhance the mathematical experience” (UCLES/MOE, 2000).

It is important to clarify that in regions where Sketchpad is used in teaching, both the G-goals as well as the T-goals can be active. The T-goals do not replace the G-goals in the classroom; rather they are in addition to the other goals of teaching geometry.
Inasmuch as teaching with Sketchpad is meant to go hand in hand with teaching geometry, the teaching goals associated with bringing Sketchpad on board [T-goals] can be expected to occur alongside the other goals of teaching [G-goals].

### 7.2 The Subordinate Goals

Similar to the g/G goal subordinations used in the earlier chapters, there are related subordinate t-goals. The overarching T-goals were those that were determined before the module began, shaped by my previous experience in teaching and research. The subordinate t-goals were goals that arose during instructional work in the module. The identification of t-goals followed the same method described in Section 4.2.2. Table 30 shows the t-goals detected at those regions of instruction where Sketchpad was used:

<table>
<thead>
<tr>
<th>tclf</th>
<th>tppd</th>
<th>tcrp</th>
<th>tcpo</th>
<th>tdrst</th>
<th>tlkg</th>
</tr>
</thead>
<tbody>
<tr>
<td>To clarify instructions for Sketchpad tasks</td>
<td>To prompt students towards productive directions in Sketchpad work</td>
<td>To teach the correct procedures for obtaining the intended onscreen results</td>
<td>To teach students to cooperate productively during pairwork while working on the computer</td>
<td>To teach the use of dragging/measure tools for investigating, checking, and testing of conjectures or construction procedures</td>
<td>To make links between onscreen phenomena/tasks and related geometrical terms, procedures, properties, and reasoning</td>
</tr>
</tbody>
</table>

Table 30: List of t-goals

While g-goals can be present at all points of teaching the geometry module, the t-goals—by virtue of their being tied to Sketchpad use—were only expressed in those regions of teaching when Sketchpad was used. The approximate duration of Sketchpad use in the entire geometry module is given in Table 31.
Table 31: Approximate lesson times in which Sketchpad was used in the module

Table 31 indicates that Sketchpad, in terms of time spent using it, featured most prominently in Lesson 9. This lesson also contained a richer variety of contexts in terms of how Sketchpad was used. The earlier lessons mainly focused on helping students gain familiarity with the software [T1], while Lesson 9—when students had become sufficiently exposed to the Sketchpad environment—addressed both T1 and T2. I will thus select snippets of Lesson 9 to illustrate how the six t-goals were carried out in my teaching.

7.2.1 Carrying Out t-goals in Lesson 9

In presenting the evidence of t-goals in the lesson, I proceed first with the main components of the lesson before zooming-in on details within each component. As for the g-goals in Chapter 4, the fine-grained analysis will only be done selectively. The purpose here is not to fully analyse the entire lesson; rather, it is to illustrate how each of the t-goals were evident during the lesson.

Lesson 9 was intended to be the culmination of the study of “special quadrilaterals,” started in Lesson 6, a development elaborated in Chapter 6. The main instructional components of Lesson 9 are shown in Table 32.
There were two main motivations behind the instructional activities in Component I. The first was to demonstrate the Sketchpad construction procedures for a square [tcrp], to prepare students to do constructions in Component II. The other goal was to connect students’ knowledge about squares to the task of constructing drag-resistant squares. I demonstrated the Sketchpad construction of a square using “circle” and “perpendicular lines”. In the process, I highlighted the corresponding critical attributes of “equal sides” and “perpendicular adjacent sides” in squares.

There are a few reasons why I do this activity …. I want to put a link between what they have been doing [in previous lessons on special quadrilaterals] and what they are going to do [in the next component of the lesson where they construct drag-resistant figures]. This is very helpful because this is to recall the geometrical properties and how the geometrical properties of a square are used in different [alternative] constructions [tlkg]. If I don’t do this at all they may just be very lost [later with the construction procedures]. So here is a demonstration of what they could possibly do when they do the first task on construct square [tcrp]. [Post-lesson reflections]

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3 The time allocated for each lesson is 70 minutes. In this lesson, there was a delay of more than 10 minutes before the lesson actually started because the teacher of the previous (music) lesson held the students back to complete the watching of a video.
In Component II of the lesson, students paired up to attempt the *Sketchpad* task of constructing four drag-resistant special quadrilaterals—square, rectangle, parallelogram, and rhombus, presented in that order in four documents within the same file\(^4\). The instructions and on-screen help for all the documents were similar. Figure 65 shows the rectangle construction task as an example of what the students were expected to do.

![Rectangle Construction Task](image)

**Figure 65:** The rectangle-construction task

The on-screen rectangle in Figure 65 was drag-resistant and was included to remind students that the rectangle they were to construct ought to be drag-resistant instead of merely a ‘by-eye’ sketch. The “Hint” at the foot of the screen gave guidance if students were ‘stuck’. The “double arrow button” provides a script that details the steps involved in a construction procedure. The details of this script function are given in Appendix 1.

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\(^4\) In the *Sketchpad* version that was used, multiple documents can be arranged in sequence within the same file.
The main goals I had in mind for Component II were captured in the lesson memo I wrote prior to the lesson:

… The Sketchpad [activity] is likely to offer tools like the drag-mode which can potentially help students to check their work on their own and allow them to retrace their construction steps so they can rectify the faults and try again … [emphases added]

I had, on numerous occasions in earlier lessons, demonstrated the practice of dragging to check for inaccuracies in construction procedures. In the students’ Sketchpad work here, I anticipated that students would use dragging during their construction procedures to “check their work”. This meant that I had tdrg in mind when I planned this construction activity. That I wanted them to “rectify the faults” also indicated that I anticipated student errors along the way. I wanted them to exploit dragging “on their own” to expose their errors as well as to find ways to rectify those construction errors. In other words, both gerr and grel goals were also intended in this lesson component. The mixture of t- and g-goals within a single component of the lesson is not surprising. As mentioned earlier, while T-goals (and hence t-goals) were only relevant in parts of the module where Sketchpad was used, G-goals (and hence g-goals) were operational throughout the entire module, including when Sketchpad was used.

In the final component of the lesson, I conducted a whole-class discussion with the students over five true/false statements about hierarchical relationships among special quadrilaterals. As t-goals were not prominent in Component III, the rest of Section 7.2 will focus on the first two components of the lesson.

7.2.2 Component I of Lesson 9
In the first five minutes of Component I, I drew/constructed three different squares on Sketchpad. I wanted to remind students of the difference between ‘by-eye’ drawing and constructing drag-resistant squares. I also intended to show an alternative correct construction. The construction processes of the second and third squares highlighted
side/angle properties and diagonal properties of squares respectively. The features of the three squares, in the order that they were constructed, are given below.

Square 1: Drawn purely ‘by-eye’.
Square 2: Used “perpendicular line” to construct perpendicular adjacent sides and “circle” tool to construct equality of sides. [Details shown in Figure 67, and the corresponding transcript]
Square 3: Used “midpoint” and “perpendicular line” to construct perpendicular and mutually bisecting diagonals; used “circle” tool to construct equality of diagonals. [Details shown in Figures 66.1 and 66.2]

I zoom-in further here to the construction of the third square. I performed the steps on the screen while providing a verbal commentary of the process. Figure 66.1 shows the on-screen visuals corresponding to the simultaneous verbalisations. Step [n] on the screen matches [Line n] of my talk.

The purpose of Lines 2-5 was to show students the Sketchpad construction procedures [tcrp]. In Line 6, there was a shift of focus from procedural steps to using the drag-mode to check the intended midpoint behaviour [tdrg]. In addition, Line 1 was intended to signal an alternative construction different from tools used in the first two square drawing/construction methods. The underlying goal was thus to consider alternatives [galt]. I continued constructing Square 3 with similar goals of
demonstrating procedure [tcrp] and utilising the drag-mode for checking [tdrg]. Figure 66.2 shows the remaining steps.

[Line 7] Select the midpoint, [pause] select the line segment.
[Line 9] Then I go to my compass,
[Line 10] put it at the centre and drag it to as long as this - half the segment.
[Line 11] Then I go to my ruler [pause] I join this, join this, join this, and join this.
[Line 12] Next drag to see if this remains a square.

Figure 66.2: Construction of the third square II

After constructing the three squares, I conducted a whole-class discussion on the unique geometrical features (or lack thereof in the case of Square 1) of each of the squares. The main purpose was to get students to articulate their observations [gart] and to connect the Sketchpad constructions to the related geometrical properties of squares [tlkg]. The transcript below illustrates how I carried out the tlkg goal in the whole-class instructional segment towards the end of Component I:

9.00   T: … in [square] number 3 I made use of what you say - the diagonals [pointing to statement “Diagonals bisect each other” written on board].
       T: I made used of the fact that the diagonals bisect [emphasis] each other. [Finger pointing at the diagonals one after the other] [pause]
       T: Not only that, [pause] they are equal [pointing to two diagonals]. I used the circle, remember, to draw [one index finger pointing to centre, the other index finger traced the movement of the circle] - so the diagonals are equal to one each other, and they are perpendicular.

7.2.3 Component II of Lesson 9

In the second component of the lesson, the students worked in their usual pairs at a computer to perform the prescribed Sketchpad constructions. I moved from pair to pair to observe their work and provide help when necessary. As described earlier, the broad goals I had in this component were tdrg, gerr, and grel. To illustrate how other emerging goals surfaced in my instruction, I zoom in to my exchanges with the pair Lyn and Shi Hui.
Before I approached Lyn and Shi Hui, I noticed that a number of students misread the instructions for the construction tasks. Instead of using the script tool merely as hints, they simply played the script to obtain the required Sketchpad figures. This was not what I intended. When I noticed that Lyn and Shi also did this, I clarified with them my expectations for the task:

17.13 T: [Looking at screen of Lyn-Shi Hui pair] How about you? Is the square working for you? This one [pointing at the square they constructed] is using this [pointing to script tool] to draw is it?
Both: [Nod head]
T: You got to try to draw it on your own.

Later in Component II, I returned to them to check if they were making progress with the square-construction task. I noticed they had attempted some constructions. Figure 67 shows their screen display after the following interaction:

30.39 T: [watches Shi Hui drags to test her ‘square’ and sees it’s not drag-resistant] Oops.
Both students: [smile sheepishly]
T: [smiling] That’s a rhombus, not a square.
[Teacher walks away because student Chai Hwa calls for his attention]
T: [returns to Lyn and Shi Hui] [sensing frustration] You need some help?
T: [takes over the mouse] Which method did you use? Method number 2 [used for Square 2] or number 3 [used for Square 3]?
Shi Hui: Perpendicular.
T: OK. You did this huh? [referring to segment KL already on their screen] Help me along, please. This one you know right? [selecting the “construct” menu] You want to construct perpendicular line [selects “perpendicular line” to construct line KM]. Would you agree with me - this one?
Both: [Nod head]
T: And then I do the same for this one right? [select “construct” menu] Construct perpendicular line [constructs line LN].
T: Then what I need is for this side [use two fingers to coincide at points K and L] to be as long as this [use same two fingers to mark on line KM] right? So I need the compass. Go to my compass [select “circle” tool].
T: You must put your pointer here right [click-once at point K]? Then you must make sure that the pointer goes to here [finger points at L] [pause, looks at Shi Hui, then at Lyn]. Then this [using two fingers to point at K and L] will be as long as this [using two fingers to mark along KM], agree? [click-once again to form circle centred at K]
Both: [Nod head]
T: I do the same. I put the pointer here [click-once at point L] and I go to the other side [click-once again at point K to form circle centred at L]
When I noticed that they had constructed a drag-resistant rhombus instead of a square, my first goal was to point out their error [gerr]. In addition, I could see from their body language that they were feeling rather frustrated at not getting it right. My post-lesson reflections reveal my observation that Lyn and Shi Hui were lagging behind the rest of the class at this stage:

When I was in the pair to pair [mode of instruction] I was trying to help. For those who were really struggling or lagging behind I tried to give more hints. Guide them along and see whether they can move towards a certain direction. … Where they are completely struggling, like Lyn and Shi Hui having their own struggles, I have to handle them one by one.

Thus, apart from gerr, I wanted to help them from being ‘stuck’ to “move towards a certain [productive] direction”, which is essentially a tppd goal. I did so by revising the key construction steps demonstrated with Square 2 in Component I. Here, I had in mind the teaching of construction procedures [tcrp]. Goal tlkg, too, was conspicuous. I was attempting to make connections between “perpendicular line” and “compass” constructions on the Sketchpad with the “perpendicular and equal adjacent sides” properties of squares. Of secondary importance, grel was also in operation, as reflected in my prompts for collaboration such as “which method”, “help me along”, and “would you agree with me”, and my stopping the demonstration halfway (at Figure 67) to leave them to complete it.
The descriptions of my goals of teaching above were based only on my interactions with Lyn and Shi Hui. One other goal that was not apparent in my discourses with them but was conspicuous when I interacted with other pairs was tcop. On numerous occasions in Component II, my interactions encouraged pairs to work cooperatively with their partners:

14.23 [Teacher stands at the front of the classroom. He sees Shi having a seemingly off-task conversation with Elsie – seated on the left of Shi but belonging to another pair]
   T: [walks over to Shi-Edward pair] Edward I think you need to give Shi more opportunity to try. She's so bored that she needs to chit-chat with Elsie along the way.
   [Edward nods while Shi gives a blank look]
   T: Can you [looking at Elsie] exchange seat with Rashid [who is seated on the left of Elsie and is her partner]?
   [Elsie and Rashid looks at one another, with reluctance]
   T: [Firmer tone, and with waving of his hand signalling switching]
   Exchange seats.
   [Rashid and Elsie get up from their chairs and exchange seats]

19.17 [Teacher notices Zhiyin is rather detached]
   T: Yuxin [Zhiyin’s partner] – Zhiyin you must be working too. Yuxin don’t dominate it. You must learn to share.
   [Both nod]

23.00 T: Wanxia … Wanxia. [in low tones, directed only to her] I already had a chat with you that day. [She was seated facing the front of the class and away from the computer screen. The teacher pushes her roller seat back to place and turn the seat so that she faces the screen] You need to put a lot more effort in your work. Even when Kai [her partner] is working you must help him along. You are a pair together so must help him along.

The analysis of Lesson 9 above reveals that all the t-goals listed in Table 30 were present. The next section examines the relationship between the t-goals and my overarching goals of teaching.

7.3 Relationship Between t-goals and the Overarching Goals

In Chapter 4, I explained how the g-goals are subordinate to my G-goals of teaching. The task here is to similarly relate the t-goals to my overarching goals.

Goals tclf, tppd, tcrp, and tcop can be seen to be subordinate to T1. These t-goals are about getting students adjusted to working with computers in general or with Sketchpad in particular. To help students become familiar with the Sketchpad environment, I need them to know clearly what is required in the Sketchpad tasks I set.
[tclf]. Students’ misinterpretation of tasks can affect their learning of Sketchpad operation. Prompting students towards productive directions in the Sketchpad work [tppd] helps students overcome their frustration of being ‘stuck’ in Sketchpad and become more familiar with the software. Similarly, the explicit teaching of Sketchpad procedures [tcrp], via mouse-clicks and specific keyboard controls, helps students become proficient with various tools in Sketchpad. Finally, teaching students to work in pairs [tcop] may not at first appear directly related to T1, yet there is a subordinate relationship: cooperation in pairs was necessary to their computer work. Since all hands-on Sketchpad work was done in pairs, a failure to work well together would jeopardise their work on the Sketchpad tasks.

Unlike the t-goals mentioned above, tlkg is about making links between on-screen work and the related geometrical terms, procedures, properties, and reasoning. It is about connecting work done in the Sketchpad environment to the “geometrical world” of objects and relationships. Thus, tlkg is directly supportive of T2.

The goal tdrg has features that relate it to both T1 and T2. The tdrg-T1 subordination is clear: dragging and measure tools are significant features of Sketchpad. However, tdrg is not merely about learning these functions per se; they allow students to check/test conjectures or construction procedures and thus think about the underlying geometrical properties [T2]. As discussed in Chapter 2, researchers like Kaput (1998), Chazan and Yerushalmy (1998) recognise the potential of dragging in helping students see the geometrical properties inherent in Sketchpad objects.

While t-goals are quite naturally related to T-goals, they can be subordinated to G-goals too. I assert that tdrg, for example, is supportive not only of T1 and T2 but also of G4. Through tdrg students come to use some Sketchpad tools to help them in investigating, checking, and testing of their work. These inductive approaches are important features of mathematical reasoning (see Chapter 2). Therefore, in teaching students to experiment with dragging and measuring [tdrg], I am guiding them to reason inductively within the geometrical domain [G4].

Another set of t-G subordinations involves tdrg, tlkg, and the goal of students’ van Hiele progression [G3]. Both tdrg and tlkg share an underlying objective of shifting
students’ object of focus from the gestalt features of objects (such as special quadrilaterals) to the inherent geometrical properties. In the case of tdrg, dragging distorts incident visual features but does not alter the invariant properties. This draws students’ attention away from non-critical features, like size and orientation, to critical attributes, like equality of sides and perpendicularity. Similarly, tlkg shifts the focus from visual phenomenon to geometrical ideas. These shifts of focus intended in tdrg and tlkg correspond to desired progressions in van Hieles’ levels of thought, particularly between Level 1 and Level 2.

Another t-G subordination is tcop-G5. One purpose of encouraging students to cooperate in pairs is to teach them the importance of bringing their partner into active productive work along with them. As seen from the vignettes in Section 7.2.3, I reminded students of their responsibility to get their partners to participate actively in the Sketchpad work. I wanted them to know that I valued not only their Sketchpad work, but also their efforts in not leaving their partners behind in the process [G5].

A summary of the goal subordinations between the t-goals to the T- and G-goals are given in Table 33.

<table>
<thead>
<tr>
<th>Overarching goals</th>
<th>T1</th>
<th>T2</th>
<th>G3</th>
<th>G4</th>
<th>G5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subordinate goals</td>
<td>tclf</td>
<td>tppd</td>
<td>tcpp</td>
<td>tdrp</td>
<td>tcop</td>
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<td>tdrp</td>
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<td>tcop</td>
</tr>
</tbody>
</table>

Table 33: Relationship between t-goals and T/G-goals

7.4 Studying Problems and Coping Strategies Involving Sketchpad Use

In the previous chapter, a chronological survey was used to investigate problems of teaching and the coping strategies by sequentially sampling the overall prominent turbulences in the module. This presents the reader with a continuing picture of the teaching problems and coping strategies set against the backdrop of module progression from lesson to lesson.
The purpose in this chapter is to analyse those regions of the module where Sketchpad was used and determine how that would add to or alter the picture of problems and coping strategies developed so far. As seen in the consideration of Lesson 9 (Section 7.2), goals-analyses showed not only t-goals at play but a combination of both g- and t-goals. Because of this overlap, the nature of goal interactions and turbulences were similar to that described in Chapter 6. A chronological re-surveying on a lesson-by-lesson basis used in Chapter 6 would thus duplicate results already found, which is not appropriate for the purpose here. Rather, only selected turbulences that significantly add to or alter the picture about problems and coping strategies will be discussed. These turbulent regions are not organised chronologically but are organised according to the type of instructional work I did while integrating Sketchpad.

7.5 Weaving Sketchpad Work into the Instructional Plan

In analysing the problems associated with Sketchpad use, actually weaving Sketchpad work into the instructional development of the whole geometry module was the most significant challenge, resulting in a substantial discussion here. I separate the analysis into two parts: the work of weaving Sketchpad before the start of the module, and during the module.

7.5.1 Weaving Sketchpad work into the Instructional Plan before the Module

The first trigger of tension I faced was the challenge of integrating Sketchpad use into the other components of the overall plan so that the development of the lessons within the module would be smooth. There were a number of developmental ‘tracks’ to consider. Firstly, I thought of the need to develop the geometrical content in a way that would preserve the logical Euclidean build-up from points, then lines, then angles, then triangles, and so forth. This intention to structure the content in such a way is captured in my reflection notes about module design:

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3 The reflection notes are part of the data collected under the category “Teacher reflections of classroom experience and planning” (see Section 3.2.5). All the other quotations in this Section 7.5.1 are taken from the same data source.
There are a few developmental ‘tracks’ to consider here. The first is the sequence of instruction in accordance to how the material is developed historically, i.e. define points, then lines, then parallel/perpendicular lines, triangles . . . . [I] see this historical track as important to be incorporated in the module as it carries the Euclidean spirit of deductive progression from axioms to derived results.

In other words, I wanted the module to reflect the development in geometrical reasoning from self-evident entities and assumptions to reasoned conclusions. Seen through my goals framework, this desire to model deductive reasoning is motivated by my G4 goal. For simplicity, I shall refer to this perspective on the module progression as the G4-track. To illustrate the impact of this G4-track on the ordering of module content, consider its influence on the sub-module of “Triangles”:

Following the [G4] track, one should attempt to link logically this idea of triangle to previously known definitions or results, then proceed to study properties of triangles based on this new definition of triangle. The progression should thus follow this rough order:

<table>
<thead>
<tr>
<th>Definitions of line segment and angles theorems (e.g., alt angle theorem)</th>
<th>Definition of a triangle</th>
<th>Examples of types of triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition of a triangle Some properties of triangles (e.g., the interior angle theorem)</td>
<td>Some properties of particular types of triangles</td>
<td></td>
</tr>
</tbody>
</table>

In addition to the G4-track, I was simultaneously mindful of whether the content progression was consistent with theoretical ideas about how students develop geometrical thinking:

Another track is one where pedagogical soundness drives the sequence of instruction. van Hiele provides a framework of dealing with geometrical objects—progressing from looking at them holistically to looking at them as holders of intrinsic properties to increasing levels of abstraction.

Having the van Hieles’ theory in mind brought the G3 goal into the module planning. Continuing the “Triangle” example, I thought of the development along the G3-track in this way:
PHASE III ANALYSIS

Examples of triangles of varying types → Explore properties of different types of triangles and relate them to one another and to previously-known results. → Lead to definitions of different types of triangles

It is clear that the two tracks do not move coherently in the same direction. They begin from different starting points. Reflecting the van Hieles’ “Recognition” stage, the G3-track starts with visual examples of different types of triangles to help students recognize the gestalt features of triangles. On the G4-track, the starting point is not a range of visual examples of triangles; rather, it is the notion that a triangle is a logical extension from the previously-studied ideas of lines and angles. The directions of movement along the tracks are also quite different. The G4-track derives properties based on definition of triangle; whereas the direction of the G3-track is somewhat reversed—from the study of properties (corresponding to “analysis” level) to arriving at a definition (corresponding to “deduction” level). Having incongruous tracks by itself does not necessarily lead to an irresolvable conflict in teaching. One way to overcome the apparent tension is to view some parts of these tracks as non-linear but cyclical and overlapping. As an example, I can teach by first moving from properties to arrive at definitions [G4-track] and then use these definitions to derive the properties [G3-track] to ‘close the loop’. In so doing, I may then be able to accommodate both of these tracks.

The difficulty, however, was not merely in the coherent sequencing of the tracks. Apart from G3 and G4 considerations, there was also the goal of covering all the required geometrical content within time limitations [G1]. This, in turn was exacerbated by the need to weave in yet another track that came about because of Sketchpad use:

There is yet the ‘DGS track’, where the unique features of the software require one who uses it to be familiar with its functions within the particular geometry domain-in-focus.

For students to use Sketchpad proficiently, I needed to familiarise them with the relevant features of the software as well as connect these skills to the “geometry domain-in-focus”. In other words, I had both T1 and T2 in mind when I thought of
the DGS-track. In goals language, the DGS-track represents the T1&T2-track. To fulfill those goals required a careful sequencing of the *Sketchpad* tasks with a view of supporting the G3- and G4-tracks of geometrical development. Guided by Laborde’s (2002) classification of DGS tasks of increasing complexities⁶, I laid out the T1&T2-track in the specific case of “Triangles”:

![Diagram of tasks](image)

While the developmental paths motivated by different goals are presented as different tracks above, it does not imply that they are altogether separate. For example, there are elements in the T1&T2-track that could support G3: dragging triangles in *Sketchpad* within the T1&T2-track is a way of exploring examples of varying types of triangles in the G3-track. This illustrates the potential usefulness of *Sketchpad* as described in the literature (see Chapter 2). However, not all elements of each track can be combined in this way. Proceeding along a single track alone would not support fully all the elements of the other tracks and would thus compromise other goals of teaching. The alternative of covering all the tracks sequentially would go against time limitations [G1]. I was confronted with the problem of not being able to completely fulfil all the goal-tracks within the allotted time.

My decision was to incorporate the flow of each of the tracks as much as feasible given the time limitations. In the process, I had to compromise certain aspects of each track. The final instructional sequence on “Triangles” appeared as follows:

---

⁶ The classification is recorded in Section 2.4.2.
In my attempt to merge the various tracks into one, I left out the build-up of triangles from lines as well as reversed the direction of development between examples of triangles and interior angle theorem of triangles in the G4-track; the formal “definitions” in both the G3 and G4 tracks were also removed; the direct step between “exploring” and “interpreting” in the T1&T2-track was broken up into two steps to accommodate the re-sequencing of instructional components; and I tried to pack in more without increasing class time, knowing that it risked the fulfilment of G1. My response at merging components from the various tracks then was not a solution to the problem. It was an attempt to accommodate each of the goals—G1, G3, G4, T1, and T2—by re-sequencing components, in hope that all the goals could at least be partially carried out. These merging and re-sequencing adaptations were coping strategies that I adopted which were not previously highlighted in earlier chapters.

7.5.2 Weaving Sketchpad into the Instructional Plan during the Module
The problem with weaving T1 and T2 goals into the other overarching goals was, however, not only confined to the module-planning stage of teaching. There were struggles during the implementation of the plan. The next trigger was detected early in the module at the end of Lesson 2. I felt that students were developing familiarity with both Sketchpad and the hardware more slowly than I expected.

Quite some time were spent on just getting them started on the computer work. These [issues] include telling them which key to press, what file to go to, getting the computer booted up. … I hope with every lesson, students will get more and more comfortable with both the software and the hardware. [Post-lesson reflections]

That awareness of their slow ‘pick-up’ triggered a desire to adjust my instructional behaviour.
... [A]fter the initial experience with them [two lessons], I think they still need a lot of guidance on Sketchpad. So I think I will demonstrate [first] how to do it on Sketchpad for them for the first task [before asking them to try on their own]. [I’ll] show what a ‘by-eye’ line is like and contrast it to a drag-resistant line, thus explaining what “drag-resistant” means. Increasingly, more time will be apportioned for their computer work ...

… [Memo for Lesson 3]

In the previous two lessons, I let the students read the instructions on the prepared sketches and work accordingly. In Lesson 3 I changed my instructional mode by demonstrating some Sketchpad features to them first, before letting them proceed with the sketches on their own. Also, I was prepared to devote more instructional time to students’ computer work. There was more time spent on Sketchpad in Lesson 3 compared to earlier lessons (see Table 31).

However, attempting to speed up the learning of Sketchpad tools did not seem to produce the intended results in Lesson 3.

I think they are still unfamiliar with some of the features of where the ‘pointer’ tool is, how to click on the ‘point’ tool to get a point, etc. [They are] still pretty much getting used to the software. That was also what I intended – to progress with the familiarity of the software [T1]. From the 1st lesson – tessellations, which was essentially just click-and-drag; to the previous lesson where they were dragging and observing numerical feedback; to this lesson, where they need to construct very basic things like parallel lines, perpendicular lines, putting in the midpoints. [Post-lesson reflections]

To my mind, the Sketchpad tasks in the first three lessons focused on progressive familiarisation with the various tools [T1] so that students can harness them to learn the intended geometry [T2] in subsequent lessons. Specifically, they would use the ‘measure’ and ‘label’ to do Template 1 in Lesson 4 (see Figure 68) and Template 2 in Lesson 5 (see Figure 69); and build on basic construction tools to do Template 3 in Lesson 5 (see Figure 70).
Figure 68: Template 1 used in Lesson 4

Figure 69: Template 2 used in Lesson 5

Figure 70: Template 3 used in Lesson 5
However, the realisation at the end of Lesson 3 that students were still unfamiliar with some Sketchpad tools created a problem for my plan for Lessons 4 and 5. If I pressed on with the original plan despite students’ inadequate Sketchpad proficiency, some slower students may not be able to perform the Sketchpad tasks [against G5] and they might miss out on learning the related geometrical ideas [against T2]. On the other hand, changing the instructional plan to accommodate the slower students risked derailing the entire developmental plan and the concomitant goals, especially the goal of content coverage [against G1].

I contemplated modifying Templates 1-3 to take into account the slower-than-anticipated pace of Sketchpad familiarisation in the earlier lessons, but this was weighed against some logistical challenges:

Based on my observations of their computer work in Lesson 3, they have yet to acquire the necessary comfort level with the software to move towards ‘free’ exploration of the templates [required in lesson 4]. But in a sense the Sketchpad progression [as spelt out in the module plan] can’t wait for them. Even if I want to modify the templates, it is not logistically easy as I have to pass them to the Lab technician (the only one who has access to saving files in the hard disk) in advance. The ‘easier’ option will be to save into the 20 diskettes and ask them to access the files from the diskette instead. That may break their routine … I guess what I will do is throw in as many helps as I can as ‘scaffolds’ to help them get going at stages where I anticipate they will meet with difficulties. [Memo for Lesson 4]

My decision was to stay with the Sketchpad templates I originally prepared for them. I intended to cope by “throw[ing] in as many helps as I can” during lesson time to bridge any gaps between their proficiency level and what was required by the Sketchpad tasks. This improvisation would not solve the problem of conflicting goals, but was conscious ‘damage control’ to help struggling students.

As it turned out, Lesson 4 went better than I had expected. I was glad the students’ overall proficiency with the Sketchpad environment improved.

I thought the students got on with the computer work rather smoothly this time. In hindsight, the time invested before sending them to the computers to show them how to
do the measures paid off. Not many students asked me how to do the necessary measures when I walk from computer to computer later. On the whole the computer part of the work is satisfactory . . . . [Post-lesson reflections]

However, another trigger surfaced at the end of Lesson 4: I was not able to finish one component of the lesson—demonstrating some textbook exercise items—thus compromising the G2 goal for the lesson. [The turbulence leading to this situation was analysed in Turbulent Region 4 in Section 6.4.] I wanted to carry that missed-out component into the next lesson, as shown in my memo for Lesson 5:

In the last lesson, I did not have the chance to show examples where the triangle theorems are applied. It is important, I believe, not to assume the students can pick up from my introduction [on these triangle theorems] and do the problems [on their own]. I will show two examples – Ex14A Qn 9(k) and 9(l).

However, inserting an unplanned-for component into Lesson 5 brought about the now-familiar problem of time pressure. At that point, instead of choosing the route of goals-dilution, I actually decided to remove an existing component in Lesson 5 to make way for the incoming component of “textbook exercises”. In other words, I had to prioritise between the goals associated with incoming component and the original goals intended for Lesson 5. I decided to leave out Template 3 (see Figure 70) to make way for “textbook exercises”. From the goals perspective, I compromised on T2 in order to prioritise G2. That choice had to do with my knowledge that Sketchpad constructions would be visited again later in the module when dealing with quadrilaterals.

Nevertheless, I was reluctant to give up the T2-goal altogether. As part of my coping strategy, I planned to partially address Template 3.

… However, students who have completed [the earlier Sketchpad tasks] and have the time, I will encourage them to proceed to try the construction of the special triangles [Template 3]. [Memo for Lesson 5]
I was aware that not all students would have the time or ability to reach Template 3 in Lesson 5. My plan was to provide some who did get there the opportunity to work on it, providing partial fulfilment of the T2-goal. [Recall a similar situation in Turbulent Region 6 in Section 6.6.] My post-lesson reflections reveal that my improvisations helped provide more quality Sketchpad time but confirmed my fears that most students would not go far enough in the Sketchpad task to get to Template 3:

The time allocation for the Sketchpad work today was good. There were some students though who seemed to rush through the initial [Template 2]. I pointed them to the more challenging [Template 3] that required them to construct the respective triangles. That proved to be too difficult for them though.

The entire turbulent region about weaving Sketchpad use into the instructional plan started with the first trigger of considering multiple developmental tracks (described in Section 7.5.1) and lasted right through to the end of Lesson 5. Issues associated with integrating Sketchpad into Lessons 6 – 9 are similar to those described here, so I will base the discussion on problems and coping strategies related to Sketchpad-integration on the turbulent region described above.

The turbulent region began with the problem of negotiating different developmental ‘tracks’, each motivated by a dominant overarching goal. I could not find a way to keep each track intact while fulfilling the goal of timely syllabus-coverage. I dealt with the problem by enmeshing the tracks into one while trying to preserve the general flow of each of the contributing strands. The next trigger arose after Lesson 2 when I sensed that students’ Sketchpad development lagged behind other elements of the developmental track. I reacted by giving more directed instructions as well as giving them more time on Sketchpad work in Lesson 3. After Lesson 3, I evaluated the situation again and was still dissatisfied at students’ level of familiarity with the software. Nevertheless, after considering a number of options, I pressed on with the original plan for Lesson 4 and tried to cope by anticipating their difficulties in advance and giving as much in-class help as possible. The post-Lesson 4 reflections reveal my satisfaction with their Sketchpad progress but another issue—that of uncompleted component in Lesson 4—triggered a re-organisation of Lesson 5 that led to the problematic situation of paring down the triangle-construction task originally
planned for the lesson. The overview of the turbulent region is presented in Figure 71.

Figure 71: Turbulent region about weaving Sketchpad use into the instructional plan

7.5.3 Problems and Coping Strategies in Relation to Weaving T1 and T2 into the Instructional Plan

Without doing a point-by-point evaluation of the conjectures (as this was amply demonstrated in the previous chapter), it can be seen that the nature of turbulence shown in Figure 71 supports P1-P6 and C1-C5. Thus, the broad-grained analysis of the goals-in-operation in integrating T1 and T2 into the instructional plan reveals many similarities with the turbulent regions reviewed earlier. The focus in this section is on other contributions to the study of problems and coping strategies that were not previously discussed.

From the account of the turbulence, the structuring of T1 and T2 goals into the curriculum was not at all straightforward. Each Sketchpad task had to be sequenced so that students could make steady progress learning the relevant software tools to fulfil my teaching goals of T1 and T2. Also, students’ development of Sketchpad proficiency cannot be considered in isolation from the other developmental tracks. Sketchpad components from the T1&T2-track had to be fitted in and timed with the flow of components from other tracks. Where relevant, Sketchpad elements were
merged with other components. Problems occurred when there was no easy way to completely merge all components and preserve the intended sequence of each track while keeping to time. I coped by retaining as many elements of the respective tracks as possible in the final plan.

The tension of balancing the goal-tracks was not done once only at the beginning of the module. Throughout the implementation of the plan, the actual instructional situation did not follow exactly the original plan. For example, the T1&T2 track sometimes lagged behind the instructional sequence. This was a problem because the pace of the T1&T2 track was tightly linked to the G-tracks. Slowing the T1&T2 track meant that the progression of the other tracks (and the concomitant goals) was affected too. The way I adjusted to students’ slow pick-up of Sketchpad was to clarify anticipated difficulties prior to students’ computer work and to give them more computer time to gain Sketchpad proficiency in order to catch up with the overall pace of geometry progression.

Including Sketchpad in the curriculum amplified the problem of time pressure in an already packed teaching schedule. The inclusion of Sketchpad components in the module exacerbate demands on this valuable resource. Thus, while Sketchpad offers potential benefits in facilitating better learning of geometrical ideas (as reviewed in Chapter 2), one real challenge is to do so without drawing substantially from the available instructional time.

It would not be an accurate picture, however, to view Sketchpad time as merely addition to instructional time. In planning, I did not just add on Sketchpad work over and above time for other module components. Rather, I attempted to weave Sketchpad tasks tightly with other instructional components so that the total time taken to carry out the merged track was less than the aggregated time for each of the component tracks. Moreover, Sketchpad time can also be seen as an ‘investment’. Students may take some time to become familiar with new software, but when they attain the required proficiency, they may be more time-efficient in their learning because Sketchpad tools afford them quicker exploration compared to conventional paper-and-pencil instruments. This seemed to be borne out in the account above. When students were familiar with Sketchpad in Lesson 5, they took only
approximately thirty minutes on the classifying activity (see Figure 69). The activity would likely take a longer time to complete in a non-Sketchpad environment. Thus, more computer time in earlier lessons paid off when Sketchpad helped save instructional time later in the module.

7.6 Students’ Hands-on Work with Sketchpad

In this section, the focus will turn to another challenge—that of helping students work directly with Sketchpad at their shared computers. While Sketchpad was sometimes used in a whole-class instructional setting, it was mostly employed as a tool for students’ hands-on work. Table 31 showed the amount of time in each lesson where Sketchpad was used; Table 34 includes an additional column to indicate the durations where Sketchpad was used by students directly for hands-on tasks.

<table>
<thead>
<tr>
<th>Lesson number</th>
<th>Duration of entire lesson (in minutes)</th>
<th>Approx duration (in min) when Sketchpad was used</th>
<th>Approx duration (in min) for students’ hands-on work</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>57</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>2</td>
<td>71</td>
<td>17</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>62</td>
<td>34</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>68</td>
<td>26</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>63</td>
<td>34</td>
<td>26</td>
</tr>
<tr>
<td>6</td>
<td>62</td>
<td>33</td>
<td>33</td>
</tr>
<tr>
<td>7</td>
<td>64</td>
<td>23</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>65</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>54</td>
<td>44</td>
<td>31</td>
</tr>
<tr>
<td>10</td>
<td>36</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>66</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>668</td>
<td>229</td>
<td>177</td>
</tr>
</tbody>
</table>

Table 34: Duration of time for Sketchpad hands-on work

The main reason in providing such a large proportion (more than three quarters) of Sketchpad time for students’ hands-on work was that this mode better accesses the potential of the software. More specifically, the useful features of Sketchpad such as “ease of experimentation” and “drag-mode” (discussed in Sections 2.4.2 and 2.4.4) are better utilised when students worked directly with the software rather than through the mediation of the teacher. Moreover, when students worked on Sketchpad they did

“Hands-on” means direct working with computers in contrast with indirect access such as viewing teacher demonstrations.
so as a pair, sharing a single computer, providing opportunities for students to articulate their observations and discuss their conjectures as a way to support reasoning.

The actual conduct of the hands-on mode was, however, not straightforward, with various problems associated with letting students work directly on Sketchpad tasks. The purpose here is not to list every problem encountered. I shall restrict the discussion to those more significant problems that occurred more than once and that provoked conspicuous tensions. I discuss the different challenges I faced—organised by the type of goals—in Sections 7.6.1 to 7.6.3 before summarising the problems and coping strategies experienced in Section 7.6.4.

7.6.1 Teaching Students Self-reliance During Their Hands-on Work with Sketchpad

In the first lesson, I focused on helping students become familiar with the basic features of the software. The main Sketchpad skill I taught them was to use the pointer tool to move objects. Figure 72 shows a sample Sketchpad task from Lesson 1. Since the students did not have prior experience with the software, I expected students to repeatedly ask for help. I was thus prepared to do the work of clarifying Sketchpad tasks to students [tclf].

Instructions:
Below are some equilateral triangles.
Assume they are the tiles that you consider using to tile your bedroom floor.
Re-arrange the tiles below to show how the tiling can be done, if possible.
[Use the pointer to move the tiles. To rotate a tile, use the pointer to rotate the vertex of the triangle]

![Sample Sketchpad task](image)

Figure 72: Sample Sketchpad task in Lesson 1
I was surprised, however, that students’ difficulties were not directly about *Sketchpad* features but something more basic. A number stared at the screen awaiting further instructions from me, even though the instructions were already written on-screen (see Figure 72). It was certainly possible that the written instructions were not clear enough for the students, but the greater problem seemed to be that students were not even reading the written instructions.

Quite a number seemed to be not very clear and asking for confirmation. Perhaps my instructions were not clear …, or simply they are not learning to read instructions? Could this be one of the problems? [Post-lesson reflections]

That students “read instructions” seemed to me a very basic self-reliant responsibility [grel]. But clarifying instructions on *Sketchpad* tasks [tclf], especially at such an early stage of *Sketchpad* use, was also my goal. I could have just asked them to read the instructions for themselves [in favour of grel], but doing so risk discouraging students at their very first attempt at hands-on work. I decided to give them the benefit of the doubt and clarified the task from table to table whenever student pairs looked uncertain [in favour of tclf]. The following vignette shows a typical exchange I had with students:

37.30  T: [saw Syarah and Zhanwong looking unsure] What happened? You try to use this [pointing to the tiles on-screen] - imagine these are tiles on your floor.
Zhanwong: Ah ah [meaning, "I'm following"]
T: You want to tile up your whole kitchen using these tiles. Can you do that? Let me give you an example [takes over the mouse] Let’s say you put this here and you [moves tiles on-screen] … Oh! You go to this and you rotate [rotates tiles]. Can you see that it rotates?
Zhanwong: Yeah.
T: You put them together [showing on-screen]…
Zhanwong: Ah ah.
T: You continue to do that [hands gesturing an expansion horizontally] …

The grel versus tclf conflict became more evident in the second lesson. The first *Sketchpad* task involved moving two lines to a stage where the sum of the interior angles bounded by the transversal equalled $180^\circ$ (see Figure 73).
Despite the instructions being clearly written, Shamu appeared not to have read or understood them:

51.52  
Shamu: [raised hand to signal for T’s attention]  
T: [walks over] Yes, Shamu?  
Shamu: Don’t know whether this angle [pointing at his screen] correct or not?  
T: You are supposed to make angles p and q add up to 180. Now the angle is 200.7. So you got to make the two lines - these two lines [pointing on the screen] such that the red angle and the yellow angle add up to 180 degrees.

A similar failure to read or comprehend instructions was detected in student Edward’s question about the second task (see Figure 74). The related transcript of my dialogue with Edward reveals the tensions I experienced.
Edward: Cher what are we supposed to do here?
T: Follow the instruction. [pause]
T: You are supposed to observe as many pairs of relationship as possible. I gave you one example here [pointing to example on screen]. I noticed that this green angle is always equal to the purple one [pointing], because -
Edward: Then what are we supposed to do?
T: They are vertically opposite angles. So give a lot of this kind of - er, make as many observations as you can.
Edward: We are supposed to write down?
T: For example, the blue angle is always equal to the yellow angle [pointing], and what is the reason? Because they are vertically opposite angles.

In my exchange with Edward I was beginning to find students’ dependence on me for readily-available instruction and confirmation rather uncomfortable. By directing him to “follow the instruction” and not answering his questions directly, I was signalling a shift towards helping students take more personal responsibilities [more grel] instead of over-relying on the teacher to tell them exactly what to do [less tclf].

The deliberate move to promote grel became more obvious in the third lesson. I introduced the idea of “drag-resistance” and demonstrated its use in checking the correctness of constructions. During the students’ hands-on work, I avoided pointing out the errors for them [gerr]; instead, I used the “drag-test” as a device to prompt students to check the constructions for themselves [tdrg]. In so doing, I meant to shift the responsibility and authority for checking from the teacher to the students themselves [grel]. In other words, tdrg [teaching the use of dragging] served the function of a stepping stone to grel. Following are some samples of such uses of dragging in Lesson 3:

34.23 T: [looks at Syarah’s work] You check whether it is drag-resistant. Remember how to check?
Syarah: [drags, revealing the error in construction]
T: Ah …

38.36 Ihsa: [raises his hand]
T: Yes? [and walks over]
Ihsa: [inaudible]
T: You follow the instructions carefully? Which one?
Ihsa: [clicks onscreen]
T: Whether it is correct or not you can check by dragging [shows the hand gestures of the dragging movement]
Ihsa: [drags accordingly]
T: Yeah, yeah, it works.
The purpose of urging students to use the “drag-test” in relation to encouraging students to be less-reliant on the teacher for answers [grel] is summarised in my post-lesson reflections:

I meant to pass the responsibility of checking drag-resistance to the students (instead of relying on me to tell them). A number of them asked me as I stop along their desks whether their constructions are correct (i.e. pass the drag-test). I threw the question back at them and ask them to check for themselves by dragging, in hope of training in them a habit of self-check instead of relying on the teacher as an external authority.

The turbulent region surrounding the struggles with grel is summarised in Figure 75.

Interestingly, beginning with Lesson 4, there seemed to be a reduction in students relying on me for simple directions. This may have to do with the instructional changes I made to help students become familiar with the software, reported in Section 7.5.2. My post-lesson reflections at the end of Lesson 4 indicated that “[t]oday, for computer work, they seemed to be progressing well and seemed to know what to do”. At the end of Lesson 6, a more balanced but still encouraging assessment emerged:

I took a risk (so to speak) to let them go into computer work straightaway [today]. It’s not the usual [way, where I conducted] whole-class giving [of] guidelines [on what to do]. I thought they have reached a level of comfort with the Sketchpad for them to just look at the instructions and do it. I have mixed feelings after this. On one hand, I think they [still] have not gotten used to reading the instructions from the template and they
just plunge into it not knowing what is expected. But [on the other hand] a few of them—in fact, a majority of them—have begun to read instructions and know what is expected and then do accordingly. [Post-lesson reflections]

7.6.2 Teaching Students to Experiment During Their Hands-on Work with Sketchpad

One of the reasons for getting students to work hands-on with Sketchpad is that it encourages students to experiment for themselves instead of following a fixed set of procedures. I thus built in flexibility into the templates. The intention was that students would not be inhibited from trying their own ways of making sense of the on-screen visual phenomena. Students could easily use tools such as retracing constructions steps [via “undo”] and testing of conjectures [via dragging and “measures”] to make multiple attempts on the Sketchpad tasks [gtry].

My first sense of the problems related to this gtry goal was in Lesson 3. Prior to giving them their hands-on task in that lesson, I demonstrated to the whole class the Sketchpad construction of parallel and perpendicular lines. They were then asked to try some Sketchpad tasks in their pairs. Screenshots of the first two tasks are given in Figure 76.

![Instructions:](image1)

- **1st task: “parallel line”**

![Instructions:](image2)

- **2nd task: “parallel line segment”**

Figure 76: Screenshots of the first two Sketchpad tasks in Lesson 3

I walked from computer to computer to monitor their work. Most of them were able to complete the first task. They had difficulty, however, with the second task. While I wanted them to make persistent attempts [gtry] at the task, I could sense that their “frustration” was building up:
PHASE III ANALYSIS

I walked through one round of helping the students at the computers. Can sense their interest level is waning because of frustration. [Post-lesson reflections]

The problem I faced was this: I knew that if I intervened at that point to show them the correct construction steps [tc rp], that would remove their frustration. However, that would go against the spirit of “keep trying” that I wanted to inculcate [counter gtry]. But letting them continue struggling may heighten their frustration, leading to lack of interest or even discouragement [counter gcon]. It was a situation of conflicting goals. I decided to let them keep trying [gtry] for a while more, but as a form of ‘damage control’. I monitored their frustration level and timed my demonstration of the second task [tc rp] only when I sensed that their frustration was almost too overwhelming. I was trying to balance the gtry and the tcrp goals between encouraging them to keep experimenting and managing their frustration. When I finally showed the construction at the front the class, I made clear that my help was meant as a form of ‘relief’ only to those who needed it, signalling to those who did not need the help to keep trying. This shows my continuing tension in balancing both the gtry and the tcrp goals.

I deliberately therefore timed the moment for me to give them some helps by way of appropriate ideas to ‘pump-prime’ their efforts. So I invited those who wanted to see to watch how I do the “transform” part. For those who have no idea at all, it is hoped that it can kickstart them into the next level. [Post-lesson reflections]

The second time I sensed the tension relating to gtry was in Lesson 5. Students were asked to work on the Sketchpad file entitled “types of triangles”. Figure 77 shows the effect when dragging is applied to objects on the screen of this template8.

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8 This template looks similar to Template 2 in Figure 69 but they are actually different tasks. Students did this template before moving on to Template 2 in Lesson 5.
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Figure 77: Screenshots of “types of triangles” before and after dragging

Again, I expected students to uncover the properties of the respective special triangles. However I noticed that their discovery routes did not quite lead to what I intended. The interaction I had with Hassan and his partner Malana illustrates the directions students took:

38.04 Hassan: These two stays the same [pointing to two vertices of a triangle on-screen]
T: Which two stays the same? For obtuse-angled triangle?
Malana: For obtuse-angled triangle the base always the same.
T: The base always the same is it - you can’t change it?
Malana: Yeah.
T: Let’s try. [Takes over the mouse] Is it true? No [as he drags]. It becomes smaller right? Can you see?

The problem was that while I wanted them to rely on themselves to experiment [grel and gtry], students who did not move productively towards the curriculum goals in the process of exploration lost time [counter gcbt] and experienced frustration [counter gcon]. I needed to balance the grel and gtry goals against the goal of helping students towards productive directions in their Sketchpad work [tppd], “that they are not taken too far away towards unproductive paths” (post-Lesson 5 reflections). In the case of Hassan and Malana, I tried to re-direct them to a more productive direction but without telling the features of “obtuse-angled triangle” directly. The purpose was to carry out tppd without completely abandoning the spirit of grel and gtry:

38.22 T: [to Hassan and Malana] What is the thing that always doesn’t change? In this obtuse-angled triangle, do you notice? Do you try to measure some of the angles and observe something? Because the name is obtuse-ANGLED [emphasis] triangle right? [pause] Maybe the trick is in the angle, right?
The turbulent region surrounding my attempts to teach gtry while monitoring students’ level of frustration is summarised in Figure 78.

Figure 78: Turbulent region relating to helping students experiment with Sketchpad

### 7.6.3 Teaching Students to Work in Pairs During their Hands-on Work with Sketchpad

During the hands-on segments, students were paired to work on the Sketchpad tasks. Apart from the logistical constraints of 20 computers in a class of 40 students, pairing them was intended to provide opportunities for students to articulate their conjectures and refutations one to another [gart]. However, using pairs also created challenges for students in learning to cooperate with one another⁹.

Up until Lesson 5, I noticed cases where one member of the pair dominated the Sketchpad work. On those occasions, I urged students to take turns as a way to learn sharing and cooperation in pairwork [tcp].

This aspect of them not sharing their time within the pair is also a problem [counter tcp]. I have chosen not to stipulate strict time shares between the partners in order not to over-mechanise the procedure. I also wanted them to learn [on their own accord] how to share their time with one another and to cooperate. [Post-Lesson 5 reflections]

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⁹ Students’ difficulties in cooperating might have to do with their unfamiliarity with working in pairs. Prior to this module, the primary mode of classroom organisation used by the resident teacher was whole-class instruction.
The problem of cooperation within pairs was noticeably more serious in Lesson 6 where the task was to classify special quadrilaterals. There were students who were visibly uninvolved with what their partners did during hands-on work. I interpreted that phenomenon to be a degeneration of intra-pair cooperation. I therefore took a more assertive stance by instructing these students to physically switch positions so that the passive partner was nearer the computer:

19.52  T: [to Edward-Shi pair] Umm … Rhombus and rhombus [referring to their labels onscreen]? [takes the mouse and drags] remember you have to give each only one name …
Edward: Square!
T: Square?
T: [notices Shi was not looking] Are you the only one doing or Shi also contributing [bends down to look at Shi]? Is she only sit and watch?
Edward: She only sit and watch
T: Huh? [to Shi] You only watch and learn?
Shi: [Looks sheepish]
T: OK I want Shi to do the next one now [as he clicks the next template]. [to Edward] Can you exchange seats and let Shi try this one?
[they move to exchange seats]

24.03  T: Yes Syarah [walks quickly over]? Oh this looks much better. Convinced? Good. Can you move on to the next one …? Very good [takes the mouse and clicks to the next file]
T: [Notices Zhanwong looking detached from what Syarah doing] Zhanwong [walks over and puts his hand on Zhanwong ‘s head], you are supposed to help … [to Syarah] You’re doing it all the time is it? Shall we exchange seat now? Come Zhanwong [putting his hands on his shoulder, as if guiding him to stand up]. [To Syarah] Exchange seat and let Zhanwong try on the computer as well. [non-movement, both reluctant to exchange seats.]
T: [Firmer tone, looking at them, with finger gestures showing circular movement] Exchange seats, c’mon. [Movement. Zhanwong looks lethargic, but moves.]

My change of mode—from instructing students to share their computer time to the rather forceful approach of getting students to change seats in Lesson 6—indicated that the problem of non-cooperation within pairs had apparently degenerated. But the challenge I faced in engaging “detached” students like Zhanwong paled in comparison to the more serious problems in Lesson 9. In that lesson, a shouting match broke out between Wanxia and Kai and the working relationship between Xiao and Karwai appeared to break down completely:

32.03  T: [Heard some commotion between Wanxia and Kai ]
Wanxia: [To Kai, in Mandarin] You don’t know how to do lah
Kai: [To Wanxia, in Mandarin] You shut up!
T: Wanxia, c’mon, you must learn to partner whoever you are with. [Notices she is
apart from her desk, and so pushes her roller chair in again] Can I see your square? Is it drag-resistant? Doesn’t look like to me …  
Wanxia: [To Kai, in Mandarin] You see! [as in blaming Kai for error]  
Kai: [In Mandarin] Shut up!  
T: You must help one another. [Pause, notice they stop shouting before moving off]

35.23 Karwai: [To teacher] You separate both of us. Everything I do— You separate us lah.  
T: You need to cooperate with each other. Xiao, what happened?  
Xiao: [Looks moody] She insult me.  
Karwai: What I do?  
T: One of the things you learn in pairs is to learn to cooperate with each other. You [looking at Karwai] finish rectangle then Xiao take over  
Karwai: I do finish square and rhombus already you know.  
T: Two of you can't cooperate then how to work together …  
Karwai: So you separate us?  
T: This is the last time we're going to work on computers\textsuperscript{10}. Separate already then what are you going to do?  
[No response]

I tried in both cases to divert their attention away from their disputes back to the \textit{Sketchpad} tasks in the hope of getting them interested in the computer activity. From the emotional outbursts (in the case of Wanxia and Kai) and the extent of the cold war (in the case of Xiao and Karwai), I realised that the problems they had were not something I could solve on the spot. What I tried was really to calm them and prevent the outbursts from degenerating further. I left them alone for the rest of the lesson and spoke to them only after the lesson. My post-lesson reflections uncovered some underlying reasons for the breakdowns as well as how I coped with the problems:

And today there was another problem: Xiao and Karwai. That was a big disappointment. They used to be a team that moved very quickly but I think the relationship broke down today. Things weren’t quite right. … This is surprising because Xiao is usually very engaged. … but I just left it and concentrated on helping Karwai instead, who was the one working on it [the computer]. To some extent I chose to do that because I didn’t want to get involved [in a] protracted [way] because I want to spend more time on others, at the same time I know Xiao, left on his own, he can take care of himself mathematically. … [Judging from previous lessons] he certainly didn’t have any problem with the \textit{Sketchpad} requirements. And of course only later that I got to realise that he was trying to help Karwai perform the constructions but she sort of spited him and said very insulting harsh words on him that totally put him off.  
Another relationship broke down was Kai and Wanxia. And it has to come at a time when it is the last [computer] lesson. As a teacher I want to handle these but only in a

\textsuperscript{10} There were no \textit{Sketchpad} components in Lesson 10 and 11.
limited fashion. Knowing at the back of my mind – “Is it worth this investment? It is already the last computer lesson [and they are not going to work in pairs again]”.

Although teaching students to work productively in their pairs was one of my goals [tcop], I acknowledged at that point that it was not worthwhile to pursue that goal further since the students were clearly not ready for it. There were also other priorities such as spending time on other students. Moreover, the knowledge that the damage would likely be limited—it was the last computer lesson and Xiao was capable on his own—contributed to the decision.

It was not clear to me, however, why cooperation difficulties appeared to be increasingly severe with time after Lesson 6. Perhaps difficulties in getting along existed from the beginning but took time to surface. Tracing how the Xiao-Karwai and Wanxia-Kai pairs might develop beyond this bust-up could possibly shed more light on the longer term implications of pairwork on Sketchpad tasks. This study over eleven lessons is admittedly inadequate in addressing issues that inherently demand scrutiny over a longer timeframe.

7.6.4 Problems and Coping Strategies in Relation to Students’ Hands-on Work with Sketchpad

In keeping with the educational potential of Sketchpad, I wanted students to use the software in a way that would help them become more independent learners [grel], support their experimentation [gtry], and provide opportunities to discuss their observations with their peers [gart]. As reported above, Sketchpad indeed provided opportunities to realise these goals. Nevertheless, carrying out these goals during students’ hands-on sessions with the software was by no means straightforward. For the first three lessons, students exhibited strong dependence on me to provide detailed instructions on the Sketchpad tasks. I had to pare down my grel expectations and focus more on tclf. Similarly, for gtry, students were observed to become “frustrated” at unsuccessful attempts; I had to monitor the “frustration level” carefully and time the hints (in the form of tppd and tcrp goals) to alleviate discouragement. The intention of using pairwork to enhance discussion [gart] also met with problems of students having difficulty cooperating with their partners. In some cases, the intra-pair conflicts completely distracted the pairs from the Sketchpad tasks.
The nature of inter-goal conflicts experienced during the hands-on segments of the module was similar to that in other parts of the module. In other words, although Sketchpad is a valuable tool for teaching geometry, it did not remove goal conflicts from teaching. In allowing students to do hands-on work with Sketchpad, teachers cannot take as ‘givens’ students’ propensity to figure out instructions, their patience at making multiple tries, nor their ability to work well with their partners. Rather, each of these required qualities will need to be deliberately considered as goals of instruction in their own right and carefully inculcated in the students over time.

But to paint Sketchpad use as presenting just problems would not be an accurate picture. Sketchpad did provide additional coping resources that were not available in conventional instructional modes. I used the Sketchpad device of dragging as a way to help students check for themselves whether their constructions or conjectures were correct. The advantage of the drag mode is that the invariant or in-built attributes of the figures become immediately and visibly exposed. Students can therefore use dragging at any point in time to check their work. This can help towards the fulfilment of goals.

This Sketchpad feature of exposing errors was not only useful for the students; it was useful for the teacher as well. Because of the convenience of detecting students’ errors via mere observation or a quick drag-check, I had a good sense of students’ approaches and that provided me with inputs to decide when to give help. Moreover, I found Sketchpad a useful medium to help me point out students’ errors without telling them the actual answers, striking a balance between gtry and tppd-terp demands. This use was evident in my interactions with the Hassan-Malana pair. Dragging provided the hints for where they went wrong but did not give the answer directly. Students still had the responsibility to work out the correct approaches.

7.7 Hardware Matters

In anticipating problems of teaching involving Sketchpad, hardware issues did not at first attract my attention at all. In using computers in my personal capacity, I had hardly encountered any difficulties with hardware malfunction, so computer glitches
did not naturally feature as a significant problem. However, upon deeper scrutiny, it became clear that since Sketchpad use was accessed through computer hardware, it was unavoidable that the work of teaching involved dealing with issues directly related to hardware. Insofar as hardware glitches interfered with teaching significantly, they constituted actual problems of teaching. Nevertheless, as the focus of this study is more directly on Sketchpad and not on the supporting hardware, only a brief treatment on the latter will be given in this section.

Throughout the entire module technical glitches surfaced and interfered with classroom work. There was, in particular, one episode associated with Lesson 4 where difficulties posed by hardware malfunction caused substantial tension. I begin the discussion here on problems and coping strategies in relation to computer hardware with a description of the turbulence surrounding that event.

As per my usual practice, I went to the classroom about 10 minutes before time to get ready for the lesson. I was disturbed, however, when the screen did not light up at the teacher’s computer monitor. I responded by checking the related switches. All those switches were in the “on” position. Suspecting that there was partial power failure to the classroom, I checked the students’ computers and was further surprised to find them not working too. At that point, I had to decide between a number of options:

Option 1: Continue fiddling with the switches, knowing I might not get the power up before the students come in, especially when the main switch may be located outside my view;
Option 2: Abandon use of computers for the lesson, knowing that would result in very significant last-minute changes to the instructional plan;
Option 3: Get help from the technical staff, knowing they are not easily reached since their office was on the second level while I was on the third level of an adjacent block.

Despite the inconvenience and time pressure, I decided on Option 3. The tension build-up and the consequences of that move were captured in my post-lesson reflections:
I am not too happy about today’s lesson really. Technical problem was one [cause] of it. Before the lesson started I thought the power supply was down for the particular [teacher’s] computer. But I tried the other neighbouring ones but they also do not work. Except for the air-con and lightings, all the power supply points do not give electricity. So I went to the next block to see the technician. I thought he would come over to help. But all he offered was suggestions like “do you see the box there …”, not volunteering to come over to the lab to help. I got to get the class going so I have to sort out the problems.

Since I was not a resident teacher of the school, I was naturally unfamiliar with the Math lab. When the technician gave verbal directions about switch boxes to try out in the classroom, I was not confident that I could find the relevant switches to rectify the technical problem. Nevertheless, seeing he was reluctant to come with me, I had no choice but to return to the classroom and continued fiddling with the switches. Thankfully, I managed to click the right master switch—unlabelled and located a distance away from the teacher’s computer—that supplied the power to the computers. The situation would have been far more tense and problematic if the glitch was not sorted out before the arrival of the students.

The hardware-related difficulty, however, did not stop there. I continued my “doubling up as technician” work during the lesson as well:

During the lesson itself, a few students approach me to say that their computer screens were not showing anything. My immediate reaction internally was, “Oh no, am I suppose to solve their technical problems as well?” I wanted initially to provide the laptop for one pair. But when I realized that the problem affected a few pairs, I thought it too much hassle to bring out laptops for a few pairs. So I decided to become a technician to try and solve the technical problems with the computers. I then realized later that the power cord leading to one of the PCs was disconnected. I then discovered that the same problem applied to the other 2 neighbouring PCs as well. I therefore did some hands-on technician work to plug the cords back for them. [Post-lesson reflections]

These “technical problems” posed challenges to me. In my case, my professional training as a teacher certainly did not include the craft of trouble-shooting hardware glitches during computer-use. The reality, though, was that when computers were
employed in teaching, I must be prepared to handle the work of a technician without getting substantial help from others. Although these interferences did not, in the end, prevent the carrying out of my teaching goals in Lesson 4, they were still problematic in the sense that when they surfaced, I needed to suspend other teaching activities—including teaching goals—to deal with them immediately.

There were other instances of technical glitches. The more significant ones are summarised in Table 35.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Change in hardware environment</th>
<th>Teacher action</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Keys on keyboard short-circuited. Typing “m” shows “n” on the monitor and vice-versa</td>
<td>Cannot resolve the problem. Advise student to adjust the typing to accommodate the problem.</td>
</tr>
<tr>
<td>3</td>
<td>Inaccessible keyboard</td>
<td>Unplug and replug cord connecting keyboard to computer.</td>
</tr>
<tr>
<td>5</td>
<td>Blank screen</td>
<td>Unknown cause. I could not resolve the problem, but the students later managed to get the screen to work.</td>
</tr>
<tr>
<td>1, 6</td>
<td>Insensitive mouse – a lot of effort to drag the cursor displayed on the screen (3 pairs of students).</td>
<td>Do not have replacement. Ask students to be patient in dragging the mouse.</td>
</tr>
<tr>
<td>3, 6, 7</td>
<td>System ‘hung’ or ‘illegal operation’ dialog box popped up. Result in loss of data (4 pairs of students altogether)</td>
<td>Advise students to reboot.</td>
</tr>
<tr>
<td>6, 9</td>
<td>Computer could not start (2 pairs of students)</td>
<td>Changed to laptop</td>
</tr>
</tbody>
</table>

Table 35: Samples of technical glitches

7.7.1 Problems and Coping Strategies in Relation to Hardware Matters
In my experience using computers in teaching, I realised that I had to contend with the hardware issues that came with computer use. Whether the technical problems were humanly-induced or otherwise, they were largely unavoidable given the unpredictability of human behaviour and the performance of the computers. I had to
confront the unanticipated disruptions in the hardware environment as best as I could to carry on with the work of teaching.

The nature of the problem with hardware glitches was different from the goal-conflict and goal-dilution problems discussed in the previous chapter. In the case of difficulties relating to hardware matters, the situation was not one where certain specific goals of teaching were directly challenged. Rather, all intended goals, whatever they were when the glitch occurred, had to be suspended until the matter was dealt with. In other words, malfunctioning computers were a problem because they interrupted me from carrying my goals of teaching in order to deal immediately with the technical glitches.

In contrast to the pattern of trigger-chains in the turbulences discussed in the previous chapter, the triggers of hardware disruptions did not form links of related perturbations; rather, each trigger appeared isolated from others, perhaps because of the nature of the technical faults which differed from computer to computer. There was no chain-like series of triggers that build up to a problematic point when I dealt with faulty hardware. Neither was there discernible a time lapse between the initial trigger and the culmination of the problem like that featured in the other forms of turbulences. All the triggers of technical glitches led immediately to intervention on my part and hence instantly to the problematic situations of suspended goals.

Although there were no clear trigger-links leading to problems, there were links in the way I coped with similar triggers in the sense that I learnt from previous experiences of dealing with technical glitches. In the case of the “blank screen” glitch in Lesson 5 (Table 35), my first reaction was captured in the transcription of the video data:

24.15 T: [walks over to Rashid and Elsie] Is there a problem? Oh because there's no power supply is it? Check whether the power is on? [stoops down to below the computer desk to check for power]

My post-lesson reflections for that part of the lesson revealed that my thoughts about the possibility of the glitch being that of disconnected power cord had to do with my experience of similar ‘symptoms’ earlier in Lesson 4. I applied the measures I took in
Lesson 4 to the situation in Lesson 5. These recalled history formed part of the resources I tapped on to cope with technical problems in the module.

Again I put on my technician’s hat to try and see how I can help. Based on historical knowledge, the first thing I checked was whether the power cord was unplugged, since the symptoms looked familiar – cannot start.

The frequent reference to my “being a technician” in my reflections also indicated another way I coped with the technical glitches. I consciously took on the role of technician in addition to my role as teacher. I did not avoid the technician job even though I was aware that I was unqualified to assume the role adequately.

Another difference in turbulent experiences with hardware was that, unlike earlier cases where the triggers and the resources I used were primarily invisible to the external observer, the constraints and affordances were very much linked directly to visible objects and phenomena. Table 35 shows that perturbations came from the detectable physical environment. Similarly, the way I coped with the external changes marked a shift from the previous dominant mode of unobservable mental tossing of options to that of observable teacher actions. In other words, and risking oversimplification\textsuperscript{11}, it can be said that computer use in teaching required more spontaneous action (such as in dealing with technical glitches) over and above the work of spontaneous thinking (such as in the constant weighing of instructional options).

7.8 Summary of Chapter

In this chapter, I looked at those parts of the geometry module where Sketchpad was used, focusing on analysing those regions in which the nature of turbulences yielded additional or different views to the picture of problems and coping strategies developed in the earlier chapters.

\textsuperscript{11} I am not suggesting that “doing” and “thinking” can be separated in teaching. Rather, it seems that the “doing” appears more at the foreground when dealing with computer glitches in teaching.
Sketchpad helped in providing additional tools to support the goals of teaching geometry. The Sketchpad environment allowed students to explore geometrical ideas and to engage in joint mathematical reasoning. In particular, the drag mode enabled students to check their conjectures [G4] and shift their focus from mere visuals to the invariant geometric properties [G3]. Seen as an instrument to support instructional goals amidst problems, the use of Sketchpad is another coping strategy.

However, actual integration of Sketchpad into classroom instruction also needs to take account some non-trivial challenges: students needed time to be familiar with the software; modification to the curriculum plan was required to preserve the flow of multiple developmental tracks; time pressure—especially in the initial lessons—was exacerbated; cooperation difficulties within pairs arose; and hardware issues were confronted. Careful attention to coping with these problems is thus necessary for a successful harnessing of Sketchpad into the geometry classroom.

The close of this chapter completes the report on the final phase of analysis. The next chapter presents an overall summary of results of this study.
Chapters 5-7 focused on problems and coping strategies involved when using Sketchpad to teach in a geometry classroom that supports reasoning, and thus addressed Research Questions 1 and 2. From the analysis of a single lesson in Chapter 5, a list of conjectures was proposed. These conjectures were examined and refined in Chapter 6 by the data from the other turbulent regions in the module. Chapter 7 reported specifically on problems and coping strategies encountered in Sketchpad use. In this chapter, I briefly review the conjectures and summarise the response to the Research Questions before taking a step back from the particularities of the case study to revisit the broader original “Question 0” which spells out the motivation of this entire research project.

8.1 Review of the Conjectures

With the exception of a refinement to Conjecture P4 (to P4a), the data at each successive phase of the analysis process increasingly strengthens the claims of the original conjectures. The conjectures are reproduced below for ease of reference.

P1. Every problem of teaching was traceable to a trigger.
P2. Triggers need not be easily detectable by an external observer.
P3. One of the ingredients for problems to occur was the attempt to improvise as a response to triggers.
P4a. Problems of teaching can involve one or more overarching goals.
P5. Studying a particular problem of teaching without considering the wider context of the region of turbulence can result in a distortion of the complex work of teaching.
P6. Using one lesson alone as the ‘unit of analysis’ of problems of teaching may be inadequate for understanding the internal tension involved when dealing with competing goals of teaching.
C1. Coping involved an ongoing monitoring of changes in the instructional situation.
C2. Many of the resources I harnessed during coping were invisible to others.
C3. While coping, I was influenced by phenomena outside the immediate context of the instructional situation.
C4. Coping involved prioritisation of goals, and the priorities need not remain the same throughout the turbulent region.
C5. Actions were taken to alleviate the potential ill effects of unfulfilled goals.

The significance of these conjectures in this study can be seen from their direct contribution in addressing the Research Questions in the next section.
8.2 Summary Response to the Research Questions

Question 1: What is the nature of problems that a Secondary teacher experiences when using dynamic geometry software to teach geometry in a way that supports students’ participation in mathematical reasoning in an Express class of a government school in Singapore?

In this study, problems are defined as interferences to the carrying out of teaching goals. Interferences have been seen to take various forms. Taking the example of just the single goal of covering the geometry syllabus [G1], problems can occur at different temporal junctures and over different time ranges in a teaching module (Conjectures P5 and P6). Before the start of the module, I faced difficulties fitting the content within the limited time of ten lessons; at various points along the module, I realized that my coverage lagged behind the module plan and I needed to make content adjustments; within lessons, I observed student responses—such as errors—which triggered (Conjecture P1) the need to modify the depth and coverage of the planned lesson content. Apart from problems along the temporal plane, a teacher also needs to handle problems of different grain-sizes along the social plane. In whole-class instructional settings, some students cannot keep pace with the intended instructional development; within pairs, students can have difficulties cooperating; in teaching individuals, personal traits—such as over-reliance on the teacher and detachment—can interfere with the goal of learning the required content. There are also internal (Conjecture P2) and external triggers that pose challenges to a straightforward implementation of syllabus. Externally, lack of suitable equipment, lesson cancellation, and technical glitches can significantly interfere with practice; internally, knowledge of content and students, and interaction of theoretical knowledge with craft knowledge throw up instructional alternatives that contribute to the complexity of in-class decision-making (Conjecture P3). Moreover, these challenges—temporal, social, external, and internal—do not occur separately; they often happen simultaneously and thus accentuate the problems of teaching.

All these problems are associated with goal G1. But teaching involves the fulfilment of other goals as well (in this case, G2-G5, T1-T2). Similar problems along multiple planes were found to apply to all the other goals of teaching. Teaching reasoning [G4] and using Sketchpad to connect to geometry [T2] are subject to issues involving time, students’ ability to learn at different levels of social organization, and influences
DISCUSSION OF RESULTS

from internal and external resources and constraints. Thus, problems of teaching are concomitantly enlarged by the need to fulfil multiple goals of teaching.

The picture of teaching problems is further complexified by the existence of interaction among goals. Teachers rarely carry out a single goal at a time; rather, a number of goals compete for attention simultaneously (Conjecture P4a). Other goals of teaching sometimes emerge to conflict, dilute, or compromise intended instructional goals, resulting in situations where not all goals-in-consideration can be fully implemented. Thus, goals such as “teaching reasoning” [G4] and “using Sketchpad to teach geometry” [T2] cannot be considered in isolation from the other goals when examining actual problems of practice. This study reveals that tensions are most frequently and intensely encountered when these goals of teaching interact problematically with other worthwhile instructional goals.

That goal interactions are so difficult to anticipate adds to their problem-causing nature. Returning to the navigation metaphor, imagine a scenario where all potential perturbations can be anticipated. The navigator can use his knowledge and navigational systems to avoid or alleviate the damage caused by these turbulences. However, in reality, this scenario does not happen in navigation, and it certainly does not apply to teaching practice. Triggers in teaching are hard to anticipate because of the dynamic interactions of classroom phenomena; even if goals that arise from triggers can be anticipated, the ensuing interactions among goals can result in a wide variety of instructional choices. Granted that all goal interaction structures can be anticipated, elements of uncertainty and novelty still remains as the same sets of goals can play out very differently under different instructional contexts. Like navigation, unanticipated triggers and goal interactions are inherent in the work of teaching. A teacher cannot avoid them but can learn to cope with the resulting turbulences.

Question 2: How does a teacher cope with the problems of practice?

As with all problems, the first step in coping with problems of teaching practice is to be aware of the nature of the problem. A teacher needs to take into account the different planes of complexities and the interacting goals that are described above. Adopting an over-simplistic instructional model to cope with problems of teaching
can omit other elements that are central to the teaching enterprise. As an example, a teacher who copes with teaching problems by consistently choosing to work on a single goal (to the exclusion of others) artificially narrows the teaching agenda.

Part of the work of coping involves monitoring the instructional situation—with all the associated triggers, emerging goals, and improvisational options. This requires careful observation of the external environment which includes students’ responses, onscreen outputs, and equipment function, together with the internal weighing of tensions and opportunities. Moreover, such monitoring efforts are ongoing (Conjecture C1). It is not a case of a one-time careful factoring of these considerations into a lesson plan and then simply executing them in the lesson. Because of the ongoing dynamic changes in the instructional situation, constant monitoring of ‘navigational dials’ and the willingness to continuously adapt are necessary in coping.

Coping also involves harnessing resources. The primary resource is the knowledge of the teacher (Conjecture C2). The knowledge base includes mental resources derived from prior teaching experience, previous instructional sequences with the class, and knowledge of individual students’ performances. In other words, coping is an act that is not disconnected from instructional history; rather, it taps into relevant data from past encounters to inform in-class decision-making. Coping also harnesses opportunities and tools as they present themselves in the classroom—such as seizing upon one student’s contribution to channel discourse to a productive direction, and using Sketchpad’s drag mode to prompt students’ conjectures. Coping even draws upon knowledge of what is to come. It anticipates available time, content development, and students’ propensity to learn further along the modular plan so as to decide current instructional courses. Mental resources are therefore not locked within a fixed temporal zone. A teacher can cope by mentally moving temporally from the current instructional situation to past experiences and projected happenings (Conjecture C3).

In relation to the problem of fulfilling multiple goals in teaching, coping involves finding ways to accommodate a plurality of goals in a single activity. Examples of this strategy were reported in this study: In planning the geometry module, the
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*Sketchpad* activity on triangles fitted into the flow of more than one goal-track; throughout the later part of the module, I used *Sketchpad* dragging for both the goals of pointing out students’ errors and of shifting the responsibility of deciding correctness to the students; in revising homework, my approach was to invite students to evaluate alternative solutions, highlight their errors, present the correct solutions, and demonstrate problem-solving heuristics, as well as encourage mathematical reasoning; geometrical constructions were introduced both for fulfilling syllabus requirement as well as bridging the “Visual” and “Analysis” levels of the van Hielean scheme.

However, there are situations where goals cannot be easily harmonized into a single activity. Where the carrying out of some goals conflicts with or dilutes the fulfilment of other goals, then a teacher must decide which goals to favour (Conjecture C4). The priorities of goals, however, do not follow a rigid order; rather, depending on the instructional setting and resources available, a goal such as “teaching reasoning” [G4] may be given higher priority in one instance but may surrender to “content coverage” [G1] at another occasion. It is important to view prioritisation of goals as a coping strategy to advance, not impede, the work of teaching. As in navigation, the pilot may not always be able to resolve all warning signals and perturbations. In such instances, he addresses those alerts that present the greatest dangers so that the plane can safely reach the destination. Likewise, the teacher focuses attention on those goals that, if ignored, can potentially derail the whole instructional course.

Prioritization of goals does not imply *abandoning* goals of lower priority. Rather, some goals are put at the foreground of the teacher’s attention during instructional practice while the other goals under consideration remain in the background—awaiting opportunities to be carried out, even if only partially (Conjecture C5). Such a coping strategy was adopted at numerous occasions in my teaching work: In Lesson 1, I resorted to direct-telling but tried to partially preserve G4 by seeking students’ agreement; in Lesson 6, I focused mainly on tasks for weaker students, but encouraged stronger ones to proceed to more challenging work when I observed that they were ready; in Lesson 8, before moving on to other lesson components, I quickly explained using *Sketchpad* to illustrate the critical differences between squares and rhombuses. Like a navigator who may not be able to keep the plane damage-free but
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nonetheless tries to minimize damages, a teacher can adopt damage-control measures to alleviate the ill-effects of goals that are not fully carried out.

8.3 Returning to Question 0

How would a teacher teach mathematics in a way that supports the reform agenda in Singapore?

While the teaching goals reported in this study are mine in the sense that they are owned by me, they are also shared by a wider community of teachers insofar as they can identify with these goals in their own practices. In this sense, the findings in this study extend beyond my teaching experience to other practitioners in the classroom.

Since Question 0 is very broad in scope, I shall again decompose it into the four field-based questions introduced in the first chapter, and replicated below, to produce manageable parts that can be integrated into an overall response.

How would a teacher teach ...?
How would a teacher teach mathematics ...?
How would a teacher . . . support reform agenda . . .?
How would a teacher teach . . . in Singapore?

8.3.1 How Would a Teacher Teach ...?

The literature reviewed in Chapter 2 characterizes teaching as a complex activity. A teacher has to attend to complexities along different planes—intellectual, social, temporal—all at the same time. Instead of having ample time to analyse options, the teacher needs to make moment-by-moment decisions and improvisations while the lesson is ongoing so as to lead the class towards worthy instructional destinations. Figure 79—a merger of sorts between Lampert’s (2001) navigation metaphor and the instructional triangle proposed by Cohen, et al. (2002)—provides a diagrammatic summary of the components involved in the complexity of teaching.
My teaching experience reflected the interacting complexities depicted in the model above. Like a navigator, I attempted to anticipate, monitor, and control the various elements that can affect the planned instructional course. I also used resources such as students’ discourses and my own knowledge to cope with the problems of teaching that I encountered. The component of “time”, that Lampert (2001) highlighted as another dimension of complexity, was similarly experienced as critical in many of my instructional dilemmas.

This study contributes to the discussion of teaching complexity by extending the navigation metaphor to include “regions of turbulence”. This sub-metaphor affords the language to describe the more subtle nuances involved in teaching—such as the duration of a turbulence, nested turbulences, triggers, relations between triggers, and damage control measures—and is more fine-grained than the general “navigation” metaphor. As demonstrated in the previous chapters, the concept of “turbulence” was useful in studying the nature and relations of various elements that occurred around problems of practice. Interactions between significant components of teaching thought and practice—such as triggers, appreciation of situations, weighing courses of action, making decisions, monitoring outcomes, and damage controls—emerge more clearly when integrated via the overarching theme of “turbulence”.

Figure 79: The instructional triangle interpreted through Lampert’s navigation metaphor

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A number of phenomena of teaching become conspicuous when viewed through this “turbulence” framework. This lens reveals that the problems of teaching occurred within the context of related triggers and tensions. Studying the problems of teaching through turbulences helps to shift the focus away from merely looking at the problematic peaks of practice to the surrounding turbulent region. It goes beyond answering where the problems occur and what they are, to how they happen amidst the actual context of teaching. It uncovers the complex interplay between various significant instructional components—students’ behaviour, teacher actions, teacher thoughts, and classroom resources—when coping with real problems of practice. In that sense, the turbulence metaphor is a suitable frame through which to study the interactions between these elements of classroom practice.

Similarly, the occurrence of interlinking triggers was easily detected when they were viewed as a build-up of tension within turbulent regions. This indicates that a trigger can sometimes be linked to an earlier trigger or be heightened in anticipation of a trigger that is about to occur. The knowledge of links between triggers is useful in guarding against an isolationistic treatment of triggers and the error of dismissing some triggers—perhaps based on ‘remoteness’—as irrelevant or unimportant in the study of teaching.

Another contribution in this study is the infusion of goal-based methodology into the existing modes of narrative research used to investigate the complexity of teaching. While a number of researchers (e.g., Ball, 2000; Fleischer, 1995; Lampert, 2001) acknowledge that the challenges for a teacher in the classroom include managing dilemmas and coping with conflicting goals of teaching—what has been called “walking the pedagogical tightrope” (Wood, et al., 1995)—there is little development in the literature about the deeper instructional issues faced by teachers as they balance goals. By analysing teaching behaviour via instructional goals, this study contributes to the knowledge of this balancing act. The triadic model (Figure 79) depicts the key components involved in instructional practice and the possible interactions between them; the goal-based perspective allows us to study these interactions in the light of the underlying motivations of the teacher. In that sense, bringing goals into the study
adds yet another dimension of complexity into the model but provides scope to study not only what the teacher does in the classroom but also why.

The large number of goals that emerge during teaching is already an indication of the complexity involved in classroom instruction; what is more illuminating is the complex interaction among the goals of teaching. As illustrated in this study, different combinations of goals and different interaction patterns are possible at different junctures. Given the spontaneous nature of classroom practice, it may appear impossible for a teacher to balance all the emerging goals simultaneously. At such times, a teacher may need to compromise one or more worthy goals of teaching. These conflicting priorities pose serious challenges to the work of teaching. Nevertheless, when a teacher needs to suppress some goals, it is not to abandon the work of teaching altogether; rather, the giving up of a goal was to allow other goals of teaching to be fulfilled.

8.3.2 How Would a Teacher Teach Mathematics ...?

The subject matter of “Mathematics” is part of the “content” component of the model of teaching illustrated in Figure 79. Unlike other views of the teaching enterprise that downplay the role of content in teaching—such as King’s (2001) metaphor of teaching as playing jazz (reviewed in Chapter 2)—subject matter occupies an essential role within the interactions in the instructional triangle. In a mathematics classroom, mathematics is not merely a resource to tap into; rather, it exerts a strong influence over the thoughts and actions of teacher and students during instruction.

That mathematics contributes substantially to classroom events is borne out in this study too. In the overall survey of the module, content-related matters featured prominently in the weighing of options and decision-making processes in every lesson. Whether it was the definition of polygon in Lesson 1, or the construction procedures in Lesson 4, or the rhombus problem in Lesson 8, mathematics shaped the coping mechanisms I used and how I actually carried out the lesson.
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Of my five overarching G-goals of teaching, four relate directly to geometry (see below). This implies that my goals are heavily influenced by the subject matter and how I intended it to be developed in the module.

G1. To cover the geometry content …
G2. To prepare students to tackle exam-type questions from … the geometry module;
G3. To help students progress to higher van Hiele levels of geometric competence;
G4. To encourage students to reason within the geometrical domain;

G1 relates to syllabus coverage and indicates that the subject matter (geometry) has a direct bearing on the actual topics taught in class. The number of topics to cover within a specified time period is somewhat beyond the direct control of a teacher as they are inherited from the school’s scheme of work. While teachers have freedom to vary the approach and sequence of coverage, it is part of their professional duty to teach the required geometry content. As seen from the accounts found in this study, this responsibility places stress on a teacher and the ongoing pressure to ‘cover content’ can affect the instructional decisions in the classroom.

The goal spelt out in G2 implies a direct link between subject matter and the standard test items associated with that particular subject matter. One of the responsibilities tied to “teaching mathematics” is that of teaching students how to tackle conventional items that go with the kind of mathematics that is being taught. Whether the “mathematics” involved is geometry, or algebra, or trigonometry, there is usually a set of questions or mastery items that test the related skills and/or conceptual abilities of students who have undergone the respective modules. The deliberate teaching of ways to solve these exercise items is conspicuous in my account of teaching. The challenge for teachers is to help develop such techniques in the students within limited lesson time and with so much to cover. The use of homework is one way to ‘extend classroom time’ in the sense that it provides students opportunities to hone their skills and concepts outside the temporal boundaries of the classroom. But, as seen here, homework revision may imply a need to devote time in class to point out error, bringing about other problematic situations such as goals-dilution.
G3 is about helping students to make progress in geometrical thinking along the stages proposed by the van Hieles’ theory. However, the need to build knowledge in developmental stages is not restricted solely to the field of geometry. Indeed, although the van Hieles’ initial research was within the domain of geometry, they subsequently broadened their theory of learning stages to apply to mathematical thinking in general (van Hiele, 1986). Thus, for each area of mathematical knowledge, learners will need to go through the associated stages of development to reach the required level of proficiency in the subject matter. Teachers must therefore plan the module development of each topic in a way that will suit this track of progress. Using this ‘track’ language, teachers need to take the G3-track into consideration when planning the lessons so that classroom instruction has the potential to fulfil the overall goal of G3 for the module.

Similarly, as this study has revealed, the demands of G4 for logical development of geometrical ideas also implies a certain track in the instructional sequence. However, in the realistic condition of classroom practice, the teacher, with limited time, cannot proceed with both G3- and G4-tracks one after the other, nor is it developmentally or mathematically necessary. As in my case, what is perhaps workable is an enmeshing of the respective tracks of development in a way that retains, as much as feasible, the sequential order in each of the tracks while necessarily compromising certain aspects within each track of development.

G1-G4 also shows that “teaching mathematics” is not restricted to any one of these purposes. It is not merely “covering the mathematics syllabus” alone [G1], nor is it merely “teaching mathematical skills and techniques” alone [G2], nor is it merely “teaching mathematics according to van Hielean-like stages” [G3], nor just “teaching mathematical reasoning” [G4]. The challenge for teachers is that all these goals have to be borne in mind in “teaching mathematics”. The teacher does not have the luxury to tackle each goal separately; nor should they, lest some significant mathematical learning be omitted. Within the limitations of classroom instruction, the teacher merges these respective goals into his/her teaching practice so that one or more of these goals can be carried out at any one time.
Reviewing the binary-talk about approaches in mathematics teaching that exists in the literature—for example, “calculationally-oriented mathematics teacher” versus “conceptually-oriented mathematics teacher” (Thompson, et al., 1994), and teaching by “procedural instruction” versus teaching by “inquiry” (Cobb, et al., 1998), the multiple-G talk presented here challenges the simplistic depiction of mathematics teaching as either of one particular mode or of another. Rather, when mathematics teaching is seen as an act of fulfilling different worthy instructional goals, elements of different modes of teaching—such as “topics-based” [G1], “technique-oriented” [G2], “theory-informed” [G3], or “by reasoning” [G4]—can co-exist within actual teaching practice. Furthermore, goals-talk avoids the need to be locked into any of the traditional theoretical streams of behaviourist, constructivist, or sociocultural for research on mathematics teaching. Thus, the goal-based approach not only shifts the talk away from simplistic contrastive modes of mathematics teaching, it is also useful towards greater cross-disciplinarity in classroom research.

8.3.3 How Would a Teacher … Support Reform Agenda …?

The part of the “reform agenda” that is the focus of this study consists of two key initiatives: “infusing thinking”, and “integrating technology”. In the context of mathematics teaching, these initiatives are interpreted as movements to encourage disciplinary reasoning and the use of relevant software in the mathematics classroom. In goals language, they are represented primarily by G4 and T1/T2 respectively.

When “reform agenda” is translated into teaching goals, it is seen as part of the total set of goals of teaching that a teacher brings into the classroom. These G4 and T1/T2 goals are then not viewed as necessarily more important than the other goals (such as G1-G3). For some teachers, regardless of reform, these goals form part of their agenda for teaching geometry. Depending on the instructional situation, these goals ‘that support reform’ can sometimes be given the priority but can at other times be suppressed for the purpose of pursuing other worthy goals. Seen in this way, “reform agenda” does not replace incumbent instructional goals but enters the classroom in the form of other goals of teaching that interacts in a complex way with other
instructional goals that a teacher possesses. Thus, “reform-oriented” teaching does not necessarily result in transformational changes of classroom practice; rather, it can intertwine dynamically with existing forms and goals of teaching.

Due to inter-goal competition, fulfilling G1-G3 alone in the classroom is a challenging task. With the addition of G4 and T1/T2, the struggles are magnified. Teaching that supports the reform agenda can thus be problematic as teachers have to cope with an enlarged pool of instructional goals during classroom practice with no corresponding enlargement of resources (such as time). These limitations act as constraints against endeavours to carry out multiple desirable goals of teaching within an intended time period.

It is, however, not accurate to view an increased number of goals as purely problem-causing and thus counter-productive to teaching. Firstly, as reviewed in Chapter 2, these reform initiatives are worthy goals of teaching and should rightfully be included as part of the teaching enterprise; secondly, including these goals into practice provides a motivation for the teacher to seriously and deeply re-consider his/her instructional priorities; thirdly, creative ways of teaching can consequently emerge to productively incorporate various goals of teaching concurrently (e.g., in weaving Sketchpad into the instructional plan as discussed at Section 7.5); fourthly, multiple goals act as different mental ‘drivers’ to engage the teacher continuously in thoughtful and balanced practice; fifthly, while some goals are compromised at some points of practice, when seen over a longer period of instruction (such as over a module), all the goals can find expression at different times. The findings reported in this study attest to all these positive benefits of including the reform agenda into teaching practice.

While “infusing thinking” and “integrating technology” both share the common label of being reform initiatives, there are differences in the nature of their influence on classroom instruction. When the former is interpreted as encouraging reasoning within the discipline of mathematics, its import in the classroom takes a rather naturalistic form in the sense that teaching mathematical reasoning is viewed not as external to but part of the work of teaching mathematics. Examining the g-goals that support mathematical reasoning [G4] would make this point clearer:
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<table>
<thead>
<tr>
<th>glan</th>
<th>To teach the language of geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>gwrs</td>
<td>To teach students how to present geometric reasons in written answers to problems</td>
</tr>
<tr>
<td>grsd</td>
<td>To teach students to reason deductively about geometrical ideas</td>
</tr>
<tr>
<td>gmrn</td>
<td>To teach the geometric meanings and relationships underlying accepted facts or solutions to problems</td>
</tr>
<tr>
<td>grel</td>
<td>To reduce students’ reliance on the teacher for answers</td>
</tr>
<tr>
<td>gtry</td>
<td>To instil a ‘give-a-good-try’ attitude in students</td>
</tr>
<tr>
<td>gart</td>
<td>To encourage students to articulate their observations, conjectures, refutations, justifications, results, and explanations.</td>
</tr>
<tr>
<td>galt</td>
<td>To encourage in students a disposition towards seeking for alternatives</td>
</tr>
</tbody>
</table>

None of the g-goals listed above require one to venture beyond the ordinary role of a mathematics teacher. In fact, each of these goals can be seen as contributing directly to the overall work of teaching mathematics. Moreover, each of the g-goals can be viewed as part of the basic model of teacher-students-content triadic interactions (see Figure 79). For example, the carrying out of glan can be viewed as interactions between the following pairs: teacher-student (teacher communicates with students about the language of geometry), teacher-(student-content) (teacher helps students to study the language of geometry), and teacher-content (teacher taps into the various forms of knowledge required to teach the language of geometry). The other g-goals can be similarly represented within the instructional triangle. That the g-goals can be incorporated within the basic model of teaching shows how “teaching mathematical reasoning” can be seen as a fundamental component of the work of “teaching mathematics”.

Although “infusing thinking” can be viewed as a natural part of teaching mathematics from the disciplinary perspective, it does not imply that a teacher will find no challenges with incorporating it into practice. For some teachers, it may require some significant dispositional changes in instructional practice. For example, a cursory look at goals grel, gtry, gart, and galt will indicate that teaching mathematical reasoning involves students’ active participation in the process of arriving at taken-as-shared knowledge. For some teachers and students who may be used to seeing the teacher as the sole knowledge-provider in the classroom, attempting these goals may represent a shift of mindset to that where students contribute to classroom discourse and reasoning. Such a shift is not straightforward and it involves persistent effort towards changes in teacher/student roles as well as classroom routines. Thus, for the reform
initiative of “infusing thinking” to be successfully implemented, one should take into account the countering forces present within existing pedagogical environments.

But the other reform initiative of “integrating technology” presents quite a different challenge. The inclusion of technology—whether hardware or software—in the classroom represents an insertion of something relatively novel both in the traditional classroom setup as well as the tradition of teaching mathematics. Classroom instruction has for a long time taken place without the use of computers; similarly, mathematics teaching has been carried out for millennia without the aid of software. Thus, compared to “infusing thinking”, “integrating technology” has a somewhat greater sense of new elements being introduced into the classroom. Indeed, making reference to the basic triadic model of teaching again, technology-use cannot be easily accounted for in that triangular framework. Rather, “technology” appears to interact with the rest of the components of teaching in a way that is more akin to a fourth vertex of an ‘instructional quadrilateral’ (see Figure 80).

The findings reported in this study depicted conspicuous interactions between teacher-technology (e.g., dealing with hardware), student-technology (e.g., students’ hands-on work), as well as technology-content (e.g., the interaction between the DGS-track and Euclidean track in module planning). The teacher-(technology-content) and teacher-(technology-student) arrows were also exemplified by my instructional attempts at linking students’ Sketchpad work to geometry (goal T2) and helping students become familiar with the software (goal T1) respectively.
If technology is another non-trivial component of the teaching enterprise, time must be allowed for teachers to become familiar with it, and to integrate its use effectively with the other essential elements of teaching. It is thus not surprising then that teachers who are told to use technology in teaching often find it a burden in an already crowded schedule (Manouchehri, 1999). As illustrated in this study, sufficiently rich understanding of both the hardware and software issues in relation to the other components of teaching is necessary to cope with the problems associated with technology-use in teaching. Surface implementation alone, such as providing schools with computer tools and scheduling training sessions for software-use, is unlikely to be sufficient to help realize the goal of “integrating technology” in mathematics classrooms. While educators expect time and actual experience to be required to gain knowledge about the other vertices of the ‘quadrilateral’—such as “students” and “content”—for teaching, it is unreasonable to expect teachers to know sufficient about the fourth vertex of “technology” over a short time to incorporate it effectively into their instructional practice. This study supports the need for time for technology integration—both as a necessary investment before benefits eventually accrue, as well as to allow the necessary process of teachers grappling with competing goals to develop workable instructional pathways.
Besides exercising patience, this study also highlights the need for attempts at “integrating technology” to be accompanied by careful weaving of technology within and across lessons as well as sustained persistence at monitoring and reflections about its effects. Too often, technology is brought into the classroom on an ad hoc basis as an add-on to a conventional lesson (Leong, 2003). Used in this way, it is no wonder that teachers detect more initial problems than usefulness and may even dismiss technology integration as unworkable or burdensome. Technology should not be viewed simply as a mere addition to or straightforward replacement of existing instructional modes; rather, deliberate planning at both the lesson and module levels is needed to fit technology-use tightly with the rest of the instructional components so that multiple goals can be implemented developmentally throughout the module. This may result in significant re-sequencing and merging of components in the development of a module.

Even with a carefully-designed plan, however, there can be unanticipated problems during implementation in class. Hardware glitches and students’ slower-than-expected rate of learning the software can pose significant challenges. Teachers need to be mentally prepared and equipped to face these challenges and make in-class decisions to cope with them. “Technology integration” will need to be monitored and reflected upon regularly so that necessary changes may be made to reduce recurrence of similar problems.

8.3.4 How Would a Teacher Teach … in Singapore?

Although the analyses in this study are based mainly on classroom phenomena, they take into account sociocultural forces at play in the wider environment of Singapore education that influence classroom instruction. I highlighted, in particular, two responsibilities that Singapore mathematics teacher are expected to fulfil: timely syllabus coverage and helping students do well in examinations. Translated into goal language, Singapore teachers generally adopt G1 and G2 as their goals of teaching mathematics. While these goals may indeed be shared by teachers in other parts of the world, the extent and degree to which these motivations drive instructional behaviour of teachers here (Kaur & Yap, 1998) seems a distinctive characteristic of Singapore’s education culture.
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As demonstrated in this study, the demands of G1 can place a strong time pressure on teachers. This constant sense of time pressure in teaching is problematic because it can result in instructional compromises. Worthy goals of teaching (including reform-oriented goals) can be sacrificed or relegated to lower priority simply because there is no easy way to fit all of these goals into a lesson and yet keep to time. The resource—time—seems insufficient to meet the ever-increasing demands— instructional goals—of teaching.

There may sometimes be a temptation by members outside the teaching community to dismiss teachers’ claims of time pressures as merely excuses for the lack of effort in improving practice. However, such a simplistic viewpoint only serves to widen the chasm threatening to separate teaching practitioners from other education non-practitioners, such as policy-makers, leaders, and researchers. The now-familiar theory-practice rhetoric (Bishop, 1998; Christiansen, 1999) about theories’ inadequacy in directly informing and reforming actual teaching practice is one manifestation of the gap between insiders and outsiders of teaching. One way to begin bridging the gap is by acknowledging the real problems of practice—such as time pressures—and to build theoretical knowledge that is grounded on these concrete experiences of teachers. Furthermore, in researching or recommending novel teaching strategies to practitioners, real constraints like shortness of lesson times should be given serious consideration.

Another major challenge for Singapore mathematics teachers is to prepare students adequately for examinations [G2]. In-class work on textbook exercises, use of worksheets, administering class tests, and homework assignments are practices that directly support this goal. Given the career implications of good examination results, it is perhaps helpful to students that moderate dosages of examination-orientedness is applied in teaching. There seems to be, however, a danger in the Singapore system that examination preparations dictate the choice of instructional modes in disproportionate ways (Lim, 2002). An almost singular pursuit of G2 could indeed contribute to good examinations scores and is perhaps one factor behind Singapore’s pole position in TIMSS rankings (MOE, 2000b). However, as is represented by the
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other overarching goals of this study, there are other very worthy (though not so easily assessable) aims of mathematics teaching that should not be neglected. Rather than focusing on G2 alone, there is a need for schools to strive for a balance between all the desirable outcomes of mathematics education.

But teachers may not find the balancing task an easy one. They are surrounded by cultural norms that are cultivated through years of examination-orientedness. The countervailing forces can include students’ low tolerance for teachers’ instructional forays into ‘non-examinable’ territory, or school managers’ linking of teachers’ performance directly with their students’ examination scores, or parents’ inquiries into their children’s poor test scores. Moreover, for a teacher, teaching according to examination techniques alone may be an easier option than attempting the problematic path (as illustrated in this study) of juggling multiple teaching goals in the classroom.

8.3.5 Teaching Mathematics in a Reform-oriented Classroom in Singapore

A mathematics teacher who intends to carry out reform initiatives in a Singapore classroom can expect to confront many challenges. The task involves handling complex interactions of different dynamic instructional elements such as students, content, time, and computers as well as the fulfilling of multiple and often conflicting goals of teaching. Many problematic situations can occur when a teacher cannot carry out all the intended goals at the same time. These problems can significantly affect the instructional course and may lead to compromises in important goals of teaching. On some occasions, when I examine deeply the multiple and interacting complexities that a teacher needs to contend with in the work of teaching, I instinctively echo Lampert’s (2001) sentiments when she wrote that “while I know the [triadic] model must be further revised and elaborated [to account for all the complexities], I recognize that this endeavour is dangerous because it forces us back on whether teaching . . . is an impossible task” (p. 448, emphasis added).

But when I look at actual workable practices of teaching throughout history, I am comforted that the bleak assessment is too pessimistic. Teachers devise coping strategies to deal with the challenging work of teaching. It involves employing
resources—both external and internal—to steer the instructional course in class towards worthy destinations. Although not all goals can be carried out at all times, some goals can still be fulfilled through prioritization and harnessing of opportunities. In addition, constant monitoring of the changing instructional situation helps to check for needed adjustments or damage control. Having a good view of temporal ranges is also a useful coping strategy. Not only should teachers be aware of ongoing moment-by-moment happenings in class, the ability to mentally tap on instructional history as well as ‘move forward’ in anticipation of planned components provides a wider temporal frame to situate and deal with problems.

My account of balancing various teaching goals reported in this study is hopefully an encouragement to teachers who hold similar beliefs about teaching. There remain opportunities to fulfil desired objectives in the classroom. I urge teachers to be unperturbed by the obstacles, but rather focus on the possibilities that are available for students to experience a range of worthwhile educational destinations.

However, teachers need not be left to cope with problems of teaching entirely on their own, nor should the implications of the complex work of managing multiple goals in teaching be discussed in the world of research alone. Other practitioners, school administrators, and teaching supervisors who observe the instructional work of teachers, especially in the context of assessing their practices, should also develop a deeper understanding of the relation between the goals of teachers and their teaching behaviour as well as the potential conflict of goals in actual practice. All too often, observers of teaching look at external instructional actions alone without attempting to understand the internal interplay of goals that lead to why teachers proceeded the way they did; or, they evaluate the work of teaching through what they thought were the teacher’s goals of teaching, which may not all cohere with the actual goals of the teacher. Moreover, such observations, like that of conventional research, commonly use a single lesson as the whole unit of analysis, leaving out the historical linkage of previous lessons altogether.

Hidden from the direct scrutiny of the classroom observers are all the inter-goal struggles that are closely tied not only to the immediate demands of instruction but
also to the wider scope of instructional development. Thus, judgements or conclusions
drawn from observations of acts of teaching alone do not do justice to the teacher’s
work. To understand why a teacher chooses to take a particular course of action in
class, there is a need to take into account the goals behind the actions, the history of
goals implementation, and the problems of implementing these goals given practical
constraints such as time limitations, the responsibility of syllabus coverage, the task of
preparing students for examinations, and others. When an observer reviews
someone’s teaching, it would certainly be more helpful if post-lesson discussions are
not merely about actions in class, but also on the teacher’s underlying goals. Such a
dialogue could potentially shift the focus from whether in-class actions were right or
wrong to one of exploring better ways to perform the balancing act amidst multiple
goals.

### 8.4 Other Potential Contributions of This Study

This study seeks to contribute to the education profession’s knowledge base (Zeichner &
Noffke, 2001). There are different ways in which the contribution can be made.
Practitioner studies such as the one undertaken here can be seen to contribute by way
of accumulating the pool of “craft knowledge” used by teachers in actual classroom
setting. As acknowledged by Hiebert, et al. (2002), craft knowledge taken together
with “traditional research knowledge”, can work towards a desired synthesis of theory
generation and practice that is informed by and rooted in the particulars of actual
practice so that the end result is one that both fulfils the requirements of the research
community and resonates with practitioners’ experiences. Doyle (1997) viewed the
contribution of narrative inquiries—which include the study reported here—as
“provisional models that seem to account, in at least a limited fashion, for how things
work” (p. 97). He argued that these provisional models can help “improve [teachers’]
understanding of events and actions in classroom settings and their ability to
recognize and produce novel arrangements. In other words, theoretical models,
although provisional, incomplete, and indeterminant, enable us to see” (p. 97). I echo
his desire to see the results of this research as worthy provisional models used in the
way that Doyle has suggested.
Cochran-Smith and Lytle (1993) proposed yet another way of looking at contributions of practitioner research to the profession’s knowledge base. Instead of seeing contribution by way of adding to knowledge, they viewed the contribution as *challenge for change* to existing forms of knowledge. They wrote that “research by teachers represent a distinctive way of knowing about teaching . . . that will *alter*—not just add to—what we know in the field” (p. 85, emphasis added). Through the findings in this study, I seek to offer alternative perspectives of questions raised about classroom teaching and reforms in Singapore.

In particular, I have attempted to inform Singapore education policy makers about problems and management matters that teachers can encounter in carrying out policy mandates so that future reform efforts do not take the mere simplistic route of changing curriculum and expecting teaching to conform easily. That this seems to be the current approach of the education authorities is a major concern I share. The lack of practice-based research about teaching in Singapore classrooms adds to the relevance of the research reported here. Fully aware that a study of such a limited scope may not by itself effect major shifts in education policy-making, I am nevertheless hopeful that it can kickstart a movement in this direction to address the concern.

On a personal note, this research enterprise has enriched both the teacher and the researcher ‘selves’ in me. The researcher-teacher way of understanding and experiencing the work of teaching merges the two selves in me to look at teaching in ways that each self in me would have missed. The teacher in me benefits from the theoretical and critical eye of the researcher in looking at my own practice; the researcher in me benefits from the contextually-rich experiences of the teacher as he engages in practice. Thus, I take comfort that even if this research is judged to have failed in reaching to those loftier aims as depicted in the above two paragraphs, it will certainly have had an educational effect on both the teacher and the researcher selves in me.


References


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The “double arrow button” on the left toolbar essentially provides a script that details the steps of a construction. When the “show script view” for rectangle construction is activated, a dialog window will pop up, as shown in Figure A.

Upon inserting two points according to the “given”, the script can begin to “play” the steps, as indicated by the emergence of the “next step” button in Figure B.
In successive clicking of the “next step” button, the “steps” in the dialog box will be played accordingly, together with the corresponding constructions that build upon the two initial points. As an example of how the script actually plays step-by-step, a screenshot of the monitor screen is given in the next Figure C after Step 5 is activated. The “next step” button is available until the final step is played. When the steps are completed, the construction lines that are meant to be hidden will disappear, leaving behind only the drag-resistant rectangle like the one shown in Figure D.

Figure C: Screenshot after Step 5 is played

Figure D: Rectangle construction script completed
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