BLACK HOLES AND
HIGHER DIMENSIONS

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ABSTRACT

Higher dimensional black holes are studied in the extra large dimensions scenario. Bulk fermion quasi-normal modes and bulk fermion Hawking emission is calculated. It is found that bulk emission dominates brane emission for \( d > 5 \).

To address the Planck phase an effective field theory is investigated. Lepton family number violating processes are elucidated and the corrections to the muon magnetic moment from these channels are calculated. Bounds are placed on the couplings of the theory.

A discrete symmetry between quarks and leptons, and left- right- chirality fields, is orbifolded in 5 dimensions. Using split fermions a one generational standard model extension is found.

An investigation of entanglement in black holes and accelerated motion is presented. It is found that in a certain system the acceleration between two spinors enhances the rate of their disentanglement.
DECLARATION

This is to certify that

(i) the thesis comprises only my original work towards the PhD except where indicated in the Preface,

(ii) due acknowledgement has been made in the text to all other material used,

(iii) the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices

I authorise the Head of the School of Physics to make of have made a copy of this thesis to any person judged to have an acceptable reason for access to the information, i.e., for research, study of instruction.

__________________________
Jason Doukas

__________________________
Date
PREFACE

The material presented in section 3 was conducted in collaboration with Hing Tong Cho, Wade Naylor and Alan Cornell. In particular, sections 3.3 and 3.4.2 were done by the mentioned others, nevertheless the results of these sections are crucial to the rest of the chapter and hence are included within this thesis for purposes of consistency and coherence.

Furthermore, the QLLR project was conducted in collaboration with Kristian McDonald and Andrew Coulthurst. Our roles were segregated where I worked out the gauge sector of the model whilst Andrew Coulthurst simultaneously worked out the fermion sector. The fermion sector has been presented within this thesis, section 4.5, for reasons of completeness.
LIST OF PUBLICATIONS

During the course of this project, a number of peer reviewed articles have been made which are based on the work presented in this thesis. They are listed here for reference in descending chronological order.

Articles


ACKNOWLEDGEMENTS

Over the last three and a half years I have had the great pleasure of doing physics with several collaborators. The work presented in chapter 3 represents the distillation of almost an entire years worth of work conducted with Alan Cornell, Wade Naylor and Hing Tong Cho all of whom I am gratefully indebted to for sharing their experience and mastery of black hole physics. In particular, I thank Alan for inviting me on board with this collaboration. It has turned out to be quite a productive alliance and one that I hope will continue into the future.

Furthermore, it was Kristian McDonald who convinced me to work on five dimensional QLLR models and I am also thankful to him for much inspiration, guidance and discussion along the way. On this project Andrew Coulthurst played an imperative role and his efforts must be acknowledged; the QLLR project certainly would not have been completed without his dedication. On top of this Andrew and I shared many useful discussions which was also extremely valuable to my own work.

It would not have been possible, of course, without my primary supervisor Girish Joshi. It was his suggestion in 2003 that I work on TeV black holes for my honours project, and I owe my current interest and enthusiasm for black hole physics to him. His unwavering belief that the study of black holes was substantiative physics, especially when things were not going so well, has left an indelible mark that will always give me reason for optimism.

I must also mention my secondary supervisors, Ray Volkas and Bruce McKellar, who could always be relied upon in times of need. I thank Robert Foot, for several useful discussions and likewise the quantum information people, Andrew Greentree, Lloyd Hollenberg, and Jarred Cole.

Many thanks also to the current and past members of room 606, Matthew Testolin, John McIntosh, Gajendran Kandasamy, Masum Rab and Ben Car-
son, and the students in the TPP group, Sandy Law, Alison Demaria, Damien George and Rhys Davies, who when asked were always kind enough to give some of their time to my questions, either on white boards or through discussion.

Producing a thesis requires a lot of dedication and much of ones time. Due to this it is easy for other aspects of ones personal life to lose precedence. I have been very fortunate in having loving parents, siblings and friends who understand these demands. I thank my Mum and Dad for their support over these demanding years, my brothers and sister, who have always displayed to some extent an interest and respect for what I do and also to Natalie Maroki my partner of three years, who patiently stood by me over this period of time and encouraged me to finish.

Lastly, to any physics students or staff I have failed to mention here but have known over the past four years. The friendly atmosphere around the school of physics has made the PhD experience much more endurable and I take from it many great memories.
### LIST OF ABBREVIATIONS

The following symbols and acronyms are used throughout this thesis.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$e$</td>
<td>The electric charge of an electron $-1.6 \times 10^{-19}$ C.</td>
</tr>
<tr>
<td>$k$</td>
<td>Boltzmann’s constant $k = 1.381 \times 10^{-23}$ J/K.</td>
</tr>
<tr>
<td>$M_{\odot}$</td>
<td>Mass of the sun, $M_{\odot} \sim 1.99 \times 10^{30}$ kg.</td>
</tr>
<tr>
<td>$M_*$</td>
<td>The fundamental scale in the ADD model $M_* \sim M_{EW}$.</td>
</tr>
<tr>
<td>$M_{EW}$</td>
<td>Mass of the electroweak scale $\sim 246$ GeV.</td>
</tr>
<tr>
<td>$M_P$</td>
<td>Observed Planck mass scale $\sim 1.22 \times 10^{19}$ GeV.</td>
</tr>
<tr>
<td>$M_{bh}$</td>
<td>Mass of the black hole under consideration.</td>
</tr>
<tr>
<td>$R$</td>
<td>Generic radius of an extra dimension.</td>
</tr>
<tr>
<td>$r_H$</td>
<td>Schwarzschild radius.</td>
</tr>
<tr>
<td>ADD</td>
<td>Referring to the theory of N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali [1, 2, 3].</td>
</tr>
<tr>
<td>AS</td>
<td>Arkani-Hamed and Schmaltz [4].</td>
</tr>
<tr>
<td>BC</td>
<td>Boundary condition.</td>
</tr>
<tr>
<td>BH</td>
<td>Black hole.</td>
</tr>
<tr>
<td>c.c</td>
<td>Complex conjugate.</td>
</tr>
<tr>
<td>COM</td>
<td>Centre of mass.</td>
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<tr>
<td>ELD</td>
<td>Extra large dimensions, also known as the ADD model.</td>
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<tr>
<td>EQG</td>
<td>Euclidean quantum gravity.</td>
</tr>
<tr>
<td>h.c</td>
<td>Hermitian conjugate.</td>
</tr>
<tr>
<td>IRF</td>
<td>Instantaneous rest frame.</td>
</tr>
<tr>
<td>KK</td>
<td>Kaluza-Klein masses or modes.</td>
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<tr>
<td>OBC</td>
<td>Orbifold boundary condition.</td>
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<td>QLLR</td>
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<tr>
<td>SM</td>
<td>Standard Model- the non-abelian gauge theory that describes the elementary particles.</td>
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<tr>
<td>SUSY</td>
<td>Supersymmetry.</td>
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<tr>
<td>VEV</td>
<td>Vacuum expectation value.</td>
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<tr>
<td>WKBJ</td>
<td>Approximation method for solving Schrödinger equations developed by Wentzel, Kramers, Brillouin and Jeffreys.</td>
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CORRECTIONS

We surmise the changes that have been made to address the reviewers suggestions.

- PAGE 29: Summary justifying scalar field theory treatment of BH’s added.
- PAGE 30: Missing terms, $\lambda \phi_{bh}$ and $\alpha \phi^3_{bh}$, are added to equation (2.10).
- PAGE 31: Instances of ‘lepton number violation’ replaced with ‘lepton flavour number violation’ and a reference to lepton family number violation in the little higgs model is given.
- PAGE 40: Further discussion of neutrino masses given. Clarification that the work presented was in establishing neutrino mass rather than constraining parameters.
- PAGE 41: Comments added in regards to pushing higher states above the cut-off. Quantitative treatment added and results discussed.
- PAGE 46: Footnote added discussing the Barbero-Immirzi parameter and it’s connection with QNM’s. Reference to literature added.
- PAGE 54: It is clarified why QNM’s on Schwarzschild backgrounds are the correct description for BH’s in split fermion models.
- PAGE 58: Mass scale used in numerical results is mentioned.
- PAGE 76: The significance of the distinct nature of brane and bulk cases is elaborated in terms of experimental signatures at the LHC.
- PAGE 132: We elaborate on the GEMS procedure and clarify that we do not use it in the work that follows.
- PAGE 134: A footnote is added clarifying that the results only pertain to spin degree’s of entanglement.
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CHAPTER 1

Introduction

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In this thesis, we consider the phenomenological consequences of extra large dimensions, placing particular emphasis on the study of higher dimensional black holes (BHs) and the quark-lepton left-right (QLLR) extension to the standard model (SM) on an extra dimensional orbifold. Further to this, we also investigate the loss of entanglement in an accelerated spin system which, as will be explained within, is intimately connected with the information paradox of BHs.

The topics covered in this thesis are quite diverse and in many ways this outcome can be attributed to the fact that quantum theory and general relativity are still not reconciled. Thus, much speculation for instance upon extra dimensions or on the quantum nature of gravitational systems is necessary. While it would be impossible to bring the reader entirely up to speed with all of these developments a brief presentation of some of the background to this multifaceted field is in order and we present in what follows an introduction to extra large dimensions, the hierarchy problem, orbifolds, split fermions, BHs and the information paradox.

1.1 Extra large dimensions

The origins of extra dimensions in physics can be traced back three quarters of a century, to when Albert Einstein attempted to unify electromagnetism with gravity by invoking the ideas of Kaluza and Klein. In these models the extra dimensions were thought to be compact spaces with size in the order of the Planck length ($10^{-35}$ m). Since then extra dimensions have played a keystone role in the development of modern theories of quantum gravity, including supergravity and string theory.

In 1998 Arkani-Hamed, Dimopoulos and Dvali proposed an extra dimensional scenario now known as the ADD model\cite{1}. Breaking away from tradition, they hypothesised that the extra dimensions might be as large as 1\footnote{Also sometimes referred to as the Extra Large Dimensions (ELD) scenario.}
1.2 The Hierarchy problem and the ADD solution

Since then it has been speculated that these dimensions might even be infinite, see for example [5, 6].

Perhaps the most striking feature of the ADD theory is that it predicts BH phenomenology operating at low energies. We provide here an overview of the ADD model as it forms the basis for much of the rest of this thesis.

1.2 The Hierarchy problem and the ADD solution

Naturally one might wonder why the energy scale of gravity \( M_P \sim 10^{19} \text{ GeV} \) is seventeen orders of magnitude larger than the electroweak scale \( M_{EW} \sim 10^2 \text{ GeV} \). Then again, numbers of different size occur commonly in physics, and one might also question, for example, why the age of the universe is twelve orders of magnitude larger than the time it takes the Earth to rotate. We know however, that any effort to answer this latter question would end in vain, since both the period of the Earth’s rotation and the age of the universe are not fundamental in any sense. Both are relevant only to the place (the Earth) and time (13 billion years after the big bang) in which humans exist. In analogy one might speculate that the strength’s of the different forces are also not fundamental but in some sense dependent on the way humans perceive them from their biased perspective of the universe.

Such discussions about the difference between the strengths of the electroweak force and that of gravity would have remained purely philosophical, had it not been that this difference is the cause of highly unnatural parameters in the theory of the elementary particles known as the Standard Model (SM). This fine tuning of the SM is known as the Hierarchy problem and we will now give some discussion as to how it arises.

Under the assumption that the SM is an effective theory that is completed
at the Planck scale, the loop processes shown in figure 1.1 impart quantum corrections to the tree level Higgs mass. After calculating these amplitudes and cutting off their momentum integrals at the Planck energy, the physical Higgs mass is found to be \( M^2_H \approx M^2_{\text{bare}} - M^2_P \), \(^\text{(1.1)}\)

where the two numbers appearing on the r.h.s are the bare mass of the Higgs particle squared and the Planck mass squared. Since the physical Higgs mass is expected to be \( \sim 120 \) GeV and the known Planck energy scale is \( \sim 10^{19} \) GeV, we need to choose in the SM a bare mass parameter of \( M^2_{\text{bare}} = 100000000000000000000000000000000001442 \), \(^\text{(1.2)}\)

in order to obtain the right physical mass. It is important to note that the last four digits of this number are extremely important as the physical Higgs mass would vanish if we were to simply ignore them. Most physicists find such a finely tuned parameter unsatisfactory. If one takes a pragmatic view to this and simply accepts this parameter within the theory then the SM matches the experimental results to extremely high precision. However, theorists suspect this absurd number might be a harbinger indicating that the SM will need to be modified as we probe higher energies, thus giving us some insight into what the next theory should be. In this sense the Hierarchy problem is more of a

---

\(^2\)Ignoring all numerical factors multiplying the energy scales.
ADD have presented a solution to the Hierarchy problem by assuming \textit{a priori} the phenomenologically incorrect equality $M_* \sim M_{\text{EW}}$, where $M_*$ is the new scale of gravity (called \textit{the fundamental scale}), thereby lowering the problematic cutoff scale appearing in equation \eqref{eq:1.1} and removing the fine tuning of the theory. However, in doing so gravity has become much stronger than what we experience. Fortunately, as these authors have found, this can be amended by invoking the existence of new dimensions. If there existed $n$ additional large extra dimensions through which gravity could propagate, the force of gravity would appear weak at large distances because the gravitational field lines would be diluted in the extra dimensional bulk. Thus, the commonly experienced and measured weakness of gravity, $M_P$, would only be illusional owing to the scale at which we live. If the new dimensions are assumed to be compact and flat and have volume $R^n$, the relationship
\begin{equation}
M_P^2 \sim M_*^{2+n} R^n, \tag{1.3}
\end{equation}
can be established\footnote{For a derivation see section \eqref{sec:1.2.1}}. The reason why we don’t actually see or experience the new dimensions is explained by assuming that all the SM fields, except gravity, are stuck on a 4 dimensional brane. Bounds on the size of the extra dimensions can then be inferred from collider limits, torsion balance experiments, supernovae cooling rates, and absence of BH effects in neutrino cosmic ray data. For a more complete list of experiments see the review by Kanti \cite{Kanti}. 

Interestingly, the model is also experimentally falsifiable. The equality of the fundamental Planck scale, $M_*$, and the electroweak scale, $M_{\text{EW}}$, makes many of the features including, as we shall see, BHs, within the view of the LHC. An absence of these signatures within some reasonable threshold ($10^{-10^2}$) above the electroweak scale, would rule this model out as a solution to the Hierarchy problem.
1.2.1 The relationship between $M_P$ and $M_*$

In the previous section we posited that gravity was weak because it was diluted in the extra dimensional bulk. This can be seen most clearly from equation (1.3). In this section we derive this expression from the Einstein-Hilbert action in $n + 4$ dimensional space-time. The action in $n + 4$ dimensional space-time, taking $c = 1$, is

$$I_{n+4} = \int \left( \frac{1}{16\pi G_{(n+4)}} R + \mathcal{L} \right) \sqrt{-gd^{n+4}x},$$

where $G_{(n+4)}$ is the gravitational constant, $R = g_{AB}R^{AB}$ is the Ricci scalar and $g$ is the determinant of the metric. Roman letters run from $1, \ldots, n + 4$.

We are interested in what is observed on “human” scales, that is, we would like to know the effective theory on scales much larger than the size of the bulk. As the matter fields are stuck on a four dimensional brane we can write $\mathcal{L} = \delta(x_\perp)\mathcal{L}_4$, where $x_\perp$ are the directions perpendicular to the brane. If we also make the approximation that the gravitational field does not vary significantly within the extra dimensional fibre then we can integrate out these perpendicular dimensions to obtain

$$I_{n+4} = \int \left( \frac{V_n}{16\pi G_{(n+4)}} R + \mathcal{L}_4 \right) \sqrt{-gd^4x},$$

but usually, in four dimensions, the action is written:

$$I = \int \left( \frac{1}{16\pi G} R + \mathcal{L}_4 \right) \sqrt{-gd^4x}.$$

Thus we obtain the relationship

$$G = \frac{G_{n+4}}{V_n}.$$  

To transform this into equation (1.3) we first clear up any ambiguities that may arise with units. It is customary to work with $\hbar = c = G = 1$. Normally in
3-spatial dimensions we can define a natural system of units (known as Planck units) out of the following combinations of $\hbar$, $c$ and $G$

\[
M_P = \left(\frac{\hbar c}{G}\right)^\frac{1}{2} = 2.18 \times 10^{-8}\text{kg}, \quad (1.8)
\]
\[
R_P = \left(\frac{\hbar G}{c^3}\right)^\frac{1}{2} = 1.62 \times 10^{-35}\text{m}, \quad (1.9)
\]
\[
E_P = M_P c^2 = 1.22 \times 10^{19}\text{GeV}, \quad (1.10)
\]
\[
t_P = \frac{\hbar}{M_P c^2} = 5.31 \times 10^{-44}\text{s}. \quad (1.11)
\]

However it is obvious from equation $(1.7)$ that the dimensions of $G$ and $G_{n+4}$ are not equal and in $n+4$ dimensions the usual Planck system does not survive. We therefore create a natural system that applies in the $n+4$ case. It is necessary that equations $(1.8)$-$(1.11)$ should be modified so that:

\[
M_{P(n+4)} = \left(\frac{\hbar^{1+n} c^{1-n}}{G_{n+4}}\right)^\frac{1}{2^{1+n}} \equiv M_{EW} = 1.78 \times 10^{-24}\text{kg} \quad (1.12)
\]
\[
R_{P(n+4)} = \left(\frac{\hbar G_{n+4}}{c^3}\right)^\frac{1}{2^{1+n}} \text{undetermined up to } n \quad (1.13)
\]
\[
t_{P(n+4)} = \left(\frac{\hbar G_{n+4}}{c^{5+n}}\right)^\frac{1}{2^{1+n}} = 6.58 \times 10^{-28}\text{s} \quad (1.14)
\]

Now keeping $G$ and $G_{n+4}$ explicit and setting $\hbar = c = 1$ (natural units) we obtain from equation $(1.8)$, $G = M_P^2$, which is the usual four dimensional expression and from equation $(1.12)$ we obtain $G_{n+4} = \frac{1}{M_{P(n+4)}^{2+n}}$. Therefore, equation $(1.7)$ becomes:

\[
M_P^2 = M_{P(n+4)}^{2+n} R^n. \quad (1.15)
\]

Now for convenience we set one fundamental energy scale, $M_* = M_{P(n+4)}$, and obtain the relation $(1.3)$ that we set out to find.
assumed equality $M_* = M_{EW}$:

$$\left( RM_{EW} \right)^n = \frac{M_P^2}{M_{EW}^2},$$  \hspace{1cm} (1.16)

$$R \sim 10^{\frac{30}{n} - 3} \text{ GeV}^{-1}. \hspace{1cm} (1.17)$$

Now putting in the units $\hbar c = 2 \times 10^{-14} \text{ GeV}.\text{cm}$, we see that the size of the extra dimension is related to the number of dimensions, by

$$R = 2 \times 10^{\frac{30}{n} - 17} \text{ cm.} \hspace{1cm} (1.18)$$

The $n = 1$ ($R \sim 10^{13} \text{ cm}$) scenario is immediately excluded since this size would change gravity so dramatically that it would catastrophically alter the planetary motion of the solar system; the theory is realisable only for $n \geq 2$. Currently, no deviations of Newton’s Law have been found to sizes as small as 160 $\mu$m \cite{8}.

1.3 Orbifolds

Formally, an orbifold is a generalisation of a manifold such that locally it looks like a quotient space of Euclidean space formed under the action of a finite group of isometries. Orbifolds were first named\footnote{In fact the name is a result of a democratic vote held in one of Thurston’s classes, see the reference for details.} and studied by William Thurston, see \cite{9} for his lecture notes. Since then they have played a large role in physics, especially in string theory and the duality theorems of M theory, see \cite{10} for a review.

Initially, it was assumed that the extra dimensional space (known as a fibre) in the ADD model was a compact and smooth manifold. Thus, such spaces would be free of singularities, edges, and boundaries. Beginning with the pioneering work of Kawamura \cite{11}, such assumptions began to be relaxed.

Kawamura considered an $SU(5)$ gauge symmetry on an extra dimensional
space that was an $S_1/Z_2$ orbifold. This orbifold has the base space of a circle, $S_1$, however, identified under the reflection in its mirror symmetry, $Z_2$, this circle reduces to a line with two end points (known as it’s physical space), see figure 1.2. Boundaries and singularities are common features of orbifolds.

It was found that the orbifold, by way of it’s reflection boundary conditions, could break the $SU(5)$ gauge symmetry down to the SM, $SU(3)_c \times SU(2)_L \times U(1)_Y$, thereby presenting an alternative symmetry breaking mechanism to the Higgs mechanism. However, one significant problem was encountered in that the fifth component of the five dimensional gauge field could not be removed from the low energy theory. This is most easily seen by observing that the fifth dimensional derivative transforms like $\partial_5 \rightarrow -\partial_5$ under the reflection symmetry. Yet the remaining derivatives, $\partial_\mu$, are unaffected. Since the covariant derivative defined by

$$D_M = \partial_M - ig A_M, \hspace{1cm} (1.19)$$

must transform properly under the reflection symmetry, then the fifth component of the gauge field must transform oppositely to the other field components. Thus, any coset of the symmetry that we break, must leave spurious fifth component fields in the low energy theory. It was quickly realised that this could be overcome by choosing instead the orbifold $S_1/Z_2 \times Z'_2$, shown in figure 1.3, where the base space is identified under a second reflection symmetry, $Z'_2$. While the physical space is topologically unchanged (i.e., it is still a line with two end points) the additional reflection symmetry allows the fifth dimensional component to be removed from the low energy theory.
Since this realisation, a plethora of SM extensions have been studied on the \( S_1/Z_2 \times Z_2' \) orbifold. In chapter 4 we study one possible extension known as the quark-lepton left-right model.

1.4 Split fermions

One poignant problem with the theories of extra large dimensions is that the proton can decay much quicker than is observed. Operators of the kind

\[
\frac{QQQL}{M_2^2}, \quad (1.20)
\]

are now enhanced with the alteration of the fundamental cut off scale. One way to suppress this rapid decay is to physically separate the quarks and leptons in the higher dimension(s). Such models are generically called split fermions models, for example see references [12, 4, 13]. Such localisation is achieved through a kink solution to a bulk scalar field, shown in figure 1.4.

Assuming that the fermion profiles along the extra dimensions are Gaussian’s of width \( \mu^{-1} \), it was shown by Arkani-Hamed and Schmaltz (AS) \([14]\) that regardless of the particular proton decay inducing operator, a quark lepton separation of \( L \), leads to a suppression of the rate of proton decay by \( \sim e^{-\mu^2L^2} \). Thus provided \( \mu \gtrsim 10/L \) the proton lifetime will be greater than

\[5\] The \( Q \) representing a quark field and the \( L \) a leptonic field.
In their original paper, AS [14] also noted that split fermions can be used to explain the hierarchy in SM fermion masses. Since the left and right handed components of a given SM fermion are in different gauge representations they can be localised at different points in the extra dimension. The Higgs Yukawa coupling in the effective 4d theory is

\[ \mathcal{L} = f \int_0^L dy F_R^c F_L \Phi = f k K \overline{f_R} f_L, \]  

where \( L \equiv \pi R/2 \) is the length of the extra dimension, \( k \) is the Higgs VEV, \( K = \int_0^L F_R(y) F_L(y) dy \sim e^{-\mu^2 r^2} \) and \( r \) is the separation between the left and right handed fields. Thus, even if the fundamental Yukawa coupling, \( f \), is of order one, the mass operator in the effective theory can be exponentially suppressed. Therefore, in this picture, it is natural to expect a hierarchy in SM fermion masses, since \( \mu \) and \( r \) will vary for different fermions.
Split fermions motivate studying bulk emission from BHs, as will be discussed in chapter 3. Furthermore, the split fermion mechanism will be employed in chapter 4 to suppress proton decay and obtain the correct SM masses for the QLLR extension to the SM.

1.5 Black holes

A BH is an object so dense that not even light can escape from within a certain distance known as the Schwarzschild radius. It has been known since the 1930’s that when a star of mass above the Chandrasekhar limit ($\sim 1.4M_\odot$) dies the electron degeneracy pressure can not support the gravitational pressure of the imploding matter and a BH forms.

Until recently, it was thought that BHs smaller than this limit could only have been formed in the beginning stages of the universe, when the conditions were hot and dense enough. However, speculation of a stronger fundamental theory of gravity like that of the ADD model opens up the possibility of actually creating BHs in the laboratory.

The LHC is expected to reach a centre of mass (COM) energy of $\sim 14$ TeV, which is greater than $M_{\text{EW}} \sim 0.246$ TeV. Thus, if the ADD model is the one realised by Nature, in the near future we may begin witnessing trans-Plankian collisions, i.e., collisions with COM energies greater than the fundamental scale, $M_\ast$. Since BHs are expected to form abundantly near the scale of gravity, we might naively speculate BH formation at the LHC. However, such speculation does require some justification, which we shall discuss in section 1.5.3.

Since light can not escape from within a BH, it was thought for a long time that they would be completely black. However, it was discovered in the 1970’s that due to quantum considerations, a BH will emit radiation with a thermal black body spectrum at a temperature related to it’s mass. In the next section we shall calculate this temperature within a non-perturbative framework of
1.5 Black holes

Figure 1.5: The spacetime manifold, $\mathcal{M}$, divided into two boundaries.

quantum gravity known as Euclidean quantum gravity.

1.5.1 Temperature and entropy

A notably quick derivation of BH thermodynamics can be made within a theory known as Euclidean Quantum Gravity (EQG) in which the Feynman sum over paths method is generalised to a sum over metrics. EQG was developed in the 70’s and 80’s by Hawking, Gibbons, and Hartle to mention a few and a good overview can be found in [15]. Their treatment proceeds as follows.

Consider two induced metrics $g_1$, $g_2$ on the 3-surfaces $S_1$ and $S_2$ respectively such that these surfaces bound a compact volume, see figure [1.5]. Then the probability amplitude for measuring a metric $g_1$ on $S_1$ and $g_2$ on $S_2$ is given by the path integral over all those metrics within the volume contained by these boundary conditions:

$$\langle g_2, S_2 \mid g_1, S_1 \rangle = \int D[g] e^{iI[g]},$$

where $I[g]$ is the action of the metric field $g$, and $D[g]$ is an appropriate measure.
on the space of metrics. It has been shown that the action of the vacuum is:

\[
I = -\frac{1}{16\pi G} \int d^4x R\sqrt{g} + \frac{1}{8\pi G} \int d^3x (K - K^0)\sqrt{h}.
\]  

(1.23)

The first volume part being identical with the usual Einstein-Hilbert action that recovers Einstein’s equations in free space \((G_{\mu\nu} = 0)\). The second part needs to be added for reasons of consistency and is essentially a surface integral of the difference between the trace of the external curvature on the boundary, \(K\), and the same trace evaluated in flat space, \(K^0\). The induced metric field on the boundary is denoted \(h\).

The deep part of the theory stems from the startling connection between these path integrals and the statistical mechanical partition function. Recall that the partition function of a Hamiltonian \(\hat{H}\) is given by:

\[
Z = \text{Tr} \, e^{-\beta \hat{H}}, \quad \beta = \frac{1}{kT}.
\]  

(1.24)

This looks remarkably similar to the trace of the evolution operator, \(e^{\frac{i}{\hbar} \hat{H}(t-t_0)}\) in quantum mechanics. Indeed if we make the identification \(\beta = \frac{i(t-t_0)}{\hbar} = \frac{\tau}{\hbar}\) (in other words a wick rotation, \(\tau = it\), followed by periodical identification in intervals \(\hbar\beta\) of time) they are equal. The wick rotation changes the metric from the usual pseudo-Riemannian signature \((-\,+,\,+,\,+)\) into a Euclidean signature \((+\,,\,+\,,\,+\,,\,+\,)\) and the periodicity needs to be determined case by case.

We now show that in the case of the Schwarzschild geometry the action is periodic in time with the period, \(\beta = 8\pi M\). The classical Schwarzschild solution to Einstein’s equation is:

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2.
\]  

(1.25)

The easiest way to see the periodicity is to change equation (1.25) into the null Kruskal coordinates. This can be done by the following change of
1.5 Black holes

variables:

\[ U = \left(1 - \frac{r}{2M}\right)^{1/2}e^{(r-t)/4M}, \quad (1.26) \]
\[ V = \left(1 - \frac{r}{2M}\right)^{1/2}e^{(r+t)/4M}. \quad (1.27) \]

Now we analytically continue \( t \) into the complex plane, \( t = \tau + i\sigma \). One then easily verifies that:

\[ U = |U|e^{-i\sigma/4M} = |U|e^{-i\sigma/4M+i2\pi n}, \quad (1.28) \]
\[ V = |V|e^{i\sigma/4M} = |V|e^{i\sigma/4M+i2\pi k}, \quad (1.29) \]

where \( k \) and \( n \) are arbitrary integers. So these coordinates and the metric are invariant under the transformation \( \sigma \rightarrow \sigma + 8\pi M \) or imaginary time is periodic in the Schwarzschild solution with period \( \beta = 8\pi M \).

The partition function in statistical mechanics can then be interpreted as the path integral starting from a given metric and returning to that state after a period \( \hbar \beta \) in imaginary time. This can be expressed:

\[ Z = \int dg \langle g | e^{-\beta H} | g \rangle \quad (1.30) \]
\[ = \int D[g(\tau)]e^{iI[x(\tau)]} \quad (1.31) \]

It is through this connection that the thermodynamics of BHs becomes apparent. Originally Hawking derived the result of BH radiation \[16\] from the semi-classical approximation of quantum field theory in curved space-time. Since thermodynamical properties usually imply some kind of course-grained micro-states, i.e., like the way that the motions of many atoms collectively make up the temperature of a room, the question remained as to what the micro-states of the BH actually were. In the present derivation since entropy can be found through the partition function and this is in turn related back to the sum over all possible metrics, it seems to be implied that the BH entropy is caused by the fluctuations of the spacetime itself. Unfortunately since EQG
is a non-perturbative theory it can not give a precise description of these gravitational micro-states. However, there have been several interesting attempts to do this in string theory [17, 18] and loop quantum gravity [19].

One can now calculate the temperature and entropy of the Schwarzschild BH to first order in EQG. The first step is to make the saddle approximation to the partition sum. As in normal quantum mechanics one expects the most significant contribution to come from the classical path

\[
Z = \int Dx e^{-\frac{\text{I}[x]}{\hbar}} \approx e^{-\frac{\text{I}^*}{\hbar}},
\]

(1.32)

where \( I^* \) is the classical action. In free space \( R = 0 \), so it is only the surface part of equation (1.23) that contributes. The \( K \) integral is given by,

\[
\int K(-h)^{1/2} d^3x = \frac{\partial}{\partial n} \int (-h)^{1/2} d^3x,
\]

(1.33)

where \( n \) is a unit normal for each point of \( \partial M \). By choosing the boundary \( \partial M \) to be the \( t \)-axis multiplied by a sphere of radius \( r_0 \) one finds,

\[
\int K(-h)^{1/2} d^3x = \frac{\partial}{\partial n} \int (-h)^{1/2} d^3x,
\]

(1.34)

\[
= \frac{\partial}{\partial n} 4\pi r_0^2 \int (1 - \frac{M}{r_0} + O(r_0^{-2}))dt,
\]

(1.35)

\[
= \int (8\pi r_0 - 4\pi M - 8\pi M + O(r_0^{-2}))dt.
\]

(1.36)

Where in equation (1.35) it has been made use of the fact that at distances far from the event horizon the Schwarzschild metric in equation (1.25) becomes:

\[
ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 + \frac{2M}{r})dr^2 + r^2d\Omega^2 + O(r^{-2}).
\]

(1.37)
For the flat space metric $K^0 = 2r^{-1}$. So,

\[
I^* = \frac{1}{8\pi G} \int d^3x (K - K^0) \sqrt{\mathcal{h}}, \quad (1.38)
\]

\[
= -\frac{M}{2G} \int dt + \mathcal{O}(r_0^{-1}), \quad (1.39)
\]

\[
= -\frac{M\beta\hbar}{2G}, \quad (1.40)
\]

where the last line was found by performing the path integral over the period, $\beta\hbar$. Therefore,

\[
I^* = -\frac{(\beta\hbar)^2}{16\pi G\hbar}. \quad (1.41)
\]

One can now use elementary thermodynamic relationships to obtain the thermodynamical properties of the BH. For instance the temperature is,

\[
T = \frac{1}{k\beta} = \frac{1}{8\pi kM}, \quad (1.42)
\]

which agrees with the semi-classical result first obtained by Hawking in \[16\].

Next one can calculate the mass\(^6\):

\[
E = -\frac{\partial \ln Z}{\partial \beta}, \quad (1.43)
\]

\[
= \frac{\hbar\beta}{8\pi G}, \quad (1.44)
\]

\[
= M, \quad (1.45)
\]

as we expect to first order, and finally the entropy,

\[
S = k(\ln Z + \beta M) = \frac{kA}{4G\hbar}. \quad (1.46)
\]

This is, of course, the celebrated semi-classical result for BH entropy.

\(^6\)Even though the classical mass is $M$, it is conceivable that higher order quantum fluctuations may alter this.
1.5.2 Hawking radiation and greybody factors

The emission from a BH, to a first approximation, follows a black body spectrum \[16\]. However, the radiation is not exactly black body and different particles will be radiated by more or lesser amounts depending on their properties \[20\], \[21\], \[22\]. In \( n + 4 \) dimensions the power emission spectrum is given by:

\[
\frac{d\mathcal{E}(E)}{dt} = \sum_{j,b} \sigma_{j,b}(E) \frac{E}{\exp(E/T_{BH}) + 1} \frac{d^{n+3}k}{(2\pi)^{n+3}},
\]

\( (1.47) \)

where \( j \) labels the total angular momentum of the particle and \( b \) labels any other quantum numbers associated with it and the \( \mp \) sign accounts for whether the emitted species is a boson or a fermion. The factor of \( \sigma_{j,b}(E) \) is known as the greybody factor, it is a frequency dependent function which arises from the back scattering of the emitted radiation off the metric back into the BH. We note that if this value were unity then \( (1.47) \) would be precisely the equation for the power emission from a black body.

![Diagram of BH with fields](image)

Figure 1.6: Emission of scalar (H), fermion (f) and spin 2 gravitons (g) into the bulk from a \( d \)-dimensional BH.

Equation \( (1.47) \) describes the situation shown in figure 1.6 where the field modes are permitted to propagate off the brane into what is called the bulk. This situation will occur when the field modes are not localised to the brane, i.e., as occurs for gravity in the ADD model, or scalars in split fermion models.\(^7\)

\(^7\)See the introduction to split fermion models in section (1.4).
1.5 Black holes

and also fermions in supersymmetric (SUSY) split fermion models\[8\].

Figure 1.7: Emission of SM fields onto the brane.

However, the majority of research performed on TeV scaled BH emission [23, 24] has focussed on brane localised emission depicted in figure 1.7. This is because within the ADD paradigm all SM fields are stuck onto a brane. In this case the phase space in equation (1.47) is reduced, and the power emission becomes:

\[
\frac{d\mathcal{E}_{\text{brane}}(E)}{dt} = \sum_{j,b} \sigma_{j,b}^{\text{brane}}(E) \frac{E}{\exp(E/T_{BH}) + 1} \frac{d^3k}{(2\pi)^3},
\]

(1.48)

Kanti et al [25, 23] have shown that the greybody factor for brane localised fermions is:

\[
\sigma_{j,b}^{\text{brane}} = \frac{\pi(2j + 1)}{E^2} |A|^2,
\]

(1.49)

where \( j = \frac{1}{2}, \frac{3}{2}, \ldots \), and \( |A|^2 \) is the absorption probability. Using the fact that \( d^3k = 4\pi E^2 dE \) for massless fields, we can then write the differential power emission as:

\[
\frac{d\mathcal{E}_{\text{brane}}(E)}{dEdt} = \frac{1}{2\pi} \sum_j D_j \frac{E}{\exp(E/T_{BH}) + 1} |A|^2,
\]

(1.50)

\[8\] For more on this see chapter [3].
where we have defined the degeneracy factor $D_j = 2j + 1$. It is convenient to redefine the sum taken here over the half integer $j$, to be taken over the non-negative integer $l = 0, 1, 2 \cdots$. Then the degeneracy factor must be written as $D_l = 2(l + 1)$.

The equations presented in this section are used in section 3.5.2 to compare the bulk emission to the brane localised emission during the evaporative decay of a $d$-dimensional BH. These rates will be important signatures in BH searches planned for the LHC. We now investigate the plausibility of creating BH’s in particle accelerators.

1.5.3 Black hole creation from a high energy collision

It is often claimed that a BH will form if a quantity of matter is squeezed to within a spherical volume defined by it’s Schwarzschild radius $2Gm/c$, but this is not always the case, for instance the angular momentum of the in falling material is also an important factor. Actually, the question of if and when under general conditions a BH will form remains an outstanding theoretical problem. One entertaining counter example to the above claim comes from the universe itself. The mass of the observable universe is estimated to be $\sim 3 \times 10^{52}$ kg giving a Schwarzschild radius of about 6 billion light years. At some time in the distant past the observable universe would have been contained within it’s Schwarzschild radius. However the Universe is not and has never been accurately described by a BH solution. The Universe is after all flat and described by the Robertson-Friedman-Walker metric. It is quite clear then that the condition for BH creation must be more involved than simply meeting the Schwarzschild density requirement.

The authors [26] consider the classical scattering of two Aichelburg-Sexl metrics (which is found by boosting the Schwarzschild solution [27]) at non-zero impact parameter $b$ and find that the cross section for BH formation is bounded by

$$\sigma > \pi b^2.$$  (1.51)
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These Aichelburg-Sexl collisions approximate the gravitational dynamics between two colliding proton beams (like in the LHC) very well, providing strong evidence that at least classically BHs do form via such collisions. Indeed some authors argue that the production rates will be so high that the LHC will become a BH factory [28, 29, 30] producing about one per second.

1.5.4 Phases of black hole decay

Once produced the BH is thought to go through four main stages of decay [30, 31, 25];

1. **Balding phase:** The BH sheds the “hair” collected from the original colliding particles via gravitational radiation.

2. **Spin-down phase:** Due to the non-zero impact parameter of the colliding particles the BHs formed in particle accelerators will have a lot of angular momentum. In this phase these Kerr BHs will lose their angular momentum via Hawking radiation [16] and possibly super radiance.

3. **Schwarzschild phase:** Once all the angular momentum has been radiated away a Schwarzschild BH will remain. In this phase the Schwarzschild BH will shrink by radiating its energy in Hawking radiation. The emission from the Schwarzschild phase is expected to greater than the emission from the other phases, since the BH during this phase is hot and large enough to abundantly produce SM particles. The emission would be isotropic and therefore have a distinctive signature in LHC searches. For calculations of scalar, fermion and vector particle emission onto the brane see [23, 24]. These results focus on brane localised emission but there is some recent motivation for studying bulk emission as we shall discuss in chapter 3, where we calculate the emission rates for bulk fermions on a Schwarzschild background.
4. **Planck phase:** As the mass of the BH approaches the fundamental scale, quantum gravity effects of the BH begin to take over. Since no completely satisfactory theory of quantum gravity has been found this phase is least understood. In Chapter 2, we present a novel model which partially handles processes that might occur within the Planck phase. In it we quantize the BH, then study it’s lowest excitation. In order not to destroy any of the well established results of the SM, we postulate that all the BH non-self interactions appear as effective non-renormalisable operators in the Lagrangian at the dimension five level. They can be understood as being mediated via some not yet understood theory of gravity.

### 1.5.5 The information paradox

Since the discovery of the Schwarzschild solution in 1916, BHs have been a source for many Gedanken experiments, often leading to new breakthroughs in theoretical physics. Perhaps none is more relevant today than that of the *information paradox*. The paradox will now be surmised.

During the seventies, in a collection of works known as the *no-hair theorems* [32, 33, 34, 35, 36] it was shown that the fields external to the event horizon of a BH are determined uniquely by only three quantities—mass, angular momentum, and charge. Aside from these quantities any other information about the material falling into a BH is lost once the material falls beyond the event horizon. For example, one would not be able to tell, after passing the event horizon, whether it was a chair or a computer (if they both contained the same mass, angular momentum and charge) that had fallen in.

Classically, this does not present a problem because the BH is eternal. One can imagine that the information, though inaccessible to outside observers, is contained forever within the event horizon. However, we now know that due to quantum effects the BH has a temperature and if this temperature is hotter than it’s surroundings the BH will eventually disappear altogether leaving be-
hind a thermal information-less product of particles. From this semi-classical viewpoint the BH has destroyed the information.

To push the implications of this idea a little further, consider what happens if we let an entangled pair of spins fall into a BH. Once the BH completely evaporates the remaining radiation is thermal and therefore in a completely mixed state. Yet the initial state of entangled spins contained quantum correlations. Thus, the interaction with the BH has destroyed the quantum information of the spins. In other words, processes involving BHs are non-unitary. This possibility is of even more importance when we consider that BHs may be operating at the TeV scale through quantum processes like those discussed in chapter 9.

Unitarity is one of the fundamental axioms of quantum mechanics and there are no known experimental violations of it. Of course, just because we have strong evidence for an idea today does not by any means make it infallible. Nevertheless most theories of modern physics accept unitarity as a fundamental principle. In string theory, BH processes must be unitary since they are related via the AdS/CFT conjecture [37] to a conformal field theory on the horizon. Recently, Stephen Hawking [38] has claimed to have shown that in EGQ, when the calculation is done correctly, the process of scattering from a BH is actually unitary. One may then think that the information paradox is resolved. There are two reasons why this author thinks that this issue of unitarity loss remains open and needs to be more thoroughly addressed. Firstly, in lack of an experimentally tested theory of quantum gravity no one can be sure that the answers provided by string theory or EQG etcetera correctly describe nature. Secondly, there are related effects [39] performed in the setting of a QFT on a Rindler background that don’t involve BHs but do show violations of unitarity.

In chapter 5 we attempt to gain further insight into the non-unitary phenomena [39] by looking at the loss of entanglement in a specific accelerating

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9However, in chapter 2 we assume that the Planck phase processes occur unitarily, as previous authors have done.
spin system. In our work we use an open quantum system and calculate the loss of quantum correlations. However, it must be admitted that the non-unitarity found in our system can be explained by the decohering interactions that our system has with the environment.

1.6 Outline of this thesis

In the next chapter we will study the effective field theory for BH process in the Planck phase, before moving on in chapter 3 to more rigorous work on bulk fermion emission in the Schwarzschild phase. After this we take leave of BHs and in chapter 4 study the QLLR extension of the SM on a five dimensional orbifold, before returning to investigations of BHs in chapter 5, where we look at the information loss of a pair of accelerating spinors.
CHAPTER 2

Black hole effective field theory

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As early as 1974 [40] it was hypothesised that 4-dimensional black holes (BHs) might be quantised in irreducible mass [1]. In the largely unknown terrain of quantum BH physics this hypothesis paints a speculative picture for what the final stage of a BH evaporation might look like: the BH will de-excite down a series of mass levels, see figure 2.1, by emitting radiation (in the form of particles) analogously to the way an electron in a hydrogen atom returns from an excited state down to the ground state by emitting photons.

One important difference between the electronic energy levels of an atom and the mass energy levels of a BH is that the latter need not have a stable

\footnote{The ordinary mass of a spinning BH can be reduced to the irreducible mass through a process known as The Penrose process [41] (and similarly for a charged BH [42]), therefore, the irreducible mass is in some sense more fundamental than the actual mass of a BH.}


**Figure 2.1:** Quantised mass levels of a black hole

Ground state and may entirely annihilate at the end of their de-excitation. However, there are a variety of models in which BH remnants remain at the end of the evaporation, see for instance [33, 44]. In what follows we will take the view that the BH can fully evaporate into SM particles and that no remnant remains.

This kind of mass quantisation differs from the usual way in which we quantise fields, because in that case we usually have a particle with a specific mass in mind. In the present hypothesis the different quantum states of the BH have different masses and thus we could think of each state as representing a quantum field itself.

In the present work we are interested in mini BHs that may form in forthcoming particle accelerators like the LHC [28, 29, 30]. We are thus motivated to extend the 4-dimensional BH quantisation procedure to higher dimensions.
In $n + 3$ spatial dimensions the full Kerr solution is not analytically solvable, but since we are building a picture for quantum BH’s in the Planck phase (see section (1.5.3)) it is expected by this time all the angular momentum will have evaporated away. Fortunately the Resissner-Nordstrom BH’s (charged but with no angular momentum) are analytically solvable in higher dimensions and we now investigate the quantisation of them. In units with $\hbar = c = 1$ we have the Einstein-Hilbert-Maxwell action:

$$S = \frac{1}{16\pi G_{n+4}} \int d^{n+4}x \sqrt{-g} \left( R - F_{\mu\nu} F^{\mu\nu} \right).$$ (2.1)

Assuming a spherically symmetric background of the form:

$$ds^2 = -f^2 dt^2 + g^2 dr^2 + r^2 d\Omega_{n+3},$$ (2.2)

the solution [45] is found to be,

$$f = g^{-1} = (1 - \frac{C}{r^{n+1}} + \frac{D^2}{r^{2(n+1)}})^{\frac{1}{2}},$$ (2.3)

where $C = \frac{16\pi G_{n+4} M_{bh}}{S_{n+3}(n+2)}$ and $D^2 = \frac{2Q^2 G_{n+4}}{(n+2)(n+1)}$.

From equation (2.3) there is an event horizon if:

$$r_{n+1}^{\pm} = \frac{1}{2} C \pm \sqrt{\frac{1}{4} C^2 - D^2}.$$ (2.4)

We now want to perform the canonical quantization on the area of the outer horizon ($A = A(r_+)$) in the same spirit as Bekenstein [40]. The irreducible mass of a BH, $M_{ir}$, is related to its area via:

$$M_{ir}^2 = \frac{A}{16\pi G_{n+4}^2}.$$ 

Quantising the irreducible mass $M_{ir} = n_h g_p$ and the charge $Q = q e$ and rear-
ranging for $M_{bh}$ we find:

$$\frac{M_{bh}}{M_\ast} = c_1(n_b g_p)^{\frac{n+1}{n+2}} \left[ 1 + \frac{1}{4} \frac{c_2 q^2 \alpha_{em}}{(n_b g_p)^{\frac{2(n+2)}{n+3}}} \right]$$

where,

$$c_1 \equiv (n + 2) \left( \frac{S_{n+3}}{16 \pi} \right)^{\frac{n+2}{n+3}}$$

$$c_2 \equiv 2 \sqrt{2(n + 2)(n + 1)} \left( \frac{S_{n+3}}{16 \pi} \right)^{\frac{2n+2}{n+3}}$$

$S_m = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the surface area of a unit $m$-sphere (i.e., in the case of an $(n+3)$-sphere $A = S_{n+3} r_{n+2}^n$) and $n_b (\in N)$ and $q (\in Z)$ are the quantization numbers for mass and charge respectively. $n_b$ is not to be confused with $n$ the number of extra dimensions\(^2\). The BH mass gap, $g_p$, is controversial. Some authors use $g_p = 0.614/\pi$, which is calculated in a loop quantum gravity framework \cite{47}.

In the $n = 0$ case, $c_1$ and $c_2$ are equal to one and we recover the 3 dimensional quantization scheme:

$$\frac{M_{bh}}{M_\ast} = (n_b g_p)^{1/2} \left[ 1 + \frac{1}{4} \frac{q^2 \alpha_{em}}{n_b g} \right]$$

We see that the lightest such excitation has zero charge and zero angular momentum and thus corresponds to a neutral scalar field. To this end the analysis that follows applies not just in the context of BHs but to any scalar gauge singlet added to the SM \cite{34}. Note we do not believe there is a need to include thermal statistical factors in the decay phase space as was done in \cite{46} since in the Planck phase Hawking radiation is believed to break down. This is the simplest extension one can make to the SM and interestingly if a discrete symmetry is added then a stable remnant particle is allowed. Such scalar gauge singlets have been extensively studied as possible dark matter

\(^2\)We have adopted the notation used in \cite{46}.

\(^3\)With the new fundamental scale $M_\ast$ setting the mass scale of effective scalar field theory.
candidates see [48, 49, 50, 51, 52].

The preceding analysis allows us to think of a BH in its lowest energy state, as a scalar particle. To emphasise this very important point we surmise the logic that lead us to this conclusion:

1. A mini BH will interact with SM particles through gravitational interactions, the BH is then assumed to have a field theory description (like all other particles in the SM), which at the low energy level will be described by an effective field theory.

2. The BH itself is a quantum object with many degrees of freedom, these are defined by the quantum numbers of spin, charge and angular momentum.

3. The lowest energy excitation of these degrees of freedom is the uncharged, spinless particle (or scalar).

4. Since this scalar particle has the smallest mass of all BH degrees of freedom it will also contribute the most to scattering cross sections and decay rates in the effective theory.

5. Therefore, the scalar particle will be the first order approximation to the entire BH dynamics.

After accepting that a scalar field is the correct first order approximation to BH dynamics it warrants asking how such a field will interact with normal matter. Originally an ansatz was constructed to describe the interaction between two charged fermions and a doubly charged BH [46]. It consisted of a Yukawa Lagrangian of the type:

\[
L_{\text{int}} = i k_{\text{eff}} M_{\text{bh}} \phi_{\text{bh}}^{++} \overline{\Psi}_f \hat{C} \Psi_f + \text{h.c.}, \tag{2.6}
\]

where \( \hat{C} = i \gamma_2 K \) is the charge conjugation operator, \( K \) being complex conjugation and \( \Psi_f \) is the fermion field. Similarly, one can write an effective
Lagrangian for a neutral scalar BH by removing the \( \hat{C} \) operator:
\[
\mathcal{L}_{\text{int}} = i k_{\text{eff}} M_{\text{bh}} \phi_{\text{bh}} \overline{\Psi}_f \Psi_f + \text{h.c.}
\] (2.7)

Photon-BH interactions have also previously been considered \cite{53}:
\[
\mathcal{L}_{\text{int}} = \frac{k}{M_s} \phi_{\text{bh}} F_{\mu\nu} F^{\mu\nu}.
\] (2.8)

The renormalisable SM Lagrangian can be written down in a few lines:
\[
\mathcal{L}_{\text{SM}} = -\frac{1}{2g^2} \text{Tr} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2g^2} \text{Tr} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4g^2} B_{\mu\nu} B^{\mu\nu} + i \frac{\theta}{16\pi^2} \text{Tr} G_{\mu\nu} \tilde{G}^{\mu\nu} + |D_\mu H|^2 + \bar{Q}_i i \not\!D Q_i + \bar{U}_i i \not\!D U_i + \bar{D}_i i \not\!D D_i + \bar{L}_i i \not\!D E_i + \text{c.c.}
\] (2.9)

Here, \( \tilde{H} = i\sigma_2 H^* \), and \( i, j = 1, 2, 3 \) are generation indices. It is quite remarkable that the nineteen physically independent parameters in these few lines explain nearly all phenomena we have observed in our universe. The most general renormalisable Lagrangian we can add by including a scalar field is
\[
\mathcal{L}_{\phi} = \frac{1}{2} \partial_\mu \phi_{\text{bh}} \partial^\mu \phi_{\text{bh}} - \lambda \phi_{\text{bh}}^2 - \frac{1}{2} m^2 \phi_{\text{bh}}^2 - \frac{k}{2} |H|^2 \phi_{\text{bh}}^2 - \alpha \phi_{\text{bh}}^3 - \frac{h}{4!} \phi_{\text{bh}}^4.
\] (2.10)

Keeping terms up to the non-renormalisable dimension five level we have the Lagrangian:
\[
\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{g'}{\Lambda} \mathcal{L}_{\text{SM} \phi_{\text{bh}}} + (D\phi_{\text{bh}})^2 - V(\phi_{\text{bh}}) + \text{h.c.},
\] (2.11)

where \( g' \mathcal{L}_{\text{SM} \phi_{\text{bh}}} \) is symbolic for all SM operators times a \( \phi_{\text{bh}} \) operator with possibly different coupling constants \( g' \) on each term and \( D \) is the covariant derivative. We expect that the theory will be cut off at the fundamental Planck scale \( \Lambda = M_s \), where a new theory that accommodates gravity will take over.
2.1 Lepton flavour number violation

It is the second term of equation (2.11) that contains the neutral BH interactions used in previous work. For example, the Lagrangian in equation (2.8) follows immediately by taking $g' = \kappa$. One also sees that the interaction in equation (2.7) can be obtained from the $\frac{g'}{M^*} \Phi e_R \phi_{bh}$ term after Higgs breaking $\langle H \rangle = v,$

$$\frac{g'}{M^*} \Phi e_R \phi_{bh} + \text{h.c.} \rightarrow k_{\text{eff}} M_{bh} \bar{\psi}_e \psi_{e \phi_{bh}}, \quad (2.12)$$

if we make the identification $k_{\text{eff}} = \frac{g'v}{M^* M_{bh}}$. Note that in order to describe interactions of the kind in equation (2.6) we would need to add to equation (2.11) all terms of dimension five consistent with SM symmetries incorporating a doubly charged BH.

2.1 Lepton flavour number violation

It has been conjectured that BH processes will violate certain approximate global symmetries like lepton number [54, 55, 56]. Presently, we wish to extend the original work [46, 53] to include lepton family number violation. There are various processes which will violate lepton family number, for some interesting examples from the Little Higgs model with T parity see [57]. In the present case, we can naturally implement lepton family number violation by including the 3 generations of fermion fields into equation (2.11). Consider the terms

$$\mathcal{L} = \ldots + \frac{1}{M^*} \bar{L}_i H g'_{ij} e_R \phi_{bh} + \bar{L}_i H \lambda_{ij} e_R,$$  

where a summation over generations is understood ($i, j \in 1, 2, 3$). Now after Higgs breaking and rotating the weak eigenstates into mass eigenstates, i.e., $m = \mathbb{U} \lambda \mathbb{V}^\dagger$, this Lagrangian becomes

$$\mathcal{L} \rightarrow \bar{e}_{Li} k_{ij} e_R \phi_{bh} + m_i \bar{e}_{Li} e_{Ri}, \quad (2.14)$$
Importantly \( k \equiv \frac{g}{M_*} \bar{u}g\phi^+\phi^+ \) is not in general diagonal. Thus lepton family violating interactions like \( k_{ee}\mu e\phi_{bh} \) arise naturally in this picture as off diagonal terms in \( \bar{e}_Li_{ij}e_Rj\phi_{bh} \).

We now consider the decay mode \( \mu^{-} \rightarrow e^{-}e^{+}e^{-} \) \[^{58}\] , which proceeds through the processes shown in figure 2.2. It has been said \[^{46}\] that the experimental bound on this process would require a Planck scale at the 100 TeV range. In our approach we have an extra parameter in \( g' \), recall \( k = \frac{g'}{M_*} \), if we want a TeV scaled Planck mass then we must also tolerate \( g' \sim 10^{-3} - 10^{-4} \) sized couplings. The decay rate is:

\[
\Gamma(\mu^{-} \rightarrow e^{-}e^{+}e^{-}) = m_{\mu}^{5}k_{ee}k_{\mu e}, \tag{2.15}
\]

which puts a bound on the product of the two couplings,

\[
k_{ee}k_{\mu e} < 1.5 \times 10^{-7} \text{ TeV}^{-2}. \tag{2.16}
\]

Figure 2.2: Feynman diagram for the muon decay via the neutral scalar BH interaction.
2.2 Muon magnetic moment correction

Since we take $M_* \sim 1 \text{ TeV}$ this means

$$g_{ee} g_{\mu e} < 1.5 \times 10^{-7}, \quad (2.17)$$

which is consistent with a TeV scaled Planck mass. It is also interesting that with tolerable tuning we could have one $g$ rather large $\sim 10^{-1}$ and the other small $\sim 10^{-6}$. Our model would therefore be able to accommodate BHs that favour certain processes rather than interacting indiscriminately with only one coupling constant as is implied in [46].

2.2 Muon magnetic moment correction

In this section we calculate the correction that the muon magnetic moment will receive from the scalar BH excitation due to the diagram shown in figure 2.3.

![Muon magnetic moment correction](image)

Figure 2.3: Muon magnetic moment correction

The matrix element for this process is:

$$\mathcal{M} = e k^2 \xi M_{bh}^2 \overline{U}(p') \Lambda^\mu U(p) A_\mu(p' - p), \quad (2.18)$$

where,

$$\Lambda^\mu = \frac{-i}{(2\pi)^4} \int \frac{dk^4}{k^2 - M_{bh}^2 + i\epsilon} \frac{p' - k + m_l}{(p' - k)^2 - m_l^2 + i\epsilon} \frac{p - k + m_l}{(p - k)^2 - m_l^2 + i\epsilon}, \quad (2.19)$$
and we have allowed for lepton number violating vertices, i.e., the internal fermion lines in figure 2.3 could be non-muonic charged leptons of mass \( m_l \).

We use dimensional regularisation [59] to remove the ultraviolet divergence in this integral. This is achieved by performing the integration over a space of fractional dimension \( 4 - D \). The integral has a finite solution in this space, and we can determine the required 4 dimensional answer by taking the limit \( D \to 0 \) at the end of the calculation. To keep the dimensions correct we introduce the irrelevant dimensionful constant \( \kappa \). First we define:

\[
\begin{align*}
a(k) &= k^2 - M_{bh}^2 + i\epsilon, \quad \text{(2.20)} \\
b(k) &= (p' - k)^2 - m_l^2 + i\epsilon, \quad \text{(2.21)} \\
c(k) &= (p - k)^2 - m_l^2 + i\epsilon, \quad \text{(2.22)} \\
N^\mu(k) &= (p' - k + m_l)\gamma^\mu(p' - k + m_l). \quad \text{(2.23)}
\end{align*}
\]

then,

\[
\Lambda^\mu = -i\kappa^{4-D} \frac{(2\pi)^4}{D} \int \frac{dk^D}{a(k)b(k)c(k)} N^\mu(k).
\]

(2.24)

This integral can be done by rewriting it using the Feynman parametrization [59]:

\[
\Lambda^\mu = -\frac{i\kappa^{4-D}}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dz \int \frac{2N^\mu(k)}{[a + (b - a)x + (c - a)z]^3}.
\]

(2.25)

where we have suppressed in our notation the dependence of \( a, b \) and \( c \) on \( k \) for simplicity. The denominator can be simplified as follows,

\[
a + (b - a)x + (c - a)z = k^2 - M_{bh}^2 + i\epsilon + (m_{\mu}^2 - m_l^2 + M_{bh}^2 - 2p'k)x \\
+ (m_{\mu}^2 - m_l^2 + M_{bh}^2 - 2pk)z,
\]

\[
= k^2 - 2k(p'x + pz) + (m_{\mu}^2 - m_l^2)(x + z) \\
- M_{bh}^2(1 - x - z) + i\epsilon.
\]

(2.26)
If we define $a^\mu = (p'x + pz)^\mu$ and $r = (m_m^2 - m_l^2)(x + z) - M_{bh}^2(1 - x - z)$, then make the change of variables $t^\mu = k^\mu - a^\mu$, we obtain,

$$a + (b - a)x + (c - a)z = t^2 - a^2 + r + i\epsilon. \quad (2.28)$$

Thus,

$$\Lambda^\mu = -i\kappa^{4-D} \frac{2N^\mu(t + a)}{(t^2 - a^2 + r + i\epsilon)^3}. \quad (2.29)$$

The numerator can be rewritten in terms of powers of $t$

$$N^\mu(t + a) = (\not{p}' - \not{p} - \not{d} + m_l)\gamma^\mu(\not{p}' - \not{p} - \not{d} + m_l), \quad (2.30)$$

$$= (\not{p}' - \not{d} + m_l)\gamma^\mu(\not{p}' - \not{d} + m_l) \quad (2.31)$$

$$-(\not{p}' - \not{d} + m_l)\gamma^\mu \not{p}' - \not{p}'\gamma^\mu(\not{p}' - \not{d} + m_l) \quad (2.32)$$

$$+ \not{p}'\gamma^\mu \not{p}'. \quad (2.33)$$

The terms linear in $t$ form odd integer and’s of equation (2.29) and therefore these integrals will be zero. We can see on dimensional grounds that the part quadratic in $t$ is logarithmically ultraviolet divergent, i.e., we have an integral of the form

$$\int dt^4 \frac{t^2}{(t^2 + \xi)^3} = \frac{1}{2} \int \frac{dk}{k^3}, \quad (2.34)$$

where in the second line we have made the change of variables $k = t^2 + \xi$. However, we see from the known integral [59]:

$$\int dt^D \frac{t^\nu t^\sigma}{(t^2 - s + i\epsilon)^3} = i\pi^{D/2} \frac{\Gamma(2 - D/2)}{2\Gamma(n)} \frac{g^{\nu\sigma}}{s^{2-D/2}}, \quad (2.37)$$
and the relationship $\gamma_\nu \gamma^\mu \gamma_\nu = -2\gamma^\mu$, that our integral
\[
\int dt^D \frac{2 \gamma^\mu \gamma_\nu}{(t^2 - a^2 + r + i\epsilon)^3} = \gamma^\mu \gamma_\nu \gamma_\sigma \int dt^D \frac{2\gamma^\nu \gamma_\sigma}{(t^2 - a^2 + r + i\epsilon)^3},
\]
is proportional to $\gamma^\mu$. From Lorentz invariance, parity conservation, and gauge invariance we can express $\bar{U}(p') \Gamma^\mu U(p)$ in terms of two parameters, $F_1$ and $F_2$, which are functions of $q^2$:
\[
\bar{U}(p') \Gamma^\mu U(p) A_\mu = \bar{U}(p') \left[ \gamma^\mu F_1(q^2) + \frac{i}{2m_\mu} \sigma^{\mu\nu} q_\nu F_2(q^2) \right] U(p) A_\mu.
\]
This is known as the Gordan decomposition. Thus, the divergence of the $\gamma^\mu \gamma_\nu \gamma_\sigma$ term is contributing to $F_1$ and not $F_2$. $F_1$ represents the strength of the electromagnetic current, $\bar{U}(p') \gamma^\mu U(p)$, which appears to be blowing up due to radiative corrections. We know however, that our procedure for including the BH scalar by adding terms at the dimension five level (equation (2.11)) was consistent with the SM symmetries, and in particular the $U(1)$ electromagnetic symmetry. Hence, we have a conserved electromagnetic current $\partial_\mu J^\mu_{\text{EM}} = 0$. Therefore it must be that other BH loop processes are operating to cancel this infinity, and ensuring the finiteness of the physical electromagnetic current. Given that the finiteness is guaranteed from this non-perturbative perspective, we do not attempt to find what the cancelling processes are but return to the task of determining the contribution to the magnetic moment.

The $F_2$ term can be shown to be related to the muon magnetic moment. The only term remaining in equation (2.32) is the part independent of $t$ and we can therefore perform the $t$ integration of equation (2.29) using the standard integral:
\[
\int dt^4 \frac{1}{(t^2 + s + i\epsilon)^3} = \frac{i\pi}{2} \frac{1}{s}.
\]
Thus,
\[
\Lambda_\mu = \frac{\pi}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dz \frac{(p^\mu - p^\nu + m_1) \gamma^\mu (p^\nu - p^\mu + m_1)}{r - a^2}.
\]
We now show that this leads to a finite correction to $F_2$. To simplify the numerator any further we make use of the fact that in equation (2.18) $\Lambda^\mu$ is wedged between the $\overline{U}$ and $U$ spinors. Letting $q = p' - p$, we can derive from the Dirac equation the relationships:

\begin{align}
\overline{U}(p') \not{\!}p' &= m_\mu \overline{U}(p'), \quad \text{(2.43)} \\
\overline{U}(p') \not{\!}p &= \overline{U}(p')(\not{\!}q - \not{\!}q) = \overline{U}(p')(m_\mu - \not{\!}q), \quad \text{(2.44)} \\
\not{\!}p U(p) &= m_\mu U(p), \quad \text{(2.45)} \\
\not{\!}p' U(p) &= (m_\mu + \not{\!}q) U(p), \quad \text{(2.46)}
\end{align}

and use them to simplify the numerator of (2.42):

\begin{align}
\overline{U}(p')(m_\mu - m_\mu x - \not{\!}p z + m_l)\gamma^\mu(m_\mu - \not{\!}p' x - m_\mu z + m_l) U(p) \\
= \overline{U}(p') \{(m_\mu - m_\mu x + m_l)\gamma^\mu(m_\mu - \not{\!}p' x + m_l) - (m_\mu - m_\mu x + m_l)\gamma^\mu \not{\!}p' x \\
- z \not{\!}p'\gamma^\mu(m_\mu - m_\mu z + m_l) + \not{\!}p z\gamma^\mu \not{\!}p' x \} U(p), \\
= \overline{U}(p') \{(m_\mu - m_\mu x + m_l)\gamma^\mu(m_\mu - \not{\!}p' x + m_l) - x m_\mu(m_\mu - m_\mu x + m_l)\gamma^\mu \\
- (m_\mu - m_\mu x + m_l)\gamma^\mu \not{\!}q x - m_\mu z(m_\mu - m_\mu x + m_l)\gamma^\mu \\
+ z \not{\!}q\gamma^\mu(m_\mu - m_\mu z + m_l) + z x(m_\mu - \not{\!}q)\gamma^\mu(m_\mu + \not{\!}q) \} U(p). \quad \text{(2.48)}
\end{align}

To put this into the Gordan form, we must appeal to the Ward identity. In the Lorentz gauge this can be stated:

\begin{equation}
q^\mu A_\mu = 0. \quad \text{(2.50)}
\end{equation}

We observe that from the anticommutation relations for the Dirac matrices \{\gamma^\mu, \not{\!}q\} A_\mu = 2\delta^{\mu\nu} q_\nu A_\mu = 2q^\mu A_\mu = 0. Therefore,

\begin{align}
\gamma^\mu \not{\!}q A_\mu &= - \not{\!}q \gamma^\mu A_\mu, \\
&= - i q_\nu \sigma^{\mu\nu} A_\mu, \quad \text{(2.51)}
\end{align}

where the appearance of the Lorentz anti-symmetric tensor, $\sigma_{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. 
in the last line, follows by observing that
\[ q_{\nu}[\gamma^\mu, \gamma^\nu]A_\mu = \gamma^\mu q A_\mu - \gamma^\mu q A_\mu = 2\gamma^\mu q A_\mu. \] (2.53)

Also, multiplying equation (2.51) from the left by \( q \) gives
\[ \gamma^\mu q A_\mu = -q^2 \gamma^\mu A_\mu, \] (2.54)

revealing that \( \gamma^\mu q \) is a negligible current term.

Thus the important terms of equation (2.49) are:
\[
\begin{align*}
\mathcal{U}(p')N^\mu U(p)A_\mu & \to \mathcal{U}(p')\{-x(m_\mu - m_\mu x + m_l) \\
& \quad -z(m_\mu - m_\mu z + m_l) + 2zx m_\mu\} \gamma^\mu q U(p)A_\mu; \\
& = -i\mathcal{U}(p'(x+z)((x+z)m_\mu - m_\mu - m_l)q \sigma^{\mu\nu}U(p)A_\mu.
\end{align*}
\] (2.55) (2.56)

Taking the limit \( p \to p' \) in equation (2.42), \( \Lambda^\mu \) becomes:
\[
\frac{\pi}{(2\pi)^4} \int_0^1 dx \int_0^{1-x} dz \frac{(x+z)^2m_\mu - (x+z)(m_\mu + m_l)}{m_\mu^2(x+z)^2 + (m_\mu^2 - m_l^2)(x+z) - M_{bh}^2(1 - (x+z))},
\] (2.57)

We notice that the integrand is only a function of \( x+z \), we can then turn this double integral into a single integral by making the change of variables
\[
\begin{align*}
u &= x+z, \\
\omega &= -x+z,
\end{align*}
\] (2.58) (2.59)

which has Jacobian \( \frac{\partial(x,z)}{\partial(u,w)} = \frac{1}{2} \). Writing the integrand of equation (2.57) as \( I(x+z) \) for brevity and converting the integration terminals, we have,
\[
\int_0^1 dx \int_0^{1-x} dz I(x+z) = \frac{1}{2} \int_0^1 du \int_{-u}^u du I(u), \quad \text{(2.60)}
\]
\[
= \int_0^1 uI(u)du. \quad \text{(2.61)}
\]
Thus,

\[ \Lambda^\mu = \frac{\pi m_\mu}{(2\pi)^4 M_{bh}^4} \int_0^1 du \frac{u^2 (1 + \frac{m_\mu}{m_\mu}) - u}{(\frac{m_\mu}{M_{bh}})^2 u^2 - (1 - \frac{m_\mu}{M_{bh}})^2 + (\frac{m_\mu}{M_{bh}})^2 u + 1}, \]  

(2.62)

and since \( M_{bh} \gg m_\mu, m_l \) we can write,

\[ \Lambda^\mu = \frac{\pi m_\mu}{(2\pi)^4 M_{bh}^4} \int_0^1 du \frac{u^2 (\beta - u)}{\alpha u^2 - u + 1}, \]  

(2.63)

where \( \beta = 1 + \frac{m_l}{m_\mu} \) and \( \alpha = (\frac{m_l}{M_{bh}})^2 \). This gives a corresponding correction to the muon magnetic moment of

\[ a_{bh} = \frac{k_\mu^2 m_\mu^2}{8\pi^2} \int_0^1 dz \frac{z^2 (\beta - z)}{(1 - z) + \alpha z^2}. \]  

(2.64)

If one assumes a \( g_{\mu\mu} \lesssim 1 \) the correction is of order \( a_{bh} \sim 10^{-10} \), which is close to the current level of deviation between SM and experiment \( |a_{\text{exp}} - a_{\text{theory}}| < 42.6 \times 10^{-10} \) \([61]\). Choosing \( g_{\mu\mu} \lesssim 1 \) is not inconsistent with the muon decay result (2.17), however it does require some tuning. Assuming that all couplings are of the same order brings us to corrections of the size \( 10^{-15} \). In any case it is not possible for perturbative \( g \) to produce corrections that would contradict experimental results. Specifically, an overestimate can be obtained by taking \( g = 1 \) and multiplying the correction for a tauon in the loop by three since this contributes the most of the three leptons. This leads to \( a_{bh} < 1.5 \times 10^{-10} \).

### 2.3 Neutrino masses

It is interesting that BH process at the TeV scale can induce small neutrino masses. Consider the process shown below:
This loop has a momentum integral:
\[
\int_0^{M_*} d^4k \frac{1}{k^2 + M_{bh}^2} \frac{1}{k},
\] (2.65)
which induces a mass \( m \sim g^2 M_* \). Taking \( g \) of the order of \( \sim 10^{-6} \) would give masses of \( \sim 1 \text{ eV} \) size.

It is interesting that the low energy effects of BH’s can produce neutrino masses consistent with the observed neutrino mass scale. This is achieved essentially through the lepton family number violating channel. However, this can not be taken too seriously, as in doing so we have had to invoke a small amount of fine tuning. Therefore, other mechanisms of neutrino mass generation are preferable (like the seesaw mechanism), nevertheless we have shown that BH processes would impart corrections to these.

2.4 Summary and comments

In this chapter we have attempted to address the Planck phase (see section 1.5.3) of the BH life cycle. We did this by quantising the higher dimensional BH itself and then incorporating the lowest mass excitation into an extended SM. In quantising, see equation (2.5), we have introduced an infinite tower of BHs labelled by the numbers \( n_b \) and \( q \) each with a definite mass, \( M_{bh}^{(n_b, q)} \), and therefore a different propagator. Thus, to calculate the \( \mathcal{M} \) matrix for any given process we would need to sum over all the BH modes that can contribute. In the current work the tower is naturally cut off at the \( M_* \) scale. The exact number of modes that can participate will therefore depend on the value of \( g_p \) and the number of extra dimensions.
If the BH is allowed to propagate in the bulk, Kaluza-Klein modes will also be present of mass, \( M_{bh}^{\vec{k}} = \sqrt{M_{bh}^2 + (k/R)^2} \), where we have assumed that all the extra dimensions are of the same size \( \sim R \). For simplicity we have only considered the lowest order BH mass mode in our discussions. It is tempting to wonder if the higher modes could be pushed above the cut-off and therefore swept away from our concerns. Indeed the criterion to achieve this result is

\[
M_{bh}^{\vec{k}} > \Lambda, \quad \forall \vec{k} \neq 0. \tag{2.66}
\]

Since higher modes are heavier than their predecessors, we only need to satisfy equation (2.66) for the first order modes, i.e., for instance \( \vec{k} = (1, 0, \cdots, 0) \), the other modes will then satisfy this relationship automatically. Thus we obtain:

\[
M_{bh}^2 > \Lambda^2 (1 - 10^{-64/n}). \tag{2.67}
\]

If there were seven dimensions the BH mass squared would need to be one billionth less than the cut-off squared in order to push the first mode out of consideration. While one might speculate along these lines, other than to simplify our analysis there exists no other argument for this ad hoc assumption.

One might also prefer to assume that the BH is only a four dimensional particle, as has been done elsewhere \[62\, 53\]. We however, prefer to take the view that the BH does possess degrees of freedom in the extra dimensions, since it is after all a solution to the generalised Einstein’s equations in higher dimensions \[63\].

Then we have no choice but to accept these higher order modes, however, the same reasoning that allowed us to neglect the higher spin and angular momentum modes -namely, that the lowest mass state gives the greatest contribution to the observables in an effective theory- also applies here. Therefore, since the KK modes are heavier than our lowest order scalar excitation our analysis remains the correct first order calculation of these BH effects.
We have focused on the observable effects that a single scalar excitation will produce. If in new experiments BH effects are discovered one can then use equation \((2.5)\) to determine both \(g_p\) and \(n\) and the related BH phenomenology.

In this work we have shown that the phenomenology of a single scalar BH excitation at the TeV scale can be introduced into the SM without spoiling current experimental results either for the muon magnetic moment or the neutrino masses. All calculations made in this paper are done within the effective non-renormalizable theory. One imagines that these operators are remnant radiative effects of some high energy theory like string theory. The true worth of our approach lies in its ability to attain sensible results that are comparable with experiment. Our approach has the flexibility to accommodate alternative BH phenomenology that would be dependent on the results of forthcoming searches planned for the LHC.

Whilst we have found that the model is extremely useful for calculating BH effects we must remember that our results are highly dependent on the assumptions of the model. It is therefore worth surmising what assumptions we have made:

1. The BH is quantised in irreducible mass.
2. There exists a lightest such mass.
3. The lightest mass state can be described by an effective quantum field theory.
4. All interactions that are consistent with SM symmetries are allowed.
5. At energies below the fundamental scale the amplitudes for BH processes can be calculated perturbatively.

In lack of a consistent theory of quantum gravity one can not say for certain whether these assumptions are valid. In our next exploration of the features
of quantum BH’s we will move back one phase from the Planck phase to
the Schwarzschild phase. Here we take a look at some of the more classical
signatures of BH’s and investigate the Hawking radiation and Quasi-Normal
Modes (QNM’s) of bulk fermions. These signatures have the advantage of
being less model dependent than that of the current study and thus more
amenable to experimental scrutiny.
CHAPTER 3

Bulk fermions on a
Schwarzschild background

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As discussed in section 1.4 split fermions present a natural means by which proton decay can be suppressed in theories of extra large dimensions. Depending on which fixed point the BH is closest to, emission onto the brane will consist either completely of quarks, or of leptons \[64, 13\]. In supersymmetric versions of this idea the localising scalars and bulk gauge fields will have fermionic bulk superpartners \[65, 66\]. Since the BH would emit into these bulk fermion modes it is important to consider their properties. Indeed such emission would significantly alter the ratio of missing energy off the brane to brane emission. However, up to now only brane localised fermions on Schwarzschild backgrounds have been calculated \[67, 68, 23, 24\]. That said, the reasons for studying the Dirac fermions in higher dimensions are also motivated from a theoretical point of view. In string theory QNMs can shed light on the AdS/CFT and ds/CFT correspondence (see, e.g., \[69, 70\]) and are relevant to the Barbero-Immirzi parameter \[14, 72, 73\] of loop quantum gravity \[74, 75, 76\]. The lack of any work done in greater than four dimensions \[77\] with Dirac fields is an omission in the literature. Our calculations serve to fill this gap.

This chapter shall be structured as follows: First we introduce the generalised WKBJ method. Then we define Quasi-Normal Modes (QNMs) and present the 3rd order results of Iyer and Will. In section 3.3 we shall discuss how a conformal transformation of the metric allows for a convenient separation of the Dirac equation into a time-radial part and a \((d-2)\)-sphere, in \(d\)-dimensions. After this we shall present the \(d\)-dimensional fermionic QNMs, along with the analytical forms of the frequencies in both the large angular momentum as well as the large mode number limits. Then in section 3.5 we plot the emission spectrum for a bulk Dirac fields on a \(d\)-dimensional Schwarzschild background. Finally, in the last section, we shall summarise this work and present some possible extensions to it.

\[^{1}\]The Barbero-Immirzi parameter in loop quantum gravity measures the size of the quantum area in Planck units. Recently QNM’s have been used to constrain this parameter, see \[71\].
3.1 The WKBJ method

The WKBJ method is a semi-analytic method that was first exploited to solve the Schrödinger equation in Quantum Mechanics, see for instance [78]. Given an equation in the Schrödinger form:

$$\epsilon^2 \frac{d^2}{dx^2} + Q(x) \psi(x) = 0,$$

(3.1)

where $\epsilon$ is a perturbation parameter used to keep track of the order of the WKBJ, the asymptotic solution is

$$\psi \sim \exp[S(x)/\epsilon].$$

(3.2)

By expanding $S$ in powers of the parameter $\epsilon$:

$$S(x) = \sum_{n=0}^{\infty} \epsilon^n S_n(x),$$

(3.3)

substituting $\psi(S)$ into equation (3.1) and then collecting terms with the same power of $\epsilon$, we obtain the fundamental equations of the WKBJ approximation presented here to third order:

$$S_0'(x) = (-Q)^{1/2},$$

(3.4)

$$S_1'(x) = -\frac{1}{4} Q'/Q,$$

(3.5)

$$S_2'(x) = -\frac{1}{8} \left[ \frac{Q''}{(-Q)^{3/2}} + \frac{5}{4} \frac{Q^2}{(-Q)^{5/2}} \right],$$

(3.6)

$$S_3'(x) = \frac{1}{16} \left( \frac{Q''}{Q^2} - \frac{5}{4} \frac{Q'^2}{Q^3} \right).$$

(3.7)

As we will see in section (3.3) the equations of BH perturbations often reduce to the Schrödinger form. The potential written in tortoise coordinates would look something like that shown in figure 3.1. Usually the domain is divided into three regions by the turning points $x_1$ and $x_2$, where the fundamental WKBJ equations are used to solve for $\psi$ in each region, these solutions are then matched across the boundaries. The approximation is valid when the
wavelength is small compared with the characteristic distance over which the potential varies.

In the case of BH perturbations one finds that the lowest lying normal modes are expected to occur for frequencies such that $x_1$ and $x_2$ are close together. In this case the two points are too close together to permit a valid WKBJ approximation on the interior region. A method which circumvents this problem was found by Iyer and Will [79] where in region III, $Q$ is expanded in a Taylor series about the maximum, $x_0$,

$$Q(z \equiv x - x_0) = k \left( z^2 - z_0^2 + \sum_{n=3} b_n z^n \right), \quad (3.8)$$

where

$$z_0^2 \equiv -2 \frac{Q_0}{Q_0''}, \quad k \equiv \frac{1}{2} Q_0'', \quad b_n \equiv \frac{2}{n! Q_0''} \left. \frac{d^n Q}{dx^n} \right|_0, \quad (3.9)$$

Note that the subscript zero in equations (3.9) indicates evaluation at $z = 0$.
(x = x₀). The solutions to this polynomial potential are parabolic cylindrical functions with index ν, where ν is defined implicitly by the equation:

\[ \nu + \frac{1}{2} = -ik^{1/2}z^2_0/2 - \Lambda(\nu + \frac{1}{2}) - \Omega(\nu + \frac{1}{2}). \]  

(3.10)

The expressions for Λ and Ω were then found in [79] by matching solutions across x₁ and x₂ simultaneously. They are:

\[ \Lambda = ik^{-1/2}[(\frac{3}{16}b_4 - \frac{7}{64}b_3^2) + (\nu + \frac{1}{2})^2(\frac{3}{4}b_4 - \frac{15}{16}b_3^2)], \]  

(3.11)

\[ \Omega = k^{-1}[(\nu + \frac{1}{2})(\frac{155}{1024}b_3^4 - \frac{459}{128}b_3^2b_4 + \frac{67}{64}b_2^2 + \frac{95}{32}b_3b_5 - \frac{25}{16}b_6) \]  

\[ + (\nu + \frac{1}{2})^2(\frac{705}{256}b_3^4 - \frac{225}{32}b_3^2b_4 + \frac{17}{16}b_2^2 + \frac{35}{8}b_3b_5 - \frac{5}{4}b_6)]. \]  

(3.12)

It was also found that the scattering amplitudes in regions I and III, see figure 3.1, are related by the matrix:

\[ \begin{pmatrix} Z^\text{III}_{\text{out}} \\ Z^\text{III}_{\text{in}} \end{pmatrix} = \begin{pmatrix} e^{i\pi \nu} & iR^2 e^{i\pi \nu}(2\pi)^{1/2}/\Gamma(\nu + 1) \\ R^{-2}(2\pi)^{1/2}/\Gamma(-\nu) & -e^{i\pi \nu} \end{pmatrix} \begin{pmatrix} Z^\text{I}_{\text{out}} \\ Z^\text{I}_{\text{in}} \end{pmatrix}, \]  

(3.13)

where Γ is the gamma function, and R is given by

\[ R = (\nu + \frac{1}{2})^{(\nu+1/2)/2}\exp[-\frac{1}{2}(\nu + \frac{1}{2}) + \mathcal{O}(\nu + \frac{1}{2})^{-1}]. \]  

(3.14)

Classically no particles can escape from the horizon of a BH. Therefore, the boundary conditions (BCs) in region III are only outgoing waves, i.e., Z^III_{in} = 0. For simplicity we write the scattering matrix as M_ij. Then from the bottom row we obtain the relationship Z^I_{out} = -\frac{M_{22}}{M_{21}}Z^I_{in} and substituting this into the top row we obtain:

\[ Z^\text{III}_{\text{out}} = \left( \frac{M_{21}M_{12} - M_{11}M_{22}}{M_{21}} \right) Z^\text{I}_{\text{in}}. \]  

(3.15)

\[ ^2 \text{We have found 4 errors in equations (4)-(7) of [80]. The errors relevant to the equations we are presenting are: equation (4) is missing a factor of one half, and the } \frac{95}{21}b_3b_5 \text{ term in equation (6) should be } \frac{95}{32}b_3b_5 \text{ as we have written here.} \]

\[ ^3 \text{In the sense that the waves are moving outward from the potential.} \]
Since $Q$ is always a real potential then $\nu + \frac{1}{2} \equiv ai$, is always imaginary. Using this result we find the following useful identities:

\[ (e^{i\pi \nu})^* = e^{-\pi a + i\pi/2} = e^{\pi i(\nu+1)}, \quad (3.16) \]
\[ = -e^{i\pi \nu}. \quad (3.17) \]

By observation it is then clear that $M_{11}^* = M_{22}$. Also,

\[ R = (ai)^{ai/2} \exp\left[-\frac{1}{2}ai + \frac{i}{48a}\right], \]
\[ \Rightarrow R^* = (-1)^{-ai/2}(ai)^{-ai/2} \exp\left[\frac{1}{2}ai - \frac{i}{48a}\right], \]
\[ = (-1)^{-ai/2}R^{-1} = e^{-\pi a/2}R^{-1}, \]
\[ = e^{-\frac{\pi}{2}(\nu+1)/2}R^{-1} \quad (3.20) \]

Since $\Gamma(\nu)^* = \Gamma(\nu^*)$ and $(\nu + 1)^* = -\nu$ it follows that $\Gamma(\nu + 1)^* = \Gamma(-\nu)$ then

\[ M_{12}^* = \frac{iR^2 e^{i\pi \nu} \sqrt{2\pi}}{\Gamma(\nu + 1)^*}, \quad (3.22) \]
\[ = \frac{i e^{-\pi(\nu+1)/2}R^{-2} e^{i\pi \nu} \sqrt{2\pi}}{\Gamma(-\nu)}, \quad (3.23) \]
\[ = M_{21}. \quad (3.24) \]

In the following we use the identity $\Gamma(\frac{1}{2} - ai)\Gamma(ai + \frac{1}{2}) = \frac{\pi}{\cosh a \pi}$, to obtain $M_{21}$:

\[ |M_{21}|^2 = M_{21} M_{12}^* = \frac{i2\pi e^{i\pi \nu}}{\Gamma(-\nu)\Gamma(\nu + 1)}, \quad (3.25) \]
\[ = 2e^{i\pi \nu} \cosh a \pi, \quad (3.26) \]
\[ = 1 + e^{2\pi(\nu+1/2)}. \quad (3.27) \]

Now since $|M_{11}|^2 = e^{2\pi(\nu+1/2)}$, we find the final identity $|M_{21}|^2 = 1 + |M_{11}|^2$. With these identities at hand we re-analyse equation (3.15) to obtain $\frac{2^{11}}{2_{11}^2} = M_{21}^1$. But $T \equiv \frac{|2_{11}^1|^2}{|2_{11}^2|^2}$ is the probability for an incoming wave to tunnel through the barrier into the BH. Therefore the transmission coefficient can be found using equation (3.27):
3.2 Quasi normal modes

QNMs are defined to be those perturbations that as well as having no waves from the horizon also have no incoming waves from region I. This is in contrast to standard BH scattering, see figure 3.1, where both ingoing, $Z_{\text{in}}^I$, and outgoing, $Z_{\text{out}}^I$, waves are present at infinity. This definition literally means that the QNM is the mode of oscillation of the BH that occurs when there is nothing falling into it. QNMs play an important role in astrophysics where the future detection of BH gravitational QNMs may allow physicists to precisely measure BH properties like mass and angular momentum, for a review see [81].

With the advent of theories postulating the existence of additional dimensions there has been much discussion in the literature related to the QNMs of BHs in $d$-dimensions, see for instance [82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93]. The QNM itself is a complex number where the real part represents the actual frequency of the oscillation and the imaginary part represents the damping. Finding QNMs numerically requires choosing a complex frequency, integrating the differential equation and then checking that the boundary conditions are satisfied. This survey of all possible complex numbers is computationally costly. Usually the WKBJ approximation developed by Iyer and Will [79] and discussed in the previous section, is used to locate the vicinity of these frequencies before the full numerical computation is performed, thereby improving the efficiency of the numerical search. We now show how the WKBJ method can be used to calculate QNMs.

At first sight it is not clear that there are solutions to equation (3.1) with
the required QNM BCs. However, we can use the scattering matrix (3.13) found using the WKBJ method to show that non-trivial solutions do in fact exist. We rewrite the scattering matrix, $M_{ij}$, as a transfer matrix so that

$$\begin{pmatrix} -\frac{M_{11}}{M_{12}} & \frac{1}{M_{12}} \\ \frac{M_{21}M_{12}-M_{22}M_{11}}{M_{12}} & \frac{M_{22}}{M_{12}} \end{pmatrix} \begin{pmatrix} Z_{\text{out}}^I \\ Z_{\text{out}}^\text{III} \end{pmatrix} = \begin{pmatrix} Z_{\text{in}}^I \\ Z_{\text{in}}^\text{III} \end{pmatrix},$$

$$= 0,$$  

(3.29)

(3.30)

where we have used the $Z_{\text{in}}^\text{III} = 0$ and $Z_{\text{in}}^I = 0$ BCs in the last line. Thus, there will be non-trivial solutions when the determinant of the transfer matrix is zero. This leads to the constraint $\frac{M_{21}}{M_{12}} = 0$, or

$$\frac{\Gamma(\nu + 1)}{\Gamma(-\nu)} = 0.$$  

(3.31)

This constraint is satisfied for $\nu = 0, 1, 2, \ldots$. Secondly, if we want to find the negative frequency QNMs the identification of “in” and “out” must be reversed in which case one obtains by a similar line of reasoning that the determinant of the inverse of the transfer matrix must equal zero or that

$$\frac{\Gamma(-\nu)}{\Gamma(\nu + 1)} = 0,$$  

(3.32)

which is true if $\nu$ is a negative integer. Thus we may rewrite equation (3.10) as

$$n + \frac{1}{2} = -iQ_0/(2Q_0'0)^{1/2} - \Lambda(n + \frac{1}{2}) - \Omega(n + \frac{1}{2}),$$

$n = \begin{cases} 0, 1, 2, \cdots, \text{Re}(E) > 0 \\ -1, -2, -3, \cdots, \text{Re}(E) < 0 \end{cases}.$  

(3.33)

When $Q(x) = E^2 - V_1(x)$, one can obtain the third order QNM frequencies, $E^2$, from equation (3.33). They are:

$$E^2 = [V_0 + (-2V_0')^{1/2}\Lambda] - i(n + \frac{1}{2})(-2V_0')^{1/2}(1 + \Omega),$$  

(3.34)
where we denote $V_0$ as the maximum of $V_1$. From equations (3.11) and (3.12) we can write $\Lambda$ and $\Omega$ as:

\[
\Lambda = \frac{1}{(-2V_0^\prime)^{1/2}} \left\{ \frac{1}{8} \left( \frac{V_0^{(4)}}{V_0^\prime} \right) \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V_0^{\prime\prime\prime}}{V_0^\prime} \right)^2 \left( 7 + 60\alpha^2 \right) \right\}, \quad (3.35)
\]

\[
\Omega = \frac{1}{(-2V_0^\prime)^{1/2}} \left\{ \frac{5}{6912} \left( \frac{V_0^{(4)}}{V_0^\prime} \right)^4 (77 + 188\alpha^2) - \frac{1}{384} \left( \frac{V_0^{\prime\prime\prime}V_0^{(4)}}{V_0^\prime} \right)(51 + 100\alpha^2) + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0^\prime} \right)^2 (67 + 68\alpha^2) + \frac{1}{288} \left( \frac{V_0^{\prime\prime\prime}}{V_0^\prime} \right)^2 \left( 19 + 28\alpha^2 \right) - \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0^\prime} \right)(5 + 4\alpha^2) \right\}. \quad (3.36)
\]

Here

\[
\alpha = n + \frac{1}{2}, \quad n = \begin{cases} 0, 1, 2, \cdots, \text{Re}(E) > 0 \\ -1, -2, -3, \cdots, \text{Re}(E) < 0 \end{cases}, \quad (3.37)
\]

and

\[
\text{and} \quad V_0^{(n)} = \left. \frac{d^n V}{dr_{r^*}^n} \right|_{r^*=r_{r_{\text{max}}}}. \quad (3.38)
\]

Note that this semi-analytic method has been used extensively in various BH cases [67, 68, 82, 94, 95, 96], where comparisons with other numerical results have been found to be accurate up to around 1% for both the real and the imaginary parts of the frequencies for low-lying modes with $n < l$ (where $n$ is the mode number and $l$ is the spinor angular momentum quantum number).

### 3.3 Spinor radial wave equation

We now consider a Dirac field on a $d$-dimensional Schwarzschild background. A similar approach has already been applied in reference [97] to the case of low energy and momentum channel ($s$-wave) absorption cross-sections; however, this has not been used yet in the context of BH QNMs.
As previously mentioned one motivation for fermions in the extra dimensions comes from split fermion models. Since the BH hole is much smaller than the size of the extra dimension then we do not need to worry about the topology of the larger space that we embed the BH within (for example the \( S_1/Z_2 \) orbifold space typical of split fermion models), since at some large distance away from the horizon the space will be flat and to a good approximation resemble the ordinary Schwarzschild background.

We shall begin our analysis by supposing a background metric which is \( d \)-dimensional and spherically symmetric, as given by:

\[
\text{d}s^2 = -f(r)\text{d}t^2 + h(r)\text{d}r^2 + r^2\text{d}\Omega^2_{d-2},
\]

(3.39)

where \( \text{d}\Omega^2_{d-2} \) denotes the metric for the \((d-2)\)-dimensional sphere.

Under a conformal transformation \([97][98]\):

\[
g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \\
\psi \rightarrow \bar{\psi} = \Omega^{-(d-1)/2}\psi, \\
\gamma^\mu \nabla_\mu \psi \rightarrow \Omega^{(d+1)/2} \bar{\gamma}^\mu \nabla_\mu \bar{\psi},
\]

(3.40, 3.41, 3.42)

where we shall take \( \Omega = 1/r \), the metric becomes:

\[
\text{d}\bar{s}^2 = -\frac{f}{r^2}\text{d}t^2 + \frac{h}{r^2}\text{d}r^2 + \text{d}\Omega^2_{d-2},
\]

where \( \bar{\psi} = r^{(d-1)/2}\psi \).

(3.43)

Since the \( t - r \) part and the \((d-2)\)-sphere part of the metric are completely separated, one can write the Dirac equation in the form:

\[
\bar{\gamma}^a \nabla_a \bar{\psi} = 0, \\
\Rightarrow \left[ (\gamma^t \nabla_t + \gamma^r \nabla_r) \otimes 1 \right] \bar{\psi} + \left[ \bar{\gamma}^5 \otimes (\gamma^a \nabla_a)_{S_{d-2}} \right] \bar{\psi} = 0,
\]

(3.44)

where \((\bar{\gamma}^5)^2 = 1\). Note that from this point on we shall change our notation by omitting the bars.

We shall now let \( \chi_{l}^{(\pm)} \) be the eigenspinors for the \((d-2)\)-sphere \([99]\), that
3.3 Spinor radial wave equation

The spinor radial wave equation is:

\[(\gamma^a \nabla_a) \chi_{l}^{(\pm)} = \pm i \left( l + \frac{d - 2}{2} \right) \chi_{l}^{(\pm)}, \]  

(3.45)

where \( l = 0, 1, 2, \ldots \). Since the eigenspinors are orthogonal, we can expand \( \psi \) as:

\[ \psi = \sum_{l} \left( \phi_{l}^{(+) \chi_{l}^{(+)}} + \phi_{l}^{(-) \chi_{l}^{(-)}} \right). \]  

(3.46)

The Dirac equation can thus be written in the form:

\[ \left\{ \gamma^t \nabla_t + \gamma^r \nabla_r + \gamma^5 \left[ \pm i \left( l + \frac{d - 2}{2} \right) \right] \right\} \phi_{l}^{(\pm)} = 0, \]  

(3.47)

which is just a 2-dimensional Dirac equation with a \( \gamma^5 \) interaction.

To solve this equation we make the explicit choice of the Dirac matrices:

\[ \gamma^t = \frac{r}{\sqrt{f}} (-i \sigma^3), \quad \gamma^r = \frac{r}{\sqrt{h}} \sigma^2, \]  

(3.48)

where the \( \sigma^i \) are the Pauli matrices:

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. 
\]  

(3.49)

Also,

\[ \gamma^5 = (-i \sigma^3) \sigma^2 = -\sigma^1. \]  

(3.50)

The spin connections are then found to be:

\[ \Gamma_t = \sigma^1 \left( \frac{r^2}{4\sqrt{f}h} \right) \frac{d}{dr} \left( \frac{f}{r^2} \right), \quad \Gamma_r = 0. \]  

(3.51)

From this point on we shall work with the + sign solution, where the – sign case would work in the same way. The Dirac equation can then be written
explicitly as:

\[
\left\{ \frac{r}{\sqrt{f}} (-i\sigma^3) \left[ \frac{\partial}{\partial t} + \sigma^1 \left( \frac{r^2}{4\sqrt{fh}} \right) \frac{d}{dr} \left( \frac{f}{r^2} \right) \right]
\right. \left. + \frac{r}{\sqrt{h}} \sigma^2 \frac{\partial}{\partial r} + (-\sigma^1)(i) \left( l + \frac{d-2}{2} \right) \right\} \phi^+ = 0,
\]

\[
\Rightarrow \sigma^2 \left( \frac{r}{\sqrt{h}} \right) \left[ \frac{\partial}{\partial r} + \frac{r}{2\sqrt{f}} \frac{d}{dr} \left( \frac{\sqrt{f}}{r} \right) \right] \phi^+_l
\]

\[
- i\sigma^1 \left( n + \frac{d-2}{2} \right) \phi^+_l = i\sigma^3 \left( \frac{r}{\sqrt{f}} \right) \frac{\partial \phi^+_l}{\partial t}.
\]

(3.52)

We shall now determine solutions of the form:

\[
\phi^+_l = \left( \frac{\sqrt{f}}{r} \right)^{-1/2} e^{-iEt} \left( \frac{iG(r)}{F(r)} \right),
\]

(3.53)

where \( E \) is the energy. The Dirac equation can then be simplified to:

\[
\sigma^2 \left( \frac{r}{\sqrt{h}} \right) \left( \frac{iG}{F} \right) - i\sigma^1 \left( l + \frac{d-2}{2} \right) \left( \frac{iG}{F} \right) = i\sigma^3 E \left( \frac{r}{\sqrt{f}} \right) \left( \frac{iG}{F} \right),
\]

or

\[
\sqrt{\frac{f}{h}} \frac{dG}{dr} - \frac{\sqrt{f}}{r} \left( l + \frac{d-2}{2} \right) G = EF,
\]

(3.55)

\[
\sqrt{\frac{f}{h}} \frac{dF}{dr} + \frac{\sqrt{f}}{r} \left( l + \frac{d-2}{2} \right) F = -EG.
\]

(3.56)

It will be convenient to define the tortoise coordinate, \( r_* \), and the function \( W \) as:

\[
\sqrt{\frac{f}{h}} \frac{d}{dr_*} \equiv \frac{d}{dr_*}, \quad W = \frac{\sqrt{f}}{r} \left( l + \frac{d-2}{2} \right).
\]

(3.57)

In which case, our equations can be expressed as:

\[
\left( \frac{d}{dr_*} - W \right) G = EF, \quad \left( \frac{d}{dr_*} + W \right) F = -EG.
\]

(3.58)

The equations can then be separated to give two equations in the Schrödinger
3.3 Spinor radial wave equation

form (3.1):

\[
\left(-\frac{d^2}{dr^2} + V_1\right) G = E^2 G \quad \text{and} \quad \left(-\frac{d^2}{dr^2} + V_2\right) F = E^2 F,
\]

(3.59)

where:

\[
V_{1,2} = \pm \frac{dW}{dr} + W^2.
\]

(3.60)

Since \(V_1\) and \(V_2\) are supersymmetric to each other, \(F\) and \(G\) will have the same spectra, both for scattering and quasi-normal. Incidentally, for \(\phi_{\nu}^{(-)}\), we have these two potentials again.

Refining our study now to a \(d\)-dimensional Schwarzschild BH, where \(f\) becomes:

\[
f(r) = h^{-1}(r) = 1 - \left(\frac{r_H^d}{r} \right)^{d-3},
\]

(3.61)

and where the horizon is at \(r = r_H\) with:

\[
r_H^{d-3} = \frac{8\pi M \Gamma((d-1)/2)}{\pi^{(d-1)/2} (d-2)}.\]

(3.62)

In this case the potential \(V_1\) can be expressed as:

\[
V_1(r) = f \frac{d}{dr} \left[ \sqrt{f} \left( \frac{l + \frac{d-2}{2}}{r} \right) \right] + f \left( \frac{l + \frac{d-2}{2}}{r} \right)^2,
\]

\[
= \left[ 1 - \left(\frac{r_H^d}{r}\right)^{d-3} \right] \frac{d}{dr} \left[ \sqrt{1 - \left(\frac{r_H^d}{r}\right)^{d-3}} \left( \frac{l + \frac{d-2}{2}}{r} \right) \right]
\]

\[
+ \left[ 1 - \left(\frac{r_H^d}{r}\right)^{d-3} \right] \left( \frac{l + \frac{d-2}{2}}{r} \right)^2.
\]

(3.63)

Next, we shall define, for notational convenience:

\[
\kappa \equiv l + \frac{d-2}{2},
\]

(3.64)

\[
\Delta \equiv r^{d-3}(r_H^{d-3} - r^{d-3}),
\]

(3.65)

with \(\kappa = \frac{d}{2} - 1, \frac{d}{2}, \frac{d}{2} + 1, \ldots\). This allows the above potential, \(V_1\), to be simplified.
to:

\[ V_1 = \frac{\kappa \Delta^{1/2}}{r^{2(d-2)}} \left[ \kappa \Delta^{1/2} - r^{d-3} + \left( \frac{d-1}{2} \right) r_H^{d-3} \right]. \tag{3.66} \]

For the case \( d = 4 \), this then becomes:

\[ V_1 = \frac{\kappa \Delta^{1/2}}{r^4} \left( \kappa \Delta^{1/2} - r + 3M \right), \tag{3.67} \]

with \( \Delta = r(r - 2M) \), which is just the radial equation for the Dirac equation in the 4-dimensional Schwarzschild BH \[100\].

### 3.4 QNMs using the Iyer and Will method

Since equation (3.59) is effectively a Schrödinger equation with \( Q = E^2 - V_1 \) we can calculate the QNMs by using the potential (3.67) in equations (3.34)–(3.36). The maximum is found using a numerical root finding algorithm. The frequencies are shown in Table 3.1 and are plotted in figures 3.2–3.3. We also plot, in figure 3.4, the fundamental mode \( (n = l = 0) \) for various dimensions. We have used \( M = 1 \).
3.4 QNMs using the Iyer and Will method

<table>
<thead>
<tr>
<th>$(l, n)$ even $d$</th>
<th>$d = 5$</th>
<th>$d = 7$</th>
<th>$d = 9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l=0$, $n=0$</td>
<td>0.725 − 0.396i</td>
<td>1.79 − 0.809i</td>
<td>2.66 − 0.999i</td>
</tr>
<tr>
<td>$l=1$, $n=0$</td>
<td>1.32 − 0.384i</td>
<td>2.73 − 0.807i</td>
<td>3.73 − 1.03i</td>
</tr>
<tr>
<td>$l=1$, $n=1$</td>
<td>1.15 − 1.22i</td>
<td>2.05 − 2.68i</td>
<td>2.30 − 3.57i</td>
</tr>
<tr>
<td>$l=2$, $n=0$</td>
<td>1.88 − 0.384i</td>
<td>3.61 − 0.817i</td>
<td>4.71 − 1.06i</td>
</tr>
<tr>
<td>$l=2$, $n=1$</td>
<td>1.75 − 1.18i</td>
<td>3.11 − 2.56i</td>
<td>3.65 − 3.35i</td>
</tr>
<tr>
<td>$l=2$, $n=2$</td>
<td>1.56 − 2.03i</td>
<td>2.27 − 4.53i</td>
<td>1.80 − 6.23i</td>
</tr>
<tr>
<td>$l=3$, $n=0$</td>
<td>2.43 − 0.384i</td>
<td>4.46 − 0.821i</td>
<td>5.64 − 1.08i</td>
</tr>
<tr>
<td>$l=3$, $n=1$</td>
<td>2.33 − 1.17i</td>
<td>4.08 − 2.52i</td>
<td>4.85 − 3.29i</td>
</tr>
<tr>
<td>$l=3$, $n=2$</td>
<td>2.17 − 1.99i</td>
<td>3.38 − 4.36i</td>
<td>3.30 − 5.84i</td>
</tr>
<tr>
<td>$l=3$, $n=3$</td>
<td>1.96 − 2.84i</td>
<td>2.46 − 6.36i</td>
<td>1.30 − 8.87i</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$(l, n)$ odd $d$</th>
<th>$d = 6$</th>
<th>$d = 8$</th>
<th>$d = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l=0$, $n=0$</td>
<td>1.28 − 0.639i</td>
<td>2.24 − 0.924i</td>
<td>3.05 − 1.05i</td>
</tr>
<tr>
<td>$l=1$, $n=0$</td>
<td>2.10 − 0.623i</td>
<td>3.27 − 0.936i</td>
<td>4.14 − 1.10i</td>
</tr>
<tr>
<td>$l=1$, $n=1$</td>
<td>1.71 − 2.04i</td>
<td>2.24 − 3.18i</td>
<td>2.26 − 3.87i</td>
</tr>
<tr>
<td>$l=2$, $n=0$</td>
<td>2.87 − 0.631i</td>
<td>4.21 − 0.956i</td>
<td>5.14 − 1.14i</td>
</tr>
<tr>
<td>$l=2$, $n=1$</td>
<td>2.11 − 3.42i</td>
<td>3.45 − 3.01i</td>
<td>3.75 − 3.59i</td>
</tr>
<tr>
<td>$l=2$, $n=2$</td>
<td>2.27 − 4.53i</td>
<td>2.15 − 5.45i</td>
<td>1.26 − 6.95i</td>
</tr>
<tr>
<td>$l=3$, $n=0$</td>
<td>3.61 − 0.632i</td>
<td>5.11 − 0.964i</td>
<td>6.08 − 1.16i</td>
</tr>
<tr>
<td>$l=3$, $n=1$</td>
<td>3.39 − 1.93i</td>
<td>4.54 − 2.96i</td>
<td>5.06 − 3.54i</td>
</tr>
<tr>
<td>$l=3$, $n=2$</td>
<td>3.00 − 3.32i</td>
<td>3.46 − 5.18i</td>
<td>2.95 − 6.34i</td>
</tr>
<tr>
<td>$l=3$, $n=3$</td>
<td>2.49 − 4.78i</td>
<td>2.04 − 7.69i</td>
<td>0.327 − 10.0i</td>
</tr>
</tbody>
</table>

Table 3.1: Massless bulk Dirac QNM frequencies (Re($E$) > 0) for a higher dimensional Schwarzschild BH with $l \geq n \geq 0$, with $d = 5, 6, 7, 8, 9$ and 10 dimensions. Given the accuracy of the 3rd order WKBJ method, results are presented up to three significant figures only.
Figure 3.2: Lines of constant $l$ for massless bulk Dirac QNM frequencies for a Schwarzschild BH in odd $d$-dimensions.
Figure 3.3: Lines of constant $l$ for massless bulk Dirac QNM frequencies for a Schwarzschild BH in even $d$-dimensions.
In the next subsection we will calculate the QNMs in the large angular momentum limit as a means of ratifying the general behaviour of our numerical results.

### 3.4.1 Large angular momentum

If we now focus on the large angular momentum limit ($\kappa \to \infty$) we can easily extract an analytic expression for the QNMs to first order:

$$E^2 \approx V_0 - i(n + \frac{1}{2})(-2V_0')^{1/2} + \ldots, \quad (3.68)$$

where $V_0$ is the maximum of the potential $V_1$, see equation (3.66). In this limit the potential now takes the form:

$$V_1 \bigg|_{\kappa \to \infty} \approx \frac{\kappa^2(r^{d-3} - r_H^{d-3})}{r^{d-1}}. \quad (3.69)$$
3.4 QNMs using the Iyer and Will method

The location of the maximum of the potential is at:

\[ r_{\text{max}} \bigg|_{\kappa \to \infty} \approx \left( \frac{d - 1}{2} \right)^{\frac{1}{d-3}} r_H. \]  

(3.70)

The maximum of the potential in such a limit is then found to be:

\[ V_0 \bigg|_{\kappa \to \infty} \approx \kappa^2 \frac{2^{\frac{2}{d-3}} (d - 3)}{2^{\frac{d-1}{d-3}} r_H^2}. \]  

(3.71)

In this case we find from the first order WKBJ approximation that:

\[ E \bigg|_{\kappa \to \infty} \approx \frac{2^{\frac{1}{d-3}} \sqrt{d - 3}}{(d - 1)^{\frac{d-1}{2(d-3)}}} \left( \kappa - i \left( n + \frac{1}{2} \right) \sqrt{d - 3} \right). \]  

(3.72)

This result agrees with the standard result in four dimensions, \( d = 4 \), for example see reference [94], and is similar to the spin-0 result given in reference [95]. These limiting values also appear to agree well with the plots made in figure 3.2 and 3.3. We have also plotted the QNM \( n = l = 0 \) dependence on dimension, \( d \), in figure 3.4. Next we look at the large mode number limit.

3.4.2 Asymptotic quasi normal frequency

We can also calculate the quasi normal frequency in the limit of large mode number, \( n \). In this case, as we can see from the results in the previous sections, that as \( n \to \infty \) the imaginary part of \( E \) tends to negative infinity, \( \text{Im} E \to -\infty \). Hence, we are really looking at the large \( |E| \) limit here. Using the method by Andersson and Howls [101], who have combined the WKBJ formalism with the monodromy method of Motl and Neitzke [102], we make our evaluations in this limit. Note that in reference [77] this method has been used to obtain the asymptotic quasinormal frequency for the four-dimensional Dirac field. Since we are following the same procedure as in references [77][101], we shall only show the essential steps, where one can consult references [77][101] for details.
To start we return to equation (3.59):
\[
\left(-\frac{d^2}{dr^2} + V_1\right) G = E^2 G \Rightarrow \frac{d^2 G}{dr^2} + \left( E^2 - V_1 \right) G = 0. \tag{3.73}
\]

Defining a new function,
\[
Z(r) = \frac{\Delta^{1/2}}{r^{d-3}} G(r), \tag{3.74}
\]
equation (3.73) can be rewritten as:
\[
\frac{d^2 Z}{dr^2} + R(r) Z = 0, \tag{3.75}
\]
with
\[
R(r) = \frac{r^{4(d-3)}}{\Delta^2} \left[ E^2 - V_1 + \frac{(d-3)(d-2)r_H^{d-3}}{2r^{d-1}} - \frac{(d-3)(d-1)r_H^{2(d-3)}}{4r^{2(d-2)}} \right]. \tag{3.76}
\]
The WKBJ solutions to this equation are:
\[
f_{1,2}^{(t)}(r) = \frac{1}{\sqrt{Q(r)}} e^{\pm i \int_t^r d\xi Q(\xi)}, \tag{3.77}
\]
where \( t \) is a reference point and
\[
Q^2(r) = R(r) - \frac{1}{4r^2} = \frac{r^{4(d-3)}}{\Delta^2} \left[ E^2 - V_1 - \frac{1}{4r^2} + \frac{(d^2 - 5d + 7)r_H^{d-3}}{2r^{d-1}} - \frac{(d - 2)^2 r_H^{2(d-3)}}{4r^{2(d-2)}} \right]. \tag{3.78}
\]

As \(|E| \to \infty\), the zeros of \( Q^2(r) \), or the turning points, \( t_n \), in this WKBJ approximation, are close to the origin in the complex \( r \)-plane. In this limit,
\[
Q^2(r) \sim \left( \frac{r_H^{d-3}}{r_H^{d-3}} \right)^2 \left[ E^2 - \frac{(d - 2)^2 r_H^{2(d-3)}}{4r^{2(d-2)}} \right], \tag{3.79}
\]
and the turning points are at:

\[ Q^2(r) = 0 \Rightarrow t^{2(d-2)} \sim \frac{(d-2)^2 r_H^{2(d-3)}}{4E^2}. \] (3.80)

Asymptotically \( E \) is very close to the negative imaginary axis, that is, \( E \sim |E|e^{-i\pi/2} \), as such, the turning points can then be represented by:

\[ t_n \sim \left[ \frac{(d-2)^2 r_H^{2(d-3)}}{4|E|^2} \right]^{1/2(d-2)} e^{i(2n-1)\pi/2(d-2)}, \] (3.81)

for \( n = 1, 2, \cdots, 2(d-2) \). Note that the equation giving the asymptotic quasinormal frequency involves two quantities, the first being the line integral from one turning point to the other. Here, we have:

\[
\gamma \equiv - \int_{t_1}^{t_2} d\xi Q(\xi), \\
= - \int_{t_1}^{t_2} d\xi \left( \frac{\xi^{d-3} E}{r_H^{d-3}} \right) \left[ 1 - \frac{(d-2)^2 r_H^{2(d-3)}}{4E^2 \xi^{2(d-2)}} \right]^{1/2}, \\
= - \frac{1}{2} \int_{\frac{y_1}{y_2}}^{1} \left( 1 - \frac{1}{y^2} \right)^{1/2}, \\
= - \frac{\pi}{2}, \] (3.82)

where we have made the change of variable \( y = 2E\xi^{(d-2)}/(d-2)r_H^{(d-3)} \). The other quantity is the closed contour integration around \( r = r_H \):

\[
\Gamma \equiv \oint_{r=r_H} d\xi Q(\xi) = \oint_{r=r_H} d\xi \frac{r^{d-3} E}{r^{d-3} - r_H^{d-3}} = \frac{2\pi i r_H E}{d-3}. \] (3.83)

The asymptotic quasinormal frequency is then given by the formula,

\[ e^{-2i\Gamma} = -(1 + 2 \cos 2\gamma) \Rightarrow e^{4\pi r_H E/(d-3)} = 1, \]

\[ \Rightarrow E_n = -i \left( \frac{d-3}{2r_H} \right) n, \] (3.84)
for large $n$. In terms of the corresponding Hawking temperature, $T_H = (d - 3)/4\pi r_H$,

$$E_n = -i2\pi T_H n.$$  \hspace{1cm} (3.85)

We therefore obtain a vanishing real part for the frequency and find that the spacing of the imaginary part goes to $2\pi T_H$ regardless of the dimension. Note that these results are in accordance with that of integral spin fields in higher-dimensional Schwarzschild spacetimes [102, 103, 104].

### 3.5 Hawking radiation from bulk fermions

In section 3.3 we applied conformal methods which allowed us to separate the Dirac equation on a higher dimensional spherically symmetric background. This allowed us to discuss the QNMs for Schwarzschild BHs in section 3.4. Now we shall use these results to calculate the greybody factors and emission rates for Dirac perturbations on a $d$-dimensional Schwarzschild background. For a treatment of brane localized Hawking emission in the case of static BHs see reference [7, 25, 24, 105, 106, 107, 108, 109, 110, 111], for rotating BHs, see reference [112, 113, 114, 115, 116, 117, 118, 119, 120] as well as references [121, 23, 122, 123].

As already stated we used a conformal transformation to separate the Dirac equation into a time-radial part and a $(d-2)$-sphere. Moreover, the radial part was reduced to a Schrödinger-like equation, (3.59), in the tortoise coordinate $r_*$:

$$\left(-\frac{d^2}{dr_*^2} + V_1\right) G = E^2 G,$$  \hspace{1cm} (3.86)

where $dr = f(r)dr_*$, and the potential is given by:

$$V_1(r) = \kappa^2 \frac{f}{r^2} + \kappa f \frac{d}{dr} \left[\sqrt{\frac{f}{r}}\right],$$  \hspace{1cm} (3.87)

where

$$\kappa = \ell + \frac{d - 2}{2} = \frac{d}{2} - 1, \quad \frac{d}{2}, \quad \frac{d}{2} + 1, \ldots.$$  \hspace{1cm} (3.88)
It may be worth mentioning that the above potential reduces to the brane-localised results found in reference [124] when we set $\kappa = \ell + 1$ and therefore provides an alternate derivation of the brane localised potential.

In the following subsection, we shall compare the absorption probability for this potential as calculated under various approximation techniques, namely, the low energy WKBJ, the first to third order intermediate WKBJ and the low energy Unruh method. Then we shall relate these quantities to the absorption cross-section and calculate the Hawking emission rate. After this we will compare bulk emission with brane emission (to third order in WKBJ) for various dimensions, before presenting our concluding remarks.

3.5.1 Absorption probabilities via the WKBJ approximation

Low Energy

In the low energy WKBJ approach the absorption probability corresponds to the probability for a particle to tunnel through the potential barrier (the barrier penetration probability) and can be found in many standard quantum mechanics text books (a derivation can also be found in reference [125]). The result to first order in the low energy WKBJ approximation is given by (with $r_* = x$):

$$|A_\kappa(E)|^2 = \exp \left[ -2 \int_{x_1}^{x_2} dx' \sqrt{V_1(x') - E^2} \right], \quad (3.89)$$

where $x_1$ and $x_2$ are the turning points for a given energy $E$ with potential $V_1$. This approximation is valid for $V_1 \gtrsim E^2$ and as long as we can solve for the turning points in $V_1(x) = E^2$. Note that we can numerically integrate equation (3.89) for each energy $E$ to obtain the absorption probability as a function of $E$.

As discussed in reference [126], for example, in order to use the WKBJ method we must rewrite the perturbations in the $(n + 2)$-dimensional tortoise
coordinate defined by:

\[ \frac{dr_\ast}{dr} = \frac{1}{f(r)} \quad \Rightarrow \quad r_\ast = r + \frac{r_H^{d-3}}{d-3} \sum_{j=0}^{d-4} \frac{\ln(r/\alpha_j - 1)}{\alpha_j^{d-4}}, \quad (3.90) \]

where

\[ \alpha_j = r_H e^{2\pi i j/(d-3)} \quad (j = 0, \ldots, d - 4) . \quad (3.91) \]

For intermediate energies (see the next subsection) we can simply work with the original radial coordinate \( r \) and convert derivatives of the potential in terms of \( r_\ast \) using equation (3.90). However, for the low energy result we need to integrate between the two turning points in terms of tortoise coordinates. The intermediate WKBJ approximation can be applied at all energies \([125]\) including low energy, and there also exists a method developed by Unruh \([21, 127]\) which leads to useful analytic results at low energies. These three techniques will be compared in the low energy regime in the next section.

**Intermediate Energy: 3rd Order WKBJ**

As discussed in section 3.1 when the scattering takes place near the top of the potential barrier, an adapted form of the WKBJ method can be employed to find the absorption probability (which is equal to the transmission probability, equation (3.28)). We obtain the third order results for the absorption coefficient by iterating equation (3.10) until terms up to order \( O(k^{-3/2} z_0^0, k^{-1/2} z_0^4, k^{1/2} z_0^8) \) converge. We found this to take 2 iterations. In the following we shall use the same notation as reference \([80]\), where we have confirmed their results to fourth order. In which case we express the absorption probability as shown in section 3.28 as:

\[ |A_\kappa(E)|^2 = \frac{1}{1 + e^{2S(E)}}, \quad (3.92) \]
where

\[
S(E) = \pi k^{1/2} \left[ \frac{1}{2} z_0^2 + \left( \frac{15}{64} b_4 - \frac{3}{16} b_4 \right) z_0^4 + \left( \frac{1}{1024} b_3^4 - \frac{459}{128} b_4 b_3^2 + \frac{95}{32} b_4 b_3 + \frac{67}{64} b_4^2 - \frac{25}{16} b_6 \right) z_0^5 \right] \\
+ \pi k^{-1/2} \left( \frac{3}{16} b_4 - \frac{7}{64} b_3^2 \right) - \pi k^{-1/2} \left( \frac{1365}{2048} b_4^4 - \frac{3245}{256} b_4 b_3^2 b_4 + \frac{93}{64} b_4 b_3 + \frac{85}{128} b_4^2 - \frac{25}{32} b_6 \right) z_0^2 \\
+ O(k^{-3/2} z_0^6, k^{-1/2} z_0^8) .
\] (3.93)

Note that the first order result comprises of just the first term in this expression, while the second order result consists of the second term on the first line and the first term on the second line.

We would like to draw the readers attention to the fact that as we go to higher order the approximation becomes valid for lower energies. However, as can be seen from figure 3.5, even orders in the intermediate WKBJ method drop back down to zero for large energy. For this reason we shall work to third and not fourth order in our calculations, as odd orders have the nice property that \( |A|^2 \to 1 \) for large energy, making numerical work with these orders easier. Also, note that the first order WKBJ under-predicts the absorption probability for \( \varepsilon \sim O(1) \) as compared to the second order WKBJ result. We have compared these results up to fourth order in the WKBJ approximation [80], and the essential features of the energy absorption profile (such as dipping back down to zero for even orders) do not change when going from second to fourth order; nor do features of the first order change when going to third order, see figure 3.5.

In terms of the WKBJ approximation, in general, it will be convenient to make a change of variables to \( x = Er \) [126]. This leads to the following form for the potential:

\[
Q(x_*) = 1 - \kappa^2 \frac{f}{x^2} - \kappa f \frac{df}{dx} \left[ \frac{\sqrt{f}}{x} \right] ,
\quad f(x) = 1 - \left( \frac{\varepsilon}{x} \right)^{d-3} , \quad (3.94)
\]

where

\[
E^2 Q(x_*) = E^2 - V_1 , \quad (3.95)
\]
As such, the Schrödinger equation, equation (3.59), takes the form:

\[
\left( \frac{d^2}{dx^2} + Q \right) G = 0 .
\] (3.96)

We can then follow the treatment given in section 3.1. Note that for our purposes it is more convenient to work in terms of \( z \), rather than \( z_\ast \). However, given that the WKBJ form of the potential is in terms of the parameter \( z_\ast \), including its derivatives, we can convert derivatives in terms of \( z \) into \( z_\ast \) via the equation \( \frac{dz_\ast}{dz} = \frac{1}{f(z)} \).

High Energy

For high energies the absorption probability tends to unity, and the cross-section reduces to that of the classical cross-section, see reference [126]. However, as discussed in reference [125], there will always be small corrections to the large energy limit. A high energy WKBJ approach can be applied in this limit, but for the purposes of this current study it will be sufficient to use \( |A_\kappa(E)|^2 = 1 \) (given that it reproduces the high energy geometric optics limit, see references [25, 25, 7, 126]).

Absorption Probability Results

The results of these analyses have been shown in figure 3.5. In this figure we presented plots for the first, second and third order WKBJ Iyer and Will method [79], which is an intermediate WKBJ approximation, as well as the low energy WKBJ result (to first order) and the analytical method of Unruh [21]. The WKBJ approximation, in general, is accurate for larger angular momentum channels, whereas the Unruh approach is valid for only the lowest angular momentum channels and \( \varepsilon \ll 1 \) (namely small BHs). However, it is interesting to note that although the low energy WKBJ result does not agree exactly with the Unruh result, they both tend to zero for \( \varepsilon \to 0 \). On the other

\footnote{Note that in reference [126] we defined the horizon \( r_H \) in terms of \( \varepsilon = r_H E^{d-3} \), however, by using this different parameterisation here we may compare this result with the low energy Unruh approach.}
3.5 Hawking radiation from bulk fermions

Figure 3.5: Plots of the absorption probability, via various approximation schemes, for \( d = 7 \) and the first two angular momentum channels: \( \ell = 0 \) (top) and \( \ell = 1 \) (bottom). Note the Unruh result, is only valid for \( \varepsilon \ll 1 \).

hand, unlike the Unruh result, the low energy WKBJ is valid for energies up to \( \varepsilon \sim \mathcal{O}(1) \).

### 3.5.2 Emission rates

As stated in section 1.5.2 the emission rate for a massless fermion from a BH is related to the greybody factor by a \( d^{d-1}k \) dimensional momentum integral times a fermionic thermal temperature distribution:

\[
\frac{dE}{dt} = \sum_{\lambda,E} \sigma_{\lambda,E} \frac{E}{e^{E/T} + 1} \frac{d^{d-1}k}{(2\pi)^{d-1}},
\]

(3.97)
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where $T_H$ is the Hawking temperature, $\sigma_{\lambda,E}$ are the greybody factors and the sum is a generic sum over all angular momentum and momentum variables. Using the method of Cardoso et al. [128] we can relate the greybody factor to the absorbtion probability [127]:

$$\sigma_{\lambda,E} = \frac{1}{2\Omega_{d-2}} \left(\frac{2\pi}{E}\right)^{d-2} \sum_{\kappa} D_\kappa |A_\kappa(E)|^2,$$  \hspace{1cm} (3.98)

where $\Omega_{d-2}$ the total volume of a unit $(d-2)$-sphere:

$$\Omega_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}.$$  \hspace{1cm} (3.99)

Given that angular integration over the momentum for a massless field ($|k| = E$) leads to the Jacobian $\int d^{d-1}k = \int \Omega_{d-2}E^{d-2}dE$, the fermion emission rate can be expressed solely in terms of the absorption probability:

$$\frac{d^2\mathcal{E}}{dEdt} = \frac{1}{2\pi} \sum_{\kappa} \frac{E}{e^{\frac{E}{T_H}}+1} D_\kappa |A_\kappa(E)|^2,$$  \hspace{1cm} (3.100)

where the sum is over $\kappa$ for $\kappa = \pm \left(\frac{d}{2} - 1\right), \pm \frac{d}{2}, \pm \left(\frac{d}{2} + 1\right), \ldots$ and the degeneracy factor $D_\kappa$, calculated in [127], is given by:

$$D_\kappa = \frac{1}{2^{(d-2)/2}} \frac{\Gamma(|\kappa| + \frac{d}{2}) \Gamma(|\kappa| + \frac{d}{2} - 1)}{\Gamma(|\kappa| - \frac{d}{2} + 2) \Gamma(d-2)}.$$  \hspace{1cm} (3.101)

However, since the integrand depends only on the absolute value of $\kappa$ we only sum for $\kappa \geq 0$ and multiply by a factor of two. We then recover a result identical to that for a scalar field [25, 23, 7], except for a difference in sign due to fermion statistics.

After changing variables to $\varepsilon = Er_H$, and using the fact that the Hawking temperature is $T_H = (d-1)/(4\pi r_H)$, we obtain:

$$\frac{d^2\mathcal{E}}{dEdt} = \frac{1}{\pi r_H} \sum_{\kappa > 0} \frac{\varepsilon}{e^{\frac{\varepsilon}{r_H}}+1} D_\kappa |A_\kappa(\varepsilon)|^2.$$  \hspace{1cm} (3.102)
3.5 Hawking radiation from bulk fermions

As such, the evaluation of the emission rate is now a simple task, as $|A_{\kappa}(\varepsilon)|^2$ can now be obtained via the WKBJ method. Note that we have presented an example of these emission rates for various values of $d$ in figure 3.6 (up to third order WKBJ), these results shall be discussed in the remarks following this section.

Importantly, when we come to consider brane-localised emissions we simply set $\kappa = \ell + 1$ in the potential (3.87) and $D_l = 2(\ell + 1)$ for the degeneracy factor, as discussed in section 1.5.2.

3.5.3 Third Order WKBJ Emission Spectrum

Since the intermediate WKBJ is accurate for $\varepsilon > 1$ we plot the emission rates in this approximation, as shown in figure 3.6. We have also calculated the total power by integrating over $\varepsilon$, the results are shown in Table 3.2. From these results we find that for $d > 5$ the emission into the bulk is greater than the emission onto the brane. Note that in order to obtain convergence in equation (3.102) we must choose some value of $\kappa_{\text{max}} > \varepsilon$ and to ensure this we have taken $\kappa_{\text{max}} = 34 + \frac{d}{2}$. 
Figure 3.6: Plots of the fermion emission rates for \( d = 5, 6, 7 \) and 8 using the third order intermediate energy WKBJ method, where we compare bulk emissions (top panel) with brane-localised emissions (bottom panel).
3.6 Remarks

In this chapter we have presented new results for massless Dirac fields on a $d$-dimensional Schwarzschild background. In particular, we have calculated the
QNM’s of these fields, see figures 3.2 and 3.3, and calculated their Hawking emission, see figure 3.6.

In the case of the QNMs, the results can be compared with the brane-localised results of reference [23], revealing that bulk fermion modes result in much larger damping rates. It was also found that the QNM damping rate increases with dimension $d$, see figure 3.4, thus, we can naively infer that the amount of energy available to be radiated as Hawking radiation increases with the number of dimensions.

Some words of caution are necessary, as can be seen for example in the $d = 10$ result, which plots the $l = 0, 1, 2$ and 3 channels. In the next angular momentum channel, $l = 4$, the point $n = l$, crosses the imaginary axis. The presence of a branch cut along this axis would force us to choose the positive imaginary value in accordance with the constraint equation (3.37). However this does not signify that there are modes which are unstable, but indicates, as can be deduced from our asymptotic large $n$ analysis, that the WKB approximation is breaking down.

We have also presented new results for the emission rate of a massless Dirac field on a bulk $d$-dimensional Schwarzschild background. The main result is that fermions are predominantly emitted into the bulk for $d > 5$, see table 3.2, which is in contrast to the scalar field case [25, 7] and for bulk to brane photons [129]. This is an example contrary to the conjecture that BHs radiate mainly on the brane [130]. These are results have practical significance in that the observation signature at the LHC will be distinctly different between the two cases. One could then use these experimental results to determine the still unsolved theoretical question of whether the emission occurred off the brane or strictly onto it.

We highlighted how semi-analytic results can be obtained by considering different versions of the WKBJ approximation, these are shown in figure 3.5.  

\footnote{This is also true of brane-localised QNMs, see [23].}
3.6 Remarks

We also used the low energy WKBJ approximation and Unruh approximation, where it should be stressed that the low energy approximation extends further than that of the Unruh result (right up to intermediate energies), see figure 3.5. The most appropriate approximation scheme for calculating the power emission and emissivity was the third order WKBJ method since this approximation gives results that are valid for intermediate energies. Although the WKBJ approximation becomes worse for low energies, there is little power emission from this region, see figure 3.6, thus the emissivity (integrated differential power) calculated will be a very good approximation to the exact result.

In figure 3.6 we have plotted the third order WKBJ approximation of the emission rates for various dimensions, $d$. As can be seen from this figure, in the bulk emission case, there is an interesting behaviour around $\varepsilon \sim \mathcal{O}(1)$ where the emission rates in different dimensions cross. The cross-over effect is confirmed by plotting the emission in the low energy region using the Unruh method, see figure 3.7 and plotting the geometric optics limit $|A_\kappa(E)|^2 = 1$ at high energy and observing an opposite ordering of lines. We are unaware of this curious feature being reported anywhere else in the literature.

The results, when compared with the brane-localised fermion results of references \cite{121,23,122,123,25} reveal that bulk fermion modes result in much larger emission rates (though in both cases larger $d$ results in greater emission rates). These results also agree qualitatively with our work for the QNMs on such a background \cite{131}, where the BH damping rate was found to increase with dimension. Interestingly, the cross-over feature is absent from the emission rates of the brane-localised fermions, as can be seen in the right panel of figure 3.6, also see figure 4 of reference \cite{25}.\footnote{Figure 4 of reference \cite{25} is a Log plot not a Linear plot, however, there is no observable cross-over behaviour at intermediate energies.} Thus, this feature appears to be specific to bulk species.\footnote{As can be observed also in plots made in references \cite{25} \cite{7}}

This ends our discussion of BH phenomenology, we shall return to the
somewhat related subject of entanglement in accelerated frames (where the problem can be recast into the physical situation of a BH) in chapter 5. In the next chapter we will study the QLLR unification model in five dimensions.
CHAPTER 4

5D Quark Lepton Left Right model

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In continuing our investigations into higher dimensional physics in this chapter we take leave from BHs to study the Quark-Lepton Left-Right (QLLR) unification model in five dimensions. The study of this model will require the uses of a rich variety of extra dimensional tools including split-fermions (which served as a motivation for considering bulk fermions in chapter 3) and orbifolds.

4.1 Quark lepton left right models

In this section we review the four dimensional QLLR model [132, 133]. To this end, let us recall some features of the SM. The one generation fermion spectrum of the SM is given by:

\[ Q_L \sim (3, 2, 1/3), \quad u_R \sim (3, 1, 4/3), \quad d_R \sim (3, 1, -2/3), \]
\[ L_L \sim (1, 2, -1), \quad e_R \sim (1, 1, -2), \]

(4.1)

where the quantum numbers label the transformation properties of the fields under \( G_{SM} \equiv SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \).

The impetus for considering a QLLR extension of the SM stems from the observation that there exist similarities between the quantum numbers of the SM fermions. One similarity is that all left- and right-chiral fields possess identical electric and colour charges; another is the similar family structure of quarks and leptons, with all left-chiral fields forming \( SU(2)_L \) doublets whilst their right-chiral partners assume singlet \( SU(2)_L \) representations.

The gauge group of the QLLR model is

\[ G_{QLLR} = SU(3)_t \otimes SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_Y \]
where the fermions are assigned to the following representations:

\[ L_L \sim (3, 1, 2, 1, -1/3), \quad L_R \sim (3, 1, 2, 1, -1/3), \]
\[ Q_L \sim (1, 3, 2, 1, 1/3), \quad Q_R \sim (1, 3, 2, 1, 1/3). \]  

The action of the discrete symmetry \( Z_2^{QL} \times Z_2^{LR} \) is defined as

\[ L_L \leftrightarrow L_R \quad V \quad G_l \quad W_L \leftrightarrow W_R \]
\[ Q_L \leftrightarrow Q_R \quad -V \quad G_q \]  

where the QL (LR) symmetry acts vertically (horizontally) and \( V \) denotes the \( U(1)_V \) boson. Note that (4.2) contains only one independent fermion field with the quantum numbers of all other fermion fields determined by the discrete symmetry.

The additional symmetry needs to be broken in order to reproduce the SM and this is achieved through the implementation of scalar fields in two steps. The first step is achieved by the introduction of the scalars

\[ \Delta_{1L} \sim (\bar{6}, 1, 3, 1, 2/3), \quad \Delta_{1R} \sim (\bar{6}, 1, 1, 3, 2/3), \]
\[ \Delta_{2L} \sim (1, \bar{6}, 3, 1, -2/3), \quad \Delta_{2R} \sim (1, \bar{6}, 1, 3, -2/3), \]  

which transform as

\[ \Delta_{1L} \leftrightarrow \Delta_{1R} \]
\[ \Delta_{2L} \leftrightarrow \Delta_{2R} \]  

under the discrete symmetries. The Yukawa Lagrangian for these fields is

\[ \mathcal{L}_\Delta = \lambda_\Delta [(\overline{L}_L)^c L_L \Delta_{1L} + (\overline{L}_R)^c L_R \Delta_{1R} + (\overline{Q}_L)^c Q_L \Delta_{2L} + (\overline{Q}_R)^c Q_R \Delta_{2R}] + \text{H.c.}, \]  

(4.6)
If the neutral component of $\Delta_{1R}$ develops a non-zero VEV the gauge symmetry will be broken as per

$$G_{QLLR} \downarrow$$

$$SU(2)_l \otimes SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_{Y'},$$

where $Y$ denotes the SM hypercharge, $Y'$ denotes some orthogonal unbroken $U(1)$ factor and $SU(2)_l \subset SU(3)_l$. The hypercharge generator is given by

$$Y = 2I_{3R} + \frac{1}{\sqrt{3}}T^8_l + V,$$

(4.7)

where $T^8_l = 1/\sqrt{3} \times \text{diag}(-2, 1, 1)$ is a diagonal generator of $SU(3)_l$ and $I_{3R} = 1/2 \times \text{diag}(1, -1)$ is the diagonal generator of $SU(2)_R$. The next stage of symmetry breaking is achieved by introducing colour triplet scalars found in QL models, namely

$$\chi_l \sim (3, 1, 1, 1, 2/3), \quad \chi_q \sim (1, 3, 1, 1, -2/3),$$

which permute under the $Z^Q_L_2$ symmetry, $\chi_l \leftrightarrow \chi_q$. The Yukawa Lagrangian for these fields is

$$L_\chi = \lambda_\chi \left[ (L_L)^c L_L \chi_l + (L_R)^c L_R \chi_l + (Q_L)^c Q_L \chi_q 
\qquad \text{+} \ (Q_R)^c Q_R \chi_q \right] + \text{H.c.}$$

(4.8)

When the electrically neutral component of $\chi_l$ develops a VEV and the following breaking takes place:

$$SU(2)_l \otimes SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_{Y'},$$

$$\downarrow$$

$$SU(2)_l \otimes SU(3)_c \otimes SU(2)_L \otimes U(1)_Y.$$
a Majorana mass for the right-handed neutrino (this Majorana neutrino then conspires with the electroweak Dirac neutrino to produce one neutrino mass of the correct size). This VEV also gives mass to the $SU(3)_l/SU(2)_l$ coset bosons and the $W_R$ boson and breaks one of the two additional neutral bosons $Z''$ making these bosons unobservable heavy.

The non zero VEV for $\chi_l$ breaks $Y'$, resulting in a massive neutral gauge boson with an order $\langle \chi_l \rangle = w_I$ mass. The symmetry breaking induced by $\chi_l$ also gives mass to the exotic fermions, known as liptons in the literature. The unbroken $SU(2)_l$ symmetry serves to confine the liptons into two-fermion bound states. These states all decay via the usual electroweak interactions into the known fermions \[133\]. The lower bound on $w_I$ is of order TeV (we provide a detailed discussion of the bound on $w_I$ in section 4.6) and the key experimental signatures for the model are the $Z'$ boson and the liptons. The liptons may be produced at the LHC via the usual electroweak interactions and via virtual $Z'$ creation.

The gauge group $G_{SM} \otimes SU(2)_l$ is broken down to $SU(3)_c \otimes U(1)_Q \otimes SU(2)_l$ by the introduction of a Higgs bidoublet

$$
\Phi \sim (1, 1, 2, 2, 0),
$$

resulting in the following electroweak Yukawa Lagrangian

\[
\mathcal{L}_\Phi = \lambda_{\Phi_1}[\bar{L}_L L_R \Phi + \bar{Q}_L Q_R \Phi] +
\lambda_{\Phi_2}[\bar{L}_L L_R \bar{\Phi} + \bar{Q}_L Q_R \bar{\Phi}] + \text{H.c.},
\]

where $\bar{\Phi} = \epsilon \Phi^* \epsilon$ (we denote the two dimensional anti-symmetric tensor as $\epsilon$) and

\[
\Phi \leftrightarrow \tilde{\Phi}
\]

under the QL symmetry. The Yukawa couplings in (4.10) give rise to phe-
nomenologically incorrect fermion Dirac masses of the type

\[ m_u = m_e, \quad m_d = m_\nu, \quad (4.12) \]

where \( m_\nu \) is the neutrino Dirac mass. The relationship \( m_d = m_\nu \) is not problematic since as explained above the light neutrinos acquire mass via the seesaw mechanism. There is also enough parameter freedom to give the right-handed neutrinos an arbitrary mass through their couplings to \( \Delta_{1R} \). The mass relations between the electrons and the up quarks are removed by introducing an additional bidoublet \( \Phi' \sim (1, 1, 2, 2, 0) \). This doubles the number of Yukawa couplings and thus reduces the predictivity of the model.

### 4.2 Introduction to QLLR in five dimensions

In this work we study the quark-lepton left-right symmetric extension to the SM in five dimensions. Given that there is no chirality operator in 5 dimensions we shall denote the gauge group of the theory as \( SU(3)_l \times SU(3)_c \times SU(2)_1 \times SU(2)_2 \times U(1)_Y \). We will be required to ensure that the low energy fermion spectrum contains the chiral fermions found in the SM. The zero mode \( SU(2)_{1(2)} \) gauge bosons will eventually be identified with the usual \( W_{L(R)} \) bosons in LR models via their action on the low energy fermion content. Thus the 5d theory is invariant under the interchange \( 1 \leftrightarrow 2 \) which will prove to be equivalent to the usual LR symmetry in the low energy theory. This matter has already been discussed in \([134]\). We shall continue to label the discrete symmetry of the 5d model as \( \mathbb{Z}_{2L}^Q \times \mathbb{Z}_{2R}^L \).

The subgroup \( SU(3)_l \times SU(2)_2 \) is broken down to \( SU(2)_l \times U(1)_l \times U(1)_2 \) upon compactification on a \( S^1/Z_2 \times Z_2' \) orbifold. The resultant four dimensional theory is then further reduced to the SM via Higgs mechanisms. In section 4.3 we perform this combination of orbifolding and Higgs mechanisms to break the five dimensional theory. The technical details of the neutral gauge boson sector are worked out in section 4.4. In section 4.5 we detail the fermion content and then look at the neutral current sector in 4.6 and derive bounds.
4.3 Symmetry breaking

We start with the 5-dimensional \((x^\mu, y)\) gauge group and impose the discrete quark-lepton left-right symmetries \(q \leftrightarrow l\) and \(1 \leftrightarrow 2\) so that the couplings are related by, \(g_q = g_l \equiv g_s\) and \(g_1 = g_2 \equiv g\).

The orbifold is constructed by identifying \(y \rightarrow -y\) under the \(Z_2\) transformation and \(y' \rightarrow -y'\) under the \(Z'_2\) transformation, where \(y' = y + \pi R/2\). Then the orbifold space is an interval \([0, \pi R/2]\).

We impose the gauge boson transformations:

\[
\begin{align*}
W_\mu(x^\mu, y) &\rightarrow W_\mu(x^\mu, -y) = PW_\mu(x^\mu, y)P^{-1}, \\
W_5(x^\mu, y) &\rightarrow W_5(x^\mu, -y) = -PW_5(x^\mu, y)P^{-1}, \\
W_\mu(x^\mu, y') &\rightarrow W_\mu(x^\mu, -y') = PW_\mu(x^\mu, y')P'^{-1}, \\
W_5(x^\mu, y') &\rightarrow W_5(x^\mu, -y') = -PW_5(x^\mu, y')P'^{-1},
\end{align*}
\]

which leave the five dimensional Lagrangian invariant under \(Z_2 \times Z'_2\). We take \(P\) and \(P'\) to be trivial for \(SU(3)_q, SU(2)_1\) and \(U(1)_V\) gauge fields.

For the five dimensional \(SU(2)_2\) gauge fields:

\[
W_2 = \frac{1}{2} \begin{pmatrix}
W_0^0 & \sqrt{2}W_2^0 \\
\sqrt{2}W_2^0 & -W_0^0
\end{pmatrix},
\]

(4.13)

we chose \(P_2 = \text{diag}(1, 1)\) and \(P'_2 = \text{diag}(-1, 1)\). For the \(SU(3)_l\) bosons:

\[
G_l = \begin{pmatrix}
-\frac{2}{\sqrt{3}}G_l^0 & \sqrt{2}Y_l^1 & \sqrt{2}Y_l^2 \\
\sqrt{2}Y_l^1 & G_l^3 + \frac{1}{\sqrt{3}}G_l^0 & \sqrt{2}G_l^l \\
\sqrt{2}Y_l^2 & \sqrt{2}G_l^l & -G_l^3 + \frac{1}{\sqrt{3}}G_l^0
\end{pmatrix},
\]

(4.14)

we choose \(P_l = \text{diag}(1, 1, 1)\) and \(P'_l = \text{diag}(-1, 1, 1)\).
These transformations give the $Z_2 \times Z'_2$ parities:

\begin{align}
G^{0}_{\mu}, & \quad G^{3}_{\mu}, \quad \tilde{G}_\mu, \quad \tilde{G}^{\dagger}_\mu, \quad W^0_{2\mu} : (+, +), \\
Y^{1}_{\mu}, & \quad Y^{2}_{\mu}, \quad Y^{1\dagger}_{\mu}, \quad Y^{2\dagger}_{\mu}, \quad W^{\pm}_{2\mu} : (+, -), \\
G^{0}_{15}, & \quad G^{3}_{15}, \quad \tilde{G}_{15}, \quad \tilde{G}^{\dagger}_{15}, \quad W^{0}_{2,5} : (-, -), \\
Y^{1}_{15}, & \quad Y^{2}_{15}, \quad Y^{1\dagger}_{15}, \quad Y^{2\dagger}_{15}, \quad W^{\pm}_{2,5} : (+, -).
\end{align}

A general five dimensional field, $\psi$, can be expanded in terms of Fourier modes in the compact dimension:

\begin{align}
\psi_{(+,+)}(x^\mu, y) &= \sqrt{\frac{2}{\pi R}} \left( \psi_{(+,+)}^{(+)}(x^\mu) + \sqrt{2} \sum_{n=1}^{\infty} \psi_{(+,+)}^{(n)}(x^\mu) \cos \frac{2ny}{R} \right), \\
\psi_{(+,-)}(x^\mu, y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \psi_{(+,-)}^{(n)}(x^\mu) \cos \frac{(2n+1)y}{R}, \\
\psi_{(-,+)}(x^\mu, y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \psi_{(-,+)}^{(n)}(x^\mu) \sin \frac{(2n+1)y}{R}, \\
\psi_{(-,-)}(x^\mu, y) &= \sqrt{\frac{2}{\pi R}} \sum_{n=0}^{\infty} \psi_{(-,-)}^{(n)}(x^\mu) \sin \frac{(2n+2)y}{R}.
\end{align}

Thus only fields with a $(+, +)$ transformation property under $Z_2 \times Z'_2$ will possess a massless zero mode. Importantly, we see that $G^{0}_{\mu}, G^{3}_{\mu}, \tilde{G}_\mu, \tilde{G}^{\dagger}_\mu, W^0_{2\mu}$ are the only such four dimensional gauge fields to do so. Effectively then our $SU(3)_l \times SU(2)_2$ bulk symmetry has been broken down to $SU(2)_l \times U(1)_l \times U(1)_2$ at the zero mode level. It is worth commenting that new heavy exotic bosons, equation (4.16), exist in Kaluza Klein states at the inverse compactification scale but since this scale is taken much larger than the others they are not phenomenologically relevant.

The zero mode gauge group is now,

$$SU(2)_l \times SU(3)_q \times SU(2)_1 \times U(1)_l \times U(1)_2 \times U(1)_V.$$
4.3 Symmetry breaking

tonic gauge sector, $SU(2)_l \times SU(3)_q \times U_Q(1)$, using the Higgs mechanism. The extra $SU(2)_l$ gauge sector acts only on the liptons (exotic leptons) which get masses at the $SU(3)_l$ breaking scale and are expected to be be confined into leptonic bound states [135]. We will not break this symmetry any further but instead postulate that it is an additional as yet unobserved symmetry of nature. This does not contradict any observations since the effect of the extra symmetry is believed to manifest only at the $SU(3)_l$ breaking scale.

We use the higgs sector,

\[
\chi_l \sim (3, 1, 1, 1)\left(\frac{2}{3}\right) \\
\chi_q \sim (1, 3, 1, 1)\left(-\frac{2}{3}\right)
\]

and a bidoublet,

\[
\Phi \sim (1, 1, 2, 2)(0).
\]

Under $Z_2 \times Z'_2$ we assume the Higgs fields transform as:

\[
\begin{align*}
\chi_2(x^\mu, y) &\quad \rightarrow \quad \chi_2(x^\mu, -y) = P_2 \chi_R(x^\mu, y), \\
\chi_2(x^\mu, y') &\quad \rightarrow \quad \chi_2(x^\mu, -y') = P'_2 \chi_R(x^\mu, y'), \\
\chi_1(x^\mu, y) &\quad \rightarrow \quad \chi_1(x^\mu, -y) = P_1 \chi_L(x^\mu, y), \\
\chi_1(x^\mu, y') &\quad \rightarrow \quad \chi_1(x^\mu, -y') = -P'_1 \chi_L(x^\mu, y'), \\
\chi_q(x^\mu, y) &\quad \rightarrow \quad \chi_q(x^\mu, -y) = P_q \chi_q(x^\mu, y), \\
\chi_q(x^\mu, y') &\quad \rightarrow \quad \chi_q(x^\mu, -y') = P'_q \chi_q(x^\mu, y'), \\
\chi_l(x^\mu, y) &\quad \rightarrow \quad \chi_l(x^\mu, -y) = P_l \chi_l(x^\mu, y), \\
\chi_l(x^\mu, y') &\quad \rightarrow \quad \chi_l(x^\mu, -y') = -P'_l \chi_l(x^\mu, y'),
\end{align*}
\]
\[ \Phi(x^\mu, y) \rightarrow \Phi(x^\mu, -y) = P_1 \Phi(x^\mu, y) P_2^{-1}, \]
\[ \Phi(x^\mu, y') \rightarrow \Phi(x^\mu, -y') = P'_1 \Phi(x^\mu, y') P'_2^{-1}, \]

Summarising our orbifold boundary conditions we have,

\[ P_l = \text{diag}(1, 1, 1), \quad P'_l = \text{diag}(-1, 1, 1), \]
\[ P_q = P'_q = \text{diag}(1, 1, 1), \]
\[ P_1 = P'_1 = \text{diag}(1, 1), \]
\[ P_V = P'_V = 1, \]
\[ P_2 = \text{diag}(1, 1), \quad P'_2 = \text{diag}(-1, 1). \]

If we use the notation, \((P, P') = (P_l \oplus P_q \oplus P_1 \oplus P_2 \oplus P_V, P'_l \oplus P'_q \oplus P'_1 \oplus P'_2 \oplus P'_V)\) then the parity assignments for the bulk scalar fields immediately follow:

\[ \chi_l = \begin{pmatrix} \chi_{1l}^0 (+, +) \\ \chi_{2l}^{1/2} (+, -) \\ \chi_{3l}^{1/2} (+, -) \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} \chi_{12}^+ (+, -) \\ \chi_{12}^0 (+, +) \end{pmatrix}, \]
\[ \Phi = \begin{pmatrix} \phi_1^0 (+, +) & \phi_2^- (+, -) \\ \phi_1^+ (+, +) & \phi_2^0 (+, -) \end{pmatrix}. \quad (4.21) \]

Only neutral components that have zero modes under the orbifold can develop a VEV. Thus, we must take the VEV’s of the Higgs fields to be:

\[ \langle \chi_l^{(0)} \rangle = \begin{pmatrix} w_l \\ 0 \\ 0 \end{pmatrix}, \quad \langle \chi_2^{(0)} \rangle = \begin{pmatrix} 0 \\ w_R \end{pmatrix}, \quad \langle \Phi^{(0)} \rangle = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.22) \]

The subscript \(R\) on the \(\chi_2\) VEV has the obvious meaning as being the VEV associated with effective 4 dimensional right boson masses. The model broken in this way possesses several new phenomenological features in the neutral boson sector that we shall now discuss.
4.4 The gauge boson sector

In this section we discuss the phenomenology of the gauge boson sector. From here on we identify, \( W_1 \leftrightarrow W_L \) and \( W_2 \leftrightarrow W_R \).

The relevant part of the covariant derivative is:

\[
D_\mu = \partial_\mu + igW_L + igW_R + \frac{g_s}{2}G_l + \frac{g_V}{2}VB_V, \tag{4.23}
\]

where \( W \) and \( G \) are the matrices defined in equations (4.13) and (4.14). The boson masses are derived from the scalar kinetic terms once the fields have been re-expressed as perturbations about their minima, \( \chi \rightarrow \langle \chi \rangle + \chi \). For example,

\[
\mathcal{L}_{\text{kinetic}} = [D_\mu \chi_l]^\dagger D^\mu \chi_l, \tag{4.24}
\]

\[
\rightarrow [D_\mu \langle \chi_l \rangle]^\dagger D^\mu \langle \chi_l \rangle + [D_\mu \chi_l]^\dagger D^\mu \chi_l + \text{other terms.} \tag{4.25}
\]

With the choice of VEV’s in equation (4.22):

\[
D_\mu = \partial_\mu + i\frac{g_s}{2} \begin{pmatrix}
-\frac{2}{\sqrt{3}} & \sqrt{2}Y_1 & \sqrt{2}Y_2 \\
\sqrt{2}Y_1^* & \frac{1}{\sqrt{3}}G_l^0 & \sqrt{2}G_l \\
\sqrt{2}Y_2^* & \sqrt{2}G_l^0 & -G_l^3 + \frac{1}{\sqrt{3}}G_l^0
\end{pmatrix} + i\frac{g_V}{2}VB_V \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}. \tag{4.26}
\]

So,

\[
D_\mu \langle \chi_l \rangle = i\nu_l \begin{pmatrix}
g_s \sqrt{2}Y_1^0 + \frac{g_V}{2}B_V \\
\frac{g_s}{\sqrt{2}}Y_1^{11} \\
\frac{g_s}{\sqrt{2}}Y_2^{11}
\end{pmatrix}, \tag{4.27}
\]

Note that taking a factor of one half explicitly out of the coupling constant in the \( G_l \) term is consistent with the definition of \( g_s \) in QCD where gluon color potentials are defined similarly. Had we ignored this convention and written, \( ig_{\text{new}}G_l \), by the QL symmetry the coupling \( g_{\text{new}} \) would be half the value of the strong coupling constant, \( g_s \).

Since there is no physical measurement of \( g_V \) we are free to normalise it by taking out a factor of one half, as we have done, this is typical when dealing with \( U(1) \) fields (the reason is that this simplifies the coupling constant relationships; namely the half appearing on the hypercharge operator in \( Q = I_3 + Y/2 \), cancels with the one defined in the covariant derivative above.)
\[ [D_\mu(\chi_l)]^\dagger D^\mu(\chi_l) = w_l^2 \left( -\frac{g_s}{\sqrt{3}} G_l^0 + \frac{g_V}{3} B_V \right)^2 + \frac{g_s^2}{2} (Y_l^1)^2 + \frac{g_s^2}{2} (Y_l^2)^2 \]. \quad (4.28)

Likewise, for the \( \chi_R \) kinetic term:

\[
D_\mu = \partial_\mu + i \frac{g}{2} \left( \begin{array}{c} W_R^0 \\ \sqrt{2} W_R^- \end{array} \right) \left( \begin{array}{c} \sqrt{2} W_R^+ \\ -W_R^0 \end{array} \right) + i \frac{g_V}{2} V B_V \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \quad (4.29)
\]

\[ D_\mu(\chi_l) = i w_R \left( \begin{array}{c} \frac{g}{\sqrt{2}} W_R^+ \\ -\frac{g}{2} W_R^0 + \frac{g_V}{2} B_V \end{array} \right), \quad (4.30) \]

\[ [D_\mu(\chi_R)]^\dagger D^\mu(\chi_R) = \frac{g^2 w_R^2}{2} W_R^+ W_R^- + w_R^2 (-\frac{g}{2} W_R^0 + \frac{g_V}{2} B_V)^2. \quad (4.31) \]

Lastly, we must consider the bidoublet, \( \Phi \) which by definition transforms like,

\[ \Phi \rightarrow U_L \Phi U_R^\dagger, \quad (4.32) \]

for some group element \( U_{L,R} \in SU(2)_{L,R} \). This in turn implies that it has a covariant derivative

\[ D_\mu = \partial_\mu + i \frac{g}{2} (W_L \Phi - \Phi W_R^+). \quad (4.33) \]

Thus,

\[ D_\mu(\Phi) = i \frac{g}{2} \left\{ \left( \begin{array}{c} W_L^0 \\ \sqrt{2} W_L^- \end{array} \right) \left( \begin{array}{c} \sqrt{2} W_L^+ \\ -W_L^0 \end{array} \right) \left( \begin{array}{cc} k & 0 \\ 0 & 0 \end{array} \right) \right\} - \left( \begin{array}{c} k \\ 0 \\ 0 \end{array} \right) \left( \begin{array}{c} W_R^0 \\ \sqrt{2} W_R^- \end{array} \right) \left( \begin{array}{cc} \sqrt{2} W_R^+ \\ -W_R^0 \end{array} \right) \left( \begin{array}{cc} k & 0 \\ 0 & 0 \end{array} \right) \right\}. \quad (4.34) \]

In this case the kinetic term requires the trace to make it a number, so the
4.4 The gauge boson sector

relevant term is:

\[ \text{Tr}[D_\mu \langle \Phi \rangle]D^\mu \langle \Phi \rangle = \frac{g^2 k^2}{4} \text{Tr} \left( \begin{pmatrix} W_L^0 - W_R^0 & \sqrt{2} W_L^+ \\ -\sqrt{2} W_R^+ & 0 \end{pmatrix} \begin{pmatrix} W_L^0 - W_R^0 & -\sqrt{2} W_R^- \\ \sqrt{2} W_L^- & 0 \end{pmatrix} \right), \]

(4.36)

\[ = \frac{g^2 k^2}{4} (W_L^0 - W_R^0)^2 + \frac{g^2 k^2}{2} W_L^+ W_L^- + \frac{g^2 k^2}{2} W_R^+ W_R^- \].

(4.37)

One sees that there is no mixing between the charged gauge bosons and they have the Kaluza-Klein (KK) mass towers:

\[ m^2_{n,W_L^\pm} = M^2_{W_L} + \left( \frac{2n}{R} \right)^2, \]

(4.38)

\[ m^2_{n,W_R^\pm} = \frac{g^2}{2} \left( k^2 + w_r^2 \right) + \left( \frac{2n - 1}{R} \right)^2, \]

(4.39)

\[ m^2_{n,Y_1} = \frac{g_s^2}{2} w_l^2 + \left( \frac{2n - 1}{R} \right)^2, \]

(4.40)

\[ m^2_{n,Y_2} = \frac{g_s^2}{2} w_l^2 + \left( \frac{2n - 1}{R} \right)^2, \]

(4.41)

where \( M^2_{W_L} = \frac{g^2}{2} k^2 \). As anticipated only \( W_L \) has a zero mode. Unlike the charged gauge boson sector the neutral gauge bosons do mix with each other, this will now be investigated.

4.4.1 Neutral gauge boson mixing

The neutral boson mass term can be written:

\[ \mathcal{L}_{\text{mass}} = \frac{1}{\sqrt{2}} \sum_n \text{V} M_n^2 v_1, ^{2} \]

(4.43)

\[^{2}\text{One may wonder about the one half appearing in the neutral boson mass term but not in the equivalent charged terms. Firstly, the half comes from the fact that the lagrangian, } \mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} m^2 A^2, \text{ gives the correct equation of motion, } \Box^2 A_{\mu} + m^2 A_{\mu} = 0. \text{ Secondly, recall that a charged field can be thought of as two independent vector fields, } W^\pm = \frac{1}{\sqrt{2}} (W_1 \pm iW_2), \text{ and thus } L_{\text{mass}} = M^2 W^+ W^- = \frac{1}{2} M^2 (W_1^2 + W_2^2). \]
where $V = (W_L^{0(n)} W_R^{0(n)} B_V^{(n)} G_l^{0(n)})$, and

$$
M_n^2 = \begin{pmatrix}
\frac{g^2 k_v^2}{2} & \frac{g^2 k_v^2}{2} & 0 & 0 \\
-\frac{g^2 k_v^2}{2} & \frac{v^2 (k_v^2 + w^2)}{2} & 0 & 0 \\
0 & -\frac{g v g_w^2}{2} & -\frac{2 g v g_w^2}{3} & -\frac{2 g v g_w^2}{3} \\
0 & 0 & -\frac{g v g_w^2}{2} & -\frac{2 g v g_w^2}{3}
\end{pmatrix}.
$$

(4.44)

In the following we focus on the $(n = 0)$ zero modes. A general procedure for relating coupling constants when the generators they correspond to are linearly dependent can be found in Appendix A. Using this procedure we find the relation

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g_Y^2} + \frac{1}{g_B^2} + \frac{1}{3 g_s^2}.$$

In order to simplify the analysis it is useful to introduce the SM $U(1)_Y$ field with coupling constant, $g_Y$, and a $U(1)_{B-L}$ field, with coupling $g_B$, used in standard Left-Right models \[134\]. In these terms the coupling constants are related by,

$$
\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g_Y^2},
\frac{1}{g_Y^2} = \frac{1}{g^2} + \frac{1}{g_B^2},
\frac{1}{g_B^2} = \frac{1}{g^2} + \frac{1}{3 g_s^2},
$$

(4.45)

and the fields

$$
A = \cos \theta \ B_Y + \sin \theta \ W_L^0,
$$

(4.46)

$$
Z = - \sin \theta \ B_Y + \cos \theta \ W_L^0,
$$

(4.47)

$$
B_Y = \cos \alpha \ B_B + \sin \alpha \ W_R^0,
$$

(4.48)

$$
Z' = - \sin \alpha \ B_B + \cos \alpha \ W_R^0,
$$

(4.49)

$$
B_B = \cos \beta \ B_V + \sin \beta \ G_l^0,
$$

(4.50)

$$
Z'' = - \sin \beta \ B_V + \cos \beta \ G_l^0,
$$

(4.51)
where the mixing angles are defined

\[
\tan \theta = \frac{g_Y}{g}, \quad \tan \alpha = \frac{g_B}{g} \quad \text{and} \quad \tan \beta = \frac{g_Y}{\sqrt{3}g_s}.
\] (4.52)

Using Equations (4.52) and (4.45) one can relate all angles to the Weinberg angle, \(\theta\). From the relationship, \(\frac{1}{g_Y^2} = \frac{1}{g^2} + \frac{1}{g_B^2}\), we can obtain, \(\frac{1}{\tan^2 \theta} = 1 + \frac{1}{\tan^2 \alpha}\), which then leads to:

\[
\sin \alpha = \tan \theta.
\] (4.53)

Also, \(\frac{1}{g_B} = \frac{1}{g_Y^2} + \frac{1}{g_V^2} = \frac{1}{g_Y^2} (\cot^2 \beta + 1) = \frac{1}{g_Y^2 \sin^2 \beta}\), and \(\frac{1}{g_B} = \frac{1}{g_Y^2} - \frac{1}{g_Y^2} = \frac{1}{g_Y^2} (\cot^2 \theta - 1) = \frac{1}{g_Y^2 \sin^2 \theta}\). Thus,

\[
\sin \beta = \frac{g}{\sqrt{3}g_s} \frac{\sin \theta}{\sqrt{\cos 2\theta}}.
\] (4.54)

We now reconsider the neutral boson masses with the relations (4.53)-(4.54) in mind. The neutral boson mass Lagrangian is:

\[
L_{\text{mass}} = \left(-\frac{g_s}{\sqrt{3}} G_l + \frac{g_Y}{3} B_V \right)^2 w_l^2 + \frac{1}{4} (g W_R^0 - g_Y B_V)^2 w_R^2 + \frac{g^2}{4} (W_L^0 - W_R^0)^2 k^2.
\] (4.55)

We eliminate the fields and couplings in favour of \(A, Z, Z', Z''\) and \(g\) and \(g_s\).

\[
-\frac{g_s}{\sqrt{3}} G_l + \frac{g_Y}{3} B_V = -\frac{g_s}{\sqrt{3}} G_l + \frac{g_s}{\sqrt{3}} \tan \beta B_V
\] (4.56)

\[
= -\frac{g_s}{\sqrt{3} \cos \beta} (\cos \beta G_l^0 - \sin \beta B_V)
\] (4.57)

\[
= -\frac{g_s}{\sqrt{3} \cos \beta} Z''.
\] (4.58)

We find by inverting equations (4.48) and (4.49) that \(W_R^0 = \sin \alpha B_Y + \cos \alpha Z'\), so

\[
W_R^0 = \tan \theta B_Y + \cos \alpha Z';
\] (4.59)
and
\[ \frac{1}{\cos \theta} Z = -\tan \theta B_Y + W_L^0, \] (4.60)
thus
\[ W_L^0 - W_R^0 = \frac{1}{\cos \theta} Z - \cos \alpha Z'. \] (4.61)
Lastly, we must consider the $gW_R^0 - g_Y B_Y$ term. From equation (4.49) we obtain
\[ gW_R^0 = \frac{g}{\cos \alpha} Z' + g_B B_B, \] (4.62)
and inverting equations (4.50) and (4.51) we find
\[ B_Y = \cos \beta B_B - \sin \beta Z'', \] (4.63)
\[ \Rightarrow g_Y B_Y = g_Y \cos \beta B_B - g_Y \sin \beta Z''. \] (4.64)
Subtracting these two equations from each other yields
\[ gW_R^0 - g_Y B_Y = \frac{g}{\cos \alpha} Z' + \frac{\sqrt{3} g_s \sin^2 \beta}{\cos \beta} Z''. \] (4.65)

Finally, we arrive at an expression for the neutral boson mass Lagrangian in terms of $Z$, $Z'$ and $Z''$. Since there are no $A$ terms, the photon field is massless. This is really only a reflection of the fact that the $U(1)_Q$ subgroup is unbroken.

The Lagrangian can be expressed as a mixing matrix between the $Z$, $Z'$ and $Z''$. Writing $\vec{Z} = (Z, Z', Z'')^T$ and $\mathcal{L}_{mass} = \frac{1}{2} \vec{Z}^T H \vec{Z}$ one has
\[ H = \begin{pmatrix}
  m_Z^2 & -m_Z^2 \cot \alpha \sin \theta & 0 \\
  -m_Z^2 \cot \alpha \sin \theta & m_{Z'}^2 & \frac{g^2 w^2}{4} \tan \beta \tan 2 \theta \\
  0 & \frac{g^2 w^2}{4} \tan \beta \tan 2 \theta & m_{Z''}^2
\end{pmatrix}, \] (4.66)
where,

\[
m_Z^2 = \frac{g^2 k^2}{2 \cos^2 \theta},
\]

\[
m_{Z'}^2 = \frac{g^2 w_R^2}{2 \cos^2 \alpha} \left( 1 + \left( \frac{k}{w_R} \right)^2 \cos^4 \alpha \right),
\]

\[
m_{Z''}^2 = \frac{2 g_t^2}{3 \cos^2 \beta} \left( w_t^2 + \frac{9}{4} w_R^2 \sin^4 \beta \right). \tag{4.67}
\]

### 4.4.2 Diagonalising the mass matrix

In order to find the masses and physical fields we must diagonalise $H$. It is very impractical to do this exactly, yet we can solve it approximately by employing time-independent non-degenerate perturbation theory [78].

Consider the eigenvalue problem

\[
(H_0 + \lambda V) |n\rangle = E |n\rangle
\]

where $H_0$ is exactly diagonalisable, $H_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle$, and $V$ is a perturbation matrix. The eigenvalues are approximately

\[
E_n = E_n^{(0)} + \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \cdots, \tag{4.68}
\]

where $V_{nm} = \langle n^{(0)} | V | k^{(0)} \rangle$. Furthermore, the eigenvectors are approximated by

\[
|n\rangle = |n^{(0)}\rangle + \lambda \sum_{k \neq n} |k^{(0)}\rangle \frac{V_{km}}{E_n^{(0)} - E_k^{(0)}} + \cdots \tag{4.69}
\]

One may wonder why we have kept terms of order $\lambda^2$ in the eigenvalues but only to order $\lambda$ in the eigenvectors, the reason is that the correction to the $Z$ mass is of order $\lambda^2$, but the analysis of the eigenvectors becomes too tedious to perform to that same order. In general, we remain consistent to order $\lambda$ in both the eigenvalues and eigenvectors and will only use the $\lambda^2$ term to calculate the correction to the $Z$ mass.
To use perturbation theory in the case at hand we observe that we can separate
the matrix into a sum of two parts; one part of order \( k \), and the other of order
\( w_l \) or \( w_R \). Since \( k \ll w_l, w_R \), this will be a reasonable strategy for all regions
of the parameter space we wish to consider.

In order to make the above statement more rigorous, consider a mass
squared scale, \( \Lambda \), such that \( m_Z^2 < \Lambda < w_l^2, w_R^2 \). The following parametriza-
tion will be found to be useful:

\[
\begin{align*}
c & = \cos^2 \alpha \cos^2 \theta, \\
d & = \frac{g^2}{2} \sec^2 \alpha \frac{w_R^2}{\Lambda}, \\
h & = \frac{3g_2^2}{2} \sin^2 \beta \tan^2 \beta \frac{w_R^2}{\Lambda}, \\
f & = \frac{2g_2^2}{3} \sec^2 \beta \frac{w_l^2}{\Lambda}, \\
p & = f + h.
\end{align*}
\]

Then the matrix, \( H \), in equation (4.66) can be rewritten as

\[
H/\Lambda = \begin{pmatrix}
0 & 0 & 0 \\
0 & d & \sqrt{dh} \\
0 & \sqrt{dh} & p
\end{pmatrix} + \lambda \begin{pmatrix}
1 & -\sqrt{c} & 0 \\
-\sqrt{c} & c & 0 \\
0 & 0 & 0
\end{pmatrix},
\]  

where \( \lambda = m_Z^2/\Lambda \) is the perturbation parameter. The \( H_0 \) part is easily dia-
goanalised with the eigenvalues, \( 0, \frac{1}{2}(d + p - \Delta) \) and \( \frac{1}{2}(d + p + \Delta) \), where we have defined

\[
\Delta = \sqrt{d^2 + 4hd - 2pd + p^2},
\]

and the normalised eigenvectors are

\[
\begin{align*}
|0^{(0)}\rangle & = (1, 0, 0), \\
|1^{(0)}\rangle & = \frac{1}{\sqrt{2\Delta(\Delta - d + p)}}(0, d - p - \Delta, 2\sqrt{dh}), \\
|2^{(0)}\rangle & = \frac{1}{\sqrt{2\Delta(\Delta + d - p)}}(0, d - p + \Delta, 2\sqrt{dh}),
\end{align*}
\]
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respectively\[4\]. The matrix elements are thus,

\[
\begin{align*}
\langle 0|V|0 \rangle &= 1, \\
\langle 0|V|1 \rangle &= \sqrt{\frac{c(\Delta - d + p)}{2\Delta}}, \\
\langle 0|V|2 \rangle &= -\sqrt{\frac{c(\Delta + d - p)}{2\Delta}}, \\
\langle 1|V|1 \rangle &= -\frac{c(d - p - \Delta)}{2\Delta}, \\
\langle 2|V|2 \rangle &= \frac{c(d - p + \Delta)}{2\Delta}, \\
\langle 1|V|2 \rangle &= -\frac{c\sqrt{hd}}{\Delta}.
\end{align*}
\]

And so to order \(O(\lambda^2)\) we have the following eigenvalues\[5\]

\[
\begin{align*}
E_0 &= \lambda + O(\lambda^2), \\
E_1 &= \frac{1}{2}(d + p - \Delta) - \frac{c(d - p - \Delta)}{2\Delta}\lambda + O(\lambda^2), \\
E_2 &= \frac{1}{2}(d + p + \Delta) + \frac{c(d - p + \Delta)}{2\Delta}\lambda + O(\lambda^2).
\end{align*}
\]

The first eigenvalue implies that \(M_Z^2 = m_Z^2\), thus we are not at a high enough order to determine the correction to the \(Z\) mass, \(\delta m_Z^2\). We now obtain this by calculating the second order term of the first eigenvalue. From the last term in equation (4.68) we find:

\[
\begin{align*}
E_0^{(2)} &= \lambda^2 \left( \frac{c(\Delta - d + p)}{\Delta(\Delta - d - p)} - \frac{c(\Delta + d - p)}{\Delta(\Delta + d + p)} \right), \\
&= \lambda^2c \left( (\Delta + p)^2 - (\Delta - p)^2 \right) \frac{\lambda^2cp}{\Delta(\Delta^2 - (p + d)^2)} = \frac{\lambda^2cp}{hd - pd}, \\
&= -\frac{\lambda^2cp}{fd}.
\end{align*}
\]

\[4\] Note that in calculating the normalisation factors we are using the relation \(4hd = (\Delta + p - d)(\Delta - p + d)\).

\[5\] As a check of these observe that \(E_1 + E_2 + E_3 = \lambda + c\lambda + d + p = \text{Tr}H\).
We proceed to find the eigenvectors

\[ |0\rangle = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) - \lambda \frac{\sqrt{c}}{\Delta (d+p+\Delta)} \left( \begin{array}{c} 0 \\ d - p - \Delta \\ 2\sqrt{dh} \end{array} \right) + \lambda \frac{\sqrt{c}}{\Delta (d+p+\Delta)} \left( \begin{array}{c} 0 \\ d - p + \Delta \\ 2\sqrt{dh} \end{array} \right) + O(\lambda^2), \] 

(4.91)

\[ = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) - \lambda \frac{\sqrt{c}}{\Delta (d+p)^2 - \Delta^2} \left( \begin{array}{c} 0 \\ d - p - \Delta \\ 2\sqrt{dh} \end{array} \right) + \lambda \frac{\sqrt{c}}{\Delta (d+p)^2 - \Delta^2} \left( \begin{array}{c} 0 \\ d - p + \Delta \\ 2\sqrt{dh} \end{array} \right) + O(\lambda^2), \] 

(4.92)

\[ = \left( \begin{array}{c} \frac{1}{\sqrt{2\Delta}} \\ -\lambda \frac{\sqrt{c}}{\sqrt{df}} \end{array} \right) + O(\lambda^2). \] 

(4.93)

To simplify the rest of the analysis we note that the zeroth order eigenvectors can be written in a more symmetric fashion:

\[ |0^{(0)}\rangle = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \quad |1^{(0)}\rangle = \frac{1}{\sqrt{2\Delta}} \left( \begin{array}{c} 0 \\ -\sqrt{\Delta} \\ 0 \end{array} \right) + \frac{1}{\sqrt{2\Delta}} \left( \begin{array}{c} \frac{\Delta}{\sqrt{\Delta^2 + d^2}} \\ \frac{\Delta}{\sqrt{\Delta^2 + d^2}} \end{array} \right), \quad |2^{(0)}\rangle = \frac{1}{\sqrt{2\Delta}} \left( \begin{array}{c} \frac{d}{\sqrt{\Delta^2 + d^2}} \\ \frac{-d}{\sqrt{\Delta^2 + d^2}} \end{array} \right). \] 

(4.95)

So,

\[ |1\rangle = \frac{1}{\sqrt{2\Delta}} \left( \begin{array}{c} 0 \\ -\sqrt{\Delta} \\ 0 \end{array} \right) + \lambda \frac{\sqrt{c}}{\sqrt{\Delta^2 + d^2}} |0^{(0)}\rangle + \lambda \frac{\sqrt{c}}{\sqrt{\Delta^2 + d^2}} \left( \begin{array}{c} 0 \\ \frac{\Delta}{\sqrt{\Delta^2 + d^2}} \\ \frac{d}{\sqrt{\Delta^2 + d^2}} \end{array} \right) + O(\lambda^2), \] 

(4.96)

\[ = \left( \begin{array}{c} \frac{\lambda \sqrt{2c(d-p)}}{\sqrt{\Delta^2 + d^2}} \\ \frac{-\lambda \sqrt{2c(d-p)}}{\sqrt{\Delta^2 + d^2}} \\ \lambda c \frac{\sqrt{2c(d-p)}}{\sqrt{\Delta^2 + d^2}} \end{array} \right) + O(\lambda^2). \] 

(4.97)

\[ \text{The relation } \Delta^2 - (p + d)^2 = (\Delta + p + d)(\Delta - p - d) = -4df \text{ is useful here.} \]
and

\[ |2\rangle = \frac{1}{\sqrt{2\Delta}} \left( \sqrt{\Delta + p} \right) \left( \sqrt{\Delta - d + p} - \frac{\lambda \sqrt{2(\Delta + d - p)}}{\sqrt{\Delta(d + p + \Delta)}} |0(0)\rangle \right) - \frac{1}{\sqrt{2\Delta}} \left( \sqrt{\Delta - d - p} + \frac{\lambda \sqrt{2(\Delta + d - p)}}{\sqrt{\Delta(d + p + \Delta)}} |0(0)\rangle \right) + \mathcal{O}(\lambda^2), \]

(4.98)

\[ \left( \begin{array}{c} -\frac{\lambda \sqrt{2(\Delta + d - p)}}{\sqrt{\Delta(d + p + \Delta)}} \\ \frac{\lambda \sqrt{2(\Delta + d - p)}}{\sqrt{\Delta(d + p + \Delta)}} \end{array} \right) + \mathcal{O}(\lambda^2). \]

(4.99)

At this stage we have approximately solved the eigenvalue equation \( H/\Lambda |n\rangle = E_n |n\rangle \), thus to find the Z mass squared values we must multiply the eigenvalues by \( \Lambda \), but this is equivalent to redefining \( d \rightarrow \Lambda d, h \rightarrow \Lambda h, f \rightarrow \Lambda f \) and \( p \rightarrow \Lambda p \).

Letting \( w \) generically denote \( w_l \) and \( w_R \), the physical mass-squared eigenvalues to \( \mathcal{O} \left( \frac{\lambda^2}{w^2} \right) \) are:

\[ M_Z^2 = m_Z^2, \]
\[ M_{Z'}^2 = M^2 - \frac{1}{2} \Delta - m_Z^2 \mu_+ \cos^2 \alpha \cos^2 \theta, \]
\[ M_{Z''}^2 = M^2 + \frac{1}{2} \Delta + m_Z^2 \mu_- \cos^2 \alpha \cos^2 \theta, \]

(4.100)

(4.101)

(4.102)

with

\[ M^2 \equiv \frac{1}{2} (d + p), \]
\[ = Aw_l^2 + Bw_R^2, \]

(4.103)

(4.104)

and

\[ \frac{1}{2} \Delta = \sqrt{A^2 w_l^4 + Cw_l^2 w_R^2 + B^2 w_R^4}, \]

(4.105)

\[ \begin{array}{c}
\text{From here onwards one should understand } d, h, f \text{ and } p \text{ to mean what they are defined in equation (4.70) but without the } \Lambda \text{ factor. For example } d = \frac{\alpha}{2} \sec^2 \omega q \text{. The introduction of the } \Lambda \text{ scale was included to make the arguments more rigorous, the author does not feel that it is necessary to introduce extra notation, for instance primes, as this would clutter the work.}
\end{array} \]
where

\[ A = \frac{f}{2w_l^2} = \frac{1}{3} g_s^2 \sec^2 \beta, \]  

\[ B = \frac{d + h}{2w_R^2} = \frac{1}{4} \left( g^2 \sec^2 \alpha + 3 g_s^2 \sin^2 \beta \tan^2 \beta \right), \]  

\[ C = A(2B - g^2 \sec^2 \alpha) = \frac{1}{2} g_s^4 \tan^4 \beta - \frac{1}{6} g^2 g_s^2 \sec^2 \alpha \sec^2 \beta, \]  

and

\[ \mu_{\pm} = \frac{1}{2} \pm \frac{p - d}{2\Delta}, \]  

\[ = \frac{1}{2} \pm \frac{1}{4} \left( 3 g_s^2 \sin^2 \beta \tan^2 \beta - g^2 \sec^2 \alpha \right) \frac{w_R^2}{\Delta} \pm \frac{1}{3} g_s^2 \sec^2 \beta \frac{w_l^2}{\Delta}. \]

The leading order correction to the $Z$ mass is obtained by retaining higher order terms, giving

\[ M_Z^2 = m_Z^2 + \delta m_Z^2, \]  

where

\[ \delta m_Z^2 = -m_Z^4 \frac{c_p}{f_d}, \]  

\[ = -\frac{2 \cos^2 \theta}{g^2} m_Z^4 \cos^4 \alpha \left( \frac{1}{w_R^2} + \frac{9}{4} \sin^4 \beta \frac{1}{w_l^2} \right), \]  

\[ = -m_Z^2 \left[ \left( \frac{k}{w_R} \right)^2 \cos^4 \alpha + \frac{g^4}{4g_s^4} \left( \frac{k}{w_l} \right)^2 \tan^4 \theta \right]. \]  

It is unnecessary to determine the higher order corrections to the $Z'$ and $Z''$ masses. We now pursue further the implications of having three neutral bosons. The physical $Z$-bosons are a 3-dimensional orthogonal rotation of the interac-
4.4 The gauge boson sector

The gauge boson sector of the theory is described by the equations:

\[
\begin{pmatrix}
Z_p \\
Z'_p \\
Z''_p
\end{pmatrix} = U^{-1} \begin{pmatrix}
Z \\
Z' \\
Z''
\end{pmatrix},
\]

where \( U = |0\rangle\langle e_0| - |1\rangle\langle e_1| + |2\rangle\langle e_2| \).

To first order, we have:

\[
U = \begin{pmatrix}
1 & -\lambda \sqrt{2} \cos(\Delta - d + p) \sqrt{2}\Delta / \sqrt{2}\Delta^{\gamma/2} & -\lambda \sqrt{2} \cos(\Delta + d - p) \sqrt{2}\Delta / \sqrt{2}\Delta^{\gamma/2} \\
\lambda \sqrt{2} \sqrt{cp} & \sqrt{2}\Delta \sqrt{2} \Delta / \sqrt{2}\Delta^{\gamma/2} - \lambda c \sqrt{2} \Delta \sqrt{2} \Delta / \sqrt{2}\Delta^{\gamma/2} & \lambda \sqrt{2} \sqrt{cp} \\
-\lambda \sqrt{2} \sqrt{cp} & \lambda \sqrt{2} \sqrt{cp} & \lambda \sqrt{2} \sqrt{cp}
\end{pmatrix} + \mathcal{O}(\lambda^2).
\]

We will find it convenient to parametrize this matrix with Euler angles. We choose the following parameterisation for \( U(\zeta, \sigma, \psi) \):

\[
\begin{pmatrix}
\cos \zeta \cos \psi - \cos \sigma \sin \zeta \sin \psi & \cos \psi \sin \zeta + \cos \zeta \cos \sigma \sin \psi & \sin \sigma \sin \psi \\
-\cos \sigma \cos \psi \sin \zeta - \cos \zeta \sin \psi & \cos \zeta \cos \sigma \cos \psi - \sin \zeta \sin \psi & \cos \psi \sin \sigma \\
\sin \zeta \sin \sigma & -\cos \zeta \sin \sigma & \cos \sigma
\end{pmatrix}.
\]

To order \( \mathcal{O} \left( \frac{\mu^4}{w^4} \right) \) we find:

\[
U_{11} = 1;
\]

\[
U_{21} = m_Z^2 \sqrt{cp} \frac{df}{dp},
\]

\[
= m_Z^2 \cos^3 \alpha \cos \theta (9w_R^2 \sin^4 \beta + 4w^2) \frac{2g^2 w_1^2 w_R^2}{2g^2 w_1^2 w_R^2};
\]

\footnote{The diagonalised matrix is \( D = U^{-1}HU \).}

\footnote{The \( |e_i\rangle \) are a Cartesian orthonormal basis. The choice of minus sign in front of the \( |1\rangle \) is taken so that the columns of the \( U \) matrix \( \{ |0\rangle, -|1\rangle, |2\rangle \} \) form a right-handed set (since \( H \) is symmetric we are guaranteed the eigenvectors will be orthogonal and as normalisation leaves a plus or minus sign degree of freedom we are free to choose the signs as we have).}

\footnote{\( U^{-1}(\zeta, \sigma, \psi) = U(-\psi, -\sigma, -\zeta) \).}
\begin{align}
U_{31} &= -m_Z^2 \frac{\sqrt{c} h}{\sqrt{d f}} , \quad (4.121) \\
&= -3 \sqrt{3} m_Z^2 \cos^2 \alpha \cos \beta \cos \theta \sin^2 \beta , \quad (4.122) \\
U_{33} &= U_{22} = \sqrt{\frac{1}{2} + \frac{p - d}{2 \Delta}} - m_Z^2 c \frac{\sqrt{h d (\Delta + d - p)}}{\sqrt{2 \Delta^{5/2}}} , \quad (4.123) \\
&= \sqrt{\mu_+} - m_Z^2 g^2 w^2 \frac{4 \Delta^2}{4 \Delta^2} \sin 2 \theta \tan \beta \sqrt{\mu_-} ; \quad (4.124) \\
U_{23} &= -U_{32} = \sqrt{\frac{1}{2} - \frac{p - d}{2 \Delta}} + m_Z^2 c \frac{\sqrt{h d (\Delta - d + p)}}{\sqrt{2 \Delta^{5/2}}} , \quad (4.125) \\
&= \sqrt{\mu_-} + m_Z^2 g^2 w^2 \frac{4 \Delta^2}{4 \Delta^2} \sin 2 \theta \tan \beta \sqrt{\mu_+} ; \quad (4.126) \\
U_{12} &= -m_Z^2 \frac{\sqrt{2 c (\Delta - d + p)}}{\sqrt{\Delta (d + p - \Delta)}} , \quad (4.127) \\
&= -m_Z^2 U_{33} \frac{2 \sqrt{c}}{d + p - \Delta} = - \left( \frac{m_Z}{M^2 - \frac{1}{2} \Delta} \right)^2 U_{33} \sqrt{c} , \quad (4.128) \\
&= - \left( \frac{m_Z}{M^2 - \frac{1}{2} \Delta} \right)^2 U_{33} \cos \alpha \cos \theta ; \quad (4.129) \\
U_{13} &= -m_Z^2 \frac{\sqrt{2 c (\Delta + d - p)}}{\sqrt{\Delta (d + p + \Delta)}} , \quad (4.131) \\
&= m_Z^2 U_{32} \frac{2 \sqrt{c}}{(d + p + \Delta)} = \left( \frac{m_Z}{M^2 + \frac{1}{2} \Delta} \right)^2 U_{32} \sqrt{c} , \quad (4.132) \\
&= \left( \frac{m_Z}{M^2 + \frac{1}{2} \Delta} \right)^2 U_{32} \cos \alpha \cos \theta . \quad (4.133)
\end{align}

We now try to express the parameters of the matrix (4.117) in terms of the
4.4 The gauge boson sector

elements $U_{ij}$. Firstly, if $\sigma \neq 0$ then

$$\sigma = \arccos[U_{33}],$$  \hspace{1cm} (4.134)

$$\zeta = \arcsin[U_{31} \csc \sigma],$$  \hspace{1cm} (4.135)

$$\psi = \arcsin[U_{13} \csc \sigma],$$  \hspace{1cm} (4.136)

but as $\sigma = 0$ equations (4.135) and (4.136) become pathological. In this case all we can constrain is their sum

$$\psi + \zeta = -\arcsin U_{21}.$$  \hspace{1cm} (4.137)

**LR and ql limits of the model**

It is interesting to stop at this point and reflect on the LR and ql limits of this sector and compare them with work that has already been done on these models separately (see references [134, 136] for LR and [137] for ql).

The additional symmetry in the LR model is $SU(2)_R \times U(1)_{B-L}$, which has the generator $Y_{LR} = 2I_{3R} + B$, but this is a subgroup of our model with $B = \frac{1}{\sqrt{3}}T_l + V$, what is more is that $B$ is unbroken by our $\chi_l$ Higgs field:

$$B\langle \chi_l^{(0)} \rangle = \frac{-2}{3} + \frac{2}{3} = 0,$$

and therefore our model should reduce to this in the limit of large $w_l$. In the ql case the additional symmetry is the group $SU(3)_q \times U(1)_X$ which again is a subgroup of our symmetry with $X = 2I_{3R} + V$. It is also unbroken by $\chi_R,$

$$X\langle \chi_R^{(0)} \rangle = -1 + 1 = 0,$$

hence is the limit of our theory for large $w_R$.

Now that we are confident that our theory should have an LR limit for large $w_l$ and a ql limit for large $w_R$ we proceed to reduce our expressions to those found previously in the separate models.
The $k < w_R << w_l$ limit:

$$\frac{1}{2} \Delta \rightarrow A w_l^2 - \frac{1}{2} C w_R^2 = M^2 - \frac{1}{2} g^2 \sec^2 \alpha w_R^2, \quad (4.138)$$

and

$$\Delta^{-1} \rightarrow \frac{1}{2 A w_l^2}, \text{ thus } \frac{p - d}{\Delta} \rightarrow (2 A w_l^2 + 2 B w_R^2 - 2 d) \frac{1}{2 A w_l^2} \rightarrow 1. \quad (4.139)$$

So the masses become$^{11}$

$$M_Z^2 = m_Z^2 + \mathcal{O}(\lambda^2); \quad (4.140)$$

$$M_{Z'}^2 = M^2 - \frac{1}{2} \Delta - \frac{c(d - p - \Delta)}{2 \Delta} \lambda + \mathcal{O}(\lambda^2), \quad (4.141)$$

$$\rightarrow M^2 - (M^2 - \frac{1}{2} g^2 \sec^2 \alpha w_R^2) + \frac{m_Z^2 c}{2} \left( \frac{p - d}{\Delta} + 1 \right), \quad (4.142)$$

$$\rightarrow \frac{g^2 w_R^2}{2 \cos^2 \alpha} \left[ 1 + \left( \frac{k}{w_R} \right)^2 \cos^4 \alpha \right]; \quad (4.143)$$

$$M_{Z''}^2 = M^2 + \frac{1}{2} \Delta + \frac{c(d - p + \Delta)}{2 \Delta} \lambda + \mathcal{O}(\lambda^2), \quad (4.144)$$

$$\rightarrow 2 M^2 - \frac{1}{2} g^2 \sec^2 \alpha w_R^2, \quad (4.145)$$

$$\rightarrow 2 A w_l^2 = \frac{2 g^2 w_l^2}{3 \cos^2 \beta}. \quad (4.146)$$

As expected, we have one very heavy boson $M_{Z''}$ with a mass going like $w_l^2$ and a light one $M_Z^2$, who’s mass agrees with that found in $^{12}$

Furthermore, $(U_{33})^2 = \frac{p - d}{2 \Delta} + \frac{1}{2} \rightarrow 1$ which implies that $\sigma = 0$. Thus the $U$

$^{11}$Note that in this case we are keeping the first order terms of $M_{Z'}^2$ and $M_{Z''}^2$. These are needed in order to attain the same accuracy as presented in the LR model.

$^{12}$In fact, there is a disagreement about the power of the $\cos^4 \alpha$ term- the authors in $^{13}$ have a power of two. We have corresponded with the authors and they confirm a misprint.
4.4 The gauge boson sector

matrix \((4.117)\) becomes

\[
U \rightarrow \begin{pmatrix}
\cos(\zeta + \psi) & \sin(\zeta + \psi) & 0 \\
-\sin(\zeta + \psi) & \cos(\zeta + \psi) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\tag{4.147}
\]

Thus, \(\xi = -(\zeta + \psi)\)\(^{13}\) is the mixing angle between the \(Z\) and the \(Z'\) boson in the LR theory. Therefore,

\[
\tan \xi = \frac{U_{21}}{U_{11}},
\tag{4.148}
\]

\[
= \frac{m_Z^2 \cos^3 \alpha \cos \theta (9 w_R^2 \sin^4 \beta + 4 w_t^2)}{2 g^2 w_t^2 w_R^2},
\tag{4.149}
\]

\[
\rightarrow \frac{2 m_Z^2 \cos \alpha \cos^2 \beta \cos \theta}{g^2 w_R^2},
\tag{4.150}
\]

and we see that this mixing angle agrees with that of the LR model:

\[
\tan \xi = \left( \frac{k}{w_R} \right)^2 \frac{\sin \alpha \cos^3 \alpha}{\sin \theta}.
\tag{4.151}
\]

Finally, we note that in this limit the correction to the \(Z\) boson mass (see equation \((4.112)\)) becomes

\[
\delta m_Z^2 = -m_Z^2 \left( \frac{k}{w_R} \right)^2 \cos^4 \alpha,
\tag{4.152}
\]

confirming that our results do reduce to those of the LR model in the large \(w_t\) limit.

The \(k < w_t << w_R\) limit:

\[
\frac{1}{2} \Delta \rightarrow B w_R^2 + \frac{1}{2} C w_t^2 = M^2 - \frac{A}{2 B} g^2 \sec^2 \alpha w_t^2.
\tag{4.153}
\]

\(^{13}\)The minus sign is coming from the fact that it is \(U^{-1}\) that mixes \(Z\) and \(Z'\).
In this limit the $Z'$ mass becomes

$$M_{Z'}^2 = M^2 - \frac{1}{2} \Delta + \mathcal{O}(\lambda^2), \quad (4.154)$$

$$\rightarrow \frac{A}{2B} g_s^2 \sec^2 \alpha w_l^2, \quad (4.155)$$

$$= \frac{\frac{2}{3} g_s^2 w_l^2}{3 g_s^2 \cos^2 \alpha \sin^4 \beta + g^2 \cos^2 \beta'}, \quad (4.156)$$

$$\approx \frac{\frac{2}{3} g_s^2 w_l^2}{(\tan^2 \theta - 1) \sin^2 \beta + 1}, \quad (4.157)$$

but $3g_s^2 \cos^2 \alpha \sin^2 \beta = g^2 \tan^2 \theta$, so

$$M_{Z'}^2 = \frac{\frac{2}{3} g_s^2 w_l^2}{(\tan^2 \theta - 1) \sin^2 \beta + 1}, \quad (4.158)$$

and using equation (4.154)\(^{14}\)

$$M_{Z'}^2 = \frac{\frac{2}{3} g_s^2 w_l^2}{1 - \tan^2 \theta \frac{g^2}{3 g_s^2}}, \quad (4.159)$$

$$\approx \frac{2}{3} g_s^2 w_l^2 (1 + \tan^2 \theta \frac{g^2}{3 g_s^2}). \quad (4.160)$$

The value quoted in [137] is in the $g \ll g_s$ limit in which case $M_{Z'}^2 \rightarrow \frac{2}{3} g_s^2 w_l^2$ and we have agreement. For completeness we check that the other boson decouples from the theory by becoming very heavy:

$$M_{Z''}^2 = M^2 + \frac{1}{2} \Delta + \mathcal{O}(\lambda^2), \quad (4.161)$$

$$\rightarrow 2Bw_R^2 = \frac{1}{2} (g^2 \sec^2 \alpha + 3g_s^2 \sin^2 \beta \tan^2 \beta)w_R^2. \quad (4.162)$$

In this limit $\psi = 0$ which follows from equation (4.136) since $U_{13} \rightarrow 0$\(^{15}\)

\(^{14}\)And using the simple trigonometric identity $1 - \tan^2 \theta = \frac{\cos 2\theta}{\cos^2 \theta}$.

\(^{15}\)Since $\Delta^{-1} \rightarrow \frac{1}{\Delta^{-1} w_R}$, we observe that $\sigma$ does not vanish as this would require $U_{33} = 1$ which is clearly not true by equation (4.123). Thus we see, using the first rule for $\zeta$ in equation (4.135), that $U_{31}$ must vanish if $\zeta$ does but since $U_{31}$ is proportional to $\frac{1}{w_l^2}$ this can not happen.
Thus $U$ becomes
\[
\begin{pmatrix}
\cos \zeta & \sin \zeta & 0 \\
-\cos \sigma \sin \zeta & \cos \zeta \cos \sigma & \sin \sigma \\
\sin \zeta \sin \sigma & -\cos \zeta \sin \sigma & \cos \sigma
\end{pmatrix}.
\]
(4.163)

This may be written as $U = R_\sigma R_\zeta$ where
\[
R_\sigma = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \sigma & \sin \sigma \\
0 & -\sin \sigma & \cos \sigma
\end{pmatrix},
\]
(4.164)
\[
R_\zeta = \begin{pmatrix}
\cos \zeta & \sin \zeta & 0 \\
-\sin \zeta & \cos \zeta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
(4.165)

so that
\[
Z_P = U^{-1} Z_I = R_\zeta^{-1} R_\sigma^{-1} Z_I = R_\zeta^{-1} \tilde{Z}_I,
\]
(4.166)

where we have redefined the interaction basis as $\tilde{Z}_I = R_\sigma^{-1} Z_I$. In this basis, one of the additional neutral gauge bosons, $Z''_{QL}$, decouples in the large $w_R$ limit. The mixing between the SM $Z$ boson and the other additional neutral gauge boson, $Z'_{QL}$, in this limit is given by $\zeta$. These extra gauge boson are
\[
Z'_{QL} = \cos \sigma Z' - \sin \sigma Z'',
\]
(4.167)
\[
Z''_{QL} = \sin \sigma Z' + \cos \sigma Z'',
\]
(4.168)

with $Z'$ and $Z''$ defined in (4.51) and the subscript emphasises that $Z'_{QL}$ has the interaction properties of the extra neutral boson found in QL symmetric models.
This means that in general the $Z$ boson mixes with both the $Z'$ and the $Z''$ bosons in this limit. It is by no accident that the $Z$ mixed only with the $Z'$ in the LR limit. While it is unfortunate that the same thing does not happen here, we should not expect it to. From the beginning we defined our $Z'$ field to be the one that is used in LR models \cite{134}, recall (4.49),

$$Z' = -\sin\alpha B_B + \cos\alpha W_R,$$

thus in taking the large $w_l$ limit this was exactly one of the bosons we expected to get. On the other hand, in the ql limit, our remnant $Z_{ql}$ boson could be an arbitrary superposition of all the $Z$ fields.

An alternative procedure can also be used. The mixing can be found from the physical masses and the squared mass correction, $\delta m_Z^2$, alone. Here we provide a general derivation of this.

Mixing

Take the the mixing matrix

$$\begin{pmatrix} m_Z^2 & b \\ b & m_{Z'}^2 \end{pmatrix}$$

(4.169)

between the $Z$ boson of the SM and another one of higher energy (for example the $Z_{ql}$), with some arbitrary mixing term $b$. Diagonalising this matrix leads to the eigenvalues $\lambda_1 = M_Z^2$ and $\lambda_2 = M_{Z'}^2$, with the characteristic equation

$$(m_{Z'}^2 - \lambda) - \frac{b^2}{m_Z^2 - \lambda} = 0,$$

and the normalised eigenvectors

$$\vec{v}_1 = N_1 \begin{pmatrix} -b \\ m_Z^2 - M_Z^2 \end{pmatrix}; \quad \vec{v}_2 = N_2 \begin{pmatrix} -b \\ m_Z^2 - M_{Z'}^2 \end{pmatrix}.$$

(4.170)
Thus,

\[
U = \begin{pmatrix}
-N_1b & -N_2b \\
-N_1\delta m_Z^2 & N_2(m_Z^2 - M_{Z'}^2)
\end{pmatrix},
\]

\[
\equiv \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix},
\]

implies that

\[
\tan^2 \theta = -\frac{\delta m_Z^2}{M_{Z'}^2 - m_Z^2}.
\]

We note that in the ql limit the correction to the Z mass is:

\[
\delta m_Z^2 = -m_Z^2 \frac{9}{4} \frac{k^2}{w_1^2} \cos^4 \alpha \sin^4 \beta,
\]

\[
= -m_Z^2 \frac{1}{4} \left( \frac{k}{w_1} \right)^2 \left( \frac{g}{g_s} \right)^4 \tan^4 \theta.
\]

So the mixing is

\[
\tan^2 X \approx -\frac{\delta m_Z^2}{M_{Z'}^2},
\]

\[
= \frac{3}{4} \frac{g^2}{g_s} \tan^2 \theta \sin^2 \theta \frac{m_4^4}{w_1^4},
\]

which implies that

\[
\tan X = \frac{\sqrt{3}}{4} \left( \frac{k}{w_1} \right)^2 \left( \frac{g}{g_s} \right)^3 \frac{\tan^2 \theta}{\cos \theta} \quad [16]
\]

\[16\text{It does not appear that this has been calculated before in any of the quark-lepton literature.}\]
4.5 QLLR with split fermions

Having discussed in some detail the symmetry breaking and gauge sectors of the model we now turn our attention to fermions. One interesting aspect of studying models in additional dimensions is the novel new mechanisms which become available to solve old problems. As we have already emphasised, the scalar content of 4d QLLR models is quite complicated with the $\Delta$ scalars of equation (4.4) included to simultaneously break the gauge symmetry and suppress neutrino masses below the electroweak scale (seesaw mechanism). These states have not been included in the 5d construct and thus we must present an alternative method of suppressing neutrino masses if we are to persist with the simplified scalar content. In doing this we will find we are also able to remove the troublesome mass relations which occur in 4d QLLR models without the need for a second Higgs bidoublet.

Since all the fermions transform non-trivially under either $SU(3)_c$ or $SU(3)_l$ and either $SU(2)_1$ or $SU(2)_2$, their $Z_2 \times Z'_2$ transformations are given by:

$$
\begin{align*}
\Psi(x^\mu, y) &\rightarrow \Psi(x^\mu, -y) = \pm \gamma_5 F_{1,2}^a P_{q,l}^a \Psi_{a,\alpha}(x^\mu, y), \\
\Psi(x^\mu, y') &\rightarrow \Psi(x^\mu, -y') = \pm \gamma_5 F_{1,2}^a P_{q,l}^a \Psi_{a,\alpha}(x^\mu, y'),
\end{align*}
$$

where $a (\alpha)$ are indices of the relevant $SU(2)$ ($SU(3)$) group. The $\pm$ signs in the two equations are independent and govern which chiral component of the fermion wave function will be odd and which even about the relevant fixed point. These orbifold boundary conditions (OBCs) force the two $SU(2)_L$ quark/lepton singlets of the SM to come from different $SU(2)_2$ doublets. We must therefore double the minimal fermion content of our model compared with 4d QLLR models. This doubling of the fermion spectrum is typically required in 5d LR [134] and QL models [138]. Thus the fermion spectrum is:
4.5 QLLR with split fermions

Here the numerical subscripts and the primes are used to label different 5d fields such that \(Q_{1L}\) and \(Q_{1R}\) \((Q'_{1L}\) and \(Q'_{1R}\)) form the left and right chiral components of the one 5d field \(Q_1\) \((Q'_1\)) etc., the superscripts \(r, b, g\) label quark colours and \(r', b', g'\) label lepton colours. We have taken \(r'\) to be the color of the

\[
\begin{align*}
Q^{r,b,g}_{1,L} &\sim (u(+,+))^{r,b,g}, & Q^{r,b,g}_{1,R} &\sim (u(+,-))^{r,b,g}, \\
Q^{r,b,g}_{1,L} &\sim (d(+,+))^{r,b,g}, & Q^{r,b,g}_{1,R} &\sim (d(+,-))^{r,b,g}, \\
Q^{r,b,g}_{2,L} &\sim (u(-,+))^{r,b,g}, & Q^{r,b,g}_{2,R} &\sim (u(-,-))^{r,b,g}, \\
Q^{r,b,g}_{2,L} &\sim (d(-,+))^{r,b,g}, & Q^{r,b,g}_{2,R} &\sim (d(-,-))^{r,b,g}, \\
L^{r}_1 &\sim (\nu(+,+))^{r}, & L^{r}_1 &\sim (\nu(+,-))^{r}, \\
L^{r}_1 &\sim (e(+,-))^{r}, & L^{r}_1 &\sim (e(+,+))^{r}, \\
L^{r'}_1 &\sim (\nu(-,+))^{r'}, & L^{r'}_1 &\sim (\nu(-,-))^{r'}, \\
L^{r'}_1 &\sim (e(-,+))^{r'}, & L^{r'}_1 &\sim (e(-,-))^{r'}, \\
L^{r'}_2 &\sim (\nu(+,-))^{r'}, & L^{r'}_2 &\sim (\nu(+,+))^{r'}, \\
L^{r'}_2 &\sim (e(+,-))^{r'}, & L^{r'}_2 &\sim (e(+,+))^{r'}. \\
\end{align*}
\]
SM leptons. Note that zero modes of some of the exotic $b'$ and $g'$ colored leptons are present. The appearance of these states is a fortunate consequence of the fermion orbifold parity structure as they are required to ensure an anomaly free zero mode fermion content \cite{139}. These states gain masses as in the 4d theory via the $\chi$ Yukawa Lagrangian which has the form (we must define the action of the discrete symmetries before we can specify its exact form):

$$
\mathcal{L}_{\text{Yuk non-EW}} \sim \sum \text{fermions} \, h_i \left( \bar{T}_i L_i \chi_l + \bar{Q}_i Q_i \chi_q \right),
$$

whilst the quarks remain massless since $\chi_q$ has a vanishing VEV. We must also define the action of the QL and $1 \leftrightarrow 2$ symmetries on the fermions. Due to the doubling of the fermion spectrum there are several ways we could do this. The various possibilities result in different phenomenology and influence the extent to which the issues of neutrino mass, proton decay and unwanted mass relations can be resolved. Below we shall investigate the two most interesting scenarios.

We structure the remainder of this section as follows. In section \ref{sec:split-fermions} we briefly re-introduce the split fermion mechanism paying more detail to it’s uses in the model at hand. In section \ref{sec:neutrino-mass-proton-decay} we discuss the nature of neutrino mass and proton decay in our model. In section \ref{sec:fermion-localisation-pattern} we explore the use of split fermions with one possible assignment for quarks, leptons and scalars under the QL and $1 \leftrightarrow 2$ symmetries. This assignment is interesting as it induces a fermion localisation pattern motivating the differences in quark and lepton masses observed in the SM. The hierarchy between, for example, the top quark and a Dirac neutrino is obtained with Yukawa couplings which vary by only a factor of five.

We investigate an alternative symmetry assignment in section \ref{sec:alternative-symmetry}. This arrangement allows one to simultaneously suppress the proton decay rate and understand the range of fermion masses in the SM. We note that the symmetries of the model highly constrain the parameters required to localise fermions. It is a non-trivial result that we are able to remove the unwanted mass rela-
tions implied by the QL symmetry, suppress the proton decay rate by spatially separating quarks and leptons and understand some of the flavour features found in the SM. To the best of our knowledge this is the first model in the literature that motivates a localisation pattern which simultaneously ensures proton longevity and addresses flavour. We demonstrate our ideas in this section with one generation examples and further work is required to ensure that these promising ideas carry over to a three generational model. A complete numerical analysis of the three generational setup is beyond the scope of the present work.

4.5.1 Split fermion mechanism

In extra dimensional models the effective 4d theory is obtained by integrating out the extra dimensions and any symmetry or naturalness arguments should be made in the fundamental extra dimensional model. As explained in section 1.4 the basic idea behind split fermion models is that by appropriately choosing the profiles of the fields in the extra dimensions, the hierarchy of fermion masses in the SM and the stability proton can be explained.

The work of AS was performed with an infinite extra dimension. We are interested in the case where the extra dimension is compactified and therefore follow [140]. To localise fermions along the extra dimension we introduce a gauge singlet bulk scalar, $\Sigma_1$, assumed to possess odd parity about both fixed points.

For a given bulk fermion, $\psi$, the Yukawa Lagrangian with the bulk scalar $\Sigma_1$ is

$$\mathcal{L} = \overline{\psi} (i\gamma^M \partial_M - f_{\psi_1} \Sigma_1) \psi + \frac{1}{2} \partial^M \Sigma_1 \partial_M \Sigma_1 - \frac{\Lambda_1}{4} (\Sigma_1^2 - v_1^2)^2,$$  \hspace{1cm} (4.183)

where $f_{\psi_1}$, $\Lambda_1$ and $v_1$ are constants. The OBCs prevent $\Sigma_1$ from developing a
constant VEV along the extra dimension and lead to a kink configuration

\[ \langle \Sigma_1 \rangle \approx v_1 \tanh (\xi_1 v_1 y) \tanh \left( \xi_1 v_1 \left( \frac{\pi R}{2} - y \right) \right) \], \quad (4.184)

where \( \xi_1^2 = \Lambda_1/2 \). Solving the Dirac equation for the fermion gives:

\[ \psi(y) = N e^{f_\psi \int_0^y \langle \Sigma_1 \rangle(y') dy'} . \quad (4.185)\]

Using (4.184) this solution is approximately a Gaussian of width \((f_\psi v_1)^{-1}\) localised around \( y = 0 \) \((y = \pi R/2)\) for \( f_\psi v_1 > 0 \) \((f_\psi v_1 < 0)\). Thus by assuming distinct couplings to \( \Sigma_1 \) for distinct SM fermion multiplets one can localise them around different fixed points with varying widths.

The fermion \( \psi \) may be shifted from the fixed points by using two localising scalars, \( \Sigma_{1,2} \), with VEVs \( v_{1,2} \) and fermion couplings \( f_{\psi_{1,2}} \). One finds that \( \psi \) is localised around \( y = 0 \) \((y = L)\) for \( f_\psi v_1, f_\psi v_2 > 0 \) \((f_\psi v_1, f_\psi v_2 < 0)\). However, if \( \text{sign}(f_\psi v_1) \neq \text{sign}(f_\psi v_2) \), the localization of \( \psi \) will depend on the relative sizes of \( f_\psi v_i \). Cases exist where the fermion is localised around one of the fixed points, within the bulk or has a bimodal profile. Fermions localised inside the bulk generally have wider profiles than those localised at a fixed point. A detailed discussion of the various cases may be found in [140].

Previous studies have confirmed that it is possible to obtain the SM masses and mixings from a split fermion setup with reduced parameter hierarchies [140, 141, 66, 142, 143]. The price we pay is that we must introduce a new free parameter for every scalar-fermion coupling. The setup therefore lacks predictivity, telling us nothing, for example, about the relative masses of the quarks and leptons, or the top and bottom quarks.

In calculating the localization of the fermions in our model below, we shall do so only classically. In taking this approach we are following previous work on split fermions [141, 66, 142, 143] in assuming that any quantum corrections will be small enough not to alter the qualitative nature of our fermion geography. Before attempting to find realistic split fermion geographies for our model, we briefly discuss the nature of neutrino mass and proton decay within it.
4.5.2 Neutrino mass and proton decay

The nature of neutrino mass and proton decay in any model are related to the status of baryon and lepton number symmetries. Our model contains accidental unbroken baryon- and lepton-number symmetries. Baryon number, $B$, takes the value $1/3$ for quarks and $-2/3$ for the colored scalar $\chi_q$. Lepton number is given by

$$L = L' - \frac{T_l^8}{\sqrt{3}},$$

(4.186)

where $L'$ is $1/3$ for leptons and $-2/3$ for $\chi_l$. It is easy to check that all renormalizable Lagrangian terms which respect the gauge symmetries also conserve $B$ and $L$. As such there is no process which will lead to Majorana neutrino mass or proton decay at any order within the model. This differs from 4d QLLR models where the larger scalar sector precludes lepton number conservation.

However, there are non-renormalizable operators which may induce these processes. The leading order effective operators resulting in Majorana neutrino masses are

$$\mathcal{O}_{\nu L}^{\text{eff}} = \frac{g}{(\Lambda \pi R)^3} \frac{(\chi_l^\dagger \chi_R \Phi L_L)^2}{\Lambda^5},$$

$$\mathcal{O}_{\nu R}^{\text{eff}} = \frac{g'}{(\Lambda \pi R)^2} \frac{(\chi_l N_R \chi_R^\dagger)^2}{\Lambda^3}.$$  \hfill (4.187)

These operators can only lead to cosmologically acceptable masses for the right-handed neutrino while keeping the left-handed neutrinos light if we take the breaking scales to be at least $w_{l,R} \gtrsim 100$ TeV. For phenomenological reasons, such a possibility is uninteresting so we do not consider it further. Instead we will assume that these Majorana masses are zero or negligibly small. This could occur either if the cut-off, $\Lambda$, is large, or if a sub-group of the $B$ and $L$ symmetries is preserved also above the cut-off so as to forbid these operators.
Proton decay occurs non-renormalizably via the effective operator

\[ \mathcal{O}_{\text{eff}}^{p} = \frac{1}{(\Lambda R)^{3/2}} \epsilon_{\alpha\beta\gamma} Q^{\alpha} Q^{\beta} Q^{\gamma} \chi_{\lambda_{\alpha}}^{\dagger} L_{\lambda'} L^{\alpha'}. \]  \hspace{1cm} (4.188)

Below we will consider two possibilities for preventing proton decay. In section 4.5.3 we will assume this operator is unimportant either because the cut-off is high, or the high energy theory respects the \( B \) symmetry evident at low energies. In section 4.5.4, we consider the case where this operator can lead to significant proton decay and must be suppressed via the split fermion mechanism. \footnote{Such an approach may be taken consistently with the requirement for negligible Majorana neutrino masses. There are values for \( w_l, w_R, \Lambda \) such that the Majorana masses will be sufficiently small to ignore but proton decay will proceed at rates above experimental bounds. Alternatively, the theory above the cut-off may preserve only \( B - L \), allowing proton decay to proceed but forbidding Majorana neutrino masses.}

### 4.5.3 Fermion mass relationships

In the original 4d QLLLR models with a single Higgs bidoublet the QL symmetry led to phenomenologically inconsistent mass relations between the quarks and leptons. Depending on how we define the action of the discrete symmetries on the fermions, these can be partially removed in our 5d model due to the doubling of the fermion spectrum. To proceed any further we must define the action of the \( Q \leftrightarrow L \) and \( 1 \leftrightarrow 2 \) on the fermions, which we take to be

\[ L_1 \leftrightarrow L_2 \quad L_1' \leftrightarrow L_2' \]
\[ Q_1 \leftrightarrow Q_2 \quad Q_1' \leftrightarrow Q_2' \]  \hspace{1cm} (4.189)

If we take the Higgs bidoublet to transform trivially under \( 1 \leftrightarrow 2 \) and as \( \Phi \leftrightarrow \tilde{\Phi} \) under QL, the resulting EW Yukawa Lagrangian is

\[ \mathcal{L}_{\text{Yuk}} = \lambda_1 (\bar{Q}_1 \tilde{\Phi} Q_2 + \bar{L}_1 \tilde{\Phi} L_2) + \lambda_2 (\bar{Q}_1 \tilde{\Phi} Q_2' + \bar{L}_1 \tilde{\Phi} L_2') + \lambda_3 (\bar{Q}_1' \Phi Q_2 + \bar{L}_1' \Phi L_2) + \lambda_4 (\bar{Q}_1' \Phi Q_2' + \bar{L}_1' \Phi L_2') + \text{H.c..} \]  \hspace{1cm} (4.190)
Note that the $1 \leftrightarrow 2$ symmetry requires

$$\lambda_1 = \lambda_1^\dagger, \quad \lambda_4 = \lambda_4^\dagger, \quad \lambda_2 = \lambda_3^\dagger. \quad (4.191)$$

The EW Yukawa Lagrangian for the SM particles, $\mathcal{L}_{\text{Yuk}}^{\text{SM}} \subset \mathcal{L}_{\text{Yuk}}^{\text{EW}}$, is

$$\mathcal{L}_{\text{Yuk}}^{\text{SM}} = \lambda_1 k^* \bar{d}_L d_R + \lambda_2 k \bar{u}_L u_R + \lambda_3 k^* \bar{\tau}_L \nu_R + \text{H.c.}. \quad (4.192)$$

Thus equation (4.191) implies $m_u = m_e$. As in the 4d case, these phenomenologically incorrect relationships can be removed by introducing a second Higgs bidoublet but at the cost of predictivity and without any explanation for the hierarchical nature of SM fermion masses.

Split fermions provide a natural alternative approach. Naively one may think that the symmetries of our model over constrain the extra dimensional fermion profiles. Indeed with only one localising scalar, the quark doublet and the right-handed down quark and charged lepton of a given generation necessarily have the same profile, albeit possibly localised around different fixed points (and similarly for the lepton doublet and right-handed up quark and neutrino). However, by using two localising scalars with different parities under the symmetries it is possible to give the fermions different profiles and also move some fermions into the bulk.

Taking $\Sigma_1$ to be even under both the $Q \leftrightarrow L$ and $1 \leftrightarrow 2$ symmetries and $\Sigma_2$ even (odd) under $Q \leftrightarrow L$ ($1 \leftrightarrow 2$) results in the Lagrangian

$$\mathcal{L}_{\text{Yuk,kink}} = f(\overline{Q}_1 Q_1 + \overline{L}_1 L_1 + \overline{Q}_2 Q_2 + \overline{L}_2 L_2)\Sigma_1$$
$$+ f'(\overline{Q}'_1 Q'_1 - \overline{L}'_1 L'_1 - \overline{Q}'_2 Q'_2 - \overline{L}'_2 L'_2)\Sigma_1$$
$$+ g(\overline{Q}_1 Q_1 + \overline{L}_1 L_1 - \overline{Q}_2 Q_2 - \overline{L}_2 L_2)\Sigma_2$$
$$+ g'(\overline{Q}'_1 Q'_1 - \overline{L}'_1 L'_1 + \overline{Q}'_2 Q'_2 + \overline{L}'_2 L'_2)\Sigma_2. \quad (4.193)$$

Taking $f, f', g, g' > 0$, the coupling of the SM quark doublets to both scalars is
positive, strongly localising them around the $y = 0$ fixed point while the lepton doublets couple negatively ensuring they are localised at the $y = L$ fixed point. The SM singlets couple to the two scalars with different signs allowing them to be localised at either fixed point or within the bulk. However the symmetries ensure that the profiles of the right-handed up quarks and neutrinos (down quarks and charged leptons) have identical extra dimensional profiles.

We note that the localisation of fermions along the extra dimension does not suppress the mass of zero mode exotic leptons relative to $\omega_l$. Inspection of equation (4.182) reveals that the mass terms generated by the $\chi_l$ VEV couple exotic leptons from the same gauge multiplet. As these fields necessarily have the same profile in the extra dimension the lightest exotic leptons are generically expected to have an order $\omega_l$ mass independent of the localization pattern required to achieve a realistic SM spectrum.

In order to give a concrete example, we consider a single generation model. To determine the localisation pattern of fermions it is only necessary to specify the bulk scalar parameters $v_{1,2}$ and the Yukawa coupling constants $f, f', g, g'$ as functions of $\xi_{1,2}$ and $L = \pi R/2$. We take these to be $v_1 = 4/(\xi_1 L)$ and $v_2 = 12/(\xi_2 L)$. With the choice of fermion-$\Sigma$ couplings $f = 28.4\xi_1, \; f' = 14.4\xi_1, \; g = 7.0\xi_2, \; g' = 6.4\xi_2$. The resulting fermion localization pattern is shown in figure 4.1 and is of interest as it allows us to explain several SM features:

- The top singlet is localised on top of the quark doublet so we expect $m_t \approx k$, while the bottom singlet is in the bulk leading us to expect $m_t > m_b$. Since $b_R$ is localised in the bulk it has a relatively large width. This ensures that the suppression of $m_b$ is not too large.

- The tau singlet is localised in the bulk close to the opposite fixed point to the lepton doublet leading to $m_t > m_\tau$. Again the large width of $\tau_R$ prevents the suppression being too large.

- The right handed neutrino is strongly localised with $t_R$ about the opposite fixed point to the lepton doublet. The strong localization of both $\nu_L$
4.5 QLLR with split fermions

Figure 4.1: Fermion profiles for the parameter values described in Section 4.5.3. Solid line = $Q_L$, short dash = $b_R, \tau_R$, long dash = $t_R, \nu_R$ and the dot-dash = $L_L$. Note that some of the fermions possess identical profiles. In particular the inset shows the identical profiles of $b_R$ and $\tau_R$ about $x/L = 1$.

and $\nu_R$ allows the neutrino mass to be tiny.

Recalling that our symmetries force the Higgs Yukawa coupling of the top and tau to be identical (Eq. (4.191)), the Yukawa couplings $\lambda_t = \lambda_\tau = \lambda_b = \lambda_\nu = 1.01$ lead to masses $^{18}$ $m_t = 169$ GeV, $m_b = 4.16$ GeV, $m_\tau = 1.77$ GeV, $m_\nu = 26$ meV. Hence we are able to obtain realistic fermion masses with fewer free parameters than previously required. Whilst a complete three generation study remains to be undertaken, this approach does appear to provide a viable and novel approach to explaining the SM fermion masses with fewer free parameters and without any parameter hierarchies. It also nullifies the phenomenologically incorrect mass relationships of previous QLLR models.

4.5.4 Simultaneously suppressing proton decay and obtaining correct fermion masses

We have shown that our 5d QLLR setup enables us to obtain realistic fermion masses. However this setup does not allow one to suppress the proton decay

$^{18}$The masses must be run to a common scale which we take to be $m_t$. This leads to running constants $m_b(m_b)/m_b(m_t) = 1.55$, $m_\tau(m_\tau)/m_\tau(m_t) = 1.02$. 

rate. Proton decay occurs in the 5d QLLR model from operators of the form

\[ O_p \sim \frac{1}{M_9^{9/2}} Q^3 L \chi_l, \tag{4.194} \]

where \( Q \) (\( L \)) denotes a quark (lepton) field and \( M_9 \) is the fundamental scale. As quarks and leptons have significant fifth dimensional wave function overlap with the setup in section 4.5.3 one must take the fundamental scale to be large or extend the model to ensure proton longevity. If one simply assumes the cutoff is large the usual fine tuning is required to stabilise the Higgs mass at the electroweak scale.

It was shown in [144] that models with a QLLR symmetry admit a split fermion setup which suppresses proton decay less arbitrarily than the split fermion implementation of the SM. This requires one of the localising scalars to be odd (even) under \( Q \leftrightarrow L \) \((1 \leftrightarrow 2)\). Unfortunately neither of the scalars in section 4.5.3 transformed in this way. If the fermion transformations of equation \( (4.189) \) are retained and one of the localising scalars of section 4.5.3 is forced to be odd (even) under the \( Q \leftrightarrow L \) \((1 \leftrightarrow 2)\) symmetry, the resulting fermionic geographies require large parameter hierarchies to produce realistic mass spectra. We instead choose \( \Phi \) to be trivial under QL and the fermions to transform as

\[
L_1 \leftrightarrow L_2 \quad L'_1 \leftrightarrow L'_2 \\
Q_1 \leftrightarrow Q'_2 \quad Q'_1 \leftrightarrow Q_2
\tag{4.195}
\]

under the QL and 1 \( \leftrightarrow 2 \) symmetries which leads to the mass relationship \( m_d = m_e \). Choosing \( \Sigma_1 (\Sigma_2) \) to be odd (odd) under the \( Q \leftrightarrow L \) symmetry and even (odd) under the 1 \( \leftrightarrow 2 \) symmetry, the localising scalar Yukawa Lagrangian is

\[
\mathcal{L}_{\text{Yuk}} \sim f(Q_1' Q_1 - L_1' L_1 + Q_2' Q_2 - L_2' L_2) \Sigma_1 \\
+ f'(Q_1'^c Q_1' - L_1'^c L_1' + Q_2'^c Q_2 - L_2'^c L_2) \Sigma_1 \\
+ g(-Q_1 Q_1 + L_1 L_1 + Q_2 Q_2 - L_2 L_2) \Sigma_2 \\
+ g'(Q_1'^c Q_1' - L_1'^c L_1' + Q_2'^c Q_2 - L_2'^c L_2) \Sigma_2.
\tag{4.196}
\]
If we take $f, f', g, g' > 0$, all the right handed fermions are localised at the ends of the extra dimension, with quarks at one end and leptons at the other. Further, we find that $u_R (d_R)$ localised about $y = 0$ has the same profile as $e_R (\nu_R)$ around $y = L$. Meanwhile the quark and lepton doublets have unrelated profiles with peaks in the bulk. This is precisely the setup advocated in [145, 146] to achieve a naturally small neutrino Dirac mass. That the leptons are lighter than the quarks now results from the lepton doublet being more strongly localised than the quark doublet. It then follows that we expect $m_\nu/m_\tau \ll m_b/m_t$ since the difference in the amplitudes of the right handed wave functions becomes more dramatic the further in to the bulk we move.

Again simplifying to the one generation case, the parameter choice $v_1 = 7.9/(\xi_1 L), v_2 = 69/(\xi_2 L)$ and $f = 15.6\xi_1, f' = 865\xi_1, g = 0.440\xi_2, g' = 33.3\xi_2$, produces the fermion localization pattern shown in figure 4.2. Note that the overlap between quarks and leptons is small enough to suppress the proton decay rate below current bounds with an order 10-100 TeV fundamental scale. If we take $\lambda_t = 1.32, \lambda_b = \lambda_\tau = 0.0713$ and $\lambda_\nu = 0.3$, the fermion masses are $m_t = 173$ GeV, $m_b = 4.13$ GeV, $m_\tau = 1.78$ GeV and $m_\nu = 77$ meV. This setup does contain some hierarchy: for $v_1 \approx v_2$ one requires $\Lambda_2 \approx 100\Lambda_1$ which leads to a hierarchy of $\mathcal{O}(10^2)$ between the smallest and largest $\Sigma$ Yukawa coupling. This remains a vast improvement over the $\sim 12$ orders of magnitude parameter hierarchy required to explain fermion masses with Dirac neutrinos in the SM. It is also, to our knowledge, the first realisation of the ideas of AS which implements both features of their proposal. Further work is required to check that this carries over to three generations and that, in particular, the SM mixing angles may be reproduced [19]. However if this is shown to be the case this would represent the first dynamical setup to produce both realistic fermion masses and suppress proton decay via the split fermion mechanism.

\(^{19}\)One complication when we begin considering a three generation model is that the fermion eigenstates which couple to $\Sigma_1$ need not coincide with the eigenstates coupling to $\Sigma_2$. Most previous split fermion work, such as that listed in [141, 142, 143], has ignored this complication and assumed the coupling matrices are simultaneously diagonalisable. The implications when such ‘twisting’ is considered are discussed in [137].
Figure 4.2: Fermion profiles for the parameter values described in Section 4.5.4. Note only the regions around the ends of the extra dimension are plotted as all fermions have miniscule amplitudes in the central region. Quarks (leptons) are shown in the left (right) plot. Solid line=$Q_L(L_L)$, short dash=$d_R(\nu_R)$, dot-dash=$u_R(e_R)$.

4.6 Neutral currents

Having specified the fermion content we now present the neutral currents of the model and obtain bounds on the symmetry breaking scales $w_R$ and $w_l$. Since we consider $1/R \gg w_l, w_R$ it shall suffice to consider the interactions of the zero mode fermions and gauge bosons. After changing to the neutral gauge boson mass eigenstate basis by diagonalising the matrix $H$ with the rotation (4.117) the neutral current interactions for the zero mode fields may be written as

$$\mathcal{L}_{NC} = [eA^\mu Q + \left(\frac{g}{\cos \theta}\right) Z_{\text{phy}}^\mu A_{NC} + \left(\frac{g}{\cos \theta}\right) Z_{\text{phy}}^\mu B_{NC} + \left(\frac{g}{\cos \theta}\right) Z_{\text{phy}}^\mu C_{NC}] \ J_{NC, \mu},$$

(4.197)

where the zero mode components of the current are

$$J_{NC, \mu}^{00} = \sum \bar{Q}_i^0 \gamma_\mu Q_i^0 + \bar{L}_i^0 \gamma_\mu L_i^0.$$  

(4.198)

The effective couplings, $A_{NC}$, $B_{NC}$ and $C_{NC}$ are found by transposing all the fields in the neutral part of the covariant derivative, equation (4.23), into the mass eigenstate fields $Z_{\text{phy}}, Z_{\text{phy}}', Z_{\text{phy}}''$.

We rewrite

$$gI_{3L}W_L^0 + gI_{3R}W_R^0 + \frac{g_V}{2}VB_V + \frac{g_s}{2}T_l G_l^0$$
4.6 Neutral currents

into $A, Z, Z'$ and $Z''$ using the pairwise matrix inversions of the coupled equations (4.46)-(4.51). Then reexpress $Z, Z'$ and $Z''$ in terms of $Z_{\text{phy}}, Z'_{\text{phy}}$ and $Z''_{\text{phy}}$ using $\vec{Z} = U \vec{Z}_{\text{phy}}$. We find that:

$$A_{NC} = I_{3L}(U_{11} + \delta Z) - Q(U_{11} \sin^2 \theta + \delta Z) + I_{3R}(U_{21} \cos \alpha \cos \theta + \delta Z) + T_1 U_{31} \frac{g_L \cos \theta}{2g \cos \beta},$$

$$B_{NC} = I_{3L}(U_{12} + \delta Z') - Q(U_{12} \sin^2 \theta + \delta Z') + I_{3R}(U_{22} \cos \alpha \cos \theta + \delta Z') + T_1 U_{32} \frac{g_L \cos \theta}{2g \cos \beta},$$

$$C_{NC} = I_{3L}(U_{13} + \delta Z'') - Q(U_{13} \sin^2 \theta + \delta Z'') + I_{3R}(U_{23} \cos \alpha \cos \theta + \delta Z'') + T_1 U_{33} \frac{g_L \cos \theta}{2g \cos \beta},$$

where

$$\delta Z = \sin \theta(U_{21} \tan \alpha + U_{31} \sec \alpha \tan \beta),$$

$$\delta Z' = \sin \theta(U_{22} \tan \alpha + U_{32} \sec \alpha \tan \beta),$$

$$\delta Z'' = \sin \theta(U_{23} \tan \alpha + U_{33} \sec \alpha \tan \beta).$$

It is instructive to check that our results have the correct LR and ql limits. In the LR case one easily verifies by taking $\sigma = 0$ and $\zeta + \psi = -\xi$ that $U_{11} = U_{22} = \cos \xi, U_{21} = -U_{12} = \sin \xi, U_{33} = 1$ and all other elements are zero. Then one finds

$$A_{NC} = \left(\frac{\sin \xi \sin^2 \theta}{(2\cos \theta)^{1/2}} + \cos \xi\right) I_{3L} + \left(\frac{\sin \xi \cos^2 \theta}{(2\cos \theta)^{1/2}}\right) I_{3R} - \sin^2 \theta \left(\frac{\sin \xi}{(2\cos \theta)^{1/2}} + \cos \xi\right) Q,$$

$$B_{NC} = \left(\frac{\cos \xi \sin^2 \theta}{(2\cos \theta)^{1/2}} - \sin \xi\right) I_{3L} + \left(\frac{\cos \xi \cos^2 \theta}{(2\cos \theta)^{1/2}}\right) I_{3R} - \sin^2 \theta \left(\frac{\cos \xi}{(2\cos \theta)^{1/2}} - \sin \xi\right) Q.$$

Note that the coupling coefficients have been written in a form that readily reveals the deviations, $\delta Z$, from the SM, i.e., the SM coupling $A^S_{NC} = (I_{3L} - \sin^2 \theta Q)$, appears from $A_{NC}$, since $U_{11} = 1$ to second order in the perturbation $\lambda$, with $\delta_z$ corrections arising due to the effects of the extended model.
These are precisely the coupling coefficients of 5 dimensional left-right models \[134\]. For completeness we mention that the $C_{NC}$ is given by\[^{21}\]:

$$C_{NC} = \frac{\sin \theta \tan \beta \cos \theta}{\sqrt{\cos 2\theta}} (J_{3L} + J_{3R} - Q) + \frac{g_s \cos \theta}{2 \cos \beta} T_l.$$ (4.207)

In the ql limit, $\psi \to 0$, implies only that $U_{13} \to 0$. One observes that

$$\frac{U_{21}}{U_{31}} = \frac{9m_Z^2 \cos^3 \alpha \cos \theta \sin^4 \beta}{2g^2 w_l^2} \frac{2g g_s w_l^2}{-3\sqrt{3}m_Z^2 \cos^2 \alpha \sin^2 \beta \cos \beta \cos \theta},$$ (4.208)

$$= -\frac{\sqrt{3}g_s \cos \alpha \sin \beta \tan \beta}{g},$$ (4.209)

$$= \tan \theta \tan \beta,$$ (4.210)

and from the form of (4.163), $\tan \sigma = -U_{31}/U_{21} = -U_{32}/U_{22} = U_{23}/U_{33}$. Thus,

$$\delta_Z = U_{21} \sin \theta \left( \tan \alpha + \frac{U_{31}}{U_{21}} \sec \alpha \tan \beta \right),$$ (4.211)

$$= U_{21} \sin \theta \left( \tan \alpha - \frac{1}{\tan \theta \cos \alpha} \right),$$ (4.212)

$$= -U_{21} (\cos 2\theta)^{1/2}.$$ (4.213)

Similarly,

$$\delta_{Z'} = -U_{22} (\cos 2\theta)^{1/2},$$ (4.214)

and

$$\delta_{Z''} = U_{23} \sin \theta \left( \tan \alpha + \frac{U_{33}}{U_{23}} \sec \alpha \tan \beta \right),$$ (4.215)

$$= U_{23} \sin \theta \left( \tan \alpha + \tan \theta \tan^2 \beta \sec \alpha \right),$$ (4.216)

$$= U_{23} \tan \theta \sin \theta \frac{\tan \theta \sec \alpha}{\cos \alpha \cos^2 \beta}. $$ (4.217)

\[^{21}\] Of course this interaction will always be suppressed by the large $M_{Z'}^2$ mass in the denominator of the $Z''_{phy}$ propagator.
Hence,

\[ A_{NC} = I_{3L} \left( \cos \zeta + \cos \sigma \sin \zeta \cos (2 \theta)^{1/2} \right) \]

\[ - Q \left( \cos \zeta \sin^2 \theta + \cos \sigma \sin \zeta \cos (2 \theta)^{1/2} \right) + T_l \sin \zeta \sin \sigma \frac{g_s \cos \theta}{2 g \cos \beta}; \quad (4.218) \]

\[ B_{NC} = I_{3L} \left( \sin \zeta - \cos \zeta \cos \sigma \cos (2 \theta)^{1/2} \right) \]

\[ - Q \left( \sin \zeta \sin^2 \theta - \cos \zeta \cos \sigma \cos (2 \theta)^{1/2} \right) - T_l \sin \zeta \sin \sigma \frac{g_s \cos \theta}{2 g \cos \beta}. \quad (4.219) \]

We can calculate the coupling of the \( Z_{phy} \) to the \( T_l \) generator and compare this with \( \Delta \) in [135].

\[ T_l \frac{g}{\cos \theta} U_{31} g_s \cos \theta = \frac{-3 \sqrt{3} m_Z^2}{2g_s w l^2} \frac{g}{2g \cos \beta} g_s, \quad (4.220) \]

\[ = \frac{-3 \sqrt{3} m_Z^2}{4gwl^2} g \sin^2 \theta, \quad (4.221) \]

\[ = \frac{\sqrt{3} m_Z^2}{4 \frac{w_l}{g_s^2} \cos \theta^2}, \quad (4.222) \]

\[ = \frac{g}{\cos \theta} \left( \frac{\sqrt{3} k^2 g^2 \tan^2 \theta}{4 \frac{2g^2 w_l^2}{g_s^2}} \right), \quad (4.223) \]

\[ = \frac{2e}{\sin 2 \theta} \left( -\sqrt{3} \frac{8}{8} \left( \frac{k}{g_s} \right)^2 \frac{g}{g_s} \tan^2 \theta \right), \quad (4.224) \]

but since the \( U(I)X \) subgroup has the generator \( X = 2I_{3R} + V \) then\(^{22}\)

\[ \frac{1}{g'^2} = \frac{1}{g^2} + \frac{1}{g_V^2} \]

i.e.,

\[ \frac{1}{g_Y^2} = \frac{1}{g^2} + \frac{1}{3g_s^2}, \]

\(^{22}g'\) is the coupling of the extra boson we are calling \( X \), but note that the original authors called this boson \( Y' \).
so

\[ g^2 \tan \theta = \frac{3g'^2s^2}{3g'^2 + g^2}, \quad(4.225) \]

\[ \approx g'^2. \quad (4.226) \]

Thus we compute

\[ \Delta = \frac{2e}{\sin 2\theta} \left( -\frac{\sqrt{3}}{8} \left( \frac{k}{w_l} \right)^2 \left( \frac{g'}{g} \right)^2 T_l \right), \quad(4.227) \]

and once we correct for the fact that they use a $3 \times 3$ antisymmetric $\chi$ field, which because of normalisation means that $w_l = \sqrt{2}w$, and that they define $T_l = T/\sqrt{3}$, we obtain

\[ \Delta = \frac{2e}{\sin 2\theta} \left( -k^2 \frac{g'^2T}{16g^2w^2} \right), \quad(4.228) \]

which agrees up to an inconsequential minus sign.  

For completeness the coupling of the very heavy third boson is

\[ C_{NC} = (I_{3L} - Q) \frac{\sin \sigma \tan \theta \sin \theta}{\cos \alpha \cos^2 \beta} + I_{3R} \left( \sin \sigma \cos \alpha \cos \theta + \frac{\sin \sigma \tan \theta \sin \theta}{\cos \alpha \cos^2 \beta} \right) + T_l \frac{g^\alpha \cos \theta}{2g \cos \beta}. \quad(4.229) \]

These couplings may now be used to bound the symmetry breaking scales $w_{R,l}$. We achieve this by performing a $\chi^2$ fit of the predictions of this model to the following electroweak precision data:

\[ R_{e,\mu,\tau,b,c}, \quad A_{e,\mu,\tau,b,c,s}, \quad A^{FB}_{e,\mu,\tau,b,c,s}, \quad Q_W(C_s), \quad Q_W(T_l), \quad g_{nL}^2, \quad g_{nR}^2, \quad \Gamma_Z, \quad \sigma_{\text{had}}. \quad(4.230) \]

Under the phenomenological necessary assumption that $M_{Z'} \gg M_Z$, the phys-

\[ ^{23} \text{The other corrections, presented very differently in another paper [133], are too cumbersome to show any analytical correspondence. Nevertheless, we have confirmed that they agree numerically.} \]
4.7 Experimental signatures

In the limit $1/R \gg w_{\text{min}}$ the key signatures of 5d QLLR models result from the additional neutral gauge bosons and the liptons. The discovery of an additional neutral gauge boson at the LHC would be via Drell-Yann processes.
$pp \rightarrow Z' \rightarrow l^+l^-$ \cite{150,151}. The lower centre-of-mass energy of a next generation linear collider (NLC) precludes the production of real $Z'$s. Thus any discovery at the NLC would be made by measuring $e^+e^- \rightarrow f \bar{f}$ and observing corrections resulting from interference between diagrams with a $Z'$ propagator, and those with a $\gamma$ or $Z$ \cite{150}, to the relevant quantities discussed in section 4.6.

Current data allows a $Z'$ at TeV energies regardless of the hierarchy between $w_l$ and $w_R$, however the discovery prospects depend strongly on this hierarchy. In the limit $w_R < w_l$ the additional boson is basically that of the LR model. Previous studies have found that $Z'_{LR}$ is discoverable up to a mass of about 5 TeV at the LHC (NLC with $\sqrt{s} = 1$ TeV) with an integrated luminosity of 100fb$^{-1}$ \cite{150,152}. If $w_l < w_R$ the light $Z'$ is that of the QL model. In this case discovery at a hadron collider such as the LHC is unlikely since the cross section of the Drell-Yann process, $\sigma(pp \rightarrow Z')B(Z' \rightarrow l^+l^-)$, will not be large. This is because $Z'$ contains a large fraction of $G^0_l$ which does not couple to the quarks, meaning the cross section $\sigma(pp \rightarrow Z')$ will be small. The discovery prospects are greater at the NLC since $G^0_l$ has large universal couplings to all leptons. These couplings are quite different from any of the canonical $Z'$s usually considered, providing a distinctive signature.

The most interesting case is when $w_R \sim w_l$. Then both $Z'$ and $Z''$ possess significant couplings to the $SU(3)_l$ and $SU(2)_R$ neutral currents and may have TeV masses. As such they could both be produced at the LHC. The prospects of discovery depend on the masses of the liptons, which are of order $w_l$. If the $Z'$ bosons can decay into liptons their branching fraction into charged leptons will reduce and hence the Drell-Yann cross section will be smaller. The presence of light liptons would afford alternative discovery channels, with liptonic final states, at the LHC. At a linear collider light liptons are unimportant since only virtual $Z'$s are produced. However, diagrams with both $Z'$ and $Z''$ propagators will interfere with those of the SM, causing corrections. Without prior knowledge of the mass and mixings of the neutral gauge bosons their couplings to fermions are unknown. Thus separating their effects and categorically identifying the nature of the bosons is difficult.
If $w_l$ is at the TeV scale then the production of liptons is also possible at the LHC. For $1/R \gg w_l$, only the zero mode liptons with orbifold parity $(+, +)$, which are in different $SU(3)_l$ multiplets to the SM leptons, can be produced. Since both the $Y_{1,2}^\pm$ and $W_R$ bosons have $(+, -)$ orbifold parity, they do not directly couple the lightest liptons to the SM leptons. As the unbroken $SU(2)_l$ is expected to be confining, these liptons will form bi-lipton bound states. These states will decay into SM fermions via creation of a $W$, $Z$ or photon. This will produce a clear experimental signature, the details of which are similar to those obtained in the 4d QLLR model [133].

4.8 Conclusion to 5D QLLR

In this chapter we have constructed and analysed the 5d QLLR model. We have shown that the higher dimensional construct permits a novel mechanism for suppressing neutrino masses below the electroweak scale and allows one to dramatically simplify the scalar sector employed in 4d constructs. This allows one to keep both the QL and the LR symmetry breaking scales low (TeV energies) so that two neutral gauge bosons may be observed at the LHC.

Split fermions were used to explain some of the features of the SM mass spectrum. Two interesting fermionic geographies were presented, each of which provided a rationale for the relationships $m_t > m_b, m_\tau$ and $m_\nu \ll m_t$. One of these had no Yukawa coupling hierarchy but to suppress the proton decay rate required either a large cut-off or that the fundamental theory observed the accidental $B$ and $L$ symmetries of the QLLR model. In the former case, fine tuning was required to stabilise the Higgs boson mass at the electroweak scale. The alternative arrangement suppressed the proton decay rate by spatially separating quarks and leptons in the extra dimension. Thus the hierarchy problem was alleviated but at the cost of introducing an order $10^2$ Yukawa coupling hierarchy. Given the extent to which the symmetries of the model constrain the Yukawa sector it is a non-trivial result that interesting fermionic geographies can be obtained with mild Yukawa coupling hierarchies.
CHAPTER 5

Entanglement and Hawking radiation

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5.1 Introduction

Since the discovery of Hawking radiation [16] in the early seventies much research into quantum field theory on general classical backgrounds has taken place. One of the most well established results of this field is that of the Unruh effect [153, 154, 155] which states that a constantly accelerated particle
detector in the Minkowski vacuum will register a thermal bath of particles at a temperature given by:

\[ kT = \frac{\hbar a}{2\pi c}, \]  

(5.1)

where \( a \) is the acceleration of the detector.

Roughly one decade after the Hawking result, Bell and Leinass took these observations and applied them to linearly and circularly moving electrons [156] addressing the anomalous spin depolarization that was observed in storage rings [1].

Today the connection between Hawking and Unruh radiation is even more striking; through a procedure known as the Global Embedding of Minkowski Space (GEMS) BHs can be imbedded into higher dimensional spaces, where the two effects are truly equivalent [158,159]. In the GEMS program it has been shown that the familiar four dimensional Schwarzschild solution has an embedding in six dimensions such that the usual radially constant detector in Schwarzschild space follows a hyperbolic Rindler worldline in the flat six dimensional space, thereby neatly explaining why the Hawking and Unruh temperatures agree. While these results illustrate the connection between Hawking and Unruh radiation in a profound way, they diverge somewhat from our discussions and so we do not consider these any further. Specifically, in our treatment we consider an accelerated Rindler observer in four dimensions.

Moving in parallel with developments in BH physics, mainly due to the drive for quantum computing, the last twenty years has also seen the rise of quantum information theory, at the heart of which lies the resource of entanglement. Entanglement is a phenomenon unique to quantum systems. Two systems are said to be entangled if they are described by a single wave function that cannot be written as a product of wave functions for each part. The entanglement between two qubits can be quantified by a function called concurrence [160], taking values on the interval [0,1], where a concurrence of zero

\[ ^1 \text{However, it appears this effect is masked by the classical Sokolov-Ternov effect, see [157].} \]
While quantum entanglement has been extensively studied over this period of time, it has only recently been shown to be frame dependent [161]. Thus the amount of entanglement between two electron spins can depend on the speed of the observer measuring these properties. However, the entanglement is not lost in this situation, but redistributed to other properties of the electrons such as their momentum.

What is more surprising is that recently it has been shown that a pair of constantly accelerating field modes in a cavity entangled with another pair at rest, possesses degraded entanglement due to the relative acceleration between the parts [39, 162]. It is still unclear whether this entanglement disappears altogether or is redistributed to some other variables for instance those related to the gravitational field. It was further shown [163] that these ideas could be applied to the case of one cavity placed at a fixed distance near the event horizon of a BH, whilst the other in free fall, with similar outcomes. Specifically, the Schwarzschild metric:

\[
\frac{ds^2}{R^2} = -(1 - 2M/R)dT^2 + (1 - 2M/R)^{-1}dR^2,
\]

(5.2)
can be turned (near the horizon) into the Rindler metric

\[
\frac{ds^2}{Ax^2} = -(Ax)^2dT^2 + dx^2,
\]

(5.3)
through the coordinate transformation \( R = x^2/8M + 2M \), where \( A \equiv 1/4M \) is the inverse acceleration. The intrinsic loss of entanglement in this system is another example of non-unitarity in relativistic quantum mechanics known as the information paradox. We find these issues of high interest as they pertain to a theory of quantum gravity which to date has escaped a consistent theoretical formulation.

An important question one may ask is: is the entanglement shared by ac-
celerated spinors degraded in a fashion similar to the pair of accelerated field modes?

In this chapter we address this issue by extending the work of Bell and Leinass [156] to include entanglement in their setup. This is done by inserting an ancillary rest particle that is entangled with the original accelerating one. Using an open quantum system we then analytically calculate the relaxation timescales and the concurrence of this system. It is found that due to the Unruh radiation that the accelerated particle experiences, the acceleration increases the rate at which the entanglement is destroyed. In the limit when the Rindler temperature (5.1) is larger than the spin energy separation scale this dependence takes on a particularly simple form. For an initially maximally entangled system the time to completely disentangle, as measured by the accelerated spinor, is found to be:

$$\tau_0 = \frac{3\pi \log 3}{8} \frac{\hbar c^6}{\mu^2 a^3},$$  \hspace{1cm} (5.4)

and in the rest frame:

$$t_0 = \frac{c}{2a} \exp\left(\frac{3\pi \log 3}{8} \frac{\hbar c^5}{\mu^2 a^2}\right).$$  \hspace{1cm} (5.5)

### 5.2 Review of accelerated paths

Consider a particle subjected to arbitrary accelerations in its Instantaneous Rest Frame (IRF), i.e., the frame that it is at rest with at time $\tau$. In that frame the particle will gain a small velocity $dv' = a(\tau)d\tau$ in a small time interval $d\tau$ due to its acceleration. A static observer will measure a velocity $v + dv = \frac{dv' + v}{1 + v dv'/c^2}$, where $v$ is the velocity of the IRF at time $t(\tau)$. Keeping terms to $O(dv')$ we find $dv = (1 - (v/c)^2)dv'$, then $\int a(\tau)d\tau = \int \frac{dv'}{1 + (v/c)^2} = \frac{3\pi \log 3}{8} \frac{\hbar c^5}{\mu^2 a^2}$

\footnote{It must be made clear that our results depend heavily on the spin-field interaction. In which case we can only talk about the effect of acceleration on spin entanglement. The author does not see any way to extend the current analysis in order to incorporate more general entangled states, i.e., like the continuous states: position, momentum etc.}
5.3 Spin evolution

\[ c \arctanh v/c. \] The particle’s trajectory in the static frame is then

\[
z(\tau) = \int \frac{dz}{dt} \frac{dt}{d\tau} d\tau, \tag{5.6}
\]

\[
= \int v \sqrt{1 - (v/c)^2} d\tau, \tag{5.7}
\]

\[
= c \int \sinh \left[ \int \frac{a(\tau)}{c d\tau} \right] d\tau. \tag{5.8}
\]

We can also find the dependence of coordinate time on the proper time, \( t(\tau) \), in flat spacetime using the Minkowski line element \( 1 = (\frac{dt}{d\tau})^2 - (\frac{dz}{cd\tau})^2 \). This implies that

\[
t(\tau) = \int \cosh \left[ \int \frac{a(\tau)}{c} d\tau \right] d\tau. \tag{5.9}
\]

For the special case where the acceleration is constant, \( a(\tau) = a \), we obtain:

\[
t(\tau) = \frac{c}{a} \sinh \frac{a}{c} \tau, \tag{5.10}
\]

\[
z(\tau) = \frac{c^2}{a} \cosh \frac{a}{c} \tau, \tag{5.11}
\]

which is the usual Rindler result.

5.3 Spin evolution

In our setup, we take two electrons keeping one of them (\( e_2 \)) stationary and isolated from any fields\(^3\) whilst the other (\( e_1 \)) is placed under a constant magnetic field and accelerated by a constant electric field, see figure 5.1. In this situation all the dynamics occurs on spin 1, with spin 2 playing a spectator role whose only function is to provide an initial source of entanglement. We would like to know what effect the acceleration has on any entanglement the two electron spins initially possess.

\(^3\)Spin two could be imagined, for example, as being localised within a charge trap, however for the sake of generality an exact physical system will not be necessary.
Figure 5.1: Two different views of the two entangled spins in our setup. In figure a) we have the Minkowski diagram with electron $e_1$ following the Rindler wordline for a constantly accelerated path while electron $e_2$ remains at rest (following the vertical dotted line) and defines the inertial frame of this diagram. Figure b) on the other hand is a physical view of the two electrons; the top spinor, $e_1$, is accelerated by an electric field and placed under a magnetic field while the bottom spinor, $e_2$, is isolated and at rest.

Let the electron mass be $m$ and the magnetic moment, $\mu$, and suppose that the fields point along the $z$-direction i.e., $\mathbf{E} = (0, 0, E_z)$ and $\mathbf{B} = (0, 0, B_z)$. If $e$ is the electric charge then the IRF acceleration of the first electron ($e_1$) is given by

$$a = \frac{e}{m} E_z. \quad (5.12)$$

Taking the Hilbert space to be $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_{\text{field}}$, where $\mathcal{H}_1$ is the subspace of the accelerated spinor, the spin-field interaction Hamiltonian is:

$$H = -\mu \sigma \otimes 1 \cdot \mathbf{B}, \quad (5.13)$$

The sigma operator is only acting on the accelerated spin Hilbert space, $\mathcal{H}_1$, as the second spinor is isolated and still. Since the action on $\mathcal{H}_2$ is always the identity, for notational simplicity we suppress the factor of $\otimes 1$ from all operators acting on the combined spin Hilbert space in what follows.

If we approximate the magnetic field as a classical field, the evolution of
5.3 Spin evolution

the spin is determined by the usual semi classical Hamiltonian

$$H_{SC} = -\mu \sigma_z B_z.$$  \hfill (5.14)

The system has two pseudo-energy eigenstates, defined by $\sigma_z = \pm 1$, with an energy gap $\Delta = 2|\mu||B_z|$ and therefore spin flipping does not occur at this level. However, quantum fluctuations of the magnetic field do cause the spin polarisation to change and in order to incorporate these fluctuations we must second quantize the magnetic field. We do this in the standard way by interpreting the magnetic field as a space-time dependent field operator. Expanding equation (5.13), and defining the field operators, $B_{\pm} = B_x \pm iB_y$, we find:

$$V = -\mu(\sigma_x B_x(x) + \sigma_y B_y(x) + \sigma_z B_z(x)),$$

$$= -\mu(\sigma_- B_+(x) + \sigma_+ B_-(x) + \sigma_z B_z(x)).$$  \hfill (5.15)

Thus, when the spin flips, a magnetic field mode\(^4\) is excited. To understand what effect the field fluctuations have on the spin entanglement it is constructive to consider the simpler spin flip interaction:

$$H_{int} = \lambda (b^{\dagger} | \downarrow \rangle \langle \uparrow | + b | \uparrow \rangle \langle \downarrow |),$$  \hfill (5.16)

where $b^{(\dagger)}$ are the annihilation (creation) operators for the field modes. Then a first order interaction will turn the state into:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (| \uparrow \uparrow \rangle + | \downarrow \downarrow \rangle) |n_0\rangle,$$

$$\rightarrow \frac{\lambda}{\sqrt{2}} (| \downarrow \uparrow \rangle |n_0 + 1\rangle + | \uparrow \downarrow \rangle |n_0 - 1\rangle).$$  \hfill (5.17)

In only looking at the entanglement between spins, tracing out over the field

\(^4\)This is just a change of basis of the standard polarization directions and should not be confused with the creation or annihilation operators which are obtained by separating the field into positive and negative frequencies.

\(^5\)These magnetic modes are related to photon modes through $B = \nabla \times A$. 


fluctuations is implied, then over time we expect
\[
\text{Tr}_B \rho(\Psi) \to \frac{\lambda^2}{2} |↑\downarrow\rangle\langle↑\downarrow| + \frac{\lambda^2}{2} |↓\uparrow\rangle\langle↓\uparrow|,
\]
(5.19)
which is a completely mixed density i.e., the spin entanglement has been completely destroyed.

To make the above argument more rigorous, we calculate the time evolution of the two electron state using a perturbative master equation \[164, 165\]. Since in the current setup the spin-field coupling strength is suitably small a perturbative expansion is acceptable. In viewing the fluctuations of the external magnetic field as an unobserved environment, tracing over it is implied and no \textit{ad hoc} dynamical modelling of the source of decoherence is necessary.

We can write the total Hamiltonian of the system as
\[
H = H_{SC} + H_B + V,
\]
(5.20)
where \(H_{SC}\) is the semiclassical evolution of the spin system, \((5.14)\), \(H_B\) is the free Hamiltonian for the magnetic field, whose exact form will not be required, and \(V\) takes into account the quantum fluctuations arising from the second quantization of the spin-field interaction, determined in equation \((5.15)\).

Let \(\rho_T\) be the total density operator of the system plus field in the interaction picture. The equation of motion in the interaction picture is:
\[
\frac{d\rho_T(\tau)}{d\tau} = \frac{1}{i\hbar}[V(\tau), \rho_T(\tau)],
\]
(5.21)
where \(\tau\) is the proper time of the accelerating electron. We assume that initially the electron spin system and field states are uncorrelated so that:
\[
\rho_T(0) = \rho(0) \otimes \rho_B,
\]
(5.22)
where \(\rho\) describes the state of the two electron spins and \(\rho_B\) describes the
state of the magnetic field. Following [165], we expand equation (5.21) in a perturbative series and noting that \( \text{Tr}_B(V(\tau)\rho_B) = 0 \) we find to second order in the perturbation:

\[
\frac{d\rho}{d\tau} = -\frac{1}{\hbar^2} \int_0^\tau d\tau_1 \text{Tr}_B[V(\tau), [V(\tau_1), \rho(\tau) \otimes \rho_B]].
\]

(5.23)

For brevity we write the interaction in (5.15) as

\[
V(\tau) = -\mu \sum_i \sigma_i B^i_1(x(\tau)),
\]

(5.24)

where \( i = +, -, z \). By rewriting this master equation in the system Schrödinger picture we obtain:

\[
\frac{d\rho}{d\tau} = \frac{1}{i\hbar}[H_{SC}, \rho] - \frac{\mu^2}{\hbar^2} \sum_{i,j} \int_0^\tau d\tau_1 \times \cdots \text{Tr}_B[\sigma_i B^i_1(x), [\sigma_j (\tau_1 - \tau) B^j_1(x_1), \rho \otimes \rho_B]].
\]

(5.25)

where \( x = x(\tau) \) and \( x_1 = x(\tau_1) \) are defined for short. From the Heisenberg equations of motion \( \sigma_j(\tau) = e^{\alpha_j \Delta \tau / \hbar} \sigma_j \) where \( \alpha_j = +1, -1, 0 \), for \( j = +, -, z \), respectively. Defining the expectation value over the field state to be \( \langle O \rangle = \text{Tr}_B(O\rho_B) \) we obtain:

\[
\frac{d\rho}{d\tau} = \frac{1}{i\hbar}[H_{SC}, \rho] - \frac{\mu^2}{\hbar^2} \sum_{i,j} \int_0^\tau d\tau_1 e^{\alpha_j \Delta i(\tau_1 - \tau) / \hbar} \times
\]

\[
\left\{ \langle \sigma_i \sigma_j \rho - \sigma_j \rho \sigma_i \rangle (B^i_1(x) B^j_1(x_1)) + \langle \rho \sigma_j \sigma_i - \sigma_i \rho \sigma_j \rangle (B^i_1(x_1) B^j_1(x)) \right\}.
\]

(5.26)

In Gaussian units the electromagnetic free field Wightman function is

\[
(0|F_{\mu\nu}(x)F_{\rho\sigma}(x')|0) = \frac{4\hbar c}{\pi} (x - x')^{-6} \left\{ (x - x')^2 (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) - 2 ((x - x')_\mu (x - x')_\rho g_{\nu\sigma} - (x - x')_\nu (x - x')_\rho g_{\mu\sigma}) - (x - x')_\mu (x - x')_\sigma g_{\nu\rho} + (x - x')_\nu (x - x')_\sigma g_{\mu\rho} \right\}.
\]

(5.27)
Given that the motion is entirely along the $z$ direction we find that the only non-zero two point correlation functions in the $x, y$ and $z$ directions are $\langle B_x(x)B_x(x') \rangle = \langle B_y(x)B_y(x') \rangle = \langle B_z(x)B_z(x') \rangle = G(x(\tau-\imath \epsilon), x(\tau'))$\footnote{Note that $\langle B_\pm(x)B_\mp(x') \rangle = \langle B_x(x)B_x(x') \rangle + \langle B_y(x)B_y(x') \rangle = 2G(x, x')$.}, where

$$G(x, x') = \frac{4hc}{\pi}(x - x')^{-4}. \quad (5.28)$$

Along the constantly accelerated path described by equations (5.10)-(5.11), this further reduces to:

$$G(\tau, \tau') = \frac{ha^4}{4\pi c^7} \left\{ \sinh \left[ \frac{a^2 c (\tau - \tau')}{\hbar} \right] \right\}^{-4}. \quad (5.29)$$

Thus we obtain the equation:

$$\frac{d\rho}{d\tau} = \frac{1}{\imath \hbar} [H_{SC}, \rho] - \frac{\mu^2}{\hbar^2} \left\{ 2(\sigma_+ \rho - \sigma_+ \rho \sigma_-) \int_0^\tau d\tau_1 e^{i\Delta(\tau_1 - \tau)/\hbar} G(\tau - \tau_1 - \imath \epsilon) \\
+2(\rho \sigma_+ \sigma_- - \sigma_- \rho \sigma_+) \int_0^\tau d\tau_1 e^{i\Delta(\tau_1 - \tau)/\hbar} G(\tau - \tau_1 + \imath \epsilon) \\
+2(\sigma_+ \sigma_- \rho - \sigma_- \rho \sigma_+) \int_0^\tau d\tau_1 e^{-i\Delta(\tau_1 - \tau)/\hbar} G(\tau - \tau_1 - \imath \epsilon) \\
+2(\rho \sigma_- \sigma_+ - \sigma_+ \rho \sigma_-) \int_0^\tau d\tau_1 e^{-i\Delta(\tau_1 - \tau)/\hbar} G(\tau - \tau_1 + \imath \epsilon) \\
+(\rho - \sigma_z \rho \sigma_z) \int_0^\tau (G(\tau - \tau_1 - \imath \epsilon) + G(\tau - \tau_1 + \imath \epsilon)) d\tau_1 \right\}. \quad (5.30)$$

Since the integrands in the above equation are sharply peaked functions about $\tau_1 = 0$ we are justified in making the Markovian approximation and extending the integration over the interval $[0, \infty]$. We define the following integrals:

$$\Gamma_{\pm} = \int_0^\infty e^{i\Delta s/\hbar} G(s \pm \imath \epsilon) ds, \quad (5.31)$$

$$\Gamma_z = \int_0^\infty \{G(s - \imath \epsilon) + G(s + \imath \epsilon)\} ds. \quad (5.32)$$
5.3 Spin evolution

Using

\[
\int \frac{ds}{\sinh^4(s + ia)} = -\frac{1}{3} \coth(s + ia) \left( \text{csch}^2(s + ia) - 2 \right),
\]

we find that \( \Gamma_z = \frac{2}{3} \hbar a^3 \). Then making the change of variables \( s = \tau_1 - \tau \) equation (5.30) becomes

\[
\frac{d\rho}{d\tau} = \frac{1}{i\hbar} \left[ H_{SC, \rho} - \frac{\mu^2}{\hbar^2} \right] \{ \nonumber \\
2(\sigma_-\sigma_+\rho - \sigma_+\rho\sigma_-)\Gamma_+ + 2(\rho\sigma_+\sigma_- - \sigma_-\rho\sigma_+)\Gamma_- \\
+2(\sigma_+\sigma_-\rho - \sigma_-\rho\sigma_+)\Gamma_+^* + 2(\rho\sigma_-\sigma_+ - \sigma_+\rho\sigma_-)\Gamma_-^* \\
+ \frac{2}{3} \hbar a^3 \left( \rho - \sigma_z \rho \sigma_z \right) \} ,
\]

where we have made use of the identity \( G(-s) = G(s) \), which follows from equation (5.29). By separating \( \Gamma_1 \) and \( \Gamma_2 \) into their real and imaginary components equation (5.34) simplifies into:

\[
\frac{d\rho}{d\tau} = \frac{1}{i\hbar} \left[ H_{SC, \rho} - \frac{\mu^2}{\hbar^2} \right] \{ \nonumber \\
2\text{Re}\Gamma_- (\sigma_-\sigma_+\rho + \rho\sigma_-\sigma_+ - 2\sigma_+\rho\sigma_-) \\
+2\text{Re}\Gamma_+ (\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_- - 2\sigma_-\rho\sigma_+) \\
+ \frac{2}{3} \hbar a^3 \left( \rho - \sigma_z \rho \sigma_z \right) + i\text{Im}(\Gamma_+ + \Gamma_-)\left[ \sigma_z, \rho \right] \} .
\]

The last term is a correction to the unperturbed energy separation \( \Delta \). It is due to the spin-field coupling and effectively renormalizes the Hamiltonian by a \textit{lamb-shift} term \( H_{LC} = \frac{\mu^2}{\hbar} \text{Im}(\Gamma_+ + \Gamma_-)\sigma_z \), reminiscent of the Lamb shift \([166]\) in atomic physics. Further defining:

\[
\gamma_\pm = \frac{4\mu^2}{\hbar^2} \text{Re}\Gamma_\pm = \frac{2\mu^2}{\hbar^2} \int_{-\infty}^{\infty} e^{\pm i\Delta s/\hbar} G(s - i\epsilon) ds,
\]

\[ (5.36) \]
we observe that the Master equation takes a manifestly Lindblad form:

\[
\frac{d\rho}{dt} = -i\frac{\hbar}{\hbar}[H', \rho] + \sum_j \left[2L_j\rho L_j^\dagger - \{L_j^\dagger L_j, \rho\}\right],
\]

(5.37)

where \(\{x, y\} = xy + yx\) denotes an anticommutator, \(H' = H_{SC} + H_{LC}\) is the coherent part of the renormalized spin Hamiltonian, and \(L_j\) are the Lindblad operators, given by:

\[
L_1 = \sqrt{\frac{\gamma_-}{2}} \sigma_-,
\]

(5.38)

\[
L_2 = \sqrt{\frac{\gamma_+}{2}} \sigma_+,
\]

(5.39)

for transitions down and up (in spin energy) respectively and

\[
L_3 = \sqrt{\frac{\gamma_z}{2}} \sigma_z,
\]

(5.40)

(a pure dephasing channel) where

\[
\gamma_z \equiv \frac{\mu^2}{\hbar^2} \Gamma_z = \frac{2}{3} \frac{\mu^2 a^3}{\hbar \pi c^6}.
\]

(5.41)

By moving into the renormalized interaction picture,

\[
\tilde{\rho}(\tau) = e^{iH'\tau/\hbar} \rho(\tau) e^{-iH'\tau/\hbar},
\]

(5.42)

we can write

\[
\frac{d\tilde{\rho}}{d\tau} = \frac{\gamma_-}{2} [2\sigma_-\tilde{\rho}\sigma_+ - \sigma_+\sigma_-\tilde{\rho} - \tilde{\rho}\sigma_+\sigma_-] \\
+ \frac{\gamma_+}{2} [2\sigma_+\tilde{\rho}\sigma_- - \sigma_-\sigma_+\tilde{\rho} - \tilde{\rho}\sigma_-\sigma_+] \\
+ \gamma_z [\sigma_z\tilde{\rho}\sigma_z - \tilde{\rho}].
\]

(5.43)

We can expand the density matrix for a two spin system in terms of sigma
matrices:

\[
\rho = \sum_{i=0}^{3} \sum_{j=0}^{3} r_{ij} \sigma_i \otimes \sigma_j,
\]

where \( \sigma_i \) are the Pauli matrices:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and for notational simplicity we have defined \( \sigma_0 = 1 \). Note also that \( r_{ij} = \frac{1}{4} \text{Tr}(\rho \sigma_i \otimes \sigma_j) \), then for instance, a maximally entangled Bell state of the kind \( \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) \) would be expressed

\[
\rho_{\text{Bell}} = \frac{1}{4} (\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3).
\]

Substituting \( \dot{\rho} = \sum_{i,j} \dot{r}_{ij} \sigma_i \otimes \sigma_j \) we get:

\[
\frac{d\dot{r}_{ij}}{d\tau} \sigma_i \otimes \sigma_j = \frac{\gamma}{2} \dot{r}_{ij} [2\sigma_- \sigma_+ - \sigma_+ \sigma_- - \sigma_- \sigma_+] \otimes \sigma_j + \frac{\gamma}{2} \dot{r}_{ij} [2\sigma_+ \sigma_- - \sigma_- \sigma_+ - \sigma_+ \sigma_-] \otimes \sigma_j + \gamma \dot{r}_{ij} [\sigma_z \sigma_0 - \sigma_0] \otimes \sigma_j,
\]

which after a little algebra gives sixteen first order linear differential equations

\[
\dot{r}_{ij}(\tau) = \begin{cases} 0, \\
-\frac{1}{2} (\gamma_+ + \gamma_+) \dot{r}_{ij}(\tau), \\
-\frac{1}{2} (\gamma_+ + \gamma_+) \dot{r}_{ij}(\tau), \\
(\gamma_- - \gamma_+) \dot{r}_{ij}(\tau) - (\gamma_- + \gamma_+) \dot{r}_{ij}(\tau),
\end{cases}
\]

where dots imply differentiation with respect to \( \tau \). The solutions to these
The equations are found to be:

\[ \tilde{r}_{0j}(\tau) = \tilde{r}_{0j}(0), \]
\[ \tilde{r}_{1j}(\tau) = \tilde{r}_{1j}(0)e^{-\frac{1}{2}(\gamma_-+\gamma_+\pm 4\gamma_z)\tau}, \]
\[ \tilde{r}_{2j}(\tau) = \tilde{r}_{2j}(0)e^{-\frac{1}{2}(\gamma_-+\gamma_+\pm 4\gamma_z)\tau}, \]
\[ \tilde{r}_{3j}(\tau) = \tilde{r}_{3j}(0)e^{-(\gamma_\pm \gamma_+ \pm 4\gamma_z)\tau} + \frac{\gamma_- - \gamma_+ \pm \gamma_0}{\gamma_- + \gamma_+} \tilde{r}_{0j}(0)(1 - e^{-(\gamma_-+\gamma_+\pm 4\gamma_z)\tau}). \]  

(5.49)

Thus the relaxation times are:

\[ T_2^{-1} = \frac{\gamma_- + \gamma_+}{2} + 2\gamma_z, \]  
\[ T_1^{-1} = \gamma_- + \gamma_+, \]  

(5.50)  
(5.51)

where \( T_2 \) is the timescale for which the off-diagonal elements of the density matrix (‘coherences’) decay and \( T_1 \) is the timescale for spin flipping. We will determine their relative magnitudes in the next section after we have calculated the spin flip rates.

### 5.4 Calculation of spin flip rates

We have succeeded in finding the time evolution for a system of spins relatively accelerating within a specific setup of constant fields. In order to analyse the effect that the acceleration has on the entanglement shared by these spins it is necessary to calculate the spin flip rates defined in equation (5.36).

Making a linear change of variables, \( s' = as/c \), the integral in (5.36) becomes

\[ \gamma_\pm = \frac{2\mu^2}{\hbar^2} \frac{ha^4}{4\pi c^2} a \int_{-\infty}^{\infty} ds' \exp \left( \mp is'c/a\hbar \right) \frac{\sinh \frac{1}{2}(s' - i\epsilon)}{\sinh \frac{1}{2}(s' - i\epsilon)}. \]  

(5.52)

The remaining integral can be solved using contour integration. As the
5.4 Calculation of spin flip rates

The integrand in equation (5.52) has the property

\[ I[s] = \frac{\exp (\mp i \Delta sc / a\hbar)}{\sinh^{4} \frac{1}{2} (s - i\epsilon)}, \]

\[ = e^{\mp 2\pi c \Delta / a\hbar} I[s + 2\pi i], \]

(5.53)

(5.54)

Thus, it is constructive to take the contour, \( C \), shown in figure (5.2).

\[
\int_{C} I[s] = \int_{l_1} I[s]ds + \int_{l_3} I[s]ds,
\]

\[ = (1 - e^{\mp 2\pi c \Delta / a\hbar}) \int_{l_1} I[s]ds. \]

(5.55)

(5.56)

Therefore,

\[
(1 - e^{\mp 2\pi c \Delta / a\hbar}) \gamma_{\pm} = \frac{\mu^2 a^3}{2\hbar c^3} 2\pi i \text{ Res}_{s \to 0} I[s],
\]

(5.57)

and since \( \text{Res}_{s \to 0} I[s] = \pm \frac{8}{3} i \frac{\Delta}{a \hbar}(\frac{\Delta^2 c^2}{a^2 \hbar^2} + 1) \), the transition rates are:

\[
\gamma_{\pm} = \frac{8 \mu^2 \Delta}{3 \hbar^2 c^3} \left( \Delta^2 + \frac{a^2 \hbar^2}{c^2} \right) (1 - e^{\pm 2\pi c \Delta / a\hbar})^{-1}.
\]

(5.58)

These are the spin flip transition rates for an accelerating electron as found by Bell and Leinass [156], they observed that the ratio of the transition rates,

\[
\frac{\gamma_{+}}{\gamma_{-}} = e^{-2\pi c \Delta / a\hbar},
\]

(5.59)
define an equilibrium ratio of populations of the upper and lower states. Thus, the equilibrium distribution over the levels has a thermal character in accordance with the Unruh temperature formula \( T = \frac{\hbar a}{2\pi kc} \). To proceed further we recall the Bose occupation number

\[
n = \frac{1}{e^{2\pi c\Delta/\hbar} - 1}.
\]  

(5.60)

We also notice from equation (5.58) that even when the acceleration goes to zero there is still an emission rate:

\[
\gamma_0 \equiv \lim_{a \to 0} \gamma_- = \frac{8\mu^2\Delta^3}{3\hbar^4c^5}.
\]  

(5.61)

This spontaneous emission is due to the quantum interactions of the magnetic moment with the magnetic field. It is much weaker than the usual atomic spontaneous emission rate, and significantly harder to detect, see [167]. Then the equations in (5.36) can be written:

\[
\gamma_+ = \gamma_0 \left( 1 + \left( \frac{a\hbar}{c\Delta} \right)^2 \right) n, \quad (5.62)
\]

\[
\gamma_- = \gamma_0 \left( 1 + \left( \frac{a\hbar}{c\Delta} \right)^2 \right) (n + 1). \quad (5.63)
\]

Furthermore, it is easily verified that

\[
\gamma_z = \frac{1}{4\pi} \gamma_0 \left( \frac{a\hbar}{c\Delta} \right)^3. \quad (5.64)
\]

Thus, the relaxation times are:

\[
T_1^{-1} = \gamma_0 (1 + \left( \frac{a\hbar}{c\Delta} \right)^2) \coth \frac{\pi c\Delta}{a\hbar}, \quad (5.65)
\]

\[
T_2^{-1} = \frac{\gamma_0}{2} \left\{ (1 + \left( \frac{a\hbar}{c\Delta} \right)^2) \coth \frac{\pi c\Delta}{a\hbar} + \frac{1}{3} \left( \frac{a\hbar}{c\Delta} \right)^3 \right\}. \quad (5.66)
\]

The relaxation times are compared in figure 5.3. It can be seen that \( T_2 > T_1 \) for all values of the acceleration. Thus, typically there will be more than one spin flip before the coherences vanish. In the next section we will use the concurrence function in order to quantify the entanglement of this system as a function of time.
5.5 Disentanglement of Spin

In the long time limit the state (5.44) becomes:

\[
\tilde{\rho}_a(\infty) = \sum_j \tilde{r}_{0j} \mathbf{1} \otimes \sigma_j + \tanh\left(\frac{c\Delta}{\hbar}a\right) \tilde{r}_{0j} \sigma_3 \otimes \sigma_j,
\]
\[= (1 + \tanh\left(\frac{c\Delta}{\hbar}a\right) \sigma_3) \otimes \sum_j \tilde{r}_{0j} \sigma_j,
\]
(5.67)
(5.68)

which is a totally mixed product state. The entanglement between the spin degrees of freedom can be obtained by calculating Wooters’ concurrence \[168\]

\[
C(\rho) = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\},
\]
(5.69)

where \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} are the square roots of the eigenvalues of the matrix

\[
M = \rho(\sigma_2 \otimes \sigma_2)\rho^*(\rho_2 \otimes \rho_2).
\]
(5.70)
Since concurrence is invariant under local unitary transformations [160] we are justified in calculating the concurrence of $\hat{\rho}$ instead of $\rho$. However, for completeness we provide the procedure for going from $\hat{\rho}$ back to $\rho$ in Appendix B.

We initialise the system into the maximally entangled Bell state (5.46) which then evolves according to equation (5.49). The time-dependent system density matrix is:

$$\hat{\rho}(\tau) = \frac{1}{4} \left\{ \sigma_0 \otimes \sigma_0 + e^{-\Gamma_2 t} \sigma_1 \otimes \sigma_1 - e^{-\Gamma_2 t} \sigma_2 \otimes \sigma_2 
+ e^{-\Gamma_1 t} \sigma_3 \otimes \sigma_3 + \tanh \left( \frac{c\pi \Delta}{a\hbar} \right) \left( 1 - e^{-\Gamma_1 t} \right) \sigma_3 \otimes \sigma_0 \right\},$$

(5.71)

where we have defined $\Gamma_1 = T_1^{-1}$ and $\Gamma_2 = T_2^{-1}$. One observes from equation (5.71) that $\hat{\rho}(\tau) = \hat{\rho}^*(\tau)$, and since the eigenvalues of $M$ are real and positive [160] we can find the concurrence of $\hat{\rho}(\tau)$ by diagonalising the matrix:

$$\hat{\rho} \sigma_2 \otimes \sigma_2,$$

(5.72)

where we take the $\{\lambda_i\}$ in equation (5.69) to be the absolute values of the eigenvalues of the matrix (5.72). We find that the concurrence is given by:

$$C(a, \tau) = \max \left\{ e^{-\tau \Gamma_2} \frac{1}{2} \left( 1 - e^{-\tau \Gamma_1} \right) \text{sech} \left( \frac{c\pi \Delta}{a\hbar} \right), 0 \right\}.$$

(5.73)

The concurrence is plotted in figure 5.4 as a function of the acceleration and the proper time. We observe that the greater the acceleration the quicker that the initial entanglement disappears. The time taken for the system to reach zero concurrence can be clearly seen by the non-smooth transition between the flat ($C = 0$) plane and the curved surface. We will use this line of vanishing concurrence to quantify the dependence of disentanglement on acceleration. For a given acceleration the time taken, $\tau_0$, for the system to completely disentangle is given by the equation:

$$e^{-m \Gamma_2} = \frac{1}{2} \left( 1 - e^{-m \Gamma_1} \right) \text{sech} \left( \frac{c\pi \Delta}{a\hbar} \right),$$

(5.74)

which follows from equation (5.73). In practice one would observe a loss of
5.5 Disentanglement of Spin

Figure 5.4: We have plotted the concurrence of two initially maximally entangled spins as a function of the acceleration, $a$, (in units of $\frac{a}{\gamma}$) and the proper time (in units of $\gamma^{-1}$). Our main result, equation (5.76), is also shown overlaid in a red line ($C' = 0$).

Concurrence before this time, however, defining the timescale in this way is preferred since it is meaningful to speak of the time taken for the spins to completely disentangle from each other (as opposed to the time taken to reach some arbitrary value of concurrence).

Defining the dimensionless parameter $\alpha \equiv \frac{a\hbar}{c\Delta}$ and taking the limit $\alpha \gg 1$ (i.e., $\Delta \ll \frac{a\hbar}{c}$) we find,

$$\Gamma_1 = \Gamma_2 = \frac{8\mu^2a^3}{3\pi\hbar c^6} + O(\alpha).$$

Equation (5.74) can now be solved for $\tau_0$:

$$\tau_0 = \frac{3\pi \log 3}{8} \frac{\hbar c^6}{\mu^2 a^3} + O(\alpha^{-5}).$$

Thus in the large acceleration small magnetic field limit the proper time taken to disentangle the two spinors is proportional to the inverse of the acceleration.
We have used this result to plot the zero concurrence (red) line in figure 5.4.

In the frame of the stationary spinor the time taken to disentangle is exponentially longer:

\[ t_0 = \frac{c}{2a} \exp \left( \frac{3\pi \log 3}{8} \frac{\hbar c^5}{\mu^2 a^2} \right). \tag{5.77} \]

If we put in the numbers for an electron we find:

\[ t_0 = \frac{c}{2a} \exp \left( \frac{3.8 \times 10^{61} \text{m}^2 \cdot \text{s}^{-4}}{a^2} \right). \tag{5.78} \]

Therefore accelerations of magnitude in the high twenties are required to observe the disentanglement on reasonable timescales. This is consistent with the thermal equilibrium timescales found in [156].

5.6 Comments

We have looked at the system of two entangled spins when one is accelerated and placed under a constant magnetic field whilst the other is at rest and isolated. Our method consisted of explicitly calculating the open quantum system where the quantised magnetic fluctuations were considered to be an unobserved environment. This generalises the linearly accelerated single electron case considered by Bell and Leniass over 25 years ago. For the first time we have found analytic expressions for the \( T_2 \) (equation (5.66)), the concurrence (equation (5.73)) and the entanglement lifetime (equation 5.77) of this system.

This system could also be reinterpreted, using equations (5.2) and (5.3), into the physical scenario of two spins near the event horizon of a BH [163], where one spin is taken some depth within the BH (but outside it’s horizon) and held at a fixed distance from its centre by a locally constant electric field.

\footnote{We note for completeness that the time taken for the concurrence to decay to the value \( C \), is given by \( \tau_C = \frac{3\pi}{8} \log \left( \frac{3}{2e+1} \right) \frac{\hbar c^3}{\mu^2 a^2}. \)
and under a locally constant magnetic field, while the second spinor is placed in free fall. Thus, this system also sheds some light into the nature of quantum information in gravitational systems; in this case the quantum correlations are destroyed more rapidly by the acceleration. However, the correlations still exist in the external magnetic field and therefore unitarity in the total system is not violated. One wonders if the model we have presented here can also be extended to include interactions with an external gravity field and if this can restore the entanglement that is apparently lost in [39].
CHAPTER 6

Conclusion

This thesis has explored several aspects of the addition of extra large dimensions including BHs, orbifolds and split fermions. Also studied was the nature of quantum information within accelerating systems.

Chapter 1 gave a brief introduction to several of the fields that featured prominently in the main body of work. We introduced extra large dimensions, orbifolds, split fermions and BHs and gave some discussion about the motivations of, and problems contained within, these fields themselves.

Chapter 2 investigated the effective field theoretic model for quantum BHs operating at the TeV scale. These are the processes that under certain assumptions occur when the BH has evaporated the majority of its mass in Hawking radiation and entered into the so called Planck phase.

We briefly explained how various effective BH Lagrangians which had previously featured in the literature could be collectively understood as arising from non-renormalisable dimension five operators that respected the SM symmetries. Furthermore, we calculated the corrections that the muon magnetic moment receives from these additional BH fields and placed bounds on the
couplings of the theory. It was shown that the measured deviation between SM and experiment would need to improve by roughly an order of magnitude before some of the parameter space for these processes could be ruled out.

Chapter 3 considered the quasi-normal modes and power emission of fermions within a $d$-dimensional spacetime. The analysis was performed using the WKBJ method. The quasi-normal modes were calculated and may have some use in string theory and loop quantum gravity.

Contrary to the usual claim, we have found that for $d > 5$, fermion emission is greater into the bulk than into the brane. This is pertinent for supersymmetric split fermion models, since a knowledge of the missing energy off the brane allows for distinction between competing models.

Chapter 4 looked at the QLLR model orbifolded in five dimensions. The model was shown to require less scalar degrees of freedom than that of the four dimensional case. This of course is the result of using an orbifold to achieve some of the symmetry breaking.

Due to the imposed discrete symmetries, phenomenologically incorrect mass relationships existed in the basic theory. Yet these were naturally dealt with by separating the fermions along the extra dimension and this maintained more predictivity than was possible in the four dimensional solution to this problem. Furthermore, a configuration was found that reproduced approximately the masses for a one generational SM and also suppressed the proton decay.

The model predicts two neutral bosons and a coloured liptonic hadron sector at low energies. The extra neutral boson can in principle distinguish this theory from the four dimensional version.

Chapter 5 furthers the study of BHs by investigating the information paradox. One manifestation of this problem is seen in the nature of quantum information under acceleration. Using an open quantum system we calculated the
time dependence of the concurrence between two maximally entangled electron spins with one accelerated uniformly in the presence of a constant magnetic field and the other at rest and isolated from fields. We found that at high Rindler temperature the proper time for the entanglement to be extinguished is proportional to the inverse of the acceleration cubed.

Our approach had consequences for entangled spinors near the event horizon of a BH \cite{163} and could possibly be used as a tool for understanding the apparent loss of entanglement in other systems \cite{39}.

The greatest function that large extra dimensions serve is to resolve the Hierarchy problem, which inherently arises from the extremely high energy scale of gravity. Furthermore, the effective field theory model described in chapter 2 and the information paradox discussed in chapter 1, both are due to the absence of a clear formulation of quantum gravity. Although the physics community today is mostly divided between string theory and loop quantum gravity, neither of these models is without their own problems; it is safe to say that no completely accepted quantum theory of gravity exists. Such a theory would need to survive experimental scrutiny and unfortunately the energy scales required for these theories are typically beyond our reach.

What is appealing of the extra large dimensions scenario is that it is testable. When the LHC comes online and has collected enough data, the theory of extra large dimensions will either be promoted to legendary status, or relegated, like so many other great theories of the past, to the ideas waste bin.

New discoveries in the growing field of relativistic quantum information theory, like those relating to entanglement under accelerated motion, can only shed more light on this complex issue of the quantum nature of gravitation. The appeal again being that this field has some reasonable prospect of being tested.
The ultimate objective of this thesis was to gain some further insight into the fascinating interplay between quantum mechanics and gravitation. By way of a tortuous journey through BHs and higher dimensions we have found several new results that may aid experimentation. The true workings of nature for now rests hidden from our scientific eyes and it remains for further experiments to be performed before the validity of our ideas can be evaluated. Until then, as theorists, we are free to speculate on the inner workings of the world around us, in the hope that through logical deduction, physical intuition and inspiration we will unravel some of the mystery.
APPENDIX A

Coupling constant relations

Suppose we have a charge operator defined in terms of a linear direct sum of several diagonal generators

\[ Q = aA + bB + cC + \cdots, \]

then the part of the covariant derivative containing these generators is

\[ D_\mu = \partial_\mu - ig_AW_AA - ig_BW_BB - ig_CW_CC + \cdots. \] (A.1)

Yet

\[ D_\mu = \partial_\mu - ieW_QQ, \]
\[ = \partial_\mu - ieaW_QA - iebW_QB - iecW_QC + \cdots. \] (A.2)

The \( W_{A,B,C,\cdots} \) fields are an orthogonal mixture of the \( W_{Q,Z_1,Z_2,\cdots} \) fields

\[
\begin{pmatrix}
W_A \\
W_B \\
W_C \\
\vdots
\end{pmatrix}
= R
\begin{pmatrix}
W_Q \\
W_Z \\
W_{Z'} \\
\vdots
\end{pmatrix},
\]
where \( R = (v_1|v_2|v_3| \cdots) \), and \( v_i \cdot v_i = 1 \). So if we write \( v_1^* = (\alpha, \beta, \gamma, \cdots) \) then

\[
\begin{align*}
W_A &= \alpha W_Q + \cdots, \\
W_B &= \beta W_Q + \cdots, \\
W_C &= \gamma W_Q + \cdots,
\end{align*}
\]

and we can compare equations (A.1) and (A.2) to find

\[
e = \frac{g_A}{a} \alpha, \quad \frac{g_B}{b} \beta = \frac{g_C}{c} \gamma, \cdots,
\]

and therefore

\[
\frac{1}{e^2} = \frac{a^2}{g_A^2} + \frac{b^2}{g_B^2} + \frac{c^2}{g_C^2} + \cdots \tag{A.3}
\]

since \( \alpha^2 + \beta^2 + \gamma^2 + \cdots = 1 \).

---

\(^1\)It is comforting to know that these relationships do not depend on the normalisation of the representation, \( \text{Tr} T^a T^b = k \delta^{ab} \), i.e., consider \( A = k A' \) then the \( A \) terms in equations (A.1) and (A.2) become \( ig_A \alpha k W_A A' \), and \( i e k W_A A' \), thus \( e = g_A \alpha / a \) is unchanged.
APPENDIX B

Interaction to Schrödinger picture

Here we outline the transformation that converts \( \rho \) (in chapter 5) from the interaction picture \( \tilde{\rho} \) back to the Schrödinger picture \( \rho \), see equation (5.42). We observe that

\[
\sum \tilde{r}_{ij} \sigma_i \otimes \sigma_j = \sum r_{ij} \tilde{\sigma}_i \otimes \sigma_j
\]

where

\[
\tilde{\sigma}_i = e^{i\Delta' \tau_3/2} \sigma_i e^{-i\Delta' \tau_3/2}
\]

where \( H' \equiv -\Delta' \sigma_3 \otimes 1 \). As \( \sigma_3 \) commutes with \( 1 \) and itself, \( \tilde{r}_{0j} = r_{0j} \) and \( \tilde{r}_{3j} = r_{3j} \), however the \( \sigma_1 \) and \( \sigma_2 \) matrices transform into themselves, and after some manipulation we find

\[
\tilde{r}_{1j} = r_{1j} \cos \Delta' \tau/\hbar + r_{2j} \sin \Delta' \tau/\hbar;
\]

\[
\tilde{r}_{2j} = -r_{1j} \sin \Delta' \tau/\hbar + r_{2j} \cos \Delta' \tau/\hbar.
\]
BIBLIOGRAPHY


