Stochastic Routing Models in Sensor Networks

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Abstract

Sensor networks are an evolving technology that promise numerous applications. The random and dynamic structure of sensor networks has motivated the suggestion of greedy data-routing algorithms.

In this thesis stochastic models are developed to study the advancement of messages under greedy routing in sensor networks. A model framework that is based on homogeneous spatial Poisson processes is formulated and examined to give a better understanding of the stochastic dependencies arising in the system. The effects of the model assumptions and the inherent dependencies are discussed and analyzed. A simple power-saving sleep scheme is included, and its effects on the local node density are addressed to reveal that it reduces one of the dependencies in the model.

Single hop expressions describing the advancement of messages are derived, and asymptotic expressions for the hop length moments are obtained. A Kullback-Leibler analysis is applied to quantify how the hop distributions vary with respect to the sink distance. Expressions for the distribution of the multihop advancement of messages are derived. These expressions involve high-dimensional integrals, which are evaluated with quasi-Monte Carlo integration methods. An importance sampling function is derived to speed up the quasi-Monte Carlo methods. The subsequent results agree extremely well with those obtained via routing simulations. A renewal process model is proposed to model multihop advancements, and is justified under certain assumptions.

The model framework is extended by incorporating a spatially dependent density, which is inversely proportional to the sink distance. The aim of this extension is to demonstrate that an inhomogeneous Poisson process can be used to model a sensor network with spatially dependent node density. Elliptic integrals and asymptotic approximations are used to describe the random behaviour of hops. The final model extension entails including random transmission radii, the effects of which are discussed and analyzed. The thesis is concluded by giving future research tasks and directions.
Declaration

This is to certify that

1. The thesis comprises only my original work towards the PhD except where indicated in the Preface.

2. Due acknowledgement has been made in the text to all other material used.

3. The thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Holger Paul Keeler
Preface

The research presented here, unless stated otherwise, is original to the best of my knowledge. Furthermore, it has not been submitted for any other degree.

Chapter 1 gives an introduction to sensor networks and the problem of modelling greedy routing algorithms in stochastic settings. Relevant and related literature is briefly reviewed. This chapter serves as a motivating force for the rest of the thesis.

Chapter 2 lightly covers some of the main mathematical techniques and results used throughout this thesis. Consequently, more comprehensive references are recommended herein.

Chapter 3 introduces the reader to the main set of model assumptions used in this thesis. Issues and considerations resulting from the inherent dependencies of the model are discussed and studied. These dependencies give rise to two distinct models, which feature heavily in the ensuing chapters. Message advancement up to three hops is covered, and the two models are compared. The work found here forms the majority of a paper by H. P. Keeler and P. G. Taylor, which is titled *A stochastic analysis of a greedy routing scheme in sensor networks*, and has been published in SIAM Journal on Applied Mathematics.

Chapter 4 extends the model to examine the $n$-hop case. A simple energy-saving scheme is included and the effects it has on the local node density is studied. High-dimensional integration is performed with quasi-Monte Carlo methods. This work has been submitted as *A model framework for greedy routing in a sensor network with a stochastic power scheme* by H. P. Keeler and P. G. Taylor.

Chapter 5 further extends the model to examine a specific case of spatially dependent node density. Methods from the previous two chapters are adapted and used to shed light on this new setting. This work serves as the material in a submitted manuscript *A stochastic analysis of greedy routing in a spatially dependent sensor network* by H. P. Keeler.

Chapter 6 returns the reader to the constant node density model and extends the model by including randomly distributed transmission radii for the sensor nodes. Subsequent communications assumptions are considered, which result in two random radius models. These models are examined and compared. This work with some additional research will lead to a manuscript by H. P. Keeler and P. G. Taylor.

Chapter 7 summarizes the results and techniques found throughout the thesis. Future research tasks and directions are given with motivating reasons.
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A grateful nod goes to my office mates for tolerating the racket from my number-crunching computer (and from me) over the years and for creating an enjoyable office environment. A very warm thanks to my friends and fellow students who have greatly enriched my lasting Melbourne experience. Many a board game session, squash game, cryptic crossword, trivia night, geography-fact-listing afternoon, or Friday drinks have blessed me with an abundance of wonderful experiences, memories and arcane pieces of knowledge (and the simple fact that only an amateur would forget a microstate nation). The combined experiences from both my old and new friends have made my decision to pursue this project so much more rewarding.

Finally, sincere thanks and acknowledgement goes to my family for their support over the years, and for inspiring me in general to further my knowledge and understanding of the world.
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Chapter 1

Introduction

1.1 Sensor networks

Advances in communications and computing have resulted in the evolving technology of wireless sensor networks. Sensor networks consist of electronic sensing devices known as sensor nodes. The nodes are positioned over an area known as a sensor field to gather environmental information (as shown in Fig. 1.1). Each node has the ability to collect and process environmental data within its sensing range, and to communicate with other nodes within its transmission range. The collected data is ultimately relayed, often via surrounding nodes, to a main station known as a sink.

The applications of sensor networks are valuable and diverse, and they include security and traffic surveillance, environmental and animal monitoring, natural disaster warning and analysis, and building and structure assessment [3, 16, 76]. The continuing reduction in cost and size of electronic components has resulted in sensor networks being more feasible and readily available with high node numbers. The existence and function of sensor networks will play a major part in our future. However, many challenges are to be met in developing sensor networks. Specifically, their performance will need to be quantified and modelled, thus helping to design more efficient sensor networks, and to implement and operate them accordingly.

Sensor networks are different from other networks in some key aspects. The surrounding environment may be adverse, causing node failure, and remote, removing the possibility of repair. Sensor nodes are often employed in large numbers, possibly resulting in dense networks. Node deployment can be in an unordered manner such as aerial dispersal. Nodes have restricted computing power and memory, depending on their purpose and type. These network properties lead
to the fact that sensor networks need to be localized, distributed, and scalable in their computing and coordinating [3, 16].

Furthermore, sensor nodes will have limited power sources, usually batteries, although gaining energy from the environment may be possible in some situations. Hence, the transmission radius of a node will be restricted, resulting in data messages needing to travel multihop paths back to the sink. Power consumption of sensor networks is a primary concern. A large amount of research has been done with the aim of reducing power consumption, and thus, lengthening the lifetime of sensor networks. Power consumption can be broadly classified into sensing, communication, and data processing [3]. Consequently, sensor network components and behaviour, such as circuits, processor architectures and communication protocols, are being designed to be more energy efficient [66]. Another option is to implement sleep schemes in which some subset of the nodes switch to a low energy-consuming sleep state, thus reducing the overall power consumption. These power schemes could be coordinated or uncoordinated among nodes, deterministic or stochastic, responding to certain events, or some combination of these.

![Sensor Network Diagram]

Figure 1.1: Data messages in sensor networks often travel via surrounding nodes to the communication sink.

Nodes will only be able to relay messages when they are in their awake state. Consequently, the topology of a sensor network is continually changing. The usual data routing approaches, such as routing addresses and tables are not suitable, owing to the limited memory of nodes and their large numbers. Hence, sensor networks are an application subclass of ad hoc networks, and routing methods in such networks need to adapt in real time. Indeed, the algorithms and technologies being applied to sensor networks are often used in other network applications, such as mobile ad hoc networks and mesh networks, and may have been originally proposed for those networks [54, 4]. Consequently, although the work presented in this thesis was originally driven by want of a better understanding of sensor network performance, it may be applied to other...
wireless network technologies.

Researchers are constantly developing and improving on a large number of routing algorithms in sensor networks, as indicated by various survey papers [5, 22, 71]. These methods are designed with the aim of optimizing some performance metric such as minimizing power consumption or maximizing the likelihood of data arrival. The ability of a sensor network to relay data from a source node to the sink may be referred to as its connectivity.

A commonly employed routing type in sensor networks, and ad hoc networks in general, is geometric or position-based routing [22, 54, 71]. The operation of these algorithms hinges upon each node knowing its geographical location in relation to the sink, and the location of neighbouring nodes within its transmission radius. Regardless of routing, such assumptions are usually needed to give the collected data geographical meaning. These assumptions are reasonable when sensor nodes have a global positioning system [16]. Alternatively, algorithms are being developed for nodes to calculate their geographical positions [13, 37] and to discover their neighbours [53, 77], thus enabling geographic routing, or close approximations, in sensor networks.

The appeal of geometric routing is its localized nature and, hence, its scalability. In fact, Stojmenovic [71] argues that it is likely that only geometric routing can offer a truly scalable solution to routing in ad hoc networks and, in particular, sensor networks. An intuitive approach for geometric routing is to have the source node forward the message to the node that is geographically the closest to the sink, and repeat this step until the message finally reaches the target sink. This approach can serve either as a simple greedy routing method by itself or as a basis for more intelligent routing methods in various wireless networks [12, 19, 21, 29, 40, 80]. The popularity of greedy routing owes to its simplicity and scalability. An additional advantage is that greedy routing is loop-free; it is not possible for a message to cycle around a set of nodes with limited memory indefinitely, unlike other routing methods [72].

Arguably, systems that have the complexity of sensor networks are best modelled as random. The unordered deployment of nodes and the inclusion of stochastic energy-saving schemes (for an example, see [18]) adds two more layers of randomness to the problem of examining sensor network performance. Many immediate questions arise from investigating this stochastic system: What is the probability that a message sent by a node will reach the sink? How do the locations of previous nodes influence the next hop? How does a power scheme affect message propagation? How densely should nodes be deployed to ensure a respectable message delivery rate? The answers to such questions give insight; sensor network abilities need to be described in probabilistic terms. We let the nature of these questions serve as the main motivation behind this thesis. We an-
analyze geometric routing, with a focus on greedy routing in multihop instances, as it is frequently used, but has had relatively little stochastic analysis applied to it. Furthermore, we examine the impact of a stochastic energy-saving sleep scheme on routing since power consumption is a major issue for sensor networks.

### 1.2 Brief history

We briefly outline the research history of greedy, and similar, routing. The intuitive approach of greedy routing, as a routing algorithm or as a mathematical model, suggests that it may be found in research fields outside of wireless networks. However, we discuss greedy routing in the context of wireless ad hoc networks and, more specifically, sensor networks.

It is commonly written [22, 40, 71] that the first greedy routing algorithm was proposed by Finn in a technical report [21], although no mathematical analysis relevant to our interests was given. However, a similar approach, *Most Forward within Radius*, was proposed earlier by Takagi and Kleinrock [74] to be used in packet radio networks. The MFR method entails having the forwarding node send the message to the node that maximizes forward progress. The corresponding probability distribution for hop advancement is simpler than that of the greedy approach. We will discuss how MFR is a limiting case of greedy routing, and, quantify the observation that nodes chosen by the two different methods can often coincide. Kleinrock and Silvester [42] observed this in an earlier paper, and stated, without showing numerical evidence, that the difference in the two approaches is negligible. In both papers, the routing methods allow for negative progress; a message can be relayed away from the target [42, 74]. The expectation of the first hop progress is given as an integral expression, under the assumption that communication nodes are scattered uniformly over a plane. In subsequent work, the message progresses only forward. For example, the work by Hou and Li [35] in which MFR is compared with other routing suggestions, but only first hop analysis is applied to these methods.

Intuitively, there is a positive probability that a message reaches a node that has no neighbouring nodes closer to the sink, hence the message fails to reach the sink. A popular solution to this problem is face routing where the message is relayed around the routing void by employing the righthand rule used in blind maze traversal. This recovery technique was first coupled with a different geometric routing method, called *compass routing*, by Kranakis, Singh and Urrutia [46]. It was combined with greedy routing by Bose et al. [12], and later a very similar method was independently proposed by Karp and Kung [40]. The use of the righthand rule requires node links to
form a planar graph, and this process requires further energy and computation in detecting and removing cross-links, though this may be reduced with more efficient techniques [30]. Regardless, these methods, assuming overall connectivity, give guaranteed message delivery [12], and hence greedy-based routing is receiving considerable attention. However, the majority of these schemes have been investigated by routing simulation.

Furthermore, where stochastic analysis has been applied, it has been mostly used for the single hop problem, and not the multihop problem. An exception is the analysis of the Geographic Random Forwarding or GeRaF scheme, which is effectively a simple greedy routing method proposed by Zorzi and Rao [80]. They suggested multihop upper and lower bounds for the average number of hops required to reach the sink. The bounds are based on the first hop at two different sink distances, and Zorzi and Rao supported their bounds well with simulation results. However, they assume that after each hop a new set of nodes is independently created, due to a power scheme, and state that the message will reach the sink with probability one. We do not think that this assumption is entirely reasonable, and we will detail our reasoning and solution at the relevant point.

1.3 Related work

Stochastic processes have been inherently linked to modelling communication systems during their history. There are a number of different techniques that can be used to examine wireless networks under stochastic assumptions. This research field has been gathering steady attention in recent years. Indeed, there is a recent special issue [32] on stochastic geometry (and related fields) in ad hoc networks, as well as a two-volume monograph by Baccelli and Blaszczyszyn [6, 7]. We briefly describe some related work in analyzing the stochastic performance of wireless networks, and refer the reader to the aforementioned references.

Perhaps the first application of stochastic geometry in a wireless network setting was by Kleinrock and Silvester [42] who used it to examine connectivity in packet radio networks in 1978. Stochastic analysis of these networks continued over the years, and arguably laid the foundations for the next research stage. More recently, the focus of stochastic analysis has shifted towards modelling wireless ad hoc networks.

A closely related concept to connectivity is the sensing coverage of a sensor network. Sensing coverage is the ability of a sensor network to successfully sense or cover the entire sensor field. Coverage stems from each node having a finite sensing or sensitivity range. Similar techniques
and mathematical analysis have been used to shed light on both the coverage and connectivity in sensor networks.

One particular set of approaches has involved analyzing the coverage and connectivity of infinite node networks. A popular tool in this area has been continuum percolation; see the text by Meester and Roy [55], and the recent thesis by Dousse [17]. This discipline appears to have started in an applied network setting by Gilbert [27] who adapted random graphs on the infinite plane. A related and emerging field is the study of random geometric graphs on finite planes, which often uses techniques from stochastic geometry and combinatorics; see the monograph by Penrose [63].

Alternatively, coverage in sensor networks has been recently examined by Lazos and Poovendran [52] from the perspective of integral geometry, a field related to stochastic geometry. They reformulated the coverage problem into a set intersection problem, and subsequently examined a finite number of sensor nodes with arbitrary coverage range on a planar sensor field of any bounded shape.

To examine routing methods, some recent and novel asymptotic results have been obtained in the purely geometric setting in which transmission radii are not present. Baccelli and Bordenave [10] proposed a mathematical object called a radial spanning tree, which is a natural basis for greedy routing. They built this tree on a realization of a homogeneous Poisson point process, and analyzed it both locally and globally. There is contrast between this research and the work covered here, which relies heavily on the nodes having transmission radii. Research on radial spanning trees should lead to some interesting applications, but is beyond the scope of the work here; see the thesis by Bordenave [11] for an introduction.

In analyzing wireless networks, signal interference from surrounding nodes is a significant factor. Based on information-theoretic arguments, the ratio of the transmission signal to interference and noise gives an upper bound to how much data can be sent across a wireless communication link. This influences the effective transmission radius of each sensor node. Although this thesis only lightly touches on this aspect, we mention recent work that uses stochastic geometry to shed light on modelling signal interference. For example, see the thesis by Ganti [24] as well as the aforementioned special issue [32].

Reducing the overall energy-consumption and maximizing the lifetime of sensor networks is an essential goal due to the limited power sources. Consequently, there have been numerous power schemes and energy-saving routing methods proposed. One that is particularly relevant to our stochastic analysis was proposed by Dousse et al. [18]. Their scheme entailed uniformly scattered
nodes randomly falling in and out of sleep states independently of each other, thus resulting in an uncoordinated blinking sleep scheme. Under this scheme, dynamic percolation theory was used to examine message-relaying latency issues in sensor networks.

In summary, the stochastic analysis and modelling of ad hoc networks is an emerging field. Any further relevant results and solution methods not listed in the last two sections will be discussed in future when appropriate.

1.4 Overview

The main focus of this thesis is to examine and study greedy routing methods in sensor networks, and to display the effects of a stochastic power scheme. To achieve this, we use methods from stochastic geometry, numerical integration, and general mathematical modelling. We support our results with computer simulations where feasible. Ideally, the mathematical framework presented in this thesis will serve future research efforts, and may be modified and extended to capture the stochastic behaviour of other geometric routing methods and power schemes.

The work behind this thesis has been an evolution of ideas and methods with the aim to understand and describe in probabilistic terms the performance of greedy routing and its ability to deliver data. This thesis is written in a manner reflecting the progressive development of our solution approach. We start with simple assumptions and gradually remove them, thus revealing and elucidating a more complex problem. We conclude by discussing possible future problems and model extensions.

The thesis is structured as follows:

- Chapter 2 lists and describes briefly the various mathematical techniques and theory used throughout this thesis. The field of stochastic geometry and in particular Poisson processes are introduced with an emphasis on modelling and simulation applicability. The rest of the chapter covers integration methods and special functions. In particular, asymptotic integral methods in one-dimension, elliptic integral functions, and Monte Carlo methods in high dimensions are covered in necessary detail with references provided therein.

- Chapter 3 is primarily concerned with laying the preliminary mathematical framework that will be used and adapted throughout this thesis. The main model assumptions, such as constant node density and transmission radii, are outlined and the problem of message routing is presented as a tractable mathematical model. Asymptotic methods are developed to give closed-form moment expressions. Two dependencies that arise in the model are
detailed and their effects on modelling message propagation are considered, thus resulting in two distinct greedy routing models. Under the two models, message advancement after two relays is quantified and coupled with proposed inequalities relating the models.

- Chapter 4 progresses to cover multiple hop propagation, which involves calculating high-dimensional integrals. Monte Carlo integration methods are detailed and applied with an emphasis on using quasi-random points. These integration methods are sped up by way of importance sampling. A simple but insightful energy-saving sleep scheme is included to see its effects on message routing. In particular, it is revealed that in a certain parameter range of the sleep scheme the two routing models closely resemble each other, which reduces computation time. Some simple heuristics are given to indicate how many relay attempts a node should make before assuming there are no nodes within its total feasible region.

- Chapter 5 is focused on a spatially dependent node deployment model. Under this model extension, analysis and techniques akin to those in the previous two chapters are successfully applied to validate the evolving mathematical model framework used in this thesis. Analytic solutions involving elliptic integral functions are compared to more amenable approximations. Quasi-Monte Carlo methods involving Halton points and lattice rules are used to evaluate high-dimensional integrals. Future spatially dependent models and their modelling considerations are discussed.

- Chapter 6 brings attention back to the original constant node density model, and extends it to include randomly distributed transmission radii. Motivation and model considerations are presented. The rest of the chapter is divided to cover two different random radius models. The characteristics of the respective two models are outlined and their mathematical tractability is examined. Asymptotic and Monte Carlo methods akin to those used in previous chapters are applied to the simpler of the two models. Simulations are used to examine the more mathematically involved model. Possible reasons are given to explain observed stochastic behaviour and future models and techniques are outlined.

- Chapter 7 details the conclusions of the previous three chapters. Important and insightful results and discoveries are emphasized. Finally, future research directions in sensor networks and their routing methods are covered with a focus on stochastic modelling.
1.5 Note on terms

Some terms used in this thesis may be ambiguous, particularly those that have different meanings depending whether it is in a sensor network or probability context. To reduce confusion, we briefly discuss our term definitions, and henceforth, use them consistently in the appropriate context.

The term distribution will only be used to refer to probability distribution, and not the physical placement, positioning or deployment of sensor nodes. We use density for node density, and probability density will denote the derivative, if it exists, of a distribution.

We use the terms relay and forward to denote a message being passed by one node to another directly. We use neighbour or neighbouring nodes, in relation to some forwarding node, to describe nodes within its transmission radius (or its range).

Finally, we note that we use the term greedy in a specific sense to refer to the routing method’s aim to minimize the distance from the forwarding node to the sink. For the greedy routing examined here, other terms such as radial or best hop routing may be used in the literature, while the term greedy is used to denote any routing method which seeks to optimize some distance when choosing the next forwarding node.

1.6 Summary of Contributions

In Chapter 3 we describe the initial model and analyze the single hop setting, thus giving the following contributions:

- Introduction of a mathematical framework for modelling greedy routing that is based on spatial Poisson processes;
- Identification and analysis of two dependencies that affect the advancement of messages, thus leading to the proposal of two distinct models;
- Application of Laplace’s method to give closed-form and accurate approximations to moments of single hop advancements under greedy routing;
- Derivation and conjecture of stochastic inequalities for single and multiple message hops, which are supported with numerical evidence;
- A fast and accurate approximation, which is based on renewal processes, is derived to give bounds for the multihop advancement of messages under greedy routing.
In Chapter 4, the previous work is extended to give contributions in the multihop setting, namely:

- Integral expressions for the distributions of multihop advancement under the two respective models;

- Application of results from quasi-Monte Carlo theory to give a relatively fast and accurate approach to estimate aforementioned integrals;

- Suggestion of a simple sleep scheme, which was analyzed to reveal that the sleep scheme reduces one of the dependencies, thus resulting in the more complex model closely resembling the other;

- Probability inequalities are given, which offer a rough guide on how many relay attempts a sensor node should make before deciding there are no nodes within its total feasible region.

In Chapter 5, the theory and techniques used thus far are adapted to examine the case of nonconstant node density. This is achieved by proposing a viable node density model that is spatially dependent, which leads to expressions involving elliptic integral functions. Under this setting, asymptotic methods are used to derive approximations for single hop moments. Recent results from the theory of lattice rules are used to estimate the integrals that arise in the multihop distributions.

In Chapter 6, under the constant node density assumption, a new model is proposed in which the transmission radii of the sensor nodes are random, hence:

- Model considerations and assumptions are outlined and discussed, which leads to the proposal of two random radius models;

- Immediate stochastic effects and the mathematical tractability of the models are examined;

- Asymptotic and numerical methods are used to give moment expressions and probability distributions respectively;

- Routing simulations are performed to reveal performance differences in the two models, and proposed explanations of the results are given.

The aim of this thesis is not to argue that quasi-Monte Carlo methods are faster in obtaining statistical results than routing simulations per se. Rather, the aim is to show that it is possible to derive expressions for multihop distributions which involve integrals, and if these integrals are
evaluated in some manner, then they give results which agree extremely well with the simulations. Moreover, recent advancements in quasi-MC methods, such as lattice rules, give hope that evaluating these integrals via these methods will be a much faster approach compared to that of simulations.

In summary, this thesis seeks to both outline a mathematical framework for modelling multi-hop advancement of messages and to give solution methods that can be used to accurately and efficiently describe the stochastic behaviour greedy routing. Furthermore, a focus has been placed on observing and quantifying the stochastic effects of sleep schemes in sensor networks.
Chapter 2

Mathematical Methods

Mathematical methods from a number of different fields are used throughout this thesis. The techniques employed include those from the areas of stochastic geometry, special functions, asymptotics, simulation, and Monte Carlo and quasi-Monte Carlo methods. We present an informal summary of relevant results and techniques from these mathematical fields that are used here, and refer the reader to references therein for more detailed information. We note in many instances notation from each section has been influenced by popular resources on the topic. Thus, notation should be treated individually in each section of this chapter.

2.1 Stochastic geometry

We are interested in examining the performance of networks of randomly positioned sensor nodes, and in particular, the interaction nodes have with neighbouring nodes. Consequently, the problem formulations and results presented throughout this thesis hinge largely upon the useful field of stochastic geometry. The results of stochastic geometry are increasingly being applied to problems in numerous scientific and technical disciplines. Although stochastic geometry has a rich theoretical side, we only discuss well-established applied results that are relevant to our problem. In particular, we informally outline key results from the tractable spatial Poisson point process, which will be the underlying stochastic process in our models.

The ensuing details and notation were mainly based on the general reference by Stoyan, Kendall and Mecke [73]. The monograph by Kingman [41] also served as a reliable source on Poisson processes.
2.1.1 Poisson point processes

The most fundamental objects in stochastic geometry are point processes. Informally, a point process can be considered as a random set of points on some space. In modelling applications, the point process is often defined on Euclidean space, henceforth we assume that a point process \( \Phi \) consists of a random set of points \( X_1, X_2, \ldots, X_n \in \mathbb{R}^d \) where \( d \geq 1 \). The order of the points is of no interest.

For a bounded Borel set \( B \subset \mathbb{R}^d \), the point process \( \Phi \) can also be regarded as a locally finite counting measure such that \( \Phi(B) \) gives the number of points falling in \( B \). The mean or intensity measure of a point process is defined as

\[
\Lambda(B) = \mathbb{E}[\Phi(B)].
\]

In other words, \( \Lambda(B) \) is the average number of points of \( \Phi \) lying in \( B \). In general, a point process is said to be stationary if it is invariant under any translation. Moreover, a point process is isotropic if it is invariant under any rotation. Stationarity and isotropy are combined to yield motion invariance.

We now examine the tractable point process known as the spatial Poisson point process, which we denote by \( \Phi \). The Poisson process is the initial cornerstone of developing stochastic models due to its convenient characteristics. The general Poisson process is defined by having the properties:

1. The number of points in a bounded Borel set \( B \) is a Poisson random variable with a mean function \( \Lambda(B) \) such that

\[
P(\Phi(B) = n) = \frac{(\Lambda(B))^n}{n!} e^{-\Lambda(B)}, \quad n = 0, 1, \ldots
\]

2. For any \( k \) bounded and disjoint Borel sets \( B_1, \ldots, B_k \) the number of points of \( \Phi \) lying in each set form \( k \) independent variables.

The Poisson process is sometimes described as completely or purely random due to property (2). This characteristic results in models often being more tractable. Another characteristic of a Poisson process is that it is simple, which entails that with probability one no points of \( \Phi \) coincide with each other.

The mean measure \( \Lambda \) is interesting when it can written be in terms of a density or intensity function \( \lambda(x) \) such that

\[
\Lambda(B) = \int_B \lambda(x)dx, \quad x \in \mathbb{R}^d.
\]
We refer to the positive function $\lambda(x)$ simply as the density, and call $\Lambda(B)$ simply a mean or parameter function. The special case when the density is constant leads to the homogeneous Poisson process with a parameter function $\Lambda(B) = \lambda \nu_d(B)$ where $\nu_d$ is the Lebesgue measure. The homogeneous Poisson process is stationary and isotropic, thus it often serves as first models for networks [42, 80] and spatial phenomena in general [73]. A spatially dependent mean measure results in what is known as an inhomogeneous or nonhomogeneous Poisson process.

As an example, problems are often formulated on the plane $\mathbb{R}^2$. Consequently, under a homogeneous Poisson process the number of points $N = \Phi(B)$ located within some bounded region $B$ of area $A$ leads to the elementary probability mass function

$$P(N = n) = \frac{(\lambda A)^n}{n!} e^{-\lambda A}. \quad (2.3)$$

### 2.1.2 Distances

Distances between points play an essential role in modelling spatial phenomena. A motivating feature of using Poisson processes is their convenient properties regarding distances. We examine this by initially considering the concept of void-probability $v_B$, which gives the probability of no points lying in some set $B$. The void-probability expression of a Poisson process quickly follows

$$P(\Phi(B) = 0) = v_B = e^{-\Lambda(B)}. \quad (2.3)$$

For a bounded convex Borel set $B$ with $\nu_d(B) > 0$, a closely related quantity known as the contact distribution function $H_B(r)$ is defined as

$$H_B(r) = 1 - P(\Phi(rB) = 0), \quad r \geq 0. \quad (2.3)$$

The contact distribution describes the first point encountered when the set $B$ is magnified. For example, it is common to take $B$ to be a unit ball at the origin to give the spherical contact distribution.

Frequently of interest is examining the nearest neighbouring point to some point of $\Phi$. The nearest neighbour distribution is the spherical contact distribution conditioned on a point of $\Phi$ existing at $x$. Fortunately, the completely random nature of the Poisson process implies that conditioning on a point at $x$ does not change the distribution. Consequently, the nearest neighbour distribution is equal to the spherical contact distribution without a point existing at $x$. 

2.1.3 Thinning

The attractive characteristics of Poisson processes result in them being used to construct spatial models via a number of operations. We only discuss two operations known as thinning and superposition, and refer the reader to Stoyan, Kendall and Mecke [73, page 146] for further details on these and other operations.

The thinning operation of a point process entails points being removed based on some rule. One simple rule known as \( p \)-thinning involves deleting a point with probability \( 1 - p \) or retaining a point with probability \( p \). Each point is removed or kept independently of all other points, hence this is a very simple thinning process. Applying this operation to a homogeneous Poison process with density \( \lambda \) gives a thinned homogeneous Poisson process with \( p\lambda \). Moreover, the number of points kept and the number of points removed form two independent Poisson random variables with densities \( p\lambda \) and \( (1 - p)\lambda \) respectively.

The thinning operation is extended by introducing a deterministic probability function \( p(x) \) defined over the space as the point process. Again, a point at \( x \) is deleted with probability \( 1 - p(x) \) or retained with probability \( p(x) \). It follows that applying the \( p(x) \)-thinning operation to a homogeneous Poisson process results in a inhomogeneous Poisson process with a parameter function

\[
\Lambda(B) = \lambda \int_B p(x)dx.
\]

Again, the number of points removed and the number of points kept form two independent Poisson random variables. Further generalizations can be considered such as having the function \( p \) to be random itself. In summary, thinning is a useful operation for constructing models.

2.1.4 Superposition

It may arise that a sensor network consists of two or more different types of nodes. Alternatively, a node may be asleep or awake for different reasons, which induces distinct thinnings and thus mean measures in the model. The superposition of Poisson processes gives a way to model these combinations.

Let \( \Phi_1 \) and \( \Phi_2 \) be two independent Poisson processes with their respective mean measures \( \Lambda_1 \) and \( \Lambda_2 \). A further point process is formed by the union

\[
\Phi = \Phi_1 \cup \Phi_2,
\]

where the points from the two processes do not overlap almost surely. The resulting Poisson
process $\Phi$ has the mean measure

$$\Phi = \Lambda_1 + \Lambda_2.$$ 

### 2.1.5 Simulation

Computer simulation of stochastic point processes is an intuitive and essential means of verifying theoretical results. There are effectively two aspects to simulating Poisson processes. Namely generating a Poisson number of independent points for the sample space, and then assigning each point a coordinate in the sample space. The first step has become straightforward since standard mathematical packages now include Poisson pseudo-random number generators. For small Poisson mean values (less than fifteen), the generator function in Matlab uses a method based on exponentially distributed waiting times. Alternatively, the generator function employs method by Ahrens and Dieter [2] as suggested by Knuth [43, page 137].

Assigning the points coordinates is an easy step under a homogeneous Poisson process. The position of a point given by independent and uniformly distributed random variables. Let the random variables $X_s$ and $Y_s$ be cartesian coordinates of a single point on the unit square, then

$$P(X_s \leq x) = x, \quad P(Y_s \leq y) = y, \quad (x, y) \in [0, 1] \times [0, 1].$$

On the unit disk a single point has the polar coordinates

$$P(R_s \leq r) = r^2, \quad P(\Theta_s \leq \theta) = \frac{\theta}{2\pi}, \quad (r, \theta) \in [0, 1] \times [0, 2\pi].$$

Assigning the coordinates under an inhomogeneous Poisson process varies in difficulty depending on the density function. A tractable density function can result in the closed-form distribution expressions for the coordinate variables. Alternatively, acceptance-rejection methods can be employed. Further details of this task are given in the relevant sections.

### 2.2 One-dimensional integration

Probability and area expressions frequently give rise to one-dimensional integrals. We consider some general evaluation techniques from the fields of special functions and asymptotic methods.

#### 2.2.1 Elliptic integrals

Elliptic integrals often arise in applications since they occur in solutions to integrals of the form

$$\int_x^y R(s, t)dt,$$
where $R$ is a rational function, and $s$ is a square root of a cubic or quartic polynomial in $t$. These integrals can be manipulated using reduction methods found in the standard reference [1]. More specifically, in this thesis we consider the incomplete elliptic integrals of the first and second kind defined in Legendre form

$$F(\phi, k) = \int_{0}^{\phi} \left(1 - k^2 \cos^2 \theta\right)^{-1/2} d\theta,$$

$$E(\phi, k) = \int_{0}^{\phi} \left(1 - k^2 \cos^2 \theta\right)^{1/2} d\theta.$$  

A good number of computer packages have pre-written routines for numerically evaluating elliptic integrals, however, care must be taken as their definitions vary depending on the package (for example, the $k = m^2$ definition may be used).

In recent years it has become standard to use symmetric Carlson [14] elliptic integrals [64]. The elliptic integrals of the first and second kind defined in Carlson form follow

$$R_F(x, y, z) = \frac{1}{2} \int_{0}^{\infty} \left[(t + x)(t + y)(t + z)\right]^{-1/2} dt,$$

$$R_D(x, y, z) = \frac{3}{2} \int_{0}^{\infty} (t + x)^{-1/2}(t + y)^{-1/2}(t + z)^{-3/2} dt,$$

where the square root is taken real and positive if $x$, $y$, and $z$ are positive and varies continuously when these variables become complex; for additional definition details and other elliptic integrals in this form see Carlson [14, 15].

A significant number of functions and integrals can be written in the form of Carlson elliptic integrals, including the incomplete elliptic integrals (2.4) and (2.5), namely

$$F(\phi, k) = R_F(c - 1, c - k^2, c),$$

$$E(\phi, k) = R_F(c - 1, c - k^2, c) - \frac{k^2}{3} R_D(c - 1, c - k^2, c).$$

where $c = 1/\sin^2 \phi$. There exist useful algorithms that consist of elementary function evaluations for quickly calculating Carlson elliptic integrals. We use an easily implementable method by Carlson [15], which works in the complex plane under specified variable and parameter regimes. Subsequently, the relevant results presented here are obtained via our purposely written elliptic integral functions based on the Carlson symmetric form.

2.2.2 Asymptotics

The assumed circular transmission range of sensor nodes implies that inverse trigonometric functions often arise throughout this thesis. Their intractable form invokes the need for more amenable
approximations. Clearly, approximating with various types of power series is easily achieved with modern algebra manipulation packages, and needs no attention. More relevant is evaluating or approximating integrals of the form

\[ I(\lambda) = \int_a^b \phi(x)e^{-\lambda A(x)} dx, \]

where \( \lambda \) is some positive constant. This expression often arises in moment expressions and transforms. We give a short informal account of applying asymptotic means to the above integral. The results given here are based on those found in the texts by Olver [60] and Wong [79].

Laplace observed that the major contribution of the above integral occurs at the maximum of \( e^{-\lambda A(x)} \) when \( \lambda \) is sufficiently large. Assume that the minimum of \( A(x) \) is at \( x_0 \in (a,b) \) such that \( A'(x_0) = 0 \), and \( \phi(x_0) \neq 0 \). Expanding the function \( A(x) \) around \( x_0 \) gives the integral approximation

\[ I(\lambda) \approx \int_a^b \phi(x_0)e^{-\lambda [A(x_0)+(x-x_0)^2 A''(x_0)/2]} dx, \]

and extending the limits of integration gives

\[ I(\lambda) \approx e^{-\lambda A(x_0)} \int_{-\infty}^{\infty} e^{-\lambda(x-x_0)^2 A''(x_0)/2} dx, \]

\[ = e^{-\lambda A(x_0)} \left[ \frac{2\pi}{\lambda A''(x_0)} \right]^{1/2}. \]

The above approach is often known as Laplace’s method or approximation. This result can be made more general and shown to be an asymptotic expression for large \( \lambda \). We make ourselves content by just stating the general result; for details see Olver [60, page 85] or Wong [79, page 58].

We outline some necessary conditions for the asymptotic result. Without a loss of generality, assume the real function \( A(x) \) has one minimum on the interval \([a,b]\) which occurs at \( x = a \). Assume that

\[ A(x) \sim A(a) + \sum_{t=0}^{\infty} a_t(x-a)^{t+\mu}, \]

and

\[ \phi(x) \sim \sum_{t=0}^{\infty} b_t(x-a)^{t+\alpha-1}, \]

as \( t \to a \) from the right, and \( \lambda \) and \( \mu \) are positive constants, where the latter allows the expansion of \( A(x) \) to have a term for \( s = 0 \) without a loss of generality. The constant \( \alpha \) can be real or complex provided that the real part is positive, and it serves the same role as \( \mu \) in the expansion of \( \phi(x) \). Furthermore, assume that the first expansion (2.10) can be differentiated

\[ A'(x) \sim \sum_{t=0}^{\infty} (t+\mu)a_t(x-a)^{t+\mu-1}, \]

(2.12)
as $t \to a$ from the right.

**Theorem 2.2.1.** Assume that $A(x) > A(a)$ for all $x \in (a, b)$; for every $c \in (a, b)$ the infimum of $A(x) - A(a)$ is positive in $[c, b)$; $A'(x)$ and $\phi(x)$ are continuous in a neighbourhood of $a$, except possibly at $a$; the expansions (2.10), (2.11), and (2.12) all hold; and the integral

$$I(\lambda) = \int_a^b \phi(x)e^{-\lambda A(x)}dx,$$

converges absolutely for all sufficiently large $\lambda$. Then

$$I(\lambda) \sim e^{-\lambda A(a)} \sum_{t=0}^{\infty} \Gamma \left( \frac{t + \alpha}{\mu} \right) \frac{c_t}{\lambda^{(t+\alpha)/\mu}}, \quad \text{as} \quad \lambda \to \infty,$$

(2.13)

where $\Gamma(\cdot)$ is the gamma function and the coefficients $c_t$ are defined in terms of $a_t$ and $b_t$.

Although we will only need the value of $c_0$, we give the first two coefficients as

$$c_0 = \frac{b_0}{\mu a_0^{\alpha/\mu}}, \quad c_1 = \left[ \frac{b_1}{\mu} - \frac{(\alpha + 1)a_1 b_0}{\mu^2 a_0} \right] \frac{1}{a_0^{(\alpha+1)/\mu}}.$$

(2.14)

### 2.3 Multi-dimensional integration

The work presented throughout this thesis often involves evaluating high-dimensional integrals. The intractable nature of the integrals eliminates using analytic and asymptotic methods. Moreover, traditional numerical means such as trapezoid or quadrature schemes cannot handle the number of integral dimensions that arise in this work. This motivates us to employ regular Monte Carlo and quasi-Monte Carlo integration methods.

In general, we wish to evaluate integrals over the $s$-dimensional interval or brick

$$\tilde{B}_s = [b_1 - a_1] \times \cdots \times [b_s - a_s] \subseteq \mathbb{R}^s$$

such that

$$I f = \int_{\tilde{B}_s} f(x)dx = \int_{a_s}^{b_s} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_s)dx_1 \cdots dx_s.$$

For simplicity of explaining the integration schemes, the integration domain will be considered on the $s$-dimensional cube $\tilde{C}_s = [0, 1]^s$ such that

$$I f = \int_{\tilde{C}_s} f(x)dx = \int_{0}^{1} \cdots \int_{0}^{1} f(x_1, \ldots, x_s)dx_1 \cdots dx_s.$$

The equivalent results for $\tilde{B}_s$ are quickly obtained and are presented when relevant.
Predominantly, we use the books of Gentle [26] and Liu [51] as general sources on Monte Carlo methods. The brief descriptions and explanations of quasi-random sequences and quasi-Monte Carlo methods presented here are based largely on the article and the thorough book by Nieddereiter [57, 58]. The complementary and more introductory article by Spanier and Maize [70] also served as a good starting point.

### 2.3.1 Monte Carlo integration

We outline the basics of the standard or crude Monte Carlo integration method by first considering the integral of \( f \) on the unit interval

\[
If = \int_0^1 f(x)dx
\]

Let \( X \) be a uniform random variable defined on the sample space \([0, 1]\). It follows that the above integral is simply the expectation of \( f \) under the uniform distribution. Hence, sampling \( N \) uniformly distributed and independent samples \( x_1, \ldots, x_N \in [0, 1] \) leads to the unbiased Monte Carlo estimate of the integral

\[
\hat{I}f = \frac{1}{N} \sum_{n=1}^{N} f(x_n) \quad (2.15)
\]

In \( s \) dimensions the estimate retains its form

\[
\hat{I}f = \frac{1}{N} \sum_{n=1}^{N} f(x_n), \quad x_1, \ldots, x_N \in \bar{O}_s. \quad (2.16)
\]

The equivalent result for \( \bar{B}_s \) follows

\[
\hat{I}f = \frac{1}{N} \prod_{i=1}^{s} (b_i - a_i) \sum_{n=1}^{N} f(x_n), \quad x_1, \ldots, x_N \in \bar{B}_s. \quad (2.17)
\]

Provided that the samples are random and independent, the strong law of large numbers guarantees that this result converges almost surely. Furthermore, the error \(|If - \hat{I}f|\) is estimated by

\[
\frac{\sqrt{V(f)}}{N^{1/2}},
\]

where

\[
V(f) = If^2 - (If)^2,
\]

is the variance of the function \( f \). It follows that the error \(|If - \hat{I}f|\) reduces at the rate \( O(N^{-1/2}) \). The Monte Carlo approach allows for continuous computation until the result is sufficiently converged. This convenience and the independence of dimension leads to the appeal of tackling high-dimensional integrals with Monte Carlo methods.
2.3.2 Quasi-Monte Carlo integration

Despite their random theoretical basis, regular Monte Carlo methods are nearly always performed with pseudo-random number sequences. Commonplace generators use deterministic algorithms, based on number theory results, to produce such sequences. Their purpose is to give the appearance of randomness by passing a number of statistical tests [26]. In recent years, the issue of whether these numbers behave convincingly randomly has been sidestepped with ‘less random’ infinitely long quasi-random sequences. In analogy to the Monte Carlo approach, these purely deterministic sequences are used in the estimate expressions (2.15) to (2.17) to give quasi-Monte Carlo estimates.

Recently, quasi-Monte Carlo methods have gained much interest, particular in financial applications [62, 38, 59], due to their higher speed and accuracy in evaluating high-dimensional integrals compared to regular Monte Carlo methods. Examples of quasi-random number or point sequences include Faure [20], Halton [34], Niederreiter [58], and Sobol [68] sequences. Mathematically, quasi-random sequences have low-discrepancy. Informally, low-discrepancy sequence of points implies that the proportion of points falling in some interval is proportional to the volume of that interval. It follows that two consecutive quasi-random numbers are less likely to be near each other compared to those from regular pseudo-random sequences. Hence, quasi-random points have less clustering than pseudo-random numbers resulting in more evenly distributed sampling over the domain.

We give a more precise definition of discrepancy, and explain how quasi-Monte Carlo methods achieve their accuracy. For more details, we refer the reader to the work of Niederreiter [57, 58].

Discrepancy

We consider the discrepancy of a point set

\[ Q = x_1, x_2, \ldots, x_N \in \bar{C}_s, \]

Let \( C_s = [0, 1)^s \) and let \( J \) be the \( s \)-dimensional subinterval of \( C_s \) such that

\[ J = [v_1, w_1) \times \cdots \times [v_s, w_s) \]

where

\[ 0 \leq v_i \leq w_i \leq 1, \quad 1 \leq i \leq s, \]

and let \( \nu(J) \) be the volume

\[ \nu(J) = \prod_{i=1}^{s} (w_i - v_i). \]
The discrepancy of a point set $Q$ in $C_s$ is defined by

$$D_N(Q) = \sup_J \left| \frac{m(J)}{N} - \nu(J) \right|,$$

where the counting function $m(J)$ gives the number of sequence points lying in $J$. The specific case known as star-discrepancy is defined by

$$D^*_N(Q) = \sup_{J^*} \left| \frac{m(J^*)}{N} - \Lambda(J^*) \right|,$$

where the supremum is taken over all the sets $J^* = [0, w_1) \times \cdots \times [0, w_s)$.

The volume $\nu(J^*)$ is the ideal proportion of points belong to $Q$ that are in $J^*$, and $m(J^*)/n$ is the actual proportion of points in $J^*$. Consequently, star-discrepancy is the supremum of the difference between these two proportions taken over all the intervals $J^*$. The two discrepancies obey the inequality

$$D^*_N \leq D_N \leq 2^s D^*_N.$$

Infinite sequences are generally considered quasi-random when their star-discrepancy behaves such that

$$D_N^* \leq k_s \frac{(\log N)^s}{N} + O \left( \frac{(\log N)^{s-1}}{N} \right), \quad s \geq 2,$$

where $k_s$ is a constant that depends on the particular sequence and the dimension $s$. Such sequences are used to evenly divide a domain and approximate the integral. However, quasi-random sequences are not necessarily needed to evenly divide a domain. For example, on the interval $[0,1]$ the lowest star-discrepancy value $D_N^* = 1/2N$ is achieved by the increasing sequence

$$x_n = \frac{(2n - 1)}{N}, \quad 1 \leq n \leq N,$$

which reduces the Monte Carlo estimate (2.15) to the classic mid-point rule. This finite sequence, however, requires a knowledge of $N$ before computing the sequence begins. In high dimensions, increasing the sampling spacing evenly in all dimensions is computationally exhaustive. Conversely, quasi-random sequences are infinite, which allows for continuous and relatively even sampling without knowing how many samples are needed beforehand. Hence, to increase $N$ from one value to another, only additional samples are required, thus allowing continuous computation of the integral estimate.
Variation of a function

As the name suggests, the variation of a function can be considered as a measure of its variability over the domain. For example, the variation of $f$ on the interval $[a, b]$ is defined by

$$V^b_a(f) = \sup_n \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

where the supremum is taken over all the partitions of the form

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b.$$ 

If the function is continuously differentiable on the interval $[a, b]$, its variation can be written as the integral

$$V^b_a(f) = \int_a^b |f'(x)| dx.$$ 

The variation can be generalized to multiple dimensions in more than one way, but in this context we assume it is in the sense of Hardy and Krause; for a precise definition and more details, see Niederreiter [58, page 19].

Koksma-Hlawka inequality

The Koksma-Hlawka inequality is arguably one of the most important results for quasi-Monte Carlo estimates and integration approximations in general that take the estimate form (2.16). For any sequence of points, the Koksma-Hlawka inequality gives an upper bound to the estimate error in relation to the star-discrepancy of the sequence.

Theorem 2.3.1. If $f$ has bounded variation $V(f)$ on $\hat{C}_s$ in the sense of Hardy and Krause, then for any sequence of points $x_1, x_2, \ldots, x_N \in C_s$, we have the inequality

$$\left| \int_{\hat{C}_s} f(x) dx - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq D_N^*(x_1, x_2, \ldots, x_N) V(f).$$

The Koksma-Hlawka inequality gives a deterministic error between the integral and its approximations (2.15) to (2.17). It follows that the integration error based on quasi-random sequences converges to zero more rapidly than regular Monte Carlo methods, provided the sequence discrepancy is sufficiently low and the function is well-behaved in terms of variation. We elaborate on this in more detail in Section 2.3.2.
Halton sequence

We illustrate quasi-random points further by considering the conceptually straightforward Halton sequence [34]. Let $r \geq 2$ be an integer. Then each integer $n \geq 0$ has a unique expansion in base $r$ such that

$$n = \sum_{i=0}^{\infty} n_i r^i.$$ 

The radical inverse function in base $r$ is defined by

$$\phi_r(n) = \sum_{i=0}^{\infty} n_i r^{-i-1},$$

for all integers $n \geq 0$. The function $\phi_r(n)$ gives a fraction by effectively ‘reflecting’ the integer $n$ at the ‘decimal point’. The Halton sequence for $s$ dimensions is then defined by

$$x_j = (\phi_{r_1}(n), \ldots, \phi_{r_s}(n)).$$

where $r_1, \ldots, r_s$ are the first $s$ primes. To be precise, only relatively prime integers are needed, as noted by Halton [34], but in general only primes are considered. A numerical example follows by writing the counting numbers $0, 1, 2, \ldots$ in base two

$$0, 1_2, 10_2, 11_2, 100_2, 101_2, \ldots,$$

then in base three

$$0, 1_3, 2_3, 10_3, 11_3, 12_3, \ldots,$$

and base five

$$0, 1_5, 2_5, 3_5, 4_5, 10_5, \ldots,$$

and so forth. The resulting sequence values are then ‘reflected at the decimal point’ to give values that fall in the interval $[0, 1)$. Hence, we obtain in base two

$$0, 0.1_2, 0.01_2, 0.11_2, 0.001_2, 0.101_2, \ldots,$$

then in base three

$$0, 0.1_3, 0.2_3, 0.01_3, 0.11_3, 0.21_3, \ldots,$$

and base five

$$0, 0.1_5, 0.2_5, 0.3_5, 0.4_5, 0.01_5, \ldots.$$
and so forth. The Halton sequence follows

\[ x_0 = (0, \ldots, 0), \]
\[ x_1 = (0.1_2, 0.1_3, \ldots, 0.1_r), \]
\[ x_2 = (0.1_2, 0.2_3, 0.2_5, \ldots, 0.2_r), \]
\[ x_3 = (0.11_2, 0.01_3, 0.3_5, \ldots, 0.3_r), \]
\[ \vdots \]
\[ x_N = (\phi_{r_1}(n), \ldots, \phi_{r_s}(n)). \]

The star-discrepancy of the first \( N \) points of the Halton sequence and the Koksma-Hlawka inequality lead to the error estimate

\[ |I_f - \hat{I}_f| \leq k_s \frac{(\log N)^s}{N} V(f) \]  \hspace{1cm} (2.18)

where \( k_s \) is some constant dependent on the Halton sequence.

**Lattice rules**

In this thesis, most of the quasi-Monte Carlo results are based on the classic Halton sequence. However, some of the results are based on a family of quasi-random points based on so-called lattice rules. These rules can produce well-behaving quasi-random sequences and have been a research focus in recent years owing to their ability to counter the curse of dimensionality [50]. The points arising from lattice rules are used in a similar manner to the quasi-Monte Carlo approach. There are many suggestions for lattice rules, but we lightly examine only one known as rank-1 lattice rule, which over the unit hyper-cube gives the integral estimate

\[ \hat{I}_f = \frac{1}{N} \sum_{n=1}^{N} f\left(\left\{ \frac{n}{N} \right\}\right), \]  \hspace{1cm} (2.19)

where the generating vector \( z \in \mathbb{Z}^s \), and the braces give the fractional part in \([0, 1)\).

Quasi-random sequences based on such a lattice rule can be clearly produced in an exceedingly fast way provided a generating vector is given. Moreover, the trick for quickly evaluating an integral is to choose a suitable generating vector. However, the drawback is that lattice rules are based on input parameters known as weights, which depend on the nature of the function. In particular, the weights depend on how the function varies with respect to all its variables. Furthermore, the choice of some lattice rules require the total number of function samples before the integral calculation starts. This differs from regular quasi-Monte Carlo methods in which the
integral estimate can be calculated continually until a sufficient number of function samples has been taken.

Under lattice rules, the number of function samples also influences the choice of the generating vector. Consequently, using the most suitable generating vector may not be a simple task as it involves performing analysis on the function of interest. However, a thorough investigation of which weights and quasi-random sequences are the most suitable in this setting is beyond the scope of this thesis. We simply give some complementary results in Chapter 5 based on the rank-1 lattice rule, and note that future work lies in investigating which quasi-random points are the optimal choice.

We stress that the research field of lattice rules has a relatively short history, and that new lattice rules are being developed continually. For more information, we refer the reader to the introductory piece by Kuo and Soan [50], and an example of lattice rules used in a financial setting [28].

Unknown performance

The quasi-Monte Carlo approach gives a deterministic error that appears to significantly outperform regular Monte Carlo, which suggests that quasi-Monte Carlo methods should be given serious consideration when performing numerical integration. However, there are varying and unknown aspects to its performance in certain settings; for a brief account see Section 3 in [70]. First, the bound $V(f)D_N^*$ is often a severe overestimate of the actual error. Consequently, even though this error bound is deterministic, its ability to predict the convergence rate can vary wildly.

Moreover, the two components of the bound $V(f)D_N^*$ individually result in difficulties. Identifying low-discrepancy infinite sequences is a difficult task in itself. More specifically, the sizes of the star-discrepancy constants $k_s$ of sequences are usually difficult to estimate in multiple dimensions. The values that are known often grow unsuitably large for reasonable values of $s$. For example, there is a known problem of Halton sequences demonstrating correlation of points in higher dimensions, thus rapidly increasing the discrepancy [44]. Consequently, the uncertainty in the discrepancy in quasi-random sequences reduces the reliability of quasi-Monte Carlo methods.

The variation of the function in the error bound $V(f)D_N^*$ poses another difficulty. The variation can be significantly large especially for larger dimensions. This may be reduced via a change of variables or other forms of variation reduction. Also, variation can differ dramatically depending on the type of function; for an extreme example see [56]. The aforementioned problems with quasi-random sequences result in them performing on the same level or possibly worse than
regular Monte Carlo methods in some situations.

We stress that quasi-random sequences and their related techniques are evolving research areas. There are emerging ways of decreasing discrepancy such as applying scrambling or permutation methods, and skipping points [44]. Despite some of their negative and unknown characteristics, quasi-random sequences have offered thought provoking solutions in recent years. Truly surprising and remarkable results have been achieved such as evaluating high-dimensional integrals with an empirical convergence rate of almost the order $N^{-1}$ in financial applications [62]. The ability of quasi-random sequences to speed up integral evaluation of certain functions in which the effect of dimension appears negligible remains an active research pursuit [67, 78].

**Implementation**

Regardless of the possible problems, the understanding and popularity of quasi-Monte Carlo methods is rising. Quasi-random sequence generators are available as open source packages [23] and are found in the standard numerical reference [64]. Moreover, Halton and Sobol sequence generators are now standard in the latest Matlab editions [75], which we use to perform all our simulations and integral calculations.

We experimented with Halton and Sobol sequences, and discovered that in general Halton sequences performed better in estimating our high-dimensional integrals. We countered the problem of increased discrepancy in higher dimensions by first discarding the initial thousands points. We then use the leaping method, which entails choosing only every $L$-th point in the Halton sequence where $L$ is a prime number which is not equal to any of the bases used in the sequence construction. We adopt this method due to the empirical studies [44], which examined the discrepancy of some quasi-random sequences in high dimensions. The Halton function (as well as the Sobol function) in Matlab allows for easily discarding and skipping sequence points.

Consequently, throughout this thesis leaped Halton sequences are used mostly to calculate integrals. However, often integrals were also calculated via regular Monte Carlo methods to validate the quasi-random result, and in Chapter 4 we experimented casually with lattice rules. Future extensions of this work may involve experimenting with other quasi-random sequences. However, we focus on reducing error irrespective of the chosen sequences by employing importance sampling.
2.3.3 Importance sampling

It was noted that the crude Monte Carlo estimate is simply the sample expectation of the function when the samples are distributed uniformly and independently. However, the behaviour of \( f \) warrants the concentration of sampling to vary over the domain. The samples should be random variables with some distribution function \( P \) that reflects the behaviour of \( f \). Sampling continuously over the domain in this manner is known as \textit{importance sampling}. In regular Monte Carlo terminology this is a variance reduction method. There is no variance in a quasi-Monte Carlo method, per se, as it is a deterministic method. However, some variance reduction methods have inspired quasi-random equivalents. Importance sampling is such a method as it can be used to reduce the variation of the function, thus lowering the error bound; see Section 4 in the article by Spanier and Maize [70] for an introduction.

We give a brief description of importance sampling in context of the regular Monte Carlo approach. The importance function \( p \) is the probability density of the importance distribution \( P \).

It follows that the integral of \( f \) on the interval \([0, 1]\) can be written as

\[
If = \int_0^1 f(x)dx, \\
= \int_0^1 f(x) p(x)dx, \\
= \mathbb{E}_P \left( \frac{f(x)}{p(x)} \right),
\]

where the subscript denotes the expectation under the importance distribution. The Monte Carlo estimate follows

\[
\hat{I}f = \frac{1}{N} \sum_{j=1}^{N} \frac{f(y_j)}{p(y_j)},
\]

where the sample points \( y_1, \ldots, y_N \) have the distribution \( P \). This result extends to the general interval \([a, b]\) such that

\[
\hat{I}f = \frac{1}{N} \sum_{j=1}^{N} \frac{f(y_j)}{p_i(y_j)}, \quad y_1, \ldots, y_N \in [P_i^{-1}(a), P_i^{-1}(b)],
\]

where \( P_i^{-1} \) is simply the inverse of the distribution \( P_i \). The \( s \)-dimensional result follows

\[
\hat{I}f = \frac{1}{N} \prod_{i=1}^{s} \left( (b_i - a_i) \sum_{n=1}^{N} f(y_n) \right), \quad (2.20)
\]

where the samples

\[
y_1, \ldots, y_N \in \prod_{i=1}^{s} \left[ P_i^{-1}(a_i), P_i^{-1}(b_i) \right]
\]
It can be shown that the importance function $p$ needs to take the form

$$p(x) = \frac{|f(x)|}{\int_{B_x} |f(x)| \, dx}$$

to minimize the variance [26]. Clearly, it is unlikely that an analytic solution to the integral $\int_{B_x} |f(x)| \, dx$ is known. In lieu of this, $p$ needs to be closely proportional to $|f|$ such that $|f|/p$ is roughly constant over the domain. Needless to say, choosing the appropriate $p$ is often a difficult task.

To calculate the Monte Carlo estimate, the random variable $Y$ with importance distribution $P$ needs to be simulated over the same sample space as $X$. We use the common inverse distribution method to simulate random variables such that they have the distribution $P$. We choose this method since it is quick to compute and does not change the order of sequence points. This second feature is important in maintaining the effectiveness of quasi-random sequences; changing the order of the sequence points is likely to increase the discrepancy. Hence, the distribution $P$ needs to be invertible if the inverse distribution technique is used to simulate (pseudo or quasi)-random variables. One approach is to use a power series to approximate $f$, and manipulate it so its integral gives a distribution.

Finally, importance sampling applies to quasi-random sequences provided that $f/p$ has finite variation in the sense of Hardy and Krause [70].
Chapter 3

Analysis of Greedy Routing

In this chapter we outline the main model assumptions and the resulting mathematical model that will be used throughout the majority of this thesis. We examine the distribution of a single message hop, and derive asymptotic moment expressions. We analyze the dependencies that arise within the model, and discuss how these affect the calculation of multihop distributions. We list stochastic bounds for hop advancements and discuss the effects of a power scheme.

3.1 Mathematical model

We present a mathematical model of a sensor network consisting of an ensemble of randomly positioned sensor nodes. We assume that all nodes are identical in capability and performance, and all operate perfectly. We assume that nodes communicate data radially, and that a node’s transmission radius clearly cuts off at some distance, thus the radiation pattern of the node’s antenna forms a perfect circle. Subsequently, a node can relay data to another node only when it is within the forwarding node’s transmission radius. Moreover, two nodes can communicate and cooperate with each other directly only when they are within mutual transmission range.

We further assume that signal interference from surrounding nodes and noise is negligible. The issue of noise can be bypassed by assuming that it is constant or that the signal is sufficiently large. However, this may not be the case for interference from surrounding nodes. For now we circumvent this issue by assuming that our sensor network model has relatively low traffic so interference is not a concern, and just examine the propagation of a single message. In Chapter 6, we propose a random transmission radius model and give a precise definition of a transmission radius distribution that takes into account the signal interference and noise. Consequently, in
this chapter the transmission radius is set to the constant \( r \). For all results, unless otherwise stipulated, the length values will be rescaled by the transmission radius, thus each node has a unit transmission radius.

We assume that the sensor nodes are scattered according to a homogeneous Poisson process. If each node is independently kept with probability \( p \) and removed otherwise, a thinned Poisson process is formed. In our analysis, we set \( p \) as the probability of a node being awake at any given time, and assume that separate transmission attempts sample independent thinnings of the underlying Poisson process. Let the constant \( \alpha \) be the density of nodes over a sensor field. It follows that the number of awake nodes in an area \( A \) is a homogeneous Poisson random variable \( N_A \), and has the probability mass function

\[
P(N_A = n) = \frac{(\lambda A)^n}{n!} e^{-\lambda A},
\]

where the new density parameter \( \lambda = p\alpha \) represents the mean number of awake nodes per unit area. In a thinned Poisson process, both the number of points kept and the number of points removed form random variables that are independent of each other [73]. Hence, in our model the number of awake nodes is independent of the number of asleep nodes. This useful property will be employed when we examine the inclusion of a simple sleep scheme in the next chapter.

We point out that even though the underlying nodes are modelled with a homogeneous Poisson process, the nodes which are transmitting (as well as being awake) depend on how the channel access is regulated by a medium access control (MAC) protocol. This issue is often sidestepped by analyzing the historically popular MAC protocol known as slotted Aloha. Under this protocol, a node sends a message with some probability \( p_S \) and delays sending it with probability \( 1 - p_S \). This completely random protocol results in the network of active nodes adhering to another homogeneous Poisson process. Recently, work has been done in analyzing the Aloha protocol and a proposed variant [8] with stochastic geometry methods [25, 9]. However, in this thesis we do not focus on channel access, hence our model does not incorporate effects from MAC protocols.

In our model there is a positive probability that no nodes, regardless of their state, lie within transmission radius of a forwarding node. Hence, there is a positive probability that a message will never reach the sink. This event never occurs in the model of Zorzi and Rao [80], who derived multihop bounds based on renewal processes for their GeRaF routing algorithm. In their model the locations of the nodes are re-sampled at each hop, and if there is no node within the transmission radius in one sample, there is a chance that there will be a node within in the radius in the next sample. In fact with probability one, there will eventually be a forwarding node, and the message will reach the sink. As \( \alpha \) approaches infinity and \( p \) approaches zero with \( \lambda \) held constant,
our model will closely approximate that of Zorzi and Rao.

We informally describe our definition of greedy routing by first introducing some notation. Let the random points $X_0$ and $X_S$ denote the respective locations of the source node and the target sink. Under greedy routing the next forwarding node is located at $X_1$, which is the neighbouring node of point $X_0$ that is closest to the sink. This procedure is repeated until the message reaches the sink.

More formally, we use $M(X)$ to denote the next forwarding node used by a message originating at point $X$. Let $\mathcal{N}(X)$ be the set of neighbouring nodes of a node at point $X$ with each neighbouring node location $n \in \mathcal{N}(X)$ iff $|n - X| \leq r$, where $|.|$ is the Euclidean metric. For the integer $i \geq 0$, greedy routing [7] is defined by taking

$$X_{i+1} = \arg\min_{n \in \mathcal{N}(X_i)} |n - X_S|,$$  \hfill (3.2)

which implies in particular that $M(X_S) = X_S$. Because the underlying node distribution is a Poisson process, the point $X_{i+1}$ is uniquely defined with probability one. For more details on routing definitions in a stochastic geometry setting, see the recent monograph [7, page 110] where greedy routing is called best hop routing.

### 3.2 Single hop distribution

Let the parameter $\gamma$ be the distance between a forwarding node and the sink (refer to Fig. 3.1). The sink distance is set to $\gamma = \ell$ when the forwarding node is the source of the message. The total feasible region (where the next forwarding can be located under greedy routing) of a node is the area formed by the intersection of the two circles of radii $r$ and $\gamma$ centered at the source node and the sink respectively, thus forming a lens.

After a single hop, let the random variable $U$ be the distance between the sink and the new forwarding node chosen by the greedy routing algorithm. To obtain the distribution of $U$ we employ a nearest neighbour approach. Consider the feasible region of the source node (shaded region in Fig. 3.1) where potential receiving nodes can be located at some distance $u$ or less from the sink (we suppress the qualifier 'total' when discussing the feasible region at some distance). For $\ell > r$, the probability that no receiving nodes exist in this feasible region of area $A_{\ell}(u)$ is equivalent to the probability that $U$ is strictly greater than $u$. The complement of this probability
Figure 3.1: Feasible region of nodes located a distance of \( u \) or less from the sink.

yields the distribution

\[
F(u) = \begin{cases} 
1 - e^{-\lambda A_\gamma(u)} & \ell - r \leq u < \ell \\
1 & u \geq \ell \\
0 & u < \ell - r.
\end{cases}
\] (3.3)

Since there is a positive probability that no nodes lie within the total feasible region, \( F(u) \) is a mixed discrete-continuous distribution, with a discontinuity at \( u = \ell \). However, this jump discontinuity is small for large \( \lambda \) values. We differentiate the distribution where it is absolutely continuous to give the probability density

\[
f(u) = \lambda A'_\gamma(u) e^{-\lambda A_\gamma(u)}, \quad \ell - r \leq u < \ell.
\] (3.4)

where the prime denotes differentiation with respect to \( u \).

We need an expression for the general feasible area function \( A_\gamma(u) \) as a function of \( u \) for a given sink distance \( \gamma \). We use a geometric approach initially and use the integral approach later. We define the angles of the two intersecting sectors as \( 2\phi_\gamma \) and \( 2\psi_\gamma \), and see that

\[
\phi_\gamma(u) = \arccos \left( \frac{r^2 + \gamma^2 - u^2}{2 r \gamma} \right),
\] (3.5)

\[
\psi_\gamma(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2 u \gamma} \right).
\] (3.6)

It follows that

\[
A_\gamma(u) = r^2 \phi_\gamma(u) + u^2 \psi_\gamma(u) - u \gamma \sin \psi_\gamma(u), \quad \gamma - r \leq u \leq \gamma,
\] (3.7)

and the derivative is given by

\[
A'_\gamma(u) = 2u \psi_\gamma(u).
\] (3.8)
3.2 Let $C = \gamma - U$ be the distance travelled towards the sink when the originating node is at a distance $\gamma$. Let $\bar{F}_\gamma$ denote the distribution of $C$, which leads to

$$
\bar{F}_\gamma(c) = \begin{cases} 
    e^{-\lambda A_\gamma (\gamma - c)} & 0 < c \leq r \\
    1 & c > r \\
    0 & c \leq 0,
\end{cases}
$$

which has a probability mass at $c = 0$. Given that $C$ is non-negative, the $m$-th moment expression follows

$$
E(C^m) = m \int_0^r c^{m-1} \mathbb{P}(C > c) dc
= r^m - m \int_0^r c^{m-1} e^{-\lambda A_\gamma (\gamma - c)} dc
$$

which gives the first moment

$$
E(C) = r - \int_0^r e^{-\lambda A_\gamma (\gamma - c)} dc,
$$

and the second moment

$$
E(C^2) = r^2 - 2 \int_0^r c e^{-\lambda A_\gamma (\gamma - c)} dc.
$$

3.2.1 Asymptotic analysis

The form of the area function (3.7) implies that it is not viable to obtain analytic expressions for integrals such as those in the moment equations (3.10) and (3.11). In lieu of this, we present an asymptotic result for these moment expressions.

**Theorem 3.2.1.** For $\gamma > r$, under greedy routing the first hop moment

$$
E(C) \sim r - \frac{\Gamma(5/3)}{(\lambda a_0)^{2/3}},
$$

and the second hop moment

$$
E(C^2) \sim r^2 - 2r \frac{\Gamma(5/3)}{(\lambda a_0)^{2/3}} + \frac{\Gamma(7/3)}{(\lambda a_0)^{4/3}},
$$

as the node density

$$
\lambda \to \infty,
$$

where $\Gamma(\cdot)$ is the gamma function, and

$$
a_0 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{4(\gamma - r)}{3} \right].
$$
Proof. Expand the area expression at \( u = \gamma - r \) (see Appendix A.1 for details) to obtain

\[
A_\gamma(u) \sim a_0(u - \gamma + r)^{3/2} + a_1(u - \gamma + r)^{5/2} + O(u^{7/2})
\]

as \( u \to \gamma - r \),

\[
(3.14)
\]

where the first two terms \( a_0 \) and \( a_1 \) are respectively

\[
a_0 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{4(\gamma - r)}{3} \right],
\]

\[
(3.15)
\]

\[
a_1 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{(-3\gamma^2 + 9\gamma r + r^2)}{15\gamma r} \right].
\]

\[
(3.16)
\]

Consider integrals of the form

\[
I(\lambda) = \int_{a}^{b} t^k e^{-\lambda A(t)} \, dt,
\]

\[
(3.17)
\]

where \( A(t) > A(a) \) for all \( t \in (a, b) \), and asymptotically

\[
A(t) \sim a_0(t - a)\mu, \quad \text{as} \quad t \to a.
\]

\[
(3.18)
\]

Applying the generalized Laplace’s method (2.13) to this integral gives the result

\[
I(\lambda) \sim \frac{\Gamma(\tau)}{\mu(\lambda a_0)^\tau}, \quad \lambda \to \infty,
\]

\[
(3.19)
\]

where

\[
\tau = \frac{2(k + 1)}{3}.
\]

\[
(3.20)
\]

However, \( A(a) \) needs to be the minimum on the integral interval, which motivates the variable \( t = u - \gamma + r = r - c \), and with a slight abuse of notation leads to the area function expansion

\[
A_\gamma(t) \sim a_0 t^{3/2} + a_1 t^{5/2} + O(t^{7/2}) \quad \text{as} \quad t \to 0.
\]

\[
(3.21)
\]

The above expansion implies \( \mu = 3/2 \). The first moment result (3.12) readily follows, and similarly for the second moment

\[
E(C^2) = r^2 - 2 \int_0^r (1 - t) e^{-A_\gamma(t)} \, dt,
\]

\[
(3.22)
\]

\[\square\]

The moment approximations agrees well with numerical results for a practical range of \( \lambda \) (see Fig. 3.4 and Fig. 3.5). The approximations begin to break down for \( \lambda \sim 1 \), particular for the second moment. However, the closed-form approximations give sufficiently accurate results for the first and second moments of \( C \). Additionally, we observe that the approximation of \( E(C) \) naturally breaks down as \( \ell \) approaches \( r \) as there is a singularity here due to the expansion term \( a_0 \), but this only occurs in close vicinity of this singularity (see Fig. 3.6).
Figure 3.2: Approximation of feasible area ($\gamma = 10$).

Figure 3.3: Approximation of distribution of $U$ ($\gamma = 10$ and $\lambda = 3$).
Figure 3.4: Approximation of $\mathbb{E}(C)$ by generalized Laplace’s Method ($\ell = 10$).

Figure 3.5: Approximation of $\mathbb{E}(C^2)$ by generalized Laplace’s Method ($\ell = 10$).
Figure 3.6: Approximation of $E(C)$ breaks down near the singularity at $\ell = 1$ ($\lambda = 3$).

Finally, we note that the next expansion term of the area function (3.14) can be found in the appendix. However, the two-term area expansion (3.14) gives reasonably accurate results regardless of the value of $\gamma$. This accuracy carries over when the above expansion is substituted into equation (3.3) to yield an approximation, which is almost indistinguishable from the true distribution (see Fig. 3.3) when $\gamma = 10$ and $\lambda = 3$.

### 3.2.2 Effect of sink distance

The area function increases with respect to the parameter $\gamma$, which implies that hop lengths increase stochastically with $\gamma$. This area behaviour can be verified geometrically by observing that the arc of the intersecting sector of radius $u$ flattens as $\gamma$ increases, hence the total feasible region approaches a semi-circle (refer to Fig. 3.1). A simple geometrical argument can be used to obtain the supremum of the area function. Alternatively, we consider the derivative of the area function (3.8) with respect to $c = \gamma - u$, that is

$$A'_1(\gamma - c) = 2 (\gamma - c) \arccos \left(1 + \frac{c^2 - r^2}{2\gamma (\gamma - c)}\right),$$  

(3.23)

and in the limit of $\gamma$ approaching infinity we obtain the function

$$A'_\infty(c) = -2\sqrt{r^2 - c^2},$$  

(3.24)
Figure 3.7: Distribution comparison for supremum area approximation and true area ($\ell = 10$ and $
abla = 3$).

which is the derivative of the supremum of the area function, thus

$$A_{\infty}(c) = r^2 \arccos(c/r) - c\sqrt{r^2 - c^2}, \quad 0 \leq c \leq r.$$  (3.25)

This is the feasible area function under the Most Forward within Radius routing model proposed by Takagi and Kleinrock [74] provided that messages only travel forward. Kleinrock and Silvester [42] stated earlier that the difference in the two area functions is negligible provided that $\gamma$ is sufficiently large.

The difference in the hop distribution parameterized by two different values, $\gamma_1$ and $\gamma_2$, of $\gamma$ can be expressed in terms of a goodness-of-fit measure. We choose the Kullback-Leibler divergence [48], which measures the difference between the ‘true’ probability measure $P$ and a proposed probability measure $Q$ over some sample space $\Omega$. In general, the Kullback-Leibler divergence is defined as

$$D(P, Q) = -\int_\Omega \log \left( \frac{dQ}{dP} \right) dP,$$  (3.26)

and is non-negative and zero when the two measures are identical. Consequently, the mixed discrete-continuous hop distribution leads to expression

$$D(\gamma_1, \gamma_2) = \int_0^r f_{\gamma_2}(c) \log \left( \frac{f_{\gamma_2}(c)}{f_{\gamma_1}(c)} \right) dc + F_{\gamma_2}(0^+) \log \left( \frac{F_{\gamma_2}(0^+)}{F_{\gamma_1}(0^+)} \right),$$  (3.27)
where the probability of a routing void or zero advancement is

\[
\bar{F}_\gamma(c) = e^{-\lambda A_{\gamma}(\gamma)}. 
\]

The Kullback-Leibler divergence is non-negative [48] and is zero when the two distributions are identical. We evaluate the integral in equation (3.27) numerically to determine how the hop distribution changes when \(\gamma_1 = \ell\) and \(\gamma_2 = \gamma\) is varied (refer to Fig. 3.9). We observe that \(D(\ell, \gamma)\) is virtually zero for the majority of \(\gamma\), then it increases markedly in the for \(\gamma \leq 3\). This implies that the algebraic dependence of \(\bar{F}_\gamma(c)\) on \(\gamma\) significantly affects the distribution of the remaining hop lengths only after this point. The negative consequence of this is that hop lengths cannot be assumed to be independent of the current sink distance for \(\gamma \leq 3\). Conversely, the hops up to this point can be modelled reasonably well with the first hop distribution. Moreover for \(\ell \geq 3\), the divergence \(D(\ell, \gamma)\) is approximately equal to \(D(\infty, \gamma)\), which suggests that the simpler area expression (3.25) may be used.
3.3 Multihop distribution

3.3.1 Dependence between successive hops

We introduce some notation to describe the locations of the forwarding nodes. Specifically, we introduce indexing for each sink distance such that $U_0 = \ell$ and $U_1 = U$, and after $i$ hops, the remaining distance to the sink is $U_i$. Let the random variable $\Theta_i$ be the angle between the $i$-th node and the previous node in relation to the sink. We note that $U_i$ is a global coordinate (in relation to the sink) and $\Theta_i$ is a local coordinate (in relation to the previous node). We assign the point $X_i = (U_i, \Theta_i)$ to the $i$-th forwarding node. The source (or zeroth) node corresponds to the point $X_0 = (U_0, 0) = (\ell, 0)$. A message travels $i$ hops along a path that corresponds to a sequence of random points $\tilde{X}_i = (X_0, X_1, \ldots, X_i)$.

The decrease in the sink distance between $i$-th and $(i-1)$-th hops is $C_i = U_{i-1} - U_i$. Each $C_i$ depends on the forwarding node’s sink distance $U_{i-1}$. We refer to this dependence of the distribution on the sink distance simply as the sink dependence. Zorzi and Rao [80] pointed out that each $C_{i+1}$ is stochastically dominated by $C_i$ in the sense that, for $i \geq 0$, there exists the stochastic ordering

$$\mathbb{P}(C_{i+1} > c) \leq \mathbb{P}(C_i > c), \quad c \in (0, r). \quad (3.28)$$

This follows directly from the observation in Section 3.2.2 that the hop distribution increases with respect to the sink distance.

Figure 3.9: Kullback-Leibler divergence quantifying the change in the distribution ($\lambda = 3$).
The Kullback-Leibler analysis suggests that the sink dependence is negligible for the majority of hop advancements. However, after the first hop there is another source of dependence between the random variables \( C_i \) for all \( i \geq 2 \). To simplify our explanation of this phenomenon we assume that all nodes within the forwarding node’s total feasible region are in their awake state. Then there can be no nodes closer to the sink in the total feasible region of the original node than the forwarding node. Hence, there is a region where there can be no potential forwarding nodes (region \( \mathcal{R}_B \) in Fig. 3.10). If this intersection did contain a node, then this node would have been chosen as the new forwarding node at the previous hop instead of the one that was chosen. This implies that \( C_2 \) is dependent on both \( U_1 \) and \( \Theta_1 \). We refer to this dependence on both the sink distance and the sink angle as path dependence, and the hop model that includes path dependence simply as the dependent model. Conversely, the independent model includes only the sink dependence.

![Figure 3.10: No nodes in the intersection region \( \mathcal{R}_B \) leads to path dependence.](image)

We use \( F \) and \( G \) to denote the distributions of \( U_i \) under the independent and dependent models respectively. The distribution of \( U_{i+1} \) under the independent model is dependent only on the sink distance of the current forwarding node, thus we can write

\[
F_i(u_{i+1}) = \mathbb{P}_I(U_{i+1} \leq u_{i+1} | U_i = u_i),
\]

while under the dependent model the distribution is dependent on entire the message path, and thus we write

\[
G_i(u_{i+1}) = \mathbb{P}_D(U_{i+1} \leq u_{i+1} | \bar{X}_i = \bar{x}_i),
\]

where the subscripts \( I \) and \( D \) indicate probability measures under the two models. Of course the distributions (3.29) and (3.30) depend on \( U_i = u_i \) and \( \bar{X}_i = \bar{x}_i \) respectively, but for convenience, we have suppressed this in the subscript notation. Further shorthand notation is used to denote the feasible area under the independent and dependent models respectively as \( A_i(u_{i+1}) \)
and $\tilde{A}_i(u_{i+1})$.

The area expression under the independent model is always given by the original area equation (3.7), and hence, the distribution and probability density of $U_{i+1}$ are obtained by setting $\ell = u_i$ in equations (3.3) and (3.4). Under the dependent model if the feasible area function is given after $i$ hops, the conditional sink distribution immediately follows

$$G_i(u_{i+1}) = \begin{cases} 1 - e^{-\lambda \tilde{A}_i(u_{i+1})} & u_i - r \leq u_{i+1} < u_i \\ 1 & u_{i+1} \geq u_i \\ 0 & u_{i+1} < u_i - r. \end{cases}$$

and its probability density on the region where it is absolutely continuous

$$g_i(u_{i+1}) = \lambda \tilde{A}_i'(u_{i+1})e^{-\lambda \tilde{A}_i(u_{i+1})}.$$  (3.31)

For $C_i$, we adopt a similar notation used for the sink distance random variables such that $\bar{F}$ and $\bar{G}$ denote the hop distributions under the two models. The complement of the sink distribution yields the conditional hop distribution under both the independent and dependent models; the latter being

$$\bar{G}_i(c_{i+1}) = \begin{cases} e^{-\lambda \Delta_i(u_i - c_{i+1})} & 0 < c_{i+1} \leq r \\ 1 & c_{i+1} > r \\ 0 & c_{i+1} \leq 0. \end{cases}$$

and its probability density on the region where it is absolutely continuous

$$\bar{g}_i(c_{i+1}) = \lambda \Delta_i'(u_i - c_{i+1})e^{-\lambda \tilde{A}_i(u_i - c_{i+1})},$$

where a simple sum relates the hop and sink distance variables

$$u_i = \ell - \sum_{j=1}^{i} c_j.$$  (3.35)

For any two node path $\tilde{X}_1 = (X_0, X_1)$, the feasible region under dependence is clearly a subset of the region under independence, hence

$$\tilde{A}_1(u_1 - c_2) \leq A_1(u_1 - c_2), \quad c_2 \in [0, r].$$  (3.36)

For specific details, a method for calculating the area $\tilde{A}_1(c_2)$ is presented in the appendix (Section A.3). Regardless, the above inequality allows us to compare the hop distributions (3.9) and (3.33) under the two models, which leads to the stochastic ordering

$$\mathbb{P}_D(C_2 > c|\tilde{X}_1 = \tilde{x}_1) \leq \mathbb{P}_I(C_2 > c|\tilde{X}_1 = \tilde{x}_1), \quad c \in (0, r).$$  (3.37)
on the conditional distributions after one hop. Hence, for any two node path, $C_2$ under the dependent model is stochastically dominated by $C_2$ under the independent model.

We have assumed for the purpose of our discussion of path dependence that all nodes within the forwarding node’s total feasible region are in their awake state. However, the analysis can be extended to the situation where a power scheme is in operation. In fact, if a proportion of nodes were asleep during the first relay attempt, then there is a chance that nodes will be awake in the intersection region ($R_B$ in Fig. 3.10) during the next relay attempt. This lessens the effect of the stochastic dependence, of which we give more precise details in the next chapter.

Let the set $I_i(u_{i+1}) \subset \mathbb{R}^2$ be the feasible region of the $i$-th forwarding node as a function of $u_{i+1}$ under the independent model. The set representing the feasible region under the dependent model follows by excluding the intersections of previous feasible regions, thus

$$D_i(u_{i+1}) = I_i(u_{i+1}) \setminus \bigcup_{j=0}^{i-1} I_j(u_{j+1}).$$

To calculate the general dependent model area $\tilde{A}_i(u_{i+1})$ seems intractable, given all the possible geometric configurations. After one hop, it is possible to calculate the feasible area function under the dependent model; see the appendix Section A.3 for details. After two hops, to calculate $\tilde{A}_2(u_3)$, the area of the region $I_2(u_3) \cap I_0(u_3) \setminus I_1(u_3)$ is needed (shaded region in Fig. 3.11). We outline a method for calculating this area in the appendix (Section A.8). We discuss the calculation of feasible regions further in the next chapter, and restrict our current analysis to only the two-hop case.

We note that the geometry of the problem leads us to use the equation of a circle and inequalities in polar coordinates to define regions and their corresponding indicator functions. In relation to the sink, an arbitrary point $(u_j, \theta_{0j})$ is in the feasible region of the $i$-th forwarding node when

$$u_i^2 + u_j^2 - 2u_i u_j \cos \theta_{ij} \leq r, \quad u_j \leq u_i,$$

where $\theta_{ij} = \theta_{0j} - \theta_{0i}$ is the angle between the point and the $i$-th forwarding node. In practice, we define indicator functions by using these inequalities, which eliminate the need to divide the sink angle domain of the feasible region into the subintervals where nodes can or cannot be present. These simple indicator functions are combined to form those representing more complex events, such as those in the next chapter that describe the different subregions under the sleep model.

We observed that there is a stochastic ordering (3.28) on single hops stemming from the sink dependence. Furthermore, there is a stochastic ordering on the second hop owing to the path dependence (3.37). We now investigate the possibility of a stochastic ordering for the multihop
Figure 3.11: The existence of a non-null $I_2(u_3) \cap I_0(u_3) \setminus I_1(u_3)$ region in a sensor network.

case. Subsequently, under the dependent model we introduce the distribution

$$G_i(u_{i+1}, \theta_{i+1}) = \mathbb{P}_D(U_{i+1} \leq u_{i+1}, \Theta_{i+1} \leq \theta_{i+1} | \vec{X}_i).$$

The respective angles $-\psi_{u_i}(u_{i+1})$ and $\psi_{u_i}(u_{i+1})$ denote the lower and upper values of the $\theta_{i+1}$ domain. The distribution $G_i(u_{i+1}, \theta_{i+1})$ is absolutely continuous on the domain $D_i(u_{i+1})$. However, it has a mass at the point $X_i$ corresponding to the event when there are no nodes in the total feasible region. In the dependent case, this means that there will never be a closer node in the total feasible region, and hence, the sink is unreachable. We denote the density function of $G_i(u_{i+1}, \theta_{i+1})$ by $g_i(u_{i+1}, \theta_{i+1})$, which is defined on the region where the distribution is absolutely continuous.

Under the dependent model the joint probability density of $U_i$ and $\Theta_i$ is

$$g_i(u_{i+1}, \theta_{i+1}) = \lambda D_i(u_{i+1}, \theta_{i+1}) u_{i+1} e^{-\lambda \tilde{A}_i(u_{i+1})},$$

(3.38)

whose derivation is given in the appendix. The spatially dependent density function $\lambda D_i(u_{i+1}, \theta_{i+1}) = \lambda I_{D_i}(u_{i+1}, \theta_{i+1})$ and the indicator function of the dependent feasible region

$$I_{D_i}(u_{i+1}, \theta_{i+1}) = \begin{cases} 1, & (u_{i+1}, \theta_{0i}) \in D_i, \\ 0, & \text{otherwise.} \end{cases}$$

The global angular coordinate $\theta_{0i}$ is simply the angle between the source node and the $i$-th forwarding node in relation to the sink

$$\theta_{0i} = \sum_{j=0}^{i} \theta_j.$$ 

The indicator function gives a zero joint probability density in the regions where there is zero node density (that is, no nodes exist owing to the path dependence).
Under the independent model we introduce a similar distribution

\[ F_i(u_{i+1}, \theta_{i+1}) = \mathbb{P}_I(U_{i+1} \leq u_{i+1}, \Theta_{i+1} \leq \theta_{i+1} | \vec{X}_i). \]

The distribution \( F_i(u_{i+1}, \theta_{i+1}) \) is absolutely continuous on the domain \( I_i(u_{i+1}) \), and similarly to the dependent model, it has a mass at the point \( X_i \). However, even if there is zero advancement, that is \( (u_{i+1}, \theta_{i+1}) = (u_i, \theta_i) \), it is still possible that \( (u_{i+2}, \theta_{i+2}) \neq (u_{i+1}, \theta_{i+1}) \) since under the independent model we effectively re-sample node positions at every hop so there is a possibility that there will be a node in the total feasible region at the next re-sampling. We also note that \( F_i(u_{i+1}, \theta_{i+1}) \) depends only on the current point \( X_i \), and not on the previous points, \( \vec{X}_{i-1} \). We denote the corresponding density function by \( f_i(u_{i+1}, \theta_{i+1}) \), which is defined on the region where the distribution is absolutely continuous.

We write the distribution and the density for the remaining distance to the sink for the dependent model respectively as

\[ G_i(u_{i+1}) = \int_{-\psi_i(u_{i+1})}^{\psi_i(u_{i+1})} G_i(u_{i+1}, \theta_{i+1}) d\theta_{i+1}, \]

\[ g_i(u_{i+1}) = \int_{-\psi_i(u_{i+1})}^{\psi_i(u_{i+1})} g_i(u_{i+1}, \theta_{i+1}) d\theta_{i+1}, \]

and similarly for the independent model

\[ F_i(u_{i+1}) = \int_{-\psi_i(u_{i+1})}^{\psi_i(u_{i+1})} F_i(u_{i+1}, \theta_{i+1}) d\theta_{i+1}, \]

\[ f_i(u_{i+1}) = \int_{-\psi_i(u_{i+1})}^{\psi_i(u_{i+1})} f_i(u_{i+1}, \theta_{i+1}) d\theta_{i+1}. \]

Finally, the stochastic dominance inequality after the first hop (3.37) is equivalent to

\[ G_1(u_2) \leq F_1(u_2), \quad u_1 - r \leq u_2 < u_1. \] (3.39)

This result leads to a stochastic bound after two hops between the respective models.

**Lemma 1.** For 2 hops the sink distances \( U_2 \) under the two models behave such that

\[ \mathbb{P}_D(U_2 \leq u_2) \leq \mathbb{P}_I(U_2 \leq u_2). \] (3.40)
Proof. This is equivalent to the inequality

\[ P_D(U_2 \leq u_2) = G_0(u_2) + \int_{u_0}^{u_2} \int_{\max(u_2, u_0 - r)}^{\min(u_1, u_0)} g_0(u_1, \theta_1) G_1(u_2) \, d\theta_1 \, du_1 \quad (3.41) \]

\[ = F_0(u_2) + \int_{u_0}^{u_2} \int_{\max(u_2, u_0 - r)}^{\min(u_1, u_0)} f_0(u_1, \theta_1) G_1(u_2) \, d\theta_1 \, du_1 \quad (3.42) \]

\[ \leq F_0(u_2) + \int_{u_0}^{u_2} \int_{\max(u_2, u_0 - r)}^{\min(u_1, u_0)} f_0(u_1, \theta_1) F_1(u_2) \, d\theta_1 \, du_1 \quad (3.43) \]

\[ \leq F_0(u_2) + \int_{u_0}^{u_2} F_1(u_2) dF_0(u_1) \quad (3.44) \]

\[ = P_I(U_2 \leq u_2), \quad (3.45) \]

where we denote \( u_i^- \) as the left limit

\[ u_i^- = \lim_{\epsilon \to 0} (u_i - \epsilon). \]

Clearly, the joint density and distribution of the first hop under the independent model are identical to those under the dependent model, that is, \( g_0(u_1, \theta_1) = f_0(u_1, \theta_1) \) and \( F_0(u_i) = G_0(u_i) \).

Initially, the probability for zero advancement is not included in the range of integration on the right hand side of the dependent model expression (3.41) as the occurrence of zero advancement under this model implies that it is not possible for any further advancement towards the sink in future hops. The step from line (3.42) to (3.43) is a direct result of inequality (3.39). Finally, the Riemann-Stieltjes type integral in line (3.44) has the upper limits of the form \( u_i \), instead of \( u_i^- \). That is, the probability for zero advancement during a hop has been included because under the independent model since it is possible for the message not to advance during one hop, then advance during the next.

The stochastic order (3.40) implies that after two hops the message has stochastically advanced more under the independent model than the dependent model. Moreover, this inequality holds even if the message fails or succeeds to advance after the first hop. Consequently, the simpler independent model is stochastically the more optimal of the two models in terms of greatest message advancement. We wish to extend the stochastic order (3.40) such that it holds for any number of hops.

**Conjecture 1.** For any number of hops \( i \) the sink distances \( U_i \) under the two models behave such that

\[ P_D(U_i \leq u_i) \leq P_I(U_i \leq u_i). \]
Numerical integration (see Fig. 3.12 and Fig. 3.13) provides support to the suggestion that $\Pr(D(U_i \leq u_i) \leq \Pr(I(U_i \leq u_i))$, but we have not been able to prove it. For $i \geq 3$ hops, deriving an analytic area expression becomes intractable and regular numerical integration is too slow. We now examine the inequality with some further detail. We express $\Pr(D(U_i \leq u_i))$ explicitly

$$
\Pr(D(U_i \leq u_i)) = G_0(u_i) + 
\int_{u_0}^{u_i} \int_{\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} g_0(u_1, \theta_1) G_1(u_2) d\theta_1 du_1 + \cdots + 
\int_{\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} \cdots \int_{\max(u_i, u_{i_1} - r)}^{\psi_{u_1}(u_{i_2})} \cdots \int_{\max(u_i, u_{i_{i-1}} - r)}^{\psi_{u_{i-1}}(u_{i_0})} g_0(u_1, \theta_1) \cdots G_{i-1}(u_i) \cdots d\theta_1 du_1.
$$

The conjecture hinges upon the truth of the inequality

$$
\Pr(D(U_i \leq u_i)) = G_0(u_i) + 
\int_{u_0}^{u_i} \int_{\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} g_0(u_1, \theta_1) G_1(u_2) d\theta_1 du_1 + \cdots + 
\int_{\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} \cdots \int_{\max(u_i, u_{i_1} - r)}^{\psi_{u_1}(u_{i_2})} \cdots \int_{\max(u_i, u_{i_{i-1}} - r)}^{\psi_{u_{i-1}}(u_{i_0})} g_0(u_1, \theta_1) \cdots G_{i-1}(u_i) \cdots d\theta_1 du_1

\leq F_0(u_i) + 
\int_{u_0}^{u_i} \int_{\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} f_0(u_1, \theta_1) F_1(u_2) d\theta_1 du_1 + \cdots + 
\int_{\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} \cdots \int_{\max(u_i, u_{i_1} - r)}^{\psi_{u_1}(u_{i_2})} \cdots \int_{\max(u_i, u_{i_{i-1}} - r)}^{\psi_{u_{i-1}}(u_{i_0})} f_0(u_1, \theta_1) \cdots F_{i-1}(u_i) \cdots d\theta_1 du_1

\leq F_0(u_i) + 
\int_{\max(u_i, u_0 - r)}^{u_0} \int_{\\max(u_i, u_0 - r)}^{\psi_{u_0}(u_1)} F_1(u_2) dF_0(u_1) + \cdots + 
\int_{\max(u_i, u_0 - r)}^{u_0} \cdots \int_{\max(u_i, u_{i_1} - r)}^{u_0} \cdots \int_{\max(u_i, u_{i_{i-1}} - r)}^{u_0} F_{i-1}(u_i) dF_{i-2}(u_{i-1}) \cdots dF_0(u_1)

= \Pr(I(U_i \leq u_i)),
$$

where we observe that $f_i(u_{i+1}, \theta_{i+1})$ independent of $\theta_i$.

For $i \geq 2$, we observe that $g_i(u_{i+1}, \theta_{i+1})$ is actually greater or equal to $f_i(u_{i+1}, \theta_{i+1})$ on the region that it is positive, but the range of $\theta_i$ over which $f_i(u_{i+1}, \theta_{i+1})$ is integrated in (3.46) is larger. The trade-off between these two competing influences is delicate.

As previously mentioned, the area $A_2(u_3)$ can be calculated after two hops. Consequently, the conjecture can be tested via regular numerical integration methods for three or less hops. Numerical investigation reveals that the difference between the two distributions is small for large $\lambda$ values. This implies that the independent model is a good approximation for the dependent model at high node density. The independent model significantly reduces the complexity of the problem of calculating the distributions. We note if using the independent model is not reasonable, it still provides a stochastic lower bound for the hop advancements if our conjecture is true.
Figure 3.12: Comparison of $P_D(U_3 \leq u_3)$ and $P_I(U_3 \leq u_3)$ ($\ell = 10$ and $\lambda = 3$).

Figure 3.13: Comparison of $P_D(U_3 \leq u_3)$ and $P_I(U_3 \leq u_3)$ ($\ell = 10$ and $\lambda = 4$).
3.4 Chapter summary

We outlined our assumptions and mathematical model. We used a homogeneous spatial Poisson process to model the distribution of nodes that may be awake or asleep. We addressed the ‘sink’ and ‘path’ dependencies that arise in our model. The sink dependence is the simpler of the two dependencies. This dependence manifests itself mathematically in the hop distribution via the sink distance parameter $\gamma$. Our Kullback-Leibler analysis showed that the sink distance has little effect on hop behaviour for approximately $\gamma \geq 3$.

The path dependence, however, is not representable as a simple parameter in the hop distribution. The path dependence is a result of the inherent nature of greedy routing, and is also directly affected by the power scheme being implemented. We examined the path dependence under the assumption that all nodes were and remained in their awake state. We established a stochastic order that relates single hop advancements under the independent and dependent models. We conjectured that a similar order holds for multiple hops. We supported this conjecture with numerical calculation of the sink distribution after two hops, which showed that the difference between the distributions of the independent and dependent models is small for large $\lambda$ value.

In our analysis, we assumed that the nodes were all awake to accentuate the stochastic dependence. However, we pointed out that if a power scheme is implemented, the stochastic dependence reduces, which leads us to the next chapter.
Chapter 4

Multihop Paths and Sleep Schemes

We extend the results from the previous chapter to examine the stochastic behaviour of multihop paths for $n$ hops. We include a energy-saving scheme in form of a sleep model, and analyze its influence on routing. More specifically, we examine the effects of having a certain proportion of nodes awake at any given time. We derive corresponding probability integral expressions, and demonstrate their viable evaluation via Monte Carlo integration methods. We give complementary results that may be used in deciding if a node should wait for surrounding nodes to awake.

4.1 After two hops

We are interested in calculating the distribution of $U_{i+1}$ or $C_i$ for $i$ hops. To achieve this under the dependent model, we need the joint probability density of $U_{i+1}$ and $\Theta_{i+1}$, namely

$$g_i(u_{i+1}, \theta_{i+1}) = \lambda D_i(u_{i+1}, \theta_{i+1})u_{i+1}e^{-\lambda A_i(u_{i+1})},$$

(4.1)

We recall that the set representing the feasible region under the dependent model follows by excluding the intersections of previous feasible regions, thus

$$D_i(u_{i+1}) = I_i(u_{i+1}) \setminus \bigcup_{j=0}^{i-1} I_j(u_{j+1}).$$

In the previous chapter, we used the area expressions for $D_1(u_2)$ and $D_2(u_3)$ (which are located in the appendix) to calculate the distribution of $U_3$.

The calculation of $\hat{A}_3(u_4)$ is more complex, as it involves calculating the area of the subregion $I_3(u_4) \cap I_0(u_4) \setminus I_1(u_4) \setminus I_2(u_4)$. In practice, however, a true analytic solution is not needed because this subregion is small compared to the total feasible region. Moreover, the exponentially
progressive nature of greedy routing implies it is extremely unlikely that the fourth node would be located in a position for which this area is nonzero, although we illustrate a contrived example (the shaded region in Fig. 4.1) to show that it is possible. In this example the first and second nodes are only slightly closer to the sink than the source node, an unlikely event. We neglect this small area contribution when calculating $\hat{A}_3(u_4)$. Thus, for $i \geq 2$, we approximate the feasible region under the dependent model by

$$D_i(u_{i+1}) \approx I_i(u_{i+1}) \cap \bigcap_{j=i-\kappa}^{j=i-1} I_j(u_{j+1})$$

(4.2)

where we set $\kappa = 2$, which can be considered as a discrete memory parameter for the path dependence. We refer to this approximation as the two-hop model and the $\kappa = 1$ case as the one-hop model.

![Figure 4.1: The existence of a non-null $I_3(u_4) \cap I_0(u_4) \setminus I_1(u_4) \setminus I_2(u_4)$ region in a sensor network (source sink distance $\ell = 10$). Sector arcs have been included to show feasible regions.](image)

### 4.2 Sleep model

In the previous chapter, we assumed that all nodes were awake to elucidate the stochastic dependence. If no sleep scheme exists no nodes exist in the intersection region ($R_B$ in Fig. 4.2). Clearly, with a sleep scheme this assertion is invalid as it is possible for previously asleep nodes to awake and become potential forwarding nodes.
Specifically, if \( \alpha \) is the actual density of nodes and \( p \) is the probability of each node being awake, an event which is sampled independently at each time step, then the nodes in the intersection region (\( \mathcal{R}_B \) in Fig. 4.2) will have a density \( \alpha(1-p)p \), whereas the nodes in the non-intersecting region (\( \mathcal{R}_A \) in Fig 4.2) will have a density \( \alpha p \).

Hence, the path dependence lessens, as a new ensemble of possible forwarding nodes will be sampled, implying the dependent model will more closely resemble the independent model. The path dependence will be reduced further as the number of nodes increases and the probability of them being awake decreases. In the limit as \( \alpha \to \infty \) and \( p \to 0 \) with \( \lambda = \alpha p \) held constant, the dependence disappears entirely and we have the independent model of Zorzi and Rao [80].

We present a sleep model such that we assume that the probability of a node being awake on each hop is \( p \) and the event that a node is awake during a transmission attempt is independent of the event that it is awake at other transmission attempts. A similar blinking sleep scheme was proposed and analyzed with continuum percolation techniques to examine latency issues [18]. In our setting, we wish to examine in what \( p \) range the dependent model closely resembles the independent model.

![Figure 4.2: Nodes can exist in intersection region \( \mathcal{R}_B \) under a sleep scheme.](image)

We gave reasoning above that led to the local node density after one hop in the two subregions \( \mathcal{R}_A = \mathcal{I}_1 \setminus \mathcal{I}_0 \) and \( \mathcal{R}_B = \mathcal{I}_1 \cap \mathcal{I}_0 \) in Fig. 4.2. It follows that the node density function after one message hop is given by

\[
\lambda_{\mathcal{D}_1}(u_2, \theta_2) = \lambda \left[ I_{\mathcal{I}_1 \setminus \mathcal{I}_0}(u_2, \theta_2) + (1-p)I_{\mathcal{I}_1 \cap \mathcal{I}_0}(u_2, \theta_2) \right],
\]

where the superscripts denote the indicator functions of the disjoint regions. As previously noted, the Poisson process assumption implies that the number of nodes in each of these two disjoint subregions are independent random variables.
After two hops similar reasoning can be applied to reveal that the intersection region $I_2 \cap I_0 \cap I_1$ has a node density $(1 - p)^2 \lambda$. Again, the regions that consist of the intersection of only two feasible regions have a node density $(1 - p) \lambda$. Our dependent model incorporates only the positions of the previous two forwarding nodes, thus we arrive at the node density function

$$\tilde{X}_{D_i}(u_{i+1}, \theta_{i+1}) = \lambda \left[ I_{x_i \setminus (x_{i-1} \cup x_{i-2})} (u_{i+1}, \theta_{i+1}) + (1 - p)^2 I_{x_i \cap x_{i-1} \cap x_{i-2}} (u_{i+1}, \theta_{i+1}) + (1 - p) \left[ I_{x_i \cap x_{i-1} \setminus x_{i-2}} (u_{i+1}, \theta_{i+1}) \right] \right].$$

(4.3)

which captures the node density information of the four subregions. To calculate the Poisson parameter or mean function over the region $D_i$, the above density function is integrated over the domain leading to

$$\tilde{\Lambda}_{D_i}(u_{i+1}) = \int_{u_i - r}^{u_{i+1}} \int_{-\psi_{u_i}(w_{i+1})}^{\psi_{u_i}(w_{i+1})} \lambda_{D_i}(w_{i+1}, \theta_{i+1}) w_{i+1} d\theta_{i+1} dw_{i+1}. \quad (4.4)$$

We introduce the rescaled mean function

$$\tilde{Q}_{D_i}(u_{i+1}) = \frac{\tilde{\Lambda}_{D_i}(u_{i+1})}{\lambda}, \quad (4.5)$$

to keep the ensuing expressions in an analogous form to those when there is no sleep scheme.

The node density function (4.3) leads to the joint probability density of $U_{i+1}$ and $\Theta_{i+1}$

$$g_i(u_{i+1}, \theta_{i+1}) = \lambda_{D_i}(u_{i+1}, \theta_{i+1}) u_{i+1} e^{-\lambda \tilde{Q}_{D_i}(u_{i+1})}. \quad (4.6)$$

### 4.3 Multihop distribution

In the last chapter, the sink distance variable was chosen to formulate this problem initially since greedy routing hinges upon it. However, the hop advancement $C_i$ is a more intuitive variable in describing message progress over a multihop path.

#### 4.3.1 Total number of hops

We give some motivation for calculating the distribution of $C_i$ in general. Let the random variable $N$ represent the total number of hops required for a message to reach the sink. We define the random variable

$$Z_n = \sum_{i=1}^{n} C_i, \quad (4.7)$$

to represent the distance advanced by a message in $n$ hops assuming the sink dependence discussed in Section 3.2.2. The random variables $N$ and $Z_n$ are naturally connected.
Theorem 4.3.1. For a sink distance \( \ell \), the number of message hops \( N \) has the distribution

\[
P_D(N = 1) = 1, \quad \text{if } \ell \leq r, \tag{4.8}
\]
\[
P_D(N \leq n) = 1 - P_D(Z_{n-1} < \ell - r), \quad \forall n \geq 1, \quad \text{if } \ell > r. \tag{4.9}
\]

Proof. Clearly, when \( \ell \leq r \) only a single hop is needed to reach the sink. For \( \ell > r \) it follows

\[
P(N > n) = P(U_{n-1} > r).
\]

Observing \( U_{n-1} = \ell - Z_{n-1} \), we see that

\[
P(N > n) = P(Z_{n-1} < \ell - r).
\]

\( \square \)

The distribution of the random variable \( N \) arguably presents more accessible information on the performance of the sensor network. For instance, if transmission errors are introduced at a given relay node with probability \( p_E \), then it is elementary to calculate the probability that errors are introduced along a path. In the next section, we outline an approach for calculating the distribution of \( Z_n \), and hence, the distribution of \( N \).

4.3.2 Distribution of \( Z_n \)

To calculate the distribution of \( Z_n \) we need the joint probability density of the random variables \( C_1 \) to \( C_n \) and \( \Theta_1 \) to \( \Theta_n \). This is written as the product of the conditional probability densities (see appendix for details), which under the dependent model without a sleep scheme is

\[
\bar{g}_{(n-1)}(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) = \prod_{i=1}^{n} \lambda_{D_{i-1}}(u_{i-1} - c_i, \theta_i) (u_{i-1} - c_i) e^{-\lambda_{D_{i-1}}(u_{i-1} - c_i)}.
\]

The joint probability density is not defined where probability masses exist, which correspond to zero advancements, hence the above expression is defined for sequences \( C_i \) such that all \( C_i \in (0, r] \).

After one hop the distribution of the advancement under the dependent model is simply

\[
P_D(Z_1 \leq z) = \int_0^z dc_1 \int_{-\psi_u(c_1)}^{\psi_u(c_1)} \bar{g}_{(0)}(c_1, \theta_1) d\theta_1 + P_D(C_1 = 0), \tag{4.11}
\]

where we adopt the shorthand notation

\[
\psi_u(c_{i+1}) = \psi_u(u_i - c_{i+1}), \tag{4.12}
\]
to denote the maximum angle for a sink distance given by the sink angle function (3.6). The feasible region of the source node is identical under both the independent and dependent models. This implies that the joint probability densities are also identical, and hence, independent of the sink angle $\theta_1$. Integrating over the angle domain simplifies the above expression

$$P_D(Z_1 \leq z) = \int_{0^+}^z f_0(c_1) dc_1 + P_I(C_1 = 0),$$

$$= P_I(Z_1 \leq z).$$

(4.13)

The distribution of message advancement in two hops under the dependent model is

$$P_D(Z_2 \leq z) = \int_{0^+}^{\min(z,r)} dc_1 \int_{-\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1$$

$$\int_{0^+}^{\min(z-c_1,r)} dc_2 \int_{-\psi_1(c_2)}^{\psi_1(c_2)} g_{(1)}(c_1, c_2, \theta_1, \theta_2) d\theta_2$$

$$+ P_D(C_1 = 0) + P_D((Z_1 \leq z) \cap (C_2 = 0)).$$

(4.15)

We note that if the two hops were completely independent of each other, the integral in the above expression would reduce to the standard convolution result for a sum of random variables. The above two-hop expression extends to the $n$-hop case

$$P_D(Z_n \leq z) = \int_{0^+}^{\min(z,r)} dc_1 \int_{-\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1 \int_{0^+}^{\min(z-c_1,r)} dc_2 \int_{-\psi_1(c_2)}^{\psi_1(c_2)} \cdots$$

$$\int_{0^+}^{\min(z-\sum_{i=1}^{n-1} c_i, r)} dc_n \int_{-\psi_n(c_n)}^{\psi_n(c_n)} \bar{g}(n-1)(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) d\theta_n$$

$$+ P_D(C_1 = 0) + P_D((Z_1 \leq z) \cap (C_2 = 0)) + \cdots$$

$$+ P_D((Z_{n-1} \leq z) \cap (C_n = 0)).$$

(4.18)

(4.19)

(4.20)

(4.21)

Similarly, the distribution of $Z_n$ under a sleep scheme is obtained by substituting the product of joint probability densities (4.6) under the sleep model.

The integral explicitly shown in the above expression is the distribution of $Z_n$ conditioned on the event that all hops $C_i$ advance some positive distance. We will often refer to this integral simply as the conditional distribution of $Z_n$, and denote it by

$$P(Z_n \leq z | +) = P(Z_n \leq z | C_1 > 0, \ldots, C_n > 0).$$

(4.22)

Under the independent model the joint probability density is not a function of any of the variables $\theta_1$ to $\theta_n$. Hence, the equivalent integral can be analytically integrated over the sink
angle domains, thus giving a simplified expression in the form of hop probability densities

\[
P_I(Z_n \leq z) = \int_{0^+}^{\min(z,r)} dc \int_{0^+}^{\min(z-c_1,r)} dc_2 \ldots
\]

\[
+ \int_{0^+}^{\min(z-\sum_{i=1}^{n-1} c_i,r)} f_0(c_1) \ldots f_{n-1}(c_n) dc_n
\]

\[
+ \int_{0^+}^{\min(z-r)} dc \int_{\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1 \ldots
\]

\[
+ \int_{\psi_{n-1}(c_n)}^{\psi_{n-1}(c_n)} g_{n-1}(c_1,\ldots,c_n,\theta_1,\ldots,\theta_n) P_I(C_{n+1} = 0) d\theta_n.
\]

(4.23)

(4.24)

(4.25)

(4.26)

(4.27)

We now investigate the event of a message not advancing. From the hop distribution (3.33), the probability of a message encountering a routing void after advancing \(i\) hops immediately follows

\[
P_D(C_{i+1} = 0 | \vec{X}_i = \vec{x}_i) = e^{-\lambda \vec{A}_i(u_i)},
\]

which is simply the probability that no nodes exist in the total feasible region under the dependent model. Hence, under our two-hop dependent model this probability is dependent on the current sink distance \(U_i\) and the previous two sink angles \(\Theta_{i-1}\) and \(\Theta_i\). The routing void probability leads to the distribution of \(Z_n\) such that the message meets a routing void on the last hop

\[
P_D((Z_n \leq z) \cap (C_{n+1} = 0)) = \int_{0^+}^{\min(z,r)} dc_1 \int_{\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1 \ldots
\]

\[
+ \int_{0^+}^{\min(z-\sum_{i=1}^{n-1} c_i,r)} f_0(c_1) \ldots f_{n-1}(c_n) dc_n \ldots
\]

\[
+ \int_{\psi_{n-1}(c_n)}^{\psi_{n-1}(c_n)} g_{n-1}(c_1,\ldots,c_n,\theta_1,\ldots,\theta_n) P_D(C_{n+1} = 0) d\theta_n.
\]

(4.28)

Similarly, the equivalent result under the independent model is obtained by using the probability of encountering a routing void under the independent model.

When a message does not advance \(X_i = X_{i+1}\). If there is no sleep scheme, there is nothing to be gained by a node attempting to relay a message more than once because if there was no nodes in the total feasible region on the first attempt, then there will be none on the second or subsequent attempts either. On the other hand, if there is a sleep scheme a forwarding node can benefit by making multiple relay attempts. This is because nodes that were asleep on the first attempt might have woken up and be available on the second attempt. The number of different possible sequences of events results in the corresponding integral expressions quickly growing to be intractable. This increase in integral complexity also occurs under the independent model. Consequently, under the independent model and dependent models we assume that a node executes only one relay attempt. In Section 4.6 we give suggestions for ‘rules of thumb’ for the number of relay attempts that a node should make.
4.4 Integration methods

To obtain the distribution of $Z_n$ under the dependent model a $2n$-fold integral (equations lines (4.18) and (4.19)) needs to be evaluated. The intractable nature of the area functions and, thus, the probability densities reduces the feasibility of analytic integration techniques. For low $n$, traditional numerical integration schemes can be used. Unfortunately, integration by these methods is too unwieldy at higher hop numbers which motivates us to employ Monte Carlo methods.

4.4.1 Monte Carlo integration

We outline the basics of the standard or crude Monte Carlo integration method by considering the conditional distribution of $Z_2$, which we recall as

$$
\mathbb{P}_D(Z_2 \leq z|+) = \int_{0^+}^{\min(z, r)} dc_1 \int_{-\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1 \int_{0^+}^{\min(z-c_1, r)} dc_2 \int_{-\psi_1(c_2)}^{\psi_1(c_2)} g_1(c_1, c_2, \theta_1, \theta_2) d\theta_2
$$

(4.29)

Given a value of $z$, an ensemble of $C_1$ values is generated (pseudo-)randomly from a uniform distribution, thus we have $m$ samples

$$
c_1^1, \ldots, c_1^m \in (0, \min(r, z)]
$$

where the superscript denotes the sample number. Based on the hop sample values, an ensemble of $\Theta_1$ values is generated such that for each sample

$$
\theta_1^j \in [-\psi_0(c_1^j), \psi_0(c_1^j)],
$$

where we use the shorthand notation again

$$
\psi_1^j(c_{i+1}^j) = \psi_{u_1^j}(u_1^j - c_{i+1}^j),
$$

and $u_1^j = \ell - c_1^j$ is the updated sink distance sample. Based on the samples $c_1^j$, the $C_2$ values are sampled such that

$$
c_2^j \in (0, \min(r, z - c_1^j)]
$$

Similarly, the $\Theta_2$ values are generated based on the previous two hop values such that

$$
\theta_2^j \in [-\psi_1(c_2^j), \psi_1(c_2^j)]
$$
where the updated sink distance samples \( u_j^i = \ell - c_j^i + c_j^2 \). Sampling the kernel in the integral (4.29) leads to the Monte Carlo estimate

\[
I_m = \frac{1}{m} \sum_{j=1}^{m} V_2(c_j^i) V_2(\theta_j) \bar{g}_2(c_j^i, c_j^2, \theta_j^1, \theta_j^2),
\]

(4.30)

where \( V_2(c_j^i) \) and \( V_2(\theta_j) \) are the volumes of the sample domains given by

\[
V_2(c_j^i) = \min(z, r) \min(z - c_j^1, r), \quad V_2(\theta_j) = 4\psi_j^0(c_j^1)\psi_j^1(c_j^2).
\]

which vary depending on the sample values. This estimate method readily extends to the \( n \)-hop integral, hence the Monte Carlo estimate

\[
I_m = \frac{1}{m} \sum_{j=1}^{m} V_n(c_j^i) V_n(\theta_j) \bar{g}_{(n-1)}(c_j^i, \ldots, c_j^n, \theta_j^1, \ldots, \theta_j^n),
\]

(4.31)

where

\[
u_j^i = \ell - \sum_{i=1}^{n} c_j^i, \quad V_n(c_j^i) = \min(z, r) \min(z - c_j^1, r) \ldots \min(z - \sum_{i=1}^{n-1} c_i, r),
\]

(4.32)

\[
V_n(\theta_j) = 2^n \prod_{i=0}^{n} \psi_i^1(c_{i+1}).
\]

(4.33)

The iterative nature of the numerical scheme, which stems from the product form of the joint probability density, allows for judicious implementation to save computing time. Subsequently, if the conditional distribution \( P_D(Z_n \leq z|+) \) has been calculated, the previously generated ensembles can be reused in addition to a new ensemble to calculate the conditional distribution \( P_D(Z_n \leq z|+) \). Similar reasoning can be applied to calculate the distribution of \( Z_n \) in general.

### 4.4.2 Importance sampling

To reduce the variance of the integral estimate we employ importance sampling; that is, suitably generate \( C_i \) values to sample the function in key regions. There is no variance in a quasi-Monte Carlo method, per se, as it is a deterministic method. However, some variance reduction methods have inspired quasi-random equivalents. Importance sampling is such a method as it can be used to reduce the variation of the function, thus lowering the error bound; see Section 4 in the article by Spanier and Maize [70] for an introduction.
We illustrate an example of importance sampling, and refer the reader to a book on Monte Carlo methods, such as [26]. Consider the single hop integral

\[ I_{\bar{g}} = \int_{0^+}^{z} \bar{g}_0(c_1) dc_1. \]

Under importance sampling, we do not choose \( C_i \) values uniformly. Instead we choose them according to some distribution \( \hat{F} \). Hence, let \( T_1 \) be a uniform random variable on the interval \((0, 1)\), then we generate \( C_1 \) with the distribution \( \hat{F} \) by setting

\[ C_1 = \hat{F}^{-1}(T_1), \quad C_1 \in (0, z), \]

where \( \hat{F}^{-1} \) is the inverse of the importance sampling distribution. The corresponding probability density \( \hat{f} \) is the weighting function, thus the Monte Carlo estimate under importance sampling follows

\[ I_m = \frac{z}{m} \sum_{j=1}^{m} \bar{g}_0(c_j^1) \hat{f}(c_j^1). \]

Provided that the sampling function \( \bar{F} \) is well chosen, importance sampling can significantly speed up Monte Carlo methods. A good importance sampling function is the dependent model hop distribution. Unfortunately, the complex nature of the area functions, especially under the dependent model, implies it is not viable to invert the dependent hop distribution function. In lieu of this we derive an importance sampling function, by way of asymptotics, that is based on the independent model hop distribution, which serves as a good first guess, particularly at high node density. The resulting function, which has an analytic inverse, is then used to sample \( C_i \) values in both Monte Carlo and quasi-Monte Carlo schemes, thus speeding up the integration procedure.

We recall that the general feasible area function (3.7) has the expansion

\[ A_\gamma(u) \sim a_0(u - \gamma + r)^{3/2} + O(u)^{5/2} \quad \text{as} \quad u \to \gamma - r, \]

where the first expansion term is

\[ a_0 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{4(\gamma - r)}{3} \right]. \]

A change of variable \( c = \gamma - u \) leads to the function

\[ \tilde{A}_\gamma(c) = a_0(r - c)^{3/2}. \]

Thus, we obtain an approximate solution to the hop distribution

\[ \hat{F}_\gamma(c) = e^{-\lambda \tilde{A}_\gamma(c)}. \]
If the higher expansion terms of the area function are used, then the resulting asymptotic expansion dramatically increases the difficulty of inverting the corresponding distribution function (a similar problem exists if the path dependence terms are included after the first hop).

For the importance sampling function to give all the possible values from the unit interval \((0, 1)\), the quantity \(\tilde{F}_\gamma(0)\) is subtracted from the above expression and the result is divided by \(\Delta \tilde{F}_\gamma = \tilde{F}_\gamma(c_{\text{max}}) - \tilde{F}_\gamma(0)\), to obtain an importance sampling function

\[
\tilde{F}_\gamma(c) = \frac{1}{\Delta \tilde{F}_\gamma} \left[ e^{-\lambda \tilde{A}_\gamma(c)} - \tilde{F}_\gamma(0) \right], \quad 0 \leq c \leq c_{\text{max}}.
\]  

(4.36)

where \(c_{\text{max}}\) is the largest hop value. The subtracting of the routing void term clearly has negligible effect for sufficiently large \(\lambda\). The corresponding derivative needs to integrate to one, hence the rescaling step. Rescaling by a constant does not influence the effectiveness of the importance sampling procedure. The derivative

\[
\hat{f}_\gamma(c) = \frac{3\lambda}{2\Delta \tilde{F}_\gamma} (r - c)^{1/2} e^{-\lambda \tilde{A}_\gamma(c)},
\]

eexists on the same domain as the sampling function. The inverse of the sampling function is given by

\[
\tilde{F}_\gamma^{-1}(t) = r - \left( \frac{-1}{\lambda \alpha_1} \ln \left[ t \Delta \tilde{F}_\gamma + \tilde{F}_\gamma(0) \right] \right)^{3/2}, \quad 0 \leq t \leq r.
\]

Substituting a random variable from a uniform distribution, say \(T \sim U(0, r)\), into the inverse of the importance sampling function gives a random variable \(C\) adhering to the importance sampling distribution (4.36).

We apply importance sampling to each hop \(C_i\) in the \(n\)-hop Monte Carlo estimate (4.31) to give

\[
I_m = \frac{1}{m} \sum_{j=1}^{m} V_n(c^j) V_{n}(\theta^j) \frac{\tilde{g}_{(n-1)}(c^j_1, ..., c^j_n, \theta^j_1, ..., \theta^j_n)}{\prod_{i=1}^{n} \hat{f}_{\ell}(c^j_i)},
\]

(4.37)

where the weighting function is the product of the importance sampling function derivatives. We have set the sink distance parameter \(\gamma = \ell\). The importance sampling function can be updated after each hop by setting \(\gamma\) to the current sink distance for each sample. However, arguably the benefit of this dynamic updating is slight, due to the relatively weak sink dependence. Furthermore, for higher node density the importance sampling step should improve as it is based on the independent model, and at high node density the intersection regions are small.
4.5 Convolution approximation

The Kullback-Leibler analysis has revealed that hops are weakly dependent on the sink distance except within the last couple hops of the sink. Furthermore, if \( p \) is small, then the path dependence is negligible. Thus, ignoring both these dependencies implies each hop is independent of all other hops. Consequently, the random variable \( Z_n \) can be approximated by

\[
Z_n(\gamma) = \sum_{i=1}^{n} C_i(\ell),
\]

where each \( C_i(\gamma) \) has the distribution of a node with a sink distance \( \gamma \) under the independent model. If we set \( \gamma = \ell \), then \( C_i(\ell) \) is stochastically the largest hop from the source node to the sink under the independent model.

We proceed by first considering the probability density of the conditional probability distribution \( P(Z_n(\gamma) \leq z|+) \). It follows from the complete hop independence that the conditional probability density of \( Z_n(\gamma) \) is calculated by a \( n \)-fold convolution of the hop probability densities

\[
f_{Z_n(\gamma)}(z) = \tilde{f}_\gamma(c_1) \ast \tilde{f}_\gamma(c_2) \ast \ldots \ast \tilde{f}_\gamma(c_n),
\]

where \( \ast \) denotes convolution. The nature of the probability densities eliminates the possibility of analytically calculating this convolution directly. Similarly, it is not viable to reduce the convolution to a product of transforms by analytically calculating the transforms of the respective probability densities. Hence, we are motivated to use numerical methods to perform both the transforms and the inversions.

4.5.1 Convolutions via fast Fourier transforms

We employ fast Fourier transforms (FFTs) to calculate the convolution numerically. To use FFTs, we approximate the largely continuous hop distribution \( F_\gamma(c) \) with a completely discrete distribution. We partition the hop interval into \( n_c \) subintervals of width \( \Delta C = 1/n_c \). We introduce the discrete random variable \( J \) to represent the \( j \)-th subinterval in which a hop value \( C(\gamma) \) may lie. We approximate the probability mass function of \( J \) by the difference relation

\[
P(J = j) \approx F_\gamma((j + 1/2)\Delta C) - F_\gamma((j - 1/2)\Delta C),
\]

where the integer \( j \in [1, n_c] \), and we approximate \( P(J = 0) \approx 0 \), thus excluding the probability mass for zero advancement.

We apply a standard FFT algorithm to give a transformed probability mass function, which is simply raised to the power of \( n \) in a pointwise manner, and transformed back to yield a final
mass function. The accumulative sum of the resulting mass function gives an approximation to the conditional distribution of $Z_n$.

The conditional distribution of $Z_n(\gamma)$ such that the $(n+1)$-th hop does not advance is calculated via multiplying the conditional distribution of $Z_n(\gamma)$ by the probability of encountering a routing void under the independent model, that is

$$
P_I((Z_n(\gamma) \leq z) \cap (C_{n+1}(\gamma) = 0)) = P_I(Z_n(\gamma) \leq z|+)P_I(C_{n+1}(\gamma) = 0).
$$

(4.41)

This leads to an approximation for $Z_n(\gamma)$, which can be used via result (4.3.1) to give an approximation of the distribution of $N$.

## 4.6 Relay attempts

The distribution of $Z_n$ is given by expression (4.18) under the assumption that a forwarding node will only make one relay attempt when a message enters a routing void. However, under a sleep scheme there is always a chance that a previously undiscovered node might wake up, and so a node should reattempt a message relay even though it was unsuccessful in advancing on a previous attempt. We address this issue by considering how many attempts a forwarding node should make until it considers alternatives (such as backtracking the message).

The details of what a node ‘knows’ about its surrounding environment affects its relaying decisions. For example, a forwarding node might have knowledge of the positions of asleep nodes in its total feasible region, thus a node can be programmed to wait until one of the nodes awakes. Conversely, nodes may only be aware of nodes that are currently awake. Alternatively, a node may know the sensor network’s overall node density, and thus use this knowledge to make more informed relaying decisions.

First we assume that a forwarding node has no knowledge of nodes within its total feasible region. Consider a node that happens to be within the total feasible region of the forwarding node. The sleep state of the node at every relay attempt is the outcome of a Bernoulli trial, thus the number of attempts until the node awakens is a geometric random variable. Let $K$ be the number of relay attempts before the node awakes, then

$$
P(K > k) = (1 - p)^{k+1}.
$$

(4.42)

Expression (4.42) gives us the probability that the forwarding node is yet to discover a given node within its total feasible region at after $k$ attempts. If this probability is sufficiently low, then a
forwarding node that has not discovered a node within its total feasible region after \( k \) attempts is entitled to assume that no such node is there. For the right hand side of equation (4.42) to be less than some value \( \epsilon \) the inequality
\[
k > \frac{\log \epsilon}{\log(1 - p)} - 1,
\]
must be satisfied. We recommend that number of attempts \( k \) should be chosen such that \( k \) is the minimum integer that satisfies this inequality.

Now we examine the situation where a forwarding node cannot sense any of the asleep nodes within its total feasible region, but it knows the overall node density. Let \( W \) be the number of relay attempts before any node awakes in some region.

**Theorem 4.6.1.** In a region of area \( A \), \( W \) has the distribution
\[
P(W \leq w) = 1 - e^{(1 - p)^w + 1 - 1} \mu A,
\]
where the density parameter \( \mu = (1 - p)\alpha \) is the mean number of asleep nodes per unit area.

**Proof.** Let \( N_S \) be the number of asleep nodes in some region of area \( A \), thus
\[
P(N_S = n) = \frac{(\mu A)^n}{n!} e^{-\mu A}.
\]

Given \( n \) asleep nodes, each node awakes after \( K_i \) relay attempts independently. Assuming a greedy routing method chooses the first node to awake, the random variable
\[
W = \min (K_1, K_2, \ldots, K_n),
\]
represents the number of relay attempts before a new forwarding node is chosen, then
\[
P(W = w | N_s = n) = p_n (1 - p_n)^w,
\]
where the parameter \( p_n \) is given by
\[
p_n = 1 - (1 - p)^n.
\]

By summing over all the events
\[
P(W = w) = \sum_{n=1}^{\infty} P(W \leq w | N_s = n)P(N_s = n),
\]
we see that

\[ P(W = w) = e^{-\mu A} \sum_{n=1}^{\infty} \frac{p_n(1 - p_n)^w(\mu A)^n}{n!}, \]

\[ = e^{-\mu A} \sum_{n=1}^{\infty} \left[ (1 - p)^{wn} - (1 - p)^{wn+n}(\mu A)^n \right], \]

\[ = e^{-\mu A} \sum_{n=0}^{\infty} \left( \frac{[(1 - p)^w\mu A]^n}{n!} - \frac{[(1 - p)^{w+1}\mu A]^n}{n!} \right), \]

\[ = e^{-\mu A} [e^{(1-p)^w\mu A} - e^{(1-p)^{w+1}\mu A}], \]

which summed accordingly gives the distribution. □

Another guide similar to the ‘no knowledge’ case is easily derived. For the probability that all nodes are still asleep to be less than some value \( \epsilon \), \( w \) attempts are required such that

\[ w > \frac{1}{\log(1-p)} \log \left[ \frac{\log \epsilon}{\mu A} + 1 \right] - 1. \] (4.47)

Consequently, a knowledge of \( W \) can be used to preset how many attempts nodes should make before they take other routing actions (such as backtracking). We do not, however, explore this further, and leave empirical validation as a future task.

Figure 4.3: The conditional distribution \( P(Z_3 \leq z|+) \) under the independent model and the two dependent models (\( \lambda = 3, \ell = 20, p=1 \)).
4.7 Results

We compared the dependent model to routing simulations of various ensemble sizes with and without a blinking sleep scheme. As noted, for multihop paths there is a positive probability at each relay that a message will encounter a routing void. Consequently, when analyzing multihop paths the node density needs to be suitably high for a given source node’s sink distance.

Monte Carlo integration and routing simulations were performed in Matlab, usually on a standard desktop machine. The built-in Halton sequence generator was employed for the quasi-Monte Carlo integration. Some of the larger integrals were evaluated on a small four-processor cluster. All calculations took no longer than an hour to complete, and usually considerably less. The routing simulations are based on the same assumptions made in the mathematical model. The empirical distributions were calculated simply by rescaling the message counts for each $z$ subinterval.

The first thousand Halton points were skipped, and then every $L$-th one was used to reduce the correlation of points that happens in higher dimensions. We originally set $L = 409$ based on the empirical findings of [44]. This gave generally good results even at higher dimensions. However, we observed that other values of $L$ worked better for some dimensions. That is, the number of samples may be sufficient for one value of $L$ but not for another value, depending on the number of jumps. For example, the results in Fig. 4.6 are better obtained for $L = 101$ than
L = 409. Although, this problem is easily rectified by increasing the number of samples, this mercurial behaviour is a slight hindrance for the quasi-Monte Carlo method. Future research lies in using better behaving quasi-random sequences or finding suitable L values, thus removing the uncertainty in integration performance.

Our experimentation revealed that quasi-Monte Carlo methods produced more accurate and smoother results than the equivalent obtained via pseudo-random numbers. We believe that they offer better means of evaluating the higher dimensional integrals presented here. However, we point out a deceptive artefact when using the quasi-Monte Carlo integration. The calculated distributions appear smooth and hence converged, however, the results may be slightly wrong, especially for low λ and high hop numbers. The errors seem relatively small and increasing \( n_{mc} \) naturally removes them, though care should be taken.

To calculate the conditional distribution \( P(Z_n \leq z|+) \) for some \( z \) value, \( n_{mc} \) points are sampled in the \( 2n \)-dimensional variable space; that is, \( n_{mc} \) function evaluations. The conditional distribution \( P(Z_n \leq z|+) \) and distribution \( P(Z_n \leq z) \) shown here were calculated by evaluating the integrals from 0 to \( z \) for each \( z \). This very wasteful approach was used to show the lack of variance in the integral result for each \( z \) value when calculated by the regular Monte Carlo method (see Fig. 4.4).

We employed a simple importance sampling function to speed up the overall Monte Carlo
process. As $\lambda$, increases importance sampling improves the results, as hops are skewed closer to their maximum value (and rendered very unsuitable for uniform sampling).

**Hop memory**

Initially, we had planned to use the two-hop dependent model to see whether it contained a sufficient level of path dependence. The two-hop dependent model was compared with the one-hop dependent model by setting the memory parameter $\kappa = 1$ in the approximation (4.2). Strikingly, without a sleep scheme there is almost no difference in the two models for the majority of $\lambda$ values (for an example, refer to Fig. 4.3). A small difference arises for very low node density (in the vicinity $\lambda = 0.5$), however, this results in a network that would be too sparse for multihop message forwarding.

Under a sleep scheme, the difference between the one-hop and two-hop model is slightly larger. A small difference arises near $\lambda = 1$, however, it still remains negligible in practical terms. We argue that the increase in the difference under a sleep scheme stems from greater geometrical area being available. Under a sleep scheme, the previous two forwarding nodes can lie in regions where they would not be able to exist if no scheme was in operation. This greater geometrical range implies that the intersection regions can be larger, hence the path dependence is greater.

The difference between the one-hop and two-hop model decreases as $\lambda$ increases, which is...
Figure 4.7: Quasi-Monte Carlo integration compared to empirical distribution of $Z_{10}$ ($\lambda = 5$, $\ell = 20$, $p=1$).

Figure 4.8: Conditional distribution $P(Z_4 \leq z | +)$ under independent and dependent models ($\lambda = 3$, $\ell = 20$).
Figure 4.9: Distribution of $Z_4$ under independent and dependent models ($\lambda = 3$, $\ell = 20$).

Figure 4.10: Independent and dependent models compared to empirical distribution $N$ without sleep scheme ($p = 1$, $\lambda = 3$ and $\ell = 10$).

achieved by increasing $\alpha$ and holding $p$ fixed. In this situation, the hops stochastically grow larger, naturally reducing intersection regions. It follows that the overwhelming majority of the path dependence is captured by incorporating only the intersection area of the current and the previous forwarding nodes. Hence, the simpler, less computationally intensive one-hop model allows for accurate hop distribution calculations.
Figure 4.11: Independent and dependent models compared to empirical distribution of $N$ with sleep scheme ($p = 0.2$, $\lambda = 3$ and $\ell = 10$).

Figure 4.12: Independent and dependent models compared to empirical distribution of $N$ with sleep scheme ($p = 0.1$, $\lambda = 3$ and $\ell = 10$).
Distributions of $Z_n$ and $N$

The resulting empirical distributions show that the model results agree very well with the simulations. For the four hop case, $10^4$ routing simulations is comparable to the $10^3$ quasi-Monte Carlo samples (the two curves are almost indistinguishable when $n_{mc} = 10^4$). This indicates that both dependencies are sufficiently captured with our one-hop dependent model. Agreeable results also applied for when sleep schemes were in operation (for example, see the distribution of $N$ in Fig. 4.10, Fig. 4.11, and Fig. 4.12).

As the hop number increases, we observed that more Monte Carlo sample points are needed. This is explained by noting how the integral error bound (2.18) is dependent on the dimension of the integral. Naturally, as the hop number increases the joint probability density exists over more dimensions. Thus, the error increases with the hop numbers. In higher dimensions, Halton sequences breakdown as they begin to show correlation between points. We attempted to counter this by skipping sequence points, with reasonable success. Moreover, for greater hop numbers a large portion of the sampling is wasted (even with importance sampling) as the conditional distribution of $Z_n$ is exponentially small for the majority of the $z$ domain.

As a sensor network grows (in size, density or a combination of both), routing simulations take longer to be completed due to the increased number of nodes. Conversely, for large $\lambda$ the quasi-Monte Carlo approach seems to perform well with relatively small $n_{mc}$. For example, when $\lambda = 5$ as few as a hundred sample points are needed to calculate the conditional distribution $P(Z_4 \leq z|+)$. A possible explanation stems from the importance sampling function, which is skewing more of the hops towards their maximum values. In this range, the independent model (which the importance sampling is based on) is a good approximation. Another explanation is that the variation of the function decreases as $\lambda$ increases (perhaps owing to the importance sampling step removing some of the $\lambda$ terms in the integral kernel). A more detailed explanation of this remains an open task. We also note that the events (and subsequent integrals) involving a message encountering a routing void become negligible for large $\lambda$.

Model comparison

The independent model distribution is naturally quicker to evaluate, thus it is important to know when it may be used without a significant loss in accuracy. We compared the independent model with the (one-hop) dependent model by varying $p$ (and accordingly $\alpha$, thus holding $\lambda$ constant). Our experimentation shows that for $\lambda = 3$ the dependent model with $p = 0.1$ agrees closely with the dependent model (see Fig. 4.8 and 4.9).
The difference is more pronounced for low $\lambda$. The greatest difference in the conditional distributions of $Z_n$ under the two models seems to be at its endpoint value $z = n$. Also, the difference of $\mathbb{P}(Z_n \leq z)$ appears smaller than of $\mathbb{P}(Z_n \leq z|+)$. This smaller difference, however, accumulates after multiple hops, and becomes significant when calculating the distribution of $N$ (see Fig. 4.10).

A smaller $p$ value results in the dependent model closely resembling the independent model for sufficiently large $\lambda$ (see Fig. 4.11 and 4.12). The number of hops does not appear to significantly affect these observations. Hence, greedy routing in a sensor network with ten to twenty percent of the nodes in sleep mode can be modelled with the independent model.

**Convolution approximation**

The previous results inspire us to set $p = 0.1$ for comparing dependent and independent models to the convolution model. For this $p$ value, the dependent and independent models are practically indistinguishable. For the conditional distribution of $Z_4$, the convolution model gives results (in Fig. 4.13) that are very close to the two dependent and independent models for sufficiently large $\ell$. Moreover, the accuracy in the convolution model carries over when the zero advancement events are included to give the distribution of $Z_4$ (see Fig. 4.14).

This accuracy diminishes slightly as the message approaches the sink as the conditional dis-
Figure 4.14: Independent, dependent, and convolution model distributions of $Z_4$ with sleep scheme ($p = 0.1$, $\lambda = 3$ and $\ell = 10$).

Figure 4.15: Independent, dependent, and convolution model conditional distributions of $Z_4$ with sleep scheme ($p = 0.1$, $\lambda = 3$ and $\ell = 4$).
Figure 4.16: Independent, dependent, and convolution model compared to empirical distribution of $N$ with sleep scheme ($p = 0.1$, $\lambda = 3$ and $\ell = 10$).

Figure 4.17: Independent, dependent, and convolution model compared to empirical distribution of $N$ with sleep scheme ($p = 0.1$, $\lambda = 5$ and $\ell = 10$).
tribution of $Z_4$ when $\ell = 4$ reveals (see Fig. 4.15). However, this accuracy is still quite good for the remarkable decrease in computation speed that the convolution method offers.

The difference between the convolution model and the independent model is more significant when calculating the distribution of $N$. We speculate that this is due to the multihop distributions that are required to calculate $P(N \leq n)$, in which the small difference accumulates to give a more noticeable contrast between the models. However, we believe that this is a very low cost given the fast computation times. Consequently, we stress that the convolution method should be used for constant $\lambda$ and transmission radius models provided that $p$ is sufficiently small.

4.8 Chapter summary

A greedy routing model was presented that incorporates both sink and path dependence. This model is an alternative means of ascertaining the probabilistic characteristics of greedy routing in sensor and ad hoc networks in general. We presented quasi-Monte Carlo schemes coupled with importance sampling as a viable method for quickly evaluating the resulting higher dimensional integrals.

Moreover, a sleep scheme has been included and its effects have been measured. The different subregions in the node’s total feasible region and the varying node densities under the power scheme have been observed and quantified. Results that demonstrated the influence of the power scheme on the dependencies were obtained. In particular, the observation that only ten to twenty percent of the nodes in sleep mode allows the dependent model to be replaced with the more tractable independent model is revealing. Finally, we demonstrated that the superbly fast convolution model can approximate the independent model with only a small loss of accuracy.
Chapter 5

Spatially Dependent Node Density

In this chapter we extend the mathematical model to examine the case of node density varying over the sensor field. We propose a simple but insightful spatially dependent node density function, which is given with motivation. Consequently, we apply the previously employed mathematical techniques to examine and model this problem. The dependencies are considered, resulting in the distribution of multihop advancements. The effects of the previously proposed power scheme are studied. We develop amenable approximations and show that they work well and agree with simulation results.

5.1 Introduction

The majority of this work done thus far has been under the assumption that sensor nodes are deployed according to a homogeneous Poisson process. Although such a tractable model might not capture the underlying node density variation, it can serve as a simple first approximation for studying network characteristics, hence its appeal. We wish to extend the standard model by examining inhomogeneous or nonhomogeneous node deployment such that the node density is spatially dependent.

There are various reasons why inhomogeneous deployment models may be necessary. The deployment of sensor nodes depends on the environment and application of the sensor network. Consequently, sensor nodes may need to be deployed more densely in important sensing regions. The obstacles in the network surroundings may prevent nodes being positioned in certain regions. Inconsistent system parameters (such as battery lifetime) and interference can reduce the effective node density in certain regions. Furthermore, the nature of the actual node deployment influences...
the node density. For example, an aerial dispersal of nodes could result in the nodes obeying some type of diffusion process. Also, there may be network protocols that require positioning the nodes in certain regions, which ultimately lead to performance advantages.

The design and deployment of sensor networks must address the problem of message collisions. The data messages in sensor networks converge towards the sink. Nodes closer to the sink need act as relay nodes more than nodes away from the sink. One possible solution to this problem is to deploy more nodes in these heavy traffic regions. Consequently, the node density would decay at some rate that is dependent on the distance to the sink.

It is in this last setting that we wish to examine in more detail. We propose to extend the mathematical model used in the previous chapters. Ideally, we want to offer a computationally quick and reliable way to obtain probabilistic descriptions of multihop paths in a sensor network with a simple stochastic sleep scheme. Moreover, we want to broaden the model, analysis, and calculation methods from the homogeneous case to the nonhomogeneous case, and demonstrate that the solutions techniques are still valid.

To achieve these goals, we analyze greedy routing in randomly deployed networks under the multihop situation. We propose a tractable spatially dependent node density function, and analyze the resulting stochastic characteristics. Furthermore, we examine the influence of a simple stochastic power scheme. More specifically, we examine the effects of having a certain proportion of nodes awake at any given time. We derive corresponding probability integral expressions, and demonstrate their evaluation via Monte Carlo integration methods.

The work presented here is focused on the stochastic behaviour of multihop paths, the calculation methods, and modelling the effects of a power scheme. Overall, we show the application of this mathematical formulation in describing the stochastic behaviour of message delivery in sensor networks with inhomogeneous node deployment. Additionally, we demonstrate an implementable calculation procedure based on quasi-Monte Carlo methods for evaluating high-dimensional integrals.

### 5.2 Background work

We briefly describe some related work in analyzing ad hoc and sensor networks. The results presented here follows on from the initial homogeneous Poisson model development and analysis presented in this thesis. Consequently, the majority of the formulation and techniques used here have been applied in the constant density setting.
5.3 More generally, Ishizuka and Aida [36] performed some preliminary analysis of node placement approaches via simulations to gauge the fault tolerance of networks against random node breakdown and battery exhaustion. In particular, they assumed that individual nodes are scattered uniformly around the sink, and that the density decreased as the distance from the sink increased. Ishizuka and Aida examined two node density models: a Gaussian function and a simple power law ( inversely proportional to the sink distance). They concluded that the simple power law placement resulted in good fault tolerance.

A concept closely related to connectivity is the sensing coverage of a sensor network, which is the ability of a sensor network to successfully sense or cover the entire sensor field. Solutions to coverage problems have been based on coverage processes such as a Boolean model; see Hall [33] and Stoyan, Kendall and Mecke [73] for more details. In the sensor network setting, a more recent example is that of Manohar et al. [61] who used coverage processes to examine the coverage of a sensor network with an exponentially decreasing node density.

The aforementioned work involving inhomogeneous node deployment cases did not examine the stochastic behaviour of any particular routing method. Furthermore, the work did not cover the effects that a sleep scheme has on stochastic dependencies in the routing model.

5.3 Mathematical model

As in previous chapters, we assume that nodes communicate data radially, and that a node’s transmission radius is constant, which implies that a node can relay data only to another node when it is within the forwarding node’s transmission radius. For calculations and simulations, we rescale all lengths with respect to the transmission radius.

We assume that nodes are scattered according to a two-dimensional Poisson process over a finite region, and that at any given time a random number of nodes are in sleep mode while the remaining are in their awake mode. Furthermore, we assume the node density is a spatially dependent function; that is, the nodes are scattered according to an inhomogeneous Poisson process [73]. We assume the nodes are scattered uniformly around the sink, but the density decreases in some way as the distance to the sink increases. Hence, the node density is a radial function of the form

\[ \lambda(u) = \lambda q(u) \quad u \in (0, \infty), \]

where \( u \) is the distance to the sink, and \( q(u) \) is some type of weighting function. The function \( \lambda(u) \) can be interpreted as the mean number of awake nodes per infinitesimal area element. We refer
to the constant $\lambda$ as the initial node density (in respect to the source node).

Ideally, the $q(u)$ function should be amenable to analytical and asymptotic methods while reflecting a realistic node distribution. We propose a node density that is inversely proportional to the sink distance

$$q(u) = \frac{1}{u},$$

which is the model proposed by Ishizuka and Aida [36]. However, we chose this function also because we believe that the positioning of nodes will better accommodate the convergence of messages near the sink.

The weighting function (5.2) implies that each node is scattered on the plane such that both its polar coordinates are uniformly distributed. Hence, we introduce a Poisson process mean or parameter measure $\Lambda(u)$, which for a bounded Borel set $B \subset \mathbb{R}^2$ with area $A$ is the density function integrated over the region in polar coordinates, thus

$$\Lambda(B) = \lambda \int_B q(u)udud\theta,$$

$$= \lambda Q(B),$$

where $Q$ is referred to as the rescaled mean, and is used such that the notation is analogous to the results based on the constant density case when $Q$ reduces to the area $A$ of the region. We emphasize that all the $Q$-type expressions we consider in this work are derived in a similar manner to the corresponding area expressions under the homogeneous model by integrating the density over specific regions.

Under a sleep model, the initial node density parameter can be written as $\lambda = p\alpha$ where $p$ is the probability of a node being awake and $\alpha$ is the underlying (that is, sleep and awake) node density parameter, hence awake nodes form a thinned Poisson process. As in the homogeneous case, both the number of points kept and the number of points removed form random variables that are independent of each other [73]. Hence, in our model the number of awake nodes is independent of the number of asleep nodes.

The number of awake nodes $N_B$ located within some region $B$ is an inhomogeneous Poisson random variable with a probability mass function

$$P(N_B = n) = \frac{(\lambda Q(B))^n}{n!}e^{-\lambda Q(B)}.$$
5.4 Single hop analysis

As in the previous chapters, let $\gamma$ be the distance between a forwarding node and the sink (refer to Fig. 5.1), and the sink distance parameter is set to $\gamma = \ell$ for the source node. Often we shall present results such that the initial node density parameter $\lambda$ is a product of the source-sink distance $\ell$ and some positive constant.

For a forwarding node with a sink distance $\gamma$, let $\mathcal{I}_\gamma(u) \subset \mathbb{R}^2$ be a portion of its feasible region as a function of $u$. This region is formed by the intersection of two circles of radii $r$ and $u$ centered at the source node and the sink respectively (as illustrated in Fig. 5.1). The area of $\mathcal{I}_\gamma(u)$ is denoted by $A_\gamma(u)$, and given by the integral

$$A_\gamma(u) = 2 \int_{\gamma-r}^{u} \int_{0}^{\psi_\gamma(w)} w \, d\theta \, dw,$$

(5.6)

where the sink angle function $\psi_\gamma(u)$ is defined as

$$\psi_\gamma(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2u\gamma} \right).$$

(5.7)

Furthermore, the mean measure of the region $\mathcal{I}_\gamma(u)$ is given by the integral

$$\Lambda(\mathcal{I}_\gamma(u)) = 2 \int_{\gamma-r}^{u} \int_{0}^{\psi_\gamma(w)} \lambda(w) w \, d\theta \, dw,$$

(5.8)

which for the functions (5.1) and (5.2) reduces to

$$\Lambda(\mathcal{I}_\gamma(u)) = 2\lambda \int_{\gamma-r}^{u} \int_{0}^{\psi_\gamma(w)} d\theta \, dw.$$

(5.9)
Henceforth, we write
\[ \Lambda_\gamma(u) = \Lambda(I_\gamma(u)), \quad Q_\gamma(u) = Q(I_\gamma(u)), \]
to refer to the mean measure and the rescaled mean measure respectively, the motivation of which will become apparent by the analogous results that follow.

After a single message hop, the distribution of \( U \) is obtained by the standard nearest neighbour method, thus
\[
F_\gamma(u) = \begin{cases} 
1 - e^{-\lambda Q_\gamma(u)} & \gamma - r \leq u < \gamma \\
1 & u \geq \gamma \\
0 & u < \gamma - r, 
\end{cases} \tag{5.10}
\]
where there is a jump discontinuity in the distribution at \( u = \gamma \). On the interval where the distribution is absolutely continuous the probability density is defined by
\[
f(u) = \lambda Q'_\gamma(u)e^{-\lambda Q_\gamma(u)}, \quad \gamma - r \leq u < \gamma, \tag{5.11}
\]
where the derivative of the rescaled mean measure
\[ Q'_\gamma(u) = 2\psi_\gamma(u). \]

For a node of sink distance \( \gamma \), it follows from integral (5.9) that the mean measure as a function of \( u \) is calculated by the integral
\[
\Lambda_\gamma(u) = 2\lambda \int_{\gamma-r}^{u} \psi_\gamma(w)dw. \tag{5.12}
\]
The corresponding indefinite integral has the analytic solution
\[
\int \psi_\gamma(w)dw = w\psi_\gamma(w) + iE \left( \sin^{-1} \left( \frac{w}{\gamma + r} \right); \frac{(\gamma + r)}{(\gamma - r)} \right) (\gamma - r) + iF \left( \sin^{-1} \left( \frac{w}{\gamma + r} \right); \frac{(\gamma + r)}{(\gamma - r)} \right) (2r), \tag{5.13}
\]
where \( i = \sqrt{-1} \), and \( F \) and \( E \) denote incomplete elliptic integrals of the first and second kind defined in Legendre form as
\[
F(\phi; k) = \int_{0}^{\phi} \left( 1 - k^2 \cos^2 \theta \right)^{-1/2} d\theta, 
\]
\[
E(\phi; k) = \int_{0}^{\phi} \left( 1 - k^2 \cos^2 \theta \right)^{1/2} d\theta. 
\]
A number of standard computer packages have pre-written routines for numerically evaluating elliptic integrals, however, care must be taken as their definitions vary depending on the package.
The results presented here are obtained via our purposely written elliptic integral functions based on Carlson symmetric elliptic integrals [15]. For more details see the papers by Carlson [14, 15], which give algorithms that are readily implementable and can handle both real and complex values under specified variable and parameter regimes.

The solution to the original integral (5.12) is

\[
\Lambda_{\gamma}(u) = 2\lambda u \psi_{\gamma}(u) + 2i\lambda E \left( \sin^{-1} \frac{u}{\gamma + r}; \frac{(\gamma + r)}{(\gamma - r)} \right) (\gamma - r)
\]

\[
-2i\lambda E \left( \sin^{-1} \frac{\gamma - r}{\gamma + r}; \frac{(\gamma + r)}{(\gamma - r)} \right) (\gamma - r)
\]

\[
+ 2i\lambda F \left( \sin^{-1} \frac{u}{\gamma + r}; \frac{(\gamma + r)}{(\gamma - r)} \right) (2r)
\]

\[
-2i\lambda F \left( \sin^{-1} \frac{\gamma - r}{\gamma + r}; \frac{(\gamma + r)}{(\gamma - r)} \right) (2r).
\]

The final result gives a real value, although in practice an insignificant imaginary part may arise from rounding errors. We found that the solution is obtained quickly owing to the speedy convergence of the Carlson algorithms.

Let \( C = \gamma - U \) be the distance advanced towards the sink when the originating node is at a distance \( \gamma \). We let \( \bar{F} \) denote the distribution of \( C \), which leads to

\[
\bar{F}_{\gamma}(c) = \begin{cases} 
  e^{-\lambda Q_{\gamma}(\gamma - c)} & 0 < c \leq r \\
  1 & c > r \\
  0 & c \leq 0.
\end{cases}
\]

Given that \( C \) is non-negative, the \( m \)-th moment expression

\[
\mathbb{E}(C^m) = m \int_0^r c^{m-1} P(C > c) dc,
\]

\[
= r^m - m \int_0^r c^{m-1} e^{-\lambda Q_{\gamma}(\gamma - c)} dc,
\]

follows; with the first moment given by

\[
\mathbb{E}(C) = r - \int_0^r e^{-\lambda Q_{\gamma}(\gamma - c)} dc,
\]

and the second moment given by

\[
\mathbb{E}(C^2) = r^2 - 2 \int_0^r ce^{-\lambda Q_{\gamma}(\gamma - c)} dc.
\]
5.4.1 Asymptotic results

We derive an asymptotic approximation to the rescaled mean measure $Q_\gamma(u)$, which gives a tractable and accurate expression. The angle function $\psi_\gamma$ expanded around $u = \gamma - r$ gives

$$
\psi_\gamma(u) \approx b_0 (u - \gamma + r)^{1/2} + b_1 (u - \gamma + r)^{3/2} + b_2 (u - \gamma + r)^{5/2},
$$

(5.20)

where the three expansion terms

$$
b_0 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2},
$$

(5.21)

$$
b_1 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{r^2 - 3r\gamma - 3\gamma^2}{12(\gamma^2r - \gamma r^2)} \right],
$$

(5.22)

$$
b_2 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{3r^4 + 25r^2\gamma^2 - 10r^3\gamma + 30\gamma^3r - 5\gamma^4}{160\gamma^2(\gamma - r)^2r^2} \right],
$$

(5.23)

follow. The asymptotic result of the rescaled mean measure

$$
Q_\gamma(u) \approx 4 \left[ \frac{b_0}{3} (u - \gamma + r)^{3/2} + \frac{b_1}{5} (u - \gamma + r)^{5/2} + \frac{b_2}{7} (u - \gamma + r)^{7/2} \right],
$$

(5.24)

immediately follows.

The second-order approximation results are accurate for a unit radius (see Fig. 5.2). Adding the third term only improves the results slightly. However, it may be needed for larger transmission radius models (as Fig. 5.3 suggests). Conversely, the two-term expansion gives accurate results (see Fig. 5.4) when substituted into the sink distribution (5.10). In fact, it appears that the
expansion (5.24), which consists only of elementary functions, can be used to give accurate results, which are clearly faster to evaluate than those based on elliptic integrals. However, we continue to use the exact solution of the integral (5.12), and later compare it and its approximation.

We now present asymptotic moment results for our spatially dependent node density model.

**Theorem 5.4.1.** For $\gamma > r$, under greedy routing the first hop moment

$$\mathbb{E}(C) \sim r - \frac{\Gamma(5/3)}{(\lambda q_0)^{2/3}},$$

and the second hop moment

$$\mathbb{E}(C^2) \sim r^2 - 2r \frac{\Gamma(5/3)}{(\lambda q_0)^{2/3}} + \frac{\Gamma(7/3)}{(\lambda q_0)^{4/3}},$$

as the initial node density $\lambda \to \infty$,

where $\Gamma(.)$ is the gamma function, and

$$q_0 = \frac{4}{3} \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2}.$$

**Proof.** As in Chapter 3, we consider integrals of the form

$$I(\lambda) = \int_a^b t^k e^{-\lambda Q(t)} dt,$$

where $Q(t) > Q(a)$ for all $t \in (a, b)$, and asymptotically

$$Q(t) \sim q_0(t - a)^\mu \quad \text{as} \quad t \to a.$$
The generalized Laplace’s method applied to this integral gives the asymptotic result

\[ I(\lambda) \sim \frac{\Gamma(\tau)}{\mu(\lambda q_0)^{\tau}} \quad \lambda \to \infty. \]

where \( \Gamma(\tau) \) is the gamma function, and

\[ \tau = \frac{2(k + 1)}{3}. \]

For Laplace’s method, \( Q(a) \) needs to be the minimum on the integral interval. Hence, the change of variable \( t = u - \gamma + r = r - c \) with a slight abuse of notation leads to the area function expansion

\[ Q_2(t) \sim q_0 t^{3/2} + q_1 t^{5/2} + O(t^{7/2}) \quad \text{as} \quad t \to 0, \quad (5.27) \]

where

\[ q_0 = \frac{4b_0}{3}, \quad \mu = \frac{3}{2}. \]

The first moment result (5.25) follows, and the change of variable applied to the second moment equation gives

\[ \mathbb{E}(C^2) = r^2 - 2 \int_0^r (r - t)e^{-\lambda Q_2(t)} dt, \]

which leads to the second result (5.26).
Figure 5.5: Numerical and asymptotic results for first moment ($r = 1$ and $\ell = 10$).

Figure 5.6: Numerical and asymptotic results for second moment ($r = 1$ and $\ell = 10$).
5.4.2 Sink dependence

The node intensity function is clearly dependent on the forwarding node sink distance. Comparing the hop distributions (in Fig. 5.7) reveals that a message is relayed farther in a single hop if the forwarding node is closer to the sink. Intuitively, hops increase stochastically as the message approaches the sink as more potential forwarding nodes are available in the forwarding region. Geometrically, we observe that the integral kernel in the mean measure equation (5.12) is the angle function $\psi_\gamma$, which decreases as $\gamma$ increases; that is

$$\psi_{\gamma_1}(u) \leq \psi_{\gamma_2}(u), \quad \gamma_1 \geq \gamma_2 \geq r.$$  

Conversely, $\psi_\gamma$ and, hence, the indefinite integral (5.16) increases as the sink distance decreases. This dependence on the sink distance of the forwarding node is simply referred to as the sink dependence.

The influence of the sink dependence can be observed by comparing the difference in two hop distributions with different sink distances $\gamma_1$ and $\gamma_2$. In Chapter 3, the hop distribution dependence on the sink distance under the homogeneous model hop was examined by a Kullback-Leibler analysis. Subsequently, the Kullback-Leibler divergence [48] applied to the mixed discrete-continuous hop distribution gives

$$D(\gamma_1, \gamma_2) = \int_0^r \frac{f_{\gamma_2}(c)}{f_{\gamma_1}(c)} dc + F_{\gamma_2}(0^+) \log \frac{F_{\gamma_2}(0^+)}{F_{\gamma_1}(0^+)},$$  

(5.29)
where the null probability is given by

\[
\bar{F}_\gamma(0^+) = \lim_{\epsilon \to 0} \bar{F}_\gamma(\epsilon),
\]

\[
= e^{-\lambda Q_\gamma(\gamma)}.
\]

The Kullback-Leibler divergence is non-negative and is naturally zero for identical distributions [48].

We calculated the integral in equation (5.29) numerically to observe how the hop distribution is influenced when we set $\gamma_1 = \ell$ and vary $\gamma_2 = \gamma$. The comparison reveals that $D(\ell, \gamma)$ is high near the sink and decreases as $\gamma$ increases (refer to Fig 5.8). This is the same expected behaviour as observed under the homogeneous model, however, $D(\gamma_1, \ell)$ decays relatively slowly under the inhomogeneous model. This contrasts starkly with the homogeneous model where $D(\gamma_1, \ell)$ is mostly zero away from the sink, and only shoots up near the sink. Moreover, $D(\gamma_1, \gamma)$ differs significantly for two different $\gamma_1$; hence, the forwarding node sink distance can differ greatly when calculating hop distributions. This distribution behaviour eliminates the possibility of using renewal processes to model message advancement over multiple hops.
5.5 Multihop analysis

We use the previous indexing for the random variables $U$ and $C$, hence $U_0 = \ell$ and $U_1 = U$ such that the $i$-th hop advancement is $C_i = U_{i-1} - U_i$. Each $C_i$ depends on the forwarding node’s sink distance $U_{i-1}$; thus, the sink distance of the source node clearly affects the first hop and, subsequently, following hops. As noted in the previous section, comparing the hop distributions demonstrates that each $C_i$ is stochastically dominated by $C_{i+1}$ in the sense that, for $i \geq 0$, we have the stochastic ordering

$$P(C_{i+1} > x) \geq P(C_i > x), \quad x \in (0, r).$$  \hspace{1cm} (5.30)

This inequality is the opposite to the equivalent result (3.28) under the homogeneous model as originally noted by Zorzi and Rao [80].

5.5.1 Path dependence

We adopt the the same notation used in previous chapters; thus, let the random variable $\Theta_i$ be the angle between the $i$-th node and the previous node in relation to the sink. We assign the point $X_i = (U_i, \Theta_i)$ to the $i$-th forwarding node, and recall the source node corresponds to $X_0 = (\ell, 0)$. A message travels $i$ hops along a path that corresponds to a sequence of random points $\bar{X}_i = (X_0, X_1, \ldots, X_i)$.

After the first hop greedy routing implies that path dependence arises in the distribution of $U_2$, which was observed under the homogeneous model in Chapter 2. Initially, we assume no sleep scheme exists to reduce the complexity of the problem. After our analysis, we include a simple sleep scheme and examine how varying $\rho$ and $\alpha$ (while fixing $\lambda$) affects the path dependence.

![Figure 5.9: No nodes in the intersection region $\mathcal{R}_B$ leads to path dependence.](image)

We use $F$ and $G$ to denote the distributions of $U_i$ under the independent and dependent mod-
els respectively. The distribution of \(U_{i+1}\) under the independent model is dependent only on the sink distance of the current forwarding node, thus we write

\[
F_i(u_{i+1}) = \mathbb{P}_I(U_{i+1} \leq u_{i+1} | U_i = u_i),
\]

while under the dependent model the distribution is dependent on the message path, and so we write

\[
G_i(u_{i+1}) = \mathbb{P}_D(U_{i+1} \leq u_{i+1} | \vec{X}_i = \vec{x}_i),
\]

where the subscripts \(I\) and \(D\) indicate probability measures under the two models, and the subscript notation has been used again to denote that the distributions (5.31) and (5.32) are dependent on \(U_i\) and \(\vec{X}_i\). The rescaled mean measure of the feasible region under the independent and dependent models are respectively denoted as \(Q_i(u_{i+1})\) and \(\hat{Q}_i(u_{i+1})\).

The rescaled mean measure under the independent model is always given by the original equation (5.16), and hence, the distribution and probability density of \(U_{i+1}\) are obtained by setting \(\gamma = u_i\) in equations (5.10) and (5.11). Under the dependent model if the rescaled mean measure is given after \(i\) hops, the sink distribution immediately follows as

\[
G_i(u_{i+1}) = \begin{cases} 
1 - e^{-\lambda \hat{Q}_i(u_{i+1})} & u_i - r \leq u < u_i \\
1 & u \geq u_i \\
0 & u < u_i - r.
\end{cases}
\]

as well as the probability density

\[
g_i(u_{i+1}) = \lambda \hat{Q}_i'(u_{i+1}) e^{-\lambda \hat{Q}_i(u_{i+1})}.
\]

Let the set \(I_i(u_{i+1}) \subset \mathbb{R}^2\) be the feasible region of the \(i\)-th forwarding node as a function of \(u_{i+1}\) under the independent model. Under the dependent model, it is possible to calculate the rescaled mean over the feasible region after one hop; see appendix. If \(i \geq 2\), we approximate the feasible region under the dependent model

\[
\mathcal{D}_i(u_{i+1}) \approx I_i(u_{i+1}) \setminus I_{i-1}(u_{j+1}),
\]

where we refer to this approximation as the one-hop model. Results in the previous chapter revealed that under the homogeneous model this approximation sufficiently captures the path dependence. Consequently, only the location of the previous node is needed to calculate the rescaled mean under the dependent model.

To describe the random behaviour of message hops we need the joint density of \(U_i\) and \(\Theta_i\). Under our inhomogeneous Poisson model, the joint probability density is

\[
g_{i+1}(u_{i+1}, \theta_{i+1}) = \lambda_{\mathcal{D}_i(u_{i+1}, \theta_{i+1})} e^{-\lambda \hat{Q}_i(u_{i+1})},
\]
where the derivation follows from that of the homogeneous model; see appendix for details.

The spatially dependent initial node density function $\lambda_{D_i}(u_{i+1}, \theta_{i+1}) = \lambda I_{D_i}(u_{i+1}, \theta_{i+1})$ and the indicator function of the dependent feasible region

$$I_{D_i}(u_{i+1}, \theta_{i+1}) = \begin{cases} 1, & (u_{i+1}, \theta_{0i}) \in D_i, \\ 0, & \text{otherwise}. \end{cases}$$

\section*{5.6 Sleep model}

We again make the assumption that the probability that a node is awake on each hop is $p$ and the event that a node is awake during a transmission attempt is independent of the event that it is awake at other transmission attempts. Consequently, the sleep scheme has a thinning effect on the local density. It follows that the node density function after one message hop is given by

$$\lambda_{D_i}(u_2, \theta_2) = \lambda \left[ I_{I_1 \setminus I_0}(u_2, \theta_2) + (1-p)I_{I_1 \cap I_0}(u_2, \theta_2) \right],$$

where the superscripts denote the indicator functions of the disjoint regions.

To calculate the Poisson mean measure over the region $D_i$, the above node density function is integrated over the domain leading to

$$\vec{\Lambda}_{D_i}(u_{i+1}) = \int_{u_{i+1}}^{u_{i+1}} \int_{w_{i+1}}^{w_{i+1}} \lambda D_i(w_{i+1}, \theta_{i+1}) q(w_{i+1}) w_{i+1} d\theta_{i+1} dw_{i+1},$$

where we define

$$\vec{Q}_{D_i}(u_{i+1}) = \frac{\vec{\Lambda}_{D_i}(u_{i+1})}{\lambda}.$$ 

The joint probability density of $U_i$ and $\Theta_i$, namely

$$g_i(u_{i+1}, \theta_{i+1}) = \lambda_{D_i}(u_{i+1}, \theta_{i+1}) e^{-\lambda \vec{Q}_{D_i}(u_{i+1})},$$

follows.

\section*{5.7 Multihop distribution}

\subsection*{5.7.1 Hop advancements}

Let the random variable $C_i$ be the advancement towards the sink in the $i$-th hop. The complement of the sink distribution yields the hop distribution under both the independent and dependent
models; the latter being

\[
\tilde{G}_i(c_{i+1}) = \begin{cases} 
  e^{-\lambda \bar{Q}_i(u_i - c_{i+1})} & 0 < c_{i+1} \leq r \\
  1 & c_{i+1} > r \\
  0 & c_{i+1} \leq 0.
\end{cases}
\] (5.41)

and its probability density defined on the absolutely continuous part of the support

\[
\bar{g}_i(c_{i+1}) = \lambda \bar{Q}_i'(u_i - c_{i+1}) e^{-\lambda \bar{Q}_i(u_i - c_{i+1})}.
\] (5.42)

### 5.7.2 Distribution of \( Z_n \)

Let the random variable \( Z_n \) represent the distance advanced by a message in \( n \) hops, thus

\[
Z_n = \sum_{i=1}^{n} C_i.
\] (5.43)

To calculate the distribution of \( Z_n \) we need the joint probability density of the random variables \( C_1 \) to \( C_n \) and \( \Theta_1 \) to \( \Theta_n \), which is similar to that under the homogeneous mode, thus

\[
\bar{g}_{(n-1)}(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) = \prod_{i=1}^{n} \lambda D_{i-1}(u_i - c_i, \theta_i) (u_{i-1} - c_{i-1}) e^{-\lambda \bar{Q}_{i-1}(u_{i-1} - c_{i-1})}.
\] (5.44)

It follows that the distribution of message advancement after \( n \) hops is expressed by

\[
\mathbb{P}_D(Z_n \leq z) = \int_{0^+}^{\min(z,r)} dc_1 \int_{-\psi_1(c_1)}^{\psi_1(c_1)} d\theta_1 \int_{0^+}^{\min(z-c_1,r)} dc_2 \int_{-\psi_1(c_2)}^{\psi_1(c_2)} d\theta_2 \cdots \int_{0^+}^{\min(z-\sum_{i=1}^{n-2} c_i,r)} dc_{n-1} \int_{-\psi_{n-1}(c_{n-1})}^{\psi_{n-1}(c_{n-1})} d\theta_{n-1} \int_{0^+}^{\min(z-\sum_{i=1}^{n-1} c_i,r)} dc_n \int_{-\psi_{n-1}(c_n)}^{\psi_{n-1}(c_n)} d\theta_n \bar{g}_{(n-1)}(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) d\theta_n
\] (5.45)

\[
+ \mathbb{P}_D(C_1 = 0) + \mathbb{P}_D((Z_1 \leq z) \cap (C_2 = 0)) + \ldots
\] (5.46)

\[
+ \mathbb{P}_D((Z_{n-1} \leq z) \cap (C_n = 0)).
\] (5.47)

where

\[
\psi_i(c_{i+1}) = \psi_{u_i}(u_i - c_{i+1}),
\] (5.49)

denotes the maximum angle for a sink distance given by the sink angle function (5.7). The distribution of \( Z_n \) under a sleep scheme is obtained by substituting the product of the joint probability densities (5.40).

Under the independent model the joint probability density is not a function of any of the variables \( \theta_1 \) to \( \theta_n \). Hence, the equivalent integral can be analytically integrated over the sink
angle domains, thus giving a simplified expression in the form of hop probability densities

\[
P_I(Z_n \leq z) = \int_{0^+}^{\min(z,r)} dc_1 \int_{0^+}^{\min(z-c_1,r)} dc_2 \ldots \int_{0^+}^{\min(z-\sum_{i=1}^{n-1} c_i,r)} f_0(c_1) \ldots f_{n-1}(c_n) dc_n \tag{5.50}
\]

\[
+ P_I(C_1 = 0) + P_I((Z_1 \leq z) \cap (C_2 = 0)) + \ldots \tag{5.51}
\]

\[
+ P_I((Z_{n-1} \leq z) \cap (C_n = 0)). \tag{5.52}
\]

The probability of a message reaching a routing void after advancing \(i\) hops

\[
P_D(C_{i+1} = 0 | \bar{X}_i = \bar{x}_i) = e^{-\lambda \bar{Q}_i(u_i)}, \tag{5.54}
\]

immediately follows. The routing void probability leads to the distribution of \(Z_n\) that conditioned on the event that the message meets a routing void on the last hop

\[
P_D((Z_n \leq z) \cap (C_{n+1} = 0)) = \int_{0^+}^{\min(z,r)} dc_1 \int_{-\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1 \ldots \int_{0^+}^{\min(z-\sum_{i=1}^{n-1} c_i,r)} dc_n \cdot \int_{-\psi_n-1(c_n)}^{\psi_n-1(c_n)} \tilde{g}_{n-1}(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) P_D(C_{n+1} = 0) d\theta_n. \tag{5.55}
\]

5.8 Integration methods

To obtain the distributions of \(Z_n\) under the dependent model, a \(2n\)-fold integral (5.45) needs to be evaluated. The form of this integral is very similar to that in the last chapter, hence we employ Monte Carlo methods.

5.8.1 Quasi-random points and lattice rules

In the previous chapter we mostly used leaped Halton sequences to calculate integrals. In this chapter, we also give some results based on lattice rules, which lead to specific cases of quasi-random sequences. These quasi-random sequences have been a very active research in recent years, which has resulted in significant advancements [50].

These types of quasi-random points are faster to compute, provided a generating vector, than those based other methods, and can result in faster convergence than other Monte Carlo methods if properly applied. However, the choice of a suitable generating vector depends on the nature of
the function, and in particular, its dependence on each integration variable. Consequently, lattice rules cannot be used blindly.

Lattice rules may offer a substantially faster way of evaluating the high-dimensional integrals presented the previous chapter. However, a thorough analysis of which generating vectors are the most suitable is beyond the scope of this work. Future extensions may involve experimenting with better behaving quasi-random sequences.

### 5.8.2 Importance sampling

We derive an importance sampling function which is very similar in form to that in the previous chapter. We recall the expansion of the $Q$ function (5.24) in which we use only the first term, hence

$$Q_\gamma(u) \sim q_0(u - \gamma + r)^{3/2} + O(u)^{5/2}$$

as $u \to \gamma - r$,  

where

$$q_0 = \frac{4}{3} \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2}.$$ 

A change of variable $c = \gamma - u$ leads to the function

$$\tilde{Q}_\gamma(c) = q_0(r - c)^{3/2},$$

which leads to an approximate solution for the hop distribution

$$\tilde{F}_\gamma(c) = e^{-\lambda \tilde{Q}_\gamma(c)}.$$ 

For the importance sampling function to give all the possible values from the unit interval $(0, 1)$, the quantity $\tilde{F}_\gamma(0)$ is subtracted from the above expression and the result is divided by $\Delta \tilde{F}_\gamma = \tilde{F}_\gamma(c_{\text{max}}) - \tilde{F}_\gamma(0)$, to obtain an importance sampling function

$$\hat{F}_\gamma(c) = \frac{1}{\Delta \tilde{F}_\gamma} \left[ e^{-\lambda \tilde{Q}_\gamma(c)} - \tilde{F}_\gamma(0) \right], \quad 0 \leq c \leq c_{\text{max}}.$$ 

(5.57)

where $c_{\text{max}}$ is the largest hop value. The subtracting of the routing void term has negligible effect for sufficiently large $\lambda$. The corresponding derivative needs to integrate to one, hence the rescaling step. The derivative

$$\hat{f}_\gamma(c) = \frac{3\lambda}{2\Delta \tilde{F}_\gamma} (r - c)^{1/2} e^{-\lambda \tilde{Q}_\gamma(c)},$$

exists on the same domain as the sampling function. The inverse of the sampling function is given by

$$\hat{F}_\gamma^{-1}(t) = r - \left( \frac{-1}{\lambda a_1} \ln \left[ t \Delta \tilde{F}_\gamma + \tilde{F}_\gamma(0) \right] \right)^{3/2}, \quad 0 \leq t \leq r.$$
Substituting a random variable from a uniform distribution, say \( T \sim U(0, r) \), into the inverse of the importance sampling function gives a random variable \( C \) adhering to the importance sampling distribution (5.57).

The importance sampling function should be updated after each hop by setting \( \gamma \) to the current sink distance for each sample. This step was not necessary under the homogeneous model in the previous chapter. However, under this inhomogeneous model each hop distribution relies more strongly on the sink distance. Furthermore, for high \( \lambda \), the importance sampling step should improve as it is based on the independent model, and at high node density the intersection regions grow stochastically smaller as argued in the previous chapters.

### 5.9 Simulation

We used routing simulations to verify our stochastic model and calculations. Given a source node sink distance \( \ell \), message relaying is simulated in a circular sensor field \( C_\ell \subset \mathbb{R}^2 \) of radius \( \ell \) with the sink located at the origin. The fact that messages only advance towards the sink under greedy routings implies that sensor field edges do not influence the message routing. The total number of nodes per simulation is a Poisson random variable with the parameter

\[
\Lambda(C_\ell) = \lambda \int_0^\ell \int_0^{2\pi} q(u) \, du \, d\theta
\]

\[
= 2\lambda \pi \ell.
\]

In simulation, a node is assigned two independent random variables \( \Theta_S \) and \( R_S \) corresponding to their polar coordinates in relation to the sink. To simulate node deployment such that the nodes adhere to the spatially dependent node density (5.2), both random variables are uniformly distributed

\[
P(\Theta_S \leq \theta) = \frac{\theta}{\pi}, \quad \theta \in [0, \pi],
\]

\[
P(R_S \leq r) = \frac{r}{\ell}, \quad r \in (0, \ell].
\]

### 5.10 Results

Monte Carlo integration and routing simulations were performed in Matlab on a standard desktop machine. The built-in Halton sequence generator was employed for the quasi-Monte Carlo integration. For a given value of \( z \), it took anywhere between \( 10^2 \) to \( 10^4 \) points to obtain a quasi-Monte Carlo estimate of the conditional distribution \( P(Z_n \leq z|+) \). The importance sampling step
improved the rate of convergence, particularly for high $\lambda$. The actual number of points depends on the number of hops; more hops require more points. All calculations took no longer than an hour to complete, and usually considerably less.

We compared the dependent model to routing simulations of various ensemble sizes with and without a blinking sleep scheme. Generally, anywhere between $10^3$ to $10^5$ routing simulations were required. Similar to the integration process, higher hop numbers required more simulations to give converged results. The routing simulations are based on the same assumptions made in the mathematical model. The empirical distributions were calculated simply by rescaling the message counts for each $z$ subinterval.

We observed under the inhomogeneous model that more points and simulations are needed to give similarly converged results compared to those obtained under the homogeneous model. An extensive empirical investigation is needed to see how fast quasi-Monte Carlo methods are compared to routing simulations. Furthermore, more empirical and theoretical evidence is needed to elucidate the advantages and disadvantages of calculating probabilistic behaviour via our model.

For a single hop, the mathematical model results are practically identical to the simulation results (see Fig. 5.10). This implies that the model adequately captures the spatially dependent node density function, and also vindicates the use of the elliptic integral functions.

The lattice rule approach was only used to calculate one set of results (see Fig. 5.12). We used
Figure 5.11: Results of $P(Z_2 \leq z|\cdot)$ via quasi-Monte Carlo integration and simulations ($p = 1, \lambda = 3\ell, r = 1$ and $\ell = 10$).

Figure 5.12: Results of $P(Z_2 \leq z|\cdot)$ via lattice rules integration (based on analytic and asymptotic expressions) and simulations ($p = 1, \lambda = 3\ell, r = 1$ and $\ell = 10$).
Despite the accuracy of elliptic integral functions, we found that the results based on them could be replaced with results that used the three-term approximations (5.20) and (5.24) with no discernible loss of accuracy (see the indistinguishable plots in Fig. 5.12). Since these results only involve elementary functions, evaluating them is considerably faster, which is naturally convenient for Monte Carlo approaches. Consequently, the remaining results are based on these approximations (Fig. 5.13 to Fig. 5.15).

We calculated the distribution of $N$, and included a simple sleep scheme to see its influence on the path dependence. We compared the independent model to the dependent model by varying $p$ (and accordingly $\alpha$, thus holding $\lambda$ constant). The difference between the two models has an accumulative effect on $N$, hence it serves as a good indicator of path dependence.

Under the independent model, the distribution $\mathbb{P}(N \leq n)$ gives greater values for each $n$ than the equivalent result under the dependent model. Also, the path dependence clearly lessens as $p$
Figure 5.14: Independent and dependent model results of $\mathbb{P}(N \leq n)$ compared to simulations ($p = 0.1, \lambda = 2\ell, r = 1$ and $\ell = 10$).

Figure 5.15: Independent and dependent model results of $\mathbb{P}(N \leq n)$ compared to simulations ($p = 1, \lambda = 3\ell, r = 1$ and $\ell = 10$).
approaches zero (compare Fig. 5.13 and Fig. 5.14). These results are analogous to those under the homogeneous density models in the previous chapter.

For large $\lambda$, the difference between the two models is less. As is the case for the homogeneous, a larger density results in the next forwarding node being closer to the sink, thus reducing the intersection of the feasible regions and lessening the path dependence. Moreover, under the inhomogeneous model hops grows stochastically larger due to the increasing node density. Thus, we believe that the path dependence continues to decrease stochastically as the message approaches the sink.

In conclusion, the results reveal that the dependent model clearly captures the path dependence. The resulting expressions, both involving the elliptic integrals and the asymptotic expansions, give results that closely agree with simulations. We believe more numerical investigation is needed to choose the most appropriate quasi-random sequences in this setting.

### 5.11 Chapter Summary

We presented a tractable inhomogeneous spatially dependent density model. Inspired by previous work, we developed and examined a greedy routing model that incorporates both sink and path dependence. Moreover, the spatially dependent density model verified that the formulation of the homogeneous model can be extended to an inhomogeneous case. This model is an alternative means of ascertaining the stochastic characteristics of greedy routing in sensor and ad hoc networks. We used quasi-Monte Carlo methods coupled with importance sampling to estimate the resulting high-dimensional integrals. For a sufficient number of function samples, all the results agreed admirably with those obtained by simulations.

Finally, we included a sleep scheme to demonstrate its effects on the local node density and the path dependence. For systems with a low $p$ the results imply that the independent model can be used, thus reducing computation time in calculating stochastic properties of the system.
Chapter 6

Random Transmission Radius

In this chapter we return to the homogeneous node deployment model and extend the model such that each node has a random transmission radius. The effects of this assumption are outlined and communication considerations are addressed. We give a preliminary introduction to a communication model and study some specific examples of random transmission models. We describe analytic, asymptotic, and numerical methods that can be applied. Similarly to previous work, we derive multiple integral expressions and evaluate them with quasi-Monte Carlo methods. Theoretical analysis is compared with routing simulations when applicable. We discuss mathematical difficulties with evaluating certain models, and give future possible model extensions.

6.1 Introduction

The model analysis and development thus far have been performed under the assumption that each node is located on a plane and has a constant transmission radius. The tractable nature of this disk model has resulted in it being used frequently in analyzing various types of wireless networks [19, 74, 80], of which an early example was a packet radio network [42]. Consequently, a large amount of analysis, both mathematical and simulation-based, has been performed under this simplifying assumption.

Unfortunately, this assumption in transmission models does not sufficiently reflect reality. Sensor nodes and their power sources are not produced in a fashion that results in them having constant transmission radii. More notably, transmission signals are susceptible to various environmental influences including the transmission medium, noise, surrounding obstacles, and interference from other transmissions. Furthermore, having a node within the transmission range
of a forwarding node does not guarantee it will successfully receive the message. These factors lead to signal fading or attenuation, reduced likelihood of message receiving, and, varying transmission radii across the network.

Criticisms of this nature are not new [69], but have become more prevalent in recent years with the emergence of general ad hoc network research [45, 47]. Consequently, network models and analysis techniques are being developed to include realistic assumptions such as fading and its effects on routing performance [31, 81]. For more details, we refer the reader to the recent article on stochastic geometry and related techniques [32], which covers network model considerations such as fading and noise.

The obvious need for more realistic transmission assumptions motivates us to adapt our model. In particular, we extend our model such that each node has a random transmission radius. However, the aim of this work is not to accurately model transmission ranges per se. Rather, we wish to present a small number of random radius models and examine their direct effects. Moreover, we wish to validate our previously developed mathematical formulation in this setting and outline considerations in constructing models. Ideally, the ensuing results will contribute to developing routing models that sufficiently capture the random behaviour of transmission radii.

### 6.2 Mathematical model

We assume that for a successful message relay or transmission to occur the signal-to-noise-and-interference-ratio (SINR) of the receiving node needs to be above a certain network-specific threshold $t_0$. For points $x, y \in \mathbb{R}^2$, if a forwarding node is at $x$, then the SINR at $y$ is a stochastic process $S$ given by

$$S(y) = \frac{P_R(d)}{P_N(y) + P_I(y)},$$

(6.1)

where $d = |x - y|$ is the distance from the source, $P_R(d)$ is the power of the received signal, $P_N(y)$ is the background-noise power, and $P_I(y)$ is the interference power experienced by the receiving node. These three power terms can be modelled as stochastic processes (though $P_N(y)$ is often assumed to be constant). Furthermore, it is usually assumed that the received power obeys some path-loss law $P_R = P_0 \rho(d)$ where the stochastic process $P_0$ is proportional to the signal power at the message source, and $\rho(d)$ is an attenuation function. The Rayleigh network model, which leads to an exponentially distributed $P_0$, is a popular approach due to its tractable characteristics [32]. The stochastic process $P_I(y)$ is a combination of signals from some or all the transmitting nodes in the network, hence it depends on the choice of the path-loss law. If the nodes adhere
to a homogeneous Poisson process, then it can be shown that $P_I(y)$ is spatially independent, and is often modelled with an alpha-stable distribution [32]. If we assume that $P_I(y)$ and $P_N(y)$ are independent of $y$, then the righthand side of expression (6.1) is only dependent on $d$, and, with an abuse of notation, we write $S(y) = S(d)$.

Based on the SINR, we define the transmission radius

$$R = \inf\{d : S(d) > t_0\}$$

to allow a message to be received by a node at $y$. Understanding and modelling SINR in wireless networks is vital as it gives an information-theoretic upper bound to the amount of data that can be sent across a communication link. However, we do not suggest or develop any particular SINR model. Rather, we suggest some random radius distributions purely for analyzing their effects on our model. On this note, we point out that recently Kaj [39] has derived random radius distributions under some SINR assumptions, but we are unaware of any published results of this kind.

Let the nonnegative random variable $R_i$ be the radius of the $i$-th forwarding node; hence the source node has a radius $R_0$. We assume that nodes communicate data radially, and that $\mathbb{E}(R_i) < \infty$. Thus, the radiation pattern of the node’s antenna is a circle with a transmission radius $R_i$. We assume that all $R_i$ are independent and identically distributed according to a distribution denoted by

$$F_{R_i}(r_i) = \mathbb{P}(R_i \leq r_i).$$

We assume that the distribution is absolutely continuous on its domain, which may be bounded or not, and denote the probability density as $f_{R_i}(r_i)$. In general, we often write the radius random variable simply as $R$. Most numerical results are given for the scenario where the expectation of $R$ is set at one.

No set of any nodes have identical transmission radii (with probability one), thus it is possible for a node $Y$ to be within transmission radius of node $X$ even when node $X$ is not within the transmission radius of node $Y$. Taking this consideration into account, for message relays to occur under a varying-radius model, two primary communication assumptions exist, which result in the models:

**One-way** Only the next forwarding node needs to be within the transmission radius of the current forwarding node.
**Two-way** Both the source node and the next forwarding node need to be within each other’s transmission radius.

Let the random variable $H_i$ be the distance that a message travels in the $i$-th hop towards the next forwarding node (not to be confused with $C_i$). Clearly $H_i$ is bounded by the forwarding node’s transmission radius $R_{i-1}$. Under the one-way model the radius of the next forwarding node is not important. This implies that all awake nodes in the total feasible region are eligible, hence, the awake node density remains a constant $\lambda$.

Under the two-way model $H_i$ is bounded by the transmission radius of the forwarding node and the next forwarding node. Consequently, not all nodes within the total feasible region of the forwarding node are eligible to become the next forwarding node, which clearly affects the node density by thinning out the nodes that do not have sufficiently large radii.

The two-way model may be a more desirable model since bilateral communication results in better cooperation in ad hoc networks. However, the one-way model is a simpler model as the node density is not affected by thinning, which increases the complexity of the governing expressions under this model. Consequently, we focus on applying previously used techniques to the one-way model, and then discuss the effects of the two-way model in Section 6.5.

Under the two-way model, there is another model consideration that warrants further attention. If a message is received by a node with some sampled transmission radius, then we assume that when the node forwards the message it has the same sampled transmission radius. Alternatively, we could assume that the transmission radii of all nodes are completely re-sampled from the radius distribution after each hop. This radius re-sampling assumption has no influence on the one-way model as the transmission radius of the forwarding node only matters when the node relays the message, and not when it receives it. However, the re-sampling assumption does have an influence on the two-way model. We refer to this model as the re-sampled two-way model and the former model as the one-sampled two-way or simply the two-way model. We study the differences of the two models further in Section 6.5.

We now introduce some possible continuous random variables for modelling transmission radii.

**Uniform distribution**

We consider the uniform distribution because of its apparent simplicity. The node transmission radius distribution

$$F_R(r) = \frac{r}{r_{\text{max}} - r_{\text{min}}}, \quad r \in [r_{\text{min}}, r_{\text{max}}],$$

where the parameters $r_{\text{min}}$ and $r_{\text{max}}$ denote the minimum and maximum transmission radius values. The probability density

$$f_R(r) = \frac{1}{r_{\text{max}} - r_{\text{min}}}, \quad r \in [r_{\text{min}}, r_{\text{max}}],$$

immediately follows.

**Weibull distribution**

We examine the Weibull distribution because it is sufficiently general and can be adjusted to behave like other distributions. The node transmission radius distribution

$$F_R(r) = 1 - e^{-(r/\beta_w)^{k_w}}, \quad r \in [0, \infty),$$

where $k_w$ and $\beta_w$ are the Weibull shape and scale parameters. The probability density

$$f_R(r) = \left(\frac{k_w}{\beta_w}\right) \left(\frac{r}{\beta_w}\right)^{k_w-1} e^{-(r/\beta_w)^{k_w}}, \quad r \in [0, \infty),$$

immediately follows. By setting the parameters accordingly, the Weibull distribution reduces to specific distributions such as the exponential distribution ($k_w = 1$) and Rayleigh distribution ($k_w = 2$).

**Truncated Weibull distribution**

To give a bounded transmission radius, we truncate and rescale the Weibull distribution, obtaining the distribution

$$F_R(r) = \begin{cases} 
  k_T \left[1 - e^{-(r/\beta_w)^{k_w}}\right], & r \in [0, r_{\text{max}}] \\
  1, & r \in [r_{\text{max}}, \infty),
\end{cases} \quad (6.2)$$

where the normalization constant is simply

$$k_T = \left[1 - e^{-(r_{\text{max}}/\beta_w)^{k_w}}\right]^{-1}.$$

The probability density

$$f_R(r) = k_T \left(\frac{k_w}{\beta_w}\right) \left(\frac{r}{\beta_w}\right)^{k_w-1} e^{-(r/\beta_w)^{k_w}}, \quad r \in [0, r_{\text{max}}], \quad (6.3)$$

immediately follows.

All the above distributions have closed-form and invertible expressions for the distribution function. This is ideal for producing random numbers that adhere to that particular distribution.
using the inverse transform method. Consequently, these distributions can be used in importance sampling to speed up Monte Carlo integration methods. The inverse method is also a good way of maintaining the order and, thus, the low-discrepancy of quasi-random sequences [70].

### 6.3 Total number of hops

We recall the random variable $Z_n$, which is the total message advancement towards the sink after $n$ hops. In Chapter 4, we gave a result (4.3.1) relating the distributions of $Z_n$ and $N$, which we extend for the random radius model.

**Theorem 6.3.1.** For a sink distance $\ell$, the number of message hops $N$ has the distribution

\begin{equation}
\Pr_D(N = 1) = \Pr(R_0 > \ell),
\end{equation}

\begin{equation}
\Pr_D(N = n|N \geq n) = \Pr_D(\ell - Z_{n-1} \leq R_{n-1}), \quad \forall n \geq 2,
\end{equation}

where

\begin{equation}
\Pr_D(\ell - Z_{n-1} \leq R_{n-1}) = \int_0^{\infty} \Pr_D(\ell - Z_{n-1} \leq r|R_{n-1} = r)f_R(r)dr.
\end{equation}

**Proof.** For $n = 1$ the first result is immediate. Furthermore, after $n \geq 2$ hops, if $\ell - Z_{n-1} \leq R_{n-1}$, then only one more hop is needed to reach the sink, and so

\[\Pr_D(N = n|Z_{n-1} < \ell) = \Pr_D(\ell - Z_{n-1} \leq R_{n-1}).\]

The rest of the proof quickly follows. \(\square\)

This result holds under both the one-way and two-way models if we assume the sink transmission radius is larger than that of the last forwarding node. However, we see in Section 6.5 that the two-way model dramatically increases the difficulty of computing the distribution of $Z_n$.

### 6.4 One-way model

#### 6.4.1 Single hop distribution

For the source node radius $R = r$, we recall that the random variable $C$ represents the advancement towards the sink under greedy routing. Consequently, the hop distribution (3.9) leads to the conditional distribution

\[\Pr(C \leq c|R = r < \gamma) = \begin{cases} 
e^{-\lambda A_r(\gamma - c)} & 0 < c \leq r \\ 1 & c > r \\ 0 & c \leq 0, \end{cases}\]

\begin{equation}
\end{equation}
Furthermore, the non-negative $C$ random variable has the $m$-th conditional moment expression given by

$$
E(C^m|R = r < \gamma) = m \int_0^r c^{m-1} P(C > c|R = r < \gamma) dc
= r^m - m \int_0^r c^{m-1} e^{-A_{\gamma,r}(\gamma-c)} dc,
$$

follows, which gives the first moment

$$
E(C|R = r < \gamma) = r - \int_0^r e^{-A_{\gamma,r}(\gamma-c)} dc, \quad (6.8)
$$

and the second moment

$$
E(C^2|R = r < \gamma) = r^2 - 2 \int_0^r c e^{-A_{\gamma,r}(\gamma-c)} dc. \quad (6.9)
$$

We recall the feasible area function

$$
A_{\gamma,r}(u) = r^2 \phi_{\gamma,r}(u) + u^2 \psi_{\gamma,r}(u) - u \gamma \sin \psi_{\gamma,r}(u), \quad \gamma - r \leq u \leq \gamma, \quad (6.10)
$$

the angles of the two intersecting sectors as $2\phi_{\gamma,r}$ and $2\psi_{\gamma,r}$, namely

$$
\phi_{\gamma,r}(u) = \arccos \left( \frac{r^2 + \gamma^2 - u^2}{2r\gamma} \right), \quad (6.11)
$$

$$
\psi_{\gamma,r}(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2u\gamma} \right). \quad (6.12)
$$

and the derivative of the area function

$$
A'_{\gamma,r}(u) = 2u \psi_{\gamma,r}(u). \quad (6.13)
$$

For distributions in which the radius can be greater than the sink distance there is a positive probability that the message will reach the sink in one hop, thus the expectation of the first moment is given by

$$
E(C) = \int_0^\gamma E(C|R = r < \gamma) dF_R(r) + \gamma(1 - F_R(\gamma)). \quad (6.14)
$$

and the second moment is given by

$$
E(C^2) = \int_0^\gamma E(C^2|R = r < \gamma) dF_R(r) + \gamma^2(1 - F_R(\gamma^2)), \quad (6.15)
$$

where for sufficiently large $\gamma$ the terms on the far right of these two expressions can be often omitted without a significant loss in accuracy.
6.4.2 Asymptotic analysis

In previous chapters we observed that moment expressions that involve one-dimensional integrals often do not lend themselves to analytic means; hence, the integrals were approximated by asymptotic methods. We use similar methods to derive moment expressions in random radii setting.

We recall the first moment asymptotic expression (3.12) such that the equivalent conditional expression

\[ E(C|R = r) \sim r - \frac{\Gamma(5/3)}{[\lambda a_0(r)]^{2/3}}, \quad \text{as} \quad \lambda \to \infty, \tag{6.16} \]

follows, and the expansion term

\[ a_0(r) = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{4(\gamma - r)}{3} \right], \tag{6.17} \]

is clearly a function of \( r \). Taking the expectation with respect to \( R \) gives the asymptotic result

\[ E(C) \sim E(R) - \frac{\Gamma(5/3)E[a_0(R)^{-2/3}]}{\lambda^{2/3}}, \quad \text{as} \quad \lambda \to \infty, \tag{6.18} \]

where

\[ a_0(r)^{-2/3} = \left[ \frac{9\gamma}{32} \right]^{1/3} \frac{1}{[r(\gamma - r)]^{1/3}}. \tag{6.19} \]

Similarly, the second moment asymptotic expression (3.13)

\[ E(C^2|R = r) \sim r^2 - \frac{2r\Gamma(5/3)}{[\lambda a_0(r)]^{2/3}} + \frac{\Gamma(7/3)}{[\lambda a_0(r)]^{4/3}}, \quad \text{as} \quad \lambda \to \infty, \tag{6.20} \]

leads to

\[ E(C^2) \sim E(R^2) - \frac{2\Gamma(5/3)E[a_0(R)^{-2/3}R]}{\lambda^{2/3}} + \frac{\Gamma(7/3)E[a_0(R)^{-4/3}]}{\lambda^{4/3}}, \quad \text{as} \quad \lambda \to \infty, \tag{6.21} \]

where

\[ a_0(r)^{-4/3} = \left[ \frac{9\gamma}{32} \right]^{2/3} \frac{1}{[r(\gamma - r)]^{2/3}}. \tag{6.22} \]

Some difficulty lies in obtaining the expectation of the \( a_0(R) \) terms. The right hand side in the expression (6.19) can be expanded around \( r = 0 \), giving the approximation

\[ \frac{1}{[r(\gamma - r)]^{1/3}} \approx \frac{1}{(\gamma r)^{1/3}} + \frac{r^{2/3}}{3\gamma^{4/3}}, \tag{6.23} \]

to simplify forthcoming expectation integrals in the next section. The above approximation leads to

\[ \frac{r}{[r(\gamma - r)]^{1/3}} \approx \frac{r^{2/3}}{\gamma^{1/3}}, \tag{6.24} \]

\[ \frac{1}{[r(\gamma - r)]^{2/3}} \approx \frac{1}{(\gamma r)^{2/3}} + \frac{2r^{1/3}}{3\gamma^{5/3}}, \tag{6.25} \]

both of which are used in the second moment approximation.
Figure 6.1: Approximation of $\mathbb{E}(C)$ ($\ell = 10$, $r_{\text{min}} = 0.5$, $r_{\text{max}} = 1.5$, uniform distribution).

Figure 6.2: Approximation of $\mathbb{E}(C^2)$ ($\ell = 10$, $r_{\text{min}} = 0.5$, $r_{\text{max}} = 1.5$, uniform distribution).
Uniform distribution

For the uniform distribution, the first two moments of the transmission radius

\[ E(R) = \frac{r_{\text{max}} + r_{\text{min}}}{2}, \]  
\[ E(R^2) = \frac{r_{\text{max}}^2 + r_{\text{max}}r_{\text{min}} + r_{\text{min}}^2}{3}, \]

follow while approximations (6.23), (6.24) and (6.25) yield

\[ E[a_0(R)^{-2/3}] \approx \left[ \frac{9\gamma}{32} \right]^{1/3} \left[ \frac{3(r_{\text{max}}^{2/3} - r_{\text{min}}^{2/3})}{2\gamma^{1/3}(r_{\text{max}} - r_{\text{min}})} + \frac{(r_{\text{max}}^{5/3} - r_{\text{min}}^{5/3})}{5\gamma^{4/3}(r_{\text{max}} - r_{\text{min}})} \right], \]
\[ E[a_0(R)^{-2/3}R] \approx \left[ \frac{9\gamma}{32} \right]^{1/3} \left[ \frac{3(r_{\text{max}}^{5/3} - r_{\text{min}}^{5/3})}{5\gamma^{4/3}(r_{\text{max}} - r_{\text{min}})} \right], \]
\[ E[a_0(R)^{-4/3}] \approx \left[ \frac{9\gamma}{32} \right]^{2/3} \left[ \frac{3(r_{\text{max}}^{1/3} - r_{\text{min}}^{1/3})}{\gamma^{2/3}(r_{\text{max}} - r_{\text{min}})} + \frac{(r_{\text{max}}^{4/3} - r_{\text{min}}^{4/3})}{2\gamma^{5/3}(r_{\text{max}} - r_{\text{min}})} \right]. \]

Truncated Weibull distribution

The form of the truncated Weibull probability density necessitates the use of the lower incomplete gamma function

\[ \Gamma^L(n, x) = \int_0^x t^{n-1}e^{-t}dt. \]
The change of variables $r = \beta_w t^{1/k}$ readily gives the two radius moments

$$
\mathbb{E}(R) = k_N \beta_w \Gamma^L(1 + 1/k_w, x_{\text{max}}), \quad (6.31)
$$

$$
\mathbb{E}(R^2) = k_N \beta_w^2 \Gamma^L(1 + 2/k_w, x_{\text{max}}), \quad (6.32)
$$

where the parameter

$$
x_{\text{max}} = (r_{\text{max}}/\beta_w)^{k_w}, \quad (6.33)
$$

follows. The approximations (6.23), (6.24) and (6.25) give

$$
\mathbb{E}[a_0(R)^{-2/3}] \approx k_N \left[ \frac{9\gamma}{32} \right]^{1/3} \frac{\Gamma^L(1 - 1/(3k_w), x_{\text{max}})}{(\gamma\beta_w)^{1/3}} + \frac{\beta_w^{2/3}\Gamma^L(1 + 2/(3k_w), x_{\text{max}})}{3\gamma^{2/3}},
$$

$$
\mathbb{E}[a_0(R)^{-2/3}R] \approx k_N \left[ \frac{9\gamma}{32} \right]^{1/3} \beta_w^{2/3} \Gamma^L(1 + 2/(3k_w), x_{\text{max}}), \quad (6.34)
$$

$$
\mathbb{E}[a_0(R)^{-4/3}] \approx k_N \left[ \frac{9\gamma}{32} \right]^{2/3} \frac{\Gamma^L(1 - 2/(3k_w), x_{\text{max}})}{(\gamma\beta_w)^{2/3}} + \frac{2\beta_w^{1/3}\Gamma^L(1 + 1/(3k_w), x_{\text{max}})}{3\gamma^{5/3}}. \quad (6.35)
$$

Naturally, for large $x_{\text{max}}$, the gamma function closely approximates the lower incomplete gamma function in the aforementioned expressions.
6.4.3 Asymptotic results

For the uniform distribution, the asymptotic moment expressions (6.18) and (6.21) work reasonably well (see Fig. 6.1 and Fig. 6.2). The first moment approximation is generally more accurate than the second moment approximation (Fig. 6.2), which is a phenomenon we observed under the constant radius model in Chapter 3. The approximation worsens when $\ell$ is small (Fig. 6.3), which is caused likely by a breakdown in the area function expansion.

There are more accurate results for the first moments under the Weibull distribution assumption (see the $k_w = 2$ or Rayleigh case in Fig. 6.4). Again, the second moment approximation is less accurate (Fig. 6.5). For $k_w = 3$, the first moment is particularly accurate (see Fig. 6.6). However, the approximation breaks down when the radius is exponentially distributed ($k_w = 1$) (see Fig. 6.7).

The asymptotic moment expressions (6.18) and (6.21) generally give good results, particularly for the first moments. We are unsure of the cause of the reduction in accuracy for certain approximations such as when the radius is exponentially distributed. Future work may involve examined these approximations in more detail and perhaps deriving better approximations in general, which may be achieved possibly by including more expansion terms or expanding functions at more suitable points.

Figure 6.5: Approximation of $E(C^2)$ ($\ell=10$, $r_{\text{max}}=1.5$, truncated Weibull distribution $\beta_w=2$ and $k_w=2$).
6.4.4 Multihop distribution

We examine the multihop distribution under the one-way random radius model. We point out that for the random radius model the feasible area function becomes more complicated. For example, it is possible that a message is relayed to a node whose transmission range is completely surrounded by the previous node, thus it has zero total feasible region under the dependent model.

For the one-hop dependent model, we outline a procedure for calculating the intersection area of two nodes with radii $r_i$ and $r_{i-1}$ (see Section A.8 for details). As for the constant radius model, this area-calculating procedure leads to the dependent joint density, and thus the conditional distribution of $Z_n$ is given by

$$
P_D(Z_n \leq z|+) = \int_0^\infty dr_0 \int_0^{\min(z, r_0)} dc_1 \int_{-\psi_0(c_1)}^{\psi_0(c_1)} d\theta_1 \int_0^\infty dr_1 \int_0^{\min(z - c_1, r_1)} dc_2 \ldots \tag{6.36}$$

$$\int_{-\psi_1(c_2)}^{\psi_1(c_2)} d\theta_2 \int_0^\infty dr_{n-1} \int_0^{\min(z - \sum_{i=1}^{n-1} c_i, r_{n-1})} dc_n \ldots \tag{6.37}$$

$$\int_{-\psi_{n-1}(c_n)}^{\psi_{n-1}(c_n)} \bar{g}_{n-1}(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) f_R(r_0) \ldots f_R(r_{n-1}) d\theta_n. \tag{6.38}$$

Figure 6.6: Approximation of $E(C)$ ($\ell = 10, r_{\text{max}} = 1.5$, truncated Weibull distribution $\beta_w = 2$ and $k_w = 3$).
which simplifies under the independent model
\[ P_I(Z_n \leq z) = \int_0^\infty dr_0 \int_0^{\min(z,r_0)} dc_1 \int_0^\infty dr_1 \int_0^{\min(z-c_1,r_1)} dc_2 \ldots \]
\[ \int_0^\infty dr_{n-1} \int_0^{\min(z-\sum_{i=1}^{n-1} c_i,r_i)} f_0(c_1) \ldots f_{n-1}(c_n)f_R(r_0) \ldots f_R(r_{n-1}) dc_n. \]  

(6.39)

(6.40)

To test these expressions, we evaluate the integrals via the quasi-Monte Carlo methods that were outlined in the last two chapters.

For three hops, the simulation and integration results agree well for uniformly and Rayleigh distributed radii (see Fig. 6.8 and 6.9). It appears that the one-hop dependent model sufficiently captures the path dependence under these random radius models in a similar manner to that of the constant radius model.

For uniformly and Rayleigh distributed radii we varied the sleep parameter \( p \) under the dependent model, and compared the results to those under the independent model (see Fig. 6.10 and 6.11). The results show that the path dependence reduces regardless of the behaviour of each transmission radius. We notice that there is a slightly larger difference between the independent model and the dependent model (with \( p = 0.1 \)) than the equivalent results under the constant radius model. This observation may stem from the larger transmission radii (and hence intersection regions) that are possible, which implies that the path dependence is larger.

The quasi-Monte Carlo and simulation results generally agree for higher hop numbers. How-
Figure 6.8: Quasi-Monte Carlo integration compared to empirical conditional distribution of $Z_3$ under one-way model (uniformly distributed radius, $\mathbb{E}(R) = 1$, $r_{\text{min}} = 0.5$, $r_{\text{max}} = 1.5$) ($\lambda = 3$, $\ell = 10$, $p=1$).

Figure 6.9: Quasi-Monte Carlo integration compared to empirical conditional distribution of $Z_3$ under one-way model (Rayleigh distributed radius, $\mathbb{E}(R) = 1$) ($\lambda = 3$, $\ell = 10$, $p=1$).
ever, the performance of the quasi-Monte Carlo scheme decreases as $n$ increases. This was witnessed under the constant radius model in Chapters 4 and 5. Under the random radius model, we observe that more function samples and routing simulations are generally needed. A more rigorous examination of this is needed, but it is possibly caused by an increase in variance of the model stemming from the random radius variable. Finally, as in the constant radius model, fewer function samples are needed to estimate the integrals when $\lambda$ is large, which is again attributable to the importance sampling.

Under the one-way random radius model, the expectation of the conditional distribution (6.7) with respect to $R$ gives the distribution of $C$. We emphasize that a worthwhile goal is to derive a closed-form expression or approximation of this distribution. Consequently, given sufficiently low $p$, the multihop advancement can be modelled by the convolution model. Provided a constant $\lambda$, this model would give a very fast and straightforward way for modelling multihop advancements; thus reducing the need for high-dimensional integration.

### 6.5 Two-way model

Under the two-way model, both the forwarding and the receiving nodes must be able to communicate with each other in order for a successful transmission to take place. Although a reasonable
Figure 6.11: Conditional distribution of $Z_4$ under one-way independent and dependent models (Rayleigh distributed radius, $\mathbb{E}(R) = 1$, $(\lambda = 3, \ell = 10)$.

requirement, we observe that this increases the model complexity by considering a potential forwarding node at the point $(u_1, \theta_1)$ in relation to the sink and the source node. In a single hop a message originating from the source node travels the distance

$$h_{01}(u_1, \theta_1) = \left( u_1^2 + \ell^2 - 2u_1\ell \cos \theta_1 \right)^{1/2},$$

(6.41)

It follows that the two-way model induces a local spatially dependent node density within the total feasible region of the source node.

**Theorem 6.5.1.** Given a source node transmission radius $R_0 = r_0$, the density of eligible awake nodes at the point $(u_1, \theta_1)$ under the two-way model is thus

$$\lambda(u_1, \theta_1) = \lambda \left[ 1 - F_R(h_{01}(u_1, \theta_1)) \right].$$

**Proof.** Given the necessary condition $R_0 > h_{01}$, the two-way model assumption implies $R_1 > h_{01}$. $R_i$ are independent and identically distributed, thus probability of a node meeting the second
condition given the first
\[
P(R_1 > h_{01} | R_0 \leq r_0) = \frac{P((R_1 > h_{01}) \cap (R_0 \leq r_0))}{P(R_0 \leq r_0)},
\]
\[
= \frac{P(R_1 > h_{01})P(R_0 \leq r_0)}{P(R_0 \leq r_0)},
\]
\[
= P(R_1 > h_{01}),
\]
(6.42)
(6.43)
follows, which gives the retention probability (or the complement of the thinning probability) of a node being able to communicate to the forwarding node.

We examine briefly how this thinning effect caused by the two-way model increases the complexity of the probability equations.

### 6.5.1 Mean measure

In general, the nodes that can be reached in a single hop transmission from a given node are distributed according to an inhomogeneous Poisson process with a density \( \lambda(u) \). For a source node's total feasible region \( I_\gamma \subset \mathbb{R}^2 \), we integrate the density function over the region in polar coordinates to obtain the mean measure
\[
\Lambda(I_\gamma) = \int_{I_\gamma} \lambda_{01}(u_1, \theta_1) u_1 du_1 d\theta_1.
\]
(6.44)
Unfortunately, the form of the thinning functions for our density expressions renders it unmanageable to analytic methods in general. However, some progress can be made by using asymptotic methods for particular cases. For example, the simple approximation
\[
h_{01}(u_1, \theta_1) = \left( (u_1 - \ell)^2 - u_1 \theta_1^2 \right)^{1/2},
\]
is rather accurate owing to the limited domain of \( \theta_1 \). Furthermore, if we set \( k = 2 \) to obtain the Rayleigh distribution, the resulting \( \lambda(r_0, u_1, \theta_1) \) approximation can be integrated over the \( \theta_1 \) domain to obtain
\[
\int_{-\psi(u_1)}^{\psi(u_1)} \lambda(u_1, \theta_1) d\theta_1 = \lambda \exp \left[ -((u_1 - \ell)/\beta_w)^2 \right] \text{erf} \left( \sqrt{\ell \psi(u_1)}/\beta_w \right) \sqrt{\ell \beta_w},
\]
where \( \text{erf}(\cdot) \) is the error function. To integrate this expression, further approximations are needed such as using the \( \psi_\gamma(u) \) expansion and an asymptotic result for the error function (for small or large \( \beta_w \) as appropriate). A possible future task lies in developing more tractable approximations to the two-way thinning functions and the subsequent rescaled mean measures.
6.5.2 Path dependence

We have observed that the node density, in relation to a forwarding node, is dependent on the distribution of the transmission radii. After the first hop, path dependence under the two-way model arises. We discuss the effects of the two-way model in the intersection of the total feasible regions of the source node and the first forwarding node, which is denoted by $I_0 \cap I_1$. In particular, we consider a potential forwarding node at the point $(u_2, \theta_2)$ and derive the probability requirements that its radius $R_2$ is sufficiently large to allow transmission under the two-way model. Naturally, the analysis that follows is similar to that for the path dependence and the effects of our sleep scheme in Section 4.2.

We introduce two node-to-node distance expressions akin to the previous distance expression (6.41). The distance between the source node and the potential forwarding node at $(u_2, \theta_2)$ is simply

$$h_{02}(u_2, \theta_2) = \left( u_2^2 + \ell^2 - 2u_2 \ell \cos(\theta_1 + \theta_2) \right)^{1/2},$$

and similarly, the distance between the first node and the potential node

$$h_{12}(u_2, \theta_2) = \left( u_2^2 + u_1^2 - 2u_2 u_1 \cos \theta_2 \right)^{1/2},$$

follows. Clearly, these expressions and the results that follow apply to the $n$-hop case under the one-hop dependent approximation.

One-sample model

After the first hop, any undetected nodes in the intersection region $I_0 \cap I_1$ must have a radius such that $R_2 < h_{02}$ as the node was not chosen as the next forwarding node during the first message relay. Conversely, for the node to be a potential forwarding node its radius $R_2 \geq h_{12}$. Hence, the potential node’s radius must satisfy the condition $(R_2 < h_{02}) \cap (R_2 \geq h_{12})$, which coupled with our sleep scheme induces a node thinning under the two-way model.

**Proposition 6.5.1.** Under the two-way model the density of potential awake nodes at the point $(u_2, \theta_2)$ in the intersection region $I_0 \cap I_1$ is thus

$$\lambda(u_2, \theta_2) = \lambda [F_R(h_{12}(u_2, \theta_2)) - F_R(h_{02}(u_2, \theta_2)) + (1 - p)].$$

**Proof.** Given that $R_2$ is absolutely continuous, the probability of the necessary conditions

$$\mathbb{P}((R_2 < h_{02}) \cap (R_2 \geq h_{12})) = \mathbb{P}(R_2 \leq h_{02}) - \mathbb{P}(R_2 \leq h_{12}),$$

$$\lambda(u_2, \theta_2) = \lambda [F_R(h_{12}(u_2, \theta_2)) - F_R(h_{02}(u_2, \theta_2)) + (1 - p)].$$
follows, which gives the retention probability of potential nodes in the intersection region owing to the two-way model. Furthermore, the probability of a node being asleep $1 - p$ (or awake $p$) is independent of the transmission radii, thus the $(1 - p)$ term stems from the sleep scheme analysis (see Section 4.2 in Chapter 4). The final result follows immediately from the supposition property of the Poisson process.

Re-sample model

The re-sample two-way model implies a forwarding node has one transmission radius for when it receives the message and another radius that is re-sampled independently for when it forwards the message; the node has two distinct transmission radii, which are independent and identically distributed. Consequently, any undetected nodes in the intersection region $I_0 \cap I_1$ previously had a radius, say $R'_2$ such that $R' < h_0$. Furthermore, for a node in the intersection region to be a potential forwarding node its current (that is, re-sampled) radius $R_2 \geq h_{12}$. Under the two-way re-sample model, these conditions induce a node thinning in the intersection region.

**Proposition 6.5.2.** Under the re-sample two-way model the density of potential awake nodes at $(u_2, \theta_2)$ in the intersection region $I_0 \cap I_1$ is thus

$$\lambda(u_2, \theta_2) = \lambda [F_R(h_{12}(u_2, \theta_2))[1 - F_R(h_{02}(u_2, \theta_2))] + (1 - p)].$$

**Proof.** Given that $R'_2$ and $R_2$ are independent, identically distributed, and absolutely continuous, the probability of the necessary conditions

$$\mathbb{P}((R'_2 < h_{02}) \cap (R_2 \geq h_{12})) = \mathbb{P}(R'_2 < h_{02})\mathbb{P}(R_2 \geq h_{12}),$$

follows, which gives the retention probability of potential nodes in the intersection region. Again, the probability of a node being awake is independent of the transmission radii, hence the $(1 - p)$ term via the supposition property of the Poisson process.

It is interesting that the thinning effect of the two-way model effectively reduces the path dependence. That is, under the two-way dependent model, the intersection between the current and previous forwarding regions may contain undetected nodes whose transmission radii were not large enough to be detected during the previous relay.

**6.5.3 Simulation results**

In lieu of deriving multihop equations, we investigate the two-way models by performing routing simulations. In particular, we examine the empirical distribution of $N$ under the uniform
and Rayleigh models. We compare the two-way models to the one-way model and the constant radius model (where the radius is set to mean of the random models). The routing simulations were performed under the same assumptions as the mathematical model. Generally, $10^4$ to $10^5$ simulations were performed in each experiment.

We use the Rayleigh distribution, thus given a unit mean the Rayleigh parameter $\sigma_r = \sqrt{2} \beta_w = \sqrt{2/\pi}$. For comparison purposes, the lower and upper values of the uniform distribution are set such that $R$ has the same second moment as that under the Rayleigh model; that is, for the uniform distribution

$$r_{\text{max}} = \mathbb{E}(R) + (1/2) \left( 12[\mathbb{E}(R^2) - \mathbb{E}(R)^2] \right)^{1/2}, \quad r_{\text{min}} = 2r_{\text{max}} - \mathbb{E}(R).$$

where $\mathbb{E}(R)$ and $\mathbb{E}(R^2)$ are the first and second moments of the Rayleigh distribution, the latter being $\mathbb{E}(R^2) = 2\sigma_r^2 = 4/\pi$. Hence, a unit mean via the Rayleigh distribution implies the uniform distribution parameters

$$r_{\text{max}} \approx 1.9054, \quad r_{\text{min}} \approx 0.0946,$$

which give the variance of the uniform distribution

$$\mathbb{V}(R) = \frac{1}{12} (r_{\text{max}} - r_{\text{min}})^2,$$

$$\approx 1.9054.$$
Our simulation results show that the constant radius model leads to messages being more likely to reach the target (see Fig. 6.12, 6.13, 6.14, and 6.15). For the constant radius model, the support of $N$ is stochastically smaller those that under the random radius models. This suggests that the constant radius model is a very optimistic model, and if such a model is to be used to
6.5

Figure 6.15: Empirical distribution of $N$ under different models (uniformly distributed radius, $\mathbb{E}(R) = 1$, $r_{\text{min}} = 0.5$, $r_{\text{max}} = 1.5$, $p = 1$, $\lambda = 5$ and $\ell = 10$).

Figure 6.16: Empirical distribution of $N$ under different models (uniformly distributed radius, $\mathbb{E}(R) = 1$, $r_{\text{min}} = 0.9$, $r_{\text{max}} = 1.1$, $p = 1$, $\lambda = 3$ and $\ell = 10$).

approximate a random radius model, perhaps the constant radius should be set to a value lower than the radius mean.

Naturally, for uniformly distributed radii, each respective random radius model continues to resemble the constant radius model as the lower and upper radius values approach each other,
which implies a reduction in the radius variance. More investigation is needed to see how the difference in the lower and upper radius values affects message propagation.

Another interesting observation is that messages under the one-sample two-way model sometimes have a better chance of reaching the sink than those under the one-way model. Initially, this may be counterintuitive because there are less two-way communication links than one-way communication links in the sensor network. However, a forwarding node that is chosen by greedy routing under the two-way model is more likely to have a larger radius than that of the chosen node under the one-way model. Thus, the total path length might be shorter.

From our simulations of uniformly distributed radius, it appears that the two-way model performs better when the radiance variance is sufficiently large. For \( i \geq 1 \), it is tempting to conjecture that under certain conditions the stochastic order

\[
P_{1W}(R_i > r) \leq P_{2W}(R_i > r),
\]

(6.45)

holds, in which the subscripts denote the probability measures under the two respective random radius models. However, it appears that this inequality does not hold in general (for a counterexample, see Fig. 6.16). For uniformly distributed radii, the one-way model appears to perform better if the difference between lower and upper radius values is small, which implies smaller variance in the transmission radius.

The two-way model induces a type of ‘blacklisting’ of nodes. The effects of blacklisting certain nodes has been examined [65]. In particular, it was observed that blacklisting nodes that are positioned a fraction of the total transmission range away from the forwarding node has mostly beneficial effects (see Fig. 9 in [65]). Although, this is a slightly different distance-based blacklisting problem (since the degree of blacklisting is parameterized, and not random like ours), it is observed that low density networks can actually suffer from any amount of distance-based blacklisting.

It appears that the two-way model performs better for random radius models that have large variance. However, considerably more investigation is required to understand the aforementioned stochastic behaviour induced by the two-way model. Ideally, this work will lead to performance guidelines for designing routing methods.

Finally, we observed that the re-sample two-way model seems to perform worse than the one-sample two-way model. Consequently, if bilateral communication is needed in relaying messages, then we speculate that the sensor nodes should be designed to reduce the variance of the transmission radii, thus resulting in better delivery rates.
6.6 Chapter summary

We examined the consequences of including random radii into our model. We discussed the differences between the one-way and two-way random radius models. Under the one-way model, asymptotic expressions for the first and second moments were derived. The quasi-Monte Carlo scheme was applied to the multihop problem under the one-way model to give agreeable results. We concluded that better performing quasi-random methods are needed, and that the independent model works well, provided that only a proportion of roughly ten percent of nodes are awake at any time.

We showed the immediate node-thinning effects that are induced by the two-way model. We discuss how the ensuing governing equations become less tractable under this model. We performed routing simulations that suggested under high node density and widely varying random radii that the two-way model significantly outperforms the one-way model. We discussed a possible explanation for this phenomenon and concluded that more research is needed to shed light on these intriguing models. Finally, we conjectured that reducing the variance in transmission radii would result in better routing performance, particularly under the two-way model.
Chapter 7

Conclusion

The fields of communications and computing are leading to numerous technologies that offer great benefits. Some of the most promising of these technologies are sensor networks in various forms and application settings. The random and dynamic network structure of sensor networks necessitates the need for innovative routing methods and suitable energy-saving schemes. For the efficient design and implementation of these communications methods, it is essential to gain deep understanding and insight into the considerations and issues that are involved in routing data message through these ad hoc networks, and the effects of energy-saving schemes.

The elegant and applicable field of stochastic geometry and point processes offer a large number of modelling applications in various areas of science, technology, and other disciplines. Their interest and application has grown at a steady pace while shedding much light on numerous problems via useful stochastic models. Moreover, the rapidly evolving field of quasi-random points and their application in integrating high-dimensional integrals has made significant advancements in recent years. Their ability to counter the curse of dimensionality remains an active research pursuit, but quasi-random points show promise in lending themselves to tackling many future research problems.

In this thesis, the greedy routing method was formulated and examined in the setting of a mathematical model by employing the previously mentioned mathematical techniques. Broadly speaking, this thesis was divided into two main focuses. The first entailed the proposal and analysis of suitable but tractable stochastic models that captured the important system characteristics and dependencies. The second consisted of devising relatively fast, reliable and accurate numerical methods to evaluate the resulting integrals. Consequently, in this chapter we summarize
the aforementioned work and results from the preceding chapters. Finally, we suggest possible research tasks and directions in sensor networks.

7.1 Summary

In Chapter 3, we proposed the network model that served as the basis for the remaining of the thesis. It was assumed that nodes were located on a plane and that each node had a constant circular transmission radius, which leads to the disk model. The nodes were scattered according to a homogeneous Poisson spatial process with a constant node density. From these assumptions we examined the first hop distribution and the effects of the sink dependence has on message advancement. Closed-form asymptotic expressions of the first and second moments of the first hop were ascertained and showed to give accurate results.

After the first hop, the emergence of sink dependence was examined with a focus on the effects of a sleep scheme. The exclusion or inclusion of the path dependence gave rise respectively to independent and dependent forwarding models. Under the independent model, it is demonstrated that messages stochastically decrease as they approach the sink. Intersection area expressions were obtained and used to relate the two models up to three-hop advancement. Stochastic orders were derived and proposed to compare the multihop advancement of messages under the two models.

In Chapter 4, we used the knowledge and insights from the previous chapter to develop a method of obtaining the stochastic behaviour of message advancement for the $n$-hop case. The $n$-hop case was related to the number of hops that a message takes to reach the sink. The ensuing $n$-dimensional integrals were evaluated with quasi-Monte Carlo methods, which were sped up by importance sampling based on the independent model.

Furthermore, the sleep scheme introduced in the previous chapter is described more thoroughly and its immediate effects on the local node density are examined. Stochastic inequalities are obtained and used to give heuristics for a forwarding node waiting for surrounding nodes to awake.

It is revealed that having a sensor network with approximately ten percent of the nodes in awake mode renders the dependent model to be almost identical to the independent model. Moreover, this proportion of nodes $p$ can be greater for larger $\lambda$. Importantly, the use of the independent model halves the integral dimensionality, thus speeding up the integration process. More notably, we demonstrated that a fast convolution model could approximate the indepen-
dent model with only a small loss in accuracy over many hops; hence reducing the need for high-dimensional integration.

In Chapter 5, the constant node density assumption is replaced with a spatially dependent node density function. The form of the function is chosen to be amenable to analytic and asymptotic means, but to also offer insights that the vary node density has on message routing in sensor networks. Methods akin to those used in the two previous chapters are used with a similar degree of success. It is revealed under the independent model that messages stochastically increase as they advance towards the sink.

The resulting mean function involved elliptic integral functions, which were evaluated via Carlson symmetric elliptic integral functions. The ensuing results agreed well with those from simulations. Moreover, it was shown that the mean function could be represented by a more tractable power series without a significant loss of accuracy. Recently-proposed lattice rules were used to evaluate some of the integrals. Finally, provided $p$ was sufficiently low, in this setting the dependent model also effectively reduced to the more efficient independent model.

In Chapter 6, we returned to the constant node density model and included the assumption of independently and identically distributed transmission radii. We consider the communication assumptions, thus yielding the one-way and two-way random radius models for message relaying. We further separated the latter model into models depending whether or not the transmission radius is re-sampled for each hop. Under the one-way model, closed-form asymptotic expressions for the first and second moments were obtained with varying success. Specifically, it was generally found that the first moment approximation agreed very well with numerical results while the second moment expressions were less accurate.

We conclude examination of the one-way model by noting the need for closed-form expression for the first hop distribution under the random radius model. Consequently, given a sufficiently low $p$, each hop could be treated independently. Hence, given a constant node density, we emphasize that the convolution model could be used to model multihop advancements in a very quick manner.

We outlined how the two-way models induce a blacklisting phenomenon, thus thinning the local node density. We examined the path dependence under these models and local spatially dependent node density functions. Subsequently, this model resulted in less tractable hop expressions leading us to study the model via simulations. It was revealed that under certain radius and density conditions, the two-way model significantly outperforms the one-way model. We gave a possible explanation for this observation, and stressed that further research is needed in
this area to reveal the local optima.

Finally, we observed that the one-sample two-way model in our results performed better than the re-sample two-way model, thus leading us to suggest that variance in transmission radii should be kept to a minimum.

### 7.2 Future work

Calculations and comparisons were given to demonstrate the validity of the models developed in this thesis, however, further investigation is needed. We emphasize the observation that for small node density (roughly, $\lambda < 5$) the path dependence has an observable effect and the Monte Carlo methods need more sample points as the hop number increases. Modifications to the Halton sequences (such as scrambling the points) may be used to reduce discrepancy at higher dimensions. Alternatively, the new and quick-to-generate quasi-random sequences based on lattice rules could be employed to see if they perform better. Furthermore, other variation and variance reduction methods (such as antithetic variates) may be used to accelerate both quasi and regular Monte Carlo approaches.

In regular Monte Carlo integration the order of integration does not influence the rate of convergence. However, this is not the case with quasi-random points as each dimension of a quasi-random point should correspond to a particular integration variable. The order of the integration variables is determined by first recasting the integral domain to the hyper-dimensional unit cube, and then performing a variable analysis. Somewhat in hindsight, we acknowledge that changing the variable order may help to speed up the integration methods. We briefly experimented with some orders in Chapter 5, and did not notice one order being greatly advantageous over another. However, we stress that a rigorous analysis of the integrals is needed to reveal the optimal order of integration.

An attractive feature of regular Monte Carlo methods is that the error is obtained by estimating the variance of the integral. Conversely, quasi-Monte Carlo methods lack a practical way of estimating the error, despite them generally have a faster convergence rate. The idea of ‘randomized’ quasi-Monte Carlo methods seeks to combine the advantages of both approaches. Consequently, a future task lies in investigating these hybrid methods in evaluating hop integrals.

Our ‘blinking’ sleep scheme has assumed awake nodes are selected with probability $p$ independently at each transmission attempt. A generalization of this, which may be realistic in some situations, is to assume that nodes fall asleep and awake up according to a discrete-time Markov
chain. This will change the expression for the density of nodes in ‘previously explored’ regions, but not fundamentally alter our analysis. Moreover, besides a node being in sleep mode, there may be other reasons (such as node blacklisting) why a node cannot (or should not) relay data. Hence, \( p \) may represent the probability of a node being able to relay for whatever reason, and, hence, the form of \( p \) becomes more complex.

For low \( p \) values, a general observation has been that the dependent model is closely approximated by the independent model. This reduction in model complexity greatly speeds up the calculations methods. In particular, provided a constant node density the efficient renewal model can be used. However, further research is needed to vindicate our model for a simple sleep scheme. That is, work needs to be done to check if this sleep scheme captures the essential features of real sleep schemes, be them stochastic, deterministic or a combination of both. Consequently, it is worth verifying if physical sensor networks can be modelled with the independent model with certain sleep schemes in operation.

In this work we did not examine how significantly the sleep scheme affects the message delivery delay. Intuitively, a blinking sleep scheme leads to a trade-off between message delay and energy-consumption in the sensor network. Future research may shed light on this problem and give an indication on how strongly \( p \) affects the message delay. Furthermore, upon obtaining an optimal value of \( p \), the issue of dependence in our model framework can be re-examined.

We gave, without experimental verification, a pair of simple heuristics or ‘rules of thumb’ for a forwarding node to decide how many relay attempts it should perform. Experimental investigation is needed to check that the rules can be used to optimize sensor network protocols. The use of such rules may aid in the design of more ‘self-aware’ sleep schemes and routing methods.

In real life settings, the constant node density assumption may often not be appropriate. Under the spatially dependent model, the density function was chosen such that it was simple enough for analytic means, while still being a plausible node placement scenario. Other suggestions exist such as the node density decaying exponentially or according to some negative power of the sink distance. Furthermore, under our model the sink was located at the maximum of the node density. Placing the sink at an arbitrary point in the sensor field results in the node density being dependent on the sink angle. This is an additional increase in the complexity of the density function. Moreover, the angle of an individual node would not be a uniformly distributed random variable, thus importance sampling might be needed when integrating over the angle domains. Consequently, these suggestions may result in analytic and asymptotic mean functions that can be used to model more realistic node deployment models.
Further investigation of the asymptotic approximations are needed. The approximation may break down when the radius is large compared to the sink distance. For a constant radius model, the lengths can always be rescaled with respect to the transmission radius. However, this may not be possible for a randomly varying radius model.

The majority of the work covered in this thesis was under the simplifying assumption of constant transmission radii. However, in Chapter 6 we listed commonly regarded reasons why this is often not a good assumption. We strongly believe that a significant amount of work remains to be invested in modelling message relaying under the random radius assumption. The accuracy of the simpler one-way model needs to be studied. Moreover, the contrasts in performance between the one-way and two-way models for certain parameter regimes is an intriguing observation. Ideally, analytic or asymptotic studies can reveal under what conditions either models performs better.

In general, the problem of blacklisting or thinning nodes (induced by either the two-way model or pre-set rules) strongly suggests the need to examine the conditions for local optima in such settings. We point out that blacklisting, in effect, reduces the path dependence as nodes with insufficiently large radii can exist in the intersection of total feasible regions under the two-model. From this investigation, better heuristics can be possibly devised in choosing potential forwarding nodes.

The insights gained in this thesis may lead to new suggestions for routing methods such as designing an algorithm that minimizes the intersection region, thus reducing the path dependence. Moreover, the message advancement or delivery is not the only metric to measure routing performance as energy-consumption is a pressing issue in devising sensor networks. Energy-consumption models can be included into the routing models outlined here, thus resulting in the need for more stochastic modelling.

Finally, we only examined the commonly proposed but simple greedy routing method. However, there is an ever increasing number of suggestions for routing methods. Arguably, these can benefit from thorough stochastic analysis and modelling similar to the techniques used here. Ideally, these techniques can be used and adapted to cast light on an array of routing methods; thus leading to more optimal design and implementation of sensor networks.
Appendix A

Derivation details

A.1 Asymptotic expansions

Consider the angle function

\[ \psi_{\gamma}(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2u\gamma} \right), \]

which we wish to expand at \( u = \gamma - r \). The function \( W(\gamma - r) = 1 \); thus, we expand the function \( \arccos x \) at \( x = 1 \) by observing

\[ -(1 - x^2)^{-1/2} = -2^{-1/2}(1 - x)^{-1/2} \left[ 1 - \frac{1}{2} (1 - x) \right]^{-1/2} \]

\[ \approx -2^{-1/2} \left[ (1 - x)^{-1/2} + \frac{1}{4} (1 - x)^{1/2} + \frac{3}{32} (1 - x)^{3/2} \right], \]

which integrated leads to the approximation

\[ \arccos x \approx 2^{1/2} \left[ 2(1 - x)^{1/2} - \frac{1}{12} (1 - x)^{3/2} + \frac{3}{160} (1 - x)^{5/2} \right]. \]

Expand the function \( W(u) \) at \( u = \gamma - r \), thus

\[ W(u) \approx 1 - \frac{r}{\gamma(\gamma - r)}(u - \gamma + r) + \frac{\gamma + r}{2(-\gamma + r)^2} (u - \gamma + r)^2 + \frac{\gamma + r}{2(-\gamma + r)^3} (u - \gamma + r)^3 \]

After some work the angle function is approximated by

\[ \psi_{\gamma}(u) \approx b_0(u - \gamma + r)^{1/2} + b_0(u - \gamma + r)^{3/2} + b_2(u - \gamma + r)^{5/2}, \]
where the expansion terms follow

\[ b_0 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2}, \]
\[ b_1 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{r^2 - 3r\gamma - 3\gamma^2}{12(\gamma^2 r - \gamma r^2)} \right], \]
\[ b_2 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{3r^4 + 25r^2\gamma^2 - 10r^3\gamma + 30\gamma^3 r - 5\gamma^4}{160\gamma^2 (\gamma - r)^2 r^2} \right]. \]

The feasible area function is given by

\[ A_\gamma(u) = 2 \int_{\gamma - r}^{u} w \psi_\gamma(w) dw, \]

hence the area expansion

\[ A_\gamma(u) \approx a_0 (u - \gamma + r)^{3/2} + a_1 (u - \gamma + r)^{5/2} + a_2 (u - \gamma + r)^{7/2}, \]

and the expansion terms

\[ a_0 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{4(\gamma - r)}{3} \right], \]
\[ a_1 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{-3\gamma^2 + 9\gamma r + r^2}{15\gamma r} \right], \]
\[ a_2 = \left[ \frac{2r}{\gamma(\gamma - r)} \right]^{1/2} \left[ \frac{-15\gamma^4 - 30\gamma^3 r - 45\gamma^2 r^2 + 10\gamma r^3 + 9r^4}{840\gamma^2 (\gamma - r)^2 r^2} \right]. \]

### A.2 Joint probability density under homogeneous model

We perform the ensuing calculations in terms of the sink distance, changing to the hop advancement variable in the final step, while assuming there is no sleep scheme. The inclusion of a sleep scheme is done by simply using the one-hop (or two-hop) node density and Poisson parameter functions. To derive the joint probability density expression, we consider the probability density of \( \Theta_i \) conditioned on the event \( U_i = u_i \). The homogeneous Poisson process assumption implies that the angle of any node is distributed uniformly around the sink (in the regions where the nodes can exist). The conditional probability density of the sink angle for the \( i \)-th forwarding node is a constant (with respect to the sink angle \( \theta_i \)), and integrates over the whole \( \theta_i \) domain to give one. The conditional probability density under the dependent model

\[ g_{u_1}(\theta_1 | U_1 = u_1) = \frac{I_{\delta}(u_1, \theta_1)}{2\psi_{\omega_1}(u_1)}, \]

follows, and this expression also applies to the independent model as it has the same feasible region. To obtain a general expression, we introduce the function \( \Psi_{\hat{\gamma}}(u_{i+1}) \) to denote the total
angular width of the feasible region given $U_{i+1} = u_{i+1}$ and the path $\vec{X}_i = \vec{x}_i$. The conditional probability density

$$g_{u_{i+1}}(\theta_{i+1}|U_{i+1} = u_{i+1}) = \frac{I_{D_i}(u_{i+1}, \theta_{i+1})}{\Psi_{\vec{x}_i}(u_{i+1})},$$

follows. Under the independent model the angular width function simplifies to

$$\Psi_{\vec{x}_i}(u_{i+1}) = 2\psi_{u_{i+1}}(u_{i+1}).$$

In general, the area of the feasible region written as an integral

$$\vec{A}_i(u_{i+1}) = \int_{u_i-r}^{u_i+r} w_{i+1} \Psi_{\vec{x}_i}(u_{i+1}) dw_{i+1},$$

gives the derivative of the area function

$$\vec{A}_i'(u_{i+1}) = u_{i+1} \Psi_{\vec{x}_i}(u_{i+1}).$$

The probability density

$$g_{\vec{x}_i}(u_{i+1}) = \lambda_{D_i} (u_{i+1}) u_{i+1} e^{-\lambda \vec{A}_i(u_{i+1})},$$

immediately follows. The joint probability density of the two random variables $U_{i+1}$ and $\Theta_{i+1}$ is given by

$$g_i(u_{i+1}, \theta_{i+1}) = g_{\vec{x}_i}(u_{i+1}) g_{u_{i+1}}(\theta_{i+1}|U_{i+1} = u_{i+1}, \vec{X}_i = \vec{x}_i)$$

$$= \lambda_{D_i}(u_{i+1}, \theta_{i+1}) u_{i+1} e^{-\lambda \vec{A}_i(u_{i+1})}$$

where the spatially-dependent node density function has been introduced

$$\lambda_{D_i}(u_{i+1}, \theta_{i+1}) = \lambda_{D_i}(u_{i+1}, \theta_{i+1}).$$

The joint probability density of the random variables $U_1$ to $U_n$ and $\Theta_1$ to $\Theta_n$

$$g_{(n-1)}(u_1, \ldots, u_n, \theta_1, \ldots, \theta_n) = \prod_{i=1}^{n} \lambda_{D_{i-1}}(u_i, \theta_i) u_i e^{-\lambda \vec{A}_{i-1}(u_i)},$$

follows. In terms of hop advancements the equivalent expression

$$\bar{g}_{(n-1)}(c_1, \ldots, c_n, \theta_1, \ldots, \theta_n) = \prod_{i=1}^{n} \lambda_{D_{i-1}}(u_{i-1} - c_i, \theta_i) (u_{i-1} - c_i) e^{-\lambda \vec{A}_{i-1}(u_{i-1} - c_i)},$$

readily follows.
A.3 Derivation of $A_{1,0}(u_2)$ for constant radii

The source and the current forwarding nodes are located at the points $X_0$ and $X_1$ respectively, and the sink is located at the point $X_S$. Symmetry allows us to assume that $X_1$ is located above the baseline which runs from $X_0$ to $X_S$ as shown in Fig. A.1. The location of $X_1$ is represented by the sink distance $u_1$, and the sink angle $\theta_1$.

The point closest to the sink where the transmission circumferences of the source and the current forwarding nodes intersect is also of importance. Denote this point by $X_{01}$, and let $u_{01}$ be the distance from this point to the sink (see Fig. A.1). To calculate $u_{01}$ we observe that an isosceles triangle with two $r$-sides is formed by the points $X_0$, $X_1$ and $X_{01}$ (see Fig. A.2). The third side of this triangle is the distance separating $X_0$ and $X_1$, hence

$$h_{01} = \left( u_1^2 + \ell^2 - 2u_1\ell\cos\theta_1 \right)^{1/2}.$$ 

Let $\delta_1$ represent the angle $X_1X_0X_{01}$, given by

$$\delta_1 = \arccos \left( \frac{h_{01}}{2r} \right),$$

and $\beta_1$ be the angle $X_1X_0X_S$ given by

$$\beta_1 = \arccos \left( \frac{h_{01}^2 + \ell^2 - u_1^2}{2h_{01}\ell} \right).$$

Represent the last angle, $X_SX_0X_{01}$, by $\eta_1 = \delta_1 - \beta_1$, thus giving the expression

$$u_{01} = \left( \ell^2 + r^2 - 2\ell r \cos \eta_1 \right)^{1/2}.$$ 

Consider the point of distance $u$ from the sink that lies on the transmission circumference of a node that has a sink distance $\gamma$. We recall that the angle that is formed by connecting this point to
the node via the sink is given by the expression

$$\psi_\gamma(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2u\gamma} \right).$$

We place an emphasis on this angle function as it appears in the kernels of the integral area expressions for both the independent and dependent cases.

We introduce the function $\Delta \psi(u_2)$ to describe the angular width of the intersection of the source and current feasible regions. That is, $\Delta \psi(u_2)$ represents the angle between the top and bottom edges of the intersection region at a distance $u_2$ from the sink (refer to Fig. A.3). Subsequently, the intersection area function

$$A_{1\setminus 0}(u_2) = \int_{\ell-r}^{u_2} w_2 \Delta \psi(w_2) dw_2,$$

which allows us to calculate the feasible area under the dependent model. Hence, we need an analytic form of $\Delta \psi(u_2)$ over the entire $u_2$ domain. The intersection area is zero on the interval $[u_1 - r, \ell - r]$, and hence, $\Delta \psi(u_2)$ is zero on this interval (refer to Fig. A.1). The boundary of the intersection region is formed by arcs from the two circles of radius $r$, and a sector of radius $u_2$. 
The shape of this boundary naturally affects the value of $\Delta \psi(u_2)$. Thus, we observe that the rest of the domain can be divided into two intervals, $[\ell - r, u_{01}]$ and $[u_{01}, u_1]$.

On the first interval, $[\ell - r, u_{01}]$, the intersection area is zero when the point $X_{01}$ is located above the baseline which extends from $X_0$ to $X_S$. Conversely, the intersection area on this interval is positive if $\delta_1 > \beta_1$. When this condition is met, we observe on the first interval that the upper and lower arcs, which form the feasible region boundary, coincide with the transmission circumference of the source node. Hence, on the first interval we have the intersection angle expression

$$\Delta \psi(u_2) = 2\psi_\ell(u_2)I_{01}^{-}, \quad \ell - r \leq u_2 \leq u_{01},$$

where $I_{01}^{-}$ is an indicator function for when $X_{01}$ is below the baseline. That is, $I_{01}^{-} = 1$ when $X_{01}$ is below the baseline, or equivalently when $\delta_1 > \beta_1$.

We observe on the second interval $[u_{01}, u_1]$ that part of the feasible region boundary coincides with the transmission circumference of the current forwarding node. Hence, on the second interval we obtain the intersection angle expression

$$\Delta \psi(u_2) = \psi_\ell(u_2) + \psi_{u_1}(u_2) - \theta_1, \quad u_{01} \leq u_2 \leq u_1.$$

To perform the integration step we note that our expression for the feasible area under the independent model gives us the general solution to the integral

$$A_\gamma(u) = 2 \int_{\gamma - r}^{u} w\psi_\gamma(w)dw,$$

$$= r^2 \phi_\gamma(u) + u^2 \psi_\gamma(u) - r\gamma \sin \phi_\gamma(u),$$

where we recall the angle functions

$$\phi_\gamma(u) = \arccos \left( \frac{r^2 + \gamma^2 - u^2}{2r\gamma} \right),$$

$$\psi_\gamma(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2u\gamma} \right).$$

Thus, on the first interval, $[\ell - r, u_{01}]$, we have the area expression

$$A_{1\setminus 0}(u_2) = 2 \int_{\ell - r}^{u_{01}} w_2\psi_\ell(w_2)dw_2I_{01}^{-},$$

$$= A_\ell(u_2)I_{01}^{-}.$$
On the second interval, \([u_{01}, u_1]\), we have the slightly more complicated area expression

\[
A_{10}(u_2) = \int_{\ell-r}^{u_2} w_2 [\psi(w_2) + \psi_{u_1}(w_2) - \theta_1] \, dw_2 + A_d(u_{01}) \Omega_{01},
\]

\[
= \frac{1}{2} \left( A_d(u_2) + A_{u_1}(u_2) + \theta_1 \left[u_{01}^2 - u_2^2\right]\right)
+ \frac{1}{2} \left( A_d(u_{01}) \left[2\Omega_{01} - 1\right] - A_{u_1}(u_{01})\right).
\]

Consequently, for \(i = 1\) the feasible area function under the dependent model is given by

\[
\vec{A}_1(u_2) = A_{u_1}(u_2) - A_{10}(u_2).
\]

### A.4 Joint probability density under inhomogeneous model

We assume there is no sleep scheme initially, and note that to include a sleep scheme entails substitution the corresponding node density function (5.37) into the joint probability density.

We consider the probability density of \(\Theta_i\) conditioned on the event \(U_i = u_i\). Under our inhomogeneous Poisson model, the angle of any node is distributed uniformly around the sink (in the regions where nodes can exist). Hence, the conditional probability density under the dependent model

\[
g_{u_i}(\theta_1|U_i = u_i) = \frac{\Omega_{u_1}(u_i, \theta_1)}{2\psi_{u_0}(u_1)},
\]

follows, and this expression also applies to the independent model as it has the same feasible region. We introduce the function \(\Psi_{\vec{x}_i}(u_{i+1})\) to denote the total angular width of the feasible region given \(U_{i+1} = u_{i+1}\) and the path \(\vec{x}_i = \vec{x}_i\). The conditional probability density

\[
g_{u_{i+1}}(\theta_{i+1}|U_{i+1} = u_{i+1}) = \frac{\Omega_{\vec{x}_i}(u_{i+1}, \theta_{i+1})}{\Psi_{\vec{x}_i}(u_{i+1})},
\]

follows. Under the independent model the angular width function simplifies to

\[
\Psi_{\vec{x}_i}(u_{i+1}) = 2\psi_{u_i}(u_{i+1}).
\]

The rescaled mean of the feasible region written as an integral

\[
\vec{Q}_i(u_{i+1}) = \int_{u_i-r}^{u_{i+1}} \Psi_{\vec{x}_i}(w_{i+1}) \, dw_{i+1},
\]

gives the derivative of the rescaled mean measure

\[
\vec{Q}_i'(u_{i+1}) = \Psi_{\vec{x}_i}(u_{i+1}).
\]
The probability density
\[ g_{\vec{x}}(u_{i+1}) = \lambda \Psi_{\vec{x}}(u_{i+1}) e^{-\lambda \vec{Q}_i(u_{i+1})}, \]
follows. The joint probability density of the two random variables \( U_{i+1} \) and \( \Theta_{i+1} \) is given by
\[ g_i(u_{i+1}, \theta_{i+1}) = g_{\vec{x}}(u_{i+1}) g_{u_{i+1}}(\theta_{i+1} | U_{i+1} = u_{i+1}, \vec{X}_i = \vec{x}_i) = \lambda \mathcal{D}_i(u_{i+1}, \theta_{i+1}) e^{-\lambda \vec{Q}_i(u_{i+1})} \]
where the spatially-dependent node density function has been introduced
\[ \lambda \mathcal{D}_i(u_{i+1}, \theta_{i+1}) = \lambda I_{\mathcal{D}_i}(u_{i+1}, \theta_{i+1}). \]

The joint probability density of the random variables \( U_1 \) to \( U_n \) and \( \Theta_1 \) to \( \Theta_n \)
\[ g(n-1)(u_1, ..., u_n, \theta_1, ..., \theta_n) = \prod_{i=1}^{n} \lambda \mathcal{D}_{i-1}(u_i, \theta_i) e^{-\lambda \vec{Q}_{i-1}(u_i)}, \tag{A.1} \]
follows, or in terms of hop advancements the equivalent expression
\[ \bar{g}(n-1)(c_1, ..., c_n, \theta_1, ..., \theta_n) = \prod_{i=1}^{n} \lambda \mathcal{D}_{i-1}(u_{i-1} - c_i, \theta_i) e^{-\lambda \vec{Q}_{i-1}(u_{i-1} - c_i)}. \]

### A.5 Derivation of \( Q_{1\setminus 0}(u_2) \) for constant radii

We outline how to calculate the rescaled mean measure for the region feasible region under the dependent model. The method is akin to calculating the area \( A_{1\setminus 0}(u_2) \). In fact, the derivation of \( Q_{1\setminus 0}(u_2) \) is included for completeness, and we refer the reader to Section A.3. We use the function \( \Delta \psi(u_2) \) again to describe the angular width of the intersection of the source and current feasible regions. Subsequently, the rescaled mean measure on this intersection region is given
\[ Q_{1\setminus 0}(u_2) = \int_{\ell - r}^{u_2} \Delta \psi(u_2) du_2, \]
which leads to the rescaled mean under the dependent model
\[ \bar{Q}_1(u_2) = Q_{u_1}(u_2) - Q_{1\setminus 0}(u_2). \]

It can be shown that the intersection angle expression is
\[ \Delta \psi(u_2) = 2 \psi_1(u_2) I_{X_{01}}, \quad \ell - r \leq u_2 \leq u_{01}, \]
where \( I_{X_{01}} \) is an indicator function for when \( X_{01} \) is below the baseline. On the second interval we obtain the intersection angle expression
\[ \Delta \psi(u_2) = \psi_1(u_2) + \psi_{u_1}(u_2) - \theta_1, \quad u_{01} \leq u_2 \leq u_1. \]
Recall under the independent model the rescaled mean integral
\[ Q_\gamma(u) = 2 \int_{\gamma-r}^{u} \psi_\gamma(w) dw, \]
where the angle function
\[ \psi_\gamma(u) = \arccos \left( \frac{u^2 + \gamma^2 - r^2}{2u\gamma} \right). \]
Thus, on the first interval, \([\ell - r, u_0]\), we have the rescaled mean
\[
Q_{1\setminus0}(u_2) = 2 \int_{\ell-r}^{u_2} \psi_\ell(w_2) dw_2 \mathbf{1}_{X_{01}} = Q_\ell(u_2) \mathbf{1}_{X_{01}}.
\]
On the second interval, \([u_0, u_1]\), we have the slightly more complicated rescaled mean expression
\[
Q_{1\setminus0}(u_2) = \int_{\ell-r}^{u_2} [\psi_\ell(w_2) + \psi_{u_1}(w_2) - \theta_1] dw_2 + Q_\ell(u_0) \mathbf{1}_{X_{01}},
\]
\[
= \frac{1}{2} (Q_\ell(u_2) + Q_{u_1}(u_2) + 2\theta_1 [u_0 - u_2])
+ \frac{1}{2} (Q_\ell(u_0) [2\mathbf{1}_{X_{01}} - 1] - Q_{u_1}(u_0)).
\]
This approach naturally extends to the rescaled mean on the intersection of any two feasible regions. Consequently, for \(i \geq 1\), under the one-hop dependent model the rescaled mean measure on the feasible region is given by
\[
\tilde{Q}_i(u_{i+1}) = Q_{u_i}(u_{i+1}) - Q_{\setminus_{i-1}}(u_{i+1}).
\]
A.6 Derivation of $A_{0 \cap 2 \setminus 1}(u_3)$ for constant radii

After two hops, it follows that an area expression for the intersection region

$$\mathcal{R}_{0 \cap 2 \setminus 1}(u_3) = \mathcal{I}_0(u_3) \cap \mathcal{I}_2(u_3) \setminus \mathcal{I}_1(u_3),$$

is needed to calculate the area function under the dependent model. Again, the angle function $\Delta \psi(u_3)$ describes the angular width of the area under consideration, thus giving the general integral expression

$$A_{0 \cap 2 \setminus 1}(u_3) = \int_{u_0 - r}^{u_3} w_3 \Delta \psi(w_3) dw_3.$$

Under the dependent model without a sleep scheme the placement of the second node is clearly limited to only regions where nodes can exist, consequently restricting possible node configurations. Under a power scheme, however, this is not the case as nodes may awake in previously node-free regions. This consideration is included in the ensuing area derivations by allowing the second node to exist in said regions.

We continue with the previous notation such that the point $X_{ij}$ denotes the intersection point of transmission circumferences of two nodes $X_i$ and $X_j$. Naturally, there can be two points that satisfy this condition, but we refer to the one that is closer the sink. Care needs to be taken as a slight change in node position can markedly swing an intersection point from one side of the baseline to the other. The sink distance of $X_{ij}$ is simply $u_{ij}$. We assume again that the first forwarding node is above the baseline, which runs from the source node to the sink. The indicator function of the event when the intersection point is above the baseline is represented by $I_{ij}^+$, while a minus subscript refers to the opposite situation.

A.6.1 Initial method

We outline a more general approach to find an analytic expression for $\Delta \psi(u_3)$ while referring to various node position configurations (see Fig. A.5 to A.12). In relation to the baseline, we shall refer to the boundary of the feasible region that is above the baseline as simply the top path. We consider a point on the top path which is a distance $u_3$ from the sink. Let $\psi_T(u_3)$ be the angle between the baseline and the line connecting the sink to this point. Likewise, we refer to the boundary below the baseline as the bottom path, and let $\psi_B(u_3)$ be its corresponding angle in relation to the sink, thus leading to the angle expression

$$\Delta \psi(u_3) = \psi_T(u_3) - \psi_B(u_3).$$
A.6

Figure A.4: The intersection region $\mathcal{R}_{0/2\setminus 1}(u_3)$ can consist of up to four arcs.

The region $\mathcal{R}_{0/2\setminus 1}(u_3)$ is enclosed by the three transmission circles and a sector of radius $u_3$. This knowledge can be used to obtain general expressions for $\psi_T(u_3)$ and $\psi_B(u_3)$. We consider the positioning of the intersection points of the transmission circumferences. For the top path, if the intersection point $X_{01}$ is above the baseline

$$\psi_T(u_3) = \min[\theta_1 - \psi_{u_1}(u_3), \psi_{u_0}(u_3)].$$

A similar approach applied to the bottom path gives

$$\psi_B(u_3) = \max[\theta_1 + \theta_2 - \psi_{u_2}(u_3), -\psi_{u_0}(u_3)].$$

Combining these results gives the angular width function

$$\Delta \psi(u_3) = \min[\theta_1 - \psi_{u_1}(u_3), \psi_{u_0}(u_3)] - \max[\theta_1 + \theta_2 - \psi_{u_2}(u_3), -\psi_{u_0}(u_3)],$$

which allows the calculation of the area of $\mathcal{R}_{0/2\setminus 1}(u_3)$, thus

$$\tilde{A}_2(u_3) = A_{u_2}(u_3) - A_{2\setminus 1}(u_3) - A_{0/2\setminus 1}(u_3).$$

However, this approach generally requires the angular width function to be evaluated, multiplied by the sink distance, and integrated numerically (with the trapezoid method, for example) to yield the area expression. In the next section we develop a more analytic approach inspired by the aforementioned method, and give the intervals that the intersection region is defined on.
A.6.2 Alternative method

The main obstacle of calculating the intersection area expression in this setting is ensuring that all the possible geometrical configurations are considered. We outline all the general three-node cases that we believe are possible under our model. An example of each case is given with the key region shaded with randomly placed points (see Fig. A.5 to A.12). A part of the sectors of radius \( u_0 \) and \( u_2 \) are also included to aid in viewing the feasible regions in relation to the intersection regions.

The defining characteristic of the configurations is the locations of the two intersection points \( X_{01} \) and \( X_{02} \) in relation to the baseline. In addition to directly influencing which area expressions to use, the placement of the intersection points governs the domain that the intersection function is defined on. Each of the two different points is able exist to exist in one of two separate region. The symmetry of the problem implies there are four general geometrical combinations:

Case 1 \( X_{01} \) below baseline and \( X_{02} \) above baseline.

Case 2 \( X_{01} \) below baseline and \( X_{02} \) below baseline

Case 3 \( X_{01} \) above baseline and \( X_{02} \) below baseline

Case 4 \( X_{01} \) above baseline and \( X_{02} \) above baseline

Under some of these cases there are more specific configurations (which are assigned with a letter after the case number) owing to the varying lower and upper integration intervals. We derive the area functions and their domains for all four general cases and their specific subcases.

Case 1

The intersection region \( \mathcal{R}_{0<2<1} \) consists only of two circles, thus the integration interval does not need to be divided into subintervals (refer to Fig. A.5). The top path consists entirely of the transmission circumference of node \( X_1 \). Similarly, bottom path consists of the transmission circumference of node \( X_0 \). The angular difference under this configuration leads to the area expression

\[
\mu[\mathcal{R}_{0<2<1}(u_3)] = \frac{1}{2} \left[ -A_{u_1}(u_3) + \theta_1 u_3^2 + A_{u_0}(u_3) \right] - \frac{1}{2} \left[ -A_{u_1}(u_{low}) + \theta_1 u_{low}^2 + A_{u_0}(u_{low}) \right], \quad u_3 \in [u_{low}, u_{up}],
\]

(A.2)

where \( u_{low} \) is the lower value of the integration interval. The lower and upper interval values are simply \( u_{low} = u_{01} \) and \( u_{up} = u_2 \).
Case 2

The intersection region $R_{0 \cap 2 \setminus 1}$ for the second case is slightly more complicated. Again, the top path consists solely of the transmission circumference of node $X_{1}$. However, the bottom path may consist of the transmission circumferences of both nodes $X_{0}$ and $X_{1}$ (refer to Fig. A.6). Thus, the integration interval of the bottom path can need to be divided into the two subintervals $[u_{\text{low}}, u_{\text{BD}}]$ and $[u_{\text{BD}}, u_{\text{up}}]$. The lower interval value is $u_{\text{low}} = u_{01}$.

The dividing point of the integration interval is $u_{\text{BD}} = u_{02}$ when $u_{02} < u_{2}$. Consequently, the angular difference on the first subinterval is the same as that of the first case

$$\mu[R_{0 \cap 2 \setminus 1}(u_{3})] = \frac{1}{2} \left[-A_{u_{1}}(u_{3}) + \theta_{1} u_{3}^{2} + A_{u_{0}}(u_{3})\right] - \frac{1}{2} \left[-A_{u_{1}}(u_{\text{low}}) + \theta_{1} u_{\text{low}}^{2} + A_{u_{0}}(u_{\text{low}})\right], \quad u_{3} \in [u_{\text{low}}, u_{\text{BD}}].$$

The bottom path on the second subinterval consists of the second node’s transmission circumference. Hence, the area of the second intersection region on the second subinterval, while excluding that of the first subinterval, follows

$$\mu[R_{0 \cap 2 \setminus 1}(u_{3})] = \frac{1}{2} \left[-A_{u_{1}}(u_{3}) + A_{u_{2}}(u_{3}) - \theta_{2} u_{3}^{2}\right] - \frac{1}{2} \left[-A_{u_{1}}(u_{\text{BD}}) + A_{u_{2}}(u_{\text{BD}}) - \theta_{2} u_{\text{BD}}^{2}\right], \quad u_{3} \in [u_{\text{BD}}, u_{\text{up}}],$$

This area expression is added to the total area contribution from the first subinterval to give the final area equation. If $u_{02} > u_{2}$, the bottom path is completely formed by the first node circumference (see Fig. A.7); thus the entire intersection area reduces to the first case equation A.2.

The upper value of the interval is usually $u_{2}$, however, when the point $X_{12}$ is below the baseline the upper interval value may be $u_{12}$ (refer to Fig. A.8). This motivates the use of indicator functions in deriving the upper value expression

$$u_{\text{up}} = \min(u_{2}, u_{12}^{-} + u_{2}^{+}).$$

Case 3

For the third case, both the top and the bottom paths of the intersection region consist of two different transmission circumferences, implying that the top path can also be split into two subintervals $[u_{\text{low}}, u_{TD}]$ and $[u_{TD}, u_{up}]$. Since two subintervals for each path implies two dividing values $u_{BD}$ and $u_{TD}$, we divide the intersection region on the first and second interval into the region above and below the baseline (the latter is a negative area). The difference between the top and bottom area contributions gives the area of that region on that subinterval.
Both paths on the two first subintervals are formed by the source transmission circumference. Consequently, the area between the top path and the baseline

\[
\mu[R_{0/2\backslash 1}^{\text{top}}(u_3)] = \frac{1}{2} A_{u_0}(u_3), \quad u_3 \in [u_{\text{low}}, u_{\text{TD}}],
\]

and similarly for the negative area below the baseline

\[
\mu[R_{0/2\backslash 1}^{\text{bot}}(u_3)] = -\frac{1}{2} A_{u_0}(u_3), \quad u_3 \in [u_{\text{low}}, u_{\text{BD}}],
\]

where the lower interval value is simply \(u_{\text{low}} = u_0 - r_0\). The two dividing values are \(u_{\text{TM}} = u_{01}\) and \(u_{\text{BM}} = u_{02}\).

The intersection region on the second interval above the baseline is formed by the first node’s transmission circumference, hence the area on the second interval, while excluding that of the first subinterval, follows

\[
\mu[R_{0/2\backslash 1}^{\text{top}}(u_3)] = \frac{1}{2} \left[-A_{u_1}(u_3) + \theta_1 u_3^2\right] - \frac{1}{2} \left[-A_{u_1}(u_{\text{TD}}) + \theta_1 u_{\text{TD}}^2\right], \quad u_3 \in [u_{\text{TD}}, u_{\text{up}}].
\]

The intersection region below the baseline stems from the second node’s transmission circumference, the hence the area on the second interval, while excluding that of the first subinterval, follows

\[
\mu[R_{0/2\backslash 1}^{\text{bot}}(u_3)] = \frac{1}{2} \left[-A_{u_2}(u_3) + (\theta_1 + \theta_2) u_3^2\right] - \frac{1}{2} \left[-A_{u_2}(u_{\text{BD}}) + (\theta_1 + \theta_2) u_{\text{BD}}^2\right], \quad u_3 \in [u_{\text{BD}}, u_{\text{up}}].
\]

The upper value of the interval is either \(u_2\) (see Fig. A.9) or \(u_{12}\) (see Fig. A.10) when the point \(X_{12}\) is below the baseline and \(u_{12} < 2\), hence

\[
u_{\text{up}} = \min(u_2, u_{12}I_{12} + u_2I_{12}).\]

Case 4

Again, the top path consists of two different transmission circumferences again, thus the integration interval becomes two subintervals \([u_{\text{low}}, u_{\text{TD}}]\) and \([u_{\text{TD}}, u_{\text{up}}]\). However, the bottom path only entails one transmission circumference of the second node. The top path is part of the source node’s transmission circumference, hence the area on the first subinterval is

\[
\mu[R_{0/2\backslash 1}(u_3)] = \frac{1}{2} \left[A_{u_0}(u_3) + A_{u_2}(u_3) - (\theta_1 + \theta_2) u_3^2\right] - \frac{1}{2} \left[A_{u_0}(u_{\text{low}}) + A_{u_2}(u_{\text{low}}) - (\theta_1 + \theta_2) u_{\text{low}}^2\right], \quad u_3 \in [u_{\text{low}}, u_{\text{BD}}].
\]
where the subinterval end values are \( u_{\text{low}} = u_{02} \) and \( u_{\text{TD}} = u_{01} \). On the second subinterval the top path entails the first node’s transmission circumference. Consequently, the area on the second interval, while excluding that of the first subinterval, follows

\[
\mu[R_{02}(u_3)] = \frac{1}{2} \left[ -A_{u_1}(u_3) + A_{u_2}(u_3) - \theta_2 u_3^2 \right] - \frac{1}{2} \left[ -A_{u_1}(u_{BD}) + A_{u_2}(u_{BD}) - \theta_2 u_{BD}^2 \right], \quad u_3 \in [u_{BD}, u_{up}].
\]

Again, the upper value of the interval is either \( u_2 \) (see Fig. A.11) or \( u_{12} \) (see Fig. A.12) when the point \( X_{12} \) is below the baseline and \( u_{12} < u_2 \), thus

\[ u_{\text{up}} = \min(u_2, u_{12}), \]

### A.6.3 Monte Carlo method

Alternatively, the area can be found by a Monte Carlo method. That is, an approximation of the area is achieved by randomly positioning points within the source feasible region and suitably applying indicator functions of the first and second nodes’ feasible region. This approach was applied and the results compared to the previous two approaches. The results completely agreed for a vast number of node configurations, thus giving credit to the aforementioned area methods.

### A.7 Configuration examples

<table>
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<th>Case</th>
<th>( u_0 )</th>
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<th>( \theta_1 )</th>
<th>( u_2 )</th>
<th>( \theta_2 )</th>
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<td>9.7</td>
<td>1.5( \pi )</td>
<td>9.6</td>
<td>-2.1( \pi )</td>
</tr>
<tr>
<td>2A</td>
<td>10</td>
<td>9.7</td>
<td>1.9( \pi )</td>
<td>9.6</td>
<td>-0.8( \pi )</td>
</tr>
<tr>
<td>2B</td>
<td>10</td>
<td>9.7</td>
<td>1.9( \pi )</td>
<td>9.6</td>
<td>-1.7( \pi )</td>
</tr>
<tr>
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<td>10</td>
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<td>1.9( \pi )</td>
<td>9.6</td>
<td>0.1( \pi )</td>
</tr>
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<td>10</td>
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<td>9.9</td>
<td>4.9( \pi )</td>
<td>9.2</td>
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<tr>
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<td>10</td>
<td>9.9</td>
<td>3.9( \pi )</td>
<td>9.4</td>
<td>2.3( \pi )</td>
</tr>
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Figure A.5: Case 1

Figure A.6: Case 2A
Figure A.7: Case 2B

Figure A.8: Case 2C
Figure A.9: Case 3A

Figure A.10: Case 3B
Figure A.11: Case 4A

Figure A.12: Case 4B
A.8 Derivation of $A_{1\setminus 0}(u_2)$ for distinct radii

Under the dependent model, the area of the feasible region of the first forwarding node is given by

$$A_1(u_2) = A_{u_1}(u_2) - A_{1\setminus 0}(u_2).$$

We extend the method of calculating the intersection area $A_{1\setminus 0}(u_2)$ to the case of distinct transmission radii $r_0$ and $r_1$. Again, we assume that $X_1$ is located above the baseline that extends from $X_0$ to $X_S$. To perform the necessary integrals herein, we recall the integral expression

$$A_{\gamma,r}(u) = 2\int_{\gamma-r}^{u} w\psi_{\gamma,r}(w)dw,$$

where

$$\phi_{\gamma,r}(u) = \arccos \left( \frac{r_0^2 + r_1^2 - u^2}{2r_0r_1} \right),$$

$$\psi_{\gamma,r}(u) = \arccos \left( \frac{u^2 + \gamma^2 - r_1^2}{2u\gamma} \right).$$

We write $u_0 = \ell$ and use the shorthand notation $A_{u_i} = A_{u_i,r_i}$ and $\psi_{u_i} = \psi_{u_i,r_i}$.

Let $X_{01}$ be the point closest to the sink where the transmission circumferences of the source and the current forwarding nodes intersect, and let $u_{01}$ be the distance from this point to the sink (see Fig. A.13). To calculate $u_{01}$, we observe that a triangle with sides of length $r_0$ and $r_1$ is formed by the points $X_0$, $X_1$ and $X_{01}$ (refer to Fig. A.14). The third side of this triangle is the distance separating $X_0$ and $X_1$, hence

$$h_{01} = (u_1^2 + u_0^2 - 2u_1u_0 \cos \theta_1)^{1/2}.$$
Let $\delta_1$ represent the angle $X_1X_0X_{01}$, given by
\[
\delta_1 = \arccos \left( \frac{h_0^2 + r_0^2 - r_1^2}{2h_0r_0} \right),
\]
and $\beta_1$ be the angle $X_1X_0X_S$ given by
\[
\beta_1 = \arccos \left( \frac{h_0^2 + u_0^2 - u_1^2}{2h_0u_0} \right).
\]
Represent the angle $X_SX_0X_{01}$ by $\eta_1 = \delta_1 - \beta_1$, thus
\[
u_{01} = \left( u_0^2 + r_0^2 - 2u_0r_0 \cos \eta_1 \right)^{1/2}.
\]
Let $X_{01}'$ be the other intersection point of the transmission circumferences of the source and the current forwarding node, which is closer to the source node. Let $\eta_1' = \delta_1 + \beta_1$ be the angle $X_SX_0X_{01}'$, which leads to the intersection point's sink distance
\[
u_{01}' = \left( u_0^2 + r_0^2 - 2u_0r_0 \cos \eta_1' \right)^{1/2}.
\]

The intersection area $A_{1\setminus 0}(u_2)$ is zero on the interval $[u_1 - r_1, u_0 - r_0]$ (for example, see Fig. A.13). To derive a general expression for $A_{1\setminus 0}(u_2)$ on the interval $[u_0 - r_0, u_1]$, let $\Delta \psi(u_2)$ be the angular width of the intersection of the source and current feasible regions (see Fig. A.15), hence
\[
A_{1\setminus 0}(u_2) = \int_{u_0 - r_0}^{u_2} \Delta \psi(u_2) dw_2, \quad u_2 \in [u_0 - r_0, u_1]. \tag{A.3}
\]
We consider three geometric configurations that are possible under the random radius model.

**A.8.1 Case 1**

The first case is simple, but undesirable from a routing perspective, as it entails the feasible region of the source node completely covering that of the first forwarding node (that is $\mathcal{I}_1(u_2) \subset \mathcal{I}_0(u_1)$).
Figure A.15: The angle function $\Delta \psi(u_2)$ describes the intersection region.

for the entire $u_2$-domain. Consequently, $\Delta \psi(u_2) = \psi_{u_1}(u_2)$, hence the intersection region has the area expression

$$A_{1\setminus 0}(u_2) = A_{u_1}(u_2), \quad u_2 \in [u_1 - r_1, u_1], \quad (A.4)$$

which clearly implies that the area of the feasible region of the forwarding node is zero under the dependent model.

### A.8.2 Case 2

Conversely, the total transmission region of the forwarding node may completely cover the feasible region of the source node (that is $I_0(u_2) \subset I_1(u_1)$ for some of $u_2$-domain), hence $\Delta \psi(u_2) = A_{u_0}(u_2)$ and the intersection region has the area expression

$$A_{1\setminus 0}(u_2) = A_{u_0}(u_2), \quad u_2 \in [u_0 - r_0, u_1]. \quad (A.5)$$

### A.8.3 Case 3

The final case is when neither region completely covers the other, and can be divided into three further general geometric configurations (see examples in Fig. A.16, A.17 and A.18 for $\ell = u_0 = 10$). The boundary of the intersection region is formed by circumferences of the two circles of radii $r_0$ and $r_1$, and a sector of radius $u_2$. As in the constant radius setting, the shape of this boundary naturally affects the form of $\Delta \psi(u_2)$. In particular, the main influencing factor is the location of the point $X_{01}$ in relation to the source-sink baseline and the line extending from $X_1$ to $X_S$ (henceforth known as the $X_1$-to-$X_S$ line).
We consider the case when X₀₁ is below the source-sink baseline (see example in Fig. A.16). On the interval \([u_0 - r_0, u_{01}]\) upper and lower boundaries of the feasible region boundary coincide with the transmission circumference of the source node, hence
\[
\Delta \psi(u_2) = 2\psi_{u_0}(u_2), \quad u_2 \in [u_0 - r_0, u_{01}].
\]

On the interval \([u_{01}, \min(u_1, u_{01}')]\) the upper and lower boundaries of the intersection region consist partially of the circumferences of both the source node and the first forwarding node, hence
\[
\Delta \psi(u_2) = \psi_{u_0}(u_2) + \psi_{u_1}(u_2) - \theta_1, \quad u_2 \in [u_{01}, \min(u_1, u_{01}']).
\]

If \(u_1 > u_{01}'\), then the upper and lower boundaries of the intersection region stem from the first forwarding node, thus
\[
\Delta \psi(u_2) = 2\psi_{u_1}(u_2), \quad u_2 \in [u_{01}', u_1].
\]

**X₀₁ above source-sink baseline and above X₁-to-Xₛ line**

Conversely, if X₀₁ is above the source-sink baseline, then \(\Delta \psi(u_2)\) can take two forms. If X₀₁ is also above the X₁-to-Xₛ line, then \(\Delta \psi(u_2) = 0\) on \([u_0 - r_0, u_1 - r_1]\), and on the interval \([u_1 - r_1, u_{01}]\) the upper and lower boundaries of the intersection region boundary coincide with the circumference of the first forwarding node (see example in Fig. A.17), hence
\[
\Delta \psi(u_2) = 2\psi_{u_1}(u_2), \quad u_2 \in [u_1 - r_1, u_{01}],
\]

Furthermore, on the interval \([u_{01}, \min(u_1, u_{01}')]\) the circumference of the source node is also partly surrounds the intersection region, thus
\[
\Delta \psi(u_2) = \psi_{u_0}(u_2) + \psi_{u_1}(u_2) - \theta_1, \quad u_2 \in [u_{01}, \min(u_1, u_{01}')]\].

As before, if \(u_1 > u_{01}'\), then the upper and lower boundaries of the intersection region stem from the first forwarding node, hence
\[
\Delta \psi(u_2) = 2\psi_{u_1}(u_2), \quad u_2 \in [u_{01}', u_1].
\]

**X₀₁ above source-sink baseline but below X₁-to-Xₛ line**

Alternatively, if X₀₁ is above the source-sink baseline, but below the X₁-to-Xₛ line (see example in Fig. A.18), then \(\Delta \psi(u_2) = 0\) on \([u_0 - r_0, u_{01}]\), and on the interval \([u_{01}, \min(u_1, u_{01}')]\) the \(\Delta \psi(u_2)\)
expression is identical to that when $X_{01}$ is below the source-sink baseline, hence

$$\Delta \psi(u_2) = \psi_{u_0}(u_2) + \psi_{u_1}(u_2) - \theta_1, \quad u_2 \in [u_{01}, \min(u_1, u_{01}')]$$.

Again, if $u_1 > u_{01}'$, then the upper and lower boundaries of the intersection region stem from the first forwarding node, hence

$$\Delta \psi(u_2) = 2\psi_{u_1}(u_2), \quad u_2 \in [u_{01}', u_1]$$.

**Combined $\Delta \psi(u_2)$ expression**

The final $\Delta \psi(u_2)$ expression is simply obtained via the linear combination of the angle expressions with the appropriate indicator functions for each situation, namely

$$\Delta \psi(u_2) = \begin{cases} 
2\psi_{u_0}(u_2)I_{01}^- + 2\psi_{u_1}(u_2)I_{01}^+, & u_2 \in [u_0 - r_0, u_{01}], \\
\psi_{u_0}(u_2) + \psi_{u_1}(u_2) - \theta_1, & u_2 \in [u_{01}, \min(u_1, u_{01}')] \\
2\psi_{u_1}(u_2), & u_2 \in [\min(u_1, u_{01}'), u_1].
\end{cases} \tag{A.6}$$

where $I_{01}^-$ and $I_{01}^+$ are the indicator functions for when $X_{01}$ is below the source-sink baseline and above the $X_1$-to-$X_S$ line respectively. This expression coupled with the integral (A.3) gives the area function for Case 3.

A further linear combination of the areas expressions under the three cases leads to a generalized area function $A_{1\setminus 0}(u_2)$ for all the possible geometric configurations, which immediately extends to the $n$-hop setting under the one-hop dependent model.
Figure A.16: Intersection point $X_{01}$ below the source-sink baseline.

Figure A.17: Intersection point $X_{01}$ above source-sink baseline and above $X_1$-to-$X_5$ line.
Figure A.18: Intersection point $X_{01}$ above source-sink baseline but below $X_{1}$-to-$X_{S}$ line.
Bibliography


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