A MODEL OF FERMION MASSES WITHOUT A HIGGS MECHANISM

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The purpose of this thesis is to understand the origin of fermion masses in a model where the electroweak symmetry is not a fundamental gauge symmetry. Electroweak symmetry is instead a global symmetry of a new strongly interacting sector, with the electromagnetic symmetry remaining as a gauged subgroup. The $W$ and $Z$-bosons are thus composite vector mesons, which obtain their mass from strong dynamics rather than the traditional Higgs mechanism in spontaneously broken gauge theories. As a result, the hierarchy problem is evaded since the compositeness scale is exponentially small compared to the Planck scale $M_P$. Fermions obtain mass by Yukawa coupling to a scalar vacuum expectation value, which breaks electroweak symmetry at the Planck scale. Interestingly, the natural scale of fermion masses is $\ll M_P$ and the hierarchical pattern of masses and mixings can be solved by Planck-scale physics.

After reviewing important background material on flavor physics, extra dimensions and branes, we describe the relationship between warping and compositeness using the holographic principle inspired by the anti de-Sitter/conformal field theory correspondence (AdS/CFT). Despite the non-perturbative nature of the physics underlying composite models, the technology of AdS/CFT is employed to construct a higher-dimensional, weakly-coupled description of fermions interacting with composite electroweak vector bosons in four spacetime dimensions. The techniques of electroweak precision analysis are explained and then applied to the model, using
the higher-dimensional model as a calculational tool. It is demonstrated that despite the composite nature of the $W/Z$-bosons, the model is consistent with electroweak precision data for the first two generations, while the third generation leads to a tension with indirect bounds. Moreover, the model is shown to predict definite deviations from the Standard Model parameters at energies that will be explored by the LHC.
DECLARATION

This is to certify that:

(i) the thesis comprises only my original work towards the MPhil except where indicated in the Statement of Contributions,

(ii) due acknowledgement has been made in the text to all other material used,

(iii) the thesis is less than 50,000 words in length, exclusive of tables, bibliographies, appendices and footnotes.

James David Eric Stokes
Statement of Contributions

Chapters 1 and 2 present an original review and extension of background material which is necessary to understand Chapter 3. Chapter 3 is based largely on the work presented in [1], which is a collaboration between the author, Tony Gherghetta and Yanou Cui. The appendices are reviews of known material. All of the calculations and numerical work in the thesis have been performed independently by the author, aided by discussions with Tony Gherghetta.
First and foremost, I would like to thank my supervisor Tony Gherghetta for his constant encouragement and support. I always enjoyed our discussions from which I have learnt a great deal of physics. I especially thank him for his patience and generosity over the course of the project.

I also deeply thank my loving parents and sister for their unfaltering belief in me. They have always encouraged me to pursue my ambitions in physics, which has made this a stronger thesis.

Finally, I would like to thank Sharman for always being there for me.
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# AdS/CFT

## B AdS/CFT

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The Standard Model presents a conundrum: despite over 30 years of electroweak data supporting a spontaneously broken gauge theory of weak interactions, the origin of electroweak symmetry breaking remains as elusive as ever.

It has been experimentally verified that three of the gauge bosons associated with the electroweak symmetry group $SU(2) \times U(1)$ are massive. If the electroweak symmetry is to remain as a fundamental gauge symmetry of nature, then the longitudinal polarization states of the massive vector fields must be obtained from the so-called Higgs mechanism. This is the process by which Nambu-Goldstone bosons of a spontaneously broken gauge symmetry are absorbed, becoming the longitudinal polarizations of the associated massive gauge fields.

The standard approach to accomplish electroweak symmetry breaking (EWSB) is to hypothesize that an elementary scalar multiplet of $SU(2) \times U(1)$ develops a nonzero vacuum expectation value (VEV), which is driven by a non-trivial potential. The nonzero VEV does not respect gauge symmetry, thereby triggering EWSB. Three Nambu-Goldstone bosons contained in the multiplet are absorbed and one physical scalar $h$ remains, known as the Higgs boson. The idea of an elementary scalar field breaking electroweak symmetry suffers from the serious theoretical deficiency that quantum corrections carry the mass of the Higgs boson — and consequently its VEV $v$ — up to a very high cut-off scale $\Lambda_{UV}$, which could be as high as the Planck scale $M_P = 2.4 \times 10^{18}$ GeV. This is in stark conflict with the Standard-Model prediction.
of $v = 250 \text{ GeV}$, which necessitates an unnatural cancelation over many orders of magnitude between the bare Higgs mass and the quantum corrections it receives. The delicate fine tuning of the bare Higgs mass parameter required to achieve this cancelation is the notorious \textit{gauge hierarchy problem}, which has motivated a large body of research into extensions of the standard electroweak theory.

Supersymmetry offers a potential mechanism to remedy the gauge hierarchy problem. If supersymmetry were an unbroken symmetry, then an exact cancelation would occur between the quantum corrections to the Higgs mass and quantum-mechanical loops induced by superpartner fields. The fact that nature does not realize supersymmetry at low energies suggests that supersymmetry must be broken at a high scale $\Lambda_{\text{SUSY}}$ beyond the reach of existing collider experiments. The corrections to the Higgs mass will thus be cancelled only above $\Lambda_{\text{SUSY}} \gg v$. If the SUSY-breaking scale lies too far beyond the electroweak scale, then the hierarchy problem is reintroduced.

An alternative approach to supersymmetrizing the SM with an elementary Higgs, is to assume that the Higgs is instead a condensate of strongly interacting states. If the Higgs is not elementary, then its mass will be proportional to a compositeness scale $\Lambda_{\text{comp}}$ which can be exponentially small compared to $M_P$. This is reminiscent of quantum chromodynamics (QCD) where the mass of the proton is proportional to the location of the Landau pole $\Lambda_{\text{QCD}}$ which is exponentially small compared to the UV scale,

$$
\Lambda_{\text{QCD}} = \Lambda_{\text{UV}} \exp \left( \frac{-4\pi}{\alpha_s b_0} \right)
$$

(1)

where $b_0 = 11N_c/3 - 2N_f/3 = 7$ for six flavours and three colours, and $\Lambda_{\text{UV}}$ is the cut-off scale for QCD which is of order $M_P$. Another approach to EWSB based on strong dynamics, which more closely parallels QCD is \textit{technicolor} [2,3]. In contrast to the previously discussed models, there is no Higgs boson and electroweak symmetry is spontaneously broken by new gauge degrees of freedom interacting with massless fermions, which are asymptotically free at high energies and become confining at the
electroweak scale. An even more radical suggestion is that the $W$ and $Z$ bosons could be resonances of a new strongly-interacting sector [4–6]. If the $W/Z$ are composite vector mesons, then a Higgs mechanism is not mandatory since the electroweak symmetry need not be a fundamental gauge symmetry. Instead, the electroweak symmetry can be a global symmetry to which the vector mesons couple. A major obstacle in the study of strongly-interacting theories of EWSB is that calculations cannot be performed using perturbative techniques. This drawback has resulted in minimal progress despite three decades of research. Strongly-coupled dynamics therefore presents us with a rich, yet unexplored arena of possibilities for the EWSB sector.

The proposal that new spatial dimensions may open up at the weak scale has shattered the existing dichotomy between supersymmetry and strong dynamics. An exponential hierarchy of mass scales can be generated geometrically due to the spatial separation of fields localized on different four-dimensional, Lorentz-invariant submanifolds called branes, which foliate a five-dimensional warped spacetime. If the intervening bulk space is anti-de-Sitter (AdS), then the Higgs VEV can be related to the Planck scale by an exponential whose argument is given by the proper distance $L$ separating the branes in units of the AdS radius $R_{AdS} \equiv 1/k$,

$$v \lesssim \Lambda_{IR} = \Lambda_{UV} \exp(-kL). \quad (2)$$

In an interesting twist of fate, the discovery of warped compactifications of AdS$_5$ [7,8] followed shortly after a conjecture that supergravity on non-compact AdS$_5$ crossed with certain compact manifolds is dual to large $N$, strongly coupled, conformally invariant field theory (CFT) in four spacetime dimensions [9–11]. It was soon realized that the introduction of branes in the warped space could be associated with breaking of conformal invariance at particular energy scales in the 4D gauge theory, leading to the formation of bound states representing the Kaluza-Klein modes in the dimensional reduction of the compactified 5D theory. The so-called \textit{holographic}
dictionary relating the two sides of the duality was subsequently developed, incorporating fermions, scalars and gauge fields in the bulk [12–16]. Using the dictionary, dual gravity theories have been engineered to mimic a composite Higgs boson [17], technicolor [18] and more recently a model of composite $W/Z$-bosons [19]. Despite the fact that the 4D physics underlying these strongly interacting theories is necessarily non-perturbative, the AdS/CFT correspondence affords a new perspective on the problem. According to the correspondence, in the limit of strong coupling in the 4D theory, the higher-dimensional gravity theory becomes weakly coupled and thus perturbative calculations can be performed on the gravity side of the duality.

The purpose of this thesis is to extend the proposal [19] by considering the generation of the fermion masses. Using the technology of AdS/CFT we will construct the gravity dual of fermions interacting with composite electroweak vector bosons. By performing calculations in the dual gravity theory, the phenomenological viability of this model will be assessed.

We first give a review of flavor and gauge hierarchy problems in particle physics. This is followed by an exposition into the extra dimension, showing how geometry offers a natural solution to both the flavour puzzle using the wavefunctions of bulk fields and the gauge hierarchy problem using warped compactifications such as the Randall-Sundrum scenario. The AdS/CFT correspondence is also discussed, emphasizing how warped spacetime provides a holographic perspective of the Standard Model. In the next chapter, the intricacies of electroweak symmetry breaking in warped compactifications are explored, and the effect of the extra dimension on electroweak precision observables is explained. Finally, the emergent model of EWSB is reviewed and an extension to give the fermions mass is presented. A perturbative calculation in 5D warped spacetime is carried out, showing that model is potentially consistent with electroweak precision constraints and displacements of gauge-boson/fermion vertices are predicted for the third generation of quarks.
1.1 The flavor puzzle and the Froggatt-Nielsen mechanism

The Standard Model (SM) is a theory of chiral fermions (matter) interacting via gauge bosons (forces) with gauge group SU(3) × SU(2) × U(1). The masses of the charged fermions, such as the electron and the up quark, originate from dimension-four interactions between chiral fermions and a scalar multiplet in the fundamental representation of SU(2) × U(1). The charged fermions in the Standard Model are comprised of three families of quarks and leptons, with identical gauge quantum numbers. The three-fold replication of the SM gauge quantum numbers results in the following general Lagrangian describing Yukawa interactions,

\[ \mathcal{L}_{\text{Yukawa}} = -Y^l_{ij} \overline{L_i} \varphi_R e_{Rj} - Y^d_{ij} \overline{Q_L i} \varphi_R d_{Rj} - Y^u_{ij} \overline{Q_L i} \tilde{\varphi}_R u_{Rj} + \text{h.c.} \]  

(1.1)

where \( \tilde{\varphi} \equiv i\sigma_2 \varphi^* \) is the charge-conjugate field and \( Y^l, Y^d, \) and \( Y^u \) are dimensionless \( 3 \times 3 \) matrices in family space which appear as free parameters in the SM. The assumption of naturalness posits that all of the dimensionless parameters should be of order unity. The experimental finding strongly disagrees with this expectation, however, indicating that only the Yukawa coupling between \( t_L \) and \( t_R \) is of order unity, with all other Yukawas of order \( \ll y_t \). In the mass eigenstate basis with the
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top quark Yukawa coupling normalized to unity, we have

\[(y_u, y_c, y_t) \approx (2 \times 10^{-5}, 8 \times 10^{-3}, 1)\] (1.2)

\[(y_d, y_s, y_b) \approx (3 \times 10^{-5}, 0.6 \times 10^{-3}, 2 \times 10^{-2})\] (1.3)

\[(y_e, y_{\mu}, y_{\tau}) \approx (0.3 \times 10^{-5}, 0.6 \times 10^{-3}, 1 \times 10^{-2}).\] (1.4)

In addition to the large hierarchy in the fermion masses, the mixing of fermions from different generations is strongly suppressed. This is reflected in the approximately diagonal structure of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix, which is clearest when cast in the Wolfenstein parametrization,

\[
V = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\rho - i \eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\
A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1
\end{pmatrix} + \mathcal{O}(\lambda^4) \quad (1.5)
\]

where \(A\) is of order unity, \(\lambda \simeq 0.22\) and \(\lambda < \rho, \eta < 1\).

While all of the observed flavor structure can be accommodated within the Standard Model with appropriate tuning of parameters, no explanation is offered for the hierarchical pattern of masses and mixings.

In an attempt to naturally obtain the flavor hierarchy, Froggatt and Nielsen [20] proposed that the small Yukawa couplings of the first and second generations originated from higher-dimensional operators which are naturally suppressed by the scale of new physics. They supposed that each quark family is separately charged under an additional \(U(1)'\) gauge symmetry. The charge assignments of each family under this additional gauge group implies that the mass term \(\overline{Q_L} \phi d R_j\) is forbidden unless the \(U(1)'\) charges satisfy \(q_R + q_H - q_L = 0\). It is assumed that \(U(1)'\) is spontaneously broken at a high scale by the VEV of a field \(\theta\) with \(q_\theta = 1\). Mass matrices arise from non-renormalizable interactions obtained after integrating out messenger fields \(\chi\) of
mass $\Lambda$. Assuming that $q_H = 0$, the effective Yukawa interactions are given by

$$Y_{ij}^{u} \overline{u}_{Li} u_{Rj} H \left( \frac{\theta}{\Lambda} \right)^{q_{Li}^u - q_{Rj}^u} + \text{h.c.}$$  \hspace{1cm} (1.6)$$

$$Y_{ij}^{d} \overline{d}_{Li} d_{Rj} H \left( \frac{\theta}{\Lambda} \right)^{q_{Li}^d - q_{Rj}^d} + \text{h.c.}$$  \hspace{1cm} (1.7)$$

If we assume that $q_{Ri}^{u,d} = -q_{Li}^{u,d}$, then a $q_L$ charge assignment of $(3, 2, 0)$ gives

$$Y^u \simeq Y^d \sim \begin{pmatrix} \epsilon^6 & \epsilon^5 & \epsilon^3 \\ \epsilon^5 & \epsilon^4 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}$$  \hspace{1cm} (1.8)$$

where $\epsilon \equiv \langle \theta \rangle / \Lambda$. It follows that the matrices transforming from the weak eigenstate basis to the mass eigenstate basis are of the form

$$U_L \simeq D_L \sim \begin{pmatrix} 1 & \epsilon & \epsilon^3 \\ \epsilon & 1 & \epsilon^2 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}$$  \hspace{1cm} (1.9)$$

and consequently the CKM is approximately reproduced if $\epsilon \simeq 0.22$.

The Froggatt-Nielsen mechanism thus provides an approximate fit to the hierarchical structure of the Yukawa couplings and the CKM matrix. The exact mass eigenvalues of the SM can be obtained by multiplying the entries of (1.8) with coefficients of order unity.

### 1.2 The hierarchy problem

In the standard model Lagrangian, gravity appears in the Einstein-Hilbert action with a coupling of order the reduced Planck mass squared $M_P^2 = \hbar / 8\pi G$. Other than
1. Background material

the quadratic coefficient $\mu_0^2$ of the Higgs potential,

$$V = -\mu_0^2 \phi^\dagger \phi + \lambda_0 (\phi^\dagger \phi)^2,$$

the Planck mass is the only other dimensional constant appearing in the standard model. The notation $\mu_0^2$ indicates the bare coupling constant as opposed to the renormalized value $-\mu^2$ which receives radiative corrections of order $\lambda \Lambda^2$ due to the quartic term in the Higgs potential, where $\Lambda$ is the UV cut-off. The value of $\mu^2$ is constrained by our knowledge of the Higgs vacuum expectation value $v = \mu/\sqrt{\lambda}$ which can be inferred experimentally from the gauge boson masses and coupling constants. If we assume that the Higgs sector is perturbative (i.e., that $\lambda \gg 1$) it follows that $\mu$ is at most $O(100\text{GeV})$ [21]. Consequently for $\Lambda \sim M_P = 2.4 \times 10^{18}\text{GeV}$, the relation $-\mu^2 = -\mu_0^2 + \lambda M_P^2$ implies a delicate fine-tuning in the bare mass parameter $\mu_0^2$ to ensure a light Higgs; in particular, $\mu^2/\Lambda^2 \sim 10^{-33}$. The Hierarchy problem is the statement that $\mu$ is unnaturally small compared to the Planck scale. This smallness is viewed as unnatural because $-\mu_0^2$ must be fine-tuned to achieve almost perfect cancelation with the radiative corrections, to an accuracy of one part in $10^{33}$. The hierarchy problem is sometimes phrased as the question of why gravity is so weak compared to the other forces which have couplings of order unity evaluated at the electroweak scale. These statements are equivalent because dimensional analysis implies the 4D gravitational coupling at the energy scale $s$ is $s/M_P^2$ and thus weakly coupled gravity at $s = M_{\text{ew}}^2$ implies that $M_{\text{ew}} \ll M_P$.

1.3 Extra dimensions

Extra dimensions have been a topic of intense research ever since the discovery that string theory predicts the number of spatial dimensions to be greater than three. The only known ways to conceal extra dimensions are compactification and localization.
of gravity. In $3+n$ spatial dimensions, the spacetime metric can be written as

$$ds^2 = g_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + g_{ij} dx^i dx^j + 2g_{\mu i} dx^\mu dx^i. \quad (1.10)$$

Here we have separated the metric into four-dimensional (4D) and extra-dimensional components, which are represented by Greek and Latin indices, respectively. The requirement of four-dimensional Poincaré invariance restricts the form of the metric to

$$ds^2 = e^{-A} \eta_{\mu\nu} dx^\mu dx^\nu + g_{ij} dx^i dx^j \quad (1.11)$$

where $e^{-A}$ is a function known as the warp factor which depends only on the extra dimensions. In the event that $e^{-A} = \text{const}$, the metric describes the Cartesian product of 4D Minkowski spacetime $M^4$ with some $n$-dimensional space $X_n$. If $X_n$ is taken to be a compact space of finite diameter $\text{diam}(X_n) < \infty$, then for distances $\gg \text{diam}(X_n)$, the product space $M^4 \times X_n$ should be indistinguishable from the observed universe, represented by $M^4$. In particular, the distance-dependence of forces will be unaffected at large distances, but will be modified at distances comparable to $\text{diam}(X_n)$ due to the leaking of flux into the volume of the extra dimensions.

The extra dimensions also have implications for particle physics. When fields occupying the higher-dimensional space have sufficient energy to resolve $X_n$, their momenta become quantized into a series of modes, analogous to the non-relativistic quantum-mechanical box problem. If the mode functions are chosen to be mass eigenstates, then the field will be represented by an infinite tower of 4D excitations with masses beginning at the compactification scale $1/\text{diam}(X_n)$ and with mass splittings of the same order. This tower can also contain massless zero modes if permitted by the boundary conditions or symmetries of the theory. It follows that if the extra dimension is to remain hidden to an observer consisting of particles $\phi$, then the diameter of $X_n$ must be smaller than the distance to which $\phi$ has been probed. The
most stringent experimental constraint currently comes from the Standard-Model particles, which have been measured to distances of order $1/\text{TeV} = 10^{-19}$ m, while the weakest constraint comes from the graviton — precision gravity measurements have only been made to distances of about 0.1 mm. Models with extra dimensions face the additional challenge of having to conceal the extra degrees of freedom associated with higher-dimensional fields such as the fifth component of the five-dimensional (5D) vector field $A_M$. The biggest difficulty lies with the metric tensor, whose fluctuations $h_{MN}$ about the classical background $g^c_{MN}$ include both four-dimensional components $h_{\mu\nu}$ as well as a set of spacetime vector fields $h_{\mu m}$ and scalar fields $h_{mn}$. While the zero mode of the spacetime tensor $h_{\mu\nu}$ provides the ordinary graviton in four dimensions, the zero modes of the vector and scalar fluctuations create unwanted long-range forces. Consequently, these additional fields must be given a mass to decouple them from the low-energy theory.

The relative ignorance about the microscopic behavior of gravity compared to the Standard Model has motivated the ADD model [22] which conjectures that the Standard-Model states are trapped to a 3-dimensional submanifold (3-brane) of small transverse extent, embedded in a spacetime with $d > 3$ spatial dimensions called the bulk. This spatial localization of the SM fields over a small characteristic distance pushes their Kaluza-Klein scale beyond the detectable limit. Unlike the SM fields, the graviton probes the full extra-dimensional space since gravity is responsible for the structure of spacetime. The asymmetry in the way that gravity is treated compared to the SM fields weakens the constraint on the size of the extra dimension to the distances at which gravity has been measured; that is, the micrometer scale. The large size of the extra dimension results in the production of very light and finely split Kaluza-Klein gravitons. These do not generally interfere with phenomenology at energies below the compactification scale, since they have the same gravitational coupling to matter as the ordinary massless mode. The ADD model has attracted considerable interest because it provides a method to solve the
hierarchy problem. The four-dimensional gravitational coupling on the 3-brane is
diluted by the volume of the $n$ extra dimensions. In particular, if $X_n = T^n$ where
the torus has radius $R$, then the gravitational force law at small distances $r \ll R$
behaves as $g(r) = m/(M_s^{2+n} r^{2+n})$ where $M_s$ is the fundamental gravity scale. At
distances large compared to the compactification scale, however, the extra dimen-
sions saturate with flux and $g(r)$ tends to $m/(M_s^{2+n} R^n r^2)$. The effective Planck
constant felt at large distances is thus

$$M_P^2 \simeq M_s^{2+n} R^n. \quad (1.12)$$

More generally, the effective Planck constant is given by the coefficient of the 4D
Ricci scalar $R^{(4)}$, obtained after integrating away the $n$ extra coordinates in the
Einstein-Hilbert action; that is,

$$M_P^2 = M_s^{d-1} \int d^n x \sqrt{|\det g_{mn}|} e^{-A} = M_s^{d-1} \text{Vol}(X_n). \quad (1.13)$$

If $M_s \sim \text{TeV}$ then one can show that $n = 1$ has already been ruled out while $n = 2$
constrains the scale of new physics to $M_s \gtrsim 3 \text{TeV}$. Since the scale of fundamental
gravity has been lowered, it is argued that gravitational interactions become strong
at the TeV scale where, for example, stringy corrections must be included. The
ADD model is naturally incorporated into string theory where the 3-brane can be a
solitonic solution (D3-brane) of closed strings propagating in 10 dimensions. The D-
brane supports gauge and matter fields which can be used to construct the standard
model.

Randall and Sundrum (RS) [7, 8] opened up a new branch of model building with
their insight that the warp factor need not be independent of the extra coordinates.
Such a spacetime cannot be factored into a Cartesian product of the extra dimensions
with $M^4$ and is said to be warped. There are several phenomenological reasons
to consider warped spacetime. Firstly, the warp factor leads to a local definition
of energy scale which varies with position in the extra dimensions. This enables hierarchies to be obtained by physically separating fields in the extra dimension. RS considered a 5D spacetime \((x^\mu, y)\) where \(y\) is compactified on the interval \([0, L]\). They found a solution to the Einstein equations with warp factor \(e^{-2ky}\) when the energy-momentum consisted of a negative cosmological constant in the bulk and gravitating end-of-the-world 3-branes placed at \(y = 0\) and \(y = L\). It follows from invariance of the interval that an observer located in a region of small warp factor \(e^{-2kL} \ll 1\) will experience an effective cut-off scale \(M_p e^{-kL}\) which can be made of order the TeV scale for \(kL = \mathcal{O}(10)\).

The second incentive for studying warping is that the extra dimensions need not be compact if the spacetime is sufficiently warped. One clue to this is the fact that the volume factor which determines the effective Planck constant involves an integral of the warp factor over the internal space. If \(e^{-A}\) decays sufficiently fast with the extra coordinates, then the extra dimensions will have finite volume even if they are topologically non-compact. If the warp factor is sharply peaked on a Minkowski 3-brane, then it is intuitively clear that observers on the brane will experience approximately four-dimensional gravity since the amplitude for propagation of gravitons almost vanishes in the bulk.

The example studied by RS was obtained by taking \(L \to \infty\) in the Randall-Sundrum metric. The gravitational potential between test masses on the \(y = 0\) brane receives corrections due the nonvanishing amplitude for propagation of gravitons in the bulk as well as the brane. The result is in accordance with the expectation that the approximation improves as the decay constant \(k\) is increased; namely,

\[
G \frac{m_1 m_2}{r} \left( 1 + \frac{1}{r^2 k^2} \right). \tag{1.14}
\]

As of 2007, the experimental constraint on the bulk curvature \(k > 1.6 \times 10^{-2} \text{eV} [23]\) is safely evaded by the expectation that \(k\) is of order the gravity scale \(M_p\).
Following the proposal of infinite warped extra dimensions, a modification of the RS setup was put forward by Gregory, Rubakov, and Sibiryakov (GRS) \[24\] where the 3-brane is embedded in a non-compact, asymptotically flat bulk with infinite volume. The GRS model supposes that the warp factor is peaked on the \(y = 0\) brane and decays to a constant \(\epsilon > 0\) away from the brane. A particular realization of this model is given by the metric 
\[
ds^2 = (e^{-2k|y|} + \epsilon)\eta_{\mu\nu} \, dx^\mu \, dx^\nu - dy^2
\]
where \(\epsilon \ll 1\). At small distances from the brane, \(\epsilon \ll e^{-2k|y|}\) and the metric behaves like the Randall-Sundrum proposal. At sufficiently large \(|y|\), we have \(e^{-2k|y|} \ll \epsilon\) and rescaling the \(x^\mu\) coordinate gives the metric for \(M^5\). The fifth dimension of space thus reveals itself in both short and long-distance tests of gravity. It was later pointed out that the GRS model is unlikely to be tenable because the energy-momentum tensor necessary to produce the background spacetime violates the null-energy condition which demands that \(A'' \leq 0\) \[25\].

A subsequent proposal of Dvali, Gabadadze and Porrati (DGP) \[26\] offers an alternative mechanism to recover four-dimensional gravity on a 3-brane embedded in the bulk of a 5D Minkowski spacetime \(M^5\). They assumed that in addition to the usual five-dimensional Einstein-Hilbert term describing gravity in the bulk, the action contains a large four-dimensional graviton kinetic term on the worldvolume of the 3-brane,

\[
S[g] = \frac{M^3}{2} \int d^5x \sqrt{|g|} R + \frac{M_5^2}{2} \int d^4x \sqrt{|g^{(\text{ind})}|} R^{(4)}.
\]

(1.15)

Note that for \(r_c \equiv M_5^2/M_3^3 \to \infty\) the action describes 4D gravity on the brane while for \(r_c \to 0\) it describes 5D bulk gravity. For \(0 < r_c < \infty\), we thus expect that gravity will be four-dimensional for distances \(\lesssim r_c\) and five-dimensional at larger distances. In summary, the universe can consistently possess one or more extra dimensions. If the extra dimensions are flat, then their size can range anywhere from \(1/M_P\) up to \(1/\text{TeV}\). In the braneworld scenario, the extra dimensions can be significantly larger than \(1/\text{TeV}\), perhaps of order a fraction of a millimeter. If gravity is localized in the
extra dimensions, then their size can range anywhere from Planckian to infinity.

1.3.1 Dimensional reduction

Field theories in $d$ spacetime dimensions can be given a $(d-1)$-dimensional description. This reduction in spatial dimension is achieved by carrying out one of the $d$ spacetime integrals ($y$ say), leaving $d-1$ integration variables in the action. In order to facilitate the integration with respect to $y$, it is necessary to separate out the $y$-dependence of the $d$-dimensional fields by expanding them into a generalized Fourier series,

$$\delta \Phi(x, y) = \sum_n \varphi_n(x) f_n(y)$$

(1.16)

where $\delta \Phi$ denotes a fluctuation of the $d$-dimensional field $\Phi$ about its classical background configuration $\Phi_c$ and where $\{f_n\}$ is any complete set of functions of $y$. The Fourier coefficients $\varphi_n$ are functions of the $d-1$ remaining coordinates $x$. There is a large arbitrariness in the choice of the expansion functions $f_n$, which is reflected by the arbitrary choice of basis for the infinite-dimensional function space over $y$. A suitable mode expansion can be motivated by requiring the expansion coefficients $\varphi_n$ to have a simple interpretation.

We now specialize to the case $d = 5$, where the fifth dimension extends the four dimensions of Minkowski space. It will often be convenient to work in the basis where the 4D fields $\varphi_n$ represent particles of definite mass $m_n$. This is referred to as the Kaluza-Klein basis for historical reasons. If $\varphi_n$ are scalar fields, for example, then the mass basis corresponds to the requirement that each coefficient function satisfies an independent Klein-Gordon equation $(\Box + m_n)\varphi_n = 0$, where $\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$. Note that since massive free fields obey linear equations of motion, it will not generally be possible to find a Kaluza-Klein basis for an interacting field theory. If the interaction terms are small, however, then it is still sensible to consider a Kaluza-Klein decomposition of the linearized equations of motion. In such a basis, small fluctuations of
the higher-dimensional field are represented by towers of 4D mass eigenstates. The Kaluza-Klein basis can be found by mode-expanding the equations of motion for linearized fluctuations about the classical background \( \Phi_c \). Substituting the mode expansion (1.16) into the linearized field equation, one finds that derivatives with respect to \( x \) and \( y \) act only on the functions \( \varphi_n \) and \( f_n \), respectively. The linearized field equation can thus be written in the general form

\[
\sum_n \left[ \sum_{i,j,k,\ldots \geq 0} X_n^{(ijk\ldots)}(x) \partial_{ijk\ldots} f_n(y) \right] = 0 \tag{1.17}
\]

where \( X_n^{(ijk\ldots)} \) represents a function of \( \varphi_n \) and its \( x \)-derivatives. By the completeness of the set of functions \( \{f_n\} \), any choice of basis implies a set of equations of motion satisfied by the coefficient functions \( \{\varphi_n\} \). Viewed another way, imposing a suitable equation of motion for each \( \varphi_n \) dictates a complete set of functions of \( y \). Imposing the equation of motion for a massive free field in the linearized mode expansion (1.17) and then independently varying the coefficient functions gives a second-order, ordinary differential equation in the basis functions \( f_n \). The basis functions typically satisfy a Sturm-Liouville equation where the eigenvalues are the Kaluza-Klein masses \( m_n \).

Once the profile functions and Kaluza-Klein masses have been fixed by boundary conditions, the dimensional reduction of the action can be obtained by substituting the mode expansion into the action and carrying out the integrals with respect to \( y \), integrating by parts if necessary to remove \( y \)-derivative terms. The appropriate normalization for \( f_n \) is then suggested by the requirement that the kinetic terms for \( \varphi_n \) are canonically normalized.

The KK towers sometimes contain massless modes, in addition to a discretum and/or continuum of massive modes. The massive KK modes typically decouple since they are either too heavy to be produced or couple negligibly to the low-energy theory. The dimensional reduction is therefore well-described by a theory of massless states.
1. Background material

This fact can be used to construct the SM as a low-energy, decoupled limit of a field theory in a space of higher dimension.

1.3.2 The 5D chirality problem and the orbifold solution

Unlike in four dimensions where the irreducible representations of the Lorentz group form Weyl spinors, the smallest representation of the five-dimensional Lorentz group induces a Dirac spinor representation of the 4D Lorentz group. This can be understood from the form of the gamma matrices in five dimensions, which satisfy the 5D Clifford algebra \( \{ \gamma_M, \gamma_N \} = 2 \eta_{MN} \) and can thus be given the explicit matrix representation

\[
\gamma_M = (\gamma_\mu, \pm i \gamma_5) \quad \text{where} \quad \gamma^\mu \quad \text{are the 4D gamma matrices satisfying} \quad \{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu\nu}. \]

Since \( \gamma_5 \) is a diagonal matrix, the 5D component of the kinetic energy acts like a mass term by mixing the right-handed (+) and left-handed (−) components of \( \Psi = \Psi_+ + \Psi_- \) where \( \gamma_5 \Psi_\pm = \pm \Psi_\pm \). Hence the gauge quantum numbers for the left and right-handed components of a 5D fermion cannot be separated. The action for a five-dimensional fermion of mass \( m_\Psi \) on the extra-dimensional interval \( \mathbb{R}^{1,3} \times [0, L] \) is

\[
S_\Psi = \int d^4x \int_0^L dy \left[ \frac{i}{2} \left( \overline{\Psi} \gamma^M \partial_M \Psi - \partial_M \overline{\Psi} \gamma^M \Psi \right) - m_\Psi \overline{\Psi} \Psi \right] \tag{1.18}
\]

where the kinetic term has been written in this form to take into account the possibility of nonvanishing total divergences. Stationary variations of the action give

\[
\delta S_{\text{bulk}} = \int d^5x \left\{ \delta \overline{\Psi}_+ \left( i \partial \Psi_+ + \partial \overline{\Psi}_- - m_\Psi \Psi_- \right) + \delta \overline{\Psi}_- \left( i \partial \Psi_- - \partial \overline{\Psi}_+ + m_\Psi \Psi_+ \right) \right\} + \text{h.c.} \tag{1.19}
\]

\[
\delta S_{\text{bdry}} = \int d^4x \left\{ \frac{1}{2} \left[ \delta \overline{\Psi}_- \Psi_+ - \delta \overline{\Psi}_+ \Psi_- \right]|_0^L \right\} + \text{h.c.} \tag{1.20}
\]

\footnote{For definiteness, we will fix \( \gamma_M = (\gamma_\mu, -i \gamma_5) \) so that \( \gamma^M = (\gamma^\mu, +i \gamma_5) \), where \( \gamma^0 = \text{diag}(1, 1) \), \( \gamma^i = \text{diag}(\sigma_i, -\sigma_i) \) and \( \sigma_i \) are the Pauli matrices \( \sigma_1 = \text{diag}(1, 1) \), \( \sigma_2 = \text{diag}(i, -i) \), \( \sigma_3 = \text{diag}(1, -1) \). The relationship with the mostly plus convention is \( \gamma^\mu_{mp} = -i \gamma^\mu_{mn} \).}
where \( \partial \equiv \gamma^\mu \partial_\mu \). Requiring \( \delta S_{\text{bulk}} \) to vanish for arbitrary variations of \( \Psi_- \) and \( \Psi_+ \) in the bulk gives the equations of motion,

\[
i \partial \Psi_\pm \pm \partial_\gamma \Psi_\mp - m_\Psi \Psi_\mp = 0. \tag{1.21}
\]

It is not possible to vary \( \Psi_- \) and \( \Psi_+ \) independently on the boundaries at \( y_* = 0, L \) because the boundary values \( \Psi_+|_{y_*} \) and \( \Psi_-|_{y_*} \) are related by the bulk equations of motion. A consistent set of boundary conditions can be obtained by considering constrained variations of the boundary field values. Imposing the constraint that \( \Psi_\pm|_{y_*} \) is independent of the transverse coordinates \( x^\mu \) so that \( \delta \Psi_\pm|_{y_*} = 0 \), we find that stationary variations of \( \Psi_\mp|_{y_*} \) imply \( \Psi_\pm|_{y_*} = 0 \). The bulk equations then determine the boundary condition for \( \Psi_\mp \) to be \( (\pm \partial_y - m_\Psi) \Psi_\mp|_{y_*} = 0 \). It will be shown later that if \( \Psi_\pm \) is chosen to have Dirichlet boundary conditions at both ends of the extra dimension, then only the Kaluza-Klein tower for \( \Psi_\mp \) will contain a zero mode, resulting in a chiral four-dimensional theory at low energies.

Before explaining the elimination of the zero mode from the \( \Psi_\pm \) tower, it is pertinent to discuss a popular alternative approach based on the orbifold. An orbifold is a space which is locally homeomorphic to the quotient space \( \mathbb{R}^n / G \) where \( G \) is a discrete subgroup of the linear isometries of \( \mathbb{R}^n \). Typically, the orbifold is taken to be an identification space \( X/G \) where \( X \) is a smooth manifold and \( G \) is a discrete subgroup of the isometries of \( X \). The prototypical example, which we follow throughout this thesis is \( S^1 / \mathbb{Z}_2 \). The circle \( S^1 \) is a smooth manifold obtained by periodically identifying points in \( \mathbb{R} \); that is \( y \sim y + 2\pi R \). The isometry group \( \mathbb{Z}_2 \) denotes reflection in \( y \).

To write down the action for the orbifold theory, recall that the circle \( S^1 \) can be obtained by from the real line \( \mathbb{R} \) by periodically identifying points. To facilitate comparison with the interval theory, we take the diameter of the circle to be \( 2L \) so that the covering space is \([-L, L] \subset \mathbb{R} \). Imposing a \( \mathbb{Z}_2 \) symmetry is equivalent to requiring the action to be invariant under \( y \rightarrow -y \). Since the orbifold is
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without boundary, the integration by parts only gives bulk terms and the behavior of the fields at \( y = 0 \) and \( L \) is determined by symmetries of the theory. Under the coordinate transformation \( x'{}^{\mu} = x^{\mu}, y' = -y \), it can be shown from the 5D Dirac equation that a fermion transforms as \( \Psi'(x', y') = \pm \gamma_5 \Psi(x, y) \). Requiring \( \mathbb{Z}_2 \) symmetry is equivalent to the requirement that \( \Psi(x, -y) = \pm \gamma_5 \Psi(x, y) \) or that \( \Psi_+(-y) = \pm \Psi_+(y) \) and \( \Psi_-(y) \). The fact that \( \Psi_\pm \) is even when \( \Psi_\mp \) is odd can also be seen from the kinetic term which contains \( \mathcal{L}_{\Psi} \). Since \( \partial_5 \) is obviously odd, \( \mathbb{Z}_2 \) symmetry implies that the combination \( \mathcal{L}_{\Psi} \) must also be odd.

Thus if \( \Psi_\pm \) is even then \( \Psi_\mp \) must be odd and consequently the mass term \( m_\Psi \) must be an odd function of \( y \). Since we are interested in mass eigenstates, we expand the 5D fields into Kaluza-Klein modes as

\[
\Psi_\pm(x, y) = \sum_n \psi^{(n)}_\pm(x) f^{(n)}_\pm(y), \quad i\partial_5 \psi^{(n)}_\pm(x) = m_n \psi^{(n)}_\pm(x).
\]

Varying the coefficient functions independently gives

\[
f^{(n)}_\pm(y) m_n + \left( \pm \frac{d}{dy} - m_\Psi(y) \right) f^{(n)}_\pm(y) = 0.
\]

Fixing \( \Psi(x, -y) = \gamma_5 \Psi(x, y) \) so that \( \Psi_+ \) is even and \( \Psi_- \) is odd, it follows that only \( \Psi_+ \) is consistent with a zero mode because the zero-mode solution

\[
f^{(0)}_\pm(y) = \exp \left( \mp \int_0^y \right)
\]

is not compatible with the fact that \( \Psi_- \) vanishes at \( y = 0 \) and \( L \). The zero mode for the odd tower is said to have been projected out by the orbifold. The zero mode contained in the tower for the even field \( \Psi_+ \) is not paired with a zero mode of opposite chirality. Using the fact that the wavefunctions \( f^{(n)}_\pm \) can be normalized to
form an orthonormal set, we obtain the effective action

\[ S = \int d^4x \left[ i\psi^{(0)} \psi^{(0)*} + \sum_{n=1}^{\infty} \psi^{(n)} \left( i\phi - m_n \right) \overline{\psi}^{(n)} \right]. \]  

(1.25)

where \( n \geq 1 \) represent the massive Kaluza-Klein levels. Consequently at energies below the compactification scale, the four-dimensional effective theory consists of a massless chiral fermion. If there is sufficient energy in the field to bridge the mass gap separating the zero mode from KK modes, then the extra dimension will reveal itself as a tower of massive particles.

The field-theoretic origin of the odd mass term is understood to be the vacuum expectation value of a topological, domain-wall soliton. Domain walls arise as non-trivial solutions of field theories with spontaneously broken discrete symmetries. As an example, consider a \( \mathbb{Z}_2 \) symmetric scalar field theory on an extra-dimensional orbifold \([-L, L]\),

\[ S = \int d^4x \left( \frac{1}{2} \partial_M \eta \partial^M \eta + \frac{m^2}{2} \eta^2 - \lambda \eta^4 \right). \]  

(1.26)

When \( L = \infty \), this theory admits kink/anti-kink solutions which interpolate between degenerate vacua at \( y = \pm \infty \),

\[ \langle \eta \rangle (y) = \pm \frac{m}{\sqrt{4\lambda}} \tanh(by) \]  

(1.27)

where \( b = m/\sqrt{2} \). When \( L \) is finite and much larger than the wall width \( b^{-1} \), the solution is well-approximated by a series of alternating kinks and anti-kinks (see Fig. 1.1); that is [27],

\[ \langle \eta \rangle (y) = \mp \frac{m}{\sqrt{4\lambda}} \tanh b(-L - y) \tanh(by) \tanh b(L - y) \]  

(1.28)

In the limit \( m \to \infty \), the zero-mode profile vanishes off the wall, forming a delta function which indicates that it is completely brane-localized, realizing the ADD limit. Taking \( m \to \infty \) keeping \( m^2/\lambda \) finite, we find that the kink becomes propor-
tional to a step function. In this limit, the zero-mode solution is de-localized from
the brane and has profile function $f^{(0)}_{\pm}(y) = e^{\mp m \pm |y|}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The vacuum expectation value of a $Z_2$ kink on the orbifold.}
\end{figure}

1.3.3 The extra dimension as a solution to the flavor puzzle

In addition to solving the gauge-hierarchy problem, extra dimensions also provide a
natural mechanism to generate hierarchies amongst the fermion Yukawa couplings
providing an alternative realization of the Frogatt-Nielsen mechanism. The mecha-
nism relies on the simple observation that the wavefunctions for fermion zero-modes
vary exponentially along the extra dimension. Effective Yukawa interactions in the
dimensionally reduced theory descend from the higher-dimensional Yukawa sector
after integrating away the extra dimension. An example Yukawa interaction in five
dimensions is

$$\int d^5x \sqrt{|g|} Y_{ij}^{(5)} \Psi_i^{(L)} \Phi_5 \Psi_j^{(R)} + \text{h.c.}$$ (1.29)

where $\Phi_5$ represents a 5D scalar field, the superscripts (R) and (L) indicate that $\Psi_i^{(R)}$
and $\Psi_i^{(L)}$ contain zero modes, $i, j$ are generation indices and $Y_{ij}^{(5)}$ has mass dimension
$-1/2$. If the Higgs field is localized on a brane so that $\Phi_5(x, y) = \phi(x) \delta(y - y_*) \sqrt{L}$,
while the fermions are de-localized, then the 4D Yukawa interactions are given by

\[
\left( \sqrt{L} Y^{(5)}_{ij} \int dy \sqrt{|g|} f_{Li-}^{(0)*}(y) f_{Rj+}^{(0)}(y) \delta(y-y_*) \right) \int d^4x \bar{\psi}^{(L)}_i \phi(x) \psi^{(R)}_j + h.c.
\]

(1.30)

While the original proposal involved trapping the fermions to a domain wall which represents a thick brane [28], it is simpler to consider de-localized fermions in the bulk [29]. Working in flat space on the orbifold \([-L, L]\) for simplicity, and normalizing the fermions as \(\int_{-L}^{L} dy (f^{(0)})^2 = 1\), so that

\[
f_{Li\pm}^{(0)}(y) = \sqrt{m_{Li}} \frac{e^{\mp m_{Li} y}}{1 - e^{-2m_{Li}L}} \]

(1.31)

\[
f_{Rj\pm}^{(0)}(y) = \sqrt{m_{Rj}} \frac{e^{\mp m_{Rj} y}}{1 - e^{-2m_{Rj}L}}
\]

(1.32)

the expressions for the effective Yukawa couplings are given by

\[
\lambda_{ij} = \sqrt{L} Y^{(5)}_{ij} \frac{\sqrt{m_{Li}m_{Rj}}}{\sqrt{(1 - e^{-2m_{Rj}L})(e^{2m_{Li}L} - 1)}} \exp \left[(m_{Li} - m_{Rj})y_\Phi\right].
\]

(1.33)

Setting \(-m_L = m_R = m\) and \(y_\Phi = L\) so that the zero modes lean away from the Higgs brane for \(m > 0\), we obtain

\[
\lambda = \sqrt{L} Y^{(5)} \frac{m}{1 - e^{-2mL} e^{-2mL}}
\]

(1.34)

where we have ignored off-diagonal terms in flavor space. It follows that the 4D Yukawa couplings depend exponentially on bulk parameters. Since the natural scale for all 5D parameters in the theory is the compactification scale \(L\), the Yukawa couplings of the Standard Model can be reproduced by choosing all dimensionless 5D parameters to be of order unity. Thus, the extra dimension offers the possibility that the unnatural flavor structure of the SM can result from a natural model with more than three spatial dimensions.
1.3.4 5D gauge fields and charge universality

While it is straightforward to localize chiral fermions onto a topological defect, the localization of gauge fields presents a more challenging problem. The only known mechanism to satisfactorily localize gauge fields onto a brane is to assume that the bulk spacetime possesses a non-abelian gauge symmetry which is in confinement phase [30]. A disadvantage of this approach is that the physics underlying the bulk is necessarily non-perturbative, which renders analytical field theory calculations difficult. This provides one incentive to study de-localized gauge fields. A second incentive is that if charged fermions occupy the bulk, then gauge fields must accompany them. The action for a gauge field in flat, five-dimensional spacetime $\mathbb{R}^{1,3} \times [0, L]$ is

$$S = \int d^5 x \left[ -\frac{1}{4} F_{MN} F^{MN} \right]$$

(1.35)

$$= \int d^5 x \left[ -\frac{1}{4} F_{\mu \nu}^2 + \frac{1}{2} (\partial_5 A_{\mu} - \partial_{\mu} A_5)^2 \right]$$

(1.36)

Varying the action gives $\delta S = \delta S_{\text{bulk}} + \delta S_{\text{bdry}}$ where

$$\delta S_{\text{bulk}} = -\int d^5 x \partial_M F^{MN} \delta A_N$$

(1.37)

$$= -\int d^5 x \left[ -\partial_M \partial^M A^N + \partial^N \left( \partial_M A^M \right) \right] \delta A_N$$

(1.38)

$$\delta S_{\text{bdry}} = -\int d^4 x \left[ \partial^5 A^\mu - \partial^\mu A^5 \right] \delta A_\mu \bigg|_0^L.$$  

(1.39)

We see that if the boundary conditions for the fifth component of the gauge field are chosen to be Dirichlet, then stationary variations of the action on the boundary ensure that the $\mu$ components have Neumann boundary conditions. In the orbifold language, this corresponds to choosing $A_5$ to be the odd field, which implies that $A_\mu$ is even as a result of the mixing term in (1.36). Conversely, if $A_\mu$ has Dirichlet boundary conditions, then $\delta A_\mu|_{0,L} = 0$ so one can consistently choose $A_5$ to be
Neumann. Alternatively, if \( A_5 \) is even on the orbifold, then \( A_\mu \) must be odd.

Under 4D Lorentz transformations, the \( \mu \) components of the 5D gauge field \( A_M(x, y) \) form a four-vector while the fifth component acts as a scalar field. This suggests that a suitable mode expansion is

\[
A_\mu(x, y) = \sum_{n=0} A^{(n)}_\mu(x)f_n(y),
\]

\[
A_5(x, y) = \sum_{n=0} A^{(n)}_5(x)g_n(y).
\]

A massless gauge field in five dimensions possesses three physical degrees of freedom as a result of gauge symmetry, corresponding to the three transverse polarization states. This implies that each Kaluza-Klein level \( n \) will possess at most three degrees of freedom, which are shared amongst the 4-vector \( A^{(n)}_\mu \) and the scalar component \( A^{(n)}_5 \). If the 4D vectors are chosen to represent massive vector fields, then the scalar fields in the massive Kaluza-Klein levels will be unphysical. This can be understood from the fact that a massive vector field in four dimensions possesses two transverse and one longitudinal polarization state; accounting for all three of the requisite degrees of freedom. The absence of the scalar fields in the massive KK levels can also understood in terms of spontaneous symmetry breaking. In the dimensionally reduced theory, the 5D gauge symmetry will appear as an infinite number of 4D gauge symmetries; one for for each KK level \( n \). At each massive KK level \( n \geq 1 \), the scalar field \( A^{(n)}_5 \) provides the Nambu-Goldstone boson which is absorbed as a longitudinal polarization state for the massive vector \( A^{(n)}_\mu \). In the zero mode level \( n = 0 \), the gauge symmetry is unbroken and consequently \( A^{(0)}_5 \) is the only mode which remains in the spectrum of \( A_5 \).

To see the explicit absorption of the Nambu-Goldstone bosons by the massive vector fields, let us mode expand the action (1.36). Guided by the fact that the modes of
\(A_\mu\) represent states of definite mass, the profile functions are chosen to satisfy

\[-\frac{d^2 f_n}{dy^2} = m_n^2 f_n(y).\] (1.42)

The mode expansion of \(A_5\) is chosen so that the mixing term \(-\partial_\mu A_5 \partial_\nu A_\nu\) is diagonal. This suggests that the wavefunctions for the massive levels should be \(g_n(y) = (1/m_n) f'_n(y)\). The zero mode is chosen to have a wavefunction orthogonal to \(\{f'_n\}_{n \geq 1}\), which will be determined later. The resulting dimensionally-reduced action is

\[S = \int d^4 x \left\{ -\frac{1}{4} (F^{(0)}_{\mu\nu})^2 + \frac{1}{2} (\partial_\mu A_5^{(0)})^2 \int dy g_0(y)^2 + \sum_{n=1} \left[ -\frac{1}{4} (\tilde{F}^{(n)}_{\mu\nu})^2 + \frac{1}{2} m_n^2 (\tilde{A}_\mu^{(n)} - \frac{1}{m_n} \partial_\mu A_5^{(n)})^2 \right] \right\},\] (1.43)

which is equivalent to the unitary gauge. The fact that the massive \(A_5\) modes are unphysical can also be seen using a generalization of the \(R_\xi\) gauge. The gauge-fixing Lagrangian is taken to be

\[\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial_\mu A_\mu - \xi \partial_\nu A_\nu)^2\] (1.47)

which is a generalization of the 5D Lorentz-invariant gauge-fixing term

\[\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial_\mu A_\mu)^2.\] (1.48)
After dimensional reduction, the gauge-fixed theory is

\[
S = \int \! \! d^4x \left\{ \frac{1}{4} (F^{(0)}_{\mu\nu})^2 - \frac{1}{2\xi} (\partial_\mu A^{(0)\mu})^2 + \frac{1}{2} (\partial_\mu A_5^{(0)})^2 \int \! \! dy g_0(y)^2 + \sum_{n=1} \left[ -\frac{1}{4} (F^{(n)}_{\mu\nu})^2 + \frac{1}{2} m_n^2 (A^{(n)}_\mu)^2 - \frac{1}{2\xi} (\partial_\mu A^{(n)\mu})^2 + \frac{1}{2} (\partial_\mu A_5^{(n)})^2 - \frac{1}{2} (\sqrt{\xi} m_n) (A_5^{(n)})^2 \right] \right\}
\]

This result is consistent with the $R_\xi$ gauge in spontaneously broken gauge theories where the Nambu-Goldstone bosons receive gauge-dependent masses, which are equal to the vector masses multiplied by $\sqrt{\xi}$. The unitary gauge corresponds to the limit $\xi \to \infty$ in which the massive $A_5$ modes decouple from the theory. The bulk equation of motion for $A_5$ obtained by varying the gauge-fixed action is

\[
\Box A_5 - \partial_\mu^2 A_5 = 0.
\]

For the zero-mode solution, $A_5(x, y) = A_5^{(0)}(x)g_0(y)$, we have \(\Box A_5^{(0)}(x) = 0\) which gives $g_0''(y) = 0$. It follows that the zero mode for $A_5$ is projected out if the boundary conditions are Dirichlet, while a constant zero mode exists only if Neumann boundary conditions are chosen.

Allowing both fermions and gauge fields in the bulk generically violates charge universality because the effective 4D gauge couplings depend on overlap integrals between the profile functions for bulk fermions with bulk gauge fields, which may not be universal. The effective 4D gauge couplings are obtained by substituting the mode-expansion for the 5D fields into the 5D interaction terms. In flat space, this is given by

\[
g_5 \int d^5x \left( \overline{\Psi} \gamma^M \Psi A_M \right) (x, y) \supset \int d^4x \overline{\psi}_n(x) \gamma^\mu \psi_m(x) A_\mu^{(0)}(x) \times \left( g_5 \int dy f^{(n)}_\psi(y) f^{(m)}_\psi(y) f^{(l)}_A \right)
\]

where we identify the term in brackets with the 4D effective coupling for the vertex $A_\mu \psi_n \psi_m$. In the zero-mode level of $\Psi$ and $A_M$, charge universality is an automatic consequence of the fact that the gauge-field zero-mode profile functions are flat in
the extra dimension so that orthonormality of the fermion profiles ensures universal couplings at low energies,

\[ g_5 \int dy f_\psi^{(n)}(y) f_\psi^{(m)}(y) f_A^{(0)} = g_5 N_0 \delta_{nm}. \]  

(1.52)

The fermion zero modes do not couple universally to the gauge-field KK modes, however, which can lead to phenomenological implications.

1.3.5 Bulk fields with brane-localized operators

We have seen that de-localizing charged matter into the bulk is phenomenologically interesting if the Yukawa interactions remain brane-localized. In the calculation of the low-energy effective Yukawa couplings, the simplifying assumption \( \sqrt{L} Y_{ij}^{(5)} \ll 1 \) was made to ensure that the back-reaction of the brane-localized operators on the bulk wavefunctions was negligible. In general, however, it is possible to include any operator on a 3-brane that is consistent with 4D Lorentz symmetry and the dimensionless couplings of bulk fields to the brane may be large. This leads to a non-trivial interplay between the de-localized bulk fields and the brane-localized operators to which they couple.

In the limit where the brane thickness vanishes, the 4D operators become delta-function localized in the extra dimension. A brane lying in the bulk acts as a junction, to either side of which the wavefunctions of bulk fields are related by the strength of their coupling to the brane. The matching condition is determined by integrating the bulk equations of motion, which will contain delta functions weighted by the brane couplings. For bosonic fields, the equations of motion are second order, so the profile functions for the 4D modes are continuous across the branes, while their first derivatives suffer discontinuities. The profile functions for the 4D fermions involve only first-order derivatives, so fermion wavefunctions incur discontinuities at the brane locations. In the special case where the brane is localized on the boundary
space, there are no discontinuities since the matching conditions are replaced by boundary conditions which are determined from stationary variations of the action on the boundary.

The distinction between bulk and boundary-localized branes can be concisely elucidated by comparing a 5D real scalar field with brane mass terms on the interval and the orbifold. The action is taken be

\[
\int d^4x \, dy \, \frac{1}{2} \left[ \partial_M \Phi \partial^M \Phi - M_5^2 \Phi^2 - M_0 \delta(y) \Phi^2 - M_1 \delta(y - L) \Phi^2 \right]
\]

where the \(y\) integral runs over \([0, L]\) in the interval theory and over the covering space \([-L, L]\) in the orbifold theory. The branes are taken to lie exactly on the boundaries of the interval, so that, for example

\[
\int_0^L f(y) \delta(y - L) = f(L)/2.
\]

In the corresponding orbifold theory, the delta function is regularized according to the convention

\[
\delta(y) \equiv \lim_{\theta \to 0} \begin{cases} 
\frac{1}{2} \delta(y + \theta), & y < 0 \\
\frac{1}{2} \delta(y - \theta), & y > 0.
\end{cases}
\]

This definition is necessary for odd fermions, which couple to the brane even though they vanish at the brane location.

Stationary variations of the fields in the bulk and on the boundaries of the interval theory gives

\[
(\Box + M_5^2) \Phi - \frac{\partial^2}{\partial y^2} \Phi = 0
\]

(1.55)

\[
\left( \frac{\partial}{\partial y} + \frac{1}{2} M_1 \right) \Phi \bigg|_L = 0
\]

(1.56)

\[
\left( \frac{\partial}{\partial y} - \frac{1}{2} M_0 \right) \Phi \bigg|_0 = 0.
\]

(1.57)

In the orbifold theory, the variation of the action gives only a bulk term, leading to
the bulk equation of motion
\[ (\Box + M_0^2)\Phi - \frac{\partial^2}{\partial y^2} \Phi + M_0 \delta(y) + M_1 \delta(y - L) = 0. \] (1.58)

The matching conditions are found by integrating the equation of motion over the intervals \([-\epsilon, \epsilon]\) and \([L - \epsilon, L + \epsilon]\), before taking the limit as \(\epsilon \to 0\)
\[ \left. \frac{\partial}{\partial y} \Phi \right|_\epsilon - \left. \frac{\partial}{\partial y} \Phi \right|_-\epsilon = M_0 \Phi(0) \] (1.59)
\[ \left. \frac{\partial}{\partial y} \Phi \right|_{L+\epsilon} - \left. \frac{\partial}{\partial y} \Phi \right|_{L-\epsilon} = M_1 \Phi(L). \] (1.60)

Taking \(\Phi\) to be an even function on the orbifold, we obtain
\[ \frac{\partial}{\partial y} \Phi(0^+) - \frac{1}{2} M_0 \Phi(0) = 0 \] (1.61)
\[ \frac{\partial}{\partial y} \Phi(L^-) + \frac{1}{2} M_1 \Phi(L) = 0 \] (1.62)

and thus the first derivative of the wavefunction undergoes a jumps at the brane locations.

We note parenthetically that unlike the bulk mass term \(M_0\), increasing the brane mass terms arbitrarily does not decouple the Kaluza-Klein modes from the theory. In this limit, the back-reaction of the brane mass term on the bulk wavefunction is sizeable, causing the boundary/jump conditions for the field to change from Neumann to Dirichlet.

### 1.3.6 Boundary kinetic terms and field normalization

Apart from brane-localized mass terms (BMTs), the most phenomenologically important operators are brane-localized kinetic terms (BKTs). The interpretation of a BKT is to enhance the probability of exciting the field on the brane, thereby lowering the probability of finding the field in the bulk. This fact was first exploited to lo-
calize gravity to a brane as was discussed in section 1.3. Boundary-localized kinetic terms have since found important phenomenological applications to bulk matter and gauge fields [31–34]. Localized kinetic terms also play an important theoretical role in any extra-dimensional theory containing lower-dimensional defects, since they are required to absorb localized divergent radiative corrections [26, 35].

When BKTs are introduced, the profile functions for the 4D fields can no longer be normalized form an orthonormal set. This can be understood by recalling that the modulus squared of the profile function represents the probability density to excite a 4D mode in the bulk, but not on the brane. The normalization of the profile function must therefore tend to zero when the BKTs dominate over the bulk kinetic terms, in order for the total probability to remain unity. For concreteness, we consider a 5D scalar field on the extra-dimensional interval \([0, L]\) with a BKT on the \(y = 0\) boundary. This can be modeled by the action

\[
S = \int d^4x \int_0^L dy \frac{1}{2} \{ (\partial_\mu \Phi)^2 [1 + \zeta(y)] - (\partial_y \Phi)^2 \}
\]

where \(\zeta : [0, L] \rightarrow \mathbb{R}\) is a function satisfying \(\int_0^L dy \zeta(y) = 1\), which is sharply peaked near \(y = L\). Choosing the usual Kaluza-Klein expansion ansatz

\[
\Phi(x, y) = \sum_n \varphi_n(x) f_n(y), \quad (\Box - m_n^2) \varphi_n(x) = 0
\]

we find that stationary variations of the action in the bulk and on the boundary give

\[
-\frac{d^2 f_n}{dy^2} (y) = [1 + \zeta(y)] m_n^2 f_n
\]

\[
\frac{df_n}{dy} \bigg|_{y=0,L} = 0.
\]
It can be shown from these equations that

\[ \int dy \left[ 1 + \zeta(y) \right] \left( m_n^2 - m_m^2 \right) f_n f_m = 0 \]  

(1.67)

and thus the appropriate normalization condition is

\[ \int dy \left[ 1 + \zeta(y) \right] f_n f_m = \delta_{nm}. \]  

(1.68)

The chiral fermion masses considered in section 1.3.3 also lead to normalization conditions which are not of Sturm-Liouville form [36]. This can be understood from the fact that the profile functions for the fields \( \Psi_\pm^{(L)} \) and \( \Psi_\pm^{(R)} \) each satisfy a first-order, linear differential equation. In the limit where the chiral mass term vanishes, the system of equations for \( \Psi^{(L)} \) decouples from \( \Psi^{(R)} \). By solving linear systems, this gives four sets of decoupled, second-order ODEs which are of Sturm-Liouville form. When the mixing is turned on, we obtain a system of four coupled first-order ODEs which can only be decoupled by invoking fourth-order derivatives.

For concreteness, we will analyze the situation with a chiral fermion mass term on the \( y = 0 \) brane and BKTs on the \( y = L \) brane. The action for the theory on the interval \([0, L]\) is \( S_5 = S_{\text{bulk}} + S_{\text{BKT}} + S_{\text{BMT}} \), where

\[
S_{\text{bulk}} = \int d^5x \left[ \frac{1}{2} \left( \bar{\Psi}_+^{(L)} \gamma^M \partial_M \Psi^{(L)}_+ - \partial_M \bar{\Psi}_+^{(L)} \gamma^M \Psi^{(L)}_+ \right) - m_L \bar{\Psi}_+^{(L)} \Psi^{(L)}_+ + (L \leftrightarrow R) \right] 
\]

(1.69)

\[
S_{\text{BKT}} = \int d^4x \int_0^L dy \left[ \bar{\Psi}_+^{(L)} \gamma^\mu \partial_\mu \Psi^{(L)}_+ + \bar{\Psi}_-^{(R)} \gamma^\mu \partial_\mu \Psi^{(R)}_- \right] \delta(y - L) 
\]

(1.70)

\[
S_{\text{BMT}} = -\int d^4x \int_0^L dy \lambda_5 \left[ \bar{\Psi}_-^{(L)} \Psi^{(R)}_+ + \bar{\Psi}_+^{(R)} \Psi^{(L)}_- \right] \delta(y) 
\]

(1.71)

where we have performed integration by parts on the boundary kinetic terms and have only included brane terms for \( \Psi_\pm^{(L)} \) and \( \Psi_\pm^{(R)} \). Stationary variations of the action in the bulk give the usual equations of motion, while stationary variations on the
boundaries give the boundary conditions

\begin{align}
\Psi_{-}^{(L)}(0) &= -\lambda_{5}\Psi_{-}^{(R)}(0), & \Psi_{+}^{(L)}(L) &= -\eta_{L}\phi\Psi_{+}^{(L)}(L), \\
\Psi_{-}^{(R)}(0) &= \lambda_{5}\Psi_{-}^{(L)}(0), & \Psi_{+}^{(R)}(L) &= \eta_{R}\phi\Psi_{+}^{(R)}(L),
\end{align}

(1.72)

(1.73)

Since the chiral fermion mass term mixes \(\Psi^{(L)}\) and \(\Psi^{(R)}\), the mass eigenstates for the system are obtained by expanding each field over the same set of 4D coefficient functions

\begin{align}
\Psi_{\pm}^{(L)}(x, y) &= \sum_{n} \psi_{\pm}^{(n)}(x)f_{L\pm}^{(n)}(y) \\
\Psi_{\pm}^{(R)}(x, y) &= \sum_{n} \psi_{\pm}^{(n)}(x)f_{R\pm}^{(n)}(y).
\end{align}

(1.74)

(1.75)

where each \(\psi_{\pm}^{(n)}\) is a mass eigenstate satisfying \(i\partial x\psi_{\pm}^{(n)}(x) = m_{n}\psi_{\pm}^{(n)}(x)\). Expanding the equations of motion into Kaluza-Klein modes and imposing the normalization conditions

\begin{align}
\int dy f_{L\pm,R\pm}^{(m)}(y)f_{L\pm,R\pm}^{(m)*}(y) + \frac{1}{2}\eta_{L\pm,R\pm} f_{L\pm,R\pm}^{(m)} f_{L\pm,R\pm}^{(m)*}\bigg|_{L} = \frac{1}{2}\delta_{nm} + \Delta_{nm}^{L\pm,R\pm},
\end{align}

(1.76)

where \(\eta_{L} = \eta_{L}^{*}, \eta_{R} = \eta_{R}^{*}\) and \(\eta_{L} = \eta_{R} = 0\), it follows from the differential equations for the profile functions that

\begin{align}
m_{n}\Delta_{nm}^{L+} - m_{m}\left(\Delta_{nm}^{L-} - \frac{1}{2}\eta_{L} f_{L+}^{(m)*} f_{L-}^{(m)}\bigg|_{L}\right) + f_{L+}^{(m)*} f_{L-}^{(m)}\bigg|_{0} = 0, \\
m_{m}\left(\Delta_{nm}^{R+} - \frac{1}{2}\eta_{R} f_{R+}^{(m)*} f_{R-}^{(m)}\bigg|_{L}\right) - m_{m}\Delta_{nm}^{R-} + f_{R+}^{(m)*} f_{R-}^{(m)}\bigg|_{0} = 0.
\end{align}

(1.77)

(1.78)

Utilizing the boundary conditions on the \(y = L\) brane we obtain

\begin{align}
m_{n}\Delta_{nm}^{L+} - m_{m}\Delta_{nm}^{L-} + \frac{1}{2} f_{L+}^{(m)*} f_{L-}^{(m)}\bigg|_{L} - f_{L+}^{(m)*} f_{L-}^{(m)}\bigg|_{0} = 0, \\
m_{m}\Delta_{nm}^{R+} - m_{m}\Delta_{nm}^{R-} + \frac{1}{2} f_{R+}^{(m)*} f_{R-}^{(m)}\bigg|_{L} - f_{R+}^{(m)*} f_{R-}^{(m)}\bigg|_{0} = 0.
\end{align}

(1.79)

(1.80)
Using the symmetry of $\Delta_{nm}^{L\pm}$ in $n$ and $m$, these expressions can be used to write the explicit formulas for $n \neq m$,

$$\Delta_{nm}^{L-} = \frac{-m_n}{m_n^2 - m_m^2} \left( f_{L+}^{(m)*} f_{L-}^{(m)} \bigg|_0 - \frac{1}{2} \frac{f_{L+}^{(m)*} f_{L-}^{(m)} |}{L} \right) + (n \leftrightarrow m) \quad (1.81)$$

$$\Delta_{nm}^{L+} = \frac{-m_n}{m_n^2 - m_m^2} \left( f_{L+}^{(m)} f_{L-}^{(m)*} \bigg|_0 - \frac{1}{2} \frac{f_{L+}^{(m)} f_{L-}^{(m)*} |}{L} \right) + (n \leftrightarrow m) \quad (1.82)$$

and similarly from $\Delta_{nm}^{R\pm}$. For $n = m$, we use the fact that the dimensional reduction of the 5D action $S_5$ should have the canonical form

$$S_4 = \int d^4 x \left( \overline{\psi}^{(n)}(\nu) \gamma^\nu \psi^{(n)}(\nu) - m_n \overline{\psi}^{(n)}(\nu) \psi^{(n)}(\nu) \right). \quad (1.83)$$

It follows that the condition $\Delta_{nm}^{L\pm} = -\Delta_{nm}^{R\pm}$ must be imposed for all $n$ and $m$. Adding together the normalization conditions for $f_{L\pm}^{(n)}$ and $f_{R\pm}^{(n)}$ gives

$$\int d^4 y \left( f_{L\pm}^{(n)} f_{L\pm}^{(n)*} + f_{R\pm}^{(n)} f_{R\pm}^{(n)*} \right) + \frac{1}{2} \left( \eta_{L\pm} f_{L\pm}^{(n)*} f_{L\pm}^{(n)} + \eta_{R\pm} f_{R\pm}^{(n)*} f_{R\pm}^{(n)} \right) = \delta_{nm}. \quad (1.84)$$

Setting $n = m$ in this expression allows the normalizations $N_{L}^{(n)}$ and $N_{R}^{(n)}$ to determined. The diagonal terms $\Delta_{nn}^{L\pm R\pm}$ can now be determined using (1.76).

### 1.4 The Randall-Sundrum models

#### 1.4.1 A slice of Anti de-Sitter space

The most simple field-theoretic origin of a warped geometry is a cosmological constant, which is represented by an action of the form

$$S = \int d^4 x d^4 y \sqrt{|g|} \left( \frac{1}{2 \kappa_5^2} R + \Lambda \right). \quad (1.85)$$
The fact than an isotropic vacuum energy can induce a metric which breaks Lorentz invariance from SO(1, 4) to SO(1, 3) has an analogy in electromagnetism: A non-conducting medium carrying a uniform electric charge density is consistent with an electric field which varies linearly with position in a chosen direction. The choice of direction for the electric field is determined by boundary conditions and this breaks rotational invariance from SO(3) to SO(2).

In order to compute the form of the warp factor, it is necessary to solve the 5D Einstein equations with the metric ansatz

$$ds^2 = e^{-A} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$$

in the presence of the source $T_{MN} = \Lambda g_{MN}$. The components of the Einstein tensor $G_{MN}$ can be found by direct computation from the metric (1.86). Alternatively, one can employ the fact that the metric is conformally equivalent to the metric of flat, five-dimensional space for which the Einstein tensor necessarily vanishes.

Using the coordinate transformation $dz = e^{A/2} dy$, the metric (1.86) can be put into the factorizable form $ds^2 = e^{-A} \eta_{MN} dx^M dx^N$. The relationship between the Einstein tensor in two conformally equivalent spacetimes $g_{MN}$ and $\tilde{g}_{MN}$ of spacetime dimension $d + 1$, is [37]

$$G_{MN} = \tilde{G}_{MN} + \frac{d-1}{2} \left[ \frac{1}{2} \tilde{\nabla}_M A \tilde{\nabla}_N A + \tilde{\nabla}_M \tilde{\nabla}_N A 
- \tilde{g}_{MN} \left( \tilde{\nabla}_K \tilde{\nabla}^K A - \frac{d-2}{4} \tilde{\nabla}_K A \tilde{\nabla}^K A \right) \right].$$

(1.87)

where $g_{MN} = e^{-A} \tilde{g}_{MN}$. Setting $\tilde{g}_{MN} = \eta_{MN}$, the covariant derivatives become partials and moreover $\partial_\mu A = 0$, so the only nonvanishing components are $G_{55}$ and $G_{\mu\nu}$. The 55 component of the Einstein equations $G_{55} = \kappa_5 T_{55} = -\kappa_5 \Lambda$ gives

$$\frac{dA}{dz} = e^{-A/2} \sqrt{- \frac{2\kappa_5 A}{3}} = 2k e^{-A/2}$$

(1.88)
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where we have defined the curvature scale \( k = \sqrt{-\kappa_5 \Lambda / 6} \). It is evident from (1.88) that the metric describes an anti de-Sitter (AdS) space since \( \Lambda \) is necessarily negative. Solving the differential equation for \( e^{-A} \) with the initial condition \( A(z = 0) = -\infty \), gives the metric

\[
\frac{1}{(k z)^2} (\eta_{\mu\nu} \, dx^\mu \, dx^\nu - dz^2) \tag{1.89}
\]

\[
e^{-2k y} \eta_{\mu\nu} \, dx^\mu \, dx^\nu - dy^2 \tag{1.90}
\]

where we have defined \( z(y = 0) = 1/k \). The divergence in the metric at the conformal boundary \( y = -\infty \) can be regulated by replacing the infinite extra dimension \( y \) by the half-line \([0, \infty)\). In the region \( y > 0 \), the metric is given by (1.90) and the energy-momentum tensor consists of a cosmological constant. In the presence of a finite spacetime boundary, the Einstein-Hilbert action must be supplemented by Gibbons-Hawking-York boundary terms. Stationary variations of the action imply that the energy-momentum tensor must include a singular sheet of positive energy density (brane tension) on the 4D subspace at \( y = 0 \). The brane tension satisfies a fine-tuning condition \( \sigma_0 = 6k/\kappa_5 \) which is determined by the bulk cosmological constant. The extra dimension can be compactified on an interval by truncating the space at some \( 0 < L < \infty \) and inserting a brane of negative tension \( \sigma_1 = -\sigma_0 \) at \( y = L \). The brane tension fine-tuning conditions ensure that the cosmological constant for the 3-branes vanishes. If the brane tensions are de-tuned from these relationships, then the branes will not describe Minkowski space [38].

The above brane constructions can be easily translated into the language of orbifolds by observing that the half-line and the interval are topologically equivalent to \( \mathbb{R}/\mathbb{Z}_2 \) and \( S^1/\mathbb{Z}_2 \), respectively, where \( \mathbb{Z}_2 \) represents reflection about \( y = 0 \). The metric in the covering space for \( \mathbb{R}/\mathbb{Z}_2 \) is given by

\[
ds^2 = e^{-2k|y|} \eta_{\mu\nu} \, dx^\mu \, dx^\nu - dy^2. \tag{1.91}
\]
The energy-momentum tensor necessary to create this background can be determined by computing the Einstein tensor (appendix A.2). The result is a negative cosmological constant $\Lambda$ separated by a positive-energy brane at $y = 0$. From the orbifold perspective, the brane tension fine-tuning condition is equivalent to the jump condition for the first derivative of the warp factor at $y = 0$. In order to compactify the extra dimension one must impose a symmetry under discrete translations. This can be achieved by taking the warp factor to be the periodic extension of $e^{-2k|y|}$ on $[-L, L]$ to the whole real line $\mathbb{R}$. The new warp factor is invariant under both $y \rightarrow -y$ and $y \rightarrow y + 2L$ and the covering space is the circle $S^1$, which is represented by $-L < y < L$. The discontinuous change in slope of the warp factor at $y = L$ necessitates the introduction of a brane of negative tension $\sigma_1 = -\sigma_0$ at that location. By $\mathbb{Z}_2$ and $2L\mathbb{Z}$ symmetry, it follows that the physical domain $[0, L]$ consists of a region of warped space delimited by gravitating branes of equal and opposite tension. The brane tensions are fine-tuned against the bulk cosmological constant which determines the curvature of the AdS$_5$ space. The remaining independent variable, namely the size of the extra dimension $L$ is left unfixed.

A comment on the relationship between the interval and orbifold pictures is in order. The brane tensions computed for the interval theory are exactly one half of those for the orbifold. This can be understood intuitively from that fact that the $S^1/\mathbb{Z}_2$ orbifold theory can be regarded as a pair of interval theories defined on $[-L, 0]$ and $[0, L]$ with their end-points attached together and with points identified under $\mathbb{Z}_2$ symmetry. The boundary terms in the interval theories therefore add to give the boundary term in the $S^1/\mathbb{Z}_2$ theory [39].

While the positive-tension brane has a natural field-theoretic interpretation as the energy density created by a domain-wall soliton, the negative-tension brane cannot be realized in this way because transverse fluctuations of the brane will possess negative kinetic terms [40]. These transverse modes will be projected out, however, if the negative-tension brane is taken to lie exactly at a fixed point of the $S^1/\mathbb{Z}_2$
orbifold. The difficulty with the negative-tension brane is likely to be an artifact of the simplistic treatment of the boundary spacetime, which is expected to be refined in a UV-complete theory such as string theory [41]. A plausible scenario is that the AdS$_5$ slice forms a subspace of a flux compactification of string theory [42] (see [43] for a review). In this proposal the spacetime is taken to be $M^4 \times Y_6$, where $Y_6$ is a six-dimensional manifold. At each point in $Y_6$, the metric can be locally cast into the form

$$ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu - dr^2 - r^2 h_{mn} dx^m dx^n$$

(1.92)

where $ds_5^2 \equiv h_{mn} dx^m dx^n$ is the metric for a compact 5-manifold $X_5$. If a stack of $N$ parallel D3-branes is introduced at this point, then the back-reaction of gravity creates the following geometry

$$ds^2 = \left(1 + \frac{r_0^4}{r^4}\right)^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu - \left(1 + \frac{r_0^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 h_{mn} dx^m dx^n\right)$$

(1.93)

where $r_0 \equiv 2\pi \ell_s N^{1/4}$. For $r \ll r_0$, the geometry describes AdS$_5 \times X_5$. In this framework, the $X_5$ regulates the positive-tension brane at large $r$, while the radial coordinate can be smoothly terminated at small $r$, providing an effective negative-tension brane.

### 1.4.2 Effective 4D gravity

The 4D effective Planck constant in the Randall-Sundrum model on the interval $[0, L]$ is given by (1.13),

$$M_P^2 = \frac{M_*^3}{k} (1 - e^{-2kL}).$$

(1.94)

This shows that for $L \gtrsim 1/k$, the gravitational coupling is relatively insensitive to the size of the extra dimension; that is, $M_P^2 \simeq M_*^3/k$ which should be compared with the result for flat extra dimensions for which $M_P^2 = M_*^3 L$. Note that the warping of the extra dimension solves the hierarchy problem even if $M_* \simeq M_P$, which will occur
when $k \simeq M_p$. The fact that the gravitational coupling (1.94) remains finite even in the decompactified limit $L \to \infty$ indicates the existence of a normalizable zero mode in the dimensionally reduced theory which can represent the four-dimensional graviton (appendix A.3). The existence of the zero-mode when $L \to \infty$ can be understood heuristically from the fact that the zero-mode is localized on the positive-tension brane, so it is not affected significantly by the removal of the negative-tension brane. An alternative viewpoint is that the warp factor makes the volume of extra-dimensional space probed by gravity effectively finite. This alone does not guarantee that gravity is four-dimensional, however, because the contribution of the KK modes to the gravitational potential has not yet been considered. If the extra dimension is compact then the first excitation of the graviton Kaluza-Klein tower is separated from the zero mode by a mass gap of order $ke^{-kL}$. The resulting Yukawa suppression of the potential created by this and subsequent KK modes ensures that only the zero-mode contributes in the low-energy theory. As $L$ increases, the KK modes bunch up until they ultimately form a continuum starting from zero mass. The absence of a mass gap in the Kaluza-Klein spectrum places the existence of four-dimensional gravity in question. On the one hand, the potential created by arbitrarily light KK modes does not experience significant Yukawa suppression so might be expected to compete with the massless zero mode. Unlike the zero mode, however, the KK modes are always localized towards the negative-tension brane. When the size of the extra dimension is taken to infinity, this leaves a negligible wavefunction overlap with observers on the positive-tension brane. Detailed calculation reveals that the KK modes are suppressed exactly when the zero mode wavefunction is normalizable so that 4D gravity is reproduced. This scenario with an infinite warped dimension is referred to as RS2 to distinguish from the compact case called RS1.
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1.4.3 Resolution of the hierarchy problem

The most attractive feature of the RS1 model is that it resolves the hierarchy between the Planck scale and the electroweak symmetry breaking scale. This is made possible because of the exponential dependence of the 4D induced metric on the position along the extra dimension. The warp factor implies that a unit of distance measured on the positive-tension brane will be exponentially dilated in units of distance on the negative-tension brane. By the same reasoning, energy scales will be exponentially decreased from their values on the positive-tension brane. In particular, if the Higgs resides on the negative-tension brane, then after canonically normalizing fields it can be shown that the physical Higgs acquires a VEV \( v \) which is related to the VEV of the fundamental theory \( v_0 \), by \( v = e^{-kL}v_0 \), where \( L \) is the radius of the orbifold and \( v_0 \) is naturally of order \( M_P \). Since physical mass scales are warped down by the same factor as the VEV, one can arrange that \( M_{ew} = e^{-kL}M_P \) by choosing

\[
kL = \ln \left( \frac{M_P}{M_{ew}} \right). \tag{1.95}
\]

We see that despite the large ratio \( M_P/M_{ew} \), the logarithm allows the brane separation \( L \) to take relatively modest values (e.g., \( L \sim O(10k^{-1}) \)). For this reason, the negative-tension brane is referred to as the infrared (IR) or TeV brane since it corresponds to the approximate scale of electroweak symmetry breaking. At the other end of the spectrum, the positive-tension brane is commonly referred to as the ultraviolet (UV), Planck or ‘gravity’ brane because it is associated with the cut-off scale at which gravitational interactions become strong.

The ability of the warped geometry to generate exponentially different mass scales from physical separation in the extra dimension offers a potential geometric solution to the hierarchy problem. This solution can only be considered satisfactory if the two-brane geometry is stable with respect to gravitational fluctuations and the required interbrane separation \( r \sim O(10k^{-1}) \) can be naturally justified. In general,
fluctuations of the 5D metric $\delta g_{MN}$ contain both a tensor component $\delta g_{\mu\nu}$ which is associated with the usual 4D graviton as well as a vector $\delta g_{\mu5}$ and a scalar fluctuation $\delta g_{55}$ called the radion. Since the radion sets the size of the extra dimension through its VEV, it must be given a non-trivial potential. In the original proposal, the massless radion has no potential so the size of the extra dimension is unfixed. Given that diffeomorphism invariance favours no coordinate system over any other, the assumption $L \sim O(10^{k-1})$ cannot be considered natural if one chooses to work in conformal coordinates.

A mechanism to stabilize the radion at a fixed VEV has been proposed whereby a bulk scalar coupling to gravity develops a $y$-dependent VEV which produces a potential for the radion [44]. After integrating out the extra dimension, it can be shown that the minimum of the resulting effective potential occurs for $L \sim O(10^{k-1})$ with minimal fine-tuning. This demonstrated that the RS1 solution to the hierarchy problem is both stable and natural. In RS2, the stabilization problem is avoided altogether because the run-away behavior of the radion does not pose any problem to the already infinite extra dimension. This comes at the expense of solving the hierarchy problem, however, since the natural location for the fields is now on the UV brane. A hierarchy can still be generated, however, if the Higgs field is localized on a zero-tension brane located at a small distance from the UV brane where gravity concentrates [45].

1.4.4 Randall-Sundrum with bulk fields

While it is natural to consider all of the SM states to be localized at the position of the Higgs, this is not a necessity in the RS1 model by virtue of the small size of the extra dimension—the proper distance $L$ separating the 3-branes is of order $1/k \simeq 1/M_P$. All mass scales in the problem, such as the Kaluza-Klein scale for the bulk fields, will be of order the IR scale $k e^{-k \ell}$ which can be chosen to exceed present detection limits of about 1 TeV. Allowing fermions to leak into the bulk is
advantageous because it allows the fermion mass hierarchy to be explained using the
mechanism described in section 1.3.3 [46, 47]. Hence the Randall-Sundrum model
can address both the hierarchy problem and the flavor puzzle geometrically.

The action for a fermion coupled to a gauge field in curved spacetime is

\[ S = \int d^5x \sqrt{|g|} \left[ \frac{i}{2} \left( \bar{\Psi} \Gamma^M D_M \Psi - D_M \bar{\Psi} \Gamma^M \Psi \right) - m_\Psi \bar{\Psi} \Psi - \frac{1}{4} F_{MN} F^{MN} \right]. \] (1.96)

In AdS$_5$, it can be shown that $\Gamma_\mu = e^{-ky} \gamma_\mu$, $\Gamma_5 = -i\gamma_5$ and

\[ D_\mu = \partial_\mu + ig A_\mu - \frac{1}{2} \Gamma_5 \Gamma_\mu k, \quad D_5 = \partial_y + ig A_5. \] (1.97)

Varying the fermion fields on a fixed spacetime background gives the equation of motion

\[ e^{ky} i \phi \hat{\Psi}_\pm + (\pm \partial_y - m_\Psi) \hat{\Psi}_\mp = 0 \] (1.98)

where $\hat{\Psi}(x, y) = e^{-2ky} \Psi(x, y)$. Mode-expanding $\Psi$ into massive states

\[ \Psi_\pm(x, y) = \sum_n \psi^{(n)}_\pm(x) f^{(n)}_\pm(y). \] (1.99)

and defining $\hat{f}^{(n)}_\pm(y) = e^{-2ky} f^{(n)}_\pm(y)$ for convenience, we obtain the first-order equations

\[ e^{ky} \hat{f}_\pm m_n + (\pm \partial_y - m_\Psi) \hat{f}_\mp = 0 \] (1.100)

which give the zero mode solution $\hat{f}_\pm^{(0)} \propto e^{\mp cy}$, where $m_\Psi = ck$; that is,

\[ \hat{f}_\pm^{(0)} \propto e^{(2\mp c)ky} \] (1.101)

which agrees with the flat space model $\hat{f}_\pm^{(0)} \to e^{\mp m_\Psi y}$ when $k \to 0$. After canonically
normalizing the kinetic term, it can be shown that for $c > 1/2$, the electron is
localized towards the UV brane and has negligible overlap with the Higgs on the IR
brane. Conversely for $c < 1/2$ the electron has significant overlap and acquires a
large 4D Yukawa coupling to the Higgs.

To obtain the massive modes, it is necessary to turn the pair of first-order coupled equations (1.98) into a pair of decoupled, second-order equations by operating on each side with $i\partial_y$,

$$\partial_y(e^{-ky}\partial_y \hat{\Psi}_\pm) + \left[ e^{ky}(i\partial)^2 - e^{-ky}(m^2_{\Psi} \pm m_{\Psi}k) \right] \hat{\Psi}_\pm = 0.$$  \hfill (1.102)

Varying the coefficient functions $\psi_{\pm}^{(n)}$ independently in the KK expansion of this expression gives

$$-\frac{d}{dy} \left[ e^{-ky} \frac{d f_{\pm}^{(n)}}{dy} \right] + e^{-ky}(m^2_{\Psi} \pm m_{\Psi}k) f_{\pm}^{(n)}(y) = m^2_n e^{ky} f_{\pm}^{(n)}(y)$$  \hfill (1.103)

which is of Sturm-Liouville form with weight function $w(y) = e^{ky}$. If the boundary conditions are of the Sturm-Liouville type, then the normalized eigenfunctions form an orthonormal set

$$\int dy e^{ky} f_{\pm}^{(n)}(y) f_{\pm}^{(m)}(y) = \delta_{nm}.$$  \hfill (1.104)

The differential equation for the profile functions (1.103) can be written in Schrödinger form using the change of coordinates $z = e^{ky}/k$,

$$-\frac{d^2 \hat{f}_{\pm}^{(n)}}{dz^2} + \frac{c^2 \pm c}{z^2} \hat{f}_{\pm}^{(n)}(z) = m^2_n \hat{f}_{\pm}^{(n)}(z),$$  \hfill (1.105)

$$\int dz \hat{f}_{\pm}^{(n)}(z) \hat{f}_{\pm}^{(m)}(z) = \delta_{nm}.$$  \hfill (1.106)

This choice of coordinates is useful since it reveals information about the profile functions based on intuition about the time-independent Schrödinger equation in non-relativistic quantum mechanics. In particular, we see that there is a pole in the equivalent potential for the right-handed field near the UV brane which is negative for $-1 < c < 0$ and positive otherwise. This indicates that the wavefunction spends more time near the IR brane when $c = -1/2$ and becomes increasingly UV localized.
as we move away in direction of positive or negative $c$. The left-handed field displays similar behavior about $c = 1/2$. The analytic solution for the wavefunction profiles are

$$\hat{f}^{(n)}(z) = (kz)^{1/2}[A_{n\pm}J_{\alpha\pm}(m_nz) + B_{n\pm}Y_{\alpha\pm}(m_nz)],$$

(1.107)

where $\alpha_{\pm} = c \pm 1/2$. It is important to realize that since the fermionic equations of motion are fundamentally first order, there is a constraint on the possible boundary conditions which can be imposed on the general solution (1.107). Substituting (1.107) into the first-order equations

$$m_n\hat{f}_{\mp} - \frac{c}{z}\hat{f}_{\mp} + \frac{d}{dz}\hat{f}_{\mp} = 0$$

(1.108)

gives $A_+ = A_-$ and $B_+ = B_- [47, 48]$, and thus

$$\hat{f}^{(n)}(z) = N_n(kz)^{1/2}[J_{\alpha\pm}(m_nz) + b_nY_{\alpha\pm}(m_nz)].$$

(1.109)

Returning to warp-factor coordinates, one finds

$$\hat{f}^{(n)}(y) = N_ne^{ky/2}\left[J_{\alpha\pm}\left(\frac{m_n}{ke-ky}\right) + b_nY_{\alpha\pm}\left(\frac{m_n}{ke-ky}\right)\right].$$

(1.110)

Turning to the gauge fields in AdS$_5$ [47,49–51], it can be shown that the equation of motion obtained from varying $A_M$ is $\partial_M \left(\sqrt{|g|}g^{MN}g^{RS}F_{NS}\right) = 0$. The differential equation satisfied by the Kaluza-Klein profile functions for $A_{\mu}$ generalizing (1.42) is

$$-\frac{d}{dy}\left[e^{-2ky}\frac{df_n}{dy}\right] = m_n^2f_n(y).$$

(1.111)

The general solution is

$$f_n(z) = N_nz[J_1(m_nz) + b_nY_1(m_nz)].$$

(1.112)

and the zero mode solution is $f_0(z) = A + Bz^2$. In the absence of brane-localized op-
The notion of a holographic principle originated with Bekenstein’s observation [52] that the maximum entropy stored by the gravitational degrees of freedom in a region of space is equal to the entropy of a black hole of the same size. According to the Bekenstein-Hawking relation, this implies that the gravitational entropy scales with the area instead of the volume which conflicts with the expectation of local quantum field theory. In order to reconcile the information-carrying capacities of quantum theory and gravity, Stephens et al. [53] and subsequently Susskind [54] were led to conjecture that gravitational theories could be described by equivalent quantum theories living in spacetimes of lower dimension. This equivalence is said to be holographic because the quantum theories typically reside on the spacetime boundary of the gravitational theories.

The existence of powerful symmetries in string theory called dualities have led to concrete realizations of holography in which gauge-field theories are related to higher-dimensional gravitational theories. One of the most celebrated examples is the AdS/CFT correspondence [9, 10] (see [55] for a review), the weakest form of which states the exact equivalence of a supergravity theory in the 10-dimensional curved spacetime $\text{AdS}_5 \times S^5$, with a supersymmetric conformally invariant field theory (CFT) in flat, four-dimensional spacetime. The holographic correspondence is clear from the fact that boundary of $\text{AdS}_5 \times S^5$ is the conformal compactification of 4D Minkowski space and moreover the isometries of $\text{AdS}_5$ act on the boundary as conformal transformations.
1. Background material

1.5.1 Motivation for the AdS/CFT correspondence

In theories of closed strings such as type IIB string theory, the closed strings have periodic boundary conditions. In open string theory, however, one can impose either Neumann or Dirichlet boundary conditions. Although the latter breaks Poincaré invariance, it is mapped to the former and vice-versa via a symmetry known as *T-duality* so cannot be neglected. The possibility of Dirichlet BCs suggests that open strings having Neumann BCs along $p$ spatial dimensions must end on $p$-dimensional hypersurfaces known as Dirichlet branes ($D_p$-branes) with $(p+1)$-dimensional world-volumes. It was shown by Polchinski [56] that D-branes can be identified with higher-dimensional analogues of black-holes in closed-string theory. This identification works because D-branes can serve as emitters of closed strings when the ends of an open string meet on the brane. Since the open strings break Poincaré invariance, they appear as non-perturbative excitations of closed strings. In particular, black $p$-branes occur as solitonic solutions of ten-dimensional type IIB supergravity which is the low-energy limit of type IIB string theory. Thus, open strings can be accommodated in type IIB string theory by introducing D-brane solutions.

It can be shown that the low-energy description ($\ell_s \to 0$) of type IIB open string theory for a set of $N$ overlapping D3-branes is given by the dimensional reduction of supersymmetric SU($N$) Yang-Mills (SYM) theory from ten dimensions to the $(3 + 1)$-dimensional D-brane world volume [57]. The effective gauge coupling of the dimensionally reduced theory is $g_{\text{eff}}^2 = N g_{\text{YM}}^2$, where $g_{\text{YM}}^2 = 4\pi g_s$ is the 10D Yang-Mills gauge coupling in terms of the closed-string coupling constant $g_s$. On the other hand, the picture which emerges in the low-energy limit of the corresponding closed string theory is a supergravity description of the curved spacetime induced by the black $p$-branes. The black-brane horizon occurs infinitely far from the branes at the end of an infinite ‘warped throat’, with near-horizon geometry $\text{AdS}_5 \times S^5$. This generalizes the near-horizon geometry $\text{AdS}_2 \times S^2$ of an extremally charged, non-rotating black hole in four dimensions (appendix B.4). The AdS radius $R \equiv 1/k$ is
quantized in units of the string length according to $R^4 = 4\pi g_s N \ell_s^4$. The closed string theory is said to be weakly coupled when stringy corrections can be ignored and the theory reduces to supergravity. This occurs when $R \gg \ell_s$, which corresponds precisely to the limit of strong coupling in the dimensionally-reduced Yang-Mills theory.

On the supergravity side, extra information can be obtained by requiring $R \gg \ell_P$ so that quantum gravity corrections can be ignored. This translates into the limit $N \to \infty$ since $\ell_P^4 = \ell_s^4 g_s$. In this limit of both a large number of colors ‘$N$’ and strong coupling, one would naively expect that coupling of the graviton to the D3-brane energy density would distort the flat-space metric of SYM theory. Near the horizon, however, the open and closed strings decouple and the open string theory continues to describe a gauge theory in flat space. One can thus expect that the supergravity theory on AdS$_5 \times S^5$ will correspond to the strong-coupling limit of the SYM theory. Whereas the supergravity theory provides a valid perturbative description of gravity below the string scale, the AdS/CFT duality provides a non-perturbative definition of gravity above the string scale as a strong CFT with $N = (R/\ell_s)^4/(4\pi g_s) = 4$ colors.

Since its conjecture, the supergravity form of the AdS/CFT correspondence has accumulated a large body of evidence including the agreement of correlation functions computed on both sides of the duality [9, 11, 58]. An unexpected application of AdS/CFT to particle phenomenology is the so-called AdS/QCD correspondence. This involves approximating QCD as a large-$N$ gauge theory and then applying a
modified version of the AdS/CFT correspondence. The use of gravity duals in this way has enabled calculations of observables in QCD, the most famous of which is the calculation of the meson mass spectrum and couplings to within an error of ten percent \[59, 60\]. Explicit string theory constructions of the gravity dual have also been realized \[61\]. More recently, there has been growing evidence of an all-encompassing \textit{gauge-string} duality \[62\], suggesting that all gauge theories have string-theory duals.

1.5.2 Conformal field theory and the operator/state correspondence

A conformal field theory (CFT) is a quantum field theory which is invariant under conformal transformations; that is, general coordinate transformations of the form $g_{\mu\nu}(x) \mapsto e^{-\sigma}g_{\mu\nu}(x)$. In flat space, this consists of Poincaré transformations, scale transformations $x \mapsto x' = \lambda x$ (dilatations) and \textit{special conformal transformations}. In $d - 1$ spacetime dimensions, the conformal group is $\text{SO}(d, 2)$. In the S-matrix formalism of quantum field theory, particles are identified with asymptotic states which solve the free-field theory. Scale invariance implies that there is no notion of particles in a CFT, rendering S-matrix techniques useless. The goal of CFT is therefore to calculate correlation functions of fields at finite spacetime insertions. In conventional QFT, the correlation functions are computed from a knowledge of the action for the theory. If the symmetry of the theory is high enough, however, then the form of the correlation functions can be deduced from symmetry principles alone, even in the absence of specified fields or an action. This is possible because if the action is invariant under a symmetry $g$, then the correlation function satisfies

$$
\langle \Phi_1(g \cdot x_1) \cdots \Phi_n(g \cdot x_n) \rangle = \langle g \cdot \Phi_1(x_1) \cdots g \cdot \Phi_n(x_n) \rangle
$$

where $\cdot$ denotes the action of $g$ on both spacetime and the fields.

In a conformal field theory, fields fall into representations of the conformal algebra. In particular, there is a certain set of operators called \textit{primary operators} which
transform under scale transformations as \( \mathcal{O}(x) \rightarrow \mathcal{O}'(x') = \lambda^{-\Delta_\mathcal{O}} \mathcal{O}(x) \) where \( \Delta_\mathcal{O} \) is called the *scaling dimension* of \( \mathcal{O} \). It can be seen that the scaling properties of \( \mathcal{O}_i \), together with the requirement of translational and rotational invariance is sufficient to determine the form of the 2-point correlation function for primary operators,

\[
\langle \mathcal{O}(r)\mathcal{O}(0) \rangle \propto r^{-2\Delta_\mathcal{O}}
\]

(1.113)

where it is sufficient to choose \( \Phi_1(x) = \Phi_2(x) \) since fields with different scaling dimensions are not correlated which can be shown from the transformation properties of the correlation function under special conformal transformations [63].

In a quantum field theory which is not scale invariant, there is a natural association between operators and the asymptotic states that they generate when acted on a Fock vacuum. These states can be interpreted as initial conditions for a field theory. For \( d \)-dimensional CFTs on \( \mathbb{R}^{1,d-1} \), this association is not well-defined so it is necessary to associate the CFT operators with states on a different spacetime background.

### 1.5.3 Witten prescription

A general QFT can be specified by functionally differentiating the correlation function \( Z[J] \) with respect to the source \( J \), evaluated at \( J = 0 \). In order to relate a QFT defined on the boundary of a spacetime to a field theory on the interior, it is sensible to express the correlation function for the boundary operators in terms of a functional integral over the bulk theory in which the bulk fields are treated as sources.

The above reasoning has motivated the following prescription [9, 11]. For each bulk field \( \varphi(x, y) \), there is a corresponding boundary operator \( \mathcal{O}_\varphi \) which couples to the boundary value of the bulk field \( \varphi_0 \) via \( \int d^4x \varphi_0 \mathcal{O}_\varphi \). It follows that \( \varphi_0 \) is the source for a generating functional \( Z[\varphi_0] \) of \( \mathcal{O}_\varphi \). Since the boundary theory has no knowledge
of the internal configuration space of fields, it is sensible to integrate away these
bulk degrees of freedom and set $Z[\phi_0]$ equal to the constrained functional integral,

$$Z[\phi_0] = \int \phi_0 D\phi e^{-S_5[\phi]}.$$  \hspace{1cm} (1.114)

Thus it can be seen that (1.114) prescribes how states of the higher dimensional
theory induce correlators of a distinct boundary theory, not to be confused with the
theory obtained by restricting the bulk theory to the boundary. It should be noted
that this prescription does not guarantee equivalence of the bulk and the boundary
theory even at the classical level.

For our purposes, it will be sufficient to assume that the correspondence holds at the
classical level of the gravity theory, where the 5D field configuration in AdS$_5$ is the
solution of the bulk Euler-Lagrange equations of motion, with boundary condition
given by the free parameter $\phi_0$. In this case, the functional integral is redundant,
and (1.114) becomes

$$Z[\phi_0] = e^{-S_{\text{eff}}[\phi_0]}.$$  \hspace{1cm} (1.115)

where $S_{\text{eff}}[\phi_0]$ is the on-shell action obtained by substituting the classical solution
with boundary value $\phi_0$, into $S_5$. Hence, the effective action is the generating
functional for the correlation functions of the boundary theory,

$$\langle O \cdots O \rangle = \frac{\delta^n S_{\text{eff}}[\phi_0]}{\delta \phi_0^n}.$$  \hspace{1cm} (1.116)

The effective action can be obtained by expressing the solution $\phi$ in terms of $\phi_0$
using the method of Green’s functions [11]. Alternatively, one can use the fact
that the action can be generically separated as $S_5[\phi] = S_{\text{bulk}}[\phi] + S_{\text{bdry}}[\phi]$, where
$S_{\text{bdry}}[\phi]$ contains, for example, boundary-mass terms and boundary terms induced
by integration by parts. The on-shell condition can be found by solving the equation
of motion for the bulk field (by e.g., Fourier analysis) and substituting into $S_5[\phi]$.
For linear theories, the bulk action vanishes after making the on-shell replacement,
leaving the on-shell boundary term $S_{\text{bdry}}[\phi_0]$. With simple manipulations, the action can be put into the form

$$\int \frac{d^4p}{(2\pi)^4} \Phi(p, z_0) \Sigma(p) \Phi(-p, z_0),$$  \hspace{1cm} (1.117)

where $\Sigma(p)$ is the Fourier transform of the 2-point correlation function.

In the case when the boundary theory is a (CFT), the scaling dimension of the CFT operator can be determined from scale invariance of the object $\int d^4 x \phi_0 O$. Applying the scaling transformation $x \rightarrow \lambda x$, gives $\text{dim } O = 4 - \text{dim } \phi_0$. The scaling dimension of the source $\phi_0$ can be deduced from the asymptotic behavior of the field near the conformal boundary. It can be shown that in order to keep the metric scale-invariant under a dilatation $x \rightarrow \lambda x$, it is necessary to shift the $y$ coordinate by $y \rightarrow y + (1/k) \ln \lambda$. Assuming a separation of variables for the bulk field of the form

$$\phi(x, y) = e^{\kappa y} \phi_0(x),$$  \hspace{1cm} (1.118)

one finds that scale-invariance of $\phi$ requires that $\text{dim } \phi_0 = \kappa$. If $\phi$ is a $p$-form field, then the components of $\phi_0$ will have scaling dimension $\text{dim } \phi_0 = \kappa + p$ because of the additional Lorentz structure $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ which has scaling dimension $-p$ [64]. Hence,

$$\text{dim } O = 4 - \kappa - p.$$  \hspace{1cm} (1.119)

The exponent $\kappa$ depends on the bulk mass of the field $\phi$ according to the asymptotic ($y \rightarrow -\infty$) form of the bulk equation of motion. This definition of $\text{dim } O$ is ambiguous, however, since $\kappa$ can be multi-valued if the equation of motion is not first order.

It is argued that the scaling dimension of the operator is determined by the bulk solution which dominates near the conformal boundary; that is, for $y \rightarrow -\infty$. Assuming asymptotic behavior of the form (1.118), the appropriate scaling dimension is thus

$$\text{dim } O = 4 - \min\{\kappa\} - p.$$  \hspace{1cm} (1.120)
Let us now apply (1.120) to the case of a massive scalar in AdS$_5$, for which the action is

$$S_{\text{scalar}} = \int d^5x \sqrt{|g|} \frac{1}{2} (g^{MN} \partial_M \phi \partial_N \phi - m^2 \phi^2).$$

(1.121)

Stationary variations with respect to $\phi$ gives the equation of motion

$$(\partial^2 + m^2 \phi^2 e^{-2ky}) \phi - e^{2ky} \partial_y (e^{-4ky} \partial_y \phi) = 0$$

(1.122)

where $\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu$. We see that at large $-y$, the warp factor suppresses the contribution of the 4D kinetic term so the general solution has asymptotic form $\phi(x, y) = e^{(2\pm\alpha)ky} \varphi_0(x)$ where $\alpha = \sqrt{4 + a}$ and we have defined $m^2 = ak^2$. Thus

$$\text{dim} \mathcal{O}_\phi = 4 - (2 - \alpha) = 2 + \alpha = 2 + \sqrt{4 + a}.$$ 

(1.123)

We will be concerned with the scenario where the conformal boundary is regulated by an ultra-violet cut-off brane. In this case, the boundary condition on the UV brane is determined by requiring boundary terms in the variation of the action to vanish. This leads to the condition $\partial_y \phi|_{UV} = 0$ which is not compatible with $\phi(x, y) = e^{(2\pm\alpha)ky} \varphi_0(x)$. In order to obtain consistent boundary conditions, one inserts the action for a mass term on the UV brane,

$$S_{\text{brane}} = -\int d^4x \sqrt{|g^{\text{ind}}|} \frac{1}{2} M_0 \phi^2|_{UV}.$$ 

(1.124)

Varying the action gives the boundary terms

$$\delta S_{\text{bdry}} = \int d^5x \sqrt{|g|} \delta \phi \left( \partial_5 \phi - \sqrt{\left|g^{\text{ind}}\right|} M_0 \phi \right)|_{UV}.$$ 

(1.125)

Requiring $\delta S_{\text{bdry}}$ to vanish for arbitrary field variations on the UV brane gives $(\partial_y - M_0) \phi|_{UV} = 0$ and thus we must choose $b = M_{UV}/k = 2 \pm \alpha$. For $b > 2$, the solution takes the negative branch and $\text{dim} \mathcal{O}_\phi = 4 - b$. Similarly, for $b > 2$ the positive
branch solution gives $\dim \mathcal{O}_\phi = b$. Combining these equations gives an expression for the scaling dimension in terms of the boundary mass

$$\dim \mathcal{O}_\phi = 2 + |b - 2|.$$ (1.126)

The derivation of the scaling dimension for fermions is more involved due to the coupled nature of the chiral components and is best determined by studying CFT correlation function directly.

### 1.5.4 Implications for Randall-Sundrum models

We have seen that the AdS/CFT correspondence relates a supergravity theory on $\text{AdS}_5 \times S^5$ to a supersymmetric CFT at strong coupling, in which the conformal transformations are realized as the isometries of $\text{AdS}_5$ acting on the boundary. In a similar way, the group $\text{SO}(6)$ of internal rotations of six scalars and four gluinos in the SYM is related to the isometries of the compact space $S^5$ in the gravity theory. The association of conformality with AdS and supersymmetry with the compact space suggests that it is possible to safely dissect the Cartesian product $\text{AdS}_5 \times S^5$ by considering the dual interpretation of $\text{AdS}_5$ without the five-sphere. In the spirit of AdS/QCD, it will be assumed that such a dual description exists and can be approximated as a strongly coupled CFT without supersymmetry.

It is useful to think of the Randall-Sundrum scenario as a regulated form of the AdS/CFT duality. In particular, pure $\text{AdS}_5$ corresponds to the case in which the branes are placed at $\pm \infty$. The presence of branes along the warped dimension of $\text{AdS}_5$ explicitly breaks isometric invariance of the gravity theory. Since the coordinate describing the extra dimension is interpreted via the duality as a renormalization scale for the CFT, the associated broken symmetries of the CFT occur at particular energy scales $\Lambda = M_* e^{-Ky}$ corresponding to the brane locations $y$.

If the UV brane is taken in from $-\infty$ and placed at finite $y_0$, then normalizable
graviton modes now exist so the bulk degrees of freedom can interact gravitationally with the CFT [65]. This implies a finite UV cut-off scale $\Lambda_0 = M_* e^{-k y_0} \approx M_P$ for the dual CFT [12], breaking conformality by the addition of operators which are assumed to be higher-dimensional. From the point of view of fields which are localized towards the Planck brane, these operators are Planck-suppressed and so the mixing with the CFT is irrelevant. As a result, they can be considered to be elementary sources probing the CFT in an effective theory valid all the way up to the ultra-violet cut-off. Due to red-shift effect of the warp factor, negative mass dimension operators are enhanced rather than suppressed. Fields which are localized further down the AdS throat (e.g., near the IR brane) therefore experience an exponentially increased coupling to the CFT. Their effective theory thus consists of an elementary state valid up to $\sim$ TeV, beyond which their compositeness due to the CFT interaction is revealed. In summary, the Planck brane introduces a $y$-dependent compositeness scale with fields being more composite the further they are located from the Planck brane.

If the IR brane is located at finite $y > y_0$, then there is a corresponding breakdown of conformal invariance at low energies $\Lambda_1 \approx$ TeV in the CFT. It has been argued that this symmetry breaking is spontaneous [12, 13], with the resulting massless Nambu-Goldstone boson being associated with the five-dimensional radion field in the gravity theory. At energies below $\Lambda_1$, the conformal symmetry is nonlinearly realized, leading to the creation of a mass gap and particles in the CFT spectrum similar to QCD.

The abrupt termination of the extra dimension at the IR boundary brane is an idealization known as the ‘hard wall’ which corresponds in the 4D theory to the spontaneous breaking of conformal symmetry by the VEV of an operator having infinite scaling dimension (and thus completely IR-localized). Models in which the metric varies smoothly away from AdS beyond the IR scale have also been considered so that the associated operator has finite scaling dimension. These ‘soft wall’ models
are useful in AdS/QCD for reproducing the observed linear mass spectrum of the hadrons [66]. They have also been applied to extensions of the standard model [67] which can help to alleviate electroweak precision constraints suffered by ordinary hard-wall models.

1.5.5 Holography of fermions

The complete holographic interpretation of bulk fermions was worked out in [15] (see appendix B.1 for the holographic interpretation of bulk scalars). If $\Psi^\pm$ is chosen to be Dirichlet on the UV brane in the gravity theory, then $\Psi^\mp$ is the appropriate choice for the source field since the boundary conditions require it to be Neumann. The appropriate correlation function is (see appendix B.2)

$$\Sigma(p) = \frac{\bar{p} f^+_\mp(z_0, p)}{p f^\mp(z_0, p)}$$  \hspace{1cm} (1.127)

where $p \equiv \sqrt{\eta^{\mu\nu}p_\mu p_\nu}$. The CFT spectrum is given by the poles which occur at $f^\pm(z_0, p) = 0$. This is reasonable since the CFT spectrum should differ from the physical Kaluza-Klein spectrum, which is determined by $f^\mp(z_0, p) = 0$. Consider the case when $\Psi^\pm$ is chosen to have a zero mode in the gravity theory. For $c > 1/2$, the mixing between the source and the CFT operator is irrelevant, so the zero mode must correspond in the dual theory to the elementary source field $\psi^0_{\pm}$. The massless eigenstate is thus predominantly source with a small admixture of CFT. For $c < 1/2$, however, the zero mode loses its interpretation as an elementary source due to mixing with the CFT sector. For $c < -1/2$, an additional elementary degree of freedom enters the theory and the source acquires a large mass, decoupling from the low-energy theory. Meanwhile the CFT sector is now in the regime where an exponentially light mode is generated. The massless eigenstate is predominantly CFT with a small admixture of the new degree of freedom.

If $\Psi^\pm$ is chosen to be Dirichlet on the IR brane, so that the gravity theory does
1. Background material

not contain a zero mode, then the CFT spectrum will be given by the Kaluza-Klein spectrum for the bulk theory which admits a zero mode. For \( c > \frac{1}{2} \), a light (vectorlike) mode is present in the physical spectrum while the CFT contains a massless chiral mode. For \( c < -\frac{1}{2} \) the CFT still contains a massless mode but this does not survive in the physical spectrum due to mixing with the source. The scaling dimension of the operator \( O_{\mp} \) sourced by \( \Psi^{0}_{\pm} \) is given by \( \dim O_{\mp} = \frac{3}{2} + |c \pm 1/2| \).

1.5.6 Holography of gauge fields

Suppose that the bulk possesses gauge fields whose gauge symmetry \( G \) survives up to arbitrarily high energies; that is, \( G \) is unbroken in the bulk and on the UV brane. Then the corresponding CFT possesses the same gauge symmetry \( G \), irrespective of the dynamics on the IR brane. If \( G \) is broken on the UV brane to subgroup \( H \leq G \), then only the subgroup \( H \) is gauged in the CFT with \( G \) remaining as a global CFT symmetry [12, 68]
We have seen that the extra dimension offers an elegant geometrical solution to both the flavour puzzle and the gauge hierarchy problem. These observations lend strong support to the idea that the electroweak symmetry breaking (EWSB) could have a higher-dimensional origin.

### 2.1 Gauge boson masses

The simplest possibility to explain the gauge boson masses is to couple the 5D fields to an IR-localized Higgs which breaks electroweak symmetry in the IR but not in the bulk. In the thin-brane limit where the Higgs is completely localized on the IR brane, the brane-localized VEV gives the zero mode a mass.

The minimal model of electroweak symmetry breaking in warped space consists of an SU(3)$_c \times$ SU(2)$_L \times$ U(1)$_Y$ gauge theory in the bulk which is broken to SU(3)$_c \times$ U(1)$_Q$ on a thin IR brane by the VEV of a localized SU(2)$_L$ doublet scalar with VEV $v/\sqrt{2}$, where $v$ has mass dimension $+1$. The action for the gauge bosons relevant to EWSB is

$$S_{\text{gauge}} = \int \! d^5 x \sqrt{|g|} \left\{ -\frac{1}{4} W^a_M W^{aM} - \frac{1}{4} W_{MN} W^{MN} \right\}$$

(2.1)

$$S_{\text{Higgs}} = \int \! d^5 x \sqrt{|g|} \left( g^2_{L5} (W^1_M W^{1M} + W^2_M W^{2M}) + \right.$$  

$$+ \left( g_{L5} W^{3M} - g_{Y5} B_M \right) \left( W^{3M} - g_{Y5} B^M \right) \right\}_{\text{IR}}.$$  

(2.3)
With an appropriate generalization of the $R_\xi$ gauge [18], the 4-vectors $A_\mu$, $Z_\mu$ and $W^\pm_\mu$ decouple from their fifth components $A_5$, $Z_5$ and $W^\pm_5$. Taking the extra dimension to be the interval $[0, L]$, the brane-localized VEV modifies the boundary conditions from their natural Neumann state to the modified-Neumann boundary conditions

$$
\begin{align*}
&y = 0 : \\
&\quad \begin{cases} \\
\quad \quad \partial_y W^\pm_\mu = 0 \\
\quad \quad \partial_y Z_\mu = 0 \\
\quad \quad \partial_y A_\mu = 0 \\
\end{cases} \\
&y = L : \\
&\quad \begin{cases} \\
\quad \quad \partial_y W^\pm_\mu + (v^2/4)g^2_{L5}W^\pm_\mu = 0 \\
\quad \quad \partial_y Z_\mu + (v^2/4)(g^2_{L5} + g^2_{Y5})Z_\mu = 0 \\
\quad \quad \partial_y A_\mu = 0 \\
\end{cases}
\end{align*}
\tag{2.4}
$$

where

$$
\begin{align*}
A_M &= \frac{g_{Y5}W^3_M + g_{L5}B_M}{\sqrt{g^2_{L5} + g^2_{Y5}}} \\
Z_M &= \frac{g_{L5}W^3_M - g_{Y5}B_M}{\sqrt{g^2_{L5} + g^2_{Y5}}} \\
W^\pm_M &= \frac{1}{\sqrt{2}} (W^1_M \mp W^2_M)
\end{align*}
\tag{2.5-7}
$$

are 5D versions of the photon, $Z$-boson and $W$-boson, respectively. The four-dimensional spectrum of fields is obtained by mode-expanding the 5D fields into an infinite tower of Kaluza-Klein resonances,

$$
\begin{align*}
A_\mu(x, y) &= \sum_n A^{(n)}_\mu(x)f^{(n)}_A(y) \\
Z_\mu(x, y) &= \sum_n Z^{(n)}_\mu(x)f^{(n)}_Z(y), \\
W^\pm_\mu(x, y) &= \sum_n W^{\pm(n)}_\mu(x)f^{(n)}_W(y),
\end{align*}
\tag{2.8-10}
$$

where each of the 4D fields $A^{(n)}_\mu$, $Z^{(n)}_\mu$ and $W^{\pm(n)}_\mu$ satisfies the Klein-Gordon equation with mass $m_{A_n}$, $m_{Z_n}$ and $m_{W_n}$, respectively. Expanding the boundary conditions into KK modes and varying the 4D coefficient functions independently gives the
boundary conditions for the profile functions $f^{(n)}_A$, $f^{(n)}_Z$ and $f^{(n)}_W$. In the limit where the VEV vanishes, the Neumann boundary conditions permit four massless zero modes $A^{(0)}_\mu$, $Z^{(0)}_\mu$ and $W^{\pm(0)}_\mu$ with constant wavefunction profiles $f^{(0)}_A$, $f^{(0)}_Z$ and $f^{(0)}_W$, respectively. A nonzero VEV distorts the profile functions and the would-be zero modes $W^{\pm(0)}_\mu$ and $Z^{(0)}_\mu$ receive masses proportional to the VEV. The exact mass eigenvalues for the would-be zero modes can be determined by substituting the general solution for the gauge-boson wavefunction into the boundary conditions for $f^{(n)}_W$ and $f^{(n)}_Z$. In the approximation $m_{Z_{\text{UV}}} \ll m_{Z_{\text{IR}}} \ll 1$, one finds that to leading order in the 5D gauge coupling, lowest-lying states have masses

$$M_{W_{\text{IR}}} = \frac{v_{\text{UV}}}{2 z_{\text{UV}}} \sqrt{\frac{g_{L5}^2}{z_{\text{UV}} \ln(z_{\text{IR}}/z_{\text{UV}})}}, \quad M_{Z_{\text{IR}}} = \frac{v_{\text{UV}}}{2} \sqrt{\frac{(g_{L5}^2 + g_{Y5}^2)}{z_{\text{UV}} \ln(z_{\text{IR}}/z_{\text{UV}})}}$$ (2.11)

It is important to realize that since $v$ and $1/g_{L5}^2$ are both of order the UV scale, the physical masses $M_W$ and $M_Z$ are of order the IR scale and thus the Hierarchy problem has been solved. This hierarchy of mass scales relied on the fact that the Higgs developed its VEV on the IR brane so that physical mass scales are warped down to the IR scale. If electroweak symmetry is broken at the Planck scale, repeating the analysis we find

$$M_{W_{\text{UV}}} = \frac{v_{\text{UV}}}{2 z_{\text{UV}}} \sqrt{\frac{g_{L5}^2}{z_{\text{UV}} \ln(z_{\text{IR}}/z_{\text{UV}})}}$$ (2.12)

which suggests that after EWSB, the zero mode obtains a large mass and decouples from the low-energy theory.

### 2.2 Fermion masses

A realistic model of EWSB must also provide masses for the elementary fermions such as the electron and the quarks. Based on the discussion in section 1.3.3, we
expect exponential hierarchies in the 4D mass spectrum to be generated if bulk fermions couple to a localized Higgs. Taking $\Psi^{(R)}_+$ and $\Psi^{(L)}_-$ to be even on the orbifold with localized Yukawa couplings at the orbifold fixed points, we obtain the boundary conditions

\begin{align}
\Psi^{(L)}_+(0) &= -\lambda_{UV} \Psi^{(R)}_+(0), \\
\Psi^{(R)}_-(0) &= \lambda_{UV} \Psi^{(L)}_-(0),
\end{align}

(2.13)

\begin{align}
\Psi^{(L)}_+(L) &= \lambda_{IR} \Psi^{(R)}_+(L), \\
\Psi^{(R)}_-(L) &= -\lambda_{IR} \Psi^{(L)}_-(L).
\end{align}

(2.14)

Letting $\lambda_{UV} = 0$ and $\lambda_{IR} \neq 0$ so that electroweak symmetry is broken only on the IR brane, we expect a light mass to be generated when the zero modes overlap negligibly with the IR brane. Choosing $-c_L = c_R = c$ so that the zero-mode wavefunctions lean away from the IR brane for $c > 1/2$, it can be shown that the lightest-mass eigenvalue ($m_0 z_{IR} \ll 1$) for $c > 1/2$ is approximately

\begin{equation}
m_0 z_{IR} \simeq 2\lambda_{IR}(c - 1/2)\varepsilon^{2(c-1/2)}
\end{equation}

(2.15)

where $\varepsilon \equiv e^{-kL}$. This shows that the fermions generate masses in two stages. In the first stage, the IR scale $z_{IR}^{-1}$ is generated from the UV scale $z_{UV}^{-1}$ by the warped geometry of the AdS space. In the second stage, light fermion masses are generated from the IR scale due to the factor $\varepsilon^{2(c-1/2)}$ which accounts for the exponentially small overlap between the fermion wavefunction and the Higgs field. This should be compared with the flat-space model in which the fermion masses are obtained directly from the UV scale as a result of wavefunction suppression. Moreover, unlike the flat-space model, the profile functions feel the effect of the exponential warp factor which tends to push the fermions toward the IR brane. This pressure accounts for the extra factor of $-1/2$ in the exponent of $\varepsilon$. While a similar expression can be obtained for $c < 1/2$, the assumption that $m_0 z_{IR} \ll 1$ is violated and the masses stabilize at the IR scale.

Repeating the analysis for the UV Yukawa coupling with the IR coupling switched
off, we find (this time for $c < -1/2$) that

$$m_0 z_{IR} \simeq 2\lambda_{UV} (1/2 - c) \varepsilon^{-2c}. \quad (2.16)$$

Interestingly, electroweak symmetry breaking on the Planck brane can generate mass eigenvalues of order $z_{IR}$. Light fermion masses can be generated as before using the wavefunction suppression factor $\varepsilon^{-2c}$.

In the limit that $\lambda_{IR} \to \infty$ ($\lambda_{UV} \to \infty$), the even fields will be forced to vanish on the IR (UV) brane. This corresponds to choosing Dirichlet-Neumann (Neumann-Dirichlet) boundary conditions for $\Psi(t)$ in the interval theory.

If $\Psi_+^{(R)}$ possesses N-D boundary conditions, then the Kaluza-Klein expansion yields $N_n(J_{c+1/2} + b_n Y_{c+1/2})|_{IR} = 0$ and $N_n(J_{c-1/2} + b_n Y_{c-1/2})|_{UV} = 0$. The mass spectrum is thus determined by the quantization condition

$$\frac{J_{c-1/2}(m_{n z_{UV}})}{Y_{c-1/2}(m_{n z_{UV}})} = \frac{J_{c+1/2}(m_{n z_{IR}})}{Y_{c+1/2}(m_{n z_{IR}})}.$$  \quad (2.17)

For $c > 1/2$, we find a light mode ($m_n z_{IR} \ll 1$) of mass

$$m_1 z_{IR} \simeq \sqrt{4c^2 - 1} \varepsilon^{-c-1/2}. \quad (2.18)$$

This mass should be associated with a Kaluza-Klein mode rather than a would-be zero mode since the boundary conditions do not permit a chiral zero mode in the spectrum. Note that this differs from the N-N boundary conditions considered earlier in which the mass of the first KK mode is of order $z_{IR}^{-1}$.

Reversing the boundary conditions to D-N we find that a light KK mode is now generated for $c < -1/2$,

$$m_t z_{IR} \simeq \sqrt{4c^2 - 1} \varepsilon^{-c-1/2}. \quad (2.19)$$

Imposing the boundary conditions and substituting (2.18) into the expression for
the Kaluza-Klein profile functions, we find the following $y$-dependence

\begin{align}
  f_+(y) &\propto e^{\frac{5k y}{2}} \sinh[(c + 1/2)k(y - L)] \quad (2.20) \\
  f_-(y) &\propto e^{\frac{5k y}{2}} \sinh[(c - 1/2)ky]. \quad (2.21)
\end{align}

We see that the exponential dependence of $m_1$ on $c$ follows from the exponentially small overlap between the left and right Weyl components of the first KK states, which are localized towards opposite ends of the extra dimension. The differing localization profiles of the Weyl components for a given fermion might provide the starting point for a mirror model in which Standard-Model states lie near the UV brane, while each of their Weyl partner fields inhabit the IR brane. Such a model faces the phenomenological challenge of hiding the mirror sector, however, since gauge fields with flat profiles couple equally to the SM and its mirror copy.

### 2.3 Matching with the Standard Model

To match with the SM gauge couplings it is useful to mode-expand the 5D gauge eigenstates instead of the mass eigenstates. Since the boundary conditions mix $W_\mu^3$ and $B_\mu$, these fields must be expanded over the same set of 4D coefficient functions; that is, we must identify $B_\mu^{(n)} = W_\mu^{3(n)}$ for all $n$. It follows that the 4D photon and the $Z$ boson of the standard model are identified with the coefficient functions $A_\mu \equiv B_\mu^{(0)}$ and $Z_\mu \equiv B_\mu^{(1)}$, respectively. The appropriate mode expansion is thus

\begin{align}
  W_\mu^{\pm}(x,y) &= \sum_n W_\mu^{(n)\pm}(x)f_W^{(n)}(y) \quad (2.22) \\
  W_\mu^3(x,y) &= f_W^{(0)}A_\mu(x) + \sum_{n=1} Z_\mu^{(n)}(x)f_W^{(n)}(y) \quad (2.23) \\
  B_\mu(x,y) &= f_B^{(0)}A_\mu(x) + \sum_{n=1} Z_\mu^{(n)}(x)f_B^{(n)}(y). \quad (2.24)
\end{align}
where \((\Box + m_{Z_n}^2)Z_{\mu}^{(n)}(x) = 0\) and \((\Box + m_{W_n}^2)W_{\mu}^{\pm(n)}(x) = 0\). The 4D gauge couplings to fermions can now be read off from the dimensional reduction of the action to be

\[
g_W^{(nm)} = g_{L5}f_W^{(nm)} \quad g_{Z_{\pm}}^{(nm)} = g_{L5}f_{Z_{\pm}}^{(nm)} T^3 + g_{Y5} \left( f_{Y_L L_{\pm}}^{(nm)} \frac{Y_L}{2} + f_{Y_R R_{\pm}}^{(nm)} \frac{Y_R}{2} \right)
\]

where \(f_{W_{\pm}}^{(nm)}\) represents the overlap between the profile function for the \(I_{L_{\pm}}\) level of the \(W_{\pm}\) generator with the \(n\)th and \(m\)th KK levels of \(\Psi^{(L)}\) and \(\Psi^{(L)}\) respectively. The other overlaps are defined similarly. For example, the expression for \(I_{L_{\pm}}^{(nm)}\) with fermions and gauge bosons in the Randall-Sundrum bulk is

\[
f_{L_{\pm}}^{(nm)} = \int dy e^{ky} f_W^{(l)}(l) f_{L_{\pm}}^{(n)} f_{L_{\pm}}^{(m)}.
\]

The coupling \(g_{Z_{\pm}}^{(nm)}\) corresponds to the gauge coupling of the 4D photon. In order to show that the 4D photon couples to a vectorial current, we use the boundary condition for \(Z_{\mu}\) on the IR brane, which gives

\[
\left( \frac{\partial}{\partial y} + \frac{v^2}{4} \right) \left( g_{L5}f_{W_3}^{(l)} - g_{Y5}f_{B}^{(l)} \right) \bigg|_{L} = 0.
\]

Since the photon has a flat profile function, this implies \(f_{B}^{(0)} / f_{W_3}^{(0)} = g_{L5} / g_{Y5}\). It follows that the photon coupling is given by

\[
e_{\psi_{\pm} \gamma}^{(nm)} = g_{L5}f_{W_3}^{(0)} \left( f_{L_{\pm} L_{\pm}}^{(nm)} + f_{R_{\pm} R_{\pm}}^{(nm)} \right) Q
\]

where we have used the fact that \(Q = T^3 + Y_L/2\) and \(Q = Y_R\). If the fermion fields satisfy an appropriate orthonormality condition, then the coupling will be universal and can be exactly matched to the SM prediction of \(eQ\). This is a consequence of the unbroken \(U(1)_Q\) gauge symmetry of the model. Since the profile functions for the broken generators are not flat, the \(W\) and \(Z\)-boson couplings receive corrections
from the extra dimension, resulting in deviations from charge universality. 

The matching of the \( Z \rightarrow f \bar{f} \) vertex to the Standard Model can be achieved using the SM relation

\[
g_V g_A = 1 - 4 |Q| \sin^2 \theta_W. \tag{2.30}
\]

The prediction for the coupling to fermion zero modes is determined by setting \( l = 1 \) and \( n = m = 0 \) in (2.26). The expressions for the vector and axial couplings in terms of overlaps are \((g_{V,A})^{th} = g_{Q \pm}Q + g_{T \pm}T^3\) where

\[
g_{Q \pm} = g_Y \left[ (I_{YR-} + I_{YL-}) \pm (I_{YR+} + I_{YL+}) \right] \tag{2.31}
\]

\[
g_{T \pm} = g_L I_{3L-} - g_Y I_{YL-} \pm (g_{L5} I_{3L+} - g_Y I_{YL+}). \tag{2.32}
\]

If \( c_L = -c_R \) and \( \eta_L = \eta_R \), then \( |f^{(0)}_{L\pm}| = |f^{(0)}_{R\pm}| \), so \( g_{Q-} = 0 \) and thus

\[
\frac{(g_V)^{th}}{(g_A)^{th}} = 1 + \varepsilon - 4 |Q| (\sin^2 \theta_W)^{th} \tag{2.33}
\]

where \( \varepsilon = g_{T+}/g_{T-} - 1 \) and \((\sin^2 \theta_W)^{th} = -g_{Q+}/(2g_{T-})\). These definitions depend on the ratio \( \beta_5 \equiv g_{L5}/g_Y \) and ratios of overlaps of the \( Z \)-boson profile functions \( f_B^{(1)} \) and \( f_W^{(1)} \). The parameter \( \varepsilon \) is naturally small since the difference between \( g_{T+} \) and \( g_{T-} \) depends on an overlap involving odd fermion fields, which are projected out in the massless limit. The equality of \((\sin^2 \theta_W)^{th}\) with the on-shell scheme value \( \sin^2 \theta_W = 1 - M_{W_1}^2/M_Z^2 \) is a more challenging requirement which will only be satisfied if the lowest-lying masses \( M_{Z_1} \) and \( M_{W_1} \) in the Kaluza-Klein spectrum are related to the 4D gauge couplings in a non-trivial way. This is equivalent to requiring that the low energy theory possesses a custodial symmetry.

In addition to the gauge-boson/fermion interactions, there are trilinear and quadrilinear gauge-boson interactions terms which must be matched to the Standard Model
values,

\[ g^{(nm)}_{WWZ} = g_L^5 I_{3WW}^{(nm)} \]  \hspace{1cm} (2.34)
\[ g^{(nmk)}_{WWWW} = g_L^5 I_{WWWW}^{(nmk)} \]  \hspace{1cm} (2.35)
\[ g^{(nmk)}_{WWZZ} = g_L^5 I_{WWWW}^{(nmk)} \]  \hspace{1cm} (2.36)

where we associate \( g^{(nm0)}_{WWZ} \), \( g^{(nm1)}_{WWZ} \), \( g^{(nm00)}_{WWZZ} \), \( g^{(nm11)}_{WWZZ} \) and \( g^{(nm10)}_{WWZZ} \) with the vertices \( WW\gamma \), \( WWZ \), \( WW\gamma\gamma \), \( WWZZ \) and \( WWZ\gamma \), respectively.

### 2.4 Electroweak precision analysis

One of the major challenges that faces the construction of realistic models for EWSB, is confronting the large amount of precisely-measured electroweak data collected by the LEP1 and SLC experiments. These were electron-positron colliders which operated at the \( Z \) resonance, where a huge number \( Z \) boson decays were observed. While working at tree-level in the Standard Model of electroweak interactions gives a reasonable fit to the data, the low uncertainties which were achieved experimentally indicate deviations from theory at a statistically unacceptable level. In order to match the accuracy of the experiments, the tree-level analysis is insufficient and one must include higher-order, quantum-mechanical terms in the perturbative expansion called radiative corrections. With the effects of radiative corrections taken into account, it has been shown that the Standard Model is consistent with all of the electroweak precision data provided that the Higgs mass lies in the region \( 114 \text{ GeV} < m_H < 219 \text{ GeV} \) [69].

#### 2.4.1 Peskin-Takeuchi and Altarelli-Barbieri parameters

In order to place constraints on the parameter space for models of EWSB, it is necessary to disentangle the effects of new physics from the radiative corrections...
which are expected in Standard Model. Since LEP and SLC were $e^+e^-$ colliders, the process under study was $e^+e^- \rightarrow f\bar{f}$ where $f$ is a final-state fermion. This process proceeds at tree level due to the exchange of a photon or $Z$ boson, and thus the primary radiative corrections affecting it are

(i) QED corrections to initial and final-state fermions

(ii) QCD corrections to the final-state fermions if the final states are quarks

(iii) Electroweak radiative corrections including corrections from vacuum polarization amplitudes (oblique corrections) and vertex corrections due to the exchange of $W/Z$ bosons in the final states

Of these corrections, the electroweak radiative corrections are believed to contain the most information about physics beyond the standard electroweak theory. Theories in which new physics couples very weakly to fermions and no additional gauge bosons are present are said to be universal since new physics enters only through vacuum polarizations of the gauge bosons. The effect of vacuum polarizations is to shift the gauge-boson propagators, resulting in different theoretical expressions for observables in terms of bare quantities. For example, after taking into account the self-energy corrections, the predictions for the physical masses of the $W$ and $Z$ bosons change from the tree-level expressions $M_W = g^2v^2/4$ and $M_Z = (g^2 + g'^2)v^2/4$ to the 1-loop expressions

\[
M_W = \frac{g^2v^2}{4} + \Pi_{WW}(M_W) \tag{2.37}
\]
\[
M_Z = \frac{(g^2 + g'^2)v^2}{4} + \Pi_{ZZ}(M_Z). \tag{2.38}
\]

The self-energies of the $W$ and $Z$ bosons can be expressed in terms of the self-energies for the SU(2)$_L$ generators, the electric charge generator $Q$ and the $W^3,Q$ mixing self-energy using the fact that $W^1,^2_\mu$ couples to the current $(e/s)J^{1,2}_\mu = gJ^{1,2}_\mu$
while the $Z_\mu$ couples to $(e/sc)(J_\mu^3 - s^2J_{Q\mu})$

$$\Pi_{WW} = g^2\Pi_{11} \quad \Pi_{ZZ} = (g^2 + g'^2) \left( \Pi_{33} - 2s^2\Pi_{3Q} + s^4\Pi_{QQ} \right) \quad (2.39)$$

where we have used $s^2 = g'^2/(g^2 + g'^2)$, $c^2 = g^2/(g^2 + g'^2)$ and $\Pi_{11} = \Pi_{22}$ since $U(1)_Q$ is unbroken. Similarly, the photon couples to the current $eJ_Q$, so the photon and $Z-\gamma$ mixing self-energies can be expressed as

$$\Pi_{\gamma\gamma} = e^2\Pi_{QQ} \quad \Pi_{\gamma Z} = gg' \left( \Pi_{3Q} - s^2\Pi_{QQ} \right) \quad (2.40)$$

Expanding the vacuum polarizations about $q^2 = 0$ up to first order gives

$$\Pi_{QQ}(q^2) \simeq q^2\Pi'_{QQ}(0) \quad (2.41)$$
$$\Pi_{3Q}(q^2) \simeq q^2\Pi'_{3Q}(0) \quad (2.42)$$
$$\Pi_{33}(q^2) \simeq \Pi_{33}(0) + q^2\Pi'_{33}(0) \quad (2.43)$$
$$\Pi_{11}(q^2) \simeq \Pi_{11}(0) + q^2\Pi'_{11}(0) \quad (2.44)$$

where the QED Ward identity has been used which implies $\Pi_{QQ}(0) = \Pi_{3Q}(0) = 0$. Thus the effects of oblique corrections have been expressed in terms of six parameters. Three of these can be absorbed into renormalizations of the input parameters $v$, $g$ and $g'$, leaving three, linearly independent parameters. A conventional choice is the Peskin-Takeuchi parameters $\hat{S}, \hat{T}, \hat{U}$, which are defined as the finite quantities $[70,71]$

$$\alpha\hat{S} = 4e^2 \left[ \Pi'_{33}(0) - \Pi'_{3Q}(0) \right] \quad (2.45)$$
$$\alpha\hat{T} = \frac{e^2}{s^2c^2M_Z^2} \left[ \Pi_{11}(0) - \Pi_{33}(0) \right] \quad (2.46)$$
$$\alpha\hat{U} = 4e^2 \left[ \Pi'_{11}(0) - \Pi'_{33}(0) \right] \quad (2.47)$$

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where \( \alpha \equiv (\frac{e^2}{4\pi})[1 + \Pi'_{\gamma\gamma}(0)] \) is fine-structure constant observable defining the Thomson limit of Compton scattering, while \( e^2, s^2 \) and \( c^2 \) are bare parameters. The Peskin-Takeuchi parameters are defined so that \( \alpha^T = \rho_s(0) - 1 \) and \( \alpha^\rho S = Z_s - 1 \) where \( \hat{Z}_s \) and \( \rho_s(0) \) are observables related to the renormalization of the Z width and the ratio of neutral current to charged current low-energy, 4-Fermi coupling, respectively. Using the fact that \( \Pi_{QQ}(0) = \Pi_{3Q}(0) = 0 \), we find

\[
\Pi_{WW}(0) = g^2 \Pi_{11}(0) \quad \Pi'_{WW}(0) = g^2 \Pi'_{11}(0) \quad (2.48)
\]

\[
\Pi_{ZZ}(0) = (g^2 + g'^2) \Pi_{33}(0) \quad \Pi'_{ZZ}(0) = (g^2 + g'^2) \left[ \Pi'_{33}(0) - 2s^2 \Pi'_{3Q}(0) + s^4 \Pi'_{QQ}(0) \right] \quad (2.49)
\]

\[
\Pi'_{\gamma\gamma}(0) = e^2 \Pi'_{QQ}(0) \quad \Pi'_{\gamma Z}(0) = gg' \left[ \Pi'_{3Q}(0) - s^2 \Pi'_{QQ}(0) \right] \quad (2.50)
\]

An equivalent set of independent parameters are the Altarelli-Barbieri parameters \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) [72, 73] where \( \epsilon_1 \) is related to the \( \hat{T} \) parameter.

In models with extra dimensions, oblique corrections arise at tree level after integrating away the extra dimension, while loop corrections from the extra dimension are usually considered to be negligible. From the holographic perspective, these tree-level corrections are associated with vacuum polarizations of the gauge-bosons due to loops of the 4D CFT. The tree-level contribution of the extra dimension to the Peskin-Takeuchi parameters can be determined by matching the dimensionally-reduced theory with the effective Lagrangian

\[
\mathcal{L}_{\text{eff}} = -\frac{1}{4} Z_\gamma F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_Z Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{2} Z_W W^+_\mu W^{-\mu} - \frac{1}{2} \Pi_{\gamma Z} F_{\mu\nu} Z^{\mu\nu} + \left( \frac{g^2 f^2}{4} + \Pi_{WW} \right) W^+_\mu W^{-\mu} + \frac{1}{2} \left( \frac{(g^2 + g'^2) f^2}{4} + \Pi_{ZZ} \right) Z_\mu Z^\mu. \quad (2.51)
\]

where \( Z_\gamma \equiv 1 - \Pi'_{\gamma\gamma}, Z_W \equiv 1 - \Pi_{WW}, Z_Z \equiv 1 - \Pi'_{ZZ} \) and \( f \) is a phenomenological parameter which represents the part of the gauge-boson masses which is obtained from sources which respect custodial isospin symmetry. Note that since this is a
tree-level analysis, the bare parameters $g, g'$ have been substituted with their tree-level values. Neglecting operators of dimension greater than four, this is the most general Lagrangian for electroweak gauge bosons.

If the electromagnetic gauge group is unbroken, then the photon can always be chosen to be canonically normalized ($\Pi'_\gamma = 0$). Moreover, the $Z-\gamma$ mixing vanishes in the tree-level calculation so that $\Pi'_{3Q} = 0$. It follows from these assumptions that

$$
\hat{S} = \frac{16\pi}{g^2 + g'^2} \Pi'_{ZZ} = \frac{16\pi}{g^2 + g'^2} (1 - Z_Z) \quad (2.52)
$$

$$
\hat{T} = \frac{4\pi}{s_W^2 c_W M_Z^2} \left( \frac{\Pi'_{WW}}{g^2} - \frac{\Pi_{ZZ}}{g^2 + g'^2} \right) \quad (2.53)
$$

$$
\hat{U} = 16\pi \left( \frac{\Pi'_{WW}}{g^2} - \frac{\Pi'_{ZZ}}{g^2 + g'^2} \right) = 16\pi \left( \frac{1 - Z_W}{g^2} - \frac{1 - Z_Z}{g^2 + g'^2} \right) \quad (2.54)
$$

where we have substituted the tree-level values for $g, g', s$ and $c$. Outside of the oblique corrections, the parameters most affected by new physics are the $Z \to f \bar{f}$ vertices. The large mass of the top quark compared to the other fermions is an indirect indication that it plays a special role in EWSB and may be a source of new physics. Unfortunately, however, the large mass of the top made it kinematically inaccessible at LEP1/SLC and thus it was not possible to directly constrain the $Z \to t \bar{t}$ vertex. The study of the bottom quark is therefore of particular importance due to its relation with the top quark, which lies in the same SU(2)$_L$ doublet as the bottom. The exchange of a virtual $W$ boson in the final states leads to a large radiative correction to the $Z \to b \bar{b}$ vertex, which is quadratically sensitive to the mass of the top quark. A fourth epsilon parameter $\epsilon_b$ is therefore introduced to parametrize this correction

$$
g^b_L \equiv \frac{g}{\cos \theta_W} \left[ -\frac{1}{2} (1 + \epsilon_b) + \frac{1}{3} \sin^2 \theta_W \right] \quad (2.55)
$$

where the top-quark contribution in the Standard Model is $\epsilon_b \simeq -\frac{\lambda^2}{2\pi^2}$. Together with $\epsilon_b$ the most sensitive observable to new physics in the top quark vertices is the $\epsilon_1$. 

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2. Electroweak symmetry breaking in warped space

Parameter. Measurements of $\epsilon_1$ and $\epsilon_b$ can therefore be used to place constraints on the anomalous top-quark couplings $W \to tb$ and $Z \to t\bar{t}$. The tree-level contribution of the anomalous couplings to $\epsilon_1$ and $\epsilon_b$ can be calculated using the non-linear electroweak chiral Lagrangian [74]

$$L = \frac{g}{2 \cos \theta_W} \left( 1 - \frac{4 \sin^2 \theta_W}{3} + \kappa_L^{NC} \right) \bar{t}_L \gamma^\mu t_L Z_\mu + \frac{g}{2 \cos \theta_W} \left( -\frac{4 \sin^2 \theta_W}{3} + \kappa_R^{NC} \right) \bar{t}_R \gamma^\mu t_R Z_\mu + \frac{g}{\sqrt{2}} (1 + \kappa_L^{CC}) \bar{t}_L \gamma^\mu b_L W^+_\mu + \frac{g}{\sqrt{2}} (1 + \kappa_L^{CC}) b_L \gamma^\mu t_L W^-_\mu + \frac{g}{\sqrt{2}} \kappa_R^{CC} \bar{t}_R \gamma^\mu b_R W^+_\mu + \frac{g}{\sqrt{2}} \kappa_R^{CC} b_R \gamma^\mu t_R W^-_\mu. \quad (2.56)$$

This is the most general Lagrangian of dimension $\leq 4$ which describes new physics in these vertices. The anomalous couplings give rise to the following contributions to $\epsilon_1$ and $\epsilon_b$:

$$\delta \epsilon_1 = \frac{3 m_t^2 G_F}{2 \sqrt{2} \pi^2} \left[ \kappa_R^{NC} - \kappa_L^{NC} + \kappa_L^{CC} - (\kappa_R^{NC})^2 - (\kappa_L^{NC})^2 + (\kappa_L^{CC})^2 + 2 \kappa_R^{NC} \kappa_L^{NC} \right] \ln \frac{\Lambda^2}{m_t^2} \quad (2.57)$$

$$\delta \epsilon_b = \frac{m_t^2 G_F}{2 \sqrt{2} \pi^2} \left( \kappa_L^{NC} - \frac{1}{4} \kappa_R^{NC} \right) (1 + 2 \kappa_L^{CC}) \ln \frac{\Lambda^2}{m_t^2} \quad (2.58)$$

where $\Lambda$ is the cut-off of the effective theory. The experimental constraints (at 68% confidence level) are [75]

$$4.4 \times 10^{-3} \leq \epsilon_1^{\text{exp}} \leq 6.4 \times 10^{-3} \quad (2.59)$$

$$-6.2 \times 10^{-3} \leq \epsilon_b^{\text{exp}} \leq -3.1 \times 10^{-3}.$$

2.4.2 Randall-Sundrum scenarios

In the Randall-Sundrum models discussed above, the contributions to the Peskin-Takeuchi parameters from the gauge-boson sector are completely determined by the
order parameter \( v \), which controls electroweak symmetry breaking. When \( v = 0 \), the gauge bosons can be canonically normalized so that \( \hat{S} = \hat{T} = 0 \), while the flat profiles of the \( W \) and \( Z \)-bosons ensure \( \Pi_{WW} = \Pi_{ZZ} = 0 \). From dimensional analysis, one naively expects that \( \hat{S}, \hat{T} \sim (v_{\text{UV}})^2 \) while \( \hat{U} = 0 \). When the full calculation is performed, however, it can be shown that the \( \hat{S} \) and \( \hat{T} \) parameters are enhanced relative to these estimates by a factor of \( kL \) \([49, 76]\). This pushes them into the excluded region for \( kL \) necessary to solve the hierarchy problem. Although the problems with the \( \hat{S} \) parameter can be cured by placing fermions into the bulk \([47, 77–79]\), the excessive contribution to the \( \hat{T} \) parameter persists unless \( L \) is decreased \([80]\) or cancelations occur due to vector bosons of additional bulk gauge symmetries \([81]\). To understand why the \( \hat{T} \) parameter is problematic in these models, it is useful to compare with the 4D Standard Model, in which the \( \rho \) parameter is exactly unity at tree level. Expanding the gauge-boson mass terms into weak eigenstates gives

\[ M_W^2 W^+ W^- + \frac{1}{2} M_Z^2 Z^2 = \frac{1}{2} M_W^2 [(W^1)^2 + (W^2)^2] + \frac{1}{2} M_Z^2 \cos^2 \theta_W [W^3 - B (g'/g)]^2. \]

(2.60)

It is clear that \( \rho = 1 \) is equivalent to the requirement that \( W^3_\mu \) has the same mass term as \( W^1_\mu \) and \( W^2_\mu \). When the \( B_\mu \) mixing is turned off (i.e., \( g' = 0 \)) the SU(2)\(_L\) gauge eigenstates will thus form a mass-degenerate triplet. Reversing the logic by imposing the mass-degeneracy of the SU(2)\(_L\) gauge eigenstates, we obtain the mass matrix

\[
\frac{1}{2g^2} M_W^2 \begin{pmatrix} W^1 & W^2 & W^3 & B \end{pmatrix} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g^2 \end{pmatrix} \begin{pmatrix} W^1 \\ W^2 \\ W^3 \\ B \end{pmatrix}
\]

(2.61)
which immediately implies $\rho = 1$. In the Standard Model, the equality of $M_{W3}$ and $M_W$ is ensured at tree level by a symmetry of the Higgs sector called custodial symmetry, under which the SU(2)$_L$ gauge bosons form a triplet. In addition, the custodial symmetry protects the $\rho$ parameter from excessive radiative corrections, resulting in a small $\hat{T}$ parameter.

The original Randall-Sundrum model enjoys all of the above successes of the Standard model since the gauge bosons are completely localized on an infinitely thin, custodially-symmetric 3-brane. If the gauge bosons probe the bulk, then the would-be gauge zero modes now obtain part of their mass from the extra dimension, in addition to the IR brane. It follows that the bulk must possess a custodial symmetry of its own, in order to ensure that the deviation from the $\rho$-parameter is sufficiently small. According to the holographic interpretation gauge fields spelled out in section 1.5.6, additional gauge symmetries must be present in the bulk which act as a global custodial symmetry in the low-energy theory [81].
Emergent Electroweak Symmetry Breaking

3.1 Gauge bosons

In the emergent model of EWSB, electroweak symmetry is broken on the UV brane by a VEV that is effectively infinite. This is equivalent to choosing boundary conditions for $Z_\mu$ and $W^{\pm}_\mu$ to be Dirichlet on the UV brane. As we have seen, the UV boundary VEV decouples the zero mode and consequently the $W$ and $Z$ bosons must now be associated with the first excited states of the Kaluza-Klein towers. At first sight, this association seems problematic since the masses of the $W$ and $Z$ bosons are considerably lighter than the IR scale which is typically of order $\gtrsim 1$ TeV to avoid direct detection bounds for gauge-boson KK modes. In order to keep the masses of the first excitations light compared to the IR scale, brane kinetic terms are inserted at the boundaries for the unbroken generators of SU(2)$_L \times$ U(1)$_Y$. The three independent brane kinetic term coefficients $\zeta_Q$, $\zeta_L$ and $\zeta_Y$ can be adjusted to fit the experimental values of $M_W$ and $M_Z$. The expressions for the vector-boson Kaluza-Klein profiles $f^{(n)}_{W3}(z)$, $f^{(n)}_B(z)$ and $f^{(n)}_W(z)$ are given by (1.112). Using the boundary conditions on the UV and IR branes, it can be shown that [19]

$$b^{W \pm}_n = \frac{J_1(m_{W_n}z_{UV})}{Y_1(m_{W_n}z_{UV})},$$

$$b^{W3,B}_n = \frac{(\zeta_L Y^k)(m_{Z_n}z_{IR})J_1(m_{Z_n}z_{IR}) - J_0(m_{Z_n}z_{IR})}{Y_0(m_{Z_n}z_{IR}) - (\zeta_L Y^k)(m_{Z_n}z_{IR})Y_1(m_{Z_n}z_{IR})},$$

$$N^B_n = \frac{g_{L5}}{g_{Y5}} \frac{J_1(m_{Z_n}z_{IR}) + b^{W3}_nY_1(m_{Z_n}z_{IR})}{J_1(m_{Z_n}z_{IR}) + b^{W3}_nY_1(m_{Z_n}z_{IR})}$$

$$N^{W3}_n = \frac{g_{L5}}{g_{Y5}} \frac{J_1(m_{Z_n}z_{IR}) + b^{W3}_nY_1(m_{Z_n}z_{IR})}{J_1(m_{Z_n}z_{IR}) + b^{W3}_nY_1(m_{Z_n}z_{IR})}.$$
where \( g_{L5} \) and \( g_{Y5} \) are the 5D SU(2)\(_L\) and U(1)\(_Y\) gauge couplings, respectively. The relationship between the zero mode profiles has the usual form \( f^{(0)}_B/f^{(0)}_{W3} = g_{L5}/g_{Y5} \). The overall normalizations \( N_n^W \), \( N_0^{W3} \) and \( N_0^{W3} = f^{(0)}_{W3} \) are determined by requiring canonically normalized kinetic terms for the photon, the \( W^\pm \) bosons and the Z-boson. To find the bulk wavefunctions for the photon \( f_\gamma \) and Z-boson \( f_Z \), which are required to canonically normalize their 4D kinetic terms, the mode expansion is substituted into the kinetic terms for \( B_\mu \) and \( W^3_\mu \), giving

\[
-\frac{1}{4}(F_{\mu\nu}^Y)^2 - \frac{1}{4}(F_{\mu\nu}^{W^3})^2 \geq \frac{1}{4} (f_0^{W3})^2 + (f_0^B)^2 F_{\mu\nu}^2 - \frac{1}{4} \sqrt{(f_1^{W3})^2 + (f_1^B)^2 Z_{\mu\nu}^2} \quad (3.4)
\]

\[
\equiv -\frac{1}{4} f_\gamma F_{\mu\nu}^2 - \frac{1}{4} f_Z Z_{\mu\nu}^2. \quad (3.5)
\]

In addition to the exact expressions above, it will be useful have an approximate expression for the \( W \)-boson profile. This can be obtained by using the fact that \( M_W z_{UV} \ll M_W z_{IR} \ll 1 \). Using the small argument approximation for the Bessel functions in the expression for \( f_n^W(z) \) and canonically normalizing the kinetic term we obtain

\[
f^{(1)}_W \simeq \sqrt{\frac{1}{\zeta_L} \left( \frac{z}{z_{IR}} \right)^2} . \quad (3.6)
\]

Unlike the minimal scenario discussed in chapter 2, the emergent model does not suffer from an excessive \( \hat{S} \) or \( \hat{T} \) parameter induced by gauge bosons in the bulk. The small contribution to the \( \hat{T} \) parameter is a consequence of the extra dimension providing a natural mechanism to ensure the mass-degeneracy of the \( W^1, W^2 \) and \( W^3 \) bosons in the hypercharge-decoupling limit. Since the massive gauge bosons in the emergent model are genuine Kaluza-Klein modes, their masses are entirely of extradimensional origin. In the hypercharge-decoupling limit, the KK modes \( W^{(n)}_\mu(x) \) for \( n = 1, 2, 3 \) form a degenerate Kaluza-Klein level and mixing of the third generator with the 4D hypercharge boson leads to the correct mass relation. From a holographic perspective, the bulk gauge symmetry SU(2)\(_L\) serves as a global custodial symmetry in the low-energy theory, protecting the \( \hat{T} \) parameter from excessive
radiative corrections [19].

3.2 Fermions

The idea of representing the $W$ and $Z$ bosons by Kaluza-Klein modes presents challenges for bulk fermions. Since the profile functions of the KK modes are not flat, the argument leading to charge universality in section 1.3.4 does not apply, so the fermion zero modes of different flavors must be taken to have identical profile functions in the extra dimension. This sacrifices the split-fermion solution to the flavor puzzle because the 5D fermion masses of different flavors must be identical to ensure universal profiles. With electroweak symmetry broken at the Planck scale, the fermion zero modes gain their masses by coupling to a Dirac mass term on the UV brane. The flavor puzzle can be solved by assuming that a Frogatt-Nielsen mechanism operates at the Planck scale, producing a hierarchical pattern of couplings to the UV brane. The back-reaction of these brane-localized operators on the bulk fermion profiles will cause slight deviations from charge universality that are significant only for the third generation. In addition to the UV mass term, fermion boundary kinetic terms are allowed on the IR brane. Large fermion boundary kinetic terms on the IR brane are required in order to ensure the correctness of the electroweak precision analysis carried out in [19], which assumed that the fermions are completely localized on the IR brane. The action for bulk fermions in this model can be determined by writing the action (1.69), (1.70) and (1.71) in generally covariant form. This is achieved by making the replacements: $\gamma^M \rightarrow \Gamma^M$, $\partial_M \rightarrow D_M$, $d^5 x \rightarrow d^5 x \sqrt{|g|}$ and $\delta(y - y_*) \rightarrow \delta(y - y_*) \sqrt{|g^{(ind)}|/|g|}$, resulting in the action

$$S_5 = S_\Psi + S_{BMT} + S_{BKT},$$

where

$$S_\Psi = \int d^5 x \sqrt{|g|} \left[ \frac{1}{2} (\bar{\Psi}^{(L)}_i \Gamma^M D_M \Psi^{(L)}_i - D_M \bar{\Psi}^{(L)}_i \Gamma^M \Psi^{(L)}_i) + m^{(i)}_L \bar{\Psi}^{(L)}_i \Psi^{(L)}_i + (L \leftrightarrow R) \right]$$

(3.7)
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is the bulk term and

\[ S_{\text{BMT}} = - \int d^4x \sqrt{|g^{(\text{ind})}|} \left[ \lambda_5^{(i)} \left( \bar{\Psi}_i^{(L)} \Psi_i^{(R)} + \text{h.c.} \right) \right]_{\text{UV}} \]

\[ S_{\text{BKT}} = \int d^4x \sqrt{|g^{(\text{ind})}|} \left[ \eta_{L\alpha} \bar{\Psi}_i^{(L)} e_\alpha^{\mu} \partial_{\mu} \Psi_i^{(L)} + \eta_{R\alpha} \bar{\Psi}_i^{(R)} e_\alpha^{\mu} \partial_{\mu} \Psi_i^{(R)} + (L \leftrightarrow R) \right]_{\text{IR}} \]

(3.8)

(3.9)

We take the extra dimension to be the interval and work in the Poincaré coordinates (1.89), where \( z \in [z_{\text{IR}}, z_{\text{UV}}] \) is related to \( y \in [0, L] \) by \( z = e^{ky}/k \). The boundary conditions for the profile functions obtained from stationary variations of the action on the boundaries are

\[ \Psi_+^{(L)}(z_{\text{UV}}) = -\lambda_5 \Psi_+^{(R)}(z_{\text{UV}}), \quad \Psi_+^{(L)}(z_{\text{IR}}) = -\eta_L e^{kL/k} \Psi_+^{(L)}(z_{\text{IR}}), \]

\[ \Psi_-^{(R)}(z_{\text{UV}}) = \lambda_5 \Psi_-^{(L)}(z_{\text{UV}}), \quad \Psi_-^{(R)}(z_{\text{IR}}) = \eta_R e^{kL/k} \Psi_-^{(R)}(z_{\text{IR}}), \]

(3.10)

(3.11)

The solution for the profile functions in the bulk are

\[ f^{(n)}_{L,\pm,R\pm}(z) = N^{(n)}_{L,R}(kz)^{5/2}[J_{\alpha_{L,\pm,R\pm}}(m_n z) + j^{(n)}_{L,R} Y_{\alpha_{L,\pm,R\pm}}(m_n z)], \]

(3.12)

where \( \alpha_{L,\pm,R\pm} \equiv 1/2 \pm c_{L,R} \) and \( c_{L,R} \equiv m_{L,R}/k \). The coefficients

\[ b^{(n)}_L = \frac{J_{\alpha_{L,\pm}}(m_n z_{\text{IR}}) + (\eta_L k) m_n z_{\text{IR}} J_{\alpha_{L,\pm}}(m_n z_{\text{IR}})}{Y_{\alpha_{L,\pm}}(m_n z_{\text{IR}}) + (\eta_L k) m_n z_{\text{IR}} Y_{\alpha_{L,\pm}}(m_n z_{\text{IR}})}, \]

(3.13)

\[ b^{(n)}_R = \frac{J_{\alpha_{R,\pm}}(m_n z_{\text{IR}}) - (\eta_R k) m_n z_{\text{IR}} J_{\alpha_{R,\pm}}(m_n z_{\text{IR}})}{Y_{\alpha_{R,\pm}}(m_n z_{\text{IR}}) - (\eta_R k) m_n z_{\text{IR}} Y_{\alpha_{R,\pm}}(m_n z_{\text{IR}})}, \]

(3.14)

are obtained by imposing the IR boundary conditions, while the overall normalizations \( N^{(n)}_{L,R} \) are determined from conditions analogous to (1.84),

\[ \int dz \left( \tilde{j}^{(n)}_{L,\pm} \tilde{j}^{(m)*}_{L,\pm} + \tilde{j}^{(n)}_{R,\pm} \tilde{j}^{(m)*}_{R,\pm} \right) + \frac{1}{2}(k z_{\text{IR}}) \left( \eta_{L,\pm} \tilde{j}^{(n)}_{L,\pm} \tilde{j}^{(m)*}_{L,\pm} + \eta_{R,\pm} \tilde{j}^{(n)}_{R,\pm} \tilde{j}^{(m)*}_{R,\pm} \right)_{\text{IR}} = \delta_{nm}. \]

(3.15)

Substituting the expressions for the profile functions into the UV boundary con-
ditions and using the small-argument approximations to the Bessel functions, the
lowest-lying state is found to be of mass
\[
m_{\text{IR}} = \frac{\lambda_5|1 - 2c|}{\sqrt{1 + \eta k(1 - 2c)^2} + 2\eta k\lambda_5^2(1 - 2c)^2(1 + 2c)^{-1}(z_{\text{UV}}/z_{\text{IR}})^{-2c}}.
\]
(3.16)
where we have assumed \(\eta_L = \eta_R \equiv \eta\) and \(c_R = -c_L \equiv c\) in order to ensure that the
zero mode profiles are universal to leading order in \(m\).
The gauge-covariant fermion boundary kinetic terms on the IR boundary give a
contribution to the 4D effective couplings in addition to the bulk overlap integral
\[
S^{(4)}_{\text{int}} = - \int d^5\sqrt{|g^{(\text{ind})}|} 2\xi_{\text{UV}}\xi_5 Q \left[ \bar{\Psi}^{(L)} \gamma^\mu A^\mu \Psi^{(L)} + \bar{\Psi}^{(R)} \gamma^\mu A^\mu \Psi^{(R)} \right]_{\text{UV}}
- \int d^5x \sqrt{|g^{(\text{ind})}|} 2\xi_{\text{IR}} \left[ \frac{1}{\sqrt{2}}g_{L5} \bar{\Psi}^{(L)} \gamma^\mu (W^{+}_\mu T^+ + W^{-}_\mu T^-) \Psi^{(L)}
+ g_{L3} \bar{\Psi}^{(L)} \gamma^\mu Y^3 T^3 \Psi^{(L)} + g_{Y5} \bar{\Psi}^{(R)} \gamma^\mu E^5 \Psi^{(R)} \right]_{\text{IR}}.
\]
(3.17)
The contributions to 4D photon, \(W\) and \(Z\) boson are found to be
\[
\Delta g_{Q^-}^{(nm)} = \xi_{\text{UV}} \xi_5 Q J^{(n)}_{L-} f_{L-}^{(n)} \bigg|_{z_{\text{UV}}} + \xi_{\text{IR}}(kz_{\text{IR}}) N_0 Q J^{(n)}_{L-} f_{L-}^{(n)} \bigg|_{z_{\text{IR}}}, \\
\Delta g_{Q^+}^{(nm)} = \xi_{\text{UV}} \xi_5 Q J^{(0)}_{R+} f_{R+}^{(0)} \bigg|_{z_{\text{UV}}} + \xi_{\text{IR}}(kz_{\text{IR}}) N_0 Q J^{(n)}_{R+} f_{R+}^{(n)} \bigg|_{z_{\text{IR}}}, \\
\Delta g_{W_{ij}}^{(nml)} = \xi_{\text{IR}}(kz_{\text{IR}}) g_{L5} f_{L}^{(l)(n)} f_{L}^{(m)} \bigg|_{z_{\text{IR}}}, \\
\Delta g_{Z^-}^{(nml)} = \xi_{\text{UV}} \xi_5 Q \left( g_{L5} f_{L}^{(l)(n)} f_{L}^{(m)} \bigg|_{z_{\text{UV}}} + \xi_{\text{IR}}(kz_{\text{IR}}) g_{L3} f_{L}^{(l)(n)} f_{L}^{(m)} \bigg|_{z_{\text{IR}}}, \\
\Delta g_{Z^+}^{(nml)} = \xi_{\text{UV}} \xi_5 Q \left( g_{L5} f_{L}^{(l)(n)} f_{L}^{(m)} \bigg|_{z_{\text{UV}}} + \xi_{\text{IR}}(kz_{\text{IR}}) g_{Y5} Y f_{L}^{(l)(n)} f_{L}^{(m)} \bigg|_{z_{\text{IR}}}, \\
(3.18, 3.19, 3.20, 3.21, 3.22)
where we have neglected the contribution of the odd fields to the brane interactions since these fields do not possess brane kinetic terms.

With the effective gauge-boson couplings to fermions at our disposal, we are now ready to match with the Standard Model. The theory predictions for the electric charge and $W/Z$-boson couplings are, respectively

$$e_{\psi \gamma} = N_0 \left( 1 - \frac{\eta}{2} (k z_{\text{IR}}) (f^{(0)}_{\text{even}})^2 \right) \bigg|_{\text{IR}} + \xi_{\text{IR}} (k z_{\text{IR}}) N_0 \left( f^{(0)}_{\text{even}} \right)^2 \bigg|_{\text{IR}},$$

$$g_{W_{ij}} = g_{L5} I_{W_{i \pm L_{j \pm}}},$$

$$g_{Z_{\pm}} = g_{L5} I_{3L_{\pm}} T^{(3)} + g_{Y5} \left( I_{Y_{L \pm}} \frac{Y_L}{2} + I_{Y_{R \pm}} \frac{Y_R}{2} \right),$$

where $i, j$ are flavor indices and the overlaps contain bulk as well as boundary contributions

$$I_{W_{i \pm L_{j \pm}}} = \int dz f^{(1)}_W (f^{(0)}_{L_{j \pm}, L_{j \pm}}) + \xi_{\text{IR}} (k z_{\text{IR}}) f^{(0)}_W (f^{(0)}_{L_{j \pm}, L_{j \pm}}) \bigg|_{z_{\text{IR}}},$$

$$I_{Y_{L \pm}, Y_{R \pm}} = \int dz f^{(1)}_B (f^{(0)}_{L_{j \pm}, R_{j \pm}}) + \xi_{\text{IR}} (k z_{\text{IR}}) f^{(1)}_B (f^{(0)}_{L_{j \pm}, R_{j \pm}}) \bigg|_{z_{\text{IR}}},$$

$$I_{3L_{\pm}} = \int dz f^{(1)}_3 (f^{(0)}_{L_{j \pm}})^2 + \xi_{\text{IR}} (k z_{\text{IR}}) f^{(1)}_3 (f^{(0)}_{L_{j \pm}}) \bigg|_{z_{\text{IR}}},$$

and we have defined $\xi_{\text{IR}^-} = \xi_{\text{IR}}, \xi_{\text{IR}^+} = 0$.

### 3.2.1 Massless fermions

For general $c$, the fermion profile functions must be expressed in terms of Bessel functions, so the overlap integrals cannot be performed analytically. The expressions simplify considerably for the case $c = 0$, corresponding to massless bulk fermions, in which case they are given by

$$f_{\text{even}}(z) = N_n (k z)^2 \left\{ \cos[m_n (z_{\text{IR}} - z)] - (\eta k) (m_n z_{\text{IR}}) \sin[m_n (z_{\text{IR}} - z)] \right\},$$

$$f_{\text{odd}}(z) = N_n (k z)^2 \left\{ \sin[m_n (z_{\text{IR}} - z)] + (\eta k) (m_n z_{\text{IR}}) \cos[m_n (z_{\text{IR}} - z)] \right\}.$$
where \( f_{\text{even}}(z) \equiv |f_{L+}^{(n)}(z)| = |f_{R-}^{(n)}(z)| \), \( f_{\text{odd}}(z) \equiv |f_{L+}^{(n)}(z)| = |f_{R-}^{(n)}(z)| \) and the overall prefactor is given by

\[
N_n \simeq \frac{1}{\sqrt{z_{\text{IR}}}} \frac{1}{\sqrt{1 + (\eta k)/2 + (\eta k)^2 (m_n z_{\text{IR}})^2}}.
\]  

In the limit \( \lambda_5 \ll 1 \), the lightest solution to the equation of motion is

\[
m_0 z_{\text{IR}} \simeq \frac{\lambda_5}{\sqrt{(1 + \eta k)^2 + 2\eta k (\lambda_5)^2}},
\]

while the next heaviest state is near \( \hat{m}_1 \sim \pi/2 \). When \( \lambda_5 = \mathcal{O}(1) \), however, the first KK mode becomes light \( \hat{m}_1 < 1 \) and is approximately given by

\[
m_1 z_{\text{IR}} \simeq \sqrt{\frac{1}{(\lambda_5)^2} + \frac{2}{(\eta k)}}.
\]

This can cause the next-to-heaviest KK mode of the top quark to be too light and violate experimental bounds. For \( z_{\text{IR}} = 1800 \text{ GeV}, \lambda_5^{(t)} = 1.15 \) and \( \eta k = 10 \) we obtain \( m_{t0} \simeq 171 \text{ GeV} \) and \( m_{t1} \simeq 1503 \text{ GeV} \). This is not a problem for the light fermions, however, for which \( \lambda_5 \ll 1 \).

### 3.2.2 Photon coupling

The most stringent constraint on the emergent model is the photon coupling to light fermions which has been extremely well measured. While the universality of the electric charge is usually guaranteed by the unbroken U(1) gauge symmetry, we will see later that it is necessary to break U(1) slightly in order achieve a perturbative 5D theory.

For \( \eta k \gtrsim 10 \), it can be shown using the analytical expression for the fermion profiles for \( c = 0 \) and the approximate expression for the W boson profile (3.6) that the 4D
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effective $W$-boson coupling to light fermions (3.24) is approximately

$$g_{W}^{\text{th}} = \frac{2g_{L5}}{\eta k \sqrt{\xi_L}} \left\{ \frac{1}{3} + \xi_{\text{IR}} k [1 - 2(\eta k) \hat{m}_i \hat{m}_j] \right\} \simeq \sqrt{2k} g_{L5} M_W z_{\text{IR}} \tag{3.34}$$

where we have assumed $\xi_{\text{IR}} = \eta$ by 5D gauge invariance. Matching to the experimental $\overline{MS}$ value of $g_{W}(M_Z) = 0.652$ we find from the 5D perturbativity bound $g_{L5} \sqrt{k} \lesssim 4\pi$ that $z_{\text{IR}}^{-1} \lesssim 2.2$ TeV. We see that for $z_{\text{IR}}^{-1} = 1.8$ TeV, which is required for consistency with the electroweak precision tests [19], the bulk lies close to the perturbativity bound with $g_{L5} \sqrt{k} = 10.3$. We therefore cannot trust the effective field theory to give a reliable description of the first KK resonances. To understand how the lack of perturbativity comes about, recall that the bulk probability density vanishes in the limit of a large boundary kinetic term. In this limit, the effective coupling is therefore dominated by the boundary contribution which is proportional to the fermion BKT coefficient $\eta$. Since the normalization factors depend on the inverse square root of the BKT, the 4D couplings of gauge fields to fermions scale as $g_5 \eta (1/\sqrt{\eta})^2 (1/\sqrt{\zeta}) = g_5/\sqrt{\zeta}$. Taking the large-BKT limit, we see that the bulk necessarily becomes non-perturbative to keep the effective couplings of order unity.

On the other hand, it is reasonable to expect that the large-BKT limit will realize a 4-dimensional gauge theory with couplings of order unity. This will be the case if the boundary value of the coupling is renormalized by a factor of $\sqrt{\zeta}$ to account for the brane dynamics of the gauge fields. While this approach suffers the disadvantage of explicitly breaking 5D gauge invariance, it is well-motivated by the renormalization principle in quantum field theory. In renormalized QED, for example, physical gauge coupling $e_{\text{phys}}$ is related to the bare coupling $e_0$ by the gauge-field renormalization factor $Z_{e_{\text{phys}}} = e_0 Z_{\psi} \sqrt{Z_A}$. Identifying the renormalization factors in the 5D model with $Z_A = 1 + 2\zeta \delta (z - z_{\text{IR}})$ and $Z_{\psi} = 1 + \eta \delta (z - z_{\text{IR}})$, the boundary contribution to the coupling satisfies

$$\int dz \, 2\xi_{\text{IR}} \delta (z - z_{\text{IR}}) = \int dz \left( Z_{\psi} \sqrt{Z_A} - 1 \right). \tag{3.35}$$
Replacing the delta functions by the cut-off scale \( \Lambda_5 \sim k \) we obtain

\[
\xi_{IR} = \frac{1}{2\Lambda_5} \left[ (1 + \eta \Lambda_5)\sqrt{1 + 2\zeta} - 1 \right] \quad (3.36)
\]

which can be approximated by \( \eta \sqrt{\zeta \Lambda_5/2} \) for \( \eta \Lambda_5, \zeta \Lambda_5 \gg 1 \). Since the gauge-boson kinetic terms in the emergent model are of order \( 10^3 \), the necessary 5D coupling is decreased from the previous estimate by a factor of \( \sqrt{\zeta \Lambda_5} \sim 30 \), which is safely within the perturbativity bound.

In addition to complying with electric charge universality, the model must reproduce the success of the Standard Model in predicting the universality of the photonic coupling to \( W \) bosons which has been accurately measured via \( W \)-pair production at LEP2. Substituting the expression for the fermion zero mode profiles (3.29) and (3.30) into (3.23), we obtain the expression for the electric charge for fermions

\[
e_{\psi\psi\gamma} = N_A^{(0)} \left[ 1 + \frac{\xi_{IR} k - \eta k/2}{1 + (\eta k)/2 + (\eta k)^2 (m_0 z_{IR})^2} \right]. \quad (3.37)
\]

Note that mass differences amongst the light fermions will therefore lead to small shifts in their electric charge. These shifts can be offset, however, by introducing a flavor-dependent \( \eta \).

Since the boundary kinetic term for the \( W \)-boson is extremely large, its electric charge is dominated by the boundary contribution of (2.34), which is given by

\[
e_{WW\gamma} = N_A^{(0)} \zeta_L \sqrt{2\zeta L \Lambda_5 \rho_W^{(1)}(z_{IR})^2} \simeq N_A^{(0)} \sqrt{2\Lambda_5 \zeta_L} \quad (3.38)
\]

Comparing \( e_{WW\gamma} \) with \( e_{\psi\psi\gamma} \) we see that if \( m_0 = 0 \), the agreement is exact in the large-BKT limit \( \eta, \zeta_L \to \infty \). This is consistent with the fact that all of the probability density lies on the IR brane, so the gauge couplings do not receive corrections from the bulk integral. For \( m_0 \neq 0 \), the contribution from the third term in the denominator of (3.37) becomes significant for \( \eta k \gtrsim 1/(m_0 z_{IR}) \). This implies that the
fermion/charged-boson universality cannot be improved by making $\eta k$ arbitrarily large. The physical reason behind this result is that for $m_0 \neq 0$, the zero mode behaves partially like a KK mode, whose bulk probability density is minimized for $\eta k \simeq 1/(m_0 z_{\text{IR}})$. In the limit $m_0 \to 0$, we recover the result that the massless fermions are completely brane-localized in the limit $\eta k \to \infty$.

An explicit numerical check shows that the boundary kinetic terms for the light fermions should be of order $\eta k \sim 100$, while the top quark favors $\eta k \sim 10$, owing to its large mass. The option of having different boundary kinetic terms for the bottom and the top is inconsistent with the SU(2)$_L$ gauge symmetry, under which $t_L$ and $b_L$ transform as a doublet. Unfortunately, however, forcing $\eta_{Lb} = \eta_{Lt}$ runs afoul with the measurement of the electric charge of the top quark at Tevatron [82].

A possible solution to this problem is to arrange for the top quark to experience a smaller characteristic length in the extra dimension than the light fermions, thereby increasing the effective IR scale for the top quark compared to the light fermions. This scenario could emerge naturally from string theory as a multi-throat background on a Calabi-Yau manifold [83, 84]. In order to avoid dealing with the added complexity of a multi-throat metric, we will approximate the situation by allowing for a different boundary kinetic term for the top and bottom quark, leading to an additional source of custodial isospin violation.

### 3.2.3 $W$-boson coupling

It can be seen from the approximate formula (3.34) that the $W$-coupling is universal to leading order in the fermion masses. Performing the numerical calculation with no approximations, one finds that the largest deviation in the $W$ vertices amongst the light fermions is between $W e \nu_e$ and $W \tau \nu_\tau$ which differ by one part in 5000. This consistent with the constraints from both lepton-lepton and lepton-quark universality which are at the level of one part in 500 and 167, respectively [85, 86]. The only significant deviation is for the $Wtb$ vertex, which differs from the light fermions by
14%. This deviation is compatible with recent measurements of single top-quark production at D0 [82], which have constrained the $Wtb$ vertex at the level of 20%.

### 3.2.4 Z-boson coupling

Numerically performing the overlap integrals (3.26), (3.27) and (3.28) one finds $1.5 \times 10^{-3} \leq \varepsilon \leq 9 \times 10^{-4}$ and $(\sin^2 \theta_W)^{th} = 0.223$, where the smallest values are for the electron and the largest are for the top quark. Using (2.57) and (2.58) we can estimate the contributions $\delta \epsilon_1$ and $\delta \epsilon_b$ from the anomalous couplings of the top quark. The results are

\[
\begin{align*}
\epsilon_1^{SM} + \delta \epsilon_1 & \simeq 19 \times 10^{-3} \quad (3.39) \\
\epsilon_b^{SM} + \delta \epsilon_b & \simeq -13 \times 10^{-3} \quad (3.40)
\end{align*}
\]

where we have taken $\Lambda = z_{IR}^{-1}$ and have neglected the contribution of the Higgs to $\epsilon_{1,b}^{SM}$ since the model is Higgsless. While these results fall outside of the experimental 68% confidence bounds (2.59), the experimental range will encompass the theoretical prediction if the confidence level is increased to 99%.

Although the $Z$-coupling to leptons is well-constrained by LEP1, the largest deviation in the emergent model is for the muon which deviates by only 0.04%. This is well within the experimental uncertainties which are of order 0.1%. In the quark sector, the only known $Z$-boson couplings are $Z \rightarrow b \bar{b}$ and $Z \rightarrow c \bar{c}$. These are also consistent with the emergent model as shown in Tab. 3.1

It is interesting to see if the predicted displacements can help to explain the anomalous $Z \rightarrow b \bar{b}$ coupling which shows a 3-sigma deviation from the SM prediction [69]. Converting the left and right-handed couplings into the vector $g_{Vb} = g_{Lb} + g_{Rb}$ and axial-vector $g_{Sb} = g_{Lb} - g_{Rb}$ couplings, we see from Fig. 3.1 that the predicted deviation is too small and of the wrong sign to account for the anomaly.
### 3. Emergent electroweak symmetry breaking

Table 3.1: Experimental uncertainties and predicted
displacements of $Z$-couplings to quarks

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Uncertainty (%)</th>
<th>Displacement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{Lb}$</td>
<td>0.4</td>
<td>0.25</td>
</tr>
<tr>
<td>$g_{Le}$</td>
<td>1.0</td>
<td>0.06</td>
</tr>
<tr>
<td>$g_{Rb}$</td>
<td>6.5</td>
<td>0.36</td>
</tr>
<tr>
<td>$g_{Re}$</td>
<td>3.2</td>
<td>0.19</td>
</tr>
</tbody>
</table>

Unlike the light fermions, the $Z \to t\bar{t}$ vertex receives a significant displacement of 27% from the SM prediction. Since this vertex has yet to be measured, this represents an interesting discovery prospect at the LHC which will measure this vertex to the required precision with $300 \text{ fb}^{-1}$ of data [87].

### 3.3 Conclusion

While gauge symmetry and the Higgs mechanism are regarded as fundamental tenets underlying the SM, the results of this work are consistent with the hypothesis that the electroweak symmetry group is not a fundamental gauge symmetry and that no Higgs mechanism operates. Working in the framework of a model with composite $W/Z$-bosons, the question of how charged matter can obtain mass has been addressed. The fermion masses in this model have a different origin than the vector bosons, which gain mass from strong dynamics. Electroweak symmetry breaking at the Planck scale produces charged fermion masses and exponential hierarchies are generated from the Frogatt-Nielsen mechanism. The absence of electroweak gauge symmetry in the model implies that the fermions must be placed precariously in the bulk in order to be consistent with the known bounds on charge universality. Despite the high scale at which the fermion masses are generated, there is suffi-
Figure 3.1: Comparison of vector and axial-vector couplings for the bottom quark predicted in the Standard Model and the emergent model.

cient back-reaction from the high-scale VEV to generate sizeable displacements in the gauge-boson-fermion vertices for the third generation. Conversely, the displacements of the light fermion vertices remain within the experimental uncertainties, ensuring agreement with electroweak precision tests. It is remarkable that despite such a drastic overhaul of fundamental principles, a model can constructed which is compatible with the stringent constraints from electroweak precision data.

Several aspects of the model require refinement in order to be truly realistic. The following issues offer potential topics for future research:

- A proper description of the top quark requires it to be treated differently from the light fermions. This will involve a more complicated metric than the simple AdS$_5$ slice. The effect of the fermion wavefunctions in this more complicated background on the electroweak observables remains to be understood.

- The renormalization procedure used to obtain a perturbative bulk theory leads to a small explicit breaking of 5D gauge invariance. A fully gauge-invariant renormalization scheme is yet to be found and will probably require the IR
brane to be regulated

- For simplicity of analysis, the bulk fermions are taken to be massless. Allowing the bulk masses to vary can significantly alter the overlap integrals which determine electroweak observables. This may induce a more tangible effect on the $Z \rightarrow b\bar{b}$ anomalous vertex

Since the LHC has yet to uncover the true nature of EWSB, one must remain open to the possibility that gauge symmetry is not fundamental. Until the origin of EWSB is discovered, the existing doctrine of gauge invariance and a Higgs mechanism should be viewed with a critical eye.
A.1 Identification spaces and orbifolds

A simple way to obtain a compact extra dimension is to subject the non-compact space $\mathbb{R}^n$ to identifications under a subgroup of its isometries. A circle of radius $R$, for example, can be obtained from $\mathbb{R}^1$ by identifying points under displacements by $2\pi R$; that is, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. It follows that there is no way to distinguish a field theory on $S^1$ from a field theory on $\mathbb{R}$ with a $2\pi R\mathbb{Z}$-invariant action [88].

In order to obtain realistic low-energy spectra, the identification space serving as the extra dimension is often taken to be an orbifold. An orbifold is a topological space which is locally homeomorphic to the quotient space $\mathbb{R}^n/G$ where $G$ is a discrete subgroup of the linear isometries $O(n)$ of $\mathbb{R}^n$. Usually the orbifolds considered in physics are of the form $X/G$ globally, where $X$ is a smooth manifold and $G$ is a discrete group of the isometries of $X$. Unlike smooth manifolds, orbifolds contain singular points where non-trivial elements of $G$ map a point on the manifold to itself. A simple example is $S^1/\mathbb{Z}_2$. To find its singular points, note that $S^1$ can be obtained from $\mathbb{R}$ by identifying points separated by $2\pi$, that is, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Applying $\mathbb{Z}_2$ to a point in $\mathbb{R}$ and making the identification given by translations one finds that there are exactly two singular points. Similarly, $T^2/\mathbb{Z}_2$ has four singular points.

The fact that an orbifold can be obtained from $\mathbb{R}^n$ often makes it easier to do field theory on the covering space $\mathbb{R}^n$ with an action that is invariant under the group $G$, rather than working with an action which is defined on $\mathbb{R}^n/G$ directly.
In this ‘upstairs’ approach, any classical solution of this theory which respects the orbifold symmetries when substituted into the action can be viewed as an orbifold theory when restricted to the set of non-identical points in $\mathbb{R}^n$. Note that it is not necessary that the solutions for the fields on $\mathbb{R}^n$ satisfy the same symmetries as the orbifold, since they are not physically observable. It is only the action which must be $G$-invariant. It is also possible to write down an action for a field theory on the orbifold space $\mathbb{R}^n/G$ which is referred to as the ‘downstairs’ approach. This is technically difficult, however, since it will involve boundary terms including Gibbons Hawking terms if gravity is included [39].

A.2 Calculation of the brane tensions in the Randall-Sundrum model

Since $S^1 = \mathbb{R}/2\pi r \mathbb{Z}$, the metric on $S^1/\mathbb{Z}_2$ is given by

$$d s^2 = e^{-2k|y|} \eta_{\mu \nu} \, d x^\mu \, d x^\nu - d y^2, \quad (A.1)$$

where $|y|$ is illustrated below. Note that each interval of the form $n\pi r < y < (n + 1)\pi r$ for integer $n$ is equivalent up to a coordinate transformation and can thus be described by the cosmological constant term in (1.85). The stress-energy...
resulting in the orbifold condition must therefore originate from sources localized at the orbifold fixed points. One can determine the nature of these sources by studying the left-hand side of Einstein’s equation, given by

\begin{align}
G_{55} &= (3/2)[A'^2/2 + A'' + 1(-A'' + A'^2/2)] = (3/2)A'^2 \quad (A.2) \\
G_{\mu\nu} &= -(3/2)\eta_{\mu\nu}(-A'' + A'^2/2) \quad (A.3)
\end{align}

where \( ' \) denotes differentiation with respect to \( z \).

One must take care in evaluating the derivatives in the Einstein tensor due to the cusps in \( A, A'' \)

\begin{equation}
A'' = -2k^2e^{-A} + 4ke^{-A/2}[\delta(z - z_{\text{UV}}) - \delta(z - z_{\text{IR}})] \quad (A.4)
\end{equation}

where we have used \(|z|^2 = 1\) and \(|z|'' = \text{sgn}(z)' = [2\theta(z) - 1]' = 2[\delta(z - z_{\text{UV}}) - \delta(z - z_{\text{IR}})]\). Substituting this into (A.3) gives

\begin{equation}
G_{\mu\nu} = -6\eta_{\mu\nu}\{k^2e^{-A} - ke^{-A/2}[\delta(z - z_{\text{UV}}) - \delta(z - z_{\text{IR}})]\}. \quad (A.5)
\end{equation}

On the other hand, the 4D component of stress-energy arising the bulk cosmological constant is \( \kappa_5 T_{\mu\nu} = \kappa_5\Lambda e^{-A}\eta_{\mu\nu} = -6\eta_{\mu\nu}k^2e^{-A} \), which exactly accounts for the first term in the Einstein tensor. The second term requires the introduction of three-volumic energy densities (brane tensions) at the locations of the \( z = 0, z_{\text{IR}} \) branes. A constant brane tension \( \sigma \) at \( z = z_* \) in the 4D theory corresponds to the following action

\begin{equation}
S_{\text{brane}}^{4} = \int d^4x \sqrt{|g^{\text{ind}}|}\sigma, \quad (A.6)
\end{equation}

where \( g^{\text{ind}}(x) = g_{MN}(X) \partial_{\mu}X^M \partial_{\nu}X^N \) is the 4D action induced by \( g_{MN} \) on the submanifold described by \( z = z_* \). We can easily promote this to a 5D action by integrating over a delta function centered at the brane location,

\begin{equation}
S_{\text{brane}}^{5} = \int d^4x dz \delta(z - z_{\text{IR}})\sqrt{|g^{\text{ind}}|}\sigma = \int d^4x dz \sqrt{|g|}L_{\text{brane}} \quad (A.7)
\end{equation}
A. Extra dimensions

where \( \mathcal{L}_{\text{brane}} = \sigma \delta(z - z_{\text{IR}}) / \sqrt{|g^{(\text{ind})}| / |g|} \). The stress-energy due to the brane tensions are thus \( \kappa_5 T_{\mu\nu}^{\text{brane}} = \kappa_5 e^{-A/2} [\sigma_1 \delta(z - z_{\text{UV}}) + \sigma_2 \delta(z - z_{\text{IR}})] \). Equating this with the singular terms in (A.5) implies

\[
\sigma_0 = -\sigma_{\pi r} = 6k / \kappa_5. \tag{A.8}
\]

A.3 Localization of gravity

In this appendix we reproduce the result [89] that the existence of a normalizable graviton zero mode is equivalent to having a finite four-dimensional Planck constant. The background spacetime is assumed to be of the non-factorizable form

\[
\text{ds}^2 = e^{-A} \eta_{\mu\nu} \text{dx}^\mu \text{dx}^\nu + g_{mn} \text{dx}^m \text{dx}^n \tag{A.9}
\]

where \( m, n \) index the extra dimensions and \( A \) depends only on the extra dimensions. Consider the metric obtained from (A.9) by deforming \( \eta_{\mu\nu} \) from flat, four-dimensional space,

\[
\text{ds}^2 = g_{MN} \text{dx}^M \text{dx}^N = e^{-A} \mathcal{g}_{\mu\nu} \text{dx}^\mu \text{dx}^\nu + g_{mn} \text{dx}^m \text{dx}^n \tag{A.10}
\]

where \( \mathcal{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) and \( h_{\mu\nu} = h_{\mu\nu}(x^\mu, x^m) \). In general, the 4D components of the \( d + 1 \)-dimensional Ricci tensor \( R_{\mu\nu} \equiv R^S_{\mu S\nu} \) contain the terms

\[
R^\sigma_{\mu\sigma\nu} = - (\partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^R_{\mu\sigma} \Gamma^\sigma_{\nu R}) + (\partial_\nu \Gamma^\sigma_{\mu\nu} + \Gamma^R_{\mu\nu} \Gamma^\sigma_{\nu R}) \tag{A.11}
\]

\[
\supset - (\partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^R_{\mu\sigma} \Gamma^\sigma_{\nu \rho}) + (\partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^R_{\mu\nu} \Gamma^\sigma_{\nu \rho}) \tag{A.12}
\]
and the 4D components of the Christoffel symbols \( \Gamma^{\kappa}_{\sigma\nu} = \frac{1}{2} g^{M\kappa} (\partial_\nu g_{\sigma M} + \partial_\sigma g_{M\nu} - \partial_M g_{\nu\sigma}) \) contain the terms

\[
\Gamma^{\kappa}_{\sigma\nu} \supset \frac{1}{2} g^{\mu\kappa} (\partial_\nu g_{\sigma\mu} + \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma}).
\]  
(A.13)

Substituting the 4D components of the metric (A.10), we find \( \Gamma^{\kappa}_{\sigma\nu} \supset \Gamma^{\kappa}_{\sigma\nu} \), where

\[
\Gamma^{\kappa}_{\sigma\nu} \equiv \frac{1}{2} g^{\mu\kappa} (\partial_\nu g_{\sigma\mu} + \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\nu\sigma}).
\]  
(A.14)

To obtain this result, we used the fact that \( g_{\mu\nu} = e^{-A} g_{\mu\nu} \), \( g^{\mu\nu} = e^{A} \tilde{g}^{\mu\nu} \) and that \( A \) is independent of \( x^m \). It follows that the 4D components of the Ricci tensor \( R_{\mu\nu} \) for (A.10) contain \( R_{\mu\nu} \), where

\[
R_{\mu\nu} \equiv - (\partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\mu\rho} \Gamma^\sigma_{\rho\nu}) + (\partial_\sigma \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\sigma\rho}).
\]  
(A.15)

The general \( d+1 \)-dimensional Ricci scalar \( R = g^{MN} R_{MN} \) contains the terms \( g^{\mu\nu} R_{\mu\nu} \) and thus the Ricci scalar for (A.10) contains \( e^A \tilde{R} \), where \( \tilde{R} \equiv \tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} \).

We now show that in fact \( \tilde{g}_{\mu\nu} \supset g^{(4)}_{\mu\nu} \) so that \( \tilde{R} \) can be identified with the effective Ricci scalar of the 4D theory. This is tantamount to showing that \( h_{\mu\nu} \) contains a massless mode which is independent of \( x^m \). Massless modes show up as zero modes of the linearized equation of motion for metric fluctuations \( h_{\mu\nu} \).

We assume that the metric can be put into the form (summation over \( m \) is implied)

\[
ds^2 = e^{-A} [(\eta_{\mu\nu} + h_{\mu\nu}) \, dx^\mu \, dx^\nu - dz^m \, dz^m]
\]  
(A.16)

so that the metric is conformally equivalent to a four-dimensional deformation of flat \( d+1 \)-dimensional space; that is, \( \eta_{MN} + h_{MN} \), where the only nonvanishing components of \( h_{MN} \) are \( h_{\mu\nu} \). The Einstein tensor for \( g_{MN} \) can be expressed in terms
of the Einstein tensor for \( \hat{g}_{MN} \equiv \eta_{MN} + h_{MN} \) using the relation (1.87), which gives

\[
G_{MN} = \hat{G}_{MN} + \frac{d-1}{2} \left[ \frac{1}{2} \partial_M A \partial_N A + \tilde{\nabla}_M \partial_N A + \right. \\
\left. - \hat{g}_{MN} \hat{g}^{KL} \left( \tilde{\nabla}_K \partial_L A - \frac{d-2}{4} \partial_K A \partial_L A \right) \right].
\]

(A.17) (A.18)

The linearization of this expression is simply obtained by substituting \( \hat{g}_{MN} = \eta_{MN} + h_{MN} \) in the right-hand side and ignoring terms higher order than quadratic in \( h_{MN} \). The first term gives

\[
\hat{G}_{AB} \equiv \delta \hat{G}_{AB} = \frac{1}{2} (\partial_C \partial_B h^C_A + \partial_C \partial_A h^C_B - \partial_A \partial_B h - \Box h_{AB} - \eta_{AB} \partial_C \partial_D h^{CD} + \eta_{AB} \Box h)
\]

(A.19)

where beginning-alphabet indices are raised and lowered using the Minkowski metric \( \eta_{AB} \) and \( \Box \equiv \eta^{AB} \partial_A \partial_B \). The first term in square brackets is zeroth order in \( h_{MN} \). The second is given by \( \partial_M \partial_N A - \Gamma^K_{MN} \partial_K A \) which contains the linear terms

\[
- \frac{1}{2} \eta^{AB} (\partial_M h_{NB} + \partial_N h_{MB} - \partial_B h_{MN}) \partial_A A.
\]

(A.20)

The third term is

\[
-(\eta_{MN} + h_{MN}) (\eta^{KL} - h^{KL}) \left( \partial_K \partial_L A - \Gamma^P_{KL} \partial_P A - \frac{d-2}{4} \partial_K A \partial_L A \right) \supset
\]

\[
(h_{MN} \eta^{KL} - h_{KL} \eta_{MN}) \left( \frac{d-2}{4} \partial_K A \partial_L A - \partial_K \partial_L A \right) + \eta^{KL} \eta_{MN} \Gamma^A_{KL} \partial_A A.
\]

(A.21)

Working in the gauge \( \partial_A h^{AB} = 0 \) and \( h \equiv h^A_A = 0 \) gives \( \delta \hat{G}_{AB} = -\frac{1}{2} \Box h_{MN} \). Using this together with the fact that only \( h_{\mu\nu} \) are nonzero and that \( \partial_\mu A = 0 \), gives

\[
G_{MN} \supset \delta G_{MN} \equiv -\frac{1}{2} \Box h_{MN} + \frac{d-1}{2} \left[ \frac{1}{2} \eta^{AB} \partial_A h_{MN} \partial_B A + \\
\quad + \frac{d-2}{4} h_{MN} \eta^{KL} \partial_K A \partial_L A - h_{MN} \Box A \right].
\]

(A.22)
If the matter Lagrangian does not depend on any derivatives of the metric, then
\( T_{MN} = g_{MN} L_M \) and thus \( \delta T_{MN} = h_{MN} L_M = T^L_M h_{NL} \). In the linear approximation, it suffices to replace \( T^L_M \) by the energy-momentum tensor for the classical background since gravitational waves induce higher-order terms in the fluctuation. Thus \( \delta T_{MN} = T^c_L M h_{NL} \). By the Einstein equation for the classical background, however we have \( T^c_{MN} = \kappa^{-1} G^c_{MN} \). The graviton equation of motion \( \delta G_{MN} = G^c_L M h_{NL} \) thus gives \( -1/2 \Box h_{MN} + (d - 1)/4 \partial^B A \partial_B h_{MN} = 0 \). The linear derivative term should be transformed away in order to obtain a canonically normalized kinetic term. This can be achieved by the change of variables \( h_{MN} = e^{(d-1)A/2} \tilde{h}_{MN} \). Looking for zero-mode solutions \( \tilde{h}_{\mu \nu} = H_{\mu \nu}(x) \psi_0(z) \) such that \( \Box H_{\mu \nu} = 0 \) we obtain \( \psi_0 = e^{-(d-1)A/4} \) where we have assumed \( A(z = 0) = 0 \). The normalization condition is thus \( \int d^n z \psi_0^2 = \int d^n z e^{-(d-1)A/2} < \infty \). Substituting into the metric gives

\[
    ds^2 = e^{-A}(\eta_{\mu \nu} + H_{\mu \nu}(x) + \cdots) dx^\mu dx^\nu + g_{mn} dx^m dx^n
\]

as we wanted to show. Since \( \sqrt{|g|} = e^{-2A}\sqrt{|\tilde{g}|} \), the gravitational action contains

\[
    S[g] = \frac{1}{2} M_{*}^{d-1} \int d^{d+1} x \sqrt{|g|} R
\]

\[
    \geq \frac{1}{2} M_{*}^{d-1} \int d^n y e^{-A} \int d^4 x \sqrt{|\tilde{g}|} R^{(4)}
\]

\[
    = \frac{1}{2} M_{*}^{d-1} \int d^n z e^{-(d-1)A/2} \int d^4 x \sqrt{|\tilde{g}|} R^{(4)}.
\]

where in the last line we have changed to conformal coordinates and used the fact that \( d = 3 + n \). Comparing with the \( d = 3 \) action shows that

\[
    M_P^2 = M_{*}^{d-1} \int d^n z e^{-(d-1)A/2}.
\]
B.1 Holography of scalars

Here we review the holographic interpretation of bulk scalar field [16]. Consider the action for a real bulk scalar in a slice of AdS$_5$ delimited by branes at $y = y_0$ and $y = y_1 > y_0$.

\[
S = \int d^4x\, dy \sqrt{|g|} \left\{ g^{MN} \partial_M \Phi \partial_N \Phi - m_\Phi^2 \Phi^2 - 2bk [\delta(y - y_0) - \delta(y - y_1)\Phi^2] \right\}. \tag{B.1}
\]

The variation of the action is given by

\[
\delta S_{\text{scalar}} = \int d^4x\, dy \{ \delta \Phi[ -e^{-2ky} \partial^2 \Phi + \partial_y(e^{-4ky} \partial_5 \Phi) - e^{-4ky} m_\Phi^2 \Phi] \} + \int d^4x\, \sqrt{|g|} \delta \Phi(\partial_y - bk)\Phi |_{y_1} + \int d^4x\, \sqrt{|g|} \delta \Phi(\partial_y - bk)\Phi |_{y_0}. \tag{B.2}
\]

According to the Witten prescription, the effective action is obtained by imposing the bulk equation of motion in $S$, leaving the UV boundary value $\Phi(x, y_0) \equiv \Phi_0(x)$ fixed. This causes the bulk term in $\delta S$ to vanish in addition to the UV boundary term since $\delta \Phi |_{y_0} = \delta \Phi_0 = 0$. In order to be consistent with the variational principle, it is necessary to further impose the IR boundary condition $(\partial_y - bk) |_{y_1} = 0$.

Substituting the classical solution into $S$ leaving the UV boundary value fixed gives

\[
S \supset S_{\text{eff}} = \int d^4x\, \sqrt{|g|} \Phi(x, y) \left( \frac{\partial}{\partial y} - bk \right) \Phi(x, y) \bigg|_{y = y_0}. \tag{B.4}
\]
Writing $\Phi(x, y)$ in terms of its 4D Fourier transform as

$$\Phi(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \tilde{\Phi}(p, y)$$  \hspace{1cm} (B.5)

we obtain, after integrating over $x$,

$$S = \int \frac{d^4 p}{(2\pi)^4} \sqrt{|g|} \tilde{\Phi}(p, y) \left( \frac{\partial}{\partial y} - bk \right) \tilde{\Phi}(-p, y)|_{y=y_0}.  \hspace{1cm} (B.6)$$

The Fourier-transformed fields obey the differential equation obtained by replacing $\partial_\mu \to -ip_\mu$

$$0 = -p^2 \tilde{\Phi} - e^{2ky} \partial_\eta (e^{-4ky} \partial_\eta \tilde{\Phi}) + e^{-2ky} m_\eta^2 \tilde{\Phi}  \hspace{1cm} (B.7)$$

which has solution $\tilde{\Phi}(p, y) = \tilde{\phi}(p)f(p, y)$ where

$$f(p, y) = e^{2ky} \left[ J_\alpha \left( \frac{|p|}{ke^{-ky}} \right) + b(p) Y_\alpha \left( \frac{|p|}{ke^{-ky}} \right) \right],$$

$\tilde{\phi}(p)$ is an arbitrary function of $|p|$ and $b(p)$ is determined by imposing the IR boundary condition $(\partial_\eta - bk)\tilde{\Phi}|_{y=y_1} = 0$. Written in Fourier space, the effective action becomes

$$S_{\text{eff}} = \int \frac{d^4 p}{(2\pi)^4} e^{-4ky} \tilde{\phi}(p)f \left( \frac{\partial}{\partial y} f - bk f \right) \tilde{\phi}(-p)|_{y=y_0} = \int \frac{d^4 p}{(2\pi)^4} \tilde{\Phi}_0(p) \Sigma(p^2) \tilde{\Phi}_0(-p)$$  \hspace{1cm} (B.8)

where

$$\Sigma(p^2) = e^{-4ky} \left[ \frac{\partial_y - bk f}{f} \right]_{y=y_0}.  \hspace{1cm} (B.9)$$

It can be shown that this quantity is the correlation function expressed in momentum space. To do this, express the effective action in terms of position space, perform
the functional derivatives and evaluate the integrals, giving

$$\frac{\delta^2 S_{\text{eff}}}{\delta \Phi(x', y_0) \delta \Phi(0, y_0)} = \int \frac{d^4 p}{(2\pi)^2} \int d^4 x \int d^4 y e^{-ip(x-y)} \Sigma(p) [\delta(x - x') \delta(y) + \delta(x) \delta(y - x')]$$

(B.10)

$$= 2\Sigma(x').$$

(B.11)

It is clear from the structure of $\Sigma(p^2)$ that the poles and the roots of the correlation function are given by the Kaluza-Klein mass eigenvalues for the bulk scalar with Dirichlet and Neumann boundary conditions, respectively.

In order to apply the strict AdS/CFT correspondence, one must consider the limits $y_0 \to -\infty$ and $y_1 \to \infty$ corresponding to pure AdS$_5$. As a check of this formula, it can be argued on scaling grounds as in section 1.5.3 that the position-space correlator $\langle O(x)O(0) \rangle$ must be of the form $x^{-2\text{dim}O}$ which implies that the momentum-space correlator is of the form $p^{2\text{dim}O-4} = p^{2\alpha}$. Indeed, the asymptotic form of the correlation function is proportional to $(p/ke^{-ky_0})^2 + (p/ke^{-ky_0})^{2\alpha} + \cdots$ from which we identify the first non-analytic term as the correlation function. There remains a problem, however, since this term vanishes in the UV limit $y_0 \to -\infty$ so it is necessary to rescale the fields by $\Phi \to \Phi e^{-\alpha k y_0}$. Although this rescues the correlation function, it comes at the expense of causing the leading analytic $p^2$ term to diverge. Such divergences are assumed to be canceled by appropriate counterterms.

Let us now relax the assumption $y_0 \to -\infty$ by considering an RS2 universe. We will assume that AdS/CFT is still valid in the sense that results following from the asymptotic form of $\Sigma(p^2)$ carry over to the truncated theory. With a finite cut-off, there are no counterterms so the analytical $p^2$ term must be given an interpretation. This term can be obtained from $\Sigma(p)$ by Taylor expanding about $p = 0$. This gives, in the negative branch,

$$\Sigma_-(p^2) = \frac{2k}{\tilde{g}^2} \frac{1}{\alpha - 1} \left( \frac{p}{2e^{-ky_0}} \right)^2 \left( e^{k(y_1 - y_0)} \right)^{2(1-\alpha)} - 1. $$

(B.12)
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The inverse Fourier transform of this term is interpreted as a kinetic energy term for the source field. Hence the effect of introducing a finite cut-off is that the source should be regarded as a dynamical field coupling to the CFT [14]. Dimensional analysis requires that the coupling is suppressed by $1/\Lambda^{{\text{dim}\mathcal{O}}-3} = 1/\Lambda^{\alpha-1}$, where $\text{dim}\mathcal{O}$ is the scaling dimension of the relevant CFT operator which depends on the localization profile of the bulk zero mode as discussed at the end of sections 1.5.2 and 1.5.3. Another way it can be determined is by counting the powers of $p$ in the inverse Fourier transform of the leading non-analytic term (i.e., the pure CFT 2-point correlator in position space), and then applying the scaling transformation $x \rightarrow \lambda x, p \rightarrow p/\lambda$.

Truncating the extra dimension at finite $y_1$ results in a discrete Kaluza-Klein spectrum. Similarly, because of the additional non-analytic terms present in $\Sigma(p^2)$, the CFT must now have massive resonances. Since the source is dynamical, however, the mass spectrum of the CFT is no longer determined by the poles of the 2-point 1PI correlator. The kinetic term $(\partial \phi^0)^2$ leads to the additional Feynman rule $1/k^2$ and the complete propagator is now given by an infinite sum of insertions of the 1PI correlator. The new mass spectrum is thus determined by the poles of

$$\frac{1}{k^2} + \frac{1}{k^2}[-i\Sigma(p^2)]\frac{1}{k^2} + \frac{1}{k^2}[-i\Sigma(p^2)]\frac{1}{k^2}[-i\Sigma(p^2)]\frac{1}{k^2} + \cdots = \frac{1}{k^2 - \Sigma(p^2)}. \tag{B.13}$$

Since we are interested in rest masses, we may set $k = 0$, in which case the mass spectrum is determined by the zeros of $\Sigma(p^2)$. These masses are identically equal to those obtained from the Kaluza-Klein spectrum. Hence, there is a correspondence between the CFT bound states and the massive KK states of the dual gravity theory.
The 4D CFT Lagrangian thus has the form

\[ \mathcal{L}_{4D} = \mathcal{L}_{\text{source}} + \mathcal{L}_{\text{CFT}} + \mathcal{L}_{\text{mixing}}, \]  
\[ \mathcal{L}_{\text{source}} = -Z_0 (\partial \varphi_0)^2 + m^2 \varphi_0^2, \]  
\[ \mathcal{L}_{\text{mixing}} = \frac{\omega}{\Lambda^0_0^{-1}} \varphi_0 \mathcal{O} \]  

where \( Z_0, \omega \) are dimensionless couplings and the exact form of \( \mathcal{L}_{\text{CFT}} \) is unknown. Redefining fields by \( \tilde{\varphi}_0 = \sqrt{Z_0} \varphi_0 \), we can define a dimensionless coupling which measures the strength of the interaction vertex between the source and the CFT,

\[ \xi = \frac{\omega}{\sqrt{Z_0}} \left( \frac{\mu}{\Lambda_0} \right)^\gamma \]  

where \( \gamma = \alpha - 1 \) and \( \omega, Z_0 \) are treated as quantities which run with the energy scale \( \mu \). Scale invariance of \( \int d^4x \mathcal{L}_{\text{source}} \) and \( \int d^4x \mathcal{L}_{\text{mixing}} \) can be used to show that \( Z_0 \) has scaling dimension \( 2\gamma \) whereas \( \omega \) has zero scaling dimension and thus does not run. If \( \gamma > 0 \) (\( \gamma < 0 \)) then \( Z_0 \) has an IR (UV) fixed point (assuming \( Z_0 \) is a monotonic function of \( \mu \)). The scaling relation \( Z_0 \sim \mu^{2\gamma} \) is thus a statement about the UV (IR) scaling behaviour of \( Z_0 \), that is,

\[ Z_0(\mu) = \begin{cases} 
\mu^{2\gamma} + \text{lower order terms}, & \gamma > 0 \\
\mu^{2\gamma} + \text{higher order terms}, & \gamma < 0 
\end{cases} \]  

which implies

\[ \lim_{\mu \to 0} \xi(\mu) = \begin{cases} 
0, & \gamma > 0 \\
\omega \Lambda^\gamma_0, & \gamma < 0 
\end{cases} \]  

Thus we see that for ‘large’ bulk masses (\( \alpha > 1 \), corresponding to either a UV- or IR-localized zero mode, the coupling between the source and the CFT is irrelevant. In the UV branch, this means that the bulk zero mode is interpreted as the elementary massless source field. At the other end of the spectrum in the IR branch, the source
acquires a large mass (of order the curvature scale) but the CFT spectrum now contains a massless mode which plays the role of the zero-mode dual.

On the other hand, for ‘small’ bulk masses ($\alpha \leq 1$) corresponding to intermediate localization profiles, the coupling to the source varies from significant to very large and hence the 4D states now consist of a mixture of elementary source and composite CFT. In the UV branch, the zero-mode dual remains massless since the massive CFT modes decouple from the massless source in the low-energy theory as a result of the mass gap. Similarly in the IR branch, the large mass of the source decouples it from the massless CFT state.

### B.2 Derivation of the CFT correlator for fermions

This appendix reviews the holography of fermions [15]. Consider the action for a bulk fermion in a slice of AdS$_5$ delimited by branes at $y = y_0$ and $y = y_1 > y_0$,

$$S = \int d^5x \sqrt{|g|} \left[ \frac{i}{2} (\bar{\Psi} \Gamma^M D_M \Psi - D_M \bar{\Psi} \Gamma^M \Psi) - m_\Psi \bar{\Psi} \Psi \right]. \quad (B.20)$$

Stationary variations of the bulk fields give the equations of motion

$$e^{ky_i} \gamma^\mu \partial_\mu \bar{\Psi}_+ + \partial_y \bar{\Psi}_- = 0 \quad (B.21)$$

$$e^{ky_i} \gamma^\mu \partial_\mu \bar{\Psi}_- - \partial_y \bar{\Psi}_+ = 0 \quad (B.22)$$

where we have set $m_\Psi = 0$ for convenience. The boundary terms in the variation of the action are

$$\delta S_{\text{bdry}} = \int d^4x \left\{ \bar{\Psi}_+ \delta \bar{\Psi}_- - \bar{\Psi}_- \delta \bar{\Psi}_+ \right\}|_{y_1}^{y_0} + \text{h.c.} \quad (B.23)$$

Only one of $\Psi_+$ or $\Psi_-$ can be taken to be the source field on the UV brane because fixing $\Psi_+|_{y_0}$ determines $\Psi_+|_{y_0}$ through the bulk equations of motion. If $\Psi_+(x, y_0) \equiv$
\( \Psi^0_\pm \) is taken to be the fixed source, then

\[
\delta S_{\text{bdry}} = \int d^4x \left\{ \pm \bar{\Psi}_\pm \delta \Psi_\pm \right\} \bigg|_{y_0}^{y_1} + \text{h.c.} \tag{B.24}
\]

The IR boundary terms in the variation of the action will vanish as long as one of \( \Psi_+ \) or \( \Psi_- \) is taken to have Dirichlet boundary conditions on the IR brane. Assuming this to be the case, we obtain simply

\[
\delta S_{\text{bdry}} = \int d^4x \left\{ \pm \bar{\Psi}_\pm \delta \Psi_\pm \right\} \bigg|_{y_0} \tag{B.25}
\]

Hence we must add the following term to the boundary action in order to keep the total variation stationary

\[
S \supset \int d^4x \left\{ \pm \bar{\Psi}_\pm \Psi_\pm \right\} \bigg|_{y_0} \tag{B.26}
\]

Substituting the classical solutions into the augmented action gives,

\[
S_{\text{eff}} = \int d^4x \left\{ \pm \bar{\Psi}_\pm \Psi_\pm \right\} \bigg|_{y_0} \tag{B.27}
\]

Fourier transforming the bulk equations of motion gives

\[
e^{k_y p} \hat{\Psi}_+ + \partial_y \hat{\Psi}_- = 0, \quad e^{k_y p} \hat{\Psi}_- - \partial_y \hat{\Psi}_+ = 0 \tag{B.28}
\]

Writing the solutions as \( \hat{\Psi}(p, y) = \psi_\pm(p) f_\pm(p, y) \) where

\[
e^{k_y p} f_+ + \partial_y f_- = 0, \quad e^{k_y p} f_- - \partial_y f_+ = 0 \tag{B.29}
\]
and \( p \equiv \sqrt{p^\mu p^\mu} \), we deduce

\[
\dot{p} \psi_+ = -\psi_- \frac{\partial_y f_-}{f_+} = p \psi_- \quad (B.30)
\]

\[
\dot{p} \psi_- = \psi_+ \frac{\partial_y f_+}{f_-} = p \psi_+ \quad (B.31)
\]

Hence \( \psi_\pm(p) = (p/\dot{p}) \psi_\pm(p) = (\dot{p}/p) \psi_\pm(p) \). The effective action is thus

\[
S = \pm \int \frac{d^4p}{(2\pi)^4} \bar{\Psi}_\pm(p, y_0) \Psi_\pm(p, y_0) \quad (B.32)
\]

\[
= \pm \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_\pm(p) \psi_\pm(p) f_+(p, y_0) f_-(p, y_0) \quad (B.33)
\]

\[
= \pm \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_\pm(p) \frac{\dot{p}}{p} \psi_\pm(p) f_+(p, y_0) f_-(p, y_0) \quad (B.34)
\]

\[
= \pm \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_0^\pm(p) \frac{\dot{p}}{p} f_+(p, y_0) f_-(p, y_0) \quad (B.35)
\]

Consider the correlation function for \( \dot{\Psi}_0^+ \). Defining \( q_* = pe^{k y_*} \) so that \( q_0 \) and \( 1/q_1 \) are small parameters in the pure AdS5 limit of \( y_1 \to \infty \) and \( y_0 \to -\infty \), we find that \( \Sigma_+ \) has the following schematic form

\[
\Sigma_+(p) = \left( \frac{\dot{p}}{\hat{p}} \right) \frac{a_1(q_1) q_0^{1/2-c} + a_2(q_1) q_0^{-(1/2-c)}}{b_1(q_1) q_0^{-(1/2+c)} + b_2(q_1) q_0^{1/2+c}} \quad (B.36)
\]

\[
= \left( \frac{\dot{p}}{\hat{p}} \right) \frac{a_1(q_1) q_0^{2c} + a_2(q_1) q_0^{1+2c}}{b_1(q_1) + b_2(q_1) q_0^{1+2c}} \quad (B.37)
\]

\[
= \left( \frac{\dot{p}}{\hat{p}} \right) \frac{a_1(q_1) q_0^{-2c} + a_2(q_1) q_0^{-1}}{b_1(q_1) q_0^{-1-2c} + b_2(q_1)} \quad (B.38)
\]

For \( c > -1/2 \), the denominator of the second expression can be expanded using the binomial approximation giving \( \Sigma_+(p) \sim \hat{p}[\text{const} + p^{2c-1} + \cdots] \), while for \( c < -1/2 \) the same can be done for the third equality giving \( \Sigma_+(p) \sim \hat{p}[1/p^2 + p^{-2c-3} + \cdots] \). It follows that the scaling dimension for the CFT operator is \( \text{dim} \mathcal{O}_\Psi = 3/2 + |c + 1/2| \).
B.3 The geometry of Anti de-Sitter space

The space $\text{AdS}_{n+1}$ is a maximally symmetric semi-Riemannian manifold. It can be defined as the subset of $(n + 2)$-dimensional Minkowski space $\mathbb{R}^{2,n}$ with constant squared norm

$$x^2 \equiv x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 - \cdots - x_n^2 = b^2 > 0$$

where we have used the ‘mostly minus’ convention, that is, $\text{AdS}_n$ is the Lorentzian analogue of an $n$-sphere. The above relation can be used to eliminate the second timelike dimension $x_{-1}$ in the metric for $\mathbb{R}^{2,n}$, thus giving the induced metric of $\text{AdS}_{n+1}$ in terms of the $n + 1$ coordinates $x_0, \ldots, x_n$. The induced metric is conformally equivalent to Minkowski space. Indeed, in appropriate coordinates, the metric for $\text{AdS}_{n+1}$ describes slices of Minkowski space, scaled by an exponential warp factor dependent only on the additional dimension. In $n = 4$, for example, we have

$$ds^2 = -dy^2 + e^{-2y/b} \eta_{\mu \nu} dx^\mu dx^\nu.$$ 

It is often said that $\text{AdS}_{n+1}$ has a Minkowski space of one lower dimension as a boundary. To understand this, consider taking $x \to \infty$ in the defining relation $x^2 = b^2$. We find that the boundary is defined by $x^2 = 0$ subject to the identification $tx \sim x \forall t \neq 0$ and is thus an $n$-dimensional space as expected. Clearly this identification is equivalent to $tx \sim x \forall t > 0$ followed by identifying points under inversion.

Let us first consider the $(n + 1)$-dimensional quadric $x^2 = 0$ which we can obviously think of as being parametrized by the $n+1$ independent parameters $(x_0, \vec{x}) \in \mathbb{R} \times \mathbb{R}^n$. Now consider the submanifold defined by $\vec{x} = \vec{r}$. Clearly, for each $\vec{r} \neq 0$, this defines an $S^1$ centered at the origin of $\mathbb{R}^2$ with radius given by the length of the position vector in $\mathbb{R}^n$. Thus the space is topologically described by $S^1 \times \mathbb{R}^n$.

Scaling $x$ by any $t > 0$ causes $\vec{x}$ to traverse an infinite radial line in $\mathbb{R}^n$ but has no effect on the remaining coordinates since it corresponds merely to a rescaling
of radius of the corresponding $S^1$ by $t$. Under the identification of this subset of rescalings, all of the $S^1$'s along a given radius $t\vec{r}$ are identified with each other, giving the $n$-dimensional manifold $S^1 \times S^{n-1}$. Finally, the boundary space is obtained by orbifolding, $(S^1 \times S^{n-1})/\mathbb{Z}^2$. By effectively doubling this space and unrolling the timelike dimension, one can think of this as $\mathbb{R} \times S^{n-1}$.

To illustrate the connection with Minkowski space, specialize to $n = 4$ and use the change of coordinates $t \pm r = \tan(\psi \pm \xi)/2$, to put the 4D Minkowski metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$ into the form

$$ds^2 = g(\psi, \xi)[-d\psi^2 + \sin^2 \xi d\Omega_2^2],$$

which is proportional to the line element for $\mathbb{R} \times S^3$. This shows that 4D Minkowski space is conformally equivalent to a subset of the boundary of AdS$_5$. This should be compared with the fact that the conformal compactification of Euclidean $n$-space is the boundary of the $(n + 1)$-dimensional open ball. In fact, if one works in imaginary time coordinates, then AdS$_{n+1}$ can be associated with the upper half plane $(x_0, x_1, \ldots, x_n)$ such that $x_n > 0$ with metric $ds^2 = (1/x_n^2) \sum_{i=0}^{n-1} dx_i^2$. Hence the bulk space is homeomorphic to the open ball $B_{n+1}$ and the boundary is given by $\mathbb{R}^n \cup \{\infty\} \cong S^n$.

Note that the AdS$_5$ metric does not induce a metric on the boundary since the warp factor is divergent there. In order to obtain an induced metric on the boundary, it is necessary to multiply the metric by a function $d\tilde{s}^2 = f^2 ds^2$. The $y$-dependence of $f$ must be such that $d\tilde{s}^2$ is finite and nonvanishing as $y \to -\infty$, which is possible if, for example, $f = e^{ky}$. After determining the $y$-dependence, we are free to multiply by any function $\lambda(x)$ of the boundary. It follows that the induced metric is only defined up to conformal transformations $d\tilde{s}^2 \to \lambda(x)^2 f$, with scaling dimension two.
The metric for a charged non-rotating black hole is

\[ ds^2 = g(r) \, dt^2 - g(r)^{-1} \, dr^2 - r^2 \, d\Omega^2 \]

where \( g(r) = 1 - 2M/r + Q^2/r^2 \) and \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 \) is the metric on the two-sphere. The solution possesses two horizons which merge at the extremal charge \( Q = M \), beyond which the electrostatic repulsion destabilises the gravitational attraction, revealing a naked singularity. Taylor expanding \( g(r) \) for an extremal black hole about the horizon gives

\[ ds^2 = \left( \frac{\delta}{M} \right)^2 \, dt^2 - \left( \frac{\delta}{M} \right)^{-2} \, d\delta^2 - M^2 \, d\Omega^2 \]

where \( \delta = r - M \). Introducing \( dy = -M \, d\delta/\delta \) and setting \( \delta(y = 0) = 1 \) we obtain

\[ ds^2 = (e^{-2y/M} \, dt^2 - dy^2) - M^2 \, d\Omega^2 \]

which is the line element for \( \text{AdS}_2 \times S^2 \).
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References


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