Stochastic Volatility Model and Option Pricing

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Submitted in total fulfilment of the requirements
of the degree of Doctor of Philosophy

February 2011

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Abstract

This thesis is concerned with stochastic volatility models and pricing options.

From empirical observation of volatility, we know that volatility varies ‘randomly’. Hence stochastic volatility models have become popular. We consider the pricing of European derivatives in a Black-Scholes model with stochastic volatility. We show how Parseval’s theorem may be used to express those prices as Fourier integrals. This is a significant improvement over Monte Carlo simulation in many cases. The main ingredient in our method is the Laplace transform of the ordinary (constant volatility) price of a put or call in the Black-Scholes model, where the transform is taken with respect to maturity ($T$); this does not appear to have been used before in pricing options under stochastic volatility. We derive these formulas and then apply them to the case where volatility is modelled as a continuous-time Markov chain, the so-called Markov regime switching model. This model has been used previously in stochastic volatility modelling, but mostly with only $N = 2$ states. We show how to use $N = 3$ states without difficulty, and how larger number of states can be handled. Numerical illustrations are given, including the implied volatility curve in two and three-state models. The curves have the smile shape observed in practice.

The square-root process which is one of the most widely used stochastic volatility models and integrated squared volatility over the time interval are studied. We derive an explicit measure change formula of the square-root process which can be interpreted as the standard absolute continuity relations between squared Bessel processes of different dimensions, following from Girsanov’s theorem. Then we propose a simple and accurate approach to simulate European option prices. Numerical results show that the measure change approach and Andersen’s quadratic exponential scheme perform very closely. We also examine the numerical behaviour of the Radon-Nikodym derivative.
Declaration

This is to certify that:

1. the thesis comprises only my original work towards the PhD except where indicated in the Preface,

2. due acknowledgement has been made in the text to all other material used,

3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Signed,

Stephen Seunghwan Chin
Preface

This thesis was completed under the supervision of Professor Daniel Dufresne in the Centre for Actuarial Studies, The University of Melbourne. Chapters 3-5 contain the original research of this thesis, except as otherwise noted.

Chapter 3 and 4 are based on the paper ‘A general formula for option prices in a stochastic volatility model’, co-authored by Daniel Dufresne and available on the Centre for Actuarial Studies’ webpage (Working paper no.182.) The research was carried out under the supervision of Daniel Dufresne.

Chapter 5 is based on the paper ‘Measure changes in the square-root process’, co-authored by Daniel Dufresne, in preparation. The research was carried out under the supervision of Daniel Dufresne.

None of the work appearing here has been submitted for any other qualifications, nor was it carried out prior to PhD candidature enrollment.
Acknowledgments

Writing the dissertation, I am indebted to many people who contributed academic and personal development. Since any list would be insufficient, I mention only those who bear in my opinion the closest relation to this work.

First of all, I would like to thank my academic teacher and supervisor Professor Daniel Dufresne. He gave me valuable advice and support throughout the completion of my thesis.

I would also like to thank Associate Professor Mark Joshi for being available for discussions and reading group.

Further, I am grateful to the members of the Centre for Actuarial Studies at the University of Melbourne: David Dickson, Richard Fitzherbert, Shuanming Li, David Pitt and Xueyuan Wu for providing an enjoyable environment within which to complete my PhD. In addition, thanks go to my colleagues from the centre, especially Chris Beveridge, Jium-Hong Chan, Nick Denson, Andrew Downes, Ash Evans, Evan Hariyanto, Pete Raymond, Robert Tang, Will Wright and Chao Yang for fruitful discussions and a pleasant working atmosphere. I very much enjoyed my time at the centre.

Thanks to Grace Baek for proof reading my thesis for grammatical errors.

Finally, my deepest gratitude goes to my Mum and Dad, and Seung Hyun for their enduring support and encouragement.
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Notation

The following is a list of notation used throughout the thesis, listed in order of first appearance with page number.

\[ W_t \quad \text{Standard Brownian motion} \quad 2 \]
\[ E \quad \text{Expectation} \quad 3 \]
\[ (x)_+ \quad \text{Max}(x,0) \quad 4 \]
\[ := \quad \text{Defining equality} \quad 19 \]
\[ \mathbb{R} \quad \text{The set of real numbers} \quad 20 \]
\[ \text{Im}(z) \quad \text{Imaginary part of } z \quad 20 \]
\[ \text{Re}(z) \quad \text{Real part of } z \quad 21 \]
\[ \Gamma(z) \quad \text{Gamma function} \quad 21 \]
\[ P \quad \text{Probability} \quad 28 \]
\[ \overset{d}{=} \quad \text{Equality in distribution} \quad 29 \]
\[ 1_A \quad \text{Indicator function of the event } A \quad 31 \]
\[ \mathbb{C} \quad \text{The set of complex numbers} \quad 41 \]
\[ \,_{0}F_{1}(a; z) \quad \text{Generalized hypergeometric function,} \]
\[ \quad \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{z^n}{n!} \quad 44 \]
(a)_n \quad \text{Rising factorial or shifted factorial,} \quad \frac{\Gamma(a+n)}{\Gamma(a)} \quad 44

\mathbb{N} \quad \text{The set of integer numbers} \quad 44

I_\nu(z) \quad \text{Modified Bessel function of the first kind of order } \nu, \quad \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\Gamma(n+\nu+1)} \quad 60

_{1}F_{1}(a, b; z) \quad \text{Confluent hypergeometric function,} \quad \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \quad 62

BES_\delta^x, BES_{x}^{(\nu)} \quad \text{Bessel processes of dimension } \delta, \text{ of index } \nu \text{ started at } x \quad 67

BESQ_\delta^x, BESQ_{x}^{(\nu)} \quad \text{Squares of Bessel processes} \quad 67

K \cdot M \quad \text{Stochastic integral of } K \text{ with respect to } M, \quad \int_{0}^{s} K_s dM_s \quad 73

\mathcal{E} \quad \text{Stochastic exponential or exponential martingale} \quad 73
Chapter 1

Introduction

In 1900, Louis Bachelier began the process of providing a fair price for the European call option by suggesting that asset prices follow a simple diffusion process. Since Bachelier’s first formal description of Brownian motion, the geometric Brownian motion was introduced in 1959 as a more refined market model for which prices cannot be negative. The volatility can be viewed as the diffusion coefficient of this random walk. Since then the Black-Scholes formula is widely used by traders because it is easy to use and understand. An important characteristic of the Black-Scholes model is the simple assumption that the volatility of the underlying asset being a constant.

However, practitioners have observed that particularly after the crash of 1987, the geometric Brownian motion and the Black-Scholes formula were unable to capture the option price in the markets. Further, it has been observed that the implied volatility is no longer constant but varies according to the ratio between the underlying price and strike price of the option. Since it is U-shape the curve is the so-called smile effect, in which options written on the same underlying asset usually trade, in Black-Scholes terms, with different implied volatilities. For example, deep in-the-money (ITM) and deep out-of-the-money (OTM) options are traded at higher implied volatility than at-the-money (ATM) options. There is also a time effect: options with longer maturities are traded at higher implied volatility than are shorter maturities in the FX market. These evidences are not consistent with
the constant volatility assumption made by Black and Scholes [18]. This is due to
the presence of fat tails, which are extreme values for the price that are more likely
in the real probability measure than in the log-normal model.

There are several ways to address this empirical issue. Merton [88] points out that
a jump-diffusion process for the underlying asset could cause such an effect. A more
commonly explored direction is the stochastic volatility model: two-dimensional
diffusion processes, one for the asset price and the second one governs the volatility
evolution. Among them, Hull and White [70] put forward a log-normal stochastic
volatility model, namely that the volatility of the underlying asset follows another
geometric Brownian motion. However, these models have a drawback since the
volatility is not the traded asset: the option price in the stochastic volatility context
actually depends on investors’ risk preferences, that is, the pricing formula is not
risk-neutral.

1.1 Arbitrage-free pricing

In this section, we briefly discuss how pricing assets using expectation are related to
the equivalent martingale measure. The terminology and approach of this section
follow the steps used in Bingham and Kiesel [16] and Glasserman [55]. A trading
strategy or portfolio is said to be self-financing if the portfolio involves only rebal-
ancing at no extra cost, which means no money-in or money-out from the portfolio
is allowed. A contingent claim is a future payoff without certainty but it depends
on the outcome of risky asset i.e., $\mathcal{F}_T$-measurable non-negative finite random vari-
able. The claim is said to be attainable if the random variable can be replicated by
self-financing strategy. Then, we call a model complete if all the derivatives in the
model are attainable. Now we face the important question of what a ‘fair’ value of
a contingent claim should be.

An arbitrage is an opportunity to make a profit without taking any risk. When
an arbitrage opportunity exists, investors take the opportunity and eventually it
becomes arbitrage-free or there is an absence of long-term arbitrage. That is, the
arbitrage opportunity does not last long in an efficient market. The key to the
no-arbitrage argument is that arbitrage-free is equivalent to the existence of an
equivalent martingale measure. This is known as the first fundamental theorem of
asset pricing.

In order to discuss when a model is arbitrage-free, we begin with a model in a
filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions where $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$
1.1. Arbitrage-free pricing

is a filtration on a standard Brownian motion $W$. Here, $\mathbb{P}$ denotes the real-world (or ‘physical’) measure. Given the asset price $S$, drift $\mu$ and volatility $\sigma$, which are all adapted to the filtration of $\mathcal{F}$, the dynamics of the underlying asset are described as

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$ 

In addition, we assume the existence of a risk-free bank account $B_t$ such that

$$dB_t = r B_t \, dt, \quad B_0 = 1,$$

where $r$ is the risk-free interest rate. The market price of risk is defined as the excess average return over the risk-free rate divided by the volatility of the asset. A sufficient condition for no-arbitrage is equality of market prices of risk for all assets.

A numéraire is an asset with a strictly positive value at all times. Asset prices may be denominated in terms of amounts of the numéraire. A martingale is a process with zero drift. Suppose we can construct a new measure (a so-called risk-neutral measure) such that all numéraire expressed asset prices become martingales. It can be shown that the assumption of the existence of such a measure automatically implies that the model is arbitrage-free. Moreover, if there exists a single unique risk-neutral measure, then it can be shown that the model is complete, that is, every derivative security is attainable by a replicating portfolio in the underlying assets.

For a detailed discussion on the existence of arbitrage-free economy, see Harrison and Kreps [66].

An equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an equivalent martingale measure if the discounted price process $Z_t = S_t/B_t$ is $\mathbb{Q}$-martingale. Given the assumption of equality of all market prices of risk, we may apply Girsanov’s theorem (see Rogers and Williams [102] and Steele [111]) to construct the risk-neutral measure. Under the risk-neutral measure, market prices of risk then turn out to disappear. Therefore, these do not affect arbitrage-free pricing. The martingale property of numéraire-relative asset prices implies that their future expectations take today’s value. If $S$ and $N$ denote prices of an asset and a numéraire, respectively, then for $t < T$

$$S(t) \over N(t) = \mathbb{E} \left[ \frac{S(T)}{N(T)} \mid \mathcal{F}_t \right],$$

where the expectation is with respect to the risk-neutral measure. Let $\theta$ be the price of a derivative. Then, we have

$$\theta(0) = N(0) \mathbb{E} \left[ \frac{\theta(T)}{N(T)} \right],$$
which is the fundamental arbitrage-free pricing formula. We note that the market price of risk does not occur in this formula. If we calculate the above formula for a call option on a stock that is modelled as a geometric Brownian motion, then we obtain the famous Black-Scholes formula. We refer to Björk [17] or Musiela and Rutkowski [90] for comprehensive discussion about arbitrage pricing.

1.2 The Black-Scholes model

In the early 1970s, Black and Scholes [18] made a major breakthrough in option pricing by deriving the partial differential equation (PDE). There are different ways to obtain the Black-Scholes formula, for instance introducing replicating portfolio to hedge an option. In this thesis, we focus on the equivalent martingale measure approach. The following argument is not new at all and can be found in the work of many researchers (see Shreve [109]). The purpose of this section is to provide an explanation about measure change with a simple and well-known model. Let \( Z_t = \frac{S_t}{B_t} \). By Itô’s formula, we have

\[
\begin{align*}
    dZ_t &= -re^{-rt}S_t dt + e^{-rt}dS_t \\
        &= -re^{-rt}S_t dt + e^{-rt}(\mu S_t dt + \sigma S_t dW_t) \\
        &= (\mu - r)Z_t dt + \sigma Z_t dW_t. \\
\end{align*}
\]

(1.1)

We want to find a new measure \( Q \) such that \( Z_t \) is a martingale under \( Q \). Define

\[
    dQ = \exp\left\{ \gamma W_t - \frac{1}{2} \gamma^2 t \right\} \cdot dP, \quad \text{on } F_t,
\]

where \( \gamma \) is to be decided so that \( Z_t \) is a martingale. By Girsanov theorem,

\[
    d\tilde{W}_t = dW_t - \gamma t
\]

(1.2)

is a \( Q \)-Brownian motion. Replacing \( W_t \) in (1.1) by (1.2) gives what is the \( \gamma \) looks like.

\[
\begin{align*}
    dZ_t &= (\mu - r)Z_t dt + \sigma Z_t d\tilde{W}_t + \gamma dt \\
        &= (\mu - r + \sigma \gamma)Z_t dt + \sigma Z_t d\tilde{W}_t.
\end{align*}
\]

For \( Z_t = S_t/e^{rt} \) to be a \( Q \)-martingale, we need

\[
    \gamma = \frac{r}{\sigma} - \frac{\mu}{\sigma},
\]

and finally we have

\[
    S_t = S_0 \exp\left\{ \left( r - \frac{1}{2} \sigma^2 \right) t + \sigma \tilde{W}_t \right\}.
\]
Next, the value of a plain vanilla call option at time 0 is given by

\[ C(0, S_0) = e^{-rT} \int (S_T - K)_+ dQ \]

\[ = e^{-rT} \int_{\{ S_T \geq K \}} S_T dQ - K e^{-rT} \int_{\{ S_T \geq K \}} dQ \]

\[ = I_1 - I_2, \]

where \((x)_+\) denotes \(\max(x, 0)\). Define a new measure \(Q^*\) by

\[ dQ^* = \frac{Z_t}{Z_0} \cdot dQ, \quad \text{on } \mathcal{F}_t. \]

Since \(Z_t = Z_0 \exp(-\sigma^2 t/2 + \sigma \tilde{W}_t)\), we conclude that \(dW_t^* = d\tilde{W}_t - \sigma t\) is a \(Q^*\)-Brownian motion by Girsanov theorem. Therefore, we have

\[ I_1 = Z_0 Q^*(S_T > K) \]

\[ = S_0 Q^*(Z_T > Ke^{-rT}) \]

\[ = S_0 Q^* \left( Z_0 \exp \left\{ -\frac{\sigma^2}{2} T + \sigma \tilde{W}_T \right\} > K e^{-rT} \right) \]

\[ = S_0 Q^* \left( \log \frac{S_0}{K} + \left( -\frac{\sigma^2}{2} T + \sigma W_T^* \right) > -rT \right) \]

\[ = S_0 Q^* \left( \log \frac{S_0}{K} + \left( \frac{\sigma^2}{2} T + \sigma W_T^* \right) > -rT \right) \]

\[ = S_0 Q^* \left( \sigma W_T^* > -\log \frac{S_0}{K} + \frac{\sigma^2}{2} T - rT \right) \]

\[ = S_0 Q^* \left( W_T^* > -\frac{\log \frac{S_0}{K} + \frac{\sigma^2}{2} T + rT}{\sigma \sqrt{T}} \right) \]

\[ = S_0 N \left( \frac{\log \frac{S_0}{K} + (r + \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} \right), \]

where

\[ N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \]

denotes cumulative standard normal distribution. Without any measure change, we
can calculate $I_2$ such as

$$I_2 = e^{-rT} K Q(S_T \geq K)$$

$$= e^{-rT} K Q(S_0 \exp\{(r - \frac{\sigma^2}{2})T + \sigma \tilde{W}_T\} > K)$$

$$= e^{-rT} K Q\left(\tilde{W}_1 > \log \frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T\right)$$

$$= e^{-rT} K Q\left(\tilde{W}_1 > \frac{\log \frac{K}{S_0} - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma \sqrt{T}}\right)$$

$$= e^{-rT} K N\left(\frac{\log \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma \sqrt{T - t}}\right).$$

Thus, we obtain the Black-Scholes formula for plain vanilla call option with no dividend

$$C(t, S_t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2), \quad (1.3)$$

where

$$d_1 = \frac{\log \frac{S_0}{K} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \quad \text{and} \quad d_2 = \frac{\log \frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma \sqrt{T - t}}.$$  

Either following the same argument or applying the put-call parity, the value of put option can be obtained

$$P(t, S_t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1). \quad (1.4)$$  

### 1.3 The biases of the Black-Scholes model

A financial derivative, for example, an option, is a contract between two parties based on some underlying asset, for instance, a stock. The simplest financial derivative is a European call option. A call option gives the holder the right, but not an obligation, to buy the underlying asset for a fixed price, the strike price, at a certain time, which is the expiration day. A European option can be priced by the Black-Scholes formula in which the risk is quantified by a constant volatility parameter. Note that the Black-Scholes formula has five parameters: the initial stock price $S_0$, the strike price $K$, the time to expiry $T$, the interest rate $r$ and the volatility $\sigma$. Among these five parameters, the volatility is the only parameter that is not known.

It has long been known that the geometric Brownian motion does not model stock prices perfectly, at least not in all cases; the conventional lore is that equity
returns are not normally distributed, but rather have a density with a higher peak
and heavier tails than the normal density (see Bakshi et al. [9] and Cont [32].) For
some purposes, this may not be a significant problem, but in derivatives pricing,
it has been noted that put or call prices do not conform to the standard constant-
volatility Black-Scholes model. In other words, not only does the physical measure
not agree with Black-Scholes, but also the risk-neutral (or pricing) measure does
not either. In particular, observed prices exhibit the now familiar smile of implied
volatilities against varying strike prices. The parabola shape with minimum around
ATM is a contradiction to the assumption of constant volatility in the Black-Scholes
model. To deal with this problem, there is one possible approach, going back to the
1980s, which consists of turning the volatility ‘σ’ of the Black-Scholes model into a
stochastic process (see Bakshi et al. [9].)

1.4 The structure of the thesis

In Chapter 2, we survey background information relevant to this thesis and at the
same time we illustrate several stochastic volatility models in more detail, including
the Hull-White, Stein-Stein and Heston models.

The original research of this thesis is contained in Chapters 3 to 5. Chapter 3
is concerned with the Fourier transform while Chapters 4 and 5 consider stochastic
volatility models. In Chapter 3, we study the Fourier transform approach in option
pricing. In particular, we consider the pricing of European derivatives with the
Laplace transform of the ordinary price of a put and a call in the Black-Scholes
model with respect to maturity. We derive these formula and then apply them to
the case where volatility is modelled as a continuous-time Markov chain (see Chapter
4) and a square-root process (see Chapter 5). Bounds for upper limit in Parseval’s
integrals are also considered while keeping an upper bound for the truncation error.
In this chapter, we consider the pricing of European derivatives in a Black-Scholes
model with stochastic volatility. The main ingredient in our method is the Laplace
transform of the ordinary (constant volatility) price of a put or call in the Black-
Scholes model, in which the transform is taken with respect to maturity $T$. We derive
these formulas and then apply them to the case in which volatility is modelled as
a continuous-time Markov chain (see Chapter 4) and the square-root process (see
Chapter 5). We show how Parseval’s theorem can be used to express option prices
as Fourier integrals. Then, we describe more generalised approaches, in particular
the derivation of the Laplace transform of the Black-Scholes prices for European
puts and calls, in terms of the maturity $T$.

In Chapter 4, we examine a continuous-time Markov chain, the so-called Markov regime switching model. While the concept of regime switching has been around for many years, these models are now gradually progressing from where the switching dates are known to the very general models now proposed. This model has been used previously in stochastic volatility modelling, but mostly with only $N = 2$ states. We show how to use $N = 3$ states without difficulty, and how a larger number of states can be handled. Numerical comparisons of European put prices computed using the Parseval formula described in Chapter 3 and Monte Carlo simulation when the stochastic volatility is a Markov chain with $N = 2$ or $N = 3$ states are given. For $N = 2$ states, we can compute the exact price using the density of integrated variance process, which is explicitly known. In addition, the implied volatility curve is shown in two and three-state Markov regime switching models to see whether the *smile* is observed in stochastic volatility models.

Using a change-of-measure technique, we study a relationship between the squared radial Ornstein-Uhlenbeck process and the square-root process in Chapter 5. This immediately gives a relationship between the squared Bessel process and the square-root process. Several auxiliary results are discussed that may also be of interest. We discuss some different European-style option pricing methods when the square-root process governs the volatility of the stock price. The numerical comparisons follow.
Chapter 2

Stochastic Volatility Models

In a stochastic volatility model, the volatility changes randomly according to stochastic processes. This additional random source helps to partially explain why options with different strikes and maturities have different implied volatilities, which have been observed in market prices. In 1987, Hull and White [70] use the geometric Brownian motion for the variance process. This method leads to the variance growing exponentially but it contrasts to the empirical results (see Cont [32] and Wiggins [114]) that claim that the variance tends to decay over time. Due to the shortcoming of the geometric Brownian motion in variance process, one may estimate fluctuation around a mean level, that is, a mean-reverting process for volatility instead of growing exponentially as geometric Brownian motion does. For example, the Ornstein-Uhlenbeck process is used in Scott [105] and Ball and Roma [12] to capture the mean-reverting behaviour. The mean-reverting Ornstein-Uhlenbeck process is more realistic choice over than geometric Brownian motion. However, the Ornstein-Uhlenbeck process can take negative values for the variance process which is obviously not the property we are looking for. In order to overcome the shortcomings of the Black-Scholes model described in the previous chapter and the negativity of the Ornstein-Uhlenbeck process, Stein and Stein [112] and Heston [69] adopt the absolute value of an Ornstein-Uhlenbeck process and the square-root process for stochastic volatility, respectively.

In contrast to the Black-Scholes model, it is complicated to obtain an analytic
solution for the price of a European option under the two dimensional diffusion processes: one for the asset price and another for the volatility of underlying asset. An approximate solution is obtained by expanding a power series, which was suggested by Hull and White [70]. This technique is based on the option’s underlying asset price distribution, conditional on the average value of the stochastic variance. According to Ball and Roma [12], the power series approximation is accurate and easy to implement in comparison to other approaches. In the remainder of this chapter, we describe three different stochastic volatility models that appear in the literature.

2.1 Reference models

2.1.1 Hull-White model

Hull and White [70] proposed a model of the form

\[
\begin{align*}
    dS_t &= \phi S_t \, dt + \sqrt{V_t} S_t \, dW_t \\
    dV_t &= \mu V_t \, dt + \xi V_t \, dZ_t,
\end{align*}
\]

where \( \langle dW, dZ \rangle_t = \rho dt \). The geometric Brownian motion is used for the variance process \( V_t \). When the volatility is uncorrelated with the underlying asset, that is, \( \rho = 0 \), the derivative price \( f \) can be written as the present value of the expectation of payoff at maturity with discounting in risk-free rate \( r \)

\[
f(S_t, V_t, t) = e^{-r(T-t)} \int f(S_T, V_T, T) \, p(S_T|S_t, V_t) \, dS_T,
\]

where \( p(S_T|S_t, V_t) \) is the conditional distribution of \( S_T \) given the underlying price and variance at time \( t \). For a European call option that has a payoff \( f(S_T, V_T, T) = (S_T - K)_+ \) with strike price \( K \geq 0 \), the property of conditional expectation makes (2.1) to rewrite as

\[
f(S_t, V_t, t) = \mathbb{E} \left[ e^{-r(T-t)} \mathbb{E} \left[ (S_T - K)_+ \right] \mid V \right]
\]

where

\[
V = \frac{1}{T-t} \int_t^T V_s \, ds.
\]

This integral of underlying asset’s volatility is so-called mean variance over the time to maturity. It can be interpreted as that the call option price is the expected value of the Black-Scholes call price over the distribution of the average volatility given the initial value \( V_0 \). When \( \mu = 0 \), the first few moments at \( t = 0 \) are given by (see
2.1. Reference models

Hull and White [70] p.287)

\[ \mathbb{E}V = V_0 \]
\[ \mathbb{E}V^2 = \frac{2(e^{2T} - \xi^2T - 1)}{\xi^2T^2} V_0^2 \]
\[ \mathbb{E}V^3 = \frac{e^{3\xi^2T} - 9e^{\xi^2T} + 6\xi^2T + 8}{3\xi^6T^3} V_0^3 \]
\[ \mathbb{E}V^4 = \frac{2(e^{6\xi^2T} - 10e^{3\xi^2T} + 54e^{\xi^2T} - 45 - 30\xi^2T)}{45\xi^8T^4} V_0^4. \]

Since there is no simple analytic expression for the conditional expectation as the Black-Scholes model represents, Hull and White [70] suggest an approximation to the call option price using a Taylor series:

\[ f(S_t, V_t) = C(\mathbb{E}V) + \frac{1}{2} \frac{\partial^2 C(\mathbb{E}V)}{\partial \mathbb{E}V^2} \text{Var}(\mathbb{V}) + \frac{1}{6} \frac{\partial^3 C(\mathbb{E}V)}{\partial \mathbb{E}V^3} \text{Skew}(\mathbb{V}) + \cdots, \]

where \( C \) is the Black-Scholes call price as a function of variance and \( \text{Var}(\mathbb{V}) \) and \( \text{Skew}(\mathbb{V}) \) are the second and third central moments of \( \mathbb{V} \). The formula is based on the moments of the distribution of the underlying asset conditional on \( \mathbb{V} \) the average of the stochastic variance. In particular, when \( \mu = 0 \), the approximation to the call option is

\[ f(S_t, V_t) \approx C(V_0) + \frac{S_t \sqrt{T - t} N'(d_1)}{8} \left[ \frac{2(e^k - k - 1)}{k^2} - 1 \right] \sqrt{V_0} \]
\[ + \frac{S_t \sqrt{T - t} N'(d_1) [(d_1d_2 - 3)(d_1d_2 - 1) - (d_1^2 + d_2^2)]}{48} \]
\[ \times \frac{e^{3k} - 9(2k + 1)e^k + 6k^3 + 18k^2 + 24k + 8}{3k^3} \sqrt{V_0} + \cdots, \]

where

\[ k = \xi^2(T - t), \quad d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2} \mathbb{V})T}{\sqrt{\mathbb{V}(T - t)}}, \quad d_2 = d_1 - \sqrt{\mathbb{V}(T - t)}, \]

and \( N \) denotes the cumulative probability distribution function for a standard normal. When \( \mu \neq 0 \), they use numerical methods. In addition, we refer to Wiggins [114] for a different treatment for the case \( \rho \neq 0 \).

2.1.2 Stein-Stein model

Stein and Stein [112] suggest that volatility has a mean reverting behaviour with dynamics

\[ dS_t = rS_t \, dt + \sigma_t S_t \, dW_t^{(1)} \]
\[ d\sigma_t = -\alpha(\sigma_t - \beta) \, dt + \gamma dW_t^{(2)}, \]
where $dW^{(1)}$ and $dW^{(2)}$ are two independent Brownian motions. A similar but the Ornstein-Uhlenbeck process for the logarithm of the volatility model is also suggested by Scott [105]. While Hull and White [70] use a power series expansion for an approximation, Stein and Stein [112] develop analytic formulas based on the Fourier transform. From independence, the price of a European call is given by

$$C(S_0, \sigma_0, K, T) = e^{-rT} \int_K^\infty (x - K) f_T(x) \, dx,$$

where $f_T(x)$ is Fourier inversion

$$\frac{1}{2\pi} \int e^{-ix\xi} I \left( \left( \xi^2 + i\xi \cdot \frac{t}{2} \right) \right) \, d\xi.$$

Here,

$$I(y) = \exp \left\{ \frac{1}{2} L \sigma_0^2 + M \sigma_0 + N \right\},$$

where

$$A = -\frac{\alpha}{\gamma^2}, \quad B = \frac{\alpha \beta}{\gamma^2}, \quad C = -\frac{y}{\gamma^2 t}, \quad a = \sqrt{A^2 - 2C}, \quad b = -\frac{A}{a}.$$

$$L = -A - a \left( \frac{\sinh(a \gamma^2 t) + b \cosh(a \gamma^2 t)}{\cosh(a \gamma^2 t) + b \sinh(a \gamma^2 t)} \right),$$

$$M = B \left( \frac{b \sinh(a \gamma^2 t) + b^2 \cosh(a \gamma^2 t) + 1 - b^2}{\cosh(a \gamma^2 t) + b \sinh(a \gamma^2 t)} - 1 \right),$$

$$N = \frac{a - A}{2a^2} \left( a^2 - AB^2 - B^2 a \right) \gamma^2 t + \frac{B^2 (A^2 - a^2)}{2a^3} \left( 2A + a \right) + \frac{(2A - a)}{A + a + (a - A) e^{2a \gamma^2 t}}$$

$$+ \frac{2AB^2 (a^2 - A^2) e^{a \gamma^2 t}}{a^3 (A + a + (a - A) e^{2a \gamma^2 t})} - \frac{1}{2} \log \left( \frac{1}{2} \left( \frac{A}{a} + 1 \right) \right) + \frac{1}{2} \left( 1 - \frac{A}{a} \right) e^{2a \gamma^2 t}.$$

Since the volatility is an Ornstein-Uhlenbeck process the volatility does not increase to infinity with time as Hull-White model does. However, Stein-Stein model is less tractable and more difficult to implement than the Hull-White model. Moreover, the main problem is that the volatility process can admit negative values. See Wong and Heyde [115] for a discussion of the arbitrage issue in the Stein-Stein model.

### 2.1.3 Heston model

Heston [69] suggests the stochastic volatility model, which substitutes the constant volatility parameter $\sigma$ with a stochastic process described by stochastic differential equations (SDEs)

$$dS_t = r S_t \, dt + \sqrt{V_t} S_t \, dW_t^{(1)}$$

$$dV_t = \kappa (\theta - V_t) \, dt + \sigma \sqrt{V_t} \, dW_t^{(2)}$$

where $dW^{(1)}$ and $dW^{(2)}$ are two independent Brownian motions.
with correlated Brownian motions \( d\langle W(1), W(2)\rangle_t = \rho \, dt \). The long run variance \( \theta \), the mean reversion rate \( \kappa \) and the volatility of the variance \( \sigma \) are all non-negative and \( |\rho| \leq 1 \). The volatility is related to a square-root process and can be interpreted as the radial distance from the origin of a multidimensional Ornstein-Uhlenbeck process. Heston [69] also develops an analytic formula based on Fourier inversion but more generally. The reason that the Heston model is more popular than others is that the volatility is always positive. Further, there is a tractable expression for European option prices that makes the calibration easy.

Numerous papers [2, 9, 20, 40, 41, 47, 50–52, 93] compared option pricing derived from this model and its extensions with empirical data on option pricing and found that the Heston model describes the empirical option prices much better than the Black-Scholes theory and that its extension even furthers the agreement. Dragulescu and Yakovenko [40] also showed that the Dow Jones actual distribution itself is close to the theoretical distribution resulting from the Heston model and analysed some of its important asymptotic properties, which turn out to be empirically verified by the index time-series. Bates [13] also suggests a model similar to the Heston model but encounters an additional jump process in the stock price. A compound Poisson process with intensity \( \lambda \) is used for the jump process and the jump size is independent of two Brownian motions, which drive stock price and volatility, say \( W_t(1) \) and \( W_t(2) \). See Bates [13] for more details. We will revisit the Heston model based on the Fourier inversion approach in Section 3.2.3.

### 2.2 Risk-neutral measure and no-arbitrage

Since the underlying asset, say stock \( S \), is tradable, the risk caused by the stock price can be removed by replicating the payoff function. However, it is not true for volatility \( V \) anymore and its risk cannot be eliminated. A stochastic volatility model introduces more random sources than traded assets. According to general market theory, the model is not complete since the number of random sources is greater than the number of underlying traded assets. Pricing in a market with stochastic volatility is thus an incomplete market problem, which means that a unique martingale measure does not exist, and the derivative cannot be perfectly hedged with only the underlying asset and a bank account.

This leads to a difficulty in applying no-arbitrage principles to option pricing, since the market, if it consists only of \( S \) and a risk-free bond \( B \), is then incomplete. There is then an infinite number of martingale (or risk-neutral) measures, and hedg-
Figure 2.1: A plot of implied volatility for different maturities and moneyness by equating European call prices obtained by Heston model to Black-Scholes formula. The parameters $\kappa = 11.35$, $\theta = 0.022$, $\sigma = 0.618$, $r = 0.143$ (shown in Dragulescu and Yakovenko [40]) along with zero-correlation and spot price of 100 are used.
ing of puts or calls is not possible with certainty. This problem may be solved in theory by assuming that volatility is traded in one way or another. For example, Bajeux-Besnainou and Rochet [8] discuss a way of completing market dynamically in the case of stochastic volatility. In particular, they claim that ‘classical European options typically become redundant with some probability’ in a discrete-time model (see Proposition 2.1) and ‘a European option is always a good instrument for completing markets’ in a continuous-time model under reasonable assumptions (see Proposition 5.1.) A year later, Romano and Touzi [103] extend these results to the case that the underlying asset and its volatility are correlated.

As our contribution lies elsewhere we will not study this problem in detail. We refer the readers to Harrison and Kreps [66], Harrison and Pliska [67] and Pham and Touzi [96] for a detailed discussion of no-arbitrage issues. Now, we will assume that

$$dS_t = rS_t \, dt + V_t S_t \, dW_t,$$

where $r$ is the risk-free rate and $W$ is a standard Brownian motion under the pricing measure. In the remainder of thesis, we use the notation $\mathbb{P}$ for a physical or real-world measure and $\mathbb{Q}$ for a risk-neutral or pricing measure.
Chapter 3

The Fourier Transform in Option Pricing

Since not every model is analytically tractable, we sometimes use numerical techniques such as the Monte Carlo simulation or finite difference methods. However, these are more time-consuming than analytical computation in general. Recently, much literature on option pricing [25, 45, 78, 80, 81, 84, 100] has applied the Fourier transform method to handle complex models. For instance, Lewis [79] derived generalised Fourier inversion formulas for options under stochastic volatility. Dufresne et al. [45] also give general Fourier inversion formulas for expectations such as $E(X - K)_+$ with weaker restriction than Lewis [80]. In particular, both use Parseval’s theorem to compute the Fourier integral. Carr and Madan [25] developed simple analytic formulas for call options in terms of the characteristic function of the logarithm of the price at maturity. Both Carr and Madan [25] and Lewis [80] focus on the Fourier transformation of the payoff function, whereas the former apply the transformation to the strike price and the latter transform with respect to the state variable. What is surprising about the approach used by Heston [69] is that the Fourier transform method is independent of the underlying process and can be applied if the particular characteristic function is known. These methods were generalised by Bakshi and Madan [10] to evaluate the characteristic function itself as a derivative contract with a trigonometric payoff. In derivatives pricing, Duffie
et al. [41] present a way in which the Fourier inversion can be used to solve derivative prices for general stochastic processes and Bakshi and Madan [10] show how the Fourier transform Arrow-Debreu securities can be used to span the underlying market and to price derivative.

We can find a technique of pricing option using the Fourier transform in Carr and Madan [25] and Dufresne et al. [45]. However, this method can often face the problem of integrability. Carr and Madan [25] and Dufresne et al. [45] use an extra exponential term to make the Fourier transform of the European options payoff integrable. As an alternative to the exponential damping factor, Dufresne et al. [45] suggest a polynomial damping factor. Once we adapt an appropriate damping factor, Parseval’s theorem can be applied. It is easier to compute the option price in the Fourier space than to integrate the product of the payoff function and its density. This is because the Fourier transform of the density is often easier to manage compared to the density itself. In addition, Dufresne et al. [45] bring up the possibility of expressing the expected payoffs of European options in terms of special functions such as Bessel or hypergeometric functions, which are used commonly in actuarial science or finance.

The remainder of this chapter is divided as follows. After a quick overview of pricing options in Section 3.1, we introduce three different Fourier transform pricing approaches in Section 3.2. Section 3.3 studies the computation of the Fourier integral by applying Parseval’s theorem. In Section 3.4, we present our main result, which is a representation for the price of a call and put option in the Black-Scholes model using Fourier transform methods, in which the transform is taken with respect to time. Then, using this, we derive a formula for the price of call and put options in models in which the volatility is constant. In Section 3.5, we show how to truncate the Fourier integral for option prices over an infinite interval while keeping an upper bound on the error. Results in Section 3.4 and 3.5 are heavily borrowed from Chin and Dufresne [27].

3.1 Overview

It is inevitable that the issues of completeness of the market and no-arbitrage option pricing arise when pricing an option. To begin with the stock price $S_t$ is assumed to satisfy

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$
where $W$ is a standard Brownian motion. However, this leads to difficulty in applying
no-arbitrage principles (see Harrison and Pliska [67]) to option pricing, since the
market would be incomplete if it consists only of $S$ and a risk-free asset such as
bond. There is then an infinite number of martingales (also known as risk-neutral
or pricing) measures, and hedging of puts or calls is not possible with certainty.
This problem may be solved in theory by assuming that volatility is traded in one
way or another. Under the equivalent martingale measure, every discounted price
should be a martingale. The risk-neutral valuation is the pricing of a contingent
claim under the equivalent martingale measure and the price may be expressed as
the expectation of the payoff.

The no-arbitrage price of a European derivative is

$$Eg(X) = \int g(x) \, d\mu_X(x),$$

(3.1)

for some function $g(\cdot)$ and $\mu_X$ is the distribution of random variable $X$. In many
financial applications, $\mu_X$ itself is either unknown or difficult to find analytically,
whereas its Fourier transform or characteristic function

$$\hat{\mu}_X(u) := E e^{iuX} = \int_{-\infty}^{\infty} e^{iuX} \, d\mu_X(x)$$

is relatively easy to find. Since there is one-to-one mapping between distribution
function and characteristic function, we can determine the distribution function from
its characteristic function uniquely by the inversion formula

$$\mu_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux} E e^{iuX} \, du.$$  

If the conditional probability density function of underlying asset at maturity is
known then one may price the path-independent option by computing a single inte-
gral. For example, a log-terminal stock price $x = \log S_T$ in the Black-Scholes model
has a normal random variable and its density under $Q$ is given by

$$\frac{1}{\sqrt{2\pi}\sigma^2T} \exp \left\{ - \frac{\left[ x - \left( x_0 + \left( r + \frac{\sigma^2}{2} \right) T \right) \right]^2}{2\sigma^2T} \right\},$$

where $x_0 = \log S_0$, risk-free interest rate $r$ and volatility $\sigma$. Thus, the characteristic
function of Black-Scholes log-terminal stock price $x$ is easily obtained as

$$\phi(u) = \exp \left\{ i \left[ x_0 + \left( r - \frac{\sigma^2}{2} \right) T \right] u - \frac{\sigma^2T}{2} u^2 \right\}.$$
The Fourier Transform in Option Pricing

### Payoff $g(x)$, Fourier transform $\hat{g}(u)$, Condition

<table>
<thead>
<tr>
<th>Payoff $g(x)$</th>
<th>Fourier transform $\hat{g}(u)$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(e^x - K)_+$</td>
<td>$-\frac{K^{1+iu}}{u^2 - iu}$</td>
<td>$\text{Im}(u) &gt; 1$</td>
</tr>
<tr>
<td>$(K - e^x)_+$</td>
<td>$-\frac{K^{1+iu}}{u^2 - iu}$</td>
<td>$\text{Im}(u) &lt; 0$</td>
</tr>
<tr>
<td>$\min(e^x, K)$</td>
<td>$\frac{K^{1+iu}}{u^2 - iu}$</td>
<td>$0 &lt; \text{Im}(u) &lt; 1$</td>
</tr>
</tbody>
</table>

Table 3.1: Fourier transforms of different type of options with respect to the log of underlying asset $x = \log S$.

### 3.2 Fourier transform approaches

#### 3.2.1 Lewis approach

Lewis [80] uses a more general setup that does not require existence of a strike in a payoff but does price derivatives starting with a Fourier transform of the payoff function with respect to the log of underlying asset. This is in contrast to the way in which Carr and Madan [25] take the Fourier transform with respect to the logarithm of strike price of a European option. A similar approach can be found in Raible [100], who takes the Fourier transform with respect to the log-forward price, whereas Lewis uses the log-spot price. Both Attari [7] and Lewis [79] give an equivalent result but the latter uses log-price and log-strike. In addition, similar formulas can be found in either Lipton [81], which are expressed in terms of the Fourier integral for various models, or Raible [100], who takes the transform with respect to the log-forward price.

As described in Section 3.1, the Fourier transformed payoff function is

$$ \hat{g}(u) := \int_{-\infty}^{\infty} e^{iux} g(x) \, dx, \quad u \in \mathbb{R}. \quad (3.2) $$

For instance, the transformed payoff function of a European call option $g(x) = (x - K)_+$ with respect to $x$ is

$$ \int_{-\infty}^{\infty} e^{iux}(x - K)_+ \, dx = -\frac{e^{iuK}}{u^2}, \quad K \geq 0 $$

with $\text{Im}(u) > 0$. Table 3.1 and 3.2 summarise the Fourier transform of various types of derivatives. (See Lewis [80] and Raible [100].)

#### 3.2.2 Carr-Madan approach: Fast Fourier transform

Carr and Madan [25] present the Fourier transform of the damped European call
Table 3.2: Fourier transforms of different type of options with respect to the underlying asset $S$.

<table>
<thead>
<tr>
<th>Payoff $g(x)$</th>
<th>Fourier transform $\hat{g}(u)$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S_T - K)_+$</td>
<td>$\frac{1}{u^2 + u}$</td>
<td>$\text{Re}(u) &lt; -1$</td>
</tr>
<tr>
<td>$(K - S_T)_+$</td>
<td>$\frac{1}{u^2 + u}$</td>
<td>$\text{Re}(u) &gt; 0$</td>
</tr>
<tr>
<td>$[(S_T - K)_+]^n$</td>
<td>$\frac{\Gamma(-u-n)\Gamma(n+1)}{\Gamma(-u+1)}$</td>
<td>$\text{Re}(u) &lt; -n$</td>
</tr>
<tr>
<td>$[(K - S_T)_+]^n$</td>
<td>$\frac{\Gamma(u)\Gamma(n+1)}{\Gamma(u+n+1)}$</td>
<td>$\text{Re}(u) &gt; 0$</td>
</tr>
</tbody>
</table>

They use fast Fourier transform (FFT) to obtain the value of a call option. Similar formulas for the transforms of option prices with respect to log-strike can be found in Lee [78] as well. The FFT technique has been also used in Chen and Scott [26] (for CIR model of the term structure) and Scott [106] (for jump-diffusion process). See Benhamou [14] for a discussion on the FFT approach in discrete Asian option pricing.

Denote $X$ the log of the terminal stock price and $k$ to be the log of the strike. The price of a European call with maturity $T$ and strike $K$ may be written

$$C(T,k) = e^{-rT} \int_k^\infty g(x) d\mu_X(x),$$

where $g(x) = (e^x - e^k)_+$. To satisfy the square-integrability of the function $C(T,k)$, Carr and Madan [25] consider an exponential damping factor, $e^{\alpha k}$, so that the damped call option price $e^{\alpha k}C(T,k)$ is integrable for $\alpha > 0$. They claim it is a good choice to use one fourth of the upper bound on $\alpha$ while Raible [100] suggests using an $\alpha$ equal to 0.75. In theory, it does not matter what value is chosen for $\alpha$ since it is a constant. However, it is important to choose the optimal value for $\alpha$ from a practical point of view. This is because an oscillation that depends upon time to maturity or exercise price may cause an extra difficulty in numerical integration, in particular for short maturities and deep OTM. We refer the readers to Lord and Kahl [84] and references therein for the issue of choosing optimal value of $\alpha$ for the exponential damping factor.
The Fourier transform of $e^{\alpha k}C(T,k)$ is

$$\psi(u) = \int_{-\infty}^{\infty} e^{iku} e^{\alpha k}C(T,k) \, dk,$$

and interchanging the order of integration in (3.3) gives

$$\psi(u) = \int_{-\infty}^{\infty} e^{iku} \int_{-\infty}^{\infty} e^{\alpha k} e^{-rT(e^x - e^k)} \mu_X(x) \, dk,$$

(3.3)

where $\phi(u) = \mathbb{E}e^{iuX}$ is the characteristic function of the log-price $X$. When $\phi(u)$ is known, the vanilla option price can be obtained using Fourier inversion formula

$$e^{-\alpha k} \frac{1}{\pi} \int_{0}^{\infty} e^{-iku} \psi(u) \, du.$$

(3.4)

Note that the integral in (3.4) can be approximated numerically as

$$\int_{0}^{\infty} e^{-iku} \psi(u) \, du \approx \sum_{j=0}^{n-1} e^{-iu_j k} \psi(u_j) h,$$

where $n$ is the number of subintervals, $h$ is the distance between the points of the integration grid and $u_j = jh$. In order to make use of the FFT method, the points $u_j$ have to be equally spaced. However, equidistant points $u_j$ makes numerical integration difficult. We refer to Chourdakis [31] for a fractional FFT scheme that accelerates the FFT pricing method.

The goal of Carr-Madan approach is to generate a valuation procedure that can include the FFT, an efficient device for computing the Fourier transform for different values of the underlying random variable. It must be noted though that they first carry out a Fourier transform on the payoff function with respect to the strike $K$. Clearly, an initial advantage of this method is for the purpose of computation, only one inverse transformation is required, seeing as we deal with just one transform operation on the option price.

However, difficulties with the Carr-Madan approach mostly occur if a Fourier transform on the payoff function with a real-valued $z \in \mathbb{R}$ is used. As Bakshi and Madan [10] acknowledge, because of the unbounded option payoff functions, the transform payoff accommodates an additional degree of freedom compared to the Heston model, as we are no longer restricted to the case of real-valued transformation variable $z$. This is worth noting in a numerical scheme for the computation of derivative prices.
3.2.3 Heston approach

As discussed in Section 2.1.3, Heston \[69\] models the variance as a square-root process that is correlated with the stock price. He shows that a Fourier inversion formula may be used to price European puts and calls. Recall two SDEs in (2.2) and (2.3). Let $C(S_0, K, V_0, t, T)$ denote the price of a European call option at time $t \in [0, T]$. Applying Itô’s lemma and standard no-arbitrage arguments, $C$ satisfies the PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2} S^2 V \frac{\partial^2 C}{\partial S^2} + r S \frac{\partial C}{\partial S} - r C$$

$$+ \left( \kappa (\theta - V) - \lambda V \right) \frac{\partial C}{\partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2} + \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} = 0,$$

(3.5)

where $\lambda$ is the market price of volatility risk. The dividend yield may be encountered in (3.5). For simplicity, we assume that there is no dividend payment in the remainder of this thesis.

Heston \[69\] suggests the solution to the PDE (3.5) in the form of Black-Scholes formula

$$C(S_t, K, V_t, t, T) = S_t P_1 - K e^{-r(T-t)} P_2,$$

(3.6)

where $P_1$ is the delta of the option and $P_2$ is the risk-neutral probability that the asset price will be greater than $K$ at maturity, which are defined in terms of the inverse Fourier transform

$$P_1(x, V_t, T, k) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{e^{-iuk f_1(x, V_t, T, u)}}{iu} \right\} du,$n

$$P_2(x, V_t, T, k) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left\{ \frac{e^{-iuk f_2(x, V_t, T, u)}}{iu} \right\} du,$$

where $x = \log S_t$, $k = \log K$ and $f_1$ and $f_2$ are defined as (3.7) below. Whereas Heston \[69\] characterises these probabilities ($P_1$ and $P_2$) using the characteristic functions indirectly, Bakshi and Chen \[11\] and Bates \[13\] solve for the delta and for the risk-neutral probability that the call option is ITM. If the characteristic function of the density $f$ is known analytically, we first derive an analytic expression for the Fourier transform of the option price and then compute the price by inversion. Since $P_j$, $j = 1, 2$ must satisfy the PDE

$$\frac{1}{2} v \frac{\partial^2 P_j}{\partial x^2} + \rho \sigma v \frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 P_j}{\partial v^2} + (r + w_j v) \frac{\partial P_j}{\partial x} + (\alpha - \beta_j v) \frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0,$$

the characteristic functions $f_1$ and $f_2$ have the form

$$f_j(x, V_t, \tau, u) = \exp \{ C_j(\tau, u) + D_j(\tau, u) V_t + iux \}, \quad j = 1, 2,$$

(3.7)
where

\[ \tau = T - t \]

\[
C(\tau; u) = rui\tau + \frac{\alpha}{\sigma^2} \left[ (\beta_j - \rho \sigma u i + d) \tau - 2 \log \left( \frac{1 - ge^{d\tau}}{1 - g} \right) \right]
\]

\[
D(\tau; u) = \frac{\beta_j - \rho \sigma u i + d}{\sigma^2} \left( \frac{1 - e^{d\tau}}{1 - ge^{d\tau}} \right)
\]

\[
g = \frac{\beta_j - \rho \sigma u i + d}{\beta_j - \rho \sigma u i - d}
\]

\[
d = \sqrt{(\rho \sigma u i - \beta_j)^2 - \sigma^2(2w_j u i - u^2)}
\]

\[
w_1 = 0.5, \quad w_2 = -0.5, \quad \alpha = \kappa \theta
\]

\[
\beta_1 = \kappa + \lambda - \rho \sigma, \quad \beta_2 = \kappa + \lambda.
\]

We refer to Gatheral [54] and in’t Hout et al. [71] for more details on the PDE method and Bakshi and Madan [10] for an alternative approach to develop the formulas using the characteristic functions.

A useful feature of the Heston model is that it gives a solution for the European vanilla option. The benefit makes computation simpler and easier than other stochastic volatility models. The previously mentioned features of the Heston model allow it to generate a different shape of distributions for stochastic volatility. As a result, the Heston model is robust and a good alternative to the Black-Scholes model.

However, there are at least two shortcomings of the Heston model. First, the calculation of the particular function \( P_j(x, t) \), \( j = 1, 2 \) depends on the type of payoff. In other words, it leads to different valuation formulas for each type of contract and causes difficulties in the computation of option price. Second, the integrand has a singularity at zero and the Fourier integral involves a complex logarithm. Recently, Kahl and Jäckel [74] and Fahrner [48] suggested a way of controlling those problems using the principal branch tracking in the characteristic function, sometimes called branch-cut. However, the programming complexity and computational cost remain problems. A more recently developed method of calculating option prices in the Heston model is to use the Lewis formula (see Lewis [80] Eq.(3.11) and an equivalent formula can also be found in Lipton [81] Eq.(6)) without branch-cut of the complex logarithm so that there is no singularity on the real axis. Note that \( P_1 \) and \( P_2 \) may be simplified further to

\[
\frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\text{Im}(f_j(u)) \cos(uk) - \text{Re}(f_j(u)) \sin(uk)}{u} \, du,
\]
3.3. Option prices as Fourier integrals

and hence the value of the European call is given by

\[
S_0 \left[ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\text{Im}(f_1(u)) \cos(uk) - \text{Re}(f_1(u)) \sin(uk)}{u} \, du \right]
- Ke^{-rT} \left[ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\text{Im}(f_2(u)) \cos(uk) - \text{Re}(f_2(u)) \sin(uk)}{u} \, du \right].
\] (3.8)

In other words, we need to compute a European call option using two one-dimensional integrals using numerical techniques. Attari [7] provides the formula similar to (3.8) but a simpler form:

\[
S_0 - Ke^{-rT} \times \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\text{Re}(f_2(u)) + \frac{\text{Im}(f_2(u))}{u}}{(1 + u^2)} \, du \right)
\] (3.9)

which only involves a single integral instead of two and the consequence is that it is computationally more efficient. In addition, the integrand in (3.9) has a quadratic term \(u^2\) in the denominator, which speeds up the rate of decay to zero instead of linear decay.

For the numerical integration for the functions \(P_1\) and \(P_2\), we may use the \texttt{NIntegrate} command (with \texttt{MaxRecursion} and \texttt{MaxErrorIncrease} options) in Mathematica, which appears to give very good results. In MATLAB, either \texttt{quad} for Simpson’s rule or \texttt{quadl} for adaptive Gauss Lobatto quadrature rule can be used for approximation as well.

3.3 Option prices as Fourier integrals

As we have seen in the previous section, several authors have proposed various Fourier integrals for prices of European options. In this section, we study how the Parseval theorem can be used in the Fourier transform and present general pricing formulas for the European-style vanilla option, which is discussed in Dufresne et al. [45].

3.3.1 Fourier-type formula

As described briefly in Section 3.2.1, Lewis [80] used Parseval’s theorem to find option pricing formulas in terms of the characteristic functions of logarithm of the underlying, \(E(e^X - K)_+\) given \(Ee^{iuX}\) is known. Similar work, Fourier inversion formulas for expectation \(E(X - K)_+\), is done by Dufresne et al. [45]. We may
compute the option prices either: (a) finding the distribution of $X$, or (b) integrating $(e^X - K)_+$ or $(X - K)_+$.

**Theorem 3.1.** Let $f, g \in L^1$. Then,

$$\hat{f}(u)\hat{g}(u) = \hat{h}(u),$$

where $\hat{f}$ denotes Fourier transform of $f$

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{ixu} f(x) \, dx,$$

and the convolution $h$ is defined

$$h(x) = \int_{-\infty}^{\infty} f(y)g(x-y) \, dy.$$ 

That is, any option pricing could be seen as a convolution of the payoff function and the underlying process. All we need is the Fourier transform of payoff and the density of the underlying process because the Fourier transform of convolution is the product of each Fourier transform. One way to compute $E g(X)$ is then to use Parseval’s theorem.

**Theorem 3.2** (Dufresne et al. [45] Appendix 1). If $g, \hat{g} \in L^1$ and $\mu$ is a signed measure such that $|\mu(\mathbb{R})| < \infty$, then

$$\int g(x) \mu(dx) = \frac{1}{2\pi}PV \int_{-\infty}^{\infty} \hat{g}(-u)\hat{\mu}(u) \, du,$$

where $\hat{g}(u)$ is the Fourier transform of $g$ and ‘$PV$’ is a principal value integral, which says that

$$PV \int_{-\infty}^{\infty} h(u) \, du = \lim_{M \to \infty} \int_{-M}^{M} h(u) \, du.$$ 

There are conditions for Parseval’s theorem to hold, among them that the function $g$ is Lebesgue integrable. Unfortunately, Parseval’s theorem cannot be applied directly to the pricing of puts and calls because the functions met in option pricing, for instance $x \mapsto (e^x - K)_+$, do not satisfy the integrability condition. Nevertheless, it is often possible to rewrite the problem in such a way that Parseval’s theorem may be applied, by introducing a damping factor, which in effect will replace $g$ with an integrable function. The idea is simple: rewrite the expectation as

$$\int g(x) d\mu_X(x) = \int e^{-\alpha x} g(x) e^{\alpha x} d\mu_X(x),$$

and apply Parseval’s theorem to $g^{(-\alpha)}(x) := e^{-\alpha x} g(x)$ and $d\mu^{(\alpha)}(x) := e^{\alpha x} d\mu(x)$ has finite mass. In many cases it is possible to find $\alpha$ such that $g^{(-\alpha)}(x)$ is Lebesgue
integrable and \( \mu^{(\alpha)} \) has finite mass. Therefore, its Fourier transform is well-defined. Lewis [80] assumes that \( X \) has a density. Dufresne et al. [45] suggest slightly modified results that involve a general probability distribution \( \mu_X \).

Theorem 3.3 (Dufresne et al. [45] Theorem 1). Let \( X \) be a random variable. For a measure \( \mu \), define

\[
g^r(x) = r(x)g(x) \quad \text{and} \quad d\mu^r_X(x) = \frac{1}{r(x)}d\mu_X(x).
\]

Assume that

(i) \( |\mu^r_X| < \infty \),

(ii) \( g^r \in L^1 \),

(iii) the function \( G(y) = \int g^r(x - y)d\mu^r_X(x) \) is continuous at the origin and \( g^r \) has bounded variation over \( \mathbb{R} \).

Then,

\[
E g(X) = \frac{1}{2\pi} PV \int \hat{g}^r(-u)\hat{\mu}^r_X(u)du. \tag{3.10}
\]

If an exponential damping factor \( r(x) = e^{-\alpha x} \) is used, then (3.10) reads

\[
E g(X) = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}^{(-\alpha)}(-u)\hat{\mu}^{(\alpha)}_X(u)du.
\]

Theorem 3.4 (Dufresne et al. [45]). (a) If there exists \( \alpha > 0 \) such that \( Ee^{\alpha X} < \infty \), then \( E(X - K)_+ < \infty \) for \( K \in \mathbb{R} \) and

\[
E(X - K)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}_3(-u + i\alpha)\hat{\mu}_X(u - i\alpha)du,
\]

where

\[
\hat{g}_3(z) = -\frac{e^{izK}}{z^2}.
\]

(b) Let \( X \geq 0 \). For any \( \alpha \in \mathbb{R} \) such that \( Ee^{\alpha X} < \infty \) (including \( \alpha = 0 \)) and \( K \geq 0 \),

\[
E(K - X)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}_4(-u + i\alpha)\hat{\mu}_X(u - i\alpha)du,
\]

where

\[
\hat{g}_4(z) = \frac{1}{z^2}(1 + izK - e^{izK}).
\]

Proof. See Dufresne et al. [45] Theorem 5. \( \square \)
While Lewis [80] discusses the way of avoiding the integrability problem using exponential damping factor, Dufresne et al. [45] suggest a polynomial damping factor $r(x)$. For example,

$$g^{[-\beta]}(x) = (1 + cx)^{-\beta}g(x), \quad d\mu^{[\beta]}_X(x) = (1 + cx)^\beta d\mu_X(x),$$

where $\beta \in \{1, 2, \ldots \}$ and $c > 0$.

**Theorem 3.5** (Dufresne et al. [45] Theorem 6). Let $K \geq 0$ and $c > 0$.

(a) If $\beta \in \{2, 3, \ldots \}$, $EX^\beta < \infty$, then

$$E(X - K)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}_3^{[-\beta]}(u) \left[ \sum_{k=0}^{\beta} \beta C_k \beta_k(-ci)^k \frac{\partial^k}{\partial u^k} \hat{\mu}_X(x) \right] du,$$

where

$$\hat{g}_3^{[-\beta]}(u) = \frac{e^{iuK}}{c^2(1 + cK)^{\beta - 2}} \Psi \left( 2, 3 - \beta; -\frac{i(1 + cK)}{c}u \right)$$

and

$$\Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} \frac{e^{-zt}t^{\alpha-1}}{(1+t)^{\alpha-\gamma+1}} dt, \quad \text{Re}(\alpha) > 0,$$

is the confluent hypergeometric function of the second kind.

(b) If $\beta \in \{0, 1, 2, 3, \ldots \}$, $X \geq 0$ and $EX^\beta < \infty$, then

$$E(K - X)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}_4^{[-\beta]}(u) \left[ \sum_{k=0}^{\beta} \beta C_k \beta_k(-ci)^k \frac{\partial^k}{\partial u^k} \hat{\mu}_X(x) \right] du,$$

where

$$\hat{g}_4^{[-\beta]}(u) = \hat{g}_3^{[-\beta]}(u) + \left( \frac{K}{c} + \frac{1}{c^2} \right) \Psi \left( 1, 2 - \beta; -\frac{i}{c}u \right) - \frac{1}{c^2} \Psi \left( 1, 3 - \beta; -\frac{i}{c}u \right).$$

### 3.3.2 Mellin-type formula

Since the option pricing models are sometimes expressed in terms of the logarithm of the stock price rather than $S$ itself, it is worthwhile to give the Fourier transform of log $S$. This explains the particular form of the formulas in Carr and Madan [25] and Lewis [80]. When the Fourier transform of log $S$ is known (and thus appears in the inversion formula), $E(S - K)_+$ can be expressed in terms of

$$Ee^{iu\log S} = ES^{iu}, \quad (3.11)$$

and (3.11) is sometimes referred to as a Mellin-type formula because it is the Mellin transform.
Theorem 3.6 (Dufresne et al. [45] Theorem 4). Let \( S \geq 0, K > 0 \) and

\[
h(u) = \frac{K^{-iu+1}}{iu(iu - 1)} \mathbb{E}(S^{iu}).
\]

If \( ES < \infty \), then

\[
\mathbb{E}(S - K)_{+} = ES - \frac{K}{2} (1 + \mathbb{P}(S = 0)) + \frac{1}{\pi} \int_{0}^{\infty} \text{Re}[h(u)] \, du. \tag{3.12}
\]

In all cases,

\[
\mathbb{E}(K - S)_{+} = \frac{K}{2} (1 + \mathbb{P}(S = 0)) + \frac{1}{\pi} \int_{0}^{\infty} \text{Re}[h(u)] \, du. \tag{3.13}
\]

Proof. See Dufresne et al. [45] Appendix 2. \( \square \)

As the distribution of \( S \) is not required to be determined ahead of computing the option price, a Mellin-type formula has a computational advantage. In particular, computing (3.12) or (3.13), which involves a single integral, is much faster than finding the distribution of \( S \) itself and then computing the expected payoff. Further, it is quicker than the Monte Carlo simulation as well. See Section 5.5 for numerical comparisons.

3.4 Fourier inversion formula in a stochastic volatility model

Since we will not study no-arbitrage principles in this thesis, we will assume that the underlying asset \( S \) satisfies the SDE

\[
dS_t = rS_t \, dt + V_t S_t \, dW_t, \tag{3.14}
\]

where \( r \) is the risk-free rate and \( W \) is a standard Brownian motion under the pricing measure \( Q \). We assume that the stochastic process \( \{V_t\} \) is independent of \( W_t \). Since the probability distribution of \( S_t \) is usually complicated or unknown, the computation of European option prices cannot be done by a simple integration with respect to the distribution of \( S_t \). Therefore, we suggest an alternative method of computing the European vanilla option using the Laplace transform of a put or call in the Black-Scholes setting with respect to maturity \( T \).

Let \( \mathcal{S}_t = e^{-rt}S_t \). Then we have

\[
d\mathcal{S}_t = -rS_te^{-rt} \, dt + e^{-rt}(rS_t \, dt + V_t S_t \, dW_t)
\]

\[= V_t S_t \, dW_t,\]
and further
\[ d \log S_t = V_t dW_t - \frac{1}{2} V_t^2 dt. \]
That is,
\[ \log S_t = \log S_0 + \int_0^t V_s dW_s - \frac{1}{2} \int_0^t V_s^2 ds. \]
Letting
\[ U_t = \int_0^t V_s^2 ds \quad (3.15) \]
gives
\[ S_t = S_0 \exp \left\{ rt - U_t + \int_0^t V_s dW_s \right\}. \]
Given \( V \), we have
\[ \int_0^t V_s dW_s \sim N \left( 0, \int_0^t V_s^2 ds \right). \]
The independence assumption between \( V \) and \( W \) leads us to write
\[ \int_0^t V_s dW_s \xrightarrow{d} \sqrt{U_t} Z, \]
where \( Z \sim N(0,1) \), and further the discounted payoff of European put option as a conditional expectation with respect to the variance process \( V \)
\[ e^{-rT} E[(K - S_T)_+] = e^{-rT} E\left[ E[(K - S_T)_+ | V] \right] \]
\[ = e^{-rT} E\left[ K - S_0 \exp \left\{ rT - \frac{1}{2} U_T + \sqrt{U_T} Z \right\} \right]_+ \]
\[ = E g(U_T), \]
where
\[ g(u) = E \left( Ke^{-rT} - S_0 \exp \left\{ - \frac{u}{2} + \sqrt{u} Z \right\} \right)_+. \]
Here, the function \( g \) will immediately be recognised as the price of a European put in the Black-Scholes model. This leads us to the problem of finding the simplified expression for the Laplace transform, in the time variable, of the price of a European call or put in the Black-Scholes model.

**Theorem 3.7** (Chin and Dufresne [27] Theorem 2.1). Suppose \( r \in \mathbb{R}, \sigma, K, S_0 > 0 \), and let
\[ \beta = \frac{\gamma + r}{\sigma^2}, \quad \rho = \frac{r}{\sigma^2} - \frac{1}{2}, \quad K = \frac{K}{S_0} \]
\[ \mu_1 = \rho + \sqrt{\rho^2 + 2\beta}, \quad \mu_2 = -\rho + \sqrt{\rho^2 + 2\beta}. \]
3.4. Fourier inversion formula in a stochastic volatility model

(a) If $\gamma > -r$, then

$$\int_0^\infty e^{-\gamma t}e^{-rt}E(K - S_0e^{(r-\frac{\sigma^2}{2})t+\sigma W_t})_+ dt$$

$$= \begin{cases} 
\frac{S_0}{\sigma^2\sqrt{\rho^2 + 2\beta\mu_1(1 + \mu_1)}}K^{1+\mu_1} & \text{if } K \leq S_0 \\
\frac{S_0}{\beta\sigma^2}\left[K - \frac{2\beta}{2\beta - 2\rho - 1} + \frac{2\beta K^{1-\mu_2}}{\sqrt{\rho^2 + 2\beta\mu_2(\mu_2 - 1)}}\right] & \text{if } K > S_0.
\end{cases}$$

(b) If $\gamma > \max(0, -r)$, then

$$\int_0^\infty e^{-\gamma t}e^{-rt}E(S_0e^{(r-\frac{\sigma^2}{2})t+\sigma W_t} - K)_+ dt$$

$$= \begin{cases} 
\frac{S_0}{\sigma^2\sqrt{\rho^2 + 2\beta\mu_1(1 + \mu_1)}} + \frac{S_0}{\gamma - \gamma + r}K & \text{if } K \leq S_0 \\
\frac{S_0}{\beta\sigma^2}\left[K - \frac{2\beta}{2\beta - 2\rho - 1} + \frac{2\beta K^{1-\mu_2}}{\sqrt{\rho^2 + 2\beta\mu_2(\mu_2 - 1)}}\right] + \frac{S_0}{\gamma - \gamma + r}K & \text{if } K > S_0.
\end{cases}$$

Proof. (a) Rewrite the integral as

$$S_0\int_0^\infty e^{-(\gamma + r)t}E(K - e^{(r-\frac{\sigma^2}{2})t+\sigma W_t})_+ dt$$

$$= \frac{S_0}{\sigma^2}\int_0^\infty e^{-(\frac{\gamma + r}{\sigma^2})s}E(K - e^{(r-\frac{\sigma^2}{2})s+W_s})_+ ds$$

$$= \frac{S_0}{\beta\sigma^2}\int_0^\infty e^{-\beta s}E(K - e^{\rho s+W_s})_+ ds$$

$$= \frac{S_0}{\beta\sigma^2}E(K - e^{\rho\tau+W_\tau})_+,$$

where $\tau \sim \text{Exp}(\beta)$ is independent of $W$. For simplicity, let $S_0 = 1$ and write $K$ for $K$ in the remainder of this proof. We will use the distribution of $\rho\tau + W_\tau$ (when $\rho = 0$ this is of course a symmetric double exponential). We find

$$Ee^{q(\rho\tau + W_\tau)} = Ee^{(\rho q + \frac{q^2}{2})\tau} = \frac{2\beta}{2\beta - 2\rho q - q^2} = \frac{2\beta}{(\mu_1 + q)(\mu_2 - q)},$$

where $-\mu_1$ and $\mu_2$ are the roots of $2\beta - 2\rho q - q^2$. Partial fractions show that the distribution of $\rho\tau + W_\tau$ is an asymmetric double exponential:

$$\frac{\beta}{\sqrt{\rho^2 + 2\beta}} \cdot \frac{1}{\mu_1 + q} + \frac{\beta}{\sqrt{\rho^2 + 2\beta}} \cdot \frac{1}{\mu_2 - q}$$

($\mu_1$ is the parameter of the exponential on $(-\infty, 0)$, and $\mu_2$ is the parameter of the one on $(0, \infty)$). The density of this distribution is

$$\frac{\beta}{\sqrt{\rho^2 + 2\beta}}e^{\mu_1 x}1_{x < 0} + \frac{\beta}{\sqrt{\rho^2 + 2\beta}}e^{-\mu_2 x}1_{x > 0}.$$
Then,

\[
E(K - e^{\rho \tau + W_{\tau}})_{+} = \frac{\beta}{\sqrt{\rho^2 + 2\beta}} \left[ \int_{-\infty}^{0} (K - e^{x})_{+} e^{\mu_{1}x} \, dx + \int_{0}^{\infty} (K - e^{x})_{+} e^{-\mu_{2}x} \, dx \right]
\]

\[
= \frac{\beta}{\sqrt{\rho^2 + 2\beta}} (I_{1} + I_{2}).
\]

The cases \(K \leq 1\) and \(K > 1\) lead to different formulas. If \(K \leq 1\) then \(I_{2} = 0\) and

\[
I_{1} = \int_{-\log K}^{\infty} (K - e^{-x})_{+} e^{-\mu_{1}x} \, dx = \frac{K^{1+\mu_{1}}}{\mu_{1}} - \frac{K^{1+\mu_{1}}}{1 + \mu_{1}} = \frac{K^{1+\mu_{1}}}{\mu_{1}(1 + \mu_{1})}.
\]

Hence, when \(K \leq 1\),

\[
\int_{0}^{\infty} e^{-(\gamma + rt)} E(K - e^{(r - \frac{\sigma^2}{2})t + W_{s_{2}t}})_{+} \, dt = \frac{1}{\sigma^{2}\sqrt{\rho^2 + 2\beta}} \frac{K^{1+\mu_{1}}}{\mu_{1}(1 + \mu_{1})}.
\]

If \(K > 1\), then

\[
E(K - e^{\rho \tau + W_{\tau}})_{+} = K - E(e^{\rho \tau + W_{\tau}} - K)_{+}
\]

\[
= K - \frac{2\beta}{2\beta - 2\rho - 1} + E(e^{\rho \tau + W_{\tau}} - K)_{+}.
\]

The last term reduces to

\[
\frac{\beta}{\sqrt{\rho^2 + 2\beta}} \int_{-\log K}^{\infty} (e^{x} - K) e^{-\mu_{2}x} \, dx = \frac{\beta}{\sqrt{\rho^2 + 2\beta}} \frac{K^{1-\mu_{2}}}{\mu_{2}(\mu_{2} - 1)}.
\]

(b) Use put-call parity to express the price of the call as the price of the put plus

\[
S_{0} - Ke^{-rt},
\]

and then multiply by \(e^{-\gamma t}\) before integrating between 0 and \(\infty\).

Next, we derive a general Fourier integral formula for European options using Theorem 3.3, which allows one to compute (3.1) directly from the Fourier transform instead of finding the distribution of \(X\) first.

**Theorem 3.8** (Chin and Dufresne [27] Theorem 2.2). Let \(U_{T}\) and \(S_{T}\) be as above, and let \(\nu\) be the distribution of \(U_{T}\), so that

\[
\hat{\nu}(\alpha) = Ee^{(\alpha + i\sigma)U_{T}}.
\]

(a) Suppose that \(Ee^{\alpha U_{T}} < \infty\) for some \(\alpha^{*} > 0\). Then, for any \(0 < \alpha < \alpha^{*}\),

\[
e^{-rt}E(K - S_{T})_{+} = \frac{1}{2\pi} \text{PV} \int_{-\infty}^{\infty} \hat{g}_{1}^{(-\alpha)}(-u) \hat{\nu}(\alpha) \, du, \tag{3.16}
\]

\[
= \frac{\beta}{\sqrt{\rho^2 + 2\beta}} \int_{-\log K}^{\infty} (e^{x} - K) e^{-\mu_{2}x} \, dx = \frac{\beta}{\sqrt{\rho^2 + 2\beta}} \frac{K^{1-\mu_{2}}}{\mu_{2}(\mu_{2} - 1)}.
\]
where, if \( k = Ke^{-rT}/S_0 \),
\[
\tilde{g}_1^{(-\alpha)}(-u) = \begin{cases} 
\frac{S_0k(1+\sqrt{1+8\alpha+8iu})/2}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} < S_0 \\
\frac{S_0(k-1)}{\alpha+iu} + \frac{S_0k(1-\sqrt{1+8\alpha+8iu})/2}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} \geq S_0.
\end{cases}
\] (3.17)

(b) Suppose that \( \mathbb{E}e^{\alpha^*U_T} < \infty \) for some \( \alpha^* > 0 \). Then, for any \( 0 < \alpha < \alpha^* \),
\[
e^{-rT}\mathbb{E}(S_T - K)^+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \tilde{g}_2^{(-\alpha)}(-u)\nu^{(\alpha)}(u) \, du,
\] (3.18)

where
\[
\tilde{g}_2^{(-\alpha)}(-u) = \begin{cases} 
\frac{S_0(1-k)}{\alpha+iu} + \frac{S_0k(1+\sqrt{1+8\alpha+8iu})/2}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} < S_0 \\
\frac{S_0k(1-\sqrt{1+8\alpha+8iu})/2}{(\alpha+iu)\sqrt{1+8\alpha+8iu}} & \text{if } Ke^{-rT} \geq S_0.
\end{cases}
\] (3.19)

Proof. (a) Define
\[
g_1(x) = \mathbb{E}(Ke^{-rT} - S_0e^{-\frac{x}{2}+\sqrt{x}W_1})^+1_{\{x \geq 0\}}.
\]

Then,
\[
e^{-rT}\mathbb{E}(K - S_T)^+ = \int g_1 \, d\nu = \int g_1^{(-\alpha)} \, d\nu^{(\alpha)}.
\]

Apply Theorem 3.3 with \( x = e^{-\alpha x} \), \( X = U_T \), and \( 0 < \alpha < \alpha^* \). That theorem lists three conditions under which
\[
\int g_1^{(-\alpha)} d\nu^{(\alpha)} = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \tilde{g}_2^{(-\alpha)}(-u)\nu^{(\alpha)}(u) \, du
\]

The \( \mu_X \) in Dufresne et al. [45] is our \( \nu \), and thus \( d\mu_X^{(\alpha)}(x) = d\nu^{(\alpha)}(x) = e^{\alpha x} d\nu(x) \).

The first condition is that (i) the total mass \( |\nu^{(\alpha)}| \) be finite; this is true because of the assumption \( \mathbb{E}e^{\alpha^*U_T} < \infty \). The second condition is (ii) that
\[
\int_{-\infty}^{\infty} |g_1^{(-\alpha)}(x)| \, dx = \int_{-\infty}^{\infty} e^{-\alpha x} |g_1(x)| \, dx < \infty;
\]

this is true because \( \alpha > 0 \) and \( 0 \leq g_1(x) \leq e^{-rT}K \). The third condition (iii) is in two parts. The first part is that the function
\[
G_1(y) = \int_{\mathbb{R}} g_1^{(-\alpha)}(x-y) \, d\nu^{(\alpha)}(x)
\]

be continuous at \( y = 0 \); this is correct by dominated convergence (see Lemma 1 in Dufresne et al. [45]). The second part of condition (iii) is verified because \( g_1 \) has
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a derivative that exists everywhere except at the origin and is integrable elsewhere (see Lemma 2 in Dufresne et al. [45]). The last step is to identify \( \hat{g}_1(-\alpha)(-u) \), from the definition of \( g_1(x) \) above. Theorem 3.7 with \( \sigma = 1 \) and \( r = 0 \) yields:

\[
\int_0^\infty e^{-zx}E(k - S_0e^{-\frac{x}{2} + \sqrt{x}W_1})_+ dx = \begin{cases} 
S_0 k^{(1 + \sqrt{1 + 8z})/2} / 2z \sqrt{1 + 8z} & \text{if } k < S_0 \\
S_0 / z \left( \sqrt{k} - 1 + \frac{\sqrt{1 - \sqrt{1 + 8z}}}{\sqrt{1 + 8z}} \right) & \text{if } k \geq S_0
\end{cases}
\]

for \( k \) and \( \text{Re}(z) > 0 \), if \( \sqrt{k} = k/S_0 \). Finally, replace \( k \) with \( Ke^{-rT} \) to obtain the formula for \( \hat{g}_1(-\alpha)(-u) \). Observe that the 'PV' is not needed in front of the integral when \( Ke^{-rT} \leq S_0 \) because

\[
\int_{-\infty}^\infty \left| \hat{g}_1(-\alpha)(-u)\nu(\alpha)(u) \right| du < \infty.
\]

(b) If

\[
g_2(x) = E(S_0e^{-\frac{x}{2} + \sqrt{x}W_1} - Ke^{-rT})_+ \mathbf{1}_{\{x \geq 0\}}
\]

then

\[
e^{-rT}E(S_T - K)_+ = \int g_2 d\nu = \int g_2(-\alpha) d\nu(\alpha)
\]

and conditions (i), (ii), (iii) are verified. Since

\[
g_2(x) = g_1(x) + (S_0 - Ke^{-rT})1_{\{x \geq 0\}},
\]

we have

\[
\widehat{g}_2(-\alpha)(-u) = \int_0^\infty e^{-(\alpha + iu)}g_1(x) dx + \frac{1}{\alpha + iu}(S_0 - Ke^{-rT})
\]

\[
= \widehat{g}_1(-\alpha)(-u) + \frac{S_0(1 - k)}{\alpha + iu},
\]

which completes the proof.

This formula applies in all cases in which the Black-Scholes ‘\( \sigma \)’ is replaced with a volatility process \( V \), the latter being independent of the Brownian motion driving the stock price. This is a significant improvement over the Monte Carlo simulation in many cases. The Fourier integral for option prices thus obtained is an integral over an infinite interval, but we show how to truncate that integral while keeping an upper bound on the error in following section.
3.5 Numerical analysis

The integrals in (3.16) and (3.18) have an infinite range, and it is useful to know how to restrict the range while ensuring an upper bound for the truncation error ($\epsilon$ in what follows). Observe that the function $\hat{\nu}(\alpha)(u)$ is uniformly bounded by $E e^{\alpha u/T}$, which is finite by assumption. As to the functions $g_1^{(-\alpha)}(-u)$ and $g_2^{(-\alpha)}(-u)$, one notices that the first one is $O(|u|^{1/2})$ when $Ke^{-rT} < S_0$, while it is only $O(|u|^{-1})$ when $Ke^{-rT} > S_0$; the reverse holds for $g_2^{(-\alpha)}(-u)$. Because of put-call parity, one may use either formula for a put or a call, and one would naturally choose to use the formula that tends to 0 more quickly.

Let us consider the formula for the put price in part (a) of the theorem, when $Ke^{-rT} < S_0$. The function $\hat{g}_1^{(-\alpha)}(-u)$ may be rewritten as

$$S_0 e^{-\frac{\ell}{2} - \frac{1}{2} \sqrt{1 + 8\alpha + 8iu}}$$

where $\ell = -\log k > 0$. It is obvious that

$$\left| \frac{S_0 e^{-\frac{\ell}{2}}}{(\alpha + iu)\sqrt{1 + 8\alpha + 8iu}} \right| \leq \frac{S_0 e^{-\frac{\ell}{4}}}{2\sqrt{2}|u|^{\frac{3}{2}}}.$$

Next, if $u > 0$ then $\theta = \arg(1 + 8\alpha + 8iu) \in (0, \frac{\pi}{2})$, and so

$$\sqrt{1 + 8\alpha + 8iu} = [(1 + 8\alpha)^2 + 64u^2]^{\frac{1}{4}} e^{i\frac{\theta}{2}},$$

with $\frac{\theta}{2} \in (0, \frac{\pi}{4})$. Then, $\Re(e^{i\frac{\theta}{2}}) \in (\frac{1}{\sqrt{2}}, 1)$, implying that

$$\Re(e^{i\frac{\theta}{2}}) \geq [(1 + 8\alpha)^2 + 64u^2]^{\frac{1}{4}} \frac{1}{\sqrt{2}} \geq 2\sqrt{|u|}.$$

We then have

$$\left| \frac{S_0 e^{-\frac{\ell}{2} - \frac{1}{2} \sqrt{1 + 8\alpha + 8iu}}}{(\alpha + iu)\sqrt{1 + 8\alpha + 8iu}} \right| \leq \frac{S_0 e^{-\frac{\ell}{4} - \ell \sqrt{\alpha}}}{2\sqrt{2}|u|^{\frac{3}{2}}},$$

and are ready to find $M > 0$ such that

$$\frac{1}{2\pi} \left| \int_M^\infty g_1^{(-\alpha)}(-u)\nu(\alpha)(u) \, du \right| \leq \frac{\epsilon}{2}. $$

From the above it is sufficient that

$$\int_M^\infty \frac{e^{-\ell \sqrt{\alpha}}}{u^{3/2}} \, du \leq \frac{2\sqrt{2}e\pi e^{\ell}}{S_0 E e^{\alpha u/T}} = \delta.$$

Now

$$\int_M^\infty \frac{e^{-\ell \sqrt{\alpha}}}{u^{3/2}} \, du = 2\ell \int_{\ell \sqrt{M}}^\infty \frac{e^{-y}}{y^{3/2}} \, dy \leq \frac{2}{\ell M} \int_{\ell \sqrt{M}}^\infty e^{-y} \, dy = \frac{2e^{-\ell \sqrt{M}}}{\ell M}. $$
The last expression is a decreasing function of \( M \), and thus we look for \( M \) such that

\[
\frac{2e^{-\ell\sqrt{M}}}{\ell M} = \delta \quad \text{or} \quad \ell M e^{\ell}\sqrt{M} = \frac{2}{\delta}.
\]

Letting \( x = \ell\sqrt{M}/2 \), this is the same as

\[
x e^{x} = \sqrt{\frac{\ell}{2\delta}}.
\]

The solution of this equation is an instance of the function \( \mathcal{W}(z) \), defined implicitly by

\[
\mathcal{W}(z)e^{\mathcal{W}(z)} = z,
\]

and called Lambert’s function (or product logarithm). Hence, we find that if

\[
M \geq M_{\epsilon} = \left(\frac{2}{\ell}\right)^2 \mathcal{W}\left(\sqrt{\frac{\ell}{2\delta}}\right)^2,
\]

then

\[
\left| e^{-rT}E(K - S_T) + \frac{1}{2\pi} \int_{-M}^{M} \hat{g}_1^{(-\alpha)}(-u)\overline{\nu^{(\alpha)}}(u) du \right| \leq \epsilon
\]

(ignoring the error made in the numerical computation of \( \int_{-M}^{M} \)). In the case of the call price formula (part (b) of the theorem), a similar calculation tells us that

\[
\left| e^{-rT}E(S_T - K) - \frac{1}{2\pi} \text{PV} \int_{-M}^{M} \hat{g}_2^{(-\alpha)}(-u)\overline{\nu^{(\alpha)}}(u) du \right| \leq \epsilon
\]

if \( Ke^{-rT} > S_0 \) and

\[
M \geq M'_{\epsilon} = \left(\frac{2}{\ell'}\right)^2 \mathcal{W}\left(\sqrt{\frac{\ell'}{2\delta'}}\right)^2, \quad \ell' = \log \frac{Ke^{-rT}}{S_0}, \quad \delta' = \frac{2\sqrt{2e\pi e^{-\frac{\ell'}{2}}}}{S_0e^{\alpha U_T}}.
\]

For instance, if \( S_0 = 100, r = .05, T = 1, K = 90, \epsilon = .01, \text{PV}e^{\alpha U_T} = 2 \), then \( M_{\epsilon} = 647 \). For \( K = 100, M_{\epsilon} = 4382 \). The formulas for \( M_{\epsilon} \) or \( M'_{\epsilon} \) yield smaller values when the strike \( K \) is further away from \( e^{rT}S_0 \). When \( K = e^{rT}S_0 \) the exponential disappears from \( \hat{g}_1^{(-\alpha)}(-u) \) and \( \hat{g}_2^{(-\alpha)}(-u) \), and the above formulas do not apply; in that case, an upper bound for the norm of

\[
\int_{M}^{\infty} \frac{S_0\overline{\nu^{(\alpha)}}(u)}{(\alpha + iu)\sqrt{1 + 8\alpha + 8iu}} du
\]

is

\[
\frac{S_0e^{\alpha U_T}}{\sqrt{2M}},
\]
and thus one may choose

\[ M \geq M''_e = \frac{1}{2} \left( \frac{S_0 e^{\alpha U_T}}{\epsilon} \right)^2. \]

If the Lambert function is not available, one may use the approximation (Corless et al. [34], Eq.(4.19))

\[ W(z) \approx \log z - \log \log z + \frac{\log \log z}{\log z}. \]

It is good for \( z \) as small as 10, and improves as \( z \) increases.
Chapter 4

The Markov Regime Switching Model

The volatility cannot be observed directly but one may estimate the volatility from stock price returns. From the empirical studies (see So et al. [110]), the volatility seems to be low for several days, then high for a period and so on. In addition, Zhang [121] shows that volatility is small when the market trends up, whereas the volatility tends to be much higher during a sharp market downturn. There is much literature [1, 21, 36, 39, 53, 60–65] about the Markov regime switching model. For example, Hamilton [63] uses the Markov regime switching model to describe the behaviour of the term structure of interest rates across economic states. Ang and Bekaert [6] found that the regime switching model in the volatility of the stock is significant and similar works can also be found in Schwert [104], Campbell [23] and Campbell and Hentschel [24].

Specifically, many references have discussed the contingent claims under a regime switching model as a component of an underlying asset price process. For instance, Guo [60] and Ching et al. [29] provide a pricing formula for European call options and Edwards [46] suggests a general method of valuing derivatives using the moment generating function of the occupation time in a certain regime. In particular, Fuh et al. [53] suggest the Markov switching option pricing with the Black-Scholes model. It is a model of an incomplete market by joining the Black-Scholes exponential
Brownian motion for stock fluctuations with a hidden Markov process. The drift and volatility parameters take different values depending on the state of a hidden Markov process. However, Fuh et al. [53] assume that nobody can observe the drift and the volatility but they have to be inferred from the past observations. This model is called a Black-Scholes model with Markov regime switching or a hidden Markov model in short. Moreover, Bollen [19] shows that the Markov regime switching model can resolve the volatility smile effect on option pricing. We refer to Yin et al. [117] and So et al. [110] for discussion about more general Markov chains.

In this chapter, we provide an analytic valuation method for European puts and calls with finite state Markov regime switching volatility model. The method is developed by exploiting not the explicit probability distribution of the occupation time for which the underlying asset is in a state over the time period, but only its Laplace transform. By extending the method, we provide valuation formulas for European puts and calls in general $N$-state Markov regime switching model.

The outline of this chapter as follows. In Section 4.1, the simplest Markov regime switching model ($N = 2$), which is to be used in a later example, is briefly introduced. Section 4.2 contains the main body of this chapter, focusing on calculating the Laplace transform of integrated squared volatility with particular cases $N = 2$ and 3. Section 4.3 is used to show the results of several numerical experiments, comparing our method with the Monte Carlo simulation for the pricing of options. We show that in this and other stochastic volatility models it is not necessary to simulate the stock price process itself, but only the integrated squared volatility process. The results in Section 4.2 and 4.3 are heavily borrowed from Chin and Dufresne [27].

4.1 The basic model

To begin with, we formally establish the simplest two-state Markov regime switching model where the volatility can only take two values, for instance, jump between high and low volatility. Assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ is a filtered probability space. The filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions and is generated by a Brownian motion $W_t$ and two Poisson processes $N_t^{(1)}$ and $N_t^{(2)}$ on $[0, T]$.

We note that the Markov regime switching model is incomplete due to the additional source of randomness. One way of completing the market is to introduce additional assets. For example, the change-of-state (COS) contract is suggested by Duffie according to Guo [60]. The idea is following: Suppose that there is a market at time $t$ which pays one unit of account when the Markov chain $u(t)$ changes state,
i.e., at the next time \( \tau = \inf\{s > t | u(s) \neq u(t)\} \). As soon as the contract becomes worthless a new contract is issued for next regime switching. This makes the market complete. The reader can refer to Guo [60] for more details. Another treatment can be found in Di Masi et al. [39] that gives a partial justification for the complete market assumption, by showing that the risk-minimising martingale measure in a model with Markov chain volatilities is precisely the one we described. See also Yao et al. [116] who use a Girsanov theorem to find a martingale measure under which to perform pricing.

The Markov chain represents a stochastic volatility, and it is assumed to be an alternating renewal process that is defined by the two Poisson processes, \( N^{(1)}_t \) and \( N^{(2)}_t \). The Markov chain moves between two regimes called state 1 and state 2. Hence, if the current volatility is in state \( i \in \{1, 2\} \), the volatility would change when the first jump time of \( N^{(i)}_t \) arrives. Therefore, the volatilities of the underlying asset must also have two states corresponding to the other states. We denote the states of volatility by \( v_1 \) for state 1 and \( v_2 \) for state 2. This continuous-time Markov chain may be described in terms of the integrated telegraph signal, also known as the Kac process. See Di Masi et al. [39] for the explicit expression of the distribution of the integrated telegraph signal and its proof based on the properties of order statistics in their appendix.

Di Masi et al. [39] present a model in which the drift \( \mu \) and the diffusion coefficients \( \sigma \) in

\[
dX_t = \mu(Y_t)X_t \, dt + \sigma(Y_t)X_t \, dW_t, \tag{4.1}
\]

are modelled by a Markov jump process, \( Y_t \), which is independent of \( W \), a standard Brownian motion. The additional uncertainties make the market incomplete and consequently we cannot hedge perfectly. A locally risk-minimising and \( H \)-admissible trading strategy is the main topic of discussion in their paper.

Fuh et al. [53] and Guo [60] introduce an explicit formula for the Black-Scholes option pricing with a Markov switching model. This model is different from other stochastic volatility models because the drift term follows the Markov process as well. In addition, the martingale approach used in Di Masi et al. [39] cannot be applied here because the drift term is non-zero. For the two-state Markov switching model, an explicit analytical formula for European call option prices is available. Fuh et al. [53] and Guo [60] use the Laplace transformation to derive the distribution of the integrated telegraph signal. For three or possibly more multi-state cases, they propose an approximation formula. The approximation can be obtained in two different ways. Firstly, through the discretise continuous market model based on
the binomial option pricing model that was proposed by Cox, Ross, and Rubinstein (1979), secondly, by replacing the distribution of occupation time in a given state by the product of its stationary distribution and the total time.

4.2 Regime switching volatility model

This section is about the particular application we have in mind, that is, volatility modelled as a continuous-time Markov chain, that has already been studied by others, including Di Masi et al. [39] and Fuh et al. [53]. We assume that the market is complete and the underlying asset follows the dynamic of

\[ dS_t = rS_t \, dt + V_t S_t \, dW_t, \]

where \( r \) is a risk-free interest rate, \( W \) is a standard Brownian motion under the risk-neutral measure and \( V_t \) is the volatility that changes value according to a continuous-time Markov chain. The stochastic volatility model consists in a Markov chain that takes values in \( \{v_1, \ldots, v_N\} \).

We focus on the Laplace transform of

\[ U_T = \int_0^T V_s^2 \, ds, \]

or integrated squared volatility instead of the distribution of \( U_T \). It is apparent that the calculations rapidly become more and more arduous as \( N \) (the number of possible states) increases. In the case of \( N = 2 \), the probability distribution of \( U_T \) has been known for a long time, and several proofs are known (see Janssen and Siebert [72], Steutel [113] and Orsingher [92]). In the financial literature, almost all papers are restricted to two states, the cases of \( N \geq 3 \) are seen as too demanding. This is unfortunate, as the Markov chain volatility model is not very realistic when there are only two states. Our method does not require the explicit probability distribution of \( U_T \), but only its Laplace transform \( \mathcal{E} e^{-rU_T} \), thus we are able to tackle the case \( N = 3 \) without any difficulty. In theory, our results apply to an arbitrary number of states, but the algebra does become more and more involved as \( N \) increases; symbolic mathematics software such as Mathematica is essential for \( N \geq 3 \).

4.2.1 Laplace transform and moments of \( U_T \)

The variable \( U_t \) is a linear combination of the occupation times

\[ J_k(T) = m\{t \in [0, T] : V_t = v_k\}, \quad k = 1, 2, \ldots, N, \]
where \( m\{\cdot\} \) is Lebesgue measure. The joint Laplace transform of the occupation times was obtained by Darroch and Morris [37]; we give a short derivation of the Laplace transform of \( U_T \) based on their arguments. The Laplace transform may also be found using martingale techniques. For instance, Rogers and Williams [102] (p.40–41), find the joint transform of the vector consisting of \( \{ J_k(T), k = 1, 2, \ldots, N \} \), of the number of transitions from one state to another and of \( V_T \); they call this the \textit{generalized Feynman-Kac formula for chains}. The explicit formula for the distribution of the occupation time \( J_1(T) \) in the case \( N = 2 \), which we present in Section 4.2.2, goes at least as far as back as Pedler [94].

We denote \( \lambda_{jk}, j, k = 1, \ldots, N \) as the intensity of the volatility switches from state \( j \) to state \( k \). Define

\[
\lambda_j = \sum_{k \neq j} \lambda_{jk},
\]

that is, \( \lambda_j \) is the rate of leaving state \( j \),

\[
L_j(r,T) = \mathbb{E}[e^{-rU_T} | V_0 = v_j^2]
\]

be the Laplace transform of \( U_T \) given the Markov chain start at state \( j \),

\[
\bar{L}(r,T) = (L_1(r,T), \ldots, L_N(r,T))'
\]

\[
\Lambda(r) = \begin{pmatrix}
-\lambda_1 - rv_1^2 & \lambda_{12} & \ldots & \lambda_{1N} \\
\lambda_{21} & -\lambda_2 - rv_2^2 & \ldots & \lambda_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N1} & \lambda_{N2} & \ldots & -\lambda_N - rv_N^2
\end{pmatrix}
\]

\[
\Lambda = \Lambda(0)
\]

\[
D = \text{diag}(v_1^2, \ldots, v_N^2)
\]

\[
\bar{1} = (1, \ldots, 1)'
\]

and \( \vec{v} = D\bar{1} = (v_1^2, \ldots, v_N^2)' \) be the vector of the values of each state. We assume that under a risk-neutral measure the volatility \( V_t \) takes one of the values \( v_1, \ldots, v_N \). We begin with the general \( N \)-state model, and then turn to the particular cases \( N = 2 \) and \( N = 3 \), where more explicit results are available.

**Proposition 4.1.** For \( r \in \mathbb{C} \),

\[
\bar{L}(r,T) = e^{\Lambda(r)T}\bar{1},
\]

and, for \( \text{Re}(s) > -\text{Re}(r) \min_j(v_j^2) \),

\[
\int_0^\infty e^{-sT}\bar{L}(r,T) \,dT = (sI - \Lambda(r))^{-1}\bar{1}.
\] (4.2)
Proof. Since $U_T$ is uniformly bounded $L_j(r, T)$ is finite for any $r \in \mathbb{C}$, and obviously
\[ |L_j(r, T)| \leq \exp\{-\Re(r) \min_k (v_k^2) T\}. \tag{4.3} \]

By the Markov property,
\[
L_j(r, T) = e^{-rT v_j^2} e^{-\lambda_j T} + \sum_{k \neq j} \lambda_{jk} \int_0^T e^{-\lambda_j (T-t) v_j^2} L_k(r, T-t) \, dt
\]
\[
= e^{-(\lambda_j + r v_j^2) T} \left( 1 + \sum_{k \neq j} \lambda_{jk} \int_0^T e^{(\lambda_j + r v_j^2) x} L_k(r, x) \, dx \right).
\]

Then,
\[
\frac{\partial}{\partial T} L_j(r, T) = -(\lambda_j + r v_j^2) L_j(r, T) + \sum_{k \neq j} \lambda_{jk} L_k(r, T), \tag{4.4}
\]
or, in matrix form,
\[
\frac{\partial}{\partial T} \vec{L}(r, T) = \Lambda(r) \vec{L}(r, T).
\]

This implies $\vec{L}(r, T) = e^{\Lambda(r) T} \vec{1}$. The Laplace transform in $T$ of $\vec{L}(r, T)$ follows from well-known properties of matrices. For a matrix $M$ and $s > \|M\|$,
\[
\int_0^\infty e^{-st} e^{Mt} \, dt = \sum_{n=0}^\infty M^n \int_0^\infty e^{-stn} \frac{t^n}{n!} \, dt = \sum_{n=0}^\infty s^{-n-1} M^n
\]
by dominated convergence. The last expression must be $(sI - M)^{-1}$, because
\[
(sI - M) \sum_{n=0}^p s^{-n-1} M^n = I - s^{-p-1} M^{p+1} \to I
\]
as $p \to \infty$. Hence, formula (4.2) is valid for $\Re(s) > \|M\|$. By (4.3) we know that the left-hand side of (4.2) is finite for $\Re(s + r \min_k (v_k^2)) > 0$, and it is a classical result that the Laplace transform is an analytic function of its argument in the region where it exists. Therefore (4.2) hold at least for $\Re(s + r \min_k (v_k^2)) > 0$, by analytic continuation. \hfill \Box

Next, we define $m_{nj}(t) = \mathbb{E}(U_t^n \mid V_0 = v_j)$ and $m_n(t) = (m_{n1}(t), \ldots, m_{nN}(t))^t$.

**Proposition 4.2.** Let $m_{0j}(t) \equiv 1$. Then, for $n = 1, 2, \ldots, T \geq 0$, we have
\[
\frac{\partial}{\partial T} m_n(T) = \Lambda m_n(T) + n D m_{n-1}(T)
\]
\[
m_n(T) = ne^{\Lambda T} \int_0^T e^{-\Lambda t} D m_{n-1}(t) \, dt,
\]
where
\[
m_1(T) = \int_0^T e^{\Lambda u} du \vec{v}.
\]
Proof. From (4.4), we can write
\[
\frac{\partial}{\partial T} \frac{\partial^n}{\partial r^n} L_j(r, T) = - (\lambda_j + rv_j^2) \frac{\partial^n}{\partial r^n} L_j(r, T) - n v_j^2 \frac{\partial^{n-1}}{\partial r^{n-1}} L_j(r, T) + \sum_{k \neq j} \lambda_{jk} \frac{\partial^n}{\partial r^n} L_k(r, T)
\]
because
\[
\frac{\partial^n}{\partial r^n} [(a + br)f(r)] = (a + br)f^{(n)}(r) + bn f^{(n-1)}(r).
\]
Next, multiply by \((-1)^n\) and set \(r = 0\) to get:
\[
\frac{\partial}{\partial T} m_{nj}(T) = - \lambda_j m_{nj}(T) + n v_j^2 m_{n-1,j}(T) + \sum_{k \neq j} \lambda_{jk} m_{nk}(T).
\]
In vector form, this is the first equation in the statement of the proposition, which implies
\[
e^{-\Lambda T} \left[ \frac{\partial}{\partial T} m_n(T) - \Lambda m_n(T) \right] = \frac{\partial}{\partial T} \left[ e^{-\Lambda T} m_n(T) \right] = ne^{-\Lambda T} D m_{n-1}(T).
\]
The other two equations then follow. \(\square\)

4.2.2 The case \(N = 2\)

Applying Proposition 4.1 gives
\[
(sI - \Lambda(r))^{-1}
\]
\[
= \begin{pmatrix}
  s + rv_1^2 + \lambda_{12} & -\lambda_{12} \\
  -\lambda_{21} & s + rv_2^2 + \lambda_{21}
\end{pmatrix}^{-1}
\]
\[
= \frac{1}{(s + rv_1^2 + \lambda_{12})(s + rv_2^2 + \lambda_{21}) - \lambda_{12}\lambda_{21}} \begin{pmatrix}
  s + rv_2^2 + \lambda_{21} & \lambda_{12} \\
  \lambda_{21} & s + rv_1^2 + \lambda_{12}
\end{pmatrix}
\]
Hence, we have
\[
\int_0^\infty e^{-sT} L_1(r, T) dT = \frac{s + rv_2^2 + \lambda_{12} + \lambda_{21}}{(s + rv_1^2 + \lambda_{12})(s + rv_2^2 + \lambda_{21}) - \lambda_{12}\lambda_{21}},
\]
and the same expression for \(L_2(r, T)\) can be obtained by just reversing the roles of subscripts ‘1’ and ‘2’. The inversion may be achieved by first observing that if one defines \(J_T\) as the amount of time the chain spends in state 1 during the period \([0, T]\), then, on the one hand,
\[
U_T = v_1^2 J_T + v_2^2(T - J_T)
\]
and, on the other hand, the double Laplace transform of \(J_T\) is the one above with \((v_1^2, v_2^2)\) replaced with \((1,0)\). Hence, one needs to invert
\[
s + \lambda_{12} + \lambda_{21} \\
(r + s + \lambda_{12})(s + \lambda_{21}) - \lambda_{12}\lambda_{21}
\]
\[
= \frac{s + \lambda_{12} + \lambda_{21}}{r(s + \lambda_{21}) + s(s + \lambda_{21} + \lambda_{12})}
\]
\[
= \frac{s + \lambda_{12} + \lambda_{21}}{s + \lambda_{21}} \frac{1}{r + s \left( 1 + \frac{\lambda_{12}}{s + \lambda_{21}} \right)}
\]
Finally, we have found that the distribution of $J_T$. We will make use of the generalised hypergeometric function

$$0F_1(c; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!(c)_n}, \quad z \in \mathbb{C}, \quad -c \notin \mathbb{N}.$$ 

Here, the shifted factorials $(c)_0 = 1, (c)_n = c(c+1) \cdots c(n-1)$, are used. See Lebedev [77] p.238, for more details.

First, we invert (4.5) with respect to $r$:

$$\int_0^\infty e^{-rx} \frac{s + \lambda_{12} + \lambda_{21}}{s + \lambda_{21}} \exp \left\{ \frac{-s(s + \lambda_{12} + \lambda_{21})}{s + \lambda_{21}} x \right\} dx$$

$$= \int_0^\infty e^{-rx} \left( 1 + \frac{\lambda_{12}}{s + \lambda_{21}} \right) \exp \left\{ -(s + \lambda_{12})x + \frac{\lambda_{12}\lambda_{21}}{s + \lambda_{21}} x \right\} dx$$

Next, split expression $1 + \frac{\lambda_{12}}{s + \lambda_{21}}$ into two parts, so that the integrand is now $e^{-rx(A+B)}$, where $\delta(\cdot)$ denote Dirac delta function,

$$A = e^{-(s+\lambda_{12})x} \sum_{n=0}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \left( \frac{\lambda_{21}}{s + \lambda_{21}} \right)^n$$

$$= \int_0^\infty e^{-sT} e^{-\lambda_{12}x} \left[ \delta_x(T) + \sum_{n=1}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \frac{\lambda_{12}\lambda_{21}}{s + \lambda_{21}} (T-x)^{(n-1)} e^{-\lambda_{21}(T-x)} (n-1)! 1_{(T>x)} \right] dT$$

and

$$B = e^{-(s+\lambda_{12})x} \sum_{n=0}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \left( \frac{\lambda_{21}}{s + \lambda_{21}} \right)^n \frac{\lambda_{12}}{s + \lambda_{21}}$$

$$= \int_0^\infty e^{-sT} e^{-\lambda_{12}x} \sum_{n=0}^{\infty} \frac{(\lambda_{12}x)^n}{n!} \frac{\lambda_{12}\lambda_{21}^{n+1} (T-x)^ne^{-\lambda_{21}(T-x)}}{n!} 1_{(T>x)} dT$$

Finally, we have found that the distribution of $J_T$ is:

$$P(J_T \in dx) = e^{-\lambda_{12}T} 1_{(x=T)} + \lambda_{12} e^{-\lambda_{12}x-\lambda_{21}(T-x)}$$

$$\times [\lambda_{21}x 0F_1(2; \lambda_{12}\lambda_{21}x(T-x)) + 0F_1(1; \lambda_{12}\lambda_{21}x(T-x)) 1_{(0<x<T)} dx]$$

This may also be written in terms of Bessel functions (see Di Masi et al. [39]). The distribution of $U_T$ may be found from that of $J_T$. 

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4.2.3 The case $N = 3$

When there are three states the algebra are more involved. However, our approach only requires the Laplace transform of the distribution of $U_T$, not the distribution itself. Apply Proposition 4.1 once again:

$$(sI - \Lambda(r))^{-1} = \left(\begin{array}{ccc}
    s + \lambda_1 + rv_1^2 & -\lambda_{12} & -\lambda_{13} \\
    -\lambda_{21} & s + \lambda_2 + rv_2^2 & -\lambda_{23} \\
    -\lambda_{31} & -\lambda_{32} & s + \lambda_3 + rv_3^2 
\end{array}\right)^{-1}
= \frac{1}{D(r,s)}M,$$

where

$$D(r,s) = [(s + rv_1^2 + \lambda_1)(s + rv_2^2 + \lambda_2) - \lambda_{12}\lambda_{21}](s + rv_3^2 + \lambda_3) - [\lambda_{13}(s + rv_2^2 + \lambda_2) + \lambda_{12}\lambda_{23}\lambda_{31}] - [\lambda_{13}\lambda_{21} + (s + rv_1^2 + \lambda_1)\lambda_{23}\lambda_{32}]
$$

and $M$ is $3 \times 3$ matrix with element such that

$$m_{ii} = (s + rv_j^2 + \lambda_j)(s + rv_k^2 + \lambda_k) - \lambda_{jk}\lambda_{kj}$$
$$m_{ij} = \lambda_{ij}(s + rv_k^2 + \lambda_k) + \lambda_{ik}\lambda_{kj},$$

for all different $i, j, k$. Multiplying $(sI - \Lambda(r))^{-1}$ by $\bar{T} = (1,1,1)'$, we obtain

$$\int_0^\infty e^{-sT}L_1(r,T)\,dT = \frac{s^2 + (rv_1^2 + rv_2^2) + \lambda_1 + \lambda_{12} + \lambda_{13} + \lambda_2 + \lambda_3) s + r^2(v_2^2v_3^2) + r(v_2^2(\lambda_1 + \lambda_3) + v_3^2(\lambda_1 + \lambda_2)) + c_1}{D(r,s)}$$

with $c_1 = \lambda_{12}(\lambda_{23} + \lambda_3) + \lambda_{13}(\lambda_{32} + \lambda_2) + \lambda_2\lambda_3 - \lambda_{23}\lambda_{32}$. The last expression may be inverted with respect to $s$, yielding a combination of exponentials times polynomials, that can be inserted into the Parseval integral. It is obvious that $\mathbb{E}e^{-rU_T}$ is finite for all $r$.

4.3 Numerical results

We show European put prices computed using the Parseval formula described in Section 3.4, when the stochastic volatility is a Markov chain with $N = 2$ or $3$ states. When $N = 2$, these may be compared with the exact price, found using the density of $U_T$, which is explicit, as shown in Section 4.2.2. When $N = 3$, a comparison is made with Monte Carlo simulation results. In the case $N = 2$, the generator of the Markov chain is

$$Q_1 = \left(\begin{array}{cc}
    -1 & 1 \\
    1 & -1 
\end{array}\right),$$
and there are two different volatility vectors, \((v_1, v_2) = (0.1, 0.3)\) for small change and \((0.1, 0.9)\) for large change between two states (see Tables 4.1 and 4.2). We use two different generators in the case \(N = 3\) (see Tables 4.3 and 4.4):

\[
Q_2 = \begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
1 & 1 & -2
\end{pmatrix}, \quad Q_3 = \begin{pmatrix}
-4 & 3 & 1 \\
2 & -4 & 2 \\
1 & 3 & -4
\end{pmatrix}
\]

and one volatility vector \((v_1, v_2, v_3) = (0.1, 0.2, 0.3)\).

European put prices are computed for various maturities \(T\) from 0.25 to 3 years and three strikes, one OTM, one ATM and one ITM. In the case of Monte Carlo simulation numbers, the margin of error for 95% confidence intervals are shown in brackets; MC 1 and MC 2 refer to the Monte Carlo simulation using one million runs where the number 1 and 2 represent the initial state of the Markov chain, respectively. As the Brownian motion and the volatility are independent, it is not required to simulate the Brownian motion, since

\[
e^{-rT}E[(K - S_T)_+] = E\left[e^{-rT}\left(K - S_0 \exp\left\{rT - \frac{UT}{2} + \sqrt{UT}Z\right\} + V\right)\right].
\]

The conditional expectation on the right is simply the Black-Scholes formula for a put with maturity \(T\), strike \(K\) and volatility \(\sigma = \sqrt{UT/T}\). Hence, it is sufficient to compute the Black-Scholes formula for such a put and take the average over all runs of \(V\). In order to simulate the process \(V\), we generate the next state where the Markov chain switches and the length of the inter-arrival time, that is, the time for a continuous process to change regime based on the intensity matrix \(Q\).

The Fourier integral performs very well compared with Monte Carlo simulation. Execution times are shown at the bottom of each column. These are the times in seconds it took to compute all 18 numbers in each column, on a MacBook Pro 2.66GHz with 4GB RAM. Parseval integrals were computed using Mathematica and Monte Carlo simulations were coded in C. It is of course much faster to compute a single integral numerically than to perform one million simulation runs, and the coding is easier as well. Using the explicit density (case \(N = 2\)) is not as fast as one would have thought, no doubt because the density is a special function. In theory, the parameter \(\alpha\) (in the exponential damping factor \(e^{-\alpha x}\)) can take any positive value, the numerical computations do not always follow the theory, as the integrand is different for different values of \(\alpha\). The Mathematica \texttt{NIntegrate} function does a very good job. However, if one uses the option \texttt{GlobalAdaptive} and higher values of the parameters \texttt{MaxErrorIncreases} and \texttt{MaxRecursion} (setting these parameters
to 500 and 100, respectively), then it allowed us to obtain the same Parseval value for $\alpha \in [0.01, 500]$ for the shorter maturities, say $T = 0.5$, with errors of less than 0.001 in all cases. For longer maturities the range of ‘good’ $\alpha$ was a little narrower, say $[0.001, 250]$. These results are shown in Figure 4.1 and 4.2. This is reassuring as far as $\alpha$ is concerned, but we did notice that for short maturities (for example $T = 0.25$) the Parseval values are slightly less accurate, which may be because the integrand is more oscillatory (as can be seen in the tables, a few of the Parseval values are outside the 95% confidence interval).

### 4.3.1 Implied volatility curves

Finally, we show some implied volatility curves obtained with a Markov regime switching volatility model. The implied volatility is the constant volatility $\sigma$ that makes an option price in a real market equal to what it should be in the ordinary Black-Scholes model. The *raison d'être* of stochastic volatility models is the well-known fact that implied volatilities from observed real-world option prices, when plotted against the strike price, are not constant, as they would be if the Black-Scholes model were correct. The curve when plotting the implied volatility against strike $K$ is called the *volatility smile*. We take the Markov chain volatility put prices as observed prices, and back out the implied volatility from the Black-Scholes formula using the Newton-Raphson method. Prices obtained using Parseval’s theorem are used for this exercise.

Figure 4.3 to 4.6 show the implied volatility in four different cases. In each figure, the maturities are 0.25, 0.50, 1, 2 and 3 years. The sharpest smile is for $T = 0.25$ and the flattest is for $T = 3$. In all cases, the volatility curves are concave. The steepest concavity is obtained for the shortest maturity (dashed line) of 0.25 year. Other volatility vectors and intensities would produce more or less different curves, but in all cases we examined, the Markov volatility model does indeed produce *volatility smiles* with shapes that are not too different from those obtained from real-world option prices.
The Markov Regime Switching Model

Table 4.1: Put option prices for $N = 2$ states, $S_0$ = 100, $r = 0.05$, and $v_1 = v_2 = 0.3$. $\lambda_{12} = \lambda_{21} = 1$.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>1.3940</th>
<th>0.4577</th>
<th>4.8433</th>
<th>1.3940</th>
<th>0.4577</th>
<th>4.8433</th>
<th>0.4577</th>
<th>4.8433</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K=80$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M=120$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>1.2542</td>
<td>1.2534</td>
<td>1.2527</td>
<td>1.2534</td>
<td>1.2542</td>
<td>1.2527</td>
<td>1.2527</td>
<td>1.2542</td>
</tr>
<tr>
<td>0.50</td>
<td>1.2542</td>
<td>1.2534</td>
<td>1.2527</td>
<td>1.2534</td>
<td>1.2542</td>
<td>1.2527</td>
<td>1.2527</td>
<td>1.2542</td>
</tr>
<tr>
<td>0.75</td>
<td>1.2542</td>
<td>1.2534</td>
<td>1.2527</td>
<td>1.2534</td>
<td>1.2542</td>
<td>1.2527</td>
<td>1.2527</td>
<td>1.2542</td>
</tr>
<tr>
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<td>1.2542</td>
<td>1.2534</td>
<td>1.2527</td>
<td>1.2534</td>
<td>1.2542</td>
<td>1.2527</td>
<td>1.2527</td>
<td>1.2542</td>
</tr>
<tr>
<td>2.00</td>
<td>1.2542</td>
<td>1.2534</td>
<td>1.2527</td>
<td>1.2534</td>
<td>1.2542</td>
<td>1.2527</td>
<td>1.2527</td>
<td>1.2542</td>
</tr>
<tr>
<td>3.00</td>
<td>1.2542</td>
<td>1.2534</td>
<td>1.2527</td>
<td>1.2534</td>
<td>1.2542</td>
<td>1.2527</td>
<td>1.2527</td>
<td>1.2542</td>
</tr>
</tbody>
</table>

| $K=100$    |        |        |        |        |        |        |        |        |
| $M=100$    |        |        |        |        |        |        |        |        |
| 0.25       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 0.50       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 0.75       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 1.00       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 2.00       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 3.00       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |

| $K=120$    |        |        |        |        |        |        |        |        |
| $M=80$     |        |        |        |        |        |        |        |        |
| 0.25       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 0.50       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 0.75       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 1.00       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 2.00       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |
| 3.00       | 1.2542 | 1.2534 | 1.2527 | 1.2534 | 1.2542 | 1.2527 | 1.2527 | 1.2542 |

Table 4.1: Put option prices for $N = 2$ states, $S_0$ = 100, $v_1 = 0$, $v_2 = 0$. $\lambda_{12} = \lambda_{21} = 1$. $r = 0.05$.
### Table 4.2: Put option prices for $N = 2$ states, $S_0 = 100$, $v_1 = 0.1$, $v_2 = 0.9$, $\lambda_{12} = \lambda_{21} = 1$ and $r = 0.05.$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$T$</th>
<th>Exact 1</th>
<th>Parseval 1</th>
<th>MC 1</th>
<th>Exact 2</th>
<th>Parseval 2</th>
<th>MC 2</th>
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</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.78430</td>
<td>0.78430</td>
<td>0.7841 (±0.00359)</td>
<td>6.71019</td>
<td>6.71019</td>
<td>6.71111 (±0.00368)</td>
</tr>
<tr>
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<td>0.50</td>
<td>2.53096</td>
<td>2.53096</td>
<td>2.52852 (±0.00774)</td>
<td>10.65711</td>
<td>10.65712</td>
<td>10.65951 (±0.00703)</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>4.53410</td>
<td>4.53410</td>
<td>4.52420 (±0.01087)</td>
<td>13.22986</td>
<td>13.22986</td>
<td>13.23922 (±0.00938)</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>6.53248</td>
<td>6.53248</td>
<td>6.52164 (±0.01312)</td>
<td>15.13610</td>
<td>15.13610</td>
<td>15.14754 (±0.01102)</td>
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<tr>
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<td>13.20313</td>
<td>13.20442 (±0.01683)</td>
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<td>17.73808 (±0.01704)</td>
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<td>3.56511</td>
<td>3.56367 (±0.00871)</td>
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<td>15.86794</td>
<td>15.87049 (±0.00574)</td>
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<tr>
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<td>6.92071</td>
<td>6.92090</td>
<td>6.91402 (±0.01439)</td>
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<td>20.57873</td>
<td>20.58338 (±0.00960)</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>10.13820</td>
<td>10.13671</td>
<td>10.15602 (±0.01806)</td>
<td>23.50175</td>
<td>23.50172</td>
<td>23.48957 (±0.01225)</td>
</tr>
<tr>
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<td>13.07611</td>
<td>13.07633</td>
<td>13.07644 (±0.02036)</td>
<td>25.61279</td>
<td>25.61279</td>
<td>25.61180 (±0.01402)</td>
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<tr>
<td></td>
<td>2.00</td>
<td>21.89143</td>
<td>21.89144</td>
<td>21.89083 (±0.02290)</td>
<td>30.83590</td>
<td>30.83590</td>
<td>30.83712 (±0.01694)</td>
</tr>
<tr>
<td></td>
<td>3.00</td>
<td>27.28129</td>
<td>27.28129</td>
<td>27.26103 (±0.02182)</td>
<td>33.88203</td>
<td>33.86987</td>
<td>33.88203 (±0.01716)</td>
</tr>
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<tr>
<td></td>
<td>0.25</td>
<td>19.83049</td>
<td>19.83047</td>
<td>19.83444 (±0.00581)</td>
<td>28.91152</td>
<td>28.91152</td>
<td>28.90798 (±0.00523)</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>21.26640</td>
<td>21.26640</td>
<td>21.25964 (±0.01219)</td>
<td>33.47179</td>
<td>33.47179</td>
<td>33.47997 (±0.00964)</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>23.30443</td>
<td>23.30441</td>
<td>23.29877 (±0.01703)</td>
<td>36.30230</td>
<td>36.30230</td>
<td>36.30913 (±0.01275)</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>25.51878</td>
<td>25.51876</td>
<td>25.51311 (±0.02042)</td>
<td>38.32272</td>
<td>38.32272</td>
<td>38.32707 (±0.01492)</td>
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<tr>
<td></td>
<td>2.00</td>
<td>33.33241</td>
<td>33.33243</td>
<td>33.32724 (±0.02531)</td>
<td>43.19882</td>
<td>43.19881</td>
<td>43.19839 (±0.01863)</td>
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<td>38.51273</td>
<td>38.50698 (±0.02467)</td>
<td>45.91883</td>
<td>45.91881</td>
<td>45.91940 (±0.01906)</td>
</tr>
</tbody>
</table>

| Time (sec) | 9.348 | 0.778 | 13.885 | 7.614 | 2.073 | 13.835 |

4.3. Numerical results
The Table 4.3: Put option prices with $N = 3$ states, $S_0 = 100$, $r = 0.05$, $\lambda_{12} = \lambda_{13} = \lambda_{21} = \lambda_{23} = \lambda_{31} = \lambda_{32} = 1$ and $K = 30$.

<table>
<thead>
<tr>
<th>Time (sec)</th>
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<tr>
<td></td>
<td>8.298</td>
</tr>
<tr>
<td>$K=30$</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>17.32979</td>
</tr>
<tr>
<td>0.50</td>
<td>19.92178</td>
</tr>
<tr>
<td>0.75</td>
<td>22.26218</td>
</tr>
<tr>
<td>1.00</td>
<td>21.90674</td>
</tr>
<tr>
<td>2.00</td>
<td>21.89507</td>
</tr>
</tbody>
</table>

| $K=60$     |                 |
| 0.25       | 17.32979        |
| 0.50       | 19.92178        |
| 0.75       | 22.26218        |
| 1.00       | 21.90674        |
| 2.00       | 21.89507        |

<p>| $K=120$    |                 |
| 0.25       | 17.32979        |
| 0.50       | 19.92178        |
| 0.75       | 22.26218        |
| 1.00       | 21.90674        |
| 2.00       | 21.89507        |</p>
<table>
<thead>
<tr>
<th>T</th>
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<th>MC 1</th>
<th>Parseval 2</th>
<th>MC 2</th>
<th>Parseval 3</th>
<th>MC 3</th>
</tr>
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<td></td>
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<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.02090</td>
<td>0.02221 (±0.00011)</td>
<td>0.04110</td>
<td>0.03877 (±0.00011)</td>
<td>0.20392</td>
<td>0.20452 (±0.00024)</td>
</tr>
<tr>
<td>0.50</td>
<td>0.17284</td>
<td>0.17241 (±0.00044)</td>
<td>0.22590</td>
<td>0.22592 (±0.00044)</td>
<td>0.58535</td>
<td>0.58580 (±0.00069)</td>
</tr>
<tr>
<td>0.75</td>
<td>0.41634</td>
<td>0.41694 (±0.00077)</td>
<td>0.47072</td>
<td>0.47125 (±0.00075)</td>
<td>0.92360</td>
<td>0.92242 (±0.00099)</td>
</tr>
<tr>
<td>1.00</td>
<td>0.68808</td>
<td>0.68893 (±0.00102)</td>
<td>0.72182</td>
<td>0.72224 (±0.00098)</td>
<td>1.22324</td>
<td>1.22200 (±0.00120)</td>
</tr>
<tr>
<td>2.00</td>
<td>1.64943</td>
<td>1.64982 (±0.00149)</td>
<td>1.56147</td>
<td>1.56266 (±0.00143)</td>
<td>2.13770</td>
<td>2.13768 (±0.00156)</td>
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<tr>
<td>3.00</td>
<td>2.30506</td>
<td>2.30372 (±0.00162)</td>
<td>2.11971</td>
<td>2.11813 (±0.00155)</td>
<td>2.72216</td>
<td>2.72260 (±0.00164)</td>
</tr>
</tbody>
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<table>
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<tr>
<th>T</th>
<th>Parseval 1</th>
<th>MC 1</th>
<th>Parseval 2</th>
<th>MC 2</th>
<th>Parseval 3</th>
<th>MC 3</th>
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<td>0.25</td>
<td>2.37380</td>
<td>2.37279 (±0.00201)</td>
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<td>3.21532 (±0.00146)</td>
<td>4.59332</td>
<td>4.59418 (±0.00169)</td>
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<td>4.22417 (±0.00217)</td>
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<td>5.63668 (±0.00234)</td>
</tr>
<tr>
<td>0.75</td>
<td>4.58457</td>
<td>4.58381 (±0.00298)</td>
<td>4.91641</td>
<td>4.91623 (±0.00254)</td>
<td>6.25834</td>
<td>6.25852 (±0.00263)</td>
</tr>
<tr>
<td>1.00</td>
<td>5.26093</td>
<td>5.26099 (±0.00307)</td>
<td>5.43337</td>
<td>5.43287 (±0.00273)</td>
<td>6.70472</td>
<td>6.70432 (±0.00277)</td>
</tr>
<tr>
<td>2.00</td>
<td>6.75391</td>
<td>6.75295 (±0.00304)</td>
<td>6.58856</td>
<td>6.59031 (±0.00290)</td>
<td>7.70916</td>
<td>7.71117 (±0.00287)</td>
</tr>
<tr>
<td>3.00</td>
<td>7.37681</td>
<td>7.37772 (±0.00290)</td>
<td>7.04584</td>
<td>7.04656 (±0.00283)</td>
<td>8.10787</td>
<td>8.10712 (±0.00280)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>T</th>
<th>Parseval 1</th>
<th>MC 1</th>
<th>Parseval 2</th>
<th>MC 2</th>
<th>Parseval 3</th>
<th>MC 3</th>
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<tbody>
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</tr>
<tr>
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<td>18.62049 (±0.00039)</td>
<td>18.71949</td>
<td>18.71997 (±0.00039)</td>
<td>19.22279</td>
<td>19.22302 (±0.00067)</td>
</tr>
<tr>
<td>0.50</td>
<td>17.80485</td>
<td>17.80504 (±0.00137)</td>
<td>18.03066</td>
<td>18.03099 (±0.00125)</td>
<td>18.94495</td>
<td>18.94408 (±0.00160)</td>
</tr>
<tr>
<td>0.75</td>
<td>17.40262</td>
<td>17.40136 (±0.00216)</td>
<td>17.61266</td>
<td>17.61112 (±0.00195)</td>
<td>18.69958</td>
<td>18.70078 (±0.00220)</td>
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<td>17.18160 (±0.00268)</td>
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<td>17.31811 (±0.00244)</td>
<td>18.48229</td>
<td>18.48321 (±0.00259)</td>
</tr>
<tr>
<td>2.00</td>
<td>16.66320</td>
<td>16.66574 (±0.00340)</td>
<td>16.47887</td>
<td>16.47864 (±0.00325)</td>
<td>17.73044</td>
<td>17.72743 (±0.00321)</td>
</tr>
<tr>
<td>3.00</td>
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<td>16.11589 (±0.00347)</td>
<td>15.72088</td>
<td>15.72269 (±0.00340)</td>
<td>16.98706</td>
<td>16.98761 (±0.00331)</td>
</tr>
</tbody>
</table>

| Time (sec) | 6.758 | 23.861 | 7.173 | 23.719 | 9.040 | 23.711 |

Table 4.4: Put option prices with $N = 3$ states, $S_0 = 100$, $v_1 = 0.1$, $v_2 = 0.2$, $v_3 = 0.3$, $\lambda_{12} = \lambda_{32} = 3$, $\lambda_{13} = \lambda_{31} = 1$, $\lambda_{21} = \lambda_{23} = 2$ and $r = 0.05$. 
The Markov Regime Switching Model

Figure 4.1: Percentage error for ATM put option price using Parseval integral (N = 2) at maturity 0.5 year. The command \texttt{NIntegrate} in Mathematica is used to evaluate Parseval integral numerically with \texttt{MaxErrorIncreases \rightarrow 100}. The error is less than 0.001 percent up to α = 500.
Figure 4.2: Percentage error for ATM put option price using Parseval integral ($N = 2$) at maturity 2 years. The command \texttt{NIntegrate} in Mathematica is used to evaluate Parseval integral numerically with \texttt{MaxErrorIncreases} $\rightarrow 100$. The error is less than 0.001 percent up to $\alpha = 212$. 
The dashed line is for the shortest maturity of 0.25 year and the solid with stars is for the longest maturity of 3 years.

The Markov Regime Switching Model

Figure 4.3: Implied volatility curves for different maturities when the volatility is modelled by 2-state Markov regime switching model with parameters $\lambda_1 = 1, \lambda_2 = 1, \gamma_1 = 0.1, \gamma_2 = 0.3$. It is designed for a mild market condition. Spot is 100 and initial volatility is $v_1$. The dashed line is for the shortest maturity of 0.25 year and line with stars (\*) is for the longest maturity 3 years.

The figure shows the implied volatility curves for different maturities when the volatility is modelled by a 2-state Markov regime switching model with parameters $\lambda_1 = 1, \lambda_2 = 1, \gamma_1 = 0.1, \gamma_2 = 0.3$. It is designed for a mild market condition. Spot is 100 and initial volatility is $v_1$. The dashed line is for the shortest maturity of 0.25 year and line with stars (\*) is for the longest maturity 3 years.
Figure 4.4: Implied volatility curves for different maturities when the volatility is modelled by 2-state Markov regime switching model with parameters $v_1 = 0.2$, $v_2 = 0.6$ and the generator matrix is $Q_1$. State 2 has very high volatility which leads to an extreme market condition. Spot is 100 and initial volatility is $v_1$. The dashed line is for the shortest maturity of 0.25 year and line with star(*) is for the longest maturity of 3 years.
The Markov Regime Switching Model

Figure 4.5: Implied volatility curves for various maturities when the volatility is modeled by a 3-state Markov regime switching model with parameters \( v_1 = 0.1, v_2 = 0.2, v_3 = 0.3 \) and the generator matrix is \( Q_2 \). Spot is 100 and initial volatility is \( v_1 \). The dashed line is for the shortest maturity of 0.25 year and line with star (*) is for the longest maturity of 3 years. For different set of parameters we observed the similar pattern of smiles.
Figure 4.6: Implied volatility curves for various maturities when the volatility is modelled by 3-state Markov regime switching model with parameters $v_1 = 0.1$, $v_2 = 0.2$, $v_3 = 0.3$ and the generator matrix is $Q_3$. Spot is 100 and initial volatility is $v_1$. The dashed line is for the shortest maturity of 0.25 year and line with star(*) is for the longest maturity of 3 years. For different set of parameters we observed the similar pattern of smiles.
Chapter 5

The Square-Root Process

In 1985, Cox et al. [35] proposed a tractable one-factor model for the spot rate of interest as a square-root process that guarantees positivity, which eventually became known as the Cox-Ingersoll-Ross (CIR) model. The square-root process has been a benchmark for many years because of its positivity and analytical tractability. Additionally, it has been widely used in financial problems as an alternative to the geometric Brownian motion. In the early 1990’s, the square-root process was introduced to form the stochastic volatility component of Heston’s asset price model. Since then, the square-root process has become popular because of the existence of an analytic solution for European-style options. This computational advantage is especially useful when calibrating the model to the market prices.

Dufresne [42] shows how to compute any moment of the square-root process, including moments of integrated variance through repeated application of Itô’s Lemma. Andersen [4] suggests various simulation schemes for the square-root process based on moment-matching. Shirakawa [108] derives an arbitrage-free bond pricing formula based on the CIR model applying a time change of the squared Bessel process. Further, a stochastic interest rate is modelled by the squared Bessel process with a jump to price zero coupon bond in Chou and Lin [30]. With a time change, the square-root process can be transformed to a squared radial Ornstein-Uhlenbeck process. Göing-Jaeschke and Yor [58] study the close relationship between (squared) Bessel processes and the square-root process, in particular the first hitting
time of the square-root process and squared Bessel processes and squared Ornstein-Uhlenbeck processes with negative dimension. Yuan and Kalbfleisch [120] discuss several different simulation schemes for the Bessel process, whereas Glasserman and Kim [56] claim that the distribution of an integrated variance process can be expressed as an infinite sum of a mixture of gamma distribution based on the works of Pitman and Yor [98].

The outline of this chapter is as follows. In Section 5.1, we study the distributional results of the square-root process and the integrated square-root process. Then, we explore some properties of the squared Bessel process in Section 5.2. After a brief introduction to the measure change technique, we present change-of-measure formulas between the squared Bessel process, squared radial Ornstein-Uhlenbeck process and square-root process in Section 5.3. In addition, we investigate the behaviour of the Radon-Nikodym derivative. Section 5.4 details different option pricing methods in which the variance is a square-root process. Section 5.5 reports the details of numerical experiments and results with a brief discussion of our numerical tests. Results from Section 5.3 also appear in Chin and Dufresne [28].

5.1 Distributional results for the square-root process

The square-root process is the unique strong solution of the SDE

\[ dV_t = (aV_t + b)\,dt + c\sqrt{|V_t|}\,dW_t, \quad V_0 = v_0 \geq 0, \]  
(5.1)

where \( a \in \mathbb{R}, b > 0, c > 0 \) and \( W \) is a standard Brownian motion. If \( a = 0 \) and \( v_0 = 0 \), then \( V_t \equiv 0 \) is the solution to the SDE (5.1). For \( b \geq 0 \), \( V_t \geq 0 \) a.s. by the comparison theorem (see Revuz and Yor [101] Chapter IX Theorem 3.7). When \( b = 0 \) the behaviour of the process \( \{V_t\} \) is sometimes called absorbing; the process remains 0 once it hits 0, a.s. if \( a \leq 0 \) and with positive probability if \( a > 0 \). If \( a < 0 \) the square-root process has the mean-reverting property with long-term mean \(-b/a\).

Denote \( Q \) the law of the square-root process solution of the (5.1). The square-root process has a transition density \( p_t(x, y) := Q(V_{t+s} \in dy | V_s = x) \) (see Feller [49] Lemma 9 or Cox et al. [35] Eq.(18)) such that

\[
p_t(x, y) = \frac{2ae^{-at}}{c^2(1 - e^{-at})} \left( \frac{y}{xe^{at}} \right)^{\nu/2} \exp \left\{ -\frac{2a\,x + e^{-at}y}{c^2} \right\} I_{\nu} \left( \frac{4a\sqrt{xy}e^{-at/2}}{c^2(1 - e^{-at})} \right),
\]  
(5.2)

where \( \nu = 2b/c^2 - 1 \) and \( I_{\nu} \) is the modified Bessel function of the first kind of order
\[ I_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |\arg z| < \pi, \quad \nu \in \mathbb{C}. \] (5.3)

That is, conditioning on \( V_s \) for \( s < t \), the square-root process \( V_t \) is distributed as a constant times a non-central chi-squared random variable. For general discussions of the square-root process, we refer to Cox et al. [35], Dassios and Nagaradjasarma [38], Dufresne [42], Glasserman [55], Göing-Jaeschke and Yor [58].

5.1.1 Moment of \( V_t \)

Given that \( \mathbb{E} V^n_t < \infty \) for all \( n, t \geq 0 \), a general formula for the moments of \( V_t \) can be obtained recursively by applying Itô’s lemma.

**Theorem 5.1** (Dufresne [42] Theorem 2.3). Suppose \( a \neq 0 \). Then, for non-negative integer \( n \), the moments of the square-root process \( V_t \) are

\[ \mathbb{E} V^n_t = \sum_{j=0}^{n} \theta_{n,j} e^{aj t}, \]

where

\[ \theta_{n,j} = \frac{j!}{i!} (\bar{y}^{n-i}) \frac{e^{a(j-i)}}{\bar{y}^{j}}, \quad 0 \leq j \leq n \]

\[ \bar{x} = \frac{c^2}{2a^2}, \quad \bar{y} = \frac{2b}{c^2} \]

\[ (y)_0 = 1, \quad (y)_n = y(y+1)\cdots(y+n-1), \quad n \geq 1. \]

**Proof.** For the proof we refer to Dufresne [42] p.7.

**Corollary 5.2** ([4, 35]). The mean and variance of \( V_t \) conditional on \( V_0 \) are given by

\[ \mathbb{E}[V_t|V_0 = v_0] = -\frac{b}{a} + \left( v_0 + \frac{b}{a} \right) e^{at}, \]

\[ \text{Var}[V_t|V_0 = v_0] = \frac{bc^2}{2a^2} + \left( \frac{c^2}{a} v_0 + \frac{bc^2}{a^2} \right) e^{at} + \left( \frac{c^2}{a} v_0 + \frac{bc^2}{2a^2} \right) e^{2at}. \]

Note that the variance of \( V_t \) increases as \( c \) increases and decreases as \( a \) increases.

**Theorem 5.3** (Dufresne [44] p.9). The Laplace transform of the distribution of \( V_t \) is

\[ \mathbb{E} e^{-\nu V_t} = \phi(p)^\nu \exp \{-\lambda_t (1 - \phi(p))\}, \]

where

\[ \phi(p) = \frac{1}{1 + p\mu_t}, \quad \mu_t = \frac{c^2}{2} \left( \frac{e^{at} - 1}{a} \right), \quad \lambda_t = \frac{2av_0}{c^2(1 - e^{-at})}. \]
Proof. See Dufresne [44]. He shows that the Laplace transform of the distribution of $V_t$ may be obtained by summing the moments of $V_t$.

**Proposition 5.4.** For non-integer $p > -b$, we have

$$E(V_t^p) = \mu_t^p \frac{\Gamma(b + p)}{\Gamma(b)} \, _1F_1(-p, b; -\lambda_t).$$

Proof. Since

$$\frac{1}{y^p} = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} e^{-sy} ds, \quad p, y > 0,$$

we have

$$E(V_t^{-p}) = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} e^{-sV_t} ds = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} \phi(s)^b e^{\lambda_t(\phi(s)-1)} ds = \frac{\mu_t^{-p} e^{-\lambda_t}}{\Gamma(p)} \int_0^1 r^{b-p-1}(1-r)^{p-1} e^{\lambda_tr} dr = \frac{\mu_t^{-p} e^{-\lambda_t}}{\Gamma(b)} \, _1F_1(b-p, b; \lambda_t),$$

where

$$1_F_1(p, q; z) = \sum_{n=0}^\infty \frac{(p)_n z^n}{(q)_n n!}, \quad p, q \in \mathbb{C}, \quad q \neq 0, -1, \ldots$$

is the confluent hypergeometric function and its integral representation (see Lebedev [77] p.266) is

$$\frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 e^{zu} u^{p-1} (1-u)^{q-p-1} du, \quad \text{Re}(q) > \text{Re}(p) > 0.$$

Applying the identity

$$1_F_1(p, q; z) = e^z 1_F_1(q-p, q; -z)$$

completes the proof.

5.1.2 Sampling from non-central chi-square distribution

According to Cox et al. [35] the square-root process is distributed as a non-central chi-square with $d$ degrees of freedom and a non-centrality parameter $\lambda$. Since the non-central chi-square random variable $\chi^2_d(\lambda)$ has distribution (see Glasserman [55] or Andersen [4])

$$P(\chi^2_d(\lambda) \leq y) = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{2^{d/2+j} \Gamma(d/2+j)} \int_0^y z^{d/2+j-1} e^{-z/2} dz, \quad y > 0,$$

(5.4)
the transition law of $V_t$ can be written as

$$\frac{c^2(1 - e^{a\tau})}{-4a} \chi_d^2 \left( \frac{-4ae^{a\tau}}{c^2(1 - e^{a\tau})} V_s \right), \quad t > s,$$

where

$$\tau = t - s, \quad d = \frac{4b}{c^2}, \quad \lambda = \frac{-4ae^{a\tau}}{c^2(1 - e^{a\tau})} V_s.$$

That is, given $V_s$, $V_t$ is distributed as a constant times a non-central chi-square random variable with $d$ degrees of freedom and non-centrality parameter $\lambda$. Thus, we can simulate the process exactly in a discrete time provided we can sample from the non-central chi-square distribution. Note that the degrees of freedom of non-central chi-square distribution are the same as the dimensionality of the squared Bessel process. When we simulate the non-central chi-square distribution it is easy to use chi-square distribution because $d \in \{1, 2, \ldots \}$. Let $X_i$ and $Z_i$ be $N(a_i, 1)$ and $N(0, 1)$, respectively, and

$$Y = X_1^2 + \cdots + X_d^2,$$

where $X_i$’s are independent. Then, for $d \in \mathbb{N}$, the distribution of $Y$

$$\sum_{i=1}^{d} (Z_i + a_i)^2$$

is non-central chi-square with $d$ degrees of freedom and non-centrality parameter $\lambda = \sum_{i=1}^{d} a_i^2$. That is, when $d$ is an integer we generate non-central chi-square distribution as

$$\chi_d^2(\lambda) = \chi_1^2(\lambda) + \chi_{d-1}^2,$$

provided that

$$P(\chi_d^2 \leq y) = \frac{1}{2^{d/2}\Gamma(d/2)} \int_0^y e^{-z^{2/2}} z^{d/2-1} dz,$$

where $\Gamma(\cdot)$ denotes the gamma function.

Simulating the process based on the (non-central) chi-squared distribution can be found in Glasserman [55] p.122. Alternatively, simulation can be done by noting that $\phi(p)$ in Theorem 5.3 is the Laplace transform of an exponential distribution with a mean of $\mu_t$. Therefore, the distribution of $V_t$ is the combination of Gamma distribution with parameters $\bar{b}$ and $\mu_t$, and compound Poisson with a mean of $\lambda_t$, which are independent. That is, the distribution of $V_t$ conditional on $V_0$ admits the representation

$$\{V_t|V_0 = v_0\} \xrightarrow{d} V_t + V_t'',$
where \( V_t' \sim \text{Gamma}(\bar{b}, \mu_t) \) and \( V_t'' \) is compound Poisson such that
\[
\sum_{i=1}^{N} U_i,
\]
with \( N \sim \text{Poisson}(\lambda_t) \) and \( U_i \sim \text{Exp}(\mu_t) \). See Dufresne [42] p.9 for more details about simulating non-central chi-square distribution. For the complete step-by-step algorithm, we refer the readers to Glasserman [55] Section 3.4.2.

### 5.1.3 The integrated square-root process

The integrated variance process
\[
U_t = \int_0^t V_s \, ds
\]
is studied in Dufresne [42]. Recently Glasserman and Kim [56] show that the exact distribution of \( U_t \) conditional on the end points \( V_0 \) and \( V_t \) can be expressed as an infinite sum of mixtures of gamma random variables.

**Laplace transform of \((U_t, V_t)\)**

The Laplace transform of the joint distribution of \( U_t \) and \( V_t \) is derived using a measure change relating to the square-root process \( V_t \) to a time-changed squared Bessel process. For readers who are interested in the time change formula, we refer them to Revuz and Yor [101] Chapter V Theorem 1.6. We will revisit the strong relationship between the processes after discussing the properties of Bessel process in Section 5.2.

**Theorem 5.5** (Dufresne [44] p.23). Denote \( P := P(p) = \sqrt{a^2 + 2c^2 p} \). Then, the Laplace transform of the joint distribution of \( U_t \) and \( V_t \) is given by
\[
E \exp \{ -pU_t - qV_t \} = \left[ A_1(p, q, t) \right]^{-5} \exp \{ A_2(p, q, t) \}, \quad (5.5)
\]
where
\[
A_1(p, q, t) = \frac{e^{at/2}}{P} \left[ P \cosh \left( \frac{Pt}{2} \right) - a \sinh \left( \frac{Pt}{2} \right) + c^2 q \sinh \left( \frac{Pt}{2} \right) \right]
\]
and
\[
A_2(p, q, t) = -\frac{v_0}{c^2} \left[ P + a - \frac{(P + a - c^2 q) e^{-(P-a)t/2}}{A_1(p, q, t)} \right].
\]

**Proof.** It can be proved using Lemma 1 in G"oing-Jaeschke and Yor [58]. See Dufresne [44] Section 2.2.
5.1. Distributional results for the square-root process

Letting $p = 0, q \neq 0$ and $q = 0, p \neq 0$ in (5.5) give the expression for the Laplace transform of the distribution of $V_t$ as discussed in Theorem 5.1 and the Laplace transform of the distribution of $U_t$, respectively.

**Corollary 5.6** (Dufresne [44] p.23).

\[
E e^{-pU_t} = \left[ \frac{e^{-at/2}}{\cosh(Pt/2) - \frac{a}{p} \sinh(Pt/2)} \right]^{\frac{2b}{p}} \exp \left\{ - \frac{pv_0}{P} \cosh(Pt/2) - \frac{2}{p} \sinh(Pt/2) \right\} .
\]

Moreover, the moments of $U_t$ can be found using the well-known fact

\[
EU_t^n = (-1)^n \left. \frac{d^n}{dp^n} E e^{-pU_t} \right|_{p=0}, \quad n = 1, 2, \ldots .
\]

The first three moments are given (Dufresne [42]) by

\[
EU_t = -\frac{v_0}{a} - \frac{b}{a^2} - \frac{b}{a} t + \left( \frac{v_0}{a} + \frac{b}{a^2} \right) e^{at}
\]

\[
EU_t^2 = \frac{v_0^2}{a^2} + \frac{v_0(2b - c^2)}{a^3} + \frac{2b^2 - 5bc^2}{2a^4} + \left( \frac{2v_0}{a^2} + \frac{2b^2 - bc^2}{a^3} \right) t + \frac{b^2}{a^2} t^2
\]

\[
- \left[ \frac{2v_0}{a^2} + \frac{4bv_0}{a^3} + \frac{2b(b - c^2)}{a^4} + \frac{2(b + c^2)}{a^2} \left( \frac{v_0 + b}{a} \right) t \right] e^{at}
\]

\[
+ \left[ \frac{v_0^2}{a^2} + \frac{2bv_0 + c^2v_0}{a^3} + \frac{2b^2 + bc^2}{a^4} \right] e^{2at}
\]

\[
EU_t^3 = -\frac{v_0^3}{a^3} - \frac{3v_0b}{a^4} - \frac{3v_0b^2}{a^5} - \frac{b^3}{a^5} + \frac{3v_0c}{a^4} + \frac{21v_0bc}{2a^5} + \frac{15b^2c^2}{2a^6} - \frac{3v_0c^4}{a^6} - \frac{11bc^4}{a^6}
\]

\[
+ t \left( \frac{3v_0b}{a^3} - \frac{6v_0b^2}{a^4} - \frac{3b^3}{a^5} + \frac{6v_0bc}{a^4} + \frac{21b^2c^2}{2a^5} - \frac{3bc^4}{a^6} \right)
\]

\[
+ t^2 \left( -\frac{3v_0b^2}{a^3} - \frac{3b^3}{a^4} + \frac{3b^2c^2}{a^5} \right) - \frac{b^3}{a^3} t^3
\]

\[
+ e^{at} \left[ \frac{3v_0^3}{a^3} + \frac{9v_0b}{a^4} + \frac{9v_0b^2}{a^5} + \frac{3b^3}{a^5} - \frac{3v_0c}{a^4} - \frac{33v_0bc}{2a^5} - \frac{27b^2c^2}{2a^6} - \frac{3v_0c^4}{2a^6} + \frac{15bc^4}{2a^6} \right.
\]

\[
+ t \left( \frac{6v_0b}{a^3} + \frac{12v_0b^2}{a^4} + \frac{6b^3}{a^5} + \frac{6v_0bc}{a^4} + \frac{9v_0bc}{a^4} - \frac{3b^2c^2}{a^5} - \frac{3v_0c^4}{a^6} - \frac{9bc^4}{a^6} \right)
\]

\[
+ t^2 \left( \frac{3v_0b^2}{a^3} + \frac{3b^3}{a^4} + \frac{6v_0bc}{a^4} + \frac{6b^2c^2}{a^5} + \frac{3v_0c^4}{a^5} + \frac{3bc^4}{a^6} \right)
\]

\[
+ e^{2at} \left[ -\frac{3v_0^3}{a^3} - \frac{9v_0b}{a^4} - \frac{9v_0b^2}{a^5} - \frac{3b^3}{a^5} - \frac{3v_0c}{a^4} + \frac{3v_0bc^2}{a^4} + \frac{9b^2c^2}{2a^5} + \frac{3v_0c^4}{a^6} + \frac{3bc^4}{a^6} \right]
\]

\[
+ t \left( -\frac{3v_0b^2}{a^3} - \frac{6v_0b^2}{a^4} - \frac{3b^3}{a^5} - \frac{6v_0c}{a^4} - \frac{15v_0bc}{2a^5} - \frac{15b^2c^2}{2a^6} - \frac{6v_0c^4}{2a^6} - \frac{3bc^4}{a^6} \right)
\]

\[
+ e^{3at} \left( \frac{v_0^3}{a^3} + \frac{3v_0^2b}{a^4} + \frac{3v_0b^2}{a^5} + \frac{b^3}{a^6} + \frac{3v_0c^2}{2a^5} + \frac{9v_0bc^2}{2a^6} + \frac{3b^2c^2}{2a^6} + \frac{3v_0c^4}{2a^6} + \frac{bc^4}{2a^6} \right) .
\]
Remark 5.1. The joint moments of \((U_t, V_t)\) can also be obtained by solving a triangular set of PDEs, see Dassios and Nagaradjasarma [38]. From a computational point of view, Dufresne [42] claims that the recursive method outperforms when differentiating the Laplace transform or solving linear ordinary differential equations with constant coefficients
\[
\frac{d}{dt}M_{jk}(t) = a_k M_{jk}(t) + b_k M_{j,k-1}(t) + j M_{j-1,k+1}(t),
\]
where \(M_{jk}(t) := \mathbb{E}[U_j^t V_k^t]\).

5.2 Bessel processes

5.2.1 Definition and some properties

Bessel processes emerge in many mathematical finance problems because not only do they have remarkable mathematical structures, but they are also related to the square-root process. Göing-Jaeschke and Yor [58] studied the first hitting time of the Bessel process, which is critical, for instance, in the case of a barrier option (see Lo et al. [83] for up-and-out options under the square-root CEV process.) For more properties of the Bessel processes, see Revuz and Yor [101] Chapter XI, Dufresne [43], Yor [119] Chapter III and Pitman and Yor [97].

First, we give the definition of a Bessel process. Denote \(\rho := |W|\) where \(W\) is \(\delta\)-dimensional Brownian motion starting at \(x\). By Itô’s formula we have
\[
\rho_t^2 = \rho_0^2 + \delta t + 2 \sum_{i=1}^{\delta} \int_0^t W_i^s dW_i^s.
\]
(5.7)
The one-dimensional process
\[
\beta_t = \sum_{i=1}^{\delta} \int_0^t \frac{W_i^s}{\rho_s} dW_i^s, \quad t \geq 0,
\]
is a Brownian motion. It is a local martingale and
\[
\langle \beta, \beta \rangle_t = \sum_{i=1}^{\delta} \int_0^t \frac{(W_i^s)^2}{\rho_s^2} ds
\]
\[
= \int_0^t \left[ \frac{(W_1^s)^2 + \cdots + (W_\delta^s)^2}{(W_1^s)^2 + \cdots + (W_\delta^s)^2} \right] ds
\]
\[
= t.
\]
We then apply:
Theorem 5.7 (Lévy characterisation theorem, Revuz and Yor [101] Chapter IV Theorem 3.6). For a \((\mathcal{F}_t)\)-adapted continuous \(d\)-dimensional process \(X\) vanishing at 0, the following three conditions are equivalent

i) \(X\) is an \(\mathcal{F}_t\)-Brownian motion;

ii) \(X\) is a continuous local martingale and \(<X^i, X^j>_t = \delta_{ij}t\) for every \(1 \leq i, j \leq d\);

iii) \(X\) is a continuous local martingale and \(f \in L^2(\mathbb{R}+)\), the process

\[
\mathcal{E}(f)_t = \exp\left\{ \int_0^t f(s) \, dX_s + \frac{1}{2} \int_0^t f^2(s) \, ds \right\}
\]

is a complex martingale.

For positive integer value \(\delta\), therefore, we can rewrite (5.7) as

\[
\rho_t^2 = \rho_0^2 + \delta t + 2 \int_0^t \rho_s \, d\beta_s.
\]

The above implies the SDE

\[
Z_t = x + \delta t + 2 \int_0^t \sqrt{|Z_s|} \, d\beta_s,
\]

for any non-negative \(\delta\). The comparison theorem helps to remove the absolute value.

Definition 5.8. For every \(x, \delta \geq 0\), the unique strong solution of the equation

\[
Z_t = x + \delta t + 2 \int_0^t \sqrt{Z_s} \, d\beta_s
\]

is called the \(\delta\)-dimensional squared Bessel process started at \(x\) and is denoted by either BESQ\(_\delta\) \(_x\) or BESQ\(_\nu\) \(_x\), where \(\nu := \delta/2 - 1\) is called index of the process. The probability law of the process is denoted by \(Q^\delta\) \(_x\) or \(Q^\nu\) \(_x\).

Definition 5.9. For \(x \geq 0\) the square root of BESQ\(_\delta\) \(_x\) is called the Bessel process of dimension \(\delta\) started at \(x\) and is denoted by BES\(_\delta\) \(_x\) or BES\(_\nu\) \(_x\). We denote the law of this process by \(P^\delta\) \(_x\) or \(P^\nu\) \(_x\).

For \(\delta \geq 2\) and \(x > 0\), BES\(_\delta\) \(_x\) is a solution to the SDE

\[
\rho_t = x + \beta_t + \frac{\delta - 1}{2} \int_0^t \rho_s^{-1} \, ds,
\]

\(\rho_0 = x\). \hspace{1cm} (5.8)

However, the same technique cannot be applied for the case \(\delta < 2\) because the squared Bessel process reaches 0. We refer to Revuz and Yor [101] p.446 and Exercise
2.14 in Chapter IX for more details. For integer dimensions \( \delta = 2, 3, \ldots \), \( \rho_t \) may be obtained by applying Itô’s formula to the square root of \( BESQ_2^\delta \), say \( \sqrt{\rho_t} \) if

\[
\rho_t = \sqrt{(x + W_t^{(1)})^2 + \cdots + (W_t^{(\delta)})^2}.
\]

This is the Euclidean norm of the \( \delta \)-dimensional Brownian motion starting at \( x > 0 \).

The behaviour of \( BESQ^\delta \) depends on \( \delta \). If \( \delta \geq 2 \), then the process \( Z_t \) will never hit zero, and transient (zero is an entrance boundary) when \( \delta \geq 3 \). If \( 0 < \delta < 2 \), then the process is recurrent or zero is known as a reflecting boundary, that is, zero acts as an instantaneously reflecting barrier although almost surely accessible. If \( \delta = 0 \) then zero is an absorbing barrier, i.e., a trap. We refer to Bertoin [15] and Nikeghbali [91] for the study of Bessel processes with \( 0 < \delta < 2 \) and Göing-Jaeschke and Yor [58] for \( \delta < 0 \) on the squared Bessel process and it’s extensions. Simulated paths of a \( BESQ^\delta_2 \) with \( \delta \geq 2 \) and \( \delta < 2 \) are illustrated in Figure 5.1 and 5.2, respectively.

As a standard Brownian motion has the scaling property (see Revuz and Yor [101] Chapter I Proposition 1.10) the same property holds for \( BESQ^\delta_2 \): (Göing-
5.2. Bessel processes

Jaeschke and Yor [58] Appendix A.3 or Dufresne [43]) for any $c > 0$, if $X$ is a $BESQ^\delta_x$ then the process $c^{-1}X_{ct}$ is a $BESQ^\delta_{x/c}$. It is also true for $BES^\delta_{x}$. For this class of diffusion processes, the following additivity relationship holds.

**Theorem 5.10** (Revuz and Yor [101] Chapter XI. Theorem 1.2). For every $\delta, \delta' \geq 0$ and $x, x' \geq 0$,

$$Q^\delta_x \ast Q^\delta_{x'} = Q^{\delta + \delta'}_{x + x'},$$

where ‘$\ast$’ denotes convolution.

**Proof.** Suppose that $\delta$ and $\delta'$ are positive integers. If $BESQ^\delta_x$ and $BESQ^\delta_{x'}$ are mutually independent then the process $BESQ^{\delta + \delta'}_{x + x'}$ can be simply written as

$$\{X^{\delta}(t)\} + \{X^{\delta'}(t)\},$$

where $X^{\delta}(t) = W^2_1(t) + \cdots + W^2_\delta(t)$ and $X^{\delta'}(t) = W^2_{\delta + 1}(t) + \cdots + W^2_{\delta + \delta'}(t)$. For the proof of non-integer dimensions, see Revuz and Yor [101] p.440 or Shiga and Watanabe [107] p.41.
5.2.2 Distributional results of $BESQ^δ_x$

Another useful result that will be used in the following sections is the Laplace transform of the transition function of $δ$-dimensional squared Bessel process $ρ^2$,

$$E e^{-λρ^2_t} = Q^δ_{x}[e^{-λX_t}],$$

where $Q^δ_{x}[\cdot]$ is the expectation under $Q^δ_{x}$. Apply the additivity property of the squared Bessel process in Theorem 5.10: (i) when $δ_1 = δ_2 = 0$, $Q^0_{x_1+x_2} = Q^0_{x_1} * Q^0_{x_2}$.

Thus $Q^0_{x}[e^{-λX_t}] = A^x$ for some $A$. (ii) similarly, when $x_1 = x_2 = 0$, $Q^δ_{0+δ} = Q^δ_{0} * Q^δ_{0}$ and we have $C^δ$ for some $C > 0$. Since $Q^δ_{x}[e^{-λX_t}] = Q^0_{0}[e^{-λX_t}] * Q^δ_{0}[e^{-λX_t}]$, we have

$$Q^δ_{x}[e^{-λX_t}] = A^x \cdot C^δ.$$

For a Brownian motion $\{W_t; t ≥ 0\}$ started at $\sqrt{x}$, we have

$$E e^{-λW^2_t} = \frac{1}{(1 + 2λt)^{1/2}} e^{-λx/(1+2λt)},$$

and in general, for $λ > 0$,

$$Q^δ_{x}e^{-λX_t} = (1 + 2λt)^{-δ/2} e^{-λx/(1+2λt)}. \quad (5.9)$$

Note that the Laplace transform of the squared Bessel processes with time-varying dimension $δ_u$, see Shirakawa [108], is

$$Q^δ_{x}[e^{-λX_t}] = \exp \left\{ -\frac{λ}{1 + 2λt} - \int_0^t \frac{λδ_u}{1 + 2λ(t-u)} du \right\}.$$ 

Further, we can find the transition density of $BESQ^δ$ (see Revuz and Yor [101] Chapter XI.) For $δ > 0$,

$$q^δ_t(x,y) = \frac{1}{2t} \left( \frac{y}{x} \right)^{δ-1/2} \exp \left\{ -\frac{x+y}{2t} \right\} I_ν \left( \frac{\sqrt{xy}}{t} \right), \quad t > 0, x > 0, \quad (5.10)$$

where $ν = δ/2 - 1$. For $x = 0$, it reads

$$q^δ_t(0,y) = \left( \frac{1}{2t} \right)^{-δ} \frac{1}{Γ(δ/2)} y^{δ-1} \exp \left\{ -\frac{y}{2t} \right\}.$$ 

Similarly, the density of the semigroup of the $BESQ^δ$ can be obtained by applying a change of variable. For $δ > 0$

$$p^δ_t(x,y) = \frac{y}{t} \left( \frac{y}{x} \right)^{δ-1} \exp \left\{ -\frac{x^2+y^2}{2t} \right\} I_ν \left( \frac{xy}{t} \right), \quad t > 0, x > 0,$$

and

$$p^δ_t(0,y) = \frac{1}{2t^{δ/2-1}} \cdot \frac{1}{tδ/2} \cdot \frac{1}{Γ(δ/2)} y^{δ-1} \exp \left\{ -\frac{y^2}{2t} \right\}.$$
More properties of Bessel distribution can be found in Yuan and Kalbfleisch [120]. In particular, the authors examine the link between the Bessel distribution and other well-known distributions such as a conditional Poisson distribution. In addition, the moments of the Bessel distribution can be expressed in terms of the Bessel quotient, which is defined as:

\[ R_\nu(a) := \frac{I_{\nu+1}(a)}{I_{\nu}(a)}. \]

**Remark 5.2.** Applying integration by parts and stochastic time change formula can explain that the square-root process \( V_t \) is related to the squared Bessel process \( X_t, t \geq 0 \) with \( \delta = 4b/c^2 \) such that

\[ V_t = e^{at} X\left(\frac{c^2}{4a}(1 - e^{-at})\right), \quad v_0 = \frac{4}{c^2}x. \]

The transition density \( p_t(x, y) \) in (5.2) may be obtained using the transition density of the squared Bessel process (5.10) and the time change formula. See Dufresne [44] or Göing-Jaeschke and Yor [58] for more discussion.

### 5.2.3 Radial Ornstein-Uhlenbeck processes

Next, we consider the squared Bessel process with linear drift.

**Definition 5.11.** For \( \lambda \in \mathbb{R}, \delta \geq 0 \) and \( y \geq 0 \) the unique and strong solution to the equation

\[ Y_t = y^2 + \int_0^t (\delta + 2\lambda Y_s) \, dt + 2 \int_0^t \sqrt{|Y_s|} \, dW_s \quad (5.11) \]

is called a squared \( \delta \)-dimensional radial Ornstein-Uhlenbeck process with parameter \( \lambda \). The probability law of the process is denoted by \( \lambda Q_{y^2}^{\delta} \) or \( \lambda Q_y^{(\nu)} \).

When \( \delta = 0 \) and \( y = 0 \), the solution of the equation is \( Y_t = 0 \) for all \( t \geq 0 \). Therefore, the absolute value may be removed as \( Y_t \geq 0 \) if both \( \delta \) and \( y \geq 0 \) through comparison theorem. If \( \lambda = 0 \) the squared \( \delta \)-dimensional radial Ornstein-Uhlenbeck process corresponds to the \( BESQ_\delta \). Moreover, if \( \delta \) is a positive integer then the process \( Y_t \) may be represented by the square of the Euclidean norm of a \( \delta \)-dimensional Ornstein-Uhlenbeck process. Since the squared \( \delta \)-dimensional Ornstein-Uhlenbeck process is a Markov process and \( Y_t \geq 0 \) a.s. for positive \( \delta \), the square root of \( Y_t \) is also a Markov process.

**Definition 5.12.** For \( y \geq 0 \) the square root of the squared radial Ornstein-Uhlenbeck process is called the \( \delta \)-dimensional radial Ornstein-Uhlenbeck process with parameter \( \lambda \) and its law is denoted by \( \lambda P_{y}^{\delta} \) or \( \lambda P_y^{(\nu)} \).
Condition Behaviour at $y = 0$

| $\delta \geq 2$ | unattainable |
| $0 < \delta < 2$ | instantaneous reflecting |
| $\delta = 0$ | absorbing |

Table 5.1: Behaviour of trajectories of $\delta$-dimensional radial Ornstein-Uhlenbeck processes

Define $T_0 := \inf\{t : Y_t = 0\}$. When $\delta \geq 2$ the process $\{Y_t; t \geq 0\}$ does not touch zero a.s. For $0 < \delta < 2$ we have $\lambda P_x^\delta(T_0 < \infty) > 0$ if $\lambda \geq 0$ and $\lambda P_x^\delta(T_0 < \infty) = 1$ if $\lambda < 0$. The behaviour of trajectories of the radial Ornstein-Uhlenbeck processes are summarised in Table 5.1. Note that in a squared radial Ornstein-Uhlenbeck process $Y_t$ also can be transformed to the squared-root process $V_t$. We will revisit this transformation in Section 5.4.5 along with measure change.

### 5.3 Measure changes

Switching between measures is a technique that is often used in financial mathematics. In importance sampling for example, we alter measures in order to assign greater weight to ‘important’ outcomes thus enhancing efficiency of sampling. Suppose that we wish to find the expected value of $f(x)$ given a probability density function $p(x)$

$$E f = \int_{\mathbb{R}} f(x) p(x) \, dx. \quad (5.12)$$

However, it is not always possible to evaluate the integral in (5.12) analytically. In such cases, we may use the Monte Carlo simulation. For any function $\tilde{p}(x)$ such that the quotient $p(x)/\tilde{p}(x)$ is well defined, (5.12) may be expressed as

$$E f = \int_{\mathbb{R}} f(x) \frac{p(x)}{\tilde{p}(x)} \tilde{p}(x) \, dx.$$

If $\tilde{p}(x)$ is a probability density function, then it is equivalent to

$$E f = \tilde{E} \left[ f \cdot \frac{p}{\tilde{p}} \right], \quad (5.13)$$

where $\tilde{E}$ denotes the expectation with respect to $\tilde{p}$. Applying the Monte Carlo method to (5.13) gives an approximation

$$E f \approx \frac{1}{N} \sum_{i=1}^{N} f(\tilde{x}_i) \frac{p(x)}{\tilde{p}(x)}.$$
where $\tilde{x}_i$ is sampled from the new measure $\tilde{p}$. Now, we generalise this method to the case of a measurable $f$ on a probability space $(\Omega, \mathcal{F}, P)$. Then,

$$E f = \int_\Omega f(\omega) dP(\omega)$$

may be approximated by

$$E f \approx \frac{1}{N} \sum_{i=1}^{N} f(\omega_i),$$

where $\omega_i$ are independent samples drawn from $P$. If there exists another probability measure $Q$ on $(\Omega, \mathcal{F})$ such that $P$ is absolutely continuous with respect to $Q$, then there exists a Radon-Nikodym derivative $dP/dQ$ such that

$$E f := \int_\Omega f(\omega) dP(\omega) = \int_\Omega f(\omega) \frac{dP}{dQ}(\omega) dQ(\omega) =: E^Q \left[ f \frac{dP}{dQ} \right],$$

where $E^Q$ denotes the expectation with respect to $Q$. Then, the importance sampling approximation is given by

$$E f \approx \frac{1}{N} \sum_{i=1}^{N} f(\tilde{\omega}_i) \frac{dP}{dQ}(\tilde{\omega}_i),$$

where the $\tilde{\omega}_i$ are drawn from $Q$. In order for this to be practical, we obviously must be able to draw samples from $Q$ and to explicitly compute the Radon-Nikodym derivative.

### 5.3.1 Girsanov’s theorem

The stochastic exponential of a continuous local martingale $M$ is defined

$$Z_t := \mathcal{E}(M)_t = \exp \left\{ M_t - M_0 - \frac{1}{2} \langle M, M \rangle_t \right\}.$$  

**Theorem 5.13** (Girsanov’s Theorem). Consider $M_t = \int_0^t b(Y_s) \, dW_s$ and $Y$ is a one-dimensional diffusion driven by a Brownian motion $W$. The martingale $\gamma \cdot W$ is called the stochastic integral of $\gamma$ with respect to $W$ such that

$$\int_0^\cdot \gamma_s \, dW_s.$$  

Assume $\gamma_s$ satisfies Novikov’s condition

$$\mathbb{E} \exp \left\{ \frac{1}{2} \int_0^T \gamma_s^2 \, ds \right\} < \infty. \quad (5.14)$$

Let $\mathcal{E}_t(\gamma \cdot W)$ be the stochastic exponential of $\gamma \cdot W$

$$\mathcal{E}_t(\gamma \cdot W) = \exp \left\{ \int_0^t \gamma_s \, dW_s - \frac{1}{2} \int_0^t ||\gamma_s||^2 \, ds \right\}, \quad t \leq T.$$
with $\mathcal{E}_\infty(\gamma \cdot W) > 0$. Define the measure $Q$ by
\begin{equation}
    dQ = \mathcal{E}_t(\gamma \cdot W) \, dP, \quad t \geq 0.
\end{equation}
Then the process
\begin{equation}
    \tilde{W}_t := W_t + \int_0^t \gamma_s \, ds
\end{equation}
is a Brownian motion with respect to the probability measure $Q$.

**Proof.** Denote $M_t = \mathcal{E}_t(\gamma \cdot W)$. For each $t$, define a new measure $Q_t$ such that
\begin{equation}
    Q_t(\Omega) = \int_\Omega M_t \, dP.
\end{equation}
For $\Omega \in \mathcal{F}_s, s \leq t$, using the property of conditional expectation, we have
\begin{equation}
    Q_t(\Omega) = \int_\Omega M_t \, dP = \int_\Omega E[M_t|\mathcal{F}_s] \, dP = \int_\Omega M_s \, dP
\end{equation}
because $M_t$ is a martingale. Hence, $Q$ is independent of $t$, that is, $Q_t$ and $Q_s$ coincide on $\mathcal{F}_s$ for $s \leq t$ and $\{Q_t\}$ is a probability measure on $\mathcal{F}_t$.

Next, $M_t \tilde{W}_t$ is a $P$-martingale since
\begin{equation}
    d(M_t \tilde{W}_t) = \tilde{W}_t \, dM_t + M_t \, d\tilde{W}_t + dM_t \cdot d\tilde{W}_t
    = -\tilde{W}_t M_t \gamma_t \, dW_t + M_t (dW_t + \gamma_t \, dt) - M_t \gamma_t \, dt
    = M_t (1 - \tilde{W}_t \gamma_t) \, dW_t.
\end{equation}
For $\Omega \in \mathcal{F}_s$,
\begin{equation}
    \int_\Omega \tilde{W}_t \, dQ = \int_\Omega \tilde{W}_t M_t \, dP = \int_\Omega \tilde{W}_s M_s \, dP = \int_\Omega \tilde{W}_s \, dQ.
\end{equation}
Thus, we have $E^Q[\tilde{W}_t|\mathcal{F}_s] = \tilde{W}_s$. Since $\tilde{W}_t$ is a martingale and $d\langle \tilde{W}, \tilde{W} \rangle_t = dt$, we conclude that $\tilde{W}_t$ is a $Q$-Brownian motion.

**Example 5.14.** Suppose that the Novikov condition (5.14) holds. Define
\begin{equation}
    M_t := \exp \left\{ -\int_0^t \gamma_s \, dW_s - \frac{1}{2} \int_0^t \gamma_s^2 \, ds \right\} = \exp \left\{ Z_t - \frac{1}{2} \langle Z, Z \rangle_t \right\},
\end{equation}
where \( Z_t = -f_0^t \gamma_s \, dW_s \). By Itô’s formula we have

\[
dM_t = M_t(-\gamma_t \, dW_t - \frac{1}{2} \gamma_t^2 \, dt) + \frac{1}{2} M_t \gamma_t^2 \, dt
\]

and

\[
d(\log M_t) = \frac{dM_t}{M_t} - \frac{1}{2} (dM_t)^2 = -\gamma_t \, dW_t - \frac{1}{2} \gamma_t^2 \, dt.
\]

Thus, \( M_t \) is an exponential martingale and further it is the Radon-Nikodym derivative \( \frac{dQ}{dP} \).

Girsanov’s theorem is an important statement that provides a translation between different probability measures on the same space \((\Omega, \mathcal{F})\) and filtration \(\{\mathcal{F}_t\}\). This assures us that when we change from the objective probability measure to an equivalent martingale measure, the drift in asset price may change but their volatilities do not.

### 5.3.2 The change-of-measure formula

In this section, we state and prove propositions relating to measure changes between squared Bessel process and square-root process. Recall that \( Q^{(\nu)}_x \) and \( \lambda Q^{(\nu)}_x \) denote the law of \( BESQ^{(\nu)}_x \) and the squared radial Ornstein-Uhlenbeck process with index \( \nu \), respectively (see Definition 5.8 and 5.11.)

To apply the measure change (for instance, see Theorem 5.13), we need to be able to verify the condition the stochastic exponential \( Z_t \) is a martingale. In tradition, the Novikov condition in (5.14) has been used widely.

**Proposition 5.15** (Revuz and Yor [101] p.332). *If \( M \) is a continuous local martingale satisfying the Novikov condition then \( E(M)_t \) is a uniformly integrable martingale.*

That is, Novikov condition is a sufficient to ensure that the stochastic exponential is a martingale. A weaker but sufficient condition for \( Z_t \) being a martingale is Kazamaki’s criterion

\[
\exp \left\{ \frac{1}{2} M \right\} \text{ is a submartingale.}
\]

**Proposition 5.16** (Revuz and Yor [101] p.331). *If \( M \) is a local martingale, such that \( e^{M/2} \) is a uniformly integrable submartingale, then \( E(M)_t \) is a uniformly integrable martingale.*
Mijatović and Urusov [89] suggest necessary and sufficient conditions for $Z$ to be a true martingale and for $Z$ to be a uniformly integrable martingale in their Theorem 2.1 and Theorem 2.3, respectively.

**Theorem 5.17.** Let $x > 0$.

(a) If $\mu, \nu \geq 0$ and $\lambda \in \mathbb{R}$, then $\lambda Q_x^{(\mu)}$ and $Q_x^{(\nu)}$ are equivalent probability measure and

$$
\frac{d\lambda Q_x^{(\mu)}}{dQ_x^{(\nu)}} \bigg|_{\mathcal{F}_t} = \mathcal{E}(\varphi)_t.
$$

(b) If $\mu = \nu \geq -1$, $\lambda \in \mathbb{R}$ then (a) holds.

(c) If $\mu \neq \nu$ and one of $\nu, \mu$ is in $[-1, 0)$, then $\mathcal{E}(\varphi)_t$ above is not a martingale, and $\lambda Q_x^{(\mu)}, Q_x^{(\nu)}$ are not equivalent.

**Proof.** Suppose that $a \equiv 2\lambda$, $b \equiv 2\mu + 2 \geq 0$, $c \equiv 2$ in (5.1). We may apply Theorem 2.1 from Mijatović and Urusov [89]. In that paper’s notation, replace the symbol “$\mu (\cdot)$”, “$\sigma (\cdot)$” and “$b (\cdot)$” with $\zeta (\cdot)$, $\eta (\cdot)$ and $\varphi (\cdot)$, respectively. That is, we let $\zeta(x) = ax + b$, $\eta(x) = 2\sqrt{x}$ and $J = (0, \infty)$. Their conditions (5) and (6) such that

$$
\eta(x) \neq 0 \quad \forall x \in J \quad \text{and} \quad \frac{1}{\eta^2}, \frac{\zeta}{\eta^2} \in L^1_{\text{loc}}(J),
$$

where $L^1_{\text{loc}}(J)$ denotes the class of locally integrable functions, are satisfied because the functions $1/\eta^2$ and $\zeta/\eta^2$ are integrable over any compact subset of $J$. The auxiliary differential (Eq.(15) of Mijatović and Urusov [89]) is

$$
d\tilde{X}_t = (2\nu + 2)\, dt + 2\sqrt{\tilde{X}_t}\, d\tilde{W}_t, \quad \tilde{X}_0 = x.
$$

Here we have

$$
\zeta + \varphi \eta = 2\nu + 2
$$

or

$$
\varphi(x) = \frac{(2\nu + 2) - (2\mu + 2)}{2\sqrt{x}} - \lambda \sqrt{x}.
$$

Now we ask whether $\mathcal{E}(\varphi)_t$ is a $\lambda Q_x^{(\mu)}$-martingale. Note that condition (8) of Mijatović and Urusov [89] is verified and that $\mathcal{E}(\varphi)_t > 0$ with probability 1 for all $t \geq 0$.

Theorem 2.1 of Mijatović and Urusov [89] says that $\mathcal{E}(\varphi)_t$ is a martingale if, and only if, at least one of their conditions (a)–(b) and one of (c)–(d) are satisfied. We consider two cases: (i) $\nu, \mu \geq 0$; (ii) one of $\nu$ or $\mu$ is in $[-1, 0)$.

In case (i), we check that conditions (a) $\tilde{X}$ does not exit $J$ at $r$ and (c) $\tilde{X}$ does not exit $J$ at $l$ hold. Note that the process $\tilde{X}$ exits its state space at the boundary point $r$ (respectively, $l$) if, and only if, $\tilde{v}(r) < \infty$ (respectively, $\tilde{v}(l) < \infty$). Here
5.3. Measure changes

$r = \infty$, and we verify that Eq.(20) in Mijatović and Urusov [89] holds. Choose $c = 1$;

\[
\tilde{\rho}(x) = \exp \left\{ - \int_1^x \frac{2(2\nu + 2)}{4y} \, dy \right\}
= x^{-(\nu + 1)},
\]

\[
\tilde{s}(x) = \int_1^x y^{-(\nu + 1)} \, dy,
\]

\[
\tilde{v}(x) = \int_1^x \frac{\tilde{s}(x) - \tilde{s}(y)}{y^{-(\nu + 1)}y} \, dy
= \frac{1}{4} \int_1^x y^{\nu} \int_y^x z^{-(\nu + 1)} \, dz \, dy
= \frac{1}{4\nu + 4} \left[ x - 1 - \frac{1}{\nu} (1 - x^{-\nu}) \right]
\]

if $\nu > 0$. The last expression tends to $\infty$ as $x \to \infty$ (same if $\nu = 0$). To check (c), simply note that $\nu \geq 0$ implies $\tilde{v}(x) \to \infty$ as $x \downarrow 0$.

Now turn to (ii): letting $\nu \in [-1, 0)$, we show that (c) and (d) do not hold. In the case of (c), it is clear that $\tilde{v}(0) < \infty$, meaning that $\tilde{Y}$ exits at 0. To show that (d) does not apply we need to show that $l = 0$ is bad, i.e., not good, if good requires (27):

\[
\tilde{s}(0) > -\infty \quad \text{and} \quad \frac{(\tilde{s} - \tilde{s}(0))\varphi^2}{\rho \eta^2} \in L_{\text{loc}}^1(0+).
\]

($L_{\text{loc}}^1(0+)$ is the set of functions that are integrable on $(0, \epsilon)$ for some $\epsilon > 0$). Here

\[
\tilde{s}(x) - \tilde{s}(0) = -\frac{1}{\nu x^{\nu}}
\]

and

\[
\tilde{s}(x) - \tilde{s}(0) \frac{\varphi^2(x)}{\rho(x)\eta^2(x)} = C \left( \frac{\mu - \nu}{\sqrt{x}} + \lambda \sqrt{x} \right)^2,
\]

where $C = -1/(4\nu)$. If $\mu \neq \nu$, then this function is not in $L_{\text{loc}}^1(0+)$, and thus $\mathcal{E}(\varphi)_t$ is not a martingale. If $\mu = \nu$ then the function is in $L_{\text{loc}}^1(0+)$, and $\mathcal{E}(\varphi)_t$ is a martingale.

The absolute continuity relation between two non-zero indexes of Bessel process, say $P_{a(\mu)}$ and $P_{a(\nu)}$, can be found in Revuz and Yor [101] Chapter XI Ex 1.22. An analogous formula for the squared Bessel process is following.

**Proposition 5.18.** For the indexes $\mu, \nu \geq 0$, fixed $T > 0$, $X$ any $\mathcal{F}_T$-measurable positive random variable, and any $X_0 = x > 0$, we have

\[
\left. \frac{dQ_{\mu}^y}{dQ_{\nu}^x} \right|_{\mathcal{F}_t} = \left( \frac{X_t}{x} \right)^{\frac{\nu^2 - \mu^2}{2}} \exp \left\{ \frac{\nu^2 - \mu^2}{2} J_t^X \right\},
\]

where $J_t^X = \int_0^t X_s \, ds$. 

\[\Box\]
where
\[ J_t^X = \int_0^t \frac{ds}{X_s}. \]  
(5.16)

Next, we extend the mutual absolute continuity relation between a squared radial Ornstein-Uhlenbeck process with parameter \( \lambda \) and squared Bessel process with the same index.

**Proposition 5.19.** Let \( \mathcal{F}_t = \sigma(X_s; 0 \leq s \leq t) \). For every \( \lambda \in \mathbb{R}, \mu \geq 0 \) and \( x > 0 \),
\[
\frac{d^{\lambda} Q_{x}^{(\mu)}}{d Q_{x}^{(\mu)}} \bigg|_{\mathcal{F}_t} = \exp \left\{ \frac{\lambda}{2} [X_t - x - (2\mu + 2)t] - \frac{\lambda^2}{2} I_t^X \right\},
\]  
(5.17)

where
\[
I_t^X = \int_0^t X_s ds.
\]  
(5.18)

**Proof.** We consider \( X \) satisfying the SDE
\[
dX_t = (2\mu + 2 + 2\lambda X_t) dt + 2\sqrt{X_t} dW_t,
\]
let
\[
\mathcal{E}(\varphi)_t := \exp \left\{ \int_0^t \varphi_s dW_s - \frac{1}{2} \int_0^t \varphi_s^2 ds \right\}.
\]
If \( \mathcal{E}(\varphi)_t \) is a true martingale then \( \tilde{W}_t := W_t - \int_0^t \varphi_s ds \) is a standard Brownian motion under the probability measure \( Q^*|_{\mathcal{F}_t} \) such that
\[
\mathcal{E}(\varphi)_t : \frac{d^{\lambda} Q_{x}^{(\mu)}}{d Q_{x}^{(\mu)}} \bigg|_{\mathcal{F}_t}
\]
by Girsanov’s theorem. We have
\[
dX_t = (2\mu + 2 + 2\lambda X_t) dt + 2\sqrt{X_t} (d\tilde{W}_t + \varphi_t dt)
\]
\[= (2\mu + 2 + 2\lambda X_t + 2\varphi_t \sqrt{X_t}) dt + 2\sqrt{X_t} d\tilde{W}_t. \]
If \( \varphi_t = -\lambda \sqrt{X_t} \) then
\[
\mathcal{E}(\varphi)_t = \exp \left\{ -\lambda \int_0^t \sqrt{X_s} dW_s - \frac{\lambda^2}{2} \int_0^t X_s ds \right\},
\]
and now \( X_t \) satisfies
\[
dX_t = (2\mu + 2) dt + 2\sqrt{X_t} d\tilde{W}_t.
\]
Since
\[
\int_0^t \sqrt{X_s} dW_s = \frac{X_t - x}{2} - \left( \frac{(2\mu + 2)t}{2} + \lambda \int_0^t X_s ds \right),
\]
we conclude that
\[ \mathcal{E}(\varphi)_t = \frac{dQ^{(\mu)}_x}{d\mathcal{L}^{(\mu)}} = \exp\left\{ \frac{\lambda(2\mu + 2)}{2} + \lambda^2 \int_0^t X_s ds - \frac{\lambda}{2}(X_t - x) - \frac{\lambda^2}{2} \int_0^t X_s ds \right\} 
= \exp\left\{ -\frac{\lambda}{2} [X_t - x - (2\mu + 2)t] + \frac{\lambda^2}{2} \int_0^t X_s ds \right\}, \]
and obtain the Radon-Nikodym derivative as required.

Similarly, the absolutely continuous relation between a radial Ornstein-Uhlenbeck process and a Bessel process can be found in Proposition 1 of Göing-Jaeschke and Yor [58].

Corollary 5.20. For \( \mu \geq 0, \lambda, \lambda \in \mathbb{R} \) and \( x > 0 \),
\[ \frac{d^{\lambda}Q^{(\mu)}_x}{d\mathcal{L}^{(\mu)}} = \exp\left( \frac{1}{2}(\lambda - \lambda)[X_t - x - (2\mu + 2)t] - \frac{1}{2}(\lambda^2 - \lambda^2)I_t^X \right). \]

Proof. Applying two measure changes such that
\[
\frac{d^{\lambda}Q^{(\mu)}_x}{d\mathcal{L}^{(\mu)}} = \frac{d^{\lambda}Q^{(\mu)}_x}{dQ^{(\mu)}_x} \frac{dQ^{(\mu)}_x}{dQ^{(\mu)}_x}
\]
completes the proof.

Theorem 5.21. For all \( \mu, \nu \geq 0, \lambda \in \mathbb{R} \) and \( x > 0 \),
\[ \frac{d^{\lambda}Q^{(\mu)}_x}{dQ^{(\mu)}_x} \bigg|_{\mathcal{F}_t} = \left( \frac{X_t}{x} \right)^{\frac{\mu-\nu}{2}} \exp\left\{ \frac{\lambda}{2}[X_t - x - (2\mu + 2)t] - \frac{\lambda^2}{2}I_t^X + \frac{\nu^2 - \mu^2}{2}J_t^X \right\}. \] (5.19)

Proof. We consider SDE
\[ dX_t = (2\nu + 2) dt + 2\sqrt{X_t} dW_t \]
\[ = (2\nu + 2 + 2\varphi_t\sqrt{X_t}) dt + 2\sqrt{X_t} d\tilde{W}_t, \]
where \( W_t := \tilde{W}_t + \int_0^t \varphi_s ds \). Define \( \varphi_t \) such that
\[ 2\nu + 2 + 2\varphi_t\sqrt{X_t} = 2\lambda X_t + (2\mu + 2) \]
or equivalently,
\[ \varphi_t = \lambda\sqrt{X_t} + \frac{\mu - \nu}{\sqrt{X_t}}. \]
Next, we let
\[ \frac{d\lambda Q^\mu_t}{dQ^\nu_t} \bigg|_{\mathcal{F}_t} = \mathcal{E}(\varphi)_t \]

\[ = \exp \left\{ \int_0^t \varphi_s dW_s - \frac{1}{2} \int_0^t \varphi_s^2 ds \right\}. \]

From (5.20) we have
\[ \lambda \int_0^t \sqrt{X_s} dW_s = \frac{\lambda}{2} [X_t - x - (2\nu + 2)t], \]
and applying Itô’s formula gives
\[ d\log X_t = \frac{1}{X_t} [(2\nu + 2) dt + 2\sqrt{X_t} dW_t] - \frac{1}{2X_t^2} (4X_t) dt \]
\[ = 2\nu dt + \frac{2}{\sqrt{X_t}} dW_t. \]

Further we have
\[ \int_0^t \frac{\mu - \nu}{\sqrt{X_s}} dW_s = \frac{\mu - \nu}{2} \left[ \log \left( \frac{X_t}{x} \right) - \int_0^t \frac{(2\nu + 2) - 2}{X_s} ds \right] \]
\[ = \log \left( \frac{X_t}{x} \right)^{\mu - \nu} - (\mu - \nu) \int_0^t ds \frac{X_s}{X_t}. \]

Hence,
\[ \mathcal{E}(\varphi)_t = \exp \left\{ \frac{\lambda}{2} [X_t - x - (2\nu + 2)t] + \log \left( \frac{X_t}{x} \right)^{\mu - \nu} - (\mu - \nu) \int_0^t ds \frac{X_s}{X_t} - \frac{1}{2} \int_0^t \left[ \lambda^2 X_s + \frac{(\mu - \nu)^2}{X_s} + 2\lambda(\mu - \nu) \right] ds \right\} \]
\[ = \left( \frac{X_t}{x} \right)^{\mu - \nu} \exp \left\{ \frac{\lambda}{2} [X_t - x - (2\mu + 2)t] - \frac{\lambda^2}{2} \int_0^t X_s ds - \frac{\mu^2 - \nu^2}{2} \int_0^t \frac{ds}{X_s} \right\}. \]

This also can be checked by applying two change of measure formulas such that
\[ \frac{d\lambda Q^\mu_t}{dQ^\nu_t} = \frac{d\lambda Q^\mu_t}{d\lambda Q^\mu_t} \frac{d\lambda Q^\mu_t}{dQ^\nu_t}, \]

In general, by Proposition 5.19, we have, for \( Z \geq 0 \)
\[ E^Q X^\nu_t Z = E^Q \exp \left\{ \frac{\lambda}{2} [X_t - x - (2\nu + 2)t] - \frac{\lambda^2}{2} \int_0^t X_s ds \right\} Z. \]

This may be useful if \( \delta = 1, 2, \ldots \) since then we use the fact that \( Q^\nu_t \) is the law of the \( \{ ||W_t||^2, t \geq 0 \} \), where \( W \) is \( \delta \)-dimensional Brownian motion starting from \( x \). With
financial data, it is unlikely that the $\delta$ would be an integer. However, we may, at least in theory, write

\[
E^Q \lambda Z = E^Q \left( Z \frac{d^1Q_x}{dQ_0} \bigg|_{\mathcal{F}_t} \right) = E^Q \left( Z \frac{d^1Q_x}{dQ_0} \bigg|_{\mathcal{F}_t} \cdot \frac{dQ_0}{d\lambda} \right),
\]

for $Z \in \mathcal{F}_t, Z \geq 0$. In particular, we will explore whether there is any computational advantage in using the change-of-measure to price European options in Section 5.4.

5.3.3 The ‘bump’: Simulation of Radon-Nikodym derivative

In this section, we investigate the numerical behaviour of the Radon-Nikodym derivatives. For example, we should check that

\[
E^{Q_x}_t \frac{dQ^{(\nu)}}{dQ^{(0)}_x} \bigg|_{\mathcal{F}_t} = 1. \tag{5.21}
\]

Simulating

\[
\frac{dQ^{(\nu)}_x}{dQ^{(0)}_x} \bigg|_{\mathcal{F}_t} = \exp \left\{ \frac{\nu}{2} \log \left( \frac{X_t}{x} \right) - \frac{\nu^2}{2} \int_0^t \frac{1}{X_s} \, ds \right\} \tag{5.22}
\]
on the computer yields a graph such as in Figure 5.3 to 5.5. The ‘bump’ in those graphs appears in all the simulations we performed although it varies in height, depending on the starting value $x$ and the step-size used in computing the Radon-Nikodym derivative. Here is the explanation we found for this phenomenon.

The Radon-Nikodym derivative is approximated as

\[
\exp \left\{ -\frac{\nu^2}{2} \sum_{k=1}^n \frac{1}{\hat{X}_{kt/n}} \cdot \frac{t}{n} \right\} \left( \frac{\hat{X}_t}{x} \right)^{\nu/2}. \tag{5.23}
\]

Here, the ‘small’ values of $\hat{X}_{kt/n}$ lead to ‘large’ values of $\sum_{k=1}^n 1/\hat{X}_{kt/n}$, that compensate for the large values of $(\hat{X}_t/x)^{\nu/2}$ and vice versa. The true integral catches the smallest values of $X_t$ but the discretised integral misses some, leading to an imperfect match, and a simulated value of (5.21) different from 1.

Another way to look at the problem is to calculate the partial derivative of (5.23) with respect to $\nu$:

\[
\left[ -\nu \sum_{k=1}^n \frac{1}{\hat{X}_{kt/n}} \cdot \frac{t}{n} + \frac{1}{2} \log \left( \frac{\hat{X}_t}{x} \right) \right] \exp \left\{ -\frac{\nu^2}{2} \sum_{k=1}^n \frac{1}{X_{kt/n}} \cdot \frac{t}{n} \right\} \left( \frac{\hat{X}_t}{x} \right)^{\nu/2}.
\]
Figure 5.3: Behaviour of the expectation of Radon-Nikodym derivative in (5.19) with $x = 2$, 128 time-steps per year and 10,000 simulations for each different $\lambda$. 

$E[RN]$ 

$\lambda = 0$ 
$\lambda = -1.0$ 
$\lambda = -2.0$ 
$\lambda = -3.0$
Figure 5.4: Behaviour of the expectation of Radon-Nikodym derivative in (5.22) with $x = 2$ and 10,000 simulations for each different time-steps per year.
Figure 5.5: Behaviour of the expectation of Radon-Nikodym derivative in (5.22) with 128 time-steps per year and 10,000 simulations.
When \( \nu \to 0^+ \), this approaches \( \frac{1}{2} \log(\hat{X}_t/x) \). Now, \( \log X_t \) is a local martingale since
\[
d\log X_t = \frac{1}{X_t} dX_t - \frac{1}{2X_t^2} \cdot 4X_t \, dt
\]
\[
= \frac{1}{X_t} \left( 2 \, dt + 2\sqrt{X_t} \, dW_t \right) - \frac{2}{X_t} \, dt
\]
\[
= \frac{2}{\sqrt{X_t}} \, dW_t.
\]
However, \( \log X_t \) is not a true martingale and its expectation under \( Q^{(0)}_x \) is not 0. It is easy to see that here \( X_t \) has the same distribution as
\[
\left( \sqrt{tN_1} + \sqrt{x} \right)^2 + tN_2^2
\]
where \((N_1, N_2)\) are independent standard normals. Numerical calculations give \( E^{Q^{(0)}_x} \log X_1 = 3.9904 \) if \( x = 2 \) and \( E^{Q^{(0)}_x} \log X_1 = 2.1010 \) if \( x = 0.1 \), for instance.

One way to prove that \( E^{Q^{(0)}_x} \log X_t \neq 0 \) (and that consequently, \( \log(X_t/x) \) is not a true martingale), is to verify that
\[
E \log X_t - \log 2 \to \Gamma'(1).
\]
First, \( E|\log X_t|^n < \infty \) for any \( n \geq 1 \). This can be seen by noting that \( X_t \overset{d}{=} \left( \sqrt{tN_1} + \sqrt{x} \right)^2 + tN_2^2 \), and
\[
E \left[ \left( \sqrt{tN_1} + \sqrt{x} \right)^2 + tN_2^2 \right]^p < \infty, \quad \forall p \geq 0,
\]
\[
E \left[ \left( \sqrt{tN_1} + \sqrt{x} \right)^2 + tN_2^2 \right]^{-p} \leq E \left( tN_2^2 \right)^{-p} < \infty, \quad -\frac{1}{2} < p < \infty.
\]
Hence \( E X_t^p = E \exp \{ p \log X_t \} \) exists in a neighbourhood of 0, it is therefore analytic in that neighbourhood and can be differentiated as many times as we want inside the expectation, yielding \( E|\log X_t|^n < \infty, \quad \forall n = 1, 2, \ldots \).

Next,
\[
N_2^2 \leq \left( N_1 + \sqrt{\frac{x}{t}} \right)^2 + N_2^2 \leq 2N_1^2 + 2x + N_2^2
\]
implies
\[
\left| \log \left( N + 1 + \sqrt{\frac{x}{t}} \right)^2 + N_2^2 \right| \leq |\log N_2^2| + |\log (2N_1^2 + 2x + N_2^2)|, \quad t \geq 1,
\]
where \( E |\log N_2^2| < \infty \) and \( E |\log (2N_1^2 + 2x + N_2^2)| < \infty \). Hence, applying the Dominated Convergence Theorem,
\[
E \log \left[ \left( \sqrt{tN_1} + \sqrt{x} \right)^2 + tN_2^2 \right] - \log t = E \log \left[ \left( N_1 + \frac{\sqrt{x}}{\sqrt{t}} \right)^2 + N_2^2 \right] - \log 2
\]
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\[ \mathbb{E} \log \left( N_1^2 + N_2^2 \right) - \log 2 = \mathbb{E} \log (e) = \int_0^\infty \log xe^{-x} \, dx = \Gamma'(1), \]

where \( e \sim \text{Exp}(1) \).

**Theorem 5.22.** If \( X \) is a \( \delta \)-dimensional squared Bessel process, then

\[ \mathbb{E} \left[ \log \left( X_{t/2}^2 \right) \right]^n \to \frac{\Gamma(n) \left( \frac{\delta}{2} \right)}{\Gamma \left( \frac{\delta}{2} \right)}, \quad t \to \infty. \tag{5.24} \]

**Proof.** We consider the moment of the reciprocal of the squared Bessel process

\[ Q_x^\delta [X_t^{-p}] = \frac{1}{\Gamma(p)} \int_0^\infty \lambda^{p-1} Q_x^\delta e^{-\lambda X_t} \, d\lambda. \]

Since we know the Laplace transform of \( X \) such that

\[ Q_x^\delta [e^{-\lambda X_t}] = (1 + 2\lambda t)^{-\delta/2} e^{-\lambda x/(1+2\lambda t)} \]

(see Revuz and Yor [101] p.441), letting \( u = 2\lambda t/(1 + 2\lambda t) \) gives

\[ Q_x^\delta [X_t^{-p}] = \frac{(2t)^{-p}}{\Gamma(p)} \int_0^1 u^{p-1} (1-u)^{\delta/2-p-1} e^{-xu/(2t)} \, du = (2t)^{-p} \frac{\Gamma(\delta/2-p)}{\Gamma(\delta/2)} \ _1F_1 \left( p, \frac{\delta}{2}; -\frac{x}{2t} \right). \]

Hence, we have

\[ Q_x^\delta \left( \frac{X_t}{2t} \right)^p = \frac{\Gamma(\delta/2 + p)}{\Gamma(\delta/2)} \ _1F_1 \left( -p, \frac{\delta}{2}; -\frac{x}{2t} \right). \]

If \( Q_x^\delta (X_t/(2t))^p \) exists for \( p \) in a neighbourhood of 0, then

\[ Q_x^\delta \left[ \log \left( \frac{X_t}{2t} \right) \right]^n = \left. \frac{\partial^n}{\partial p^n} Q_x^\delta \left( \frac{X_t}{2t} \right)^p \right|_{p=0} \tag{5.25} \]

Let \( n = 1 \) and take limit on both sides of (5.25):

\[ \lim_{t \to \infty} Q_x^\delta \log \left( \frac{X_t}{2t} \right) = \lim_{t \to \infty} \left. \frac{\partial}{\partial p} Q_x^\delta \left( \frac{X_t}{2t} \right)^p \right|_{p=0} \]

Then we reach

\[ Q_x^\delta \left( \frac{X_t}{2t} \right)^p = Q_x^\delta (\epsilon X_t)^p, \quad \epsilon = \frac{1}{2t}, \]

\[ = \frac{\Gamma(\delta/2 + p)}{\Gamma(\delta/2)} \ _1F_1 \left( -p, \frac{\delta}{2}; -\epsilon x \right). \]
5.4. The computation of European option prices

So we want

\[ L = \lim_{\epsilon \to 0} \frac{\partial}{\partial p} g(p; \epsilon) \bigg|_{p=0} \]

where

\[ g(p; \epsilon) = \frac{\Gamma(\delta/2 + p)}{\Gamma(\delta/2)} _1F_1 \left(-p, \frac{\delta}{2}; -\epsilon x \right) \]

is complex analytic in both \( p \) and \( \epsilon \). Hence we have

\[ L = \lim_{\epsilon \to 0} \frac{\partial}{\partial p} g(p; \epsilon) \bigg|_{p=0} = \frac{\partial}{\partial p} g(p; 0) \bigg|_{p=0}, \]

and

\[ \lim_{t \to \infty} E \log \left( \frac{X_t}{2t} \right) = \frac{\partial}{\partial p} \frac{\Gamma(\delta/2 + p)}{\Gamma(\delta/2)} \bigg|_{p=0} = \frac{\Gamma'(\delta/2)}{\Gamma(\delta/2)}. \]

In general, as \( t \to \infty \),

\[ E \left[ \log \left( \frac{X_t}{2t} \right) \right]^n \to \frac{\Gamma^{(n)}(\frac{\delta}{2})}{\Gamma(\frac{\delta}{2})}. \]

\[ \square \]

5.4 The computation of European option prices

In financial mathematics, a stochastic volatility model in vogue is the square-root process (5.1), the squared volatility of the stock price

\[ dS_t = rS_t dt + \sqrt{V_t} S_t dB_t, \quad (5.26) \]

where \( B_t = \rho W_t(1) + \sqrt{1 - \rho^2} W_t(2) \), \( W_t(1) = W_t \) and \( W_t(2) \) are two independent standard Brownian motions and \(-1 \leq \rho \leq 1\). Now, introducing another standard Brownian motion \( W_t(2) \) results in us allowing dependence between the underlying asset and its volatility process. The solution of (5.26) is

\[ S_t = S_0 \exp \left\{ rt - \frac{1}{2} U_t + \int_0^t \sqrt{V_s} d \left( \rho W_t(1) + \rho W_t(2) \right) \right\}, \]

where \( \bar{\rho} = \sqrt{1 - \rho^2} \). In order to simulate this process, at least from this explicit expression for \( S_t \), one needs both \( U_t \) and \( V_t \). There is a joint Laplace transform for the distribution of \((U_t, V_t)\), but no simple inversion method is known to date, although at least Broadie and Kaya [22] have inverted it step by step. Lord et al. [86] likewise examine the Euler discretisation scheme. The authors concentrate on the technique in which to avoid the negative values of the variance process. They assert that the basic Euler scheme, called full-truncation performs better than the other approaches including Broadie-Kaya method when considering computational
efficiency. Still we need to be vigilant to select the number of time-step to attain the target error.

In contrast, Andersen [4] formulates approximations to the exact scheme modelled on moment-matching. He approximates the non-central chi-square distribution by locally matching the moments of an associated distribution with the moments of the exact distribution. As sampling from the approximated distribution simply requires transformations to uniform and normal draws, this technique can be applied relatively efficiently. In addition to this approximation, in order to approximate the integrated variance process $U_t$, drift interpolation is utilised rather than inverting Fourier integral. The consequent moment-matched scheme surpasses all existing schemes to date in regards to computational efficiency. For alternative methods, we refer to Alfonsi [3], which provides an overview on the discretisation of the square-root process, Glasserman and Kim [56] for an exact representation of distribution of $U_t$ as an infinite sum of mixtures of gamma random variable and the references therein.

As explained in Section 5.3, the square-root process is intimately related to the squared Bessel process (i.e., the squared Bessel process is one special case of the square-root process), and the same idea, namely of simulating the process for an integer-valued dimension $\delta$ and then applying a measure change formula, may be used to price options under square-root volatility. In the remainder of this section, we describe and compare various approaches in literature. Then, we demonstrate a new and efficient simulation scheme for European vanilla option price based on the change of measure technique we discussed earlier.

5.4.1 Euler scheme

The direct discretisation technique is broadly applied in practice. In situations in which the path is required to be sampled at finer space points for approximating a path-dependent payoff, it is generally computationally quicker than simulating the transition density. Direct discretisation must have a single normal sample per step. Otherwise, instead of using the exact transition density with the technique in Dufresne [42] or Glasserman [55], sampling a chi-square random variable can be used for example. Consider two SDEs

$$S_{t+\Delta} = S_t + rS_t\Delta + \sqrt{V_t}S_t\sqrt{\Delta} Z_S$$  \hspace{1cm} (5.27)
$$V_{t+\Delta} = V_t + (aV_t + b)\Delta + c\sqrt{V_t}\sqrt{\Delta} Z_V,$$  \hspace{1cm} (5.28)
where $Z_S$ and $Z_V$ are standard normal random variables with correlation $\rho$. That is, it can be generated using

$$Z_S = \rho Z_V + \sqrt{1 - \rho^2} Z^{(2)},$$

where $Z_V := Z^{(1)}$ and $Z^{(2)}$ are two independent draws from the standard normal distribution. However, the variance process in (5.28) has to be modified because it has undesirable behaviour, in which may hit negative value with positive probability

$$P(V_t + \Delta > 0) = P\left(Z_V < \frac{-V_t - (aV_t + b)\Delta}{c\sqrt{V_t}\sqrt{\Delta}}\right).$$

Either absorption at zero or reflection are one of possibility to overcome negativity. For example, Lord et al. [86] choose absorption at zero such as $V_t + \Delta := \max(V_t, 0)$ and then (5.28) reads

$$V_{t+\Delta} = V_t + (aV_t^+ + b)\Delta + c\sqrt{V_t^+}\sqrt{\Delta}Z_V,$$

which is sometimes called full truncation method.

Once the variance process is obtained, the underlying asset process can be simulated in Euler discretising fashion as above. For computational efficiency, log-price may be considered. While the log-Euler scheme does not involve any discretisation error in the underlying price itself, it does display biases in the Euler discretisation of the variance process. See also Kahl and Jäckel [75] and Lord et al. [86] for alternative discretisation scheme for the square-root process.

5.4.2 Broadie-Kaya scheme: Another Fourier inversion technique

Broadie and Kaya [22] show how to simulate an exact sample from the distribution of $S_t$ conditional on the variance process $V_t$. As the conditional distribution of $U_t$ is unknown, they use its characteristic function first then apply Fourier inversion numerically to obtain the distribution function of $U_t$. The characteristic function of $U_t$ is given by

$$\Phi(u) = \mathbb{E}[e^{iuS_t} | V_s, V_t]$$

$$= \exp\left\{\frac{V_s + V_t}{c^2} \left(-a\frac{1 + e^{a(t-s)}}{1 - e^{a(t-s)}} - \frac{\gamma(u)(1 + e^{-\gamma(u)(t-s)})}{1 - e^{-\gamma(u)(t-s)}}\right)\right\}$$

$$\times \frac{\gamma(u)e^{-\frac{1}{2}(\gamma(u)+\gamma(t-s))e^{a(t-s)}}}{-a(1 - e^{-\gamma(u)(t-s)})} \times I_{\frac{d}{2}-1} \left[\sqrt{V_s V_t} \frac{\gamma(u) e^{\frac{1}{2}\gamma(u)(t-s)}}{\gamma(t-s)(1 - e^{-\gamma(u)(t-s)})}\right]$$

$$I_{\frac{d}{2}-1} \left[\sqrt{V_s V_t} \frac{4a e^{\frac{1}{2}\gamma(u)(t-s)}}{\gamma(t-s)(1 - e^{-\gamma(u)(t-s)})}\right]$$

(5.29)
where $\gamma(u) = \sqrt{a^2 - 2c^2u}$ and $d = 4b/c^2$. The derivation of (5.29) is based on the results in Pitman and Yor [98] and the derivation of the Laplace transform can be found in the Appendix of Broadie and Kaya [22]. Applying Fourier inversion sounds simple and straightforward but note that the characteristic function involves two modified Bessel functions. That is, Fourier inversion is involved in each time-step, which, of course, is numerically expensive to generate $U_t$. Since the characteristic function depends on the endpoints $V_0$ and $V_t$, computing the inversion at each step in the simulation makes Broadie-Kaya methods rather slow.

Once $V_t$ and $U_t$ are calculated the rest is trivial. The stock price process can be easily obtained using the fact that

$$\log \frac{S_t}{S_0} = rt - \frac{1}{2} U_t + \frac{\rho}{c} (V_t - v_0 - aU_t - bt) + \rho \int_0^t \sqrt{V_s} dW_s$$

is normally distributed conditioned on $V_t$ and $U_t$ with mean of

$$rt - \frac{1}{2} U_t + \frac{\rho}{c} (V_t - v_0 - aU_t - bt)$$

and a variance of

$$\rho^2 U_t.$$

We then be able to sample non-central chi-square random variable. Finally the probability function can be computed using Fourier inversion formula

$$P(U_{t-s} \leq x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin ux}{u} \text{Re}[\Phi(u)] \, du.$$

In short, the simulation procedure at each path has three parts: (1) sampling the end-point of the variance process conditional on the initial point, (2) sampling the integrated variance process conditional on the initial and end-point of the variance process, (3) sampling the share price process conditional on the integrated variance process $U_t$ and the variance process $V_t$.

**Remark 5.3.** We may consider an alternative way of computing $U_t$; approximate the integrated variance process $U_t$ using drift interpolation such that

$$\int_t^{t+\Delta} V(x) \, dx \approx \Delta(\eta_1 V(t) + \eta_2 V(t + \Delta))$$

for some constants $\eta_1$ and $\eta_2$. For example $(\eta_1, \eta_2)$ may be chosen either $(1, 0)$ for the simplicity or $(1/2, 1/2)$ for the central discretisation. Alternatively, numerical integration such as trapezium rule or Gaussian quadrature can be used. Mathematica provides various numerical techniques along with optional commands in `NIntegrate`. 
5.4.3 Other Fourier formulas

Mellin-type formula

The damping factor is one way of satisfying the integrability of function g such that
\[ \int g(x) \, d\mu_X(x) = \int e^{-\alpha x} g(x) e^{\alpha x} \, d\mu_X(x). \]

However, it is not necessarily true that we can find \( \alpha \) such that \( g(-\alpha)(x) = e^{-\alpha x} g(x) \) is Lebesgue integrable and \( \mu^{(\alpha)} \) has finite mass. Next, we revisit Mellin transform of the distribution of \( S, \) \( E^{iu \log S} = ES^{iu} \) so that European puts and calls are computed without multiplying damping factor.

For instance, if \( ES < \infty \) with \( S \geq 0 \) and \( K > 0 \)
\[ E(S - K)_+ = ES - \frac{K}{2} \left[ 1 + \mathbb{P}(S = 0) \right] + \frac{1}{\pi} \int_0^\infty \text{Re}[h(u)] \, du, \tag{5.31} \]
where
\[ h(u) = \frac{K^{-iu+1}}{iu(iu-1)} E(S^{iu}). \]

(See Theorem 3.6.) Since integrating the square-root process (5.1) gives
\[ \int_0^t \sqrt{V_s} \, dW_s^{(2)} = \frac{1}{c} \left( V_t - v_0 - aU_t - bt \right), \]
we have
\[ S_t = S_0 \exp \left\{ rt - \frac{1}{2} U_t + \frac{\rho}{c} (V_t - v_0 - aU_t - bt) + \sqrt{U_t} Z \right\}, \tag{5.32} \]
\[ = \mathcal{S} \exp \left\{ - \left( \frac{1}{2} + \frac{a}{c} \rho \right) U_t + \frac{\rho}{c} V_t + \sqrt{U_t} Z \right\}, \tag{5.33} \]
where
\[ \mathcal{S} = S_0 e^{rt - \rho (v_0 + bt)/c}. \]

and \( Z \) is a standard normal distribution, independent of \( V. \) Now, the \( n \)-th moment of \( S_t \) can be computed
\[ ES^n_t = E \left[ \mathcal{S} \exp \left\{ \gamma_1 U_t + \gamma_2 V_t + \sqrt{U_t} Z \right\} \right]^n \]
\[ = E(\mathcal{S})^n \mathbb{E} \left[ \exp \left\{ \gamma'_1 U_t + \gamma'_2 V_t \right\} \cdot \exp \left\{ n\sqrt{U_t} Z \right\} | V \right] \]
\[ = (\mathcal{S})^n \mathbb{E} \left[ \exp \left\{ \gamma'_1 U_t + \gamma'_2 V_t \right\} \right] \mathbb{E} \left[ \exp \left\{ \frac{1}{2} n^2 \sqrt{U_t} Z \right\} \right], \]
where
\[ \gamma_1 = - \left( \frac{1}{2} + \frac{a}{c} \rho \right), \quad \gamma'_1 = n \gamma_1, \]
\[ \gamma_2 = \frac{\rho}{c}, \quad \gamma'_2 = n \gamma_2. \]

A similar formula is obtained for European puts.
Parseval integral

We derived the damped Fourier transforms for a European put and call denoted $\hat{g}_1^{(-\alpha)}$ and $\hat{g}_2^{(-\alpha)}$ in Theorem 3.8 and the characteristic function of the distribution of $U_t$ when $V_t$ is the square-root process is also known in (5.6). Thus, we can apply the general formula discussed in Section 3.4, which incorporates the Parseval integral in the case $\rho = 0$, that is $V$ is independent of $W$. For instance, a European call price is

$$e^{-rT}E(S_T - K)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \widehat{\hat{g}_2^{(-\alpha)}}(-u) \widehat{\nu^{(\alpha)}}(u) \, du,$$

where

$$\widehat{\nu^{(\alpha)}}(u) = \left[ \frac{e^{-at/2}}{\cosh(Dt/2) - aD^{-1} \sinh(Dt/2)} \right]^{2b/c^2} \times \exp \left\{ \frac{(\alpha + iu)v_0}{D} - \frac{2 \sinh(Dt/2)}{\cosh(Dt/2) - aD^{-1} \sinh(Ut/2)} \right\},$$

$$D := D(u) = \sqrt{a^2 - 2c^2(\alpha + iu)},$$

and $g_2^{(-\alpha)}(-u)$ is as in (3.19).

Fast Fourier Transform

Since the characteristic function of the log-price is known in closed form, Fourier inversion method such as FFT discussed in Section 3.2 may be used to price vanilla options.

5.4.4 Quadratic exponential scheme: moment-matching based

Andersen [4] suggests the quadratic exponential (QE) scheme; the sampling from the non-central chi-square distribution is approximated by a draw from a related distribution, which is moment-matched to the first two moments of non-central chi-square distribution. From the fact that $V_{t+\Delta}$ is proportional to a non-central chi-squared distribution, the level of $V_t$ decides which distribution is used for generating $V_{t+\Delta}$. For large non-centrality parameter $\lambda$ of non-central chi-square distribution or equivalently, large $V_t$, a quadratic representation

$$V_{t+\Delta} = \alpha(\beta + Z)^2 \quad (5.34)$$

is chosen for approximating the variance process. A cubic transformation of Gaussian variable is another choice but it may lead to the negative values of $\{V_t\}$. Here,
α and β are constants that are based on the first two moments matching and Z is a standard normal random variable. Conversely, when \( V_t \) is small, the density of \( V_{t+\Delta} \) is approximated as

\[
P(V_{t+\Delta} \in [x, x + dx]) \approx (p\delta_0 + \zeta(1-p)e^{-\zeta x}) \, dx, \quad x \geq 0,
\]

where \( \delta \) denotes Dirac delta function and \( p \) and \( \zeta \) are non-negative constants. Inverting the distribution function enables us to simulate \( V_{t+\Delta} \) by generating a uniform random number

\[
V_{t+\Delta} = \begin{cases} 
0 & 0 \leq u \leq p, \\
\frac{1}{\xi} \log \left( \frac{1-x}{1-\xi} \right) & p < u \leq 0.
\end{cases}
\]

(5.35)

Let \( \psi := \text{Var} V_t / (\text{EV}_t)^2 \). Then, the constants \( \alpha = \frac{\text{EV}_t}{1 + \beta^2}, \quad \beta^2 = \frac{2}{\psi} - 1 + \sqrt{\frac{2}{\psi}} \sqrt{\frac{2}{\psi} - 1}, \quad p = \frac{\psi - 1}{\psi + 1}, \quad \zeta = \frac{2}{\text{EV}_t(\psi + 1)} \)

are computed based on moment-matching. (See Andersen [4] appendix for the proof.) Since \( \psi \) is inversely proportional to the non-centrality parameter of non-central chi-square, Andersen suggested that moment-matching with quadratic sampling and moment-matching with exponential scheme should be used for the case \( \psi \leq 2 \) and \( \psi \geq 1 \), respectively. For example, assigning the critical value \( \psi_c \) to 1.5 is reasonable and use (5.34) if \( \psi \leq \psi_c \) or (5.35) otherwise.

From (5.30), log-price at \( t + \Delta \) can be expressed as

\[
\log S_t + r\Delta + \frac{\rho}{c}(V_{t+\Delta} - V_t - b\Delta) - \left( \frac{1}{2} + \frac{a \rho}{c} \right) \int_t^{t+\Delta} V_s \, ds + \rho \int_t^{t+\Delta} \sqrt{V_s} \, dW_s^{(1)},
\]
given \( V_t, V_{t+\Delta} \) and \( S_t \). Then, we are able to obtain a discretisation scheme

\[
\log S_{t+\Delta} = \log S_t + K_0 + K_1 V_t + K_2 V_{t+\Delta} + \sqrt{K_3 V_t + K_4 V_{t+\Delta}} \, Z,
\]

(5.36)

where \( Z \) is a standard normal, independent of \( V \) and

\[
K_0 = -\frac{\rho b}{c} \Delta, \quad K_1 = -\eta_1 \Delta \left( \frac{a \rho}{c} + \frac{1}{2} \right) - \frac{\rho}{c}, \quad K_2 = -\eta_2 \Delta \left( \frac{a \rho}{c} + \frac{1}{2} \right),
\]

\[
K_3 = \eta_1 \Delta (1 - \rho^2), \quad K_4 = \eta_2 \Delta (1 - \rho^2).
\]

Note that \( \int_t^{t+\Delta} V_s \, ds \) can be also interpolated using one of the approximation in Remark 5.3, for example with the predictor-corrector coefficients \( \eta_1 = \eta_2 = 0.5 \). Andersen [4] claims that the QE method is the best among alternative approximation such as truncated Gaussian scheme.
Martingale correction

The discounted stock price $S_t$ in continuous time will always be a martingale under
the risk-neutral measure $Q$. However, the discretised stock price denoted $	ilde{S}_t$ in
the scheme outlined above does not satisfy the martingale condition

$$E^Q[e^{-r\Delta \tilde{S}_{t+\Delta}}|\mathcal{F}_t] \neq \tilde{S}_t.$$ 

However, for the cases in which the error $\epsilon = ES_{t+\Delta} - ES_t$ is analytically computable,
it is straightforward to remove the bias by simply adding $-\epsilon$ to the sample value for
$S_{t+\Delta}$. Define

$$\epsilon = K_2 + \frac{1}{2}K_4 = \frac{\rho}{\sigma} (1 + \kappa \gamma_2 \Delta) - \frac{1}{2} \gamma_2 \Delta \rho^2$$

$$M = E[e^{\epsilon V_t + \Delta |V_t}] > 0$$

$$K_0^* = -\log M - (K_1 + \frac{1}{2}K_3)V_t.$$ 

Then, Andersen [4] shows for the finite $M$,

$$E^Q[\tilde{S}_{t+\Delta} | \tilde{S}_t] < \infty,$$

and

$$\tilde{S}_{t+\Delta} = \tilde{S}_t \exp \{ K_0^* + K_1 V_t + K_2 V_{t+\Delta} + \sqrt{K_3 V_t + K_4 V_{t+\Delta}} Z \}, \quad (5.37)$$

satisfies the martingale condition. We refer to Andersen [4] Proposition 7 for proof

5.4.5 Measure change

The law of $\delta$-dimensional squared Bessel process $X_t$ coincides with that of

$$\left( W^{(1)} \right)^2 + \cdots + \left( W^{(\delta)} \right)^2,$$

when $\delta \in \{1, 2, \ldots \}$. This leads to the idea of simulating the squared Bessel process
as the norm of a $\delta$-dimensional Brownian motion when $\delta$ is a positive integer, and
using a change-of-measure for other $\delta$. Now, we describe how the change-of-measure
formula derived in Section 5.3 can be applied to price European vanilla option. For
instance, the payoff of a European call option can be expressed as

$$E(S_T - K)^+ = Q_x^{(0)} [(S_T - K)^+ \cdot R],$$

where $Q_x^{(0)}$ is the measure of the squared Bessel process with index $\nu = 0$ starting
from $x$ and $R$ is a Radon-Nikodym derivative.
5.4. The computation of European option prices

**Theorem 5.23.** Let \( Y_t \) be a squared radial Ornstein-Uhlenbeck process with index \( \mu \) that satisfies the SDE

\[
dY_t = (2\mu + 2 + 2\lambda Y_t) \, dt + 2\sqrt{Y_t} \, dW_t, \quad Y_0 = y > 0,
\]

and let

\[
\lambda = \frac{a}{2}, \quad q = \left( \frac{c}{2} \right)^2, \quad \mu = \frac{1}{2} \left( \frac{b}{q} - 2 \right).
\]

Then, we have

\[
E^{Q^{(\mu)}} \left[ \frac{Y_t}{y} \right] = E^{Q^{(0)}} \left[ R \cdot Z \right],
\]  

(5.38)

where

\[
R = \left( \frac{Y_t}{y} \right)^{\mu/2} \exp \left\{ \frac{\lambda}{2q} [Y_t - y - q(2\mu + 2)t] - \frac{\lambda^2}{2q} \int_0^t Y_s \, ds \right\}.
\]

**Proof.** Consider the process \( V_t = qY_t \) satisfies

\[
dV_t = [q(2\mu + 2) + 2\lambda V_t] \, dt + 2\sqrt{q} \sqrt{V_t} \, dW_t.
\]

If \( c = 2\sqrt{q} \), then \( 2\lambda = a \) and \( 2\mu + 2 = bq \) in (5.1). Applying Theorem 5.21 completes the proof. \( \square \)

This enables us to compute any European option price with \( q = c^2/4 \),

\[
e^{-rT} E g (V_t; 0 \leq t \leq T)
\]

may be expressed as

\[
e^{-rT} E g \left( \frac{Y_t}{q}; 0 \leq t \leq T \right) = e^{-rT} E Z.
\]

We show European call prices computed using the methods in described as Section 5.4, when the stochastic volatility is a square-root process \( V_t \). From (5.32), the expected payoff of a European call option can be written as conditional expectation

\[
E(S_T - K)_+ = e^{-\frac{\xi}{2}(v_0 + bT)} E \left\{ e^{\left( \frac{b^2 - 1}{2} - \frac{\xi}{2} \right) V_T + \frac{\xi}{2} V_T} E \left[ \left( S_0 e^{rT - \frac{1}{2} b^2 V_T + \xi \sqrt{V_T} Z} - K' \right)_+ \right] \right\}
\]

\[
= e^{-\frac{\xi}{2}(v_0 + bT)} E \left[ e^{\xi_1 V_T + \xi_2 V_T} g (U_T) \right],
\]

where

\[
\xi_1 = \frac{b^2 - 1}{2} - a\xi_2, \quad \xi_2 = \frac{\rho}{c}, \quad K' = \frac{K}{\exp\{\xi_1 U_T + \xi_2 (V_T - v_0 - bT)\}},
\]

\[
g(u) = E \left( S_0 \exp \left\{ rT - \frac{b^2}{2} u + \xi \sqrt{u} Z \right\} - K' \right)_+.
\]
Here, the function $g(u)$ is the price of a European call in the Black-Scholes model with strike $K'$, maturity $T$ and volatility $\overline{\rho}\sqrt{u}$. Applying measure change formula, for instance Theorem 5.23, makes the simulation of the paths of the square-root process for various $\nu \geq 0$ simple. First, generate two-dimensional Brownian motion using two independent standard normal random variables. Secondly, all we need for European option prices is to simulate the Black-Scholes and take the average. Note that the conditional expectation is simply the Black-Scholes model with $\sigma$ and $K$ are replaced by $\overline{\rho}\sqrt{U_T/T}$ and $K'$, respectively. That is, generating the square-root process suffices to simulate two-dimensional Brownian motion. In point of efficiency measure change approach sounds attractive.

### 5.5 Numerical results

#### 5.5.1 Parameter estimation

The mean-reversion rate $a$ in the square-root process (5.1) determines the degree of volatility clustering, that is, large price variations are more likely to be followed by large price variations (see Cont [32]). The volatility of the square-root process $c$ affects the kurtosis of the distribution. The effect of changing the kurtosis of the distribution affects the implied volatility. Higher $c$ makes the skew more prominent. This makes sense relative to the leverage effect as higher $c$ means that the volatility is more volatile. This means that the market has a greater chance of extreme movements. Therefore, writers of puts must charge more and those of calls, less, for a given strike. Another parameter to be estimated is the correlation $\rho$ between underlying asset and volatility. There is empirical evidence that the correlation is negative and sometimes it is referred to as leverage effect. Intuitively, if $\rho > 0$, then volatility will increase and if $\rho < 0$, then volatility will decrease as the stock price increases. In the latter case, the upper tail is heavier and the lower tail is squeezed. Thus, the correlation $\rho$ controls the skewness of the distribution. The effect of changing the skewness of the distribution also affects the shape of the implied volatility surface. Figure 5.6–5.8 shows the effect of the varying $\rho$. The model can imply a variety of volatility surfaces and hence addresses another shortcoming of the Black-Scholes model, namely constant volatility across different strike levels. We summarise the set of estimated parameters for the square-root process in Table 5.2 from recent literature. Moreover, we refer to Cont [32] for more empirical studies and references therein.

For the square-root process it is guaranteed to be greater or equal to zero. If the
5.5. Numerical results

![Implied volatility surfaces](image)

Figure 5.6: Implied volatility surfaces of the square-root process when $\rho = 0$ with fixed parameters $a = -3.4$, $b = 0.08$, $c = 0.4$, $v_0 = 0.04$, $r = 0.0$, and $S_0 = 100$.

Feller condition $2b > c^2$ (or equivalently $2\kappa\theta > \sigma^2$, if one uses notations in Heston [69]) is satisfied, then the process never reaches zero and for $2b < c^2$ the origin is accessible and reflecting. Andersen [4], p.4 claims that ‘in typical application, $2\kappa\theta$ is often significantly below $\sigma^2$, thus the likelihood of hitting zero is often quite significant’. Similarly, Broadie and Kaya [22] claim that ‘there are some cases for which the bias of the Euler discretisation is very large even if a large number of time steps are used. This is especially true when $2\kappa\theta/\sigma^2 \leq 1$ and $\sigma$ is large relative to $\theta$’ on p.222. Since $\delta = 4\kappa\theta/\sigma^2$, it is equivalent to $\delta < 2$, which is the condition for negative values of $V_t$ in the square-root process. However, Table 5.2 demonstrates that $\delta$ is likely to be greater than 2 in many empirical works and we can still use the change-of-measure formula to price of a European vanilla and Asian options when the square-root process is used to represent stochastic volatility.

5.5.2 European call option prices

We begin by comparing change-of-measure formula (see Theorem 5.23) to the various Fourier formulas and Andersen’s method described in Section 5.4. We use three different sets of parameters summarised in Table 5.4. The relevant results are con-
Figure 5.7: Implied volatility surfaces of the square-root process when positive correction, say $\rho = 0.8$, with fixed parameters $a = -3.4$, $b = 0.08$, $c = 0.4$, $v_0 = 0.04$, $r = 0.0$ and $S_0 = 100$. 
Figure 5.8: Implied volatility surfaces of the square-root process when negative correction, say $\rho = -0.8$, with fixed parameters $a = -3.4$, $b = 0.08$, $c = 0.4$, $v_0 = 0.04$, $r = 0.0$ and $S_0 = 100$. 
Table 5.2: Set of estimated parameters for the square-root process in literature. If one follows the notation in Heston [69], \(a = -\kappa\), \(b = \kappa \theta\) and \(c = \sigma\).

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>(\delta)</th>
<th>(\rho)</th>
<th>Underlying</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.146</td>
<td>0.076</td>
<td>0.579</td>
<td>0.90</td>
<td>0.00</td>
<td>Deutsch Mark/USD [20]</td>
</tr>
<tr>
<td>-6.210</td>
<td>0.118</td>
<td>0.610</td>
<td>1.27</td>
<td>-0.70</td>
<td>S&amp;P 500 option [41]</td>
</tr>
<tr>
<td>-3.400</td>
<td>0.082</td>
<td>0.390</td>
<td>2.15</td>
<td>-0.64</td>
<td>S&amp;P 500 index option [52]</td>
</tr>
<tr>
<td>-2.180</td>
<td>0.174</td>
<td>0.530</td>
<td>2.48</td>
<td>-0.70</td>
<td>S&amp;P 500 option [9]</td>
</tr>
<tr>
<td>-11.350</td>
<td>0.250</td>
<td>0.618</td>
<td>2.62</td>
<td>0.00</td>
<td>Dow Jones index [40]</td>
</tr>
<tr>
<td>-5.300</td>
<td>0.128</td>
<td>0.380</td>
<td>3.55</td>
<td>-0.57</td>
<td>S&amp;P 500 index option [93]</td>
</tr>
<tr>
<td>-5.070</td>
<td>0.232</td>
<td>0.480</td>
<td>4.02</td>
<td>-0.77</td>
<td>S&amp;P 500 option [2]</td>
</tr>
<tr>
<td>-0.017</td>
<td>0.015</td>
<td>0.107</td>
<td>5.26</td>
<td>-0.37</td>
<td>S&amp;P 500 option [47]</td>
</tr>
<tr>
<td>-5.480</td>
<td>0.789</td>
<td>0.719</td>
<td>6.11</td>
<td>0.16</td>
<td>ASX option [50]</td>
</tr>
</tbody>
</table>

tained in Tables 5.5–5.7. In each of these tables, Euler is used to indicate that the Euler discretisation methods in Lord and Kahl [84] (full-truncation), CM to indicate the change-of-measure approach, and QE to indicate the quadratic exponential scheme in Andersen [4], respectively. In the case \(\rho = 0\), we may add two more columns, Parseval and constant volatility. The latter uses constant volatility over the period \([0, T]\) and is plugged into the Black-Scholes formula directly. The margin of error for 95% confidence intervals are provided in brackets for the simulated prices. Execution times in seconds at the bottom represents total times for all 21 numbers in each column. In each case considered, we compute European call options with spot of 100, maturities from time 0.25 to 5 years. All simulations are run with one million paths unless stated and all computations were done on a MacBook Pro 2.66GHz with 4GB RAM. The columns labelled in Heston, Mellin and Parseval were computed using Mathematica and Monte Carlo simulations were coded in C.

Mellin formula gives almost exactly the same result that Heston formula produces, but is two times faster than Heston formula in general. When \(\rho = 0\), the Parseval integral also works well even though the execution time is slightly slower than Heston and Mellin formula. In addition, the Euler scheme performs poorly (as can be seen in the tables, most of the Euler values are outside the 95% confidence interval), especially when the stock price is correlated to its volatility in all moneyness. In general, both CM and QE performs very closely, the former has shorter execution time though. In particular, the higher the dimension is and the longer the maturity is, CM will slightly outperforms QE. Figure 5.9 and 5.10 show the convergence of the Euler, CM and QE methods across different moneynesses.
Table 5.3: First three moments of distribution of the square-root process $V_t$, log $V_t$ and its integrated process $U_t$ for different dimensions ($\delta = 2, 5, 8$) and maturities ($T = 1, 5$) based on 100 time-steps per year and 100,000 paths of simulation.
and maturities. For the short maturity ($T = 0.25$ year), both CM and QE indicate similar rate of convergence in price, the short stepping seems to be avoided for all moneyness though. In the case of longer maturity ($T = 5$ years) CM produces overprice when short stepping is used. To make use of the measure change formula we need to approximated two integrals, $I^X_t$ and $J^X_t$ (in (5.18) and (5.16).) In short-stepping we do not get the good approximation and even worse when the square-root process ends up near zero. Starting at small value usually results in the expected value of Radon-Nikodym derivative being away from one.

Similar results (see Tables 5.8 and 5.9) are produced in pricing Asian option as well. In addition, as Hull and White [70] and Henderson and Hobson [68] observed, we conclude that the option price along with stochastic volatility is lower than the Black-Scholes price for near-the-money options when the volatility is not correlated with the stock price. (See the column ConstVol in Table 5.5.)

**Volatility smile**

As observed in Section 4.3.1, similar implied volatility curves are produced when the volatility is modelled by the square-root process. However, the shape depends on the set of parameters and for the Case II and Case III, the skew instead of smile is observed (see Tables 5.11 to 5.13.)
Figure 5.9: Convergence of the call option price as the number of time-step per year increases, given parameters $T = 0.25$, $a = -0.017$, $b = 0.015$, $c = 0.107$, $v_0 = 0.15$, $r = 0.0$ and $S_0 = 100$. 
Figure 5.10: Convergence of the call option price as the number of time-step per year increases, given parameters $T = 5$, $a = -0.017$, $b = 0.015$, $c = 0.107$, $v_0 = 0.15$, $r = 0.0$ and $S_0 = 100$. 
## Table 5.5: Vanilla call option prices with 64 time-steps per year. Parameters in Case I are used.

<table>
<thead>
<tr>
<th>T</th>
<th>Heston</th>
<th>Mellin</th>
<th>Parseval</th>
<th>Euler</th>
<th>CM</th>
<th>QE</th>
<th>ConstVol</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>K=80</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>20.0459</td>
<td>20.0459</td>
<td>20.0459</td>
<td>20.0559 ±0.0151</td>
<td>20.0311 ±0.0254</td>
<td>20.0548 ±0.0152</td>
<td>20.0233</td>
</tr>
<tr>
<td>0.50</td>
<td>20.2346</td>
<td>20.2346</td>
<td>20.2346</td>
<td>20.2341 ±0.0199</td>
<td>20.2399 ±0.0354</td>
<td>20.2316 ±0.0200</td>
<td>20.1728</td>
</tr>
<tr>
<td>1.00</td>
<td>20.7156</td>
<td>20.7156</td>
<td>20.7156</td>
<td>20.7061 ±0.0260</td>
<td>20.6973 ±0.0356</td>
<td>20.7140 ±0.0262</td>
<td>20.6584</td>
</tr>
<tr>
<td>2.00</td>
<td>21.7917</td>
<td>21.7917</td>
<td>21.7917</td>
<td>21.7856 ±0.0345</td>
<td>21.7938 ±0.1012</td>
<td>21.8016 ±0.0346</td>
<td>21.7965</td>
</tr>
<tr>
<td>3.00</td>
<td>22.8878</td>
<td>22.8878</td>
<td>22.8879</td>
<td>22.8614 ±0.0410</td>
<td>22.7319 ±0.1764</td>
<td>22.8743 ±0.0411</td>
<td>22.9293</td>
</tr>
<tr>
<td>4.00</td>
<td>23.9410</td>
<td>23.9410</td>
<td>23.9413</td>
<td>23.8954 ±0.0466</td>
<td>23.9680 ±0.3056</td>
<td>23.9162 ±0.0468</td>
<td>24.0012</td>
</tr>
<tr>
<td>5.00</td>
<td>24.9384</td>
<td>24.9384</td>
<td>24.9390</td>
<td>24.8614 ±0.0518</td>
<td>24.7319 ±0.5043</td>
<td>24.8743 ±0.0519</td>
<td>25.0081</td>
</tr>
<tr>
<td></td>
<td>K=100</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>0.25</td>
<td>3.6203</td>
<td>3.6203</td>
<td>3.6203</td>
<td>3.5848 ±0.0096</td>
<td>3.6186 ±0.0036</td>
<td>3.6239 ±0.0096</td>
<td>3.7215</td>
</tr>
<tr>
<td>0.50</td>
<td>4.8476</td>
<td>4.8476</td>
<td>4.8476</td>
<td>4.8000 ±0.0133</td>
<td>4.8462 ±0.0070</td>
<td>4.8423 ±0.0133</td>
<td>5.0234</td>
</tr>
<tr>
<td>1.00</td>
<td>6.5137</td>
<td>6.5137</td>
<td>6.5137</td>
<td>6.4642 ±0.0183</td>
<td>6.5051 ±0.0145</td>
<td>6.5039 ±0.0184</td>
<td>6.7371</td>
</tr>
<tr>
<td>2.00</td>
<td>8.9219</td>
<td>8.9219</td>
<td>8.9224</td>
<td>8.8990 ±0.0258</td>
<td>8.9205 ±0.0367</td>
<td>8.9338 ±0.0259</td>
<td>9.1385</td>
</tr>
<tr>
<td>3.00</td>
<td>10.8138</td>
<td>10.8138</td>
<td>10.8153</td>
<td>10.7866 ±0.0319</td>
<td>10.7456 ±0.0749</td>
<td>10.8158 ±0.0320</td>
<td>11.0106</td>
</tr>
<tr>
<td>4.00</td>
<td>12.4220</td>
<td>12.4220</td>
<td>12.4244</td>
<td>12.3813 ±0.0373</td>
<td>12.4187 ±0.1430</td>
<td>12.4115 ±0.0374</td>
<td>12.6014</td>
</tr>
<tr>
<td>5.00</td>
<td>13.8425</td>
<td>13.8425</td>
<td>13.8457</td>
<td>13.7753 ±0.0424</td>
<td>13.9066 ±0.2574</td>
<td>13.8023 ±0.0425</td>
<td>14.0075</td>
</tr>
<tr>
<td></td>
<td>K=120</td>
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<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.1394</td>
<td>0.1394</td>
<td>0.1394</td>
<td>0.1419 ±0.0021</td>
<td>0.1392 ±0.0002</td>
<td>0.1400 ±0.0021</td>
<td>0.0981</td>
</tr>
<tr>
<td>0.50</td>
<td>0.5229</td>
<td>0.5229</td>
<td>0.5229</td>
<td>0.5152 ±0.0049</td>
<td>0.5223 ±0.0007</td>
<td>0.5192 ±0.0049</td>
<td>0.4553</td>
</tr>
<tr>
<td>1.00</td>
<td>1.3457</td>
<td>1.3457</td>
<td>1.3457</td>
<td>1.3227 ±0.0093</td>
<td>1.3429 ±0.0026</td>
<td>1.3383 ±0.0093</td>
<td>1.3232</td>
</tr>
<tr>
<td>2.00</td>
<td>2.9823</td>
<td>2.9823</td>
<td>2.9823</td>
<td>2.9697 ±0.0162</td>
<td>2.9800 ±0.0104</td>
<td>2.9923 ±0.0163</td>
<td>3.0423</td>
</tr>
<tr>
<td>3.00</td>
<td>4.5273</td>
<td>4.5273</td>
<td>4.5276</td>
<td>4.5081 ±0.0222</td>
<td>4.5202 ±0.0271</td>
<td>4.5312 ±0.0223</td>
<td>4.6218</td>
</tr>
<tr>
<td>4.00</td>
<td>5.9558</td>
<td>5.9558</td>
<td>5.9565</td>
<td>5.9211 ±0.0276</td>
<td>5.9440 ±0.0598</td>
<td>5.9490 ±0.0277</td>
<td>6.0642</td>
</tr>
<tr>
<td>5.00</td>
<td>7.2787</td>
<td>7.2787</td>
<td>7.2799</td>
<td>7.2231 ±0.0327</td>
<td>7.3257 ±0.1208</td>
<td>7.2448 ±0.0372</td>
<td>7.3919</td>
</tr>
</tbody>
</table>

| Time (sec) | 0.206   | 0.119   | 0.519   | 505.82  | 560.99  | 683.47  | < 10⁻³  |

Table 5.5: Vanilla call option prices with 64 time-steps per year. Parameters in Case I are used.
<table>
<thead>
<tr>
<th>T</th>
<th>Heston</th>
<th>Mellin</th>
<th>Euler</th>
<th>CM</th>
<th>QE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>21.133</td>
<td>21.134</td>
<td>20.705</td>
<td>(±0.0292)</td>
<td>21.127</td>
</tr>
<tr>
<td>0.50</td>
<td>23.002</td>
<td>23.006</td>
<td>22.191</td>
<td>(±0.0394)</td>
<td>23.007</td>
</tr>
<tr>
<td>1.00</td>
<td>26.329</td>
<td>26.329</td>
<td>24.794</td>
<td>(±0.0541)</td>
<td>26.325</td>
</tr>
<tr>
<td>2.00</td>
<td>31.643</td>
<td>31.643</td>
<td>28.769</td>
<td>(±0.0780)</td>
<td>31.628</td>
</tr>
<tr>
<td>3.00</td>
<td>35.958</td>
<td>35.958</td>
<td>31.628</td>
<td>(±0.0991)</td>
<td>35.952</td>
</tr>
<tr>
<td>4.00</td>
<td>39.684</td>
<td>39.684</td>
<td>33.924</td>
<td>(±0.1192)</td>
<td>39.679</td>
</tr>
<tr>
<td>5.00</td>
<td>43.083</td>
<td>43.083</td>
<td>35.853</td>
<td>(±0.1402)</td>
<td>43.011</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.25</td>
<td>7.738</td>
<td>7.738</td>
<td>7.472</td>
<td>(±0.0103)</td>
<td>7.734</td>
</tr>
<tr>
<td>0.50</td>
<td>10.960</td>
<td>10.960</td>
<td>10.408</td>
<td>(±0.0297)</td>
<td>10.962</td>
</tr>
<tr>
<td>1.00</td>
<td>15.546</td>
<td>15.546</td>
<td>14.397</td>
<td>(±0.0442)</td>
<td>15.540</td>
</tr>
<tr>
<td>2.00</td>
<td>22.099</td>
<td>22.099</td>
<td>19.764</td>
<td>(±0.0683)</td>
<td>22.086</td>
</tr>
<tr>
<td>3.00</td>
<td>27.183</td>
<td>27.183</td>
<td>23.526</td>
<td>(±0.0898)</td>
<td>27.179</td>
</tr>
<tr>
<td>4.00</td>
<td>31.499</td>
<td>31.499</td>
<td>26.475</td>
<td>(±0.1104)</td>
<td>31.494</td>
</tr>
<tr>
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<td>35.315</td>
<td>35.315</td>
<td>28.948</td>
<td>(±0.1318)</td>
<td>35.313</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>1.923</td>
<td>1.923</td>
<td>1.824</td>
<td>(±0.0103)</td>
<td>1.922</td>
</tr>
<tr>
<td>0.50</td>
<td>4.469</td>
<td>4.469</td>
<td>4.175</td>
<td>(±0.0195)</td>
<td>4.470</td>
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<tr>
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<td>8.719</td>
<td>8.719</td>
<td>7.931</td>
<td>(±0.0341)</td>
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<tr>
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<td>15.305</td>
<td>13.478</td>
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<tr>
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<td>20.597</td>
<td>17.557</td>
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<tr>
<td>4.00</td>
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<td>25.158</td>
<td>20.818</td>
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<td>25.151</td>
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<td>5.00</td>
<td>29.223</td>
<td>29.223</td>
<td>23.593</td>
<td>(±0.1237)</td>
<td>29.218</td>
</tr>
</tbody>
</table>

Table 5.6: Vanilla call option prices with 64 time-steps per year. Parameters in Case II are used.
## 5.5. Numerical results

Table 5.7: Vanilla call option prices with 64 time-steps per year. Parameters in Case III are used.

<table>
<thead>
<tr>
<th>K</th>
<th>T</th>
<th>Heston</th>
<th>Mellin</th>
<th>Euler</th>
<th>CM</th>
<th>QE</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.25</td>
<td>24.1136</td>
<td>24.1126</td>
<td>24.5176 (±0.0430)</td>
<td>24.0920 (±0.0372)</td>
<td>24.1035 (±0.0447)</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>27.9637</td>
<td>27.9637</td>
<td>24.9817 (±0.0578)</td>
<td>27.9690 (±0.0503)</td>
<td>27.9790 (±0.0618)</td>
</tr>
<tr>
<td></td>
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| Time (sec) | 2209.66 | 1957.88 | 566.49 |

Table 5.8: Asian call option prices with monthly averaging. The parameters are chosen as described in Case I.
Table 5.9: Asian call option prices with monthly averaging. The parameters are chosen as described in Case II.
Figure 5.11: Implied volatility curves for different maturities when the volatility is modelled by the square-root process with parameters described in Case I. European vanilla call option prices are obtained by CM.
Figure 5.12: Implied volatility curves for different maturities when the volatility is modelled by the square-root process with parameters described in Case II. European vanilla call option prices are obtained by CM.
Figure 5.13: Implied volatility curves for different maturities when the volatility is modelled by the square-root process with parameters described in Case I. European-style Asian call option prices are obtained by CM.
Bibliography


[91] Nikeghbali, A., 2006. Some random times and martingales associated with $BES_0(\delta)$ processes ($0 < \delta < 2$). ALEA 2, 67–89.


