

Special Function Aspects of Macdonald Polynomial Theory

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Dedicated to my Grandpa

Abstract

The nonsymmetric Macdonald polynomials generalise the symmetric Macdonald polynomials and the nonsymmetric Jack polynomials. Consequently results obtained in nonsymmetric Macdonald polynomial theory specialise to analogous results in the theory of these other polynomials. The converse, however, does not always apply. Thus some properties of symmetric Macdonald, and nonsymmetric Jack, polynomials have no known nonsymmetric Macdonald polynomial analogues in the existing literature. Such examples are contained in the theory of Jack polynomials with prescribed symmetry and the study of the Pieri-type formulas for the symmetric Macdonald polynomials. The study of Jack polynomials with prescribed symmetry considers a special class of Jack polynomials that are symmetric with respect to some variables and antisymmetric with respect to others. The Pieri-type formulas provide the branching coefficients for the expansion of the product of an elementary symmetric function and a symmetric Macdonald polynomial. In this thesis we generalise both studies to the theory of nonsymmetric Macdonald polynomials.

In relation to the Macdonald polynomials with prescribed symmetry, we use our new results to prove a special case of a constant term conjecture posed in the late 1990s. A highlight of our study of Pieri-type formulas is the development of an alternative viewpoint to the existing literature on nonsymmetric interpolation Macdonald polynomials. This allows for greater extensions, and more simplified derivations, than otherwise would be possible. As a consequence of obtaining the Pieri-type formulas we are able to deduce explicit formulas for the related generalised binomial coefficients.

A feature of the various polynomials studied within this thesis is that they allow for explicit computation. Having this knowledge provides an opportunity to experimentally seek new properties, and to check our working in some of the analytic work. In the last chapter a computer software program that was developed for these purposes is detailed.

Declaration

This is to certify that

1. The thesis comprises only my original work towards the PhD except where indicated in the Preface.
2. Due acknowledgement has been made in the text to all other material used.
3. The thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Wendy Catherine Baratta

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Contents

1	Introduction	1
1.1	A Brief History	1
1.2	Symmetric Polynomials	3
1.2.1	Preliminaries	3
1.2.2	Some special symmetric polynomials	10
1.3	Nonsymmetric Polynomials	16
1.3.1	Preliminaries	16
1.3.2	Some special nonsymmetric polynomials	22
1.4	Interpolation Polynomials	24
1.4.1	Preliminaries	24
1.4.2	Some special nonsymmetric interpolation polynomials	27
2	Foundational Theory	31
2.1	Nonsymmetric Macdonald Polynomials	31
2.1.1	Introduction	31
2.1.2	The eigenoperators	32
2.1.3	Recursive generation	36
2.1.4	Rodrigues formula	37
2.1.5	Orthogonality	39
2.1.6	Evaluation formula	40
2.1.7	A Cauchy formula	41
2.2	Nonsymmetric Interpolation Macdonald Polynomials	42
2.2.1	Introduction	42
2.2.2	The vanishing condition	43
2.2.3	The eigenoperators	44
2.2.4	Recursive generation	47
2.2.5	Rodrigues formula	50

2.2.6	Evaluation formula	50
2.2.7	The extra vanishing theorem	52
2.2.8	Binomial coefficients	53
3	Macdonald Polynomials with Prescribed Symmetry	55
3.1	A Brief History	56
3.2	Preliminaries	56
3.2.1	Symmetrisation	56
3.2.2	Prescribed symmetry operator	57
3.2.3	The polynomial $S_{\eta^*}^{(I,J)}(z)$	58
3.3	Antisymmetric Macdonald Polynomials	63
3.4	Prescribed Symmetry Macdonald Polynomials with Staircase Components	67
3.4.1	Special form with $\eta^* = \eta_{(n_0; N_p)}$	69
3.4.2	Special form with $\eta^* = \eta_{(N_p; n_0)}$	70
3.5	The Inner Product of Prescribed Symmetry Polynomials and Constant Term Identities	73
3.5.1	The inner product of prescribed symmetry polynomials	73
3.5.2	Special cases of the prescribed symmetry inner product	75
3.5.3	The constant term identities	77
3.6	Further Work	83
4	Pieri-Type Formulas for Nonsymmetric Macdonald Polynomials	85
4.1	A Brief History	86
4.2	Preliminaries	87
4.2.1	General Pieri-type formulas	87
4.2.2	Structure of Pieri-type expansions	88
4.3	Pieri-type Formulas for $r = 1$	90
4.3.1	Non-zero coefficients	91
4.3.2	The intertwining formula	92
4.3.3	The product $z_i E_\eta$	93
4.3.4	The main result	98
4.3.5	The classical limit	98
4.3.6	Pieri-type formula for $r = 1$ and the generalised binomial coefficient	100
4.3.7	The Pieri-type formula for $r = n - 1$	102

4.3.8	Lascoux's derivation of the Pieri-type formula for $r = 1$	105
4.4	The General Pieri-Type Formulas	107
4.4.1	An alternative derivation of the Pieri-type formulas for $r = 1$	107
4.4.2	The general Pieri-type formula coefficients	111
4.4.3	Simplifying the Pieri coefficients	114
4.4.4	Reclaiming the symmetric Macdonald Pieri formulas	115
4.5	Further Work	116
5	Computer Programming	117
5.1	A Brief History	117
5.2	Preliminaries	118
5.2.1	Recursively generating compositions	118
5.3	Recursively generating polynomials	121
5.4	Runtimes and Software	123
5.4.1	Algorithm runtimes	123
5.4.2	Mathematica notebook	124
5.5	Further Work	127
	Appendix A Some basic q-functions	139
	Appendix B Symmetric polynomials	141
B.1	Triangular structure	141
B.2	Orthogonality	142
B.3	Eigenfunctions	143
B.4	Examples	143
	Appendix C Nonsymmetric Jack polynomials	145
C.1	Special function properties	145
C.2	Recursive generation formulas	145
C.3	Examples	146
	Appendix D Nonsymmetric Interpolation Jack polynomials	147
D.1	Special function properties	147
D.2	Recursive generation formulas	147
D.3	Examples	148

Chapter 1

Introduction

It is the main aim of this chapter to provide motivation and background for the theory of the nonsymmetric Macdonald polynomials and nonsymmetric interpolation Macdonald polynomials required for the main studies of the thesis. The nonsymmetric Macdonald polynomials and nonsymmetric interpolation Macdonald polynomials are generalisations of older and more established classes of polynomials. We begin with a brief history of the relevant polynomial theory to motivate the mathematics in the following sections. We use the historic development to order our remaining sections, first considering the symmetric polynomials, secondly the nonsymmetric polynomials and lastly the nonsymmetric interpolation polynomials. Each section begins with the required preliminary material and then moves onto specific families highlighting their special function properties. The key references for each polynomial class are also provided.

1.1 A Brief History

Symmetric functions originated due to their connection with the representation theory of the symmetric group. However in the last thirty years they have found applications in combinatorics, mathematical physics and classical analysis. The history of the subject dates back to the mid-nineteenth century when Jacobi [38] singled out a family of symmetric functions, now called the Schur functions. The functions are named after Issai Schur as it was he who found the connection between the functions and the representation theory of the symmetric and general linear groups [70].

A major development occurred in the early 1960's with the work of Hall [33] and Littlewood [51] who independently found a one-parameter extension of the Schur function, referred to naturally as the Hall-Littlewood polynomials. It was the work of Green [29]

and Macdonald [52] that identified a connection between these polynomials and the representation theory of GL_n over finite and p -adic fields.

Around the same time the zonal polynomials were introduced independently by James [39] and Hua [35]. Although like the Schur and Hall-Littlewood polynomials the zonal polynomials arose from studies in representation theory, the literature is predominantly focussed on their applications in statistics [20, 75, 62].

Later that decade Henry Jack [36, 37] discovered another one-parameter generalisation of the Schur functions that was, besides its limiting properties, very different from the Hall-Littlewood polynomials. The now called Jack polynomial limits to both the Schur and zonal polynomials, unifying theories that were originally perceived to be disconnected. Like the zonal polynomials the Jack polynomials have applications in both representation theory and statistics [39].

It was not until the 1980's that all polynomial classes were unified. Macdonald identified a two parameter family of symmetric polynomials, now named Macdonald polynomials [53], that generalised both the Hall-Littlewood and Jack polynomials. The Hall-Littlewood polynomials were obtained by setting one of the parameters to zero and the Jack polynomials by limiting both parameters to one. The Macdonald polynomials generalise the symmetric characters of compact groups. Credit also goes to Kadell who independently discovered the polynomials through a connection with his studies on the Selberg integral [41]. More recently another application to pure mathematics of Macdonald polynomials was found in the studies of the geometry of the Hilbert scheme of points in the plane [31].

Due to their relation with symmetric characters it was quite unexpected when the polynomials naturally gave rise to a family of nonsymmetric polynomials. The nonsymmetric Macdonald polynomials were first introduced by Cherednik [15], six years after their symmetric counterpart. This came about because of the identification of an algebraic structure referred to as the double affine Hecke algebra. The symmetric Macdonald polynomials could be obtained from the nonsymmetric Macdonald polynomials via symmetrisation. Consequently, it was possible to develop the theory of the symmetric polynomials from that of their nonsymmetric analogues. This approach was found to be simpler and more accessible than the previous derivations. Furthermore, due to the limiting properties of the Macdonald polynomials nonsymmetric analogues of the other symmetric polynomial classes were automatically obtained. Nonsymmetric Macdonald polynomials, although rather natural in the Macdonald theory, are still somewhat mysterious, as unless some parameters are specified they do not have a clear representation-theoretic meaning. The

nonsymmetric polynomials have found application in physics in the study of an ideal gas [32, 72, 7].

The nonsymmetric polynomials were found themselves to be generalisable to the so-called interpolation polynomials. These were first found independently by Knop [44] and Sahi [67] in the field of Capelli identities. They were observed to be the inhomogeneous generalisation of the symmetric and nonsymmetric polynomial classes and were defined by rather simple vanishing properties. The interpolation polynomials were able to simplify the derivation of many polynomial properties more significantly than what was done by the nonsymmetric polynomials.

The definitive reference on symmetric functions and their applications is Macdonald's book "Symmetric functions and Hall polynomials" [54]. For further details on the history of symmetric polynomials, bibliographical notes on the key contributors and reprints of [51] and [36] we refer the reader to the proceedings of the 2003 workshop on Jack, Hall-Littlewood and Macdonald polynomials [47].

1.2 Symmetric Polynomials

1.2.1 Preliminaries

In this section we provide the basic properties of permutations and partitions required for the introduction of the symmetric polynomial classes and their special function properties. Unless otherwise stated we are following [54].

Permutations and their Decompositions

A permutation of $\mathbb{N}_n := \{1, \dots, n\}$ is a bijection $\sigma : \mathbb{N}_n \rightarrow \mathbb{N}_n$. Commonly we write σ as the n -tuple $\sigma := (\sigma(1), \dots, \sigma(n))$. An inversion of a permutation σ is a pair (i, j) where $i < j$ and $\sigma(i) > \sigma(j)$. With $K_j(\sigma)$ given by

$$K_j(\sigma) := \# \{i : (i, j) \text{ is an inversion of } \sigma\} \quad (1.1)$$

the length of a permutation σ is defined to be

$$\begin{aligned} l(\sigma) &:= \sum_{j=1}^n K_j(\sigma) \\ &= \#\{(i, j) : i < j, \sigma(i) > \sigma(j)\}. \end{aligned}$$

The set of permutations, with multiplication defined by composition, form a group S_n of size $n!$ called the symmetric group of order n . For example when $n = 3$ we have

$$S_3 = \{(1, 2, 3), (2, 1, 3), (1, 3, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}. \quad (1.2)$$

The transposition s_{ij} is a permutation that interchanges the i^{th} and j^{th} components, defined by

$$s_{ij}(k) := \begin{cases} j, & k = i \\ i, & k = j \\ k, & k \neq i, j. \end{cases}$$

From this we can introduce the simple transpositions s_i , $i = 1, \dots, n - 1$,

$$s_i := s_{i,i+1},$$

which interchanges the positions i and $i + 1$ in the n -tuple. The simple transpositions satisfy the algebraic relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n - 2 \\ s_i s_j &= s_j s_i, & |i - j| > 1 \\ s_i^2 &= 1, \end{aligned} \quad (1.3)$$

the first two of which are referred to as the braid relations. These algebraic relations generate the symmetric group [59] and therefore any permutation $\sigma \in S_n$ can be written as a product of simple transpositions. The products of smallest length are referred to as reduced decompositions and are of length $l(\sigma)$. A reduced decomposition is denoted by s_σ and given explicitly by

$$s_\sigma := s_1 \cdots s_{2-K_2(\sigma)} \cdots s_{n-2} \cdots s_{n-1-K_{n-1}(\sigma)} s_{n-1} \cdots s_{n-K_n(\sigma)},$$

where $K_j(\sigma)$ is given by (1.1). To avoid unnecessary ambiguity in the interpretation of the reduced decomposition we state that the product is read from left to right, that is $s_\sigma = s_{i_1} s_{i_2} \cdots s_{i_l}$. For example with $\sigma = (3, 2, 4, 1)$ we have a reduced decomposition $s_\sigma = s_1 s_3 s_2 s_1$. We note that a reduced decomposition is generally not unique, for example, another reduced decomposition for $\sigma = (3, 2, 4, 1)$ is $s_\sigma = s_3 s_2 s_1 s_2$.

The simple transpositions act on functions by interchanging the variables z_i and z_{i+1}

$$(s_i f)(z_1, \dots, z_i, z_{i+1}, \dots, z_n) := f(z_1, \dots, z_{i+1}, z_i, \dots, z_n).$$

For example

$$s_2 s_1 f(z_1, z_2, z_3) = s_2 f(z_2, z_1, z_3) = f(z_3, z_1, z_2).$$

More generally the action of permutations on polynomials is specified by

$$s_\sigma f(z_1, \dots, z_n) = s_{i_1} \dots s_{i_2} s_{i_1} f(z_1, \dots, z_n) = \sigma f(z_1, z_2, \dots, z_n) = f(z_{\sigma(1)}, \dots, z_{\sigma(n)}). \quad (1.4)$$

Throughout this thesis we introduce many operators that are labelled by the integers $1, \dots, n-1$. It is regularly required that these operators act on polynomials in a sequence specified by a permutation. For example if we have the operators F_1, \dots, F_{n-1} and a permutation represented by the reduced decomposition $s_\sigma = s_{i_1} \dots s_{i_\ell}$ we define

$$F_\sigma := F_{i_1} \dots F_{i_\ell} \quad (1.5)$$

and specify the action of F_σ on polynomials according to

$$F_\sigma f(x) = F_{i_1} \dots F_{i_\ell} f(x).$$

Partitions, diagrams and orderings

The polynomials in this section are labelled by partitions. A partition is defined to be any (finite or infinite) sequence of non-negative and weakly decreasing integers

$$\kappa = (\kappa_1, \kappa_2, \dots), \quad \kappa_1 \geq \kappa_2 \geq \dots$$

containing only finitely many non-zero terms. In this thesis we consider only partitions containing finitely many integers. Each κ_i is referred to as a component of κ and the length of a partition, denoted by n , is defined to be the number of components of κ . We denote the number of non-zero components of a partition by $\ell(\kappa)$. The modulus of a partition is denoted by $|\kappa|$ and is equal to the sum of its components. That is, if we take $\kappa = (4, 2, 0)$ we have $n = 3$, $\ell(\kappa) = 2$ and $|\kappa| = 6$.

The first ordering we define on partitions is the lexicographic ordering \prec . The lexicographic ordering compares partitions of the same modulus and is defined by $\mu \prec \kappa$ if

the first non-vanishing difference $\kappa_i - \mu_i$ is positive. The lexicographic ordering is a total ordering, meaning all partitions of a particular modulus are comparable. For example, the partitions of modulus 4 are ordered as

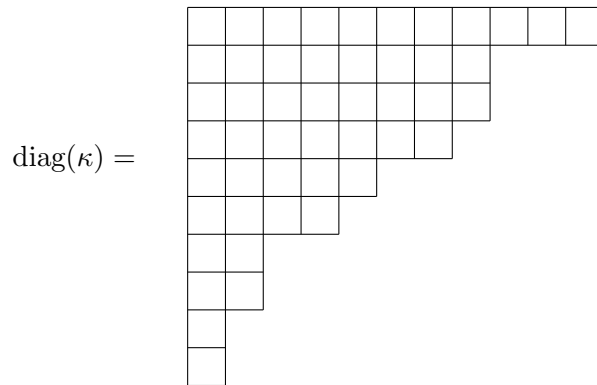
$$(4, 0, 0, 0), (3, 1, 0, 0), (2, 2, 0, 0), (2, 1, 1, 0), (1, 1, 1, 1).$$

The second ordering is the dominance ordering. The dominance ordering is an ordering on partitions of the same modulus and defined by

$$\mu \leq \kappa \quad \text{iff} \quad \sum_{i=1}^p (\kappa_i - \mu_i) \geq 0 \quad \text{for all} \quad 1 \leq p \leq n.$$

This is only a partial ordering, as for example the partitions $(4, 1, 1)$ and $(3, 3, 0)$ are incomparable under \leq .

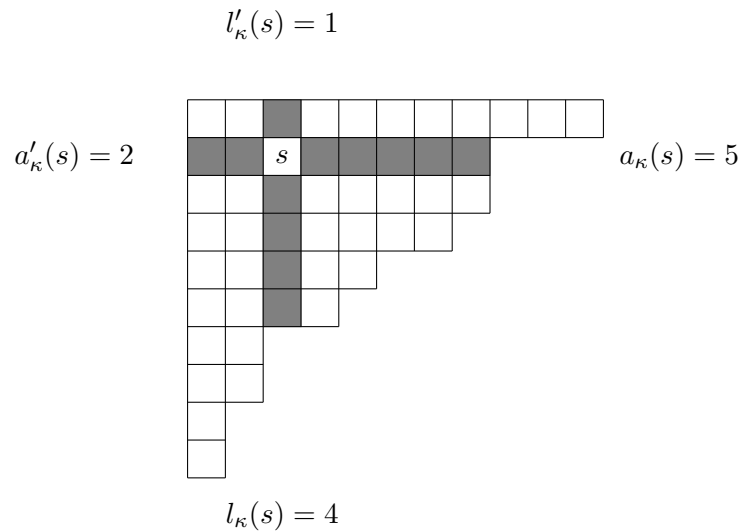
Each partition κ has a corresponding diagram defined to be the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \kappa_i$, ordered as for a matrix (row 1 along the top). For convenience we draw the diagram by replacing the elements of \mathbb{Z}^2 with boxes, the box in the i^{th} row and j^{th} column is labelled by (i, j) . We denote the diagram of a composition κ by $\text{diag}(\kappa)$. For example, with $\kappa = (11, 8, 8, 7, 5, 4, 2, 2, 1, 1)$ we have



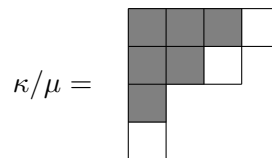
For each box $s = (i, j)$ in a partition κ , the number of boxes to the right, left, below and above s are the arm length $a_\kappa(s)$, arm colength $a'_\kappa(s)$, leg length $l_\kappa(s)$ and leg colength $l'_\kappa(s)$, respectively. These are formally defined by

$$\begin{aligned} a_\kappa(s) &:= \kappa_i - j, & a'_\kappa(s) &:= a'_\kappa(j) := j - 1, \\ l_\kappa(s) &:= \#\{k : k > i, j \leq \kappa_k\}, & l'_\kappa(s) &:= l'_\kappa(i) := i - 1. \end{aligned} \tag{1.6}$$

For example, if we have the κ defined as above and take the box $s = (2, 3)$ we have



For two partitions such that $\mu \subset \kappa$ we have a skew diagram κ/μ which consists of those boxes of κ which are not in μ . A skew diagram is said to be a vertical m -strip if κ/μ consists of m boxes, all of which are in distinct rows. For example, with $\kappa = (4, 3, 1, 1)$ and $\mu = (3, 2, 1)$, κ/μ is a vertical 3-strip. Pictorially we have



Symmetric polynomials

We consider the set of polynomials in n independent variables $z := (z_1, \dots, z_n)$ with rational integer coefficients, denoted $\mathbb{Q}[z_1, \dots, z_n]$. Equation (1.4) shows that permutations act on polynomials by permuting variables. We classify a polynomial as symmetric if it is invariant under the action of any permutation, that is

$$s_i f(z) = f(z), \quad \text{for all } i = 1, \dots, n-1.$$

For example, the polynomial $z_1 + z_2$ is considered symmetric if $n = 2$ but is not symmetric if $n = 3$.

As a consequence of this property any symmetric polynomial can be expressed as a linear combination of $e_\kappa(z) := e_{\kappa_1}(z) \dots e_{\kappa_n}(z)$ where $e_{\kappa_i}(z)$ are the elementary symmetric functions defined by

$$e_r(z) := \sum_{1 \leq i_1 < \dots < i_r \leq n} z_{i_1} \dots z_{i_r}.$$

Other classes of polynomials that are considered to be building blocks of all symmetric functions are the complete symmetric functions $h_r(z)$

$$h_r(z) := \sum_{1 \leq i_1 \leq \dots \leq i_r \leq n} z_{i_1} \dots z_{i_r}, \quad (1.7)$$

the symmetric monomial functions $m_\kappa(z)$

$$m_\kappa(z) := \sum_{\sigma} z^{\sigma\kappa}, \quad (1.8)$$

and the power sums $p_\kappa(z)$

$$p_\kappa(z) := p_{\kappa_1}(z) \dots p_{\kappa_n}(z),$$

where

$$p_r(z) := \sum_{i=1}^n z_i^r = m_{(r,0,\dots,0)}(z).$$

We note that the power sums are not linearly independent unless the number of variables is large enough. In (1.8) the sum is over all distinct permutations σ of κ and z^κ denotes the monomial corresponding to the partition κ and is defined by

$$z^\kappa := z_1^{\kappa_1} \dots z_n^{\kappa_n}. \quad (1.9)$$

The symmetrisation operator

$$\text{Sym} := \sum_{\sigma \in S_n} s_\sigma$$

can transform any polynomial to a symmetric polynomial. For example, with $n = 3$

$$\begin{aligned} \text{Sym}(z_1 + z_2) &= (z_1 + z_2) + s_1(z_1 + z_2) + s_2(z_1 + z_2) \\ &\quad + s_2s_1(z_1 + z_2) + s_1s_2(z_1 + z_2) + s_1s_2s_1(z_1 + z_2) \\ &= 4(z_1 + z_2 + z_3), \end{aligned}$$

a symmetric polynomial in z_1, z_2 and z_3 .

Related to the symmetric polynomials are the antisymmetric polynomials. These polynomials change sign upon the interchange of any two variables and hence are defined by

$$s_i f(z) = -f(z), \quad \text{for all } i = 1, \dots, n-1.$$

The antisymmetrisation operator is defined to be

$$\text{Asym} := \sum_{\sigma \in S_n} (-1)^{l(\sigma)} s_\sigma,$$

and works in an analogous way to the Sym operator. Furthermore,

$$\text{Asym } f(z) = \Delta(z)g(z) \tag{1.10}$$

where

$$\Delta(z) := \prod_{i < j} (z_i - z_j) \tag{1.11}$$

is the Vandermonde product and $g(z)$ is symmetric.

***t*-symmetric polynomials**

For reasons to be made clear in Chapter 2, *t*-analogues of Sym and Asym are required. To construct such operators we first require the *t*-analogue of s_i and the definitions of *t*-symmetric and *t*-antisymmetric.

The operators T_i , referred to as the Demazure-Lustig operators, that generalise s_i are realisations of the type-*A* Hecke algebra. The type-*A* Hecke algebra is an associative unital algebra over $\mathbb{Q}(t)$ generated by elements h_1, \dots, h_{n-1} and subject to the relations

$$\begin{aligned} h_i h_{i+1} h_i &= h_{i+1} h_i h_{i+1}, & \text{for } 1 \leq i \leq n-2 \\ h_i h_j &= h_j h_i, & |i-j| > 1 \\ (h_i + 1)(h_i - t) &= 0. \end{aligned} \tag{1.12}$$

The operators T_i are defined by

$$T_i := t + \frac{tz_i - z_{i+1}}{z_i - z_{i+1}}(s_i - 1). \tag{1.13}$$

We note that in the limit $t \rightarrow 1$ (1.12) reduces to (1.3) and T_i reduces to s_i .

From this operator we define a polynomial to be *t*-symmetric if

$$T_i f(z) = t f(z), \quad \text{for all } i = 1, \dots, n-1,$$

and t -antisymmetric if

$$T_i f(z) = -f(z), \quad \text{for all } i = 1, \dots, n-1.$$

We note that by the definition of T_i being t -symmetric is an equivalent condition to being symmetric, consequently when we speak of a symmetric polynomial we are not required to specify the nature of the symmetry.

The t -analogues of Sym and Asym are given respectively by

$$U^+ := \sum_{\sigma \in \mathcal{S}_n} T_\sigma, \quad (1.14)$$

and

$$U^- := \sum_{\sigma \in \mathcal{S}_n} \left(-\frac{1}{t}\right)^{l(\sigma)} T_\sigma, \quad (1.15)$$

where the operator T_σ is as in (1.5).

For example, with $n = 3$

$$\begin{aligned} U^+(z_1 + z_2) &= (z_1 + z_2) + T_1(z_1 + z_2) + T_2(z_1 + z_2) \\ &\quad + T_2T_1(z_1 + z_2) + T_1T_2(z_1 + z_2) + T_1T_2T_1(z_1 + z_2) \\ &= (1+t)^2(z_1 + z_2 + z_3), \end{aligned}$$

a t -symmetric polynomial in the variables z_1, z_2 and z_3 .

The t -analogue of (1.10) is

$$U^- f(z) = \Delta_t(z)g(z) \quad (1.16)$$

where

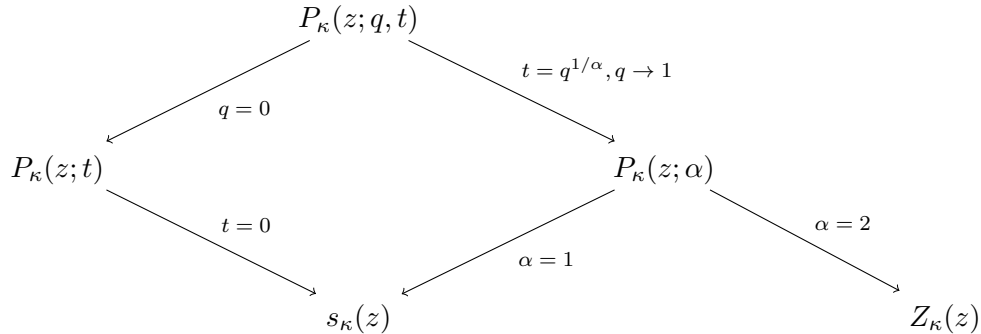
$$\Delta_t(z) := \prod_{i < j} (z_i - t^{-1}z_j), \quad (1.17)$$

is the t -Vandermonde product and $g(z)$ is a symmetric polynomial.

1.2.2 Some special symmetric polynomials

Over the years, various applied and theoretical advances have isolated some special classes of symmetric polynomials beyond the simple building blocks introduced in the previous section. These are the Schur polynomials $s_\kappa(z)$, zonal polynomials $Z_\kappa(z)$, Hall-Littlewood polynomials $P_\kappa(z; t)$, symmetric Jack polynomials $P_\kappa(z; \alpha)$ and symmetric Macdonald

polynomials $P_\kappa(z; q, t)$. As detailed in the following diagram, it turns out that by taking different limits of the two parameters in the symmetric Macdonald polynomials, we can obtain all the other special symmetric polynomials just listed.



Here we discuss some fundamental properties of the symmetric Macdonald polynomial $P_\kappa(z; q, t)$. The corresponding properties of the other special symmetric polynomials follow by the limits of the diagram. For convenience, the latter are listed in Appendix B along with the relevant references.

Triangular structure

The symmetric Macdonald polynomials are labelled by a partition κ and have the triangular structure

$$P_\kappa(z; q, t) = m_\kappa(z) + \sum_{\mu < \kappa} K_{\kappa\mu} m_\mu(z), \quad (1.18)$$

for some coefficients $K_{\kappa\mu} \in \mathbb{Q}(q, t)$. Since all $\mu < \kappa$ satisfies $|\mu| = |\kappa|$ we also have that $P_\kappa(z; q, t)$ is a homogeneous polynomial, meaning each term in the polynomial is of the same degree.

The following examples highlight the triangular structure of the symmetric Macdonald polynomials,

$$\begin{aligned}
 P_{(3,0,0)}(z; q, t) &= m_{(3,0,0)}(z) + \frac{(q^2+q+1)(t-1)}{q^2t-1} m_{(2,1,0)}(z) + \frac{(q+1)(q^2+q+1)(t-1)^2}{(qt-1)(q^2t-1)} m_{(1,1,1)}(z) \\
 P_{(2,1,0)}(z; q, t) &= m_{(2,1,0)}(z) + \frac{(t-1)(2qt+q+t+2)}{qt^2-1} m_{(1,1,1)}(z) \\
 P_{(1,1,1)}(z; q, t) &= m_{(1,1,1)}(z).
 \end{aligned} \quad (1.19)$$

Eigenfunctions

The symmetric Macdonald polynomials are eigenfunctions of the operator

$$D_n^1 := \sum_{i=1}^n \prod_{i \neq j} \frac{tz_i - z_j}{z_i - z_j} \tau_i, \quad (1.20)$$

where τ_i is a q -shift operator and defined by

$$(\tau_i f)(z_1, \dots, z_i, \dots, z_n) := f(z_1, \dots, qz_i, \dots, z_n). \quad (1.21)$$

Explicitly

$$D_n^1 P_\kappa(z; q, t) = M_\kappa P_\kappa(z; q, t), \quad (1.22)$$

where $M_\kappa := \sum_{i=1}^n q^{\kappa_i} t^{n-i}$ and is an eigenvalue.

The symmetric Macdonald polynomials can be defined solely by their triangular structure and eigenfunction properties. This is a consequence of the triangular action of D_n^1 on the symmetric monomial functions, specified by

$$D_n^1 m_\kappa(z) = M_\kappa m_\kappa(z) + \sum_{\mu < \kappa} N_{\kappa\mu} m_\mu(z), \quad (1.23)$$

where M_κ is defined above and $N_{\kappa\mu} \in \mathbb{Q}(q, t)$. The eigenfunction property, triangular structure and the triangular action of the eigenoperator on the monomial functions provides a user-friendly method for constructing the polynomials. Before providing the details of this construction we consider one last property of the symmetric Macdonald polynomials.

Orthogonality

A family of polynomials, F_κ say, is said to be orthogonal if there exists an inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle F_\kappa, F_\mu \rangle = \mathcal{L}_\kappa \delta_{\kappa, \mu} \quad (1.24)$$

for some positive constant \mathcal{L}_κ . As an inner product $\langle \cdot, \cdot \rangle$ is symmetric, bilinear and positive definite. In (1.24) $\delta_{\kappa, \mu}$ is the Kronecker delta and is defined to be 1 if $\kappa = \mu$ and zero otherwise. Macdonald [54] defines an inner product, of which the symmetric Macdonald polynomials are orthogonal with respect to, by the power sum orthogonality

$$\langle p_\kappa(z), p_\mu(z) \rangle_{q,t}^p = \delta_{\kappa, \mu} c_\kappa \prod_{i=1}^{\ell(\kappa)} \frac{1 - q^{\kappa_i}}{1 - t^{\kappa_i}}, \quad (1.25)$$

where $c_\kappa := \prod_i i^{m_i} m_i!$, with $m_i := m_i(\kappa)$ being the number of components in κ equal to i . The number of variables in the power sums in (1.25) is equal to the maximum of $|\kappa|$ and $|\mu|$.

The symmetric Macdonald polynomials are also orthogonal with respect to another inner product. With $z_i = e^{2\pi i x_i}$ we introduce the general inner product [14]

$$\langle f, g \rangle_{q,t}^{\text{Int}} = \int_0^1 dx_1 \dots \int_0^1 dx_n f(z; q, t) g(z^{-1}; q^{-1}, t^{-1}) W(z), \quad (1.26)$$

where $W(z)$ is some weight function (defined explicitly in (2.20)). Alternatively, this inner product can be expressed as

$$\langle f, g \rangle_{q,t}^{(\text{CT})} = \text{CT}(f(z; q, t) g(z^{-1}; q^{-1}, t^{-1}) W(z)), \quad (1.27)$$

where $\text{CT}(f)$ denotes the constant term with respect to z of any Laurent polynomial $f(z)$ where $z \in [z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]$. The formula (1.27) is particularly useful when $W(z)$ itself is a Laurent polynomial.

A concept naturally arising within inner product theory is the notion of an adjoint operator. We say \mathcal{H}^* is the adjoint operator of \mathcal{H} with respect to the inner product $\langle \cdot, \cdot \rangle$ if

$$\langle f, \mathcal{H}g \rangle = \langle \mathcal{H}^* f, g \rangle. \quad (1.28)$$

It is fundamental that D_n^1 is self adjoint with respect to both the inner products (1.26) and (1.27).

These results lead us to our first theorem.

Theorem 1.2.1. *We have*

$$\langle P_\kappa(z; q, t), P_\mu(z; q, t) \rangle_{q,t}^p = \hat{\mathcal{L}}_\kappa \delta_{\kappa,\mu}, \quad (1.29)$$

for normalisations $\hat{\mathcal{L}}_\kappa \in \mathbb{Q}(q, t)$.

Proof. By the eigenfunction properties of the symmetric Macdonald polynomials and the bilinearity of $\langle \cdot, \cdot \rangle_{q,t}^p$ we can use the fact that

$$\langle D_n^1 P_\kappa(z; q, t), P_\mu(z; q, t) \rangle_{q,t}^p = \langle P_\kappa(z; q, t), D_n^1 P_\mu(z; q, t) \rangle_{q,t}^p$$

to show

$$(M_\kappa - M_\mu) \langle P_\kappa(z; q, t), P_\mu(z; q, t) \rangle_{q,t}^p = 0.$$

When $\kappa \neq \mu$, $M_\kappa \neq M_\mu$ and therefore $\langle P_\kappa(z; q, t), P_\mu(z; q, t) \rangle_{q,t}^p = 0$. Furthermore, it follows from the definition of $\langle \cdot, \cdot \rangle_{q,t}^p$ that $\langle P_\kappa(z; q, t), P_\kappa(z; q, t) \rangle_{q,t}^p \neq 0$. These results together show (1.29) holds true. \square

The triangular structure and the orthogonality of the symmetric Macdonald polynomials allows the coefficients $K_{\kappa\mu}$ in (1.18) to be constructed recursively by the Gram-Schmidt process. We provide a detailed description of the construction of $P_{(2,1,0)}(z; q, t)$ in the following section.

Generation

We first show how to generate the general symmetric Macdonald polynomial $P_\kappa(z; q, t)$ using the eigenfunction properties.

Suppose we wish to generate the symmetric Macdonald polynomial $P_\kappa(z; q, t)$ for which there are k partitions μ , denoted $\mu^1, \mu^2, \dots, \mu^k$, such that $\mu < \kappa$. From (1.18) we have

$$P_\kappa(z; q, t) = m_\kappa(z) + K^1 m_{\mu^1}(z) + K^2 m_{\mu^2}(z) + \dots + K^k m_{\mu^k}(z), \quad (1.30)$$

for some coefficients $K^1, \dots, K^k \in \mathbb{Q}(q, t)$. Acting upon $P_\kappa(z; q, t)$ with D_n^1 according to (1.22) gives

$$D_n^1 P_\kappa(z; q, t) = M_\kappa(m_\kappa(z) + K^1 m_{\mu^1}(z) + K^2 m_{\mu^2}(z) + \dots + K^k m_{\mu^k}(z)). \quad (1.31)$$

Alternatively, using (1.23) we obtain

$$\begin{aligned} D_n^1 P_\kappa(z; q, t) &= M_\kappa m_\kappa(z) + N_{\kappa\mu^1} m_{\mu^1}(z) + N_{\kappa\mu^2} m_{\mu^2}(z) + \dots + N_{\kappa\mu^k} m_{\mu^k}(z) \\ &\quad + K^1 (M_{\mu^1} m_{\mu^1}(z) + N_{\mu^1\mu^2} m_{\mu^2}(z) + \dots + N_{\mu^1\mu^k} m_{\mu^k}(z)) \\ &\quad + K^2 (M_{\mu^2} m_{\mu^2}(z) + \dots + N_{\mu^2\mu^k} m_{\mu^k}(z)) \\ &\quad \vdots \\ &\quad + K^k (M_{\mu^k} m_{\mu^k}(z)). \end{aligned} \quad (1.32)$$

We can represent the sets of equations given in (1.31) and (1.32) as the matrix equation

$$\begin{bmatrix} M_\kappa & N_{\kappa\mu^1} & N_{\kappa\mu^2} & \cdots & N_{\kappa\mu^k} \\ 0 & M_{\mu^1} & N_{\mu^1\mu^2} & \cdots & N_{\mu^1\mu^k} \\ 0 & 0 & M_{\mu^2} & \cdots & N_{\mu^2\mu^k} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & M_{\mu^k} \end{bmatrix} \begin{bmatrix} 1 \\ K^1 \\ K^2 \\ \vdots \\ K^n \end{bmatrix} = M_\kappa \begin{bmatrix} 1 \\ K^1 \\ K^2 \\ \vdots \\ K^n \end{bmatrix}. \quad (1.33)$$

It therefore follows that the eigenvector corresponding to eigenvalue M_κ will provide the coefficients K^1, \dots, K^n in (1.30) giving an explicit formula for $P_\kappa(z; q, t)$. Moreover, if κ is chosen to be the highest partition with respect to the partial ordering $<$ the system (1.33) will have $k + 1$ eigenvectors, one for each eigenvalue $M_\kappa, M_{\mu^1}, \dots, M_{\mu^k}$, giving the coefficients of each of $P_\kappa(z; q, t), P_{\mu^1}(z; q, t), \dots, P_{\mu^k}(z; q, t)$.

An alternate method for generating the symmetric Macdonald polynomials is using their orthogonality properties with respect to $\langle \cdot, \cdot \rangle_{q,t}^p$ and the Gram-Schmidt procedure. We first show how one would construct $P_{(2,1,0)}(z; q, t)$ and then describe the general approach.

By the triangular structure of the symmetric Macdonald polynomials we have

$$P_{(1,1,1)}(z; q, t) = m_{(1,1,1)}(z) \quad \text{and} \quad P_{(2,1,0)}(z; q, t) = m_{(2,1,0)}(z) + K_{(2,1,0),(1,1,1)} m_{(1,1,1)}(z).$$

We can determine the coefficient $K_{(2,1,0),(1,1,1)} = K_{(1,1,1)}$ by solving

$$\langle P_{(1,1,1)}(z; q, t), P_{(2,1,0)}(z; q, t) \rangle_{q,t}^p = 0. \quad (1.34)$$

To solve the equation we first need to express the symmetric Macdonald polynomials in terms of the power sums. To do this we require the expansions,

$$\begin{aligned} m_{(1,1,1)}(z) &= (p_{(1,1,1)}(z) - 3p_{(2,1,0)}(z) + 2p_{(3,0,0)}(z))/6 \\ m_{(2,1,0)}(z) &= p_{(2,1,0)}(z) - p_{(3,0,0)}(z). \end{aligned}$$

Rewriting $P_{(1,1,1)}(z; q, t)$ and $P_{(2,1,0)}(z; q, t)$ in terms of the power sums allows us to use the definition of $\langle \cdot, \cdot \rangle_{q,t}^p$ to simplify (1.34) to

$$\frac{K_{(1,1,1)}(1-q)^3}{6(1-t)^3} + \frac{(2-K_{(1,1,1)})(1-q)(1-q^2)}{2(1-t)(1-t^2)} + \frac{(K_{(1,1,1)}-3)(1-q^3)}{3(1-t^3)} = 0.$$

The solution to this equation is

$$K_{(1,1,1)} = \frac{(t-1)(2qt+q+t+2)}{qt^2-1},$$

which is consistent with (1.19).

In the general case one can construct the set of symmetric Macdonald polynomials of modulus k using the Gram-Schmidt method by first taking $P_{(1^k)}(z; q, t)$ and sequentially generating the polynomials according to the lexicographic order, completing the process at $P_{(k,0,\dots,0)}(z; q, t)$.

Notes: Further families of symmetric polynomials are the Hermite, Laguerre and Jacobi polynomials. These families are related to the Jack polynomials but are not directly connected to the Macdonald polynomials. Consequently we do not provide the details of these polynomials in this thesis. Interested readers can find the details in [23, Chap. 13]. Two families directly relating to the Macdonald polynomials are the symmetric Al-Salam and Carlitz polynomials [8]. Both polynomials are defined in terms of the Macdonald polynomials, have eigenfunction properties and form an orthogonal set with respect to an inner product. Additional applications of the symmetric Jack polynomials are in random matrix theory and we refer the reader to [17] for an overview of studies in the area.

We also mention that the Schur and Hall-Littlewood polynomials can simply be obtained using the explicit formulas

$$s_\kappa(z) := \frac{\text{Asym}(z^{\kappa+\delta})}{\text{Asym}(z^\delta)} \quad \text{and} \quad P_\kappa(z; t) := \prod_i \frac{1-t}{1-t^{m(i)}} \text{Sym}\left(z^\kappa \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\right),$$

where $m(i)$ is defined just below (1.25).

1.3 Nonsymmetric Polynomials

1.3.1 Preliminaries

Here we present the preliminary material on compositions required for the theory of nonsymmetric polynomials. Unless otherwise stated a reference containing the material in this section is [57].

Compositions, diagrams and orderings

Each nonsymmetric polynomial is labelled by a composition. We define a composition to be an n -tuple $\eta := (\eta_1, \dots, \eta_m)$ of non-negative integers. As with partitions each η_i is called

a component, the length of a composition is the number of components it contains and the sum of the components is called the modulus and denoted $|\eta|$. We define permutations to act on compositions by

$$s_\sigma(\eta_1, \dots, \eta_m) = s_{i_l} \dots s_{i_2} s_{i_1}(\eta_1, \dots, \eta_m) = \sigma(\eta_1, \dots, \eta_m) := (\eta_{\sigma^{-1}(1)}, \dots, \eta_{\sigma^{-1}(m)}). \quad (1.35)$$

According to this

$$\begin{aligned} s_i \sigma \eta &= (\eta_{\sigma^{-1} s_i(1)}, \dots, \eta_{\sigma^{-1} s_i(i)}, \eta_{\sigma^{-1} s_i(i+1)}, \dots, \eta_{\sigma^{-1} s_i(n)}) \\ &= (\eta_{\sigma^{-1}(1)}, \dots, \eta_{\sigma^{-1}(i+1)}, \eta_{\sigma^{-1}(i)}, \dots, \eta_{\sigma^{-1}(n)}) \end{aligned}$$

and thus s_i acts on compositions by always interchanging the i^{th} and $(i+1)^{\text{th}}$ components regardless of their value. For example

$$s_2 s_1(\eta_1, \eta_2, \eta_3) = s_2(\eta_2, \eta_1, \eta_3) = (\eta_2, \eta_3, \eta_1).$$

We note that it is the difference between how s_i acts on polynomials and compositions that accounts for the variation between (1.4) and (1.35). A further important operator that acts on compositions is the raising operator Φ which has the action

$$\Phi \eta := (\eta_2, \dots, \eta_m, \eta_1 + 1).$$

It is clear that all compositions can be recursively generated from $\eta = (0, \dots, 0)$ using the two operators s_i and Φ . An explanation of a computational approach to generating any composition most efficiently is given in Chapter 5.

Each composition η corresponds to a unique partition η^+ obtained by suitably rearranging the components of η . We define ω_η to be the shortest permutation such that

$$\omega_\eta^{-1}(\eta) = \eta^+. \quad (1.36)$$

We denote the reverse composition by η^R and the composition where each component is increased by k by $\eta + (k^n)$, explicitly

$$\eta^R := (\eta_m, \eta_{m-1}, \dots, \eta_1) \quad (1.37)$$

and

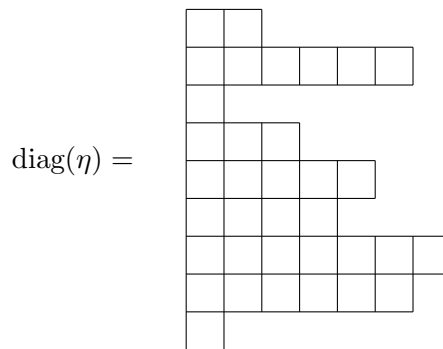
$$\eta + (k^n) := (\eta_1 + k, \dots, \eta_m + k). \quad (1.38)$$

There are two important partial orderings on compositions. The dominance ordering \leq from the previous section and a further ordering \preceq . The partial order \preceq is defined on compositions of the same modulus so that

$$\lambda \preceq \eta \quad \text{iff} \quad \lambda^+ < \eta^+ \quad \text{or in the case} \quad \lambda^+ = \eta^+, \quad \lambda \leq \eta. \quad (1.39)$$

For example two compositions that lie below $(0, 1, 2, 1)$ according to \preceq are $(0, 1, 1, 2)$ and $(1, 1, 1, 1)$.

The diagrams of compositions are defined the same as diagrams for partitions. For example, with $\eta = (2, 6, 1, 3, 5, 4, 7, 6, 1)$ we have

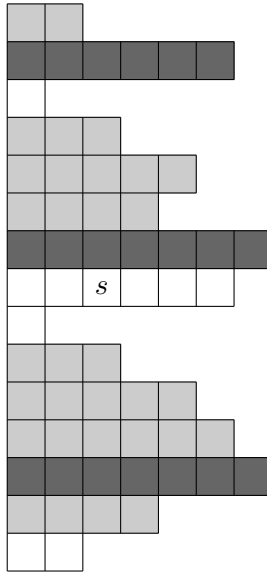


In the previous section we gave formulas for the arm length, arm colength, leg length and leg colength of a box in a partition. Sahi [68] generalised these quantities to apply to compositions. For compositions the definition of arm length and arm colength remain the same while leg length and leg colength are defined by

$$l_\eta(s) := \#\{k < i : j \leq \eta_k + 1 \leq \eta_i\} + \#\{k > i : j \leq \eta_k \leq \eta_i\} \quad (1.40)$$

$$l'_\eta(s) := l''_\eta(i) := \#\{k < i : \eta_k \geq \eta_i\} + \#\{k > i : \eta_k > \eta_i\}.$$

For example, for the composition $\eta = (2, 6, 1, 3, 5, 4, 7, 6, 1, 3, 5, 6, 7, 4, 2)$ and the box $s = (8, 3)$ we have the diagram



showing $l_\eta(s) = 8$ and $l'_\eta(s) = 3$.

The concept of a skew diagram is identical for compositions. For example, with $\eta = (3, 4, 1, 1)$ and $\lambda = (3, 3, 1)$, η/λ is a vertical 2-strip. Pictorially we have

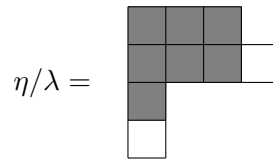


Diagram dependent constants

The following quantities

$$d_\eta := d_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} (1 - q^{a_\eta(s)+1} t^{l_\eta(s)+1}),$$

$$d'_\eta := d'_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} (1 - q^{a_\eta(s)+1} t^{l'_\eta(s)}), \quad (1.41)$$

$$e_\eta := e_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} (1 - q^{a'_\eta(s)+1} t^{n-l'_\eta(s)}), \quad (1.42)$$

$$e'_\eta := e'_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} (1 - q^{a'_\eta(s)+1} t^{n-1-l'_\eta(s)}),$$

which are constants independent of the polynomial variable z , appear regularly in Macdonald polynomial theory. Most important for our studies is knowing how these constants transform under the action of s_i and Φ on η . We will first consider the action of s_i . For this we require the quantity

$$\delta_{i,\eta} := \delta_{i,\eta}(q, t) = \bar{\eta}_i / \bar{\eta}_{i+1}. \quad (1.43)$$

The symbol $\bar{\eta}_i$ denotes the eigenvalue of the nonsymmetric Macdonald polynomial and is defined by $\bar{\eta}_i := q^{\eta_i} t^{-l'_\eta(i)}$. This quantity plays a crucial role throughout our studies and will be given a more thorough introduction in Chapter 2.

Lemma 1.3.1. [68] We have

$$e_{s_i\eta}(q, t) = e_\eta(q, t) \quad (1.44)$$

$$\frac{d_{s_i\eta}(q, t)}{d_\eta(q, t)} = \begin{cases} \frac{1-t\delta_{i,\eta}(q,t)}{1-\delta_{i,\eta}(q,t)}, & \eta_i > \eta_{i+1} \\ \frac{1-\delta_{i,\eta}(q,t)}{t-\delta_{i,\eta}(q,t)}, & \eta_i < \eta_{i+1}, \end{cases} \quad (1.45)$$

where $\delta_{i,\eta}(q, t)$ is given by (1.43).

Proof. Consider first $e_\eta(q, t)$. Taking $s = (i, j)$ we have $q^{a'_\eta(s)+1} = q^j$ and $l'_\eta(s) = l'_\eta(i)$ allowing (1.42) to be rewritten as

$$e_\eta(q, t) = \prod_{i=1}^n \prod_{j=1}^{\eta_i} (1 - q^j t^{n-l'_\eta(i)}).$$

Since $s_i\eta$ is a permutation of η , it follows that $e_{s_i\eta}(q, t)$ corresponds to a reordering of terms in (1.3.1) only and thus $e_\eta(q, t) = e_{s_i\eta}(q, t)$. In relation to $d_\eta(q, t)$, when $\eta_i > \eta_{i+1}$, we must first recognise that switching rows i and $i + 1$ increases the leg of the point $(i, \eta_{i+1} + 1)$ by one while all other arms and legs stay the same. Hence,

$$\frac{d_{s_i\eta}(q, t)}{d_\eta(q, t)} = \frac{1 - q^{a_\eta(s)+1} t^{l_\eta(s)+2}}{1 - q^{a_\eta(s)+1} t^{l_\eta(s)+1}}, \quad (1.46)$$

where $s = (i, \eta_{i+1} + 1)$. We simplify this by observing that for $s = (i, \eta_{i+1} + 1)$ we have $\eta_i - \eta_{i+1} = a_\eta(s) + 1$ and $l'_\eta(i+1) - l'_\eta(i) = l_\eta(s) + 1$. The latter is true since with $\eta_i > \eta_{i+1}$ we have

$$\begin{aligned} \#\{k < i + 1 : \eta_k \geq \eta_{i+1}\} - \#\{k < i : \eta_k \geq \eta_i\} &= \#\{k < i : \eta_i > \eta_k \geq \eta_{i+1}\} + 1 \\ &= \#\{k < i, \eta_{i+1} + 1 \leq \eta_k + 1 \leq \eta_i\} + 1 \end{aligned}$$

and

$$\#\{k > i + 1 : \eta_k > \eta_{i+1}\} - \#\{k > i : \eta_k > \eta_i\} = \#\{k > i : \eta_{i+1} + 1 \leq \eta_k \leq \eta_i\}.$$

Consequently, $q^{a_\eta(s)+1} t^{l_\eta(s)+1} = \bar{\eta}_i / \bar{\eta}_{i+1} = \delta_{i,\eta}(q, t)$. Substituting this into (1.46) gives the required result. The case where $\eta_i < \eta_{i+1}$ can be derived immediately from the previous

result by considering the ratio $d_\eta(q, t)/d_{s_i\eta}(q, t)$ where $(s_i\eta)_i > (s_i\eta)_{i+1}$ and recalling that $\delta_{i,s_i\eta}(q, t) = \delta_{i,\eta}^{-1}(q, t)$. \square

The quantities $e'_\eta(q, t)$ and $d'_\eta(q, t)$ have a similar dependence on $a_\eta(s)$ and $l_\eta(s)$ to $e_\eta(q, t)$ and $d_\eta(q, t)$. Consequently the relations equivalent to (1.44) and (1.45) can be deduced in an analogous way.

Lemma 1.3.2. [68] We have

$$\begin{aligned} e'_{s_i\eta}(q, t) &= e'_\eta(q, t), \\ \frac{d'_{s_i\eta}(q, t)}{d'_\eta(q, t)} &= \begin{cases} \frac{1-\delta_{i,\eta}(q, t)}{1-t^{-1}\delta_{i,\eta}(q, t)}, & \eta_i > \eta_{i+1} \\ \frac{t^{-1}-\delta_{i,\eta}(q, t)}{1-\delta_{i,\eta}(q, t)}, & \eta_i < \eta_{i+1}. \end{cases} \end{aligned} \quad (1.47)$$

Consider now the action of Φ .

Lemma 1.3.3. [68] We have

$$\frac{d_{\Phi\eta}(q, t)}{d_\eta(q, t)} = 1 - q^{\eta_1+1}t^{n-l'_\eta(1)} = \frac{e_{\Phi\eta}(q, t)}{e_\eta(q, t)}. \quad (1.48)$$

Proof. The diagram of $\Phi\eta$ is obtained from η by removing the top row, fixing an additional box to it, moving it to the bottom row and shifting the entire diagram up one unit. To show the first equality we consider the new point to be prefixed to the beginning of the row and so at $s = (n, 1)$. From their definitions (1.6) and (1.40) one sees that the arms and legs of all other points are unchanged. The new point s has $a(s) = \eta_1$ and $l(s) = \#\{k < n : 1 \leq (\Phi\eta)_k + 1 \leq \eta_1 + 1\}$. By observing that $(\Phi\eta)_k = \eta_{k+1}$ for $k = 1, \dots, n-1$ we can rewrite $l(s)$ as $l(s) = \#\{k > 1 : \eta_k \leq \eta_1\} = n-1-l'(s)$ which gives the required result. For the second equality we consider the new point to be appended to the end of the row and so at $s = (n, \eta_1 + 1)$. This time the coarms and the colegs of the other points are unchanged and for the new point s we have $a'(s) = \eta_1$ and $l'(s) = \#\{k > 1 : \eta_k > \eta_1\}$ and the result follows. \square

As before we state the results equivalent to (1.48) for $e'_\eta(q, t)$ and $d'_\eta(q, t)$ separately.

Lemma 1.3.4. [68]

$$\frac{d'_{\Phi\eta}(q, t)}{d'_\eta(q, t)} = 1 - q^{\eta_1+1}t^{n-1-l'_\eta(1)} = \frac{e'_{\Phi\eta}(q, t)}{e'_\eta(q, t)}. \quad (1.49)$$

Two ratios that regularly appear in Macdonald polynomial theory are

$$\frac{d_\eta(q, t)}{d_\eta(q^{-1}, t^{-1})} = (-1)^{|\eta|} q^{\sum s a_\eta(s) + |\eta|} t^{l(\eta) + |\eta|} \quad \text{and} \quad \frac{d'_\eta(q, t)}{d'_\eta(q^{-1}, t^{-1})} = (-1)^{|\eta|} q^{\sum s \in \eta a_\eta(s) + |\eta|} t^{l(\eta)}, \quad (1.50)$$

where $l(\eta)$ is defined by

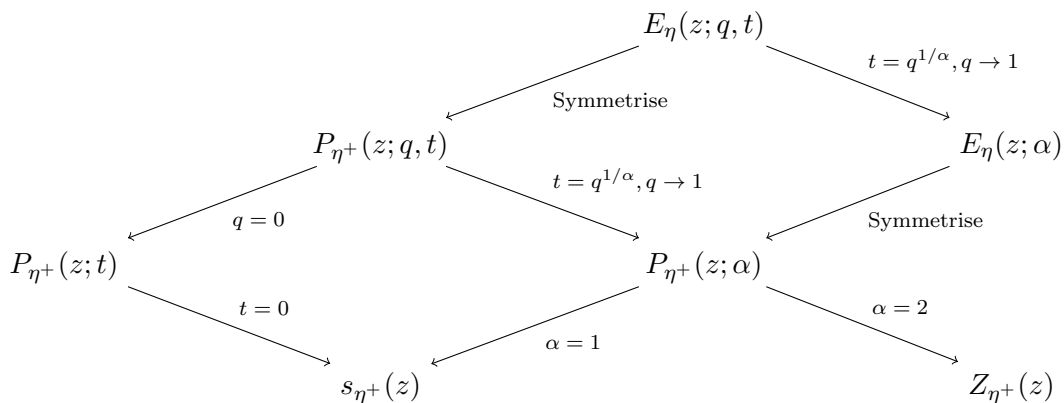
$$l(\eta) := \sum_{s \in \eta} l_\eta(s).$$

The ratios in (1.50) follow immediately from the definitions of $d_\eta(q, t)$ and $d'_\eta(q, t)$ and the identity

$$\frac{1-x}{1-x^{-1}} = -x. \quad (1.51)$$

1.3.2 Some special nonsymmetric polynomials

The nonsymmetric Macdonald polynomials $E_\eta(z; q, t)$ are the nonsymmetric generalisations of the symmetric Macdonald polynomials $P_\kappa(z; q, t)$. The nonsymmetric Macdonald polynomials are considered as building blocks of the symmetric Macdonald polynomials as t -symmetrisation of $E_\eta(z; q, t)$ by U^+ (1.14) gives $P_{\eta^+}(z; q, t)$. Consequently many of the fundamental properties of the symmetric Macdonald polynomials have nonsymmetric analogues. In this section we highlight the similarities between the special function properties of the nonsymmetric Macdonald polynomials and the symmetric Macdonald polynomials. The full details of the theory are not presented until the following chapter. As in the symmetric theory taking limits of the parameters in the nonsymmetric Macdonald polynomials gives more specialised families of polynomials. A polynomial that regularly appears in the theory due to its applications in quantum many body systems is the nonsymmetric Jack polynomial $E_\eta(z; \alpha)$ [22]. Due to their importance in the literature we provide the details of the nonsymmetric Jack polynomial theory in Appendix C. Inter-relations between the various nonsymmetric and symmetric polynomials are given by the following diagram.



Triangular structure

As their name suggests the nonsymmetric Macdonald polynomials are not symmetric and thus cannot be expressed as a linear combination of monomial symmetric functions. Rather they are presented as a linear combination of monomials (1.9). Although the nonsymmetric Macdonald polynomials are summed over a different basis to the symmetric Macdonald polynomials they still display a triangular structure.

The nonsymmetric polynomials are labelled by compositions η and have the triangular structure

$$E_\eta(z; q, t) = z^\eta + \sum_{\lambda \prec \eta} \hat{K}_{\eta\lambda} z^\lambda,$$

for some coefficients $\hat{K}_{\eta\lambda} \in \mathbb{Q}(q, t)$. Once again since all $\lambda \prec \eta$ satisfy $|\lambda| = |\eta|$ we also have that $E_\eta(z; q, t)$ is a homogeneous polynomial.

These properties of the nonsymmetric Macdonald polynomials can be seen in the following examples

$$\begin{aligned} E_{(0,3)}(z; q, t) &= z^{(0,3)} + \frac{t-1}{q^2t-1} z^{(2,1)} + \frac{(q+1)(t-1)}{q^2t-1} z^{(1,2)} \\ E_{(2,1)}(z; q, t) &= z^{(2,1)} + \frac{q(t-1)}{qt-1} z^{(1,2)} \\ E_{(1,2)}(z; q, t) &= z^{(1,2)}. \end{aligned}$$

Eigenfunctions

Unlike the symmetric Macdonald polynomials, which only require the single eigenoperator D_n^1 to be uniquely determined, the nonsymmetric Macdonald polynomials require a set of n commuting eigenoperators,

$$Y_i E_\eta(z; q, t) = \bar{\eta}_i E_\eta(z; q, t), \quad \text{for all } i = 1, \dots, n,$$

where $\bar{\eta}_i \in \mathbb{Q}(q, t)$ is referred to as an eigenvalue, given explicitly by (2.10), and the eigenoperator Y_i is specified by (2.2). Furthermore, the nonsymmetric Macdonald polynomials are referred to as simultaneous eigenfunctions of the Y_i .

Like D_n^1 , the eigenoperator Y_i has a triangular action on monomials (refer to (2.6)). Consequently the eigenfunction properties joined with the triangular structure once again provides a method of constructing the polynomials.

Orthogonality

The orthogonality of the nonsymmetric Macdonald polynomials is identical to their symmetric counterparts with respect to the power sums and as such is regarded as a defining property. The nonsymmetric Macdonald polynomials are orthogonal with respect to the integral inner product (1.27), however the normalisation is different.

Generation

The methods used to construct the symmetric Macdonald polynomials specified in the previous section can be generalised to the nonsymmetric case. An alternate approach is via recursive generation. It was stated earlier in this section that each composition η can be recursively generated from $(0, \dots, 0)$ using two elementary operators, namely Φ and s_i . In the same way the nonsymmetric Macdonald polynomials labelled by η can be recursively generated from $E_{(0, \dots, 0)}(z; q, t) := 1$ using two elementary operators; a switching operator that can be used to obtain $E_{s_i \eta}(z; q, t)$ from $E_\eta(z; q, t)$ and a raising operator that relates $E_{\Phi \eta}(z; q, t)$ to $E_\eta(z; q, t)$.

Notes: The generalised symmetric Hermite, Laguerre and Jacobi polynomials have nonsymmetric counterparts, for details on their theory please refer to [23, Chap. 13]. Similarly, nonsymmetric analogues exist for the symmetric Al-Salam and Carlitz polynomials (see e.g. [7]).

1.4 Interpolation Polynomials

1.4.1 Preliminaries

We now introduce two further partial orderings on compositions and present the theory required to state the extra vanishing conditions of the interpolation polynomials. The original references for the interpolation polynomials are [44] and [67].

Further partial orderings and successors

The nonsymmetric interpolation Macdonald polynomials are inhomogeneous polynomials and hence to express their triangular structure we require a partial ordering that is defined on compositions of different modulus. We define the partial ordering \triangleleft to be between any

two compositions by

$$\lambda \triangleleft \eta \quad \text{iff} \quad \lambda^+ < \eta^+ \quad \text{or in the case} \quad \lambda^+ = \eta^+, \quad \lambda \leq \eta. \quad (1.52)$$

Here we extend the meaning of $\lambda^+ < \eta^+$ to compositions of different modulus. We observe that the definition of \triangleleft is identical to \preceq but without the requirement that $|\lambda| = |\eta|$. For example $(1, 2, 0)$ lies below $(1, 1, 2)$ with respect to \triangleleft but are incomparable with respect to \leq .

We now introduce a further partial ordering that plays a fundamental role in the theory of interpolation polynomials and also the study of the Pieri-type formulas. We write $\eta \preceq' \lambda$, and say λ is a successor of η , if there exists a permutation σ such that

$$\eta_i < \lambda_{\sigma(i)} \quad \text{if} \quad i < \sigma(i) \quad \text{and} \quad \eta_i \leq \lambda_{\sigma(i)} \quad \text{if} \quad i \geq \sigma(i).$$

We call σ a defining permutation of $\eta \preceq' \lambda$ and write $\eta \preceq' \lambda; \sigma$. It is important to note that defining permutations are not unique. For example $(1, 2, 1) \prec' (1, 2, 2)$ has defining permutations $(1, 2, 3)$ and $(3, 2, 1)$. However, there is only one defining permutation such that for all i satisfying $\eta_i = \lambda_i$ we have $\sigma(i) = i$; this defining permutation is to be denoted $\widehat{\sigma}$.

Important to our studies are the minimal elements lying above η , the λ such that $\eta \preceq' \lambda$ and $|\lambda| = |\eta| + 1$. Knop [44] denoted these compositions as $\lambda = c_I(\eta)$ and defined them by

$$(c_I(\eta))_j = \begin{cases} \eta_{t_k+1}, & j = t_k, \text{ if } k = 1, \dots, s-1 \\ \eta_{t_1} + 1, & j = t_s \\ \eta_j, & j \notin I, \end{cases} \quad (1.53)$$

where $I = \{t_1, \dots, t_s\}$ is a non-empty subset of $\{1, \dots, n\}$ with $1 \leq t_1 < \dots < t_s \leq n$. More explicitly

$$c_I(\eta) = (\eta_1, \dots, \eta_{t_1-1}, \eta_{t_2}, \eta_{t_1+1}, \dots, \eta_{t_2-1}, \eta_{t_3}, \eta_{t_2+1}, \dots, \\ \eta_{t_{s-1}-1}, \eta_{t_s}, \eta_{t_{s-1}+1}, \dots, \eta_{t_s-1}, \eta_{t_1} + \mathbf{1}, \eta_{t_s+1}, \dots, \eta_n),$$

(the 1 added to η_{t_1} has been set in bold to highlight its location). Each successor $c_I(\eta)$ can be recursively generated from η using the switching and raising operators. We show how this can be done in the following proposition.

Proposition 1.4.1. *With $c_I(\eta)$ defined as above we have*

$$c_I(\eta) = \sigma_{t_1+1} \dots \sigma_n \Phi s_1 \dots s_{t_1-1} \eta,$$

where

$$\sigma_i = \begin{cases} 1, & i \in I \\ s_{i-1}, & i \notin I. \end{cases}$$

Proof. The operators to the right of Φ move η_{t_1} to the first position, thus enabling Φ to increase its value by 1. Each s_{i-1} on the left hand side moves each η_i , for $i \notin I$, back to its original position, automatically placing the η_i with $i \in I$ into the correct position. \square

Example 1.4.1. *Take $\eta = (1, 3, 5, 7, 9, 11, 13, 15)$ and $I = \{3, 4, 7, 8\}$. We then have*

$$\begin{aligned} s_2 \eta &= (1, \mathbf{5}, \mathbf{3}, 7, 9, 11, 13, 15) \\ s_1 s_2 \eta &= (\mathbf{5}, \mathbf{1}, 3, 7, 9, 11, 13, 15) \\ \Phi s_1 s_2 \eta &= (1, 3, 7, 9, 11, 13, \mathbf{15}, \mathbf{6}) \\ s_5 \Phi s_1 s_2 \eta &= (1, 3, 7, 9, \mathbf{13}, \mathbf{11}, 15, 6) \\ s_4 s_5 \Phi s_1 s_2 \eta &= (1, 3, 7, \mathbf{13}, \mathbf{9}, 11, 15, 6) \\ &= c_I(\eta). \end{aligned}$$

By the definition of $c_I(\eta)$ it is clear that $\eta \prec' c_I(\eta)$. The following lemma considers the other direction.

Lemma 1.4.2. [44] *If $|\lambda| = |\eta| + 1$ and $\eta \prec' \lambda$ then there exists a non-empty set $I = \{t_1, \dots, t_s\} \subseteq \{1, \dots, n\}$ with $1 \leq t_1 < \dots < t_s \leq n$ such that $c_I(\eta) = \lambda$.*

Proof. Since $|\lambda| = |\eta| + 1$ and $\eta \prec' \lambda$ the defining permutation $\hat{\sigma}$ must satisfy $\lambda_{\hat{\sigma}(i)} = \eta_i$ for all but one i , say $i = k$, in which case $\lambda_{\hat{\sigma}(k)} = \eta_k + 1$. By the definition of \prec' we must have $i \geq \hat{\sigma}(i)$ for $i \neq k$. It follows that with $I = \{i; \hat{\sigma}(i) \neq i\} = \{t_1, \dots, t_s\}$ we must have $\hat{\sigma}$ specified by

$$\hat{\sigma}(i) = \begin{cases} i, & i \notin I \\ t_{j-1}, & i = t_j \in I, \text{ if } j = 2, \dots, s \\ t_s, & i = t_1 \in I. \end{cases} \quad (1.54)$$

Combining (1.54) with (1.53) shows $\lambda = c_I(\eta)$. \square

We can use this lemma to show that one does not need to check all permutations to establish that $\eta \preceq' \lambda$. This result is originally due to Knop [44], however here it is derived using alternative methods.

Lemma 1.4.3. *For λ such that $|\lambda| = |\eta| + 1$ the defining permutation $\widehat{\sigma}$ of $\eta \preceq' \lambda$ is $\widehat{\sigma} = \omega_\lambda \omega_\eta^{-1}$, where ω_η^{-1} is defined by (1.36).*

Proof. From the previous lemma we can replace λ by $c_I(\eta)$ and specify the defining permutation $\widehat{\sigma}$ by (1.54). From the definition of $c_I(\eta)$ it is clear that

$$(\eta^+)_i \leq (c_I(\eta)^+)_i, \quad \text{for all } i \in \{1, \dots, n\}.$$

Manipulating this using (1.36) and (1.35) shows

$$\eta_i \leq c_I(\eta)_{\omega_{c_I(\eta)} \omega_\eta^{-1}(i)}, \quad \text{for all } i \in \{1, \dots, n\}.$$

By (1.36) it can be deduced that $t_{j-1} = \omega_{c_I(\eta)} \omega_\eta^{-1}(t_j)$ for $j = 2, \dots, s$, $t_s = \omega_{c_I(\eta)} \omega_\eta^{-1}(t_1)$ and $\omega_{c_I(\eta)} \omega_\eta^{-1}(i) = i$ if $\eta_i = (c_I(\eta))_i$. These properties of $\omega_{c_I(\eta)} \omega_\eta^{-1}$ show it to be identically equal to (1.54), thus concluding the proof. \square

From Lemma 1.4.3 we have the following corollaries.

Corollary 1.4.4. *We have $\eta \preceq' c_{I_1}(\eta); \sigma$ and $c_{I_1}(\eta) \preceq' c_{I_2} c_{I_1}(\eta); \rho$ if and only if $\eta \preceq' c_{I_2} c_{I_1}(\eta); \rho \circ \sigma$.*

Corollary 1.4.5. [44] *For λ such that $|\lambda| = |\eta| + r$ the defining permutation of $\eta \preceq' \lambda$ where $\widehat{\sigma}(i) = i$ if $\eta_i = \lambda_i$ is $\widehat{\sigma} = \omega_\lambda \omega_\eta^{-1}$.*

Corollary 1.4.6. *If $|\lambda| = |\eta| + r$ and $\eta \prec' \lambda$ then there exists sets $\{I_1, \dots, I_r\}$, $I_k = \{^k t_1, \dots, ^k t_s\} \subseteq \{1, \dots, n\}$ with $1 \leq ^k t_1 < \dots < ^k t_s \leq n$ such that $c_{I_r} \dots c_{I_1}(\eta) = \lambda$.*

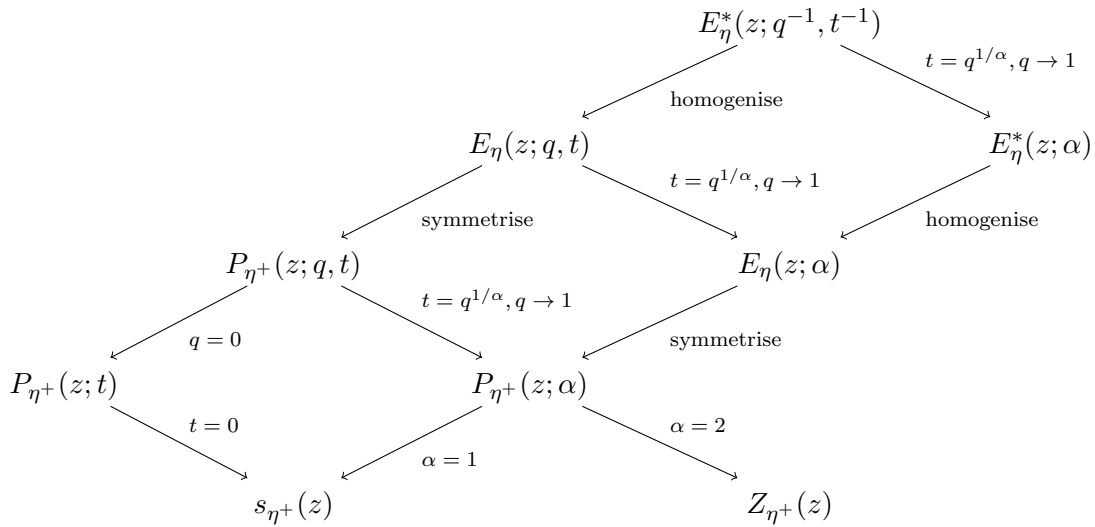
This concludes the preliminary material and we continue with the special function properties of the interpolation polynomials.

1.4.2 Some special nonsymmetric interpolation polynomials

The nonsymmetric interpolation Macdonald polynomials, denoted by $E_\eta^*(z; q, t)$, are the most general of all the polynomial classes to be considered. Once again, as a generalisation of the previous classes of polynomials many of the special function properties introduced so far have interpolation analogues. We introduce the main properties of the nonsymmetric

interpolation Macdonald polynomials in this section and provide the full details in the following chapter. Although not as prominent in the literature as the nonsymmetric Jack polynomials, the theory of the nonsymmetric interpolation Jack polynomials $E_\eta^*(z; \alpha)$ are of some interest (e.g. [58]) and we present the details in Appendix D.

The nonsymmetric polynomials relate to the nonsymmetric interpolation polynomials via a process of homogenisation, that is taking only the terms of top degree from the interpolation polynomial extracts the nonsymmetric polynomial. This, and further limiting properties are indicated in the following diagram.



Triangular structure

Like the nonsymmetric Macdonald polynomials the nonsymmetric interpolation Macdonald polynomials $E_\eta^*(z; q, t)$ are a linear combination of monomials and have the triangular structure

$$E_\eta^*(z; q, t) = z^\eta + \sum_{\lambda \triangleleft \eta} \tilde{K}_{\eta\lambda} z^\lambda, \quad (1.55)$$

for some coefficients $\tilde{K}_{\eta\lambda} \in \mathbb{Q}(q, t)$. Since the partial ordering \triangleleft is over compositions of modulus $\leq |\eta|$ the monomials in $E_\eta^*(z; q, t)$ are not all of the same degree making the polynomial $E_\eta^*(z; q, t)$ inhomogeneous.

For example we have

$$\begin{aligned} E_{(1,1)}^*(z; q, t) &= z^{(1,1)} - \frac{1}{t} z^{(1,0)} - \frac{1}{t} z^{(0,1)} + \frac{1}{t^2} \\ E_{(1,0)}^*(z; q, t) &= z^{(1,0)} + \frac{t-1}{qt-1} z^{(0,1)} - \frac{qt^2-1}{t(qt-1)} \\ E_{(0,1)}^*(z; q, t) &= z^{(0,1)} - \frac{1}{t}. \end{aligned}$$

Vanishing

Unlike the symmetric and nonsymmetric Macdonald polynomials the nonsymmetric interpolation Macdonald polynomials can be defined by certain vanishing properties. It was observed independently in [44] and [67] that the interpolation Macdonald polynomials could be defined to be polynomials with triangular structure (1.55) satisfying the properties

$$E_{\eta}^*(\bar{\eta}; q, t) \neq 0 \quad \text{and} \quad E_{\eta}^*(\bar{\lambda}; q, t) = 0, \quad \text{for all } |\lambda| \leq |\eta|, \lambda \neq \eta. \quad (1.56)$$

In (1.56) $\bar{\eta} := (\bar{\eta}_1, \dots, \bar{\eta}_n)$, where $\bar{\eta}_i$ is the i^{th} eigenvalue of the nonsymmetric Macdonald polynomial $E_{\eta}(z; q, t)$.

Furthermore, it was found that $E_{\eta}^*(\bar{\lambda}; q, t) \neq 0$ for $|\lambda| > |\eta|$ if and only if λ is a successor of η , that is $\eta \preceq' \lambda$. It is this property of the nonsymmetric interpolation Macdonald polynomial that plays the most crucial role in our study of the Pieri-type formulas.

Eigenfunctions

As in the case of the nonsymmetric Macdonald polynomials, the nonsymmetric interpolation Macdonald polynomials are simultaneous eigenfunctions of a family of n commuting eigenoperators. Together with the triangular structure the eigenfunction property provides a second definition for the interpolation Macdonald polynomials.

Generation

The nonsymmetric interpolation Macdonald polynomials further mirror the nonsymmetric Macdonald polynomials in the way they can be recursively generated from $E_{(0, \dots, 0)}^*(z; q, t) := 1$ using two elementary operators.

Notes: Interpolation polynomial theory exists for symmetric polynomials also; details of such theory can be found in [44].

Chapter 2

Foundational Theory

This chapter follows on from the previous by adding in the details to the theory of the special function properties of the main classes of polynomials to be studied: the nonsymmetric Macdonald polynomials and the nonsymmetric interpolation Macdonald polynomials. We begin with the details of the theory leading to the definitions and constructions of these polynomials. Properties are then derived to the extent needed in the later chapters, when we develop the theory of Macdonald polynomials with prescribed symmetry, and study Pieri-type formulas for nonsymmetric Macdonald polynomials.

2.1 Nonsymmetric Macdonald Polynomials

2.1.1 Introduction

The original reference on nonsymmetric Macdonald polynomials is Cherednik [15] and there is a further early work by Macdonald [55]. Two more accessible references are Kirillov [43] and Marshall [57]. Marshall gave an even more detailed account in his MSc thesis [56]. Here we will outline the development of the theory in many cases referring to [56] for the full proofs.

The nonsymmetric Macdonald polynomials $E_\eta(z) := E_\eta(z; q, t)$ are polynomials of n variables $z = (z_1, \dots, z_n)$, having coefficients in the field $\mathbb{Q}(q, t)$ of rational functions of the indeterminates q and t and labelled by compositions η . We proceed to define the polynomials by their eigenfunction properties.

2.1.2 The eigenoperators

We wish to construct from the T_i (1.13) a family of commuting operators which permit polynomial eigenfunctions. This requires the q -shift operator τ_i (1.21), and more particularly the operator $\omega := s_{n-1} \dots s_1 \tau_1$. The operator ω intertwines T_i and T_{i+1} ,

$$\omega T_{i+1} = T_i \omega. \quad (2.1)$$

Combining (1.12) with (2.1) gives the algebraic relations which specify the type- A affine Hecke algebra. With this notation a commutative subalgebra is constructed from a family of commuting operators, referred to as the Cherednik operators, defined by

$$Y_i := t^{-n+i} T_i \dots T_{n-1} \omega T_1^{-1} \dots T_{i-1}^{-1}, \quad 1 \leq i \leq n. \quad (2.2)$$

Note that T_i^{-1} is derived from the quadratic relation of (1.12) and given explicitly by

$$T_i^{-1} := t^{-1} - 1 + t^{-1} T_i.$$

To show that the Y_i commute we require one further operator, defined by

$$T_0 := t + \frac{qtz_n - z_1}{qz_n - z_1} (s_0 - 1),$$

where $s_0 := s_{in} \tau_1 \tau_n^{-1}$. We note that the indices $0, 1, \dots, n-1$ are understood as elements of \mathbb{Z}_n and so we may write $T_0 = T_n$.

Lemma 2.1.1. [12, 13] We have

$$Y_i Y_j - Y_j Y_i = 0. \quad (2.3)$$

Proof. Without loss of generality assume $i < j$. We use (2.1) to write

$$Y_i Y_j = t^{-2n+i+j} T_i \dots T_{n-1} T_0^{-1} \dots T_{i-2}^{-1} T_{j-1} \dots T_{n-2} T_{n-1}^{-1} T_0^{-1} \dots T_{j-3}^{-1} \omega^2 \quad (2.4)$$

and

$$Y_j Y_i = t^{-2n+i+j} T_j \dots T_{n-1} T_0^{-1} \dots \mathbf{T}_{i-1}^{-1} \dots \mathbf{T}_{j-2}^{-1} \mathbf{T}_{i-1} \dots \mathbf{T}_{j-2} \dots T_{n-2} T_{n-1}^{-1} T_0^{-1} \dots T_{i-3}^{-1} \omega^2, \quad (2.5)$$

where $T_{i-1}^{-1} \dots T_{j-2}^{-1} T_{i-1} \dots T_{j-2}$ has been set in bold in (2.5) to highlight the repeated sequence of subscripts. We transform (2.4) into (2.5) beginning from the centre by alternatively moving T_{j-1}, T_j, \dots to the left and moving $T_{i-2}^{-1}, T_{i-3}^{-1}$ to the right using the first two relations of (1.12). At one point in the procedure the subscripts of the most central T^{-1} and T will equal. We note that cancelling these terms reduces the number of operators by two. The overall process concludes when

$$Y_i Y_j = t^{-2n+i+j} T_j \dots T_{n-1} T_0^{-1} \dots T_{i-2}^{-1} T_i T_{i-1}^{-1} T_{i+1} T_i^{-1} \\ \dots T_{j-3} T_{j-4}^{-1} T_{j-2} T_{j-3}^{-1} T_{j-1} \dots T_{n-2} T_{n-1}^{-1} T_0^{-1} \dots T_{i-3}^{-1} \omega^2.$$

To transform the central operators $T_i T_{i-1}^{-1} T_{i+1} T_i^{-1} \dots T_{j-3} T_{j-4}^{-1} T_{j-2} T_{j-3}^{-1}$ to match those of (2.5), and thus establish (2.3), we append $T_{i-1}^{-1} T_{i-1}$ to the left of T_i and again use the first two relations of (1.12). We remark that the two additional operators $T_{i-1}^{-1} T_{i-1}$ replace those that were cancelled earlier. \square

The Y_i relate with the T_i according to the algebraic relations

$$T_i Y_i = Y_{i+1} T_i + (t-1) Y_i \\ T_i Y_{i+1} = t Y_i T_i^{-1} = Y_i T_i - (t-1) Y_i \\ T_i Y_j = Y_j T_i, \quad \text{if } j \neq i, i+1,$$

which give rise to the notion of the double affine Hecke algebra.

The fact that the Y_i commute suggests we seek simultaneous eigenfunctions. By first showing that the Cherednik operators have a triangular action on the monomials z^η (1.9) we show that the simultaneous eigenfunctions of the Y_i are the nonsymmetric Macdonald polynomials.

Proposition 2.1.2. [55] For $1 \leq i \leq n$,

$$Y_i z^\eta = q^{\eta_i} t^{-l'_\eta(i)} z^\eta + \sum_{\lambda < \eta} \hat{N}_{\eta\lambda}^i z^\lambda, \quad \hat{N}_{\eta\lambda}^i \in \mathbb{Q}(q, t). \quad (2.6)$$

Proof. (outline) The result is most simply observed by introducing

$$T_{k-1}^{(i)} := s_{i,k-1} T_{k-1} s_{i,k-1} s_{ik}$$

and

$$T_k^{-1(i)} := s_{ik} s_{i,k+1} T_k^{-1} s_{i,k+1}.$$

The Cherednik operator decomposes in terms of these operators and τ_i according to

$$Y_i = t^{-n+i}T_i^{(i)} \dots T_{n-1}^{(i)}\tau_i T_1^{-1(i)} \dots T_{i-1}^{-1(i)}.$$

By computing the action of these operators on the monomials $z_k^a z_i^b$ one can determine the explicit formula for $Y_i z^\eta$. \square

The following proposition is considered to be a definition for the nonsymmetric Macdonald polynomials.

Proposition 2.1.3. [55] For each composition η there exists a unique polynomial $E_\eta(z)$ which is a simultaneous eigenfunction of each Y_i , $1 \leq i \leq n$, and has the form

$$E_\eta(z) = z^\eta + \sum_{\lambda \prec \eta} \hat{K}_{\eta\lambda} z^\lambda, \quad \hat{K}_{\eta\lambda} \in \mathbb{Q}(q, t). \quad (2.7)$$

For $1 \leq i \leq n$, the respective eigenfunction equations satisfied by the polynomial $E_\eta(z)$ are

$$Y_i E_\eta(z) = q^{\eta_i} t^{-l'_\eta(i)} E_\eta(z), \quad (2.8)$$

where $l'_\eta(i)$ is given by (1.40). The polynomials $E_\eta(z)$ are called the nonsymmetric Macdonald polynomials.

Proof. Let $\{u_i\}$ be independent indeterminants. From (2.6) it follows that

$$\prod_{i=1}^n (1 + u_i Y_i) z^\eta = \prod_{i=1}^n (1 + u_i q^{\eta_i} t^{-l'_\eta(i)}) z^\eta + \sum_{\lambda \prec \eta} \hat{N}_{\eta\lambda}^i(q, t, \{u_i\}) z^\lambda.$$

If $\lambda \neq \eta$ then $\prod_{i=1}^n (1 + u_i q^{\lambda_i} t^{l'_\lambda(i)}) \neq \prod_{i=1}^n (1 + u_i q^{\eta_i} t^{l'_\eta(i)})$. It follows that there exists a unique eigenfunction of $\prod_{i=1}^n (1 + u_i Y_i)$ satisfying

$$E_\eta(z; \{u_i\}) = z^\eta + \sum_{\lambda \prec \eta} \hat{K}_{\eta\lambda}(q, t, \{u_i\}) z^\lambda. \quad (2.9)$$

Since being an eigenfunction of $\prod_{i=1}^n (1 + u_i Y_i)$ is equivalent to being a simultaneous eigenfunction of each Y_i it only remains to show that (2.9) is independent of $\{u_i\}$. Since the Cherednik operators commute, $\prod_{i=1}^n (1 + u_i Y_i)$ and $\prod_{i=1}^n (1 + v_i Y_i)$ must also commute, where $\{v_i\}$ are independent from $\{u_i\}$. It follows that $\prod_{i=1}^n (1 + v_i Y_i) E_\eta(z; \{u_i\})$ is an eigenfunction of $\prod_{i=1}^n (1 + u_i Y_i)$ and hence is a scalar multiple of $E_\eta(z; \{u_i\})$. It follows that $E_\eta(z; \{u_i\}) = E_\eta(z; \{v_i\})$. Hence $E_\eta(z)$ does not depend on $\{u_i\}$. \square

We note that it is the presence of the operator T_i in the defining eigenoperator and the underlying algebra of the nonsymmetric Macdonald polynomials that indicates the latter are compatible with t -symmetrisations.

In relation to the eigenvalue and its related n -tuple, define

$$\bar{\eta}_i := q^{\eta_i} t^{-l'_\eta(i)} \quad (2.10)$$

$$\bar{\eta} := (\bar{\eta}_1, \dots, \bar{\eta}_n). \quad (2.11)$$

The n -tuple $\bar{\eta}$ is most simply obtained by first superscripting the components of η by $l'_\eta(i)$ as specified by (1.40). To avoid individually computing the leg colength of each component one could superscript the components by the numbers $0, 1, \dots, n-1$, reading by decreasing values, beginning from the largest value, from left to right. For example the superscripted form of $\eta = (3, 6, 4, 4, 6, 9, 2, 4)$ is

$$(3^6, 6^1, 4^3, 4^4, 6^2, 9^0, 2^7, 4^5),$$

and from this we read off $\bar{\eta}$ to be

$$\bar{\eta} = (q^3 t^{-6}, q^6 t^{-1}, q^4 t^{-3}, q^4 t^{-4}, q^6 t^{-2}, q^9, q^2 t^{-7}, q^4 t^{-5}).$$

The following two lemmas give useful properties of the nonsymmetric Macdonald polynomials.

Lemma 2.1.4. [56] We have

$$z^k E_\eta(z) = E_{\eta+(k^n)}(z), \quad (2.12)$$

where $\eta + (k^n)$ is given by (1.38) and $z^k = (z_1 \dots z_n)^k$.

Proof. From the definition of Y_i we can show $Y_i z^k = q^k z^k Y_i$. Then using (2.8) we see that

$$Y_i z^k E_\eta(z) = q^{\eta_i+k} t^{-l'_\eta(i)} z^k E_\eta(z).$$

Since $l'_{\eta+(k^n)}(i) = l'_\eta(i)$ the result follows from Proposition 2.1.3. \square

Note that Lemma 2.1.4 allows meaning to be given to $E_\beta(z)$ where β is an array of integers not necessarily a composition. This is useful in stating our next result.

Lemma 2.1.5. [56] We have

$$E_\eta(z^{-1}) = E_{-\eta^R}(\underline{z}) \quad (2.13)$$

where η^R is given by (1.37) and $\underline{z} := (z_n, \dots, z_1)$.

The derivation of the above result requires the notion of the dual nonsymmetric Macdonald polynomials, related to the nonsymmetric Macdonald polynomials through the involutions $x_i \rightarrow x_{n-i+1}$, $q \rightarrow q^{-1}$, $t \rightarrow t^{-1}$, refer to [56] for the proof.

2.1.3 Recursive generation

The nonsymmetric Macdonald polynomials are most commonly defined as the simultaneous eigenfunctions of the commuting operators Y_i . Here we present an alternative definition by showing how each $E_\eta(z)$ can be recursively generated using two elementary operators.

In Section 1.2.1 we noted that every composition η can be recursively generated from $(0, \dots, 0)$ using the operators s_i and Φ . In a similar way each polynomial $E_\eta(z)$ can be recursively generated from $E_{(0, \dots, 0)}(z) = 1$ using T_i (1.13) and Φ_q , (2.15) below. We begin by showing how T_i relates $E_{s_i\eta}(z)$ and $E_\eta(z)$. We again refer the reader to [56] for the proof of the following proposition.

Proposition 2.1.6. [60] For $1 \leq i \leq n-1$ we have

$$T_i E_\eta(z) = \begin{cases} \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_\eta(z) + t E_{s_i\eta}(z), & \eta_i < \eta_{i+1} \\ t E_\eta(z), & \eta_i = \eta_{i+1} \\ \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_\eta(z) + \frac{(1-t\delta_{i,\eta}(q,t))(1-t^{-1}\delta_{i,\eta}(q,t))}{(1-\delta_{i,\eta}(q,t))^2} E_{s_i\eta}(z), & \eta_i > \eta_{i+1}, \end{cases} \quad (2.14)$$

where $\delta_{i,\eta}(q,t) = \bar{\eta}_i / \bar{\eta}_{i+1}$.

The second operator Φ_q , a raising-type operator, relates $E_\eta(z)$ and $E_{\Phi\eta}(z)$. The operator Φ_q is defined by [5]

$$\Phi_q := z_n T_{n-1}^{-1} \dots T_1^{-1} = t^{i-n} T_{n-1} \dots T_i z_i T_{i-1}^{-1} \dots T_i^{-1}, \quad (2.15)$$

and has the following explicit action on the $E_\eta(z)$. Please refer to [5] for the proof.

Proposition 2.1.7. [5] The operator Φ_q acts on nonsymmetric Macdonald polynomials according to

$$\Phi_q E_\eta(z) = t^{-\#\{i > 1: \eta_i \leq \eta_1\}} E_{\Phi\eta}(z). \quad (2.16)$$

To use these formulas one would first consider how η is constructed from $(0, \dots, 0)$ and then apply the same sequence of operators to generate $E_\eta(z)$ from $E_{(0, \dots, 0)}(z) = 1$. The details of such a procedure is given in Section 5. In the following section we provide an algorithm that can be used, in theory at least, to compute any $E_\eta(z)$. The method exploits the eigenfunction and recursive generation properties of the nonsymmetric Macdonald polynomials.

Table 2.1.3 shows the recursive generation of some nonsymmetric Macdonald polynomials.

η	$E_\eta(z; q, t)$
0	1
$0^{n-1}1$	z_n
$0^{n-2}10$	$z_{n-1} + \frac{qt^{n-2}(t-1)}{qt^{n-1}-1}z_n$
$0^{n-3}10^2$	$z_{n-2} + \frac{qt^{n-3}(t-1)}{qt^{n-2}-1}(z_{n-1} + z_n)$
\vdots	\vdots
10^{n-1}	$z_1 + \frac{q(t-1)}{qt-1}(z_2 + \dots + z_n)$
$0^{n-2}1^2$	$z_n z_{n-1}$
$0^{n-3}101$	$z_n \left(z_{n-2} + \frac{qt^{n-2}(t-1)}{qt^{n-1}-1}z_{n-1} \right)$
$0^{n-4}10^21$	$z_n \left(z_{n-3} + \frac{qt^{n-3}(t-1)}{qt^{n-2}-1}(z_{n-2} + z_{n-1}) \right)$
\vdots	\vdots
$10^{n-2}1$	$z_n \left(z_1 + \frac{qt(t-1)}{qt^2-1}(z_2 + \dots + z_{n-1}) \right)$
$0^{n-1}2$	$z_n \left(z_n + \frac{t-1}{qt-1}(z_1 + \dots + z_{n-1}) \right)$

Table 2.1: Recursive generation of some nonsymmetric Macdonald polynomials

2.1.4 Rodrigues formula

The Rodrigues formulas algebraically compute polynomials. Examples of families for which Rodrigues formulas exist are the symmetric Macdonald polynomials [48], nonsymmetric Jack polynomials [74] and other classes of orthogonal polynomials like the nonsymmetric Hermite [76] and nonsymmetric Laguerre polynomials [65]. In this section

we provide the Rodrigues formulas for the nonsymmetric Macdonald polynomials [64]. These formulas are not the most efficient way to compute the nonsymmetric Macdonald polynomials but their structure is neat.

To present the Rodrigues formulas we require some new operators, the braid exclusion operators [74, 76, 63]

$$X_i := Y_i T_i - T_i Y_i, \quad (2.17)$$

and constituent operators [64]

$$\begin{aligned} B_i &:= X_i \dots X_{n-1} \Phi_q, & i = 1, \dots, n-1 \\ B_n &:= \Phi_q. \end{aligned} \quad (2.18)$$

From these we define a new raising operator for the nonsymmetric Macdonald polynomial as

$$A_i := (B_i)^i.$$

We show how these operators can be used to construct the nonsymmetric Macdonald polynomials in the following theorem. The result is originally due to [64] however here an alternative method is used.

Theorem 2.1.8. *With $\kappa = \eta^+$, $|0\rangle := 1$ and X_σ as in (1.5) we have*

$$E_\eta(z) = a_\eta X_{\omega_\eta} (A_1)^{\kappa_1 - \kappa_2} (A_2)^{\kappa_2 - \kappa_3} \dots (A_n)^{\kappa_n} |0\rangle, \quad a_\eta \in \mathbb{Q}(q, t).$$

Proof. We first prove

$$E_\kappa(z) = a_\eta^\dagger (A_1)^{\kappa_1 - \kappa_2} (A_2)^{\kappa_2 - \kappa_3} \dots (A_n)^{\kappa_n} |0\rangle, \quad a_\eta^\dagger \in \mathbb{Q}(q, t) \quad (2.19)$$

inductively. By Proposition 2.1.7 we have that

$$(A_n)^{\kappa_n} |0\rangle = a_\eta' E_{(\kappa_n, \dots, \kappa_n)}(z),$$

for some computable $a_\eta' \in \mathbb{Q}(q, t)$, which we consider to be our base case. Suppose we have

$$(A_{i-1})^{\kappa_{i-1} - \kappa_{i-2}} \dots (A_n)^{\kappa_n} |0\rangle = a_\eta'' E_{(\kappa_{i-1}, \dots, \kappa_{i-1}, \kappa_{i-2}, \dots, \kappa_n)}(z),$$

with $a_\eta'' \in \mathbb{Q}(q, t)$, once again computable. Proposition 2.1.7 shows $\Phi_q E_\eta(z)$ to be a multiple of $E_{\Phi\eta}(z)$ and together Propositions 2.1.3 and 2.1.6 imply $X_i E_\eta(z)$ is a multiple of

$E_{s_i\eta}(z)$. It is therefore straightforward to see that $(A_i)^{\kappa_i - \kappa_{i-1}}$ is able to transform the nonsymmetric Macdonald polynomial labelled by $(\kappa_{i-1}, \dots, \kappa_{i-1}, \kappa_{i-2}, \dots, \kappa_n)$ to some multiple of the nonsymmetric Macdonald polynomial labelled by $(\kappa_i, \dots, \kappa_i, \kappa_{i-1}, \kappa_{i-2}, \dots, \kappa_n)$, thus proving (2.19) by induction. The final operator X_{ω_η} transforms $E_\kappa(z)$ to $E_\eta(z)$ and the a_η provides the normalisation to ensure that the coefficient of the leading term is unity. \square

2.1.5 Orthogonality

The final alternative characterisation of the nonsymmetric Macdonald polynomials is as multivariate orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle_{q,t} := \langle \cdot, \cdot \rangle_{q,t}^{(\text{CT})}$, defined previously in (1.27). The weight function $W(z)$ in (1.27) is defined by

$$W(z) := W(z; q, t) = \prod_{1 \leq i < j \leq n} \frac{(\frac{z_i}{z_j}; q)_\infty (q \frac{z_j}{z_i}; q)_\infty}{(t \frac{z_i}{z_j}; q)_\infty (qt \frac{z_j}{z_i}; q)_\infty}, \quad (2.20)$$

where the Pochhammer symbol $(a; q)_\infty$ is given by (A.3).

Lemma 2.1.9. [55] Y_i^{-1} is the adjoint operator of Y_i with respect to $\langle \cdot, \cdot \rangle_{q,t}$ for each $i \in \{1, \dots, n\}$. That is

$$\langle f, Y_i g \rangle_{q,t} = \langle Y_i^{-1} f, g \rangle_{q,t}.$$

Proof. Direct calculations show that T_i^{-1} and ω^{-1} are the adjoint operators of T_i and ω_i respectively. The result follows immediately from this. \square

Proposition 2.1.10. [55] The nonsymmetric Macdonald polynomials $\{E_\eta(z)\}$ form an orthogonal set with respect to $\langle \cdot, \cdot \rangle_{q,t}$. That is

$$\langle E_\eta(z), E_\lambda(z) \rangle_{q,t} = \delta_{\eta,\lambda} \mathcal{N}_\eta(q, t), \quad (2.21)$$

where $\mathcal{N}_\eta(q, t) \in \mathbb{Q}(q, t)$, (explicitly given in (2.22)).

Proof. It follows from Proposition 2.1.3 and Lemma 2.1.9 that $\prod_{i=1}^n (1 + u_i Y_i^{-1})$ is the adjoint operator of $\prod_{i=1}^n (1 + u_i Y_i)$ with respect to $\langle \cdot, \cdot \rangle_{q,t}$. Therefore

$$\begin{aligned} \prod_{i=1}^n (1 + u_i q^{-\lambda_i} t^{\lambda'_i(i)}) \langle E_\eta(z), E_\lambda(z) \rangle_{q,t} &= \left\langle E_\eta(z), \prod_{i=1}^n (1 + u_i Y_i) E_\lambda(z) \right\rangle_{q,t} \\ &= \left\langle \prod_{i=1}^n (1 + u_i Y_i^{-1}) E_\eta(z), E_\lambda(z) \right\rangle_{q,t}, \end{aligned}$$

and consequently

$$\prod_{i=1}^n (1 + u_i q^{-\lambda_i} t^{l'_\lambda(i)}) \langle E_\eta(z), E_\lambda(z) \rangle_{q,t} = \prod_{i=1}^n (1 + u_i q^{-\eta_i} t^{l'_\eta(i)}) \langle E_\eta(z), E_\lambda(z) \rangle_{q,t}.$$

Hence if $\lambda \neq \eta$ then $\langle E_\eta(z), E_\lambda(z) \rangle_{q,t} = 0$. \square

We will have future use for the explicit value of $\mathcal{N}_\eta(q, t)$ in (2.21). The quantity $\mathcal{N}_\eta(q, t)$ was initially derived by Macdonald [55] and Cherednik [15]. The form used in our study is due to Marshall [56] and given by

$$\mathcal{N}_\eta(q, t) := \langle E_\eta(z), E_\eta(z) \rangle_{q,t} = \frac{d'_\eta(q, t) e_\eta(q, t)}{d_\eta(q, t) e'_\eta(q, t)} \langle 1, 1 \rangle_{q,t}. \quad (2.22)$$

By the definition of $\langle \cdot, \cdot \rangle_{q,t}$ it is clear the $\mathcal{N}_\eta(q, t) = \mathcal{N}_{\eta+(1^n)}(q, t)$. Also, by (1.51) we have $\mathcal{N}_\eta(q, t) = \mathcal{N}_\eta(q^{-1}, t^{-1})$. It is due to the infinite nature of $W(z)$ in $\langle \cdot, \cdot \rangle_{q,t}$ that we require the multiplication of $\langle 1, 1 \rangle_{q,t}$ to provide an explicit formula for $\langle E_\eta(z), E_\eta(z) \rangle_{q,t}$.

In some applications it is useful to set $t = q^k$ for some positive integer k . In such cases

$$W(z; q, q^k) = \prod_{1 \leq i < j \leq n} \begin{pmatrix} z_i \\ z_j \end{pmatrix}_k \begin{pmatrix} q \frac{z_j}{z_i} \\ q \end{pmatrix}_k$$

and [78]

$$\langle 1, 1 \rangle_{q, q^k} = \frac{[nk]_{q^k}!}{[k]_{q^k}! n}, \quad (2.23)$$

where the q -factorial is given by (A.1).

2.1.6 Evaluation formula

We present here the evaluation formula for the Macdonald polynomials.

Proposition 2.1.11. [15] Let $t^\delta := (1, t, \dots, t^{n-1})$. We have

$$E_\eta(t^\delta) = t^{l(\eta)} \frac{e_\eta(q, t)}{d_\eta(q, t)}, \quad (2.24)$$

where $l(\eta)$ is given by (1.3.1).

Proof. As stated, the operators s_i and Φ can generate all compositions recursively, starting with $(0, \dots, 0)$, and hence allow (2.24) to be proved inductively. Clearly, when $\eta = (0, \dots, 0)$ (2.24) holds, which establishes the base case. Assume for η general that

(2.24) holds. Our task is to deduce from this that

$$E_{s_i\eta}(t^{\underline{\delta}}) = t^{l(s_i\eta)} \frac{e_{s_i\eta}(q, t)}{d_{s_i\eta}(q, t)} \quad (2.25)$$

and

$$E_{\Phi\eta}(t^{\underline{\delta}}) = t^{l(\Phi\eta)} \frac{e_{\Phi\eta}(q, t)}{d_{\Phi\eta}(q, t)}. \quad (2.26)$$

To show (2.25) we must consider the cases $\eta_i < \eta_{i+1}$ and $\eta_i > \eta_{i+1}$ separately. We begin with the case $\eta_i < \eta_{i+1}$. Using (2.14) we have

$$E_{s_i\eta}(t^{\underline{\delta}}) = t^{-1}T_i E_{\eta}(t^{\underline{\delta}}) - \frac{t-1}{t(1-\delta_{i,\eta}^{-1}(q, t))} E_{\eta}(t^{\underline{\delta}}). \quad (2.27)$$

From the definition of T_i (1.13) it follows that for any function $f(z)$ we have

$$T_i f(t^{\underline{\delta}}) = t f(t^{\underline{\delta}}). \quad (2.28)$$

By substituting (2.28) and (2.24) into (2.27) and simplifying with the results of Lemma 1.3.1 we obtain (2.25). The case $\eta_i > \eta_{i+1}$ is obtained similarly. We now consider (2.26). Using (2.28) we see that

$$\begin{aligned} (\Phi_q E_{\eta}(z)) \Big|_{z=t^{\underline{\delta}}} &= (t^{1-n} T_{n-1} \dots T_1 z_1 E_{\eta}(z)) \Big|_{z=t^{\underline{\delta}}} \\ &= E_{\eta}(t^{\underline{\delta}}). \end{aligned}$$

This result combines with (2.16) to show

$$E_{\Phi\eta}(t^{\underline{\delta}}) = t^{\#\{i>1: \eta_i \leq \eta_1\}} E_{\eta}(t^{\underline{\delta}}). \quad (2.29)$$

Substituting (2.24) in (2.29) and using Lemma 1.3.3 gives

$$E_{\Phi\eta}(t^{\underline{\delta}}) = t^{\#\{i>1: \eta_i \leq \eta_1\}} t^{l(\eta)} \frac{e_{\Phi\eta}(q, t)}{d_{\Phi\eta}(q, t)}. \quad (2.30)$$

The result follows from above since $\#\{i > 1 : \eta_i \leq \eta_1\} = l_{\Phi\eta}(n, 1)$. \square

2.1.7 A Cauchy formula

Here we state the results from the theory of the Cauchy formula for the nonsymmetric Macdonald polynomials required for our studies; the reader is referred to [56] for a full explanation.

Lemma 2.1.12. [60] Let

$$\Omega(x, y; q, t) := \sum_{\eta} \frac{d_{\eta}}{d'_{\eta}} E_{\eta}(x; q, t) E_{\eta}(y; q^{-1}, t^{-1}). \quad (2.31)$$

We have

$$T_i^{(x)} \Omega(x, y^{-1}; q, t) = T_i^{(y)} \Omega(x, y^{-1}; q, t),$$

where the superscripts denote which variables the respective operators act upon.

Using Lemma 2.1.12 and (2.14) we are able to deduce an explicit formula for $T_i E_{\eta}(z^{-1}; q^{-1}, t^{-1})$.

Proposition 2.1.13. For $1 \leq i \leq n-1$ and $E_{\eta}(z^{-1}) := E_{\eta}(z^{-1}; q^{-1}, t^{-1})$ we have

$$T_i E_{\eta}(z^{-1}) = \begin{cases} \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_{\eta}(z^{-1}) + \frac{d_{s_i \eta} d'_{\eta}}{d'_{s_i \eta} d_{\eta}} \frac{(1-t\delta_{i,\eta}(q,t))(1-t^{-1}\delta_{i,\eta}(q,t))}{(1-\delta_{i,\eta}(q,t))^2} E_{s_i \eta}(z^{-1}), & \eta_i < \eta_{i+1} \\ t E_{\eta}(z^{-1}), & \eta_i = \eta_{i+1} \\ \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_{\eta}(z^{-1}) + t \frac{d_{s_i \eta} d'_{\eta}}{d'_{s_i \eta} d_{\eta}} E_{s_i \eta}(z^{-1}), & \eta_i > \eta_{i+1}. \end{cases} \quad (2.32)$$

Proof. By Lemma 2.1.12

$$\begin{aligned} T_i^{(x)} \left(\frac{d_{\eta}}{d'_{\eta}} E_{\eta}(x) E_{\eta}(y^{-1}) + \frac{d_{s_i \eta}}{d'_{s_i \eta}} E_{s_i \eta}(x) E_{s_i \eta}(y^{-1}) \right) \\ = T_i^{(y)} \left(\frac{d_{\eta}}{d'_{\eta}} E_{\eta}(x) E_{\eta}(y^{-1}) + \frac{d_{s_i \eta}}{d'_{s_i \eta}} E_{s_i \eta}(x) E_{s_i \eta}(y^{-1}) \right). \end{aligned}$$

Using (2.14) and equating coefficients of like terms gives (2.32) . \square

2.2 Nonsymmetric Interpolation Macdonald Polynomials

2.2.1 Introduction

The nonsymmetric interpolation Macdonald polynomials $E_{\eta}^*(z) := E_{\eta}^*(z; q, t)$ are the inhomogeneous analogues of the nonsymmetric Macdonald polynomials $E_{\eta}(z)$ and were introduced by Knop and Sahi in [44] and [67], respectively. Although the interpolation Macdonald polynomials are not homogeneous, like the Macdonald polynomials they permit a number of distinct characterisations which can be taken as their definition. Here we present the details of such definitions, important to our subsequent workings. The first definition characterises the polynomials according to their vanishing properties.

2.2.2 The vanishing condition

The interpolation Macdonald polynomials in one variable generalise the monomial z^p to the polynomial

$$f_p(z) := z(z-1)\dots(z-p+1).$$

This is the unique polynomial, up to normalisation, of degree at most p that vanishes at $z = 0, 1, \dots, p-1$ and is non-zero at $z = p$. We now show how the interpolation Macdonald polynomials in n variables can also be defined by vanishing conditions.

Lemma 2.2.1. [44] With $\eta_n \neq 0$ we have $\overline{\Phi^{-1}\eta} = (\overline{\eta}_n/q, \overline{\eta}_1, \dots, \overline{\eta}_{n-1})$.

Proof. Follows from the definition of $\overline{\eta}$, (2.11). □

Theorem 2.2.2. [44, 67] Let $d \in \mathbb{N}$ and $S(n, d)$ be the set of all $\overline{\eta}$ where $|\eta| \leq d$. Let the values of $\overline{f}(\overline{\eta})$ be prescribed. Then there exists a unique polynomial $f(z)$ of degree at most d such that $f(z) = \overline{f}(z)$ for all $z \in S(n, d)$.

Proof. Due to the bijection between compositions η and monomials z^η the size of $S(n, d)$ equals the dimension of the space of polynomials of degree at most d . Hence showing $f(z)$ exists will imply its uniqueness. To show existence, we proceed by induction on n and d . It is true that every polynomial can be uniquely written as

$$f(z_1, \dots, z_n) = g(z_1, \dots, z_{n-1}) + (z_n - t^{-n+1})h(z_n/q, z_1, \dots, z_{n-1}). \quad (2.33)$$

Let S_0 be the set of $\overline{\eta} \in S(n, d)$ with $\eta_n = 0$, or equivalently, $\overline{\eta}$ such that $\overline{\eta}_n = t^{-n+1}$. As z runs through S_0 , $z' := (z_1, \dots, z_{n-1})$ will run through $S(n-1, d)$. By induction, one can choose $g(z)$ such that $f(z)$ takes the required values at S_0 . Consider now the set S_1 of remaining points $\overline{\eta}$ with $\eta_n \neq 0$. By Lemma 2.2.1, as z runs through S_1 , $z'' := (z_n/q, z_1, \dots, z_{n-1})$ will run through $S(n, d-1)$. By appropriately choosing q and t we can ensure the factor $z_n - t^{-n+1} = \overline{\eta}_n - t^{-n+1}$ is non-zero. By induction, we can find $h(z)$ of degree at most $d-1$ with arbitrary values at $S(n, d-1)$. Consequently $f(z)$ exists and the theorem holds. □

Theorem 2.2.3. [44, 67] For every composition η there is a unique polynomial $E_\eta^*(z)$ with triangular structure

$$E_\eta^*(z) = z^\eta + \sum_{\substack{\lambda: |\lambda| \leq |\eta| \\ \lambda \neq \eta}} \tilde{K}_{\eta\lambda} z^\lambda, \quad \tilde{K}_{\eta\lambda} \in \mathbb{Q}(q, t) \quad (2.34)$$

that satisfies (1.56).

With the normalisation specified by (2.34), the polynomial $E_\eta^*(z)$ is referred to as the nonsymmetric interpolation Macdonald polynomial.

Proof. By Theorem 2.2.2, there is a unique polynomial $E_\eta^*(z)$ satisfying the vanishing condition with $E_\eta^*(\bar{\eta}) \neq 0$. We have to show that it contains z^η with a non-zero coefficient. Consider (2.33) with $f(z) = E_\eta^*(z)$. If $\eta_n = 0$, then $g = E_{\eta'}^*(z)$ with $\eta' := (\eta_1, \dots, \eta_{n-1})$. By induction, $g(z)$, and therefore $E_\eta^*(z)$, contain z^η . If $\eta_n \neq 0$ then $g(z) = 0$ and $h(z) = E_\eta^*(z)$. We conclude again by induction. \square

2.2.3 The eigenoperators

The second definition of the interpolation polynomials classifies them as the unique simultaneous eigenfunctions of the operators Ξ_i defined by

$$\Xi_i := z_i^{-1} + z_i^{-1} H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1}. \quad (2.35)$$

Here

$$H_i := \frac{(t-1)z_i}{z_i - z_{i+1}} + \frac{z_i - tz_{i+1}}{z_i - z_{i+1}} s_i \quad (2.36)$$

$$= t + \frac{z_i - tz_{i+1}}{z_i - z_{i+1}} (s_i - 1) \quad (2.37)$$

(cf. (1.13)) is the Hecke operator and Φ_q^* is the interpolation analogue of Φ_q (2.15) defined

$$\Phi_q^* := (z_n - t^{-n+1})\Delta, \quad (2.38)$$

where $\Delta f(z_1, \dots, z_n) = f(z_n/q, z_1, \dots, z_{n-1})$. We note that the Hecke operator is another realisation of the type- A Hecke algebra (1.12).

To show the eigenoperator properties of the interpolation Macdonald polynomials we must first give explicit formulas to show how the Hecke operator H_i and the raising operator Φ_q^* act on a general function $f(z)$ evaluated at $\bar{\eta}$.

Lemma 2.2.4. [44] Let $f(z)$ be a Laurent polynomial. Then $H_i f(\bar{\eta})$ is a linear combination of $f(\bar{\eta})$ and $f(\bar{s}_i \bar{\eta})$ where the coefficients are independent of $f(z)$.

Proof. By (2.36) we have

$$H_i f(\bar{\eta}) = \frac{(t-1)\bar{\eta}_i}{\bar{\eta}_i - \bar{\eta}_{i+1}} f(\bar{\eta}) + \frac{\bar{\eta}_i - t\bar{\eta}_{i+1}}{\bar{\eta}_i - \bar{\eta}_{i+1}} f(s_i \bar{\eta}).$$

The result follows since either $\eta_i \neq \eta_{i+1}$ and $s_i \bar{\eta} = \overline{s_i \eta}$, or, $\eta_i = \eta_{i+1}$ and $\bar{\eta}_i - t \bar{\eta}_{i+1} = 0$. \square

Lemma 2.2.5. [44] Let $f(z)$ be a Laurent polynomial. We have

$$\Phi_q^* f(\overline{\Phi \eta}) = (q \bar{\eta}_1 - t^{-n+1}) f(\bar{\eta}).$$

Proof. Follows from the definition of Φ_q^* . \square

After showing that the interpolation Macdonald polynomials are eigenfunctions of Ξ_i we use the operators Ξ_i to determine the relationship between the nonsymmetric Macdonald polynomials and the interpolation Macdonald polynomials.

Theorem 2.2.6. [44] For all $i = 1, \dots, n$ we have

$$\Xi_i E_\eta^*(z) = \bar{\eta}_i^{-1} E_\eta^*(z). \quad (2.39)$$

Proof. We begin by rewriting the left hand side of (2.39) as

$$z_i^{-1} (E_\eta^*(z) + H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1} E_\eta^*(z)). \quad (2.40)$$

By Lemmas 2.2.4 and 2.2.5 and the vanishing properties of $E_\eta^*(z)$ (2.40) will vanish at $\bar{\lambda}$ such that $|\lambda| \leq |\eta|$, $\lambda \neq \eta$. This implies that $\Xi_i E_\eta^*(z) = c E_\eta^*(z)$ for some $c \in \mathbb{Q}(q, t)$. Evaluation at $z = \bar{\eta}$ implies $c = \bar{\eta}_i^{-1}$. \square

Corollary 2.2.7. [44] The operators Ξ_1, \dots, Ξ_n commute pairwise.

In [44] Knop showed the top homogeneous component of $E_\eta^*(z; q, t)$ to be $E_\eta(z; q^{-1}, t^{-1})$. To define the eigenoperator of $E_\eta(z; q^{-1}, t^{-1})$ we require a further realisation of the type- A Hecke algebra, \overline{H}_i , given by

$$\begin{aligned} \overline{H}_i &:= \frac{(t-1)z_{i+1}}{z_i - z_{i+1}} + \frac{z_i - tz_{i+1}}{z_i - z_{i+1}} s_i \\ &= H_i + 1 - t \end{aligned}$$

The operator \overline{H}_i is seen to be a realisation of the type- A Hecke algebra by observing that (1.12) is invariant under the mapping $h_i \mapsto h_i + 1 - t$. The \overline{H}_i appear in the eigenoperator of $E_\eta(z; q^{-1}, t^{-1})$ according to [44]

$$\xi_i^{-1} := \overline{H}_i \dots \overline{H}_{n-1} \Delta H_1 \dots H_{i-1}.$$

Theorem 2.2.8. [44] The top homogeneous component of $E_\eta^*(z; q, t)$ is $E_\eta(z; q^{-1}, t^{-1})$.

Proof. Straight forward manipulations show that

$$\Xi_n = \xi_n^{-1} + z_n^{-1}(1 - t^{-n+1}\xi_n^{-1}),$$

that is $\Xi_n = \xi_n^{-1} +$ degree lowering operators. It follows from this that $\Xi_i = \xi_i^{-1} +$ degree lowering operators and therefore $E_\eta^*(z; q, t) = E_\eta(z; q^{-1}, t^{-1}) +$ lower order terms. \square

Theorem 2.2.8 allows us to introduce an isomorphism Ψ mapping each nonsymmetric Macdonald polynomial to its corresponding interpolation polynomial,

$$\Psi E_\eta(z; q^{-1}, t^{-1}) = E_\eta^*(z; q, t). \quad (2.41)$$

This relationship is fundamental to our study of the Pieri-type formulas, allowing us to use interpolation Macdonald polynomial theory to derive explicit formulas for coefficients in expansions involving nonsymmetric Macdonald polynomials.

We conclude this section by showing how the eigenoperator Ξ_i can be used to give a more precise structure to (2.34). The result is due to Knop [44], however below an alternative derivation is given.

Lemma 2.2.9. *We have*

$$\Xi_i z^\eta = \bar{\eta}_i^{-1} z^\eta + \sum_{\lambda \triangleleft \eta} \tilde{N}_{\eta\lambda} z^\lambda, \quad \tilde{N}_{\eta\lambda} \in \mathbb{Q}(q, t), \quad (2.42)$$

where \triangleleft is specified by (1.52)

Proof. The left hand side of (2.42) is written explicitly as

$$\Xi_i z^\eta = z_i^{-1} z^\eta + z_i^{-1} H_i \dots H_{n-1} (z_n - t^{n+1}) \Delta H_1 \dots H_{i-1} z^\eta. \quad (2.43)$$

The composition labelling the first component of (2.43) is $\lambda = (\eta_1, \dots, \eta_i - 1, \dots, \eta_n)$, and clearly $\lambda \triangleleft \eta$. As for the second component of (2.43) we reduce the problem further by only considering the expansion of

$$z_i^{-1} H_i \dots H_{n-1} z_n \Delta H_1 \dots H_{i-1} z^\eta. \quad (2.44)$$

This is possible since any z^λ in the expansion of (2.44) will correspond to a $z^{\lambda'}$ in the expansion of $z_i^{-1}H_i \dots H_{n-1}t^{-n+1}\Delta H_1 \dots H_{i-1}z^\eta$ and the substitution of z_n for t^{-n+1} forces $\lambda' \triangleleft \lambda$.

With $z_n\Delta z^\eta = z^{\Phi\eta}$ and the Hecke operator H_i acting on monomials according to

$$H_i(z^\eta) = \begin{cases} z^{s_i\eta}, & \eta_i < \eta_{i+1} \\ tz^\eta, & \eta_i = \eta_{i+1} \\ tz^{s_i\eta} + (t-1)z^\eta, & \eta_i > \eta_{i+1}, \end{cases} \quad (2.45)$$

it can be seen with some work that the monomials z^λ appearing in the the expansion of (2.44) will be such that

$$\lambda_j \leq \eta_j \text{ for } j \neq i. \quad (2.46)$$

To show $\lambda \triangleleft \eta$ we consider the following two cases. Let η_k be the component of η that is raised by the action of $z_n\Delta$. If $\eta_k + 1 \geq \max\{\eta_{i+1}, \dots, \eta_n\}$ then $\lambda_i = \eta_k$, $\lambda^+ = \eta^+$ and by (2.46) we have $\lambda \triangleleft \eta$. Alternatively if $\eta_k + 1 < \max\{\eta_{i+1}, \dots, \eta_n\}$ some of the monomials z^λ appearing in the the expansion of (2.44) will have $\lambda = \sigma(\eta_1, \dots, \eta_k + 1, \dots, \eta_i, \dots, \eta_j - 1, \dots, \eta_n)$, for some permutation σ , where $\eta_j > \eta_k$. In this case $\lambda^+ \neq \eta^+$ and $\lambda^+ \leq \eta^+$. It follows from (2.45) and the action of Δ that the coefficient of z^η is $\bar{\eta}_i^{-1}$. \square

Theorem 2.2.10. [44] The interpolation nonsymmetric Macdonald polynomials are of the form

$$E_\eta^*(z) = z^\eta + \sum_{\lambda \triangleleft \eta} \tilde{K}_{\eta\lambda} z^\lambda, \quad \tilde{K}_{\eta\lambda} \in \mathbb{Q}(q, t). \quad (2.47)$$

Proof. By the triangularity of Ξ_i there must be an eigenfunction of the form (2.47) with eigenvalue $\bar{\eta}_i^{-1}$. Thus, it is equal to $E_\eta^*(z)$. \square

2.2.4 Recursive generation

Another defining characteristic of the interpolation Macdonald polynomials relates to the recursive generation (cf (2.14), (2.15)). The interpolation Macdonald polynomials can be recursively generated by two types of elementary operators, the Hecke operator given by (2.36) and the raising operator Φ_q^* specified by (2.38).

We first consider how the Hecke operator relates $E_\eta^*(z)$ to $E_{s_i\eta}^*(z)$.

Lemma 2.2.11. [44] If $\eta_i = \eta_{i+1}$ then $H_i E_\eta^*(z) = t E_\eta^*(z)$.

Proof. With $E := H_i E_\eta^*(z)$ and λ such that $|\lambda| \leq |\eta|$, $\lambda \neq \eta$ we have $E_\eta^*(\bar{\lambda}) = E_\eta^*(\overline{s_i \lambda}) = 0$. By Lemma 2.2.4 $E(\bar{\lambda}) = 0$ and hence E is a multiple of $E_\eta(z)$. Evaluation at $z = \bar{\eta}$ implies that the factor is t . \square

Lemma 2.2.12. [67] When $\eta_i \neq \eta_{i+1}$, we have

$$\left(H_i + \frac{1-t}{1-\delta_{i,\eta}^{-1}} \right) E_\eta^*(z) = k E_{s_i \eta}^*(z), \quad k \in \mathbb{Q}(q, t) \quad (2.48)$$

where $\delta_{i,\eta}$ is given by (1.43).

Proof. Evaluating the right hand side of (2.48) at $\bar{\lambda}$ and using Lemma 2.2.4 gives

$$\left(H_i + \frac{1-t}{1-\delta_{i,\eta}^{-1}} \right) E_\eta^*(\bar{\lambda}) = \frac{(t-1)\bar{\lambda}_i}{\bar{\lambda}_i - \bar{\lambda}_{i+1}} E_\eta^*(\bar{\lambda}) + \frac{\bar{\lambda}_i - t\bar{\lambda}_{i+1}}{\bar{\lambda}_i - \bar{\lambda}_{i+1}} E_\eta^*(s_i \bar{\lambda}) + \frac{1-t}{1-\bar{\eta}_{i+1}/\bar{\eta}_i} E_\eta^*(\bar{\lambda}). \quad (2.49)$$

By the vanishing properties of $E_\eta^*(z)$ (2.49) will vanish for all $\bar{\lambda}$ where $|\lambda| \leq |\eta|$ and $\lambda \neq \eta, s_i \eta$. In the case where $\lambda = \eta$ the inner term vanishes and the outer terms are equal, consequently (2.49) vanishes. On the other hand if $\lambda = s_i \eta$, evaluation of (2.49) gives

$$E_{s_i \eta}^*(\overline{s_i \eta}) = \frac{(\overline{s_i \eta})_i - t(\overline{s_i \eta})_{i+1}}{(\overline{s_i \eta})_i - (\overline{s_i \eta})_{i+1}} E_\eta^*(\bar{\eta}),$$

since $\eta_i \neq \eta_{i+1}$ and $(\overline{s_i \eta})_i - t(\overline{s_i \eta})_{i+1} \neq 0$ it follows that $E_{s_i \eta}^*(\overline{s_i \eta}) \neq 0$. These vanishing conditions imply (2.48). \square

The following results - due to Sahi [67] and Knop [44], respectively, but to be derived differently below - determine the value of k in Lemma 2.2.12.

Theorem 2.2.13. *When $\eta_i < \eta_i + 1$, we have*

$$\left(H_i + \frac{1-t}{1-\delta_{i,\eta}^{-1}} \right) E_\eta^*(z) = E_{s_i \eta}^*(z). \quad (2.50)$$

Proof. Due to Lemma 2.2.12 we are only required to show that the coefficient of $z^{s_i \eta}$ in the left hand side of (2.50) is unity. The triangular structure of $E_\eta^*(z)$ (2.47) implies that the monomial $z^{s_i \eta}$ does not appear in the expansion of $E_\eta^*(z)$ and the coefficient of z^η is 1. Therefore, we only need to consider the action of $H_i + (1-t)/(1-\delta_{i,\eta}^{-1})$ on z^η . Using the identity

$$\frac{z_i^m - z_j^m}{z_i - z_j} = z_i^{m-1} + z_i^{m-2} z_j + \dots + z_j^{m-1}, \quad (2.51)$$

the coefficient of $z^{s_i \eta}$ is found to be 1, from which the theorem follows. \square

Theorem 2.2.14. *When $\eta_i > \eta_{i+1}$, we have*

$$\left(H_i + \frac{1-t}{1-\delta_{i,\eta}^{-1}} \right) E_\eta^*(z) = \frac{(1-t\delta_{i,\eta}(q,t))(t-\delta_{i,\eta}(q,t))}{(1-\delta_{i,\eta}(q,t))^2} E_{s_i\eta}^*(z). \quad (2.52)$$

Proof. As in the previous theorem we seek the coefficient of $z^{s_i\eta}$ in the left hand side of (2.52). By viewing η with $\eta_i > \eta_{i+1}$ as $s_i\eta'$ with η' such that $\eta'_i < \eta'_{i+1}$ we can use the previous theorem to show that the coefficient of $z^{s_i\eta}$ in $E_\eta^*(z)$ is $(1-t)/(1-\delta_{i,\eta}^{-1})$. Consequently the coefficient of $z^{s_i\eta}$ in the left hand side of (2.52) is found by computing the action of $H_i + (1-t)/(1-\delta_{i,\eta}^{-1})$ on $z^\eta + (1-t)/(1-\delta_{i,\eta})z^{s_i\eta}$. Using straight forward calculations and the identity (2.51) the desired result is obtained. \square

We now consider the relationship between $E_\eta^*(z)$ and $E_{\Phi\eta}^*(z)$.

Proposition 2.2.15. [44] We have $\Phi_q^* E_\eta^*(z) = q^{-\eta_1} E_{\Phi\eta}^*(z)$.

Proof. By Lemma 2.2.5 we have $\Phi_q^* E_\eta^*(z)(\overline{\Phi\lambda}) = (q\overline{\lambda}_1 - t^{-n+1})E_\eta^*(z)(\overline{\lambda})$. The vanishing properties of $E_\eta^*(z)$ imply $\Phi_q^* E_\eta^*(z)$ vanishes for all $v = \Phi\lambda$ such that $|v| \leq |\eta| + 1$ and $v \neq \Phi\eta$. Thus $\Phi_q^* E_\eta^*(z) = cE_{\Phi\eta}^*(z)$ for some constant c . Since $z^{\Phi\eta} = z_1^{\eta_2} \dots z_{n-1}^{\eta_n} z_n^{\eta_1+1}$ is the leading term of $E_{\Phi\eta}^*(z)$ and the leading term of $\Phi_q^* E_\eta^*(z)$ is

$$(z_n - t^{-n+1})q^{-\eta_1} z_n^{\eta_1} z_1^{\eta_2} \dots z_1^{\eta_n} = q^{-\eta_1} z^{\Phi\eta},$$

we have that $c = q^{-\eta_1}$. \square

With $E_{(0,\dots,0)}^*(z) := 1$ the results of this section provide us with a third definition for the interpolation Macdonald polynomials, namely the recursive generation according to the formulas

$$H_i E_\eta^*(z) = \begin{cases} \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_\eta^*(z) + E_{s_i\eta}^*(z), & \eta_i < \eta_{i+1} \\ t E_\eta^*(z), & \eta_i = \eta_{i+1} \\ \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_\eta^*(z) + \frac{(1-t\delta_{i,\eta}(q,t))(t-\delta_{i,\eta}(q,t))}{(1-\delta_{i,\eta}(q,t))^2} E_{s_i\eta}^*(z), & \eta_i > \eta_{i+1}, \end{cases} \quad (2.53)$$

$$\Phi_q^* E_\eta^*(z) = q^{-\eta_1} E_{\Phi\eta}^*(z). \quad (2.54)$$

2.2.5 Rodrigues formula

Since the interpolation Macdonald polynomials have similar recursive generation formulas to the nonsymmetric Macdonald polynomials it follows naturally that there exists Rodrigues formulas that can also be used to compute them. The Rodrigues formulas for the interpolation polynomials can be found in the literature as they are a limit of the Rodrigues formulas for the Al-Salam Carlitz polynomials [7]. Here we provide an alternative derivation of the Rodrigues formulas for the interpolation polynomials using only the theory contained in this section.

Following the methods in Section 2.1.4 we begin by defining some new operators. We have the braid exclusion operator

$$X_i^* := \Xi_i H_i - H_i \Xi_i,$$

and the raising-type operator

$$\begin{aligned} B_i^* &:= X_i^* \dots X_{n-1}^* \Phi_q^*, & i = 1, \dots, n-1 \\ B_n^* &:= \Phi_q^*, \end{aligned}$$

(cf. (2.17) and (2.18)). The final operator is the raising-type operator A_i^* , and is defined to be

$$A_i^* := (B_i^*)^i.$$

Corollary 2.2.16. *With $\kappa = \eta^+$, $|0\rangle := 1$ and X_σ^* as in (1.5) we have*

$$E_\eta^*(z; q, t) = b_\eta X_{\omega_\eta}^* (A_1^*)^{\kappa_1 - \kappa_2} (A_2^*)^{\kappa_2 - \kappa_3} \dots (A_n^*)^{\kappa_n} |0\rangle, \quad b_\eta \in \mathbb{Q}(q, t).$$

Proof. Since the algebraic relations of Ξ_i, H_i and Φ_q^* mirror those of Y_i, T_i and Φ_q , respectively the proof follows using the same arguments as Theorem 2.1.8. \square

2.2.6 Evaluation formula

The following proposition uses the discussed properties of the interpolation Macdonald polynomials to give an evaluation formula for $E_\eta^*(\vec{\eta})$. This evaluation is quite valuable in the Pieri formula study. An inductive approach is used in a similar way to Proposition 2.1.11.

Proposition 2.2.17. *We have*

$$E_\eta^*(\bar{\eta}) := k_\eta = d'_\eta(q^{-1}, t^{-1}) \prod_{i=1}^n \bar{\eta}_i^{\eta_i}. \quad (2.55)$$

Proof. Clearly, when $\eta = (0, \dots, 0)$ we have $k_\eta = 1 = E_\eta^*(\bar{\eta})$, which establishes the base case. Assume for η general $E_\eta^*(\bar{\eta}) = k_\eta$. Our task is to deduce from this that

$$E_{s_i\eta}^*(\overline{s_i\eta}) = k_{s_i\eta} \quad (2.56)$$

and

$$E_{\Phi\eta}^*(\overline{\Phi\eta}) = k_{\Phi\eta}. \quad (2.57)$$

To show (2.56) we must consider the cases $\eta_i < \eta_{i+1}$ and $\eta_i > \eta_{i+1}$ separately. We begin with the case $\eta_i < \eta_{i+1}$. To relate $E_{s_i\eta}^*(\overline{s_i\eta})$ to $E_\eta^*(\bar{\eta})$ we consider two different perspectives on the computation of $H_i E_\eta^*(z)$. From (2.53) we have

$$H_i E_\eta^*(z) = \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_\eta^*(z) + E_{s_i\eta}^*(z), \quad (2.58)$$

while alternatively from definition (2.36)

$$H_i E_\eta^*(z) = \frac{(t-1)z_i}{z_i - z_{i+1}} E_\eta^*(z) + \frac{z_i - tz_{i+1}}{z_i - z_{i+1}} E_\eta^*(s_i z). \quad (2.59)$$

Equating the right hand sides of (2.58) and (2.59) and evaluating at $z = \overline{s_i\eta}$ we obtain

$$\frac{1 - t\delta_{i,s_i\eta}^{-1}(q,t)}{1 - \delta_{i,s_i\eta}^{-1}(q,t)} = \frac{1 - t\delta_{i,\eta}^{-1}(q^{-1}, t^{-1})}{1 - \delta_{i,\eta}^{-1}(q^{-1}, t^{-1})} = \frac{E_{s_i\eta}^*(\overline{s_i\eta})}{E_\eta^*(\bar{\eta})}.$$

Since $\eta_i < \eta_{i+1}$ we can use (1.47) to show

$$\frac{E_{s_i\eta}^*(\overline{s_i\eta})}{E_\eta^*(\bar{\eta})} = \frac{d'_{s_i\eta}(q^{-1}, t^{-1})}{d'_\eta(q^{-1}, t^{-1})} = \frac{k_{s_i\eta}}{k_\eta}.$$

Hence $E_\eta^*(\bar{\eta}) = k_\eta$ implies $E_{s_i\eta}^*(\overline{s_i\eta}) = k_{s_i\eta}$. The case where $\eta_i > \eta_{i+1}$ is proven similarly.

We now consider (2.57). Proposition 2.2.15 states

$$\Phi_q^* E_\eta^*(z) = q^{-\eta_1} E_{\Phi\eta}^*(z). \quad (2.60)$$

By evaluating (2.60) at $z = \overline{\Phi\eta}$, using Lemma 2.2.4 and rearranging we obtain

$$\frac{E_{\Phi\eta}^*(\overline{\Phi\eta})}{E_{\eta}^*(\overline{\eta})} = q^{\eta_1}(q\overline{\eta}_1 - t^{-n+1}). \quad (2.61)$$

By using (1.49) and the definition of $\overline{\eta}$ we can simplify (2.61) to

$$\frac{E_{\Phi\eta}^*(\overline{\Phi\eta})}{E_{\eta}^*(\overline{\eta})} = q^{2\eta_1+1}t^{-l_{\eta}(1)} \frac{d'_{\Phi\eta}(q^{-1}, t^{-1})}{d'_{\eta}(q^{-1}, t^{-1})} = \frac{k_{\Phi\eta}}{k_{\eta}}. \quad (2.62)$$

Where the final equality follows from

$$\frac{\prod_{i=1}^n \overline{\Phi\eta}_i^{(\Phi\eta)_i}}{\prod_{i=1}^n \overline{\eta}_i^{\eta_i}} = \frac{\overline{\Phi\eta}_n^{(\Phi\eta)_n}}{\overline{\eta}_1^{\eta_1}} = \frac{(q\overline{\eta}_1)^{\eta_1+1}}{\overline{\eta}_1^{2\eta_1}} = q^{\eta_1+1}t^{-l_{\eta}(1)}.$$

This completes the proof by induction. \square

2.2.7 The extra vanishing theorem

It was found by Knop [44] that the interpolation polynomials $E_{\eta}^*(z)$ vanish on a larger domain than λ such that $|\lambda| \leq |\eta|$, $\lambda \neq \eta$. To state this domain we use the partial ordering \preceq' defined in Section 1.4.2.

In [44] Knop shows that if $\eta \not\preceq' \lambda$ then $E_{\eta}^*(\overline{\lambda}) = 0$ using eigenoperator and the defining vanishing properties of the interpolation polynomials. Here we prove his result employing an alternative method that uses the polynomials recursive generation properties as well. Furthermore it is the main result of this section that $E_{\eta}^*(\overline{\lambda}) = 0$ if and only if $\eta \preceq' \lambda$.

We now work towards showing the main result by first considering λ such that $|\lambda| = |\eta| + 1$. The results in this section depend heavily on the theory of successors contained in Section 1.4.2.

Proposition 2.2.18. *For λ such that $|\lambda| = |\eta| + 1$ we have $E_{\eta}^*(\overline{\lambda}) = 0$ if and only if $\eta \preceq' \lambda$.*

Proof. Rewriting the eigenoperator Ξ_i as

$$z_i \Xi_i - 1 = H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1}$$

and making note of the recursive generation formulas (2.54) and (2.53) shows us that

$$\begin{aligned} (z_i \Xi_i - 1)E_\eta^*(z) &= H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1} E_\eta^*(z) \\ (z_i \bar{\eta}_i^{-1} - 1)E_\eta^*(z) &= \sum_{\nu: \nu = c_I(\eta)} C_{\eta\nu} E_\nu^*(z), \quad C_{\eta\nu} \in \mathbb{Q}(q, t), \end{aligned} \quad (2.63)$$

where the summation restriction in (2.63) to $\nu = c_I(\eta)$ is a consequence of Proposition 1.4.1. By evaluating (2.63) at $\bar{\lambda}$ and using the vanishing conditions of the interpolation polynomials we see that $E_\eta^*(\bar{\lambda}) = 0$ if and only if $\lambda \neq c_I(\eta)$, that is, if and only if $\eta \not\preceq' \lambda$. \square

Proposition 2.2.19. *We have $E_\eta^*(\bar{\lambda}) = 0$ if and only if $\eta \not\preceq' \lambda$.*

Proof. Here we prove the equivalent statement $E_\eta^*(\bar{\lambda}) \neq 0$ if and only if $\eta \preceq' \lambda$. We begin with λ such that $|\lambda| = |\eta| + 2$. By Corollary 1.4.6 we know that $\lambda = c_{I_2} c_{I_1}(\eta)$ for some sets $\{I_1, I_2\}$, $I_k = \{^k t_1, \dots, ^k t_s\} \subseteq \{1, \dots, n\}$ with $1 \leq ^k t_1 < \dots < ^k t_s \leq n$.

Taking $i = {}^1 t_1$ in (2.63) gives

$$(z_{{}^1 t_1} \bar{\eta}_{{}^1 t_1}^{-1} - 1)E_\eta^*(z) = \sum_{\nu: \nu = c_I(\eta)} \hat{C}_{\eta\nu} E_\nu^*(z), \quad \hat{C}_{\eta\nu} \in \mathbb{Q}(q, t).$$

Proposition 1.4.1 can be used to show $E_{c_{I_1}(\eta)}^*(z)$ is in the summation and by Proposition 2.2.18 we know $E_{c_{I_1}(\eta)}^*(\bar{\lambda}) \neq 0$. We can be sure that $(\bar{\lambda}_{{}^1 t_1} \bar{\eta}_{{}^1 t_1}^{-1} - 1) \neq 0$ since even if $\lambda_{{}^1 t_1} = \eta_{{}^1 t_1}$ we would still have either $\bar{\lambda} = \bar{\eta}/t$ or $\bar{\lambda} = \bar{\eta}/t^2$ due to the increased value of $l'_\eta({}^1 t_1)$. These results together show $E_\eta^*(\bar{\lambda}) \neq 0$ if $\eta \preceq' \lambda$. For the converse we use an evaluated form of (2.63) and let i be the position of the leftmost component of η that does not occur with the same frequency in λ ,

$$(\bar{\lambda}_i \bar{\eta}_i^{-1} - 1)E_\eta^*(\bar{\lambda}) = \sum_{\nu: \nu = c_I(\eta)} \tilde{C}_{\eta\nu} E_\nu^*(\bar{\lambda}), \quad \tilde{C}_{\eta\nu} \in \mathbb{Q}(q, t). \quad (2.64)$$

By the assumption $E_\eta^*(\bar{\lambda}) \neq 0$ and as before $(\bar{\lambda}_i \bar{\eta}_i^{-1} - 1) \neq 0$, consequently there exists a $E_\nu^*(\bar{\lambda})$ on the right hand side of (2.64) that does not vanish and by Proposition 2.2.18 we have $\nu \preceq' \lambda$. Since $\nu = c_I(\eta)$ implies $\eta \preceq' \nu$ we can use Corollary 1.4.4 to show $\eta \preceq' \lambda$, which completes the proof for the case $|\lambda| = |\eta| + 2$. Applying this procedure iteratively shows the result holds for general λ . \square

2.2.8 Binomial coefficients

In Section 2.2.1, equation (2.2.2), we introduced the one variable interpolation polynomials $f_p(z)$. These polynomials evaluated at l and p such that $l > p$ can be used to define the

classical binomial coefficients

$$\binom{l}{p} := \frac{l!}{(l-p)!p!} = \frac{f_p(l)}{f_p(p)}.$$

The first generalisation of these coefficients stems from the one variable q -interpolation polynomials [66] defined by

$$E_k^*(z; q) := (z-1) \dots (z-q^{k-1}),$$

where we have the one variable q -binomial coefficients (c.f. A.2)

$$\binom{l}{p}_q := \frac{(q^l-1) \dots (q^l-q^{p-1})}{(q^p-1) \dots (q^p-q^{p-1})} = \frac{E_p^*(\bar{l}; q)}{E_p^*(\bar{p}; q)}.$$

Here use has been made of the fact that for one variable partitions $\kappa = m$, $\bar{m} = q^m$. This pattern motivated Sahi [69] to introduce the generalised nonsymmetric q -binomial coefficients

$$\binom{\lambda}{\eta}_{q,t} := \frac{E_\eta^*(\bar{\lambda})}{E_\eta^*(\bar{\eta})}. \quad (2.65)$$

The q -binomial coefficients play a role in our study of the Pieri-type formulas in Chapter 4.

We now proceed to our first main study, an investigation of Macdonald polynomials with prescribed symmetry.

Chapter 3

Macdonald Polynomials with Prescribed Symmetry

The Macdonald polynomials with prescribed symmetry are obtained from the nonsymmetric Macdonald polynomials via the operations of t -symmetrisation, t -antisymmetrisation and normalisation. The study of Macdonald polynomials with prescribed symmetry began in 1999 [2, 56] and was initially motivated by the analogous results in Jack polynomial theory [4]. Jack polynomials are eigenfunctions of an operator that stems from the type- A Calogero-Sutherland quantum many body system (see e.g. [23, Chap. 11]). There are cases where the system is multicomponent, containing both bosons and fermions. In such cases the system requires eigenfunctions that are symmetric or antisymmetric, with respect to certain sets of variables. This requirement lead naturally to the introduction of Jack polynomials with prescribed symmetry [4, 42].

We seek a more comprehensive theory of Macdonald polynomials with prescribed symmetry, in keeping with the more extensively studied Jack analogues. Our first result is an expansion formula for the prescribed symmetry Macdonald polynomials in terms of the nonsymmetric Macdonald polynomials (Proposition 3.2.1). Following this we determine the normalisation required to obtain the prescribed symmetry Macdonald polynomial from the symmetrisation of the nonsymmetric polynomial (Proposition 3.2.2). In Section 3.4.1 eigenoperator methods are used to relate the symmetric and antisymmetric Macdonald polynomials (Proposition 3.3.1), thus providing an alternate proof for a result of Marshall [57]. We also give an explicit formula for prescribed symmetry polynomials with antisymmetric components in specific decreasing block form in terms of Vandermonde products and symmetric Macdonald polynomials (Theorem 3.4.4). Our final investigation is of the inner product of prescribed symmetry Macdonald polynomials, giving general explicit

formulas (Theorem 3.5.2) and then again considering the cases where the antisymmetric components are in decreasing block form (Theorem 3.5.3). It turns out that these results have applications in q -constant terms identities, which are discussed in the second half of Section 3.5. We complete the chapter with a discussion of possible extensions of the theory.

3.1 A Brief History

Multivariable polynomials with prescribed symmetry have a short history only dating back to the late 1990's. There exists prescribed symmetry theory on Jack polynomials, the generalised Hermite, Laguerre and Jacobi polynomials [4] and Macdonald polynomials [2]. For each class it is known how to construct the polynomials and also which operator they are eigenfunctions of. The prescribed symmetry theory of Jack polynomials is the most well developed and we refer the reader to [25] for many key results. The Macdonald polynomials with prescribed symmetry were first introduced by Marshall [56] two years after the introduction of Jack polynomials with prescribed symmetry. Marshall determined the operator required to transform nonsymmetric Macdonald polynomials to Macdonald polynomials with prescribed symmetry and the polynomial's eigenfunction properties.

3.2 Preliminaries

3.2.1 Symmetrisation

We have stated that symmetric Macdonald polynomials $P_\kappa(z)$ can be obtained from the nonsymmetric polynomials using U^+ (1.14). Similarly, the antisymmetric Macdonald polynomials, denoted $S_{\kappa+\delta}(z)$ where $\delta := (n-1, \dots, 1, 0)$, are obtained from the nonsymmetric Macdonald polynomials using U^- (1.15). In this chapter we generalise these actions by applying a combination of t -symmetrising and t -antisymmetrising operators to $E_\eta(z)$ to generate the Macdonald polynomials with prescribed symmetry.

To do this the notions of symmetric and antisymmetric polynomials from Section 1.2 need to be made more specific by considering the symmetry of particular variables. We say a polynomial $f(z)$ is symmetric with respect to z_i if

$$s_i f(z) = f(z)$$

and antisymmetric with respect to z_j if

$$s_j f(z) = -f(z).$$

These symmetries naturally generalise to t -symmetries as follows. A polynomial $f(z)$ is t -symmetric with respect to z_i if

$$T_i f(z) = t f(z)$$

and t -antisymmetric with respect to z_j if

$$T_j f(z) = -f(z).$$

3.2.2 Prescribed symmetry operator

We begin our investigation into the Macdonald polynomials with prescribed symmetry by defining the required symmetrising operator $O_{I,J}$, first introduced in [56]. In $O_{I,J}$ the subscripts I, J represent the sets of variables which the operator symmetrises and antisymmetrises, respectively. Explicitly

$$T_i[O_{I,J}f(z)] = tO_{I,J}f(z) \text{ for } i \in I \tag{3.1}$$

and

$$T_j[O_{I,J}f(z)] = -O_{I,J}f(z) \text{ for } j \in J. \tag{3.2}$$

For $O_{I,J}$ to be well defined I and J must be disjoint subsets of $\{1, \dots, n-1\}$ satisfying

$$i-1, i+1 \notin J \text{ for } i \in I \text{ and } j-1, j+1 \notin I \text{ for } j \in J.$$

In many cases we require the set J to be decomposed into disjoint sets of consecutive integers, to be denoted J_1, J_2, \dots, J_s . For example, with $J = \{1, 2, 5, 6, 7\}$, $J_1 = \{1, 2\}$ and $J_2 = \{5, 6, 7\}$. Related to this we also require sets $\tilde{J}_j := J_j \cup \{\max(J_j) + 1\}$ and $\tilde{J} := \cup \tilde{J}_j$. For the previous example we have $\tilde{J}_1 = \{1, 2, 3\}$, $\tilde{J}_2 = \{5, 6, 7, 8\}$ and $\tilde{J} = \{1, 2, 3, 5, 6, 7, 8\}$.

Since we are symmetrising with respect to a subset of variables, in contrast to the construction of U^+ and U^- , we do not want to sum over all $\sigma \in S_n$. To define the restricted set of permutations we use the notation

$$\langle s_i : i \in \{i_1, \dots, i_k\} \rangle,$$

which is read to be the set of all permutations that can be constructed using $\{1, s_{i_1}, \dots, s_{i_k}\}$. For example $\langle s_i : i \in \{1, 2\} \rangle = S_3$ (cf. (1.2)). We introduce $W_{I \cup J} := \langle s_k : k \in I \cup J \rangle$, a subset of S_n such that each $\omega \in W_{I \cup J}$,

$$\omega = \omega_I \omega_J, \quad \text{with } \omega_I \in W_I \text{ and } \omega_J \in W_J, \quad (3.3)$$

has the property that $\omega(i) = i$ if $i \notin \tilde{I} \cup \tilde{J}$.

The operator $O_{I,J}$ is then specified by

$$O_{I,J} := \sum_{\omega \in W_{I \cup J}} \left(-\frac{1}{t}\right)^{l(\omega_J)} T_\omega,$$

where T_ω is as in (1.5).

3.2.3 The polynomial $S_{\eta^*}^{(I,J)}(z)$

To motivate the introduction of the Macdonald polynomials with prescribed symmetry we first consider the symmetric and antisymmetric Macdonald polynomials, denoted by $P_\kappa(z)$ and $S_{\mu+\delta}(z)$, respectively, where κ and μ are partitions. To generate $P_\kappa(z)$ one would t -symmetrise any $E_\eta(z)$ for which there exists a permutation $\sigma \in S_n$ such that $\sigma\eta = \kappa$. Similarly, to generate $S_{\mu+\delta}(z)$ one would t -antisymmetrise any $E_\lambda(z)$ such that there exists a permutation $\rho \in S_n$ where $\rho\lambda = \mu + \delta$. Explicitly

$$U^+ E_\eta(z) = b'_\eta P_\kappa(z) \quad \text{and} \quad U^- E_\lambda(z) = b''_\mu S_{\mu+\delta}(z),$$

for some non-zero $b'_\eta, b''_\lambda \in \mathbb{Q}(q, t)$. The importance of $\mu + \delta$ in the antisymmetric case is that $U^- E_\mu(z) = 0$ if $\mu_i = \mu_j$ for some $i \neq j$. From the definitions of $P_\kappa(z)$ and $S_{\mu+\delta}(z)$ it follows that the Macdonald polynomial with prescribed symmetry, denoted by $S_{\eta^*}^{(I,J)}(z; q, t) := S_{\eta^*}^{(I,J)}(z)$, a polynomial t -symmetric with respect to the set I and t -antisymmetric with respect to the set J , will be labelled by a composition η^* such that

$$\eta_i^* \geq \eta_{i+1}^* \text{ for all } i \in I \text{ and } \eta_j^* > \eta_{j+1}^* \text{ for all } j \in J. \quad (3.4)$$

The polynomial can be generated by applying our prescribed symmetry operator $O_{I,J}$ to any $E_\eta(z)$ such that there exists a $\sigma \in W_{I \cup J}$ with $\sigma\eta = \eta^*$. That is

$$O_{I,J} E_\eta(z) = a_\eta^{(I,J)} S_{\eta^*}^{(I,J)}(z), \quad (3.5)$$

for some non-zero $a_\eta^{(I,J)} \in \mathbb{Q}(q, t)$. This uniquely specifies $S_{\eta^*}^{(I,J)}(z)$ up to normalisation; for the latter we require that the coefficient of z^{η^*} in the monomial expansion equals unity as in (2.7).

Our first task is to find the explicit formula for the proportionality $a_\eta^{(I,J)}$ in (3.5). We do this by first computing the expansion formula of $S_{\eta^*}^{(I,J)}(z^{-1}; q^{-1}, t^{-1}) := S_{\eta^*}^{(I,J)}(z^{-1})$ in terms of $E_\eta(z^{-1})$, a result which is of independent interest.

Proposition 3.2.1. *Let $\omega \in W_{I \cup J}$ be decomposed as in (3.3). Let $\omega \eta^* = \lambda$ and $\omega_I \eta^* = \lambda_I$. The coefficients in*

$$S_{\eta^*}^{(I,J)}(z^{-1}) = \sum_{\lambda \in W_{I \cup J}(\eta^*)} \hat{L}_{\eta^* \lambda} E_\lambda(z^{-1}), \quad \hat{L}_{\eta^* \eta^*} = 1, \quad (3.6)$$

are specified by

$$\hat{L}_{\eta^* \lambda} = (-1)^{l(\omega_J)} t^{l(\omega_I)} \frac{d'_{\eta^*} d_\lambda}{d'_{\lambda_I} d_{\lambda_I}}.$$

Similarly, the coefficients in

$$S_{\eta^*}^{(I,J)}(z) = \sum_{\lambda \in W_{I \cup J}(\eta^*)} \tilde{L}_{\eta^* \lambda} E_\lambda(z), \quad \tilde{L}_{\eta^* \eta^*} = 1, \quad (3.7)$$

are specified by

$$\tilde{L}_{\eta^* \lambda} = \left(-\frac{1}{t}\right)^{l(\omega_J)} \frac{d'_{\eta^*} d_\lambda}{d'_{\lambda_I} d_{\lambda_I}}. \quad (3.8)$$

Proof. We first consider (3.6) and begin by writing

$$\begin{aligned} & \sum_{\lambda \in W_{I \cup J}(\eta^*)} \hat{L}_{\eta^* \lambda} E_\lambda(z^{-1}) \\ &= \sum_{\substack{\lambda \in W_{I \cup J}(\eta^*) \\ \lambda_i \leq \lambda_{i+1}}} \chi_{i,i+1} \left(\hat{L}_{\eta^* \lambda} E_\lambda(z^{-1}) + \hat{L}_{\eta^* s_i \lambda} E_{s_i \lambda}(z^{-1}) \right) \end{aligned}$$

where $\chi_{i,i+1} = 1/2$ if $\lambda = s_i \lambda$ and 1 otherwise. For $i \in I$ we require

$$T_i S_{\eta^*}^{(I,J)}(z^{-1}) = t S_{\eta^*}^{(I,J)}(z^{-1}). \quad (3.9)$$

If $\lambda_i = \lambda_{i+1}$ (3.9) holds due to the relation in (2.32). Hence we consider the case where $\lambda_i < \lambda_{i+1}$. Expanding the left hand side of (3.9) using (2.32) gives simultaneous equations and solving these show

$$\frac{\hat{L}_{\eta^* \lambda}}{\hat{L}_{\eta^* s_i \lambda}} = t \frac{(\delta_{i,\lambda} - 1) d_\lambda d'_{s_i \lambda}}{(\delta_{i,\lambda} - t) d'_\lambda d_{s_i \lambda}} \quad \text{for all } i \in I. \quad (3.10)$$

Since $\lambda_i < \lambda_{i+1}$ (1.45) can be used to rewrite (3.10) as

$$\frac{\hat{L}_{\eta^* \lambda}}{\hat{L}_{\eta^* s_i \lambda}} = t \frac{d'_{s_i \lambda}}{d'_{\lambda}}. \quad (3.11)$$

By noting $\eta^* = \omega_I^{-1} \lambda_I = s_{i_1} \dots s_{i_{l(\omega_I^{-1})}} \lambda_I$ where each s_i interchanges increasing components we can apply (3.11) repeatedly to obtain

$$\frac{\hat{L}_{\eta^* \lambda_I}}{\hat{L}_{\eta^* \eta^*}} = \hat{L}_{\eta^* \lambda_I} = t^{l(\omega_I)} \frac{d'_{\eta^*}}{d'_{\lambda_I}}, \quad (3.12)$$

where the first equality follows from the normalisation $\hat{L}_{\eta^* \eta^*} = 1$.

To complete the derivation we require a formula for the ratio $\hat{L}_{\eta^* \lambda} / \hat{L}_{\eta^* \lambda_I}$. Since for all $j \in J$ we have

$$T_j S_{\eta^*}^{(I, J)}(z^{-1}) = -S_{\eta^*}^{(I, J)}(z^{-1}),$$

applying the above methods, with (1.47) in place of (1.45) gives $\hat{L}_{\eta^* \lambda} / \hat{L}_{\eta^* s_j \lambda} = -d_{\lambda} / d_{s_j \lambda}$, and consequently

$$\frac{\hat{L}_{\eta^* \lambda}}{\hat{L}_{\eta^* \lambda_I}} = (-1)^{l(\omega_J)} \frac{d_{\lambda}}{d_{\lambda_I}}. \quad (3.13)$$

Combining (3.13) with (3.12) we obtain (3.2.1).

The derivation of (3.8) is as above, only replacing (2.32) with (2.14). \square

We now use Proposition 3.2.1 to determine $a_{\eta}^{(I, J)}$. To present the result requires some notation. Write $\eta^{(\epsilon_I, \epsilon_J)}$, where $\epsilon_I, \epsilon_J \in \{+, 0, -\}$, to denote the element of $W_{I \cup J}(\eta)$ with the properties that $\eta^{(+, \cdot)}$ ($\eta^{(\cdot, +)}$) has $\eta_i^{(+, \cdot)} \geq \eta_{i+1}^{(+, \cdot)}$ for all $i \in I$ ($\eta_j^{(\cdot, +)} > \eta_{j+1}^{(\cdot, +)}$ for all $j \in J$), $\eta^{(-, \cdot)}$ ($\eta^{(\cdot, -)}$) has $\eta_i^{(-, \cdot)} \leq \eta_{i+1}^{(-, \cdot)}$ for all $i \in I$ ($\eta_j^{(\cdot, -)} < \eta_{j+1}^{(\cdot, -)}$ for all $j \in J$), and $\eta^{(0, \cdot)}$ has $\eta_i^{(0, \cdot)} = \eta_{i+1}^{(0, \cdot)}$ for all $i \in I$. For example with $\lambda = \omega_I \omega_J \eta^*$ we have $\omega_I \eta^* = \lambda^{(0, +)}$ and $\omega_J \eta^* = \lambda^{(+, 0)}$. Also introduce

$$M_{I, \eta} := \sum_{\substack{\sigma' \in W_I \\ \sigma'(\eta) = \eta^{(+, 0)}}} t^{l(\sigma')}.$$

Proposition 3.2.2. *The proportionality constant $a_{\eta}^{(I, J)}$ in (3.5) is specified by*

$$a_{\eta}^{(I, J)} = (-1)^{l(\omega_J)} M_{I, \eta} \frac{d'_{\eta} d'_{\eta^{(-, +)}} d_{\eta^{(-, +)}}}{d'_{\eta^{(0, +)}} d_{\eta^{(0, +)}} d'_{\eta^{(-, -)}}} \quad (3.14)$$

where ω_J is such that $\omega_J \eta^{(+, +)} = \eta^{(+, 0)}$.

Proof. Let $G(x, y)$ be defined by

$$G(x, y) = \sum_{\eta \in W_{I \cup J}(\eta^*)} \frac{d_\eta}{d'_\eta} E_\eta(x) E_\eta(y^{-1}). \quad (3.15)$$

It follows from (2.14) and (2.32) that $T_i^{(x)} G(x, y) = T_i^{(y)} G(x, y)$ for $i \in I \cup J$, and hence

$$O_{I,J}^{(x)} G(x, y) = O_{I,J}^{(y)} G(x, y). \quad (3.16)$$

By (3.1) and (3.2) we have $O_{I,J}^{(y)} E_\eta(y^{-1}) = b_\eta^{(I,J)} S_{\eta^*}^{(I,J)}(y^{-1})$ for some $b_\eta^{(I,J)} \in \mathbb{Q}(q, t)$.

Hence substituting (3.15) into (3.16) and recalling (3.5) shows

$$S_{\eta^*}^{(I,J)}(x) \sum_{\eta \in W_{I \cup J}(\eta^*)} \frac{d_\eta}{d'_\eta} a_\eta^{(I,J)} E_\eta(y^{-1}) = S_{\eta^*}^{(I,J)}(y^{-1}) \sum_{\eta \in W_{I \cup J}(\eta^*)} \frac{d_\eta}{d'_\eta} b_\eta^{(I,J)} E_\eta(x).$$

Using separation of variables it follows that

$$S_{\eta^*}^{(I,J)}(y^{-1}) = \tilde{a}_{\eta^*} \sum_{\eta \in W_{I \cup J}(\eta^*)} \frac{d_\eta}{d'_\eta} a_\eta^{(I,J)} E_\eta(y^{-1}) \quad (3.17)$$

for some constant $\tilde{a}_{\eta^*} \in \mathbb{Q}(q, t)$. Equating coefficients for $\eta = \omega_I \omega_J \eta^*$ in (3.17) and (3.6) shows

$$\tilde{a}_{\eta^*} \frac{d_\eta}{d'_\eta} a_\eta^{(I,J)} = (-1)^{l(\omega_J)} t^{l(\omega_I)} \frac{d'_{\eta^*} d_\eta}{d'_{\eta(0,+)} d_{\eta(0,+)}}. \quad (3.18)$$

The identity (3.18) must hold for all $\eta \in W_{I \cup J}(\eta^*)$, and in particular for $\eta = \eta^{(-,-)}$. For such a composition we can use (2.14) to show

$$O_{I,J} E_{\eta^{(-,-)}}(x) = (-1)^{l(\omega_{J'})} M_{I,\eta} S_{\eta^*}^{(I,J)}(x),$$

where $\omega_{J'}^{-1} \eta^{(-,-)} = \eta^{(-,+)}$. Consequently

$$a_{\eta^{(-,-)}}^{(I,J)} = (-1)^{l(\omega_{J'})} M_{I,\eta}. \quad (3.19)$$

Substituting (3.19) into (3.18) with $\eta = \eta^{(-,-)}$ implies

$$\tilde{a}_{\eta^*} = \frac{t^{l(\omega_I)} d'_{\eta^*} d'_{\eta^{(-,-)}}}{M_{I,\eta} d'_{\eta^{(-,+)}} d_{\eta^{(-,+)}}}. \quad (3.20)$$

Substituting (3.20) in (3.18) gives the desired result. \square

As a corollary of this result we are able to provide an evaluation formula for $S_{\eta^*}^{(I, \emptyset)}(t^\delta)$. To state the evaluation we require knowledge of how the q -factorial relates to the length of permutations.

Theorem 3.2.3. [73] For all positive integers $n \geq 2$, we have

$$L_n(q) := \sum_{\sigma \in S_n} q^{l(\sigma)} = [n]_q! \quad (3.21)$$

where $[n]_q!$ denotes the q -factorial (A.1).

Proof. To prove (3.21) we go one further and show that each of the $n!$ terms in the expansion of $L_n(q)$ corresponds to exactly one permutation in S_n , that is the i^{th} term corresponding to the permutation σ_i will be of the form $q^{K_1(\sigma_i)} \dots q^{K_n(\sigma_i)}$, where K_j is given by (1.1). Starting with $n = 2$ we prove (3.21) inductively. For $n = 2$ there are 2 permutations, one of length zero and one of length one and hence $L_2(q) = 1 + q$ as required. We now assume that the statement is true for $n - 1$. Let $\omega = (\omega(1), \dots, \omega(n - 1)) \in S_{n-1}$. By inserting n somewhere in ω we form a new permutation $\omega' \in S_n$. If n is inserted in the k^{th} position then for $j = k, \dots, n - 1$, $K_{j+1}(\omega') = K_j(\omega) + 1$, hence increasing the length of the permutation by $n - k$. Therefore $L_n(q) = L_{n-1}(q)(1 + q + \dots + q^{n-1})$, which proves (3.21). \square

We now present the corollary.

Corollary 3.2.4. *We have the evaluation formula*

$$S_{\eta^*}^{(I, \emptyset)}(t^\delta) = \frac{n_I}{a_{\eta^*}^{(I, \emptyset)}} E_{\eta^*}(t^\delta) = \frac{n_I}{M_{I, \eta^*}} \frac{t^{l(\eta)} e_{\eta^*}}{d_{\eta^*}^{(-, 0)}}, \quad (3.22)$$

where $n_I := \sum_{\sigma \in W_I} t^{l(\sigma)} = \Pi_s[|\tilde{I}_s|]_t!$.

Proof. Using (2.28) the first equality of (3.22) can be derived immediately from (3.5).

With $J = \emptyset$ and $\eta = \eta^*$ Proposition 3.2.2 gives

$$a_{\eta^*}^{(I, \emptyset)} = M_{I, \eta^*} \frac{d_{\eta^*}^{(-, 0)}}{d_{\eta^*}}.$$

Substituting this and (2.24) into

$$\frac{n_I}{a_{\eta^*}^{(I, \emptyset)}} E_{\eta^*}(t^\delta)$$

gives the final equality. \square

We now move on to our first related result, deducing the form of Macdonald polynomials with prescribed symmetry in specific cases.

3.3 Antisymmetric Macdonald Polynomials

In this section we show how one can express an antisymmetric Macdonald polynomial in terms of a symmetric Macdonald polynomial and the t -Vandermonde product. This result is originally due to Marshall [57] and was proven using the polynomials orthogonality properties and Kadell's Lemma [40]. Kadell's Lemma states that for any antisymmetric function $h(z)$

$$\text{CT}\left(\prod_{i<j}(z_i - az_j)h(z)\right) = \frac{[n]_a!}{n!} \text{CT}\left(\prod_{i<j}(z_i - z_j)h(z)\right). \quad (3.23)$$

We first present Marshall's derivation and then proceed, in the second proof, to use an eigenoperator method to reclaim the result. It appears that the eigenoperator method can be adapted to a more general form of prescribed symmetry polynomial; we give the details of this in the following section.

Proposition 3.3.1. [57] *We have*

$$S_{\kappa+\delta}(z; q, t) = \Delta_t(z) P_\kappa(z; q, qt). \quad (3.24)$$

Proof 1. To employ Kadell's Lemma (3.23) we begin by writing the inner product $\langle \Delta_t(z) P_\kappa(z; q, qt), \Delta_t(z) P_\mu(z; q, qt) \rangle_{q,t}$ as

$$\langle \Delta_t(z) P_\kappa(z; q, qt), \Delta_t(z) P_\mu(z; q, qt) \rangle_{q,t} = \text{CT}\left(\prod_{i<j}(z_i - t^{-1}z_j)h(z)\right), \quad (3.25)$$

where

$$\begin{aligned} h(z) &= \prod_{i<j} (z_i^{-1} - tz_j^{-1}) \frac{(z_i z_j^{-1}; q)_\infty (q z_j z_i^{-1}; q)_\infty}{(t z_i z_j^{-1}; q)_\infty (q t z_j z_i^{-1}; q)_\infty} P_\kappa(z; q, qt) P_\mu(z^{-1}; q^{-1}, (qt)^{-1}) \\ &= \prod_{i<j} (z_i^{-1} - z_j^{-1}) \frac{(q z_i z_j^{-1}; q)_\infty (q z_j z_i^{-1}; q)_\infty}{(q t z_i z_j^{-1}; q)_\infty (q t z_j z_i^{-1}; q)_\infty} P_\kappa(z; q, qt) P_\mu(z^{-1}; q^{-1}, (qt)^{-1}) \end{aligned}$$

and is an antisymmetric polynomial. Note that the second equality in the above equation is due to the simplification

$$(z_i^{-1} - tz_j^{-1}) \frac{(1 - z_i z_j^{-1})}{(1 - z_i z_j^{-1})} = z_i^{-1} - z_j^{-1}.$$

Applying Kadell's lemma twice to the right hand side (3.25) we obtain

$$\begin{aligned} \langle \Delta_t(z) P_\kappa(z; q, qt), \Delta_t(z) P_\mu(z; q, qt) \rangle_{q,t} &= \frac{[n]_{t^{-1}}!}{[n]_{qt}!} \left(\prod_{i < j} (z_i - qt z_j) h(z) \right) \\ &= \frac{[n]_{t^{-1}}!}{[n]_{qt}!} \langle P_\kappa(z; q, qt), P_\mu(z; q, qt) \rangle_{q,qt} \\ &= \frac{[n]_{t^{-1}}!}{[n]_{qt}!} \langle P_\kappa(z; q, qt), P_\kappa(z; q, qt) \rangle_{q,qt} \delta_{\kappa\mu}, \end{aligned} \quad (3.26)$$

here, the second equality is a consequence of

$$\frac{(z_i - qt z_j)(z_i^{-1} - tz_j^{-1})}{(1 - tz_i z_j^{-1})(1 - qt z_i z_j^{-1})} = 1.$$

It follows from (3.26) that the polynomials $\Delta_t(z) P_\kappa(z; q, qt)$ form an orthogonal set with respect to $\langle \cdot, \cdot \rangle_{q,t}$. Since the polynomials are also of the form

$$S_{\kappa+\delta}(z; q, t) = m'_{\kappa+\delta}(z) + \sum_{\mu < \kappa+\delta} \bar{K}_{\kappa\mu} m'_\mu(z), \quad \bar{K}_{\kappa\mu} \in \mathbb{Q}(q, t),$$

where $m'_{\kappa+\delta} := U^-(z^{\kappa+\delta})$, we must have (3.24). \square

Proof 2. Since the unique symmetric eigenfunction of $D_n^1(q, qt)$ (1.20) with leading term $m_\kappa(z)$ (1.8) is $P_\kappa(z; q, qt)$, we can show (3.24) holds by proving the more general statement

$$D_n^1(q, t) \Delta_t(z) f(z) = \Delta_t(z) D_n^1(q, qt) f(z), \quad (3.27)$$

where $f(z)$ is any symmetric function. We begin by rewriting $D_n^1(q, t)$ as [43]

$$D_n^1(q, t) := t^{n-1} \sum_{i=1}^n Y_i$$

and deriving a more explicit form for the left hand side of (3.27). Since $\Delta_t(z) f(z)$ is t -antisymmetric the left hand side can be rewritten as

$$\left(T_1 \dots T_{n-1} \omega - t T_2 \dots T_{n-1} \omega + \dots + (-t)^{n-1} \omega \right) \Delta_t(z) f(z)$$

which, by the definition of ω is equal to

$$\left(T_1 \dots T_{n-1} - tT_2 \dots T_{n-1} + \dots + (-t)^{n-1}\right) \Delta_t(qz_n, z_1, \dots, z_{n-1}) f(qz_n, z_1, \dots, z_{n-1}).$$

For simplicity we let $\Theta_m := T_m \dots T_{n-1}$, $g_k(z) := \Delta_t(qz_k, z_1, \dots, z_n) f(qz_k, z_1, \dots, z_n)$ and

$$T_i = \frac{(1-t)z_{i+1}}{z_i - z_{i+1}} + \frac{tz_i - z_{i+1}}{z_i - z_{i+1}} s_i. \quad (3.28)$$

We begin by deducing the coefficient, $\widehat{c}[k, m]$ say, of each $g_k(z)$ after being operated on by Θ_m . Note that by the definition of Θ_m we have $\widehat{c}[k, m] = 0$ if $m > k$. The structure of T_i in (3.28) indicates that there will be 2^{n-m} terms in the expansion of $\Theta_m g_n(z)$. We begin by focusing on the coefficient of $\widehat{c}[k, k]$. To obtain $g_k(z)$ from $g_n(z)$ we must take the term corresponding to the action of

$$\frac{tz_i - z_{i+1}}{z_i - z_{i+1}} s_i$$

for each $i = n-1, \dots, k+1$, therefore

$$\widehat{c}[k, k] = \prod_{i=k+1}^n \frac{tz_k - z_i}{z_k - z_i}.$$

It can be shown by (backward) induction on m that for $m < k$

$$\widehat{c}[k, m] = (-1)^{k-m} \frac{(t-1)z_k}{z_m - z_k} \prod_{i=m+1}^{k-1} \frac{tz_i - z_k}{z_i - z_k} \prod_{i=k+1}^n \frac{tz_k - z_i}{z_k - z_i}.$$

We note that an important part of the inductive proof is to keep $\Delta_t(z)f(z)$ expressible in terms of $g_k(z)$. This is done by observing that $s_i f(z) = f(z)$ and

$$s_i \Delta_t(z) = \frac{tz_{i+1} - z_i}{tz_i - z_{i+1}} \Delta_t(z).$$

To derive the coefficient of $g_k(z)$ in the overall operator we must evaluate

$$\sum_{m=1}^k (-t)^{m-1} \widehat{c}[k, m].$$

To do this we prove

$$\sum_{m=j}^k (-t)^{m-1} \widehat{c}[k, m] = (-t)^{j-1} \prod_{i=j}^{k-1} \frac{z_k - tz_i}{z_i - z_k} \prod_{i=k+1}^n \frac{tz_k - z_i}{z_k - z_i}$$

inductively with a base case of $j = k - 1$. It follows that the coefficient of $f(qz_k, z_1, \dots, z_n)$ in $D_n^1(q, t) \Delta_t(z) f(z)$ is

$$\prod_{i=1}^{k-1} \frac{z_k - tz_i}{z_i - z_k} \prod_{i=k+1}^n \frac{tz_k - z_i}{z_k - z_i} \Delta_t(qz_k, z_1, \dots, z_n). \quad (3.29)$$

By noting

$$\Delta_t(qz_k, z_1, \dots, z_n) = \prod_{i=1}^{k-1} \frac{qz_k - t^{-1}z_i}{z_i - t^{-1}z_k} \prod_{i=k+1}^n \frac{qz_k - t^{-1}z_i}{z_k - t^{-1}z_i} \Delta_t(z)$$

we simplify (3.29) to

$$\Delta_t(z) \prod_{\substack{i=1 \\ i \neq k}}^n \frac{qtz_k - z_i}{z_k - z_i},$$

and hence

$$D_n^1(q, t) \Delta_t(z) f(z) = \Delta_t(z) \sum_{k=1}^n \prod_{\substack{i=1 \\ i \neq k}}^n \frac{qtz_k - z_i}{z_k - z_i} f(qz_k, z_1, \dots, z_n).$$

We now simplify the right hand side of (3.27). We have

$$\begin{aligned} D_n^1(q, qt) f(z) &= (qt)^{n-1} \sum_{i=1}^n Y_i^{(q, qt)} f(z) \\ &= (qt)^{n-1} \left((qt)^{1-n} T_1^{(q, qt)} \dots T_{n-1}^{(q, qt)} \omega + \dots + \omega T_1^{-1(q, qt)} \dots T_{n-1}^{-1(q, qt)} \right) f(z) \end{aligned} \quad (3.30)$$

where $Y_i^{(q, qt)}$, $T_i^{(q, qt)}$, $T_i^{-1(q, qt)}$ are the operators Y_i , T_i , T_i^{-1} with t replaced by qt . Since $f(z)$ is symmetric we have $T_i^{-1(q, qt)} f(z) = (qt)^{-1} f(z)$. Using this and the action of ω (3.30) simplifies to

$$(T_1^{(q, qt)} \dots T_{n-1}^{(q, qt)} + \dots + T_{n-1}^{(q, qt)} + 1) f(qz_n, z_1, \dots, z_{n-1}).$$

We define $\Theta_m^{(q, qt)} := T_m^{(q, qt)} \dots T_{n-1}^{(q, qt)}$ and denote the coefficient of each $f(qz_k, z_1, \dots, z_n)$ by $\widehat{c}_q[k, m]$ for each $m \leq k$. Similarly to before

$$\widehat{c}_q[k, k] = \prod_{i=k+1}^n \frac{qtz_k - z_i}{z_k - z_i}$$

and, by induction,

$$\widehat{c}_q[k, m] = (-1)^{k-m} \frac{(qt-1)z_k}{z_m - z_k} \prod_{i=m+1}^{k-1} \frac{qtz_k - z_i}{z_i - z_k} \prod_{i=k+1}^n \frac{qtz_k - z_i}{z_k - z_i}.$$

For use in (3.30) we require $\sum_{m=1}^k \widehat{c}_q[k, m]$. This is found by induction on

$$\sum_{m=j}^k \widehat{c}_q[k, m] = \prod_{\substack{i=j \\ i \neq k}}^n \frac{qtz_k - z_i}{z_k - z_i}.$$

Therefore

$$\Delta_t(z) D_n^1(q, qt) f(z) = \Delta_t(z) \sum_{k=1}^n \prod_{\substack{i=1 \\ i \neq k}}^n \frac{qtz_k - z_i}{z_k - z_i} f(qz_k, z_1, \dots, z_n),$$

which shows (3.27), and consequently (3.24), to be true. \square

3.4 Prescribed Symmetry Macdonald Polynomials with Staircase Components

In this section we consider prescribed symmetry Macdonald polynomials labelled by a composition comprising of a partition κ and p blocks of staircase partitions, denoted δ_{N_p} below. These polynomials are symmetric with respect to variables corresponding to κ and antisymmetric for those corresponding to the staircase components.

In the literature such polynomials appear in two different forms. One with the symmetric variables preceding the antisymmetric where η^* , I and J are given explicitly by

$$\begin{aligned} \eta^* = \eta_{(n_0; N_p)} &:= (\kappa, \delta_{N_p}) = (\kappa_1, \dots, \kappa_{n_0}, \delta_{n_1}, \dots, \delta_{n_p}) \\ &= (\kappa_1, \dots, \kappa_{n_0}, n_1 - 1, n_1 - 2, \dots, 1, 0, \dots, n_p - 1, \dots, 1, 0), \end{aligned} \quad (3.31)$$

$$J = J^{n_0, N_p} := \bigcup_{\alpha=1}^p \left\{ \left(\sum_{k=1}^{\alpha} n_{k-1} \right) + 1, \dots, \left(\sum_{j=1}^{\alpha+1} n_{k-1} \right) - 1 \right\} \quad (3.32)$$

and

$$I = I^{n_0, N_p} := \{1, \dots, n_0 - 1\},$$

and the other having the opposite ordering with the antisymmetric variables first, for which we have

$$\begin{aligned} \eta^* = \eta_{(N_p; n_0)} &:= (\delta_{N_p}, \kappa) = (\delta_{n_1}, \dots, \delta_{n_p}, \kappa_1, \dots, \kappa_{n_0}) \\ &= (n_1 - 1, n_1 - 2, \dots, 1, 0, \dots, n_p - 1, \dots, 1, 0, \kappa_1, \dots, \kappa_{n_0}), \\ J := J^{N_p, n_0} &= \bigcup_{\alpha=1}^p \left\{ \left(\sum_{k=1}^{\alpha-1} n_{k-1} \right) + 1, \dots, \left(\sum_{k=1}^{\alpha} n_{k-1} \right) - 1 \right\} \end{aligned} \quad (3.33)$$

and

$$I = I^{N_p, n_0} := \{n - n_0 + 1, \dots, n\},$$

where in each case

$$N_p := \{n_1, \dots, n_p\}. \quad (3.34)$$

For example with $\kappa = (4, 4, 2)$ and $N_p = \{4, 2\}$ we have $\eta_{(n_0; N_p)} = (4, 4, 2, 3, 2, 1, 0, 1, 0)$, $I^{n_0, N_p} = \{1, 2\}$ and $J^{n_0, N_p} = \{4, 5, 6, 8\}$ and; $\eta_{(N_p; n_0)} = (3, 2, 1, 0, 1, 0, 4, 4, 2)$, $I^{N_p, n_0} = \{7, 8\}$ and $J^{N_p, n_0} = \{1, 2, 3, 5\}$.

Before stating our major goal for the section we introduce the final ingredients; the generalised Vandermonde product

$$\Delta^J(z) := \prod_{\alpha=1}^p \prod_{\substack{\min(\tilde{J}_\alpha) \leq i \\ < j \leq \max(\tilde{J}_\alpha)}} (z_i - z_j),$$

(cf. (1.11)), and the generalised t -Vandermonde product $\Delta_t^J(z)$

$$\Delta_t^J(z) := \prod_{\alpha=1}^p \prod_{\substack{\min(\tilde{J}_\alpha) \leq i \\ < j \leq \max(\tilde{J}_\alpha)}} (z_i - t^{-1}z_j),$$

(cf. (1.17)).

In this section we aim to express $S_{\eta^*}^{(I, J)}(z; q, t)$, where η^*, I and J are specified by either of the above forms, in terms of the generalised Vandermonde product $\Delta_t^J(z)$ and a symmetric Macdonald polynomial. Such an expression was motivated by results in both [4] and [2] and is a generalisation of (1.16).

In [4] Baker and Forrester show that with $\kappa_1 < \min\{n_1, \dots, n_p\}$ the prescribed symmetry Jack polynomial, denoted $S_{\eta^*}^{(I, J)}(z; \alpha)$, can be expressed

$$S_{\eta_{(n_0; N_p)}^{(I^{n_0, N_p}, J^{n_0, N_p})}}(z; \alpha) := S_{\eta_{(n_0; N_p)}^{(I, J)}}(z; \alpha) = \Delta^{J^{n_0, N_p}}(z) P_{\kappa}(z_1, \dots, z_{n_0}; \alpha + p). \quad (3.35)$$

In (3.35) $P_\kappa(z; \alpha)$ is the symmetric Jack polynomial (refer Appendix B). We note that in [4] the composition labelling the prescribed symmetry polynomial in (3.35) is incorrectly documented as

$$(\kappa_1, \dots, \kappa_{n_0}, n_1 - 1, \dots, 2, 1, \dots, n_p - 1, \dots, 2, 1).$$

A prescribed symmetry polynomial labelled by such a composition does not satisfy (3.35) and a close inspection of their proof indicates the composition is required to be of the form specified by (3.31).

The structure of the prescribed symmetry Jack polynomial $S_{\eta(n_0; N_p)}^{(I, J)}(z; \alpha)$ was obtained using properties of the eigenoperator in the Jack theory. Due to the different structure of the Macdonald eigenoperator the methods used in the Jack case can not be generalized. However, within [4] a brief note is made on how one may show that

$$S_{\kappa+\delta}(z; \alpha) = \Delta(z)P_\kappa(z; \alpha/(1 + \alpha)),$$

(cf. (3.24)), where $S_{\kappa+\delta}(z; \alpha)$ is the antisymmetric Jack polynomial, using the fact that

$$\tilde{H}_\alpha^{(C, Ex)} \Delta(z)f(z) = \Delta(z)\tilde{H}_{(\alpha/(1+\alpha))}^{(C, Ex)} f(z),$$

(cf. (3.27)). We refer the reader to [4] for the definition of the Jack polynomial eigenoperator $\tilde{H}_\alpha^{(C, Ex)}$ and further details of the suggested method.

Low order cases indicate that this method can be generalised to prove (3.35). Here we explain how the eigenoperator proof of Proposition 3.3.1 could be suitably generalised to prove the Macdonald case of (3.35). Following this we consider the other form of the special prescribed symmetry polynomials where the antisymmetric components precede the symmetric. Our motivation is the formula for Macdonald polynomials in the case $p = 1$ from [2] that states if $\kappa_1 < n_1$ then

$$S_{\eta(\{n_1\}; n_0)}^{(I, J)}(z; q, t) = \Delta_t^{J^{\{n_1\}, n_0}}(z) P_\kappa(z_{n_1+1}, \dots, z_n; qt, t). \quad (3.36)$$

We generalise the methods used in [2] to show how this result can be extended beyond the case $p = 1$.

3.4.1 Special form with $\eta^* = \eta(n_0; N_p)$

Computational evidence suggests the following conjecture.

Conjecture 3.4.1. *With $\kappa_1 < \min\{n_1, \dots, n_p\}$, $(I, J) = (I^{n_0, N_p}, J^{n_0, N_p})$ and $\eta^* = \eta_{(n_0; N_p)}$ we have*

$$S_{\eta^*}^{(I, J)}(z; q, t) = \Delta_t^J(z) P_\kappa(z_1, \dots, z_{n_0}; qt^p, t). \quad (3.37)$$

The proof is yet to be found. It appears to be related to the following conjecture.

Conjecture 3.4.2. *Let $f(z_1, \dots, z_{n_0})$ be a symmetric function with leading term $m_\kappa(z)$ and $\kappa_1 < \min(n_1, \dots, n_p)$. We have*

$$D_n^1(q, t) \Delta_t^{J^{n_0, N_p}}(z) f(z_1, \dots, z_{n_0}) = \Delta_t^{J^{n_0, N_p}}(z) D_n^1(qt^p, t) f(z_1, \dots, z_{n_0}). \quad (3.38)$$

We believe it to be possible to prove (3.37) by first proving (3.38). At this stage however it is not clear how one would keep track of the blocks of variables within the anti-symmetrising set, making the strategy used in the proof of Proposition 3.3.1 problematic to generalise.

3.4.2 Special form with $\eta^* = \eta_{(N_p; n_0)}$

In this section we generalise (3.36) to η^* specified by (3.33) and show that the condition $\kappa_1 < n_1$ can be weakened in general to $\kappa_1 \leq \min\{n_1, \dots, n_p\}$. We begin with the specific case where $\kappa = (0, \dots, 0)$ and then consider the more general setting.

Theorem 3.4.3. *With $\eta^* = (\delta_{N_p}, 0^{n_0})$ and $(I, J) = (\emptyset, J^{N_p, n_0})$ we have*

$$S_{\eta^*}^{(I, J)}(z; q, t) = \Delta_t^J(z).$$

Proof. The result follows from $S_{\eta^*}^{(I, J)}(z; q, t)$ having leading term $z^{(\delta_{N_p}, 0^{n_0})}$ and the requirement that $S_{\eta^*}^{(I, J)}(z; q, t)$ be t -antisymmetric with respect to J . \square

To prove the general result we follow the methods of [2, Sect. 3] and hence require explicit formulas for the action of T_i^{-1} on the monomial $z_i^a z_{i+1}^b$. After this has been presented we proceed with the theorem.

We have [2]

$$T_i^{-1} z_i^a z_{i+1}^b = \begin{cases} (t^{-1} - 1) z_i^a z_{i+1}^b + (t^{-1} - 1) z_i^{a-1} z_{i+1}^{b+1} \\ \quad + \dots + (t^{-1} - 1) z_i^{b+1} z_{i+1}^{a-1} + t^{-1} z_i^b z_{i+1}^a, & a > b \\ t^{-1} z_i^a z_{i+1}^a, & a = b \\ (1 - t^{-1}) z_i^{a+1} z_{i+1}^{b-1} + \dots + (1 - t^{-1}) z_i^{b-1} z_{i+1}^{a+1} \\ \quad + z_i^b z_{i+1}^a, & a < b. \end{cases} \quad (3.39)$$

Theorem 3.4.4. *With $\kappa_1 \leq \min(n_1, \dots, n_p)$, $(I, J) = (I^{N_p, n_0}, J^{N_p, n_0})$ and $\eta^* = \eta_{(N_p; n_0)}$ we have*

$$S_{\eta^*}^{(I, J)}(z; q, t) = \Delta_t^J(z) P_{\kappa}(z_{n-n_0+1}, \dots, z_n; qt^p, t). \quad (3.40)$$

Proof. Using (3.5) we construct $S_{\eta^*}^{(I, J)}(z; q, t)$ with $\eta^* = \eta_{(N_p; n_0)}$ by

$$\frac{1}{a_{\eta}^{(I, J)}} O_{I, J} E_{\eta}(z; q, t),$$

where $\eta = (\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)$. Such a polynomial is an eigenfunction of the operators

$$\mathcal{H}_1 = \sum_{i=1}^{n_1} Y_i, \dots, \mathcal{H}_p = \sum_{i=n_1+\dots+n_{p-1}+1}^{n-n_0} Y_i, \quad \mathcal{H}_0 = \sum_{i=n-n_0+1}^n Y_i,$$

where each Y_i acts on all variables z_1, \dots, z_n . By the definition of the partial order \prec given in (1.39), the Macdonald polynomial $E_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)}(z; q, t)$ is of the form

$$\begin{aligned} E_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)}(z; q, t) &= z^{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)} + \sum_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \eta) \prec (\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)} \hat{K}_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \eta)} z^{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \eta)} \\ &+ \sum_{\gamma \prec (\delta_{n_1}^R, \dots, \delta_{n_p}^R, \eta)} \hat{K}_{\gamma} z^{\gamma} \end{aligned} \quad (3.41)$$

for coefficients $\hat{K}_{\eta} \in \mathbb{Q}(q, t)$, where any $\gamma \prec (\delta_{n_1}^R, \dots, \delta_{n_p}^R, \eta)$ contains a pair $i, j \in \tilde{J}_{n_k}$, for some $k = 1, \dots, p$, such that $\gamma_i = \gamma_j$. Consequently when we apply $O_{I, J}$ to $E_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)}(z; q, t)$ the terms in the latter sum of (3.41) vanish, that is

$$\begin{aligned} O_{I, J} E_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \kappa^R)}(z; q, t) &= O_I(O_J z^{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, 0^{n_0})}) \\ &\times \left(z^{(0, \dots, 0, \kappa^R)} + \sum_{\eta \prec \kappa^R} \hat{K}_{(\delta_{n_1}^R, \dots, \delta_{n_p}^R, \eta)} z^{(0, \dots, 0, \eta)} \right) \\ &= \Delta_t^J(z) g(z_{n-n_0+1}, \dots, z_n), \quad c_{\delta_{N_p}} \in \mathbb{Q}(q, t) \end{aligned}$$

where g is a symmetric polynomial with leading term $m_{\kappa}(z_{n-n_0+1}, \dots, z_n)$. To complete the proof we must show that $g(z_{n-n_0+1}, \dots, z_n)$ is an eigenfunction of

$$\mathcal{H}'_0 = \sum_{i=n-n_0+1}^n Y_i^{(qt^p, t)},$$

where each Y_i acts only on the variables z_{n-n_0+1}, \dots, z_n . This statement is equivalent to

$$\mathcal{H}_0 \Delta_t^J(z) g(z_{n-n_0+1}, \dots, z_n) \propto \Delta_t^J(z) \mathcal{H}'_0 g(z_{n-n_0+1}, \dots, z_n)$$

which is further simplified by

$$Y_i^{(q,t)} \Delta_t^J(z) g(z_{n-n_0+1}, \dots, z_n) \propto \Delta_t^J(z) Y_i^{(qt^p, t)} g(z_{n-n_0+1}, \dots, z_n),$$

where $i = n - n_0 + 1, \dots, n$. Since T_i^{-1} does not depend on q and g is symmetric with respect to the variables z_{n-n_0+1}, \dots, z_n we are only required to show

$$\omega T_1^{-1} \dots T_{n-n_0}^{-1} \Delta_t^J(z) g(z_{n-n_0+1}, \dots, z_n) \propto \Delta_t^J(z) \omega_{qt^p}^{n_0} g(z_{n-n_0+1}, \dots, z_n), \quad (3.42)$$

where $\omega_{qt^p}^{n_0} := s_{n-1} \dots s_{n-n_0+1} T_{n-n_0+1}^{qt^p}$, that is ω restricted to act on the variables

$$z_{n-n_0+1}, \dots, z_n,$$

with q replaced by qt^p . Since for $1 \leq i \leq n - n_0 - 1$,

$$T_1^{-1} \dots T_{n-n_0}^{-1} T_i = T_{i+1} T_1^{-1} \dots T_{n-n_0}^{-1}$$

it follows that

$$T_1^{-1} \dots T_{n-n_0}^{-1} O_J = O_{J'} T_1^{-1} \dots T_{n-n_0}^{-1},$$

where $J' = \{2, \dots, n_1\} \cup_{i=2}^p \{n_{i-1} + 2, \dots, n_i\}$, and so

$$T_1^{-1} \dots T_{n-n_0}^{-1} \Delta_t^J(z) z_{n-n_0+1}^{\kappa_1} = T_1^{-1} \dots T_{n-n_0}^{-1} z_{n-n_0+1}^{(\delta_{n_1}, \dots, \delta_{n_p})} z_{n-n_0+1}^{\kappa_1} \quad (3.43)$$

$$= O_{J'} T_1^{-1} \dots T_{n-n_0}^{-1} z_{n-n_0+1}^{(\delta_{n_1}, \dots, \delta_{n_p})} z_{n-n_0+1}^{\kappa_1}. \quad (3.44)$$

Due to the action of T_i^{-1} on monomials $z_i^a z_{i+1}^b$ specified by (3.39) we have

$$T_1^{-1} \dots T_{n-n_0}^{-1} z_{n-n_0+1}^{(\delta_{n_1}, \dots, \delta_{n_p})} z_{n-n_0+1}^{\kappa_1} = t^{-(n-n_0+1)+p\kappa_1} (z_2 \dots z_{n-n_0} z_{n-n_0+1})^{(\delta_{n_1}, \dots, \delta_{n_p})} z_1^{\kappa_1} + \dots \quad (3.45)$$

Also by the action of T_i^{-1} we observe that the additional terms in (3.45) will have terms containing $(z_2 \dots z_{n-n_0} z_{n-n_0+1})^\gamma$ where $\gamma_i = \gamma_j$ for some $i, j \in \tilde{J}_{n_k}$ for some $i \neq j$ and $k = 1, \dots, p$. Since such terms will vanish under the action of $O_{J'}$ we can rewrite (3.44) as

$$O_{J'} T_1^{-1} \dots T_{n-n_0}^{-1} z_{n-n_0+1}^{(\delta_{n_1}, \dots, \delta_{n_p})} z_{n-n_0+1}^{\kappa_1} = t^{-(n-n_0+1)} \Delta^{J'}(z_2, \dots, z_{n-n_0}, z_{n-n_0+1}) (t^p z_1)^{\kappa_1}. \quad (3.46)$$

Since g is a symmetric function it follows from (3.46) and (3.43) that (3.42), and consequently (3.40) holds. \square

A major result in the theory of Jack polynomials with prescribed symmetry is the evaluation $U_{\eta^*}^{(I,J)}(1^n; \alpha)$ [18], where $U_{\eta^*}^{(I,J)}(z; \alpha)$ is defined by

$$U_{\eta^*}^{(I,J)}(z; \alpha) := \frac{S_{\eta^*}^{(I,J)}(z; \alpha)}{\Delta^J(z)}.$$

The method contained in [18] does not naturally extend to Macdonald theory and the explicit evaluation of $U_{\eta^*}^{(I,J)}(t^{\delta}; q, t)$, where

$$U_{\eta^*}^{(I,J)}(z; q, t) := \frac{S_{\eta^*}^{(I,J)}(z; q, t)}{\Delta_t^J(z)}$$

remains an open problem. However, for the cases where the prescribed symmetry Macdonald polynomials are of the form specified by (3.40) use of the evaluation formula for the symmetric Macdonald polynomials [54]

$$P_{\kappa}(t^{\delta}; q, t) = t^{l(\kappa)} \prod_{s \in \text{diag}(\kappa)} \frac{1 - q^{a'_\eta(s)} t^{n - l'_\eta(s)}}{1 - q^{a_\eta(s)} t^{l_\eta(s) + 1}} \quad (3.47)$$

gives the following as a corollary to Theorem 3.4.4

$$U_{\eta(N_p; n_0)}^{(I,J)}(t^{\delta}; q, t) = t^{l(\kappa)} \prod_{s \in \text{diag}(\eta(N_p, n_0))} \frac{1 - (qt^p)^{a'_\eta(s)} t^{n - l'_\eta(s)}}{1 - (qt^p)^{a_\eta(s)} t^{l_\eta(s) + 1}}. \quad (3.48)$$

3.5 The Inner Product of Prescribed Symmetry Polynomials and Constant Term Identities

3.5.1 The inner product of prescribed symmetry polynomials

We begin this section by finding the explicit formulas for the inner product of the prescribed symmetry polynomials $S_{\eta^*}^{(I,J)}(z) := S_{\eta^*}^{(I,J)}(z; q, t)$

$$\langle S_{\eta^*}^{(I,J)}(z), S_{\eta^*}^{(I,J)}(z) \rangle_{q,t}, \quad (3.49)$$

in terms of the inner product of nonsymmetric Macdonald polynomials. We then proceed to show how these formulas can be used to prove specialisations of certain constant term conjectures. We first consider the inner product of $O_{I,J}E_\eta(z)$.

Lemma 3.5.1. *Let I_i and J_j denote the decomposition of I and J as a union of sets of consecutive integers, then with $K_{I,J}(t) := \prod_{i=1}^s [|\tilde{J}_j|]_t! \prod_{i=1}^r [|\tilde{I}_i|]_{t^{-1}}!$ we have*

$$\langle O_{I,J}E_\eta(z), O_{I,J}E_\eta(z) \rangle_{q,t} = K_{I,J}(t) \langle O_{I,J}E_\eta(z), E_\eta(z) \rangle_{q,t}. \quad (3.50)$$

Proof. We begin by rewriting the left hand side of (3.50) as

$$\begin{aligned} & \langle O_{I,J}E_\eta(z), \sum_{\omega \in W_{I \cup J}} (-t)^{-l(\omega_J)} T_\omega E_\eta(z) \rangle_{q,t} \\ &= \sum_{\omega \in W_{I \cup J}} \left\langle O_{I,J}E_\eta(z), (-t)^{-l(\omega_J)} T_\omega E_\eta(z) \right\rangle_{q,t}. \end{aligned} \quad (3.51)$$

Since T_i^{-1} is the adjoint operator of T_i (1.28), it follows that (3.51) is equal to

$$\sum_{\omega \in W_{I \cup J}} \left\langle T_\omega^{-1} O_{I,J}E_\eta(z), (-t)^{-l(\omega_J)} E_\eta(z) \right\rangle_{q,t}. \quad (3.52)$$

Using (3.1) and (3.2) we rewrite (3.52) as

$$\sum_{\omega \in W_{I \cup J}} \left\langle (-1)^{l(\omega_J)} t^{-l(\omega_I)} O_{I,J}E_\eta(z), (-t)^{-l(\omega_J)} E_\eta(z) \right\rangle_{q,t}.$$

By definition of the inner product $\langle \cdot, t^{-1} \cdot \rangle_{q,t} = \langle t \cdot, \cdot \rangle_{q,t}$ and therefore we have

$$\sum_{\omega \in W_{I \cup J}} \left\langle t^{l(\omega_J)} t^{-l(\omega_I)} O_{I,J}E_\eta(z), E_\eta(z) \right\rangle_{q,t} = \sum_{\omega \in W_{I \cup J}} \frac{t^{l(\omega_J)}}{t^{l(\omega_I)}} \langle O_{I,J}E_\eta(z), E_\eta(z) \rangle_{q,t}.$$

The final result is obtained by employing (3.21). \square

We can now show how $\langle S_{\eta^*}^{(I,J)}(z), S_{\eta^*}^{(I,J)}(z) \rangle_{q,t}$ relates to $\langle E_\eta(z), E_\eta(z) \rangle_{q,t}$.

Theorem 3.5.2. *We have*

$$\langle S_{\eta^*}^{(I,J)}(z), S_{\eta^*}^{(I,J)}(z) \rangle_{q,t} = K_{I,J}(t) \frac{\tilde{L}_{\eta^* \eta}}{a_\eta(q^{-1}, t^{-1})} \langle E_\eta(z), E_\eta(z) \rangle_{q,t}, \quad (3.53)$$

where $\tilde{L}_{\eta^* \eta}$ and $a_\eta(q^{-1}, t^{-1})$ are given by (3.8) and (3.14) respectively.

Proof. Using (3.5) we are able to rewrite (3.49) as

$$\langle a_\eta^{(I,J)}(q, t)^{-1} O_{I,J}E_\eta(z), a_\eta^{(I,J)}(q, t)^{-1} O_{I,J}E_\eta(z) \rangle_{q,t}$$

by the definition of the inner product and Lemma 3.5.1 we write this as

$$\frac{K_{I,J}(t)}{a_\eta^{(I,J)}(q,t) a_\eta^{(I,J)}(q^{-1}, t^{-1})} \langle O_{I,J} E_\eta(z), E_\eta(z) \rangle_{q,t}. \quad (3.54)$$

Again using (3.5) we rewrite (3.54) as

$$\frac{K_{I,J}(t)}{a_\eta^{(I,J)}(q^{-1}, t^{-1})} \langle S_{\eta^*}^{(I,J)}(z), E_\eta(z) \rangle_{q,t}.$$

By (3.7) and the orthogonality of the Macdonald polynomials we get the desired result. \square

3.5.2 Special cases of the prescribed symmetry inner product

Following the theory of Jack polynomials [2] we were lead to finding explicit formulas for (3.49) in the special case where

$$\eta^* = \eta_{(0_{n_0}; n_1)}^* := (0^{n_0}, \delta_{n_1}) = (0, \dots, 0, n_1 - 1, \dots, 1, 0),$$

$I = \emptyset$ and $J = J^{n_0, \{n_1\}}$ (cf. (3.31)). Upon further inspection of this formula it was observed that in the limit $t \rightarrow q^k$ the result could be used to provide an alternative derivation of a specialisation of a constant term conjecture from [3]. Whilst working with the constant term identities it became apparent that a further conjecture in [3] was related to the more general inner product formula where

$$\eta^* = \eta_{(0_{n_0}; N_p)}^* := (0^{n_0}, \delta_{N_p}) = (0, \dots, 0, n_1 - 1, \dots, 1, 0, n_2 - 1, \dots, 1, 0, \dots, n_p - 1, \dots, 1, 0), \quad (3.55)$$

$I = \emptyset$ and $J = J^{n_0, N_p}$. Note that these compositions and sets are like those in Section 3.4.1 where the symmetric components precede the antisymmetric components.

Using the theory developed in the previous section we give an explicit formula for

$$\langle S_{\eta_{(0_{n_0}; N_p)}^*}^{(I,J)}(z), S_{\eta_{(0_{n_0}; N_p)}^*}^{(I,J)}(z) \rangle_{q, q^k},$$

deriving the more specific formula with $\eta^* = \eta_{(0_{n_0}; n_1)}^*$ as a corollary.

Theorem 3.5.3. *With $\eta^* = \eta_{(0_{n_0}; N_p)}^*$ as defined by (3.55), $I = \emptyset$ and $J = J^{n_0, N_p}$ we have*

$$\langle S_{\eta^*}^{(I,J)}(z), S_{\eta^*}^{(I,J)}(z) \rangle_{q, q^k} = \prod_{i=1}^p [n_i]_{q^k}! q^{-k \sum_{i=1}^p \frac{n_i(n_i-1)}{2}} \frac{[kn + \max(N_p)]_q! (1-q)^{\max(N_p)}}{[k]_q! n \prod_{j=1}^{\max(N_p)} (1 - q^j q^{k(n-\tilde{m}(j))})}. \quad (3.56)$$

In (3.56) $\tilde{m}(j) := \sum_{k=i_j}^p (n_k^+ - j)$, where $i_j := \#\{n_k^+ \in N_p^+ : n_k^+ < j\} + 1$, with $N_p^+ := \{n_1^+, \dots, n_p^+\} = \sigma(N_p)$ and $n_1^+ \geq \dots \geq n_p^+$.

Proof. Using (3.53) we see that the task is to simplify

$$K_{\emptyset, J}(q^k) \frac{\tilde{L}_{\eta^* \eta}}{a_{\eta}(q^{-1}, q^{-k})} \langle E_{\eta}(z), E_{\eta}(z) \rangle_{q, q^k}.$$

For simplicity we take $\eta = \eta^*$, and hence $\widehat{c}_{\eta^* \eta^*} = 1$. Since $I = \emptyset$ and ω such that $\omega \eta^* = \eta$ is $\omega = 1$ we have

$$a_{\eta^*}(q^{-1}, q^{-k}) = \frac{d'_{\eta^*}(q^{-1}, q^{-k})}{d'_{\eta^*(-, -)}(q^{-1}, q^{-k})}.$$

The p disjoint sets in J indicate that

$$K_{I, J}(q^k) = \prod_{i=1}^p [n_i]_{q^k}!$$

and lastly, by (2.22) and (2.23), we have

$$\left\langle E_{\eta^*}(z; q, q^k), E_{\eta^*}(z; q, q^k) \right\rangle_{q, q^k} = \frac{d'_{\eta^*}(q, q^k) e_{\eta^*}(q, q^k) [nk]_{q^k}!}{d_{\eta^*}(q, q^k) e'_{\eta^*}(q, q^k) [k]_{q^k}! n}.$$

Putting this all together allows us to rewrite (3.56) as

$$\prod_{i=1}^p [n_i]_{q^k}! \frac{d'_{\eta^*(-, -)}(q^{-1}, q^{-k}) d'_{\eta^*}(q, q^k) e_{\eta^*}(q, q^k) [nk]_{q^k}!}{d'_{\eta^*}(q^{-1}, q^{-k}) d_{\eta^*}(q, q^k) e'_{\eta^*}(q, q^k) [k]_{q^k}! n}.$$

We begin by simplifying

$$\frac{d'_{\eta^*(-, -)}(q^{-1}, q^{-k}) d'_{\eta^*}(q, q^k)}{d'_{\eta^*}(q^{-1}, q^{-k}) d_{\eta^*}(q, q^k)}. \quad (3.57)$$

In comparison with $\text{diag}(\eta^*)$, $\text{diag}(\eta^{*(-, -)})$ has one additional empty box above each row. Hence the leg length of each $s \in \text{diag}(\eta^{*(-, -)})$ is one greater than its corresponding box in $\text{diag}(\eta^*)$. It follows from this and the definition of d_{η} and d'_{η} that $d'_{\eta^{*(-, -)}} = d_{\eta^*}$. Using this and the identity (1.51) we can write (3.57) as

$$\frac{d_{\eta^*}(q^{-1}, q^{-k}) d'_{\eta^*}(q, q^k)}{d'_{\eta^*}(q^{-1}, q^{-k}) d_{\eta^*}(q, q^k)} = q^{k \sum_{i=1}^p \frac{n_i(n_i-1)}{2}}.$$

We now consider the simplification of the ratio of e_{η} and e'_{η} . Explicitly we have

$$\frac{e_{\eta^*}(q, q^k)}{e'_{\eta^*}(q, q^k)} = \frac{\prod_s \left(1 - q^{a'_{\eta^*}(s)+1} q^{k(n-l'_{\eta^*}(s))}\right)}{\prod_r \left(1 - q^{a'_{\eta^*}(r)+1} q^{k(n-1-l'_{\eta^*}(r))}\right)}. \quad (3.58)$$

By Lemma 1.3.1 and Lemma 1.3.2 we observe that the products in (3.58) are independent of the row order. For simplicity we take $\eta^* = \eta^{*+}$. For this composition we have

$$n - l'_{\eta^{*+}}((i+1, j)) = n - 1 - l'_{\eta^{*+}}((i, j))$$

and consequently most terms in (3.58) cancel. The terms unique to the numerator correspond to the boxes in the top row of η^{*+} , and therefore the terms remaining in the denominator will correspond to the bottom box of each column. The leg colengths of the latter set are given by $\tilde{m}(j) - 1$, (for $j = 1 \dots \max(N_p) - 1$). Hence the ratio of e_η and e'_η in our expansion is given by

$$\begin{aligned} \frac{e_{\eta^*}(q, q^k)}{e'_{\eta^*}(q, q^k)} &= \frac{(1 - q^{kn+1})(1 - q^{kn+2}) \dots (1 - q^{kn+\max(N_p)-1})}{\prod_{j=1}^{\max(N_p)-1} (1 - q^j q^{k(n-\tilde{m}(j))})} \\ &= \frac{(1 - q)^{\max(N_p)} [kn + \max(N_p)]_q!}{\prod_{j=1}^{\max(N_p)-1} (1 - q^j q^{k(n-\tilde{m}(j))}) (1 - q^{\max(N_p)} q^{kn}) [kn]_q!} \\ &= \frac{(1 - q)^{\max(N_p)} [kn + \max(N_p)]_q!}{\prod_{j=1}^{\max(N_p)} (1 - q^j q^{k(n-\tilde{m}(j))}) [kn]_q!}. \end{aligned}$$

The second equality follows from (A.5) and the final expression is arrived at by noting $\tilde{m}(\max(N_p)) = 0$. Substituting each simplification into (3.5.2) gives the required result. \square

We now give the analogous result for $\eta^* = \eta^*_{(0_{n_0}; n_1)}$.

Corollary 3.5.4. *With $\eta^*_{(0_{n_0}; n_1)}$, $I = \emptyset$ and $J^{n_0, \{n_1\}}$ we have*

$$\left\langle S_{\eta^*_{(0_{n_0}; n_1)}}^{(I, J)}(z), S_{\eta^*_{(0_{n_0}; n_1)}}^{(I, J)}(z) \right\rangle_{q, q^k} = [n_1]_{q^k}! q^{-\frac{k(n_1-1)n_1}{2}} \frac{[n_1 + nk]_q! (1 - q)^{n_1}}{[k]_q!^n (q^{n_1(k+1)+n_0k}; q^{-(k+1)})_{n_1}}. \quad (3.59)$$

Proof. The result follows immediately from Theorem 3.5.3 by substituting $p = 1$ into the right hand side of (3.56) and simplifying using (A.3). \square

3.5.3 The constant term identities

We now show how the formulas for the inner product of prescribed symmetry Macdonald polynomials derived in the previous section can be used to prove special cases of two q -constant term identities. The two q -constant term identities were introduced by Baker and Forrester in the mid-1990s [3] and are generalisations of older constant term identities

dating back to the 1960s. We will state Baker and Forrester's q -constant term identity after providing the historic details of the constant terms identities preceding it.

The most specialised form of Baker and Forrester's q -constant term identity is the Dyson identity [19] which states

$$\text{CT} \left(\prod_{1 \leq i < j \leq n} \left(1 - \frac{z_i}{z_j}\right)^k \left(1 - \frac{z_j}{z_i}\right)^k \right) = \frac{(kn)!}{k!^n}.$$

This identity was first q -generalised by Andrews [1] to

$$\text{CT} \left(\prod_{1 \leq i < j \leq n} \left(\frac{z_i}{z_j}; q\right)_k \left(q \frac{z_j}{z_i}; q\right)_k \right) = \frac{[kn]_q!}{[k]_q!^n},$$

(cf. (2.23)). These constant terms were extended to [61]

$$\begin{aligned} \text{CT} \left(\prod_{l=1}^n (1 - z_l)^a \left(1 - \frac{1}{z_l}\right)^b \prod_{1 \leq i < j \leq n} \left(1 - \frac{z_i}{z_j}\right)^k \left(1 - \frac{z_j}{z_i}\right)^k \right) \\ = \frac{\Gamma(a + b + 1 + kl) \Gamma(1 + k(l + 1))}{\Gamma(a + 1 + kl) \Gamma(b + 1 + kl) \Gamma(1 + k)}, \end{aligned}$$

referred to as the Morris constant term identity and

$$\begin{aligned} \text{CT} \left(\prod_{l=1}^n (z_l; q)_a \left(\frac{q}{z_l}; q\right)_b \prod_{1 \leq i < j \leq n} \left(\frac{z_i}{z_j}; q\right)_k \left(q \frac{z_j}{z_i}; q\right)_k \right) \\ = \frac{\Gamma_q(a + b + 1 + kl) \Gamma_q(1 + k(l + 1))}{\Gamma_q(a + 1 + kl) \Gamma_q(b + 1 + kl) \Gamma_q(1 + k)}, \end{aligned}$$

where Γ_q is the q -gamma function (A.4), which was independently obtained by Habsieger, Kadell and Zeilberger [30, 40, 78]. We note that many of these identities were proved using the theory of Selberg integrals (see, e.g., [27]).

Baker and Forrester extended the Morris and q -Morris identity given in [3] to contain a Vandermonde-type product. Unlike the earlier constant term identities, this identity is yet to be proved in its full generality and thus remains a conjecture. We now state Baker and Forrester's constant terms conjectures related to our work on the inner product of Macdonald polynomials with prescribed symmetry.

Conjecture 3.5.5. [3, Conj 2.1] We have

$$\begin{aligned}
D_1(n_0; n_1; a, b, k; q) &= \text{CT} \left(\prod_{n_0+1 \leq i < j \leq n} (z_i - q^{k+1} z_j)(z_i^{-1} - q^k z_j^{-1}) \prod_{i < j} \left(\frac{z_i}{z_j}; q \right)_k \left(q \frac{z_j}{z_i}; q \right)_k \right. \\
&\quad \left. \times \prod_{i=1}^n (z_i; q)_a \left(\frac{q}{z_i}; q \right)_b \right) \\
&= \frac{\Gamma_{q^{k+1}}(n_1 + 1)}{(\Gamma_q(1 + k))^n} \prod_{l=0}^{n_0-1} \frac{\Gamma_q(a + b + 1 + kl) \Gamma_q(1 + k(l + 1))}{\Gamma_q(a + 1 + kl) \Gamma_q(b + 1 + kl)} \\
&\quad \times \prod_{j=0}^{n_1-1} \frac{\Gamma_q((k + 1)j + a + b + kn_0 + 1) \Gamma_q((k + 1)(j + 1) + kn_0)}{\Gamma_q((k + 1)j + a + kn_0 + 1) \Gamma_q((k + 1)j + b + kn_0 + 1)}.
\end{aligned}$$

Within [3] Baker and Forrester were able to prove Conjecture 3.5.5 for the cases $a = k$ and $n_1 = 2$. In a related work [6] they also proved the case where $a = b = 0$. In both cases a combinatorial identity of Bressaud and Goulden [11] is used. Following this Hamada [34] confirmed the general cases $n_1 = 2$ and $n_1 = 3$ using a q -integration formula of Macdonald polynomials and Gessel [28] showed the conjecture to be true for $n_1 = 2, n - 1, 3$ and also for the cases where $n \leq 5$.

We now state Baker and Forrester's other q -constant term identity.

Conjecture 3.5.6. [3, Conj 2.2] Define $D_p(n_1, \dots, n_p; n_0; a, b, k; q)$ by

$$\begin{aligned}
D_p(n_1, \dots, n_p; n_0; a, b, k; q) &:= \text{CT} \left(\prod_{\alpha=1}^p \prod_{\substack{\min(\tilde{J}_\alpha) \leq i < \\ j \leq \max(\tilde{J}_\alpha)}} (z_i - q^{k+1} z_j)(z_i^{-1} - q^k z_j^{-1}) \right. \\
&\quad \left. \times \prod_{i < j} \left(\frac{z_i}{z_j}; q \right)_k \left(q \frac{z_j}{z_i}; q \right)_k \prod_{i=1}^n (z_i; q)_a \left(\frac{q}{z_i}; q \right)_b \right),
\end{aligned}$$

where \tilde{J}_α is given by (3.32). Then for $n_p > n_j$, $j = 1, \dots, p - 1$, we have

$$\begin{aligned}
\frac{D_p(n_1, \dots, n_{p-1}, n_p + 1; n_0; a, b, k; q)}{D_p(n_1, \dots, n_{p-1}, n_p; n_0; a, b, k; q)} &= \frac{[n_p + 1]_{q^{k+1}}}{[k]_q!} \\
&\quad \times \frac{\Gamma_q((k + 1)n_p + a + b + k \sum_{j=0}^{p-1} n_j + 1) \Gamma_q((k + 1)(n_p + 1) + k \sum_{j=0}^{p-1} n_j)}{\Gamma_q((k + 1)n_p + a + k \sum_{j=0}^{p-1} n_j + 1) \Gamma_q((k + 1)n_p + b + k \sum_{j=0}^{p-1} n_j + 1)}.
\end{aligned} \tag{3.60}$$

This more general constant term identity has not been verified in any cases beyond what is contained within Conjecture 3.5.5.

We begin by using the further generalisation of Kadell's result, Lemma 3.5.7 below, to reclaim the result for the $a = b = 0$ case of Conjecture 3.5.5 proved by Baker and Forrester

in [6]. Unlike the methods in [6] we are able to extend ours to prove the more general case of Conjecture 3.5.6 with $a = b = 0$.

Lemma 3.5.7. *Let $J = \{r, \dots, r+s\} \subseteq \{1, \dots, n\}$ and $h(z)$ be antisymmetric with respect to z_j , $j \in J$ then*

$$\text{CT} \left(\prod_{r \leq i < j \leq r+s} (z_i - az_j) h(z) \right) = \frac{[s]_a!}{s!} \text{CT} \left(\prod_{r \leq i < j \leq r+s} (z_i - z_j) h(z) \right), \quad (3.61)$$

(cf. (3.23)).

Proof. Let $S_J = \langle s_j : j \in \{r, \dots, r+s-1\} \rangle$. For any permutation $\sigma \in S_J$, the operation $z \rightarrow \sigma z$ leaves the constant term on the left hand side of (3.61) unchanged. Hence

$$\text{CT} \left(\prod_{r \leq i < j \leq r+s} (z_i - az_j) h(z) \right) = \text{CT} \left(\prod_{r \leq i < j \leq r+s} (z_{\sigma(i)} - az_{\sigma(j)}) h(\sigma z) \right).$$

Since $h(z)$ is antisymmetric $h(\sigma z) = (-1)^{l(\sigma)} h(z)$, summing over all permutations gives

$$\begin{aligned} \text{CT} \left(\prod_{r \leq i < j \leq r+s} (z_i - az_j) h(z) \right) \\ = \frac{1}{s!} \text{CT} \left(\left(\sum_{\sigma \in S_J} (-1)^{l(\sigma)} \prod_{r \leq i < j \leq r+s} (z_{\sigma(i)} - az_{\sigma(j)}) \right) h(z) \right). \end{aligned}$$

Since

$$\prod_{1 \leq i < j \leq n} (z_i - az_j) = \sum_{\omega \in S_n} (-a)^{l(\omega)} z^{\omega\delta}$$

we can write

$$\sum_{\sigma \in S_J} (-1)^{l(\sigma)} \prod_{r \leq i < j \leq r+s} (z_{\sigma(i)} - az_{\sigma(j)}) = \sum_{\sigma \in S_J} (-1)^{l(\sigma)} \sigma \sum_{\omega \in S_J} (-a)^{l(\omega)} z^{\omega\delta}. \quad (3.62)$$

Letting $\omega = \sigma^{-1}$ we see that the coefficient of z^δ in (3.62) is $\sum_{\sigma \in S_J} a^{l(\sigma)}$. The result follows from the identity (3.21) \square

As previously stated the following result is due to Baker and Forrester [6], however here an alternative method is used.

Theorem 3.5.8. *We have*

$$D_1(n_1; n_0; 0, 0, k; q) = \frac{\Gamma_{q^{k+1}}(n_1 + 1)}{(\Gamma_q(1 + k))^n} \prod_{j=0}^{n_1-1} \frac{\Gamma_q((k+1)(j+1) + kn_0)}{\Gamma_q((k+1)j + kn_0 + 1)} \\ \times \prod_{l=0}^{n_0-1} \frac{\Gamma_q(1 + k(l+1))}{\Gamma_q(1 + kl)}. \quad (3.63)$$

Proof. Let $(I, J) = (\emptyset, J^{n_0, \{n_1\}})$ and $\eta^* = \eta_{(0_{n_0}; n_1)}^*$. By Theorem 3.4.3 we have

$$S_{\eta^*}^{(I, J)}(z) = \Delta_t^{J^{n_0, \{n_1\}}}(z),$$

and hence

$$\left\langle S_{\eta^*}^{(I, J)}(z), S_{\eta^*}^{(I, J)}(z) \right\rangle_{q, q^t} = \text{CT} \left(\prod_{n_0+1 \leq i < j \leq n} (z_i - q^{-k} z_j)(z_i^{-1} - q^k z_j^{-1}) \right. \\ \left. \times \prod_{1 \leq i < j \leq n} \begin{pmatrix} z_i \\ z_j \end{pmatrix}_k \begin{pmatrix} q \frac{z_j}{z_i} \\ q \end{pmatrix}_k \right). \quad (3.64)$$

To apply Lemma 3.5.7 we view (3.64) as

$$\text{CT} \left(\prod_{n_0+1 \leq i < j \leq n} (z_i - q^{-k} z_j) h(z) \right)$$

where $h(z)$ is antisymmetric with respect to z_{n_0+1}, \dots, z_n . Using Lemma 3.5.7 twice we see that

$$D_1(n_1; n_0; 0, 0, k; q) = \frac{[n_1]_{q^{k+1}}!}{[n_1]_{q^{-k}}!} \left\langle S_{\eta^*}^{(I, J)}(z), S_{\eta^*}^{(I, J)}(z) \right\rangle_{q, q^t}.$$

From (1.51) we have

$$\frac{[x]_{q^k}!}{[x]_{q^{-k}}!} = q^{\frac{k(x-1)x}{2}}. \quad (3.65)$$

Using this and (3.59) it follows that

$$D_1(n_1; n_0; 0, 0, k; q) = \frac{[n_1]_{q^{k+1}}! [n_1 + nk]_q! (1 - q)^{n_1}}{[k]_q!^n (q^{n_1(k+1) + n_0 k}; q^{-(k+1)})_{n_1}}. \quad (3.66)$$

To show (3.66) is equivalent to (3.63) we firstly use the identity [46] $\Gamma_q(1 + n) = [n]_q!$ to show

$$\frac{[n_1]_{q^{k+1}}!}{[k]_q!^n} = \frac{\Gamma_{q^{k+1}}(n_1 + 1)}{(\Gamma_q(1 + k))^n}$$

and

$$\prod_{l=0}^{n_0-1} \frac{\Gamma_q(1 + k(l+1))}{\Gamma_q(1 + kl)} = [n_0 k]_q!.$$

Lastly,

$$\frac{[n_1 + nk]_q! (1 - q)^{n_1}}{(q^{n_1(k+1)+n_0k}; q^{-(k+1)})_{n_1}} = [n_0k]_q! \prod_{j=0}^{n_1-1} \frac{[(k+1)j + kn_0 + k]_q!}{[(k+1)j + kn_0]_q!},$$

is seen to be true by expanding both sides and comparing terms. \square

Our final result in this section is proving the specialisation $a = b = 0$ of Conjecture 3.5.6, which until now has not been done. We do this by finding the analogue of Theorem 3.5.8 for $\eta_{(n_0, N_p)}^*$ and stating the result as a corollary. We begin with a generalisation of Lemma 3.5.7.

Lemma 3.5.9. *Let $J \subseteq \{1, \dots, n\} = J_1 \cup \dots \cup J_s$ and $h(z)$ be antisymmetric with respect to z_j , $j \in J$ then*

$$\text{CT} \left(\prod_{\alpha=1}^s \prod_{\substack{\min(\tilde{J}_\alpha) \leq i \\ < j \leq \max(\tilde{J}_\alpha)}} (z_i - az_j) h(z) \right) = \prod_{\alpha=1}^s \frac{[|\tilde{J}_\alpha|]_a!}{|\tilde{J}_\alpha|!} \text{CT} \left(\prod_{\alpha=1}^s \prod_{\substack{\min(\tilde{J}_\alpha) \leq i \\ < j \leq \max(\tilde{J}_\alpha)}} (z_i - z_j) h(z) \right).$$

Theorem 3.5.10. *With N_p as defined by (3.34) we have*

$$\begin{aligned} D_p(N_p; n_0; 0, 0, k; q) &= \prod_{\alpha=1}^p \left([n_\alpha]_{q^{k+1}}! \right) \frac{[kn + \max(N_p)]_q!}{[k]_q!^n} \\ &\times \frac{(1 - q)^{\max(N_p)}}{\prod_{j=1}^{\max(N_p)} (1 - q^j q^{k(n - \tilde{m}(j))})}, \end{aligned} \quad (3.67)$$

where $\tilde{m}(j)$ is specified just below (3.56).

Proof. With $(I, J) = (\emptyset, J^{n_0, N_p})$ and $\eta^* = \eta_{(0, n_0; N_p)}^*$, Theorem 3.4.3 implies

$$S_{\eta^*}^{(I, J)}(z) = \Delta_t^{J^{n_0, N_p}}(z)$$

and hence the inner product $\langle S_{\eta^*}^{(I, J)}(z), S_{\eta^*}^{(I, J)}(z) \rangle_{q, q^k}$ can be written as

$$\begin{aligned} \langle S_{\eta^*}^{(I, J)}(z), S_{\eta^*}^{(I, J)}(z) \rangle_{q, q^k} &= \text{CT} \left(\prod_{\alpha=1}^p \prod_{\substack{\min(\tilde{J}_\alpha) \leq i < \\ j \leq \max(\tilde{J}_\alpha)}} (z_i - q^{-k} z_j) (z_i^{-1} - q^k z_j^{-1}) \right. \\ &\quad \left. \times \prod_{1 \leq i < j \leq n} \begin{pmatrix} z_i \\ z_j \end{pmatrix}_k \begin{pmatrix} q \frac{z_j}{z_i} \\ q \end{pmatrix}_k \right). \end{aligned}$$

Applying Lemma 3.5.9 twice to (3.56) and again using (3.65) the result is obtained as in Theorem 3.5.8. \square

The $a = b = 0$ case of Conjecture 3.5.6 is verified by substituting (3.67) into the left hand side of (3.60) and making the obvious simplifications.

3.6 Further Work

The theory in this section leads most naturally to an investigation of interpolation polynomials with prescribed symmetry. Since the operator H_i (2.36) satisfies the Hecke algebra (1.12) and plays the same role in interpolation theory as T_i does in nonsymmetric Macdonald polynomial theory, it seems natural to define the symmetrising and antisymmetrising operators for the interpolation polynomials as

$$U_*^+ := \sum_{\sigma \in S_n} H_\sigma, \quad U_*^- := \sum_{\sigma \in S_n} \left(-\frac{1}{t}\right)^{l(\sigma)} H_\sigma,$$

where H_σ is as in (1.5). Mirroring the work in Section 3.2 we use these operators to build the prescribed symmetry operator $O_{I,J}^*$. Explicitly we have

$$O_{I,J}^* := \sum_{\omega \in W_{I \cup J}} \left(-\frac{1}{t}\right)^{l(\omega)} H_\omega.$$

From this set up one is able derive the interpolation polynomial analogues of Proposition 3.2.1 and Proposition 3.2.2. Trial computations also suggest that it may be possible to extend Proposition 3.3.1, Conjecture 3.4.1 and Theorem 3.4.3 to interpolation polynomial theory. Trial computations suggest that the antisymmetric interpolation polynomial $S_{\kappa+\delta}^*(z; q, t)$ and the symmetric interpolation polynomial $P_\kappa^*(z; q, t)$ are closely related by a Vandermonde-type product, however it is not immediately obvious how to transform the parameters in each polynomial. In Section 5 we give further details of these formulas. Such relationships are also evident between the prescribed symmetry interpolation polynomials labelled by $\eta^* = \eta_{(n_0; N_p)}$ or $\eta^* = \eta_{(N_p, n_0)}$ and the corresponding symmetric interpolation polynomial $P_\kappa^*(z_1, \dots, z_{n_0})$ or $P_\kappa^*(z_{n-n_0-1}, \dots, z_n)$.

Chapter 4

Pieri-Type Formulas for Nonsymmetric Macdonald Polynomials

The Pieri formulas give the branching coefficients for the product of the r^{th} elementary symmetric function $e_r(z)$ and a symmetric polynomial. The original Pieri formula considered the product between $e_r(z)$ and $s_\kappa(z)$; more recently the Pieri formulas for the symmetric Macdonald polynomials were obtained. Until now little work has been done on the nonsymmetric analogues of the Pieri formulas. In this chapter we consider the product of $e_r(z)$ and $E_\eta(z; q^{-1}, t^{-1})$ and the closely related generalised binomial coefficients $\binom{\lambda}{\eta}_{q,t}$ (2.65).

The chapter begins with a brief history on the Pieri formulas, stating the known results and applications of the theory. The Mathematical study commences with determining which coefficients in the expansion will be non-zero. The evaluation of the Pieri-type coefficients for the nonsymmetric Macdonald polynomials is separated into two studies. The first considers the expansion of $e_1(z)E_\eta(z; q^{-1}, t^{-1})$ and the second the general Pieri-type formulas for $e_r(z)E_\eta(z; q^{-1}, t^{-1})$. Although both studies depend heavily on the theory of interpolation polynomials the overall approaches are quite different.

The first study is carried out by first determining the coefficients in the expansion of $z_i E_\eta(z; q^{-1}, t^{-1})$ (Proposition 4.3.7). This result is used to compute the branching coefficients for $e_1(z)E_\eta(z; q^{-1}, t^{-1})$ (Proposition 4.3.8). Upon obtaining the Pieri-type coefficients in the case $r = 1$ we are able to provide an evaluation of the generalised binomial coefficients $\binom{\lambda}{\eta}_{q,t}$ in the case $|\lambda| = |\eta| + 1$ (Proposition 4.3.13) and the Pieri-type coefficients in the expansion of $e_{n-1}(z)E_\eta(z; q^{-1}, t^{-1})$ (Propositions 4.3.17 and 4.3.15). We

conclude the study with a discussion of an alternative derivation of the coefficients in the expansion of $e_1(z)E_\eta(z; q^{-1}, t^{-1})$.

The second study begins with another alternative derivation of the Pieri-type coefficients for the case $r = 1$ (Proposition 4.4.1 and Corollary 4.4.2). The significance of this alternative method is its ability to be generalised to any value of r . We provide the structure of the general Pieri-type formulas (Theorem 4.4.4) and explicit formulas for the coefficients (Proposition 4.4.6). As in the previous study we use the formulas for the Pieri-type coefficients to evaluate the generalised binomial coefficients $\binom{\lambda}{\eta}_{q,t}$ where $|\lambda| = |\eta| + r$ (Corollary 4.4.5). The study concludes with a discussion of the considered simplifications of the formulas.

4.1 A Brief History

The Pieri formulas were first considered by Italian geometer Mario Pieri around the early 1900's and provided a formula for expressing the product of a Schur polynomial and a complete symmetric function $h_r(z)$ (1.7) in terms of Schur polynomials of higher degree. Over time the Pieri formulas have evolved to consider the product of polynomials with the elementary symmetric function $e_r(z)$. The Pieri formula for Schur functions, see for example [54], states

$$e_r(z)s_\kappa(z) = \sum_{\mu: \mu/\kappa \text{ is a vertical } r \text{ strip}} s_\mu(z).$$

This was later generalised to the Pieri formula for symmetric Macdonald polynomials, which can be expressed as [54, Section VI. 6]

$$e_r(z)P_\kappa(z; q, t) = \sum_{\mu: \mu/\kappa \text{ is a vertical } r \text{ strip}} \psi_{\mu/\kappa}(q, t)P_\mu(z; q, t) \quad (4.1)$$

where $\psi_{\mu/\kappa}(q, t)$ is given by

$$\psi_{\mu/\kappa}(q, t) := t^{l(\mu)-l(\kappa)} \frac{P_\kappa(t^\delta)}{P_\mu(t^\delta)} \prod_{1 \leq i < j \leq n} \frac{1 - q^{\kappa_i - \kappa_j} t^{j-i+\theta_i - \theta_j}}{1 - q^{\kappa_i - \kappa_j} t^{j-i}}, \quad (4.2)$$

and $\theta = \mu - \kappa$. The evaluation $P_\kappa(t^\delta)$ is given explicitly by (3.47).

The Pieri formulas found application in studies of certain vanishing properties of Macdonald polynomials at $t^{k+1}q^{r-1} = 1$ [21]. The dual of (4.1) has also found application in the study of certain probabilistic models related to the Robinson-Schensted-Knuth correspondence [26]. In 2002 Marshall [58] derived the Pieri-type coefficients in the expansion

of $e_1(z)E_\eta(z; \alpha)$ finding the nonsymmetric Jack analogue of (4.1) in the case $r = 1$. Marshall's result was reclaimed by Schützer [71] in 2004 using combinatorial methods. A further alternate derivation of Marshall's result and the Pieri-type coefficients in the expansion of $e_{n-1}(z)E_\eta(z; \alpha)$ can be found in [24]. Most recently Pieri type formulas have been obtained for the generalised Hall-Littlewood function [77].

4.2 Preliminaries

4.2.1 General Pieri-type formulas

We set up our study of Pieri-type formulas for the nonsymmetric Macdonald polynomials by considering a more general situation.

If we take a linear basis of polynomials $\{G_\eta(z)\}$ of degree $|\eta|$, satisfying

$$G_\eta(\bar{\lambda}) = 0, \quad \text{for all } |\lambda| \leq |\eta|, \quad \lambda \neq \eta \quad (4.3)$$

and a polynomial $f_r(z)$ of degree r such that $f_r(\bar{\eta}) = 0$ we have

$$f_r(z)G_\eta(z) = \sum_{|\eta|+1 \leq |\lambda| \leq |\eta|+r} \hat{A}_{\eta\lambda} G_\lambda(z),$$

for some coefficients $\hat{A}_{\eta\lambda}$. Examples of polynomials satisfying (4.3) are the interpolation Macdonald polynomials (Section 2.2), the Schubert polynomials [50] and the interpolation Jack polynomials (Appendix D), with the appropriate meaning of $\bar{\lambda}$.

We seek the most simplified expressions for the coefficients in the expansion for the case where $f_r(z) = (e_r(z) - e_r(\bar{\eta}))$ and $G_\eta(z) = E_\eta^*(z; q, t)$. Following the strategies of [58] we then use the fact that the top homogeneous component of $E_\eta^*(z; q, t)$ is $E_\eta(z; q^{-1}, t^{-1})$ to conclude that the coefficients $A_{\eta,\lambda}^{(r)}(q, t)$ in

$$(e_r(z) - e_r(\bar{\eta}))E_\eta^*(z; q, t) = \sum_{|\eta|+1 \leq |\lambda| \leq |\eta|+r} A_{\eta\lambda}^{(r)}(q, t)E_\lambda^*(z; q, t) \quad (4.4)$$

are the same as those in

$$e_r(z)E_\eta(z; q^{-1}, t^{-1}) = \sum_{\lambda: |\lambda|=|\eta|+r} A_{\eta\lambda}^{(r)}(q, t)E_\lambda(z; q^{-1}, t^{-1}). \quad (4.5)$$

It is the relationship between (4.4) and (4.5) that allows us to use the theories of both the nonsymmetric Macdonald polynomials and the interpolation Macdonald polynomials to derive and simplify the coefficients $A_{\eta\lambda}^{(r)}(q, t)$.

4.2.2 Structure of Pieri-type expansions

We commence the derivation of the Pieri-type formulas for the nonsymmetric Macdonald polynomials by determining the necessary conditions for the coefficients (4.5) to be non-zero.

We begin using the interpolation polynomials to restrict the summation in both (4.4) and (4.5). The extra vanishing conditions of the interpolation polynomials (Proposition 2.2.19) states that $E_{\eta}^*(\bar{\lambda}; q, t) = 0$ if $\eta \not\leq' \lambda$ allowing us to write

$$(e_r(z) - e_r(\bar{\eta}))E_{\eta}^*(z; q, t) = \sum_{\substack{|\eta|+1 \leq |\lambda| \leq |\eta|+r \\ \eta \leq' \lambda}} A_{\eta\lambda}^{(r)}(q, t)E_{\lambda}^*(z; q, t) \quad (4.6)$$

and

$$e_r(z)E_{\eta}(z; q^{-1}, t^{-1}) = \sum_{\substack{|\lambda|=|\eta|+r \\ \eta \leq' \lambda}} A_{\eta\lambda}^{(r)}(q, t)E_{\lambda}(z; q^{-1}, t^{-1})$$

respectively. Following the methods of Forrester and McAnally [24] we exploit the orthogonality of the nonsymmetric Macdonald polynomials to identify further zero coefficients in the expansions.

Proposition 4.2.1. *We have*

$$A_{\eta\lambda}^{(r)}(q, t) = A_{\lambda, \eta+(1^n)}^{(n-r)}(q^{-1}, t^{-1}) \frac{\mathcal{N}_{\eta}(q, t)}{\mathcal{N}_{\lambda}(q, t)}, \quad (4.7)$$

where $\mathcal{N}_{\eta}(q, t)$ and $\eta + (k^n)$ are given by (2.22) and (1.38) respectively.

Proof. With λ such that $|\lambda| = |\eta| + r$ and $\eta \leq' \lambda$ by (4.5) and the orthogonality and linearity properties of $\langle \cdot, \cdot \rangle_{q, t}$ we have

$$\begin{aligned} & \langle e_r(z)E_{\eta}(z; q^{-1}, t^{-1}), E_{\lambda}(z; q^{-1}, t^{-1}) \rangle_{q, t} \\ &= A_{\eta\lambda}^{(r)}(q, t) \langle E_{\lambda}(z; q^{-1}, t^{-1}), E_{\lambda}(z; q^{-1}, t^{-1}) \rangle_{q, t}. \end{aligned} \quad (4.8)$$

Using (1.26) we can write the left hand side of (4.8) as

$$\text{CT}(e_r(z)E_{\eta}(z; q^{-1}, t^{-1})E_{\lambda}(z^{-1}; q, t)W(z)). \quad (4.9)$$

Replacing z with z^{-1} and then multiplying $e_r(z^{-1})$ by $(z_1 \dots z_n)$ and $E_\eta(z^{-1}; q^{-1}, t^{-1})$ by $(z_1 \dots z_n)^{-1}$ allows us to write (4.9) as

$$\text{CT}(e_{n-r}(z)E_{\eta+(1^n)}(z^{-1}; q^{-1}, t^{-1})E_\lambda(z; q, t)W(z^{-1})),$$

where (2.12) has been used to simplify $E_\eta(z^{-1}; q^{-1}, t^{-1})$. By again interchanging z and z^{-1} we recognise this as

$$\langle E_{\eta+(1^n)}(z; q^{-1}, t^{-1}), e_{n-r}(z)E_\lambda(z; q^{-1}, t^{-1}) \rangle_{q,t}.$$

After further minor manipulation, (4.8) shows this reduces to

$$A_{\lambda, \eta+(1^n)}^{(n-r)}(q^{-1}, t^{-1}) \langle E_{\eta+(1^n)}(z; q^{-1}, t^{-1}), E_{\eta+(1^n)}(z; q^{-1}, t^{-1}) \rangle_{q,t}.$$

Since $\mathcal{N}_\eta(q, t) = \mathcal{N}_\eta(q^{-1}, t^{-1})$ and $\mathcal{N}_\eta(q, t) = \mathcal{N}_{\eta+(1^n)}(q, t)$ it follows that (4.7) is true. \square

Corollary 4.2.2. *We have*

$$A_{\eta\lambda}^{(r)}(q, t) = 0 \quad \text{if } \eta \not\leq' \lambda \text{ or } \lambda \not\leq' \eta + (1^n),$$

and therefore

$$e_r(z)E_\eta(z; q^{-1}, t^{-1}) = \sum_{\substack{|\lambda|=|\eta|+r \\ \eta \leq' \lambda \leq' \eta+(1^n)}} A_{\eta\lambda}^{(r)}(q, t)E_\lambda(z; q^{-1}, t^{-1}). \quad (4.10)$$

In [24] Forrester and McAnally gave further structure to the λ in (4.10), showing that compositions λ satisfying $\eta \leq' \lambda \leq' \eta + (1^n)$ and $|\lambda| = |\eta| + r$ are characterised by the properties that there are sets $\{i_1, \dots, i_r\} \subset \{1, \dots, n\}$ and $\{j_1, \dots, j_{n-r}\} = \{1, \dots, n\} \setminus \{i_1, \dots, i_r\}$ such that

$$\begin{aligned} \lambda_{\sigma(i_p)} &= \eta_{i_p} + 1, & \text{for } i_p \leq \sigma(i_p), & \quad p = 1, \dots, r, \\ \lambda_{\sigma(j_p)} &= \eta_{j_p}, & \text{for } j_p \geq \sigma(j_p), & \quad j = 1, \dots, n-r, \end{aligned} \quad (4.11)$$

for some defining permutation σ . By Corollary 1.4.5 a defining permutation of $\eta \leq' \lambda$ is the inverse of a defining permutation of $\lambda \leq' \eta + (1^n)$. Therefore if σ is a defining permutation of $\eta \leq' \lambda$ we have

$$\eta_i < \lambda_{\sigma(i)}, \quad \text{if } i < \sigma(i), \text{ and } \eta_i \leq \lambda_{\sigma(i)}, \quad \text{if } i \geq \sigma(i),$$

and

$$\lambda_{\sigma(i)} < \eta_i + 1, \quad \text{if } \sigma(i) < i, \quad \text{and } \lambda_{\sigma(i)} \leq \eta_i + 1, \quad \text{if } \sigma(i) \geq i.$$

Combining these together and implementing the requirement $|\lambda| = |\eta| + r$ gives (4.11).

We now work towards finding the explicit formulas for the coefficients $A_{\eta\lambda}^{(r)}(q, t)$ in the case $r = 1$.

4.3 Pieri-type Formulas for $r = 1$

In this section we provide the explicit branching coefficients for the Pieri-type formulas for the nonsymmetric Macdonald polynomials in the case $r = 1$. We do this by first considering the branching coefficients for the product $z_i E_\eta(z; q^{-1}, t^{-1})$ in terms of higher order nonsymmetric Macdonald polynomials.

That such branching formulas can be derived is suggested by Jack polynomial theory. Marshall [58] derived the branching coefficients for the products $z_i E_\eta(z; \alpha)$ and $e_1(z) E_\eta(z; \alpha)$ following a strategy of Knop and Sahi [45], which proceeds by exploiting the theory of interpolation Jack polynomials.

Similarly to the Jack case, the interpolation polynomials, introduced in Section 2.2, play a key role in deriving the branching coefficients for the product $z_i E_\eta(z; q^{-1}, t^{-1})$.

The overall strategy for finding the branching coefficients is to use the mapping Ψ (2.41) between $E_\eta(z; q^{-1}, t^{-1})$ and $E_\eta^*(z; q, t)$ to intertwine the actions of multiplication by z_i on $E_\eta(z; q^{-1}, t^{-1})$ and a certain operator Z_i on $E_\eta^*(z; q, t)$. Hence, by first determining an explicit form for the coefficients $\tilde{A}_{\eta\lambda}^{\{i\}}(q, t)$ in the expansion

$$Z_i E_\eta^*(z; q, t) = \sum_{\lambda} \tilde{A}_{\eta\lambda}^{\{i\}}(q, t) E_\lambda^*(z; q^{-1}, t^{-1})$$

we can apply the mapping Ψ to obtain an explicit form of the coefficients of $z_i E_\eta(z; q^{-1}, t^{-1})$ in terms of the $E_\lambda(z; q^{-1}, t^{-1})$ (Section 4.3.3). Using this result we can derive the explicit formula for the expansion of $e_1(z) E_\eta(z; q^{-1}, t^{-1})$ (Section 4.3.4). As a check on our result we prove that the classical limit reclaims Marshall's analogous result for nonsymmetric Jack polynomials (Section 4.3.5). As a corollary of obtaining the Pieri-type coefficients for the case $r = 1$ we evaluate the generalised binomial coefficients $\binom{\lambda}{\eta}_{q,t}$ in the case $|\lambda| = |\eta| + 1$ (Section 4.3.6) and provide two methods for finding the Pieri-type formulas for the case $r = n - 1$ (Section 4.3.7); one using the identity (2.13) and the other using orthogonality properties.

A recent manuscript of Lascoux [49], available only on his website, contains results equivalent to our propositions 4.3.7 and 4.3.8. The polynomials used in his study differ from ours by a change of indeterminate. We conclude our study on Pieri-type formulas for $r = 1$ by presenting Lascoux's derivation (Section 4.3.8).

Before deriving the explicit formulas for the Pieri-type formulas for $r = 1$ we give more structure to the sets restricting the summation in (4.10).

4.3.1 Non-zero coefficients

We denote the set of λ in the summation from (4.10), in the case $r = 1$, by $\mathbb{J}_{\eta,1}^\lambda$, that is

$$\mathbb{J}_{\eta,1}^\lambda := \{\lambda : |\lambda| = |\eta| + 1, \eta \preceq' \lambda \preceq \eta + (1^n)\}. \quad (4.12)$$

The following proposition shows how we can write $\mathbb{J}_{\eta,1}^\lambda$ in terms of $\lambda = c_I(\eta)$ by introducing a restriction on the sets $I \subseteq \{1, \dots, n\}$ that ensures we never have $c_I(\eta) = c_J(\eta)$ for $I \neq J$.

Proposition 4.3.1. [44] Let $I = \{t_1, \dots, t_s\}$ with $1 \leq t_1 < \dots < t_s \leq n$ and $I \neq \emptyset$. We call I maximal with respect to η iff

- (1) $\eta_j \neq \eta_{t_u}, \quad j = t_{u-1} + 1, \dots, t_u - 1$ ($u = 1, \dots, s; t_0 := 0$);
- (2) $\eta_j \neq \eta_{t_1} + 1, \quad j = t_s + 1, \dots, n$.

We have

$$\mathbb{J}_{\eta,1}^\lambda = \{\lambda : \lambda = c_I(\eta), I \text{ maximal with respect to } \eta\}.$$

Proof. By Lemma 1.4.2 we know that any λ such that $\eta \preceq' \lambda$ can be expressed as $c_I(\eta)$ for some set I . It is straight forward to show that $c_I(\eta) \preceq' \eta + (1^n)$ by taking the defining permutation to be the inverse of that of $\eta \preceq' \lambda; \hat{\sigma}$. It follows from (1.53) that restricting the sets to those maximal with respect to η ensures $c_I(\eta) \neq c_J(\eta)$ for $I \neq J$. \square

For example with $\eta = (1, 1, 1)$ the sets $\{1\}$, $\{1, 2\}$ and $\{1, 2, 3\}$ are maximal and all other subsets of $\{1, 2, 3\}$ are not.

We denote the maximal sets by

$$\mathbb{J}_{\eta,1}^I := \{I : I \text{ maximal with respect to } \eta\}.$$

4.3.2 The intertwining formula

In order to use the theory of the interpolation polynomials to assist in our derivation of the branching coefficients in the product of $z_i E_\eta(z; q^{-1}, t^{-1})$ we must first introduce the intertwining formula equation (4.14) below. This is due to Knop [44], however in the following an alternative proof is given. The corollary after this proposition leads us nicely to the beginnings of the derivation of the first Pieri-type formula.

Proposition 4.3.2. [44, Theorem 5.1] Define

$$Z_i := t^{\binom{n}{2}} (z_i \Xi_i - 1) \Xi_1 \dots \widehat{\Xi}_i \dots \Xi_n, \quad (4.13)$$

where the hat superscript on $\widehat{\Xi}_i$ denotes the absence of Ξ_i in the product of operators $\prod_{j=1}^n \Xi_j$. With Ψ as defined in (2.41) and M an operator which acts on the subspace of homogeneous polynomials of degree d by multiplication with $q^{-\binom{d}{2}}$ we have

$$Z_i \Psi M = \Psi M z_i. \quad (4.14)$$

Proof. First consider the action of Z_i on $E_\eta^*(z; q, t)$. By the commutativity of the Ξ_i we have

$$Z_i E_\eta^*(z; q, t) = (z_i - \Xi_i^{-1}) t^{-\binom{n}{2}} \Xi_1 \dots \Xi_n E_\eta^*(z; q, t). \quad (4.15)$$

Using (2.39), (2.10), then the identity $\Sigma_i l'_\eta(i) = \binom{n}{2}$ we can simplify (4.15) to

$$q^{-|\eta|} (z_i - \bar{\eta}_i) E_\eta^*(z; q, t). \quad (4.16)$$

Since (4.16) vanishes for all $z = \bar{\lambda}$, $|\lambda| \leq |\eta|$ and has degree $|\eta| + 1$ we must have

$$Z_i E_\eta^*(z; q, t) = q^{-|\eta|} \sum_{\lambda: |\lambda|=|\eta|+1} \tilde{A}_{\eta\lambda}^{\{i\}}(q, t) E_\lambda^*(z; q, t), \quad (4.17)$$

for some coefficients $\tilde{A}_{\eta\lambda}^{\{i\}}(q, t) \in \mathbb{Q}(q, t)$. Equating the leading terms of (4.16) and the right hand side of (4.17) using Theorem 2.2.8 gives

$$z_i E_\eta(z; q^{-1}, t^{-1}) = \sum_{\lambda: |\lambda|=|\eta|+1} \tilde{A}_{\eta\lambda}^{\{i\}}(q, t) E_\lambda(z; q^{-1}, t^{-1}). \quad (4.18)$$

Applying the action of ΨM to both sides of (4.18) and using (2.41) shows

$$\Psi M z_i E_\eta(z; q^{-1}, t^{-1}) = q^{-\binom{|\eta|+1}{2}} \sum_{\lambda: |\lambda|=|\eta|+1} \tilde{A}_{\eta\lambda}^{\{i\}}(q, t) E_\lambda^*(z; q, t). \quad (4.19)$$

Using (4.17) and the identity

$$\binom{x+1}{2} - x = \binom{x}{2}$$

the right hand side of (4.19) can be simplified to

$$q^{-\binom{|\eta|}{2}} Z_i E_\eta^*(z; q, t). \quad (4.20)$$

By recalling the action of M and again using (2.41) we obtain

$$\Psi M z_i E_\eta(z; q^{-1}, t^{-1}) = Z_i \Psi M E_\eta(z; q^{-1}, t^{-1}).$$

Finally, since the $\{E_\eta(z; q^{-1}, t^{-1})\}$ form a basis for analytic functions in $\{z^\eta\}$ it follows that the intertwining property (4.14) holds generally. \square

Corollary 4.3.3. *We have*

$$z_i E_\eta(z; q^{-1}, t^{-1}) = q^{|\eta|} \Psi^{-1} Z_i E_\eta^*(z; q, t).$$

Proof. Follows from (4.19) and (4.20). \square

4.3.3 The product $z_i E_\eta$

The previous corollary indicates that the next step towards finding the decomposition of $z_i E_\eta(z; q^{-1}, t^{-1})$ is to determine an explicit formula for $Z_i E_\eta^*(z; q, t)$. The latter can be deduced as a corollary of the following lemma, specifying the expansion of $(z_i \Xi_i - 1)f(z)$, where according to (2.35) $z_i \Xi_i - 1 := H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1}$.

Lemma 4.3.4. *Let $\tilde{Z}_i = H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1}$. The action of \tilde{Z}_i on $f(z)$ is given by*

$$\tilde{Z}_i f(z) = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ i \in I}} r_I^{\{i\}}(z) f(Iz).$$

Here the rational function $r_I^{\{i\}}(z)$ can be expressed as

$$r_I^{\{i\}}(z) = \chi_I^{\{i\}}(z) A_I(z) B_I(z), \quad (4.21)$$

where

$$I = \{t_1, \dots, t_s\}, 1 \leq t_1 < \dots < t_s \leq n, \quad (4.22)$$

$$A_I(z) = \widehat{a}\left(\frac{z_{t_s}}{q}, z_{t_1}\right) \prod_{u=1}^{s-1} \widehat{a}(z_{t_u}, z_{t_{u+1}}), \quad (4.23)$$

$$B_I(z) = (z_{t_s} - t^{-n+1}) \prod_{j=1}^{t_1-1} \widehat{b}\left(\frac{z_{t_s}}{q}, z_j\right) \\ \times \prod_{u=1}^s \prod_{j=t_u+1}^{t_{u+1}-1} \widehat{b}(z_{t_u}, z_j), \quad t_{s+1} := n+1, \quad (4.24)$$

$$\chi_I^{\{i\}}(z) = \begin{cases} \frac{1}{\widehat{a}(z_{t_{k-1}}, z_i)}, & i = t_k, k = 2, \dots, s \\ \frac{1}{\widehat{a}\left(\frac{z_{t_s}}{q}, z_i\right)}, & i = t_1, \end{cases} \quad (4.25)$$

and Iz is defined as

$$(Iz)_i = \begin{cases} z_{t_{u-1}}, & i = t_u, \text{ if } u = 2, \dots, s \\ z_{t_s}/q, & i = t_1 \\ z_i, & i \notin I. \end{cases} \quad (4.26)$$

The quantities $\widehat{a}(x, y)$ and $\widehat{b}(x, y)$ are defined in (4.27) below.

Proof. Using (2.36) the action of H_i on $f(z)$ can be expressed as

$$H_i f(z) = \widehat{a}(z_i, z_{i+1})f(z) + \widehat{b}(z_i, z_{i+1})s_i f(z),$$

where

$$\widehat{a}(x, y) := \frac{(t-1)x}{x-y}, \quad \widehat{b}(x, y) := \frac{x-ty}{x-y}. \quad (4.27)$$

Hence \widetilde{Z}_i can be written as

$$(\widehat{a}(z_i, z_{i+1}) + \widehat{b}(z_i, z_{i+1})s_i) \dots (\widehat{a}(z_{n-1}, z_n) + \widehat{b}(z_{n-1}, z_n)s_{n-1}) \\ \times \Phi_q^*(\widehat{a}(z_1, z_2) + \widehat{b}(z_1, z_2)s_1) \dots (\widehat{a}(z_{i-1}, z_i) + \widehat{b}(z_{i-1}, z_i)s_{i-1}). \quad (4.28)$$

Let

$$K_I^{\{i\}} := s_i \dots \widehat{s}_{t_{r+1}-1} \dots \widehat{s}_{t_s-1} \dots s_{n-1} \Delta s_1 \dots \widehat{s}_{t_1} \dots \widehat{s}_{t_r-1} \dots s_{i-1}, \text{ for } i \in I,$$

where $1 \leq t_1 < \dots < t_r = i < t_{r+1} < \dots < t_s \leq n$, the hat superscript denotes the absence of the corresponding operators and I as defined in the statement of the result. It is clear that the expansion of \tilde{Z}_i will be of the form

$$\tilde{Z}_i = \sum_{\substack{I \subseteq \{1, \dots, n\} \\ i \in I}} r_I^{\{i\}}(z) K_I^{\{i\}}$$

for coefficients $r_I^{\{i\}}(z)$ involving $\hat{a}(x, y)$ and $\hat{b}(x, y)$. Further, it is easily verified that $K_I^{\{i\}} f(z) = f(Iz)$. The coefficients $r_I^{\{i\}}(z)$ are found by considering the individual terms in the expansion of (4.28). Due to the need to commute the transposition operators s_i through to the right the final formula is more simply obtained by expanding (4.28) termwise from the right. Inevitably, the exercise is rather tedious, however it can be structured somewhat by considering four disjoint classes of sets I

$$\begin{aligned} I_1 &= \{i\}, \\ I_2 &= \{\dots, i\}, \\ I_3 &= \{i, \dots\}, \\ I_4 &= \{\dots, i, \dots\}, \end{aligned}$$

which exhaust all possibilities. This cataloguing allows the coefficients of the corresponding four forms of $K_I^{\{i\}}$ to be considered separately and the result is more easily observed. Explicitly, the four forms of $K_I^{\{i\}}$ are

$$\begin{aligned} K_{I_1}^{\{i\}} &= s_i \dots s_{n-1} \Delta s_1 \dots s_{i-1}, \\ K_{I_2}^{\{i\}} &= s_i \dots s_{n-1} \Delta s_1 \dots \hat{s}_{t_1} \dots \hat{s}_{t_{r-1}} \dots s_{i-1}, \\ K_{I_3}^{\{i\}} &= s_i \dots \hat{s}_{t_{r+1}-1} \dots \hat{s}_{t_s-1} \dots s_{n-1} \Delta s_1 \dots s_{i-1}, \\ K_{I_4}^{\{i\}} &= s_i \dots \hat{s}_{t_{r+1}-1} \dots \hat{s}_{t_s-1} \dots s_{n-1} \Delta s_1 \dots \hat{s}_{t_1} \dots \hat{s}_{t_{r-1}} \dots s_{i-1}. \end{aligned}$$

In relation to K_{I_2} , K_{I_4} the coefficient of $s_1 \dots \hat{s}_{t_1} \dots \hat{s}_{t_{r-1}} \dots s_{i-1}$ in the partial expansion, that is terms to the right of Φ_q^* , in (4.28) is

$$\hat{a}(z_1, z_{t_1+1}) \prod_{u=1}^{r-2} \hat{a}(z_{t_u+1}, z_{t_{u+1}+1}) \prod_{j=1}^{t_1-1} \hat{b}(z_1, z_{j+1}) \prod_{u=1}^{r-1} \prod_{j=t_u+1}^{t_{u+1}-1} \hat{b}(z_{t_u+1}, z_{j+1}).$$

Hence the coefficient of $\Delta s_1 \dots \widehat{s}_{t_1} \dots \widehat{s}_{t_{r-1}} \dots s_{i-1}$ will be

$$(z_n - t^{-n+1}) \widehat{a}\left(\frac{z_n}{q}, z_{t_1}\right) \prod_{u=1}^{r-2} \widehat{a}(z_{t_u}, z_{t_{u+1}}) \times \prod_{j=1}^{t_1-1} \widehat{b}\left(\frac{z_n}{q}, z_j\right) \prod_{u=1}^{r-1} \prod_{j=t_u+1}^{t_{u+1}-1} \widehat{b}(z_{t_u}, z_j).$$

Similarly, for K_{I_1}, K_{I_3} , the coefficient of $s_1 \dots s_{i-1}$ and $\Delta s_1 \dots s_{i-1}$ are respectively

$$\prod_{j=1}^{i-1} \widehat{b}(z_1, z_{j+1})$$

and

$$(z_n - t^{-n+1}) \prod_{j=1}^{i-1} \widehat{b}\left(\frac{z_n}{q}, z_j\right).$$

The final coefficients $r_{I_j}(z)$ are found by continuing the expansion of (4.28) from the right and considering the four forms of K_I separately. Thus we find that

$$\begin{aligned} r_{I_1}^{\{i\}}(z) &= \frac{A_I(z)B_I(z)}{\widehat{a}\left(\frac{z_i}{q}, z_i\right)} \\ r_{I_2}^{\{i\}}(z) &= \frac{A_I(z)B_I(z)}{\widehat{a}(z_{t_{r-1}}, z_{t_r})} \\ r_{I_3}^{\{i\}}(z) &= \frac{A_I(z)B_I(z)}{\widehat{a}\left(\frac{z_{t_s}}{q}, z_{t_1}\right)} \\ r_{I_4}^{\{i\}}(z) &= \frac{A_I(z)B_I(z)}{\widehat{a}(z_{t_{r-1}}, z_{t_r})}, \end{aligned}$$

where $A_I(z)$ and $B_I(z)$ are defined by (4.23) and (4.24), respectively. After recalling the definition of $\chi_I^{\{i\}}(z)$ given above, the sought explicit formula (4.21) follows. \square

Corollary 4.3.5. *We have*

$$Z_i E_\eta^*(z; q, t) = q^{-|\eta|} \bar{\eta}_i \sum_{\substack{I \subseteq \{1, \dots, n\} \\ i \in I}} r_I^{\{i\}}(z) E_\eta^*(Iz; q, t). \quad (4.29)$$

Proof. Follows after recalling (4.13). \square

Together Proposition 4.3.2 and Corollary 4.3.5 allow us to derive an initial expansion of $z_i E_\eta(z; q^{-1}, t^{-1})$ in terms of the Macdonald polynomials of degree $|\eta| + 1$.

Proposition 4.3.6. *We have*

$$z_i E_\eta(z; q^{-1}, t^{-1}) = \bar{\eta}_i q^{-|\eta|} \sum_{|\lambda|=|\eta|+1} \sum_{\substack{I \subseteq \{1, \dots, n\} \\ i \in I}} \frac{r_I^{\{i\}}(\bar{\lambda}) E_\eta^*(I\bar{\lambda}; q, t)}{E_\lambda^*(\bar{\lambda}; q, t)} E_\lambda(z; q^{-1}, t^{-1}). \quad (4.30)$$

Proof. By the vanishing properties of the interpolation polynomials, when the right hand sides of (4.17) and (4.29) are equated and evaluated at $z = \bar{\lambda}$ we obtain

$$\tilde{A}_{\lambda\eta}^{\{i\}}(q, t) = \bar{\eta}_i \sum_{\substack{I \subseteq \{1, \dots, n\} \\ i \in I}} \frac{r_I^{\{i\}}(\bar{\lambda}) E_{\eta}^*(I\bar{\lambda}; q, t)}{E_{\lambda}^*(\bar{\lambda}; q, t)}.$$

Substituting this back into (4.17) and applying Corollary 4.3.3 gives (4.30). \square

The formula (4.30) can be improved as follows.

Proposition 4.3.7. *Let*

$$\tilde{B}_I(z) := (qz_{t_1} - t^{-n+1}) \prod_{j=t_s+1}^n \hat{b}(qz_{t_1}, z_j) \prod_{u=1}^s \prod_{j=t_{u-1}+1}^{t_u-1} \hat{b}(z_{t_u}, z_j), \quad (4.31)$$

$t_0 := 0$ and

$$\tilde{\chi}_I^{\{i\}}(z) := \begin{cases} \frac{z_i}{\bar{a}(z_i, z_{t_{k+1}})}, & i = t_k, k = 1, \dots, s-1 \\ \frac{z_i}{\bar{a}(z_i, qz_{t_1})}, & i = t_s, \end{cases}$$

where $I = \{t_1, \dots, t_s\} \subseteq \{1, \dots, n\}$, with $1 \leq t_1 < \dots < t_s \leq n$ and $I \neq \emptyset$. We have

$$z_i E_{\eta}(z; q^{-1}, t^{-1}) = \sum_{\substack{I \in \mathbb{J}_{\eta}^I, i \in I \\ c_I(\eta) = \lambda}} \frac{\tilde{\chi}_I^{\{i\}}(\bar{\eta}) A_I(\bar{\eta}) \tilde{B}_I(\bar{\eta}) k_{\eta}}{k_{c_I(\eta)}} E_{c_I(\eta)}(z; q^{-1}, t^{-1}). \quad (4.32)$$

Proof. By the definition of Iz (4.26) we have $E^*(I\bar{\lambda}) \neq 0$ if and only if $c_I(\eta) = \lambda$. Therefore by Proposition 4.3.1 we can restrict the summation in (4.30) to $I \in \mathbb{J}_{\eta}^I$, giving

$$z_i E_{\eta}(z; q^{-1}, t^{-1}) = \bar{\eta}_i \sum_{\substack{I \in \mathbb{J}_{\eta}^I, i \in I \\ c_I(\eta) = \lambda}} \frac{r_I^{\{i\}}(\overline{c_I(\eta)}) E_{\eta}^*(\bar{\eta})}{E_{c_I(\eta)}^*(\overline{c_I(\eta)})} E_{c_I(\eta)}(z; q^{-1}, t^{-1}). \quad (4.33)$$

It can be seen that for I maximal with respect to η we have $\tilde{\chi}_I^{\{i\}}(\bar{\eta}) = \bar{\eta}_i \chi_I^{\{i\}}(\overline{c_I(\eta)})$ and $B_I(\overline{c_I(\eta)}) = \tilde{B}_I(\bar{\eta})$. Since $A_I(\overline{c_I(\eta)}) = A_I(\bar{\eta})$, it follows that

$$\bar{\eta}_i r_I^{\{i\}}(\overline{c_I(\eta)}) = \tilde{\chi}_I^{\{i\}}(\bar{\eta}) A_I(\bar{\eta}) \tilde{B}_I(\bar{\eta}). \quad (4.34)$$

By substituting (4.34) into (4.33) and making use of the evaluation formula (2.55) we arrive at our final decomposition (4.32). \square

4.3.4 The main result

Using Proposition 4.3.7 we now provide an explicit formula for the coefficients $A_{\eta\lambda}^{(1)}(q, t)$ in (4.5).

Proposition 4.3.8. *We have*

$$e_1(z)E_\eta(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\eta^I} A_{\eta, c_I(\eta)}^{(1)}(q, t) E_{c_I(\eta)}(z; q^{-1}, t^{-1}), \quad (4.35)$$

where $A_{\eta, c_I(\eta)}^{(1)}(q, t)$ is defined by

$$A_{\eta, c_I(\eta)}^{(1)}(q, t) := \frac{(1-q)d'_\eta(q^{-1}, t^{-1})A_I(\bar{\eta})\tilde{B}_I(\bar{\eta})}{q^{n_{t_1}+1}(t-1)d'_{c_I(\eta)}(q^{-1}, t^{-1})}. \quad (4.36)$$

Proof. Summing (4.32) over all i and then reversing the order of summation gives

$$e_1(z)E_\eta(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\eta^I} \sum_{i \in I} \frac{\tilde{\chi}_I^{\{i\}}(\bar{\eta})A_I(\bar{\eta})\tilde{B}_I(\bar{\eta})k_\eta}{k_{c_I(\eta)}} E_{c_I(\eta)}(z; q^{-1}, t^{-1}). \quad (4.37)$$

We have

$$\sum_{i \in I} \tilde{\chi}_I^{\{i\}}(\bar{\eta}) = \frac{\bar{\eta}_{t_1}(1-q)}{(t-1)} \quad (4.38)$$

and analogous to (2.62) we have

$$\frac{k_\eta}{k_{c_I(\eta)}} = \frac{d'_\eta(q^{-1}, t^{-1})}{q^{2n_{t_1}+1}t^{-l'_\eta(t_1)}d'_{c_I(\eta)}(q^{-1}, t^{-1})}. \quad (4.39)$$

Substituting (4.38) and (4.39) into (4.37) gives the required result. \square

4.3.5 The classical limit

We provide a check on our formula by showing how the classical limit of Proposition 4.3.8 reclaims the Pieri-type formula for the nonsymmetric Jack polynomial [58].

Proposition 4.3.9. *We have*

$$e_1(z)E_\eta(z; \alpha) = \sum_{I \in \mathbb{J}_\eta^I} A_{\eta, c_I(\eta)}^{(1)}(\alpha) E_{c_I(\eta)}(z; \alpha),$$

where

$$A_{\eta, c_I(\eta)}^{(1)}(\alpha) = \frac{-\alpha d'_\eta(\alpha)A_{\alpha, I}(\bar{\eta}^\alpha/\alpha)\tilde{B}_{\alpha, I}(\bar{\eta}^\alpha/\alpha)}{d'_{c_I(\eta)}(\alpha)}. \quad (4.40)$$

The quantities in (4.40) are specified by

$$A_{\alpha, I}(z) := a(z_{t_s} - 1, z_{t_1}) \prod_{u=1}^{s-1} a(z_{t_u}, z_{t_{u+1}}) \quad (4.41)$$

$$\begin{aligned} \tilde{B}_{\alpha, I}(z) &:= (z_{t_1} + 1 + (n-1)/\alpha) \prod_{j=t_s+1}^n b(z_{t_1} + 1, z_j) \\ &\times \prod_{u=1}^s \prod_{j=t_{u-1}+1}^{t_u-1} b(z_{t_u}, z_j), \quad t_0 := 0 \end{aligned} \quad (4.42)$$

with

$$a(x, y) := \frac{1}{\alpha(x-y)}, \quad b(x, y) := \frac{x-y-1/\alpha}{x-y}$$

and

$$d'_\eta(\alpha) := \prod_{(i,j) \in \eta} (\alpha(a_\eta(i, j) + 1) + l_\eta(i, j)), \quad (4.43)$$

where $a_\eta(i, j)$ and $l_\eta(i, j)$ are defined by (1.6) and (1.40) and I by (4.22). The eigenvalue $\bar{\eta}^\alpha$ is defined in Appendix C.

Proof. Our task is to show that

$$\lim_{t=q^{1/\alpha}, q \rightarrow 1} A_{\eta, c_I(\eta)}^{(1)}(q, t) = A_{\eta, c_I(\eta)}^{(1)}(\alpha). \quad (4.44)$$

Comparing (1.41) with (4.43), it is immediate that

$$\lim_{t=q^{1/\alpha}, q \rightarrow 1} \frac{(1-q)d'_\eta(q^{-1}, t^{-1})}{q^{n_{t_1}+1}(t-1)d'_{c_I(\eta)}(q^{-1}, t^{-1})} = -\alpha \frac{d'_\eta(\alpha)}{d'_{c_I(\eta)}(\alpha)}.$$

To proceed further, note from (4.27) and (4.3.9) that

$$\lim_{t=q^{1/\alpha}, q \rightarrow 1} \widehat{a}(q^m t^n, q^{m'} t^{n'}) = a(m+n/\alpha, m'+n'/\alpha)$$

and

$$\lim_{t=q^{1/\alpha}, q \rightarrow 1} \widehat{b}(q^m t^n, q^{m'} t^{n'}) = b(m+n/\alpha, m'+n'/\alpha).$$

Using a term-by-term comparison of (4.23) and (4.31) with (4.41) and (4.42) also allows us to conclude that

$$\lim_{t=q^{1/\alpha}, q \rightarrow 1} A_I(\bar{\eta}) = A_{\alpha, I}(\bar{\eta}/\alpha)$$

and

$$\lim_{t=q^{1/\alpha}, q \rightarrow 1} \tilde{B}_I(\bar{\eta}) = \tilde{B}_{\alpha, I}(\bar{\eta}/\alpha).$$

This establishes (4.44), thus exhibiting Proposition 4.3.9 as a corollary of Proposition 4.3.8. \square

We now show how Proposition 4.3.8 can be used to obtain an evaluation for the generalised binomial coefficient $\binom{\lambda}{\eta}_{q,t}$.

4.3.6 Pieri-type formula for $r = 1$ and the generalised binomial coefficient

The following Lemma, due to Lascoux [49], shows how the Pieri-type coefficients $A_{\eta\lambda}^{(1)}(q, t)$ relate to the generalised binomial coefficients $\binom{\lambda}{\eta}_{q,t}$ (2.65).

Lemma 4.3.10. [49] We have

$$(z_1 + \dots + z_n - |\bar{\eta}|)E_{\eta}^*(z; q, t) = \sum_{\substack{\lambda: |\lambda|=|\eta|+1 \\ \lambda=c_I(\eta)}} \frac{(|\bar{\lambda}| - |\bar{\eta}|)E_{\eta}^*(\bar{\lambda}; q, t)}{E_{\lambda}^*(\bar{\lambda}; q, t)} E_{\lambda}^*(z; q, t), \quad (4.45)$$

where $|\bar{\eta}| = \sum_i \bar{\eta}_i$.

Proof. The left hand side of (4.45) is a polynomial of degree $|\eta| + 1$, and vanishes for every \bar{v} with $|v| \leq |\eta|$, where the linear factor implies vanishing at $\bar{\eta}$. The right hand side therefore belongs to the span of $E_{\lambda}^*(z)$ with $|\lambda| = |\eta| + 1$. The coefficients are found by evaluating both sides at $\bar{\lambda}$. Using Proposition 2.2.18 we restrict the summation to successors of η . \square

Corollary 4.3.11. We have

$$(z_1 + \dots + z_n)E_{\eta}(z; q^{-1}, t^{-1}) = \sum_{\substack{\lambda: |\lambda|=|\eta|+1 \\ \lambda=c_I(\eta)}} \binom{\lambda}{\eta}_{q,t} \frac{(|\bar{\lambda}| - |\bar{\eta}|)k_{\eta}}{k_{\lambda}} E_{\lambda}(z; q^{-1}, t^{-1}). \quad (4.46)$$

Proof. By Proposition 2.2.17 and (2.65) we have

$$\frac{E_{\eta}^*(\bar{\lambda}; q, t)}{E_{\lambda}^*(\bar{\lambda}; q, t)} = \binom{\lambda}{\eta}_{q,t} \frac{k_{\eta}}{k_{\lambda}}.$$

The final result is obtained by employing Theorem 2.2.8. \square

Alternatively, Baker and Forrester [7] use the following generating function to relate (4.35) to the generalised binomial coefficients.

$$E_{\eta}(z; q^{-1}, t^{-1}) \prod_{i=1}^n \frac{1}{(z_i; q)_{\infty}} = \sum_{\lambda} \binom{\lambda}{\eta}_{q,t} t^{l(\lambda)-l(\eta)} \frac{d'_{\eta}(q, t)}{d'_{\lambda}(q, t)} E_{\lambda}(z; q^{-1}, t^{-1}), \quad (4.47)$$

where $(z_i; q)_\infty$ is given by (A.3).

By equating the homogeneous terms of degree $|\eta| + 1$ in (4.47) we have

$$(z_1 + \dots + z_n)E_\eta(z; q^{-1}, t^{-1}) = \sum_{|\lambda|=|\eta|+1} \binom{\lambda}{\eta}_{q,t} \frac{t^{l(\lambda)-l(\eta)}(1-q)d'_\eta(q,t)}{d'_\lambda(q,t)} E_\lambda(z; q^{-1}, t^{-1}). \quad (4.48)$$

The detailed derivation of (4.47) and (4.48) would require theory not relevant to the main concepts contained in this thesis so we rely on (4.46) to avoid having to give such a derivation. However it is worthwhile to check that (4.46) and (4.48) are consistent. Note that by the vanishing conditions of the interpolation polynomials we have $\binom{\lambda}{\eta}_{q,t} = 0$ for $|\lambda| = |\eta| + 1$ if λ is not a successor of η and hence when proving that (4.46) and (4.48) are consistent we only need to consider the successors of η .

Proposition 4.3.12. *Let λ be a successor of η , then we have*

$$\frac{(|\bar{\lambda}| - |\bar{\eta}|)k_\eta}{k_\lambda} = \frac{t^{l(\lambda)-l(\eta)}(1-q)d'_\eta(q,t)}{d'_\lambda(q,t)}. \quad (4.49)$$

Proof. Using (2.55) and rearranging we rewrite (4.49) as

$$\frac{d'_\lambda(q,t)d'_\eta(q^{-1}, t^{-1})}{t^{l(\lambda)-l(\eta)}d'_\lambda(q^{-1}, t^{-1})d'_\eta(q,t)} = \frac{(1-q)\prod_{i=1}^n \bar{\lambda}_i^{\lambda_i}}{(|\bar{\lambda}| - |\bar{\eta}|)\prod_{i=1}^n \bar{\eta}_i^{\eta_i}}. \quad (4.50)$$

Using the second equation in (1.50) we can rewrite the left hand side of (4.50) as

$$\frac{1}{t^{l(\lambda)-l(\eta)}} (-1)^{|\lambda|-|\eta|} \frac{q^{\sum_{s \in \lambda} (a_\lambda(s)+1)} t^{l(\lambda)}}{q^{\sum_{s \in \eta} (a_\eta(s)+1)} t^{l(\eta)}} = -q^{\eta_{t_1}+1}.$$

We also have

$$|\bar{\lambda}| - |\bar{\eta}| = \overline{(c_I(\eta))_{t_s}} - \bar{\eta}_{t_1} = (q-1)\bar{\eta}_{t_1} \quad (4.51)$$

and

$$\frac{\prod_{i=1}^n \bar{\lambda}_i^{\lambda_i}}{\prod_{i=1}^n \bar{\eta}_i^{\eta_i}} = \bar{\eta}_{t_1} q^{\eta_{t_1}+1}.$$

Substituting these into the right hand side of (4.50) and appropriately cancelling completes the proof. \square

By (4.46) and (4.48), on obtaining the Pieri-type formula for $r = 1$ we are naturally lead to finding an explicit formula for $\binom{\lambda}{\eta}_{q,t}$ when $|\lambda| = |\eta| + 1$.

Proposition 4.3.13. *Suppose $|\lambda| = |\eta| + 1$. Then*

$$\binom{\lambda}{\eta}_{q,t} = -\frac{A_I(\bar{\eta})\tilde{B}_I(\bar{\eta})}{(t-1)},$$

where $\lambda = c_I(\eta)$. If there is no such I such that $\lambda = c_I(\eta)$ then $\binom{\lambda}{\eta}_{q,t} = 0$.

Proof. A comparison of (4.48) with (4.36) gives

$$\binom{\lambda}{\eta}_{q,t} = \frac{d'_\eta(q^{-1}, t^{-1})d'_\lambda(q, t)A_I(\bar{\eta})\tilde{B}_I(\bar{\eta})}{d'_\eta(q, t)d'_\lambda(q^{-1}, t^{-1})q^{n_1+1}(t-1)t^{l(\lambda)-l(\eta)}}. \quad (4.52)$$

The final result is obtained by appropriately using the second formula from (1.50) to reduce (4.52). \square

Comparing (2.65) with (4.3.13) and making use of (2.55) gives a new evaluation formula for $E_\eta^*(\bar{\lambda})$ where $|\lambda| = |\eta| + 1$.

Corollary 4.3.14. *Suppose $|\lambda| = |\eta| + 1$. Then*

$$E_\eta^*(\bar{\lambda}; q, t) = -\frac{A_I(\bar{\eta})\tilde{B}_I(\bar{\eta})k_\eta}{(t-1)}$$

where $\lambda = c_I(\eta)$. If there is no such I such that $\lambda = c_I(\eta)$ then $E_\eta^*(\bar{\lambda}; q, t) = 0$.

4.3.7 The Pieri-type formula for $r = n - 1$

In this section we provide two explicit formulas for the coefficients $A_{\eta\lambda}^{(n-1)}(q, t)$ in (4.5). For the first we manipulate $e_1(z)E_\eta(z; q^{-1}, t^{-1})$ using the identity (2.13).

Proposition 4.3.15. *With $\eta' := \eta - (\min(\eta)^n)$ we have*

$$e_{n-1}(z)E_\eta(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\lambda^I} A_{\lambda, c_I(\lambda)}^{(1)}(q^{-1}, t^{-1})E_{\nu+(\min(\eta)^n)}(z; q^{-1}, t^{-1}), \quad (4.53)$$

where $A_{\lambda, c_I(\lambda)}^{(1)}(q, t)$ is defined by (4.36),

$$\lambda := (-\eta' + (\max(\eta')^n))^R \text{ and } \nu := -c_I(\lambda)^R + ((\max(\lambda) + 1)^n). \quad (4.54)$$

Proof. By Proposition 4.3.8 we have

$$e_1(z)E_\lambda(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\lambda^I} A_{\lambda, c_I(\lambda)}^{(1)}(q, t)E_{c_I(\lambda)}(z; q^{-1}, t^{-1}).$$

Replacing z with z^{-1} and using (2.13) we obtain

$$e_1(z^{-1})E_{-\lambda R}(\underline{z}; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\lambda^I} A_{\lambda, c_I(\lambda)}^{(1)}(q, t) E_{-c_I(\lambda)R}(\underline{z}; q^{-1}, t^{-1}). \quad (4.55)$$

Since $e_1(z)$ is a symmetric function we can replace \underline{z} with z in (4.55) giving

$$e_1(z^{-1})E_{-\lambda R}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\lambda^I} A_{\lambda, c_I(\lambda)}^{(1)}(q, t) E_{-c_I(\lambda)R}(z; q^{-1}, t^{-1}).$$

Multiplying both sides by $z_1 \dots z_n$ and using (2.12) gives

$$e_{n-1}(z)E_{-\lambda R}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\lambda^I} A_{\lambda, c_I(\lambda)}^{(1)}(q, t) E_{-c_I(\lambda)R+(1^n)}(z; q^{-1}, t^{-1}). \quad (4.56)$$

Since $\lambda = (-\eta' + (\max(\eta')^n))^R$ we have $\eta' = -\lambda^R + (\max(\lambda)^n)$, and hence, by multiplying both sides of (4.56) by $(z_1 \dots z_n)^{\max(\lambda)}$ we obtain

$$e_{n-1}(z)E_{\eta'}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathbb{J}_\lambda^I} A_{\lambda, c_I(\lambda)}^{(1)}(q, t) E_\nu(z; q^{-1}, t^{-1}), \quad (4.57)$$

where ν is defined in (4.54). The final decomposition (4.53) is now obtained by multiplying both sides of (4.57) by $(z_1 \dots z_n)^{\min(\eta)}$. \square

Alternatively the coefficients could be determined using the results from Section 4.2. Such a derivation requires us to specify the set of compositions in the restricted summation of (4.10) in the case $r = n - 1$. As in the case $r = 1$ we introduce some notation to denote the set and consider a more precise specification of the set elements.

Proposition 4.3.16. [24] Let $I = \{t_1, \dots, t_s\}$ with $1 \leq t_1 < \dots < t_s \leq n$ and $I \neq \emptyset$. We call I alternatively maximal with respect to η iff

- (1) $\eta_j \neq \eta_{t_u}, \quad j = t_u + 1, \dots, t_{u+1} - 1 \quad (u = 1, \dots, s; t_{s+1} := n + 1);$
- (2) $\eta_j \neq \eta_{t_s} - 1, \quad j = 1, \dots, t_1 - 1.$

We have

$$\begin{aligned} \mathbb{J}_{\eta, n-1}^\lambda &:= \{\lambda : |\lambda| = |\eta| + n - 1, \eta \preceq' \lambda \preceq \eta + (1^n)\} \\ &= \{\lambda : \lambda = c_I^{-1}(\eta) + (1^n), I \text{ alternatively maximal } \eta\} \end{aligned}$$

Proof. We first observe $\eta \preceq' c_I(\eta)$ implies $c_I^{-1}(\eta) \preceq' \eta$, and hence $c_I^{-1}(\eta) + (1^n) \preceq' \eta + (1^n)$. By taking the inverse of the defining permutation of $c_I^{-1}(\eta) \preceq' \eta + (1^n)$ we obtain a defining permutation for $\eta \preceq' c_I(\eta) + (1^n)$. Therefore

$$\mathbb{J}_{\eta, n-1}^\lambda = \{\lambda : \lambda = c_I^{-1}(\eta) + (1^n), I \text{ maximal with respect to } c_I^{-1}(\eta)\}.$$

The final result is obtained from the definitions of maximal and alternatively maximal. \square

We denote the alternatively maximal sets by

$$\mathbb{J}_{\eta, n-1}^I := \{I : I \text{ alternatively maximal with respect to } \eta\}.$$

For simplicity we let the symbol $\widehat{c}_I(\eta)$ denote $c_I^{-1}(\eta) + (1^n)$ and note the explicit formula

$$(\widehat{c}_I(\eta))_j = \begin{cases} \eta_{t_s}, & j = t_1 \\ \eta_{t_{u-1}} + 1, & j = t_u, \text{ if } u = 2, \dots, s \\ \eta_j + 1, & j \notin I. \end{cases}$$

By the definitions of the compositions $c_I(\eta)$ and $\widehat{c}_I(\eta)$ it can be seen that they are related according to

$$\eta + (1^n) = c_I(\lambda), \text{ with } I \in \mathbb{J}_{\lambda, 1}^I \text{ iff } \lambda = \widehat{c}_I(\eta), \text{ with } I \in \mathbb{J}_{\eta, n-1}^I.$$

We can now show how Proposition 4.2.1 can be used to derive the Pieri-type coefficients in the case $r = n - 1$ from the case $r = 1$.

Proposition 4.3.17. *We have*

$$e_{n-1}(z)E_\eta(z; q, t) = \sum_{I \in \mathbb{J}_{\eta, n-1}^I} A_{\widehat{c}_I(\eta), \eta + (1^n)}^{(1)}(q, t) \frac{\mathcal{N}_\eta}{\mathcal{N}_{\widehat{c}_I(\eta)}} E_{\widehat{c}_I(\eta)}(z; q, t), \quad (4.58)$$

where $A_{\widehat{c}_I(\eta), \eta + (1^n)}^{(1)}(q, t) (\mathcal{N}_\eta / \mathcal{N}_{\widehat{c}_I(\eta)})$ can be rewritten to read

$$\frac{(1-q)A_I(\bar{\eta})B_I(\overline{\eta + (1^n)})(\bar{\eta}_{t_s}qt^n - 1)d_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})}{q^{\eta_{t_1}+1}(t-1)(\bar{\eta}_{t_s}qt^n - t)d_{\eta + (1^n)}(q^{-1}, t^{-1})}. \quad (4.59)$$

Proof. The initial form (4.58) follows immediately from Propositions 4.2.1 and 4.3.16. Using (4.36) and (2.22) we write the branching coefficients as

$$\frac{(1-q)d'_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})A_I(\overline{\widehat{c}_I(\eta)})\widetilde{B}_I(\overline{\widehat{c}_I(\eta)})}{q^{n_{t_1}+1}(t-1)d'_{\eta+(1^n)}(q^{-1}, t^{-1})} \frac{d'_\eta(q, t)e_\eta(q, t)}{d_\eta(q, t)e'_\eta(q, t)} \frac{d_{\widehat{c}_I(\eta)}(q, t)e'_{\widehat{c}_I(\eta)}(q, t)}{d'_{\widehat{c}_I(\eta)}(q, t)e_{\widehat{c}_I(\eta)}(q, t)}. \quad (4.60)$$

To reduce (4.60) to its simplest form we require the following 3 simplifications. First, by recalling $\mathcal{N}_\eta(q, t) = \mathcal{N}_\eta(q^{-1}, t^{-1}) = \mathcal{N}_{\eta+(1^n)}(q^{-1}, t^{-1})$ we can replace $\mathcal{N}_\eta(q, t)$ with $\mathcal{N}_{\eta+(1^n)}(q^{-1}, t^{-1})$ and $\mathcal{N}_{c_I(\eta)}(q, t)$ with $\mathcal{N}_{c_I(\eta)}(q^{-1}, t^{-1})$. After making the obvious cancellations we obtain

$$\frac{(1-q)A_I(\overline{\widehat{c}_I(\eta)})\widetilde{B}_I(\overline{\widehat{c}_I(\eta)})}{q^{n_{t_1}+1}(t-1)} \frac{d_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})e_{\eta+(1^n)}(q^{-1}, t^{-1})e'_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})}{d_{\eta+(1^n)}(q^{-1}, t^{-1})e_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})e'_{\eta+(1^n)}(q^{-1}, t^{-1})}.$$

Secondly, using Lemmas 1.3.1, 1.3.3, 1.3.2 and 1.3.4 we can simplify the ratios involving e_η and e'_η to

$$\frac{e'_{\eta+(1^n)}(q^{-1}, t^{-1})}{e'_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})} = 1 - (\overline{\eta}_{t_s} q t^{n-1})^{-1}$$

and

$$\frac{e_{\eta+(1^n)}(q^{-1}, t^{-1})}{e_{\widehat{c}_I(\eta)}(q^{-1}, t^{-1})} = 1 - (\overline{\eta}_{t_s} q t^n)^{-1}.$$

We recall from Proposition 4.3.7 that $B_I(\overline{c_I(\lambda)}) = \widetilde{B}_I(\overline{\lambda})$ and $A_I(\overline{c_I(\lambda)}) = A_I(\overline{\lambda})$. Lastly, replacing $A_I(\overline{\eta+(1^n)})$ with $A_I(\overline{\eta})$ gives (4.59). \square

Due to the different forms of expansions, $e_{n-1}(z)E_\eta(z; q^{-1}, t^{-1})$ and $e_{n-1}(z)E_\eta(z; q, t)$, and the different sets they are summed over, $I \in \mathbb{J}_{\nu, 1}^I$ and $I \in \mathbb{J}_{\eta, n-1}^I$ it is not practical to compare the two forms. Both derivations are included to display the different strengths of the theory derived so far.

4.3.8 Lascoux's derivation of the Pieri-type formula for $r = 1$

We conclude our initial study with an outline of the alternative derivation of the Pieri-type coefficients in the case $r = 1$ provided by Lascoux. Lascoux's result was obtained independently and posted on his website [49] just prior to the publication [9] containing Proposition 4.3.8. Lascoux's approach involves an interpolation polynomial slightly different from those introduced in Section 2.2. We begin with the introduction of the interpolation polynomials used by Lascoux.

Lascoux defines a new vector $\langle \eta \rangle := (\langle \eta \rangle_1, \dots, \langle \eta \rangle_n)$, where $\langle \eta \rangle_i$ is given by

$$\langle \eta \rangle_i := q^{\eta_i} t^{\eta'_i(i)},$$

(cf. (2.10)) and a new polynomial $M_\eta(z)$ satisfying $M_\eta(\langle \lambda \rangle) = 0$ for all λ where $|\lambda| \leq |\eta|$, $\lambda \neq \eta$, having leading term $z^\eta q^{-\Sigma_i \binom{\eta_i}{2}}$ (cf. (1.56)). It follows that

$$M_\eta(z) = q^{-\Sigma_i \binom{\eta_i}{2}} t^{(n-1)|\eta|} E_\eta^*(zt^{1-n}).$$

By writing

$$(z_1 + \dots + z_n - |\langle \eta \rangle|)M_\eta(z) = \sum_{\substack{\lambda: |\lambda|=|\eta|+1 \\ \lambda=c_I(\eta)}} A'_{\eta\lambda}{}^{(1)}(q, t)M_\lambda(z), \quad (4.61)$$

where $|\langle \eta \rangle| := \Sigma_i \langle \eta \rangle_i$, and evaluating at $\langle \lambda \rangle$ Lascoux expresses the coefficients $A'_{\eta\lambda}{}^{(1)}(q, t)$ as

$$A'_{\eta\lambda}{}^{(1)}(q, t) = \frac{(|\langle \lambda \rangle| - |\langle \eta \rangle|)M_\eta(\langle \lambda \rangle)}{M_\lambda(\langle \lambda \rangle)}.$$

Lascoux finds the explicit formula of $M_\eta(\langle \lambda \rangle)/M_\lambda(\langle \lambda \rangle)$ to be

$$\frac{M_\eta(\langle \lambda \rangle)}{M_\lambda(\langle \lambda \rangle)} = t^{1-n} \delta'(\eta, \lambda) \beta'(\eta, \lambda). \quad (4.62)$$

In (4.62) $\delta'(\eta, \lambda)$ is given by

$$\delta'(\eta, \lambda) := \frac{1}{1-t} \prod_i \frac{1-t}{\langle \lambda \rangle_i \langle \eta \rangle_i^{-1} - 1},$$

where the product is over all i such that $\langle \lambda \rangle_i \neq \langle \eta \rangle_i$ and $\beta'(\eta, \lambda)$ is defined to be

$$\beta'(\eta, \lambda) := \prod \frac{(X'(i, j) - t)(tX'(i, j) - 1)}{(X'(i, j) - 1)^2},$$

with the product over all $i < j$ such that $\lambda_i < \lambda_j$ and $\langle \lambda \rangle_i, \langle \lambda \rangle_j$ is a subword of $\langle \lambda \rangle$ but not of $\langle \Phi \eta \rangle$ and $X'(i, j) := \langle \lambda \rangle_j \langle \lambda \rangle_i^{-1}$.

Lascoux's proof of (4.62) uses the fact that each successor λ can be recursively generated from η according to $s_i, \dots, s_j \Phi \eta$, where s_i, \dots, s_j is a subword of one of $s_{n-1} \dots s_1, s_{n-2} \dots s_1 s_0, \dots, s_0 s_{n-1} \dots s_2$, where here s_0 acts on compositions and is defined by $s_0 := \Phi s_1 \Phi^{-1}$. He proves the equality inductively by showing the two sides are invariant

under the action of Φ_q^* and the constants appearing in (2.53) correspond to the variation of $\delta^L(\eta, \lambda)\beta^L(\eta, \lambda)$ under the action of H_1, \dots, H_{n-1} . Lascoux's use of the recursive generation of successors to prove the result inspired an alternative derivation of the evaluation of the ratio $E_\eta^*(\bar{\lambda})/E_\lambda^*(\bar{\lambda})$ and also gave hope to the possibility of deriving explicit formulas for the Pieri-type coefficients for general r . With that said, we move onto our second study.

4.4 The General Pieri-Type Formulas

In this section we provide the explicit branching coefficients for the general Pieri-type formula, the products $e_r(z)E_\eta(z; q^{-1}, t^{-1})$. As in the previous study the coefficients are obtained via the exploitation of the theory of the interpolation Macdonald polynomials.

We see from (4.45) and (4.61) that the key to providing explicit formulas for the coefficients $A_{\eta\lambda}^{(1)}(q, t)$ is evaluating

$$\frac{E_\eta^*(\bar{\lambda}; q, t)}{E_\lambda^*(\bar{\lambda}; q, t)}. \quad (4.63)$$

We begin this study with a further alternative derivation of the Pieri-type formulas for $r = 1$ by first evaluating (4.63) (Section 4.4.1). We then proceed with the general formulas and their connection to the generalised binomial coefficients $\binom{\lambda}{\eta}_{q,t}$ (Section 4.4.2). Following this we consider the possible simplifications of the coefficient formulas (Section 4.4.3) and then show how the results contained in Section 4.4.2 can be used to reclaim the Pieri formulas for the symmetric Macdonald polynomial (Section 4.4.4). We complete the chapter on Pieri-type formulas with suggestions of further work relating to the study (Section 4.5).

4.4.1 An alternative derivation of the Pieri-type formulas for $r = 1$

To derive an explicit formula for (4.62) Lascoux uses the fact that successors can be recursively generated by s_i and Φ , however he does not use the sequence specified by Proposition 1.4.1. Here we show how using this specific derivation it is possible to make use of the interpolation polynomials eigenoperator properties to derive an explicit form of (4.63).

We begin in a similar way to Lemma 4.3.4.

Proposition 4.4.1. *With $I = \{t_1, \dots, t_s\}$ maximal with respect to η , $\lambda = c_I(\eta)$ and $E_\eta^*(z) := E_\eta^*(z; q, t)$ we have*

$$\frac{E_\eta^*(\bar{\lambda})}{E_\lambda^*(\bar{\lambda})} = \frac{\delta(\eta, I)\beta(\eta, I)}{q^{\eta_{t_1}}(1-t)}, \quad (4.64)$$

where

$$\delta(\eta, I) := \prod_{u=1}^s \frac{t-1}{1-\bar{\lambda}_{t_u}\bar{\eta}_{t_u}^{-1}}$$

and

$$\beta(\eta, I) := \prod_i \frac{(1-tX(i))(t-X(i))}{(1-X(i))^2}, \quad (4.65)$$

with $X(i) := \bar{\eta}_i\bar{\eta}_{t_u(i)}^{-1}$, where $t_u(i)$ is the first element in I above i and if $i > t_s$ then $\bar{\eta}_{t_u(i)} = q\bar{\eta}_{t_1}$. The product (4.65) is over all $i \notin I$ with $\eta_i > \eta_{t_u(i)}$.

Proof. We first show that

$$H_{t_1} \dots H_{n-1} \Phi_q^* H_1 \dots H_{t_1-1} E_\eta^*(\bar{\lambda}) = \frac{\delta(\eta, I)\beta(\eta, I)(1-\bar{\lambda}_{t_1}\bar{\eta}_{t_1}^{-1})}{q^{\eta_{t_1}}(t-1)} E_\eta^*(\bar{\lambda}). \quad (4.66)$$

With I maximal with respect to η , Proposition 1.4.1 can be used to show the polynomial $E_\lambda^*(z)$ occurs exactly once in the expansion of

$$H_{t_1} \dots H_{n-1} \Phi_q^* H_1 \dots H_{t_1-1} E_\eta^*(z). \quad (4.67)$$

Since all polynomials $E_\nu^*(z)$ appearing in the full expansion of (4.67) are of size $|\eta| + 1$, by the vanishing properties of the interpolation polynomial, evaluating (4.67) at $\bar{\lambda}$ will reduce it to some multiple of $E_\lambda^*(\bar{\lambda})$.

We begin by expanding (4.67) from the right using the recursive generation formulas to determine the coefficient of each $E_{\lambda(j)}^*(z)$, where $\lambda(j)$ represents the transformed η after the j^{th} step in the transformation from η to λ . Since the operators H_i and Φ_q^* commute through constants we consider the coefficient contribution of each operator on the $E_{\lambda(j)}^*(z)$ to observe the result more easily. First consider the expansion of

$$H_1 \dots H_{t_1-1} E_\eta^*(z).$$

We know from Proposition 1.4.1 that we require the s_i to act on η at every stage to move η_{t_1} to the first position before acting upon by Φ_q^* . Therefore we must take the coefficient of $E_{s_i\lambda(t_1-i)}^*(z)$ when H_i acts on each $E_{\lambda(t_1-i)}^*(z)$. At each stage the switching operator swaps η_i with η_{t_1} , where i runs from $t_1 - 1$ to 1. By (2.53) the coefficient contribution will

be 1 if $\eta_i < \eta_{t_1}$, and

$$\frac{(1 - tX(i))(t - X(i))}{(1 - X(i))^2}$$

where $X(i) = \bar{\eta}_i \bar{\eta}_{t_1}^{-1}$ if $\eta_i > \eta_{t_1}$. Multiplying these terms together gives the coefficient of $E_{\lambda(t_1-1)}^*(z)$ where

$$\lambda(t_1 - 1) = (\eta_{t_1}, \eta_1, \dots, \eta_{t_1-1}, \eta_{t_1+1}, \dots, \eta_n).$$

Next we act upon $H_1 \dots H_{t_1-1} E_{\eta}^*(z)$ with Φ_q^* . By (2.54) when Φ_q^* acts on $E_{\lambda(t_1-1)}^*(z)$ the coefficient contribution is $q^{-\eta_{t_1}}$ and the new polynomial is $E_{\lambda(t_1)}^*(z)$ where

$$\lambda(t_1) = (\eta_1, \dots, \eta_{t_1-1}, \eta_{t_1+1}, \dots, \eta_n, \eta_{t_1} + 1).$$

We proceed by considering the coefficients of $E_{\lambda(j)}^*$ for $j > t_1$ in the expansion of

$$H_{t_1} \dots H_{n-1} E_{\lambda(t_1)}^*(z).$$

At this stage particular attention must be paid to the set I to know whether we want to extract the coefficient and polynomial of $E_{\lambda(t_1+j)}^*(z)$ or $E_{s_{n-j}\lambda(t_1+j)}^*(z)$ from the action of H_{n-j} on $E_{\lambda(t_1+j)}^*(z)$. First consider the action of H_{i-1} for $i \in I$, $i > t_1$. From Proposition 1.4.1 we know that if $i \in I$ we do not require the switch s_{i-1} in the generation of λ . Therefore, when H_{i-1} acts on $E_{\lambda(t_1+n-i+1)}^*(z)$, we take the coefficient of $E_{\lambda(t_1+n-i+1)}^*(z)$. By (2.53) this is given by

$$\frac{t-1}{1 - \delta_{i-1, \lambda(t_1+n-i+1)}^{-1}}.$$

To determine the value of $\delta_{i-1, \lambda(t_1+n-i+1)}^{-1}$ we consider the $(i-1)^{th}$ and i^{th} value of $\lambda(t_1+n-i+1)$. Since we do not need to swap the components we must have $\lambda(t_1+n-i+1)_i = \lambda_i$. Also, at this stage $\lambda(t_1+n-i+1)_{i-1}$ is equal to η_i since the $(i-1)^{th}$ component hasn't changed since $\lambda(t_1)$. Hence the coefficient of the polynomial $E_{\lambda(t_1+n-i+1)}^*(z)$ in $H_{i-1} E_{\lambda(t_1+n-i+1)}^*(z)$ with $i \in I$ is

$$\frac{t-1}{1 - \bar{\lambda}_i \bar{\eta}_i^{-1}}. \quad (4.68)$$

It is important to note here that $\bar{\lambda}_i \neq \bar{\eta}_i$ for $i \in I$ since even if $\lambda_i = \eta_i$ we would still have $\bar{\lambda}_i = \bar{\eta}_i/t$. The total contribution of these terms is the product of (4.68) as i runs from 2 to s .

Lastly we consider the case where $i \notin I$, where in the expansion of $H_{i-1} E_{\lambda(t_1+n-i+1)}^*(z)$ we take the coefficient of the $E_{s_{i-1}\lambda(t_1+n-i+1)}^*(z)$. For $i > t_s$ we use s_{i-1} to move $\eta_{t_1} + 1$ to the t_s^{th} position, each time swapping $\eta_{t_1} + 1$ with η_i . For $t_1 < i < t_s$ each s_{i-1} is used to

move $\eta_{t_u(i)}$ to the t_{u-1}^{th} position, swapping η_i and $\eta_{t_u(i)}$, where $t_u(i)$ is defined above. By (2.53) when either $\eta_i > \eta_{t_1} + 1$, for $j = t_s + 1, \dots, n$ or $\eta_i > \eta_{t_u(i)}$, for $t_1 < i < t_s$ we have the coefficient

$$\frac{(1 - tX(i))(t - X(i))}{(1 - X(i))^2}$$

where $X(i) = \bar{\eta}_i \bar{\eta}_{t_u(i)}^{-1}$. Combining all coefficients gives (4.66). By (2.35) and (2.39) we have

$$(z_{t_1} \bar{\eta}_{t_1}^{-1} - 1)E_\eta^*(z) = H_{t_1} \dots H_{n-1} \Phi_q^* H_1 \dots H_{t_1-1} E_\eta^*(z). \quad (4.69)$$

By (4.66) and the vanishing properties of the interpolation polynomials, evaluating (4.69) at $\bar{\lambda}$ gives (4.64). \square

Corollary 4.4.2. *We have*

$$e_1(z)E_\eta(z; q^{-1}, t^{-1}) = \sum_{\lambda: \lambda = c_I(\eta)} \frac{(|\bar{\lambda}| - |\bar{\eta}|)\delta(\eta, I)\beta(\eta, I)}{q^{\eta_{t_1}}(1-t)} E_\lambda(z; q^{-1}, t^{-1}). \quad (4.70)$$

Of course we would like to show that the coefficients found in (4.70) are consistent with (4.35).

Proposition 4.4.3. *We have*

$$\frac{(|\bar{\lambda}| - |\bar{\eta}|)\delta(\eta, I)\beta(\eta, I)}{q^{\eta_{t_1}}(1-t)} = \frac{(1-q)d'_\eta(q^{-1}, t^{-1})A_I(\bar{\eta})\tilde{B}_I(\bar{\eta})}{q^{\eta_{t_1}+1}(t-1)d'_\lambda(q^{-1}, t^{-1})}.$$

Proof. Firstly,

$$A_I(\bar{\eta}) = A_I(\overline{c_I(\eta)}) = \delta(\eta, I).$$

Following the methods of the previous proof using (1.47) and (1.49) we see that

$$\begin{aligned} \frac{d'_\eta(q^{-1}, t^{-1})}{d'_\lambda(q^{-1}, t^{-1})} &= (1 - q^{-\eta_{t_1}-1} t^{-n+1+l'_\eta(t_1)})^{-1} \\ &\times \prod_{\substack{i \notin I \\ \eta_i < \eta_{t_u(i)}}} \frac{\bar{\eta}_{t_u(i)} \bar{\eta}_i^{-1} - 1}{\bar{\eta}_{t_u(i)} \bar{\eta}_i^{-1} - t} \prod_{\substack{i \notin I \\ \eta_i > \eta_{t_u(i)}}} \frac{t \bar{\eta}_{t_u(i)} \bar{\eta}_i^{-1} - 1}{\bar{\eta}_{t_u(i)} \bar{\eta}_i^{-1} - 1}. \end{aligned}$$

By writing $\tilde{B}_I(\bar{\eta})$ more explicitly as

$$\tilde{B}_I(\bar{\eta}) = (t^{-l'_\eta(t_1)} q^{\eta_{t_1}+1} - t^{-n+1}) \prod_{i \notin I} \frac{\bar{\eta}_{t_u(i)} \bar{\eta}_i^{-1} - t}{\bar{\eta}_{t_u(i)} \bar{\eta}_i^{-1} - 1},$$

we see that

$$\frac{\widetilde{B}_I(\overline{\eta})d'_\eta(q^{-1}, t^{-1})}{d'_\lambda(q^{-1}, t^{-1})} = \frac{(t^{-l'_\eta(t_1)}q^{\eta t_1+1} - t^{-n+1})}{(1 - q^{-\eta t_1-1}t^{-n+1+l'_\eta(t_1)})}\beta(\eta, I).$$

Using these simplifications along with (4.51) and then observing that

$$\frac{(t^{-l'_\eta(t_1)}q^{\eta t_1+1} - t^{-n+1})}{q^{\eta t_1+1}(1 - q^{-\eta t_1-1}t^{-n+1+l'_\eta(t_1)})} = \frac{1}{t'^{\eta}(t_1)}$$

gives the desired result. \square

We now show how this procedure can be extended to determine the general Pieri-type coefficients.

4.4.2 The general Pieri-type formula coefficients

To determine explicit formulas for the $A_{\eta\lambda}^{(r)}(q, t) := A_{\eta\lambda}^{(r)}$ in (4.10) we once again return to the theory of the interpolation polynomials. We begin by rewriting (4.6) as

$$(e_r(z) - e_r(\overline{\eta}))E_\eta^*(z; q, t) = \sum_{i=1}^r \sum_{\eta \preceq' \lambda^i} A_{\eta\lambda^i}^{(r)} E_{\lambda^i}^*(z; q, t), \quad (4.71)$$

where we've introduced the notation λ^i to denote a composition of modulus $|\eta| + i$.

Since the sum in (4.71) is over compositions of varying modulus we cannot just evaluate at each $\overline{\lambda}$ to obtain the coefficient of $E_\lambda^*(z; q, t)$ like we did in the proof of Proposition 4.4.1. Here, the coefficients must be generated recursively beginning with λ^1 , that is λ such that $|\lambda| = |\eta| + 1$. The details are provided in the following theorem.

Theorem 4.4.4. *For $\eta \preceq' \lambda^i$ the coefficients $A_{\eta\lambda^i}^{(r)}$ in (4.71) are recursively generated as*

$$A_{\eta\lambda^1}^{(r)} = \frac{(e_r(\overline{\lambda^1}) - e_r(\overline{\eta}))E_\eta^*(\overline{\lambda^1})}{E_{\lambda^1}^*(\overline{\lambda^1})} \quad (4.72)$$

$$A_{\eta\lambda^2}^{(r)} = \frac{(e_r(\overline{\lambda^2}) - e_r(\overline{\eta}))E_\eta^*(\overline{\lambda^2})}{E_{\lambda^2}^*(\overline{\lambda^2})} - \sum_{\lambda^1: \eta \preceq' \lambda^1 \preceq' \lambda^2} A_{\eta\lambda^1}^{(r)} \frac{E_{\lambda^1}^*(\overline{\lambda^2})}{E_{\lambda^2}^*(\overline{\lambda^2})} \quad (4.73)$$

and in general

$$A_{\eta\lambda^i}^{(r)} = \frac{(e_r(\overline{\lambda^i}) - e_r(\overline{\eta}))E_\eta^*(\overline{\lambda^i})}{E_{\lambda^i}^*(\overline{\lambda^i})} - \sum_{k=1}^{i-1} \sum_{\lambda^k: \eta \preceq' \lambda^k \preceq' \lambda^i} A_{\eta\lambda^k}^{(r)} \frac{E_{\lambda^k}^*(\overline{\lambda^i})}{E_{\lambda^i}^*(\overline{\lambda^i})}. \quad (4.74)$$

If $\eta \not\preceq' \lambda^i$ we have $A_{\eta\lambda^i}^{(r)} = 0$.

Proof. We first consider the structure of the coefficients. By the vanishing properties of the interpolation polynomials evaluating (4.71) at $\overline{\lambda^1}$ gives (4.72). When we evaluate (4.71) at $\overline{\lambda^2}$ we obtain

$$(e_r(\overline{\lambda^2}) - e_r(\overline{\eta}))E_\eta^*(\overline{\lambda^2}) = A_{\eta\lambda^2}^{(r)}E_{\lambda^2}^*(\overline{\lambda^2}) + \sum_{\lambda^1: \eta \preceq' \lambda^1 \preceq \lambda^2} A_{\eta\lambda^1}^{(r)}E_{\lambda^1}^*(\overline{\lambda^2}), \quad (4.75)$$

since any $E_{\lambda^1}^*(z)$ such that $\lambda^1 \preceq \lambda^2$ will not vanish when evaluated at $\overline{\lambda^2}$. Rearranging (4.75) gives (4.73). The general coefficient formula (4.74) is derived using the same methods, recursively generating $A_{\eta\lambda^1}^{(r)}, \dots, A_{\eta\lambda^{i-1}}^{(r)}$ to determine $A_{\eta\lambda^i}^{(r)}$.

If $\eta \not\preceq' \lambda^i$ the leading term of (4.74) would be zero due to the vanishing properties of $E_\eta^*(z)$, likewise, all terms in the following sum would be zero as by Corollary 1.4.4 there would be no such λ^k satisfying $\eta \preceq' \lambda^k \preceq' \lambda^i$. Consequently if $\eta \not\preceq' \lambda^i$ it follows that $A_{\eta\lambda^i}^{(r)} = 0$. \square

Corollary 4.4.5. *The coefficients $A_{\eta\lambda^r}^{(r)} := A_{\eta\lambda^r}^{(r)}$ in (4.10), where $\eta \preceq' \lambda^r \preceq' \eta + (1^n)$, satisfy the recursion (4.74) with $i = r$, explicitly*

$$A_{\eta\lambda^r}^{(r)} = \frac{(e_r(\overline{\lambda^r}) - e_r(\overline{\eta}))E_\eta^*(\overline{\lambda^r})}{E_{\lambda^r}^*(\overline{\lambda^r})} - \sum_{k=1}^{r-1} \sum_{\lambda^k: \eta \preceq' \lambda^k \preceq' \lambda^r} A_{\eta\lambda^k}^{(r)} \frac{E_{\lambda^k}^*(\overline{\lambda^r})}{E_{\lambda^r}^*(\overline{\lambda^r})}. \quad (4.76)$$

To use this to obtain explicit formulas for the coefficients $A_{\eta\lambda^i}^{(r)}$ we require formulas for the evaluation of $E_{\lambda^i}^*(\overline{\lambda^j})$, where $j > i + 1$. The evaluations of $E_{\lambda^i}^*(\overline{\lambda^i})$ and $E_{\lambda^i}^*(\overline{\lambda^{i+1}})$ are specified in Propositions 2.2.17 and 4.4.1.

Proposition 4.4.6. *Let*

$$c(\eta, m) := \{\nu : \nu = c_I(\eta); I \text{ maximal with respect to } \eta \text{ and } t_1 \leq m\}$$

and $\lambda^i \preceq' \lambda^j$ with $j > i + 1$. We have

$$E_{\lambda^i}^*(\overline{\lambda^j}) = \sum_{\substack{\nu \in c(\lambda^i, k), \\ \nu \preceq' \lambda^j}} \frac{(\overline{\nu}_k \overline{\lambda}_k^{i-1} - 1)E_{\lambda^i}^*(\overline{\nu})}{(\overline{\lambda}_k^j \overline{\lambda}_k^{i-1} - 1)E_\nu^*(\overline{\nu})} E_\nu^*(\overline{\lambda^j}),$$

where k is the position of the leftmost component of λ^i that does not occur with the same frequency in λ^j .

Proof. We begin with $j = i + 2$. Manipulating Ξ_k (2.35) and acting on $E_{\lambda^i}^*(z)$ gives

$$\begin{aligned} (z_k \Xi_k - 1)E_{\lambda^i}^*(z) &= H_k \dots H_{n-1} \Phi_q^* H_1 \dots H_{k-1} E_{\lambda^i}^*(z) \\ &= \sum_{\nu \in c(\lambda^i, k)} c_{\eta\nu}^k E_{\nu}^*(z), \end{aligned} \quad (4.77)$$

where k is specified above. Evaluating at $\bar{\nu}$ for ν a particular composition in the sum shows

$$c_{\eta\nu}^k = (\bar{\nu}_k \bar{\lambda}_k^{i-1} - 1) \frac{E_{\lambda^i}^*(\bar{\nu})}{E_{\nu}^*(\bar{\nu})}.$$

Substituting back in (4.77) shows

$$E_{\lambda^i}^*(z) = \sum_{\nu \in c(\lambda^i, k)} \frac{(\bar{\nu}_k \bar{\lambda}_k^{i-1} - 1) E_{\lambda^i}^*(\bar{\nu})}{(z_k \bar{\lambda}_k^{i-1} - 1) E_{\nu}^*(\bar{\nu})} E_{\nu}^*(z). \quad (4.78)$$

We compute $E_{\lambda^i}^*(\overline{\lambda^{i+2}})$ as follows. With k as specified we can be sure there is at least one $\nu \in c(\lambda^i, k)$ such that $E_{\nu}^*(\overline{\lambda^{i+2}}) \neq 0$. Evaluating (4.78) at $\overline{\lambda^{i+2}}$ gives

$$E_{\lambda^i}^*(\overline{\lambda^{i+2}}) = \sum_{\substack{\nu \in c(\lambda^i, k) \\ \nu \preceq' \lambda^{i+2}}} \frac{(\bar{\nu}_k \bar{\lambda}_k^{i-1} - 1) E_{\lambda^i}^*(\bar{\nu})}{(\overline{\lambda^{i+2}}_k \bar{\lambda}_k^{i-1} - 1) E_{\nu}^*(\bar{\nu})} E_{\nu}^*(\overline{\lambda^{i+2}}), \quad (4.79)$$

where the further restriction on the summation to $\nu \preceq \lambda^{i+2}$ is due to the vanishing conditions of the interpolation polynomials. Since Proposition 4.4.1 gives an explicit formula for each $E_{\nu}^*(\overline{\lambda^{i+2}})$ (4.79) does indeed give us an explicit formula for $E_{\lambda^i}^*(\overline{\lambda^{i+2}})$.

One can then evaluate (4.78) at $\overline{\lambda^{i+3}}$ and use $E_{\lambda^i}^*(\overline{\lambda^{i+2}})$ to find an explicit formula for $E_{\lambda^i}^*(\overline{\lambda^{i+3}})$. This process can be extended to allow any $E_{\lambda^i}^*(\overline{\lambda^j})$ where $j > i + 1$ to be broken down into a combination of evaluations of the form $E_{\lambda^i}^*(\overline{\lambda^{l+1}})$, which in turn can be explicitly evaluated using (4.64) and (2.55).

□

This result leads us very nicely to a consequence for the generalised binomial coefficients (2.65) for general λ .

Corollary 4.4.7. *With $j > i + 1$ we have*

$$\binom{\lambda^j}{\lambda^i}_{q,t} = \frac{E_{\lambda^i}^*(\overline{\lambda^j})}{E_{\lambda^i}^*(\overline{\lambda^i})} = \sum_{\substack{\nu \in c(\lambda^i, k) \\ \nu \preceq' \lambda^j}} \frac{(\bar{\nu}_k \bar{\lambda}_k^{i-1} - 1) E_{\lambda^i}^*(\bar{\nu}) E_{\nu}^*(\overline{\lambda^j})}{(\overline{\lambda^j}_k \bar{\lambda}_k^{i-1} - 1) E_{\nu}^*(\bar{\nu}) E_{\lambda^i}^*(\overline{\lambda^i})},$$

where k and $c(\eta, m)$ are as in the previous proposition.

Clearly the explicit formulas for the Pieri-type coefficients are rather complex. We now consider possible simplifications.

4.4.3 Simplifying the Pieri coefficients

First a coefficient of unity in each Pieri-type formula is identified. An analogous result was observed earlier by Forrester and McAnnaly [24] within Jack polynomial theory, and identical principles apply for Macdonald polynomials.

Forrester and McAnnaly found that with $\eta + \chi_r$ given by

$$(\eta + \chi_r)_i := \begin{cases} \eta_i, & l'_\eta(i) \geq r \\ \eta_i + 1, & l'_\eta(i) < r, \end{cases}$$

we have

$$A_{\eta, \eta + \chi_r}^{(r)} = 1.$$

We first give an explicit derivation in the case $r = 1$ and then state their reasoning in the general case.

Proposition 4.4.8. *We have*

$$A_{\eta, \eta + \chi_1}^{(1)} = 1. \quad (4.80)$$

Proof. Let η_i be such that $l'_\eta(i) = 0$. By (2.53) and (2.54) the coefficient of $E_{\eta + \chi_1}^*(z)$ in the expansion of $H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1} E_\eta^*(z)$ will be $q^{-\eta_i}$. Using the vanishing properties of the interpolation polynomials and

$$z_i \Xi_i - 1 = H_i \dots H_{n-1} \Phi_q^* H_1 \dots H_{i-1}$$

we have

$$\frac{E_\eta^*(\overline{\eta + \chi_1})}{E_{\eta + \chi_1}^*(\overline{\eta + \chi_1})} = \frac{1}{(q-1)q^{\eta_i}} = \frac{1}{e_1(\overline{\eta + \chi_1}) - e_1(\overline{\eta})},$$

which upon substitution in (4.72) implies (4.80) □

Proposition 4.4.9. [24] We have

$$A_{\eta, \eta + \chi_r}^{(r)} = 1.$$

Proof. By definition of the Macdonald polynomials the coefficient of z^η in $E_\eta(z)$ is unity, and consequently the coefficient of $z^{\eta + \chi_r}$ in $e_r(z)E_\eta(z)$ is unity also. Since $\lambda^r \prec \eta + \chi_r$ for all $\lambda^r \neq \eta + \chi_r$ such that $\eta \preceq' \lambda^r \preceq' \eta + (1^n)$ the triangular structure of Macdonald

polynomials (2.7) ensures that the monomial $z^{\eta+\chi_r}$ will only occur in $E_{\eta+\chi_r}(z)$, forcing $A_{\eta,\eta+\chi_r}^{(r)}$ to be unity. \square

When $r > \lceil \frac{n}{2} \rceil$ less steps are required to obtain $A^{(n-r)}$ than $A^{(r)}$ and consequently we greatly simplify the computation of $A_{\eta\lambda^r}^{(r)}$ by employing (4.7).

There is some freedom in the implementation of the recurrences, and we have investigated ways to reduce the required number of calculations. For example, the k specified in the formulas of Proposition 4.4.6 is not the only such k that will provide a pathway to the explicit formula of $E_{\lambda^i}^*(\overline{\lambda^j})$, where $j > i + 1$. The only requirement on k is that $c(\lambda^i, k)$ contains a ν such that $\nu \preceq \lambda^j$. If one was to compute $E_{\lambda^i}^*(\overline{\lambda^j})$ it would be most efficient to choose k such that the number of $\nu \in c(\lambda^i, k)$ such that $\nu \not\preceq \lambda^j$ is maximised. This maximises the number of vanishing terms and hence minimises the number of computations. At this stage there doesn't seem to be an obvious way of choosing such a k and the problem remains open.

4.4.4 Reclaiming the symmetric Macdonald Pieri formulas

We have discussed how the nonsymmetric Macdonald polynomials can be symmetrised to obtain the symmetric Macdonald polynomials. Consequently we can symmetrise the Pieri-type formulas for the nonsymmetric Macdonald polynomials to reclaim their symmetric counterpart.

Symmetrising both sides of our Pieri-type formula (4.10) with coefficients specified by (4.76) and using (3.5) we obtain

$$a_{\eta}^{\{1,\dots,n\},\emptyset} e_r(z) P_{\eta^+}(z; q, t) = \sum_{\substack{\mu: |\mu|=|\eta^+|+r \\ \mu/\eta^+ \text{ is a} \\ \text{vertical } r \text{ strip}}} \sum_{\lambda: \lambda^+ = \mu} a_{\lambda}^{\{1,\dots,n\},\emptyset} A_{\eta\lambda}^{(r)}(q, t) P_{\lambda^+}(z; q, t),$$

where here the restriction on μ to partitions such that μ/η^+ is a vertical r strip follows from (4.11). Explicit low order calculations suggest this does indeed reclaim (4.2), that is

$$\psi_{\mu/\eta^+}(q, t) = \frac{1}{a_{\eta}^{\{1,\dots,n\},\emptyset}} \sum_{\lambda: \lambda^+ = \mu} a_{\lambda}^{\{1,\dots,n\},\emptyset} A_{\eta\lambda}^{(r)}(q, t). \quad (4.81)$$

An explicit proof of (4.81) is problematic as the grouping of λ such that $\lambda^+ = \mu$ is most challenging.

4.5 Further Work

It is clear from trial computations that the Pieri-type coefficients can generally be expressed as a product. Unfortunately our recursive formulas shed no light on the general requirement for a product formula to hold true. As such it seems that there would exist an expression for the Pieri-type coefficients much simpler than the ones obtained in these studies.

Forrester and McAnally derived product expressions for the Pieri-type coefficients for the nonsymmetric Jack polynomials for $r = 1$ in [24]. Unpublished work by McAnally containing formulas determined by an Ansatz analysis also suggest a product formula is possible. To further motivate a study on product formulas for the general Pieri-type formulas we present one more expression of the Pieri-type formulas for the nonsymmetric Macdonald polynomials in the case $r = 1$, the analogous formula to that from [24]. We note also that the formulas given in [24] contain some typographical errors. They can be corrected following the derivation provided here.

We proceed to give an alternate expression for the coefficients specified by (4.36), an expression we claim to be suitable for generalisation.

By introducing the sets G_0 and G_1

$$\begin{aligned} G_0 &:= G_0(\eta, \lambda) = \{i \in \{1, \dots, n\} : \lambda_{\sigma(i)} = \eta_i\}, \\ G_1 &:= G_1(\eta, \lambda) = \{i \in \{1, \dots, n\} : \lambda_{\sigma(i)} = \eta_i + 1\}, \end{aligned}$$

where $\sigma := \hat{\sigma}$ is the defining permutation of $\eta \preceq' \lambda$, we can rewrite the product of $A_I(\bar{\eta})$ and $\tilde{B}_I(\bar{\eta})$, originally given in (4.23) and (4.31), as

$$\begin{aligned} A_I(\bar{\eta})B_I(\bar{\eta}) &= \prod_{\sigma(j) < j} \frac{(t-1)\bar{\eta}_{\sigma(j)}}{\bar{\eta}_{\sigma(j)} - \bar{\eta}_j} \left(\prod_{j \in G_1} \frac{(t-1)\bar{\eta}_{\sigma(j)}}{\bar{\eta}_{\sigma(j)} - q\bar{\eta}_j} (q\bar{\eta}_j - t^{-n+1}) \right) \\ &\quad \times \prod_{\sigma(j) < k < j} \frac{\bar{\eta}_j - t\bar{\eta}_k}{\bar{\eta}_j - \bar{\eta}_k} \prod_{j \in G_1, k < j} \frac{\bar{\eta}_j - t\bar{\eta}_k}{\bar{\eta}_j - t\bar{\eta}_k} \prod_{\substack{k \in G_0, j \in G_1 \\ \sigma(j) < k}} \frac{q\bar{\eta}_j - t\bar{\eta}_k}{q\bar{\eta}_j - \bar{\eta}_k}. \end{aligned}$$

These formulas can be substituted in (4.36) to give a new viewpoint on the Pieri-type coefficients for $r = 1$. It is hoped that these formulas provide an insight into the possible structure of the product formulas for the general Pieri-type formulas. Examples of Pieri-type coefficients in the case $r = 2$ are given in Section 5.5.

A further extension of this study is to consider other methods of simplifying the coefficients obtained in Section 4.4.2.

Chapter 5

Computer Programming

Throughout the research period the computer algebra system Mathematica has been used extensively. The primary use of the system was to generate families of multivariate polynomials. These polynomials were used to gain deeper understanding of known results, assist in conjecture formulation and testing and also as a means of sharing information with others.

In this chapter we present an algorithm that specifies the smallest sequence of switching and raising operators required to recursively generate any composition from the all zero composition (Proposition 5.2.1). We extend this algorithm to one that generates families of polynomials that can be constructed via switching and raising operators (Proposition 5.3.1). We then provide the details of the Mathematica notebook mirroring the work contained in this thesis. We state the main functions within the notebook and briefly describe what they do (Section 5.4.2). Following this we analyse the efficiency of our recursive generation algorithm against the Rodrigues formulas by comparing run times of the generation of specific nonsymmetric and nonsymmetric interpolation Macdonald polynomials. (Table 5.1). We conclude the chapter with a discussion of how our Mathematica notebook could be used to aid the further work suggestions of the previous two chapters.

5.1 A Brief History

Computer software that generates polynomial families discussed in this thesis are quite limited. One of the few programs available is designed to compute symmetric Jack polynomials $C_\kappa(z; \alpha)$ [17]. The $C_\kappa(z; \alpha)$ relate to $P_\kappa(z; \alpha)$ (refer Appendix C) according to

$$C_\kappa(z; \alpha) = \frac{\alpha^k k!}{\prod_{s \in \kappa} h_*^\kappa(z)} P_\kappa(z; \alpha), \quad h_*^\kappa(s) := l_\kappa(s) + 1 + \alpha a_\kappa(s).$$

The software, written in Maple, uses recurrence formulas to evaluate the coefficients $K'_{\kappa\mu}(\alpha) \in \mathbb{Q}(\alpha)$ in the expansion

$$C_{\kappa}(z; \alpha) = \sum_{\mu \leq \kappa} K'_{\kappa\mu}(\alpha) m_{\mu}(z).$$

The study in [17] discusses the implementation of the software and the run times of particular functions. They also provide details for the generation of the generalised classical Hermite, Laguerre and Jacobi polynomials.

A further study in the generation of symmetric Jack polynomials is by Demmel and Koev [16]. They use the expansion formula for a Jack polynomial in one of its variables to obtain a more efficient evaluation for sets of Jack polynomials than was known previously.

Although there are known methods for generating the nonsymmetric polynomials, for example the Rodrigues formulas, software that generates nonsymmetric and interpolation polynomials appears to be nonexistent in the literature. It is our aim to initiate momentum in this area.

5.2 Preliminaries

As stated in Section 1.2.1, every composition η can be recursively generated from the all zero composition $(0, \dots, 0)$ using a (non-unique) sequence of raising and switching operators. For example, $\eta = (0, 2, 1)$ can be generated from $(0, 0, 0)$ by $s_1 s_2 s_1 s_2 \Phi s_2 \Phi s_2 \Phi(0, 0, 0)$ or more efficiently by $s_2 \Phi s_1 \Phi \Phi(0, 0, 0)$. In polynomial classes for which switching and raising operators exist analogous methods can be used to generate polynomials. Consequently once we construct an algorithm that generates any composition recursively we automatically obtain an algorithm for the polynomials.

5.2.1 Recursively generating compositions

Our aim is to construct an algorithm to recursively generate any composition η from $(0, \dots, 0)$ using the least number of operators. We first observe that the raising operator Φ must be used $|\eta|$ times. Since the raising operator acts on a composition by increasing the value of the component in the first position by one, appending it to the end of the composition and shifting each other component back one position, to minimise the number of operators we must always increase the value of the leftmost component requiring raising. A systematic way of doing this is to apply the raising operator to build all components greater or equal to a specific size only using the switchings to move the leftmost

component needing raising to the first position. We note that this method is quite similar to the Rodrigues formulas construction. Using this method we naturally construct the composition $(\eta^+)^R$. Due to the nature of Φ there is no possible way to construct a composition containing each component of η using fewer operators.

To reorder $(\eta^+)^R$ minimally we switch each component into its correct position beginning with either η_1, η_2, \dots or $\eta_n, \eta_{n-1}, \dots$. We choose to start with repositioning η_n . By always choosing the closest component of the unordered composition we ensure that the number of switches is minimal.

Proposition 5.2.1. *Define $l_{\eta,i} := \#\{\eta_j < i\}$, $g_{\eta,i} := \#\{\eta_j \geq i\}$ and*

$$\sigma(\eta, i) := (l_{\eta,i-1}, \dots, 1, l_{\eta,i-1} + 1, \dots, 2, \dots, l_{\eta,i-1} + g_{\eta,i} - 1, \dots, g_{\eta,i}).$$

Define

$$r_{\eta,i} := \begin{cases} \Phi^{g_{\eta,i}}, & i = 1 \\ \Phi^{g_{\eta,i}} s_{\sigma(\eta,i)}, & i > 1, \end{cases}$$

where $s_{(i_l, \dots, i_1)} := s_{i_l} \dots s_{i_1}$. Define

$$p_{\eta,i} := \max \left\{ j \leq i : \prod_{k=1}^{i-1} s_{\sigma'(\eta,k)} (\eta^+)_j^R = \eta_i \right\}$$

and

$$s_{\sigma'(\eta,i)} := \begin{cases} s_{(p_{\eta,i}, \dots, i-1)}, & p_{\eta,i} < i \\ 1, & p_{\eta,i} = i. \end{cases}$$

The minimal length sequence of operators that transforms $(0, \dots, 0)$ to η is

$$s_{\sigma'(\eta,2)} \dots s_{\sigma'(\eta,n)} r_{\eta, \max(\eta)} \dots r_{\eta,1}.$$

That is

$$s_{\sigma'(\eta,2)} \dots s_{\sigma'(\eta,n)} r_{\eta, \max(\eta)} \dots r_{\eta,1} (0, \dots, 0) = \eta. \quad (5.1)$$

Proof. We prove (5.1) two steps. We first show by induction that

$$r_{\eta, \max(\eta)} \dots r_{\eta,1} (0, \dots, 0) = (\eta^+)^R. \quad (5.2)$$

By the definition of $r_{\eta,1}$ it is clear that $r_{\eta,1}(0, \dots, 0)$ produces a composition of the form $(0, \dots, 0, 1, \dots, 1)$ where the number of 1's is equal to the number of components of η that are greater or equal to 1. Suppose before applying $r_{\eta,k+1}$ we have generated the

correct number of components with value $0, 1, \dots, k-1$, that is we have constructed a composition of the form $((\eta^+)_1^R, \dots, (\eta^+)_j^R, k \dots, k)$ where the number of k 's is equal to the number of components of η that are greater or equal to k . Quite obviously $r_{\eta, k+1}((\eta^+)_1^R, \dots, (\eta^+)_j^R, k \dots, k) = ((\eta^+)_1^R, \dots, (\eta^+)_j^R, k \dots, k, k+1, \dots, k+1)$ where the number of k 's equals the number of components of η equal to k and the number of $(k+1)$'s equals the number of components greater than or equal to $k+1$. By induction (5.2) holds. The final task is to show that

$$s_{\sigma'(\eta, 2)} \dots s_{\sigma'(\eta, n)} (\eta^+)^R = \eta. \quad (5.3)$$

This result follows immediately from the definition of $p_{\eta, i}$ as quite clearly $\sigma'(\eta, i)$ successively permutes each η_i into the correct position. The fact that the total sequence of operators is of minimal length follows from the action of Φ and the inability to generate a composition with components η_1, \dots, η_n more economically than what is specified by (5.2), the permutation that places each component into its correct position can not be improved either. \square

We note that further evidence showing the permutation in (5.3) is minimal is the comparison of its length to the minimal permutation $\omega_\eta \omega_{(\eta^+)^R}^{-1}$. Due to the different structures of the permutations we use the computational evidence to support our claim.

To provide additional clarity to the algorithm we show how $(4, 1, 2, 1)$ is generated using the above operators.

Example 5.2.1. *We construct the composition $(4, 1, 2, 1)$ recursively from $(0, 0, 0, 0)$. We first construct $(1, 1, 2, 4)$ using the operators $r_{\eta, i}$.*

$$\begin{aligned} r_{\eta, 4} r_{\eta, 3} r_{\eta, 2} r_{\eta, 1} (0, 0, 0, 0) &= r_{\eta, 4} r_{\eta, 3} r_{\eta, 2} \Phi^4 (0, 0, 0, 0) \\ &= r_{\eta, 4} r_{\eta, 3} r_{\eta, 2} (1, 1, 1, 1) \\ &= r_{\eta, 4} r_{\eta, 3} \Phi^2 (1, 1, 1, 1) \\ &= r_{\eta, 4} r_{\eta, 3} (1, 1, 2, 2) \\ &= r_{\eta, 4} \Phi s_1 s_2 (1, 1, 2, 2) \\ &= r_{\eta, 4} (1, 1, 2, 3) \\ &= \Phi s_1 s_2 s_3 (1, 1, 2, 3) \\ &= (1, 1, 2, 4). \end{aligned}$$

We complete the generation by permuting each component into its correct position

$$\begin{aligned}
s_{\sigma'(\eta,2)}s_{\sigma'(\eta,3)}s_{\sigma'(\eta,4)}(1, 1, 2, 4) &= s_{\sigma'(\eta,2)}s_{\sigma'(\eta,3)}s_3s_2(1, 1, 2, 4) \\
&= s_{\sigma'(\eta,2)}s_{\sigma'(\eta,3)}(1, 2, 4, 1) \\
&= s_{\sigma'(\eta,2)}s_2(1, 2, 4, 1) \\
&= s_{\sigma'(\eta,2)}(1, 4, 2, 1) \\
&= s_1(1, 4, 2, 1) \\
&= (4, 1, 2, 1).
\end{aligned}$$

We now move onto the major goal of our study, the recursive generation of various polynomial classes.

5.3 Recursively generating polynomials

There are many families of polynomials that can be recursively generated via switching and raising type operators. Some examples are the nonsymmetric Macdonald polynomial (Section 2.1.3), nonsymmetric interpolation Macdonald polynomials (Section 2.2.4), nonsymmetric Jack polynomials (Appendix C), nonsymmetric interpolation Jack polynomials (Appendix D) and the generalised nonsymmetric Hermite and Laguerre polynomials (see e.g. [23, Chap. 13]). In this section we show how the algorithm developed in Proposition 5.2.1 can be employed to recursively generate any of the polynomials in these families.

For simplicity in this section we use $F_\eta(z)$ to denote any of the aforementioned families of polynomials. We show how the recursive generation formulas work in the general setting and then provide a specific example for the nonsymmetric Macdonald polynomials.

To most simply express the sequence of operators required to recursively generate a composition according to Proposition 5.2.1 we use the numbers $1, \dots, n-1$ to represent the allowable switching operators, 0 to represent the raising operator and denote the required sequence by $R(\eta)$. For Example 5.2.1 in the previous section we observe that $R((4, 1, 2, 1)) = \{0, 0, 0, 0, 0, 0, 2, 1, 0, 3, 2, 1, 0, 2, 3, 2, 1\}$.

Proposition 5.3.1. *Define*

$$\text{RG}_j(F_{\eta(j)}(z), \eta(j), R(\eta)_j) := \begin{cases} \{F_{s_i\eta(j)}(z), s_i\eta(j)\}, & i = R(\eta)_j = 1, \dots, n-1 \\ \{F_{\Phi\eta(j)}(z), \Phi\eta(j)\}, & i = R(\eta)_j = 0, \end{cases} \quad (5.4)$$

where $\eta(j)$ represents the composition obtained after j transformations from $(0, \dots, 0)$ to η specified by $R(\eta)$ and $F_{s_i\eta}(z)$ and $F_{\Phi\eta}(z)$ are obtained from $F_\eta(z)$ using known formulas. With initial input $RG_1(1, (0, \dots, 0), R(\eta)_1)$ and each subsequent polynomial derived from the previous by entering the newly obtained polynomial and composition along with the next number in $R(\eta)$ in RG the polynomial $F_\eta(z)$ will be obtained after $|R(\eta)|$ steps.

Proof. By Proposition 5.2.1 we know that the sequence specified by $R(\eta)$ will recursively generate the composition η from $(0, \dots, 0)$. Consequently RG_j will recursively generate $F_\eta(z)$ from $F_{(0, \dots, 0)}(z)$. \square

We note that we keep track of the composition labelling the polynomial at each stage due to requirements of the formulas transforming $F_\eta(z)$ to $F_{s_i\eta}(z)$ and $F_{\Phi\eta}(z)$.

Example 5.3.1. To recursively generate a nonsymmetric Macdonald polynomial using the methods in the previous proposition we begin by rewriting the formulas of Section 2.1.3 as

$$E_{s_i\eta}(z) = \begin{cases} t^{-1}T_i E_\eta(z) - \frac{t-1}{t(1-\delta_{i,\eta}^{-1}(q,t))} E_\eta(z), & \eta_i < \eta_{i+1} \\ E_\eta(z), & \eta_i = \eta_{i+1} \\ \frac{(1-\delta_{i,\eta}(q,t))^2}{(1-t\delta_{i,\eta}(q,t))(1-t^{-1}\delta_{i,\eta}(q,t))} \left(T_i E_\eta(z) \frac{t-1}{1-\delta_{i,\eta}^{-1}(q,t)} E_\eta(z) \right), & \eta_i > \eta_{i+1}, \end{cases}$$

$$E_{\Phi\eta}(z) = t^{\#\{i > 1: \eta_i \leq \eta_1\}} \Phi_q E_\eta(z).$$

To generate the polynomial $E_{(2,1)}(z; q, t)$ from $E_{(0,0)}(z; q, t) = 1$ using RG we first compute $R((2, 1))$ using Proposition 5.2.1 to be $R(2, 1) = \{0, 0, 0, 1\}$ we then proceed recursively

$$\begin{aligned} RG_1(1, (0, 0), 0) &= \{E_{(0,1)}(z), (0, 1)\} \\ RG_2(E_{(0,1)}(z), (0, 1), 0) &= \{E_{(1,1)}(z), (1, 1)\} \\ RG_3(E_{(1,1)}(z), (1, 1), 0) &= \{E_{(1,2)}(z), (1, 2)\} \\ RG_4(E_{(1,2)}(z), (1, 2), 1) &= \{E_{(2,1)}(z), (2, 1)\}. \end{aligned}$$

In the cases where $F_\eta(z)$ is homogeneous we can greatly reduce the number of operators required to generate polynomials labelled by compositions with no zero components by making use of the relationship

$$F_{\eta+(k^n)}(z) = (z_1 \dots z_n)^k F_\eta(z), \quad (5.5)$$

(c.f. (2.1.4)). This result allows us to omit the first $n \times \min\{\eta\}$ zeros from $R(\eta)$, forming a new set $R'(\eta)$, and begin our recursive process with

$$\{(z_1 \dots z_n)^{\min\{\eta\}}, (\min\{\eta\}, \dots, \min\{\eta\}), R'(\eta)_1\}$$

rather than $\{1, (0, \dots, 0), R(\eta)_1\}$.

Example 5.3.2. *Using (5.5) we recursively generate $E_{(2,1)}(z; q, t)$ from $E_{(1,1)}(z; q, t)$. With $R'((2,1)) = \{0, 1\}$ we obtain*

$$\begin{aligned} \text{RG}_1(z_1 z_2, (1, 1), 0) &= \{E_{(1,2)}(z), (1, 2)\} \\ \text{RG}_2(E_{(1,2)}(z), (1, 2), 1) &= \{E_{(2,1)}(z), (2, 1)\}. \end{aligned}$$

5.4 Runtimes and Software

5.4.1 Algorithm runtimes

We now have two methods for recursively generating the nonsymmetric Macdonald and nonsymmetric interpolation Macdonald polynomials; the Rodrigues formulas (Sections 2.1.4 and 2.2.5) and the algorithm specified by Proposition 5.3.1. We examine the efficiency of our algorithm by comparing generation runtimes against the Rodrigues methods for various polynomials.

To provide a thorough analysis we must select polynomials of varying degrees of complexity. That is, varying the number of variables, the maximum degree and of these polynomials selecting those requiring the least and most number of operators to generate. Of the polynomials with n variables and maximum degree k those that take the least and most number of operators to generate using the algorithm of Proposition 5.3.1 are labelled by compositions $(q^{n-r}, (q+1)^r)$ and $(k, 0, \dots, 0)$, where $k = qn + r$, and require k and nk operators to generate, respectively. Similarly, using the Rodrigues methods, of the polynomials with n variables and maximum degree k those that take the least and most number of operators to generate are labelled by compositions $((q+1)^r, q^{n-r})$ and $(0, \dots, 0, k)$ and require $nq + (n - (r - 1))r$ and $n(k + 1) - 1$ operators to generate, respectively. We note that we have employed (5.5) into our algorithm and polynomials labelled by compositions with no zero components will be generated more efficiently than what is specified above, for example the nonsymmetric Macdonald polynomials labelled by $\eta = (k^n)$ would be generated almost instantly.

Table 5.1 shows the runtimes of the generation of different nonsymmetric Macdonald polynomials using the recursive generation algorithm of Proposition 5.3.1 and the Rodrigues formulas. The computer these runtimes were observed on was an iMac 2.4GHz Intel core 2 duo processor in version 7.01.0 of Mathematica.

$ \eta $	η	$E_\eta(z)$		$E_\eta^*(z)$	
		Recursive Generation	Rodrigues	Recursive Generation	Rodrigues
4	(0, 4)	0.0308	6.6782	0.7792	205.972
	(1, 3)	0.0064	1.1256	0.0759	11.4942
	(2, 2)	0.0011	0.0648	0.0009	0.1281
	(3, 1)	0.0177	0.6116	0.6081	5.0752
	(4, 0)	0.0674	3.6920	12.3216	109.839
	(0, 0, 4)	228.33	15.9176	4.54	4458.24
	(1, 1, 2)	0.0014	2.6550	0.0010	11.1048
	(2, 1, 1)	0.0105	1.0142	0.3108	3.3316
	(4, 0, 0)	0.6337	92.8975	65.7861	2330.89
	(1, 3, 0)	0.3171	66.1389	30.3733	3201.16
7	(0, 7)	0.2677	127.8250	34.4630	25888.1
	(3, 4)	0.0019	0.5252	0.001648	7.2465
	(4, 3)	0.0070	0.2074	0.1471	1.6495
	(7, 0)	0.4744	54.3738	68.2642	9887.15

Table 5.1: Runtimes in seconds of Rodrigues formulas and recursive generation algorithm

It is the number and complexity of operators required to generate the polynomials using the Rodrigues formulas that make them considerably more time consuming to use than the algorithm we have constructed. However, one advantage of the Rodrigues formulas is that it is much easier to implement into computer software. We also note that it is the inhomogeneity of the interpolation polynomials that makes their construction time longer than the nonsymmetric polynomials.

5.4.2 Mathematica notebook

We now provide the details of the Mathematica notebook containing the polynomials and operators contained in this thesis. The Mathematica notebook, titled SpecialFunctions.nb, can be found at www.ms.unimelb.edu.au/~wbaratta/index.html and

ran on Mathematica 7. The original purpose of the Mathematica notebook was to efficiently generate nonsymmetric and nonsymmetric interpolation Macdonald polynomials to develop an understanding of their known theory. Surpassing this motivation it has been extensively used throughout the research period to assist with conjecture formulation and testing. We now present a table containing the key functions defined in the Mathematica notebook. The notations used in the table are consistent with those used throughout the thesis.

Syntax	Description
CompositionModulus[η]	Computes $ \eta $
SwitchComposition[η, i]	Computes $s_i\eta$
RaiseComposition[η]	Computes $\Phi\eta$
cI[η, I]	Computes $c_I(\eta)$
Dominance[η, λ]	Determines whether $\eta \leq \lambda$ or $\lambda \leq \eta$
PartialOrder[η, λ]	Determines whether $\eta \preceq \lambda$ or $\lambda \preceq \eta$
PartialOrder2[η, λ]	Determines whether $\eta \triangleleft \lambda$ or $\lambda \triangleleft \eta$
Successor[η, λ]	Determines whether $\eta \preceq' \lambda$ or $\lambda \preceq' \eta$
ArmLength[η, i, j]	Computes $a_\eta(i, j)$
ArmCoLength[η, i, j]	Computes $a'_\eta(i, j)$
LegLength[η, i, j]	Computes $l_\eta(i, j)$
LegCoLength[η, i]	Computes $l'_\eta(i)$
Md[η]	Computes $d_\eta(q, t)$
MdDash[η]	Computes $d'_\eta(q, t)$
Me[η]	Computes $e_\eta(q, t)$
MeDash[η]	Computes $e'_\eta(q, t)$
R[η]	Computes the set $R(\eta)$, specifying the sequence of operators required to generate η from $(0, \dots, 0)$

Table 5.2: Mathematica functions relating to compositions

Syntax	Description
PermutationOnComposition[σ, η]	Computes $\sigma(\eta)$
DecompositionOnComposition[$\{s_{i_1}, \dots, s_{i_l}\}, \eta$]	Computes $s_{i_1} \dots s_{i_l} \eta$
SwitchingOperator[f, i, j]	Computes $s_{ij}f(\dots, z_i, \dots, z_j, \dots)$
PermutationOnPolynomial[σ, f]	Computes $\sigma f(z_1, \dots, z_n)$
ShortestPermutation[η]	Computes ω_η , the shortest permutation such that $\omega_\eta^{-1}(\eta) = \eta^+$
RequiredPermutation[η, λ]	Computes the permutation σ such that $\sigma(\eta) = \lambda$

Table 5.3: Mathematica functions relating to permutations

Syntax	Description
Monomial[η]	Computes z^η
ElementarySymmetricFunction[r, n]	Computes $e_r(z)$ in n variables
CompleteSymmetricFunction[r, n]	Computes $h_r(z)$ in n variables
SymmetricMonomialFunction[κ]	Computes $m_\kappa(z)$
Vandermonde[n]	Computes $\Delta(z)$
VandermondeJ[J]	Computes $\Delta^J(z)$
tVandermonde[n]	Computes $\Delta_t(z)$
tVandermondeJ[J]	Computes $\Delta_t^J(z)$
Schur[κ]	Computes $s_\kappa(z)$
Zonal[κ]	Computes $Z_\kappa(z)$
HallLittlewood[κ]	Computes $P_\kappa(z; t)$

Table 5.4: Mathematica functions relating to miscellaneous polynomials

Syntax	Description
SymJack[κ]	Computes $P_\kappa(z; \alpha)$
NSJack[η]	Computes $E_\eta(z; \alpha)$
IntJack[η]	Computes $E_\eta^*(z; \alpha)$
JEvalue[η]	Computes $\bar{\eta}^\alpha$, where $\bar{\eta}_i := \alpha\eta_i - l'_\eta(i)$
EOpSymJack[f, n]	Computes $D_2(\alpha)f$
EOpNSJack[f, n, i]	Computes $\xi_i f$
EOpIntJack[f, n, i]	Computes $\Xi_i^\alpha f$
JInnerProduct[f, g, n, k]	Computes $\langle f, g \rangle_{1/k}$ for polynomials f, g of n variables

Table 5.5: Mathematica functions relating to Jack polynomials

Syntax	Description
NSMac[η]	Computes $E_\eta(z; q, t)$
SymMac[κ]	Computes $P_\kappa(z; q, t)$
ASymMac[κ]	Computes $S_\kappa(z; q, t)$
IntMac[η]	Computes $E_\eta^*(z; q, t)$
SymIntMac[κ]	Computes $P_\kappa^*(z; q, t)$
ASymIntMac[κ]	Computes $S_\kappa^*(z; q, t)$
MEvalue[η]	Computes $\bar{\eta}$, where $\bar{\eta}_i := q^{\eta_i} t^{l'_\eta(i)}$
EOpSymMac[f, n]	Computes $D_n^1(q, t)f$
EOpNSMac[f, i]	Computes $Y_i f$
EOpIntMac[f, i]	Computes $\Xi_i f$
Ti[f, i]	Computes $T_i f$
TiInv[f, i]	Computes $T_i^{-1} f$
Phiq[f, n]	Computes $\Phi_q f$
Hi[f, i]	Computes $H_i f$
PhiqInt[f, n]	Computes $\Phi_q^* f$
MInnerProduct[f, g, n, k]	Computes $\langle f, g \rangle_{q, q^k}$, for polynomials f, g of n variables

Table 5.6: Mathematica functions relating to Macdonald polynomials

Syntax	Description
UPlus[f, n]	Computes $U^+ f$, where f is a function of n variables
UMinus[f, n]	Computes $U^- f$, where f is a function of n variables
UPlusInt[f, n]	Computes $U_*^+ f$, where f is a function of n variables
UMinusInt[f, n]	Computes $U_*^- f$, where f is a function of n variables
OIJ[f, n, I, J]	Computes $O_{I,J} f$, where f is a function of n variables
OIJInt[f, n, I, J]	Computes $O_{I,J}^* f$, where f is a function of n variables
PreSymMac[η^*, I, J]	Computes $S_{\eta^*}^{(I,J)}(z; q, t)$
PreSymIntMac[η^*, I, J]	Computes $S_{\eta^*}^{*,(I,J)}(z; q, t)$
aEta[η, I, J]	Computes the normalisation $a_{\eta}^{(I,J)}(q, t)$

Table 5.7: Mathematica functions relating to prescribed symmetry polynomials

Syntax	Description
JEta[η, r]	Produces the set of λ such that $ \lambda = \eta + r$ and $\eta \preceq' \lambda \preceq' \eta + (1^n)$
Pieri[η, λ, r]	Computes the Pieri-type coefficient $A_{\eta,\lambda}^{(r)}(q, t)$
GeneralisedBinomialCoefficient[λ, η]	Computes $\binom{\lambda}{\eta}_{q,t}$

Table 5.8: Mathematica functions relating to Pieri-type coefficients

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5.5 Further Work

It is hoped that the notebook SpecialFunctions.nb will continue to assist researchers with conjecture formulation and testing in areas relating to the work within this thesis. In fact this has already shown itself to be the case in a study of special vanishing properties of Jack polynomials for $\alpha = -(r-1)/(k+1)$ and Macdonald polynomials with $t^{k+1}q^{r-1} = 1$ [10]. Below are some examples of computations that aim to provide additional motivation for two of the problems suggested in the further work sections of the previous chapters.

Interpolation polynomials with prescribed symmetry

In Section 3.6 we suggested that theory of prescribed symmetry Macdonald polynomials could be extended to interpolation Macdonald polynomials. Here we provide some explicit

computations that may assist with the identification of the relationship between the anti-symmetric interpolation polynomial $S_{\kappa+\delta}^*(z)$ and the symmetric interpolation polynomial $P_{\kappa}^*(z)$.

In[1] SymIntMac[{1,0}]

$$\text{Out[1]} \quad \frac{t(x_1 + x_2) - t - 1}{t}$$

In[2] ASymIntMac[{2,0}]

$$\text{Out[2]} \quad \frac{(x_1 - tx_2)(t(x_1 + x_2) - qt - 1)}{t}$$

In[1] SymIntMac[{1,1}]

$$\text{Out[1]} \quad \frac{(tx_1 - 1)(tx_2 - 1)}{t^2}$$

In[2] ASymIntMac[{2,1}]

$$\text{Out[2]} \quad \frac{(x_1 - tx_2)(tx_1 - 1)(tx_2 - 1)}{t^2}$$

In[1] SymIntMac[{1,0,0}]

$$\text{Out[1]} \quad \frac{t^2(x_1 + x_2 + x_3) - t^2 - t - 1}{t^2}$$

In[2] ASymIntMac[{3,1,0}]

$$\text{Out[2]} \quad \frac{(x_1 - tx_2)(x_1 - tx_3)(x_2 - tx_3)(t^2(x_1 + x_2 + x_3) - qt(qt + 1) - 1)}{t^2}$$

These computations show a clear relationship between $S_{\kappa+\delta}^*(z)$ and $P_{\kappa}^*(z)$, highlighting that the difficulty in identifying the relationship lies in the transformations of the parameters. It also appears that the t -Vandermonde product that relates the polynomials is not of standard form.

Pieri-type formulas for nonsymmetric Macdonald polynomials

It is stated in Section 4.5 that trial computations show that the Pieri-type formulas for the nonsymmetric Macdonald polynomials can generally be expressed as products. In Section 4.5 we provided an explicit formula for the $r = 1$ case and left the remaining cases as an open problem. Here we present explicit formulas for the Pieri-type coefficients for the case $r = 2$.

In[1]	Pieri[{1,0,1,0},{2,0,2,0},2]	In[2]	Pieri[{1,0,1,0},{2,1,1,0},2]
Out[1]	1	Out[2]	$\frac{q-1}{qt-1}$
In[3]	Pieri[{1,0,1,0},{1,1,2,0},2]	In[4]	Pieri[{1,0,1,0},{2,0,1,1},2]
Out[3]	$\frac{t(q-1)(qt^3-1)}{(qt^2-1)^2}$	Out[4]	$\frac{t(q-1)(qt^3-1)}{(qt^2-1)^2}$
In[5]	Pieri[{1,0,1,0},{1,1,1,1},2]	In[6]	Pieri[{1,0,1,0},{1,2,0,1},2]
Out[5]	$\frac{(q-1)(qt-1)(qt^4-1)}{(qt^2-1)^3}$	Out[6]	$\frac{q^2t^2(q-1)(t-1)^4}{(qt-1)^3(qt^2-1)^2}$
In[7]	Pieri[{1,0,1,0},{1,2,1,0},2]	In[8]	Pieri[{1,0,1,0},{2,1,0,1},2]
Out[7]	$-\frac{qt(q-1)(t-1)^2}{(qt-1)^2(qt^2-1)}$	Out[8]	$-\frac{qt(q-1)(t-1)^2}{(qt-1)^2(qt^2-1)}$
In[9]	Pieri[{1,0,1,0},{0,2,2,0},2]	In[10]	Pieri[{1,0,1,0},{2,0,0,2},2]
Out[9]	$\frac{qt^2(q-1)(t-1)}{(qt-1)(qt+1)(qt^2-1)}$	Out[10]	$\frac{qt^2(q-1)(t-1)}{(qt-1)(qt+1)(qt^2-1)}$
In[11]	Pieri[{1,0,1,0},{1,0,2,1},2]		
Out[11]	$\frac{(q-1)t^2(q(t(q((q-2)qt^5+(3q-1)t^4+(2-3q)t^3-2t+1)+3(t-1))+2)-1)}{(qt-1)(qt^2-1)^2(q^2t^3-1)}$		

In the above formulas for the Pieri-type coefficients $A_{(1,0,1,0),\lambda}^{(2)}(q,t)$ a simple product structure in q and t , with roots in t being simple fractional powers of q for example, is exhibited in all cases except $\lambda = (1,0,2,1)$. Additional trial computations suggest that it is always the successor of the form $\Phi\Phi\eta$ that cannot be expressed as a simple product.

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Appendix A

Some basic q -functions

The q -factorial, q -binomial coefficient, q -Pochhammer symbol and q -gamma function, denoted $[k]_q!$, $\binom{n}{k}_q$, $(a; q)_k$ and $\Gamma_q(z)$ respectively are given by

$$[k]_q! := 1(1+q)(1+q+q^2)\dots(1+q+\dots+q^{k-1}), \quad (\text{A.1})$$

$$\binom{n}{k}_q := \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad (\text{A.2})$$

$$(a; q)_k := \begin{cases} \prod_{j=0}^{k-1} (1 - aq^j), & k > 0 \\ \prod_{j=0}^{\infty} (1 - aq^j), & k = \infty, \end{cases} \quad (\text{A.3})$$

$$\Gamma_q(z) := \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1-q)^{1-z}, \quad (\text{A.4})$$

each reducing to their classical counterparts when $q \rightarrow 1$. Note that $(a; q)_{\infty}$ requires $|q| \rightarrow 1$ to be convergent. Like their classical counterparts there are alternative ways to express the q -factorial and q -binomial coefficient. We have [46]

$$[k]_q! = \frac{(q; q)_k}{(1-q)^k} = \Gamma_q(k+1),$$

where the first equality follows from the identity

$$(1-q^k) = (1-q)(1+\dots+q^{k-1}) \quad (\text{A.5})$$

and the second from (A.4), and we also have

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

The q -gamma function satisfies the recurrence relation

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q} \Gamma_q(z)$$

thus generalising the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ of its classical counterpart.

Appendix B

Symmetric polynomials

B.1 Triangular structure

Class	Notation and structure
Schur polynomials;	$s_\kappa(z) = m_\kappa(z) + \sum_{\mu < \kappa} K_{\kappa\mu} m_\mu(z)$
zonal polynomials;	$Z_\kappa(z) = m_\kappa(z) + \sum_{\mu < \kappa} K'_{\kappa\mu} m_\mu(z)$
Hall-Littlewood polynomials;	$P_\kappa(z; t) = m_\kappa(z) + \sum_{\mu < \kappa} K_{\kappa\mu}(t) m_\mu(z)$
Symmetric Jack polynomials;	$P_\kappa(z; \alpha) = m_\kappa(z) + \sum_{\mu < \kappa} K_{\kappa\mu}(\alpha) m_\mu(z)$
Symmetric Macdonald polynomials;	$P_\kappa(z; q, t) = m_\kappa(z) + \sum_{\mu < \kappa} K_{\kappa\mu}(q, t) m_\mu(z)$

Class	Coefficients
Schur polynomials;	$K_{\kappa\mu} \in \mathbb{Z}$
zonal polynomials;	$K'_{\kappa\mu} \in \mathbb{Q}$
Hall-Littlewood polynomials;	$K_{\kappa\mu}(t) \in \mathbb{Q}(t)$
Symmetric Jack polynomials;	$K_{\kappa\mu}(\alpha) \in \mathbb{Q}(\alpha)$
Symmetric Macdonald polynomials;	$K_{\kappa\mu}(q, t) \in \mathbb{Q}(q, t)$

B.2 Orthogonality

Class	Alternative inner product
Schur polynomials;	$\langle p_\kappa(z), p_\mu(z) \rangle_1 = \delta_{\kappa\mu} c_\kappa$
zonal polynomials;	$\langle p_\kappa(z), p_\mu(z) \rangle_2 = \delta_{\kappa\mu} c_\kappa 2^n$
Hall-Littlewood polynomials;	$\langle p_\kappa(z), p_\mu(z) \rangle_q = \delta_{\kappa\mu} c_\kappa \prod_{i=1}^n (1 - t^{\kappa_i - 1})$
Symmetric Jack polynomials;	$\langle p_\kappa(z), p_\mu(z) \rangle_\alpha = \delta_{\kappa\mu} c_\kappa \alpha^n$
Symmetric Macdonald polynomials;	$\langle p_\kappa(z), p_\mu(z) \rangle_{q,t} = \delta_{\kappa\mu} c_\kappa \prod_{i=1}^n \frac{1 - q^{\kappa_i}}{1 - t^{\kappa_i}}$

Class	Explicit formula for the inner product
Schur polynomials;	$\text{CT}(s_\kappa(z) s_\mu(z^{-1}) W(z; 1))$
Symmetric Jack polynomials;	$\text{CT}(P_\kappa(z; \alpha) P_\mu(z^{-1}; \alpha) W(z; \alpha))$
Symmetric Macdonald polynomials;	$\text{CT}(P_\kappa(z; q, t) P_\mu(z^{-1}; q^{-1}, t^{-1}) W(z; q, t))$

Class	Weight function
Schur polynomials;	$W(z; 1) := \prod_{i < j} z_i - z_j ^2$
Symmetric Jack polynomials;	$W(z; \alpha) := \prod_{i < j} z_i - z_j ^{2/\alpha}$
Symmetric Macdonald polynomials;	$W(z; q, t) := \prod_{i < j} \frac{(z_i/z_j; q)_\infty (qz_i/z_j; q)_\infty}{(tz_i/z_j; q)_\infty (qtz_i/z_j; q)_\infty}$

B.3 Eigenfunctions

Class	Eigenoperator
Schur polynomials;	$D_2(1) := \sum_{i=1}^n z_i^2 \frac{\partial^2}{\partial z_j^2} + 2 \sum_{i \neq j} \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i}$
zonal polynomials;	$D_2(2) := \sum_{i=1}^n z_i^2 \frac{\partial^2}{\partial z_j^2} + \sum_{i \neq j} \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i}$
Symmetric Jack polynomials;	$D_2(\alpha) := \sum_{i=1}^n z_i^2 \frac{\partial^2}{\partial z_j^2} + \frac{2}{\alpha} \sum_{i \neq j} \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i}$
Symmetric Macdonald polynomials;	$D_n^1(q, t) := \sum_{i=1}^n \prod_{j \neq i} \frac{tz_i - z_j}{z_i - z_j} \tau_i,$ $\tau_i f(z) := f(z_1, \dots, qz_i, \dots, z_n)$

Class	Eigenvalue
Schur polynomials;	$\sum_{i=1}^n \kappa_i (\kappa_i + 1 - 2i) + \kappa (n - 1)$
zonal polynomials;	$\sum_{i=1}^n \kappa_i (\kappa_i - i) + \kappa (n - 1)$
Symmetric Jack polynomials;	$\sum_{i=1}^n \kappa_i (\kappa_i - 1 - \frac{2}{\alpha} (i - 1)) + \kappa (n - 1)$
Symmetric Macdonald polynomials;	$\sum_{j=1}^n q^{\kappa_j} t^{n-j}$

B.4 Examples

Class	Polynomial
Schur polynomials;	$s_{(2,0)}(z) = z_1^2 + z_1 z_2 + z_2^2$
zonal polynomials;	$Z_{(2,0)}(z) = z_1^2 + \frac{2}{3} z_1 z_2 + z_2^2$
Hall-Littlewood polynomials;	$P_{(2,0)}(z; t) = z_1^2 + (1 - t) z_1 z_2 + z_2^2$
Symmetric Jack polynomials;	$P_{(2,0)}(z; \alpha) = z_1^2 + \frac{2}{\alpha+1} z_1 z_2 + z_2^2$
Symmetric Macdonald polynomials;	$P_{(2,0)}(z; q, t) = z_1^2 + \frac{(q+1)(t-1)}{(qt-1)} z_1 z_2 + z_2^2$

Remark: The details of the triangular structure, the defining properties of the inner product and the explicit formulas for the Schur and Hall-Littlewood polynomials can be found in [54]. The alternative inner product formulas for the symmetric Jack and symmetric Macdonald can be found in [23, 57], respectively. The explicit formula for the inner product of the Schur polynomials is obtained by taking $\alpha \rightarrow 1$ in the Jack formula. The details of the eigenoperator and eigenvalue for the Jack and zonal polynomials is in [62], and once again taking the limit $\alpha \rightarrow 1$ to obtain the Schur details from the Jack. The eigenoperator and eigenvalue of the Macdonald polynomial is found in [57].

Appendix C

Nonsymmetric Jack polynomials

C.1 Special function properties

Notation and structure;	$E_\eta(z; \alpha) = z^\eta + \sum_{\lambda < \eta} \hat{K}_{\eta\lambda}(\alpha) z^\lambda,$
Coefficients;	$\hat{K}_{\eta\lambda}(\alpha) \in \mathbb{Q}(\alpha)$
Inner product;	$\langle f(z), g(z) \rangle_\alpha := \text{CT} \left(f(z) g(z^{-1}) \prod_{i < j} z_i - z_j ^{2/\alpha} \right)$
Eigenoperator;	$\xi_i := \alpha z_i \frac{\partial}{\partial x_i} + \sum_{i < j} \frac{z_i}{z_i - z_j} (1 - s_{ij})$ $+ \sum_{j < i} \frac{z_j}{z_i - z_j} (1 - s_{ij}) + 1 - i$
Eigenvalue;	$\bar{\eta}_i^\alpha := \alpha \eta_i - l'_\eta(i)$
Switching operator;	s_i
Raising operator;	$\Phi_\alpha := z_n s_{n-1} \dots s_1$

C.2 Recursive generation formulas

$$E_{s_i \eta}(z; \alpha) = \begin{cases} s_i E_\eta(z; \alpha) - \frac{1}{\bar{\eta}_i^\alpha - \bar{\eta}_{i+1}^\alpha} E_\eta(z; \alpha), & \eta_i < \eta_{i+1} \\ E_\eta(z; \alpha), & \eta_i = \eta_{i+1} \\ \left(1 - \frac{1}{(\bar{\eta}_i^\alpha - \bar{\eta}_{i+1}^\alpha)^2}\right)^{-1} \left(s_i E_\eta(z; \alpha) - \frac{1}{\bar{\eta}_i^\alpha - \bar{\eta}_{i+1}^\alpha} E_\eta(z; \alpha)\right), & \eta_i > \eta_{i+1} \end{cases}$$

$$E_{\Phi_\eta}(z) = \Phi_\alpha E_\eta(z)$$

C.3 Examples

η	$E_\eta(z; \alpha)$	η	$E_\eta(z; \alpha)$
$(0, 0, 0)$	1	$(0, 1, 1)$	$z_2 z_3$
$(0, 0, 1)$	z_3	$(1, 0, 1)$	$z_3(z_1 + \frac{1}{\alpha+2}z_2)$
$(0, 1, 0)$	$z_2 + \frac{1}{\alpha+2}z_3$	$(0, 0, 2)$	$z_3(z_3 + \frac{1}{\alpha+1}(z_1 + z_2))$
$(1, 0, 0)$	$z_1 + \frac{1}{\alpha+1}(z_2 + z_3)$		

Remark: The details within this appendix can be found in [23, Chap. 12].

Appendix D

Nonsymmetric Interpolation Jack polynomials

D.1 Special function properties

Notation and structure;	$E_\eta^*(z; \alpha) = z^\eta + \sum_{\lambda \triangleleft \eta} \tilde{K}_{\eta\lambda}(\alpha) z^\lambda,$
Coefficients;	$\tilde{K}_{\eta\lambda}(\alpha) \in \mathbb{Q}(\alpha)$
Eigenoperator;	$\Xi_i^\alpha := z_i - \sigma_i \dots \sigma_{n-1} \left(z_n - \frac{n-1}{\alpha} \right) \Delta^{(n)} \sigma_1 \dots \sigma_{i-1}$
Eigenvalue;	$\bar{\eta}_i^{\alpha,*} := \eta_i - \frac{1}{\alpha} l'_\eta(i)$
Switching operator;	$\sigma_i := s_i + \frac{1}{\alpha} \frac{1-s_i}{z_i - z_{i+1}}, \quad i = 1, \dots, n-1$
Raising operator;	$\Phi_\alpha^* := \left(z_n + \frac{n-1}{\alpha} \right) \Delta^{(n)},$ $\Delta^{(n)} f(z) := f(z_n - 1, z_1, \dots, z_{n-1})$

D.2 Recursive generation formulas

$$E_{s_i \eta}^*(z; \alpha) = \begin{cases} \sigma_i E_\eta^*(z; \alpha) - \frac{1}{\bar{\eta}_i^\alpha - \bar{\eta}_{i+1}^\alpha} E_\eta^*(z; \alpha), & \eta_i < \eta_{i+1} \\ E_\eta^*(z; \alpha), & \eta_i = \eta_{i+1} \\ \left(1 - \frac{1}{(\bar{\eta}_i^\alpha - \bar{\eta}_{i+1}^\alpha)^2} \right)^{-1} \left(\sigma_i E_\eta^*(z; \alpha) - \frac{1}{\bar{\eta}_i^\alpha - \bar{\eta}_{i+1}^\alpha} E_\eta^*(z; \alpha) \right), & \eta_i > \eta_{i+1} \end{cases}$$

$$E_{\Phi_\eta}^*(z) = \Phi_\alpha^* E_\eta^*(z)$$

D.3 Examples

η	$E_\eta^*(z; \alpha)$	η	$E_\eta^*(z; \alpha)$
(0, 0, 0)	1	(0, 1, 1)	$z_2 z_3 + \frac{2}{\alpha}(z_2 + z_3) + \frac{4}{\alpha^2}$
(0, 0, 1)	$z_3 + \frac{2}{\alpha}$	(1, 0, 1)	$z_3(z_1 + \frac{1}{\alpha+2}z_2) + \frac{2}{\alpha}z_1 + \frac{2}{\alpha(\alpha+2)}z_2$
(0, 1, 0)	$z_2 + \frac{1}{\alpha+2}z_3 + \frac{4+\alpha}{\alpha(\alpha+2)}$		$+ \frac{4+\alpha}{\alpha(\alpha+2)}z_3 + \frac{2(4+\alpha)}{\alpha^2(\alpha+2)}$
(1, 0, 0)	$z_1 + \frac{1}{\alpha+1}(z_2 + z_3) + \frac{3}{\alpha(\alpha+1)}$	(0, 0, 2)	$z_3(z_3 + \frac{1}{\alpha+1}(z_1 + z_2)) + \frac{2}{\alpha(\alpha+1)}(z_1 + z_2)$
			$+ \frac{5+\alpha-\alpha^2}{\alpha(\alpha+1)}z_3 + \frac{2(3-\alpha-\alpha^2)}{\alpha^2(\alpha+1)}$

Remark: The details within this appendix can be found in [23, Chap. 12].



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