STABILITY IMPROVEMENT
OF PERIODIC MOTION
OF HELICOPTER ROTOR BLADES

JOHN SIMON FARAGHER
B. ENG.

A thesis submitted in total fulfilment of
the requirements of the degree of
Doctor of Philosophy

January 1996

Department of Mechanical and Manufacturing Engineering
The University of Melbourne

Produced on Acid Free Paper.
PREFACE

The thesis is less than 100,000 words in length.

Many machines and devices are designed to perform periodic motion (engines, rotors, vehicles, etc.). From an engineering point of view, the stability of this periodic motion is crucial. A lack of stability can lead to poor performance, damage or even destruction. Helicopter rotor blades are prone to various kinds of instability that can cause excessive vibration and lead to structural failure.

For many systems, an accurate mathematical model cannot be produced because of the complexity of the system and poorly known or varying parameters. Such systems are called uncertain systems.

The thesis presents and investigates a new control method for stabilising the periodic motion of uncertain systems - with particular application to helicopter rotor blades. The control method uses proportional displacement and velocity feedback with a time delay in the feedback path which is synchronised with the period of the motion being stabilised. This synchronisation is achieved in practice by using a phase-locked oscillator circuit.

No knowledge of the dynamics of the system being controlled or the desired trajectory is required for this control method. So long as a signal having the required period can be fed to the phase-locked oscillator, the time-delayed feedback can be synchronised with the period. The system can be treated as a "black box" and the parameters of the control method can be adjusted in a trial-and-error fashion until the best performance is obtained. This is demonstrated in the experimental work described
A further property of the control method is that it does not alter the steady-state motion of the uncontrolled system, but improves its stability. This is important for helicopters, where the steady-state motion of the rotor blades determines the flight path of the helicopter.

As an initial investigation, the control method is applied to mathematical models of linear and non-linear one-degree-of-freedom systems. A computer program is developed to determine the stability margin by finding the dominant root of a characteristic equation with an infinite number of roots. The optimal values of the control parameters are found to vary in a clear pattern as a function of the period of the motion being stabilised. The control method is shown to enlarge greatly the region of asymptotic stability of the periodic steady-state motion of the non-linear system, and make it much less sensitive to disturbances.

The effectiveness of the control method is verified experimentally using a multi-degree-of-freedom laboratory installation, employing appropriate hardware and software developed for this purpose. The existence of optimal values of the control parameters and their variation as a function of the period of the motion being stabilised is demonstrated. The pattern of the variation of the optimal values of the control parameters in the experiments is shown to agree with that found for the one-degree-of-freedom mathematical model, but it is more complex because of the additional degrees-of-freedom.

The control method is shown to improve the stability of the equilibrium position of a helicopter rotor blade in hovering flight and the steady-state periodic motion of a helicopter rotor blade in forward flight, using a two-degree-of-freedom
mathematical model, including the aerodynamic forces. This means that when the control method is applied, larger values of the chordwise centre of gravity offset and more flexible mechanical control linkages can be tolerated before the pitch-flap flutter instability occurs.
ACKNOWLEDGEMENTS

The author acknowledges with gratitude:

Dr Janusz Krodkiewski and Professor Zdzislaw Parszewski, the supervisors of the author, for their guidance and assistance throughout this project;

Dr Tom Chalko for his kindly advice and assistance;

Mr Shafiul Alam, the technical officer, for his help at various stages in the laboratory work and for other technical assistance;

Mr Linxiang Sun for the benefit of his advice, assistance and encouragement;

and

Miss Thitima Jintanawan for useful discussions in certain technical matters.

Thanks are also due to the following organizations:

The Aeronautical and Maritime Research Laboratory of the Defence Science and Technology Organisation of the Department of Defence for granting the author a cadetship to make this research possible; and particularly to Mr D. Glenny, Mr N. Swansson and Dr W. Schofield for supporting his application, and to Mr N. Swansson, who, acting as mentor during his cadetship, assisted the author with his encouragement and advice; and

The Department of Mechanical and Manufacturing Engineering, of the University of Melbourne, for providing facilities for this project.
# Table of Contents

## Abstract

## Acknowledgements

## Table of Contents

## Nomenclature

## Chapter 1 - Introduction

1.1 Basics of Helicopter Rotors

1.2 Stability of Motion of Helicopter Rotor Blades

## Chapter 2 - Literature Review

Introduction

**Part A: Active Control with Emphasis on Delayed Feedback**

2.A.1 Introduction to Part A

2.A.2 Review of Relevant Control Methods for Mechanical Systems

2.A.2.1 Time Delay Control

2.A.2.2 Sliding Mode Control

2.A.2.3 Repetitive Control

2.A.2.4 Iterative Learning Control
<table>
<thead>
<tr>
<th>Table of Contents</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2.A.2.5 Adaptive Control</td>
<td>22</td>
</tr>
<tr>
<td>2.A.2.5.1 Gain Scheduling</td>
<td>23</td>
</tr>
<tr>
<td>2.A.2.5.2 Self-Tuning Control</td>
<td>24</td>
</tr>
<tr>
<td>2.A.2.5.3 Model Reference Adaptive Control</td>
<td>25</td>
</tr>
<tr>
<td>2.A.2.6 Optimal Control</td>
<td>26</td>
</tr>
<tr>
<td>2.A.2.7 Summary of Section 2.A.2</td>
<td>27</td>
</tr>
<tr>
<td>2.A.3 The Delayed Resonator</td>
<td>28</td>
</tr>
<tr>
<td>2.A.4 Control of Chaotic Systems</td>
<td>33</td>
</tr>
<tr>
<td>2.A.4.1 The OGY Control Method</td>
<td>35</td>
</tr>
<tr>
<td>2.A.4.2 The Pyragas Methods for Control of Chaotic Systems</td>
<td>38</td>
</tr>
<tr>
<td>2.A.4.2.1 External Force Control</td>
<td>39</td>
</tr>
<tr>
<td>2.A.4.2.2 Delayed Feedback Control</td>
<td>41</td>
</tr>
<tr>
<td>2.A.4.2.3 Pyragas and Tamasevicius (1993)</td>
<td>43</td>
</tr>
<tr>
<td>2.A.4.2.4 Schöll and Pyragas (1993)</td>
<td>44</td>
</tr>
<tr>
<td>2.A.4.2.5 Reznik and Schöll (1993)</td>
<td>45</td>
</tr>
<tr>
<td>2.A.5 Delayed Feedback in Chemical Systems</td>
<td>45</td>
</tr>
<tr>
<td>2.A.5.1 Weiner, Schneider and Bar-Eli (1989)</td>
<td>46</td>
</tr>
<tr>
<td>2.A.5.1.1 Method A</td>
<td>47</td>
</tr>
<tr>
<td>2.A.5.1.2 Method B</td>
<td>48</td>
</tr>
<tr>
<td>2.A.5.1.3 Method A Results</td>
<td>48</td>
</tr>
<tr>
<td>2.A.5.1.4 Method B Results</td>
<td>52</td>
</tr>
<tr>
<td>2.A.5.2 Weiner, Holz, Schneider and Bar-Eli (1992)</td>
<td>54</td>
</tr>
<tr>
<td>2.A.5.3 Holz and Schneider (1993)</td>
<td>57</td>
</tr>
<tr>
<td>2.A.5.4 Schneider, Blittersdorf, Förster, Hanck, Lebender &amp; Müller (1993)</td>
<td>58</td>
</tr>
</tbody>
</table>
### CHAPTER 4 - STABILITY IMPROVEMENT OF THE PERIODIC MOTION OF ONE-DEGREE-OF-FREEDOM SYSTEMS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1. Introduction</td>
<td>111</td>
</tr>
<tr>
<td>4.2. Description of the Physical Model</td>
<td>112</td>
</tr>
<tr>
<td>4.3. Analysis of the Linear System</td>
<td>113</td>
</tr>
<tr>
<td>4.3.1 Equation of Perturbations</td>
<td>115</td>
</tr>
<tr>
<td>4.3.2 Solution of the Characteristic Equation</td>
<td>117</td>
</tr>
<tr>
<td>4.3.3 Optimal Values of the Control Parameters</td>
<td>118</td>
</tr>
<tr>
<td>4.3.4 Response of the Controlled System</td>
<td>124</td>
</tr>
<tr>
<td>4.4 Analysis of the Non-Linear System</td>
<td>126</td>
</tr>
<tr>
<td>4.4.1 Non-Linear Equation of Motion</td>
<td>126</td>
</tr>
<tr>
<td>4.4.2 Dynamic Behaviour of the Uncontrolled Object</td>
<td>127</td>
</tr>
<tr>
<td>4.4.3 Equation of Perturbations</td>
<td>129</td>
</tr>
<tr>
<td>4.4.4 Optimal Values of the Control Parameters</td>
<td>130</td>
</tr>
<tr>
<td>4.4.5 Results of Numerical Computation using the Non-Linear Mathematical Model</td>
<td>132</td>
</tr>
<tr>
<td>4.4.5.1 Influence of the Control on the Region of Asymptotic Stability</td>
<td>132</td>
</tr>
<tr>
<td>4.4.5.2 Influence of the Control on the Response of the Non-Linear System</td>
<td>133</td>
</tr>
<tr>
<td>4.5 Conclusions</td>
<td>134</td>
</tr>
</tbody>
</table>
CHAPTER 5 - APPLICATION OF THE CONTROL METHOD TO A
LABORATORY INSTALLATION

5.1 Introduction 135
5.2 Hardware Setup 135
5.3 Software 139
5.4 Automatic Detection of the Excitation Frequency 142
5.5 Experimental Determination of the Optimal Values of the Control Parameters 142
5.6 Experimental Results for an Excitation Frequency of 26 Hz 147
  5.6.1 Equally Dominant Characteristic Roots 149
  5.6.2 Extreme Variation in Stability 150
5.7 Experimental Results for an Excitation Frequency of 16 Hz 152
5.8 Case 1: \(a = -0.4, b = -0.25\) s, \(c = 75\) 157
5.9 Case 2A: \(a = -0.1, b = -0.2\) s, \(c = 75\) 160
5.10 Case 2B: \(a = -0.1, b = -0.2\) s, \(c = 150\) 163
5.11 Case 3: \(a = 0.6, b = 0.1\) s, \(c = 0\) 171
5.12 Case 4A: \(a = 0.16, b = -0.06\) s, \(c = 215\) 174
5.13 Case 4B: \(a = 0.16, b = -0.06\) s, \(c = 350\) 185
5.14 Case 4C: \(a = 0.16, b = -0.06\) s, \(c = 300\) 187
5.15 Cases 5, 6, 7 and 8 190
5.16 Case 5A: \(a = 0.35, b = -0.14\) s, \(c = 575\) 191
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.17</td>
<td>Case 5B: $a = 0.35, b = -0.14 , s, c = 575-610$</td>
<td>194</td>
</tr>
<tr>
<td>5.18</td>
<td>Case 6A: $a = 0.35, b = -0.15 , s, c = 430$</td>
<td>198</td>
</tr>
<tr>
<td>5.19</td>
<td>Case 6B: $a = 0.35, b = -0.15 , s, c = 460$</td>
<td>201</td>
</tr>
<tr>
<td>5.20</td>
<td>Case 6C: $a = 0.35, b = -0.15 , s, c = 430-460$</td>
<td>204</td>
</tr>
<tr>
<td>5.21</td>
<td>Comparison of Case 5 with Case 6</td>
<td>206</td>
</tr>
<tr>
<td>5.22</td>
<td>Case 7: $a = 0.35, b = -0.145 , s, c = 470$</td>
<td>206</td>
</tr>
<tr>
<td>5.23</td>
<td>Case 8A: $a = 0.35, b = -0.11 , s, c = 520$</td>
<td>209</td>
</tr>
<tr>
<td>5.24</td>
<td>Case 8B: $a = 0.35, b = -0.11 , s, c = 400$</td>
<td>212</td>
</tr>
<tr>
<td>5.25</td>
<td>Conclusions</td>
<td>214</td>
</tr>
</tbody>
</table>

**CHAPTER 6 - THE CONTROL METHOD APPLIED TO A HELICOPTER ROTOR BLADE**

6.1 Introduction | 215
6.2 Non-Linear Mathematical Model | 218
6.3 Linear Mathematical Model | 220
6.3.1 Flight Conditions | 220
6.3.2 Equations of Perturbations for the Controlled System | 221
6.3.2.1 Equations for Hovering Flight Without Control | 221
6.3.2.2 Equations for Hovering Flight With Control | 222
6.3.2.3 Equations for Forward Flight Without Control | 224
6.3.2.4 Equations for Forward Flight With Control | 225
6.3.3 Physical Model | 227
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.3.4 Systems of Coordinates</td>
<td>229</td>
</tr>
<tr>
<td>6.3.5 Flap and Pitch Natural Frequencies</td>
<td>230</td>
</tr>
<tr>
<td>6.3.6 Non-Dimensional Time</td>
<td>231</td>
</tr>
<tr>
<td>6.3.7 Centre of Gravity Offset</td>
<td>231</td>
</tr>
<tr>
<td>6.3.8 Control Moment</td>
<td>232</td>
</tr>
<tr>
<td>6.4 Motion of the Rotor Blade in Hovering Flight without Control</td>
<td>233</td>
</tr>
<tr>
<td>6.4.1 Equations of Perturbations</td>
<td>233</td>
</tr>
<tr>
<td>6.4.2 Parameter Values</td>
<td>235</td>
</tr>
<tr>
<td>6.4.3 State-Space Formulation</td>
<td>235</td>
</tr>
<tr>
<td>6.4.4 Stability Analysis</td>
<td>236</td>
</tr>
<tr>
<td>6.4.4.1 Pitch Divergence</td>
<td>237</td>
</tr>
<tr>
<td>6.4.4.2 Pitch-flap Dynamic Instability</td>
<td>242</td>
</tr>
<tr>
<td>6.5 Rotor Blade with Control in Hovering Flight</td>
<td>246</td>
</tr>
<tr>
<td>6.5.1 State-Space Formulation</td>
<td>249</td>
</tr>
<tr>
<td>6.5.2 Stability Analysis</td>
<td>252</td>
</tr>
<tr>
<td>6.5.3 Pitch Divergence</td>
<td>254</td>
</tr>
<tr>
<td>6.5.4 Optimal Values of the Control Parameters</td>
<td>256</td>
</tr>
<tr>
<td>6.5.5 An Illustrative Example</td>
<td>264</td>
</tr>
<tr>
<td>6.6 Rotor Blade with Control in Forward Flight</td>
<td>270</td>
</tr>
<tr>
<td>6.6.1 Equations of Perturbations for Forward Flight</td>
<td>270</td>
</tr>
<tr>
<td>6.6.2 Forward Flight without Control</td>
<td>274</td>
</tr>
<tr>
<td>6.6.3 Forward Flight with Control</td>
<td>276</td>
</tr>
<tr>
<td>6.7 Practical Determination of Parameters a and b</td>
<td>277</td>
</tr>
</tbody>
</table>
NOMENCLATURE

\( A_m \) \hspace{1cm} \text{pitch moment of inertia of the rotor blade}
\( a \) \hspace{1cm} \text{adjustable control parameter}
\( a_f \) \hspace{1cm} \text{lift curve slope of the airfoil}
\( a_{opt} \) \hspace{1cm} \text{optimal value of control parameter} \ a

\( B_m \) \hspace{1cm} \text{flap moment of inertia of the rotor blade}
\( b \) \hspace{1cm} \text{adjustable control parameter}
\( b_{opt} \) \hspace{1cm} \text{optimal value of control parameter} \ b

\([C]\) \hspace{1cm} \text{damping matrix}
\( c_1, c_2, c_3 \) \hspace{1cm} \text{damping coefficients}
\( c_h \) \hspace{1cm} \text{chord dimension of the rotor blade}
\( C_{max} \) \hspace{1cm} \text{amount of negative damping that could be added to \textit{controlled} system before periodic motion became unstable}
\( C_{ref} \) \hspace{1cm} \text{amount of negative damping that could be added to \textit{uncontrolled} system before periodic motion became unstable}
\( \Delta c \) \hspace{1cm} \text{percentage increase (or decrease) in the amount of negative damping that could be added to the \textit{controlled} system before the motion became unstable, compared with the amount of negative damping that could be added to the \textit{uncontrolled} system before the motion became unstable}
Nomenclature

\( D \) amplitude of the periodic particular solution

\( e \) \( \approx 2.71828 \)

\( F_d \) damping force

\( f_{ex} \) the frequency of the sinusoidal excitation provided by synthesiser

\( f_o \) the frequency of the perturbations around the periodic steady-state motion that grew exponentially

\( g \) acceleration due to gravity

\( I_{3x}, I_{3y}, I_{3z} \) moments of inertia of the rotor blade

\( i \) \( \sqrt{-1} \)

\([K]\) stiffness matrix

\( k, k_l \) spring stiffness in one-degree-of-freedom system (in Chapter 4)

\( k^2 \) square of non-dimensional radius of gyration of chordwise element of rotor blade about elastic axis (in Chapter 6)

\( k_{lin} \) stiffness of linear spring element in Figure 6.3

\( k_{rot} \) effective torsional stiffness of linear spring element in Figure 6.3

\( l_1, l_2 \) rotor blade dimensions

\( l_\alpha, l_\theta \) aerodynamic derivative coefficients
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[M]$</td>
<td>inertia matrix</td>
</tr>
<tr>
<td>$M_b$</td>
<td>mass of the rotor blade</td>
</tr>
<tr>
<td>$M_y$</td>
<td>control moment exerted on rotor blade by a displacement $y$ of the lower end of the spring</td>
</tr>
<tr>
<td>$m$</td>
<td>mass in physical model shown in Fig. 4.1</td>
</tr>
<tr>
<td>$m_3$</td>
<td>rotor blade mass in a non-linear physical model</td>
</tr>
<tr>
<td>$m_6$</td>
<td>aerodynamic derivative coefficient</td>
</tr>
<tr>
<td>$n$</td>
<td>Lock Number + 8</td>
</tr>
<tr>
<td>$R$</td>
<td>real part of complex root</td>
</tr>
<tr>
<td>$R_{dom}$</td>
<td>real part of dominant characteristic root</td>
</tr>
<tr>
<td>$R_o$</td>
<td>root cut-out dimension of the rotor blade</td>
</tr>
<tr>
<td>$R_s$</td>
<td>span dimension of the rotor blade</td>
</tr>
<tr>
<td>$r'$</td>
<td>aspect ratio of the rotor blade</td>
</tr>
<tr>
<td>$r_g$</td>
<td>spanwise location of the centre of gravity of the rotor blade</td>
</tr>
<tr>
<td>$r_{mom}$</td>
<td>dimension of the moment arm of the pitch link</td>
</tr>
<tr>
<td>$T$</td>
<td>period</td>
</tr>
<tr>
<td>$T_{ex}$</td>
<td>period of the sinusoidal excitation force</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$U$</td>
<td>imaginary part of complex root</td>
</tr>
</tbody>
</table>
Nomenclature

$U_{dom}$ imaginary part of dominant characteristic root

$V$ velocity of the helicopter relative to the surrounding air

$X$ initial function

$x$ displacement

$x_0$ equilibrium position or steady-state solution

$\Delta x$ perturbations of displacement about steady-state

$y$ displacement

$y_c$ control input

$y_{ex}$ harmonic excitation

$y_0$ amplitude of harmonic excitation (Chapter 4)

in Figure 6.3, displacement of lower end of spring caused by pilot (Chapter 6)

$\Delta y$ Phase-Locked Delayed Feedback control input to flexible mechanical control linkage

$\{\Delta Z\}$ state vector of perturbations

* * *

$\alpha$ pitch angle in non-linear mathematical model

$\alpha_0$ pitch angle input from pilot in non-linear mathematical model
Δα  pitch angle control input in non-linear mathematical model

β  flap angle of the rotor blade

β₀  steady-state flap angle

Δβ  perturbations of flap angle about steady-state

γ  Lock Number of the rotor

θ  pitch angle of the rotor blade

θ₀  steady-state pitch angle

Δθ  perturbations of pitch angle about steady-state

λ  complex characteristic root or eigenvalue

λ₁Ω  flap natural frequency of rotor blade

μ  tip-speed ratio of rotor blade

ν₁  non-dimensional pitch natural frequency of rotor blade

ν₁Ω  pitch natural frequency of rotor blade

ρ  air density

σ  non-dimensional chordwise centre of gravity offset of rotor blade

φ  phase of the periodic particular solution
\( \psi \)  
non-dimensional time, which is the angle of rotation of the rotor shaft about the \( Z \) axis (which is called the "azimuth angle")

\( \Omega \)  
angular velocity of the rotor shaft

\( \omega \)  
frequency of harmonic excitation

---

A/D  
analogue/digital

C  
name of computer programming language

CF  
centrifugal force

CF\(_n\)  
component of centrifugal force normal to rotor blade

CRO  
cathode ray oscilloscope

D/A  
digital/analogue

DC  
direct current

ISR  
interrupt service routine

LHP  
left half-plane

PC-30DS  
model number of the analogue input/output board

PLO  
phase-locked oscillator

RHP  
right half-plane

---
Nomenclature

Eq.4.1: Equations are denoted by "Eq." and two numbers, the first denoting the
    Chapter and the second the number of the equation within the Chapter (eg.
    Eq.4.1 is the first equation in Chapter 4)

Figure 1.1: Figures are numbered within each Chapter, and thus Figure 1.1 is the first figure in Chapter 1

**   second derivative with respect to non-dimensional time ($\psi$)

*    first derivative with respect to non-dimensional time ($\psi$)

[1]  unity matrix

det   determinant of a matrix

Because different authors use different notations, the nomenclature for the literature review in Chapter 2 has not been listed, but is defined as needed when it is referred to.
This thesis is concerned with a control method for stabilising periodic motion, which was developed for application to helicopter rotor blades. Juan de la Cierva, the pioneer of the autogyro, which was the predecessor of the helicopter, discovered that unless the rotor blades were free to flap up and down, unequal lift forces would be generated on the two sides of the rotor when the aircraft moved forward, causing it to overturn on the runway. The flapping motion of the rotor blades created in-plane Coriolis forces which were alleviated by introducing hinges to allow the blades to move freely in their plane of rotation.

Control of the flight path of a helicopter in both the vertical and horizontal directions is achieved by adjusting the angle of attack of the rotor blades using a translating swashplate mechanism. Therefore hinges which allow the blades to rotate about their own longest axis are also required.

In all, three rotational degrees-of-freedom must be provided for each rotor blade.

This mechanical design lies at the heart of the helicopter’s ability to fly as it does. It will be unknown to anyone who is not familiar with helicopters, and therefore will be explained in more detail below.
1.1 Basics of Helicopter Rotors

Schematic diagrams of the mechanical control system of a helicopter rotor are shown in Figures 1.1, 1.2 and 1.3.

The rotor shaft is driven by the engine of the helicopter and rotates about a vertical axis. It has one rotational degree-of-freedom relative to the fuselage. The rotor hub is rigidly attached to the rotor shaft and rotates with it.

The rotor blade, the rotor blade cuff and the pitch horn form one body. In the schematic diagram presented above, the rotor blade cuff is attached to the rotor hub.
Chapter 1 Introduction

by a spherical bearing which allows three rotational degrees-of-freedom. In an “articulated” helicopter rotor, these three degrees-of-freedom are provided by three separate hinges, each of which allows a single degree-of-freedom. In “hingeless” helicopter rotors, two of the three hinges are replaced by flexible structural elements. In “bearingless” helicopter rotors, all three degrees-of-freedom are provided by flexible structural elements. The fact that the rotor blade has three rotational degrees-of-freedom relative to the rotor hub, and is therefore free to rotate in any direction, is one of the major differences between the rotor blade and the wings of a fixed-wing aircraft. The movement of the rotor blade (relative to the hub) in the plane of rotation of the rotor is called “lead-lag” motion. The movement of the rotor blade perpendicular to
the plane of rotation of the rotor is called “flap” motion. The rotation of the rotor blade about its own longest axis to vary its angle-of-attack, is called “pitch” motion.

The only thing which keeps the blades of a helicopter rotor in an approximately horizontal position is the enormous centrifugal force that acts on them when the rotor is turning at its operating speed. This fact is illustrated by the following. Rotor blades are fairly large. For example, the Sikorsky Black Hawk helicopter rotor has four blades. Each blade is about 8 m long, and 0.5 m wide, and weighs about 100 kg. The normal operating speed of the rotor is 258 rpm (which is about 27 rad/s).

So, the centrifugal force is approximately

\[ CF = M_b \Omega^2 R_c = 100 \text{ kg} \times (27 \text{ rad/s})^2 \times 4 \text{ m} = 292 \text{ kN}. \]

The helicopter mass \( \approx 10,000 \) kg. Therefore the weight \( \approx 100 \) kN.
Therefore in hover the required lift $= 25 \text{kN}$ per blade.

The weight of the blade $= 100 \text{ kg} \times 10 \text{ N/kg} = 1 \text{kN}$.

![Diagram of forces acting on the rotor blade]

If the rotor blade flap angle increases, the moment arm of the centrifugal force about the hinge is increased; and if the flap angle decreases, this moment arm is decreased.

So the centrifugal force acts like a powerful spring keeping the blade in position.

The pitch horn (shown in Figures 1.1 and 1.3) provides a moment arm which gives leverage to the control system in order to control the pitch angle of the rotor blade. The pitch link is simply a straight rod that has constraints at each end which allow three rotational degrees-of-freedom. One end is connected to the pitch horn and the other end is connected to the rotating swashplate.

The rotating swashplate has one rotational degree-of-freedom relative to the stationary swashplate. It is constrained to tilt and move vertically with the stationary swashplate. The constraint between the two is a circular ring of ball bearings. The rotating swashplate is connected to the rotor shaft by a series of linkages which are not shown in the schematic diagrams in Figures 1.1, 1.2 and 1.3. The linkages constrain the rotating swashplate to rotate with the rotor shaft, but they do not constrain the
vertical motion or tilting of the rotating swashplate at all.

The stationary swashplate is given this name to emphasise that it does not rotate with the rotor shaft, in contrast to the rotating swashplate. The stationary swashplate actually has three degrees-of-freedom relative to the fuselage: it can tilt about its two horizontal axes; and it can translate parallel to the rotor shaft.

Three bell cranks provide independent inputs to control the three degrees-of-freedom of the stationary swashplate. Each bell crank has one rotational degree-of-freedom relative to the fuselage, and is connected to the stationary swashplate by an input rod.

If the swashplate is in a horizontal position and not tilted at all then the pitch angle of each rotor blade will be constant as it rotates with the rotor shaft, and the pitch angle of all the rotor blades will be the same. (This is typically the case in hovering or vertical flight.) If the swashplate is moved vertically while it is still in a horizontal position and not tilted, the angle of attack of all the blades will be changed simultaneously, and the angle of attack of the blades will still be constant as the blades rotate with the rotor shaft. Control inputs of this kind are called "collective pitch"; and they are used to increase or decrease the thrust of the rotor.

If the stationary swashplate is tilted, then as the rotating swashplate rotates with the rotor shaft, the vertical position of the pitch link will vary cyclically once per revolution of the rotor shaft. The pitch angle of each rotor blade will also vary cyclically. Inputs to the swashplate that cause it to tilt are called "cyclic pitch" inputs.

The two tilting degrees-of-freedom of the swashplate are controlled by the joystick which the pilot moves with his right hand. With this control stick he controls the direction of the thrust of the rotor. In addition to this, he has a second control
stick which he moves with his left hand. This controls the vertical translation of the swashplate, which controls the amount of thrust produced by the rotor.

Active control methods typically use small modifications to the motion of the swashplate to modify the behaviour of the rotor blade by changing its pitch angle. However, the maximum number of rotor blades which can be controlled independently by using the swashplate is three, because the swashplate has only three independent inputs. To control independently the blades of a four-bladed rotor, the pitch links of each rotor blade can be replaced by actuators. This is called “Individual Blade Control”. Alternatively, a swashplate can be used with a four-bladed rotor to superimpose additional rotor blade pitch inputs on the pilot’s inputs at discrete frequencies which are integer multiples of the rotor speed. This is called “Higher Harmonic Control”. Both Individual Blade Control and Higher Harmonic Control are discussed in Chapter 2. As yet, neither of these control methods has been incorporated in a production helicopter.

1.2 Stability of Motion of Helicopter Rotor Blades

In many technical applications it is necessary to improve the stability of the periodic motion of mechanical systems.

- Helicopter rotor blades are prone to various kinds of instabilities that can cause excessive vibration and lead to structural failure. Therefore it is desirable to develop a control method that will stabilise periodic motion with particular application to helicopter rotor blades.
- The state-of-the-art mathematical models of helicopter rotors are very complex and
not entirely accurate, because the complexity of the aerodynamics and structural dynamics renders helicopter rotors extremely difficult to model mathematically. Therefore it is desirable to develop a control method that does not require a mathematical model of the system being controlled, or any knowledge of the dynamics of the system.

- A consequence of the intractability of helicopter rotor analysis is that the periodic steady-state motion of the helicopter rotor blades cannot be predicted accurately. It is not possible to use one of the existing control methods that will make the system track a particular trajectory if the trajectory is not known. Therefore it is desirable to develop a control method that does not require the desired motion of the system to be known.

- The steady-state motion of the helicopter rotor blades determines the flight path of the helicopter. Therefore it is desirable to develop a control method that improves the stability of the steady-state motion without changing it at all.

- In helicopters the rotational speed of the rotor may vary slightly in flight. Therefore it is desirable to develop a control method that does not require the period of the motion being stabilised to be known beforehand, but can detect the period in real-time and adjust itself accordingly.

The above criteria are satisfied by the control method which has been developed for investigation in this thesis. The method is a new kind of active control which has been called "Phase-Locked Delayed Feedback". It combines proportional displacement and velocity feedback which is delayed in time with a phase-locked oscillator (PLO), which allows the time-delay in the feedback to be synchronised with the period of a periodic trajectory. It is not necessary to know the trajectory which is
to be stabilised or the dynamics of the system involved. It is only necessary to know
the period. Furthermore, the period does not have to be known in advance. As long
as a signal having the required period can be fed to the phase-locked oscillator, the
time-delayed feedback can be synchronised with this period.

A diagram of the Phase-Locked Delayed Feedback control method applied to a
one-degree-of-freedom system is shown in Figure 1.5.

The input $y$ to the object is the sum of the external input $y_{ex}$ and the control
input $y_c$ where:

$$y_c = a(x(t) - x(t - T)) + b(\dot{x}(t) - \dot{x}(t - T));$$

and

\[ t = \text{time}; \]

\[ T = \text{delay time which is set equal to the period of the motion being stabilised}; \]

\[ x = \text{output}; \text{ and} \]

\[ a, b = \text{parameters of the control law which are adjusted to achieve optimal} \]

\[ \text{performance.} \]
A most important feature of the Phase-Locked Delayed Feedback control method is that the periodic steady-state motion of the controlled system is identical with that of the uncontrolled system but will have increased stability if the control parameters are set correctly. This is an extremely useful property for many applications, and in particular for the stabilisation of the motion of helicopter rotor blades. The pilot controls the flight path of the helicopter by directly controlling the motion of the rotor blades using “cyclic” and “collective” rotor blade pitch angle (angle-of-attack) controls. It is important that the steady-state motion of the rotor blades, and hence the flight path of the helicopter, is not affected by the controller. The controller aims to provide increased stability and decreased response to disturbances.
CHAPTER 2

LITERATURE REVIEW

INTRODUCTION

The subject of this thesis brings together two quite different areas of research in a way that has not been done before: namely, the area of active control using time-delayed feedback; and the area of active control applied to helicopter rotors. These two areas are treated separately in the literature review. It will be seen that prior to this thesis they did not overlap.

PART A: ACTIVE CONTROL WITH EMPHASIS ON DELAYED FEEDBACK

2.A.1 Introduction to Part A

The first part of the literature review is divided into four sections:

- The first section reviews relevant active control methods for mechanical systems where delayed feedback is used, or there is imprecise knowledge of the dynamics of the system being controlled.
• The second section describes an active vibration absorber whose resonant frequency and stability margin can be tuned by adjusting the delay time and feedback gain of a delayed feedback signal. This invention is very different from the Phase-Locked Delayed Feedback control method, but has in common with it that delayed feedback is used to modify the stability margin of a system.

• The third section centres around the Delayed Feedback Control method of Pyragas(1992) which is the closest method to the Phase-Locked Delayed Feedback control method that was found in the literature. This method was developed for stabilising the unstable periodic orbits of chaotic systems.

• The fourth section describes research into delayed feedback in periodically oscillating chemical systems. Some of this research is based on the Delayed Feedback Control method of Pyragas (1992) for chaotic systems described in the third section.

2.A.2 Review of Relevant Control Methods for Mechanical Systems

2.A.2.1 Time Delay Control

Youcef-Toumi and Wu (1992a, 1992b) designed a robust controller for complicated nonlinear systems with various kinds of uncertainties. The uncertainties may be internal or external to the control system. External uncertainties are unpredictable disturbances from the surrounding environment. Internal uncertainties may be due to:

(i) poor estimates of the values of the parameters in the mathematical model of the
• The second section describes an active vibration absorber whose resonant frequency and stability margin can be tuned by adjusting the delay time and feedback gain of a delayed feedback signal. This invention is very different from the Phase-Locked Delayed Feedback control method, but has in common with it that delayed feedback is used to modify the stability margin of a system.

• The third section centres around the Delayed Feedback Control method of Pyragas (1992) which is the closest method to the Phase-Locked Delayed Feedback control method that was found in the literature. This method was developed for stabilising the unstable periodic orbits of chaotic systems.

• The fourth section describes research into delayed feedback in periodically oscillating chemical systems. Some of this research is based on the Delayed Feedback Control method of Pyragas (1992) for chaotic systems described in the third section.

2.A.2 Review of Relevant Control Methods for Mechanical Systems

2.A.2.1 Time Delay Control

Youcef-Tourni and Wu (1992a, 1992b) designed a robust controller for complicated nonlinear systems with various kinds of uncertainties. The uncertainties may be internal or external to the control system. External uncertainties are unpredictable disturbances from the surrounding environment. Internal uncertainties may be due to:

(i) poor estimates of the values of the parameters in the mathematical model of the
system being controlled;

(ii) the nature of the system being controlled varies with time but the values of the parameters in the mathematical model do not - therefore the accuracy with which the mathematical model represents the real system varies with time;

(iii) the mathematical model is too simple (of low order) to describe accurately the behaviour of the system.

The control method developed by Youcef-Toumi and Wu (1992a, 1992b) is called Time Delay Control (TDC). This method uses information from the recent past, i.e. output signals and control actions slightly delayed in time, to make estimates of the effect of the uncertain quantities (see Figure 2.1). This includes both internal uncertainties and external disturbances. The unwanted dynamic effects can then be cancelled and the desired dynamics can be created. The TDC method is based on the assumption that the unwanted dynamic effects do not vary significantly during the delay time. TDC does not require an explicit plant model, nor does it depend on the estimation of specific plant parameters. The controller is linear and has fixed gains. It has a simple algorithm that is easy to implement. Unlike full-state feedback methods, the TDC method does not need full access to the system state-variables or precise knowledge of the system dynamics. The TDC controller asymptotically achieves global input/output linearisation using output feedback. A linear compensator can then easily be designed for output tracking.
The main weakness of the TDC method is its sensitivity to measurement noise. This is due to the need to estimate numerically the derivatives of the system state variables.

Time Delay Control has been successfully applied to active magnetic bearings (Youcef-Toumi and Reddy 1992a, Youcef-Toumi and Bobbett 1991), a two-degree-of-freedom robot manipulator and a servo motor system (Youcef-Toumi and Ito 1990, Youcef-Toumi and Huang 1993), and an automotive cruise control system (Youcef-Toumi et al. 1992).

The important difference between Time Delay Control and the control method presented in this thesis is that TDC uses a reference model to produce the input to the control system or generate the desired trajectory; whereas the control method presented in this thesis is designed for systems in which the desired trajectory is known to be periodic with known period and the desired trajectory is unknown.
2. A. 2. 2 Sliding Mode Control

Sliding Mode Control is another control method developed to cope with various uncertainties. It has been applied to trajectory control in the field of robotics (see Utkin (1977), Young (1978), Slotine and Sastry (1983), Asada and Slotine (1986)).

The uncertainties include:

1. uncertainty in the parameters of the mathematical model due to imprecise knowledge of the mass properties and geometry of the manipulator and the load, the position of the load in the end-effector, friction, etc.;
2. the presence of high frequency dynamics not included in the mathematical model, such as structural resonant modes and neglected time delays;
3. external disturbances.

Sliding Mode Control assumes that the extent of the imprecision in the mathematical model, the size of the disturbances and the control gains are not known but they have known upper bounds.

The aim of this control method is to make the trajectory of the manipulator track a desired trajectory. An acceptable value of tracking error must be chosen. This defines a "boundary layer" around the desired trajectory within which the manipulator must stay. If this value is large the control is smooth but tracking accuracy is poor. If the boundary layer thickness is set to too small a value, the control is discontinuous. So a compromise must be made between tracking accuracy and undesirable high frequency chatter.

A recently introduced modification to this control method is called Sliding Mode Control with Perturbation Estimation (Olgac and Moura, 1993). Instead of
working with fixed values for the upper bounds of the modelling uncertainties and external disturbances this method estimates them on-line. While this method performed very well in experiments at low speeds the authors refer to it as “the motion control algorithm of the future” because the electronic hardware required to test it at high speeds is not yet available.

While Sliding Mode Control can cope with modelling uncertainties and external disturbances it differs from the control method presented in this thesis in that it uses a desired trajectory that is known, and an explicit model of the plant.

2.1.2.3 Repetitive Control

Repetitive Control is used where the reference commands to be tracked or the disturbances to be rejected are periodic signals with a fixed period. It has been shown to be useful for practical applications such as mechanical systems involved in repetitive operations, e.g. industrial robots and Numerically Controlled (NC) machines, and for systems with disturbances depending on the frequency of the power supply (Hara et al. 1988, Chew and Tomizuka 1990, Inoue et al. 1981).

Repetitive Controllers are based on the Internal Model principle (Francis and Wonham 1975). This principle says that the controlled output will track a class of reference commands, without a steady-state error, against unknown and unmeasurable disturbance inputs, if the generator for the reference commands is included in the stable closed-loop system. A Repetitive Controller is defined to be a controller which includes a periodic signal generator in the closed-loop system. The periodic signal generator, as shown in Figure 2.2, includes a time-delay element with delay equal to
Repetitive Control is a simple kind of learning control because it uses information from previous periods to determine the control input to the plant, and the performance of the system is improved from period to period. It can also be applied to slowly varying periodic systems, as it will keep reducing the error between the desired and actual trajectories in spite of the variations in the system.

Curtelin et al. (1994) successfully applied a Repetitive Control method to an experimental magnetic bearing rig. They developed a control method that was designed to improve the performance of closed-loop systems already stabilized by classical means such as PID (proportional-plus-integral-plus-derivative) controllers. Their Repetitive Control method differed from that of Hara et al. (1988) in that the delay loop came after the controller rather than before it, so that the signal fed to the delay loop was the controller output, and not the error signal. This can be seen by
comparing Figures 2.3 and 2.4. The control input signal to the plant was a summation of the controller output signal and a signal that was a function of the past periodic control signals. During the first period of operation the additional signal from the delay loop was zero. Although they found their Repetitive Control method to be robust for certain kinds of non-repetitive perturbations, they suggest that further work could be done on robustness analysis with regard to the neglected dynamics and to other kinds of non-repetitive perturbations.

Figure 2.3. Block Diagram for Repetitive Control. [After Hara et al. (1988)]

Figure 2.4. Block Diagram for Repetitive Control. [After Curtelin (1994)]
Repetitive Control is similar to the control method presented in this thesis in that it applies to systems with periodic motion where the period is known, but it is different in that it tracks a desired trajectory rather than improving the stability of an unknown periodic trajectory.

2.A.2.4 Iterative Learning Control

A system is called a “learning control system” if information pertaining to the unknown features of the process or its environment is acquired by the system and the obtained experience is used in such a way that the performance of the system is improved. Using conventional closed-loop control, a robot manipulator (or any plant) performing a repetitive task would have the same performance at each repetition. Iterative Learning Control (ILC) uses information from the past to improve the performance of the system from repetition to repetition. It is easy to design and implement. It gives computational economy and requires no knowledge of the plant parameters. It is especially suited for implementation on digital computers.

ILC is aimed at producing a sequence of control input trajectories that cause the controlled output to converge asymptotically to a desired trajectory (see Arimoto 1990, Arimoto et al. 1984, Moore 1993, Craig 1984). The periodic reference command is corrected using the error signal from the previous trial. In other words, the proportional difference between the desired output and the actual output is superimposed on the control input from one trial to form the control input for the next trial. Figure 2.5 shows a block diagram for ILC. This may be contrasted to the block diagram for a fixed control method in Figure 2.6.
Figure 2.5. Block Diagram for Iterative Learning Control. [After Arimoto et al. (1984)]

Figure 2.6. Block Diagram for a Fixed Control method.

The difference between Repetitive Control and ILC is as follows. In Repetitive Control the initial state at the start of each period is equal to the final state at the end of the previous period. So it may vary from period to period. Thus the process is continuous and the closed-loop system is a retarded or neutral type time-delay system and it is not easy to stabilize the system. In ILC the same initial state is assumed at the start of every trial. Hence the process is discrete and both the stability and the convergence of the error to zero is assured.
ILC has been applied to robot manipulators in the manufacturing industry (Arimoto 1990, Arimoto et al. 1984, Moore 1993, Craig 1984). It is also applicable to batch processing in the chemical and bio-technological industries. Zhang et al. (1994) found that the ILC method in combination with a neural network model of the plant was very suitable and convenient for the control of an industrial fed-batch fermentation process. Conventional feedback control and adaptive control methods were not suitable for this application.

Buchheit et al. (1994) showed that a modified version of ILC called Optimal ILC was superior to conventional PID control when applied to the control of the output temperature of an aluminium extrusion plant. The desired output trajectory was a constant output temperature of 540°C in the face of nonlinearity, boundedness of the control action ($U_{\text{min}}, U_{\text{max}}$), and variation of the plant with time during each cycle.

Unlike ILC, Optimal ILC incorporates identification of the plant and an adaptation algorithm. The implementation of the method is as follows. The initial trial uses a starting trajectory for the control input to the plant and the output from the plant is observed. The following procedure is then repeated for every trial:

1. identification of the plant by calculation of the step response in the time domain;
2. determination of the new control input trajectory by minimization of a performance functional through simulation;
3. application of the new control input trajectory to the plant and observation of the output.
Figure 2.7. Block Diagram for Optimal Iterative Learning Control. [After Buchheit et al. (1994)]

ILC aims to make the error between the desired and actual output trajectories equal to zero (see Figure 2.5). Optimal ILC (see Figure 2.7) aims to minimize a cost functional of the output error and the control action which represents the system performance. So it tries to smooth the control input trajectory as well as to minimize the output error.

2.4.2.5 Adaptive Control

Adaptive controllers, as the name suggests, are designed for situations where the nature of the plant or the disturbances changes with time, and where a fixed controller is inadequate. As the plant or the environment changes the adaptive algorithm adapts the controller to suit the new conditions. See for example Åström and Wittenmark (1989) and Slotine and Li (1991). Three types of adaptive control are described below. These are Gain Scheduling, Self-Tuning Control, and Model Reference Adaptive Control.
2.A.2.5.1 Gain Scheduling

Gain Scheduling may also be called Piecewise Adaptive Control (see Figure 2.8). It is the least sophisticated type of adaptive control but it can be very effective. It is the most widely used type of adaptive control and is the method usually used in flight control systems where the plant varies greatly.

One of a small number of predefined fixed controllers is automatically selected according to the schedule based on some measured operating condition. Each of the fixed controllers and the schedule are designed “off-line” in advance. The control system may be described as open-loop or closed-loop depending on whether an auxiliary measurement or the output state is used as the basis for switching between the fixed controllers. The name “Gain Scheduling” is given to this method because it was originally used only to cope with changes in the gain of the plant. This method is only suitable where the plant variation is not too complex or unpredictable, in which case a more sophisticated method is needed where the controller is designed “on-line”.

![Figure 2.8. Block Diagram for Gain Scheduling or Piecewise Adaptive Control.](image-url)
2.A.2.5.2 Self-Tuning Control

Self-Tuning Control is used for a plant with unknown parameters. The parameters may be constant or they may vary slowly with time. An identification algorithm is used to estimate recursively the unknown plant parameters. The controller parameters are tuned automatically, using the estimated values of the plant parameters (see Figure 2.9). This idea was first suggested by Kalman (1958) but was not significantly developed until the 1970s, when the development of microprocessors made the implementation of Self-Tuning Control comparatively easy. In the special case in which the plant parameters are constant, after the controller has run for long enough in its given application to be tuned, the outer loop in theory could be disconnected and the inner loop left as a fixed controller. The nature of the plant, the inputs and the disturbances, are not expected to change with time; so that after an initial tuning period the outer loop does not serve any purpose.

![Block Diagram for Self-Tuning Control](image)

Figure 2.9. Block Diagram for Self-Tuning Control. [After Kalman (1958)]

More commonly, the plant parameters are not constant; and when this is so, if the identification algorithm of the Self-Tuning Controller is modified so that it forgets old information as it acquires new information, it can be used to track the variations in
the plant parameters. If this is done, Self-Tuning Control and Model Reference Adaptive Control (see below) are very closely related.

2.A.2.5.3 Model Reference Adaptive Control

Model Reference Adaptive Control uses a reference model which is known to behave in the way the designer wants the controlled plant to behave (see Figure 2.10). The reference command signal is fed to the reference model as well as to the controller. The output of the reference model is compared with the output of the plant to give the error. Then the adaptation loop adjusts the parameters of the controller to reduce this error. The reference model is generally linear even though the actual plant may be non-linear.

![Block Diagram for Model Reference Adaptive Control](image-url)
Adaptive Control is a very active research area and much work remains to be done. However, fixed controllers can often cope well with a certain amount of variation of the plant parameters, and adaptive control is much more complex and expensive to implement. So the costs and benefits need to be estimated before choosing to use adaptive control for an application. This is why the simplest kind of adaptive control, Gain Scheduling, has found the widest application so far.

2. A. 2. 6 Optimal Control

The concept of Optimal Control is simple but its realization requires a computer in the loop. The aim of Optimal Control is to find the optimal input $u_{opt}$ such that a performance index is minimised.

The performance index $J$ may have components for:

1. minimisation of error;
2. minimisation of control energy;
3. minimisation of transient times between changes of state;
4. etc. ....

For example,

$$J = \int_{t_0}^{t_f} (\|y(t) - r(t)\|^2 + \|\lambda \cdot u(t)\|^2) dt$$

where $y(t)$ is the actual output;

$r(t)$ is the desired output;
\[ \| y(t) - r(t) \|^2 \] expresses the desire to minimise the error between the desired output and the actual output;

\( \lambda \) is a diagonal weighting matrix;

\( u(t) \) is the controller output which is the input signal to the plant;

\[ \| \lambda \cdot u(t) \|^2 \] expresses the desire to minimise the control input to the plant.

Choosing the formula for the performance index is not easy. Substantial computational power is required to find the optimal control input \( u(t) \) that will give the lowest value of the performance index \( J \). The search for the optimal control input may find zero, one or more than one optimal control inputs. Finding an admissible optimal control input does not necessarily mean that satisfactory control will be achieved. The situation when the optimal solution cannot be realised and approximations must be made is called Sub-Optimal Control.

2.A.2.7 Summary of Section 2.A.2

**Time Delay Control** is designed for complicated non-linear systems with various kinds of internal uncertainties and external disturbances. This description fits helicopter rotors. However, Time Delay Control differs from the Phase-Locked Delayed Feedback control method in that it uses a reference model of the system being controlled, and is not designed for periodic motion. **Sliding Mode Control** is designed to cope with imprecise knowledge of the mathematical model of the system and external disturbances, which are also problems in active control of helicopter rotors. It differs from Phase-Locked Delayed Feedback control in that it uses an explicit mathematical model of the system being controlled, requires the desired trajectory to be known, and does not incorporate delayed feedback. Both Repetitive
Control and Iterative Learning Control are similar to the Phase-Locked Delayed Feedback control method developed in this thesis in that they apply a form of delayed feedback to systems with periodic motion. They differ from it in that they track a desired trajectory rather than improve the stability of an unknown periodic trajectory. Adaptive Control and Optimal Control are very popular modern control methods and are the objects of much current research, but neither uses delayed feedback. They are included here because of their importance, and for comparison with the other methods discussed.

2.A.3 The Delayed Resonator

Figure 2.11. A Mass-Spring-Dumper Trio as a Passive Vibration Absorber. [After Olgac & Holm-Hansen (1993)]
Olgac and Holm-Hansen (1993a, 1993b, 1994, 1995) used time-delayed feedback of the displacement of an absorber mass to create a tunable active vibration absorber (called the Delayed Resonator). A conventional and well-known solution to the problem of vibration suppression for harmonically excited systems is the tuned passive absorber/resonator as shown in Figure 2.11.

Typically $c_x = 0$ and the mass-spring system excited at its natural frequency exerts a harmonic force on the primary system which is equal and opposite to the excitation force. The passive absorber is very effective at the operating frequency for which it is tuned. Its effectiveness decreases rapidly away from this frequency. The Delayed Resonator overcomes this drawback. Its resonant frequency can be tuned in real time by adjusting the delay time and the gains in the feedback loop (see Figure 2.12). So if the excitation frequency varies, the resonant frequency of the Delayed Resonator can be tuned in real time to match it.

![Figure 2.12. The Delayed Resonator. [After Olgac & Holm-Hansen (1993)]](image-url)
The delayed feedback force which is added to a passive absorber to create the Delayed Resonator is given by

\[ f_a = g x_a(t - \tau) \]

where \( x_a \) is the displacement of the absorber mass;

\( g \) is the feedback gain;

\( \tau \) is the delay time.

The introduction of time delay into the feedback loop causes the characteristic equation of the Delayed Resonator to be transcendental and have an infinite number of roots. The delay time and feedback gain are selected so that the two dominant roots (complex conjugates) lie on the imaginary axis of the complex plane while all the other roots are in the stable left half plane (LHP). This means that the absorber is stable (not asymptotically stable) and behaves like a resonator. By adjusting the delay time and the feedback gain the position of the dominant roots on the imaginary axis of the complex plane can be controlled. Hence the resonant frequency of the Delayed Resonator can be controlled.

In other words, by adjusting the delay time and the feedback gain the real part of the complex conjugate dominant roots can be made equal to zero and the imaginary part can be controlled to achieve the desired resonant frequency.

The equation of motion for the Delayed Resonator is

\[ m_a \ddot{x}_a + c_a \dot{x}_a + k_a x_a + g x_a(t - \tau) = 0 \]

where \( m_a \) is the absorber mass;

\( c_a \) is the damping coefficient of the absorber dashpot;

\( k_a \) is the stiffness of the absorber spring.
This yields the characteristic equation

\[ n_c s^2 + c_s s + k_s + g e^{-\tau} = 0 \]

where: \( s = a + \omega i \)

which is a transcendental equation with an infinite number of roots.

The root locus plot for varying gain and a constant delay time is shown in Figure 2.13.

As mentioned above, the Delayed Resonator is tuned by setting the real part of the dominant complex conjugate roots equal to zero and the imaginary part equal to the desired resonant frequency. Thus the desired roots are given by \( s = \pm \omega_c i \)

where \( \omega_c \) is called the “crossing frequency” and refers to the frequency where the root locus crosses the imaginary axis in the complex plane.
Substituting these desired roots into the characteristic equation yields analytical expressions for the delay time and the feedback gain as a function of the crossing frequency. These are

\[ \tau_c = \frac{1}{\omega_c} \left\{ \tan^{-1}\left( \frac{c_s \omega_c}{m_s \omega^2_c - k_c} \right) + 2\pi l \right\}, \quad l = 0, 1, 2... \]

\[ g_c = \pm \sqrt{(c_s \omega_c)^2 + (m_s \omega^2_c - k_c)^2} \]

Provided that the excitation frequency of the primary system is known, these equations can be used to tune the Delayed Resonator in real time. The results of computer simulations were used to demonstrate the effectiveness of the Delayed Resonator for vibration absorption. A conceptual set-up for practical implementation of the Delayed Resonator was proposed. This included an on-line FFT analyzer for continuous detection of the excitation frequency. However no experimental work was reported to have been done.

The Delayed Resonator concept differs fundamentally from the Phase-Locked Delayed Feedback control method. The Phase-Locked Delayed Feedback control method endeavours to make a system as stable as possible by setting the time delay equal to the period of the excitation force, and adjusting the control parameters (or gains) to the optimal values. The Delayed Resonator does not set the delay time equal to the period of the excitation force, but varies both the delay time and the feedback gain to make the resonator just stable, with resonant frequency equal to the excitation frequency, so that it will absorb the vibration of the primary system.
2.A.4 Control of Chaotic Systems

The control of chaotic systems is included in this literature review because:

1. the aim of most control methods for chaotic systems is to stabilise a periodic trajectory, which is the same as the aim of the Phase-Locked Delayed Feedback control method presented in this thesis;

2. the effect of delayed feedback on chaotic systems (particularly chemical ones) has received a significant amount of attention recently;

3. delayed feedback can induce chaotic behaviour e.g. in active automotive suspension control (Palkovics and Venhovens 1992) and in chemical systems (Chevalier et al. 1991);

4. some of the control methods for chaotic systems use delayed feedback, and one in particular (Pyragas 1992) sets the delay time equal to the period of the trajectory being stabilised;

5. these methods, like the Phase-Locked Delayed Feedback control method, can stabilise unstable periodic orbits without changing them at all; and

6. like the Phase-Locked Delayed Feedback control method, these methods do not require any knowledge of the dynamics of the system, or of the equations of motion.

Chaos occurs in a wide variety of physical systems. It restricts the operating range of many mechanical and electronic devices. The erratic behaviour of a chaotic system is explained as follows. There is an infinite number of unstable periodic orbits present in the chaotic motion. If the system was on any one of these periodic orbits and there was no disturbance, it would stay on it. However, the fact that the periodic orbits are unstable means that any deviation from the periodic orbit will grow
exponentially with time. So the system never stays on any one of the infinite number of periodic orbits because in practice there are always disturbances.

Lorenz was the first person to find that the long-term prediction of the behaviour of chaotic systems is practically impossible\(^1\). The cause of the unpredictability is that initially close trajectories tend to separate exponentially. Since in practice the initial conditions can only be set with finite accuracy the errors will increase exponentially with time. Lorenz called this the "butterfly effect" because his long term forecasts of the weather could be changed significantly by a butterfly flapping its wings.

Until recently the research into chaos could be divided into three areas:
1. searching for experimental systems and theoretical models demonstrating chaotic behaviour;
2. analysis of routes to chaos and the development of bifurcation theory;
3. analysis of chaotic states and the development of the theory of chaotic time series analysis.

A fourth area, the control of chaos, has only emerged since the work of Hübler and Lüscher (1989) and Ott, Grebogi & Yorke (1990). The method of Ott, Grebogi & Yorke (1990) is so well known and so often referred to that it has been given the acronym OGY.

One characteristic of chaotic systems is that they are very sensitive to very small disturbances. This characteristic of chaotic systems distinguishes them from non-chaotic systems where to achieve a large change a large control action is necessary. As mentioned above, this characteristic is called the "butterfly effect". It is

generally thought of as undesirable. However OGY (1990) have used this characteristic as the basis of a method to control chaotic systems using only very small perturbations.

2.A.4.1 The OGY Control Method

The OGY method for controlling chaotic systems does not involve time delayed feedback but is included here for three reasons:

1. although it was not the first and is not necessarily the best method, it is by far the best known method for controlling chaotic systems;

2. it is a method for stabilising periodic trajectories and is therefore relevant to the Phase-Locked Delayed Feedback control method presented in this thesis;

3. it is useful to contrast it with the Delayed Feedback Control method of Pyragas (1992) for stabilising unstable periodic orbits in chaotic systems.

The method of chaos control developed by OGY (1990) is based on the stabilisation of unstable periodic orbits (UPOs). As there is an infinite number of UPOs embedded in a chaotic (strange) attractor, the most desirable one may be chosen for stabilisation. However, the OGY (1990) method is not continuous but uses discrete changes in time of an accessible system parameter based on the Poincaré map. A consequence of this is that the method can only stabilise periodic orbits whose maximal Lyapunov exponent is small compared with the reciprocal of the time interval between parameter changes. In addition, noise can cause occasional bursts of the system into a region far from the desired periodic orbit.
The OGY method has been applied successfully in experiments to an inverted magneto-elastic ribbon buckling under the influence of gravity (Ditto et al. 1990), laser systems (Gills et al. 1992), electrical circuits (Hunt, 1991), thermal convection (Singer et al. 1991), and arrhythmically oscillating cardiac tissue controlled by a small electrical stimulus (Garfinkel et al. 1992).

The OGY control method works as follows:

(i) examine the unstable periodic orbits (UPOs) embedded in the chaotic motion;
(ii) for each UPO ask whether the system performance would be improved if it followed that orbit;
(iii) select a desirable UPO;
(iv) wait until the chaotic wandering of the system closely approaches the selected UPO;
(v) apply small perturbations to an accessible system parameter to direct the motion of the system on to the desired UPO;
(vi) if noise is present, repeatedly apply the small perturbations to keep the system trajectory on the desired UPO.

The OGY method is like balancing a ruler vertically on the palm of your hand — you know the position you want it to be in and as long as it is close to that position you can keep it there by applying small movements of your hand. Similarly, in the OGY method the desired position or trajectory is known and a linear model is assumed in the vicinity close to this desired trajectory. When the system is far from the desired

---

trajectory no control is applied. When the control is first turned on there is a waiting period where nothing is done until the chaotic wanderings of the system bring it close to the desired trajectory. Once the system is close to the desired trajectory it is kept there by repeated application of small perturbations which are calculated using the assumed linear model. Since the desired trajectory (UPO) is unstable, the system would stay on it only if it was perfectly on it and there were no disturbances. In practice, there are always disturbances which cause the system to move away from the desired trajectory, so the small perturbations must be applied repeatedly to push it back there before it moves too far away. This is very similar to the problem of balancing a ruler on your hand. The small control perturbations used in the OGY method are not applied continuously but at discrete intervals, as the method is based on the Poincaré surface of section technique. This feature in particular distinguishes it from the chaos control method of Pyragas (1992) which is continuous.

Starobinets and Pikovsky (1993) proposed a multistep method for controlling chaos which is an extension of the OGY method. Like the OGY method, their method uses small external perturbations to stabilise an unstable fixed point embedded within a chaotic attractor. However, the OGY method involves waiting until the system trajectory falls within a small neighbourhood of the fixed point before applying the control. No control is applied while the trajectory is far away from the chosen fixed point. In contrast, the method of Starobinets and Pikovsky (1993) tries to control the system even when it is far away from the fixed point. They show that, while their method is rather crude and may be improved, it significantly reduces the mean time to achieve control compared to the OGY method. This method is similar to the “targeting” procedure of Shinbrot (1990, 1992). It is a simple method of targeting.
Petrov et al. (1993) successfully controlled chaotic behaviour by stabilising unstable periodic orbits in the chaotic regime of an oscillatory chemical system. They studied the Belousov-Zhabotinsky reaction in a continuous-flow stirred-tank reactor (CSTR). They used a map-based algorithm which is a simplified version of the OGY method. It is more convenient than the OGY method for experimental applications because it minimises the targeting procedures. They noted that with both the OGY method and the map-based algorithm the UPO is found experimentally, so no mathematical model of the system is needed. No knowledge of the underlying dynamics of the system or the governing differential equations is necessary for stabilising a periodic orbit.

This last feature of the OGY method is also a feature of the Pyragas (1992) method for stabilising periodic orbits, and also of the Phase-Locked Delayed Feedback (PLDF) control method presented in this thesis. However, the OGY method relies on finding the periodic orbit experimentally before it can be stabilised. The Phase-Locked Delayed Feedback method only requires that the period of the motion to be stabilised is known.

2.A.4.2 The Pyragas Methods for Control of Chaotic Systems

Pyragas (1992) considered a dynamic system that, without feedback and without any input signal, had a strange (chaotic) attractor. A scalar variable was available for measurement as the output of the system. Pyragas developed two control methods for stabilising the unstable periodic orbits (UPOs) embedded in the uncontrolled chaotic attractor. He called the methods External Force Control and Delayed Feedback Control.
2.A.4.2.1 External Force Control

![Diagram of Chaotic System](image)

The External Force Control method works as follows.

1. Applying a standard method of delay coordinates\(^3\) to the experimentally measured system output, it is possible to identify a large number of distinct UPOs embedded in the chaotic attractor.

2. A desirable UPO is selected for stabilisation.

3. A special external oscillator is designed to produce a reference signal that is equal to the selected UPO.

4. The difference between the generated signal and the measured system output is used as the control signal,

\[ F(t) = K[y_i(t) - y(t)] = KD(t) \]

where: \( y_i(t) \) is the generated reference signal;

\( y(t) \) is the measured system output; and

\( K \) is a weighting factor.

---

This control method, unlike the Phase-Locked Delayed Feedback control method, does not use delayed feedback.

This method was verified using numerical simulations for many different chaotic systems. After an initial transient stage the system output followed the previously unstable periodic orbit, and the control signal became very small. The method was tested for sensitivity to noise. Unlike the OGY method even large amounts of noise did not cause the system output to deviate greatly from the desired periodic trajectory. Whereas the OGY method involves a waiting period and does not apply any control until the system output is close to the desired periodic orbit, the Pyragas External Force Control method can be switched on at any time. However, this means that the initial control perturbation can be large, which may be undesirable. In addition, some systems display multistability where alternative stable solutions correspond to different initial conditions. To solve both these problems a limit can be placed on the maximum allowable size of the control perturbations. When this is done the duration of the transient stage before the system synchronises with the external oscillator is increased because the limited control perturbation is only sufficient to cause synchronisation when the system trajectory comes close to the desired periodic orbit. So a waiting period similar to the OGY method occurs. The length of this waiting period grows rapidly as the allowable size of the control perturbations is decreased.

The effectiveness of the control method for stabilising the periodic orbit was evaluated by calculating the maximal Lyapunov exponent for the linearised system. This was then plotted as a function of the weighting factor $K$ in the control law. When the control perturbation was applied to only one variable of the system the value of the
Lyapunov exponent had a minimum for a certain value of $K$. Below a certain (lower) value of $K$ the Lyapunov exponent was positive, which meant that the periodic orbit was unstable. This occurred because the control perturbations, with small values of $K$, were not large enough to compensate for the exponential divergence resulting from any deviation from the periodic orbit. For large values of $K$ it was suggested that the one variable to which the control perturbations were applied was changing so quickly that the other variables had no time to follow these changes. To confirm this suggestion the control was added to each equation of the system simultaneously. In this case the Lyapunov exponents decreased monotonically for increasing values of $K$. So that as $K$ increased the periodic orbit became steadily more stable.

2.A.4.2.2 Delayed Feedback Control

![Figure 2.15 Block Diagram for Delayed Feedback Control Method. [After Pyragas (1992)]](image)

Whereas the External Force Control method described above uses the difference between the desired trajectory and the actual trajectory to produce the control signal, the Delayed Feedback Control method uses the difference between the system trajectory at some earlier time and the current system trajectory to produce the control signal. So the control law is
Chapter 2 Literature Review

\[ F(t) = K[y(t-\tau) - y(t)] = KD(t) \]

where: \( \tau \) is the delay time;
\( y \) is the system trajectory; and
\( K \) is the weighting factor.

Note that this control method uses only displacement feedback. The Phase-Locked Delayed Feedback control method uses both displacement and velocity feedback.

To stabilise a UPO the delay time is set to be equal to the period of the UPO. So the trajectory of the UPO does not need to be known. Only the period needs to be known. If the system is following exactly a trajectory corresponding to a stabilised UPO with period equal to the delay time, then \( D(t) \) will be zero. Thus the control method stabilises the unstable periodic orbit of the uncontrolled system without changing it. To realise the Delayed Feedback Control method experimentally, all that is needed is a method of producing the delayed feedback. This can be done by storing values in the memory of a digital computer. It can also be done using an analogue delay-line, in which case no computer is required at all. Both the value of the delay time \( \tau \) and the value of the weighting factor \( K \) must be adjusted to achieve stabilisation of a UPO embedded in the chaotic system. The range of values of \( K \) for which stabilisation can be achieved with this Delayed Feedback Control is quite narrow. Note that using an analogue delay-line is not an option for the Phase-Locked Delayed Feedback control method when applied to mechanical systems, because the time-delay required is longer than can be provided by an analogue-delay line, so a digital computer must be used. An analogue-delay line can be used only for very high frequency systems where the delay time required is very short.
Since $D(t)$ tends to zero when a UPO is stabilised, this can be used as a criterion of the stabilisation. The value of $D^2(t)$ is averaged over a time interval excluding the initial transient stage. This average value is called the "dispersion". If the dispersion is calculated for many different values of the delay time, the resulting plot shows deep minima in the dispersion for periods of the UPOs that are embedded in the chaotic system. The plot also shows minima intervals which correspond to the stabilisation of unstable fixed points. For certain values of the delay time instead of chaotic motion the system follows a periodic trajectory with a period that is different from the delay time. In these cases the value of the dispersion is large. Hence the control perturbations are large and they cause the system to follow a new periodic trajectory that was not embedded in the uncontrolled system.

Multi-stability can occur in chaotic systems when Delayed Feedback Control is used. The system may have different stable solutions for the same value of the delay time with different initial conditions. However, if the allowable size of the control perturbations is limited, then the asymptotic behaviour of the system becomes unambiguous for all values of the delay time.

2.A.4.2.3 Pyragas and Tamasevicius (1993)

Pyragas and Tamasevicius (1993) described the first experimental realisation of the Delayed Feedback Control method of Pyragas (1992). The experimental system was an electronic circuit designed to behave as an externally driven nonlinear oscillator. The external drive frequency was 3.8 MHz. The delayed feedback was created using an analogue delay-line with a maximum delay time of 1200 ns. In the
experiment the adjustment of the parameters to achieve stabilisation of the unstable periodic orbits was found to be very easy. The periods of the desired periodic orbits were known to be integer multiples of the period corresponding to the external drive frequency; so, firstly, the time delay was set close to one of these periods. Then the weighting factor $K$ was increased until stabilisation was achieved. The delay time could then be adjusted slightly and the procedure repeated until the control signal for the stabilised orbit, which ideally would tend to zero, was minimised.

2.A.4.2.4 Schöll and Pyragas (1993)

A stable and tunable semiconductor oscillator can be useful in practical applications. In certain circumstances high purity semiconductors produce self-generated chaotic current or voltage oscillations. Schöll and Pyragas (1993) suggested a way to exploit this chaotic behaviour by making use of the fact that any chaotic attractor of a non-linear dynamic system contains an infinite set of unstable periodic orbits (UPOs). They theoretically demonstrated that both the External Force Control and Delayed Feedback Control methods of Pyragas (1992) could be used to stabilise the UPOs and create a simple and stable tunable semiconductor oscillator.

They state that their method of continuous control is superior to the discrete-time control method of OGY (1990) for two reasons:

1. the time-continuous control is more robust with respect to noise than the OGY method;
2. the OGY method requires extensive on-line computer analysis which the Pyragas methods do not require.
2.4.2.5 Reznik and Schöll (1993)

Reznik and Schöll (1993) conducted a numerical investigation into the nonlinear dynamics of a semiconductor oscillator. The method of continuous Delayed Feedback Control proposed by Pyragas (1992) was used to stabilise unstable periodic orbits of a chaotic repeller. This provided a widely tunable semiconductor oscillator, based on stabilised transient chaos, which may have useful applications in semiconductor microelectronic devices. This could be achieved in practice by applying the control signal as a voltage to a gate electrode on one layer of the semiconductor.

The control term used was

\[ f(t) = K(\Phi_p(t - \tau) - \Phi_p(t)) \]

where: \( \tau \) was the delay time;

\( \Phi_p \) was the interface potential barrier; and

\( K \) was the coupling constant.

To preserve the symmetry properties of the system the method of Pyragas (1992) was modified slightly in that the same control term was added to two equations of the model. The Delayed Feedback Control provided an extremely efficient and fast converging control method.

2.5 Delayed Feedback in Chemical Systems

The effect of time delayed feedback on oscillating chemical systems has recently been the subject of much research. This research is reviewed here because it is relevant to the Phase-Locked Delayed Feedback control method presented in this thesis. Some of this research is based on the Delayed Feedback Control method of
Pyragas (1992) described in section 2.A.4.2, which is the control method closest to the Phase-Locked Delayed Feedback control method developed in this thesis.

2.A.5.1 Weiner, Schneider and Bar-Eli (1989)

Weiner et al. (1989) studied the oscillations of the concentrations of the reactor species in a single continuous-flow stirred-tank reactor (CSTR) where the input (the rate of in-flow) was controlled by delayed feedback of the output (the concentration of one of the reactor species). This method of control is in contrast to coupled chemical oscillators where the output of each reactor controls the input to the other reactor, either with or without time delay. The effect of the delayed feedback was investigated experimentally on a minimal bromate oscillator (MBO); and theoretically with calculations using the mechanism for the MBO, consisting of seven reversible chemical reactions, devised by Noyes, Field and Thompson (NFT). A schematic diagram of the experimental set up is shown in Figure 2.16. Two methods were used for the calculations, which corresponded to the two procedures used for the experiments. These were called Method A and Method B and are described below. The two methods were not equivalent, and they did not produce the same results.

---

4 See section 2.A.5.2.
5 Noyes, R.M., Field, R.J. and Thompson, R.C., 1971, "Mechanism of reaction of bromine (V) with weak one-electron reducing agents," Journal of the American Chemical Society, 93(26), 7315-16.
2.A.5.1.1 Method A

The control law relating the delayed output to the input was

\[ k_f = k_o + k_o \beta \frac{[Ce^{4+}](t-D) - [Ce^{4+}]_{av}}{[Ce^{4+}]_{av}} \]

where: \([Ce^{4+}]_{av}\) was the average ceric ion concentration for the free running (uncontrolled) system at a constant \(k_o\);

\(k_o\) was the flow rate constant;

\(k_f\) was the flow rate for the controlled system;

\([Ce^{4+}](t-D)\) was the ceric ion concentration from \(D\) seconds earlier;

\(\beta\) was a weighting factor.

This control law was different from the Phase-Locked Delayed Feedback control method. It compared the delayed state to an average state and not to the current state.
So if the delayed ceric ion concentration rose above \([Ce^{++}]_m\), the flow rate was increased, and conversely if it dropped below \([Ce^{++}]_m\), the flow rate was decreased. The initial conditions for this method were zero control and constant flow rate \((k_f = k_0\) as for the free-running oscillator). The control remained zero until \(D\) seconds had elapsed. The value of \(D\) was constant in this method.

2.A.5.1.2 Method B

In the second method the delay time \(D\) was not constant. The control law was the same as for Method A (see equation above) but the initial value of \(D\) was zero. The delay time was increased in steps of \(\Delta D\) up to a maximum value and then decreased back to zero.

The period of the oscillations in the uncontrolled CSTR varied between about 150 and 1000 seconds depending on the flow rate. The length of the experiments was limited to about five hours due to the equipment used. This meant that some of the experimental runs finished after only 20 oscillations of the system.

2.A.5.1.3 Method A Results

The period of the oscillations of the system was determined for a range of constant values of the delay time \(D\). The period was found to increase steadily as the delay time increased, then suddenly drop back to a lower value. As the delay time was further increased the period steadily rose again followed by a sudden drop. This pattern kept repeating itself as the delay time was increased. Due to the shape of the plot of controlled period as a function of delay time this effect was called a "sawtooth"
pattern (see Figures 2.17 and 2.18).

As the delay was increased the size of each subsequent rise and fall of the period decreased, the slope of the rising values became less steep and the period approached the period of the uncontrolled system. The agreement between the calculated and experimental results was very close. In the plots in Figures 2.17 and 2.18 both the period of the controlled system and the delay time have been "reduced" to non-dimensional quantities by dividing them by the period of the uncontrolled system.

They also found (see Figure 2.19) that the exact delay time at which the transitions occurred varied slightly for different initial conditions. The initial conditions corresponded to the phase of the oscillation; that is, the point during a single periodic oscillation of the chemical concentrations at which the calculation was begun (it was not possible to control exactly the phase in the experiments). This variation in the delay time at which transition occurred meant that for a particular delay time the system could oscillate with either a long or a short period depending on the initial conditions. This feature is called "birythmicity". It occurred only in the range of delay times for which transition could occur. Birythmicity also occurred in the experimental results in the transition regions because different experimental runs began with different initial conditions.

The birythmicity in the results for the calculations and experiments using Method A was not a hysteresis phenomenon. Whether the system oscillated with a high or low period in the transition region depended clearly on the initial conditions.
Figure 2.17 Reduced period (in units of free-running oscillator) vs reduced delay (method A):

- exp (o) $k_0 = 3.12 \times 10^3 s^{-1}$,
- exp (□) $k_0 = 3.42 \times 10^3 s^{-1}$;
- calc (- - -) $k_0 = 4 \times 10^3 s^{-1}$,
- calc (-----) $k_0 = 4.32 \times 10^3 s^{-1}$, near the upper limit of oscillations.

[After Weiner et al. (1989)]

Figure 2.18. Same as in Figure 2.17 (calc $k_0 = 4 \times 10^3 s^{-1}$) with longer delays. Approach of the average transition to unit reduced period is seen (-----).

[After Weiner et al. (1989)]
Figure 2.19. Calculated \( (k_0 = 4 \times 10^{-3} \text{s}^{-1}) \) reduced transition time (method A) vs the phase of the initial point on the free-running limit-cycle. Upward arrow: maximum of the marked species. Downward arrow: minimum of marked species. The line is a polynomial fit to the points. [After Weiner et al. (1989)]

Figure 2.20. Reduced period vs reduced delay (method B). Calcd (---) \( k_0 = 4 \times 10^{-3} \text{s}^{-1} \); expd (\( \Delta = \) increasing delay, \( \Theta = \) decreasing delay) \( k_0 = 3.12 \times 10^{-3} \text{s}^{-1} \). Dotted lines show regions of birhythmicity. [After Weiner et al. (1989)]
2.A.5.1.4 Method B Results

For both the calculations and the experiments using Method B the plot of controlled period against the delay time showed a sawtooth pattern. Birythmicity occurred in the transition regions. In contrast to the results for Method A, this birythmicity was a hysteresis phenomenon. Whether the system oscillated with a high or low period depended on whether the delay time was being increased or decreased.

A second difference between these results and those for Method A was that the maximum and minimum values of the period did not change from one tooth of the sawtooth pattern to the next. However, the slope of the rising part of each tooth did decrease for higher values of delay (as it did for Method A). This meant that the overlap of the second and third teeth was much greater than the overlap of the first and second teeth, and therefore so was the birythmicity region (see Figure 2.20). This birythmicity region was also much larger than for Method A.

The step size used for increasing and decreasing the delay time in Method B also played a significant role. A change in the delay time resulted in a change in the controlled period and hence the system moved to a different limit-cycle. It was found that while the system was on the upper branch in the birythmicity region (oscillating with high period), if the delay was increased by a large step the system would not move further up the upper branch but would make the transition to the lower branch. If the delay time was increased in small steps the system would stay on the upper branch in the birythmicity region right up to the maximum value possible on that branch before transition to the lower branch occurred. This showed that the limit-cycle on the upper branch was becoming less stable and the limit-cycle's "basin of attraction" or "region of asymptotic stability" (the range of initial conditions which will
result in this limit-cycle) was decreasing in size as the system approached more closely the end of the upper branch.

The dependence of the transition point on step size for Method B occurred in both the calculations and the experiments. This effect of step size explains why the regions of birhythmicity were much smaller for Method A than for Method B. In Method B a small step size was used to find the maximum size for the birhythmicity region. In Method A the control law with a fixed value of delay time was applied to the uncontrolled system. This meant that the system went from zero delay to the fixed delay in a single step when the control was applied. The larger the delay time the bigger the step. As mentioned above, the limit-cycles on the upper branch of the birhythmicity region with higher periods had small basins of attraction and were not very stable. They could only be reached using many small steps in the delay time. Hence these limit-cycles could not be reached using Method A.

For both Method A and Method B, the amplitude of the oscillations increased and decreased in the same sawtooth pattern as the period. The transitions from high amplitude to low amplitude occurred at the same values of delay time as the period transitions. For Method B the same hysteresis phenomenon occurred with the amplitude as with the period.

An important conclusion from this work is that dangerous situations may occur in systems controlled in this way. The system may behave smoothly over a wide operating range. However, if the delay time is such that the system is operating in a birhythmic region, a small increase in the delay time may cause a sudden and large change in the period and amplitude of the oscillation. This could have catastrophic results.
Weiner et al. (1989) expected to find regions of multi-rythmicity for large values of delay time, but were not able to show this as the approach of the system to the limit-cycle became extremely slow for large delay times.

The control with delayed feedback was implemented using an MS-DOS standard personal computer via a parallel input/output board. The ceric ion concentrations were stored in the computer memory for use in the control equations with time delay. The computer controlled the movement of three syringes which fed the chemicals into the CSTR.

2.A.5.2 Weiner, Holz, Schneider and Bar-Eli (1992)

Weiner et al. (1992) investigated the behaviour of two identical coupled chemical oscillators where the output of each reactor controlled the input to the other reactor with a time delay in the control path. The experiments used continuous-flow stirred-tank reactors (CSTRs) and the chemical system was the minimal bromate oscillator (MBO). The calculations used the NFT mechanism (Noyes et al., 1971). The MBO and the NFT mechanism were also used in Weiner et al. (1989) (see above) where a single oscillator with delayed feedback was studied. Both control Method A and control Method B described in Weiner et al. (1989) were used in this paper for the calculations. However, Method B was not used in the experiments because of the limited run times possible with the experimental equipment. The period of the limit-cycle was determined from the time series using a calculated Fourier spectra.
The mutual coupling of the two CSTRs was achieved by using the output (the measured ceric ion concentration \([Ce^{4+}]\)) of each reactor to control the input (the flow rate) of the other. The following equations were used:

\[
k_{f,1} = k_0 + k_3 \beta_2 \frac{[Ce^{6+}]_2 (t - D_2) - [Ce^{4+}]_{av}}{[Ce^{4+}]_{av}}
\]

\[
k_{f,2} = k_0 + k_3 \beta_1 \frac{[Ce^{4+}]_1 (t - D_1) - [Ce^{4+}]_{av}}{[Ce^{4+}]_{av}}
\]

The terms in these equations are the same as those described above for Weiner et al. (1989) except that the subscripts refer to reactor 1 and reactor 2. A schematic diagram of the experimental set up is shown in Figure 2.21.
For small coupling strengths $\beta$ and small delay times $D$ the period and amplitude of the oscillations varied as a function of the delay time in the same sawtooth pattern that was found for the single oscillator with delayed feedback in Weiner et al. (1989). The slope of the rising part of subsequent "teeth" on the plots became less steep as the delay time increased. Overlap of the branches on the plots, and hence regions of brythmicity, also occurred. No hysteresis occurred with increasing and decreasing delay times. This was in contrast to Weiner et al. (1989) where hysteresis was found when Method B was used.

For high coupling strengths $\beta$ and large delay times $D$ multi-mode oscillations occurred. It was sometimes very difficult to determine the frequency unambiguously from the power spectrum. Two different frequencies could be present simultaneously.

The experimental and calculated values of the period of the controlled system agreed well only for small values of delay time. Elsewhere the agreement was only qualitative, i.e. similar patterns occurred but not at the same values. Due to the limitation of the experimental equipment the experimental runs may not have been long enough for the system to reach the stable limit-cycle when large delay times were used. Thus the transient behaviour may not have had time to die out. The complex oscillations which occurred at large delay times were checked for the existence of deterministic chaotic behaviour but none was found.

The method of time-delayed coupling used in Weiner et al. (1992) and in Holz and Schneider (1993) and for the single oscillator with delayed feedback in Weiner et al. (1989), used the difference between the delayed output signal and an average output signal because the latter is convenient to realise experimentally. This is slightly different from the Delayed Feedback Control method used by Pyragas (1992) where
the difference between the delayed output signal and the current output signal was used. It is also different from the Phase-Locked Delayed Feedback control method developed in this thesis in which the differences between the delayed states and the current states are used.

2.A.5.3 Holz and Schneider (1993)

![Diagram of experimental setup used by Holz and Schneider (1993).](Figure 2.22)

Figure 2.22. Schematic diagram of the experimental setup used by Holz and Schneider (1993). [After Holz & Schneider (1993)]

The work of Holz and Schneider (1993) is an extension of the work performed by Weiner et al. (1992) (see above). It used the same method of time delayed mutual flow-rate coupling of two minimal bromate oscillators (MBOs). The experiments used continuous-flow stirred-tank reactors (CSTRs) (see Figure 2.22) and the calculations used the Noyes-Field-Thompson (NFT) mechanism. In the control equations \( \tau \) was used for the time delay, instead of \( D \) which was used by Weiner et al. (1992). No chaotic behaviour was observed.
Two special phenomena were found. These were "rythmogenesis" and "oscillator death". As the names suggest, these two phenomena are the reverse of each other. Oscillator death describes the case where two uncoupled systems which are oscillating on stable limit-cycles cease to oscillate when mutual flow rate coupling without time delay is introduced with high coupling strength. The coupling leads to a 180 degree out-of-phase shift of the two oscillators. This phenomenon is also sometimes called "phase death". Rythmogenesis refers to the creation of oscillations in two reactors in identical steady-states when mutual flow-rate coupling with time delay is introduced. When the uncoupled reactors were in focal steady-states (after a perturbation the response decays with damped oscillation to its steady-state) the introduction of mutual flow-rate coupling either with or without time delay led to rythmogenesis. When the uncoupled reactors were in nodal steady-states (after a perturbation the response decays exponentially to its steady-state), only the introduction of mutual flow-rate coupling with time delay led to rythmogenesis. When rythmogenesis occurred in the coupled reactors, the period of the limit-cycle oscillations was equal to the sum of the two delay times plus a constant $T = \tau_1 + \tau_2 + \Delta$. The periods in each reactor were identical and the oscillations were in phase.

2.A.5.4 Schneider, Blittersdorf, Förster, Hanck, Lebender & Müller (1993)

Schneider et al. (1993) applied the Pyragas (1992) Delayed Feedback Control method to the Belousov-Zhabotinsky (BZ) reaction. Experiments were conducted in a continuous-flow stirred-tank reactor (CSTR). Calculations were performed using the
four-variable model of Györgyi and Field (1991). The Pyragas (1992) method was also applied to the calculations using the biochemical model of Aguda and Larter (1991) for the aerobic oxidation of nicotinamide adenine dinucleotide (NADH) catalysed by the enzyme horseradish peroxidase (HRP), which exhibits complex and chaotic oscillations.

The feedback function was defined as

\[ F(t, \tau) = K(y(t - \tau) - y(t)) = KD(t, \tau) \]

where:
- \( y \) is a time dependent species concentration;
- \( \tau \) is the delay time;
- \( K \) is the feedback strength; and
- \( D(t, \tau) \) is the difference between the time delayed signal and the actual signal.

This notation should not be confused with that of Weiner et al. (1992) where \( D \) was used for the delay time. In the experiments and the calculations the range of \( K \) was limited because negative flow rates were not possible. A diagram of the time delayed feedback control is shown in Figure 2.23.

![Figure 2.23. The Delayed Feedback Control Method. (After Schneider et al. (1993))](image)
When the delay time $\tau$ in the feedback function is chosen to be equal to the period of an unstable periodic orbit (UPO), (as it is in the Phase-Locked Delayed Feedback control method) in the chaotic (strange) attractor, then $D(t, \tau)$ may tend to zero and the particular orbit becomes stabilised. For other values of the delay time $\tau$, where $D(t, \tau)$ becomes larger, new dynamic states are generated, as demonstrated in Pyragas (1992) for the Rössler model.

For convenience in the experiments the feedback function was used to modulate the inflow rate of all the reactants. In the calculations the feedback function was used to modulate the inflow rate of a single species. It was found that these two methods led to similar results for stabilising unstable periodic orbits.

In two separate experiments a period-2 orbit with a period of 230 seconds and a period-3 orbit with a period of 371 seconds were stabilised from the same chaotic state by setting the delay time equal to each period respectively. Stabilisation of these orbits took a great deal of experimental effort for two reasons. Firstly, the range of values of $K$ in the feedback function for which stabilisation could be achieved was relatively narrow and was unknown. Secondly, the actual period of the oscillations fluctuated in the experiments.

To distinguish between the stabilisation of an unstable periodic orbit embedded in the chaotic attractor and the creation of a new orbit, the "dispersion" was plotted as a function of the delay time $\tau$. The dispersion was calculated by taking the average of the squared value of $D(t, \tau)$ over a time interval of 10,000 seconds for a given delay time. The dispersion became small when the delay time was close to the period of an unstable periodic orbit in the chaotic attractor. When new periodic orbits were created which were not part of the original strange attractor, their dispersion values were
relatively high and their period was different from the delay time.

2.A.5.5 Zeyer, Holz and Schneider (1993)

Zeyer et al. (1993) used the same method of time delayed mutual flow-rate coupling of two chemical oscillators as was used in Weiner et al. (1992) and Holz and Schneider (1993) (see above). Instead of the minimal bromate oscillators (MBO) the method was applied to two Belousov-Zhabotinsky (BZ) oscillators. The flow-rate into each BZ oscillator was modulated according to the difference between the time-delayed ceric ion concentration and the average ceric ion concentration of the other reactor. The control equations were the same as those used in Weiner et al. (1992) and Holz and Schneider (1993).

A schematic diagram of the experimental setup is shown in Figure 2.24. The experiments were performed using two identical continuous-flow stirred-tank reactors (CSTRs). The calculations used the 4-variable-model of the BZ reaction from Györgyi and Field (1991).

The results of both the experiments and the calculations showed that as the delay time was increased the period of the oscillations rose and fell in a sawtooth pattern similar to that found for the MBO in Weiner et al. (1992). The calculated periods rose more smoothly and fell more sharply than the periods found from the experiments. As was found by Weiner et al. (1992), the influence of the delay time on the period was found to diminish for large delay times. The influence of coupling strength \((\beta)\) and delay time \((D)\) on the period of the oscillation was much less pronounced in the BZ system than in the MBO because the MBO is more sensitive to flow rate changes.
2.A.5.6 Chevalier, Freund and Ross (1991)

Chevalier et al. (1991) applied nonlinear delayed feedback to a chemical system in order to induce complex dynamical behaviour, and use this to gain a better understanding of the chemical mechanism of the system without delay. They conducted experiments on the minimal bromate oscillator (MBO) in a continuous-flow stirred-tank reactor (CSTR) controlled by an IBM PC-XT computer. A schematic diagram of the experimental setup is shown in Figure 2.25. For comparison they integrated numerically the seven step mechanism devised by Noyes, Field and Thompson (NFT) (1971) and found good agreement with the experimental results.

The nonlinear delayed feedback used the measured bromide concentration to modulate the inflow rate of bromide with the equation

\[ f_{Br}(t) = \kappa (1 + \epsilon \sin(\omega \tau + \phi)) \]
where: \( x_r = x(t - \tau) \) represents the voltage proportional to \(-\log\) (bromide ion concentration) measured \( \tau \) seconds earlier;

\( \kappa \) is the baseline level for the flow;

\( \varepsilon \) controls the amplitude of the nonlinear feedback;

\( \omega \) represents a frequency in the concentration space;

\( \phi \) is the phase of the feedback.

They noted that the introduction of a delay does not alter the possible steady-states of the system, but it may alter the stability properties of the steady-states. This feature of delayed feedback is an essential feature of the Phase-Locked Delayed Feedback control method developed in this thesis.

They showed that the introduction of nonlinear delayed feedback to a system in a stable steady-state can cause it to begin to oscillate. This is called rhythmogenesis or the birth of a limit-cycle. The originally stable steady-state becomes unstable and the system is attracted to a limit-cycle.

They showed the first experimental evidence for a limit-cycle undergoing period-doubling bifurcations in a chemical system with delayed feedback.

They also showed multistability where, depending on the initial conditions, the asymptotic behaviour of the system may be: a simple stationary state; a limit-cycle; a quasiperiodic trajectory; or a chaotic trajectory. In the MBO with delayed feedback they observed a variety of types of multistability: a stationary state coexisting with another stationary state; a stationary state with a limit-cycle; a stationary state with a chaotic attractor; a limit-cycle with another limit-cycle (birythmic); and a limit-cycle with a chaotic attractor. They also showed the first experimentally observed instance of a "crisis" of chaos in a homogenous chemical system. This refers to the sudden
disappearance of a chaotic attractor upon collision with an unstable fixed point (a saddle point).

Figure 2.25. A schematic diagram of the experimental setup used by Chevalier et al. (1991). The CSTR was fed by three gear pumps. The bromide concentration, measured at time $t$, was used to control the (Br$^-$) inlet pump at time $t + \tau$, where $\tau$ was the delay time. Solid lines represent electrical signals. Dashed lines represent chemical flows. [After Chevalier, Freund & Ross (1991)]
2.A.5.7 Zimmerman, Schell and Ross (1984)

Zimmerman, Schell and Ross (1984) applied a simple linear delayed feedback to a bistable illuminated thermochemical system to stabilise unstable steady-states. The thermochemical system is held far from equilibrium by the illumination. The system has an internal positive feedback mechanism which works as follows. One of the reactants absorbs light of a particular wavelength which is turned into heat. This causes a rise in temperature which causes an increase in the concentration of this light absorbing reactant. So more light is absorbed, and more heat produced, and so on. When the incident power is slowly increased and then decreased a hysteresis loop occurs in the plot of steady-state absorption versus incident power.

![Diagram](image)

Figure 2.26. Schematic diagram of delayed feedback applied to an illuminated thermochemical system. Dashed lines represent the light path; solid lines represent electrical signals. [After Zimmerman, Schell & Ross (1984)]

The delayed feedback was applied using a photodetector, a variable analogue delay-line and a modulator (see Figure 2.26). The light power incident on the reaction cell was the output of the modulator $\Phi_0(t)$. 
The light power transmitted through the cell was

\[(1 - A(t))\Phi_L(t),\]

where \(A(t)\) was the partial absorption.

The output voltage of the linear photodetector was

\[V_v(t) = C_1 + C_2 (1 - A(t)) \Phi_v(t)\]

where \(C_1\) and \(C_2\) were the variable output offset and gain.

The delay line provided pure delay of \(\tau\) seconds and the modulator responded linearly to the applied voltage. So the modulator output was given by

\[\Phi_v(t) = \kappa \Phi_L (C_1 + C_2 (1 - A(t - \tau)) \Phi_v(t - \tau))\]

where:

- \(\Phi_L\) was the constant light power provided by a laser to the modulator; and
- \(\kappa\) was a constant.

So the power of the light incident on the reaction cell was determined by the amount of light transmitted through the reaction cell (which was a measure of the state of the system) at an earlier time. The delayed feedback did not alter the possible steady-states of the system but did alter their stability properties. This is also an important feature of the Phase-Locked Delayed Feedback control method developed in this thesis. By using a short delay time and adjusting \(C_1\) and \(C_2\), the intermediate steady-states of the system that were unstable without feedback were stabilised and hence made accessible to experimental observation.
2.A.5.8 Kramer and Ross (1985)

Kramer and Ross (1985) applied the same method of delayed feedback as Zimmerman, Schell and Ross (1984) to an acid-base illuminated thermochemical bistable system. The delayed feedback stabilised the unstable branch of stationary states and made them experimentally accessible.

2.A.5.9 Schell and Ross (1986)

Schell and Ross (1986) investigated the behaviour of homogenous chemical reaction mechanisms describable by ordinary differential equations, subjected to a delayed feedback. They extended the work of Zimmerman, Schell and Ross (1984) on illuminated thermochemical systems with delayed feedback. They illustrated the utility of imposing delayed feedback on a chemical system in order to explore nonequilibrium states. At short delay times the delayed feedback was shown to stabilise different types of unstable stationary states as well as induce bistability. At longer delay times transitions into chaos were possible. They also discussed the effects of time delay in chemical systems possessing natural time delays.

2.A.5.10 Roesky, Doumbouya and Schneider (1993)

Roesky, Doumbouya and Schneider (1993) presented experimental evidence that simple linear delayed feedback can generate complex dynamic behaviour, including period doubling bifurcation, \( P_3 \) sequences and deterministic chaos in the Belousov-Zhabotinsky (BZ) reaction. The experiments were conducted in a continuous-flow
stirred-tank reactor (CSTR) controlled by an IBM PC-XT computer with a 12 bit A/D converter. The delayed feedback used the same equation as Weiner et al. (1989) to modulate the inflow rate. When the delay time was varied there was no change of the period in a "sawtooth" pattern as found by Weiner et al. (1989) for the MBO system.

2.A.6 Conclusions to Part A

From the literature review it can be seen that the control method that has the most in common with the Phase-Locked Delayed Feedback control method developed in this thesis is the Delayed Feedback Control method of Pyragas (1992).

The similarities between the two methods are:

- They both employ delayed feedback.
- They are both designed to stabilise a periodic trajectory.
- The time delay is set equal to the period of the motion being stabilised.
- No knowledge is required of the dynamics of the system being controlled or of the equations of motion.
- The desired trajectory being stabilised does not need to be known.
- The adjustable parameters in the control law can be varied manually in an experiment until the best values are found.
- The stabilised periodic motion of the controlled system is identical with the unstable periodic solution of the uncontrolled system.

The differences between the two methods are:

- The Pyragas Delayed Feedback Control method uses only displacement feedback (or an electrical or chemical equivalent) and has only one adjustable weighting
factor in the control law, whereas the Phase-Locked Delayed Feedback control method uses both displacement and velocity feedback, and has two adjustable weighting parameters in the control law.

- The Pyragas Delayed Feedback Control method was developed for stabilising unstable periodic orbits in chaotic systems with applications in electronic circuits, semiconductor microelectronic devices and chemical systems; whereas the Phase-Locked Delayed Feedback control method was developed for stabilising periodic motion in mechanical systems, with particular application to helicopter rotor blades. (Note, however, that there is no reason why these control methods should be limited to the applications for which they were developed. The Pyragas Delayed Feedback Control method could be applied to mechanical systems and the Phase-Locked Delayed Feedback control method could be applied to chaotic electrical and chemical systems.)

- The Pyragas Delayed Feedback Control method does not use a phase-locked oscillator, but relies on the period of the unstable periodic orbit being known. This period must be used to set the delay time in the control system manually. Hence, the Pyragas method is designed for systems where the period of the orbit to be stabilised is constant. On the other hand, the Phase-Locked Delayed Feedback control method is designed for systems where the period of the motion to be stabilised is not known beforehand and may vary slowly with time. In particular, in helicopters in flight the rotational speed of the rotor may vary. The phase-locked oscillator in the Phase-Locked Delayed Feedback control method can automatically detect the rotational speed of the helicopter rotor and synchronize the delayed feedback with the corresponding period.
The Pyragas method has been applied to very high frequency oscillating electrical systems using an analogue delay-line to provide continuous delayed feedback; whereas the Phase-Locked Delayed Feedback control method uses discrete samples because it was developed for low frequency mechanical systems, using a digital computer and analogue/digital and digital/analogue converters to provide delayed feedback.
PART B: ACTIVE CONTROL APPLIED TO HELICOPTER ROTORS

2.B.1 Introduction to Part B

The first section of Part B of this literature review contains comment from leading researchers in this field emphasising how difficult the mathematical modelling of helicopter rotors has proved to be. The application of active control methods to helicopter rotors has been the subject of significant research for more than twenty years. In this period of time, the methods of active control of helicopter rotors which have received significant attention are primarily Higher Harmonic Control and, to a lesser extent, Individual Blade Control. The research in each of these two areas is discussed in separate sections. Higher Harmonic Control has been developed chiefly for attenuation of helicopter rotor vibration and noise; whereas Individual Blade Control has been developed with respect to a wider range of undesirable helicopter rotor phenomena, including stability problems.

Research which employs methods of active control that are different from the Higher Harmonic Control and Individual Blade Control methods is then treated in three separate sections: active control of resonance problems; active control of instability problems; and active control for attenuation of the response to gusts.

2.B.2 Intractability of Helicopter Rotor Analysis

Researchers in almost every field of engineering seem to be caught in a situation where, on the one hand, they must convince the people paying them that they...
have achieved a great deal and are giving value for money, and, on the other hand, they must convince the same people that there is still much room for improvement and more research needs to be done. The writings referred to in this section belong to the second category; that is, they emphasise that there is room for improvement in the mathematical modelling and analysis of helicopters, and, in particular, of helicopter rotors.

Ham (1972) described how at the NATO (North Atlantic Treaty Organisation) AGARD (Advisory Group for Aerospace Research and Development) Specialists' Meeting on Aerodynamics of Rotary Wings two distinct and opposed schools of thought emerged on the present and future aerodynamic research needs of the helicopter industry. One of the two groups represented "Development" and was largely made up of those research workers associated with the helicopter manufacturing industry. This group argued that enough was known about the fundamental aspects of helicopter aerodynamics and that research in this area was "learning more and more about less and less". The other group represented "Research" and largely consisted of non-industrial research workers in the field of helicopter aerodynamics. However, there was unanimous agreement that one of the prime deficiencies of the helicopter was vibration induced by aerodynamic effects. This influences every aspect of helicopter operations including: first cost; operating cost; productivity; reliability; safety; and comfort. Ham concluded by saying that the papers presented at the meeting indicated that not enough was known about the fundamental aspects of helicopter aerodynamics. He then listed eleven areas, starting with the wake of a hovering rotor and the wake of a translating rotor, where analysts were not able to describe in sufficient detail, much less model, the physical processes involved.
Ormiston (1973) described how leading companies and researchers produced significantly different results when all were given the same routine problem to determine the natural frequencies of a theoretical rotor blade rotating in a vacuum. This highlighted the lack of standardisation of analysis methods across the industry and the difficulty in what at first sight appeared to be a straightforward problem. It also highlighted the fact that not only aerodynamics but also structural dynamics is an intractable part of helicopter rotor analysis (since the blade was in a vacuum).

Niebanck (1984) found that a rotor blade with analytically optimized mass distribution (for minimum vibration) in flight tests produced higher vibration than the standard blade on an S-76 helicopter. This paper demonstrated the need for more accurate analysis techniques.

Loewy (1984) concluded that the problem of helicopter rotor aerodynamics was not sufficiently understood to allow the prediction of rotor exciting forces with any degree of confidence. He also stated that, "...whatever the aerodynamics are, they are periodic."5

Bousman and Mantay (1987) said of computer codes available for helicopter rotor loads analysis that:

- Oscillatory loads could not be predicted with any more confidence in 1987 than in 1973.

- The basic physics of the problem were not accurately modelled by any of the rotor loads analyses.

- There had been improvement in the analysis of the separate parts of the

---

problem but the various pieces of the problem had not gone back together correctly. There had been no improvement in the synthesis.

- The existing rotor loads analyses came into place in the early 1970's. Since then there had been minor improvements but, with the exception of the addition of the calculation of fuselage induced inflow to Y201 and CAMRAD, there had been no change to these analyses to improve their ability to represent the physics of the rotor loads problem.

- One factor that had limited advances in the prediction of rotor loads was the perception that vibratory loads could not be predicted by anyone in the foreseeable future.

Ormiston et al. (1987) rebuffed the many people who blamed the unsatisfactory results of analysis solely on the poor understanding of helicopter aerodynamics. They were concerned that while it was accepted among rotorcraft researchers that because of the complexity of the flow fields, an adequate description of rotary wing aerodynamics was well beyond the current state of the art, however, it was sometimes assumed that rotorcraft structural dynamics was an exact science. They pointed out that even though the mechanics of rotating structures was considerably less difficult than the aerodynamic problem, structural dynamics analysis for helicopter rotor blades was not understood adequately and the problem of predicting rotating beam dynamics had not been solved.

Kvaternick et al. (1987) stated that excessive vibration had plagued virtually all new rotorcraft developments in the forty years since the first American helicopter, the Sikorsky R-4, had gone into production. Both the Sikorsky UH-60 Black Hawk and the McDonnell Douglas AH-64 Apache had problems in early flight tests. Vibration
levels on the prototype aircraft were significantly above U.S. Army specifications throughout the flight envelope. Most of the vibration reduction over the past forty years had been achieved by add-on vibration control devices. These were effective but they weighed up to 2.5 percent of the gross weight of the helicopter, which represented a reduction of 10 to 15 percent in primary mission payload.

Crawford (1990) reinforced the statements of Kvaternick et al. (1987) by saying that the reduction in vibration levels over the last forty years had been due to brute force devices rather than improved prediction methodology. Virtually every new helicopter had higher than predicted vibration levels in early flight tests. Extensive trial-and-error efforts were necessary during flight tests to reduce the vibrations to quasi-acceptable levels. Fuselage dynamic responses, rotor blade dynamics and powerplant dynamics were difficult areas in themselves but the interaction of these disciplines placed the problem beyond the present capabilities of analysts.

Figures 2.27, 2.28 and 2.29 show three different ways of viewing the complexity of rotorcraft aeroelasticity and the interactions between the different parts of the problem.
Figure 2.27. The aeroelastic triangle of forces. [After Loewy (1984)]

Figure 2.28. The helicopter as might be viewed by a dynamicist. [After Kvaternick (1987)]
2.B.3 The Higher Harmonic Control Method

In 1955, helicopter vibration levels at the pilot's seat were typically about 0.4 g. By 1975 the typical vibration levels had been reduced to about 0.1 g. However, the reduction of vibration levels reached an asymptote at this point and further reduction was not possible with the current technology. This technology comprised passive means of vibration control such as vibration absorbers and vibration isolators. U.S. Army specifications which required vibration levels not to exceed 0.02 g in the mid 1970's could not be met by any manufacturer and had to be relaxed.

This situation led to the investigation of active vibration control concepts. The mechanism used to control a helicopter to fly in any direction other than vertically is called "cyclic pitch". This involves varying the angle-of-attack (or pitch angle) of the rotor blades in a cyclic or periodic fashion at the frequency of the rotation of the rotor,
i.e. once-per-revolution. This is the frequency of the first harmonic of the rotor speed.

The Higher Harmonic Control concept involves superimposing additional periodic changes in rotor blade pitch angle upon the once-per-revolution changes required for the directional control of the helicopter. These additional active control inputs must be a linear combination of components at discrete frequencies which are multiples of, or "higher harmonics" of, the rotor speed. The limitation on the frequencies of the active control inputs means that each rotor blade follows an identical pitch angle schedule as the rotor turns. This limitation is necessary so that the Higher Harmonic Control method can be implemented on rotors with any number of rotor blades by using hydraulic actuators in the non-rotating system to modify the motion of the swashplate.

In addition, the Higher Harmonic Control method uses only measurements taken from the fuselage. No measurements are taken from the rotor blades. No signals have to be transferred between the rotating and the non-rotating systems.

Higher Harmonic Control is an adaptive control strategy which combines recursive parameter estimation with linear optimal control theory. Linear, quasistatic, frequency domain representations of the helicopter response to control are used. Least squares or Kalman filter type identification of the helicopter control parameters has been used together with a minimum variance or quadratic performance function type controller to determine the optimal control harmonics for vibration alleviation. A summary of the work done on Higher Harmonic Control follows.

M'Ccloud and Kretz (1974) and Kretz et al. (1973a, 1973b) tested the Higher Harmonic Control concept on a full-scale jet-flap rotor in a wind-tunnel. They examined the effect of Higher Harmonic Control on blade loads and vibration. They introduced the concept of a linear, quasistatic representation of the rotor vibration
response to changes in the control inputs, including the name of the “T-matrix” for the transfer function relating the changes in vibration to changes in the controls. The T-matrix was calculated from the wind-tunnel data by the least-squares method. Then the open loop control required to minimize a quadratic performance function was calculated. McCloud (1975) applied this method to data obtained by theoretical analysis of a Multicyclic Controllable Twist Rotor (MCTR) (“multicyclic” means the same as “higher harmonic”). McCloud and Weisbruch (1978) applied the method to data from a wind-tunnel test of a full-scale MCTR. Brown and McCloud (1980) used data from the wind-tunnel test of the MCTR to examine the effects of weighting factors in the performance function and the influence of rotor lift, propulsive force and speed on the open-loop control.

Sissingh and Donham (1974), Powers (1978) and Wood et. al. (1980) all tested model rotors in wind-tunnels and measured the response of the oscillatory forces to swashplate control. They then used direct inversion of the T-matrix to find the control required to eliminate the vibration.

McHugh and Shaw (1978) tested a model hingeless rotor in a wind-tunnel. The input required to eliminate the vibratory loads was calculated by extrapolation and interpolation of the test data. They did not succeed in eliminating completely more than one component of the vibratory hub loads at a time but they did reduce all three components simultaneously.

Shaw and Albion (1980) tested a model hingeless rotor in a wind-tunnel. They examined closed-loop feedback control using gains calculated by direct inversion of the T-matrix. Using this controller, the vibratory loads were eliminated at one speed; at higher or lower speeds the vibratory loads were reduced but not eliminated, because
the changes in angle-of-attack required exceeded the available control authority.

Shaw (1980) conducted a theoretical investigation of the closed-loop feedback control of vibratory hub loads. The control gains were calculated by direct inversion of the T-matrix. The effect of errors in the estimate of the T-matrix on the stability of the controller was investigated. For an abrupt change in speed with a fixed-gain matrix, the controller became unstable. This pattern was overcome by using a Kalman filter for on-line identification of the T-matrix.

Taylor, Farrar and Miao (1980) and Taylor et al. (1980) used a numerical simulation to study the influence of the relative weighting factors given to the measurements from a number of fuselage mounted accelerometers, on the control of the helicopter fuselage vibration. They used closed-loop feedback control with a Kalman filter for on-line identification of the T-matrix and calculated the feedback gains to minimize a quadratic performance function. This control system achieved a significant reduction in vibration levels.

Hammond (1980) tested a model articulated rotor in a wind-tunnel. The response of the vibratory hub loads to swashplate control was measured and a Kalman filter was used to identify the T-matrix on-line. Closed-loop feedback was used to minimize a quadratic performance function. A stochastic (or cautious) controller algorithm succeeded in reducing the vertical force significantly and the pitch moment to some extent, but little reduction of the roll moment was achieved.

Molusis, Hammond and Cline (1981) extended this investigation and used vertical, longitudinal and lateral acceleration as the measurands. They varied the wind-tunnel speed and the collective pitch. The cautious control algorithm performed well in all the conditions tested. The vertical and longitudinal accelerations were
reduced significantly, but the lateral acceleration was increased at low speed.

Shaw and Albion (1981) tested Higher Harmonic Control on a four-bladed hingeless rotor in a wind-tunnel using fixed-gain closed-loop control. They demonstrated simultaneous suppression of three vibratory hub load components which was maintained as operating conditions were changed. The closed-loop bandwidth was adequate for suppression in manoeuvres.

Johnson (1982) compared and discussed most of the viable algorithms for Higher Harmonic Control characterized by: a linear, quasi-static, frequency-domain model of the helicopter vibratory response as a function of the control inputs; identification of the helicopter model by least-squared-error or Kalman filter methods; and a minimum variance or quadratic performance function controller. His conclusions were intended to provide guidance in the selection of algorithms. However, his analysis and numerical simulations were limited to a single-input and single-output system.

Davis (1984) refined, evaluated and compared a range of controller algorithms for multiple-input and multiple-output linear systems. Deterministic, cautious and dual controllers were investigated. For each controller type both a local linear system model and a global linear system model were evaluated. The controllers were evaluated with a helicopter vibration simulation computer program called G400 which modelled a four-bladed rotor mounted on the NASA Ames Rotor Test Apparatus which represented the fuselage. This study was intended as a step towards selecting the best active controller for alleviating helicopter vibration. The best controller configurations in terms of overall performance were identified.

Wood et al. (1985) and Straub and Byrns (1986) describe the joint U.S. Army-NASA-Hughes Helicopter Company (now McDonnell Douglas Helicopter Company)
program in which Higher Harmonic Control was applied to an OH-6A helicopter. The control algorithm included: a linear model of the relationship between the change in the vibration and a change in the control inputs; a Kalman filter for on-line identification of the T-matrix; and a minimum variance controller to minimize a quadratic performance function. The Higher Harmonic Control system superimposed four-per-revolution periodic inputs upon the non-rotating swashplate. The helicopter was flown from zero airspeed to 100 knots and very substantial reductions in vibration were achieved without serious adverse affects on blade loads or aircraft performance. Although the vibration levels were greatly reduced at the pilot's seat, where the feedback accelerometers were located, (which was the aim of the project), the vibration levels at the aircraft centre of gravity were slightly increased. Initial tests showed the OH-6A control system was incapable of transmitting high frequency feathering motion to the main rotor blades. This led to modifications to the control system which resulted in a 75 percent reduction in freeplay and a 90 percent increase in end-to-end control system stiffness.

Miao, Kottapalli and Frye (1986) and O'Leary, Kottapalli and Davis (1984) described flight tests of an experimental Higher Harmonic Control system on a Sikorsky S-76 helicopter. The control algorithms used in this system were similar to those used in the OH-6A tests.

Achache and Polychroniadis (1986, 1987) described flight tests of an experimental Higher Harmonic Control system on an Aerospatiale SA 349 Gazelle helicopter in France. The control algorithms used were similar to those used previously on the OH-6A helicopter. A reduction of cabin vibration levels by 80 percent was achieved at an airspeed of 250 km/h.
Shaw et al. (1989) tested Higher Harmonic Control on a one-sixth scale 10.5 foot diameter model of the three-bladed, fully articulated CH-47D Chinook rotor for Boeing Helicopter Company. The Higher Harmonic Control system was found to be very effective in simultaneously suppressing vertical and in-plane three-per-revolution hub forces across a wide test envelope including forward flight at up to 188 knots. This system used a fixed-gain feedback control law in contrast to the adaptive control laws used by other researchers. Three-per-revolution vertical force and two-per-revolution and four-per-revolution in-plane shears were suppressed simultaneously by 90 percent in almost all of the test envelope, including quasi-steady manoeuvres. This suppression was also maintained as rotor operating conditions were changed as rapidly as possible. In this test the standard method of supporting the swashplate at three points (not equally spaced), which leaves one half of the swashplate unsupported, had to be replaced by four equally spaced actuators to avoid undesirably large bending of the swashplate.

Lehmann (1985) and Lehmann and Kube (1990) of the German DFVLR Institute for Flight Mechanics, tested a four-bladed hingeless model rotor in the German-Dutch Wind-Tunnel (DNW). The rotor was a model of the Messerschmitt-Bolkow-Blohm (MBB - now Eurocopter Deutschland) BO-105 helicopter rotor. The relationship between the change in the vibration and a change in the control inputs was assumed to be linear. A Kalman filter was used for on-line identification of the T-matrix. A minimum variance controller was used to minimize a quadratic quality criterion (performance function). Effective vibration reduction (up to 80 percent) was achieved across the entire test envelope.

Robinson and Friedmann (1989a, 1989b) developed a computer simulation of a
four-bladed helicopter rotor attached to a fixed, rigid fuselage and used it to study Higher Harmonic Control. Both articulated and hingeless rotors were modelled. The rotor blades were treated as flexible and three flap modes, two lag modes and the fundamental torsional mode were taken into account. Forward flight and unsteady aerodynamics were included. A minimum variance controller was used to minimize a quadratic performance function. The relationship between the change in the vibration and a change in the control inputs was assumed to be linear. Both global and local controllers were evaluated and the global controller was found to be more effective in reducing vibration. It was also found that when applying Higher Harmonic Control to the hingeless rotor much larger control angles and greater power to operate the Higher Harmonic Control actuators were required than for the articulated rotor.

Nguyen and Chopra (1989a, 1989b) also developed a computer simulation. Compared with that of Robinson and Friedmann it had an identical Higher Harmonic Control algorithm and a similar structural model, but it modelled a three-bladed rotor and had a more refined aerodynamic model. This simulation was validated against wind-tunnel tests of the one-sixth dynamically scaled three-bladed articulated model rotor conducted by Shaw et al. (1989) for Boeing Helicopter Company. Good correlation with the experimental data was obtained. The conclusions of this study were similar to those of Robinson and Friedmann. Application of Higher Harmonic Control to both articulated and hingeless rotors was studied and it was found that greater control angles and power were required for hingeless rotors. Global controllers were found to be superior to local controllers.

Spletstoesser et al. (1994) tested the same four-bladed hingeless model BO-105 helicopter rotor used by Lehmann and Kube (1990) in the German-Dutch Wind-
Tunnel. They examined the benefit of Higher Harmonic Control to reduce blade-vortex interaction (BVI) impulsive noise. The test was a cooperative effort between Germany, France and the United States. BVI noise reduction was achieved but undesirable increases in low frequency loading noise and vibrational loads were produced. The authors concluded that Individual Blade Control was highly desirable for simultaneous reduction of vibration and noise.

2.B.4 Individual Blade Control

In contrast to the Higher Harmonic Control method, the Individual Blade Control method has no limitation on the frequency of the active control inputs. Each rotor blade is controlled independently, and may follow a different pitch angle schedule as the rotor turns.

The concept of Individual Blade Control involves controlling the pitch angle (angle of attack) of each helicopter rotor blade individually. For helicopter rotors with two or three blades this can be done using a conventional swashplate. The swashplate has three degrees-of-freedom: two rotational or tilting in the lateral and longitudinal directions and one translational in the vertical direction. These correspond to the lateral cyclic, longitudinal cyclic and collective blade pitch controls of the rotor. Hence the swashplate has three independent inputs and can individually control the blade pitch angles for a two-bladed or three-bladed rotor. For a rotor with four or more blades Individual Blade Control requires the pitch links that connect the swashplate to the rotor blades to be replaced by electrohydraulic actuators. This adds the complexity of having electrical and hydraulic systems rotating with the rotor. This is a difficult and
expensive design problem, and it may lead to reliability problems. Each rotor blade also requires its own sensor and feedback control loop. This either means having computers in the rotating system or transferring measurement and control signals between the fuselage and the rotating system. These complexities associated with the implementation of an Individual Blade Control system have made it less popular than the Higher Harmonic Control concept until recently. Higher Harmonic Control has been the subject of much more research and development than Individual Blade Control. However, despite the many investigations of Higher Harmonic Control, including flight tests of Higher Harmonic Control systems on several different helicopters in the mid-1980s, they have not been put into production. The proponents of Individual Blade Control (e.g. Jacklin and Leyland (1993)) suggest that the constraints inherent in the Higher Harmonic Control system limit its effectiveness to such an extent that the benefits are outweighed by the costs of implementation. Therefore the Individual Blade Control concept has been attracting more interest in recent years. The research on the Individual Blade Control method is described below.

The first work on the Individual Blade Control concept was done over twenty years ago by Kretz et al. (1973a, 1973b). A jet-flap rotor design was used in wind-tunnel tests to control individually the angle of attack of each blade to alleviate blade stresses and fuselage vibration.

Guinn (1982) described a feasibility study of an Individual Blade Control system performed by Bell Helicopter Textron for their Model 412 helicopter. The system was called "Individual Blade Control Independent of a Swashplate" and was given the acronym "IBIS". The system had four actuators per blade, each of which was powered by a hydraulic power supply. The actuators, electrical and hydraulic
power supplies, and control computers, were all located in the rotating system. The power supplies were driven by stationary gears attached to two standpipes - one inside the mast and one outside the mast. The pilot's control commands were passed to the rotating system through optical fibres and optical slip rings. The actuators were mounted at the rotor hub and the swashplate and pitch links were completely eliminated. This resulted in a 40 per cent reduction in the aerodynamic drag caused by the mechanical control system at maximum cruise speed and a 40 per cent reduction in the weight of the mechanical control system. The IBIS system had higher reliability than a conventional mechanical control system. It also allowed Individual Blade Control with both harmonic and non-harmonic control inputs. The result of the study was that IBIS was not only feasible but desirable.

Ham et al. (1983) used Individual Blade Control to achieve helicopter rotor blade lead-lag damping augmentation. Forces were created on the rotor blade which opposed the lead-lag motion and were proportional to the lead-lag velocity. These forces were Coriolis forces created by rotor blade flapping which was controlled by adjusting the blade pitch angle. A servomotor was used to control the blade pitch angle. The blade lead-lag acceleration was measured using two accelerometers. These measurements were integrated to give the lead-lag velocity which was the feedback signal.

The control system was tested on a model rotor in a wind-tunnel. The test rotor had only one blade, and two counter weights were used to achieve dynamic balance. The rotor blade was fully articulated and was 21.2 inches long with a 2 inch chord. One drawback of the tests was that the rotor was mounted on its side so that it rotated in a vertical plane. The rotor shaft was equipped with slip-rings to provide
electrical power to the servomotor and to transfer the data from the sensors. Problems were encountered because of excessive friction in the lead-lag hinge. This caused a large amount of lead-lag motion damping which tended to obscure the increase in lead-lag damping due to the control system. Nevertheless, data were obtained that were conclusive in demonstrating an increase in lead-lag damping due to the control system.

Quackenbush (1984) used Individual Blade Control to alleviate the rotor blade first torsion mode oscillations associated with stall flutter in helicopters. Stall flutter is an aeroelastic instability. An airfoil whose angle of attack is oscillating rapidly can operate transiently at angles of attack well in excess of its static stall angle without flow separation taking place. However, at sufficiently high angles of attack dynamic stall will occur. Dynamic stall differs considerably from static stall, and is characterized by a strong nose-down pitching moment on the airfoil. The motion associated with stall flutter is generally unstable only on the retreating side of the azimuth and damps out rapidly as the blade moves around towards the advancing side. However, even the one or two cycles of large amplitude torsional motion that do occur are sufficient to put extreme loads on the rotor control system. The stall flutter situation is equivalent to a one-degree-of-freedom oscillating system with negative damping. The method used to eliminate the negative damping was to provide a pitch rate (angular velocity) feedback from the blade to the blade pitch angle control actuator. Wind-Tunnel tests were performed to evaluate the Individual Blade Control system performance. The model rotor used was almost identical to that used by Ham et al. (1983). Appropriately oriented accelerometers were mounted on the model rotor blade to provide a signal proportional to the torsional angular acceleration. This signal was integrated to give the pitch rate (angular velocity). The results of the
wind-tunnel tests showed that the Individual Blade Control system was capable of alleviating some of the worst effects of the stall flutter instability. The component of the rotor blade motion at the torsional natural frequency was decreased by up to 90%.

McKillip (1985) investigated the use of Individual Blade Control for helicopter rotor blade flapping stabilisation in forward flight. He also investigated methods of controller design using modern control methods for linear systems governed by equations with periodic coefficients. Standard linear quadratic regulator (LQR) approaches to penalise excursions in blade flapping response result in closed-loop systems with a high bandwidth which is not desirable. To avoid this a model-following controller design was used. This entailed expressing in the cost function a desire for the plant being controlled to possess dynamics similar to a reference model. The plant was forced to emulate the model through liberal weighting of the difference between the two in the cost function. Analogue simulation and wind-tunnel tests were conducted. The model rotor used for the wind-tunnel tests was the same as that used by Ham (1983) except that it was mounted vertically so that the rotor rotated in a horizontal plane. It was shown that modal control using the Individual Blade Control concept is possible over a large range of advance ratios.

Ham (1985) discussed how certain Individual Blade Control techniques could be applied through actuators controlling a conventional swashplate instead of using actuators to replace the pitch links in the rotating system. This concept is discussed for a four-bladed rotor. As mentioned above, a conventional swashplate has only three degrees-of-freedom. Therefore, it is only possible to control independently the pitch angle of each rotor blade for rotors having three blades or less. So Individual Blade Control through a conventional swashplate for a four-bladed rotor would appear to be
a contradiction in terms. However, the paradox is explained by the fact that the
dynamic phenomena of the helicopter rotor that can be controlled by this method occur
at harmonics of the rotor rotational speed. Thus the control actions that are applied
are at certain discrete frequencies which are integer multiples of the rotor rotational
speed. In general, the 0 P, 1 P, N P, and (N±1) P harmonics of an N-bladed rotor can
be controlled through a conventional swashplate (where P stands for "per revolution").
This method of control is similar to the Higher Harmonic Control control method in
which the control actions are applied at discrete frequencies through a conventional
swashplate. However, there is a significant difference. In Higher Harmonic Control
the measurements are taken from sensors mounted on the fuselage. In Individual
Blade Control applied through a conventional swashplate the measurements are still
taken from accelerometers mounted on the rotor blades. Therefore some means is still
required to transmit these signals from the rotating system to the fuselage. The
dynamic phenomena that can be controlled using Individual Blade Control through a
conventional swashplate are: gust-induced flapping; motion-induced flapping;
airload-induced vibration; air resonance; and ground resonance.

For helicopter gust response alleviation using Individual Blade Control the
change in the rotor blade pitch angle was proportional to the rotor blade flapping
acceleration and displacement,

$$\Delta \theta = -\kappa \left( \frac{\ddot{\beta}}{\Omega^2} + \beta \right)$$

where $\beta$ is the rotor blade flapping angle;

$\Omega$ is the rotor rotational speed;

$\kappa$ is a constant;
$\Delta \theta$ is the change in the rotor blade pitch angle.

For suppression of ground and air resonance by the augmentation of rotor blade lead-lag damping using Individual Blade Control the change in the rotor blade pitch angle was proportional to the rotor blade lead-lag velocity,

$$\Delta \theta = -\kappa \frac{\ddot{\zeta}}{\Omega};$$

where $\dot{\zeta}$ is the rotor blade lead-lag velocity;

$\Omega$ is the rotor rotational speed;

$\kappa$ is a constant;

$\Delta \theta$ is the change in the rotor blade pitch angle.

For helicopter longitudinal attitude stabilisation using Individual Blade Control the change in the rotor blade pitch angle was proportional to the rotor blade flapping acceleration, velocity and displacement,

$$\Delta \theta = -\kappa_a \frac{\ddot{\beta}}{\Omega^2} - \kappa_p \frac{\dot{\beta}}{\Omega} - \kappa_p \beta$$

where $\beta$ is the rotor blade flapping displacement;

$\Omega$ is the rotor rotational speed;

$\kappa_a$, $\kappa_p$ and $\kappa_p$ are constants;

$\Delta \theta$ is the change in the rotor blade pitch angle.

The Individual Blade Control system for helicopter longitudinal attitude stabilisation also causes roll attitude stabilisation. It may be desirable to reduce the roll stabilisation. This could be done by varying the gains in the control law with the rotor blade azimuth angle.

It may also be necessary to prevent the attenuation of the helicopter response to the pilot's control actions. This could be done by biasing the feedback signals by a
signal proportional to the displacement of the pilot's control stick.

For helicopter vibration alleviation the control law was designed to control the rotor blade flatwise bending response. This was based on the fact that a major source of helicopter higher harmonic vertical vibration is due to the rotor blade flatwise bending response to the impulsive loading due to blade-vortex or blade-fuselage interaction. The change in the rotor blade pitch angle was proportional to the rotor blade flatwise bending acceleration, velocity and displacement,

$$\Delta \theta = -\kappa \left( \frac{\ddot{g}}{\Omega^2} + \frac{\dot{g}}{\Omega} + g \right)$$

where

- $g$ is the first elastic flatwise bending mode displacement;
- $\Omega$ is the rotor rotational speed;
- $\kappa$ is a constant;
- $\Delta \theta$ is the change in the rotor blade pitch angle.

Ham (1987) summarised the research on helicopter Individual Blade Control performed at the VTOL (Vertical Take-Off and Landing) Technology Laboratory, in the Department of Aeronautics and Astronautics, at Massachusetts Institute of Technology (MIT) between 1977 and 1985. The applications discussed were: gust response alleviation; attitude stabilization; vibration alleviation; rotor blade lead-lag damping augmentation; stall flutter suppression; rotor blade flapping stabilization; stall alleviation; and performance enhancement.

Ham and McKillip (1991) performed wind-tunnel tests on a full-size four-bladed hingeless rotor in the NASA Ames 40 by 80 Foot Wind-Tunnel. The tests had two aims. The first aim was to verify that blade-mounted accelerometers could be used to measure helicopter rotor blade lead-lag angle. The second aim was to verify that
the blade-mounted accelerometers could be used as the sensors for the feedback control of a full size helicopter rotor blade lead-lag response using Individual Blade Control. Since this was a hingeless rotor there were no hinges at which to measure either the flap angle or the lead-lag angle. However, the lead-lag angle was measured using a linear potentiometer located near the root of the blade.

The first aim was achieved by comparing the results from the accelerometers with the measurements from the conventional angle transducer. The reason given for using accelerometers was that they are more reliable than conventional angle transducers, and easier to mount - particularly on hingeless rotors. The second aim was achieved by using the recorded accelerometer signals as inputs to an Individual Blade Control system in a laboratory.

Wasikowski et al. (1989) and Wasikowski (1989) applied Individual Blade Control to mathematical models of both rigid articulated rotor blades and elastic hingeless rotor blades. They studied in-plane stability augmentation in forward flight, as well as ground resonance suppression. The controller design was based on an extension of optimal output feedback control theory to the design of fixed gain controllers for periodic systems. This design method is claimed to have many advantages over existing Linear Quadratic Gaussian (LQG) design methods. The feedback control was based only on the available sensor measurements rather than an estimate of the entire state vector. Therefore, an observer/estimator was not required. Also the feedback gains were not periodic but constant. The results of this analytical study showed that Individual Blade Control with constant gain output feedback was effective for the applications considered.
Jacklin and Leyland (1993) described the preparations for a joint United States/German wind-tunnel test program to investigate a helicopter Individual Blade Control system. A full-size Messerschmitt-Boelkow-Blohm (now Eurocopter Deutschland) BO-105 rotor system was to be tested in the 40 by 80 Foot Wind-Tunnel at the NASA Ames Research Centre. The BO-105 rotor is a four-bladed hingeless rotor system. The Individual Blade Control system used one actuator per blade in the rotating system to replace the pitch links that connected the swashplate to the rotor blades. A hydraulic slipring was used to transmit hydraulic power to the rotating system. The control computers were in the non-rotating system and the control signals were transmitted to the rotating system by an electrical slipring.

The actuators were developed by Henschel Flugzeug Werke (HFW), a subsidiary of Eurocopter-Germany. Their development began ten years earlier with low-speed actuators for individual blade tracking control. A system capable of introducing significant Individual Blade Control pitch angle inputs up to 5 P (where P stands for "cycles per revolution of the rotor") was successfully flight tested at Eurocopter's Ottobrun facility. The flight tests in 1990 used control authorities of ±0.19 degrees, and the flight tests in 1991 used control authorities of ±0.42 degrees. The first of these flight tests was described by Richter and Eibbrecher (1990). The actuators to be used in the wind-tunnel tests were capable of introducing Individual Blade Control inputs up to 12 P with control authorities up to ±3.0 degrees.

Two stages of wind-tunnel tests were planned. The first stage was planned to evaluate the open-loop effects of Individual Blade Control on rotor performance, acoustics, vibration and loads. For open-loop testing, the amplitude and phase of each harmonic (1 P to 12 P) were converted to a time domain control through Fourier
synthesis. In previous flight tests the sine and cosine amplitudes of 2P, 3P, 4P and 5P were introduced through a hand-held box of potentiometers. Since the new system could accept inputs of up to 12P that approach was too cumbersome; so a personal computer (PC) was to be used to introduce the harmonics of sine and cosine from a pre-programmed spreadsheet of commands.

In the second stage of wind-tunnel tests the PC spreadsheet approach was to be replaced by a self-adaptive, closed-loop controller. This controller was developed by DLR (German Aerospace Research Establishment) in cooperation with NASA, HFW, Eurocopter-Germany, and the U.S. Army Research Laboratories. The closed-loop controller would automatically choose the correct harmonic amplitudes and phases needed to suppress vibration, noise and loads, and to improve rotor performance.

Reichert and Arnold (1990) applied Individual Blade Control to a mathematical model corresponding to a MBB BO-105 helicopter.

2.B.5 Active Control of Helicopter Resonance Problems

Air resonance and ground resonance are resonance problems and not stability problems. Air resonance occurs in flight. Ground resonance occurs on the ground, typically when the rotor is being brought up to its operating speed. Air resonance is much less severe than ground resonance. They both occur when the frequency \(|\Omega - \omega_L|\) is near a natural frequency of the fuselage in pitch or roll, where: \(\Omega\) is the rotor speed; and \(\omega_L\) is the lead-lag natural frequency of the rotor blade.

Both air resonance and ground resonance involve the shifting of the centre of

---

6 See section 2.B.5 on control of aeromechanical instabilities.
gravity of the rotor away from the centre of the rotor hub, followed by rotation of the
centre of gravity around the rotor hub relative to the rotating system due to the lead-
lag motion of the rotor blades in the plane of the rotor. This creates forces in the
rotating system of frequency $\omega_r$.

Figure 2.30 shows how the lead-lag motion of the rotor blades of a four-bladed
rotor causes the centre of gravity to rotate relative to the rotating system (note that in
this figure the observer is rotating with the rotor at the rotor speed $\Omega$; and that the
period $\tau = 2\pi/\omega_r$ is not the period of one revolution of the rotor, but the period of
one lead-lag oscillation of the rotor blades).

The forces at frequency $\omega_r$ in the rotating system create forces at frequencies
of both $|\Omega + \omega_r|$ and $|\Omega - \omega_r|$ in the frame of the fuselage. Large oscillations of the
fuselage in pitch or roll will result if it has a natural frequency close to $|\Omega - \omega_r|$.

For the energy from the mechanical rotor drive system to drive the lead-lag
oscillation of the rotor blades the rotor must be a "soft-in-plane" rotor. This means
that the lead-lag natural frequency must be lower than the rotor speed, i.e. $\omega_r < \Omega$. If
the lead-lag natural frequency is higher than the rotor speed, \( \omega_r > \Omega \), then the rotor is called "stiff-in-plane" and these resonance phenomena do not occur.

For soft-in-plane rotors the two modes of oscillation associated with the frequencies \( |\Omega + \omega_r| \) and \( |\Omega - \omega_r| \) in the frame of the fuselage are progressing modes, which means that the rotation of the centre of gravity around the centre of the rotor hub is in the same direction as the rotation of the rotor. This is because the frequency of the rotation of the centre of gravity relative to the rotating system is equal to \( \omega_r \), which is less than the rotor speed.

For most helicopters with articulated rotors resting on their landing gear on the ground, the natural frequencies of the fuselage in pitch and roll are below the frequency of the forces at \( |\Omega - \omega_r| \) when the rotor is at its operating speed. Therefore when the rotor is being brought up to speed ground resonance can occur. In flight the restoring forces from the landing gear are not present and so the natural frequencies of the fuselage in pitch and roll are greatly reduced. The frequency of the forces at \( |\Omega - \omega_r| \) is then well above these natural frequencies. For this reason air resonance almost never occurs in helicopters with articulated rotors.

Helicopters with hingeless rotors, on the other hand, very rarely have problems with ground resonance. This is because the frequency of the forces at \( |\Omega - \omega_r| \) is below the fuselage natural frequencies in pitch and roll when the rotor is at its operating speed and the helicopter is resting on its landing gear on the ground. However, the fuselage natural frequencies in pitch and roll are lower in flight than on the ground and air resonance can be a problem for helicopters with soft-in-plane hingeless rotors. As mentioned above, resonance does not occur for stiff-in-plane rotors when \( |\Omega - \omega_r| \) is
close to a fuselage natural frequency. However, stiff-in-plane rotors can experience more complex and more severe aeroelastic problems such as the flap-lag-torsion instability. For this reason most hingeless rotors tend to be soft-in-plane.

Young et al. (1974) studied the feedback control of ground and air resonance for a helicopter in hover. A mathematical model of a coupled rotor-fuselage system was used. The rotor blades had flap and lag degrees-of-freedom. The fuselage had five degrees-of-freedom. Quasi-steady two-dimensional aerodynamics were used. The control was implemented in a purely theoretical way by setting the roll moment in the equations of motion equal to a function of the roll angle and angular velocity. By using this control method ground and air resonance instabilities were successfully suppressed.

Straub and Warmbrodt (1985) also used a coupled rotor-fuselage mathematical model of a helicopter in hover to study the active control of ground resonance. The rotor blades had flap and lag degrees-of-freedom and quasi-steady aerodynamics were used. A three-bladed hingeless rotor was modelled. The fuselage had four degrees-of-freedom. These were pitch and roll rotations and lateral and longitudinal translations. Small active pitch control inputs to the rotor blades were implemented through a conventional swashplate. Various degrees-of-freedom were used for the feedback signal. The most effective degrees-of-freedom for the control of ground and air resonance were found to be the blade lead-lag motion and the fuselage roll motion.

Straub (1987) extended the work of Straub and Warmbrodt (1985) by employing multivariable optimal control theory techniques to control ground resonance. The same mathematical model was used, except that a four-bladed articulated rotor was modelled. This model was designed to simulate a full-size H-34
rotor mounted on the Rotor Test Apparatus support structure in the NASA Ames 40 by 80 Foot Wind-Tunnel. Active control of the rotor blade pitch angles was applied through a conventional swashplate. An optimal controller with full state feedback with gain scheduling was shown to improve the stability of the system at all rotor speeds and to suppress ground resonance. The cyclic control inputs to the swashplate were obtained by minimizing a quadratic cost function. As full state feedback is not practical, partial state feedback was investigated. Feedback of only the lateral and longitudinal degrees-of-freedom of the support structure was shown to be sufficient to control ground resonance and eliminate the need for lead-lag dampers on the rotor.

Reichert and Arnold (1990) used a coupled rotor-fuselage mathematical model corresponding to an MBB BO-105 helicopter to study the active control of ground and air resonance. The model was similar to that of Straub and Warmbrodt (1985) using spring restrained rigid blades with flap and lag degrees-of-freedom and quasi-steady aerodynamics. The fuselage model had six degrees-of-freedom and the rotor model was a four-bladed soft-in-plane hingeless rotor. The control was implemented by modifying the rotor blade pitch angles. Both conventional swashplate control and Individual Blade Control techniques were shown to improve the stability of the system.

Takahashi (1988) and Takahashi and Friedmann (1991) studied the active control of air resonance. A coupled rotor-fuselage mathematical model was used to represent a helicopter similar to the MBB BO-105 with a four-bladed soft-in-plane hingeless rotor. Certain parameters were modified in order to induce an unstable air resonance mode. Rigid, spring restrained, hinged blades were combined with a rigid fuselage having five degrees-of-freedom. In addition to flap and lag degrees-of-freedom the rotor blades had a torsional degree-of-freedom. The aerodynamics
included low frequency unsteady effects. The inclusion of the rotor blade torsional flexibility and unsteady aerodynamics differentiated this study from previous ones which used a less sophisticated mathematical model.

Emphasis was placed on keeping the control method simple to implement, as a complex control system would be impractical for real helicopters. Moreover, a complex system would not be chosen in preference to the simple mechanical solution to air resonance in hingeless rotors which involves attaching lead-lag dampers to each blade. To this end the control was implemented through a conventional swashplate. Full state feedback was deemed to be impractical due to the large number of sensors that would be required. The controller used a single roll rate (angular velocity) measurement which was taken from the fuselage. This avoided the need to send signals from the rotating rotor to the non-rotating (fuselage) frame of reference. Optimal control theory and multivariable frequency domain design methods were used to produce an LQG controller based on the optimal estimator-regulator structure. This controller was shown to stabilise the system throughout a wide range of forward flight speeds and loading conditions.
2.B.6 Active Control of Instabilities

In addition to the active control of air resonance, which is a fairly mild problem, Takahashi (1988) investigated the active control of two more severe problems. The first of these was the fuselage induced flap-lag instability. The rotor blade lead-lag angle had to be used as the feedback signal for the controller because this instability does not produce substantial fuselage motion. The instability was easily suppressed throughout a wide range of operating conditions. The second instability investigated was the flap-lag-torsion instability of a stiff-in-plane hingeless rotor in forward flight. This instability was far more difficult to stabilise than the air resonance or the fuselage induced flap-lag instability. A controller was produced that stabilised the system at the design point, but it was highly sensitive to changes in the advance ratio. Small increases or decreases in the advance ratio could destabilise the system.

Peebles (1977) used an isolated rotor blade mathematical model to study the active control of the pitch-flap instability of a rotor in hover. The rotor blade was rigid with spring restraints and flap and pitch degrees-of-freedom. Quasi-steady aerodynamics were used. The system was destabilised by varying the rotor speed and the position of the centre of gravity of the rotor blade in the chordwise direction. The control was implemented by applying moments to the rotor blade about the flap and pitch axes. By using optimal control theory an improvement in the stability of the system was demonstrated.

Calico and Wiesel (1986) applied a control method based on Floquet theory to the stabilisation of the flapping motion of a helicopter rotor blade in forward flight. They had previously applied the method to problems in orbit stationkeeping and satellite attitude control. The control was applied to the rotor blade pitch angles
through a conventional swashplate. Stability improvements were demonstrated using mathematical models of one-bladed and two-bladed rotors.

2.B.7 Active Control for Helicopter Gust Response Alleviation

Briczinski and Cooper (1975) performed flight tests on a CH-53 helicopter to evaluate the use of rotor blade flapping angle feedback for gust response alleviation. The collective and cyclic blade pitch angles were varied in proportion to the rotor flapping. The helicopter response to vertical gusts was reduced by 30 to 50 per cent.

Briczinski (1976) performed an analytical investigation of a range of proportional feedback control schemes for gust response alleviation for the CH-53 helicopter. The Sikorsky GENHEL nonlinear helicopter simulation computer program was used to evaluate the schemes. Rigid rotor blades and a rigid fuselage were assumed. Using the rotor blade angle of attack, the collective flapping angle and the collective flapping angular velocity as the feedback signals it was shown that the helicopter response to gusts could be attenuated by 75 per cent.

Taylor et al. (1980) used linear, quadratic synthesis to design a feedback controller for gust response alleviation based on a linear time-invariant model of a Sikorsky Black Hawk helicopter. The fuselage pitch and roll angles, the pitch, roll and yaw angular velocities, and the main rotor collective flapping angle and angular velocity were used as the feedback signals. In forward flight at 150 knots the root-mean-square (rms) heave velocity, which is the most sensitive to gusts, was reduced by 75 per cent.
Saito (1984) analytically investigated the use of an adaptive blade pitch control system for helicopter gust response alleviation. The control inputs were superimposed on the basic collective and cyclic rotor blade pitch angle control inputs through a conventional swashplate. An isolated rotor model of a four-bladed articulated rotor was used. The blades had full flap, lead-lag and torsion elastic deflection. The aim of the controller was to reduce the thrust response of the rotor to the gust. It was assumed that the frequency of the gust was known. Therefore this type of controller does not apply to random gust responses. The measured quantities were the oscillatory hub forces and moments at the gust frequency. A Kalman filter was used for identification. The control algorithm was based on the minimization of a performance index that was a quadratic function of the input and output variables. By using numerical simulations it was shown that the rotor thrust response to a sinusoidal gust could be reduced by more than 50 per cent and the response to a step gust could be reduced by almost 100 per cent.

Ham (1987) and Ham (1985) described research performed at M.I.T. on the application of Individual Blade Control to helicopter gust response alleviation. Fuller (1991) investigated the use of active control to attenuate unwanted flapping responses of helicopter rotor blades to gusts and to suppress vibration. The control was implemented through a conventional swashplate. The rotor was fully articulated. The feedback signals were the rotor blade flap angles. Two rotor models were used for the design of active controllers: a periodic design model; and an averaged design model. The periodic design model included only the rigid flapping

\[\text{See section 2.8.4 on Individual Blade Control.}\]
degree-of-freedom of the rotor blades. It was linear and periodically time-varying. The rigid lead-lag degree-of-freedom was not included on the grounds that its effect on rotor thrust and blade flapping is small. The averaged design model was a time-invariant approximation to the periodic design model. It was determined by averaging the coefficients of the periodic design model. The averaged design model allowed the use of the sophisticated techniques that have been developed for the analysis and synthesis of multivariable controllers. The periodic design model required the use of less sophisticated controller synthesis techniques.

Fuller (1991) used the same LQG controller structure as Takahashi (1988)

which consisted of an estimator coupled with a regulator. Takahashi (1988) used a constant coefficient model to design a controller to stabilise air resonance because preliminary studies showed that the periodic terms in the model played only a small role for advance ratios up to $\mu = 0.4$.

Fuller (1991) found that up to an advance ratio of $\mu = 0.4$ the responses of the periodic model and the averaged model controllers were essentially the same except for the $b$/revolution response, where $b$ was the number of blades. The averaged model was used to produce controllers that met difficult gust response attenuation goals. However, the periodic model was recommended for the design of controllers to produce simultaneous gust response alleviation and vibration suppression.

---

8 see section 2.B.5
2.B.8 Conclusions to Part B

The first section of Part B highlighted the difficulties associated with helicopter rotor analysis and mathematical modelling which are due to the complexity of the problem. This emphasises the desirability of developing a control method which does not require a mathematical model of the system being controlled.

The two main areas of research in this field are Higher Harmonic Control and Individual Blade Control. The proponents of Higher Harmonic Control claim that the disadvantage of the Individual Blade Control concept is that for helicopter rotors having more than three blades it requires hydraulic actuators to replace the pitch links that control the rotor blade angle-of-attack in the rotating system. This is a difficult and expensive design problem and may have reliability problems. The proponents of Individual Blade Control claim that this same feature of their method is an advantage because helicopter control through a conventional swashplate is fundamentally limited for rotors with more than three blades. However, in spite of the large amounts of time and money which have been spent on research in these areas over the last twenty years, neither control method has been employed by a manufacturer in a production helicopter.

Research is continuing on these active control methods, and they may yet be adopted for production helicopters. Particularly this could be so because the helicopter designs are moving towards replacing the mechanical linkages which connect the pilot’s control levers to the rotor with actuators at the rotor which are connected to the pilot’s control levers by either electrical (fly-by-wire) or optical (fly-by-light) transmission paths.

No research has been done on the active control of helicopter rotors using
delayed feedback. Therefore the Phase-Locked Delayed Feedback control method developed in this thesis is the first study to bring together these two previously unrelated areas of research.
3.1 Formulation of the Thesis

From the Literature Review in Chapter 2, it is clear that helicopter rotor blades can be subject to a range of stability problems. The overall aim of this thesis is to investigate a new approach to stabilising the motion of helicopter rotor blades.

Researchers are agreed that helicopter rotor aerodynamics are extremely complex. However, as one leading researcher stated, "...whatever the aerodynamics are, they are periodic." Furthermore, the motion of the rotor blades determines the flight path of the helicopter. Therefore the aim of this thesis is to present, and to investigate the effectiveness of, a new control method to stabilise periodic motion in such a way that the periodic particular solution of the controlled system is identical with the periodic particular solution of the uncontrolled system but more stable. This method has been called the "Phase-Locked Delayed Feedback" control method. As it was noted in Chapter 2, this control method has much in common with the Delayed Feedback Control method developed by Pyragas (1992); but the Phase-Locked

2 See section 2.A.6.
Delayed Feedback method has been developed quite independently, in a different context, with a different purpose, and for a different application; and the approach differs from that taken by Pyragas (1992), as has been explained in Chapter 2. In particular, the Pyragas method was not developed for mechanical systems and does not employ the derivative of the output (which in a mechanical system is the velocity).

3.2 Specific Aims

A first step towards verifying its effectiveness will be to apply the Phase-Locked Delayed Feedback control method to both linear and non-linear mathematical models of a one-degree-of-freedom mass-spring-dashpot system; and to determine whether optimal values of the adjustable parameters in the control method exist, and if so how they vary as a function of the period of the motion being stabilised.

The next step will be to apply the Phase-Locked Delayed Feedback control method to a multi-degree-of-freedom laboratory installation; and, by a simple trial and error procedure, both to find the optimal values of the control parameters without modelling the system, and also to investigate how these experimentally determined optimal values of the control parameters vary as a function of the period of the motion being stabilised.

The final step will be to use a two-degree-of-freedom mathematical model, including aerodynamic forces to determine whether a helicopter rotor blade instability can be stabilised by using the Phase-Locked Delayed Feedback control method in both

---

1 See Chapter 2, section 2.A.6.
hovering and forward flight.

3.3 Overview of the Thesis

Chapter 1 has provided an introduction to helicopter rotors and the mechanical control system used by the pilot to determine the flight path of the helicopter. The stability of the motion of the rotor blades has also been discussed.

Chapter 2 has presented a review of the literature which is relevant to the Phase-Locked Delayed Feedback control method presented and investigated in this thesis. The chapter has been divided into two parts in which the literature relating to active control incorporating delayed feedback and the literature relating to active control of helicopter rotors are treated separately.

This chapter, Chapter 3, has formulated the aims of the thesis and outlines its presentation.

In Chapter 4, as an initial investigation into the concept of stability improvement using the Phase-Locked Delayed Feedback control method, it is applied to mathematical models of linear and non-linear one-degree-of-freedom systems. A computer program is developed to determine the stability margin by finding the dominant characteristic root, and the effectiveness of the control method is assessed.

Chapter 5 describes the application of the Phase-Locked Delayed Feedback control method to a multi-degree-of-freedom laboratory installation which was designed and constructed for this purpose. The hardware configuration that was assembled, and the software that has been written to achieve this, are both explained. The existence of optimal values of the control parameters and their variation as a
function of the period of the motion being stabilised is investigated experimentally.

In Chapter 6 the effectiveness of the Phase-Locked Delayed Feedback control method to stabilise the motion of a helicopter rotor blade in both hovering and forward flight is investigated using a two-degree-of-freedom mathematical model, including the aerodynamic forces.

In Chapter 7, the conclusions are summarised, and suggestions for further research are presented.
CHAPTER 4

STABILITY IMPROVEMENT OF THE PERIODIC MOTION OF ONE-DEGREE-OF-FREEDOM SYSTEMS

4.1. Introduction

In order to investigate the effectiveness of the Phase-Locked Delayed Feedback control method in improving the stability of periodic motion, the control method is applied first of all to mathematical models of very simple one-degree-of-freedom systems. Both linear and non-linear mathematical models are used.

The root of the characteristic equation which has the largest real part is called the "dominant" characteristic root, and it determines the margin of stability of the periodic particular solution of the equation of motion. The presence of time-delay terms in the equation of motion leads to a characteristic equation that has an infinite number of roots. Their values were obtained by a specially developed computer program. The results are checked by numerical integration of the equations of motion.

In addition, the following questions are addressed:

- Do optimal values of the adjustable parameters in the control law exist, and, if so, do they vary as a function of the period of the excitation force?

- Can the Phase-Locked Delayed Feedback control method stabilise the unstable
solution of a linear system with negative damping?

- What is the influence of the Phase-Locked Delayed Feedback control method on the size of the region of asymptotic stability in the state-plane for the periodic solution of the non-linear mathematical model?

- How does the sensitivity of the controlled non-linear system to external disturbances compare with that of the uncontrolled system?

4.2. Description of the Physical Model

The analysis provided in this chapter is limited to dynamic systems which can be approximated by the physical model shown in Figure 4.1. The object (Figure 4.1(a)) consists of a mass $m$, two springs $k$ and $k_1$, and a dashpot $c$. The motion of the system is described in the absolute system of coordinates $x$ and $y$. The restoring force of the springs and the damping force of the dashpot are, in general, non-linear functions of the coordinates $x$ and $y$ and their first derivatives.

Figure 4.1(b) shows a block diagram of the Phase-Locked Delayed Feedback control system. The controller senses the displacement output signal $x$, differentiates it to obtain the velocity $\dot{x}$, and stores these values in memory. A control input $y_c$ is produced which is proportional to the difference between the current values of $x$ and $\dot{x}$, and those stored $T$ seconds earlier (where $T$ is the period of the motion being stabilised). The control input $y_c$ is added to the excitation input, $y_{ex}$, to give the input $y$. The controller has no information about the object except the period $T$. 
Chapter 4: Stability Improvement of the Periodic Motion of One-Degree-of-Freedom Systems

4.3. Analysis of the Linear System

If the stiffness of the springs $k$ and $k_1$ and the damping coefficient $c_1$ are constant, then the motion of the system with control can be approximated by the following equation:

$$\ddot{x} + C\dot{x} + Kx = Y_0 \sin \omega t + A(x(t) - x(t - T)) + B(\dot{x}(t) - \dot{x}(t - T)),$$

where

$$C = \frac{c_1}{m} \quad K = \frac{k + k_1}{m} \quad Y_0 = \frac{ky_0}{m} \quad A = \frac{ka}{m} \quad B = \frac{kb}{m},$$

and the period of the harmonic excitation $y_0 = y_0 \sin \omega t$ is

$$T = \frac{2\pi}{\omega}.$$

Eq. (4.1) is the equation of motion of the controlled system. If the time-delayed terms were not present this equation would be a straightforward non-homogenous
Chapter 4 Stability Improvement of the Periodic Motion of One-Degree-of-Freedom Systems

second-order differential equation, representing the forced vibration of a simple, one-degree-of-freedom system. Its characteristic equation would have only two roots, and only two initial conditions would need to be specified in order to find any particular solution.

In general, a particular solution of Eq.(4.1), \( x(t) \) for \( t \geq 0 \), is determined by the initial function which defines, for example, the position of the system for the period of time \(-T \leq t \leq 0\) and is given by:

\[
x(t) = X(t); \quad -T \leq t \leq 0.
\]  
(4.4)

For analysis of Eq.(4.1) the following set of numerical values was adopted:

\[ m = 1 \text{ kg}, \quad k_1 = 1 \text{ N/m}, \quad k = 1 \text{ N/m}, \quad y_0 = 1 \text{ m}. \]

Eq.(4.1) has one periodic particular solution which does not depend on the control parameters \( a \) and \( b \):

\[
x(o)(t) = x(o)(t + T) = D \sin(\omega t - \phi),
\]  
(4.5)

where

\[
D = \frac{Y_o}{\sqrt{(K - \omega^2)^2 + (C\omega)^2}} \quad \phi = \arctan \frac{-C\omega}{K - \omega^2}.
\]  
(4.6)

The periodic solution Eq.(4.5) is identical with the periodic solution of the uncontrolled system. This is one of the most important features of this control method. It does not change the periodic solution of the uncontrolled system, while it improves its stability.
4.3.1 Equation of Perturbations

To analyse the stability of the solution Eq.(4.5), let us introduce the perturbed motion as follows

\[ \dot{x} = x_0 + \Delta x, \]  

(4.7)

where \( \Delta x \) are the perturbations. The perturbed solution Eq.(4.7) is the sum of the periodic particular solution \( x_0 \) Eq.(4.5) and the perturbations \( \Delta x \). The perturbed solution represents the motion that can be measured in practice. The periodic particular solution can never be seen or measured in practice because the perturbations are never zero.

To obtain the equation of perturbations, the perturbed solution Eq.(4.7) was introduced into the equation of motion Eq.(4.1):

\[ \ddot{\Delta x} + C\dot{\Delta x} + K\Delta x = A(\Delta x(t) - \Delta x(t - T)) + B(\Delta \dot{x}(t) - \Delta \dot{x}(t - T)). \]  

(4.8)

The particular solution of the equation of perturbations can be predicted as

\[ \Delta x = e^{(R+iU)t}. \]  

(4.9)

Substituting Eq.(4.9) into Eq.(4.8) and separating the real and imaginary parts of the characteristic equation gives

\[ R^2 - U^2 + CR + K - A + Ae^{-RT} \cos(UT) - BR + e^{-RT}(BR \cos(UT) + BU \sin(UT)) = 0 \]  

(4.10)

and

\[ 2RU + CU - Ae^{-RT} \sin(UT) - BU + e^{-RT}(BU \cos(UT) - BR \sin(UT)) = 0. \]  

(4.11)

The values of \( R \) and \( U \) which simultaneously satisfy both these equations define the characteristic roots of the equation of perturbations. The increase or decrease of
the perturbations with time can be determined from the roots of the characteristic equation Eqs.(4.10 & 4.11) of the perturbation equation Eq.(4.8). The roots are complex numbers and correspond to the real and imaginary parts \((R + iU)\) of the exponent in the predicted particular solution Eq.(4.9) of the perturbation equation Eq.(4.8). Clearly perturbations \(\Delta x\) in the predicted particular solution Eq.(4.9) will decrease with time \(t\) if the value of the real part \(R\) of the exponent is negative. Thus, if all the roots of the characteristic equation Eqs.(4.10 & 4.11) lie in the left half of the complex plane, then the perturbations will decrease with time and the periodic solution Eq.(4.5) of the equation of motion Eq.(4.1) is said to be “asymptotically stable”. If one or more roots have a positive real part \(R\), then the corresponding perturbations Eq.(4.9) will grow with time and the periodic solution Eq.(4.5) is said to be “unstable”.

The root of the characteristic equation Eq.(4.10 & 4.11) which has the largest real part \(R\) is called the “dominant” root. The position of this root in the complex plane determines the behaviour of the system. If the dominant root is in the left half-plane (LHP) then all the roots are in the left half-plane and have negative real parts. So the periodic solution Eq.(4.5) is asymptotically stable. If the dominant root lies on the imaginary axis, i.e. \(R = 0\), then the perturbations Eq.(4.9) neither increase nor decrease in amplitude but remain constant, and the periodic solution Eq.(4.5) is said to be simply “stable”. If the dominant root is in the right half-plane (RHP), then it has a positive real part and the periodic solution Eq.(4.5) is unstable.

The values of the control parameters \(a\) and \(b\) in the Phase-Locked Delayed Feedback control method which move the dominant root of the characteristic equation
Eq. (4.10 & 4.11) furthest to the left in the complex plane are called the "optimal values" of the control parameters.

4.3.2 Solution of the Characteristic Equation

To solve the real and imaginary parts of the characteristic equation a computer program was developed. It found the values of $R$ and $U$ which satisfied both Eq. (4.10) and Eq. (4.11). Figure 4.2(a) shows a plot of the zero points for Eq. (4.10) and Eq. (4.11) for $a = 0.0$, $b = -0.1$ s, period of excitation $T = 6$ s and damping coefficient $c_1 = 0$ Nsm$^{-1}$. The points where the lines intersect, where both equations are satisfied, are the characteristic roots of the equation of perturbations. This figure shows only a limited region of the complex plane. If the entire plane could be seen it would show an infinite number of characteristic roots. It can be seen that all the characteristic roots in Figure 4.2(a) have a negative real part. Hence the particular solution Eq. (4.5) is asymptotically stable. To confirm this result the equation of perturbations Eq. (4.8) was integrated numerically with the initial function $\Delta x(t) = 0.6m$ for $-T \leq t \leq 0$. The result of this integration is shown in Figure 4.2(b). It shows that the perturbations decrease the amplitude with time, which confirms that the particular solution is asymptotically stable. Since the control parameters influence the stability of the motion, it is important to find out whether optimal values of the control parameters exist and, if so, what they are. This is addressed in the next section.
Figure 4.2: Results for the linear object with control and zero damping. (a) + - zero points of Eq. (4.10), x - zero points of Eq. (4.11), • - solution of Eqs. (4.10) & (4.11), \( R_{dom} = -0.06 \text{s}^{-1} \); (b) Perturbations as a function of time.

4.3.3 Optimal Values of the Control Parameters

The above method of solution of the characteristic equation Eq. (4.10 & 4.11) was used to search for the control parameters giving the most stable periodic solution; that is, when the real part of the dominant characteristic root is smallest. The results of the search are shown in Figure 4.3 where the real part of the dominant characteristic root \( R_{dom} \) is plotted as a function of the control parameters \( a \) and \( b \). The continuous lines represent loci of points \((a,b)\) yielding the same value for the real part of the dominant characteristic root \( R_{dom} \). The search was carried out initially for only four different frequencies of excitation which correspond to periods \( T \) equal to 6 seconds, 5 seconds, 4 seconds and 3 seconds. The natural frequency of the system corresponded to a period of 4.44 seconds.
Figure 4.3: The highest real part of the characteristic roots as a function of $a$ and $b$.

On each plot in Figure 4.3 it can be seen that the contour separating the positive and negative values of the real part of the dominant characteristic root $R_{dom}$ passes through the point at which $a = b = 0.0$. This is consistent with the fact that when the values of the control parameters are zero, the system reduces to a simple
undamped system \((c = 0.0)\) which is stable (not asymptotically stable) and thus the real parts of the characteristic roots are equal to zero. It can be seen clearly from the four plots that the optimal values of the control parameters vary with the value of \(T\). For example: for a frequency of excitation lower than the natural frequency \((T = 6\) seconds, \(T = 5\) seconds) the optimal value of \(a\) is negative; and for a frequency of excitation higher than the natural frequency \((T = 4\) seconds, \(T = 3\) seconds) the optimal value of \(a\) is positive.

In order to gain a clearer understanding of how the optimal values of the control parameters varied as a function of the frequency of the periodic motion being stabilised, the optimal values were calculated for a wide range of excitation frequencies. The time delay in the control law in each case was set equal to the period of the periodic motion being stabilised, which was equal to the period of the excitation. The results are shown in Figure 4.4, where the optimal values of the control parameters \(a\) and \(b\) are plotted as a function of the period \(T\) of the particular solution Eq.(4.5) that represents the periodic steady-state motion of the physical system. Figure 4.4 shows that the optimal values of the control parameters \(a\) and \(b\) that were found vary a great deal as a function of the period \(T\). There is a very clear pattern in this variation of the optimal values of \(a\) and \(b\), with the significant points being at \(T = 4.44\) seconds, \(T = 8.88\) seconds and \(T = 13.33\) seconds. The first point at \(T = 4.44\) seconds corresponds to the natural frequency of the system. The other two significant points are at periods which are multiples of this period. The significance of these three points is also evident in Figure 4.5. In this figure the real and imaginary parts of the dominant characteristic root \((R_{dom} \text{ and } U_{dom})\) for the controlled system with the optimal values of the control parameters are plotted as a function of the period \(T\) of the
Chapter 4. Stability Improvement of the Periodic Motion of One-Degree-of-Freedom Systems

particular solution Eq.(4.5). The real part of the dominant characteristic root \( R_{dom} \) is a measure of the stability margin of the periodic particular solution Eq. (4.5). As mentioned above, the value of \( R_{dom} \) for the uncontrolled system with zero damping was zero, which means that the periodic particular solution Eq. (4.5) was only stable, as expected, and not asymptotically stable. For almost every value of the period \( T \) in Figure 4.5 the value of \( R_{dom} \) is negative, which means that the periodic particular solution for the controlled system with the optimal values of the control parameters is more stable than the periodic particular solution for the uncontrolled system. This shows that the Phase-Locked Delayed Feedback control method is effective in improving the stability margin. However, at the three significant values of the period \( T \) mentioned above - \( T = 4.44 \) seconds, \( T = 8.88 \) seconds and \( T = 13.33 \) seconds - the value of \( R_{dom} \) is zero. Therefore at these values of \( T \) the increase in the stability margin of the periodic particular solution relative to the uncontrolled system is negligible, and the control method is not effective. The control method was most effective mid-way between these values of \( T \). A similar pattern of results to this is obtained in the experimental work described in section 5.5 in Chapter 5.

The value of the imaginary part of the dominant characteristic root \( U_{dom} \) shown in Figure 4.5 varies slightly around an average value of about \( 1.4 \) s\(^{-1}\). The characteristic root discussed here relates to the equation of perturbations, not to the equation of motion. Therefore the imaginary part of this characteristic root indicates the frequency of the oscillation of the perturbations around the periodic particular solution of the equation of motion. The value of \( U_{dom} = 1.4 \) s\(^{-1}\) yields \( 2\pi/1.4 \approx 4.5 \) seconds, which is approximately the period of the natural frequency of the system.
Figure 4.4: Optimal values of the control parameters.
Figure 4.5: Real and imaginary parts of the dominant characteristic root.
4.3.4 Response of the Controlled System

The particular periodic solution Eq.(4.5) for a case when the damping coefficient is negative \( c_i = -0.04 \text{ Nsm}^{-1} \), amplitude of excitation \( y_0 = 1 \text{ m} \), period of excitation \( T = 6 \text{ seconds} \) and initial function \( X(t) = 0.0 \) for \(-T \leq t \leq 0\), is shown in Figure 4.6(a). For the uncontrolled system \((a = b = 0.0)\) this solution is globally unstable and the disturbed motion grows to infinity with time as shown in Figure 4.6(b). Therefore to find the unstable periodic particular solution shown in Figure 4.6(a), it was necessary to numerically integrate the equation of motion for the uncontrolled system backwards in time. The result of numerically integrating the equation of motion Eq.(4.1) for the controlled system with the optimal values of the control parameters found for \( T = 6 \text{ seconds} \) \((a = -0.105, b = -0.080 \text{ s})\) forward in time is shown in Figure 4.6(c). It can be seen that the perturbations, in this case, die out very quickly and the system settles into periodic steady state motion. Both solutions, Figure 4.6(b) and Figure 4.6(c), correspond to the same amplitude of excitation \( y_0 = 1 \text{ m} \), period of excitation \( T = 6 \text{ seconds} \) and initial function \( X(t) = 0.0 \) for \(-T \leq t \leq 0\).
Figure 4.6: Results for the linear object with negative damping. (a) The particular periodic solution; (b) without control the disturbed motion grows to infinity; (c) with control the perturbations die out quickly.
Chapter 4 Stability Improvement of the Periodic Motion of One-Degree-of-Freedom Systems 126

The periodic motion of the controlled system shown in Figure 4.6(c) is identical with the unstable periodic particular solution of the uncontrolled system shown in Figure 4.6(a). This shows that the control method has not created a new periodic particular solution that is stable, but has actually stabilised the unstable periodic particular solution of the uncontrolled system without changing it. This is an important feature of the Phase-Locked Delayed Feedback control method. In helicopter applications the pilot uses the periodic particular solution of the uncontrolled system to determine the flight path of the helicopter. So this must not be changed by the active control method.

4.4 Analysis of the Non-Linear System

4.4.1 Non-Linear Equation of Motion

A non-linear system was created by adding non-linear damping to the linear system which was considered previously.

If one adopts the mathematical model of the object shown in Figure 4.1(a) in the form

\[ m\ddot{x} + c_1\dot{x} + c_2x^3 + c_3\dot{x}^5 + (k + k_1)x = ky, \] (4.12)

the controlled system will be governed by Eq.(4.13).

\[ \ddot{x} + C_1\dot{x} + C_2x^3 + C_3\dot{x}^5 + Kx = Y_0 \sin \omega t + A(x(t) - x(t - T)) + B(\dot{x}(t) - \dot{x}(t - T)) \] (4.13)

where

\[ C_1 = \frac{c_1}{m}, \quad C_2 = \frac{c_2}{m}, \quad C_3 = \frac{c_3}{m}, \quad K = \frac{k + k_1}{m}, \quad Y_0 = \frac{ky_0}{m}, \quad A = \frac{ka}{m}, \quad B = \frac{kb}{m}, \] (4.14)
For $c_1 = 0.1 \text{ Ns}^{-1}$, $c_3 = -0.5 \text{ Ns}^3 \text{m}^{-3}$, $c_s = 0.4 \text{ Ns}^5 \text{m}^{-5}$ the non-linear damping characteristic is shown in Figure 4.7.

![Figure 4.7: The non-linear damping characteristic.](image)

4.4.2 Dynamic Behaviour of the Uncontrolled Object

With the adopted damping characteristic and $m = 1 \text{ kg}$, $k = 1 \text{ Nm}^{-1}$, $k_s = 1 \text{ Nm}^{-1}$, and free vibration with zero excitation (amplitude of excitation $y_0 = 0 \text{ m}$), the object has one stable equilibrium position at $x = \dot{x} = 0$, and around that one unstable limit cycle of period $T = 4.44 \text{ seconds}$, and around that one stable limit cycle of period $T = 4.44 \text{ seconds}$, as shown in Figure 4.8(a). If the initial conditions lie inside the unstable limit-cycle, then the free vibration of the uncontrolled system will decrease towards the equilibrium position. However, if the initial conditions lie outside the unstable limit-cycle then, even though there is no external excitation ($y_0 = 0 \text{ m}$), the uncontrolled system will move towards the stable limit-cycle and the system will continue to oscillate indefinitely. This limit-cycle behaviour (or self-excited oscillation) occurs because of the nature of the non-linear damping characteristic (see Figure 4.7). For a certain range of velocities the dashpot is putting energy into the system rather than taking energy out of it. For the stable limit-cycle oscillation the energy put into
the system is exactly equal to the energy taken out of the system in each cycle.

The forced response of the object when it is excited with amplitude of excitation \( y_0 = 0.1 \) m and period of excitation \( T = 6 \) seconds, is shown in Figure 4.8(b). The behaviour of the system depends on the initial conditions. There are two possible responses. The first is a small amplitude stable periodic solution \( x_o(t) = x_o(t + T) \) with period \( T = 6 \) seconds. This solution is labelled (1) on the state-plane in Figure 4.8(b).
Chapter 4 Stability Improvement of the Periodic Motion of One-Degree-of-Freedom Systems

The line labelled (2) defines its region of asymptotic stability. If the initial conditions are within this region, then the system will perform this small amplitude stable periodic motion. If the initial conditions are outside the region of asymptotic stability (2), the object performs the large amplitude chaotic motion labelled (3). The following analysis shows the effect of the Phase-Locked Delayed Feedback control method on the size of the region of asymptotic stability of the periodic solution $x_0$.

4.4.3 Equation of Perturbations

To analyse the influence of the control method on the stability margin of the periodic solution labelled (1) in Figure 4.8(b), the perturbed motion

$$\ddot{x} = x_0 + \Delta x,$$  \hspace{1cm} (4.15)

was introduced into Eq. (4.13) which yields the differential equation of perturbations for the non-linear system in the following form

$$\Delta \ddot{x} + C\Delta \dot{x} + K\Delta x = A(\Delta x(t) - \Delta x(t - T)) + B(\Delta \dot{x}(t) - \Delta \dot{x}(t - T)),$$  \hspace{1cm} (4.16)

where

$$C = C_1 + 3C_2x_0^2 + 5C_3x_0^4.$$  \hspace{1cm} (4.17)

Since the solution $x_0$ is a periodic function of time, the function $C$ is periodic too, as is shown in Figure 4.9. To permit the optimisation method developed in section 4.3 using Eq. (4.10 and 4.11) to be applied, the periodic function of time $C$ was replaced by its integral average $C_{av} = 0.09 \text{ Nsm}^{-1}\text{kg}$. The optimal values of the control parameters found with this method were then used in the numerical computations with the non-linear mathematical model described in section 4.4.5.
4.4.4 Optimal Values of the Control Parameters

The optimal values of the control parameters for the non-linear mathematical model with amplitude of excitation $y_0 = 0.1 \, \text{m}$ and period of excitation $T = 6 \, \text{seconds}$ were found as follows.

For each combination of the control parameters $a$ and $b$, the value of the real part of the dominant root $R_{dom}$ of the characteristic equation Eq.(4.10 & 4.11) is shown in Figure 4.10(a). In this plot the contour for $R_{dom} = 0.0$, which defines the points in the $a, b$ plane for which the controlled system is only stable (not asymptotically stable), does not pass through the point $a = b = 0.0$ as it did on the plots in Figure 4.3. This is because in this case the damping is not zero but is positive, so the periodic steady-state motion of the uncontrolled system ($a = b = 0.0$) is asymptotically stable and not just stable. The optimal values of the control parameters were identified as $a_{opt} = -0.1$, $b_{opt} = -0.07 \, \text{s}$. For the optimal values of the control parameters the characteristic roots of the equation of perturbations are shown in Figure 4.10(b).
Figure 4.10: The optimal control parameters. (a) The real part of the dominant characteristic root as a function of $a$ and $b$; (b) The characteristic roots of the equation of perturbations for $a_{opt} = -0.1$, $b_{opt} = -0.07$ s.
4.4.5 Results of Numerical Computation using the Non-Linear Mathematical Model

4.4.5.1 Influence of the Control on the Region of Asymptotic Stability

Figure 4.11 shows the massive increase in the size of the region of asymptotic stability that is achieved by applying the control method with the optimal values of the control parameters to the non-linear mathematical model with $y_o = 0.1$ m and $T = 6$ seconds. The area labelled (a) is the size of the region for the non-linear system without control ($a = b = 0.0$). This is the same region that was labelled (2) in Figure 4.8(b). The area labelled (b), which is very many times larger, is the size of the region when $a = -0.1$ and $b = -0.07$ s. This means that for the controlled non-linear system (with the optimal values of the control parameters, and with amplitude of excitation $y_o = 0.1$ m and period of excitation $T = 6$ seconds) if the initial conditions are anywhere within the region labelled (b), then the forced response will be the small amplitude periodic steady-state motion labelled (1) in Figure 4.8(b). Furthermore, it means that the size of the disturbance that would be required to push the system out of
the small amplitude periodic motion is greatly increased. This is demonstrated in the next section.

4.4.5.2 Influence of the Control on the Response of the Non-Linear System

![Graph showing the response to an impulse of force](image)

Figure 4.12: Response to an impulse of force. (a) The impulse of force; (b) the response for the uncontrolled system; (c) the response for the controlled system.

The beneficial effects of the increase in the stability margin of the periodic motion when the Phase-Locked Delayed Feedback control method is applied can be seen in Figure 4.12. Figure 4.12(a) shows an impulse of force which is applied to the non-linear system. Figure 4.12(b) shows that without control the effect of the impulse of force is to cause the system to move into motion with a large amplitude. However, if the control method is operating (with the optimal values of the control parameters), then when the impulse of force is applied, the system oscillates briefly with a large amplitude but very quickly returns to the periodic motion that existed before the impulse of force was applied, as shown in Figure 4.12(c). These plots were obtained
by numerical integration of Eq.(4.13) with initial function $X(t) = 0.0$ for $-T \leq t \leq 0$.

4.5 Conclusions

A computer program has been developed in order to find the dominant root of the characteristic equation, which has an infinite number of roots. Using this program, optimal values of the control parameters have been shown to exist.

The optimal values of the control parameters have been found to vary significantly as a function of the period of the motion being stabilised. This variation has a clear pattern, and has been shown to correspond to the period of the natural frequency of the system, and to multiples of that period. A similar pattern will be seen to occur in the experimental results presented in section 5.5 in Chapter 5.

The Phase-Locked Delayed Feedback control method has been applied to a linear system with negative damping, and has successfully stabilised the unstable periodic particular solution without changing it at all.

The control method has been shown to increase greatly the size of the region of asymptotic stability for the periodic motion of a non-linear system, and to make the system much less sensitive to external disturbances.
5.1 Introduction

This chapter describes the experimental application of the Phase-Locked Delayed Feedback control method to a multi-degree-of-freedom laboratory installation. The hardware configuration that has been used, and the software that has been written to achieve this, are explained. The existence of optimal values of the control parameters and their variation as a function of the period of the motion being stabilised were investigated experimentally.

5.2 Hardware Setup

Figure 5.1 is a schematic diagram of the experimental setup. In the middle of the figure is the mechanical part of the laboratory installation. It consisted of a steel bar, rigidly supported at each end, with three electromechanical exciters attached to it. The steel bar was 890 mm long, 74 mm wide and 3.0 mm thick. The first natural frequency of the system was about 24 Hz, and the second natural frequency was about 59 Hz. The locations of the exciters relative to the steel bar are shown in Figure 5.2.
Chapter 5. Application of the Control Method to a Laboratory Installation

Figure 5.1 Schematic diagram of the experimental setup.

Figure 5.2 Location of the three exciters.
The three exciters each performed a different function.

The middle exciter provided a steady sinusoidal excitation to the steel bar. It was driven by a power amplifier which was driven by a sine wave from the synthesiser.

The left-hand exciter provided a force which was proportional to the velocity of the steel bar. The displacement of the steel bar at the location of the left-hand exciter was sensed by a displacement transducer and then electronically differentiated by the Support Simulator Unit to give the velocity. This was used to apply a negative damping force in order to make the motion of the system unstable. The amplitude of the output of the Support Simulator Unit relative to the input could be adjusted by a potentiometer. The position of the potentiometer was shown on an indicator as a value between zero and 999. This value is denoted by $c$ in this chapter, and it provided a measure, on a linear scale, of how much negative damping was added to the system. The value was used to determine the percentage increase or decrease (denoted by $\Delta c$) in the amount of negative damping that could be added to the controlled system before the periodic motion became unstable (denoted by $c_{\text{max}}$), relative to the amount of negative damping that could be added to the uncontrolled system before the periodic motion became unstable (denoted by $c_{\text{ref}}$).

The right-hand exciter provided the control force. The Phase-Locked Delayed Feedback control method was implemented using a personal computer with an analogue input/output board. The time-delay in the feedback was synchronised with the period of the excitation force applied to the system by the middle exciter. This was achieved by using a phase-locked oscillator. The input to the phase-locked oscillator was the same sine wave from the synthesiser that was used to drive the power amplifier for the middle exciter. The phase-locked oscillator produced a square wave
with a frequency that was sixty-four times the frequency of the sine wave. Thus, if the frequency of the sine wave changed, the frequency of the square wave also changed. The square wave from the phase-locked oscillator was used to trigger the analogue/digital converter on the input/output board connected to the personal computer; so for every one period of the sinusoidal excitation of the steel bar, the analogue/digital converter was triggered sixty-four times, regardless of how the excitation frequency varied. Therefore, by using data stored in the computer memory sixty-four samples earlier to calculate the delayed feedback, the time-delay was synchronised with the period of the excitation force.

The input to the analogue/digital converter was the signal from the right-hand displacement transducer after the DC component had been removed by a DC cut-off circuit. The input/output board was configured to generate an interrupt each time the analogue/digital converter completed a conversion. The personal computer responded to the interrupt by executing an “interrupt service routine”. The interrupt service routine involved: numerically differentiating the displacement to obtain the velocity; reading from memory the displacement and velocity recorded sixty-four samples earlier; calculating the control signal; and sending the control signal to the digital/analogue converter for conversion to an analogue signal to drive the right-hand exciter via the power amplifier.

After each interrupt, the personal computer had to complete the interrupt service routine before the next interrupt came. The phase-locked oscillator constructed

---

1 Throughout the thesis the term "control input" is used. This chapter, however, is describing a specific experiment in which the control input was an electrical signal which drove an exciter via a power amplifier. Furthermore it was this electrical signal that was used as a measure of the control input and is shown in the plots in this chapter. Therefore throughout this chapter the control input is referred to as the "control signal".
for these experiments was accurate up to a maximum excitation frequency of 50 Hz. With that input frequency, the frequency of the output square wave from the phase-locked oscillator was 3200 Hz. With the analogue/digital converter being triggered and generating interrupts at a frequency of 3200 Hz, the personal computer was easily able to complete the interrupt service routine before the next interrupt came. In contrast with this, the Sikorsky Black Hawk helicopter rotor rotates at about 258 rpm, which is only about 4.3 Hz. Thus it would be very easy, in terms of computer processing power, to implement the Phase-Locked Delayed Feedback control method on a helicopter.

5.3 Software

The driver software supplied with the analogue input/output board to enable the personal computer to control it, was written in the C language. The software which was written to implement the Phase-Locked Delayed Feedback control method made use of the driver software, and was also written in the C language. The flow charts for the real-time control program and the interrupt service routine are shown in Figures 5.3 and 5.4. The analogue input/output board is referred to in these flow charts by its model number, which was PC-30DS. The “circular buffer” mentioned in the flow chart for the interrupt service routine refers to the method of storing the displacement and velocity data from the previous sixty-four iterations. A group of memory locations was set aside for this purpose, and counters were used to keep track of the locations to which new data should be written, and from which data stored sixty-four iterations earlier should be read. When the counters reached the end of the buffer, they were reset to the beginning again.
Flow Chart for Real Time Control Program

Set base address for PC-30DS (analogue input/output board)
Run diagnostic routine to check that PC-30DS is functioning properly
Initialise PC-30DS

Ask user to enter excitation frequency. (This is needed for the numerical differentiation. In a more refined version of this program it would be determined automatically. It is not needed to synchronise the control, because that is done by the phase-locked oscillator).

Ask user to enter values for the control parameters to be used in calculating the control signal.

**Disable the computer’s real time clock.**

Enable the PC-30DS interrupts, so that an interrupt will be generated at the completion of each A/D conversion.
The A/D conversions are triggered by the phase locked oscillator via the external trigger input to the PC-30DS board.

**Has an interrupt been generated?**

**Execute Interrupt Service Routine (ISR)**

Has a key been hit to quit real time control?

**Disable PC-30DS interrupts**
**Re-enable computer’s real time clock**

Does user want to enter new values of the control parameters?

**YES**

**NO**

Terminate program

Figure 5.3 Flow Chart for Real Time Control Program.
Chapter 5 Application of the Control Method to a Laboratory Installation

Interrupt Service Routine (ISR)

START

Increment read and write memory location counters for "circular buffer"

Has either counter gone beyond the end of the buffer?

NO

YES

Reset counter to beginning of buffer

Read displacement from A/D converter

Differentiate displacement to get velocity

Write current displacement and velocity to "circular buffer"

Read displacement and velocity from 64 steps earlier from "circular buffer"

Calculate control signal

Send control signal to D/A converter

Return to main program

Figure 5.4 Flow Chart for Interrupt Service Routine (ISR).
5.4 Automatic Detection of the Excitation Frequency

In the version of the real-time control program that was used in the experiments, the user was asked to enter the value of the excitation frequency. This was needed so that the time between samples could be calculated and used in the numerical differentiation of the displacement. It was not needed to synchronise the time-delay in the feedback with the period of the excitation force because that was done by the phase-locked oscillator.

A more refined version of the real-time control program was designed but not implemented. This would read the "uncommitted counter" on the analogue input/output board after each sample, and use these readings to determine the elapsed time for sixty-four samples, which would give the period of the excitation force.

5.5 Experimental Determination of the Optimal Values of the Control Parameters

In the experiments, the frequency of the sinusoidal excitation provided by the synthesiser was varied over a wide range. At each excitation frequency (denoted by \( f_{ex} \)), the values of the control parameters \( a \) and \( b \) were varied in a trial-and-error fashion in order to find the most effective values for stabilising the periodic motion of the system, that is, the optimal values. For each combination of values of \( a \) and \( b \), the amount of negative damping added to the controlled system was increased until it caused instability. This gave the value of \( c_{max} \), the maximum amount of negative damping which could be added to the controlled system before the periodic motion became unstable. The percentage increase in the maximum amount of negative damping which could be added to the controlled system relative to the uncontrolled
system before the periodic motion became unstable, was denoted by $\Delta c$ and was given by $\Delta c = 100 \times (c_{\text{max}} - c_{\text{ref}}) / c_{\text{ref}}$. This was a measure of the effectiveness of the control method. At each excitation frequency, the values of the control parameters $a$ and $b$ which gave the highest value of $\Delta c$ were the optimal values. For an example of a plot of $\Delta c$ as a function of $a$ and $b$ at an excitation frequency of 26 Hz, see Figure 5.6(a). When instability occurred the frequency of the perturbations around the periodic steady-state motion that grew exponentially was denoted by $f_u$. An example of a plot of $f_u$ as a function of $a$ and $b$ can be seen in Figure 5.6(b).

The optimal values of $a$ and $b$ plotted as a function of the period of the sinusoidal excitation force ($T_{\alpha}$) are shown in Figure 5.5. The value of $\Delta c$ corresponding to the optimal values of $a$ and $b$ is plotted as a function of the period of the sinusoidal excitation force in Figure 5.5(c).

These results were similar to the results for the one-degree-of-freedom system shown in Figures 4.4 and 4.5 in Chapter 4. However, in the experimental results the variation was more complex because the experimental system had more than one degree-of-freedom. At least the first two natural frequencies of the experimental system had a significant effect on the results.
Figure 5.5 Results as a function of $T_{ex}$: (a) optimal $a$; (b) optimal $b$; (c) $\Delta c$. 
The significant values of the period of the motion being stabilised in the results in Figures 4.4 and 4.5 in Chapter 4 were: the period corresponding to the natural frequency of the system; and integer multiples of that period. At these points the control method was not able to improve the stability of the periodic motion at all. In the experimental results shown in Figure 5.5, the periods corresponding to the first two natural frequencies of the system and integer multiples of these periods were significant.

That is,

1st Nat. Freq. = 24 Hz \hspace{1cm} \text{period} = 0.042 \text{ seconds}

\[ 2 \times \text{period} = 0.084 \text{ seconds} \]

2nd Nat. Freq. = 59 Hz \hspace{1cm} \text{period} = 0.017 \text{ seconds}

\[ 2 \times \text{period} = 0.034 \text{ seconds} \]
\[ 3 \times \text{period} = 0.051 \text{ seconds} \]
\[ 4 \times \text{period} = 0.068 \text{ seconds} \]
\[ 5 \times \text{period} = 0.085 \text{ seconds} \]

These significant points are marked on the plots in Figure 5.5.

The value of \( c_{\text{max}} \) for the controlled system with the optimal values of the control parameters \( a_{\text{opt}} \) and \( b_{\text{opt}} \) was greater than or equal to the value of \( c_{\text{ref}} \) for all the values of the excitation frequency. In other words, the amount of negative damping that could be added to the controlled system before the periodic motion became unstable was greater than or equal to the amount of negative damping that could be added to the uncontrolled system before the periodic motion became unstable for all the values of the excitation frequency.
Therefore, for almost all values of the excitation frequency the control method was able to improve the stability of the periodic motion of the system. The values of the excitation frequency for which the control method was not able to improve the stability of the periodic motion of the system at all ($c_{\text{max}}$ was equal to $c_{\text{ref}}$, i.e. $\Delta c = 0.0$) were the significant points mentioned above, where the period of the excitation force, (and therefore the time-delay in the feedback), was a multiple of the period of the second natural frequency. However, when the period of the excitation force was a multiple of the period of the first natural frequency the control method did improve the stability of the periodic motion of the system.

The value of $T_{ex} = 0.042$ seconds corresponds to the first natural frequency of 24 Hz; but the percentage increase in $c$ is not zero at this point, as it is at the significant points for the second natural frequency. This appears to be inconsistent with the general pattern of the results, but in fact it is not. This is explained as follows.

To stabilise the periodic steady-state motion it is necessary to stabilise the natural mode of vibration which loses its stability first. When negative damping was added to the uncontrolled system the second mode became unstable before the first mode for all values of the excitation frequency. This occurred because the negative damping was applied through the left-hand exciter (which was located about where the maximum displacement for the second mode would occur), and not the middle exciter (where there is a node for the second mode). Therefore at any excitation frequency, stabilising the second mode stabilised the periodic steady-state motion - regardless of whether the first mode was stabilised or not. As it was mentioned above, the only points where the second mode could not be stabilised at all were those for which the period of the excitation force was a multiple of the period of the second natural
frequency. When the excitation frequency was equal to the first natural frequency of 24 Hz (i.e. $T_{ex} = 0.042$ seconds) the control method did stabilise the second mode. It is expected that at this excitation frequency the control method did not stabilise the first mode and that the amount of negative damping required to destabilise the first mode was exactly the same in the controlled system as in the uncontrolled system; but this could not be demonstrated because the second mode became unstable before the first mode in the uncontrolled system. Therefore it was not possible to determine the amount of negative damping required to destabilise the first mode in the uncontrolled system.

5.6 Experimental Results for an Excitation Frequency of 26 Hz

For an excitation frequency of 26 Hz the plot of the percentage increase in the maximum amount of negative damping that could be added to the controlled system relative to the uncontrolled system as a function of the values of the control parameters, i.e. $\Delta c$ as a function of $a$ and $b$, is shown in Figure 5.6(a). The corresponding plot of the frequency of the perturbations that grew exponentially due to the instability as a function of the values of the control parameters, i.e. $f_u$ as a function of $a$ and $b$, is shown in Figure 5.6(b).
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.6(a) $\Delta c$ as a function of $a$ and $b$, for $f_c = 26$ Hz.

Figure 5.6(b) $f_u$ is a function of $a$ and $b$, for $f_c = 26$ Hz.
5.6.1 Equally Dominant Characteristic Roots

The optimum values of the control parameters \((a\ and\ b)\) are, by definition, where the highest values of \(\Delta c\) occur. This can be seen in Figure 5.6(a) to be a ridge with the values of \(\Delta c\) dropping gently on either side of it. By comparing Figure 5.6(a) and Figure 5.6(b) it can be seen that the values of \(a\ and\ b\) for which the highest values of \(\Delta c\) occur in Figure 5.6(a) are the same values of \(a\ and\ b\) for which in Figure 5.6(b) there is a boundary between two regions where \(f_c\) is 25 Hz and 54 Hz respectively².

The results in Figures 5.6(a) and 5.6(b) indicate that if the equations of motion of the experimental system were created, the position of the characteristic roots in the complex plane as a function of the values of the control parameters \(a\ and\ b\) would behave as follows. The dominant characteristic root, which corresponds to the least stable mode of oscillation, is moving to the left, and thus becoming more stable, as the values of \(a\ and\ b\) approach the optimal values from any particular direction. But, as the dominant characteristic root is moving to the left in the complex plane, another characteristic root which is slightly more stable, i.e. further to the left, is moving to the right and thus is becoming less stable. The optimal values of the control parameters occur for the values of \(a\ and\ b\) where the two characteristic roots have the same value of the real part and are thus equally dominant. The imaginary parts of these two characteristic roots are not the same. So the characteristic roots correspond to different frequencies of oscillation. Adjusting the control parameters away from this optimal point will cause one of the characteristic roots to move to the right in the complex plane and the other to move to the left (except in the special case to be

² This phenomenon is also observable in the results discussed in Chapter 6 (see sections 6.5.4 and 6.5.5), and also in case 7 in section 5.22 of this chapter.
described below where both roots may move to the right together). Which of the two characteristic roots moves to the right and which moves to the left depends on how the values of the control parameters are changed. The frequency of the perturbations which grow exponentially will correspond to the imaginary part of the characteristic root which moves to the right and becomes dominant. This explains why the optimal values of the control parameters occur at a point in the \( a, b \) plane where there is a boundary between two distinct values of \( f_u \), which in Figure 5.6(b) are 25 Hz and 54 Hz.

As mentioned above, there is a possibility that if the values of the control parameters are adjusted in a certain way from the optimal values, then neither characteristic root will move to the left in the complex plane, but both will move to the right and the values of the real parts of the two characteristic roots will continue to be equal. Thus they will continue to be equally dominant but not as stable as they were for the optimal values of the control parameters. This corresponds to moving in the \( a, b \) plane in Figure 5.6(b) along the boundary between the two distinct frequencies of oscillation at 25 Hz and 54 Hz.

5.6.2 Extreme Variation in Stability

From the contour plot of \( \Delta c \) as a function of \( a, b \) in Figure 5.6(a) it can be seen that the region in the \( a, b \) plane where high values of \( \Delta c \) occur is bounded on three sides by regions where the value of \( \Delta c \) drops extremely rapidly. This occurs for high and low values of \( b \) and for high values of \( a \). On the fourth side, as the value of \( a \) is
decreased the decrease in the value of \( \Delta c \) is less extreme but is still quite significant.

The most rapid drop in the value of \( \Delta c \) occurs for high values of \( a \). For example, for a value of \( b = -0.68 \) s changing the value of \( a \) by a very small amount from 0.7 to 0.75 results in a drop in the value of \( \Delta c \) from +43\% to -100\%. This means that instead of being considerably more stable than the periodic motion of the uncontrolled system, the periodic motion of the controlled system becomes very much less stable than that of the uncontrolled system: in fact, so much so that even with no negative damping at all added to the controlled system, it is unstable. In practice it would be very important to be aware of this boundary and to keep away from it.

A similar extreme drop in the value of \( \Delta c \) occurs at low values of \( b \). It is not quite so extreme as the one at high values of \( a \), but for some values of \( a \) and \( b \) it can be almost as dangerous. For example at \( a = 0.45 \) changing \( b \) from -0.65 s to -0.7 s results in a change in \( \Delta c \) from +40\% to -50\%. So again it would be very important in practice to keep away from this region.

In this example it is clear that if the values of \( a \) and \( b \) are not the optimal values, it is much safer for them to be smaller in magnitude (that is, to be closer to zero) than the optimal values than it is for them to be larger in magnitude. Values of \( a \) and \( b \) which are smaller in magnitude than the optimal values result in slightly reduced performance, whereas values which are greater in magnitude may result in enormously reduced performance.

The two huge drops in the value of \( \Delta c \) in the contour plot of \( \Delta c \) as a function of \( a,b \) which have been described above also correspond to large changes in the value of \( f_u \). In one case for very negative values of \( b \), the value of \( f_u \) changes from 25 Hz to 70 Hz; and in the other case, for very positive values of \( a \), the value of \( f_u \) changes...
from 55 Hz to 10 Hz. This can be seen by comparing the contour plots for both $\Delta c$ as a function of $a,b$, and $f_a$ as a function of $a,b$ in Figures 5.6(a) and 5.6(b). So, as was the case for the frequency boundary where the optimal values of the control parameters occurred, at these frequency boundaries also, the results indicate that there are two equally dominant characteristic roots in the complex plane for the values of $a,b$ on the boundary of the distinct regions of $f_a$. However, in these two cases a small change in the values of the control parameters causes a very large change in the stability of the periodic motion of the system. So it must also cause a very large change in the value of the real part of one of the characteristic roots.

5.7 Experimental Results for an Excitation Frequency of 16 Hz

Fifteen specific “cases” are discussed in this chapter. The frequency of the sinusoidal excitation for these fifteen cases was 16 Hz. Each case corresponds to particular values of the control parameters $a$ and $b$. Therefore the time series and frequency spectrum plots for the fifteen cases discussed in this chapter (in Figs 5.9 to 5.48 inclusive) should be examined in conjunction with the contour plots in Figures 5.7 and 5.8 showing the maximum amount of negative damping that could be added to the controlled system before it became unstable ($c_{max}$), and the frequency of the perturbations which grew exponentially due to the instability ($f_a$), as a function of the control parameters $(a,b)$. For each of the fifteen cases discussed, specific values of $\epsilon$, the amount of negative damping added to the system, were used. The values of $a, b$ and $c$ used in each case were chosen to illustrate particular aspects of the behaviour of the controlled system. The values used were as follows:
The value of $c_{ref} = 205$ was the same for all fifteen cases because it was the maximum amount of negative damping that could be added to the uncontrolled system ($a = b = 0.0$) before it became unstable. The value of $c_{\text{max}}$ was not the same for all of the fifteen cases because it was the maximum amount of negative damping that could be added to the controlled system before it became unstable, and was therefore a function of the values of $a$ and $b$ as shown in Figure 5.7(a).

The time series plots each show two traces e.g. Figure 5.11. The upper trace is the displacement of the steel bar in mm as measured by the right-hand displacement transducer. This signal was the input to the real time control program in the personal computer via the analogue/digital converter.
The lower of the two traces shows the output of the real time control program which ran on the personal computer. This is the control signal which was sent to the right-hand exciter via a power amplifier.

In the discussion of the experimental results the motion of the system is described as having become unstable either:

- when the amplitude of the oscillations of the steel bar grew to the point where the violence of the vibration could have damaged the rig and the experiment had to be stopped; or

- when the amplitude of the control signal grew until it was "saturated", that is, until it reached the limits of the digital/analogue converter which was ±10 volts.

This second criteria of instability is explained as follows. The control signal was calculated using the difference between the current displacement and velocity and those from one period earlier of the excitation force. If the amplitude of the control signal grew this indicated that the oscillation of the steel bar was no longer simply periodic at the frequency of the excitation force, but an additional component of oscillation at a different frequency was growing. If the additional component grew in amplitude until the control signal was saturated and then stopped growing, it was assumed that this additional component of oscillation of the steel bar would have continued to grow indefinitely if the control signal had been able to grow indefinitely (without being bounded by the output capabilities of the digital/analogue converter).
Figure 5.7(a). $c_{max}$ as a function of $a$ and $b$ for $f_{ex} = 16$ Hz. Cases 1 to 4 are labelled. Approximate optimal values of $a$ and $b$ are indicated. See Figure 5.8.

Figure 5.7(b). $f_c$ as a function of $a$ and $b$ for $f_{ex} = 16$ Hz. Cases 1 to 4 are labelled. Approximate optimal values of $a$ and $b$ are indicated. See Figure 5.8.
Figure 5.8(a) $c_{max}$ as a function of $a$ and $b$ for $f_{ext} = 16$ Hz (detail from Fig.5.7(a)). Points labelled 1, 2, 3 and 4 are the optimal values of $a$ and $b$.

Figure 5.8(b) $f_u$ as a function of $a$ and $b$ for $f_{ext} = 16$ Hz (detail from Fig.5.7(b)). Points labelled 1, 2, 3 and 4 are the optimal values of $a$ and $b$. 

Chapter 5. Application of the Control Method to a Laboratory Installation
5.8 Case 1: \(a = -0.4, \ b = -0.25\) s, \(c = 75\).

The contours on Figure 5.7(a) show that for \(a = -0.4, \ b = -0.25\) s the amount of negative damping that could be added to the system with control before it became unstable (denoted by \(c_{\text{max}}\)) was between 100 per cent and 50 per cent less than the amount of negative damping that could be added to the system without control \((a = b = 0.0)\) before it became unstable (denoted by \(c_{\text{ref}}\)). Without control the system was stable up to \(c_{\text{ref}} = 205\), but with control it was stable only up to about \(c_{\text{max}} = 25\). Therefore, for these values of the control parameters, the system was actually less stable with the control than without it.

The time series plot in Figure 5.11 shows that before about \(t = 0.5\) seconds (where the control had been applied for some time before \(t = 0.0\) seconds but no negative damping had been applied) the amplitude of the displacement was periodic with a frequency of 16 Hz and the control signal was almost zero. At about \(t = 0.5\) seconds the negative damping with \(c = 75\) was added to the system. From this time on the control signal grew until it became saturated at about \(t = 3.5\) seconds. The displacement did not grow to become violent but had a significant component at about 62 Hz added to it. This can be seen from the frequency spectrum plot in Figure 5.9 which corresponds to \(t = 3.5\) seconds in Figure 5.11. The frequency spectrum plot in Figure 5.9 can also be compared with Figure 5.10, which shows the frequency spectrum plot for the system without control and with no negative damping \((a = b = c = 0.0)\), and which does not have the peak at 62 Hz.
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.9 Frequency Spectrum for \( a = -0.4, b = -0.25 \) s and \( c = 75 \) and corresponding to \( t = 3.5 \) seconds in Figure 5.11.

Figure 5.10 Frequency Spectrum for the system without control and with no negative damping, \((a = b = c = 0.0)\).
Figure 5.11 Results for case 1.
5.9 Case 2A: \( a = -0.1, \ b = -0.2 \) s, \( c = 75 \).

The value of \( C_{ref} = 205 \) was the same for this case as for case 1 because it was the maximum amount of negative damping that could be added to the system without control \( (a = b = 0.0) \) before it became unstable. Therefore it was only a function of the excitation frequency which, as mentioned above, was 16 Hz for all the fifteen cases described in this chapter. For \( a = -0.1 \) and \( b = -0.2 \) s the value of \( C_{max} \), the maximum amount of negative damping that could be added to the system with control before it became unstable, was found to be \( C_{max} = 95 \). This agrees with Figure 5.7(a) where it can be seen that the point \( a = -0.1, \ b = -0.2 \) s lies just outside the \( C_{max} = 100 \) contour.

The time series plot in Figure 5.13 began at \( t = 0 \) seconds with no control and no negative damping. At about \( t = 0.2 \) seconds the control was added to the system. A large transient oscillation in the control signal can be seen at this point, but it quickly died away and the control signal settled down to a very small amplitude. The reason for this transient oscillation in the control signal is that all the values in the memory locations of the "circular buffer" which stores the past displacement and velocity are zero when the real time control program is started. At about \( t = 1.0 \) seconds the negative damping with \( c = 75 \) was added to the system. A small transient in the control signal occurred at this point but it quickly settled back to a small amplitude. The control signal did not grow to saturation in this case as it did in case 1 for \( a = -0.4, \ b = -0.25 \) s. Hence the system is more stable with the values of the control parameters used in this case.

The frequency spectrum plot (Figure 5.12) corresponding to \( t = 2.0 \) seconds in Figure 5.13 shows only a very small peak at about 63 Hz. Thus the control had suppressed the component of the oscillation at this frequency that was present in case 1.
(see Figure 5.9).

Figure 5.12 Frequency spectrum for $a = 0.1$, $b = 0.2$, $c = 75$ corresponding to $t = 2.0$ seconds in Figure 5.13.
Figure 5.13 Results for case 2A.
5.10 Case 2B: $a = -0.1, b = -0.2 \text{ s}, c = 150$.

The values of $c_{\text{ref}} = 205$ and $c_{\text{max}} = 95$ were the same for this case as for case 2A, as the same excitation frequency and the same values of the control parameters were used. The plots for this case, which are discussed below, show that the control with these values of the control parameters, when combined with the negative damping that was added to the system, made the system unstable. This is as expected because the amount of negative damping that was added to the system $c = 150$ was greater than the value of $c_{\text{max}} = 95$.

Of the four frequency spectrum plots shown in Figures 5.10, 5.14, 5.15 and 5.16, only one shows a large component of oscillation at a frequency other than 16 Hz. This is Figure 5.14 which corresponds to the system with both negative damping and control added.

The fact that Figure 5.15, which corresponds to the system with the negative damping added ($c = 150$) but without control, does not show a large peak around 60 Hz reflects the fact that the system was more stable without control than with control, with the values of the control parameters used in this case i.e. $c_{\text{ref}}$ was greater than $c_{\text{max}}$. Although the negative damping did not cause the system without control to be unstable, it did cause the amplitude of the 16 Hz component of the oscillation to grow slightly. This can be seen by comparing Figure 5.15 with Figure 5.10.

The fact that Figure 5.16, which corresponds to the system with control but without negative damping, does not show a large peak around 60 Hz is consistent with $c_{\text{max}}$ being greater than zero. In other words, the control made the system less stable but a certain amount of negative damping still had to be added to the system to make it unstable. This was not always the case. In particular the results shown in Figure 5.21.
for case 3 and the results for an excitation frequency of 26 Hz (discussed in section 5.6) show that for certain values of the control parameters the control made the system unstable even when no negative damping was added, i.e. $c_{\text{max}} = 0.0$. This is also consistent with the results of Olgac and Holm-Hansen (1994) who used delayed displacement feedback to create a system that was only stable, not asymptotically stable, to act as a resonator for vibration absorption. By adjusting the delay time and the feedback gain they could manipulate both the resonant frequency and the stability properties of their system.

There are three time series plots for this case. The first of these is Figure 5.17. At $t = 0.0$ seconds the control had been added to the system but no negative damping had been added. The negative damping was added at about $t = 0.3$ seconds. The amount of negative damping used in this case, $c = 150$, was greater than $c_{\text{max}} = 95$. Thus, as expected, when the negative damping was added to the system with control, the control signal grew to saturation. The displacement was no longer simply periodic at 16 Hz but had a significant component at about 60 Hz added to it. This component can be seen in the frequency spectrum plot in Figure 5.14. As for Figure 5.11 in case 1 the amplitude of the displacement did not grow to violent proportions.

The second time series plot for this case is shown in Figure 5.18. The negative damping was added to the system before $t = 0.0$ seconds. The control was added at about $t = 0.9$ seconds and removed at about $t = 3.3$ seconds. This plot shows clearly that while the control was acting it was saturated and the displacement had the 60 Hz component in addition to the 16 Hz component. However, before the control was

---

3 See Chapter 2, section 2.A.3.
for case 3 and the results for an excitation frequency of 26 Hz (discussed in section 5.6) show that for certain values of the control parameters the control made the system unstable even when no negative damping was added, i.e. $c_{\text{max}} = 0.0$. This is also consistent with the results of Olgac and Holm-Hansen (1994) who used delayed displacement feedback to create a system that was only stable, not asymptotically stable, to act as a resonator for vibration absorption. By adjusting the delay time and the feedback gain they could manipulate both the resonant frequency and the stability properties of their system.

There are three time series plots for this case. The first of these is Figure 5.17. At $t = 0.0$ seconds the control had been added to the system but no negative damping had been added. The negative damping was added at about $t = 0.3$ seconds. The amount of negative damping used in this case, $c = 150$, was greater than $c_{\text{max}} = 95$. Thus, as expected, when the negative damping was added to the system with control, the control signal grew to saturation. The displacement was no longer simply periodic at 16 Hz but had a significant component at about 60 Hz added to it. This component can be seen in the frequency spectrum plot in Figure 5.14. As for Figure 5.11 in case 1 the amplitude of the displacement did not grow to violent proportions.

The second time series plot for this case is shown in Figure 5.18. The negative damping was added to the system before $t = 0.0$ seconds. The control was added at about $t = 0.9$ seconds and removed at about $t = 3.3$ seconds. This plot shows clearly that while the control was acting it was saturated and the displacement had the 60 Hz component in addition to the 16 Hz component. However, before the control was

---

5See Chapter 2, section 2.A.3.
added to the system the displacement was almost purely 16 Hz despite the negative damping, which can be seen from the frequency spectrum plot in Figure 5.15. After the control was removed the 60 Hz component of the displacement quickly disappeared, even though the negative damping was still present. This shows clearly that in this case the negative damping without the control did not cause the system to become unstable.

The third time series plot for this case is shown in Figure 5.19. No negative damping was added to the system but the control was added at about \( t = 0.35 \) seconds and removed at about \( t = 3.5 \) seconds. After a large but brief initial transient the control signal became almost zero as expected. This shows clearly that in this case the control without the negative damping did not cause the system to become unstable. The absence of any significant component of the displacement at a frequency of 60 Hz for the system with control but without negative damping, is clearly shown in the frequency spectrum plot in Figure 5.16.
Figure 5.14 Frequency spectrum for $a = -0.1$, $b = -0.2$, $c = 150$.

Figure 5.15 Frequency spectrum for $a = b = 0.0$, $c = 150$. 
Figure 5.16 Frequency spectrum for $a = -0.1$, $b = -0.2$, $c = 0.0$. 
Figure 5.17 Results for case 2B with control added before negative damping.
Figure 5.18 Results for case 2B with negative damping added before control.
Figure 5.19 Results for case 2B with control but no negative damping.
5.11 Case 3: \( a = 0.6, b = 0.1 \text{ s}, c = 0.0 \).

In this case \( c_{\text{ref}} = 205 \) as before, but \( c_{\text{max}} = 0.0 \). This means that without control the system remained stable until an amount of negative damping corresponding to \( c = 205 \) had been added. However, with control the system was unstable with no negative damping added. Therefore the system was much less stable with control than without it. This can be seen from Figure 5.7(a). The point \( a = 0.6, b = 0.1 \text{ s} \) is right on the edge of the region of the \( a, b \) plane shown in this plot. From the results shown in Figure 5.7(a) it is clear that only for a small subset of all possible values of \( a, b \) does the addition of control make the system more stable rather than less stable (i.e. \( c_{\text{max}} > c_{\text{ref}} \)).

The time series plot for this case is shown in Figure 5.21. The results shown in this plot were obtained from the system with no negative damping added. The control was added at about \( t = 0.7 \) seconds and removed at about \( t = 3.0 \) seconds. The control became saturated almost as soon as it was added. However the frequency of oscillation of the control signal is clearly much lower in this case than it was in case 1 and case 2B. The difference between the frequency of the oscillation of the steel bar shown in Figure 5.21 while the control was saturated and that shown for case 1 and case 2B in Figures 5.9 and 5.17 and 5.18 is also very marked. The low frequency component is shown in the frequency spectrum plot in Figure 5.20 which corresponds to the displacement while the control is saturated. This figure shows a significant component of the oscillation at about 21 Hz. It is this component which caused the control to become saturated.

The contrast between this case and the other cases discussed so far is also
shown in Figure 5.7(b). This figure shows, for an excitation frequency of 16 Hz, the frequency of the component of the oscillation which either causes the control to become saturated or itself grows to violent proportions, for a range of values of the control parameters $a, b$. The values $a = 0.6, b = 0.1 \text{s}$ lie in the 20-30 Hz region, whereas the values $a = -0.4, b = -0.25 \text{s}$ and the values $a = -0.1, b = -0.2 \text{s}$ lie in the 60-70 Hz region.

![Graph showing frequency spectrum](image)

**Figure 5.20** Frequency spectrum for $a = 0.6, b = 0.1 \text{s}, c = 0.0$. 
Figure 5.21 Results for case 2.
5.12 Case 4A: $a = 0.16$, $b = -0.06$ s, $c = 215$.

In contrast to the previous cases, the control in this case made the system more stable. In this case $c_{\text{max}} = 316$ and $c_{\text{ref}} = 205$. This can be seen on Figure 5.7(a) where the point $a = 0.16$, $b = -0.06$ s lies between the $c_{\text{max}} = 300$ and $c_{\text{max}} = 400$ contours.

The frequency of the most significant component of the oscillation apart from the 16 Hz component can be read from Figure 5.7(b). The point $a = 0.16$ $b = -0.06$ s lies in the 50-60 Hz region.

The time series plot in Figure 5.26 shows what happened to the system when the negative damping $c = 215$ was added to the system without control. Since the amount of negative damping added, $c = 215$, was greater than $c_{\text{ref}} = 205$, the system without control became unstable. The negative damping was added at about $t = 1.2$ seconds and had to be removed at about $t = 3.7$ seconds because the oscillation became so violent. Figures 5.22, 5.23 and 5.24 show the frequency spectrum plots corresponding to the rapidly growing oscillation in the time series plot in Figure 5.26. A component of the oscillation at a frequency of about 59 Hz can be seen to grow very large.

The effect of adding the control to the system without any negative damping is shown in Figure 5.27. The control was added at about $t = 1.4$ seconds and removed at about $t = 3.4$ seconds. When the control was switched on there was a large transient oscillation of the control signal but this died out quickly and the amplitude of the control signal settled down to approximately zero. This was as expected because the control increased the stability of the system and no negative damping was added. There was no significant frequency component of the oscillation other than the one at 16 Hz and the time delay was synchronised with this frequency. Therefore the control
signal was expected to settle down to zero. The displacement was only slightly affected by the transient that occurred in the control signal when the control was switched on. Switching off the control had no effect on the displacement since the control signal was almost zero.

The effect of adding the negative damping to the controlled system is shown in Figure 5.28. The control was switched on well before $t = 0.0$ seconds. The negative damping with $c = 215$ was added at about $t = 1.0$ seconds and removed at about $t = 2.65$ seconds. The control signal was almost zero before the negative damping was added because the displacement was almost purely a 16 Hz component. A small transient occurred in the control signal when the negative damping was applied. It died out in about half a second and can be seen to include oscillations at a frequency significantly higher than 16 Hz. These oscillations correspond to the component of the oscillation at about 59 Hz shown in Figures 5.22, 5.23 and 5.24 which in this case was being suppressed by the control. The control did not become saturated in this case as it did in the earlier cases. While the negative damping was applied, after the initial transient had died out, between $t = 1.5$ seconds and $t = 2.65$ seconds in Figure 5.28 the amplitude of the control signal was very small, but not quite as small as it was before the negative damping was applied; and the very small amplitude higher frequency oscillations in the control signal are just discernible on the plot.

The frequency spectrum plot corresponding to the region of Figure 5.28 between about $t = 1.5$ seconds and $t = 2.65$ seconds is shown in Figure 5.25. It is interesting to observe that while for the uncontrolled system with negative damping added the unstable oscillation component was at about 59 Hz (see Figures 5.22, 5.23 and 5.24), there is no component at this frequency in the plot for the controlled system.
However, there is a very small peak at about 62 Hz. So adding the control caused the frequency of this component of the oscillation to change slightly. The other small peaks in Figure 5.25 at 32 Hz, 48 Hz and 80 Hz, and the very tiny peaks at 64 Hz and 96 Hz, are all at integer multiples of the excitation frequency of 16 Hz and therefore did not affect the control signal at all.

In the plot in Figure 5.28 the control was added to the system well before the negative damping. When the negative damping was added to the controlled system, the control was able to suppress the component of the oscillation with frequency of about 59 Hz which in the uncontrolled system grew very large. The effectiveness of the control when it was added to the system after the negative damping had been added and the 59 Hz component of the oscillation had already begun to grow, was also examined and is discussed next.

The time series plot in Figure 5.29 shows what happened when the negative damping was added to the uncontrolled system at about \( t = 1.4 \) seconds and then the control was switched on about 1.1 seconds later at \( t = 2.5 \) seconds. The 59 Hz component of the oscillation can be seen in the displacement trace when the control was switched on. The control signal was very large at first and oscillated at about 59 Hz, but its amplitude gradually decreased until it was very small. Similarly the 59 Hz component disappeared from the displacement. Thus the control was able to suppress the 59 Hz component of the displacement and stabilise the system.

A different outcome is shown in Figure 5.30. This time the control was not added until about 1.55 seconds after the negative damping had been added, and it was not able to stabilise the system because of the limitations of the digital/analogue converter, i.e. it became saturated. The negative damping was added to the
uncontrolled system at about $t = 0.8$ seconds and the control was added at about $t = 2.35$ seconds. The control became saturated and the 59 Hz component of the displacement grew to violent proportions so that the negative damping had to be removed from the system at about $t = 3.15$ seconds to prevent damage to the equipment.

As mentioned above, the system was stable without negative damping, and with control with the values of the control parameters $a, b$ used in this case, i.e. $c_{\text{max}} > 0$ for $a = 0.16, b = -0.06$ s. So when the negative damping was removed from the controlled system at $t = 3.15$ seconds in Figure 5.30 the 59 Hz component quickly disappeared from the displacement. As the displacement returned to almost pure 16 Hz oscillation the amplitude of the control signal decreased from being saturated to almost zero.

These results show that if the 59 Hz component of the oscillation was allowed to grow too much before the control was applied, the control became saturated as soon as it was switched on and was not able to suppress this component of the oscillation.
Figure 5.22 Frequency spectrum for \( a = 0.0, b = 0.0 \text{ s}, c = 215 \) corresponding to approximately \( t = 3.1 \text{ seconds} \) in Figure 5.26.

Figure 5.23 Frequency spectrum for \( a = 0.0, b = 0.0 \text{ s}, c = 215 \) corresponding to approximately \( t = 3.2 \text{ seconds} \) in Figure 5.26.
Figure 5.24 Frequency spectrum for $a = 0.0$, $b = 0.0$ s, $c = 215$ corresponding to approximately $t = 3.3$ seconds in Figure 5.26.

Figure 5.25 Frequency spectrum for $a = 0.16$, $b = -0.06$ s, $c = 215$ corresponding to approximately $t = 2.0$ seconds in Figure 5.28.
Figure 5.26 Results for case 4A with negative damping but no control.
Figure 5.27 Results for case 4A with control but no negative damping.
Figure 5.28 Results for case 4A with control added before negative damping.
Figure 5.29 Results for case 4A with control added 1.1 seconds after negative damping.
Figure 5.30 Results for case 4A with control added 1.55 seconds after negative damping.
5.13 Case 4B: $a = 0.16$, $b = -0.06$ s, $c = 350$

In this case the amount of negative damping added to the system $c = 350$ was greater than the amount which could be added to the controlled system with these values of the control parameters before it became unstable, $c_{\text{max}} = 316$. The result is shown in Figure 5.31. The control was added before $t = 0.0$ seconds. The negative damping was added to the controlled system at about $t = 0.7$ seconds. The component of the displacement at about 59 Hz and the control signal both grew slowly until, at about $t = 3.1$ seconds, the control signal reached saturation and the displacement grew rapidly to violent proportions. The negative damping was removed from the system at about $t = 3.5$ seconds.
Figure 5.31 Results for case 4B.
5.14 Case 4C: $a = 0.16$, $b = -0.06$, $c = 300$

In this case the amount of negative damping that was added to the system, $c = 300$, was slightly less than the maximum amount which could be added to the controlled system before it became unstable, $c_{\text{max}} = 316$. The result is shown in Figure 5.33. Both the control and the negative damping were applied to the system before $t = 0$ seconds, and the control was applied before the negative damping. The control signal is significantly greater than zero in this plot which indicates that a frequency component other than 16 Hz was present in the displacement (and velocity). This component can be seen as a small peak at about 58 Hz on the frequency spectrum plot in Figure 5.32. The small peak at 32 Hz on this plot would not have affected the control signal as it is at an integer multiple of the excitation frequency. The amplitude of the control signal and the 58 Hz component of the displacement did not grow but remained constant in amplitude, indicating that the controlled system was not unstable.
Figure 5.32 Frequency spectrum for $a = 0.16$, $b = -0.06$ s, $c = 300$. 
Figure 5.33 Results for case 4C.
5.15 Cases 5, 6, 7 and 8

Cases 5, 6, 7 and 8 all have the same value of $a = 0.35$. They each have a different value of $b$, but these values all lie within a very small range between -0.11 s and -0.15 s. One of the interesting phenomena demonstrated by these cases is that in this particular region of the $a, b$ plane small changes in the values of the control parameters caused large changes in the properties and behaviour of the system. This was also the region in the $a, b$ plane where the optimal values of the control parameters occurred for the excitation frequency of 16 Hz: that is, the values of the control parameters that gave the greatest value of $C_{\text{max}}$, the amount of negative damping that could be added to the controlled system before it became unstable.

This can be seen in Figure 5.8(a). The greatest value of $C_{\text{max}}$ obtained for the excitation frequency of 16 Hz was exactly 600. The amount of negative damping that could be added to the uncontrolled system before it became unstable was $C_{\text{ref}} = 205$. The $C_{\text{max}} = 550-600$ region in Figure 5.8(a) can be seen to consist of a long, thin region running diagonally in the $a, b$ plane. The optimal value of $C_{\text{max}} = 600$ was found to occur at four points in this region and hence there are four peaks on this plot. These four points are labelled 1, 2, 3 and 4 in Figure 5.8(a). These points in the $a, b$ plane where the highest values of $C_{\text{max}}$ occurred, and which therefore define the optimal values of $a$ and $b$, are also labelled in Figure 5.8(b). This figure shows the frequency of the component of the oscillation that either grew to violent proportions or caused the control to become saturated for a range of values of the control parameters $a, b$.

For the $a$ value of $a = 0.35$ there are three quite distinct frequency regions. For $b = -0.15$ s the frequency was 60 Hz; for $b = -0.14$ s the frequency was 54 Hz; and for
If the experimental search for the optimal values of $a$, $b$ had been continued using a finer grid of values it is possible that a single optimal point in the $a$, $b$ plane may have been found, but this would have taken an inordinate amount of time and achieved very little benefit.

It is clear from Figure 5.8(a) that if either the $a$ value or the $b$ value was decreased from the optimal values, the value of $c_{\text{max}}$ fell very rapidly indeed. It would be important in practice to avoid this situation. If either of these values was increased from the optimal values, the value of $c_{\text{max}}$ fell more gently.

The results discussed in cases 5, 6, 7 and 8 illustrate the behaviour of the controlled system with the values of the control parameters in this region.

5.16 Case 5A: $a = 0.35$, $b = -0.14$ s, $c = 575$

For the values of the control parameters used in this case the amount of negative damping that could be added to the controlled system before it became unstable was found to be $c_{\text{max}} = 590$. This was almost three times the amount of negative damping that could be applied to the uncontrolled system before it became unstable. On Figure 5.8(a) the point $a = 0.35$, $b = -0.14$ s lies in the $c_{\text{max}} = 550$-600 region. The time series plot in Figure 5.35 shows that adding negative damping with $c = 575$, which is less than $c_{\text{max}}$, to the controlled system did not make the system unstable. The control was switched on before $t = 0.0$ seconds. The negative damping
was added at about $t = 0.35$ seconds. The displacement was significantly disturbed by the addition of the negative damping. This caused the large transient oscillation in the control signal which died away to a small amplitude over a period of about two seconds. The reduction in the amplitude of the control signal indicates that the control successfully suppressed the components of the oscillation at frequencies other than the excitation frequency of 16 Hz. This is verified by the frequency spectrum plot shown in Figure 5.34, which corresponds to the displacement several seconds after the negative damping was added, by which time the transient effects had died away and the oscillation was almost purely 16 Hz.

Figure 5.34 Frequency spectrum for $a = 0.35$, $b = -0.14$ s, $c = 575$ corresponding to approximately $t = 3.5$ seconds in Figure 5.35.
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.35 Results for case 5A.
5.17 Case 5B: $a = 0.35$, $b = -0.14$ s, $c = 575-610$

In this case the system became unstable when the amount of negative damping added to the controlled system was increased to a value greater than $c_{\text{max}}$. In the time series plot in Figure 5.38 the state of the system at $t = 0$ seconds was the same as the state of the system at $t = 4.0$ seconds in Figure 5.35 in case 5A: that is, the negative damping with $c = 575$ had been applied to the controlled system for long enough for the transient effects to die away. The amplitude of the control signal was very small and the displacement was almost purely 16 Hz oscillation. Then, at about $t = 0.3$ seconds, the negative damping was increased from $c = 575$ to $c = 610$. For about one second after this increase the high frequency component of the control signal grew slowly. Over the next 0.5 seconds the control signal grew more rapidly until it became saturated. At this point, about $t = 1.8$ seconds, the amplitude of the displacement had not increased significantly but a higher frequency component can be seen to be present. At about $t = 2.1$ seconds the amplitude of the displacement grew very rapidly, and at $t = 2.7$ seconds the negative damping had to be removed to avoid damage to the equipment. After the negative damping was removed, the displacement returned quickly to its former amplitude and the control signal decreased to almost zero.

The frequency spectrum plot corresponding to the displacement at about $t = 1.8$ seconds in Figure 5.38 when the control was saturated is shown in 5.36. This plot shows that the component of the oscillation that caused the saturation of the control was at a frequency of 54 Hz. This corresponds to the results shown for the values of the control parameters used in this case ($a = 0.35$, $b = -0.14$ s) in Figure 5.8(b). There was also a smaller but significant component at 22 Hz.
The frequency spectrum plot shown in Figure 5.37 corresponds to about \( t = 2.3 \) seconds in Figure 5.38 when the oscillation had become very violent. This plot shows that when the amplitude of the displacement grew very rapidly after the control was saturated, the dominant component was at a frequency of about 59 Hz, which was the second natural frequency of the system.

---

**Figure 5.36** Frequency spectrum for \( a = 0.35, b = -0.14 \) s, \( c = 610 \) corresponding to approximately \( t = 1.8 \) seconds in Figure 5.38.
Figure 5.37 Frequency spectrum for \( a = 0.35 \), \( b = -0.14 \), \( c = 6.10 \) corresponding to approximately \( t = 2.3 \) seconds in Figure 5.38.
Figure 5.38: Results for case 5B.
5.18 Case 6A: $a = 0.35, b = -0.15 \text{s}, c = 430$

The value of $c_{\text{max}}$ for this case was $c_{\text{max}} = 450$. This was very much less than the value for case 5. A very small change in the value of the control parameter $b$ from $-0.14 \text{s}$ to $-0.15 \text{s}$ caused a large decrease in the amount of negative damping that could be added to the controlled system before it became unstable. This can be seen in Figure 5.8(a).

It can also be seen in Figure 5.8(b) that the small change in the value of $b$ from $-0.14 \text{s}$ to $-0.15 \text{s}$ caused the frequency of the unstable component of the oscillation to change from 54 Hz to 60 Hz.

In the time series plot in Figure 5.40 the control was switched on before $t = 0.0$ seconds, and the negative damping with $c = 430$ was added at about $t = 0.1$ seconds. Before the negative damping was added the amplitude of the control signal was very close to zero. When the negative damping was added a large transient oscillation occurred in the control signal. Over the next 0.8 seconds the amplitude of the control signal decreased, but it then remained constant at about 8 volts peak-to-peak, which was about 40 percent of its saturated value. The displacement trace shown in Figure 5.40 is clearly not pure 16 Hz oscillation as the height of the peaks is not constant. The oscillation of the control signal is clearly at a much higher frequency than 16 Hz.

The frequency spectrum plot in Figure 5.39 corresponds to the displacement at about three seconds after the negative damping had been added to the controlled system in Figure 5.40 and so the amplitude of the control signal was constant. The peak at about 60 Hz was the component of the oscillation that was causing the amplitude of the control signal to be significantly greater than zero. Since the amount
of negative damping added to the controlled system in this case was less than \( c_{\text{max}} \) the system was stable. If the amount of negative damping was increased above \( c_{\text{max}} \) this 60 Hz component of the oscillation would cause the control to become saturated and the system to become unstable. This situation is shown in case 6B and in case 6C.

![Frequency spectrum](image)

Figure 5.39 Frequency spectrum for \( a = 0.35, b = -0.15 \) s, \( c = 430 \) corresponding to approximately \( t = 3.1 \) seconds in Figure 5.40.
Figure 5.40: Results for case 6A.
5.19 Case 6B: \( a = 0.35, b = -0.15 \text{ s, } c = 460 \)

The time series plot for this case in Figure 5.42 may be compared with the one in Figure 5.40 in case 6A. The conditions at \( t = 0 \) seconds were the same: the control had been switched on sometime earlier but the negative damping had not been added. The difference between the two plots was that in this case slightly more negative damping was added than in case 6A. The negative damping was added at about \( t = 0.25 \text{ seconds} \). The high frequency component of the control signal can be seen to grow slowly until it became saturated. At about \( t = 3.2 \text{ seconds} \) the amplitude of the high frequency component of the displacement grew rapidly to become very violent. At about \( t = 3.65 \text{ seconds} \) the negative damping was removed to prevent damage to the equipment.

The frequency spectrum plot shown in Figure 5.41 corresponds to about \( t = 2.8 \text{ seconds} \) in Figure 5.42 when the control had become saturated but the oscillation had not become violent. It shows that the component of the oscillation that caused the control to become saturated was at a frequency of about 60 Hz.
Figure 5.41 Frequency spectrum for $a = 0.35$, $b = -0.15$ s, $c = 460$ corresponding to approximately $t = 2.8$ seconds in Figure 5.42.
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.42: Results for case 6B.
Case 6C: $a = 0.35$, $b = -0.15$ s, $c = 430 - 460$

The difference between this case and case 6B was that in case 6B the negative damping with $c = 460$ was added to the system which previously had no negative damping, so the negative damping was effectively increased from $c = 0$ to $c = 460$ suddenly. In case 6C the negative damping was simply increased from $c = 430$ to $c = 460$. This procedure was similar to that used in case 5B to create Figure 5.38.

In this case, as shown in Figure 5.43, the state of the system at $t = 0.0$ seconds was the same as the state of the system at $t = 4.0$ seconds in Figure 5.40 in case 6A. The negative damping with $c = 430$ had been added to the controlled system sometime earlier and the transient effects had had time to die away. The amplitude of the control signal was constant at about 8 volts peak-to-peak. The displacement was not pure 16 Hz oscillation but contained a small 60 Hz component.

At about $t = 0.4$ seconds the negative damping was increased from $c = 430$ to $c = 460$. The control signal grew steadily until it became saturated. The 60 Hz component of the displacement grew slowly until at about $t = 2.9$ seconds it grew rapidly to a violent amplitude. The negative damping was removed at about $t = 3.2$ seconds.
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.43 Results for case 6C.
5.21 Comparison of Case 5 with Case 6.

The very small change in the value of the control parameters from $a = 0.35$, $b = -0.14\ \text{s}$ in case 5 to $a = 0.35$, $b = -0.15\ \text{s}$ in case 6 led to two things:

- A great decrease in the amount of negative damping that could be added to the controlled system before it became unstable from $C_{\max} = 590$ in case 5 to $C_{\max} = 450$ in case 6, e.g. in case 5A Figure 5.35 shows stable oscillation for $c = 575$, whereas in case 6B Figure 5.42 shows unstable oscillation for $c = 460$. Also Figure 5.8(a) indicates that a further very small change in the control parameters to $a = 0.35$, $b = -0.16\ \text{s}$ would result in a further large decrease in the value of $C_{\max}$.

- The frequency of the component of the oscillation (of displacement and velocity) that leads to saturation of the control changed from 54 Hz in Figure 5.36 for case 5B to 60 Hz in 5.41 for case 6B. This change can also be seen in Figure 5.8(b).

5.22 Case 7: $a = 0.35$, $b = -0.145\ \text{s}$, $c = 470$

This case shows the borderline situation in which the values of the control parameters are between those used for case 5 and those used for case 6 and the high frequency components from both of these cases are present. The frequency spectrum plot in Figure 5.44 shows two peaks of equal size at 53.5 Hz and 58.8 Hz, in addition to the large peak at 16 Hz.

The corresponding time series plot is shown in Figure 5.45. In this plot negative damping with $c = 470$ had been added to the controlled system. This was not
enough negative damping to make the system unstable. However, due to the two high
frequency components in the displacement the amplitude of the control signal was very
large and varied in a "beat" pattern.

![Graph showing frequency spectrum for a = 0.35, b = -0.145 s, c = 470.](image)

**Figure 5.44** Frequency spectrum for $a = 0.35, b = -0.145 \text{ s}, c = 470$. 
5.23 Case 8A: \( a = 0.35, b = -0.11 \text{ s}, c = 520 \)

In this case the value of the control parameter \( a = 0.35 \) was the same as for cases 5, 6 and 7 but the value of \( b = -0.11 \text{ s} \) was very slightly different. This small difference in the value of \( b \) had two significant effects. The most significant effect was that the frequency of the component of the oscillation that caused saturation of the control was changed from 54 Hz to 21 Hz. This can be seen on Figure 5.8(b). The second effect was that the amount of negative damping that could be added to the controlled system before the control became saturated was decreased relative to case 5, to \( c_{\text{max}} = 510 \). This decrease can be seen in Figure 5.8(a).

The time series plot in Figure 5.47 shows what happened when negative damping with \( c = 520 \) was added to the controlled system at about \( t = 1.0 \text{ seconds} \). Since the amount of negative damping added to the controlled system was greater than \( c_{\text{max}} \) the control became saturated. However, a very marked difference can be seen between the saturated contra 1 signal in this plot and that in earlier plots, such as Figure 5.42. The difference is that in this plot the control signal is oscillating at a much lower frequency. This indicates that, in addition to the 16 Hz component of the displacement oscillation, another component is present with a relatively low frequency. This can be seen in the frequency spectrum plot in Figure 5.46. There is a significant peak at about 21 Hz. If this plot is compared with Figure 5.36 it is clear that changing the value of \( b \) from -0.14 s to -0.11 s caused the frequency of the component that caused saturation of the control to change from 54 Hz to 21 Hz. In this case, even though the control became saturated the oscillation did not grow to a violent amplitude.
Figure 5.46 Frequency spectrum for $a = 0.35$, $b = -0.11$ s, $c = 520$. 
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.47 Results for case 8A.
5.24 Case 8B: $a = 0.35$, $b = -0.11 \text{ s}$, $c = 400$

In this case the amount of negative damping added to the system was less than in case 8A. The control was switched on before $t = 0.0 \text{ seconds}$ in Figure 5.48 and the negative damping was added at about $t = 0.65 \text{ seconds}$. A large transient occurred when the negative damping was added but because the amount of negative damping was less than $c_{\text{max}}$ the control did not become saturated. The transient oscillation in the control signal died out quite slowly in this case. If the amount of negative damping was increased closer to the value of $c_{\text{max}}$ the transient died out even more slowly. The frequency of the transient oscillation of the control signal in this case was 21 Hz as for case 8A. This behaviour of the control signal indicates that a 21 Hz component was added to the 16 Hz oscillation of the steel bar and then slowly died out.
Chapter 5 Application of the Control Method to a Laboratory Installation

Figure 5.46 Results for case 3B.

[Diagram showing waveforms]
5.25 Conclusions

This chapter describes the application of the Phase-Locked Delayed Feedback control method to an experimental installation. The necessary software was written and hardware was assembled, including the construction of a phase-locked oscillator circuit to synchronise the control with the periodic motion being stabilised.

The optimal values of the control parameters were found experimentally with no knowledge of the dynamics of the system or the equations of motion. The effectiveness of the Phase-Locked Delayed Feedback control method in stabilising the unstable motion of a system with negative damping was verified experimentally. The variation of the optimal values of the control parameters as a function of the period of motion being stabilised was shown to be similar to the results obtained in Chapter 4, but with added complexity due to the increased number of degrees-of-freedom.

The optimal values of the control parameters were shown to occur at a point in the $a,b$ plane where there was a boundary between distinct regions of $f_u$. This phenomenon is further illustrated by the results in Chapter 6.
6.1 Introduction

In this chapter the ability of the Phase-Locked Delayed Feedback control method to stabilise the motion of a helicopter rotor blade in hover and forward flight is investigated. A linear two-degree-of-freedom mathematical model of a helicopter rotor blade is used. There are various kinds of problems which can affect the motion of helicopter rotor blades. These can be divided into two classes: resonance problems and stability problems. The problems are shown in Figure 6.1.

It is important to distinguish between resonance and instability.

Resonance occurs when an oscillating external force has the same frequency as a natural frequency of the system. In this case the response of the system has a frequency equal to the frequency of the excitation.

Instability occurs when motion-induced forces (i.e. terms in the equations of motion that contain the state-variables) feed energy into the system. In this case the response of the system has a frequency equal to one of its natural frequencies, regardless of the excitation frequency.
Most helicopter stability problems are analysed using linear mathematical models. If the system is linear, the analysis of stability problems involves finding the solution of homogeneous equations of motion (free vibration solution). The particular solutions of non-homogeneous linear equations of motion are used to analyse resonance problems.

The form of instability considered in this chapter is called "pitch-flap flutter". This instability is similar to the classical torsion flutter that occurs in fixed-wing aircraft in that it involves the combined flapping and twisting motion of the rotor blade. However, the rotation of the rotor blade with the rotor shaft and the significantly

different aerodynamic environment cause several phenomena that make the pitch-flap flutter which occurs in helicopter rotor blades significantly different from the flutter that occurs in fixed-wing aircraft. The main difference is caused by the enormous centrifugal force which acts on the rotor blade. This force couples the flap and pitch motion when the centre of gravity of the rotor blade does not coincide with the aerodynamic centre of the rotor blade, which is the case in this analysis\(^2\). The aerodynamic centre and the elastic axis are assumed to coincide with the pitch axis of the rotor blade. This is consistent with the analyses of Bramwell (1976), Johnson (1980) and Chopra (1990)\(^3\). The pitch axis is also called the feathering axis, because “feathering” is another word for adjusting the angle of attack (i.e. pitch angle) of the rotor blade.

The pitch-flap flutter instability is one of the major concerns in the design of articulated helicopter rotors and is also significant for hingeless helicopter rotors. It greatly restricts how far behind the leading edge of the rotor blade the centre of gravity can be located. Most rotor blade designs use balance weights at the leading edge to shift the centre of gravity forward.

The occurrence of the pitch-flap flutter instability does not depend on the thrust level of the rotor. Therefore it can occur both in the rotor of an airborne helicopter and in the rotor of a helicopter on the ground producing zero thrust.

\(^2\) See section 6.4.4.1 and Figures 6.4 and 6.5.


6.2 Non-linear Mathematical Model

A non-linear mathematical model was developed for the one-degree-of-freedom physical model of a helicopter rotor blade shown in Figure 6.2. In this physical model the pitch angle $\alpha$ was a known input and not a degree-of-freedom. The only degree-of-freedom was the flap angle $\beta$. The feedback control input was introduced by adding an angular displacement $\Delta \alpha$ to the pitch angle input from the pilot $\alpha_0$.

![Figure 6.2. The physical model for the one-degree-of-freedom non-linear mathematical model of the rotor blade.](image)
The symbols in the non-linear mathematical model have the following reference:

\( \alpha \)  pitch angle  
\( \beta \)  flap angle  
\( l_1 \)  distance from rotor shaft to flap hinge  
\( l_2 \)  distance from flap hinge to centre of gravity of rotor blade  
\( m_3 \)  mass of rotor blade  
\( \Omega \)  angular velocity of rotor shaft  
\( I_{3x3} \)  moment of inertia of rotor blade about \( x_3 \) axis  
\( g \)  gravity  
\( a \)  adjustable parameter in the control law  
\( b \)  adjustable parameter in the control law  
\( \alpha_0 \)  pitch angle input from pilot  
\( \Delta \alpha \)  additional pitch angle input from control system.
Eq.(6.1) is the non-linear equation of motion for the controlled system.

\[
\ddot{\beta}(m_3 l_2^2 + I_{3x3}) = m_3 \left[2m_3 l_2^3 \Omega (\dot{\alpha}_0 + \Delta \alpha) \cos(\alpha_0 + \Delta \alpha) \sin \beta + l_2 \Omega^2 (\dot{\alpha}_0 + \Delta \alpha)^2 \cos \beta \sin \beta - l_2^2 \Omega^2 \cos^2(\alpha_0 + \Delta \alpha) \cos \beta \sin \beta - l_2 \Omega \sin \beta \right] + \\
+I_{3y3} \Omega (\dot{\alpha}_0 + \Delta \dot{\alpha}) \cos(\alpha_0 + \Delta \alpha) + \\
\{I_{3y5} - I_{3x3}\}[\Omega \cos(\alpha_0 + \Delta \alpha) \cos \beta - (\dot{\alpha}_0 + \Delta \dot{\alpha}) \sin \beta] \\
\times((\dot{\alpha}_0 + \Delta \dot{\alpha}) \cos \beta + \Omega \cos(\alpha_0 + \Delta \alpha) \sin \beta) \\
- m_3 g l_2 \cos(\alpha_0 + \Delta \alpha) \cos \beta \\
+ 191.35 \Omega^2 \left[ (\alpha_0 + \Delta \alpha) - 0.030022 \sqrt{1 + 66.778(\alpha_0 + \Delta \alpha)^2} - 1 \right] - 1.00175 \beta / \Omega 
\]

(6.1)

\[
\Delta \alpha = a \left[ \beta(t) - \beta(t - T) \right] + b \left[ \dot{\beta}(t) - \dot{\beta}(t - T) \right] 
\]

(6.2)

\[
\Delta \dot{\alpha} = a \left[ \ddot{\beta}(t) - \ddot{\beta}(t - T) \right] + b \left[ \dddot{\beta}(t) - \dddot{\beta}(t - T) \right] 
\]

(6.3)

It can be seen that if all the brackets in Eq.(6.1) were expanded \( \Delta \dot{\alpha}^2 \) would occur in two places. Eq(6.3) shows that due to the nature of the control law the expression for \( \Delta \dot{\alpha} \) contains \( \dot{\beta} \). Therefore the resulting equation of motion for the controlled system contains \( \dddot{\beta} \) terms which make it a differential equation of second degree, i.e. the highest derivative (second derivative) appears not only in linear form but also squared. This type of equation cannot be solved using standard numerical integration methods because they use a state-space formulation of the equations of motion, and it is not possible to put a differential equation of second degree into state-space form. Therefore, to avoid obtaining differential equations of second degree, a linear mathematical model was adopted as described below.
Chapter 6 The Control Method applied to a Helicopter Rotor Blade

6.3 Linear Mathematical Model

6.3.1 Flight Conditions

The case of hovering flight and the case of forward flight are both considered, in that order. In forward flight the steady-state flap and pitch motion of the rotor blade is periodic. The flap and pitch angles (denoted by $\beta$ and $\theta$) are periodic functions of time with a period corresponding to one revolution of the rotor shaft. However in hovering flight the steady-state pitch and flap angles are constant. The control method investigated in this thesis is designed to stabilise periodic motion. Nevertheless, the case of hovering flight is considered for two reasons:

1. The pitch-flap flutter instability is more severe at hover than in forward flight\(^4\); so if the control method is effective in stabilising the motion of the rotor blade in hover, it will be effective throughout the flight envelope.

2. It is desired to investigate the effectiveness of the control method in improving the stability of an equilibrium position. At hover the swashplate is not tilted and the steady-state solution of the equations of motion of the rotor blade is constant $\theta = \theta_0$, $\beta = \beta_0$. This can be regarded as periodic motion with an undefined period. If the swashplate is tilted the steady-state solution will have a period equal to the period of the rotation of the rotor shaft. So the time-delay in the control method is set equal to this period.

See the development of this theme in section 6.6 of this chapter.
6.3.2 Equations of Perturbations for the Controlled System

In this chapter the stability of the steady-state motion of the rotor blade is investigated by determining whether perturbations around the steady-state motion will grow or decay with time. To enable this analysis the differential equations of perturbations for the rotor blade with control are obtained. For the case of hovering flight the characteristic roots of the equations of perturbations are used to determine the stability margin of the steady-state solution of the equations of motion. For the case of forward flight the equations of perturbations are integrated numerically to determine whether the perturbations grow or decay with time.

6.3.2.1 Equations for Hovering Flight Without Control

The equations of motion for the rotor blade in hovering flight without control are differential equations with constant coefficients

\[
[M]{\ddot{x}} + [C]{\dot{x}} + [K]{x} = {y(t)} \quad (6.4)
\]

where \( \{x\} = \begin{bmatrix} \beta \\ \theta \end{bmatrix} \) is a column vector containing the flap and pitch angles;

\( [M], [C] \) and \( [K] \) are the inertia, damping and stiffness matrices respectively;

\( \{y(t)\} \) is the vector of external moments acting on the rotor blade.

The equilibrium position for the uncontrolled system is determined by setting

\[
\{\ddot{x}\} = \{\dot{x}\} = 0 \quad \text{and} \quad \{y(t)\} = \{y_0\} . \quad (6.5)
\]

This gives

\[
[K]{x_0} = \{y_0\} \quad (6.6)
\]
where \( \{ x \} = \{ x_0 \} \) is the equilibrium position.

The perturbed motion about the equilibrium position is given by
\[
\{ x \} = \{ x_0 \} + \{ \Delta x \}.
\]

(6.7)

Introducing the perturbed motion into the equations of motion Eq.(6.4) gives
\[
[M]\{ \ddot{x}_0 \} + [M]\{ \Delta \ddot{x} \} + [C]\{ \dot{x}_0 \} + [C]\{ \Delta \dot{x} \} + [K]\{ x_0 \} + [K]\{ \Delta x \} = \{ y_0 \}
\]

(6.8)

but \( \{ \ddot{x}_0 \} = \{ \dot{x}_0 \} = 0 \) and subtracting Eq.(6.6) from Eq.(6.8) gives the equations of perturbations
\[
[M]\{ \Delta \ddot{x} \} + [C]\{ \Delta \dot{x} \} + [K]\{ \Delta x \} = 0.
\]

(6.9)

6.3.2.2 Equations for Hovering Flight With Control

The equations of motion for the rotor blade in hovering flight with control are differential equations with constant coefficients
\[
[M]\{ \ddot{x} \} + [C]\{ \dot{x} \} + [K]\{ x \} = \{ y(t) \} + [A]\{ x(t) - x(t - T) \} + [B]\{ \dot{x}(t) - \dot{x}(t - T) \}
\]

(6.10)

where
\[
[A] = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]
\[
[B] = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\]

are the adjustable parameters in the control law.

These differential equations Eq.(6.10) are simply the differential equations for the uncontrolled system Eq.(6.4) with the control terms added to the right hand side.

The equilibrium position for the controlled system is determined by setting
\[
\{ \ddot{x} \} = \{ \dot{x} \} = 0 \text{ and } \{ y(t) \} = \{ y_0 \}; \text{ and the control terms in Eq.(6.10) will be equal to}
\]
zero, because when the system is at rest at the equilibrium position, obviously
\[ x(t) = x(t - T). \]

Therefore the equilibrium position is determined by
\[ [K][x_0] = [y_0], \quad (6.11) \]
where \( \{x\} = \{x_0\} \) is the equilibrium position; which, as noted in earlier chapters, yields an identical equilibrium position to that for the uncontrolled system.

The perturbed motion about the equilibrium position is given by
\[ \{x\} = \{x_0\} + \{Ax\}. \quad (6.12) \]

Introducing the perturbed motion into the equations of motion for the controlled system Eq.(6.10) gives
\[
[M][\dot{x}_0] + [M][\Delta \dot{x}] + [C][\dot{x}_0] + [C][\Delta \dot{x}] + [K][x_0] + [K][Ax] = \{y_0\} + \\
+ [A]\{x_0(t) - x_0(t - T)\} + [A]\{\Delta x(t) - \Delta x(t - T)\} \\
+ [B]\{\dot{x}_0(t) - \dot{x}_0(t - T)\} + [B]\{\Delta \dot{x}(t) - \Delta \dot{x}(t - T)\} \quad (6.13)
\]
but \( \{\dot{x}_0\} = \{\dot{x}_0\} = 0 \) and \( \{x_0(t) - x_0(t - T)\} \) and \( \{\dot{x}_0(t) - \dot{x}_0(t - T)\} \) are equal to zero because \( x_0 \) is constant. Then subtracting Eq.(6.11) from Eq.(6.13) gives the equations of perturbations
\[
[M][\Delta \ddot{x}] + [C][\Delta \dot{x}] + [K][\Delta x] = [A]\{\Delta x(t) - \Delta x(t - T)\} + [B]\{\Delta \dot{x}(t) - \Delta \dot{x}(t - T)\}. \quad (6.14)
\]

The matrices \([M], [C], [K]\) in the equations for the controlled system in hovering flight are identical with those in the equations for the uncontrolled system in hovering flight. Therefore, for the purpose of the analysis for the case of hovering flight in sections 6.4 and 6.5, these coefficient matrices are taken from the literature;
the inertia, damping and stiffness matrices used by Bramwell\(^5\) to analyse the pitch-flap flutter stability of a helicopter rotor blade were adopted.

6.3.2.3 Equations for Forward Flight Without Control

The equations of motion for the rotor blade in forward flight without control are differential equations with periodic coefficients

\[
[M(t)]\{\dot{x}\} + [C(t)]\{\dot{x}\} + [K(t)]\{x\} = \{y_0(t)\}
\]

(6.15)

The steady-state solution for the uncontrolled system is determined by setting \(\{y_0(t)\} = \{y_0(t+T)\}\) where \(\{y_0(t)\}\) is a harmonic function of time with period \(T\).

The periodic steady-state solution is given by

\[
[M(t)]\{\dot{x}_0(t)\} + [C(t)]\{\dot{x}_0(t)\} + [K(t)]\{x_0(t)\} = \{y_0(t)\}
\]

(6.16)

where

\(\{x_0(t)\} = \{x_0(t+T)\}\) is a periodic function of time with period \(T\).

The perturbed motion around the periodic steady-state solution is given by

\(\{x\} = \{x_0\} + \{\Delta x\}\).  \hspace{1cm} (6.17)

Introducing the perturbed motion into the equations of motion for the uncontrolled system Eq.(6.15) gives

\[
[M(t)]\{\dot{x}_0\} + [M(t)]\{\Delta \dot{x}\} + [C(t)]\{\dot{x}_0\} + [C(t)]\{\Delta \dot{x}\} + [K(t)]\{x_0\} + [K(t)]\{\Delta x\} = \{y_0(t)\}
\]

(6.18)

Then subtracting Eq.(6.16) from Eq.(6.18) gives the equations of perturbations for the rotor blade in forward flight without control

\[ [M(t)]\{\Delta \ddot{x}\} + [C(t)]\{\Delta \dot{x}\} + [K(t)]\{\Delta x\} = 0. \]  

\((6.19)\)

6.3.2.4 Equations for Forward Flight With Control

The equations of motion for the rotor blade in forward flight with control are differential equations with periodic coefficients

\[ [M(t)][\ddot{x}] + [C(t)][\dot{x}] + [K(t)]\{x\} = \{y(t)\} + [A]\{x(t) - x(t - T)\} + [B]\{\dot{x}(t) - \dot{x}(t - T)\} \]

\((6.20)\)

In the case of forward flight

\[ \{y_0(t)\} = \{y_0(t + T)\} \]

\((6.21)\)

where \(\{y_0(t)\}\) is harmonic with period T.

The steady-state solution for the controlled system is the same as that for the uncontrolled system because the time-delay used in the control law is equal to the period of the steady-state solution, and therefore the control terms are equal to zero.

The periodic steady-state solution is given by

\[ [M(t)][\ddot{x}_0(t)] + [C(t)][\dot{x}_0(t)] + [K(t)][x_0(t)] = \{y_0(t)\} \]

\((6.22)\)

where \(\{x_0(t)\} = \{x_0(t + T)\}\).

The perturbed motion about the steady-state solution is given by

\[ \{x(t)\} = \{x_0(t)\} + \{\Delta x(t)\} \]

\((6.23)\)
Introducing the perturbed motion Eq.(6.23) into the equations of motion for the
controlled system in forward flight Eq.(6.20) gives

\[
[M(t)]\{\dot{x}_0\} + [M(t)]\{\Delta \dot{x}\} + [C(t)]\{\dot{x}_0\} + [C(t)]\{\Delta \dot{x}\} + [K(t)]\{x_0\} + [K(t)]\{\Delta x\} = \\
= \{y_0(t)\} + [A]\{x_0(t) - x_0(t - T)\} + [A]\{\Delta x(t) - \Delta x(t - T)\} + \\
+ [B]\{\dot{x}_0(t) - \dot{x}_0(t - T)\} + [B]\{\Delta \dot{x}(t) - \Delta \dot{x}(t - T)\}
\] (6.24)

but \{x_0(t) - x_0(t - T)\} and \{\dot{x}_0(t) - \dot{x}_0(t - T)\} are equal to zero because \(x_0\) and \(\dot{x}_0\)
are periodic with period \(T\).

Then subtracting Eq.(6.22) from Eq.(6.24) gives the equations of perturbations for the
controlled system in forward flight

\[
[M(t)]\{\Delta \dot{x}\} + [C(t)]\{\Delta \dot{x}\} + [K(t)]\{\Delta x\} = [A]\{\Delta x(t) - \Delta x(t - T)\} + [B]\{\Delta \dot{x}(t) - \Delta \dot{x}(t - T)\}.
\] (6.25)

The matrices \([M],[C],[K]\) in the equations for the controlled system in
forward flight are identical with those in the equations for the uncontrolled system in
forward flight. Therefore, for the purpose of the analysis in section 6.6, these
coefficient matrices are taken from the literature. For the case of forward flight the
inertia, damping and stiffness matrices used by Stammers\(^6\) to analyse the pitch-flap
flutter stability of a helicopter rotor blade were adopted.

---

6.3.3 Physical Model

The linear mathematical models of both Bramwell and Stammers\(^7\) were based on the physical model shown in Figure 6.3. The displacements that define the equilibrium position or steady-state motion \(\theta_0, \beta_0\) and \(y_0\) were assumed to be small. Therefore \(\theta, \beta\) and \(y\) are all small.

---

The assumptions made in producing the physical model of the rotor blade (which is shown in Figure 6.3), are as follows:

1. The fuselage and the rotor hub are fixed in space, and the only motion is that of the rotor blade.
2. The rotor blade is rigid and has flap and pitch degrees-of-freedom.
3. There is no lead-lag degree-of-freedom.
4. The flap and pitch axes both pass through the origin of the systems of coordinates.
5. The hinge offset from the rotor shaft is equal to zero.
6. The pitch axis ($x_3$ axis of the body system of coordinates) coincides with the elastic axis of the rotor blade.
7. The $x_3$ axis is not one of the principal axes of the rotor blade; so the products of inertia will not be zero.
8. The elastic axis of the rotor blade coincides with the aerodynamic centre.
9. The angular speed of the rotor shaft $\Omega$ is known and is constant.
10. The flapping natural frequency is $\lambda_1 \Omega$.
11. The torsional natural frequency is $\nu_1 \Omega$.
12. Aerodynamic forces are included in the model, but unsteady aerodynamic effects are neglected.
13. The chordwise displacement of the centre of gravity from the aerodynamic centre of a blade element is constant along the span of the rotor blade.
14. The effect of aeroelastic coupling between the degrees-of-freedom is included, but there is no kinematic coupling.
15. The pitch hinge is outboard of the flap hinge (in terms of the sequence of rotations).
16. The dimension $R_0$ is very small compared with the dimension $R_s$ and is treated as approximately equal to zero.

6.3.4 Systems of Coordinates

Three systems of coordinates are shown in Figure 6.3. They all have their origin at the same point, which is at the centre of the rotor hub on the axis of rotation of the rotor shaft. The axis of the flap hinge and the axis of the pitch hinge also pass through this point. The $z_1$ axis is coincident with the $Z$ axis of the absolute system of coordinates $XYZ$, and the $x_1$ and $y_1$ axes are rotated by an angle $\Omega r$ in the $XY$ plane from the $X$ and $Y$ axes.

The $y_1$ axis is the axis of the flap hinge, and the $y_2$ axis is coincident with the $y_1$ axis. The $x_2$ and $z_2$ axes are rotated by an angle $\beta$ in the $x_1 z_1$ plane relative to the $x_1$ and $z_1$ axes. The angle $\beta$ is the flap angle of the rotor blade. The angle $\beta_0$ defines the equilibrium position of the rotor blade. The angle $\Delta \beta$ defines the displacement of the rotor blade from the equilibrium position. So $\beta = \beta_0 + \Delta \beta$. All three angles $\beta$, $\beta_0$, and $\Delta \beta$ are assumed to be small.

Axis $x_3$ is the pitch axis of the rotor blade. It is coincident with axis $x_2$ and is the axis of the pitch hinge. Axes $y_3$ and $z_3$ are rotated by an angle $\theta$ in the $y_2 z_2$ plane relative to the $y_2$ and $z_2$ axes. The $x_3 y_3 z_3$ system of coordinates is rigidly attached to the rotor blade. The angle $\theta_0$ defines the equilibrium position of the rotor blade. The angle $\Delta \theta$ defines the displacement of the rotor blade from the equilibrium position. So $\theta = \theta_0 + \Delta \theta$. All three angles $\theta$, $\theta_0$, and $\Delta \theta$ are assumed to be small.
The displacement of the lower end of the flexible pitch link is denoted by \( y \). The displacement of \( y_0 \) is dictated by the pilot and corresponds to the equilibrium position of the uncontrolled system. The displacement \( \Delta y \) is the active control input, and \( y = y_0 + \Delta y \). All three displacements \( y, y_0 \) and \( \Delta y \) are assumed to be small.

6.3.5 Flap and Pitch Natural Frequencies

A torsional spring element (which is not shown in Figure 6.3) was added at the flap hinge to achieve the appropriate natural frequency of the flap motion.

The natural frequency of the pitch motion was a function of the flexibility of the mechanical control linkages used by the pilot to control the angle of attack of the rotor blade. This flexibility is represented in Figure 6.3 by the spring which is connected to the pitch horn of the rotor blade, where the pitch link, that connects the pitch horn to the rotating part of the swashplate, would normally be connected. The introduction of this flexibility to the mechanical control system means that while the displacement of the lower end of the spring \( y \) is a control input, the pitch angle of the rotor blade \( \theta \) is a degree-of-freedom. Using a rigid rotor blade connected to a flexible mechanical control system provides a good representation of the dynamics of the torsional motion of the rotor blade, because in practice the stiffness of the control system is usually less than the torsional stiffness of the rotor blade.

The pitch and flap natural frequencies \( \nu, \Omega \) and \( \lambda, \Omega \) respectively) are included

---

8 See Figure 1.1 in Chapter 1.
explicitly in the coefficient matrices of the equations of perturbations. Note that the variable \( v_1 \) is the non-dimensional pitch natural frequency; but it will simply be referred to as the "pitch natural frequency". The dimensional pitch natural frequency is obtained by multiplying by the angular speed of the rotor shaft - i.e. \( v_1 \Omega \).

6.3.6 Non-Dimensional Time

The independent variable is changed from time \( t \) to non-dimensional time \( \psi \) using the relations

\[
\psi = \Omega t, \quad \frac{d}{dt} = \Omega \frac{d}{d\psi}, \quad \frac{d^2}{dt^2} = \Omega^2 \frac{d^2}{d\psi^2}
\]

(6.26)

where \( \psi \) is the non-dimensional time, which is the angle of rotation of the rotor shaft about the \( Z \) axis (which is called the "azimuth angle"); and

\[ \Omega = \frac{d\psi}{dt}, \] which is the constant angular velocity of the rotor shaft about the \( Z \) axis.

6.3.7 Centre of Gravity Offset

The rotor blade is divided into "blade elements" which are slices parallel to the \( \gamma_3z_3 \) plane, which is perpendicular to the \( x_3 \) axis or "span" of the rotor blade. The distance of the centre of gravity of each rotor blade element behind the elastic axis of the rotor blade (i.e. in the \(-j_3\) direction from the \( x_3 \) axis) is assumed to be constant along the span of the rotor blade. This distance is denoted by \( \alpha \sigma_h \), where \( c_h \) is the chord dimension of the rotor blade (the dimension in the \( j_3 \) direction), and \( \sigma \) is a
fraction called the "centre of gravity offset". The value of \( \sigma \) is positive when the centre of gravity is behind the elastic axis of the rotor blade, i.e. in the \(-j_3\) direction which is towards the trailing edge of the rotor blade.

6.3.8 Control Moment

It has been established in section 6.3.5 that the mechanical control system of linkages which the pilot uses to manipulate the angle of attack of the rotor blade and so achieve the desired flight path of the helicopter, is not rigid but incorporates a certain amount of flexibility. In the inertia, damping and stiffness matrices, the stiffness of the mechanical control linkages is expressed in terms of the pitch natural frequency of the rotor blade. The relationship between the undamped natural frequency \( \nu_1 \Omega \) and the effective torsional stiffness \( k_{rot} \) is given by

\[
\frac{k_{rot}}{A_m} = (\text{natural frequency})^2 = \nu_1^2 \Omega^2
\]

where

\[
A_m = \text{the pitch moment of inertia of the rotor blade} = \frac{1}{12} M_b c_a^2;
\]

\[
M_b = \text{the mass of the rotor blade};
\]

\[
c_a = \text{the chord dimension of the rotor blade}.
\]

The relationship between the effective torsional stiffness \( k_{rot} \) and the stiffness \( k_{lin} \) of the linear spring element in Figure 6.3 acting with moment arm equal to \( r_{\text{mom}} \) is

\[
k_{lin} r_{\text{mom}}^2 = k_{rot}
\]
Therefore the stiffness of the linear spring element can be expressed in terms of the pitch natural frequency of the rotor blade as

\[
k_{\text{lin}} = \frac{A_m \Omega^2 \nu_1^2}{r_{\text{mean}}^2}.
\]  

(6.29)

So the control moment \( M_y \) exerted on the rotor blade by a displacement \( y \) of the lower end of the spring element is equal to

\[
M_y = k_{\text{lin}} y r_{\text{mean}} = \frac{A_m \Omega^2 \nu_1^2}{r_{\text{mean}}} y
\]  

(6.30)

The values of the parameters used in this equation were

\[
\Omega = 258 \text{ rpm} = 27.0 \text{ rad/s} \quad r_{\text{mean}} = 0.50 \text{ m}
\]

\[
A_m = 0.0833 \quad M_{bc_n}^2 = 0.0833 \times 112 (0.527)^2 = 2.59 \text{ kgm}^2.
\]  

(6.31)

Inserting these values into Eq. (6.30) reduces the expression for the control moment as a function of \( y \) to

\[
M_y = y \nu_1^2 3776.
\]  

(6.32)

6.4 Motion of the Rotor Blade in Hovering Flight without Control

6.4.1 Equations of Perturbations

From now on, an asterisk over a variable will be used to denote differentiation with respect to the non-dimensional time \( \psi \). The coupled differential equations of perturbations for the flap and pitch motion of a helicopter rotor blade in hovering flight with the inertia, dampness and stiffness matrices used by Bramwell\(^{10}\) can be written in

matrix form as

\[
[M]\begin{bmatrix} \Delta^* x \\ \Delta^* \beta \end{bmatrix} + [C]\begin{bmatrix} \Delta x \\ \Delta \beta \end{bmatrix} + [K]\{\Delta x\} = \{0\}
\]

where

\[
\{\Delta x\} = \begin{bmatrix} \Delta \theta \\ \Delta \beta \end{bmatrix}, \quad \{\Delta x\} = \begin{bmatrix} \frac{\partial \Delta \theta}{\partial \psi} \\ \frac{\partial \Delta \beta}{\partial \psi} \end{bmatrix}, \quad \{\Delta x\} = \begin{bmatrix} \frac{\partial^2 \Delta \theta}{\partial \psi^2} \\ \frac{\partial^2 \Delta \beta}{\partial \psi^2} \end{bmatrix}
\]

\[
[M] = \begin{bmatrix} 1 & -12\sigma r_g \\ c_h & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} \gamma & 0 \\ \frac{8}{\gamma} & 0 \end{bmatrix}, \quad [K] = \begin{bmatrix} \frac{v_i^2}{c_h} & -12\sigma r_g \\ c_h & \lambda_i^2 \end{bmatrix}
\]

where

\( r_g \) = the spanwise location of the centre of gravity of the rotor blade, i.e. the distance in \( \delta \) direction from the rotor shaft (which is where the origin of the system of coordinates is located in Figure 6.3) to the centre of gravity;

\( \gamma \) = the Lock Number of the rotor = \( \rho \alpha_f c_h R_s^4 / B_m \);

\( R_s \) = the span dimension of the rotor blade;

\( c_h \) = the chord dimension of the rotor blade;

\( \alpha_f \) = the lift curve slope for the airfoil;

\( B_m \) = the flap moment of inertia of the rotor blade; and

\( \rho \) = air density.
6.4.2 Parameter Values

The values of the parameters used in the equations of motion were chosen to represent realistic values for a Sikorsky Black Hawk helicopter. The two parameters in the equations of motion that affect the pitch-flap stability and can readily be varied in the design of a rotor are the chordwise blade centre of gravity offset $\sigma$ and the pitch natural frequency $\nu_t$. Therefore the stability results shown below were calculated as a function of these two parameters.

The other parameter values were as follows:

\[
\begin{align*}
&c_h = 0.527 \text{ m}, \quad r_s = 4.10 \text{ m}, \quad \lambda_1 = 1.1, \quad \rho = 1.284 \text{ kg/m}^3, \quad a_t = 5.73, \quad R_s = 8.18 \text{ m}, \\
&M_b = 112 \text{ kg}, \quad B_m = \frac{1}{3} M_s R_s^2 = 2498 \text{ kgm}^2, \quad \gamma = \rho a\omega R_s^3/B_m = 6.95
\end{align*}
\]

6.4.3 State-Space Formulation

The equation of perturbations Eq.(6.33) can be written in terms of state-space coordinates

\[
\{\Delta x_1\} = \{\Delta x\}_{2 \times 1} \quad \{\Delta x_2\} = \{\Delta x^*\}_{2 \times 1} = \{\Delta x^*\}_{2 \times 1}
\]

Then

\[
\{\Delta Z\} = \begin{bmatrix}
\{\Delta x_1\} \\
\{\Delta x_2\}
\end{bmatrix} = \begin{bmatrix}
\{\Delta x\} \\
\{\Delta x^*\}
\end{bmatrix} = \begin{bmatrix}
\Delta \theta \\
\Delta \beta \\
\frac{d\Delta \theta}{d\psi} \\
\frac{d\Delta \beta}{d\psi} \\
\frac{d^2 \Delta \theta}{d\psi^2} \\
\frac{d^2 \Delta \beta}{d\psi^2}
\end{bmatrix}
\]

\[
\{\Delta Z\} = \begin{bmatrix}
\{\Delta x\} \\
\{\Delta x^*\}
\end{bmatrix} = \begin{bmatrix}
\frac{d\Delta \theta}{d\psi} \\
\frac{d\Delta \beta}{d\psi} \\
\frac{d^2 \Delta \theta}{d\psi^2} \\
\frac{d^2 \Delta \beta}{d\psi^2}
\end{bmatrix}
\]

\[
\text{(6.37)}
\]
Equation (6.33) can then be written in state-space form as

\[
\begin{bmatrix}
\Delta Z \\
\lambda Z
\end{bmatrix}_{4 \times 1} = [E]_{4 \times 4} \begin{bmatrix}
\Delta Z \\
\lambda Z
\end{bmatrix}_{4 \times 1}
\]  
\hspace{1cm} (6.38)

where

\[
[E]_{4 \times 4} = \begin{bmatrix}
[0]_{2 \times 2} & [1]_{2 \times 2} \\
-[M]^{-1}[K] & -[M]^{-1}[C]
\end{bmatrix}
\]  
\hspace{1cm} (6.39)

where \([1]_{2 \times 2}\) is the identity matrix.

6.4.4 Stability Analysis

In this section the eigenvalues of the matrix \([E]\) are used to investigate the stability of the coupled pitch-flap motion of the rotor blade without control.

The solution of Eq.(6.38) is assumed to have the form

\[
\begin{bmatrix}
\Delta Z \\
\lambda Z
\end{bmatrix} = \{\Delta Z_0\} e^{i\omega} \hspace{1cm} \begin{bmatrix}
\Delta Z \\
\lambda Z
\end{bmatrix} = \{\Delta Z_0\} \lambda e^{i\omega}
\]  
\hspace{1cm} (6.40)

where \(\lambda = R + iU\) is the “eigenvalue” or “characteristic root” which is a complex number. Substituting Eq.(6.40) into Eq.(6.38) and rearranging produces

\[
[[E] - \lambda[1]]\{\Delta Z_0\} e^{i\omega} = \{0\}
\]  
\hspace{1cm} (6.41)

This equation always has the trivial solution \(\{\Delta Z_0\} = \{0\}\). It only has a non-trivial solution if the characteristic determinant is zero

\[
\det[[E] - \lambda[1]] = 0
\]  
\hspace{1cm} (6.42)

The characteristic equation Eq.(6.42) is a fourth-order polynomial in \(\lambda\), which has four roots. Therefore the matrix \([E]\) has four eigenvalues. The characteristic roots can be either complex conjugate pairs or purely real roots with zero imaginary part. Thus there are two different kinds of behaviour that can occur when the dominant
characteristic root crosses the imaginary axis and moves into the right half of the complex plane, and the solution of the equations of motion becomes unstable. Which kind of behaviour occurs depends on whether the root in question is a real root or a complex root. If it is a real root, the instability is called "pitch divergence". If it is a complex root, the instability is called "pitch-flap dynamic instability" or "pitch-flap flutter". These two kinds of behaviour are discussed below.

6.4.4.1 Pitch Divergence

Pitch divergence occurs when the dominant characteristic root has a positive real part and an imaginary part equal to zero. It is therefore a "static" instability, where the response of the system does not oscillate but just grows larger and larger with time until structural failure occurs. Dynamic forces do not play any role. Therefore the inertia and damping matrices in the equations of motion are irrelevant. Only the "spring forces" (forces proportional to displacement) play a part. The physical interpretation of pitch divergence can be understood from Figures 6.4 and 6.5. When the rotor blade flaps upwards with flap angle $\beta$ the centrifugal force acting on the centre of gravity has a component parallel to the rotor blade and a component normal to the rotor blade (see Figure 6.4). The component normal to the rotor blade is given by $C_{F_n} = CF \sin \beta = CF \cdot \beta$. If the centre of gravity of the rotor blade is not coincident with the aerodynamic centre of the rotor blade, then a couple is produced by the aerodynamic force acting at the aerodynamic centre and the normal component of the centrifugal force acting at the centre of gravity (see Figure 6.5).
Chapter 6 The Control Method applied to a Helicopter Rotor Blade

centre of gravity of rotor blade

Figure 6.4 Components of the centrifugal force acting on the rotor blade.

The component of centrifugal force normal to $x_3$ axis

$= CF \sin \beta \equiv CF, \beta$

aerodynamic lift force

centre of gravity

the component of the centrifugal force normal to the $x_3$ axis

Figure 6.5. Two forces creating a couple.
If the centre of gravity is further from the leading edge of the rotor blade than the aerodynamic centre, this couple causes the rotor blade to twist so that the angle of attack is increased. This increases the aerodynamic force on the rotor blade, which causes both the couple to increase and the rotor blade to flap upwards further. The increased flap angle means that the normal component of the centrifugal force is increased, which in turn increases the couple, and this process continues, leading to pitch divergence. Whether pitch divergence occurs, depends upon the relative size of the couple which is acting to increase the angle of attack of the rotor blade and the torsional restoring moment exerted on the rotor blade by the mechanical control system linkage. The size of the couple is a function of the distance of separation between the aerodynamic centre and the centre of gravity of the rotor blade. This distance is denoted by the centre of gravity offset $\sigma$ in the mathematical model. If the centre of gravity offset is increased, the size of the couple is increased. The torsional restoring moment acting on the rotor blade is proportional to the square of the pitch natural frequency of the rotor blade $\nu_1^2$ because this is proportional to the mechanical control system stiffness. If the pitch natural frequency is increased, the torsional restoring moment acting on the rotor blade is increased. Thus the pitch divergence stability depends on both the centre of gravity offset $\sigma$ and the pitch natural frequency $\nu_1$. This relationship can be seen by examining the criterion for the pitch divergence stability boundary, which is found by setting the determinant of the stiffness matrix from the equations of motion equal to zero, as follows:

$$\det[K] = \begin{vmatrix} \nu_1^2 & -12\sigma \gamma c_k \\ -\gamma & \lambda_1^2 \end{vmatrix} = 0$$

(6.43)
which gives

\[ v_1^2 \lambda_1^2 - \frac{12 \gamma \sigma_1}{8 c_n} = 0. \]  

(6.44)

This expression gives the criterion for stability, which is the boundary between the stable and unstable regions of the parameter space. It corresponds to the dominant real root lying on the imaginary axis at the origin of the complex plane. The stability is increased if the pitch natural frequency is increased or the centre of gravity offset is decreased. Therefore the pitch divergence will not occur if

\[ v_1^2 \lambda_1^2 > \frac{12 \gamma \sigma_1}{8 c_n}. \]  

(6.45)

The form of the expression for the pitch divergence stability boundary shows that the boundary is a straight line if plotted with the centre of gravity offset \( \sigma \) as the \( x \)-axis (abscissa) and the square of the pitch natural frequency as the \( y \)-axis (ordinate).

Figure 6.6 Stability Boundaries Without Control.
This is confirmed by the results shown in Figure 6.6, which shows the divergence stability boundary that was determined by finding the eigenvalues of the matrix \([E]\) defined in Eq.(6.39) using the Black Hawk helicopter rotor parameter values defined in Eq.(6.35). The region where pitch-flap divergence will occur corresponds to the values of \(\sigma\) and \(V_t^2\) for which there exists an eigenvalue of matrix \([E]\) with positive real part and zero imaginary part. The divergence stability boundary can be seen to be a straight line in the plot in Figure 6.6. Consistent with the discussion above, the pitch divergence instability occurs for high values of centre of gravity offset and low values of pitch natural frequency. Increasing the pitch natural frequency by increasing the stiffness of the mechanical control system, or decreasing the centre of gravity offset from the aerodynamic centre, will both increase the pitch divergence stability of the helicopter rotor.

Johnson presents the pitch divergence stability boundary as a straight line when he plots it as a function of the square of the pitch natural frequency and the centre of gravity offset\(^{11}\). Bramwell, on the other hand, plots the pitch divergence stability boundary as a function of the same variables which Johnson uses, but presents it as a parabolic curve\(^{12}\). Bramwell’s plot appears to be in error here because elsewhere he describes the pitch divergence stability boundary as a straight line in the \(V_t^2 - \sigma\) plane\(^{13}\). The parabolic shape of the pitch divergence stability boundary which Bramwell presents is similar to that plotted by Chopra, who used the pitch natural frequency (not

\[\text{References:}\]

12 Bramwell, A.R.S., 1976, *Helicopter Dynamics*, Edward Arnold, London, p.371, Figure 11.9
squared) for the y-axis\textsuperscript{14}. The same shape also appears in Figure 12-9 of Johnson where the pitch natural frequency (not squared) is used for the y-axis\textsuperscript{15}. Either Bramwell's pitch divergence stability boundary should be a straight line, or the label on the y-axis and the description of it in the text\textsuperscript{16} should read $v_1$ instead of $v_1^2$. This is relevant to the present investigation; because in it the pitch natural frequency squared is used for the y-axis and the pitch divergence stability boundary is a straight line.

One significant difference between pitch divergence in helicopters and in fixed-wing aircraft is that in fixed-wing aircraft the wing divergence stability depends on the distance between the aerodynamic centre and the elastic axis of the wing. In helicopter rotor blades the distance between the aerodynamic centre and the centre of gravity is important because of the very large centrifugal force that acts on the centre of gravity. The position of the elastic axis is not important in helicopter rotor blades, and pitch divergence can occur even when the aerodynamic centre and the elastic axis are coincident.

6.4.4.2 Pitch-flap Dynamic Instability

Pitch-flap dynamic instability or pitch-flap flutter occurs when the dominant characteristic root is one of a complex conjugate pair having a positive real part and non-zero imaginary part. The positive real part of the dominant characteristic roots means that the amplitude of the motion of the rotor blade grows exponentially with


The non-zero imaginary part of the dominant characteristic roots means that the frequency of the motion is not zero as it was for the pitch divergence instability; so the motion is oscillatory.

As was the case for the pitch divergence stability, the two most significant parameters that affect the pitch-flap flutter stability of the rotor blade are the pitch natural frequency and the centre of gravity offset from the aerodynamic centre. In Figure 6.6 the pitch-flap flutter stability boundary is plotted as a function of these two parameters. These results were calculated by finding the eigenvalues of the matrix $[E]$ with the parameter values for the Black Hawk helicopter shown in Eq.(6.35). The real part of the dominant complex eigenvalue determines whether the pitch-flap flutter instability occurs. From Figure 6.6 the pitch-flap flutter can be seen to occur for high values of the centre of gravity offset and low values of the pitch natural frequency (although for values of the pitch natural frequency close to zero pitch-flap flutter does not occur). Decreasing the centre of gravity offset or increasing the pitch natural frequency by increasing the mechanical control system stiffness increases the stability of the pitch-flap motion of the rotor blade.

Whether the dominant characteristic root is real or complex, and thus whether pitch divergence or pitch-flap flutter occurs, depends upon the values of the pitch natural frequency and the centre of gravity offset. In Figure 6.6, where both the pitch-flap flutter stability boundary and the pitch divergence stability boundary are plotted together, it can be seen that for low values of the pitch natural frequency $v_1$, if the centre of gravity offset $s$ is increased from zero, the pitch divergence stability boundary is encountered before the pitch-flap flutter stability boundary. On the other hand, for higher values of pitch natural frequency, if the centre of gravity offset is
increased from zero then the pitch-flap flutter stability boundary is encountered before the pitch divergence stability boundary. This behaviour of the system is further illustrated by the root locus plots shown in Figures 6.7 and 6.8 and described below. These plots were calculated by finding the eigenvalues of the matrix \([E]\), using the parameters for the Black Hawk helicopter given in Eq.(6.35).

Figure 6.7. Root locus for \(v_1^2 = 1.5\).

Figure 6.7 shows the locations of the four characteristic roots in the complex plane as a function of the centre of gravity offset \(\sigma\). The numbers which have been used to annotate the plot indicate the values of the centre of gravity offset \(\sigma\) that correspond to particular locations of the characteristic roots. The arrows which have been drawn next to the markers on the plot indicate the direction in which each characteristic root moves in the complex plane as the centre of gravity offset is increased. The values of the characteristic roots were calculated for fifty values of the
centre of gravity offset between 0.000 and 0.098 in steps of 0.002. The value of the square of the pitch natural frequency was held constant for this plot at $v_1^2 = 1.5$. This is a relatively low value compared with the range of values used for the $y$-axis in Figure 6.6. The first characteristic root to cross the imaginary axis into the right half of the complex plane as the centre of gravity offset was increased in Figure 6.7 was a real root (with zero imaginary part). It crossed the imaginary axis when $\sigma = 0.024$. The complex conjugate roots did not cross the imaginary axis until $\sigma = 0.036$. This showed that the pitch divergence instability occurred before the pitch-flap flutter instability as the centre of gravity offset was increased. This conclusion is consistent with the pitch-flap flutter and pitch divergence stability boundary locations for $v_1^2 = 1.5$ shown in Figure 6.6 where the pitch divergence stability boundary would be encountered before the pitch-flap flutter stability boundary if $\sigma$ was increased from zero while $v_1^2$ was held constant at $v_1^2 = 1.5$.

The root locus plot shown in Figure 6.8 was calculated in the same way as the plot shown in Figure 6.7, and for the same range of values of the centre of gravity offset. It also has similar annotation to indicate the values of the centre of gravity offset that correspond to particular characteristic root locations in the complex plane. The difference between the two plots is that in the plot shown in Figure 6.8 the value of the square of the pitch natural frequency was $v_1^2 = 4.1$ as opposed to $v_1^2 = 1.5$ in the plot shown in Figure 6.7. In the plot shown in Figure 6.8, two complex conjugate characteristic roots crossed the imaginary axis into the right half of the complex plane when the centre of gravity offset had been increased to a value of $\sigma = 0.030$. The characteristic root on the real axis did not cross the imaginary axis until the centre of gravity offset had been increased to a value of $\sigma = 0.062$. Therefore the pitch-flap
flutter instability occurred before the pitch divergence instability as the centre of gravity offset was increased from zero. These results confirm the locations of the pitch-flap flutter and pitch divergence stability boundaries for $v_1^2 = 4.1$ shown in Figure 6.6.

![Root locus for $v_1^2 = 4.1$.](image)

6.5 Rotor Blade with Control in Hovering Flight

It has been explained earlier in the chapter that the pilot uses the mechanical control system of linkages to manipulate the angle of attack of the rotor blade in order to achieve the desired flight path of the helicopter. The inputs from the pilot to the rotor blade are displacements of the lower end of the spring shown in the physical model in Figure 6.3. The magnitude of these displacements is denoted by the variable $y_0$ in Figure 6.3. The Phase-Locked Delayed Feedback control method is implemented
by superimposing small additional inputs on the pilot's inputs. The additional feedback control displacements of the lower end of the spring element in Figure 6.3 are denoted by $\Delta y$. The total input to the rotor blade is given by

$$y = y_0 + \Delta y.$$  \hspace{1cm} (6.46).

The Phase-Locked Delayed Feedback control method was applied to the helicopter rotor blade mathematical model according to the schematic diagram shown in Figure 6.9.

![Schematic diagram of Phase-Locked Delayed Feedback control method applied to the helicopter rotor blade.](image-url)
In the general case, both the flap angle and the pitch angle can be used in the feedback to determine the control inputs, but in this thesis it was found to be sufficient to use only the flap angle. Also, while it is theoretically possible to exert control moments about both the pitch axis and the flap axis, in practice it is only possible to exert a control moment about the pitch axis using the existing mechanical control linkages. Therefore only a control moment about the pitch axis was used in the mathematical model. So the control inputs $\Delta y$ were formulated on the basis of the measured flap angle $\beta(\psi)$ and the known period $\psi = 2\pi$ as follows:

$$
\Delta y = 0.0001a(\beta(\psi) - \beta(\psi - 2\pi)) + 0.0001b(\dot{\beta}(\psi) - \dot{\beta}(\psi - 2\pi))
$$  \hspace{1cm} (6.47)

where

- $a$ and $b$ are adjustable parameters;
- $\beta(\psi)$ is the flap angle;
- $\beta(\psi - 2\pi)$ is the flap angle recorded one period earlier.

As only very small values of $\Delta y$ are required to produce significant control moments, the values of the control parameters $a$ and $b$ in the feedback control law are very small. To avoid working with such small values of the control parameters $a$ and $b$, a constant factor of 0.0001 was introduced into the feedback control law. This constant has no effect other than to increase the size of the optimal values of the control parameters by a factor of 10,000. It was introduced solely for convenience.

The time delay in the feedback loop was set equal to the period of one revolution of the helicopter rotor $T = \frac{2\pi}{\Omega}$ or $\psi = 2\pi$. 


Therefore using Eq.(6.14) and the inertia, damping and stiffness matrices of Bramwell\textsuperscript{17} the equations of perturbations for the controlled system in hovering flight are

\[
[M]\{\Delta x^{**}\} + [C]\{\Delta x^{*}\} + [K]\{\Delta x\} = [A]\{\Delta x(\psi) - \Delta x(\psi - 2\pi)\} + [B]\{\Delta x^{*}(\psi) - \Delta x^{*}(\psi - 2\pi)\}
\]

(6.48)

where

\[
\{\Delta x\} = \begin{bmatrix} \Delta \theta \\ \Delta \beta \end{bmatrix}, \quad \{\Delta x^{*}\} = \begin{bmatrix} \frac{d\Delta \theta}{d\psi} \\ \frac{d\Delta \beta}{d\psi} \end{bmatrix}, \quad \{\Delta x^{**}\} = \begin{bmatrix} \frac{d^2\Delta \theta}{d\psi^2} \\ \frac{d^2\Delta \beta}{d\psi^2} \end{bmatrix}
\]

\[
[M] = \begin{bmatrix} 1 & -\frac{12\sigma_z}{c_z} \\ 0 & 1 \end{bmatrix}, \quad [C] = \begin{bmatrix} \gamma & 0 \\ \frac{8}{8} & \gamma \end{bmatrix}, \quad [K] = \begin{bmatrix} \frac{v_1^2}{c_a} & -\frac{12\sigma_z}{c_z} \\ \frac{-\gamma}{8} & \lambda_1^2 \end{bmatrix}
\]

\[
[A] = \begin{bmatrix} 0 & 0.3776v_1^2a \\ 0 & 0 \end{bmatrix}, \quad [B] = \begin{bmatrix} 0 & 0 \\ 0 & 0.3776v_1^2b \end{bmatrix}
\]

(6.49)

6.5.1 State-Space Formulation

The equations of perturbations can be written in terms of state-space coordinates

\[
\{\Delta x_1\} = \{\Delta x\}_{2x1}, \quad \{\Delta x_2\} = \{\Delta x^{*}\}_{2x1} = \{\Delta x^{**}\}_{2x1}
\]

(6.50)

Then

\[
\{\Delta Z\} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta x \end{bmatrix} \times 1
\]

\[
\begin{bmatrix}
\Delta \theta \\
\Delta \beta \\
\frac{d\Delta \theta}{d\psi} \\
\frac{d\Delta \beta}{d\psi}
\end{bmatrix}
\]

\[
(6.51)
\]

Then

\[
\{\Delta x\} = \{\Delta x\}
\]

\[
(6.52)
\]

\[
\{\Delta x\} = -[M]^{-1}[C]\{\Delta x\} - [M]^{-1}[K]\{\Delta x\} + [M]^{-1}[A]\{\Delta x(\psi) - \Delta x(\psi - 2\pi)\}
\]

\[
+ [M]^{-1}[B]\{\Delta x(\psi) - \Delta x(\psi - 2\pi)\}
\]

\[
(6.53)
\]

Expanding and collecting terms gives

\[
\{\Delta x\} = -[M]^{-1}[C]\{\Delta x\} - [M]^{-1}[K]\{\Delta x\}
\]

\[
+ 0.3776v_1^2 [M]^{-1}\begin{bmatrix}
\alpha(\beta(\psi) - \beta(\psi - 2\pi)) + b(\beta(\psi) - \beta(\psi - 2\pi)) \\
0
\end{bmatrix}
\]

\[
(6.54)
\]
Equation (6.48) can then be written in state-space form as

\[
\begin{bmatrix}
\Delta Z^* \\
\end{bmatrix}_{4\times 1} = \begin{bmatrix} [E]_{4\times 4} \{\Delta Z\}_{4\times 1} + [N]_{4\times 4} \end{bmatrix} \begin{bmatrix} 0 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix} a(\beta(\psi) - \beta(\psi - 2\pi)) + b\left(\Delta \beta^* e^{i(\psi - 2\pi)}\right) \\
\end{bmatrix}_{4\times 1}
\]

(6.55)

where

\[
[E]_{4\times 4} = \begin{bmatrix}
[0]_{2\times 2} & [1]_{2\times 2} \\
-[M]'[K] & -[M]'[C]
\end{bmatrix}
\]

\[
[N]_{4\times 4} = \begin{bmatrix}
[0]_{2\times 2} & [0]_{2\times 2} \\
[0]_{2\times 2} & 0.3776\nu_1^2[M]^{-1}_{2\times 2}
\end{bmatrix}
\]

(6.56)

where \([1]_{2\times 2}\) is the identity matrix.

The solution of Eq.(6.55) is assumed to have the form

\[
\begin{bmatrix}
\Delta Z \\
\end{bmatrix} = \{\Delta Z_0\} e^{i\psi} \\
\begin{bmatrix}
\Delta Z^* \\
\end{bmatrix} = \{\Delta Z_0\} \hat{\lambda} e^{i\psi}
\]

(6.57)

Substituting Eq.(6.57) into Eq.(6.55) gives

\[
\begin{bmatrix}
\Delta Z_0 \hat{\lambda} e^{i\psi} = [E]\{\Delta Z_0\} e^{i\psi} + [N]\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} + b\left(\Delta \beta_0 e^{i(\psi - 2\pi)}\right)
\]

(6.58)

The expression in Eq.(6.58) can be written as

\[
\begin{bmatrix}
\Delta Z_0 \hat{\lambda} e^{i\psi} = [E]\{\Delta Z_0\} e^{i\psi} + [N][W]\{\Delta Z_0\} e^{i\psi}
\]

(6.59)
Chapter 6 The Control Method applied to a Helicopter Rotor Blade

where

\[
[W] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & P & 0 & Q \\
0 & 0 & 0 & 0 
\end{bmatrix}
\]  \hspace{1cm} (6.60)

\[P = a(1 - e^{-2\pi i}) \]  \hspace{1cm} (6.61)

\[Q = b(1 - e^{-2\pi i}) \]  \hspace{1cm} (6.62)

Rearranging gives

\[
[[E] + [N][W] - \lambda [I]_{4\times4}] [Z_0] e^{\lambda \nu} = [0]_{4\times4}
\]  \hspace{1cm} (6.63)

6.5.2 Stability Analysis

The non-trivial solution of Eq.(6.63) exists for values of \( \lambda \) which satisfy the characteristic equation

\[
\text{det}[[E] + [N][W] - \lambda [I]_{4\times4}] = 0
\]  \hspace{1cm} (6.64)

However, in contrast to the characteristic equation for the rotor blade without Phase-Locked Delayed Feedback control, which had four characteristic roots, this characteristic equation has an infinite number of characteristic roots. This is a consequence of the presence of the time-delay terms in the control law. It is the same phenomenon that occurred for the one-degree-of-freedom system with Phase-Locked Delayed Feedback control discussed in Chapter 4\textsuperscript{18}. In this case, the computer-based eigenvalue extraction routine could not be used to find the eigenvalues of the matrix

\textsuperscript{18} Chapter 4 section 4.3.2.
So a “chicken-and-egg” situation existed where the eigenvalue extraction routine could not be used until the eigenvalues had been found already. One way to deal with this situation was to use a method of “subsequent approximation”. This iterative method involved firstly finding the four characteristic roots of the uncontrolled system by finding the eigenvalues of the matrix \([E]\); and then choosing the eigenvalue with the highest real part from these four and substituting it into the matrix \([W]\). The four eigenvalues of the matrix \([E]+[N][W]\) were then found and the one that was closest to the last one which was substituted into the matrix \([W]\) was selected. This process was repeated until the value of the selected eigenvalue converged to a constant value. This method proved to be very fast, but had the drawback that it did not always converge.

It was decided, therefore, to use the much slower (in terms of computer processing time required) but more reliable method that was used for the one-degree-of-freedom system in Chapter 4: that is, searching the complex plane for locations where the loci of zero points of the real and imaginary parts of the characteristic equation intersected. This method also had its drawbacks. Only a finite region of the complex plane could be searched. If the size of the region selected was very large, it would take a very large amount of computer processing time to search it for characteristic roots. If the region was small, there was a risk that it might not contain the dominant characteristic root. So a compromise had to be made in the selection of this region; and the result of the search had to be checked by another method, such as numerically integrating the equations of motion.
A second drawback of the method adopted was that a compromise had to be made in the selection of the resolution with which to search the complex plane for intersections of the loci of the zero points of the real and imaginary parts of the characteristic equation. If the resolution was high, a very large amount of computer processing time was required. However, if the resolution was low, places where the loci passed close together but did not actually intersect could be identified mistakenly as intersections.

6.5.3 Pitch Divergence

First of all, the effect of the control method on the pitch divergence stability boundary was considered. The effect of the values of the control parameters on the real characteristic roots (with zero imaginary part) was investigated without any regard for the locations of the complex conjugate characteristic roots. The result of this investigation was that for values of the pitch natural frequency and centre of gravity offset for which the uncontrolled system had a positive real root, the control method was able to move this root almost to the origin of the complex plane, but it was not able to make it negative. In other words, if the uncontrolled system was unstable, the controlled system could be made almost stable but never asymptotically stable. This is illustrated in Figure 6.10 where the value of the real part of the dominant real root is plotted as a function of the value of the control parameter $a$ (the value of the control parameter $b$ is zero in this plot). It can be seen that as the value of $a$ becomes more negative, the value of $R_{dom}$ asymptotically approaches zero. No matter how negative the value of $a$ becomes, the value of $R_{dom}$ never reaches zero. No values of the control
parameters $a$ and $b$ were able to make $R_{\text{dom}}$ negative.

![Graph](image)

Figure 6.10 The real part of the dominant real root as a function of $a$, with $b = 0$, $\sigma = 0.07$, $\nu_l^2 = 3.6$.

The consequence of these results was that the pitch divergence stability boundary was in exactly the same place in the $\sigma - \nu_l^2$ plane for the controlled system as it was for the uncontrolled system. The practical implication of this is that very flexible mechanical control linkages must always be avoided because the occurrence of pitch divergence at low values of the pitch natural frequency cannot be prevented by the control method.

Searching for the optimal values of the control parameters presents a dilemma, because the optimal values for maximising the pitch divergence stability and the optimal values for maximising the pitch-flap flutter stability are likely to be different. Values of the control parameters that move the complex conjugate roots to the left in the complex plane and thus improve the pitch-flap flutter stability, may move a real
root to the right and thus decrease the pitch divergence stability and vice-versa. Thus both kinds of roots must be considered together when searching for the optimal values of the control parameters.

6.5.4 Optimal Values of the Control Parameters

The optimal values of the control parameters $a$ and $b$ for specific values of the pitch natural frequency and centre of gravity offset were the values that resulted in the lowest value of the real part of the dominant characteristic root $R_{dom}$. These were the values that produced the greatest increase in the stability margin of the equilibrium position of the rotor blade.

In the search for the optimal values of the control parameters both the real characteristic roots and the complex conjugate characteristic roots were considered. A computer program to search for the optimal values of the control parameters was developed. For particular values of $a$ and $b$ the computer program determined the location of the dominant characteristic root $(R_{dom}, U_{dom})$ in the complex plane using the method discussed in section 6.5.2. The values of $a$ and $b$ were then changed slightly and the process was repeated. The computer program would then move across the $a,b$ plane in the direction in which it found the value of the real part of the dominant characteristic root $R_{dom}$ to decrease most rapidly, until it could not find a lower value in any direction. Thus the lowest achievable value of $R_{dom}$ and the optimal values of $a$ and $b$ were found. To illustrate how the location of the dominant characteristic root in the complex plane varied with the values of the control parameters, some examples of contour plots of $R_{dom}$ and $U_{dom}$ as a function of $a$ and $b$ for five different combinations
of values of centre of gravity offset and pitch natural frequency are shown in Figures 6.11 to 6.15.

The contour plots of $U_{dom}$ as a function of $a$ and $b$ show that the optimal values of $a$ and $b$ tend to occur at a boundary between two distinct regions corresponding to significantly different values of $U_{dom}$. This means that the optimal values of the control parameters occur where two characteristic roots with different imaginary parts have equal real parts. There are therefore two equally dominant characteristic roots. (This is illustrated in section 6.5.5.) Changing the values of the control parameters in one direction causes one of the two characteristic roots to move to the right in the complex plane and become dominant. Changing the values of the control parameters in the opposite direction causes the other one of the two characteristic roots to move to the right in the complex plane and become dominant. This phenomenon was described in the experimental results in Chapter 5\textsuperscript{19}.

\textsuperscript{19} Chapter 5 section 5.6.1
optimal value of $b$

$b$ [s]

Figure 6.11 (a) $R_{dem}$ as a function of $a$ and $b$: $\sigma = 0.05$, $\nu_1^2 = 3.6$.

Figure 6.11 (b) $U_{dem}$ as a function of $a$ and $b$: $\sigma = 0.05$, $\nu_1^2 = 3.6$. 
Chapter 6. The Control Method applied to a Helicopter Rotor Blade

Figure 6.12 (a) $R_{dom}$ as a function of $a$ and $b$: $\sigma = 0.09$, $v_1^2 = 4.4$.

Figure 6.12 (b) $U_{dom}$ as a function of $a$ and $b$: $\sigma = 0.09$, $v_1^2 = 14.4$. 
Figure 6.13 (a) $R_{dom}$ as a function of $a$ and $b$: $\sigma = 0.07, \nu_1^2 = 10.8$.

Figure 6.13 (b) $U_{dom}$ as a function of $a$ and $b$: $\sigma = 0.07, \nu_1^2 = 10.8$. 
Figure 6.14 (a) $R_{\text{dom}}$ as a function of $a$ and $b$: $\sigma = 0.04$, $\nu_1^2 = 3.6$.

Figure 6.14 (b) $U_{\text{dom}}$ as a function of $a$ and $b$: $\sigma = 0.04$, $\nu_1^2 = 3.6$. 
Chapter 6. The Control Method applied to a Helicopter Rotor Blade

Figure 6.15 (a) $R_{dom}$ as a function of $a$ and $b$: $\sigma = 0.08$, $v_1^2 = 10.8$.

Figure 6.15 (b) $U_{dom}$ as a function of $a$ and $b$: $\sigma = 0.08$, $v_1^2 = 10.8$. 
When the optimal values of the control parameters and the corresponding lowest values of $R_{dom}$ had been found for each combination of the pitch natural frequency and the centre of gravity offset, the values of $R_{dom}$ were plotted on the $\sigma - v_1^2$ plane. Where the value of $R_{dom}$ is positive, the equilibrium position of the controlled system is unstable and either pitch divergence or pitch-flap flutter occurs. Where the value of $R_{dom}$ is negative, the equilibrium position is stable. The boundary between the stable and unstable regions is shown in Figure 6.16. The combined pitch divergence and pitch-flap flutter stability boundary for the uncontrolled system which is taken from Figure 6.6 is also plotted in Figure 6.16 for comparison.

![Figure 6.16 Stability Boundaries with and without control.](image)
In Figure 6.16, it is clear that the entire region where the pitch divergence instability occurred for the uncontrolled system is also part of the combined instability region for the controlled system. It was noted above that it was not possible to make the motion of the system stable for the values of pitch natural frequency and centre of gravity offset in this region. However, it can be seen in Figure 6.16 that in a significant part of the region in Figure 6.6 where the pitch-flap flutter instability occurred but the pitch divergence instability did not occur, the control method was able to stabilise the unstable pitch-flap flutter motion. This was true for the region between the two stability boundaries in Figure 6.16. As an example, a point in this region is used below to illustrate the effectiveness of the control method. The selected point is where \( \sigma = 0.08 \) and \( v_1^2 = 10.8 \).

6.5.5 An Illustrative Example

In this section the results for the two-degree-of-freedom mathematical model of a helicopter rotor blade for the case with \( \sigma = 0.08 \) and \( v_1^2 = 10.8 \) are discussed in greater detail.

The contour plots showing the values of the real and imaginary parts of the dominant characteristic root \( (R_{dom} \text{ and } U_{dom}) \) plotted as a function of the values of the control parameters \( a \) and \( b \) for \( \sigma = 0.08 \) and \( v_1^2 = 10.8 \) were shown in Figure 6.15 above. The value of \( R_{dom} \) for the uncontrolled system \((a = b = 0.0)\) is \( R_{dom} = 0.108s^{-1} \) which is positive and so the equilibrium position of the system is unstable. The imaginary part of the dominant root for the uncontrolled system is not zero, so the instability is pitch-flap flutter, not pitch divergence. The dominant characteristic root
can be seen in Figure 6.17, which shows the loci of zeros of the real and imaginary parts of the characteristic equation in the complex plane. The characteristic roots are given by the intersections of the two loci, which is where the values of $R$ and $U$ cause both parts of the characteristic equation simultaneously to equal zero (note that where two branches of the same locus intersect with each other a root does not exist). The uncontrolled system has four characteristic roots. Figure 6.17 shows two of these characteristic roots. The other two characteristic roots are the complex conjugates of the two shown in this figure.

Figure 6.17 Zeros of the characteristic equation for $\sigma = 0.08$, $\nu_r = 10.8$ and $\alpha = \theta = 0.0$. 
The lowest value of $R_{dom}$ for the case when $\sigma = 0.08$ and $v_1^2 = 10.8$ occurs when $a = 6.75$ and $b = 0.6$ s, and is equal to $R_{dom} = -0.097$ s$^{-1}$. These values of $a$, $b$ and $R_{dom}$ were found using the computer program written to search for the optimal values of the control parameters $a$ and $b$. Figure 6.15 agrees with these results, although it does not show sufficient detail to determine the results so precisely. To give a more accurate picture of the behaviour of the dominant characteristic root in the neighbourhood of the optimal values of $a$ and $b$, the values of $R_{dom}$ and $U_{dom}$ were determined over a finer grid of $a$ and $b$ values than was used in Figure 6.15. These results are shown in Figure 6.18. In this figure the optimal values of $a$ and $b$ can be seen to lie precisely on the boundary between two regions with very different values of $U_{dom}$. This means that for the optimal values of $a$ and $b$ there are two equally dominant characteristic roots with equal real parts and different imaginary parts.
Figure 6.18  (a) $R_{dom}$ as a function of $a$ and $b$: $\sigma = 0.08$, $v_f^2 = 3.6$ on a finer grid.
(b) $U_{dom}$ as a function of $a$ and $b$: $\sigma = 0.08$, $v_f^2 = 3.6$ on a finer grid.
Figure 6.19 shows the loci of zeros of the real and imaginary parts of the characteristic equation for a region of the complex plane containing the dominant characteristic root for the controlled system with the optimal values of the control parameters given above. If the entire complex plane could be plotted, it would be seen to contain an infinite number of characteristic roots, as noted in Chapter 4\textsuperscript{20}. Figure 6.19 shows that there are two equally dominant characteristic roots with real parts equal to -0.097, and with imaginary parts corresponding to the two adjacent regions at the point where $a = 6.75$ and $b = 0.6$ s in the plot of $U_{dom}$ as a function of $a, b$ in Figure 6.18(b), i.e. $U_{dom} = 0.36$ s\(^{-1}\) and 2.39 s\(^{-1}\).
The information about the stability of the motion of the system with and without control which was obtained by finding the dominant characteristic root for a range of values of $a$ and $b$ was then checked by integrating numerically the coupled pitch-flap equations of perturbations for the helicopter rotor blade, using the Runge-Kutta method. The initial functions were $\Delta \theta(\omega t) = 0.0$, $\Delta \beta(\omega t) = 0.1$, for $-2\pi \leq \omega t \leq 0$. The results are shown in Figures 6.20 and 6.21. As expected, the amplitude of the perturbations for the uncontrolled system increased with time because the equilibrium position was unstable; and the amplitude of the perturbations for the controlled system with the optimal values of the control parameters, decreased with time because the equilibrium position was stable. Figure 6.21 shows clearly that for the optimal values of the control parameters when there are two equally dominant characteristic roots with different imaginary parts, the oscillation of the rotor blade about the equilibrium position is the superposition of a low frequency mode and a high frequency mode.

Figure 6.20 Perturbations of the uncontrolled system.
6.6 Rotor Blade with Control in Forward Flight

6.6.1 Equations of Perturbations for Forward Flight

The effectiveness of the control method in improving the stability of the periodic steady-state motion of the helicopter blade in forward flight was investigated briefly, using Eq. (6.25) and the inertia, damping and stiffness matrices developed by Stammers\(^{21}\). These coefficient matrices were derived by Stammers for the two-degree-of-freedom physical model of a rotor blade shown in Figure 6.3. The equations Eq. (6.25) are perturbation equations which describe small motions around the periodic steady-state motion of the rotor blade in forward flight. Therefore when the periodic

steady-state motion is stable the particular solution of these equations, after an initial 
disturbance, decreases to zero with time. The equations of perturbations for forward 
flight with control are

\[
\begin{bmatrix} M(\psi) \Delta^* x^* \end{bmatrix} + \begin{bmatrix} C(\psi) \end{bmatrix} \begin{bmatrix} \Delta^* x^* \end{bmatrix} + \begin{bmatrix} K(\psi) \end{bmatrix} \begin{bmatrix} \Delta x \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \Delta x(\psi) - \Delta x(\psi - 2\pi) \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} \Delta x(\psi) - \Delta x(\psi - 2\pi) \end{bmatrix}
\]

(6.65)

where

\[
\begin{bmatrix} \Delta x \end{bmatrix} = \begin{bmatrix} \Delta \theta \\ \Delta \beta \end{bmatrix}, \quad \begin{bmatrix} \Delta^* x^* \end{bmatrix} = \begin{bmatrix} \frac{d \Delta \theta}{d \psi} \\ \frac{d \Delta \beta}{d \psi} \end{bmatrix}, \quad \begin{bmatrix} \Delta x^{**} \end{bmatrix} = \begin{bmatrix} \frac{d^2 \Delta \theta}{d \psi^2} \\ \frac{d^2 \Delta \beta}{d \psi^2} \end{bmatrix}
\]

(6.66)

The equations of perturbations for forward flight have periodic coefficients which are 
proportional to the forward speed of the helicopter. The coefficient matrices are as 
follows:

\[
[M(\psi)] = \begin{bmatrix} 1 & \frac{r' \sigma}{k^2} \\ 0 & 1 \end{bmatrix}
\]

(6.67)

\[
[C(\psi)] = \begin{bmatrix} \frac{nm_x}{8k^2} \left(1 + \frac{4}{3} \mu \sin \psi \right) & 0 \\ 0 & nl_z \left(1 + \frac{4}{3} \mu \sin \psi \right) \end{bmatrix}
\]

(6.68)
In the forward flight analysis, the Phase-Locked Delayed Feedback control method was applied to the mathematical model of the helicopter rotor blade in exactly the same way as it was applied to the mathematical model of the rotor blade in hovering flight in section 6.5. Therefore the control parameters are given by

\[ [A] = \begin{bmatrix} 0 & 0.3776v_1^2 a \\ 0 & 0 \end{bmatrix} \quad [B] = \begin{bmatrix} 0 & 0.3776v_1^2 b \\ 0 & 0 \end{bmatrix} \]  

(6.70).

The value of the time-delay in the control method was set equal to \(2\pi\), as it was in section 6.5. Since non-dimensional time was used, this delay is equivalent to the period of one revolution of the rotor shaft. This is the fundamental period of the periodic motion of the rotor blade in forward flight.

The parameters used in this analysis were:

- \(r' = 14.14\) = aspect ratio (i.e. span/chord)
- \(k^2 = 0.05\) = square of non-dimensional radius of gyration of chordwise element of rotor blade about elastic axis
- \(n = 1.667\) = Lock Number + 8
- \(l_t = l_o = 0.5\) = aerodynamic derivative coefficients
- \(m_o = 0.5\) = aerodynamic derivative coefficient
- \(\psi\) = non-dimensional time = \(\Omega t\)
- \(\sigma\) = non-dimensional centre of gravity offset
- \(v_1\) = non-dimensional pitch natural frequency
\( \mu = \text{tip-speed ratio (see explanation in section 6.6.2 below)} \) \hspace{1cm} (6.71)

Substituting the numerical values from Eq.(6.71) into Eqs.(6.67), (6.68) and (6.69) gives,

\[
[M(\psi)] = \begin{bmatrix} 1 & 283\sigma \\ 0 & 1 \end{bmatrix} \hspace{1cm} (6.72)
\]

\[
[C(\psi)] = \begin{bmatrix} 2.084 \left(1 + \frac{4}{3}\mu \sin \psi \right) & 0 \\ 0 & 0.8335 \left(1 + \frac{4}{3}\mu \sin \psi \right) \end{bmatrix} \hspace{1cm} (6.73)
\]

\[
[K(\psi)] = \begin{bmatrix} v_1^2 & 283\sigma \\ 0.8335 \left(0.8 + 2\mu \sin \psi + \frac{2}{3}\mu^2 (1 - \cos 2\psi) \right) \left[1 + 1.111\mu \cos \psi + 0.8335\mu^2 \sin 2\psi \right] \end{bmatrix} \hspace{1cm} (6.74)
\]

The values of the parameters used in this analysis were changed from Stammers' values. The change was made because in Stammers' results the pitch divergence stability boundary was reached before the pitch-flap flutter stability boundary as the value of the centre of gravity offset \( \sigma \) was increased for constant values of pitch natural frequency \( v_1 \). With the parameter values used in this analysis, the pitch-flap flutter stability boundary was encountered before the pitch divergence stability boundary as the value of \( \sigma \) was increased for a constant value of \( v_1 \). This enabled the effect of the control method on the pitch-flap flutter stability boundary to be investigated.

To investigate the stability of the motion, the linear differential equations of perturbations with periodic coefficients were integrated numerically. The results are shown in Figures 6.22 - 6.25 and are described below.
6.6.2 Forward Flight without Control

The "tip-speed ratio" $\mu$ is the ratio of the velocity of the helicopter relative to the surrounding air $V$, to the velocity of the tip of the rotor blade relative to the helicopter $r\Omega$. Since the rotational speed of the helicopter rotor $\Omega$ varies only slightly, the tip-speed ratio $\mu$ is indicative of the forward speed of the helicopter. In hovering flight the tip-speed ratio is equal to zero. The maximum forward flight speed of helicopters is generally limited to a tip-speed ratio of about 0.5. As it has been mentioned above, Stammers\textsuperscript{22} found that the pitch-flap flutter stability of a helicopter rotor blade was greater in forward flight than in hovering flight. The same behaviour was found also in the present study, and can be seen by comparing Figures 6.22 and 6.23. These results were found by integrating numerically Eq.(6.65) without control ($a = b = 0$).

The initial functions for the numerical integration for all four plots presented here (Figures 6.22 - 6.25) were

$$\Delta \theta(Qt) = 0.0, \quad \Delta f(Qt) = 0.1 \text{ for } -2\pi \leq \Omega t \leq 0.$$ \hspace{1cm} (6.75)

The results in all four plots were calculated for a value of non-dimensional pitch natural frequency of $\nu_1^2 = 14.4$.

The results in both Figures 6.22 and 6.23 were calculated for a value of centre of gravity offset of $\sigma = 0.064$. Figure 6.22 shows the results for the case of hovering flight ($\mu = 0$), whereas Figure 6.23 shows the results for the case of forward flight ($\mu = 0.3$). It is clear from these two figures that forward flight has a stabilising effect on the motion of the rotor blade.
Figure 6.24. Helicopter in forward flight without control: $\mu = 0.3$, $\sigma = 0.069$, $a = b = 0$.

The results in Figure 6.24 were calculated for the same forward flight condition as the results in Figure 6.23, i.e. a tip-speed ratio of $\mu = 0.3$. The value of the centre of gravity offset, however, was increased from $\sigma = 0.064$ to $\sigma = 0.069$. From Figure 6.24 it is clear that this increase in the value of $\sigma$ caused the periodic steady-state motion of the rotor blade to become unstable and the perturbations to grow with time. Therefore, for the value of pitch natural frequency being considered ($v_1^2 = 14.4$), the pitch-flap flutter stability boundary lies between $\sigma = 0.064$ and $\sigma = 0.069$.

6.6.3 Forward Flight with Control

The results in Figure 6.25 were calculated for the same forward flight condition as those in Figure 6.24 ($\mu = 0.3$) and for the same value of centre of gravity offset ($\sigma = 0.069$). In Figure 6.25, however, the Phase-Locked Delayed Feedback control method was applied to the rotor blade by setting $a = 7.0$ and $b = -5.0$ s. These values
of the control parameters were found by a trial-and-error process using the results of the numerical integration. From Figure 6.25 it is clear that the addition of the control method with these values of the control parameters has caused the periodic steady-state motion of the rotor blade that was unstable without control to become stable and the perturbations to decay to zero with time. This demonstrates the effectiveness of the control method in stabilizing the periodic motion of the rotor blade in forward flight.

Figure 6.25. Helicopter in forward flight with Phase-Locked Delayed Feedback control method applied: $\mu = 0.3$, $\sigma = 0.069$, $a = 7.0$, $b = -5.0$ s.

6.7 Practical Determination of Parameters $a$ and $b$

The optimal values of the control parameters for the two-degree-of-freedom mathematical model of the helicopter rotor blade in hovering flight were found by using the characteristic roots of the equations of perturbations as a measure of the stability margin of the equilibrium position. However, when applying this control method to a real helicopter, it is anticipated that the optimal values of the control
parameters will be found experimentally. The control law was designed specifically for systems which are too complex to allow an accurate mathematical model to be constructed. Thus the optimal values of the control parameters for such a system cannot be found analytically or numerically, but must be found experimentally. That it is possible to find the optimal values of the control parameters experimentally was demonstrated in the experimental work described in Chapter 5. In that work the amount of negative damping that could be added to the system before the periodic motion became unstable was used as the measure of the stability margin for determining the optimal values of the control parameters. In helicopter rotors the pitch-flap flutter boundary places a restriction on the maximum rotor speed $\Omega$. Helicopter rotors are generally tested for pitch-flap flutter by increasing the rotor speed in a full-scale test rig until the stability boundary is reached. The onset of pitch-flap flutter is indicated by an increase in the oscillatory loads on the mechanical pitch control mechanism. Therefore the maximum helicopter rotor speed obtainable before the onset of pitch-flap flutter instability could be used as a measure of the stability margin of a helicopter rotor in experimental tests in order to determine the optimal values of the control parameters.

6.8 Conclusions

In this chapter, the Phase-Locked Delayed Feedback control method has been applied to a two-degree-of-freedom mathematical model of a helicopter rotor blade in both hover and forward flight. The computer program developed in Chapter 4 to search the complex plane for the dominant characteristic root of the one-degree-of-
freedom system with an infinite number of characteristic roots was extended to the
two-degree-of-freedom system for the case of hovering flight. The results of this
computer program were checked by integrating numerically the equations of
perturbations.

The controlled system with the optimal values of the control parameters was
shown to have two equally dominant characteristic roots. The corresponding
oscillation was shown to be a combination of two distinct frequencies.

The Phase-Locked Delayed Feedback control method stabilised the unstable
equilibrium position of the helicopter rotor blade in hovering flight so that larger values
of the centre of gravity offset and smaller values of the pitch natural frequency could
be tolerated before the pitch-flap flutter instability occurred.

The pitch-flap flutter instability was shown to be less severe in forward flight
than in hovering flight. The effectiveness of the Phase-Locked Delayed Feedback
control method in improving the stability of the periodic steady-state motion of the
helicopter rotor blade in forward flight was demonstrated.

Thus the control method would give the designer of the rotor blades more
freedom to optimise them for other criteria than pitch-flap stability. It would also
make the helicopter rotor less sensitive to changes in the design parameters which may
occur over time, such as: increased flexibility in the mechanical control system due to
wear; and shifts of the centre of gravity of the rotor blade due to ice accumulating on
the rotor blade and being shed.
CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

A new active control method called Phase-Locked Delayed Feedback has been developed to stabilise periodic motion - with particular application to helicopter rotor blades. The control input is calculated by using the difference between the current states (displacement and velocity) and time-delayed states where the time-delay is equal to the period of the motion being stabilised. In an experimental implementation of this control method a phase-locked oscillator circuit was used to synchronise the time-delay with the period of the motion being stabilised.

In this thesis the Phase-Locked Delayed Feedback control method has been applied to: linear and non-linear one-degree-of-freedom mathematical models of a mass-spring-dashpot system; a laboratory installation; and a two-degree-of-freedom mathematical model of a helicopter rotor blade in hover and forward flight.

A computer program has been developed to find the dominant root of a characteristic equation with an infinite number of roots. The computer program was used to show that optimal values of the control parameters existed. The optimal values of the control parameters were found to vary with a clear pattern as a function of the period of the motion being stabilised. The significant points in this pattern, where the control method did not improve the stability of the periodic motion for the
one-degree-of-freedom system presented in Chapter 4, were shown to coincide with the period of the natural frequency of the uncontrolled system and integer multiples of this period. However, this does not make the control method less suitable for helicopter applications because helicopter rotors are carefully designed so that their natural frequencies are not very close to the operating speed of the rotor.

The control method was shown to improve the stability of the periodic particular solution without changing it at all. This is important for the helicopter application where the periodic motion of the rotor blades determines the flight path of the helicopter.

The Phase-Locked Delayed Feedback control method has been shown to increase greatly the size of the region of asymptotic stability for the periodic motion of a non-linear one-degree-of-freedom system, and to make the system much less sensitive to external disturbances.

The effectiveness of the control method was verified experimentally. The optimal values of the control parameters were found experimentally with no knowledge of the dynamics of the system or the differential equations that govern its behaviour. This is important for complex systems for which it is not feasible to produce an accurate mathematical model.

The pattern of the variation of the optimal values of the control parameters as a function of the period of the motion being stabilised in the experiments agreed with that found for the one-degree-of-freedom mathematical model but was more complex as a result of the additional degrees-of-freedom.

The control method has been shown to improve the stability of the equilibrium position of a helicopter rotor blade in hovering flight and the steady-state periodic
motion of a helicopter rotor blade in forward flight. This means that when the control method is applied, larger values of the chordwise centre of gravity offset and more flexible mechanical control linkages can be tolerated before the pitch-flap flutter instability occurs.

Both the experimental results and the mathematical results showed that, for the optimal values of the control parameters, two of the infinite number of characteristic roots were equally dominant. With the optimal values of the control parameters, the perturbations around the steady-state periodic motion in the experimental results, and the perturbations around the equilibrium position for the rotor blade in hovering flight, were both shown to contain two components at distinct frequencies, which is consistent with two of the characteristic roots being equally dominant.

7.2 Recommendations for Further Work

In the experiments described in Chapter 5 the phase-locked oscillator synchronised the control with the period of the motion being stabilised without determining what the actual value of the period was. The value of the period to be used in the numerical differentiation had to be entered into the real-time computer program by the user. In addition, the values of the control parameters to be used in calculating the control input had to be entered by the user. A method for determining automatically the value of the period of the motion being stabilised was conceived and described in Chapter 5, but was not implemented. Therefore the period of the motion being stabilised, which was determined by the excitation frequency in the experimental runs, was held constant in each run, and so were the values of the control parameters.
In an operational system such as a helicopter, the rotor speed might vary slightly during flight. Therefore the optimal values of the control parameters, which are a function of the period of the motion being stabilised, would also vary. Further work could be done to implement the method for determining the period automatically. Then the system could be extended so that the optimal values of the control parameters that were found experimentally for the entire range of operating frequencies were stored in the computer, and the appropriate values selected automatically based on the detected value of the period.

The experimental installation described in Chapter 5 was designed in such a way that the second mode became unstable first when negative damping was added to the uncontrolled system. The design could be modified so that this was not the case. It would be extremely interesting and could be fruitful to apply the control method to experimental installations with a variety of designs.

During the experiment, problems were encountered with noise in the signals from the displacement transducers. These problems were compounded when the displacement was differentiated to give the velocity. If small, lightweight accelerometers were used, the signal would be integrated to give the velocity, and noise would not be a problem.

The values of the control parameters have been optimised only for the case where the period of the motion being stabilised is constant. If the period were to vary with time, for example in a sinusoidal manner or in a step manner, the optimal values of the control parameters for each value of the period may be different from the optimal values when the period is constant. The behaviour of the control system under such conditions should be examined.
The application of the Phase-Locked Delayed Feedback control method to a mathematical model of a helicopter rotor blade could include types of instabilities different from the one considered in Chapter 6, and attenuation of the helicopter rotor response to gusts of wind. A relatively simple mathematical model has been employed: fruitful results would be obtained if the control method were applied to a more complex mathematical model including unsteady aerodynamics and rotor blade flexibility, or to a coupled helicopter rotor-fuselage model.

A method needs to be developed to deal with the second degree differential equations that were produced when the control method was applied to a non-linear mathematical model of a helicopter rotor blade.

In the case when the object is approximated by a linear mathematical model which is not parametric, the corresponding equations of perturbations have coefficients which are independent of time. The problem of finding optimal values of the control parameters for this case has been solved in Chapter 4, section 4.3 and Chapter 6, section 6.5. Work remains to be done to solve the problem of finding optimal values of the control parameters for the case when the equations of perturbations have time-dependent coefficients. This occurs if the mathematical model used to approximate the object is non-linear (as was the case in Chapter 4, section 4.4) or parametric (as was the case in Chapter 6, section 6.6).

The iterative method for solving the characteristic equation for the two-degree-of-freedom system described in section 6.5.2 in Chapter 6 was not used because it did not always converge. However, when it did converge it produced the solution very quickly indeed. Analysis of the behaviour of the controlled system, and in particular the determination of the stability margin and finding the optimal values of the control
parameters, would be greatly accelerated if the convergence problem of this method were solved, or another method for solving the characteristic equation were developed.

It would be of value to investigate whether better performance of the controller could be achieved by using non-linear control laws based on the Phase-Locked Delayed Feedback principle of synchronising the time-delay in the feedback with the period of the motion being stabilised. For example, the control input could be formulated as

\[ y_c = a(x(t) - x(t-T))^2 + b(\dot{x}(t) - \dot{x}(t-T))^2. \]

The usefulness of the control method could be assessed by analysing the sensitivity of the optimal values of the control parameters to variations of the system parameters.
Abdallah, C., Dorato, P., Benitez-Read, J., and Byrne, R., 1993, "Delayed positive feedback can stabilize oscillatory systems," Proceedings of the American Control Conference pp.3106-3107


Äström, K. J. and Wittenmark, B., 1989, Adaptive Control, Addison Wesley, Reading, Massachusetts


Ham, N.D., 1972, “Technical evaluation report on fluid dynamics panel specialists meeting on aerodynamics of rotary wings,” AGARD Advisory Report No.61


Lebender D., and Schneider F. W., 1993, "Neural nets and the local predictor method used to predict the time series of chemical reactions," Journal of Physical Chemistry Vol97, pp.8764-8769


Nguyen, K., and Chopra, I., 1989b, "Application of higher harmonic control (HHC) to rotors operating at high speed and maneuvering flight," Proc. of the 45th Annual Forum of the American Helicopter Society, Boston, MA., pp.81-96

Noyes, R.M., Field, R.J., and Thompson, R.C. 1971, “Mechanism of reaction of bromine (V) with weak one-electron reducing agents,” Journal of the American Chemical Society, 93(26), pp.7315-16


Straub, F.K., and Byrns, E.V., Jr., 1986, Application of higher harmonic blade feathering on the OH-6A helicopter for vibration reduction, NASA CR-4031


Takahashi, M.D., 1988, Active control of helicopter aeromechanical and aeroelastic instabilities, Ph.D. Dissertation, Mechanical, Aerospace and Nuclear Engineering Department, University of California, Los Angeles, June


Author/s:
Faragher, John Simon

Title:
Stability improvement of periodic motion of helicopter rotor blades

Date:
1996

Citation:

Publication Status:
Unpublished

Persistent Link:
http://hdl.handle.net/11343/36400

File Description:
Stability improvement of periodic motion of helicopter rotor blades

Terms and Conditions:
Terms and Conditions: Copyright in works deposited in Minerva Access is retained by the copyright owner. The work may not be altered without permission from the copyright owner. Readers may only download, print and save electronic copies of whole works for their own personal non-commercial use. Any use that exceeds these limits requires permission from the copyright owner. Attribution is essential when quoting or paraphrasing from these works.