Optimal Curvature and Gradient-constrained Paths with Anisotropic Costs

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Abstract

A SERIES of mathematical problems motivated by the design of underground mine access in bad ground are studied in this thesis. A key property of underground mine declines is that they satisfy vehicle navigability constraints. More specifically, the decline must not exceed a prescribed maximum gradient anywhere (typically 1:7) and in the horizontal projection, must satisfy a minimum turning circle radius constraint (typically 25 - 30m). Furthermore, the cost of developing the decline depends on the geological properties of the rock. A decline can span over distinct geological domains and rock is an anisotropic material, this means that the cost function we are minimising is not simply the length of the decline, but depends on the position and direction of each point along the decline. We focus on the anisotropic behaviour of development cost in this thesis.

Initially, we study the planar version of the problem within a particular geological domain. This means that the curvature constraint and anisotropic cost behaviour are relevant. The anisotropic cost behaviour is modelled mathematically by applying a directional cost function when evaluating the cost of a path. This generalises the problem of shortest curvature-constrained paths studied by Dubins [33] in 1957. We prove that Dubins’ theorem for shortest paths is a special case of a more general problem where we apply a directional cost function which has a convex reciprocal in polar coordinates. We then relax this condition and show that for any directional cost function, there exists a path of the form $CSCSC$ or a degeneracy which is optimal, where $C$ denotes a continuous subset of the circle with minimum turning circle radius, and $S$ represents a straight line segment. We also present an algorithm which efficiently constructs such paths for a simple class of directional cost functions suitable for underground mine decline design.

We then extend the results for the planar case to 3-dimensional space and incorporate
the gradient constraint. By assuming that the directional-cost behaviour can be accurately approximated as a planar property, we transform the 3-dimensional problem with a maximum gradient constraint into a planar problem with a minimum length constraint. We prove that there exists a path of the form CSCSCSC (or a degeneracy) with additional loops which is optimal, where loops are additional unit circles traversed anywhere along the path, in order to satisfy the lower bound on length.

The mathematical results developed in this thesis can be applied to existing decline design optimisation software such as in [15] to be able to optimise for a network of declines in anisotropic ground conditions. The results are also potentially useful for other applications involving curvature-constrained paths, such as road design and 2 or 3-dimensional path planning in anisotropic media.
Declaration

This is to certify that

1. the thesis comprises only my original work towards the PhD,
2. due acknowledgement has been made in the text to all other material used,
3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

____________________________________
Alan J. Chang, December 2011
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Preface

This thesis contains new results on geometrically constrained paths, subject to costs which depend on instantaneous direction along the path. The required level of mathematical knowledge to understand and appreciate the results is basic undergraduate vector calculus. The results are mostly geometric and able to be interpreted visually via diagrams and examples which assist in their explanation. While the application of optimal underground mine design motivated the study of these problems, this thesis illustrates that the careful, rigorous mathematical approach of formulation and abstraction allows us to develop theory which is applicable to a much wider set of applications, and perhaps, even applications yet to be discovered.

The techniques used in the proofs of these results come from geometric optimisation, optimal control theory, and basic optimisation principles, but also borrow concepts from various mathematical disciplines. The level of mathematical knowledge required to fully appreciate the details of the proofs again does not extend far beyond basic undergraduate mathematics, as the more advanced tools employed in proofs are explained and classical references are provided. Fully understanding the algorithms developed in this thesis also does not require any knowledge beyond basic understanding of complexity theory and basic optimisation algorithms, as they are based on the developed mathematical theory within this thesis.
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Chapter 1
Introduction

MATHEMATICS plays a central role in improving the efficiency of mining operations - from the early stages of planning, through to the everyday scheduling of operations and plant. Rigorous mathematical analysis and optimisation allows mining companies to make informed decisions. A significant milestone in the history of research into mine planning was the development of the Lerchs Grossmann Algorithm [44] in 1965 which efficiently solved the optimal pit problem for open pit mines. Software packages based on the algorithm have provided open pit mine planners with very efficient tools for optimising the design of open pit mines.

Due to the inherent differences between open pit and underground mines, the algorithms for open pit mine planning are not applicable to underground mines. The access for an underground mine typically consists of a network of declines, or tunnels, which allow mining equipment and haulage trucks to travel between the surface and the orebodies or stockpiles underground. The most important vehicle navigability constraints are determined by the minimum turning circle radius of the vehicle and the maximum gradient at which the vehicle is able to travel when fully loaded. A path is curvature-constrained if it satisfies a minimum turning circle radius, typically 25 - 30m, in its horizontal projection. A path is gradient-constrained if it satisfies a maximum gradient, typically 1:7. The declines in an underground mine must be curvature and gradient-constrained in order for the haulage trucks to be able to navigate in the mine.

In [15], an algorithm for optimising the access design of underground mines is presented. Since the problem involves geometric constraints, standard non-linear optimisation or statistical simulations are not suitable for solving the problem. The model in [15]
accounts for the vehicle navigability constraints, while minimising development and haulage costs. The development cost is assumed to be isotropic (independent of direction) and homogeneous (independent of position) and hence only dependent on length. This however is an over-simplifying assumption as the cost of supporting a tunnel depends on the strength of the rock, which varies between different geological domains. Weaker rock generally requires more reinforcement and takes longer to develop through as a result. Hence, there is an associated increase in cost of the support, as well as cost to the entire project due to the slower decline development. Furthermore, within a geological domain, the rock strength can exhibit anisotropic behaviour due to directional faulting. Hence, the real cost of a decline (including both development and haulage) is actually heterogeneous and anisotropic.

This thesis addresses the gap in mathematical theory required to design an algorithm which produces an optimal path which minimises an anisotropic cost, while satisfying curvature and gradient constraints. While the motivation for developing this theory stems from the underground mine design problem, curvature-constrained paths in anisotropic media are important in other fields such as path planning and road design. The mathematical theory developed in this thesis can potentially be applied to other areas. This thesis also illustrates how mathematical problems formulated from practical applications can yield surprisingly elegant mathematical results. The main theorems in this thesis are relatively simple to state and visually interpret, despite using a diverse range of mathematical techniques to prove.

1.1 Literature Overview

In this section, a survey of the mathematical literature on underground mine design is presented. Since a key focus of the thesis is studying curvature-constrained paths, we also summarise relevant results on path planning. We also briefly present some background from which the model of heterogeneous and anisotropic development costs of tunnels is established.
1.1 Literature Overview

1.1.1 Underground Mine Networks

A detailed introduction to the problem of optimisation of underground mine design can be found in [4]. The main areas of optimisation in underground mine design are stope\(^1\) optimisation, scheduling and access design. In this thesis, we focus on the access design. The fundamental theory for the optimisation of underground mine networks began with the development of theory on gradient-constrained networks in [20], [22], [23], [52], [53] and [64]. This research branched out from research on the Steiner tree problem [41]. The results were then applied to the optimisation of underground mine network design in [16], [17], [21], [24] and [25]. The theory was extended to optimising flow-dependent networks in [66], which handles haulage cost very well since the amount of haulage on each tunnel depends on the size of the orebodies it connects to. However, the curvature constraint was not incorporated in this research and development cost was assumed to be isotropic and homogeneous.

The curvature constraint was incorporated into the models in [14], [15], [18], [19] and [63]. This work extended classical theory of curvature-constrained paths, which began with Dubins [33] in 1957. It was shown that the 3-dimensional problem of constructing a gradient and curvature-constrained network of declines can be reduced to studying the optimal structure of curvature-constrained paths in the plane. First, the possible locations and directions of access points level are discretised into a set of feasible directed points. For a particular choice of directed points on two adjacent levels, the optimal planar path is constructed and lifted into 3-dimensional space by applying a constant gradient to the path. If the gradient constraint is violated, this path is extended in order to satisfy the gradient constraint. A dynamic programming framework is applied which optimises the entire decline subject to the resolution of the discretisation of feasible directed points. The details of this procedure can be found in [4] and [14]. As with the previous work, the development cost was still assumed to be isotropic and homogeneous.

In the following subsections, we will expand on research on curvature-constrained paths from other application areas, and present relevant geotechnical literature to justify the model for development costs which will be used in the subsequent chapters.

\(^1\)A region which contains ore to be mined
1.1.2 Curvature-constrained Paths in the Plane

Paths subject to curvature constraints are studied for many different applications, with varying formulations of the curvature constraint. In [30], gradient and curvature constraints are considered in the context of road, rail and pipeline route selection. In [46], curvature-constrained paths modified by constant wind are considered for aerial navigation. In [50], curvature-constrained path planning for underwater vehicles in the presence of currents is considered. Curvature-constrained paths, where the maximum curvature is dependent on the direction, are studied for naval path planning in [32].

In this thesis, we consider curvature-constrained paths in the plane, as presented by Dubins in [33] in 1957. A path is curvature-constrained if it is $C^1$, piecewise $C^2$, and its curvature is bounded above by a specified constant $1/R$ where it is $C^2$. The problem considered was that of finding the shortest curvature-constrained path between two given directed points (points in the plane with a prescribed heading) in the plane. We assume $R = 1$ in this thesis, except where stated otherwise, without loss of generality since the problem can be rescaled accordingly. It was shown using analytical methods that the shortest path consisted of only straight line segments, labelled $S$, and arcs of radius 1, labelled $C$. Geometric arguments were then applied to show that the shortest path had the form $CSC$, $C\overline{C}$, or a degeneracy (a form obtained by removing one or more of the labels), where $\overline{C}$ represents a $C$ arc which is of length greater than $\pi$. Paths with these forms are commonly referred to as Dubins paths. There are at most 6 possible distinct Dubins paths for a given pair of direction points, as the arcs can be left or right turning. The generalised result where the path is allowed to have cusps where the path corresponds to a vehicle which is allowed to reverse, was obtained by Reeds and Shepp [55]. This result and other work such as [62] done on the case which permits cusps are not relevant for the application considered in this thesis.

The result that Dubins paths are shortest paths was later proved again, but this time using optimal control theory by Boissonnat in [11]. By applying Pontryagin’s Minimum Principle [51], the result was proven in an entirely different approach to Dubins’ original method in [33]. This approach has since been adopted by many researchers in order to study extensions of the shortest curvature-constrained path problem such as in [10],
1.1 Literature Overview

[32], [57] and [59]. The mathematical formulation in [32] is the most relevant to this thesis. Results similar to those in Chapter 2 were obtained concurrently in [32]. This is discussed in more detail in that chapter.

The problem of constructing shortest curvature-constrained paths in the presence of boundaries or obstacles has been studied extensively such as in [1], [2], [3], [5], [8], [9], [31] and [35]. Obstacles are relevant to the problem of underground mine design as there are no-go regions which the decline may not enter, such as orebodies. Other relevant work to the underground mine design problem include [48] and [60] which identify which Dubins path is the shortest between a given pair of points. Since the underground mine provides access to multiple orebodies, the problem of identifying shortest curvature-constrained paths passing through waypoints (points that the path must visit along the way to the destination) in [45] and [58] are also relevant. In this thesis, we do not focus on these particular aspects. However, the results obtained in this thesis can potentially be combined with existing results to incorporate anisotropic and heterogeneous development costs.

1.1.3 Heterogeneity and Anisotropy of Rock Strength

The background in this section is based on a combination of communication with members of the mining industry as well as classical texts including [36], [54], [39] and [13]. We first describe the heterogeneous behaviour of rock. We then discuss the anisotropic behaviour and explain how this thesis will model these mathematically.

In the early stages of planning a mine and assessing its feasibility, drill core samples are taken at various points from the surface. Using this limited information, geologists are able to estimate the size, location, and grade of orebodies. Additionally, they can identify the different types of rock that the decline would be developed through. This builds a preliminary map of the different geological domains (the distinct regions of rock within which the composition is relatively uniform). As a mine is developed further, more information is obtained as core samples can be obtained by drilling from within the mine rather than from the surface. This means the map of the geological domains and orebodies gets more accurate as the operation progresses.
Besides having a uniform rock composition, it is also common for each geological domain to exhibit directional behaviour. This is well documented in the aforementioned texts, as well as in [43] and [42]. This is a result of directional faulting and other geological features which formed over long periods of time. In the simple scenario where there is one primary direction of vertical faulting, tunnelling in a direction perpendicular to the fault planes will generally cause less rock fracturing than tunnelling at an oblique angle to the plane. Using this information, the geotechnical team would be able to develop specifications for the tunnel support for each of the different regions, and potentially also for different ranges of directions within a region. While it is common practice to simply take the worst-case direction in a region, this is overly conservative and does not fully utilise the information that is available. We demonstrate in this thesis that there are significant potential cost savings in applying different levels of support for the respective directions.

In this thesis, we focus on optimal paths within a particular geological domain, taking anisotropic behaviour into account. As long as the path remains within the geological domain, we can ignore the heterogeneous behaviour across various geological domains. In early chapters, we formulate the problem mathematically with a very general family of directional cost functions. Using those results, we produce an exact and efficient algorithm which is suitable for the application. More specifically, the algorithm constructs a minimum cost curvature-constrained path between given directed points, where the directional cost dependence is piecewise constant. A piecewise constant function is most practical as it corresponds to a uniform support cost applying over a range of directions.

1.2 Thesis Structure

This thesis follows a linear structure in that each chapter builds upon the preceding chapter. We begin the study of curvature-constrained paths subject to anisotropic cost in Chapter 2. We first adopt the approach of Boissonnat [11], reformulating the problem as a minimal time optimal control problem where the velocity of the vehicle is given by the reciprocal of the directional-cost. By applying Pontryagin’s Minimum Principle, we
show that an optimal path can always be constructed using straight lines and circular arcs of maximal curvature. We then restrict the velocities to being polarly convex and show that the characterisation of optimal forms of shortest curvature-constrained paths given by Dubins in [33] can be generalised to any polarly convex velocity function.

The restriction on the velocity function assumed in proving the main result of Chapter 2 is relaxed in Chapter 3. We use the result from Chapter 2 that an optimal path can be constructed using straight lines and circular arcs to restrict the set of paths we need to consider. We then introduce the concept of reversible deformations for such paths, and prove that there always exists an optimal path of a specific form. These optimal forms are more general than the forms of Dubins paths but still contain a finite number of straight line segments and arcs. This result is a large step in the direction of designing an algorithm which can construct an optimal curvature-constrained path between given starting and end points, for a specified anisotropic cost behaviour.

In Chapter 4, we consider the problem of implementing an algorithm which can efficiently construct an optimal path. Unlike Dubins paths which are easily constructed, the optimal curvature-constrained path with anisotropic cost is harder to construct. As such, we restrict ourselves to considering piecewise constant directional-cost functions, which is a logical choice for the application of underground mine access support costs. We then prove a fundamental result on double tangents of planar curves in order to come up with an $O(n^2)$ algorithm which exactly constructs the optimal path for any given start and end points, subject to a directional-cost function with $n$ discontinuities.

In Chapter 5, we introduce the gradient constraint into the model by considering the problem of finding the optimal path as before but between directed points in 3 dimensions. We reduce the problem down to the 2 dimensional version studied in previous chapters except now with a minimum length constraint. This transformation is possible as we know that the optimal path will have a constant gradient which must not exceed the maximum gradient. Combining the maximum gradient with the vertical difference in position between the given points in 3 dimensions then allows us to project the problem back into the plane with a minimum length constraint. This incorporating all major aspects of constructing an optimal decline in a geological domain - the curvature constraint,
gradient constraint, and anisotropy.

The results from the four main chapters are summarised in Chapter 6. The new contributions of this thesis are also listed for each chapter. Further possible work to extend the mathematical theory developed are outlined.
Chapter 2
Generalisation of the Dubins Result

In this chapter, we begin our study on the problem of finding the minimum cost curvature-constrained path between two directed points where the cost at every point along the path depends on the instantaneous direction. This generalises the results obtained by Dubins for curvature-constrained paths of minimum length, commonly referred to as Dubins paths. We conclude that if the reciprocal of the directional cost function is strictly polarly convex, then the forms of the optimal (minimum cost) paths are of the same forms as Dubins paths. If we relax the strict polar convexity to weak polar convexity, then we show that there exists a Dubins path which is optimal. The results obtained can be applied to optimising the development of underground mine networks, where the paths need to satisfy a curvature constraint and the cost of development of the tunnel depends on the direction due to the geological characteristics of the ground.

2.1 Introduction

This chapter studies the problem of finding an optimal path which we define as a minimum cost curvature-constrained path between two directed points where the cost at every point along the path depends on the instantaneous direction. The problem is formally defined in section 2.2.3. In short, we wish to solve the problem of:

$$\min_{E \in \mathcal{P}_{pq}} \int_{E} c(\alpha) \, ds$$

for some given directional cost function $c(\alpha)$, where $\mathcal{P}_{pq}$ denotes the set of all curvature-constrained paths between two directed points $p$ and $q$.

The motivation for studying this problem stems from the issue of directional faulting in the ground where an underground mine is to be developed. The resulting mathematical problem is also interesting in its own right. The costs associated with an underground
mine tunnel are mostly made up of development, haulage, and other maintenance costs. The faulting results in regions of ground where the cost of development (both tunnelling and support) is significantly more in certain directions than others due to a required increase in time and resources to safely tunnel and provide extra support. Haulage and maintenance on the other hand is generally independent of the faulting, and simply depends on the length. The formulation we adopt in this thesis is able to handle all of these costs, as the effect of haulage and maintenance would be just adding a constant to the directional cost function of development per unit length.

Empirical studies of the relationship between rock strength and fault orientation relative to the tunnelling direction, such as [42] and [43], demonstrate that there is a relationship between the required support cost and the direction in which the tunnel is developed. As the problem of finding shortest underground mine networks with this additional consideration has not been considered until this thesis, current practice of the mining engineers is to use experience and intuition to account for this when designing the mines. Understanding exactly how to solve this problem mathematically will enable the geological information to be incorporated into existing optimisation tools such as in [15]. By incorporating directional dependency of cost into the algorithms, the resulting output will be much more accurate and helpful for the strategic and tactical designs of underground mine networks.

The major tools employed in tackling this problem are adapted from Dubins [33] and Boissonnat [11]. By transforming the minimum weighted length problem into a minimum time problem, it becomes clear that the reciprocal of the directional cost function becomes the object of interest in characterising the forms of optimal paths that have to be considered. We denote this reciprocal as the velocity function, and introduce the concepts of strictness and polar convexity. A polarly convex (PC) velocity function means that the triangle inequality holds for cost of paths (in other words, the cost of a straight path between two points is the no greater than the cost of any other path between the same two points). A formal definition and geometric interpretation of polar convexity is provided in Section 2.4. This chapter focuses on the forms of optimal paths that result when we consider PC velocity functions. A strict velocity function is one where we do not have an
interval of directions where the triangle inequality is satisfied by equality. We distinguish
between strictly polarly convex (SPC) where the strict triangle inequality holds, and weakly
turally convex (WPC) where the triangle inequality could be satisfied by equality.

The structure of this chapter is as follows. In Section 2.2 we define the main con-
cepts and introduce the notation that will be used throughout the rest of the chapter. In
section 2.3, we apply Pontryagin’s Minimum Principle as in [11] to conclude that any
optimal path must only consist of maximum curvature arcs and straight segments, given
any strict velocity function. We then prove some basic properties of the cost of paths sub-
jected to SPC velocity functions. Applying these properties allows us to generalise the
results of [33] to conclude that the forms of optimal paths are the same as Dubins paths,
if the velocity function is SPC. Finally, we show that there always exist an optimal path
which is a Dubins path, if the velocity function is WPC. An alternative proof of this result
is provided in the next chapter, using a more general and powerful technique which is
also used to solve the case where the velocity function is not PC.

This chapter expands on the details of the paper [28] published by the author and
co-authors in the Journal of Global Optimization. During the course of writing that pa-
per, it was brought to our attention that similar work was being done concurrently by
Dolinskaya in [32]. The application in their case was for naval path planning but is math-
ematically formulated in similar manner. In their work, they consider an anisotropic
minimum radius of curvature function, as well as an anisotropic velocity function. The
motivation and future direction in which their work is being developed is different from
our work.

2.2 Background

We are interested in studying paths which are navigable by vehicles in an underground
mine. In the plane, this corresponds to curvature-constrained paths since vehicles have a
minimum turning circle radius. An example of a curvature constrained path between
two directed points (points in the plane with a specified direction) is shown in Figure 2.1.
We formally define this concept in the following subsection. We will then introduce the
control formulation which will be necessary for applying Pontryagin’s Minimum Principle to curvature-constrained paths. Finally, we will formalise the concept of directional cost which we use to model the anisotropic costs of development in underground mines.

### 2.2.1 Curvature-constrained paths

In the following, $a, b \in \mathbb{R}^2$ and $p, q \in \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}$. The points $p$ and $q$ are referred to as directed points where the component in $\mathbb{R}/2\pi\mathbb{Z}$ represents the direction. A $C^k$ function is a function whose first $k$ derivatives exist and are continuous.

A path from $a$ to $b$ is a directed piecewise $C^1$ curve from $a$ to $b$. We parametrise our paths by arc-length, $E : [0, t_f] \to \mathbb{R}^2$ with $E(0) = a$ and $E(t_f) = b$. The direction $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ at a differentiable point of a path $E(t)$ is given by the polar angle of the tangent at the point in the direction of increasing $t$. At the endpoints, the directions are taken as the respective limits of $\alpha$.

Given a minimum turning circle radius $R > 0$, a curvature-constrained path between two directed points $p, q$ is a $C^1$ path that is piecewise $C^2$, where the absolute curvature everywhere along the path is bounded above by $1/R$, and the given directions at the start and end points coincide with the directions of $p$ and $q$ respectively. Without loss of generality, we assume the minimum turning circle radius $R = 1$ throughout this thesis.
2.2 Background

Let $C$ be a label denoting a continuous subset of the unit circle. Let $S$ be a label denoting a straight line segment.

A CS-path is a curvature-constrained path $E : [0, t_f] \rightarrow \mathbb{R}^2$ such that there exist $t_0, \ldots, t_n$ such that $t_0 = 0$ and $t_n = t_f$, $t_{i-1} < t_i$ for $i = 1, \ldots, n$, where $E$ is not twice differentiable at $t_i$ for $i = 1, \ldots, n$, and each subpath $E_i : [t_{i-1}, t_i] \rightarrow \mathbb{R}^2$ is either a $C$ arc or $S$ segment. The form of such a CS-path is then the sequence of $C$ and $S$ labels in ascending order of $i$. The sense of a $C$ can be further specified using the labels $L$ and $R$ for left-turning and right-turning arcs respectively. In the form of a CS-path, any consecutive $C$ arcs must be of opposite sense due to the condition that the CS-path is differentiable but not twice differentiable at the point in between two consecutive labels. This condition also implies that there will never be consecutive $S$ labels in the form of a CS-path.

Dubins [33] first studied the problem of finding a shortest curvature-constrained path between any two given directed points in $\mathbb{R}^2$ and proved the following theorem.

**Theorem (Dubins, 1957).** Given any two directed points $p, q \in \mathbb{R}^2 \times \mathbb{R} / 2\pi \mathbb{Z}$ and a prescribed minimum radius of curvature $R$, a shortest curvature-constrained path is a CS-path with one of the following forms:

1. $CSC$;
2. $C\overline{C}C$;
3. any degeneracies of the two forms,

where $\overline{C}$ denotes a $C$ arc of length greater than $\pi R$.

CS-paths of the forms described in Theorem 2.2.1 will be referred to as Dubins paths. Note that there are at most 6 distinct Dubins paths for any given pair $p, q$ because of the different sense of the $C$ arcs, but they do not all necessarily exist and degeneracies may occur.

Theorem 2.2.1 was later proved again by Boissonnat et al. [11] by formulating it as a control problem and applying Pontryagin’s Minimum Principle. A more detailed version of [11] can be found in [62]. We summarise the key ideas here as we will need to use the control formulation for one of the main results.
2.2.2 Control Formulation

We summarise the necessary background for applying Pontryagin’s Minimum Principle from [51]. Note that we are presenting a simplified version of the principle as that is sufficient for this chapter. We also present the minimum principle as opposed to the maximum principle originally formulated in [51], since it is commonly stated in the minimum principle form in the current literature.

Consider a system of differential equations with prescribed boundary conditions as in (2.1), where \( x(t) \) denotes the states of the system, and the piecewise continuous function \( u: [0, T] \to U \) is the control we want to choose, for some convex and compact \( U \). The functions \( f_j \) are continuous in the variables \( x, u \) and continuously differentiable with respect to \( x \).

\[
\dot{x}_j = f_j(x(t), u(t)), \ x(0) = x^i, x(T) = x^f \text{ for } j = 1, \ldots, n
\]

where \( x(t) = (x_1(t), \ldots, x_n(t)) \) and \( \dot{z} \) denotes \( dz/dt \) \hspace{1cm} (2.1)

We restrict ourselves to only considering the controls that satisfy the boundary conditions. Such controls will be referred to as admissible controls. Since we are interested in time optimality, the relevant cost function \( J(u) \) is of the form shown in (2.2) below.

\[
J(u) = \int_0^T dt
\]

We aim to find an admissible control which minimises \( J \), the total time to start from \( x^i \) and end at \( x^f \). To state the minimum principle, we need to introduce another system of equations in the auxiliary variables \( \psi_1, \ldots, \psi_n \) as follows.

\[
\dot{\psi}_j = - \sum_{k=1}^n \frac{\partial f_j}{\partial x_k} \psi_k \text{ for } j = 1, \ldots, n
\]

Let the Hamiltonian \( H \) be defined as follows.

\[
H(\psi, x, u) = \sum_{k=1}^n \psi_k f_k(x, u)
\]

(2.4)

For fixed \( \psi, x \), \( H \) is a function of \( u \). Let \( M(\psi, x) \) denote the lower bound of the values
2.2 Background

of $H$.

$$M(\psi, x) = \inf_{u \in U} H(\psi, x, u) \quad (2.5)$$

We can then state the minimum principle from [51] as follows.

**Theorem 2.1.** If $u(t)$ is an optimal admissible control, there exists a nonzero continuous vector function $\psi(t)$ satisfying (2.3), such that $\forall t \in [0, T]$,

1. $H(\psi(t), x(t), u(t)) = M(\psi(t), x(t)) =$ constant;
2. $M(\psi(t), x(t)) \leq 0$.

### 2.2.3 Directional Cost

We now introduce directional cost to curvature-constrained paths which we use to model the anisotropic behaviour of development cost. Some basic terms which will be used consistently throughout this chapter are defined in this section. Let $\mathbb{R}_+$ denote the set of all strictly positive real numbers $\{x \in \mathbb{R} : x > 0\}$.

A *directional cost function* is a continuous, piecewise $C^2$ function $c : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_+$. The corresponding *velocity function* $v : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_+$ is given by $v(\alpha) = 1/c(\alpha)$, which is also continuous and piecewise $C^2$. The interpretation of the velocity function in the context of directional cost is a measure of the distance that can be travelled in the direction $\alpha$ at the cost of one unit. For example, if for a particular direction $\alpha_1$, $c(\alpha_1) = 2$, then we can travel a distance of $1/2$ in the direction $\alpha_1$ at the cost of one unit.

Given a path $E$ and a directional cost function $c$,

- the *length* of $E$, $L(E)$, is given by $L(E) = \int_E ds$;
- $E$ is *degenerate* if it has zero length;
- the *cost* of $E$, $T(E)$, is given by $T(E) = \int_E c(\alpha) ds$.

Note that the symbol $T$ is intentionally used to denote both the cost of a path in the definition above as well as the final time in Section 2.2.2 previously. This is because we will later formulate the problem such that these two quantities are in fact the same. Let $\mathbb{P}_{pq}$ denote the set of all curvature-constrained paths between two directed points $p$ and $q$. An *optimal path* from $p$ to $q$ is a path $E \in \{P \in \mathbb{P}_{pq} : T(P) \leq T(Q) \forall Q \in \mathbb{P}_{pq}\}$. 

Generalisation of the Dubins Result

Our problem involves identifying an optimal path between given start and end directed points, subject to a given velocity function. It can be seen that by the way the problem has been posed, the problem of finding the minimum total cost path is equivalent to a problem of finding the minimum total time for a vehicle to travel from \( p \) to \( q \) if the velocity of the vehicle depends on the direction it is facing at any point in time. This is the motivation for calling the reciprocal of the directional cost function a velocity function.

2.3 Application of Pontryagin’s Minimum Principle

In [11], the standard Dubins problem of finding a shortest curvature-constrained path between two directed points is formulated as a minimum time problem by having a cost function that is simply the integral of 1 over the path. We could simply modify the cost function to be the integral of \( c(\alpha) \) instead, and indeed this will yield the result we require for the anisotropic case. However, we choose to formulate the problem differently, keeping the cost function the same as in [11], while incorporating the directional cost by setting the velocity of the vehicle to be \( v(\alpha) = 1/c(\alpha) \). Both these approaches are mathematically identical, but the latter provides additional insight which is helpful for motivating the later sections. We first consider velocity functions that are strict as defined below.

A velocity function is strict if there are no (non-trivial) intervals \( [\alpha_1, \alpha_2] \) where \( K(\alpha) = 0 \) for all \( \alpha \in [\alpha_1, \alpha_2] \).

**Lemma 2.2.** If the velocity function is strict, every optimal curvature-constrained path is a CS-
2.3 Application of Pontryagin’s Minimum Principle

path.

Proof. First, let us consider the case where \( v \) is strict and \( C^2 \) everywhere. Let the states \( x(t), y(t) \) and \( \alpha(t) \) represent the coordinates and direction of the path at time \( t \). Let \( u(t) \) be the control variable that governs the instantaneous curvature at time \( t \). Recall that without loss of generality, we let the minimum radius of curvature be 1 and hence, \( u \in [-1, 1] \). Let \( p = (x^i, y^i, \alpha^i) \) and \( q = (x^f, y^f, \alpha^f) \). Our problem can then be formulated in the following manner.

\[
\dot{x} = v(\alpha) \cos(\alpha), \quad x(0) = x^i, \quad x(T) = x^f \quad (2.6)
\]
\[
\dot{y} = v(\alpha) \sin(\alpha), \quad y(0) = y^i, \quad y(T) = y^f
\]
\[
\dot{\alpha} = v(\alpha) u, \quad \alpha(0) = \alpha^i, \quad \alpha(T) = \alpha^f
\]
\[
J(u) = \int_0^T dt
\]

Let \( \psi_x, \psi_y \) and \( \psi_\alpha \) be the corresponding auxiliary variables. From (2.3) we get the following.

\[
\dot{\psi}_x = 0
\]
\[
\dot{\psi}_y = 0
\]
\[
\dot{\psi}_\alpha = \psi_x v(\alpha) \sin \alpha - \psi_x v'(\alpha) \cos \alpha - \psi_y v(\alpha) \cos \alpha - \psi_y v'(\alpha) \sin \alpha - v'(\alpha) \psi_\alpha u \quad (2.8)
\]

From (2.4) we get an expression for the Hamiltonian \( H \) as follows.

\[
H = \psi_x v(\alpha) \cos(\alpha) + \psi_y v(\alpha) \sin(\alpha) + v(\alpha) \psi_\alpha u
\]

Define \( \lambda \) and \( \phi \) by \( \lambda = \sqrt{\psi_x^2 + \psi_y^2} \geq 0 \), \( \tan \phi = \psi_y / \psi_x, \phi \in [0, 2\pi) \) so that \( \psi_x = \lambda \cos \phi, \psi_y = \lambda \sin \phi \). Observe that \( \psi_x \) and \( \psi_y \) are constant on \([0, T]\), so it follows that \( \lambda \) and \( \phi \) are also constant on \([0, T]\). We can then rewrite \( H \) and \( \dot{\psi}_\alpha \) as follows.

\[
H = \lambda v(\alpha) \cos(\alpha - \phi) + v(\alpha) \psi_\alpha u \quad (2.10)
\]
\[
\dot{\psi}_\alpha = \lambda v(\alpha) \sin(\alpha - \phi) - \lambda v'(\alpha) \cos(\alpha - \phi) - v'(\alpha) \psi_\alpha u \quad (2.11)
\]
By the minimum principle (Theorem 2.1), in order for the control \( u \) to be optimal, it is necessary that \( u \) is either a local minimiser of \( H \) or takes the values of its upper/lower bounds. We therefore consider the following two cases:

1. \( \partial H / \partial u \neq 0 \Rightarrow u = \pm 1 \); or
2. \( \partial H / \partial u = v(\alpha) \psi_\alpha = 0 \Rightarrow \psi_\alpha = 0. \)

If (1) holds, we obtain an arc of unit radius. We now show that if (2) holds, then \( u = 0 \) and we obtain a straight line segment. Since \( \psi_\alpha = 0 \), we have \( \dot{\psi}_\alpha = 0 \) as well. Note that \( \lambda \neq 0 \) since \( \lambda = 0 \) leads to \( (\psi_x, \psi_y, \psi_\alpha) = (0, 0, 0) \), which violates the condition that this vector must be non-zero. Rearranging (2.11) then gives the following

\[
\alpha = \begin{cases} 
\phi + \arctan(v'(\alpha)/v(\alpha)) & \text{or} \\
\phi + \arctan(v'(\alpha)/v(\alpha)) + \pi
\end{cases}
\]

Differentiating either of the two possibilities for \( \alpha \) with respect to \( t \) and recalling \( \dot{\alpha} = v(\alpha)u \) gives the following

\[
u(v^2 + 2v'^2 - vv'') = 0 \tag{2.12}
\]

Recall that \( K = v^2 + 2v'^2 - vv'' \) where the sign of \( K \) gives the sign of the curvature of \( v \). When \( K \neq 0 \), it follows that \( u = 0 \). Where we are unable to guarantee \( u = 0 \) is at some time \( t^* \) when \( K(\alpha(t^*)) = 0 \). Since \( v \) is strict, for any \( \alpha^* \) where \( K(\alpha^*) = 0 \), there exists \( \varepsilon > 0 \) such that \( K(\alpha) \neq 0 \) for all \( \alpha \in (\alpha^* - \varepsilon, \alpha^* + \varepsilon) \setminus \{\alpha^*\} \). If \( u(t^*) \neq 0 \), \( \alpha \) is changing when \( t = t^* \) since \( \dot{\alpha} = u \). Hence, there exists \( \delta > 0 \) such that for any \( t \in (t^*, t^* + \delta) \), \( K(\alpha(t)) \neq 0 \). Hence, \( u \) is never non-zero for time intervals of non-zero length. Alternatively, \( u(t^*) = 0 \) would correspond to an \( S \) subpath in the direction \( \alpha(t^*) \). Hence, any optimal path is a CS-path if \( v \) is \( C^2 \) everywhere and there are only a finite number of points where \( K = 0 \).

Suppose that \( v \) is piecewise \( C^2 \) instead of \( C^2 \) everywhere. It follows that there are only a finite number of directions \( \alpha_i \) where \( v(\alpha_i) \) is not twice differentiable. By applying a similar argument as above for the directions where \( K = 0 \), we conclude that any optimal path is a CS-path if \( v \) is a strict velocity function.
2.4 Polarly Convex Velocity Function

In the previous section, we established that any optimal path is a CS-path, given a strict velocity function \( v \). We are interested in establishing the forms of optimal paths for a given \( v \). In [33], Dubins paths are established to be optimal through Euclidean geometric arguments. Here, these arguments are no longer available to us due to the nature of the velocity function. However, if we assume that the velocity function \( v \) is strictly polarly convex, by which we mean that \( v \) is strict and it maps out the boundary of a convex region in \( \mathbb{R}^2 \) (in polar coordinates), then it can be shown that the strict triangle inequality holds, which is sufficient to re-establish the main results of Dubins.

In this section, we show that strict polar convexity of the velocity function implies that optimal curvature-constrained paths are Dubins paths (as defined in Section 2.2.1). However, an example is given to show that a shortest curvature-constrained path is not necessarily an optimal path for a particular choice of start and end directed points, even if the velocity function is strictly polarly convex. In other words, the minimum cost and minimum length Dubins paths do not necessarily coincide. Furthermore, we show that by relaxing the strict convexity to weak convexity, there still always exists a Dubins path which is an optimal path.

The practical implication of these results is that the optimal path can be constructed by computing the cost of the (up to 6) Dubins paths between any two directed points, and selecting the path of lowest cost among the Dubins paths. For the original Dubins problem of constructing a shortest path, there has been work done to exactly determine which of the Dubins paths are optimal based on the relative positions and orientations of directed points, such as in [48] and [60]. Based on their results, the partitioning of the configuration space is non-trivial, and hence, if we consider the extension to their problem where a polarly convex velocity function is introduced, it would be difficult to obtain an elegant partitioning of the configuration space in general.
2.4.1 Polar convexity

A straight path $ab$ is the directed straight line segment from $a$ to $b$. Clearly, the direction along $ab$ is constant and equal to $a_{ab}$, where $a_{ab}$ is the direction from $a$ to $b$. A polygonal path is a path $a_1 \cdots a_{k+1}$ made up of $k$ straight paths $a_1a_2, a_2a_3, \cdots, a_ka_{k+1}$.

Given a velocity function $v$, the velocity set $V$ is the subset of $\mathbb{R}^2$ enclosed by the polar plot of $v$ represented in polar coordinates by

$$V = \{(e(\alpha), \alpha) : e(\alpha) \in [0, v(\alpha)] \ \forall \ \alpha \in [0, 2\pi]\}$$

The velocity set can be interpreted in the context of directional cost as the set of all points that can be reached by a single straight path from the origin at the cost of no greater than one unit. An example of a directional cost function with its corresponding velocity function and velocity set are shown in Figure 2.2.

Let $\text{conv}\,(S)$ denote the convex hull of the set $S$. A velocity function $v$ is

- polarly convex (PC) if $V = \text{conv}\,(V)$;
  - strictly polarly convex (SPC) if it is PC and strict;
  - weakly polarly convex (WPC) if it is PC but not strict;
- polarly non-convex (PNC) if it is not PC.

For example, the velocity function shown in Figure 2.2(b) is PNC since $V \neq \text{conv}\,(V)$.

Let $\mathbb{E}^2 = (\mathbb{R}^2, \| \cdot \|_e)$ where $\| \cdot \|_e$ is the Euclidean norm. Suppose $v$ is symmetric i.e. $v(\alpha) = v(\alpha + \pi)$. We can then let $\| \cdot \|_v$ be a norm on $\mathbb{R}^2$ that has a unit ball $V$. The cost $T(ab)$ of a straight path $ab$ in $\mathbb{E}^2$ subject to a PC function $v$ is the same as the norm of $ab$ in a normed space $\mathbb{R}^2$ equipped with a norm $\| \cdot \|_v$ since $T(ab) = L(ab)/v(a_{ab}) = \|b - a\|_v$. By relaxing the symmetry property of the normed space but retaining the triangle inequality property, it can be seen that the cost of straight paths and polygonal paths subject to any PC velocity function $v$ satisfies the triangle inequality

$$T(ab) \geq T(aeb), \text{ for any } e \in \mathbb{R}^2$$

Similarly, the cost of straight paths and polygonal paths subject to an SPC velocity func-
2.4 Polarly Convex Velocity Function

![Diagram](image-url)

(a) Directional cost function $c(\alpha)$

(b) Corresponding velocity function $v(\alpha)$ and velocity set $V$

Figure 2.2: An example of a directional cost function $c(\alpha)$ is shown in (a) where $O$ is the origin and some points on the function are labelled in polar coordinates. The corresponding velocity function $v(\alpha) = 1/c(\alpha)$ is shown in (b) with the corresponding points labelled again for comparison. The velocity set $V$ is the shaded region enclosed by the velocity function as shown.

It can be shown using basic properties of curvature and convexity of functions that $v(\alpha)$ is PC iff

1. $K(\alpha) \geq 0$ where $v(\alpha)$ is twice differentiable; and
2. $v'(\alpha_i - 0) \geq v'(\alpha_i + 0)$ at the points $\alpha_i$ where $v(\alpha)$ is not twice differentiable

Equivalently, in terms of $c = 1/v$, we have that $v(\alpha)$ is PC iff

1. $c''(\alpha) + c(\alpha) \geq 0$ where $c(\alpha)$ is twice differentiable; and

For any function $f$, let $f(z - 0) = \lim_{x \to z^-} f(x)$ and $f(z + 0) = \lim_{x \to z^+} f(x)$. Recall that $K = v^2 + 2v'^2 - vv''$. 

Property 2.1. It can be shown using basic properties of curvature and convexity of functions that $v(\alpha)$ is PC iff

$$T(aeb) > T(ab) \text{, for any } e \in \mathbb{R}^2 \text{ s.t. } e \notin ab \quad (2.13)$$
2. \( c'(\alpha_i - 0) \leq c'(\alpha_i + 0) \) at the points \( \alpha_i \) where \( c(\alpha) \) is not twice differentiable

We now prove some basic properties of paths subject to an SPC velocity function. Let \( \text{bd}(S) \) denote the boundary of \( S \). In the following, we refer to points in \( \mathbb{R}^2 \) as being represented in polar coordinates. For a given velocity function \( v \), a direction \( \alpha' \) is a convex direction if \( (v(\alpha'), \alpha') \in \text{bd}(\text{conv}(V)) \). A direction \( \alpha^* \) is a concave direction if \( (v(\alpha'), \alpha') \notin \text{bd}(\text{conv}(V)) \).

**Lemma 2.3.** \( v \) is SPC iff the cost of the straight path between any two points is less than the cost of any other path between the two points.

**Proof.** (\( \Leftarrow \)) The contrapositive is that there exists a path between two points that is of lower or equal cost than the straight path between the two points if \( v(\alpha) \) is PNC or WPC. First, suppose that it is PNC. Let \( a \) be the origin, and \( b \) be an arbitrary distance in a concave direction from \( a \). Choose \( e \in \mathbb{R}^2 \) so that \( \alpha_{ae} \) and \( \alpha_{eb} \) are the first convex directions encountered when searched for in the anticlockwise and clockwise directions from \( \alpha_{ab} \) (clearly such a \( e \) exists). Let \( g(\alpha) \) be a PC velocity function such that its velocity set \( G = \text{conv}(V) \). Since \( \alpha_{ab} \) is a concave direction, \( g(\alpha_{ae}) = v(\alpha_{ae}) \) and \( g(\alpha_{eb}) = v(\alpha_{eb}) \) while \( g(\alpha_{ab}) > v(\alpha_{ab}) \). Let \( T_v \) and \( T_g \) denote the costs of paths subjected to velocity functions \( v \) and \( g \) respectively. Since \( \alpha_{ae} \) and \( \alpha_{eb} \) are the first convex directions encountered, it follows that \( g(\alpha) \) is a straight line from \( \alpha_{ae} \) to \( \alpha_{ae} \). Therefore, \( T_g(aeb) = T_g(ab) \) which gives \( T_v(aeb) = T_v(ab) < T_v(ab) \). On the other hand if \( v(\alpha) \) is WPC, we choose \( \alpha_{ab} \) to lie in the interior of an interval where \( K(\alpha) = 0 \). It is then clear that there exists \( e \notin ab \) such that \( T(\alpha_{eb}) = T(ab) \).

(\( \Rightarrow \)) Since \( v \) is SPC, the strict triangle inequality (2.13) holds. Let \( E(t) \) be a path with \( E(0) = a \) and \( E(t_f) = b \) that is not \( ab \). Let \( e = E(t_f/2) \) so if \( e \) lies on \( ab \) then \( T(aeb) = T(ab) \), otherwise \( T(aeb) > T(ab) \). Hence in general, \( T(aeb) \geq T(ab) \). This bisection argument can be repeated infinitely on the new straight paths generated to show that \( T(E) \geq T(ab) \). The first few steps of this procedure are illustrated diagrammatically in Figure 2.3. The convergence of this polygonal path to \( E(t) \) is guaranteed by the properties of Euclidean space. Observe that since \( T(ab) = T(E) \) can only occur if the inequality is an equality at every step, that would imply that \( E = ab \) which is a contradiction and hence, \( T(ab) < T(E) \). \( \square \)
A convex path is a piecewise $C^2$ path that lies on the boundary of its own convex hull. A supporting line to a convex path at a point $e$ along the convex path, is any supporting line of the convex hull of the path at $e$. Given a convex path represented by $E : [0, t_f] \rightarrow \mathbb{R}^2$, an outer normal $n_e$ of a point $e = E(t_e)$ for some $t_e \in (0, 1)$, is the normal unit vector to a supporting line at $e$ that points away from the convex hull. The outer normal also divides the supporting line at $e$ into two halves, referred to as being the supporting line on a particular side of the outer normal. Note that the outer normal is not unique if $e$ is not a differentiable point. For the following arguments, this will not be an issue, as if there are multiple choices of $n_e$, picking any one will work.

A path lies on the outer side of a convex path if the path is formed by continuously deforming the convex path while fixing the endpoints, without entering the convex hull of the convex path, and is not the convex path itself. It follows from these definitions that a supporting line at any point along a convex path $E$ never intersects the interior of the convex hull of $E$ and must always intersect a path that lies on the outer side of $E$ on both sides of the outer normal at that point.
Lemma 2.4. \( v \) is SPC iff the cost of any path lying on the outer side of a convex path is greater than the cost of the convex path.

Proof. (\( \Leftarrow \)) The contrapositive is that there exists a path lying on the outer side of a convex path that is of lower or equal cost than the convex path if \( v \) is PNC or WPC. This follows immediately by applying the proof of Lemma 2.3 in the (\( \Leftarrow \)) direction by taking the straight path to be the convex path.

(\( \Rightarrow \)) Let \( E(t) \) be a convex path with \( E(0) = a \) and \( E(t_f) = b \). Let \( e = E(t_f/2) \) and \( G(e) \) be a supporting line at \( e \). Let \( E_1(t) \) be a path lying on the outer side of \( E(t) \) with \( E_1(0) = a \) and \( E_1(t_f) = b \).

By definition, \( G(e) \) does not intersect the interior of the convex hull of \( E \), but intersects \( E_1 \) at least once on each side of \( n_e \). Pick an intersection point from each side of the supporting line which is closest to \( e \), \( E_1(t_1) \) and \( E_1(t_2) \), with \( t_2 \geq t_1 \) as illustrated in Figure 2.4. Note that in the degenerate case of \( E' \) coinciding locally with \( E \) at \( e \), these two points are the same.

Let \( \hat{E} \) denote the subpath of \( E_1 \) from \( E_1(t_1) \) to \( E_1(t_2) \). By Lemma 2.3, \( T(E_1(t_1)E_1(t_2)) < T(\hat{E}) \) if \( \hat{E} \neq E_1(t_1)E_1(t_2) \). Hence, the cost of the new path \( E_2 \) formed by the subpath of \( E_1 \) from \( a \) to \( E_1(t_1) \), followed by \( E_1(t_1)E_1(t_2) \), and then the subpath of \( E_1 \) from \( E_1(t_2) \) to \( q \), is less than or equal to the cost of \( E_1 \).

This bisection argument can now be repeated on the subpaths of \( E_2 \) lying on the outer side of the subpaths of \( E \) from \( a \) to \( e \) and \( e \) to \( b \). By infinitely iterating this construction, the path converges to a path \( E^* \) which is either the same as \( E \) or contains \( E \) as a subpath, and is of lesser or equal cost to \( E_1 \). Hence, \( T(E_1) \geq T(E) \). The first few steps of this procedure are illustrated diagrammatically in Figure 2.4. The convergence is guaranteed by properties of Euclidean space. Observe that since \( T(E_1) = T(E) \) can only occur if the inequality was satisfied by equality at every step, that would imply that \( E_1 = E \) which is a contradiction and hence, \( T(E_1) > T(E) \).

In terms of lengths in \( E^2 \) or equivalently, if \( v(\alpha) = 1 \), this is a trivial observation, as it can be thought of physically as a string lying above a convex curve being deformed without penetrating the curve, which obviously only ever lengthens the string. This lemma generalises the concept to any arbitrary SPC \( v(\alpha) \).
2.4 Polarly Convex Velocity Function

2.4.2 Optimal forms

Lemma 2.4 gives us the fundamental tool for working with SPC velocity functions. We are now able to identify the forms of optimal paths. Recall that the form of a CS-path is given by a string of $C$ and $S$ labels representing arcs of radius 1 and straights respectively. We let $C$ denote a $C$ arc of length greater than $\pi$ and $\underline{C}$ denote a $C$ arc of length no greater than $\pi$.

By adopting the methodology of Dubins [33], we will show that the optimal forms are Dubins paths ($CS\underline{C}$, $C\underline{C}C$ or degeneracies) by showing that each of the following 4 types of paths can be continuously deformed to a different path of lower cost:

1. $SC\underline{S}$;
2. $CCS$;
3. $CCC$;
4. $CCCC$.

The proofs of Lemma 2.5 and Corollaries 2.6, 2.7 below are extensions of Dubins’
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proofs where \( v(\alpha) = 1 \) to a general SPC \( v(\alpha) \). Note that since Dubins’ arguments use the fact that a subpath of an optimal path must itself be optimal, we are able to arbitrarily shrink the lengths of the first and last arc or line segment of any given path to simplify the proofs.

Given a CS-path, we denote by \( C_k \) a particular subpath of the form \( C \) where \( k \in \mathbb{Z}_+ \) is simply an index to distinguish different subpaths. \( C_k, \overline{C}_k, L_k, \overline{L}_k, R_k, \overline{R}_k, S_k \) are defined similarly. A CS-path can be written as a concatenation of its subpaths traversed in order. A direction plot of a CS-path is a plot in polar coordinates of all the directions traversed during that path, plotted at increasing radii for different \( C \) arcs in the order traversed, with radial line segments representing \( S \) segments. This contains all the information required to compute the total cost of that path, provided the lengths of all \( S \) segments are separately specified. We introduce direction plots to make the ideas in the following proofs clearer.

O denotes the origin in all direction plots.

Given directed points \( p = (x_p, y_p, \alpha_p), q = (x_q, y_q, \alpha_q) \), we let \( pq \) denote a straight path from \( (x_p, y_p) \) to \( (x_q, y_q) \), and likewise for polygonal paths.

**Lemma 2.5.** If \( v \) is SPC, then any SCS path is not optimal.

**Proof.** To prove the result, it needs to be shown that a path of the form SCS can be improved. There are only two possible cases to consider, depending on the length of the \( C \) arc.

1. **SCS**

   Without loss of generality, consider a path \( S_1R_2S_3 \) between directed points \( p \) and \( q \) as shown in Figure 2.5. Let the \( RSR \) path from \( p \) to \( q \) be \( R_4S_5R_6 \). It follows by Lemma 2.4 that \( T(S_1R_2S_3) > T(R_4S_5R_6) \) since \( R_4S_5R_6 \) is a convex path with \( S_1R_2S_3 \) lying on its outer side.

2. **SCS**

   Without loss of generality, consider a path \( S_1\overline{R}_2S_3 \) between directed points \( p \) and \( q \) with sufficiently short \( S \) segments of appropriate lengths so that \( R_4L_6 \), the \( R\overline{L} \) path from \( p \) to \( q \), exists as shown in Figure 2.6. Let \( S_1 \) be \( \hat{p}\overline{p} \) and \( S_3 \) be \( \hat{q}q \). Since, \( L(R_4) > L(\overline{R}_2) \), the subpath \( R_5L_6 \) from \( r \) to \( q \), where \( L(R_5) = L(R_4) - L(\overline{R}_2) \), exists. Since translation of a path does not affect its cost, the result is proven if
2.4 Polarly Convex Velocity Function

\[ T(R_5) + T(L_6) < T(S_1) + T(S_3) \] — see Figure 2.7. Note that since the direction at \( r \) and \( q \) along \( R_5L_6 \) are both \( \alpha_{\hat{qq}} \) it follows that \( L(R_5) = L(L_6) \). It then follows by symmetry that \( rq \) intersects \( R_5L_6 \) at \( s \), where \( R_5 \) ends at \( s \) and \( L_6 \) starts at \( s \). Let the direction of \( R_5L_6 \) at \( s \) be \( \alpha_s \). From the direction plot of \( R_5L_6 \) shown in Figure 2.8, it is clear that \( T(R_5) = T(L_6) \) since \( R_5 \) and \( L_6 \) both traverse the same set of directions. Let \( t \) be the midpoint of \( \alpha_s \). By translating \( S_1 \) so that it now begins at \( r \), a polygonal path \( \hat{qq} \) is formed where \( T(\hat{qq}) = 2T(stq) \) by similar triangles. By applying Lemma 2.4 to the convex arc \( L_6 \), it follows that \( T(L_6) < T(stq) \). The result then follows since \( T(R_5) + T(L_6) = 2T(L_6) < 2T(stq) = T(r\hat{qq}) = T(S_1) + T(S_3) \).

Corollary 2.6. If \( v \) is SPC, then any CCS path is not optimal.

Proof. This result follows from applying similar arguments to those in Lemma 2.5. If the path is of the form \( CCS \), the argument for the \( SCS \) case can be applied by taking a sufficiently small subpath of the \( CCS \) path. Otherwise, consider a path \( L_1R_2S_3 \) and apply the argument for the \( SCS \) case, replacing \( S_1 \) with \( L_1 \). □

Corollary 2.7. If \( v \) is SPC, then any CCC path is not optimal.

Proof. This result follows from similar applying arguments to those in Lemma 2.5 for the \( SCS \) case by taking a sufficiently small subpath of the \( CCC \) path. □

Lemma 2.8. If \( v \) is SPC, then any CCCC path is not optimal.

Proof. From Corollary 2.7, only \( CCCC \) cases need to be considered. Without loss of generality, consider \( LRLR \) paths from \( p \) to \( q \). We will refer to the polygonal path \( abcd \) as a valid encoding if it corresponds to the centres of the circles corresponding to the arcs of a \( LRLR \) path from \( p \) to \( q \) traversed in order. Observe that there is no unique \( LRLR \) path from \( p \) to \( q \), because even though \( a \) and \( d \) are fixed, the positions of \( b \) and \( c \) have one degree of freedom. Consider a path \( E_0 = L_1R_2L_3R_4 \) with \( b = b_0 \) and \( c = c_0 \) as shown in Figure 2.9. By translation, rotation, reflection, and reversing of the path, and corresponding rotation and possibly reflection of \( v(a) \), we can assume that \( \alpha_{bc} = 0 \), \( a = (0,0) \), \( d = (2h,2k) \) and \( \alpha_{abc} \geq \alpha_{cad} \).
Figure 2.5: $SCS$ case: $S_1R_2S_3$ shown in lighter grey, $R_4S_5R_6$ shown in darker grey

Figure 2.6: $SCS$ case: $S_1R_2S_3$ shown in lighter grey, $R_4L_6$ shown in darker grey

Figure 2.7: The $SCS$ case: Enlarged diagram with $S_1S_3$ shown in lighter grey, $R_5L_6$ shown in darker grey

Figure 2.8: The $SCS$ case: Direction plot of $R_5L_6$ showing $R_5$ and $L_6$ traversing the same directions
2.4 Polarly Convex Velocity Function

We first consider the case when \( \alpha_{ab} \neq \alpha_{cd} \) so \( \alpha_{ab} > \alpha_{cd} \) and look at the effect of a positive increase on \( \alpha_{ab} \). It can be seen that any valid encoding \( abcd \) must satisfy the conditions in (2.14)

\[
\begin{align*}
\cos \alpha_{ab} + \cos \alpha_{bc} + \cos \alpha_{cd} &= h \\
\sin \alpha_{ab} + \sin \alpha_{bc} + \sin \alpha_{cd} &= k
\end{align*}
\] (2.14)

Since we are interested in how \( \alpha_{bc} \) and \( \alpha_{cd} \) change as a result of a positive increase in \( \alpha_{ab} \), we implicitly differentiate (2.14) to obtain (2.15), where \( d\alpha_{ab} > 0 \).

\[
\begin{align*}
d\alpha_{ab} \sin \alpha_{ab} + d\alpha_{bc} \sin \alpha_{bc} + d\alpha_{cd} \sin \alpha_{cd} &= 0 \\
d\alpha_{ab} \cos \alpha_{ab} + d\alpha_{bc} \cos \alpha_{bc} + d\alpha_{cd} \cos \alpha_{cd} &= 0
\end{align*}
\] (2.15)

When \( b = b_0, c = c_0 \), (2.15) yields (2.16) and (2.17).

\[
\begin{align*}
d\alpha_{ab} \sin \alpha_{ab} + d\alpha_{cd} \sin \alpha_{cd} &= 0 \\
\Rightarrow \quad d\alpha_{cd} &= -\frac{\sin \alpha_{ab} \sin \alpha_{cd}}{\sin \alpha_{cd}} d\alpha_{ab} \\
\end{align*}
\] (2.16)

\[
\begin{align*}
d\alpha_{ab} \cos \alpha_{ab} + d\alpha_{bc} + d\alpha_{cd} \cos \alpha_{cd} &= 0 \\
\Rightarrow \quad d\alpha_{bc} \cos \alpha_{ab} + d\alpha_{bc} - \cot \alpha_{cd} \sin \alpha_{ab} d\alpha_{ab} &= 0 \\
\Rightarrow \quad d\alpha_{bc} &= \sin \alpha_{ab} (\cot \alpha_{cd} - \cot \alpha_{ab}) d\alpha_{ab} \\
\] (2.17)

First, observe from (2.16) that since \( \alpha_{ab} \), \( \alpha_{cd} \in (0, \pi) \) and \( d\alpha_{ab} > 0 \), we have that \( d\alpha_{cd} < 0 \). Next, since cot is strictly monotonically decreasing on \( (0, \pi) \) and \( \alpha_{ab} > \alpha_{cd} \), it follows from (2.17) that \( d\alpha_{bc} > 0 \). Note that the conditions for this argument to hold are that \( \alpha_{ab} - \alpha_{bc}, \alpha_{cd} - \alpha_{bc} \in (0, \pi) \) and \( \alpha_{ab} - \alpha_{bc} > \alpha_{cd} - \alpha_{bc} \). Also, if it is met for some \( b_1, c_1 \), then it is true for some \( b_2, c_2 \) where \( \alpha_{ab_2} = \alpha_{ab_1} + \delta \) and \( \delta > 0 \) since \( d\alpha_{ab} > 0 \) and \( d\alpha_{cd} < 0 \).

Since \( \text{LR}_1L_2L_3R_4 \) has non-degenerate components, \( \exists \gamma_{ab} > 0 \) such that \( ab\gamma c_2d \) is a valid encoding, where \( \alpha_{ab\gamma} = \alpha_{ab} + \gamma_{ab} \) as shown in Figure 2.10. Let the path corresponding to the valid encoding \( ab\gamma c_2d \) be \( E_{\gamma} = L_5R_6L_7R_8 \). Since the conditions are met when \( b = b_0 \) and \( c = c_0 \), it then follows that \( \gamma_{ab}, \gamma_{bc} > 0 \) and \( \gamma_{cd} < 0 \) where \( \gamma_{bc} = \alpha_{b_1} - \alpha_{b_0} \) and \( \gamma_{cd} = \alpha_{c_1} - \alpha_{c_0} \).
Figure 2.9: $E_0 = L_1 \overline{R}_2 \overline{L}_3 R_4$ with corresponding valid encoding $ab_0c_0d$
2.4 Polarly Convex Velocity Function

Figure 2.10: $E_\gamma = L_5R_6\overline{L}_7R_8$ with corresponding valid encoding $ab_\gamma c_\gamma d$

Figure 2.11: Direction plot of $E_0$

Figure 2.12: Direction plot of $E_\gamma$
Now we need to show that $T(E_γ) < T(E_0)$. Let $L_{i,j}$ represent $L_i \setminus L_j$ and $R_{i,j}$ represent $R_i \setminus R_j$. Note that $L_{3,7}$ contains two disjoint $L$ arcs as shown in Figure 2.14. By comparing the direction plots shown in Figures 2.11, 2.12 and 2.13, it can be seen that what needs to be shown is $T(L_{5,1}) + T(R_{6,2}) < T(R_{2,6}) + T(L_{3,7}) + T(R_{4,8})$ as shown in Figure 2.14. It is also clear from the direction plot that $T(R_{6,2}) = T(L_{5,1})$ and $T(R_{2,6}) + T(R_{4,8}) = T(L_{3,7})$. Since translating paths do not change their costs, it can be seen from Figure 2.14 that Lemma 2.4 can now be applied to obtain $T(L_{5,1}) < T(L_{3,7})$, which gives $T(E_γ) < T(E_0)$.

Angle $α_{ab}$ can continue to be increased until either $R_4$ becomes degenerate or $R_2$ or $L_3$ traverses an angle of $π$.

Clearly, it follows that deforming the path in the opposite direction which deforms the path towards achieving $α_{ab} = α_{cd}$ never decreases the cost. Hence, the configuration where $α_{ab} = α_{cd}$ is a local maximum among $CCCC$ paths with fixed endpoints, by symmetry.

The case where $ν(α) = 1$ was proven by Dubins [33] and Boissonnat et al. [11] using different techniques. However, both proofs were not suitable for extending to the SPC $ν$ case because there was no easy way to make use of the strict triangle inequality. The proof above makes use of the strict triangle inequality to extend the result to a general SPC $ν$ case.

Finally, putting all of the results together gives the following theorem.

**Theorem 2.9.** If $ν$ is SPC, then any optimal path is a Dubins path.

In [32], a result similar to Theorem 2.9 was obtained using a similar application of Pontryagin’s Minimum Principle but different techniques in establishing the optimal path forms. We can now apply Theorem 2.9, to obtain the following result for weakly polarly convex (WPC) functions.

**Corollary 2.10.** If $ν$ is WPC, then an optimal curvature-constrained directional cost path can be found by considering only Dubins paths.

**Proof.** Consider a WPC velocity function $ν$, where $c = 1/ν$. Define a new velocity function $ν_ε = 1/c_ε$ by $c_ε = c + ε$. Using Property 2.1, we are able to show that $ν_ε$ is SPC for any
2.4 Polarly Convex Velocity Function

\[ \alpha_{c,d} + \frac{\pi}{2} \]

\[ \alpha_{b,c} + \frac{\pi}{2} \]

\[ \gamma_{ab} \]

\[ \gamma_{ac} \]

\[ \gamma_{bc} \]

\[ \gamma_{cd} \]

\[ \alpha_{b,c} - \frac{\pi}{2} \]

\[ \alpha_{b,c} - \frac{\pi}{2} \]

\[ \alpha_{b,c} - \frac{\pi}{2} \]

\[ \alpha_{b,c} - \frac{\pi}{2} \]

Figure 2.13: Direction plot of the difference between \( E_0 \) and \( E_\gamma \)

Figure 2.14: Difference between \( E_0 \) and \( E_\gamma \)
Generalisation of the Dubins Result

\( \varepsilon > 0 \) by using the fact that \( v \) is WPC as follows. Clearly \( c \) and \( c_\varepsilon \) are twice differentiable on the same intervals of \( \alpha \), and where they are twice differentiable,

\[ c''_\varepsilon + c_\varepsilon = c'' + c + \varepsilon \geq \varepsilon > 0 \]

The second condition obviously holds since \( c'(\alpha) = c'_\varepsilon(\alpha) \). This shows that \( v_\varepsilon \) is PC and strict, and hence is SPC. Let \( T(E) \) and \( T_\varepsilon(E) \) denote the costs of path \( E \) with respect to the directional cost functions \( c \) and \( c_\varepsilon \) respectively. Let \( p, q \) be any arbitrary start and end directed points. Let \( \mathbb{D}_{pq} \) denote the set of all Dubins paths from \( p \) to \( q \) as described in Section 2.2, and recall that \( \mathbb{P}_{pq} \) is the set of all curvature-constrained paths from \( p \) to \( q \).

Suppose \( \exists E \in \mathbb{P}_{pq} \setminus \mathbb{D}_{pq} \) such that \( T(E) < T(D) \forall D \in \mathbb{D}_{pq} \). Then,

\[
T_\varepsilon(E) = T(E) + \varepsilon L(E) < T(D), \text{ by choosing } \varepsilon \in (0, (T(D) - T(E))/(L(E))) < T_\varepsilon(D), \forall D \in \mathbb{D}_{pq}
\]

However, Theorem 2.9 states that there cannot exist such an \( E \) since \( v_\varepsilon \) is SPC and \( E \notin \mathbb{D}_{pq} \). Hence, by contradiction, we know that there cannot exist a non-Dubins path which is of less cost than the Dubins paths, for any WPC velocity function. \( \square \)

From Theorem 2.9, it follows that given directed points \( p, q \), the minimum length curvature-constrained path and the optimal curvature-constrained directional cost path are both Dubins paths if \( v \) is SPC. However, Example 2.1 illustrates that they do not have to be the same path.

**Example 2.1.** Let \( v(\alpha) \) be defined as in (2.18) (shown in Figure 2.15).

\[
v(\alpha) = \begin{cases} 
\frac{4}{\sqrt{\pi}} \alpha^3 + 1 & , \quad \alpha \in [0, \frac{3\pi}{8}] \\
\frac{4}{\sqrt{\pi}} \left(\frac{3\pi}{4} - \alpha\right)^3 + 1 & , \quad \alpha \in \left(\frac{3\pi}{8}, \frac{3\pi}{4}\right) \\
1 & , \quad \alpha \in \left(\frac{3\pi}{4}, 2\pi\right)
\end{cases} \quad (2.18)
\]
By Property 2.1, it is easily checked that \( v(\alpha) \) is SPC. Let \( p = (0, 0, \pi) \) and \( q = (-2\sqrt{2}, 0, 0) \) be the start and end directed points respectively. The resulting paths \( L_1S_2R_3 \) and \( R_4S_5L_6 \) are of equal length as shown in Figure 2.16. However, the direction plots shown in Figures 2.17 and 2.18 illustrate that they traverse different directions. Hence, \( T(L_1S_2R_3) < L(L_1S_2R_3) = L(R_4S_5L_6) = T(R_4S_5L_6) \). Clearly, there exists \( \epsilon > 0 \) such that the shortest curvature-constrained path from \( \hat{p} = (0, \epsilon, \pi) \) to \( q \) is an \( RSL \) path while the optimal path from \( \hat{p} \) to \( q \) is an \( LSR \) path.

2.5 Conclusion

The mathematical problem studied was motivated by the effects of directional faulting on development cost of underground mine declines. In particular, this meant extending Dubins [33] result of minimal length paths, to incorporate a directional cost element. It was shown that if the velocity function is strictly polarly convex, any optimal path (minimum cost) is a Dubins path. If the velocity function is polarly convex, then there exists an optimal path which is a Dubins path. The results proved in this chapter lay the foundation for future work developing the theory necessary for the design of underground mines taking into consideration anisotropic development and support costs. It has also been seen that it is a useful problem to consider for other practical applications such as naval path planning [32].

From a theoretical viewpoint, this result provides a more general context for the Dubins result, in that Dubins paths are actually optimal paths for a much more general problem where the velocity depends on the direction, provided that the velocity function is polarly convex.

This chapter is the first step in developing the theory necessary to implement an efficient algorithm for constructing optimal underground mine network designs in anisotropic ground conditions. The most immediate question which has not been addressed in this chapter is establishing the forms of optimal paths when subject to a polarly non-convex velocity function. In order to construct feasible paths for underground mine networks, the problem of lifting these planar paths into 3-dimensional space while satisfying a gradient constraint such as in [15], needs to be studied. The results of these extensions are
Figure 2.15: $v(\alpha)$ as specified in (2.18)

Figure 2.16: $L_1S_2R_3$ and $R_4S_5L_6$

Figure 2.17: Direction plot of $L_1S_2R_3$

Figure 2.18: Direction plot of $R_4S_5L_6$
2.5 Conclusion

presented in the following chapters.
Chapter 3
Optimal Forms for General Anisotropic Cost

In this chapter, we extend the results from Chapter 2 to establish the optimal forms for general anisotropic cost functions. We show that there always exists a path of the form CSCSC or a degeneracy which is optimal, where C denotes a continuous subset of the unit circle, and S represents a straight line segment. This result is also extended to the case where there is not only a directional cost, but the cost of curved sections are scaled up by a given factor $w^C \geq 1$. The results obtained in this chapter are useful for underground mine design since polarly convex cost functions are insufficient to model practical anisotropic support cost behaviour and curved sections may incur more cost due to additional support and ventilation.

3.1 Introduction

In Chapter 2, we introduced the problem of finding a minimum cost curvature-constrained path between two directed points in $\mathbb{R}^2$ where the cost at every point along the path depends on the instantaneous direction. As mentioned, the principal motivation for studying this problem stems from the problem of designing a minimum cost underground transport tunnel, such as a haulage tunnel in an underground mine, in the presence of directional faulting in the ground. The problem can be stated in short as follows:

$$\min_{E \in \mathcal{P}_{pq}} \int_E c(\alpha) \, ds$$

for some given directional cost function $c(\alpha)$, where $\mathcal{P}_{pq}$ denotes the set of all curvature-constrained paths between two directed points $p$ and $q$. 
The forms of optimal paths were shown to be the same as Dubins paths [33] in Chapter 2 (also see [28]) and by Dolinskaya in [32] where the velocity function (reciprocal of the directional cost function) is strictly polarly convex. It was also shown in Chapter 2 that if the velocity function is weakly polarly convex, there exists a Dubins path which is optimal. This chapter further generalises the problem by establishing the forms of optimal paths when the velocity function is polarly non-convex.

The main result we prove in this chapter is that there exists a curvature-constrained path of the form $CSCSC$ which is an optimal path, for any continuous piecewise $C^2$ $2\pi$-periodic positive directional cost function, where $C$ denotes a continuous subset of the unit circle, and $S$ represents a straight line. This is first proved for a slightly restricted set of cost functions (whose reciprocals are strict) in Theorem 3.17, and then extended to the more general statement above in Corollary 3.18. The strategy employed in proving this result is to make use of the CS-path deformations introduced in Section 3.3. The basic idea is that if the path has sufficient flexibility, we are able to deform it into a different curvature-constrained path between the same directed points. If the deformation can be performed in both directions (extension and contraction), the effect on the total cost of the path is either strictly negative in one of the two directions, or the deformation can be performed until the path reduces to a simpler form which is no longer flexible enough to reduce further. For some cases, we also need to consider deformations that can only be performed in the extension direction. We also demonstrate how the techniques developed in this chapter can be used to provide an alternative proof to Theorem 2.9 from Chapter 2.

Finally, we consider the case where we also impose a scaling factor $w^C \geq 1$ on the cost of curved sections, and show that the main result still holds with very simple modifications to the original arguments. This particular extension is motivated by the underground mine development application, where curved sections may incur a greater cost in terms of development as well as increased ventilation costs due to shock losses around bends [47].

The main result in this chapter will form the basis for the next chapter which focuses on designing an efficient algorithm which can be implemented into the existing software.
so that the model reflects the true ground conditions more realistically. The major results proven in this chapter have been published in Optimisation and Engineering [29].

3.2 Background

The background introduced in Sections 2.2.1 and 2.2.3 for curvature-constrained paths and directional cost is important for this chapter. We also refer the reader to Sections 2.3 and 2.4.1 for definitions of $K(\alpha)$, strict and weak velocity functions, and velocity sets.

Recall from Lemma 2.2 in Chapter 2 that if the velocity function is strict, every optimal curvature-constrained path is a CS-path. We will need this the result in Section 3.4.

3.3 CS-Path Deformations

In this section, we will introduce deformations to CS-paths which will be used for proving the results in Section 3.4.

3.3.1 Reversible and irreversible deformations

Let $E$ be a CS-path from $p$ to $q$. Let $r$ be a point along a $C$ subpath of $E$, with direction $\alpha_r$ (see Figure 3.1). We can form a new CS-path $E'$ by fixing the path from $p$ to $r$, introducing an $S$ subpath of length $\delta$ in the direction $\alpha_r$, and then translating the remainder of the original path (from $r$ to $q$) by $\delta$ in the direction $\alpha_r$ so that the new path ends up at $q'$. We refer to this action as an irreversible extension. Clearly, the resulting position of $q'$ is a distance of $\delta$ away from $q$, in the direction $\alpha_r$. This action is irreversible because it is not possible to deform the original $C$ subpath so that $q'$ ends up translated by $\delta$ in the direction $\alpha_r + \pi$.

On the other hand, if $r$ is a point on an $S$ subpath of $E$, then this action is equivalent to extending the $S$ subpath as shown in Figure 3.2. Furthermore, $r$ being a point on an $S$ subpath means that we could contract the length of that $S$ subpath. The resulting endpoints of the paths after extending and contracting are shown as $q'$ and $q''$ respectively. Since this action can performed as both an extension and a contraction, we will refer to
this as a *reversible deformation*. Note that the contraction has exactly the opposite effect on $q$ as the extension.

We also consider reversible deformations taking place at inflection points in $\mathcal{CC}$ subpaths. Let $r$ be an inflection point in an $\mathcal{RL}$ subpath as shown in Figure 3.3. We can form a new CS-path by fixing the path from $p$ to $r$, then introducing an $\mathcal{L}$ arc of length $2\gamma$ starting with direction $\alpha_r$, followed by an $\mathcal{R}$ arc of length $2\gamma$ starting with direction $\alpha_r + 2\gamma$ (see Figure 3.3(b)). The new path ends up at $q'$, a directed point with the same direction as $q$, but translated $4\sin\gamma$ away from $q$, in the direction $\alpha_r + \gamma$. We can perform an analogous contraction by shortening the existing $\mathcal{LR}$ subpath starting from $r$, until both arcs are $2\gamma$ shorter. The contraction to the same original path is shown separately in Figure 3.3(c) for clarity. Observe that the effect on the resulting position of $q'$ is equal and opposite in these two deformations. By considering the mirror images of these figures, we get the corresponding actions for the situation where, instead, $r$ is an inflection point in an $\mathcal{LR}$ subpath. Note that the reversibility of these deformations at an inflection point refers to the symmetry of the effect on the translation of the point $q'$. However, in terms of the forms of the resulting paths, a contraction does not change the form of the path while extending introduces two more $\mathcal{C}$ arcs (see for example Figure 3.4). This is important to take note of, as the extension we introduced is not the same as simply increasing the lengths of the existing arcs at the inflection point.

Observe that it is the properties of the path at $r$, rather than the location of $r$, that are relevant when considering the effect on the resulting position $q'$. The types of deformations that can be performed only depend on $\alpha_r$, $\delta$ (or $\gamma$) and whether $r$ was a point on a $\mathcal{C}$, $\mathcal{S}$ or the inflection point of a $\mathcal{CC}$. Furthermore, there are only a finite number of straight line segments and inflection points on a given CS-path. Hence, there are only a finite number of essentially different choices for reversible deformations, whereas irreversible extensions can be performed in any direction $\alpha$, provided there exists a point $r$ which has direction $\alpha$. Note that the size of these deformations were greatly exaggerated in Figures 3.1 - 3.3 for clarity. We now formalise the previous discussion as follows.
3.3 CS-Path Deformations

Figure 3.1: Irreversible extension at a point $r$ along a $C$ subpath by introducing an $S$ subpath of length $\delta$. The resulting CS-path ends at $q'$ which is a distance $\delta$ away from $q$, in the direction $\alpha_r$.

Figure 3.2: Reversible deformation at a point $r$ along an $S$ subpath. The original subpath ends at $q$. The extension can be seen by considering the path ending at $q'$, while the contraction can be seen by considering the path ending at $q''$.

Figure 3.3: The original path from $p$ to $q$ with an inflection point at $r$ is shown in (a). This path is deformed reversibly as illustrated in (b) and (c). In (b), two arcs of length $2\gamma$ are added at $r$, while in (c), the existing two arcs are shortened by $2\gamma$ each, around $r$. 
3.3.2 Reversible directions

The degree $d$ of a CS-path is the number of straight line segments and inflection points between two consecutive arcs in the path. Each of these represents a distinct interval or point at which a reversible CS-path deformation can be performed. A CS-path is reducible if it can be deformed to a CS-path of strictly lower degree with cost no greater than the original CS-path.

**Lemma 3.1.** The degree $d$ can be expressed as $d = C + S_{\text{end}} - 1$ where $C$ is the number of $C$ labels in the CS-path and $S_{\text{end}}$ is the number of $S$ labels in the multi-set consisting of the first and last labels of the CS-path.

**Proof.** Suppose the CS-path begins and ends with label $C$. Observe that in between every consecutive $C$ label in the CS-path, there is exactly one of either an inflection point, or a straight line segment. Hence, in this case, $d = C - 1$. For a general CS-path, it is clear that having a first label or last label as $S$ increases the number of straights by one each, and hence $d = C + S_{\text{end}} - 1$.

$\square$

Note that if the CS-path is composed of only an $S$, then both its first and last labels are $S$, and hence $S_{\text{end}} = 2$.

**Definition 3.1.** Given a CS-path with degree $d$, let $\alpha_i$ for $i = 1, \cdots, d$ denote the directions of the straights and inflection points in the order traversed by the path. We define $A_i = (\alpha_i, \rho_i)$ for $i = 1, \cdots, d$ to be reversible directions, where

$$
\rho_i = \begin{cases} 
-1 & \text{if it corresponds to the inflection point in an } L'R \text{ subpath} \\
0 & \text{if it corresponds to a straight } S \text{ subpath} \\
1 & \text{if it corresponds to the inflection point in an } R'L \text{ subpath}
\end{cases}
$$

Given a CS-path, we denote by $C_k$ a particular subpath of the form $C$ where $k \in \mathbb{Z}_+$ is simply an index to distinguish different subpaths. $L_k, R_k, S_k$ are defined similarly.

Consider a CS-path $E$ from $p$ to $q$ which has a reversible direction $A_i = (\alpha_i, \rho_i)$. We can create a new CS-path $E'$ which starts from $p$ and ends at $q'$ by performing a number of
small reversible deformations. In the following, \( \gamma_i \) is a parameter controlling the size of a reversible deformation. We will consistently use the following notation in the chapter for brevity:

- \( \delta_i = 4 \sin \gamma_i \);
- \( \theta_i = \alpha_i + \rho_i |\gamma_i| \);
- \( \vec{\xi}_i = (\cos \theta_i, \sin \theta_i) \).

Given a reversible direction \((\alpha_i, \rho_i)\), there exists \( \varepsilon > 0 \) such that for \( \gamma_i \in (-\varepsilon, \varepsilon) \) we can find a CS-path from \( p \) to the directed point \( q + \delta_i(\vec{\xi}_i, 0) \). Let \( c(\alpha) \) denote the given directional cost function, and recall the definition of the cost of a path in Section 2.2.3. If \( \rho_i = 0 \), the resulting change in cost is \( \delta_i c(\alpha_i) \), otherwise it is \( 2 \int_0^{2\pi} c(\alpha_i + 2\rho_i |\phi|) d\phi \). Note that \( \gamma_i > 0 \) (extension) leads to an increase in cost, while \( \gamma_i < 0 \) (contraction) leads to a decrease in cost.

Given a point on a path \( E \) with a corresponding reversible direction \( A_i \), applying a deformation at that point for which \( \gamma_i > 0 \) will be called extending along \( A_i \) while applying a deformation with \( \gamma_i < 0 \) will be called contracting along \( A_i \). Note that in Section 3.3.1, \( \delta \) and \( \gamma \) were introduced as positive quantities. We now let \( \delta_i < 0 \) and \( \gamma_i < 0 \) represent the corresponding contractions of length \( |\delta_i| \) and \( |\gamma_i| \), but continue to label distances and angles on diagrams with \( \delta_i \) and \( \gamma_i \) even when they take negative values, since it is now clear what they represent.

A direction plot of a CS-path is a polar plot of all the directions traversed along that path, plotted at increasing radii for different \( C \) arcs in the order traversed, with radial line segments representing \( S \) segments. This contains all the information required to compute the total cost of that path, provided the lengths of all \( S \) segments are separately specified.

The example depicted in Figure 3.4 demonstrates the purpose of considering reversible directions. The reversibility of the deformations is explicitly illustrated in Figures 3.4(b) and 3.4(c). The direction plot of the CCSCC path from Figure 3.4(a) shown in Figure 3.5, also shows the direction plot of the contracted path from Figure 3.4(c) by considering only the lighter grey arrows. In Figure 3.6, \( \gamma_1 \) and \( \gamma_3 \) have been exaggerated to clearly illustrate the geometric meaning of \( \gamma_i \) and \( \theta_i \). In this example, \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are chosen
such that the net result of the deformation results in \( q' = q \). This will be one of the main strategies employed in applying reversible directions to formulate arguments in the next section.

For brevity, we say that a set of directions spans \( \mathbb{R}^2 \) if their corresponding unit vectors span \( \mathbb{R}^2 \). Likewise, a set of directions positively span \( \mathbb{R}^2 \) if their corresponding unit vectors positively span \( \mathbb{R}^2 \). In order to identify what the appropriate combination of CS-path deformations to consider are, we need to consider whether a set of reversible directions positively spans \( \mathbb{R}^2 \). For most cases, a set of reversible directions \( A_1, \ldots, A_n \) positively spans \( \mathbb{R}^2 \) if and only if \( \alpha_1, \ldots, \alpha_n \) positively spans \( \mathbb{R}^2 \). However, this is not always the case as the extension or contraction of a reversible direction \( A_i \), where \( \rho_i \neq 0 \), is not exactly in the direction \( \alpha_i \) even for an arbitrarily small deformation. This is why we need to formally define positive spanning for reversible directions as follows:

**Definition 3.2.** A set of reversible directions \( A_1, \ldots, A_n \)

- positively span \( \mathbb{R}^2 \) if there exists \( \epsilon > 0 \) such that for all \( \eta_1, \ldots, \eta_n \in (0, \epsilon) \), \( \alpha_1 + \rho_1 \eta_1, \ldots, \alpha_n + \rho_n \eta_n \) positively span \( \mathbb{R}^2 \).
- lie in an open half-space of \( \mathbb{R}^2 \) if there exists \( \epsilon > 0 \) such that for all \( \eta_1, \ldots, \eta_n \in (0, \epsilon) \), \( \alpha_1 + \rho_1 \eta_1, \ldots, \alpha_n + \rho_n \eta_n \) lie in an open half-space of \( \mathbb{R}^2 \).

Clearly, any set of reversible directions satisfies at most one of these two properties. A pair of reversible directions \( A_i, A_j \) \( (i \neq j) \) are

- repeated if \( A_i = A_j \) \( (i.e. \, \alpha_i = \alpha_j \) and \( \rho_i = \rho_j \)).
- opposite if \( \alpha_i = \alpha_j + \pi \) and \( \rho_i = \rho_j \).

In a set of reversible directions, the reversible directions are said to be distinct if there are no repeated reversible directions in the set.

It is clear by this definition that if each pair of directions \( \alpha_j, \alpha_k \) are linearly independent for \( j, k = 1, \ldots, n \) and \( j \neq k \), then \( A_1, \ldots, A_n \) positively span \( \mathbb{R}^2 \) if and only if \( \alpha_1, \ldots, \alpha_n \) positively span \( \mathbb{R}^2 \) and likewise for lying in an open half-space of \( \mathbb{R}^2 \). However, the following example illustrates the difference when a pair of directions are linearly dependent.

**Example 3.1.** Consider a CS-path with reversible directions \( A_i = (\alpha_i, \rho_i) \) for \( i = 1, 2, 3 \), where:
Figure 3.4: Example of reversible deformations performed on a $CCSCC$ path from $p$ to $q$ while keeping the endpoint as $q$. The 3 darker grey portions of the original path in (a) are replicated in (b) and removed in (c), as a result of the respective extending and contracting along the 3 reversible directions.
Figure 3.5: Direction plot of \( CCSCC \) path from Figure 3.4(a)

Figure 3.6: Schematic diagram of components from Figure 3.4(a)
3.3 CS-Path Deformations

- $A_1 = (\pi, 0)$;
- $A_2 = (\pi/2, 0)$;
- $\alpha_3 = 0$.

It is clear that $\alpha_1, \alpha_2, \alpha_3$ do not positively span $\mathbb{R}^2$ nor lie in an open half-space of $\mathbb{R}^2$ and $\alpha_1, \alpha_3$ are linearly dependent. However, by Definition 3.2,

- if $\rho_3 = -1$, $A_1, A_2, A_3$ positively span $\mathbb{R}^2$.
- if $\rho_3 = 0$, $A_1$ and $A_3$ are opposite.
- if $\rho_3 = 1$, $A_1, A_2, A_3$ lie in an open half-space of $\mathbb{R}^2$.

Example 3.1 shows that for some cases, knowing $\alpha_1, \ldots, \alpha_n$ alone without information of $\rho_1, \ldots, \rho_n$ is insufficient to tell us exactly whether or not $A_1, \ldots, A_n$ positively span $\mathbb{R}^2$, lie in an open half-space of $\mathbb{R}^2$, or contain a pair of repeated or opposite reversible directions. We will show later in Section 3.4 that knowing which of these properties hold will be important for determining what combination of deformations we should consider in order to show that the given path is not optimal, or can be reduced to a simpler form.

In the example illustrated by Figure 3.4, there exist $\gamma_1, \gamma_2, \gamma_3$ such that the deformed paths are distinct from original path but $q' = q$. We need to show that this is true given any 3 reversible directions, since our arguments rely on the existence of such values. Furthermore, we need to show that we can construct this new path arbitrarily close to the original path. The following lemmas and corollaries in this section will be prove fundamental results relating to combinations of reversible deformations where $q' = q$.

These results will be used in Section 3.4 for showing that certain types of paths are not optimal, or can be reduced to a simpler form.

**Lemma 3.2.** Let $A_i, A_j, A_k$ be three reversible directions of a CS-path from $p$ to $q$. Given any $\varepsilon > 0$, there exist $\gamma_i, \gamma_j, \gamma_k \in (-\varepsilon, \varepsilon)$ such that the resulting path from the corresponding deformations is distinct from the original but $q' = q$.

**Proof.** Recall that $\delta_i = 4 \sin \gamma_i, \theta_i = \alpha_i + \rho_i |\gamma_i|$ and $\tilde{\xi}_i = (\cos \theta_i, \sin \theta_i)$.

Since all reversible deformations preserve direction, the resulting path terminates at $q$ if $\gamma_i, \gamma_j, \gamma_k$ satisfy the following condition.
\[ \delta_i \vec{\xi}_i + \delta_j \vec{\xi}_j + \delta_k \vec{\xi}_k = 0 \] (3.1)

If there exists a pair of repeated reversible directions, without loss of generality let them be \( A_i \) and \( A_j \). Since there exists \( \epsilon_1 > 0 \) such that the deformations are valid for \( \gamma_i, \gamma_j \in (-\epsilon_1, \epsilon_1) \), we can simply choose \( \gamma_i = (\min\{\epsilon_1, \epsilon\})/2 \), \( \gamma_j = -(\min\{\epsilon_1, \epsilon\})/2 \), \( \gamma_k = 0 \) to satisfy (3.1).

Similarly, if there are a pair of opposite reversible directions, (3.1) is easily satisfied.

Suppose that the directions \( \alpha_i, \alpha_j, \alpha_k \) span \( \mathbb{R}^2 \). Without loss of generality, let \( \alpha_i \) and \( \alpha_j \) span \( \mathbb{R}^2 \).

Consider \( F : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined as follows:

\[ F(\gamma_i, \gamma_j) = \delta_i \vec{\xi}_i + \delta_j \vec{\xi}_j \]

The Jacobian of \( F \) can be simplified as follows:

\[
J(\gamma_i, \gamma_j) = 4 \begin{bmatrix}
\cos(\alpha_i + 2\rho_i|\gamma_i|) & \cos(\alpha_j + 2\rho_j|\gamma_j|) \\
\sin(\alpha_i + 2\rho_i|\gamma_i|) & \sin(\alpha_j + 2\rho_j|\gamma_j|)
\end{bmatrix}
\]

The determinant of \( J \) evaluated at the origin is given by

\[ \det(J) = 16 \sin(\alpha_j - \alpha_i) \]

which is nonzero since \( \alpha_i \) and \( \alpha_j \) span \( \mathbb{R}^2 \). Since \( J \) is continuous at the origin, by the Inverse Function Theorem, there exists \( \epsilon_2 > 0 \) such that for all \( \gamma_k \in (-\epsilon_2, \epsilon_2) \), there exist \( \gamma_i, \gamma_j \) satisfying (3.1).

This leaves only one case to consider, which is when \( \alpha_i, \alpha_j, \alpha_k \) do not span \( \mathbb{R}^2 \), and there are no repeated reversible directions, nor opposite reversible directions. This is only possible if all three \( \rho \) values are different. Without loss of generality, let \( \rho_i = -1, \rho_j = 0, \rho_k = 1 \). Since there exists \( \epsilon_3 > 0 \) such that the deformations are valid for \( \gamma_i, \gamma_j, \gamma_k \in (-\epsilon_3, \epsilon_3) \), let \( \epsilon_4 = (\min\{\epsilon_3, \epsilon\})/2 \) and choose \( \gamma_i = \cos(\alpha_i - \alpha_j)\epsilon_4 \), \( \gamma_j = -2\epsilon_4 \), \( \gamma_k = \cos(\alpha_k - \alpha_j)\epsilon_4 \) to satisfy (3.1).
Let $A_i, A_j, A_k$ be three reversible directions of a CS-path. If $A_i, A_j, A_k$ positively span $\mathbb{R}^2$ then there exist $\gamma_i, \gamma_j, \gamma_k < 0$ such that condition (3.1) is satisfied.

**Proof.** By Definition 3.2, there exists $\varepsilon \in (0, 2\pi)$ such that for all $\eta_i, \eta_j, \eta_k \in (0, \varepsilon)$, $\alpha_i + \rho_i \eta_i$, $\alpha_j + \rho_j \eta_j$, $\alpha_k + \rho_k \eta_k$ positively span $\mathbb{R}^2$. By Lemma 3.2, there exist $\gamma_i, \gamma_j, \gamma_k \in (-\varepsilon, \varepsilon)$ satisfying (3.1) while $(\gamma_i, \gamma_j, \gamma_k) \neq (0, 0, 0)$. Hence, $\theta_i, \theta_j, \theta_k$ positively span $\mathbb{R}^2$.

Since $(\gamma_i, \gamma_j, \gamma_k) \neq (0, 0, 0)$ and $|\gamma_i|, |\gamma_j|, |\gamma_k| < 2\pi$, we know $(\delta_i, \delta_j, \delta_k) \neq (0, 0, 0)$. Since $\xi_i, \xi_j, \xi_k$ positively span $\mathbb{R}^2$ and $(\delta_i, \delta_j, \delta_k) \neq (0, 0, 0)$, this means either $\delta_i, \delta_j, \delta_k > 0$ or $\delta_i, \delta_j, \delta_k < 0$. These two cases correspond to $\gamma_i, \gamma_j, \gamma_k > 0$ and $\gamma_i, \gamma_j, \gamma_k < 0$ respectively. Hence by choosing $\delta_i, \delta_j, \delta_k < 0$, there exist $\gamma_i, \gamma_j, \gamma_k < 0$ satisfying condition (3.1).

Given a set of distinct reversible directions $A_{i_1}, \cdots, A_{i_n}$ which lie in an open half-space of $\mathbb{R}^2$, a reversible direction $A_{i_j}$ in this set is an

- **interior reversible direction** if there exists $\varepsilon > 0$ such that for all $\eta_{i_1}, \cdots, \eta_{i_n} \in (0, \varepsilon)$, $\alpha_{i_j} + \rho_{i_j} \eta_{i_j}$ lies strictly inside the cone positively spanned by $\alpha_{i_1} + \rho_{i_1} \eta_{i_1}, \cdots, \alpha_{i_n} + \rho_{i_n} \eta_{i_n}$;
- **exterior reversible direction** if it is not an interior reversible direction.

**Corollary 3.4.** Let $A_i, A_j, A_k$ be three distinct reversible directions which lie in an open half-space of $\mathbb{R}^2$. Exactly one of these reversible directions, say $A_j$, is an interior reversible direction and there exist $\gamma_i, \gamma_k < 0$ and $\gamma_j > 0$ such that condition (3.1) is satisfied.

**Proof.** It follows by the definition of an interior reversible direction that exactly one of the three reversible directions is an interior direction. The rest of the result follows in a very similar manner to the proof of Corollary 3.3, where the sign of $\gamma_j$ is now different to $\gamma_i, \gamma_k$ because $A_j$ is the interior reversible direction.

Now that we have established results relating to the relative signs of the deformations required to satisfy condition (3.1), we wish to consider the directional cost function as well. It is clear that given reversible directions $A_1, \cdots, A_n$, the most relevant values of the directional cost function to consider are $c(\alpha_1), \cdots, c(\alpha_n)$. As observed in Chapter 2, it
is more convenient to work with the velocity function \( v(\alpha) = 1/c(\alpha) \). We introduce the following term for the point on the velocity function in the direction \( \alpha_i \) as it will be used frequently in the rest of this chapter.

**Definition 3.3.** The velocity point of a reversible direction \( A_i \) is the point in \( \mathbb{R}^2 \) which in polar coordinates is \((v(\alpha_i), \alpha_i)\).

**Lemma 3.5.** Let \( A_i, A_j, A_k \) be distinct reversible directions which lie in an open half-space of \( \mathbb{R}^2 \) where \( A_i \) is the interior reversible direction. Let \( \rho_i, \rho_j, \rho_k = 0 \). Let the straight line passing through the velocity points of \( A_i \) and \( A_k \) be given in polar coordinates by the function \( v_i^k(\alpha) \). There exist \( \delta_i, \delta_k < 0 \) and \( \delta_j > 0 \) satisfying (3.1) such that:

1. if \( v(\alpha_j) > v_i^k(\alpha_j) \), then \( \delta_i c(\alpha_i) + \delta_j c(\alpha_j) + \delta_k c(\alpha_k) < 0 \);
2. if \( v(\alpha_j) < v_i^k(\alpha_j) \), then \( \delta_i c(\alpha_i) + \delta_j c(\alpha_j) + \delta_k c(\alpha_k) > 0 \);
3. if \( v(\alpha_j) = v_i^k(\alpha_j) \), then \( \delta_i c(\alpha_i) + \delta_j c(\alpha_j) + \delta_k c(\alpha_k) = 0 \).

**Proof.** The existence of \( \delta_i, \delta_k < 0 \) and \( \delta_j > 0 \) satisfying (3.1) is given by Corollary 3.4. We start by showing property (3). Recall that since \( \rho_i, \rho_j, \rho_k = 0 \), \( \xi_i, \xi_j, \xi_k \) denote unit vectors in the directions \( \alpha_i, \alpha_j, \alpha_k \) respectively. Consider a norm \( \| \cdot \| \) on \( \mathbb{R}^2 \) such that the boundary of its unit ball contains a straight line from \((v(\alpha_i), \alpha_i)\) to \((v(\alpha_k), \alpha_k)\) in polar coordinates. It is well known (see for example [68]) that in this case, \( \| \delta_i \xi_i \| + \| \delta_k \xi_k \| = \| \delta_j \xi_j \| \). Since \( \| \delta_i \xi_i \| = \delta_i / v(\alpha_i) = \delta_i c(\alpha_i), \| \delta_k \xi_k \| = -\delta_k c(\alpha_k) \) and \( \| \delta_j \xi_j \| = \delta_j c(\alpha_j) \), this proves property (3) holds. We can then restate property (3) as \( \delta_i c(\alpha_i) + \delta_j c_i^k(\alpha_j) + \delta_k c(\alpha_k) = 0 \) where \( c_i^k(\alpha_j) = 1/v_i^k(\alpha_j) \), from which properties (1) and (2) clearly follow.

We need the following more general result which is not restricted to deformations on straight paths.

**Lemma 3.6.** Let \( A_i, A_j, A_k \) be distinct reversible directions which lie in an open half-space of \( \mathbb{R}^2 \) where \( A_i \) is the interior reversible direction. Let the straight line passing through the velocity points of \( A_i \) and \( A_k \) be given in polar coordinates by the function \( v_i^k(\alpha) \). There exist \( \gamma_i, \gamma_k < 0 \) and \( \gamma_j > 0 \) satisfying (3.1) such that:

1. if \( v(\alpha_j) > v_i^k(\alpha_j) \), then \( dT_i + dT_j + dT_k < 0 \);
2. if $v(\alpha_j) < v^k_i(\alpha_j)$, then $dT_i + dT_j + dT_k > 0$;
3. if $v(\alpha_j) = v^k_i(\alpha_j)$, and if $v$ is strict, and exactly one of $\rho_i, \rho_j, \rho_k$ is nonzero, then $dT_i + dT_j + dT_k \neq 0$,

where $dT_i < 0$, $dT_j > 0$, $dT_k < 0$ are the respective changes in cost resulting from $\gamma_i, \gamma_j, \gamma_k$ depending on $\rho_i, \rho_j, \rho_k$.

Proof. The existence of $\gamma_i, \gamma_k < 0$ and $\gamma_j > 0$ satisfying (3.1) is given by Corollary 3.4. Properties (1), (2) follow by applying Lemma 3.2 with sufficiently small $\varepsilon$ so that the results hold in a similar manner to (1), (2) in Lemma 3.5.

Let $v(\alpha_j) = v^k_i(\alpha_j)$ and exactly one of $\rho_i, \rho_j, \rho_k$ be nonzero. Let $m \in \{i, j, k\}$ denote the reversible direction with nonzero $\rho_m$ and without loss of generality, let $\rho_m = 1$. It is only possible to have $dT_j = dT_i + dT_k$ for arbitrarily small $\gamma_i, \gamma_j, \gamma_k \neq 0$ if there exists a $\varepsilon_m > 0$ such that for all $a \in [a_m, a_m + \varepsilon_m]$, $v(a) = v^k_i(a)$. If $v$ is strict, there exists no such $\varepsilon_m > 0$, and hence, $dT_j \neq dT_i + dT_k$. 

3.4 Necessary and Sufficient Path Sets

From Lemma 2.2, we know that every optimal path is a CS-path if the velocity function is strict. In this section, we will first use the tools developed in Section 3.3 to show that for strict velocity functions, all CS-paths with $d \geq 3$ are either non-optimal or reducible to a CS-path with $d \leq 2$ of equal cost. We will establish a sufficient set of CS-paths which always contains an optimal path, and a (larger) necessary set of CS-paths which contains all optimal paths if the velocity function is strict. In other words, it suffices to consider the sufficient set to find an optimal path, while any optimal path necessarily belongs to the necessary set. We then extend the results to weak velocity functions by a simple argument which employs the fact that any weak velocity function is arbitrarily close to a strict velocity function. This proves that the result holds for any velocity function. Finally, we show that this method of proof easily generalises the result to incorporate an additional operational requirement where the cost of curved sections of the path are scaled up by a constant factor $w^C \geq 1$. 

The following example illustrates the difference between the necessary set and the sufficient set. The path shown in Figure 3.7(a) is an SCCS path with $d = 3$ where the two straight line segments have the same direction. This path belongs only to the necessary set, but for any directional cost function, it is of the same cost as the CCS path in Figure 3.7(b) with $d = 2$. The CCS path belongs to both the necessary and sufficient sets.

We break up all CS-paths with $d \geq 3$ into smaller cases, and prove non-optimality or reducibility for each of these cases individually. To assist the reader in following the flow of the proofs, Tables 3.1 and 3.2 have been included which summarise Lemmas 3.7 - 3.15 graphically. These also serve to provide a visual check that all cases have been covered,
and to keep track of which cases are non-optimal, and which are reducible.

### 3.4.1 Non-optimal path forms

**Lemma 3.7.** Any CS-path with opposite reversible directions is non-optimal.

*Proof.* Let $E$ be a CS-path from $p$ to $q$ with opposite reversible directions. Contracting along both reversible directions by equal amounts results in a new path $E'$ with $q' = q$ and $T(E') < T(E)$.

**Lemma 3.8.** Any CS-path with 3 distinct reversible directions which positively span $\mathbb{R}^2$ is non-optimal.

*Proof.* Let $E$ be a CS-path from $p$ to $q$ with $d \geq 3$ and 3 distinct reversible directions $A_i, A_j, A_k$ which positively span $\mathbb{R}^2$. By Corollary 3.3, there exist $\gamma_i, \gamma_j, \gamma_k < 0$ satisfying condition (3.1). Hence, the new CS-path $E'$ formed by contracting along $A_i, A_j, A_k$ by the respective amounts has $q' = q$ and $T(E') \neq T(E)$. By choosing either $E'$ or the new path obtained by the reverse deformations, we get a new path with cost strictly less than $T(E)$.

The example in Figures 3.4(a) and (c) illustrates Lemma 3.8.

**Lemma 3.9.** Let $E$ be a CS-path from $p$ to $q$ such that:

- all the reversible directions of $E$ lie in an open half-space of $\mathbb{R}^2$;
- $E$ has at least 3 distinct reversible directions which have non-collinear velocity points.

Then $E$ is non-optimal.

*Proof.* Select 3 distinct reversible directions $A_i, A_j, A_k$ which have velocity points that are not collinear. By properties (1) and (2) of Lemma 3.6 there exist $\gamma_i, \gamma_j, \gamma_k \neq 0$ satisfying (3.1). Hence, there exists a path formed by deformations along $A_i, A_j, A_k$ such that $q' = q$ and $T(E') < T(E)$.

**Lemma 3.10.** Let $E$ be a CS-path from $p$ to $q$ such that:

- $E$ has at least 3 distinct reversible directions;
Table 3.1: Cases covered by Lemmas 3.7 - 3.11

<table>
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<tr>
<th>Lemma</th>
<th>Reversible directions</th>
<th>Non-optimal or Reducible</th>
<th>Condition on velocity function (if any)</th>
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<td>opposite</td>
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<td></td>
</tr>
<tr>
<td>3.8</td>
<td></td>
<td>N</td>
<td></td>
</tr>
<tr>
<td>3.9</td>
<td></td>
<td>N</td>
<td>velocity points not collinear</td>
</tr>
<tr>
<td>3.10</td>
<td>{ at least one ( \rho \neq 0 ) }</td>
<td>N</td>
<td>strict</td>
</tr>
<tr>
<td>3.11</td>
<td>repeated</td>
<td>N</td>
<td>strict</td>
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### Table 3.2: Cases covered by Lemmas 3.12 - 3.16

<table>
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<th>Lemma</th>
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<th>Non-optimal or Reducible</th>
<th>Condition on velocity function (if any)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.12</td>
<td>$\rho = 0$</td>
<td>N</td>
<td>strict</td>
</tr>
<tr>
<td>3.13</td>
<td></td>
<td>N</td>
<td>strict</td>
</tr>
<tr>
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</tr>
<tr>
<td>3.16</td>
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<td>R</td>
<td>velocity points collinear</td>
</tr>
</tbody>
</table>
• E has at least 1 reversible direction with $\rho \neq 0$.

If $v$ is strict, then $E$ is non-optimal.

Proof. From Lemmas 3.7 - 3.9, only the case where all the reversible directions lie in an open half-space of $\mathbb{R}^2$, and their velocity points are all collinear, needs to be considered. Select any 3 distinct reversible directions $A_i, A_j, A_k$ such that at least one of $\rho_i, \rho_j, \rho_k$ is nonzero. Without loss of generality, let $A_j$ be the interior direction.

If $\rho_j = 0$ for 2 of the 3 reversible directions, then by property (3) of Lemma 3.6, there exist $\gamma_i, \gamma_j, \gamma_k \neq 0$ satisfying (3.1) such that $dT_i + dT_j + dT_k \neq 0$ where $dT_i, dT_j, dT_k$ denote the respective changes in cost due to deforming along $A_i, A_j, A_k$. Since this deformation is reversible, it is therefore possible to construct a new path $E'$ with $q' = q$ so that $T(E') < T(E)$.

If $\rho_j \neq 0$, without loss of generality, let $\rho_j = 1$ (Recall from Definition 3.1 that this means $A_j$ corresponds to an inflection point in a RL subpath). Let the line passing through the collinear velocity points be given in polar coordinates by the function $\vec{v}(\alpha)$. For a sufficiently small $\varepsilon > 0$, if there exists a direction $\alpha^* \in [\alpha_j, \alpha_j + \varepsilon]$ such that $v(\alpha^*) > \vec{v}(\alpha^*)$, introducing a straight in direction $\alpha^*$ in either of the two arcs corresponding to $A_j$ while contracting along $A_i$ and $A_k$ so that condition (3.1) is satisfied will produce a new CS-path $E'$ from $p$ to $q$ with $T(E') < T(E)$. If no such $\alpha^*$ exists, then contracting along $A_j$ while introducing straight line segments of appropriate length in the directions $\alpha_i$ and $\alpha_k$, will result in a new CS-path $E''$ from $p$ to $q$ with $T(E'') \leq T(E)$ since $v(\alpha) \leq \vec{v}(\alpha) \forall \alpha \in [\alpha_j, \alpha_j + \varepsilon]$. However, in this case $T(E'') = T(E)$ would imply that $K(\alpha) = 0 \forall \alpha \in [\alpha_j, \alpha_j + \varepsilon]$ since this argument holds for arbitrarily small $\gamma_j$. Since $v$ is strict, this isn’t possible and hence $T(E'') < T(E)$.

This leaves only one case to consider, which is when $\rho_i, \rho_k \neq 0$, while $\rho_j = 0$. By a slight modification of Corollary 3.4, the following CS-paths exist for sufficiently small $\varepsilon > 0$.

Let $E_0$ be a CS-path from $p$ to $q$ formed by the reversible deformation of

• extending along $A_i$;
• contracting along $A_j$ by $\varepsilon$; and
• extending along $A_k$. 

Let $E_i$ be a CS-path from $p$ to $q$ formed by the CS-path deformation of

- introducing a straight in the direction $\alpha_i$;
- contracting along $A_j$ by $\varepsilon$; and
- extending along $A_k$.

Let $E_k$ be a CS-path from $p$ to $q$ formed by the CS-path deformation of

- extending along $A_i$;
- contracting along $A_j$ by $\varepsilon$; and
- introducing a straight in the direction $\alpha_k$.

Let $E_{ik}$ be a CS-path from $p$ to $q$ formed by the CS-path deformation of

- introducing a straight in the direction $\alpha_i$;
- contracting along $A_j$ by $\varepsilon$; and
- introducing a straight in the direction $\alpha_k$.

These four deformations are illustrated in Figure 3.8 where + denotes extending if $\theta$ is displayed or introducing a straight if $\alpha$ is displayed, while − denotes contracting.

Let $dT_m = T(E_m) - T(E)$ for $m = 0, i, k, ik$. Based on the deformations, it can be seen that $dT_0 + dT_{ik} = dT_i + dT_k$. Since the velocity points are collinear, $dT_{ik} = 0$. Now suppose $dT_0 = 0$, which implies that $dT_i = -dT_k$. However, since $E_i$ involves deformations where only 2 of the 3 deformations are straights, it follows that $dT_i \neq 0$ for a sufficiently small $\varepsilon$ by property (3) of Lemma 3.6. Hence, either $dT_i < 0$ or $dT_k < 0$. If $dT_i < 0$ then $T(E_i) < T(E)$ while if $dT_k < 0$ then $T(E_k) < T(E)$. If on the other hand $dT_0 \neq 0$, then since the corresponding deformation is reversible, the result is shown.

\[ \Box \]

**Lemma 3.11.** Let $E$ be a CS-path from $p$ to $q$ which has a pair of repeated reversible directions $A_i, A_j$ where $\rho_i \neq 0$. If $v$ is strict, then $E$ is non-optimal.

**Proof.** Without loss of generality, let $\rho_i = 1$. Since $A_i, A_j$ are repeated, $\rho_j = 1$.

Let $E_1$ be a CS-path formed from $E$ by:

- contracting along $A_i$ by $\gamma_i$;
- introducing an $S$ subpath of length $4 \sin \gamma_i$ in the direction $\theta_i$ in either of the 2 arcs of the $CC$ subpath corresponding to $A_j$. 


Let $E_2$ be a CS-path formed from $E$ by:
- contracting along $A_i$ by $\gamma_i$;
- introducing an $S$ subpath of length $2\tan\gamma_i$ in the direction $\alpha_i + 2\gamma_i$ in either of the 2 arcs of the $CC$ subpath corresponding to $A_j$;
- introducing an $S$ subpath of length $2\tan\gamma_i$ in the direction $\alpha_i$ at the inflection point corresponding to $A_j$.

Let $q_1$ and $q_2$ denote the endpoints of the CS-paths $E_1$ and $E_2$ respectively. Clearly, $q_1 = q_2 = q$ by construction. Recall that $K = \nu^2 + 2\nu'^2 - \nu\nu''$ from Section 2.3 and the sign of $K$ represents the sign of the curvature of the polar function $\nu$. Since $\nu$ is continuous
piecewise $C^2$ and strict, we can choose $\gamma_i > 0$ sufficiently small so that either

1. $K > 0$ for all $\alpha \in (\alpha_i, \alpha + 2\gamma_i)$; or
2. $K < 0$ for all $\alpha \in (\alpha_i, \alpha + 2\gamma_i)$.

If case (1) is true, then let the line tangential to $v(\alpha)$ at $\alpha = \theta_i$ be given by the polar function $v(\alpha)$. Let $c = 1/v$. Since $K > 0$ for all $\alpha \in (\alpha_i, \alpha + 2\gamma_i)$, it follows that $v(\alpha) < v(\alpha)$ for all $\alpha \in (\alpha_i, \alpha + 2\gamma_i)$. Hence,

$$T(E_1) = T(E) + 4 \sin \gamma_i c(\theta_i) - 2 \int_{-\gamma_i}^{\gamma_i} c(\theta_i + \phi) d\phi$$

$$< T(E) + 4 \sin \gamma_i c(\theta_i) - 2 \int_{-\gamma_i}^{\gamma_i} c(\theta_i + \phi) d\phi$$

$$= T(E) + 4 \sin \gamma_i c(\theta_i) - 2 \int_{-\gamma_i}^{\gamma_i} c(\theta_i + \phi) d\phi$$

$$= T(E)$$

The final step is due to the triangle inequality being satisfied by equality, since $v(\alpha)$ is a straight line in polar coordinates.

If case (2) is true, then let the line passing through the points $(v(\alpha_i), \alpha_i)$ and $(v(\alpha_i + 2\gamma_i), \alpha_i + 2\gamma_i)$ in polar coordinates be given by the polar function $\overline{v}(\alpha)$. Let $c = 1/\overline{v}$. Since $K < 0$ for all $\alpha \in (\alpha_i, \alpha + 2\gamma_i)$, it follows that $v(\alpha) < \overline{v}(\alpha)$ for all $\alpha \in (\alpha_i, \alpha + 2\gamma_i)$. Hence,

$$T(E_2) = T(E) + 2 \tan \gamma_i (c(\alpha_i) + c(\alpha_i + 2\gamma_i)) - 2 \int_{-\gamma_i}^{\gamma_i} c(\theta_i + \phi) d\phi$$

$$< T(E) + 2 \tan \gamma_i (\overline{v}(\alpha_i) + 2\overline{v}(\alpha_i + 2\gamma_i)) - 2 \int_{-\gamma_i}^{\gamma_i} \overline{v}(\theta_i + \phi) d\phi$$

$$= T(E)$$

Hence, $E$ is non-optimal in both cases. \qed

For the proofs of the next five lemmas, it is helpful to distinguish between a number
of different types of \( S \) segments.

**Definition 3.4.** In a CS-path,

- a terminal \( S \) segment, denoted \( \tilde{S} \), is an \( S \) segment at the start or end of the CS-path;
- an inserted \( S \) segment, denoted \( \tilde{S} \), is an \( S \) segment preceded and followed by \( C \) arcs of opposing sense;
- a linked \( S \) segment, denoted \( \mathring{S} \), is an \( S \) segment preceded and followed by \( C \) arcs of the same sense;
  - a proper linked \( S \) segment, denoted \( \mathcal{S} \), is an \( \mathring{S} \) segment where the preceding and following arcs are of combined length less than \( 2\pi \);
  - an improper linked \( S \) segment, denoted \( \mathcal{S} \), is an \( \mathring{S} \) segment that is not an \( S \) segment.

The key difference between these \( S \) labels is that contracting an \( \tilde{S} \) segment until it degenerates causes an inflection points to be created, while contracting any other type of \( S \) segment does not. While contracting an \( \mathcal{S} \) segment does not introduce an inflection point, it does however create a loop (a full unit circle).

**Lemma 3.12.** Let \( E \) be a CS-path from \( p \) to \( q \) such that:

- \( d \geq 4 \);
- no 3 reversible directions of \( E \) are distinct.

If \( v \) is strict, then \( E \) is non-optimal.

**Proof.** We can assume in \( E \) that \( \alpha_i \neq \alpha_{i+1} \) for all \( i < d \) since CS-paths do not contain loops by definition. Hence, for the path \( E \), \( A_m = A_l \) for \( m, l \) odd, while \( A_m = A_l \) for \( m, l \) even, and \( \alpha_1 \neq \alpha_2 \). If \( \rho_1 \neq 0 \) or \( \rho_2 \neq 0 \), the path is non-optimal by Lemma 3.11. Hence we simply need to consider the case of \( \rho_m = 0 \) for all \( m \in \{1, \ldots, d\} \). If \( A_2 \) corresponds to an \( S \), the combined lengths of the arcs preceding and following this straight must be of length \( 2\pi \) because \( A_1 = A_3 \). Since \( d \geq 4 \), it is possible to contract along \( A_2 \) until it degenerates while extending along \( A_4 \) by an equal amount. This would result in a new path \( E' \) from \( p \) to \( q \) where \( T(E') = T(E) \). However, \( E' \) has an arc of length \( 2\pi \) and hence both \( E' \) and \( E \) are non-optimal.
If $A_2$ corresponds to an $\tilde{S}$, contracting along $A_2$ until it degenerates while extending along $A_4$ by an equal amount would result in a $CC$ subpath being formed where $A_2$ has degenerated to a point of inflection, which has a new reversible direction distinct to $A_1$ and $A_4$ since $\rho_2 \neq 0$. The result then follows from Lemma 3.10.

\[\square\]

**Lemma 3.13.** Let $E$ be a CS-path from $p$ to $q$ which has a pair of repeated reversible directions $A_i, A_j$ such that:

- $\rho_i = 0$;
- at least one of $A_i, A_j$ corresponds to an $\tilde{S}$ or $\tilde{S}$.

If $v$ is strict, then $E$ is non-optimal.

**Proof.** Clearly, there must exist a reversible direction $A_k$ which is distinct from $A_i$ for such a path to exist. Without loss of generality, let $A_i$ correspond an $\tilde{S}$ or $\tilde{S}$. By contracting the straight edge in each of these 2 cases, the result follows in a similar manner to the proof of Lemma 3.12.

\[\square\]

**Lemma 3.14.** Let $E$ be a CS-path such that:

- $E$ has at least 3 distinct reversible directions;
- all the reversible directions of $E$ lie in an open half-space of $\mathbb{R}^2$;
- at least one of the interior reversible directions of $E$ corresponds to an $\tilde{S}$ or $\tilde{S}$.

If $v$ is strict, then $E$ is non-optimal.

**Proof.** From Lemmas 3.9 and 3.10, only the case where the velocity points of all the reversible directions are all collinear, and $\rho_1, \cdots, \rho_d = 0$ needs to be considered. Let $A_i, A_k$ be the exterior reversible directions, and $A_j$ be an interior direction corresponding to an $\tilde{S}$ or $\tilde{S}$.

Since $\rho_i, \rho_j, \rho_k = 0$, $A_j$ can be contracted until it degenerates, while $A_i$ and $A_k$ are extended to satisfy (3.1). The new path formed $E'$ has $q' = q$ and $T(E') = T(E)$ since the velocity points are collinear, but in a similar manner to the proof of Lemma 3.12, $E'$ and $E$ can be shown to be non-optimal by considering the 2 cases of $A_j$ corresponding to an $\tilde{S}$ or $\tilde{S}$.
3.4.2 Reducible path forms

Lemma 3.15. Let $E$ be a CS-path from $p$ to $q$ which has a pair of repeated reversible directions $A_i, A_j$ such that:

- $\rho_i = 0$;
- at least one of $A_i, A_j$ corresponds to an $\dot{S}$ or $S$.

Then $E$ is reducible.

Proof. Without loss of generality, let $A_i$ correspond to an $\dot{S}$ or $S$. Contracting along $A_i$ until it degenerates while extending along $A_j$ by an equal amount results in a new path $E'$ with $q' = q$ where $T(E') = T(E)$ but $E'$ is of degree 1 lower than $E$. Note that if $A_i$ instead corresponded to an $\dot{S}$, the degree would not decrease and if $A_i$ corresponded to an $S$, a loop would be created, meaning that it is no longer a CS-path.

Lemma 3.16. Let $E$ be a CS-path such that:

- it has at least 3 distinct reversible directions;
- all its reversible directions lie in an open half-space of $\mathbb{R}^2$;
- $\rho_1, \cdots, \rho_d = 0$;
- at least one of its interior reversible directions corresponds to an $\dot{S}$ or $S$;
- the velocity points of all of its reversible directions are collinear.

Then $E$ is reducible.

Proof. Let $A_i, A_k$ be the exterior reversible directions, and $A_j$ be an interior direction corresponding to an $\dot{S}$ or $S$. Since $\rho_i, \rho_j, \rho_k = 0$, $A_j$ can be contracted until it degenerates, while $A_i$ and $A_k$ are extended to satisfy (3.1). The new path formed $E'$ has $q' = q$ where $T(E') = T(E)$ but $E'$ is of degree 1 lower than $E$. 

□
3.4 Necessary and Sufficient Path Sets

3.4.3 Main result

Let \( \Pi \) denote a special case of a \( C \) arc which is of length \( \pi \).

**Definition 3.5.** Let the sufficient path set \( \mathcal{D}^S_{pq} \) be the set of all CS-paths from \( p \) to \( q \) each of which has degree \( d \leq 2 \) which do not have an SIIS subpath. This is equivalent to defining \( \mathcal{D}^S_{pq} \) as the set of all CSCSC paths and degeneracies, which do not have an SIIS subpath.

Let the necessary path set \( \mathcal{D}^N_{pq} \) be the set of all CS-paths from \( p \) to \( q \) each of which either

1. belong to \( \mathcal{D}^S_{pq} \);
2. have \( d = 3 \), \( \alpha_1 = \alpha_3 \), \( \rho_1 = \rho_3 = 0 \), where both \( A_1 \) and \( A_3 \) correspond to an \( \dot{S} \) or \( S \), and \( A_2 \neq (\alpha_1 + \pi, 0) \);
3. have \( d \geq 3 \), at least 3 distinct reversible directions, \( \rho_1, \ldots, \rho_d = 0 \) and \( A_1, \ldots, A_d \) lie strictly in a half-space of \( \mathbb{R}^2 \), with each interior reversible direction corresponding to an \( \dot{S} \) or \( S \).

Let \( \mathcal{D}_{pq} \) denote the set of all Dubins paths from \( p \) to \( q \). Let \( \mathcal{C}_{pq} \) denote the set of all CS-paths from \( p \) to \( q \). Note that by Definition 3.5, \( \mathcal{D}_{pq} \subset \mathcal{D}^S_{pq} \subset \mathcal{D}^N_{pq} \subset \mathcal{C}_{pq} \subset \mathcal{P}_{pq} \).

**Theorem 3.17.** If \( v \) is strict, every optimal curvature-constrained path from \( p \) to \( q \) belongs to \( \mathcal{D}^N_{pq} \). Furthermore, there exists a path in \( \mathcal{D}^S_{pq} \) which is optimal.

**Proof.** It follows from Lemmas 3.7 - 3.14 that every optimal curvature-constrained path from \( p \) to \( q \) belongs to \( \mathcal{D}^N_{pq} \). Applying Lemmas 3.9, 3.15 and 3.16 to this result tells us that any optimal path of types (2) and (3) in \( \mathcal{D}^N_{pq} \) is reducible to a path belonging to \( \mathcal{D}^S_{pq} \). Hence, there exists a path in \( \mathcal{D}^S_{pq} \) which is optimal.

**Corollary 3.18.** Given any velocity function \( v \), there exists a path of the form CSCSC (or degeneracy) which is an optimal curvature-constrained path from \( p \) to \( q \).

**Proof.** If \( v \) is strict, the result follows from Theorem 3.17. Hence, only weak (not strict) velocity functions need to be considered.

Let \( v \) be a weak velocity function, where \( c = 1/v \). Let \( \{I_j\} \) be the set of all maximal open intervals \( I_j = (\alpha_j - \beta_j, \alpha_j + \beta_j) \), where \( K(\alpha) = 0 \) for all \( \alpha \in I_j \), \( \forall j \) (and each \( \beta_j \) is
as large as possible). Let $h_j(\alpha)$ be the distance of $\alpha$ from the complement of $I_j$ defined as follows:

$$h_j(\alpha) = \begin{cases} \beta_j - |\alpha - \alpha_j| & \text{if } \alpha \in I_j \\ 0 & \text{otherwise} \end{cases}$$

Consider a new velocity function $v_\varepsilon(\alpha) = 1/c_\varepsilon(\alpha)$ where $c_\varepsilon(\alpha) = c(\alpha) + \varepsilon \sum_j h_j(\alpha)$. By construction, $v_\varepsilon$ is strict for any $\varepsilon > 0$. Let $T(E)$ and $T_\varepsilon(E)$ denote the costs of path $E$ with respect to the directional cost functions $c$ and $c_\varepsilon$ respectively. Let $H(E) = \int_E \sum_j h_j(\alpha) ds$.

Suppose \( \exists E \in \mathbb{P}_{pq} \setminus \mathbb{D}_{pq}^N \) such that $T(E) < T(D)$ for all $D \in \mathbb{D}_{pq}^N$. Then,

$$T_\varepsilon(E) = T(E) + \varepsilon H(E) < T(D), \text{ by choosing } \varepsilon \in (0, (T(D) - T(E))/H(E))$$

However, Theorem 3.17 states that there cannot exist such a $E$ since $v_\varepsilon$ is strict and $E \notin \mathbb{D}_{pq}^N$. Hence, by contradiction, there cannot exist a path in $\mathbb{P}_{pq} \setminus \mathbb{D}_{pq}^N$ which is of less cost than all the paths in $D \in \mathbb{D}_{pq}^N$. Finally, by applying Lemmas 3.9, 3.15 and 3.16, we get the result.

\[ \square \]

### 3.4.4 Alternate proof for Theorem 2.9

Earlier in Chapter 2, we stated that an alternate proof of Theorem 2.9 will be provided using the techniques developed in this chapter. Let $\text{conv}(V)$ denote the convex hull of the velocity set $V$, of a velocity function $v$, as defined in Section 2.2.3. Recall that $v$ is strictly polarly convex (SPC) if $v$ is strict and $V = \text{conv}(V)$.

We proved in Theorem 2.9 that if $v$ is SPC, any optimal path is a Dubins path using a direct method that did not make use of reversible deformations. The strategy we employed to prove this result was to show that the following forms are not optimal:
We will now show that the proof can be simplified by first starting with the more general result that for any velocity function, there exists an optimal path of the form \( \text{CSCSC} \) or degeneracy. Note that this immediately eliminates \( \text{CCCC} \), leaving only three forms to consider. We start by presenting a corollary obtained by restricting \( v \) to be SPC in Lemma 3.6.

**Corollary 3.19.** Let \( v \) be an SPC velocity function and \( A_i, A_j, A_k \) be distinct reversible directions which lie in an open half-space of \( \mathbb{R}^2 \) where \( A_j \) is the interior reversible direction. Let the straight line passing through the velocity points of \( A_i \) and \( A_k \) be given in polar coordinates by the function \( v_k^i(\alpha) \). There exist \( \gamma_i, \gamma_k < 0 \) and \( \gamma_j > 0 \) satisfying (3.1) such that \( dT_i + dT_j + dT_k < 0 \) where \( dT_i < 0, dT_j > 0, dT_k < 0 \) are the respective changes in cost resulting from \( \gamma_i, \gamma_j, \gamma_k \) depending on \( \rho_i, \rho_j, \rho_k \).

**Proof.** Since \( v \) is SPC, it follows that \( v(\alpha_j) > v_k^i(\alpha_j) \). The result then follows immediately from Lemma 3.6.

Rather than considering each of the 5 subcases \( \text{SCS}, \text{SCS}, \text{CCS}, \text{CCC} \) separately, we can prove that the first 4 are not optimal at once with the following lemma. The key idea of the proof is that since we restrict the problem to SPC velocity functions, we no longer require the deformations to be performed in a reversible manner.

**Lemma 3.20.** Let \( E \) be a CS-path from \( p \) to \( q \) which has distinct reversible directions \( A_i, A_k \) where \( \rho_i = 0 \). If \( v \) is strict, then \( E \) is non-optimal.

**Proof.** Without loss of generality, let \( \alpha_k \in (\alpha_i, \alpha_i + \pi) \). We can introduce a new interior reversible direction \( A_j = (\alpha_i, 1) \) which is valid as long as we restrict the deformation of \( \gamma_j \) to be positive i.e. only extend \( A_j \) since it is not possible to contract it in the absence of an inflection point. The result then follows from Corollary 3.19.
This leaves only $CCC$ paths which needs to be handled separately. We prove a result which shows that $SCS, CCS$ and $CCC$ are not optimal i.e. both lemmas apply to $SCS$ and $CCS$ paths.

Lemma 3.21. Let $E$ be a CS-path from $p$ to $q$ which has distinct reversible directions $A_i, A_k$ such that $\alpha_k \in (\alpha_i, \alpha_i + \pi)$ and there exists $\alpha_j \in (\alpha_i, \alpha_k)$ such that a point on $E$ has direction $\alpha_j$. If $\nu$ is strict, then $E$ is non-optimal.

Proof. In a similar manner to the proof of Lemma 3.20, we can introduce a new interior reversible direction $A_j = (\alpha_j, 0)$ which is valid as long as we restrict the deformation of $\gamma_j$ to be positive i.e. only extend $A_i$ since it is not possible to contract it in the absence of a straight path. The result then follows from Corollary 3.19.

Theorem 2.9 then follows by Theorem 3.17 and Lemmas 3.20 and 3.21.

3.4.5 Extension - the weighted arcs problem

The motivation of this extension is related to the underground mine development application, where curved sections may require additional support and hence incur a higher development cost. There are also additional ventilation costs due to shock losses around bends [47]. In this extension, we model these effects by imposing a scaling factor $w^C \geq 1$ on $C$ subpaths, as well as having a directional cost. This problem is only well-posed if we restrict ourselves to considering only CS-paths.

Definition 3.6. Let $E$ be a CS-path from $p$ to $q$, which is made up of subpaths $E_1, \ldots, E_n$, corresponding to its $n$ labels. Let $w^C \geq 1$ be a given scaling factor. We define the weighted-arc cost of $E$ to be:

$$T^C(E) = \sum_{i=1}^{n} w_i T(E_i), \text{ where } w_i = \begin{cases} w^C & \text{if } E_i \text{ is a } C \text{ arc} \\ 1 & \text{if } E_i \text{ is an } S \text{ straight} \end{cases}$$

A path $E$ is an optimal weighted-arc CS-path from $p$ to $q$ if $E \in \{ P \in C_{pq} : T^C(P) \leq T^C(Q) \ \forall \ Q \in C_{pq} \}$. 
3.5 Conclusion

Corollary 3.22. Given any velocity function $v$, there exists a path in $D_{pq}^S$ which is an optimal weighted-arc CS-path from $p$ to $q$. Furthermore, if $v$ is strict, every optimal weighted-arc CS-path belongs to $D_{pq}^N$.

Proof. The equivalent results to Lemmas 3.7 - 3.9 and 3.15 hold without any modification to the proofs. Lemmas 3.10 and 3.16 hold by simply replacing velocity point with equivalent velocity point defined as follows. The equivalent velocity point of a reversible direction $A_i$ is the point in $\mathbb{R}^2$ which in polar coordinates is:

- $(w^C v(\alpha_i), \alpha_i)$ if $\rho_i = 0$;
- $(v(\alpha_i), \alpha_i)$ if $\rho_i \neq 0$.

Lemma 3.11 applies because of the condition that $w^C \geq 1$. Lemmas 3.12 - 3.14 then hold as their arguments make use of the preceding lemmas. Finally, similar proofs to Theorem 3.17 and Corollary 3.18 hold by replacing $P_{pq}$ with $C_{pq}$.

Remark: It is clear that since Lemma 3.11 only holds when $w^C \geq 1$, the problem is not well-posed if we consider $w^C < 1$. This is because any straight subpath $E_i$ could be replaced by a sequence of infinitesimally short arcs approximating $E_i$, which in the limit, cost $w^C T(E_i)$.

3.5 Conclusion

This chapter shows that given any velocity function and pair of directed points, there exists a path of the form $CSCSC$ or a degeneracy which is optimal. Furthermore, if we impose a scaling factor $w^C \geq 1$ on the cost of $C$ subpaths, this result still holds. This result tells us that if we wish to construct an optimal path based on a given velocity function, $w^C \geq 1$, together with starting and end points, we need only consider a small subset of the entire set of curvature-constrained paths. The next step is to use the ideas developed in this chapter to come up with necessary conditions for optimality of a given path in relation to the velocity function, and design an algorithm which can efficiently construct optimal paths. These issues are addressed in the next chapter.
Chapter 4
Algorithm for Constructing Optimal Path

In this chapter, we develop an algorithm which explicitly constructs a minimum cost curvature-constrained path between two directed points for a given directional cost function. We use the result from Chapter 3 that there always exists a path of the form CSCSC or a degeneracy which is optimal, where C denotes a continuous subset of the unit circle, and S represents a straight line segment. We prove necessary conditions for optimality based on information of the directional cost function. For the case when both S segments are non-degenerate, we establish a connection between the optimality condition and double tangents of star convex closed planar curves, and extend classical theory proving bounds for the number of different types of double tangents in terms of the number of inflection points. We apply this result to bound the number of pairs of directions we need to consider when constructing the optimal path. We then narrow our focus down to piecewise constant directional cost functions which are useful for the intended application of constructing optimal underground mine networks under anisotropic ground conditions. We provide an $O(n^2)$ exact algorithm for constructing such a path subject to this specific type of directional cost function, where $n$ is the number of discontinuities in the piecewise constant function. We also extend this to a heuristic for a general directional cost function.

4.1 Introduction

This chapter builds upon the results from the previous chapters. There, we established the forms of optimal curvature-constrained paths between two directed points in the plane where the cost at every point along the path depends on the direction. In [32], the equivalent problem was studied for naval path planning, where the velocity of the vehicle depends on the direction. The problems are equivalent by simply consid-
ering the reciprocal of the direction-cost function as the velocity function. We define a velocity function to be polarly convex if it bounds a convex region when plotted in polar coordinates. We proved in Theorem 2.9 and Corollary 2.10 that if the velocity function is polarly convex, then we can construct an optimal path by comparing the costs of the different Dubins paths. Since there are at most only 6 distinct Dubins paths, this can be done easily.

In Theorem 3.17 and Corollary 3.18, we showed that for a general velocity function, we need to consider paths of the form \( \text{CSCSC} \) or a degeneracy of \( \text{CSCSC} \). Unlike Dubins paths, there are an infinite number of \( \text{CSCSC} \) paths between any two directed points. This is because a CS-path with 5 components has too many degrees of freedom to be uniquely specified by fixing the start and end directed points alone. This causes the construction problem to differ greatly from the Dubins path construction problem studied in [60] and [61], where there are at most 6 distinct Dubins paths between any two directed points. In this chapter, necessary conditions for optimality based on information of the velocity function are found. These necessary conditions are useful for designing an algorithm which efficiently constructs an optimal curvature-constrained path.

Observe that paths of the form \( \text{CSCSC} \), or a degeneracy of this, can be categorised based on the following mutually exclusive properties:

- **Type 1**: has a non-degenerate \( \text{SCS} \) subpath
- **Type 2**: has a non-degenerate \( \text{CCS} \) or \( \text{SCC} \) subpath
- **Type 3**: is a \( \text{CSC} \), \( \text{CCC} \) path or degeneracy of either.

For a general directional cost function, we show that if there exists an optimal path of Type 1, we can very efficiently narrow down the possibilities that the pair of directions of the \( S \) components can take. To achieve this result, we prove a general result relating to double tangents of a *star convex* closed planar curve. A curve is star convex if there exists a point such that a line drawn from that point to any point on the curve does not intersect the curve anywhere else. This general result extends classical theory on the number of double tangents of a closed planar curve and it can be applied to other problems in-
volving star convex planar curves. In our application, we apply this result, narrowing
down the set of candidate optimal Type 1 paths to a finite number of possibilities, with-
out knowledge of the two directed points. This is useful for developing an algorithm
which needs to compute optimal paths between many pairs of directed points, subject to
the same directional cost behaviour.

However, if the optimal path is of Type 2, the optimality condition is weaker and is,
in general, insufficient to give us as good a result as the case where the optimal path is of
Type 1. For Type 2, the $S$ direction cannot be predetermined to belong to a finite set, and
in general, depends on the pair of directed points.

If the optimal path is of Type 3, then there are only a finite number of distinct paths
to consider (similar to Dubins paths). This means that Type 2 is the only case where the
locally minimal paths can not be easily constructed in general. Alternatively, by restrict-
ing the cost functions to some specific families, it is possible to exactly construct all the
locally minimal paths containing non-degenerate $CCS$ or $SCC$ subpaths without using a
local descent method.

The class of directional cost functions we focus on are based on the underground
mine design application. In an underground mine, networks of tunnels provide access
to the ore from the surface and passages for hauling the ore out of the mine. Due to the
anisotropic characteristics of ground, different support structures, varying in cost due
to strength of support required, are used for different directions of development. It is
practical to represent these development costs by a directional cost function which is
piecewise constant, taking the value of the lower cost at the discontinuity. The recipro-
cal of this function gives a piecewise constant velocity function, with the higher velocity
value at the discontinuities. It is easily seen that the optimality conditions for continuous
velocity functions still hold by simply considering a piecewise constant velocity function
as the limit of a sequence of continuous velocity functions. For this family of piecewise
constant directional cost functions, we provide an exact algorithm which returns the opti-
mal curvature-constrained path between any two directed points in $O(n^2)$ computations
where $n$ is the number of discontinuities. We also describe modifying this exact algorithm
for piecewise constant velocity functions into a heuristic for general velocity functions,
which could be useful for other applications.

In the Section 4.2, we introduce relevant terminology from the previous chapters, listing the key results required by this chapter. We then prove necessary conditions for optimality and present trimming procedures which apply to general directional cost functions in Section 4.3. We then restrict ourselves to considering piecewise constant directional cost functions in Section 4.4 for which we present an exact algorithm for constructing optimal paths. A heuristic for general directional cost functions is presented based on the exact algorithm. A MATLAB GUI was created which implemented the exact algorithm. Examples of the algorithm output and screenshots of the GUI are presented.

The main results of this chapter have been submitted to Computational Geometry Theory and Applications [26].

4.2 Background

Similarly to Chapter 3, the background introduced in Sections 2.2.1, 2.2.3, 2.3 and 2.4.1 is important for this chapter.

Recall from Corollary 3.18 in Chapter 3 that given any velocity function $v$, there exists a path of the form CSCSC (or degeneracy) which is an optimal curvature-constrained path from $p$ to $q$. We will rely on this fundamental the result in this entire chapter.

Suppose $v(\alpha)$ is a velocity function where $\alpha_2 \in (\alpha_1, \alpha_1 + \pi)$ and $K = 0$ for all $\alpha \in [\alpha_1, \alpha_2]$. Geometrically, this corresponds to $v(\alpha)$ consisting of a straight line segment in polar coordinates from $(v(\alpha_1), \alpha_1)$ to $(v(\alpha_2), \alpha_2)$. A path $P$ is constrained to $[\alpha_1, \alpha_2]$ if the direction along the path $P$ only takes values in $[\alpha_1, \alpha_2]$.

4.3 Necessary Conditions for Optimality

In this section, we aim to make use of knowledge of the velocity function in order to narrow down the paths we need to consider for optimality to a finite number of paths.

Recall that a path of the form CSCSC or degeneracy can be categorised based on the following mutually exclusive properties:
Type 1: has a non-degenerate SCS subpath
Type 2: has a non-degenerate CCS or SCC subpath
Type 3: is a CSC, CCC path or degeneracy of either.

As stated previously, only the first two cases need to be discussed, as the last case has a finite number of possibilities for a given pair of directed points.

The following notation will be used in the following lemmas. For a given velocity function \( v(\alpha) \) and pair of directions \( \alpha_1, \alpha_2 \), let \( \overline{v}(\alpha) \) denote the straight line in polar coordinates passing through \((v(\alpha_1), \alpha_1)\) and \((v(\alpha_2), \alpha_2)\). Let \( \alpha_\perp \) be the direction from the origin which is perpendicular to \( \overline{v}(\alpha) \) i.e. the outward normal to \( \overline{v}(\alpha) \) (for example, see Figure 4.2(b)). It is clear that \( \overline{v}(\alpha) \) is defined for \( \alpha \in (\alpha_\perp - \pi/2, \alpha_\perp + \pi/2) \).

The following property will be a useful tool for the proofs in this section.

**Lemma 4.1.** Let \( v(\alpha) \) be a velocity function where \( \alpha_2 \in (\alpha_1, \alpha_1 + \pi) \) and \( K = 0 \) for all \( \alpha \in [\alpha_1, \alpha_2] \). Let \( ab \) denote the straight line segment from \( a \in \mathbb{R}^2 \) to \( b \in \mathbb{R}^2 \), where the direction of \( ab \) is in \( [\alpha_1, \alpha_2] \). Let \( P \) be a path from \( a \) to \( b \). If \( P \) is constrained to \([\alpha_1, \alpha_2] \), then \( T(P) = T(ab) \), where \( T(E) \) denotes the cost of the path \( E \).

**Proof.** This result follows simply by repeatedly applying the triangle equality that \( T(ab) = T(ac) + T(cb) \), provided the directions of \( ab, ac, cb \) lie in \( [\alpha_1, \alpha_2] \). \( \square \)

We now introduce deformations which will be used to prove necessary optimality conditions for Type 1 and Type 2 paths separately.

### 4.3.1 Type 1: SCS subpath

A deformation referred to as a **slide** is a translation of the central circle (corresponding to the \( C \) arc in an SCS path) in the direction of an adjacent straight line segment (or in its reverse direction) so the remaining parts of the path deform accordingly. For example, we can locally deform an existing SCS path into a CSCS or SSSC path by sliding the central circle corresponding to the \( C \) arc, along one of the two directions of the straight line segments (see Figure 4.1(a)). We will show in Lemma 4.2, using these slides, that the
Algorithm for Constructing Optimal Path

possible pairs of directions of straight line segments \( \alpha_1 \) and \( \alpha_2 \) in an optimal path are determined by the velocity function as illustrated in Figure 4.1(d). Note that \( v(\alpha) \) and \( \overline{v}(\alpha) \) are plotted in polar coordinates with \( O \) representing the origin. Each slide \( \Delta_1^-, \Delta_1^+, \Delta_2^-, \Delta_2^+ \) in Figure 4.1(a) preserves one of the two directions of the straight line segments and perturbs the other into the corresponding regions illustrated in Figure 4.1(d). The regions give the range of possible angles for the perturbed direction. For example, \( \Delta_1^- \) preserves \( \alpha_2 \) while perturbing \( \alpha_1 \) to take values in \( [\alpha_1 - \varepsilon, \alpha_1) \), for some \( \varepsilon > 0 \).

Lemma 4.2. If an optimal path contains an S\( \mathcal{S} \) path with the directions of the straight line segments being \( \alpha_1 \) and \( \alpha_2 \), where without loss of generality, \( \alpha_2 \in (\alpha_1, \alpha_1 + \pi) \), then there exists \( \varepsilon > 0 \) such that \( v(\alpha) \leq \overline{v}(\alpha) \) for all \( \alpha \in [\alpha_1 - \varepsilon, \alpha_2 + \varepsilon] \).

Proof. Suppose that for any \( \varepsilon > 0 \), there exists \( \varepsilon_1 \in (0, \varepsilon) \) such that \( v(\alpha) > \overline{v}(\alpha) \) for all \( \alpha \in [\alpha_1 - \varepsilon_1, \alpha_1) \). Without loss of generality, let the first straight line segment traversed be in direction \( \alpha_1 \) so the path is as shown in Figure 4.1(a). Consider the result of slide \( \Delta_1^- \) that increases the length of the straight line segment in the direction \( \alpha_2 \) such that the direction of the new straight line segment is \( \alpha_1^* \in [\alpha_1 - \varepsilon_1, \alpha_1) \) as shown in Figure 4.1(b). Figure 4.1(c) shows the subpaths \( P_1, Q_1^*, Q_2 \) we need to consider in comparing the cost of the new path to that of the original. Note that \( Q_1^* \) is a \( \mathcal{CSC} \) subpath while \( P_1 \) and \( Q_2 \) are \( \mathcal{S} \) subpaths. Let \( T(E) \) denote the cost of \( E \) subject to velocity function \( v \) and \( \overline{T}(E) \) denote the cost of \( E \) subject to velocity function \( \overline{v} \). The change in cost, \( dT \) of this action is given by:

\[
dT = T(Q_1^*) + T(Q_2) - T(P_1) \]
\[
< \overline{T}(Q_1^*) + T(Q_2) - T(P_1), \text{ since } v(\alpha) > \overline{v}(\alpha) \forall \alpha \in [\alpha_1 - \varepsilon_1, \alpha_1) \]
\[
= 0 \text{ by Lemma 4.1}
\]

This contradicts the optimality of the original S\( \mathcal{S} \) path. Hence, by symmetry, if we consider slide \( \Delta_2^+ \) in Figure 4.1(a), there exists \( \varepsilon > 0 \) such that \( v(\alpha) \leq \overline{v}(\alpha) \) for all \( \alpha \in [\alpha_1 - \varepsilon, \alpha_1] \cup [\alpha_2, \alpha_2 + \varepsilon] \).

Now, suppose there exists \( \alpha^* \in (\alpha_1, \alpha_2) \) such that \( v(\alpha^*) > \overline{v}(\alpha^*) \). By performing slide \( \Delta_2^- \) in Figure 4.1(a), we can construct a new path with straight line segments in
4.3 Necessary Conditions for Optimality

(a) $SCS$ subpath

(b) Deformed path from slide $\Delta_1^-$

(c) Path components $P_1, Q_1^*, Q_2$ from (b) re-arranged

(d) Necessary optimality condition of the directions of the straight line segments $\alpha_1, \alpha_2$ in terms of $v(\alpha)$ with respect to $\varpi(\alpha)$ illustrating Lemma 4.2. $\alpha_i$ is the direction of the straight line segment when $\Delta_{1i}^+$ or $\Delta_{2i}^-$ is performed until the subpath degenerates into the form $CSC$.

Figure 4.1: $SCS$ subpath with directions of the straight line segments $\alpha_1, \alpha_2$ shown in (a), with $\Delta_1^-, \Delta_1^+, \Delta_2^-, \Delta_2^+$ representing the four slides which can be performed. The effects of slide $\Delta_1^-$ are shown in (b) and (c) with darker grey indicating new subpaths added and lighter grey indicating old subpaths remaining. (d) illustrates the corresponding optimality conditions for $\alpha_1, \alpha_2$ in terms of the velocity function $v(\alpha)$ (thick solid curve) with respect to $\overline{\varpi}(\alpha)$ (solid line).
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the directions \( \alpha_1 \) and \( \alpha^* \), if \( \alpha^* \in [\alpha_s, \alpha_2] \) where \( \alpha_s \) is the direction of the new straight line segment when slide \( \Delta^-_2 \) is performed as much as possible. Similarly, we can construct a new path with straight line segments in the directions \( \alpha^* \) and \( \alpha_2 \) by performing slide \( \Delta^+_1 \), if \( \alpha^* \in [\alpha_1, \alpha_s] \). Since \( v(\alpha^*) > v(\alpha^*) \), it follows that the new path is of lower cost, again contradicting the optimality of the original path.

The strategy employed for proving Lemma 4.2 is illustrated in Figures 4.1(a) and (d). The sliding deformations are shown as well as the corresponding regions (range of directions) over which the optimality conditions hold. The remaining cases for Type 1 and Type 2 can be proved using a similar strategy, with analogous figures summarising the deformations used and corresponding regions shown where the optimality conditions apply.

We consider the case of \( \mathcal{S} \mathcal{C} \mathcal{S} \) and obtain an analogous condition. The necessary optimality condition is illustrated in Figure 4.2. Recall that \( \alpha_\perp \) is the outward normal direction for a given \( v(\alpha) \).

**Lemma 4.3.** If an optimal path contains an \( \mathcal{S} \mathcal{C} \mathcal{S} \) path with the directions of the straight line segments being \( \alpha_1 \) and \( \alpha_2 \), such that without loss of generality, \( \alpha_2 \in (\alpha_1, \alpha_1 + \pi) \), there exists \( \varepsilon > 0 \) such that \( v(\alpha) \leq v(\alpha) \) for all \( \alpha \in (\alpha_\perp - \pi/2, \alpha_1 + \varepsilon] \cup [\alpha_2 - \varepsilon, \alpha_\perp + \pi/2) \)

**Proof.** This result can be proven in a similar manner to the proof of Lemma 4.2. The deformations which need to be considered in order to obtain the result are shown in Figure 4.2(a). The domain on which the condition holds is different from the domain in Lemma 4.2 as the central arc is \( \mathcal{C} \) instead of \( \mathcal{C} \).

4.3.2 **Type 2: \( \mathcal{C} \mathcal{C} \mathcal{S} \) or \( \mathcal{S} \mathcal{C} \mathcal{C} \) subpath**

Without loss of generality, consider \( \mathcal{C} \mathcal{C} \mathcal{S} \) subpaths only since the equivalent condition for \( \mathcal{S} \mathcal{C} \mathcal{C} \) can be obtained by simply considering the path in reverse. Unlike \( \mathcal{S} \mathcal{C} \mathcal{S} \) paths which have the directions of two straight line segments \( \alpha_1, \alpha_2 \), \( \mathcal{C} \mathcal{C} \mathcal{S} \) paths have the inflection direction \( \alpha_1 \) (the direction of the path at the inflection point between the consecutive \( \mathcal{C} \) arcs) and the direction of the straight line segment \( \alpha_2 \) (see for example Figure 4.4(a)). Hence, we are unable to deform an existing \( \mathcal{C} \mathcal{C} \mathcal{S} \) path by performing the four slides introduced
4.3 Necessary Conditions for Optimality

\[ \alpha_1 \Delta^+ + \Delta^- + \Delta^+ \Delta^- \]

\( (a) \) \( S\mathcal{T}S \) subpath

\[ (b) \) Necessary optimality condition of the directions of the straight line segments \( \alpha_1, \alpha_2 \) in terms of \( v(\alpha) \) with respect to \( \pi(\alpha) \). \( \alpha_\perp \) is the outward normal direction for \( \pi(\alpha) \)

Figure 4.2: \( S\mathcal{T}S \) subpath with the directions of the straight line segments \( \alpha_1, \alpha_2 \) shown in (a), with \( \Delta^-_1, \Delta^+_1, \Delta^-_2, \Delta^+_2 \) representing the four sliding deformations which can be performed. (b) illustrates the corresponding optimality conditions for \( \alpha_1, \alpha_2 \) in terms of the velocity function \( v(\alpha) \) (thick solid curve) with respect to \( \pi(\alpha) \) (solid line).
Figure 4.3: A roll $\Gamma_2^-$ performed on the lighter grey path to form the darker grey path for SCS paths. We consider a new type of deformation known as a roll, which is performed by rolling the central circle about the circle of the adjacent arc. The results in this section are obtained by considering two slides and one roll. Note that unlike a slide, a roll changes the directions of the inflection point and straight line segment simultaneously, which is why the approach is to consider the slides before finally considering the effect of performing the roll. An example of a roll is illustrated in Figure 4.3.

First consider the case of \textit{CCS}. For brevity, we only state the condition for $\textit{RLS}$ as the analogous condition can be obtained for $\textit{LRS}$ by simply reflecting the path. The necessary optimality condition is illustrated in Figure 4.4.

**Lemma 4.4.** If an optimal path contains an $\textit{RLS}$ path with the directions of the inflection and straight line segment being $\alpha_1$ and $\alpha_2$ respectively, there exists $\epsilon > 0$ such that $v(\alpha) \leq \overline{v}(\alpha)$ for all $\alpha \in [\alpha_1, \alpha_1 + \epsilon] \cup [\alpha_2 - \epsilon, \alpha_2 + \epsilon]$.

**Proof.** Performing deformation $\Delta_2^+$ in Figure 4.4(a) shows that $v(\alpha) \leq \overline{v}(\alpha)$ for all $\alpha \in [\alpha_2, \alpha_2 + \epsilon]$ in a similar manner to the proof of Lemma 4.2. The proof differs slightly for deformations $\Delta_1^+$ and $\Gamma_2^-$. Suppose that for any $\epsilon > 0$, there exists $\epsilon_2 \in (0, \epsilon)$ such that $v(\alpha) > \overline{v}(\alpha)$ for all $\alpha \in (\alpha_1, \alpha_1 + \epsilon_2]$. It follows by the piecewise continuity of $v(\alpha)$ that there exists $\alpha_1^* \in (\alpha_1, \alpha_1 + \epsilon_2]$ such that $v(\alpha) < \overline{v}^*(\alpha)$ for all $\alpha \in (\alpha_1, \alpha_1^*)$ where $\overline{v}^*(\alpha)$ is defined similarly.
4.3 Necessary Conditions for Optimality

\[ \alpha_2 \delta + \alpha_1 \Gamma - \alpha_2 \]

(a) \( RL S \) subpath
(b) Deformed path from slide \( \Delta_1^+ \)
(c) Path components \( P_1, P_2, Q_1^* \) from (b) rearranged

(d) Necessary optimality condition of inflection directions \( \alpha_1 \) and direction of the straight line segment \( \alpha_2 \) in terms of \( v(\alpha) \) with respect to \( \overline{v}(\alpha) \)

Figure 4.4: \( RL S \) subpath with inflection direction \( \alpha_1 \) and direction of the straight line segment \( \alpha_2 \) shown in (a), with \( \Delta_1^+, \Delta_2^+ \) representing the two sliding deformations and \( \Gamma_2^- \) the rolling deformation. The effects of deformation \( \Delta_1^+ \) are shown in (b) and (c) with darker grey indicating new subpaths added and lighter grey indicating old subpaths remaining. (d) illustrates the corresponding optimality conditions for \( \alpha_1, \alpha_2 \) in terms of the velocity function \( v(\alpha) \) (thick solid curve) with respect to \( \overline{v}(\alpha) \) (solid line).
to $\bar{v}$ except for directions $\alpha_1^*, \alpha_2$ instead of $\alpha_1, \alpha_2$. We perform deformation $\Delta_1^*$ until the direction of the new straight line segment is $\alpha_1^*$ as shown in Figure 4.4(b). Figure 4.4(c) shows the subpaths $P_1, P_2, Q_1^*$ that we need to consider in comparing the cost of the new path to that of the original. Let $T(E)$ denote the cost of $E$ subject to velocity function $v$ and $T^*(E)$ denote the cost of $E$ subject to velocity function $\bar{v}$. The change in cost, $dT$ of this action is given by:

$$dT = T(Q_1^*) - T(P_1) - T(P_2)$$

$$< T^*(Q_1^*) - T^*(P_1) - T^*(P_2), \text{ since } v(\alpha) < \bar{v}^*(\alpha) \forall \alpha \in (\alpha_1, \alpha_1^*)$$

$$= 0 \text{ by Lemma 4.1}$$

This contradicts the optimality of the original $\mathcal{RLS}$ path. Hence, there exists $\varepsilon > 0$ such that $v(\alpha) \leq \bar{v}(\alpha)$ for all $\alpha \in [\alpha_1, \alpha_1 + \varepsilon]$. Applying this result while considering the rolling deformation $\Gamma_2^-$ then gives us the final result.

For the $\mathcal{CLS}$ case, we present the necessary optimality condition for $\mathcal{LRS}$ since we can obtain the condition for $\mathcal{RLS}$ by simply reflecting the path. The condition is illustrated in Figure 4.5.

**Lemma 4.5.** If an optimal path contains an $\mathcal{LRS}$ path with the directions of the inflection and straight line segment being $\alpha_1$ and $\alpha_2$ respectively, there exists $\varepsilon > 0$ such that $v(\alpha) \leq \bar{v}(\alpha)$ for all $\alpha \in [\alpha_1 - \varepsilon, \alpha_1] \cup [\alpha_2 - \varepsilon, \alpha_\perp + \pi/2]$.

**Proof.** This result can be proven in a similar manner to the proof of Lemma 4.4. The deformations which need to be considered in order to obtain the result are shown in Figure 4.5(a). The domain on which the condition holds is different as the central arc is $\overline{C}$ instead of $\overline{C}$. In particular, deformation $\Delta_2^+$ can be performed arbitrarily far, which makes the optimality condition hold on $[\alpha_2, \alpha_\perp + \pi/2)$ instead of just $[\alpha_2, \alpha_2 + \varepsilon)$.

Lemmas 4.2 – 4.5 provide useful conditions for narrowing down the possibilities of paths which need to be considered when we wish to construct an optimal path of Type 1 or 2. We now direct our focus to paths with an $\mathcal{SCS}$ subpath (Type 1) in the following Subsections 4.3.3 and 4.3.4.
4.3 Necessary Conditions for Optimality

\[ \alpha^2 \Delta^+ + \alpha^1 \Delta^- - \Gamma^- 2 \alpha^1 \]

\[ (a) \text{LRS subpath} \]

\[ \text{v}(\alpha) \]

\[ \alpha_\perp + \pi/2 \]

\[ O \]

\[ \alpha_2 - \epsilon \]

\[ \Gamma^-_2 \]

\[ \Delta^+_2 \]

\[ \alpha_1 - \epsilon \]

\[ \Delta^-_1 \]

\[ (b) \text{Necessary optimality condition of } \text{v}(\alpha) \]

Figure 4.5: LRS subpath with inflection direction \( \alpha_1 \) and direction of the straight line segment \( \alpha_2 \) shown in (a), with \( \Delta^-_1, \Delta^+_2 \) representing the two sliding deformations and \( \Gamma^-_2 \) the rolling deformation. (b) illustrates the corresponding optimality conditions for \( \alpha_1, \alpha_2 \) in terms of the velocity function \( \text{v}(\alpha) \) (thick solid curve) with respect to \( \text{v}(\alpha) \) (solid line).
4.3.3 Double Tangents of Star Convex Planar Curves

Earlier in this section, we proved necessary optimality conditions for Type 1 (contains SCS subpath) paths. In this subsection, we establish a result which extends fundamental theory on evaluating the number of double tangents of planar curves, first studied by Fabricius-Bjerre [34]. These results provide a useful bound on the number of pairs of directions of the straight line segments $a_1, a_2$ when constructing candidate optimal paths of Type 1.

A double tangent $r$ to a closed differentiable curve in the plane is either exterior or interior depending on whether the convex arcs in the neighbourhoods of the points through which $r$ is tangential to, are on the same or opposite sides of $r$ respectively. See Figure 4.6 for an example illustrating the difference between external and internal double tangents. It is shown in [34] that given a closed differentiable curve in the plane with $f$ inflection points, and $x$ crossing points (points where the curve crosses itself), the number of external double tangents $e$ and the number of internal double tangents $i$, satisfy the equality $e = i + x + f/2$.

The fundamental result shown in [34] has been extended to polygonal curves in [6] and [37], while a spherical version is given in [67], and anti-convex curves are considered in [65]. While the aforementioned papers provide identities relating the number of external double tangents to the number of internal double tangents and other values such as inflection and crossing points, they do not provide bounds for the number of the external or internal double tangents individually. Other work on double tangents prove inequalities for the number of external and internal double tangents under specific conditions such as curves with no inflection points in [38] and [49]. However, examples are provided in these papers which illustrate that it is possible to have an unbounded number of double tangents even if there are no crossing points. A curve is star convex if there exists a point such that a line drawn from that point to any point on the curve does not intersect the curve anywhere else. In this chapter, we provide a sharp bound for the number of external (and internal) double tangents which applies to star convex closed curves.

For the purpose of this chapter, we relax assumptions commonly employed in the literature on double tangents and adjust our definitions accordingly as follows:
1. Instead of assuming that no triple tangents exist, we identify a double tangent by an unordered pair of points on the curve through which it is tangential. This means that if the curve has a triple tangent through \( a, b, c \), we consider it as three distinct double tangents \( \{a, b\}, \{b, c\}, \{a, c\} \);

2. Instead of assuming that no double tangent is tangent at an inflection point, we exclude these from the definition of a double tangent, so even if they do exist, they do not affect the bound we establish in this section.

An immediate advantage of considering a star convex closed planar curve is that there are no crossing points, and we can define a sign convention for curvature with respect to a choice of origin for which the curve is star convex. For convenience, we restrict ourselves to considering strict \( C^2 \) curves so that the signed curvature \( K \) of the curve can be defined, and this only changes sign at inflection points. This assumption can be easily relaxed to \( C^0 \) piecewise \( C^1 \) curves through appropriate modifications to the arguments in this section involving \( K \) and allowing supporting hyperplanes to be counted as tangents. In particular, our assumption allows us to easily specify whether an external double tangent is a Positive External Tangent (PET) or Negative External Tangent (NET), based on whether the curvature of the curve in the neighbourhoods of the contact points is positive or negative respectively. See Figure 4.6 for an example illustrating the difference between PETs and NETs.

If we treat the plot of \( v(\alpha) \), in polar coordinates, as a planar curve \( V \) parametrised such that \( V(\alpha) = (v(\alpha), \alpha) \), it follows that \( V \) is a star convex, closed planar curve (in particular, it is star convex with respect to the origin). Since we identify a double tangent by a pair of points on the curve, we can also identify a double tangent to a velocity function by the pair of directions \( \{a_1, a_2\} \) which corresponds to the points \( \{V(a_1), V(a_2)\} \). Without loss of generality, we can order the pair of directions such that \( a_2 \in (a_1, a_1 + \pi) \). We will assume this ordering throughout the remainder of this section. This allows us to state the following result which follows from Lemmas 4.2 and 4.3.

**Corollary 4.6.** Let \( E \) be an optimal CSCS (or degeneracy) path with an SCS subpath with the directions of the straight line segments being \( a_1 \) and \( a_2 \). Then, \( \{a_1, a_2\} \) is a PET.

We can now prove the bound on the number of PETs for a star convex closed, planar
Since $\alpha$ for example. $j$ and $K$ and $K$ ordered in anticlockwise ascending order such that $K(\alpha) \geq 0 \forall \alpha \in [\alpha_0, \alpha_1]$. See Figure 4.7 for example.

For $j = 1, \cdots, n$ we will refer to the $n$ regions where $K \geq 0$ as the convex regions $\beta_{2j-2} = [\tilde{\alpha}_{2j-2}, \tilde{\alpha}_{2j-1}]$ and the $n$ regions where $K \leq 0$ as the concave regions $\beta_{2j-1} = [\tilde{\alpha}_{2j-1}, \tilde{\alpha}_j]$ where $\tilde{\alpha}_f = \tilde{\alpha}_0$ and $\beta_j = \beta_0$.

Observe that if we choose a pair of concave regions (not necessarily distinct) $(\beta_{2j-1}, \beta_{2k-1})$ for $j, k = 1, \cdots, n$, then it is impossible to have both $\tilde{\alpha}_{2j} \in (\tilde{\alpha}_{2j-1}, \tilde{\alpha}_{2j-1} + \pi)$ and $\tilde{\alpha}_{2j} \in (\tilde{\alpha}_{2k-1}, \tilde{\alpha}_{2k-1} + \pi)$. If at least one of these holds, then suppose without loss of generality that $\tilde{\alpha}_{2k} \in (\tilde{\alpha}_{2j-1}, \tilde{\alpha}_{2j-1} + \pi)$. A PET $\{a_1, a_2\}$ corresponds to this choice of unordered pair $\{j, k\}$ if $a_1 \in \beta_{2j-2}, a_2 \in \beta_{2k}$, where $a_2 \in (a_1, a_1 + \pi)$.

Now we show that each PET (that exists) corresponds to $\{j, k\}$ for some $j, k = 1, \cdots, n$. Suppose there exists a PET which does not correspond to any $\{j, k\}$. The unordered pairs cover all possible scenarios of having a PET that is tangential to two points in convex regions, except if $K(\alpha) \geq 0 \forall \alpha \in [a_1, a_2]$. We will show this leads to a contradiction. Note that the ordering of $a_1, a_2$ is essential in this statement since it can be seen in Figure 4.7 that $\{a'_1, a'_2\}$ is a PET tangential to two points in the same convex region and so $K(\alpha) \geq 0 \forall \alpha \in [a'_2, a'_1]$, but not for $\alpha \in [a'_1, a'_2]$. Suppose $K(\alpha) \geq 0 \forall \alpha \in [a_1, a_2]$ and let $\tau(\alpha) \in \mathbb{R}/2\pi\mathbb{Z}$ denote the direction of the forward tangent at $V(\alpha)$. Since $v$ is star convex, $\tau(\alpha) \in (\alpha, \alpha + \pi) \forall \alpha \in \mathbb{R}/2\pi\mathbb{Z}$. Since $\alpha$ is strict and $\tau(\alpha_1) = \tau(\alpha_2)$, there must exist $\alpha^* \in (a_1, a_2)$ such that $\tau(\alpha^*) = \tau(\alpha_1) + \pi$. Since $\alpha^* \in (a_1, a_2)$, it follows by star convexity that $\tau(\alpha^*) \in (a_1, a_2 + \pi)$.

Since $\tau(\alpha_1) = \tau(\alpha_2)$ and $a_2 \in (a_1, a_1 + \pi)$, it follows that $\tau(\alpha_1) \in (a_2, a_1 + \pi)$. Since $\tau(\alpha^*) = \tau(\alpha_1) + \pi$, this gives $\tau(\alpha^*) \in (a_2 + \pi, a_1)$ which contradicts $\tau(\alpha^*) \in (a_1, a_2 + \pi)$ obtained by considering star convexity. Hence, there cannot exist a PET $\{a_1, a_2\}$ such that $K(\alpha) \geq 0 \forall \alpha \in [a_1, a_2]$.

Next, we show that there can be at most one PET corresponding to an unordered pair.
4.3 Necessary Conditions for Optimality

Figure 4.6: A closed planar curve star convex with respect to $O$, with $2n = 6$ inflection points marked by the solid dots. An example of each distinct type of double tangent considered is labelled on the diagram, with crosses marking the contact points. The solid and dashed lines are different types of external double tangents while the dotted line is an internal double tangent. This example has the maximum number of double tangents possible with 6 inflection points, $e^+ = 6, e^- = 3, i = 6$.

Figure 4.7: A velocity function with origin at $O$ and $2n = 4$ inflection points in the directions $\tilde{a}_1, \ldots, \tilde{a}_4$, dividing the domain into convex regions $\beta_0, \beta_2$ and concave regions $\beta_1, \beta_3$. The PET $\{a_1, a_2\}$ corresponds to the choice of concave regions $\{\beta_1, \beta_1\}$, with forward tangent direction $\tau(a_1) = \tau(a_2)$. The two other PETs $\{a'_1, a'_2\}$ and $\{a''_1, a''_2\}$ are shown in grey, corresponding to the choice of concave regions $\{\beta_1, \beta_3\}$ and $\{\beta_3, \beta_3\}$ respectively. This example has the maximum number of double tangents possible with 4 inflection points, $e^- = 3, e^+ = 1, i = 2$. 
{j, k}. Suppose the distinct PETs \{a_1, a_2\} and \{a_1', a_2'\} both correspond to \{j, k\}. Note that the ordering of \(a_1, a_2\) and \(a_1', a_2'\) are essential again at this step, since it is clearly possible as seen in Figure 4.7 to have two PETs between the same convex regions (\{a_1, a_2\} and \{a_1'', a_2''\} are both PETs between \(\beta_0\) and \(\beta_2\)). However, due to the enforced ordering, any other PET between the same convex regions will correspond to a different unordered pair (in the example, \{a_1, a_2\} corresponds to \{1, 1\} while \{a_1'', a_2''\} corresponds to \{2, 2\}).

Let \(H, \hat{H} \subset \mathbb{R}^2\) denote the half-planes containing the origin which are divided by the PETs \{a_1, a_2\} and \{a_1', a_2'\} respectively. Let \(\beta_1' = (\tilde{a}_{2k} - \pi, \tilde{a}_{2j-1})\) and \(\beta_2' = (\tilde{a}_{2k}, \tilde{a}_{2j-1} + \pi)\). Since \(a_1, a_1' \in \beta_{2j-2}\) and \(a_2, a_2' \in \beta_{2k}\), it follows that \(a_1, a_1' \in \beta_1'\) and \(a_2, a_2' \in \beta_2'\). Since \(v(a) > 0\ \forall \ a \in \mathbb{R}^2/2\pi\) and the size of the intervals \(\beta_1', \beta_2'\) are less than \(\pi\), it follows from \(K(a) \geq 0\ \forall \ a \in \beta_1' \cup \beta_2'\) that \(\forall \ a \in H \cap \hat{H}\) \(\forall \ a \in \beta_1' \cup \beta_2'\).

If \(H = \hat{H}\), then both tangents are collinear. Without loss of generality, let \(a_2 \in (a_2', a_2 + \pi)\) so \(\{a_2, a_2'\}\) is also a PET. Since \(a_2, a_2' \in \beta_{2k}\), it follows that \(K(a) \geq 0\ \forall \ a \in (a_2, a_2')\) which is a contradiction since we showed earlier in this proof that there cannot exist such a PET.

Hence, we can assume \(H \neq \hat{H}\). Since either \(a_1 \in (a_1, \tilde{a}_{2j-1})\) or \(a_1 \in (a_1', \tilde{a}_{2j-1})\), we can assume without loss of generality that \(a_1 \in (a_1, \tilde{a}_{2j-1})\). Since \(K(a) \geq 0\ \forall \ a \in (a_1, \tilde{a}_{2j-1})\), it follows that \(\tau(a_1) \in (a_1, \tilde{a}_{2j-1} + \pi)\). This would mean the PET \{\(a_1, a_2'\}\} divides the plane such that \(V(a_1)\) and \(V(a_2)\) lie on different sides, so either \(V(a_1) \not\in H\) or \(V(a_2) \not\in \hat{H}\). This leads to a contradiction since \(V(a) \in H \cap \hat{H}\ \forall \ a \in \beta_1' \cup \beta_2'\).

Since there are \(\binom{n+1}{2}\) unordered pairs \(\{j, k\}\) for \(j, k = 1, \ldots, n\), this concludes the proof.

Remark: A similar approach can be applied to show that for a star convex closed planar curve with \(2n\) inflection points, the number of NETs \(e^- \leq \binom{n}{2}\) and the number of internal double tangents \(i \leq 2\binom{n}{2}\). NETs and internal double tangents do not have a particular use for the application being considered in this chapter, but will be used in Section 5.4 of the following chapter and may be useful for other applications where star convex planar curves are used.

Note that the bounds given by Theorem 4.7 and Remark 4.3.3 are sharp bounds since it is possible to construct a velocity function \(v\) such that \(e^+ = \binom{n+1}{2}, e^- = \binom{n}{2}, i = 2\binom{n}{2}\) for any \(n\). Such a velocity function can be constructed by placing all the inflection points in a
half-plane so that the convex and concave regions are all in one half-plane. For example, see Figures 4.6 and 4.7 where \( n = 3 \) and \( n = 2 \) respectively.

4.3.4 Form reduction based on \( \alpha_p, \alpha_q \)

In this subsection, we reduce the number of forms of paths that need to be considered in order to improve the efficiency of an algorithm to construct optimal paths. Consider Type 1 paths with an SCS subpath for a fixed pair of directed points \( p, q \). If we further specify that the two straight line segments must be a chosen pair of fixed directions \( \alpha_1, \alpha_2 \) (in either order), it is possible to construct at most 16 distinct Type 1 paths, though not all may exist for some \( p, q \). The 16 possibilities arise from the 4 independent binary choices: the senses of the three arcs and the order of the directions of the straight line segments \( \alpha_1, \alpha_2 \). Note that the order of the straight line segments is equivalent to the choice of the middle arc being \( C \) or \( \bar{C} \). The exception occurs when the first or last arc is degenerate, resulting in 8 distinct paths (and only 4 if both arcs are degenerate). From this observation and Theorem 4.7, it follows that given a \( v \) with \( 2n \) inflection points, we can construct the optimal Type 1 path between any two directed points by computing the costs of \( 16 \binom{n+1}{2} \) paths. However, we will show in this subsection that we can narrow this down to computing the costs of \( 4 \binom{n+1}{2} \) paths at most by only considering the directions of the given start and end directed points \( \alpha_p \) and \( \alpha_q \).

A direction plot of a CS-path is a polar plot of all the directions traversed along that path, plotted at increasing radii for different \( C \) arcs in the order traversed, with radial line segments representing \( S \) segments. This contains all the information required to compute the total cost of that path, provided the lengths of all \( S \) segments are separately specified. A direction plot can be collapsed by plotting the directions traversed at the lowest levels available without overlap. See Figures 4.8 (a), (c) for an example of a direction plot and a corresponding collapsed direction plot.

Observe that the angles spanned by the direction plot of a Type 1 path are determined entirely by the directions \( \alpha_p, \alpha_q \) relative to the directions of the straight line segments \( \alpha_1, \alpha_2 \) and which of the 16 path forms we are considering. This observation means that we can eliminate most of the 16 path forms by comparing the direction plots of the path
forms for the different cases of $\alpha_p, \alpha_q$ relative to $\alpha_1, \alpha_2$. More specifically, some path forms will be obviously non-optimal as they contain redundant components compared other path forms. Furthermore, we can apply Lemmas 4.2 and 4.3 to further improve the efficiency of the algorithm by checking whether each PET intersects the velocity function anywhere other than the double tangent points. This will be explained in more detail later in this section. We present the results in Table 4.1, and present the proof of one case to illustrate the technique used to obtain the results.

In full generality, there are 18 cases to consider, depending on where $\alpha_p, \alpha_q$ lie relative to the directions of the straight line segments $\alpha_1, \alpha_2$. Recall that without loss of generality, $\alpha_2 \in (\alpha_1, \alpha_1 + \pi)$. We denote each case using the notation $b_pb_q$, where $b_p, b_q \in \{1, +, 2, -\}$. $b_p = 1$ if $\alpha_p = \alpha_1$, $b_p = 2$ if $\alpha_p = \alpha_2$, $b_p = +$ if $\alpha_p \in (\alpha_1, \alpha_2)$ (positively spanned direction), $b_p = -$ if $\alpha_p \in (\alpha_2, \alpha_1)$, and likewise conditions on $b_q$ based on $\alpha_q$. This gives 16 cases from the 4 independent choices of each $b_p, b_q$, but $++$ and $--$ give two further sub-cases to consider based on the ordering of $\alpha_p$ and $\alpha_q$ within the open interval $(\alpha_1, \alpha_2)$. This gives us 18 cases each with their own set of potential optimal path forms.

By considering symmetry of swapping $\alpha_1$ with $\alpha_2$ and flipping the path, as well as the symmetry of swapping $\alpha_p$ with $\alpha_q$ and reversing the path, we need to consider only 7 cases where $(\alpha_q - \alpha_1) \geq (\alpha_p - \alpha_1)$ if we take the values within the brackets to be in $[0, 2\pi)$, and extrapolate the remaining cases by appropriate transformations of the optimal path forms.

For each of these 7 cases, a path form of specified left or right-turning senses for the $C$ components and specified order of the $\alpha_1, \alpha_2$ can be eliminated if it satisfies one of the following conditions:

1. The collapsed direction plot of a path of this form is identical to the collapsed direction plot of a $CCSCSC$ (or $CSCSCC$) path and hence, is not optimal by Theorem 3.17;

2. The collapsed direction plot is identical to the collapsed direction plot of a different $CSCSC$ path but in addition has an extra circle of directions.

For some of these cases, we find that a few distinct path forms share the same directional plot and are therefore equal in cost for any velocity function. These are denoted by
4.3 Necessary Conditions for Optimality

~ in Table 4.1 and can be treated as one candidate path form since only the cost of one of the forms needs to be computed. The result is that the worst case becomes computing the costs of 4 candidate paths, as opposed to 16, which significantly improves the efficiency of the algorithm.

The information obtained from this trimming procedure can be combined with Lemmas 4.2 and 4.3 to potentially further reduce the number of path forms which need to be computed. These lemmas place additional necessary optimality conditions on the PET apart from simply satisfying Corollary 4.6. For instance, if the PET \( \{a_1, a_2\} \) intersects the velocity function in \((a_1, a_2)\) then any Type 1 path with an \(S\) subpath is either not optimal, or is of equivalent cost to another Type 1 path corresponding to a different PET (this only occurs when there is a triple tangent). This means we do not need to compute the cost of these paths and can ignore them.

Let \( G \) denote the set of all PETs \( \{a_1, a_2\} \) which do not intersect the velocity function in \((a_1, a_2)\). Let \( a_\perp \) be the outward normal direction of a given PET. Let \( \overline{G} \) denote the set of all PETs \( \{a_1, a_2\} \) which do not intersect the velocity function in \((a_\perp - \pi/2, a_1) \cup (a_2, a_\perp + \pi/2)\). Let \( G = G \cup \overline{G} \). Note that this means if a PET intersects the velocity function in both \((a_1, a_2)\) and \((a_\perp - \pi/2, a_1) \cup (a_2, a_\perp + \pi/2)\), it does not belong to any of these sets, because we do not need to consider this PET at all. For an example of these different types of PETs, see Figure 4.9(a).

Table 4.1 shows the path forms which need to be considered for a PET depending on the particular case of \( b_p b_q \), and whether the PET belongs in \( G, \overline{G}, \overline{G} \).

Recall conditions (1) and (2) stated earlier in this subsection as follows:

1. The collapsed direction plot of a path of this form is identical to the collapsed direction plot of a \(CCSCSC\) (or \(CSCSCC\)) path and hence, is not optimal by Theorem 3.17;

2. The collapsed direction plot is identical to the collapsed direction plot of a different \(CSCSC\) path but in addition has an extra circle of directions.

We now provide an example of where each technique can be applied, corresponding to the case \(+−\). Figure 4.8 illustrates how condition (1) above can be applied to show \(LS\) is not optimal. Since the direction plots of the \(LS\) and \(LR\) paths
Table 4.1: Path forms which need to be considered for each of the 7 cases of $b_p b_q$

<table>
<thead>
<tr>
<th>$b_p b_q$</th>
<th>$G$</th>
<th>$\overline{G}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$SLS \sim LSLS \sim STSR \sim RSRS$</td>
<td>$SLSR \sim LSRS$</td>
</tr>
<tr>
<td>12</td>
<td>$SLS$</td>
<td>$SRS$</td>
</tr>
<tr>
<td>1+</td>
<td>$SLSR$</td>
<td>$S\overline{RSR}$</td>
</tr>
<tr>
<td>1−</td>
<td>$SLSL$</td>
<td>$S\overline{LSR}$</td>
</tr>
<tr>
<td>++</td>
<td>$RSLSR$</td>
<td>$RS\overline{LSR}$</td>
</tr>
<tr>
<td>−−</td>
<td>$LSRSL$</td>
<td>$LS\overline{SR}$</td>
</tr>
<tr>
<td>−−</td>
<td>$RSRSL$</td>
<td>$RS\overline{LSR}$</td>
</tr>
</tbody>
</table>

shown in Figures 4.8 (a) and (b) share the same collapsed direction plot shown in (c), the paths are of equal cost. Since $LR\overline{SRS}$ is not optimal, this means that $LS\overline{SL}$ is also not optimal. The direction plots can also be used to apply condition (2) above to show that $LS\overline{LSR}$ is not optimal when compared with $LSRSL$.

4.4 Algorithm

In this section, we first restrict ourselves to considering piecewise constant velocity functions as these are a suitable choice for the application of modelling support costs in different ranges of directions for developing an underground mine decline. For each range of directions, a constant cost is considered. We present an algorithm which exactly constructs the optimal path by simply computing a finite set of candidate paths in $O(n^2)$ operations where $n$ is the number of discontinuities of the velocity function. In particular, we show that there are at most $10n^2 + 26n + 8$ paths to compare. We then present a heuristic for general velocity functions by modifying the exact algorithm for piecewise constant velocity functions. Finally we provide a sample output of the algorithm.
Figure 4.8: Direction plots illustrating a technique used to rule out having to consider $\mathcal{LSL}$ paths for the case $+-+-$. The directions of the straight line segments are denoted by $\alpha_1, \alpha_2$ with lengths $l_1, l_2$ respectively. The start and end directions are given by $\alpha_p, \alpha_q$ respectively.
Given a piecewise constant velocity function $v$ (for example see Figure 4.9(a)), we can divide its domain $\mathbb{R}^2/2\pi$ into regions where $v$ is constant with $n$ directions where $v$ is discontinuous $\hat{\alpha}_0, \ldots, \hat{\alpha}_{n-1}$, ordered in anticlockwise ascending order. It is easily shown that since $v$ is the limit of a sequence of $C^2$ SPNC velocity functions with $2n$ inflection points, all the results proven in Section 4.3 hold with very slight modifications. In particular, we relax the condition of tangency to include supporting hyperplanes, and require that $v$ take the higher value of the neighbouring constant values at discontinuities.

Recall that we have the following distinct types of paths to consider when constructing the optimal path:

Type 1: has a non-degenerate SCS subpath
Type 2: has a non-degenerate CCS or SCC subpath
Type 3: is a CSC, CCC path or degeneracy of either.

We now describe the algorithm OPTIMALPATH which solves the problem for piecewise constant velocity functions. This algorithm works by computing (exactly) the cost of each of the candidate optimal paths of each type individually and selecting the path of minimum cost. We describe the procedure for generating all candidate optimal paths for each type. We also provide a bound for the number of candidate optimal paths for each type.

4.4.1 Type 1

The first step of the algorithm is to compute the sets of PETs $\mathcal{G}, \mathcal{G}, G$. As explained in Section 4.3.4, this can be performed without any knowledge of the start and end directed points $p, q$. Hence, the sets of PETs can be computed in a preprocessing phase. Since we consider a piecewise constant velocity function $v$, the sets $\mathcal{G}, \mathcal{G}, G$ can be computed exactly in $O(n^2)$ steps where $n$ is the number of discontinuities of $v$. By Corollary 4.6, we need only consider Type 1 paths with the directions of its straight line segments in $\alpha_1$ and $\alpha_2$ where $\{\alpha_1, \alpha_2\}$ is a PET. Then, for the given pair of start and end directed points, we apply Table 4.1 to narrow down the path forms which need to be computed.
and compared. This results in at most $4\binom{n+1}{2}$ candidate optimal paths.

### 4.4.2 Type 2

Type 2 presents the most difficulty when attempting to construct the optimal path for a general cost function. However, since we are considering piecewise constant velocity functions, it can be shown by applying Lemmas 4.4 and 4.5 that there are no more than $16(n + \frac{n+1}{2})$ Type 2 paths satisfying the respective necessary optimality conditions. The $16n$ paths arise from choosing the direction of the straight line segment to be one of the directions of discontinuity $\hat{a}_0, \ldots, \hat{a}_{n-1}$, and the 4 binary choices: the senses of the start and end arcs, whether the central arc is $\mathcal{C}$ or $\overline{\mathcal{C}}$ and whether the path is a $\mathcal{CSC}$ or $\mathcal{CSC}$.

It is easily shown that it is not possible for a Type 2 path with a $\mathcal{C}$ central arc to satisfy Lemma 4.4 unless the direction of the straight line segment is the direction of a discontinuity. It is however possible for a Type 2 path with a $\overline{\mathcal{C}}$ central arc to satisfy Lemma 4.5 without the direction of the straight line segment being a direction of a discontinuity. The remaining $16\binom{n+1}{2}$ paths correspond to this case and is obtained by considering the choices of unordered pairs (not necessarily distinct) $\beta_1, \beta_2 \in \{\hat{a}_0, \hat{a}_{n-1}\}$. For each of these pairs, we then choose a form out of the 8 possible forms with a $\mathcal{C}$ central arc. It can be shown in a similar manner to the proof of Theorem 4.7 that since $v$ is piecewise constant, there can exist at most 2 distinct paths of a specified form (out of the 8) satisfying Lemma 4.5 and each of these paths must correspond to a choice of unordered pair $\beta_1, \beta_2$.

### 4.4.3 Type 3

These paths are trivial and can be constructed and computed exactly. There are 8 such paths at most, though not all may exist, depending on the given start and end directed points. Additional trimming procedures can be applied here which can improve the average efficiency of the algorithm, but do not improve the worst case complexity.

**Theorem 4.8.** The algorithm $\text{OPTIMALPATH}$ correctly constructs an optimal path for a given pair of directed points and piecewise constant velocity function in $O(n^2)$ steps where $n$ is the number of discontinuities in the piecewise constant function.
Proof. The result follows from Theorem 3.17, Lemmas 4.2 - 4.5 and Theorem 4.7.

Remark: The exact algorithm described above can be easily modified into a heuristic for a general velocity function.

The step for generating Type 1 paths will differ in that the double tangents can’t be determined exactly, but Theorem 4.7 guarantees that the optimal Type 1 path can be found subject to the accuracy of determining the double tangents.

For Type 2 paths, it is difficult to solve for the optimality conditions and feasibility for given directed points. Furthermore, there is no bound established for the number of candidate optimal paths that need to be considered. Hence, we approximate the given velocity function with a piecewise constant velocity function with \( n \) discontinuities for which we can easily construct the optimal Type 2 path in \( O(n^2) \) steps. The accuracy of this step is subject to the accuracy of the piecewise constant approximation.

Type 3 paths are simply constructed in the same way since they do not depend on the velocity function.

4.4.4 Examples

The exact algorithm for piecewise constant velocity functions was implemented in MATLAB to generate the optimal path for the examples in this subsection. The velocity function is shown in Figure 4.9(a) where the PETs are illustrated with solid, dotted and dashed denoting whether they belong to \( G \) or exclusively to \( G_\cap \overline{G} \) respectively. The optimal (minimum cost) path is shown in solid line in Figure 4.9(b) while the shortest (minimum length) path is shown in dashed for comparison. This example has a Type 2 path as the optimal path for the particular choice of start and end directed points.

A graphical user interface (GUI) was also developed in MATLAB to generate optimal paths for user specified inputs which can be easily entered and modified. A simple example is shown in Figure 4.10 where two optimal paths with equal cost exist. The example in Figures 4.11 and 4.12 illustrate that there still exists an optimal path of the form \( CSCSC \) (or degeneracy) which can be very easily computed regardless of how complicated the velocity function is. These two examples have 16 discontinuities in the velocity function.
and have 38 and 24 PETs respectively. Figures 4.13 - 4.17 show an example of a 9.16% total cost saving of the optimal path compared with the shortest path, passing through a series of specified directed waypoints A to F. This is shown in separate figures for clarity of displaying the progress of the algorithm as it is applied to each consecutive pair of directed points. This illustrates how the algorithm can be incorporated into a procedure which generates the optimal decline in an underground mine.

4.5 Conclusion

In this chapter, we presented an $O(n^2)$ exact algorithm which constructs the optimal curvature-constrained directional cost path between any pair of directed points, for a given piecewise constant directional cost function with $n$ discontinuities. This algorithm can be modified into a heuristic for a general directional cost function, by approximating the function with a piecewise constant function. In Chapter 3, it was shown that there exists a $CSCSC$ (or degeneracy) path that is an optimal curvature-constrained directional cost path. However, this result was insufficient to construct an optimal path. In this chapter, we have found necessary optimality conditions that can be applied to efficiently construct optimal paths. A sharp bound for the number of double tangents of star convex closed planar curves is proved which is applied to bound the number of paths with $SCS$ subpaths that satisfy the corresponding optimality condition. The results in this chapter can be applied to improve underground mine planning algorithms as they can account for the differing costs of access development in ground with anisotropic geological properties.
Algorithm for Constructing Optimal Path

(a) Velocity function with solid, dotted and dashed PETs denoting whether they belong to $G$ or exclusively to $G_r$ respectively. The direction of the straight line segment $\alpha_S$ and inflection $\alpha_f$ are shown in grey satisfying Lemma 4.5.

Figure 4.9: Example of the MATLAB output produced by the algorithm for piecewise constant velocity functions, with the optimal path shown in (b) for the velocity function in (a).

(b) Optimal path shown in solid, shortest path shown in dashed. The optimal path in this example is an $LSLR$ Type 2 path, while the shortest path is an $RSL$ path.
4.5 Conclusion

Figure 4.10: Screenshot of the MATLAB GUI implementing **OPTIMALPATH**. The input parameters include the velocity function which can be entered and manipulated with the top left input boxes and buttons. The velocity function is displayed in the bottom left with the relevant PETs coloured differently depending on whether they belong to $G$ or exclusively to $G \setminus G$. The input parameters also include the directed points $p$ and $q$ where direction is measured in degree anticlockwise from the positive x-axis, while the minimum radius of curvature can be arbitrarily specified by $R$. The optimal paths are shown in solid line, and the shortest path in dotted, with their respective lengths and cost listed in the top right box.
Figure 4.11: Screenshot of the MATLAB GUI implementing OPTIMALPATH. The velocity function has been randomly generated with $n = 16$ discontinuities, resulting in 38 PETs. The optimal path is shown in solid line, and the shortest path in dotted, with their respective lengths and cost listed in the top right box. The optimal path in this example is of the form $RLSR$ while the shortest path is of the form $RLR$. 
4.5 Conclusion

Figure 4.12: Screenshot of the MATLAB GUI implementing OPTIMALPATH. The velocity function has been randomly generated with $n = 16$ discontinuities, resulting in 24 PETs. The optimal path is shown in solid line, and the shortest path in dotted, with their respective lengths and cost listed in the top right box. The optimal path in this example is of the form $RSLSL$ while the shortest path is of the form $LRL$. 
Figure 4.13: Screenshot of the MATLAB GUI implementing OPTIMALPATH from A to B. A minimum radius of curvature of 25 units is applied in this example. The optimal path is shown in solid line, and the shortest path in dotted. The subsequent steps of this example are shown in Figures 4.14 - 4.17.
Figure 4.14: Screenshot of the MATLAB GUI implementing OPTIMALPATH from A to C.
Figure 4.15: Screenshot of the MATLAB GUI implementing OPTIMALPATH from A to D. The optimal path and the shortest path coincide between C and D.
Figure 4.16: Screenshot of the MATLAB GUI implementing OPTIMALPATH from A to E.
Figure 4.17: Screenshot of the MATLAB GUI implementing OPTIMALPATH. The optimal path passing through the specified directed waypoints A,B,C,D,E,F is shown in solid line, and the shortest path in dotted, with a total saving in cost of 9.16%. In between E and F, there are two optimal paths of the same cost and both are shown (in a similar manner to the example in Figure 4.10). For clarity, the steps of producing this path are shown in Figures 4.13 - 4.16.
Chapter 5
Incorporating the Gradient Constraint

In this chapter, we consider the problem of constructing an optimal curvature-constrained path in 3-dimensional space that is subject to a gradient constraint as well as anisotropic cost. This extends the work from earlier chapters, incorporating more practical constraints into the model of optimal declines in an underground mine. In the previous chapters, we developed theory on curvature-constrained paths with anisotropic cost. We introduce the gradient constraint to this problem by formulating it as a lower bound on length, thereby treating the problem as still being planar. The main result we prove in this chapter is that there exists an optimal path of the form CSCSCSC (or a degeneracy) with loops, where $C$ represents a continuous subset of the unit circle, $S$ represents a straight line segment, and loops are unit circles. This generalises the existing result that without the gradient constraint, there always exists a path of the form CSCSC or a degeneracy which is optimal. We also prove results which are useful for creating an algorithm which can construct an optimal path between a given pair of directed points with a prescribed directional cost function.

5.1 Introduction

The primary motivation for the mathematical problems studied in this thesis is to develop the necessary theory required to efficiently design a network of underground mine declines which captures the geometric constraints and models the cost of development in anisotropic ground conditions. In the previous chapters, we focused on curvature-constrained paths in the plane which are subject to anisotropic development costs. In order for the curvature-constrained path to be useful for underground mine declines, this path must be lifted into 3-dimensional space with a constant gradient that is no greater than a prescribed maximum gradient. In practice, this value is typically around 1 in 7. Since the maximum gradient is relatively shallow, the curvature constraint
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is commonly accepted to apply to the planar projection as opposed to the curve in 3-dimensional space.

We assume that the anisotropic behaviour is only dependent on the direction in the horizontal projection. This is a reasonable assumption if the faulting occurs along planes which are close to vertical. This allows us to reformulate the maximum gradient constraint of the 3-dimensional problem into a minimum length constraint on the 2-dimensional problem where we project the directed points onto the horizontal plane. In [15], extensions to Dubins paths are studied for this purpose. However, the methods introduced are insufficient to handle anisotropic cost since [15] only dealt with shortest curvature-constrained paths.

In this chapter, we apply some new techniques as well as generalise techniques from earlier chapters to establish the forms of paths which are optimal after incorporating the gradient constraint. In Chapter 3, we showed that there exists an optimal path of the form $CSCSC$ (or degeneracy) for the problem without the gradient constraint. For the problem without the gradient constraint, a path which contains a loop (the limit of $C$ arc as its length approaches $2\pi$, forming a unit circle) is clearly not optimal since we can construct a better path by removing the loop. Hence, we disregarded any paths which contained loops in the previous chapters. However, loops are important for the gradient-constrained problem as we may require that a path contain additional loops in order to satisfy the minimum length constraint. We will show that if the minimum length constraint is inactive, there exists an optimal path of the form $CSCSC$ (or degeneracy) with loops (if and when needed). We also show that if the minimum length constraint is active, there exists an optimal path of the form $CSCSCSC$ (or degeneracy) with loops. Given a CS-path $E$ and lower bound on length $L_{lb}$, it is clear that the correct number of additional loops to consider is the lowest number of loops required to make the CS-path with loops satisfy the minimum length constraint.

Constructing the optimal path for the gradient-constrained problem is easy if we know that there exists an optimal path such that the minimum length constraint is inactive, since the optimality conditions from Chapter 4 will still apply. The only modification to the algorithm OPTIMALPATH would be to include the minimum amount of loops
in order to satisfy the minimum length constraint. However, it is much harder to con-
stuct the path if the minimum length constraint is active since the optimality conditions
no longer apply. Furthermore, it is hard in general to know whether or not there exists
an optimal path such that the minimum length constraint is inactive. This motivates the
application of Lagrangian duality [12] to this problem.

The last major section of this chapter is to study the Lagrangian dual problem to the
gradient-constrained problem. Since our problem is not convex, we are unable to apply
strong Lagrangian duality. However, it is still very useful to study the dual problem
as we will show that it is much easier to solve. In the event that it does not provide
the optimal solution, it will still provide a useful lower bound on optimal cost of the
gradient-constrained problem. We show that in order to solve the dual problem, we
need to extend results from previous chapters in order to be able to handle directional
cost functions which take both positive and negative values.

The results proven in this chapter are useful for developing an algorithm which ef-
ficiently constructs an optimal network of underground mine declines in 3-dimensions.
The most important geometric constraints of the decline are the curvature and gradient
constraints, and within each geological domain, anisotropic cost behaviour can be ob-
served. This theory enables us to incorporate anisotropic cost behaviour into the existing
optimisation framework in [15] while still accounting for the gradient and curvature con-
straints.

5.2 Background

Apart from the background introduced in Sections 2.2.1, 2.2.3, 2.3 and 2.4.1, the following
additions are important for this chapter.

To simplify a proof of a key result, we will add an additional condition that the size
of the derivative of the directional cost function (where differentiable) is bounded.

We refer to maximal closed intervals \([a_1, a_2]\) where \(K(\alpha) \geq 0\) for all \(\alpha \in [a_1, a_2]\) as convex regions. Similarly, maximal closed intervals \([a_1, a_2]\) where \(K(\alpha) \leq 0\) for all \(\alpha \in [a_1, a_2]\)
will be referred to as concave regions. This means that given a strict velocity function, we
can divide the domain \( \mathbb{R}/2\pi \mathbb{Z} \) into convex and concave regions which overlap only at the \( 2n \) inflection points in the directions \( \tilde{a}_i \) where \( K(\tilde{a}_i) = 0 \) for \( i = 1, \cdots, 2n \).

Recall from Corollary 3.18 in Chapter 3 that given any velocity function \( v \), there exists a path of the form \( CSCSC \) (or degeneracy) which is an optimal curvature-constrained path from \( p \) to \( q \). We will generalise this result in this chapter to the case where a lower bound on length is applied.

### 5.2.1 Reversible directions

This chapter makes use of CS-path deformations and reversible directions introduced in Chapter 3. We state the critical definitions here, but refer the reader to Section 3.3 for details and proofs for the key results which will be used.

The degree \( d \) of a CS-path is the number of straight line segments and inflection points between two consecutive arcs in the path. Each of these represents a distinct interval or point at which a reversible CS-path deformation can be performed. A CS-path is reducible if it can be deformed to a CS-path of strictly lower degree with cost no greater than the original CS-path.

**Definition 5.1.** Given a CS-path with degree \( d \), let \( a_i \) for \( i = 1, \cdots, d \) denote the directions of the straights and inflection points in the order traversed by the path. We define \( A_i = (a_i, \rho_i) \) for \( i = 1, \cdots, d \) to be reversible directions, where

\[
\rho_i = \begin{cases} 
-1 & \text{if it corresponds to the inflection point in an } \mathcal{L}\mathcal{R} \text{ subpath} \\
0 & \text{if it corresponds to a straight } \mathcal{S} \text{ subpath} \\
1 & \text{if it corresponds to the inflection point in an } \mathcal{R}\mathcal{L} \text{ subpath} 
\end{cases}
\]

A pair of reversible directions \( A_i, A_j \) (\( i \neq j \)) are repeated if \( A_i = A_j \) (i.e. \( a_i = a_j \) and \( \rho_i = \rho_j \)). A set of reversible directions are said to be distinct if there are no repeated reversible directions in the set. For brevity, we adopt the following notation:

- \( \delta_i = 4 \sin \gamma_i \)
- \( \theta_i = a_i + \rho_i |\gamma_i| \)
- \( \tilde{\xi}_i = (\cos \theta_i, \sin \theta_i) \).
Given a reversible direction \((\alpha_i, \rho_i)\), there exists \(\varepsilon > 0\) such that for \(\gamma_i \in (-\varepsilon, \varepsilon)\) we can find a CS-path from \(p\) to \(q + \delta_i(\vec{e}_i, 0)\). Let \(c(\alpha)\) denote the given directional cost function, and recall the definition of the cost of a path in Section 2.2.3. If \(\rho = 0\), the resulting change in cost is \(\delta_i c(\alpha_i)\), otherwise it is \(2 \int_0^{2\gamma_i} c(\alpha_i + 2\rho_i |\phi|) d\phi\). Note that \(\gamma_i > 0\) (extension) leads to an increase in cost, while \(\gamma_i < 0\) (contraction) leads to a decrease in cost.

Let \(A_i, A_j, A_k\) be three reversible directions of a CS-path from \(p\) to \(q\). Recall from Lemma 3.2 in Chapter 3 that given any \(\varepsilon > 0\), there exist \(\gamma_i, \gamma_j, \gamma_k \in (-\varepsilon, \varepsilon)\) such that the resulting path is distinct from the original but \(q' = q\).

### 5.3 Optimal Path Forms

In this section, we establish the forms of paths which are optimal for the gradient-constrained version of the problem studied in Chapters 2 - 4. This is essential for the application of underground mine design since we need to lift the optimal path in the plane into 3-dimensional space and require that this lifted path satisfies a maximum gradient constraint. Since we simply apply a constant gradient to this lifted path, the maximum gradient constraint is equivalent to a lower bound on the length of the planar path. This is because we require the planar length of the path \(L\) to satisfy the inequality \(z/L \leq m\) where \(z\) is the vertical distance between the start and end directed points and \(m\) is the maximum gradient. This can be rearranged to give \(L \geq z/m\) which is a lower bound on planar length of \(L_{lb} = z/m\).

This means that we can formulate our gradient-constrained problem as follows:

\[
\min_{E \in \mathbb{P}_{pq}} \int_E c(\alpha) ds \text{ subject to } \int_E ds \geq L_{lb} \tag{5.1}
\]

for some given directional cost function \(c(\alpha)\) and \(L_{lb} \geq 0\), where \(\mathbb{P}_{pq}\) denotes the set of all curvature-constrained paths between two directed points \(p\) and \(q\).

For brevity, unless otherwise specified, we will refer to an optimal path in this section as a path which is optimal for Problem (5.1). Due to the introduction of the minimum length constraint, it is possible that an optimal path may contain a unit circle. Since these are excluded from the definition of a CS-path, we define a loop to be such an arc. Note
that the location of a loop (or whether it is left or right-turning) does not affect the cost or feasibility of a CS-path with loops, and thus we do not need to specify where these loops occur along a particular curvature-constrained path. Given a CS-path \( E \) and lower bound on length \( L_{lb} \), it is clear that the correct number of additional loops to consider is the lowest number of loops required to make the CS-path with loops satisfy the minimum length constraint. This is the number of loops we imply when we refer to a CS-path with loops, which means that a CS-path with loops will not have any loops if the CS-path itself already satisfies the prescribed minimum length constraint. It is important to keep in mind that loops do not correspond to any of the \( C \) labels in the form of a CS-path. For example, a path consisting of a straight line segment, followed by a loop, and then another straight line segment is simply a CS-path of the form \( S \) with one additional loop (see for example Figure 5.9).

Our approach in establishing the optimal forms will be similar to previous chapters. We first show that there exists an optimal CS-path (with loops), and then proceed to consider CS-path deformations in order to identify the optimal forms. Unlike previous chapters however, we will not have a strong result which states that the optimal paths must be of a particular form if the velocity function is strict. The main result we prove in this section is that for any velocity function, there exists an optimal path of the form \( CSCSCSC \) (or degeneracy) with loops.

In Chapter 2, we applied Pontryagin’s Maximum Principle in order to show that there exists an optimal CS-path. However, it is inconvenient to formulate (5.1) as a control problem since there is a minimum length constraint. We therefore consider an alternative approach which requires us to first focus on proving some basic properties of length preserving reversible CS-path deformations. While these results initially appear to only be applicable if we assume an optimal CS-path exists, we will actually be able to apply them to show that there always exists an optimal CS-path.

### 5.3.1 Length-Preserving Reversible Deformations

Recall that the degree of a CS-path is the number of reversible directions it has. Also note that the addition of loops to a CS-path does not affect its degree. In Chapter 3,
we introduced the technique of applying reversible deformations in order to establish which forms are not optimal when there is no minimum length constraint. However, Lemmas 3.7 - 3.16 are inapplicable when the minimum length constraint is active since they involve deformations which potentially shorten the path. Hence, we need to consider a length-preserving combination of reversible deformations. In Lemma 3.2, three reversible directions were shown to be sufficient to perform a combination of reversible deformations which resulted in a distinct path from the original while preserving both \( p \) and \( q \). We now show that four reversible directions are sufficient to perform such a combination of deformations which also preserves the length.

Recall that \( \delta_i = 4 \sin \gamma_i \theta_i = \alpha_i + \rho_i|\gamma_i| \) and \( \tilde{\xi}_i = (\cos \theta_i, \sin \theta_i, \mu_i|\gamma_i|) \), where \( \mu_i = \sin \gamma_i \) if \( \rho_i = 0 \) and \( \mu_i = \gamma_i \) if \( \rho_i \neq 0 \). We introduce \( \tilde{\xi}_i \) for convenience, as we will need an additional equation for the length-preserving condition.

Let \( A_i, A_j, A_k, A_l \) be four reversible directions of a CS-path from \( p \) to \( q \). Consider a new CS-path formed by performing deformations of size \( \gamma_i, \gamma_j, \gamma_k, \gamma_l \) along the respective reversible directions. The resulting path terminates at \( q \) with the same length as the original path if \( \gamma_i, \gamma_j, \gamma_k, \gamma_l \) satisfy the following condition:

\[
\delta_i \tilde{\xi}_i + \delta_j \tilde{\xi}_j + \delta_k \tilde{\xi}_k + \delta_l \tilde{\xi}_l = 0 \quad (5.2)
\]

**Lemma 5.1.** Let \( A_i, A_j, A_k, A_l \) be four reversible directions of a CS-path from \( p \) to \( q \). Given any \( \epsilon > 0 \), there exist \( \gamma_i, \gamma_j, \gamma_k, \gamma_l \in (-\epsilon, \epsilon) \) such that the resulting path is distinct from the original but \( q' = q \), and the total length of the resulting path is the same as the original.

**Proof.** If there exists a pair of repeated reversible directions, without loss of generality let them be \( A_i \) and \( A_j \). Since there exists \( \epsilon_1 > 0 \) such that the deformations are valid for \( \gamma_i, \gamma_j \in (-\epsilon_1, \epsilon_1) \), we can simply choose \( \gamma_i = (\min\{\epsilon_1, \epsilon\})/2, \gamma_j = -(\min\{\epsilon_1, \epsilon\})/2, \gamma_k = 0, \gamma_l = 0 \) to satisfy (5.2).

Suppose that \( \alpha_i, \alpha_j, \alpha_k \) are distinct. Note that this is stronger than stating that there are no repeated reversible directions, since a pair of repeated reversible directions requires that both \( \alpha \) and \( \rho \) are the same. Consider \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) defined as follows:

\[
F(\gamma_i, \gamma_j, \gamma_k) = \delta_i \tilde{\xi}_i + \delta_j \tilde{\xi}_j + \delta_k \tilde{\xi}_k
\]
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The Jacobian of $F$ can be simplified as follows:

$$J(\gamma_i, \gamma_j, \gamma_k) = 4 \begin{bmatrix}
\cos(\alpha_i + 2\rho_i|\gamma_i|) & \cos(\alpha_j + 2\rho_j|\gamma_j|) & \cos(\alpha_k + 2\rho_k|\gamma_k|) \\
\sin(\alpha_i + 2\rho_i|\gamma_i|) & \sin(\alpha_j + 2\rho_j|\gamma_j|) & \sin(\alpha_k + 2\rho_k|\gamma_k|) \\
\partial\mu_i / \partial\gamma_i & \partial\mu_j / \partial\gamma_j & \partial\mu_k / \partial\gamma_k
\end{bmatrix}$$

where $\partial\mu_i / \partial\gamma_i = \cos \gamma_i$ if $\rho_i = 0$ and $1$ if $\rho_i \neq 0$.

The determinant of $J$ evaluated at the origin is given by

$$\det(J) = 64(\sin(\alpha_k - \alpha_j) - \sin(\alpha_k - \alpha_i) + \sin(\alpha_j - \alpha_i))$$

which can be simplified into the following form:

$$\det(J) = 256 \sin((\alpha_k - \alpha_j)/2) \sin((\alpha_j - \alpha_i)/2) \sin((\alpha_k - \alpha_i)/2)$$

which is nonzero since $\alpha_i, \alpha_j, \alpha_k$ are all distinct. Since $J$ is continuous at the origin, by the Inverse Function Theorem, there exists $\varepsilon_2 > 0$ such that for all $\gamma_l \in (-\varepsilon_2, \varepsilon_2)$, there exist $\gamma_i, \gamma_j, \gamma_k$ such that (5.2) is satisfied.

This leaves only one case to consider, which is when $\alpha_i, \alpha_j, \alpha_k$ are not all distinct. However, since we can exclude the case where there are repeated reversible directions, it follows from the expression of $J$ given above that there exists $\varepsilon_3 > 0$ such that $\det(J) \neq 0$ for all $\| (\gamma_i, \gamma_j, \gamma_k) \| \in (0, \varepsilon_3)$, where $\| \cdot \|$ denotes the Euclidean norm. Since $F(\gamma_i, \gamma_j, \gamma_k)$ is an odd function (antipodal with respect to the origin), it then follows that the inverse exists, and hence there exists $\varepsilon_2 > 0$ such that for all $\gamma_l \in (-\varepsilon_2, \varepsilon_2)$, there exists $\gamma_i, \gamma_j, \gamma_k$ such that (5.2) is satisfied.

Now that we have proved that such a combination of four reversible deformations exists, we wish to apply these deformations to paths of degree $d \geq 4$ to show that they are either not optimal or can be reduced to a path with degree $d \leq 3$. In Chapter 3, we proved Theorem 3.17 which states that the optimal paths must be of particular forms if the velocity function is strict, and then relaxed the strict condition in Corollary 3.18, to
obtain the weaker but more general result that there exists an optimal path of degree \( d \leq 2 \). It is harder to prove an analogous result to Theorem 3.17 for the problem (5.1) due to the minimum length constraint. However, by generalising the approach of Corollary 3.18, we will prove that there exists an optimal path of degree \( d \leq 3 \). In particular, rather than restricting ourselves to strict velocity functions, we will consider all velocity functions, but directly apply a generalised version of the argument in Corollary 3.18 if the reversible deformations we are considering do not change the cost of the path.

We first consider the case where all four reversible directions are distinct and correspond to \( S \) segments. The result we obtain will illustrate the technique which can then be generalised to the other cases. A sequence of \( n \) distinct reversible directions \( A_1, \ldots, A_n \) are in ascending order if there exists \( \varepsilon > 0 \) such that \( \forall \eta_1, \ldots, \eta_n \in (0, \varepsilon) \), \( \alpha_i + \rho_i \eta_i \in (\alpha_{i-1} + \rho_{i-1} \eta_{i-1}, \alpha_{i+1} + \rho_{i+1} \eta_{i+1}) \) for \( i = 1, \ldots, n \) where \( \alpha_0 + \rho_0 \eta_0 = \alpha_n + \rho_n \eta_n \) and \( \alpha_{n+1} + \rho_{n+1} \eta_{n+1} = \alpha_1 + \rho_1 \eta_1 \). Note that if \( \rho_1 = \cdots = \rho_n = 0 \), this definition simply reduces to the directions \( \alpha_1, \ldots, \alpha_n \) being in ascending order.

**Lemma 5.2.** Let \( A_i, A_j, A_k, A_l \) be four distinct reversible directions of a CS-path such that \( \rho_i, \rho_j, \rho_k, \rho_l = 0 \), and \( A_i, A_j, A_k, A_l \) are in ascending order. If \( (\delta_i, \delta_j, \delta_k, \delta_l) \neq 0 \) satisfies condition (5.2), then \( \text{sgn}(\delta_i) = -\text{sgn}(\delta_j) = \text{sgn}(\delta_k) = -\text{sgn}(\delta_l) \).

**Proof.** Since \( \rho_i, \rho_j, \rho_k, \rho_l = 0 \), we can simplify the length preserving condition (3rd equation in the system of equations (5.2)) to the following:

\[
\delta_i + \delta_j + \delta_k + \delta_l = 0 \tag{5.3}
\]

It is clear that it is impossible to satisfy (5.3) with the signs of \( \delta_i, \delta_j, \delta_k, \delta_l \) being the same since \( (\delta_i, \delta_j, \delta_k, \delta_l) \neq 0 \).

Now suppose that three of \( \delta_i, \delta_j, \delta_k, \delta_l \) have the same sign. Without loss of generality, we can let \( \delta_i, \delta_j, \delta_k \geq 0 \) and \( \delta_l \leq 0 \). From the first two equations of (5.2), we get that \( \delta_i \overrightarrow{\xi_i} + \delta_j \overrightarrow{\xi_j} + \delta_k \overrightarrow{\xi_k} = |\delta_l| \overrightarrow{\xi_l} \). From (5.3), we get \( \delta_i + \delta_j + \delta_k = |\delta_l| \). However, this is only possible if either \( \overrightarrow{\xi_l} = \overrightarrow{\xi_i}, \overrightarrow{\xi_i} = \overrightarrow{\xi_j}, \overrightarrow{\xi_j} = \overrightarrow{\xi_k} \) or \( \overrightarrow{\xi_l} = \overrightarrow{\xi_k} \) which contradicts the fact that the four reversible directions are distinct. Hence, it is not possible for three of the signs of \( \delta_i, \delta_j, \delta_k, \delta_l \) to be the same.
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$$X_{ij}$$ and $$X_{kl}$$ are disjoint since $$\alpha_i, \alpha_j, \alpha_k, \alpha_l$$ are arranged in ascending order. This basic property of circles is used in the proof of Lemma 5.2 to show that it is not possible for (5.2) to be satisfied with $$\delta_i, \delta_j \geq 0$$ and $$\delta_k, \delta_l \leq 0$$ where $$\alpha_i, \alpha_j, \alpha_k, \alpha_l$$ are arranged in ascending order.

Now suppose that $$\delta_i, \delta_j \geq 0$$ and $$\delta_k, \delta_l \leq 0$$. Let $$L = \delta_i + \delta_j$$ which by (5.2) implies that $$L = |\delta_k| + |\delta_l|$$. Let $$X_{ij}$$ denote the line segment in $$\mathbb{R}^2$$ from $$L\xi_i$$ to $$L\xi_j$$ and $$X_{kl}$$ denote the line segment in $$\mathbb{R}^2$$ from $$L\xi_k$$ to $$L\xi_l$$. From (5.2), we know that $$\delta_i \xi_i + \delta_j \xi_j$$ and $$|\delta_k| \xi_k + |\delta_l| \xi_l$$ are equal, and belong in $$X_{ij}$$ and $$X_{kl}$$ respectively. By definition, it can be seen that $$X_{ij}$$ and $$X_{kl}$$ are chords of a circle of radius $$L$$ centred at the origin as shown in Figure 5.1. Since $$\alpha_k \notin (\alpha_i, \alpha_j)$$ and $$\alpha_l \notin (\alpha_i, \alpha_j)$$, it follows by basic properties of circles that the chords $$X_{ij}$$ and $$X_{kl}$$ are disjoint. This implies that it is not possible for condition (5.2) to be satisfied. Hence, it is not possible to have the signs of both $$\delta_i, \delta_j$$ or both $$\delta_j, \delta_k$$ to the same.

This leaves the only possible scenarios which satisfy condition (5.2) to be $$\delta_i, \delta_k \geq 0$$ and $$\delta_j, \delta_l \leq 0$$, or $$\delta_i, \delta_k \leq 0$$ and $$\delta_j, \delta_l \geq 0$$, proving the result.  

Recall that the velocity point of a reversible direction $$A_i$$ is the point in $$\mathbb{R}^2$$ which in polar coordinates is $$(v(\alpha_i), \alpha_i)$$. For clarity, refer to Figure 5.2(a) for a diagram illustrating $$\alpha_{\text{diag}}$$ defined in the following lemma.

**Lemma 5.3.** Let $$A_i, A_j, A_k, A_l$$ be four distinct reversible directions of a CS-path such that
Figure 5.2: In (a), the vectors shown satisfy condition (5.2). A simple method for constructing a line in the direction $\alpha_{\text{diag}}$ is shown in (b) by locating the intersection of the two chords $X_{ik}$ and $X_{jl}$. 
\(\rho_i, \rho_j, \rho_k, \rho_l = 0\), and \(A_i, A_j, A_k, A_l\) are in ascending order. Let \(\alpha_{\text{diag}}\) denote the direction of the vector \((|\delta_i|\vec{\xi}_i + |\delta_k|\vec{\xi}_k)\) where \((\delta_i, \delta_j, \delta_k, \delta_l) \neq 0\) satisfies condition (5.2). Let the straight line passing through the velocity points of \(A_i\) and \(A_k\) be given in polar coordinates by the function \(v_i^k(\alpha)\), with \(v_i^j(\alpha)\) defined analogously for \(A_j\) and \(A_l\). Extending along \(A_i\) and \(A_k\) while contracting along \(A_j\) and \(A_l\) in such a manner that condition (5.2) is satisfied will:

- reduce the total cost of the path if \(v_i^k(\alpha_{\text{diag}}) > v_i^j(\alpha_{\text{diag}})\);
- increase the total cost of the path if \(v_i^k(\alpha_{\text{diag}}) < v_i^j(\alpha_{\text{diag}})\);
- not change the total cost of the path if \(v_i^k(\alpha_{\text{diag}}) = v_i^j(\alpha_{\text{diag}})\).

Proof. Firstly, it follows from Lemma 5.2 that \(|\delta_i|\vec{\xi}_i + |\delta_k|\vec{\xi}_k = |\delta_j|\vec{\xi}_j + |\delta_l|\vec{\xi}_l\) when condition (5.2) is satisfied. Let \(\delta_{\text{diag}} = |||\delta_i|\vec{\xi}_i + |\delta_k|\vec{\xi}_k||\) where \(\parallel \cdot \parallel\) denotes the Euclidean norm as shown in Figure 5.2(a).

By applying property (3) of Lemma 3.5, we know that the change in total cost of the path due to extending along \(A_i\) and \(A_k\) is equal to \(\delta_{\text{diag}} / v_i^k(\alpha_{\text{diag}})\). Similarly, the change in total cost of path due to contracting along \(A_j\) and \(A_l\) is equal to \(-\delta_{\text{diag}} / v_i^j(\alpha_{\text{diag}})\). The result then follows by summing these two changes in cost. \(\Box\)

It is worth noting that \(\alpha_{\text{diag}}\) can be obtained geometrically via the construction shown in Figure 5.2(b). \(X_{ik}\) and \(X_{jl}\) are line segments in \(\mathbb{R}^2\) from \(\vec{\xi}_i\) to \(\vec{\xi}_k\) and from \(\vec{\xi}_j\) to \(\vec{\xi}_l\) respectively. It can be shown that \(\alpha_{\text{diag}}\) is the direction from the origin \(O\) to the intersection of the two chords \(X_{ik}\) and \(X_{jl}\).

The next step is to generalise Lemma 5.3 by removing the restriction of \(\rho_i, \rho_j, \rho_k, \rho_l = 0\) in a similar manner to how we obtained Lemma 3.6 from Lemma 3.5 in Chapter 3.

**Lemma 5.4.** Let \(A_i, A_j, A_k, A_l\) be four distinct reversible directions of a CS-path such that \(A_i, A_j, A_k, A_l\) in ascending order. Let \(\alpha_{\text{diag}}\) denote the direction of the vector \((|\delta_i|\vec{\xi}_i + |\delta_k|\vec{\xi}_k)\). Let the straight line passing through the velocity points of \(A_i\) and \(A_k\) be given in polar coordinates by the function \(v_i^k(\alpha)\), with \(v_i^j(\alpha)\) defined analogously for \(A_j\) and \(A_l\). There exists \(\gamma_i, \gamma_k > 0\) and \(\gamma_j, \gamma_l < 0\) satisfying condition (5.2) such that extending along \(A_i\) and \(A_k\) while contracting along \(A_j\) and \(A_l\) by the respective quantities will:

- reduce the total cost of the path if \(v_i^k(\alpha_{\text{diag}}) > v_i^j(\alpha_{\text{diag}})\);
- increase the total cost of the path if \(v_i^k(\alpha_{\text{diag}}) < v_i^j(\alpha_{\text{diag}})\).
Proof. The existence of $\gamma_i, \gamma_k < 0$ and $\gamma_j, \gamma_l > 0$ satisfying (5.2) is given by Lemma 5.2. The properties then follow by applying Lemma 3.2 with sufficiently small $\epsilon$ so that the results hold in a similar manner to Lemma 5.3. \qed

Note that there is a subtle difference between the conditions in Lemmas 5.3 and 5.4 above since $\alpha_i, \alpha_j, \alpha_k, \alpha_l$ are not necessarily distinct when $A_i, A_j, A_k, A_l$ are distinct. This however does not affect the validity of this result as if any of the directions $\alpha_i, \alpha_j, \alpha_k, \alpha_l$ are not distinct, then either $v^k_\gamma(\alpha_{\text{diag}}) = v^l_\gamma(\alpha_{\text{diag}})$, or $\alpha_{\text{diag}}$ is not unique since the intersection is the entire chord, and any choice of $\alpha_{\text{diag}}$ will still satisfy $v^k_\gamma(\alpha_{\text{diag}}) = v^l_\gamma(\alpha_{\text{diag}})$.

### 5.3.2 Existence of Optimal CS-path

We have now developed the main tools required to prove that there exists an optimal CS-path solving (5.1). We first prove the following lemma which shows that given a convex curvature-constrained path from $p$ to $q$, we can construct a CS-path of the same length between from $p$ to $q$. We will use this result to construct an arbitrarily close CS-path approximation to any given curvature-constrained path with the same length.

**Lemma 5.5.** Let $E$ be a convex left-turning curvature-constrained path from $p$ to $q$ such that the start and end directions $\alpha_p$ and $\alpha_q$ satisfy the condition that $\alpha_q \in (\alpha_p, \alpha_p + \pi)$. If $E$ is not a CSC or SCS path, then there exists a CSCSC path $D$ from $p$ to $q$ such that $L(D) = L(E)$ and the direction of every point on $D$ lies in $[\alpha_p, \alpha_q]$.

**Proof.** Let $D_1$ be the $LSL$ path from $p$ to $q$ and $D_2$ be the $SLS$ path from $p$ to $q$. Note that $E, D_1, D_2$ are convex arcs and $D_2$ lies on the outer side of $E$, which lies on the outer side of $D_1$. From Lemma 2.4, it then follows that $L(D_1) < L(E) < L(D_2)$.

Let $O_p$ and $O_q$ denote the centres of the left-turning circles of unit radius corresponding to $p$ and $q$ respectively. Let $r$ denote the intersection of the ray from $O_p$ with direction $\alpha_p$ and the ray from $O_q$ with direction $\alpha_q + \pi$. Let $O_prO_q$ denote the polygonal path from $O_p$ to $r$ to $O_q$. It is clear that $L(D_1) = L(O_pO_q) + (\alpha_q - \alpha_p)$ and $L(D_2) = L(O_prO_q) + (\alpha_q - \alpha_p)$. Consider a family of polygonal paths made up of two straight lines, obtained by continuously deforming $O_pO_q$ into $O_prO_q$ while fixing the start and end
Incorporating the Gradient Constraint

points at $O_p$ and $O_q$ respectively. It is clear that each of these polygonal paths $D_p$ corresponds to a CSCSC path $D$ from $p$ to $q$ with length $L(D_p) + (a_q - a_p)$ where the direction of every point on $D$ lies in $[a_p, a_q]$. It then follows by the fact that $L(D_1) < L(E) < L(D_2)$ that such a path $D$ exists with $L(D) = L(E)$.

For clarity of conveying the idea of the proof that there exists an optimal CS-path to (5.1), we will break this result up into two main steps. We first focus on showing that if there is an optimal convex curvature-constrained path from $p$ to $q$, then there exists an optimal CS-path of degree $d \leq 3$ from $p$ to $q$ with the same length as the convex path. We will then generalise this result to apply to any curvature-constrained path that is not necessarily a convex path.

Let the span of a path denote the smallest closed interval of directions which contains the directions of all points on the path. As stated in Section 5.2, we will assume that the size of the derivative of the directional cost function is bounded in order to simplify the following proof. In fact, we could easily even extend this to piecewise continuous function directional cost functions such that the derivative is bounded on each piece. This would simply involve breaking up the path into subpaths which only span a set of directions on which the directional cost function is continuous and applying the following proof. This makes it a reasonable condition to assume for any practical velocity function used in application.

Let $C_{pq}$ denote the set of all CS-paths with loops from $p$ to $q$. Let $C_{pq}^3 \subset C_{pq}$ denote the subset of paths with degree $d \leq 3$. As mentioned previously, it is only relevant to consider loops in this chapter. Hence, while it was sufficient to consider CS-paths without loops in the previous chapters, here we need to let $C_{pq}$ include CS-paths with loops.

**Lemma 5.6.** Let $E$ be an optimal curvature-constrained path from $p$ to $q$. If $E$ is convex, then there exists an optimal path from $p$ to $q$ in $C_{pq}^3$.

**Proof.** Suppose to the contrary that there is no optimal path in $C_{pq}^3$. Since $E$ is a convex arc, given any $\nu > 0$, we can divide $E$ into $N_{\nu}$ subpaths $E_i$ for $i = 0, \cdots, N_{\nu}$, such that each subpath $E_i$ is a convex arc with the size of its span no greater than $\nu$. It is also clear that there exists $\hat{\nu} > 0$ such that $N_{\hat{\nu}} \geq 4$ and $\hat{\nu} < \pi$. 
Let $M$ be an upper bound on the derivative of the cost function $c(\alpha)$. By symmetry, we can assume that the subpath $E_i$ is left-turning. If $E_i$ is a CSC or SCS path then let $\hat{E}_i = E_i$. Otherwise, as long as $\nu < \pi$, we can apply Lemma 5.5 to show that there exists a CSCSC path $\hat{E}_i$ with the same length and span as $E_i$. This implies that $|T(\hat{E}_i) - T(E_i)| \leq \nu ML(E_i)$.

We can apply this construction to all of the $N_\nu$ subpaths and construct a CS-path $E_\nu$ and obtain $|T(E) - T(E_\nu)| \leq \sum_{i=1}^{N_\nu} \nu ML(E_i) = \nu ML(E)$.

Since there is no optimal path in $C_{3pq}^3$ given any $D \in C_{3pq}^3$, we know that $T(D) - T(E) > 0$. We choose $\nu = \min\{\hat{\nu}, (T(D) - T(E))/(2ML(E))\}$ so that $E_\nu$ is a CS-path with at least four distinct reversible directions $A_i, A_j, A_k, A_l$ where $\rho_i = \rho_j = \rho_k = \rho_l = 0$ and $T(E_\nu) < T(D)$. Without loss of generality, we can assume $A_i, A_j, A_k, A_l$ are arranged in ascending order. By applying Lemma 5.3 to $T(E_\nu)$, we know that regardless of whether $v^k_i(\alpha_{\text{diag}}) \neq v^l_j(\alpha_{\text{diag}})$ or $v^k_i(\alpha_{\text{diag}}) = v^l_j(\alpha_{\text{diag}})$, we can construct a new CS-path of the same length and of no greater cost since the deformations considered are reversible. Since $E_\nu$ is itself a convex arc by construction, performing this deformation until one of the straight line segments degenerates results in a new CS-path with 1 degree less. This procedure can be repeated, preserving the total length while never increasing the cost, until there are only 3 reversible directions remaining. Let $\hat{E}$ be this path, which by definition belongs in $C_{pq}^3$. Hence, we have shown that for any path $E_\nu \in C_{pq} \setminus C_{pq}^3$, there exists a path $\hat{E} \in C_{pq}^3$ such that $T(\hat{E}) \leq T(E_\nu)$. This contradicts our earlier statement that $T(E_\nu) < T(D)$, given any $D \in C_{pq}^3$.

Hence, there exists an optimal path from $p$ to $q$ in $C_{pq}^3$.

A curvature-constrained path could, in general, have an infinite number of inflection points. In this chapter, we only consider curvature-constrained paths with a finite number of inflection points. However, in [40], it is shown that we can always perturb a path with an infinite number of inflection points into an arbitrarily close path with a finite number of inflection points. The techniques from that paper could be applied to generalise the following lemma to the case where we consider curvature-constrained paths with an infinite number of inflection points.
Lemma 5.7. Given any pair of directed points \( p, q \), there exists an optimal path from \( p \) to \( q \) which is a CS-path with loops.

Proof. Let \( E \) be an optimal path from \( p \) to \( q \). We can assume that \( E \) is not a CS-path since the result follows trivially otherwise. Since \( E \) is an optimal path from \( p \) to \( q \) which satisfies the minimum length constraint, we know that any subpath \( E_i \) from \( p_i \) to \( q_i \) must also be an optimal path from \( p_i \) to \( q_i \) subject to a minimum length constraint equal to \( L(E_i) \). As explained previously, we can assume that \( E \) has a finite number of inflection points \([40]\).

We can therefore choose to break \( E \) into a finite number of subpaths \( E_i \) separated by inflection points, for \( i = 0 \cdots n \), where \( n \) is the number of inflection points.

We can now apply Lemma 5.6 to each of the subpaths \( E_i \) to deduce that there exists a path \( D_i \in C^3_{pq} \) which is also an optimal path from \( p_i \) to \( q_i \), with \( L(D_i) \). Performing this on all the subpaths \( E_i \) then results in a CS-path \( D \) where \( T(D) \leq T(E) \) and \( L(D) = L(E) \). Hence, there exists an optimal CS-path with loops from \( p \) to \( q \).

\[ \square \]

5.3.3 Sufficient Path Set

Having proved that there exists an optimal CS-path, we now identify a sufficient set of CS-paths which contains an optimal path. This is analogous to \( D^5_{pq} \) in Chapter 3. We therefore expect to find a larger set of paths which contains \( D^5_{pq} \). We use the results developed in Section 5.3.1 to prove that there exists an optimal path with degree \( d \leq 3 \). Note that the addition of loops does not affect the degree of a CS-path. Hence, a path with \( d \leq 3 \) is equivalent to a path of the form \( CSCSCSC \) (or degeneracy) with loops.

Recall that \( C^3_{pq} \) denotes the set of all CS-paths with loops between two directed points \( p \) and \( q \) with \( d \leq 3 \).

Theorem 5.8. Given any pair of directed points \( p, q \in \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z} \) and \( C^0 \) cost function \( c : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_+ \) with bounded derivative, there exists an optimal path of the form \( CSCSCSC \) (or degeneracy) with loops.

Proof. From Lemma 5.7, we know that there exists a CS-path \( E \) which is optimal. Recall that a path in \( C^3_{pq} \) has degree \( d \leq 3 \) and hence, is a path of the form \( CSCSCSC \) (or
degeneracy) with loops. Suppose to the contrary that $T(E) < T(D) \forall D \in C_{pq}^3$. Since $E$ does not belong to $C_{pq}^3$, it has degree $d \geq 4$. We now break the proof up into the following 3 subcases:

1. $E$ has 4 distinct reversible directions $A_i, A_j, A_k, A_l$ arranged in ascending order such that $\alpha_i, \alpha_j, \alpha_k, \alpha_l$ are distinct;
2. $E$ has 4 distinct reversible directions $A_i, A_j, A_k, A_l$ arranged in ascending order such that $\alpha_i, \alpha_j, \alpha_k, \alpha_l$ are not distinct;
3. $E$ has a pair of repeated reversible directions $A_i, A_j$.

We explain case (1) in detail, after which cases (2) and (3) will hold via slightly modified arguments.

If (1) holds, and (adopting the notation used in Lemma 5.4 for velocity function $v = 1/c$) $v^k(\alpha_{\text{diag}}) \neq v^l(\alpha_{\text{diag}})$, then it follows by Lemma 5.4 that $E$ is not optimal since there exists a path of lower cost than $E$ obtained through performing a combination of reversible deformations. Hence we can assume that the change in cost of any reversible deformations satisfying condition (5.2) is zero. This necessarily implies that $v^k(\alpha_{\text{diag}}) = v^l(\alpha_{\text{diag}})$. Note that unlike in Chapter 3 where this behaviour is associated with weak velocity functions, it is not as easy here to state the analogous condition for describing which type of velocity function permits this behaviour. The exact relationship will be proven later in Lemma 5.10.

Note that the following notation is related, but not exactly the same as in Corollary 3.18. For the reversible direction $A_i$, let $I_i = (\alpha_i - \beta_i, \alpha_i + \beta_i)$ where $\beta_i > 0$ is arbitrarily chosen to be sufficiently small so that $\alpha_j, \alpha_k, \alpha_l \notin I_i$. Let $h_i(\alpha)$ be defined as follows:

$$h_i(\alpha) = \begin{cases} \beta_i - |\alpha - \alpha_i| & \text{if } \alpha \in I_i \\ 0 & \text{otherwise} \end{cases}$$

Consider a new directional cost function $\hat{c}(\alpha) = c(\alpha) + \epsilon h_i(\alpha)$ with corresponding velocity function $\hat{\phi} = 1/\hat{c}$. By construction, the velocity points of $A_j, A_k, A_l$ are the same for both $v$ and $\hat{\phi}$ while the velocity point of $A_i$ is different. This implies that $\hat{\phi}^k(\alpha_{\text{diag}}) \neq \hat{\phi}^l(\alpha_{\text{diag}})$, using analogous notation for velocity points corresponding to $\hat{\phi}$. Hence, $E$ is not
an optimal path for $\hat{c}$, for any $\varepsilon > 0$. Let $T(E)$ and $\hat{T}(E)$ denote the costs of path $E$ with respect to the directional cost functions $c$ and $\hat{c}$ respectively. Let $H(E) = \int_E h_i(\alpha) ds$.

$$
\hat{T}(E) = T(E) + \varepsilon H(E)
$$

$$
< T(D), \forall D \in C_{pq}^3 \text{ by choosing } \varepsilon \in (0, (T(D) - T(E))/(H(E)))
$$

$$
< \hat{T}(D), \forall D \in C_{pq}^3
$$

This results in a contradiction since $E$ is not an optimal path for the directional cost function $\hat{c}$.

If case (2) holds, we are unable to apply the exact same argument as for case (1), since introducing $h_i(\alpha)$ will not result in $\hat{v}_i^k(\alpha_{\text{diag}}) \neq \hat{v}_j^l(\alpha_{\text{diag}})$. Without loss of generality, let $\alpha_i = \alpha_j$ and $\rho_i \neq 0$. If $\rho_i = 1$, let $I_i^+ = (\alpha_i, \alpha_i + \beta_i)$ and define $h_i^+(\alpha)$ as follows:

$$
h_i^+(\alpha) = \begin{cases} 
\beta_i/2 - |\alpha - (\alpha_i + \beta_i/2)| & \text{if } \alpha \in I_i^+ \\
0 & \text{otherwise}
\end{cases}
$$

Using $h_i^+(\alpha)$ in place of $h(\alpha)$ now still gives $\hat{v}_i^k(\alpha_{\text{diag}}) = \hat{v}_j^l(\alpha_{\text{diag}})$ however there now exist sufficiently small reversible deformations satisfying condition (5.2) such that the change in cost $\hat{T}$ using $\hat{c}$ is non-zero since $h_i^+(\alpha)$ only affects the cost of the extension (or contraction) along $A_i$ and does not affect the cost of the other reversible deformations. The rest of the argument then follows as before since $E$ is not optimal for $\hat{c}$. It is then clear that if $\rho_i = -1$ we can let $I_i^- = (\alpha_i - \beta_i, \alpha_i)$ and define $h_i^-\alpha)$ in a similar manner to achieve the same result.

If case (3) holds, we first suppose that $\rho_i, \rho_j = 0$. We can then simply extend the path along $A_i$ and contract it along $A_j$ in equal amounts until one of the straight line segments degenerates, resulting in a new distinct reversible direction or reducing the degree of the path by 1, while keeping the cost and length of the path constant. This procedure can be repeated until there are no pairs of repeated reversible directions $A_i, A_j$ where $\rho_i, \rho_j = 0$. On the other hand, if $\rho_i, \rho_j \neq 0$, we can extend along $A_i$ and contract along $A_j$.
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in equal amounts without degenerating any of the existing arcs, in order to form a new path which still has the same pair of repeated reversible directions $A_i, A_j$ but with two additional distinct reversible directions. This procedure can then be repeated to produce a new path with the same length and cost as $E$ but with 4 distinct reversible directions, from which we can apply the arguments for case (1) or (2).

Hence, there exists an optimal path in $C_{pq}^3$.

The approach used in handling case (3) in the proof of Theorem 5.8 is weaker than Lemma 3.11 since it does not show that a CS-path with a pair of reversible directions $A_i, A_j$ such that $\rho_i, \rho_j \neq 0$ is not optimal if the velocity function is strict. However, it is more general as it is applicable to the version of the problem studied in this chapter, with a minimum length constraint, whereas Lemma 3.11 cannot be applied in general, as it potentially requires a length-reducing deformation.

Lemmas 5.1 - 5.7 and Theorem 5.8 all involve length preserving deformations. This guarantees that the length constraint is never violated. However, if the length constraint is inactive, then we are able to state more about the path as we can perform arbitrarily small deformations which change the length. In this case, we obtain the following result:

**Corollary 5.9.** Let $E$ be an optimal path such that $L(E) > L_{lb}$. There exists an optimal path of the form CSCSC (or degeneracy) with loops.

**Proof.** Since $L(E) > L_{lb}$, we are able to perform CS-path deformations locally, even if the deformation is length reducing. This means that Lemmas 3.7 - 3.11 can be directly applied to this problem. The deformations from Lemmas 3.12 - 3.14 are applicable, but need to be modified for this problem. The possible outcomes from performing the respective deformations were either that an arbitrarily small combination of CS-path deformations reduced the cost of the path, or that the deformation could be performed until a loop was formed without reducing its length. For the problem without a minimum length constraint, the formation of the loop meant that the path was not optimal. However, it is clear that for this corollary, we simply need to change this to meaning that the path can be reduced to a CS-path of lower degree, with more loops, whilst not increasing
the cost. Likewise, Lemmas 3.15 and 3.16 still hold to show that the respective types of CS-paths considered can be reduced to a CS-path of lower degree without increasing its length. Putting these results together in a similar manner to that of Theorem 3.17 and then Corollary 3.18 gives the desired result.

5.3.4 Necessary Optimality Condition

In Theorem 5.8, we were not able to restrict ourselves to strict velocity functions and then generalise results to weak velocity functions. We now explain this in more detail. Let $A_i, A_j, A_k$ be three reversible directions deformed by quantities $\gamma_i, \gamma_j, \gamma_k$ respectively such that (3.1) is satisfied and the change in cost is zero. It was shown in Chapter 4 that this necessarily implies the velocity function $v(\alpha)$ is a straight line passing through the three velocity points for $\alpha \in \Gamma_i \cup \Gamma_j \cup \Gamma_k$ where $\Gamma_i$ (and $\Gamma_j, \Gamma_k$ similarly) is defined as follows:

$$\Gamma_i = \begin{cases} 
(\alpha_i - 2|\gamma_i|, \alpha_i) & \text{if } \rho_i = -1 \\
\{\alpha_i\} & \text{if } \rho_i = 0 \\
(\alpha_i, \alpha_i + 2|\gamma_i|) & \text{if } \rho_i = 1
\end{cases}$$

This was also used to prove the necessary optimality conditions in Chapter 4 and the relevance of studying double tangents (the PETs corresponded to this straight line of zero change in cost). We now prove that for the problem (5.1) where we consider length-preserving deformations, the velocity points lie on a tangential conic section. Instead of a double tangent, this could be an ellipse (see for example Figure 5.4(a)), hyperbola (see for example Figure 5.4(b)), parabola or a simpler conic section such as a circle or straight line). It is then clear why it is difficult to consider an analogous version of strict velocity functions for the problem studied in this section, since that would require excluding all functions which contain a continuous subset of a conic section in polar coordinates. While it is possible to define such a class of functions, it is unnecessarily complicated, which is why we chose to ignore this property in proving that there exists an optimal path of the form $\text{CSCSCSC}$ (or degeneracy) with loops. However, in this section, we prove an analogous necessary optimality condition for a path of the form $\text{CSCSCSC}$ with
loops and non-degenerate straight line segments, and illustrate the difficulty involved in constructing an optimal path of this form.

Any conic section can be represented in polar coordinates in the form
\[ r(\alpha) = \frac{1}{A + B \cos(\alpha) + C \sin(\alpha)} \]
with the origin at a focal point. It will become clear in the proof of Lemma 5.10 that there exists a unique conic section (with the origin at a focal point) passing through three given points in space, provided that the points are specified in polar coordinates i.e. the points \((r, \alpha)\) and \((-r, \alpha + \pi)\) are distinct. This removes the possibility of having the negative branch of a hyperbola passing through any velocity points of a given velocity function \(v\) since \(v(\alpha) > 0\) for all \(\alpha \in \mathbb{R}/2\pi\mathbb{Z}\). An example illustrating the importance of this distinction is shown in Figure 5.3 where the ellipse is the unique conic section passing through the three velocity points. The hyperbola is invalid as it passes through the point \((-v(\alpha_3), \alpha_3 + \pi)\) instead of \((v(\alpha_3), \alpha_3)\). This is why we require the values of the conic section to be positive when represented in the form
\[ r(\alpha) = \frac{1}{A + B \cos(\alpha) + C \sin(\alpha)} \]
in the following lemma.

**Lemma 5.10.** Let \(A_i, A_j, A_k, A_l\) be distinct reversible deformations such that \(\alpha_i, \alpha_j, \alpha_k, \alpha_l\) are distinct. Let \(\Gamma_i\) be defined as in (5.4) above (and \(\Gamma_j, \Gamma_k, \Gamma_l\) similarly). Given a velocity function \(v = 1/c\), the change in cost of the reversible deformations \(\gamma_i, \gamma_j, \gamma_k, \gamma_l\) satisfying (5.2) is zero if the points of \(v(\alpha)\) for \(\alpha \in \Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_l\) lie on the conic section with the focal point at the origin, represented in polar coordinates so that the values of the conic section are positive and uniquely determined by three of the velocity points.

**Proof.** We will first show that if \(\rho_i = \rho_j = \rho_k = \rho_l = 0\), then the result holds. Suppose that \(v(\alpha_i), v(\alpha_j), v(\alpha_k)\) are given but we are free to choose \(v(\alpha_l)\). We wish to show that \(v(\alpha_l)\) necessarily equals \(\overline{v}(\alpha_l)\), where \(\overline{v}(\alpha)\) is the conic section with a focal point at the origin that passes through the velocity points of \(A_i, A_j, A_k\). By basic properties of conic sections, it follows that \(\overline{v}(\alpha)\) is uniquely defined since \(\alpha_i, \alpha_j, \alpha_k\) are distinct. For brevity, let \(c_i\) denote \(c(\alpha_i)\) and likewise for \(j, k, l\). Since the change in cost is zero and (5.2) is satisfied, we obtain the following system of equations, where we arbitrarily set \(\delta_l = 1\) without loss of generality:
Figure 5.3: A unique conic section (ellipse) with focal point at the origin passing through the three velocity points in the directions $\alpha_1, \alpha_2, \alpha_3$. The hyperbola in the figure is the unique conic section passing through the velocity points in the directions $\alpha_1, \alpha_2$ and the point $(-v(\alpha_3), \alpha_3 + \pi)$ as opposed to the velocity point $(v(\alpha_3), \alpha_3)$. This illustrates the importance of specifying that the velocity points are represented in polar coordinates, and not simply as points in $\mathbb{R}^2$. 
\[
\begin{align*}
\delta_i \cos \alpha_i + \delta_j \cos \alpha_j + \delta_k \cos \alpha_k + \cos \alpha_l &= 0 \\
\delta_i \sin \alpha_i + \delta_j \sin \alpha_j + \delta_k \sin \alpha_k + \sin \alpha_l &= 0 \\
\delta_i c_i + \delta_j c_j + \delta_k c_k + c_l &= 0 \\
\delta_i + \delta_j + \delta_k + 1 &= 0
\end{align*}
\]

Note that the unknown parameters in the above system are \(\delta_i, \delta_j, \delta_k, \alpha_l, c_l\) as everything else is assumed to be given. This can be simplified into the following system:

\[
\begin{align*}
\cos \alpha_l &= \delta_i (\cos \alpha_k - \cos \alpha_i) + \delta_j (\cos \alpha_k - \cos \alpha_j) + \cos \alpha_k \\
\sin \alpha_l &= \delta_i (\sin \alpha_k - \sin \alpha_i) + \delta_j (\sin \alpha_k - \sin \alpha_j) + \sin \alpha_k \\
c_l &= \delta_i (c_i - c_k) + \delta_j (c_j - c_k) - c_k
\end{align*}
\]

Since a conic section has the general form of \(r(\alpha) = 1/(A + B \cos(\alpha) + C \sin(\alpha))\), it is sufficient to show that there exist coefficients \(A, B, C\) which are independent of \(\delta_i, \delta_j, \delta_k, \alpha_l, c_l\), such that the following equation is satisfied:

\[
c_l = A + B \cos \alpha_l + C \cos \alpha_l
\]

Substituting (5.5), (5.6) and (5.7) into (5.8) gives:

\[
\begin{bmatrix}
1 & \delta_i & \delta_j
\end{bmatrix}
\begin{bmatrix}
-c_k \\
c_j - c_k \\
c_i - c_k
\end{bmatrix}
= \begin{bmatrix}
1 & \delta_i & \delta_j
\end{bmatrix}
\begin{bmatrix}
A + B \cos \alpha_k + C \sin \alpha_k \\
B(\cos \alpha_k - \cos \alpha_i) + C(\sin \alpha_k - \sin \alpha_i) \\
B(\cos \alpha_k - \cos \alpha_j) + C(\sin \alpha_k - \sin \alpha_j)
\end{bmatrix}
\]

Since \(\alpha_i, \alpha_j, \alpha_k\) are distinct, we can solve for \(A, B, C\) uniquely via the following system.
of equations:

\[
\begin{bmatrix}
1 & \cos \alpha_k & \sin \alpha_k \\
0 & \cos \alpha_k - \cos \alpha_i & \sin \alpha_k - \sin \alpha_i \\
0 & \cos \alpha_k - \cos \alpha_j & \sin \alpha_k - \sin \alpha_j
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
= 
\begin{bmatrix}
-c_k \\
-c_j - c_k \\
-c_l - c_k
\end{bmatrix}
\]  

(5.10)

It is easily checked that the conic section \( r(\alpha) = 1/(A + B \cos(\alpha) + C \sin(\alpha)) \) does indeed pass through the velocity points of \( A_i, A_j, A_k \) and hence \( v(\alpha_l) = v(\alpha_i) \).

The final step is to remove the restriction that \( \rho_i, \rho_j, \rho_k, \rho_l = 0 \). It is sufficient to consider the cost of a convex subpath \( E \) from \( a \in \mathbb{R}^2 \) to \( b \in \mathbb{R}^2 \), and show that if the velocity function is a conic section, then given any polygonal path \( \hat{E} \) from \( a \) to \( b \) such that \( L(\hat{E}) = L(E) \), we obtain \( T(\hat{E}) = T(E) \). From the proofs of Lemmas 5.5 and 5.6, it follows that we can construct a polygonal path which is arbitrarily close to \( E \). We can then apply the result we have just proved for \( \rho_i = \rho_j = \rho_k = \rho_l = 0 \) repeatedly to any of these polygonal paths to show that they all have the same cost. The result then follows in a similar manner to Lemma 4.1 showing that in the limit, as the polygonal path approximations approach \( E \) itself, we obtain \( T(E) = T(\hat{E}) \).

We can now consider sliding deformations as in Lemmas 4.4 and 4.5 to obtain the following necessary optimality condition for a \( CSCSCSC \) path with loops and non-degenerate straight line segments:

**Corollary 5.11.** If an optimal path contains an \( SCSCS \) path with the directions of the straights being \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), then there exists \( \epsilon > 0 \) such that \( v(\alpha) \leq \overline{v}(\alpha) \) for all \( \alpha \in [\alpha_1 - \epsilon, \alpha_1 + \epsilon] \cup [\alpha_2 - \epsilon, \alpha_2 + \epsilon] \cup [\alpha_3 - \epsilon, \alpha_3 + \epsilon] \), where \( \overline{v}(\alpha) \) is the unique conic section with a focal point at the origin that passes through the velocity points of \( \alpha_1, \alpha_2, \alpha_3 \).

**Proof.** Consider a combination of sliding deformations (see Section 4.3) which perturbs \( \alpha_3 \) while preserving \( \alpha_1, \alpha_2 \) and the total length of the path. The result then follows from Lemma 5.10 and repeating the argument on all 3 directions \( \alpha_1, \alpha_2, \alpha_3 \).

This corollary is the length-constrained analogue to Corollary 4.6. As illustrated in Figures 5.4, unlike double tangents, it is much harder to find the number of triplets of
points on the velocity function satisfying this condition, as conic sections could potentially be hyperbolas or ellipses. This makes the problem of constructing an optimal path of this form much more difficult. Furthermore, we have to also consider the cases where some of the straight line segments of the path are degenerate, resulting in necessary optimality conditions analogous to Lemmas 4.4 and 4.5. This further complicates the procedure of constructing an optimal path of the form $CSCSCSC$ (or degeneracy), even if we choose to restrict ourselves to considering piecewise constant velocity functions.

This motivates the following section, which studies the Lagrangian dual of our problem (5.1). This is a much easier problem to solve as we will show we can reduce it to a generalised version of the problem studied in earlier chapters which did not involve a length constraint. While the Lagrangian dual approach is unable to produce paths of the form $CSCSCSC$, it is the logical first step of an algorithm attempting to solve (5.1) since it is much easier to solve, and in the event that it does not solve the problem, still provides a lower bound on the minimum cost of an optimal path.

### 5.4 Dual Problem

In this section, we apply Lagrangian duality (see [12]) to study the dual problem of (5.1). We will see that this dual problem is easier to solve and will also provide us with a lower bound on the optimal solution to the primal problem (5.1), by applying weak duality.

The dual problem can be stated as the following maximisation problem:

$$
\max_{\lambda \geq 0} \min_{E \in \mathcal{P}_{pq}} \int_{E} c(\alpha) ds + \lambda (L_{lb} - \int_{E} ds) \tag{5.11}
$$

We first focus on the minimisation problem for a particular choice of $\lambda \geq 0$:

$$
\min_{E \in \mathcal{P}_{pq}} \int_{E} c(\alpha) ds + \lambda (L_{lb} - \int_{E} ds) \tag{5.12}
$$

The problem in (5.12) can be simplified into the following form which is similar to the problem without a lower bound on length, except with $c(\alpha) - \lambda$ as the new effective directional cost function. The term $\lambda L_{lb}$ can be dropped since it does not depend on $E$. 
(a) Plot of $v(\alpha)$ in thick solid line for neighbourhoods of size $\epsilon$ about $\alpha_1, \alpha_2, \alpha_3$ with the ellipse shown passing through the velocity points.

(b) Plot of $v(\alpha)$ in thick solid line for neighbourhoods of size $\epsilon$ about $\alpha_1, \alpha_2, \alpha_3$ with the hyperbola shown passing through the velocity points.

Figure 5.4: Illustration of Corollary 5.11 for an ellipse and a hyperbola with an exaggerated value of $\epsilon$ for clarity.
\[
\min_{E \in \mathcal{F}_{\text{pq}}} \int_{E} [c(\alpha) - \lambda] \, ds
\]  
(5.13)

To avoid confusion with the definition of an optimal path in this chapter, we will refer to a solution to (5.13) as a \(\lambda\)-optimal path for a given \(\lambda \geq 0\). Note that even though (5.13) does not have a lower bound of length, the problem is different from the one considered in previous chapters. While \(c(\alpha) > 0 \quad \forall \quad \alpha \in \mathbb{R}/2\pi\mathbb{Z}\), it is not necessarily true that \(c(\alpha) - \lambda > 0 \quad \forall \quad \alpha \in \mathbb{R}/2\pi\mathbb{Z}\). Hence, we first have to re-establish the key result that given any \(\lambda \geq 0\) there either exists a CSC (or a degeneracy of CSC) path which is \(\lambda\)-optimal, or the objective function is unbounded and no solution exists. Intuitively, we expect the latter case to occur when the value of \(\lambda\) is too high. We will therefore assume that a \(\lambda\)-optimal path does exist when proving results relating to the forms of \(\lambda\)-optimal paths.

In this section, a \textit{generalised directional cost function} is a continuous, piecewise \(C^2\) function \(c : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}\). This differs from previous chapters in that we allow \(c\) to take negative values. The corresponding \textit{generalised velocity function} \(v : \mathbb{R}/2\pi\mathbb{Z} \to (-\infty, \infty) \cup \mathbb{R} \setminus \{0\}\) is given by \(v(\alpha) = 1/c(\alpha)\), which is piecewise \(C^2\).

A generalised velocity function \(v = 1/c\) is \textit{regular} if there are no intervals \([\alpha_1, \alpha_2]\) with \(\alpha_1 \neq \alpha_2\) where \(c = 0\). Since \(v\) is no longer necessarily continuous nor bounded, it is convenient to initially restrict our focus to generalised velocity functions which are strict and regular. This allows us to easily handle the directions \(\alpha\) where \(c(\alpha) = 0\).

### 5.4.1 \(\lambda\)-Optimal Path Forms

The proof of Lemma 5.7 is actually still applicable to this \(\lambda\)-optimal path problem since it did not require \(c(\alpha) > 0\). However, the result only proved the existence of an optimal CS-path and did not show that any optimal path is a CS-path when we restrict ourselves to strict velocity functions. We can obtain a stronger result for the \(\lambda\)-optimal path problem since we do not have to restrict ourselves to length-preserving deformations and show that if the generalised velocity function is strict and regular, paths which are not CS-paths are not optimal.

In Lemma 2.2 we showed by applying Pontryagin’s Minimum Principle that if the
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velocity function is strict, every optimal curvature-constrained path is a CS-path. Since
the velocity function is no longer necessarily continuous, it is inconvenient to formulate
it as a vehicle control problem with anisotropic velocity as in Chapter 2. However, the
insight gained from earlier results motivates the following geometric proof. This method
does not have a restriction that \( c(\alpha) > 0 \), and also makes use of the significance of \( K(\alpha) \)
established in earlier chapters. The idea for this proof was adopted from [40]. Recall
that as mentioned in Section 5.3.2, we can assume that the path has a finite number of
inflection points since the result can be generalised by applying arguments from [40].

Lemma 5.12. If the generalised velocity function is strict and regular and there exists \( \lambda \)-optimal
path, every \( \lambda \)-optimal path is a CS-path.

Proof. Let \( E : [0, t_f] \rightarrow \mathbb{R}^2 \) be the parametrisation of a curvature-constrained path from \( p \)
to \( q \). As explained in the previous discussion, we can assume that \( E \) has a finite number
of inflection points. Since \( v \) is piecewise \( C^2 \), strict and regular, we can break \( E \) into convex
subpaths (i.e. do not contain inflection points) \( E_i : [t_{i-1}, t_i] \) for \( i = 1, \cdots, n \) where \( 0 = t_0 \leq t_1, \cdots \leq t_n = t_f \) such that each subpath \( E_i \) satisfies the following properties.

For all the subpaths which are not already CS-paths, we compare them with the \( \mathcal{CSC} \)
and \( \mathcal{SCS} \) paths between the same directed points. It then follows in a similar manner to
the proof of Lemma 3.11 that by the properties above, one of these two paths is necessarily
of lower cost than the original subpath. For example, if \( c(\alpha(t)) \leq 0 \) and \( K(\alpha(t)) \geq 0 \), then
the \( \mathcal{SCS} \) path is of lower cost.

Next, we explain how the results from Chapter 3, showing certain path forms are

\( \square \)
non-optimal, can be extended to allow for cost functions which can take negative values. While it is possible to modify the Lemmas in Chapter 3 in order to accommodate negative velocity values, it is more convenient to simply perform the following transformation from which the results can be applied in their current state. The required transformation to reversible directions $A_i = (\alpha_i, \rho_i)$ (see Definition 3.1) is described as follows:

If $c(\alpha_i) < 0$, we transform $A_i$ into an auxiliary reversible direction $\tilde{A}_i = (\alpha_i + \pi, \rho_i)$ with a corresponding velocity point given by $(|v(\alpha_i)|, \alpha_i + \pi)$. Furthermore, when contracting along an auxiliary reversible direction $\tilde{A}_i$, we instead perform an extension along $A_i$, the original reversible direction. By applying this transformation, we reduce the generalised problem to the simpler problem of only having positive velocity points from which the lemmas from Chapter 3 can be naturally extended. If $c(\alpha_i) = 0$, we can treat this as the limit of either $c(\alpha_i) \to 0^+$ or $c(\alpha_i) \to 0^-$ (corresponding to $v(\alpha_i) \to +\infty$ or $v(\alpha_i) \to -\infty$ respectively). We will illustrate later in an example that both interpretations are valid and yield the same result.

We now provide examples to illustrate the transformation procedure described above. First, suppose we are given distinct reversible directions $A_i, A_j, A_k$ which positively span $\mathbb{R}^2$ where $c(\alpha_i), c(\alpha_j) > 0$ and $c(\alpha_k) < 0$. If the costs were all positive, we would apply Lemma 3.8 to show that a CS-path with these reversible directions is not optimal. However, we are unable to apply that result so instead replace $A_k$ with $\tilde{A}_k$ as shown in Figure 5.5. Now since $A_i, A_j, \tilde{A}_k$ lie in an open half-space, we instead apply Lemma 3.9, 3.10 or 3.16 depending on the velocity points (see Table 3.1 and 3.2), in order to show that a CS-path with these reversible directions is not optimal.

Next, suppose we are given distinct reversible directions $A_i, A_j, A_k$ which lie in an open half-space, where $c(\alpha_i), c(\alpha_j) > 0$, $c(\alpha_i) = 0$, and $A_j$ is the interior reversible direction. Let the straight line passing through the velocity points of $A_i$ and $A_k$ be given in polar coordinates by the function $v^k_j(\alpha)$ as shown in 5.6(a). By considering the velocity point of $A_j$ as the limit of $c(\alpha_j) \to 0^+$, we then deduce by Lemma 3.6, that the appropriate deformations to consider are extending along $A_j$ while contracting along $A_i$ and $A_k$. On the other hand, if we consider the velocity point of $A_j$ as the limit of $c(\alpha_j) \to 0^-$, we replace $A_j$ with the corresponding auxiliary reversible direction $\tilde{A}_j$ as shown in Fig-
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Figure 5.5: Example of an auxiliary reversible direction $\tilde{A}_k$ introduced because $c(\alpha_k) < 0$. The velocity point of $\tilde{A}_k$ is given by $(|v(\alpha_k)|, \alpha_k + \pi)$ which is used with the velocity points of $A_i$ and $A_j$ to determine which set of deformations can be employed to reduce the cost of the original CS-path.

It then follows from the fact that $A_j$ was the interior direction that $A_i, \tilde{A}_j, A_k$ positively span $R^2$, which means by Lemma 3.8 we should contract along all the directions. Since $\tilde{A}_j$ is an auxiliary reversible direction, contracting along $\tilde{A}_j$ means extending along $A_j$, which gives us the same deformations as when considering $c(\alpha_j) \to 0^+$.

Finally, suppose we are given distinct reversible directions $A_i, A_j, A_k$ which lie in an open half-space, where $c(\alpha_i), c(\alpha_j) > 0$, $c(\alpha_k) = 0$ and $A_j$ is the interior reversible direction. By considering the velocity point of $A_k$ as the limit of $c(\alpha_k) \to 0^+$ it is clear that $v_i^k(\alpha)$ approaches a line parallel to $\alpha_k$ passing through the velocity point of $A_i$ as shown in Figure 5.6(c). On the other hand, if we consider the velocity point of $A_k$ as the limit of $c(\alpha_k) \to 0^-$, the interior direction is instead $A_i$ after transforming $A_k$ into $\tilde{A}_k$. We then consider instead $v_j^k(\alpha)$, the line parallel to $\alpha_k$ passing through the velocity point of $A_j$ as shown in Figure 5.6(c). Clearly since $v_i^k(\alpha)$ and $v_j^k(\alpha)$ are parallel, both cases result in the same deformations being considered.

The examples shown in Figures 5.5 and 5.6 summarise the transformations required in order for the results from Chapter 3 to follow. This allows us to state the following result:
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(a) Limit of velocity point of $A_j$ as $c(\alpha_j) \to 0^+$

(b) Limit of velocity point of $\tilde{A}_j$ as $c(\alpha_j) \to 0^-$

(c) Limit of line passing through the velocity points of the exterior reversible directions as $c(\alpha_k) \to 0^+$

(d) Limit of line passing through the velocity points of the exterior reversible directions as $c(\alpha_k) \to 0^-$

Figure 5.6: Examples of velocity points corresponding to directions where $c(\alpha) = 0$. The limits of $c(\alpha) \to 0$ from the right and left are shown in (a) and (b) respectively for an example where the interior reversible direction has zero cost. The example illustrated in (c) and (d) consider the case when an exterior reversible direction has zero cost.
Theorem 5.13. Given any directional cost function and \( \lambda \geq 0 \), if there exists a \( \lambda \)-optimal path from \( p \) to \( q \), then there exists a path of the form \( \text{CSCSC} \) (or degeneracy) which is a \( \lambda \)-optimal path from \( p \) to \( q \).

Proof. A \( \lambda \)-optimal path for a given directional cost function \( c \) corresponds to an optimal path (without a minimum length constraint) for the generalised cost function \( c - \lambda \). By Lemma 5.12 and introducing auxiliary reversible directions, the result holds for generalised velocity functions which are strict and regular. We can relax the restriction of the generalised velocity function being strict and regular, by applying a similar argument to Corollary 3.18.

5.4.2 Necessary Conditions for Optimality

The obvious next step is to extend the results of Chapter 4 to the \( \lambda \)-optimal path problem in order to obtain an algorithm which can solve the problem efficiently. It is clear that 4.2 - 4.5 still hold when generalised velocity functions are considered. However, since the generalised velocity function can take negative values, Corollary 4.6, which describes the Positive External Tangent (PET) condition, no longer holds, and we require a larger bound than the bound established in Theorem 4.7 for the number of pairs of directions which need to be considered for Type 1 (contain \( \text{SCS} \) subpaths) paths. This will involve making use of the bound for the number of Negative External Tangents (NETs), as well as a modification of the number of internal double tangents. In this section, we will use notation introduced in Section 4.3.3, with slight modifications.

Let \( c(\alpha) \) be a generalised directional cost function with a corresponding generalised velocity function \( v(\alpha) \) which is strict and regular. Let \( \beta_+, \beta_- \subseteq \mathbb{R}/2\pi\mathbb{Z} \) be defined as \( \beta_+ = \{ \alpha | c(\alpha) > 0 \} \) and \( \beta_- = \{ \alpha | c(\alpha) < 0 \} \). Note that since \( v \) is regular, the closure of \( \beta_+ \cup \beta_- \) is identically \( \mathbb{R}/2\pi\mathbb{Z} \). Let \( V^+ \) denote the collection of planar curves which correspond to the points \( V(\alpha) \ \forall \ \alpha \in \beta_+ \). In a similar manner to previously, we wish to introduce auxiliary planar curves in order to avoid dealing with negative valued points in polar coordinates. Let \( V^- \) denote the collection of auxiliary planar curves which correspond to the points \( (|v(\alpha)|, \alpha + \pi) \ \forall \ \alpha \in \beta_- \).
Let $\beta_0^+$ denote the set of roots of $c(\alpha)$ where $c(\alpha) \geq 0$ in a neighbourhood of the root. Similarly, let $\beta_0^-$ denote the set of roots of $c(\alpha)$ where $c(\alpha) \leq 0$ in a neighbourhood of the root. Let $\beta_0$ denote the set of roots which do not belong in $\beta_0^+ \cup \beta_0^-$. These points are classified in this manner as they have significantly different properties when evaluating the number of feasible pairs of straight directions.

For convenience, we restrict ourselves to considering $C^2$ directional cost functions so that $K(\alpha)$ is defined on $\beta_+ \cup \beta_-$. Recall from Section 4.3.3 that a PET is a double tangent with positive curvature in the neighbourhoods of the contact points, and a NET is the same with positive replaced with negative. Also, recall that an internal tangent is a double tangent with opposing signs of curvature in the neighbourhoods of the contact points. As in the previous chapter, we shall use the notation $\{\alpha_1, \alpha_2\}$ where $\alpha_2 \in (\alpha_1, \alpha_1 + \pi)$ to denote a double tangent between the corresponding velocity points. However, if $\alpha_1$ (or $\alpha_2$) belongs to $\beta_-$ then it instead corresponds to the auxiliary velocity point of $V^-$, $|(v(\alpha)), \alpha + \pi\rangle$. We also allow $\{\alpha_1, \alpha_2\}$ to denote special objects which are not double tangents, if $c(\alpha_1) = 0$ or $c(\alpha_2) = 0$, as we will show that objects other than double tangents can satisfy the optimality conditions under this special circumstance.

By considering the implications of Lemmas 4.2 and 4.3, we deduce the following generalised version of Corollary 4.6.

**Corollary 5.14.** Given a generalised velocity function which is strict and regular, let $E$ be an optimal CSCSC (or degeneracy) path with an SCS subpath with the directions of the straights being $\alpha_1$ and $\alpha_2$ where $\alpha_2 \in (\alpha_1, \alpha_1 + \pi)$. Then, $\{\alpha_1, \alpha_2\}$ is one of the following:

1. a PET where $\alpha_1, \alpha_2 \in \beta_+$
2. a NET where $\alpha_1, \alpha_2 \in \beta_-$
3. an internal tangent with one direction in $\beta_+$ with positive curvature in the neighbourhood around the corresponding velocity point, and the other direction in $\beta_-$ with negative curvature in the neighbourhood around the corresponding auxiliary velocity point
4. a tangent to a point in $\beta_+$ with positive curvature in the neighbourhood of that point parallel to a direction in $\beta_0^+ \cup \beta_0$
5. a tangent to a point in $\beta_-$ with negative curvature in the neighbourhood of that point parallel to a direction in $\beta_0^+ \cup \beta_0$
\( a_1, a_2 \in \beta_0^+ \)

**Proof.** The first three possibilities follow easily by considering the different possibilities of signs of \( c(a_1) \) and \( c(a_2) \) and applying Lemmas 4.2 and 4.3. These double tangents cover all possibilities for when \( c(a_1) \neq 0 \) and \( c(a_2) \neq 0 \). This leaves only the cases where at least one of the directions has zero cost to be considered.

The less intuitive part of the result is in the objects described in (4),(5) and (6), which are not double tangents. For clarity, we first introduce an alternative interpretation to the double tangent result from Corollary 4.6 and then show how this method of viewing the problem enables us to easily deduce the result for the case where the cost in one of the directions is zero.

Consider a PET \( \{a_1, a_2\} \) as shown in Figure 5.7(a), with the corresponding directional cost function shown in Figure 5.7(b). The PET is denoted by the function \( v(\alpha) = 1/\cos(\alpha - \alpha_\perp) \) defined on \( \alpha \in \mathbb{R}/2\pi\mathbb{Z} \) (note that this function is negative valued and maps out the same curve in polar coordinates for \( \alpha \in (\pi/2 + \alpha_\perp, 3\pi/2 + \alpha_\perp) \)). Since \( c = 1/v \), it is clear that \( v(\alpha') < v(\alpha^*) \) if and only if \( c(\alpha') > c(\alpha^*) \). Hence, we obtain the analogous condition shown in Figure 5.7(b) with \( \tau(\alpha) = 1/\tau(\alpha) = \cos(\alpha - \alpha_\perp) \). This is less informative since there is no explicit condition on the curvature of the corresponding curve, but will be a useful interpretation later on.

Now consider the velocity function shown in Figure 5.7(c) with corresponding directional cost function shown in Figure 5.7(d). In this case, \( a_1 \in \beta_0 \) and \( a_2 \in \beta_+ \), so it necessarily follows that \( \tau \) is tangential to the velocity point at \( a_2 \) and is parallel to \( a_1 \). The curvature of \( v(\alpha) \) must also be positive in a neighbourhood about \( a_2 \). Because of the asymptotic behaviour of \( \tau \), the same conditions do not apply to \( a_1 \). However, by considering the cost function and recalling that the condition of \( c(\alpha) \) being continuous, it is clear that all we need is that \( c(\alpha) \geq \tau(\alpha) \forall \alpha \in (a_1 - \varepsilon, a_1 + \varepsilon) \) for some \( \varepsilon > 0 \). This condition can be satisfied if \( a_1 \in \beta_0 \cup \beta_0^+ \) as shown in Figure 5.7(d) but is not possible if \( a_1 \in \beta_0^- \).

In a similar manner, it follows trivially that if \( c(a_1) = c(a_2) = 0 \), then \( a_1, a_2 \in \beta_0^+ \). \( \square \)

Note that for the simpler result of Corollary 4.6, a PET necessarily corresponded to a feasibly optimal pair of straight directions. In this generalised result, all the possibilities correspond to a feasibly optimal pair of straight directions with the exception of the
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(a) A PET $\{\alpha_1, \alpha_2\}$ denoted by $\nabla(\alpha)$

(b) $\tau(\alpha) = 1/\nabla(\alpha)$ and $c(\alpha) = 1/\nabla(\alpha)$ from (a)

(c) A tangent to $\nabla(\alpha)$ in the direction $\alpha_2$

(d) $\tau(\alpha) = 1/\nabla(\alpha)$ and $c(\alpha) = 1/\nabla(\alpha)$ from (a), illustrating that the necessary optimality condition is satisfied in a neighbourhood around $\alpha_2$

Figure 5.7: Examples of directional cost function $\tau(\alpha)$ in (b) and (d) corresponding to $\nabla(\alpha)$ in (a) and (c) respectively (all shown in solid). The velocity functions $\nabla(\alpha)$ and directional cost functions are shown satisfying the necessary optimality condition in both cases (thick solid curve).
tangents parallel to a direction in $\beta^+_0 \cup \beta_0$ since they actually depend on the relative behaviour of the limit of the values of $c(\alpha)$ in the neighbourhood of the root, rather than just the signs, as described in the proof.

The next step is to provide a bound for the possible pairs $\{a_1, a_2\}$ listed in Corollary 5.14. In Theorem 4.7, all we needed to know was the number of inflection points in the velocity function. However, a generalised velocity function can potentially be more complicated, and hence, requires quantification of more features in order to obtain a useful bound in terms of the number of inflection points. For simplicity, we shall present a weaker bound which is easier to state. We avoid stating the more complicated bound which is given in terms of the number of inflection points as it also requires more information on the signs of curvature of the velocity function on both sides of each direction where $c(\alpha) = 0$.

**Theorem 5.15.** Given a generalised velocity function which is strict and regular, the number of distinct pairs of directions $\{a_1, a_2\}$ satisfying Corollary 5.14 is bounded above by

$$(h^+ + h^- + g^+) + g(h^+ + h^-) + h^+(h^- + g^+ + 1)$$

where each quantity is defined as follows:

- $h^+$, the number of distinct convex regions ($K(\alpha) \geq 0$) in $\beta^+$
- $h^-$, the number of distinct concave regions ($K(\alpha) \leq 0$) in $\beta^-$
- $g = |\beta_0|$
- $g^+ = |\beta^+_0|$

**Proof.** In order to prove this bound, we will bound each of the 6 items listed in Corollary 5.14 individually and sum the bounds to obtain the result. The respective bounds are listed alongside the item description, followed by a brief outline of how they are obtained.

1. a PET where $a_1, a_2 \in \beta^+ : \binom{h^+ + 1}{2}$.

   This can be easily bounded by applying Theorem 4.7 to $V^+$, except that now instead of selecting pairs of concave regions (with repetition allowed), we select regions in between distinct convex regions. The rest of the proof then follows easily.

2. a NET where $a_1, a_2 \in \beta^- : \binom{h^-}{2}$. 
The result for the bound on NETs can be applied in a similar manner to $V^-$. 

(3) an internal tangent with one direction in $\beta_+$ with negative curvature (convex) in the neighbourhood around the corresponding velocity point, and the other direction in $\beta_-$ with positive curvature (concave) in the neighbourhood around the corresponding auxiliary velocity point: $2h^+h^-$. 

This bound can be proven in a similar manner to the arguments used in the proof of Theorem 4.7. The key idea is that there can only be at most two such internal tangents for a given choice of convex region in $\beta_+$ and concave region in $\beta_-$. 

Let $(\tilde{a}_1, \tilde{a}_2) \subset \beta_-$ denote the chosen concave region and $(\tilde{a}_1, \tilde{a}_2) \subset \beta_+$ denote the chosen convex region. Note that this corresponds to an auxiliary planar curve in the region $(\tilde{a}_1 + \pi, \tilde{a}_2 + \pi)$. It follows that since this auxiliary planar curve is concave that the forward tangent $\tau(a) \in (\tilde{a}_1 + \pi, \tilde{a}_1) \forall a \in (\tilde{a}_1, \tilde{a}_2)$. First, suppose that the contact point with the positive convex planar curve is in the forward tangent direction. This implies that the contact point lies in $(\tilde{a}_1 + \pi, \tilde{a}_1)$. Since $\beta_+$ and $\beta_-$ are disjoint, $(\tilde{a}_1 + \pi, \tilde{a}_1) \cap (\tilde{a}_3, \tilde{a}_4)$ is a connected set. We can then apply a similar argument to that in the proof of Theorem 4.7 to show that there can only be one such internal tangent, since we need only consider the convex curve on the region $(\tilde{a}_1 + \pi, \tilde{a}_1) \cap (\tilde{a}_3, \tilde{a}_4)$ which is of size less than $\pi$. Repeating the argument for when the contact point with the positive convex planar curve is in the backwards tangent direction gives us the desired result. 

(4) a tangent to a point in $\beta_+$ with positive curvature (convex) in the neighbourhood of that point parallel to a direction in $\beta^+_0 \cup \beta_0$: $h^+(g + 2g^+)$. 

Given a direction $\tilde{a}_0$ where $c(\tilde{a}_0) = 0$, this implies that any convex positive region $(\tilde{a}_1, \tilde{a}_2)$ must not contain $\tilde{a}_0$. Hence, this restricts the direction of the tangent to at most taking the value of $\tilde{a}_0$ once, and taking the value of $\tilde{a}_0 + \pi$ once, giving a bound of $2h^+(g + g^+)$. The bound can then be tightened by considering the fact that if $\tilde{a}_0 \in \beta_0$, then the contact point must lie in $(\tilde{a}_0, \tilde{a}_0 + \pi)$ if $c'(\tilde{a}_0) > 0$ (or in $(\tilde{a}_0 + \pi, \tilde{a}_0)$ if $c'(\tilde{a}_0) < 0$. This makes it impossible for the tangent direction to take the value of $\tilde{a}_0$, reducing the total number of possible tangents to one, for $g$. 


(5) a tangent to a point in $\beta_{-}$ with negative curvature (concave) in the neighbourhood of that point parallel to a direction in $\beta_{0}^{+} \cup \beta_{0}^{0}$: $h^{-}(g + g^{+})$.

This result follows in a similar manner to (4), since concave regions are always of size less than $\pi$.

(6) $\alpha_{1}, \alpha_{2} \in \beta_{0}^{+}$: $(g^{+})$.

This bound simply follows by choosing two distinct directions in $\beta_{0}^{+}$.

Recall that in Chapter 4, we were able to bound the number of Type 1 paths (contain a $SCS$ subpath) but unable to do the same for Type 2 paths, which contain a $CCS$ (or $SCC$) subpath. However, when we restricted the cost functions to piecewise constant, we were able to bound the number of Type 2 paths as well. Lemmas 4.4 and 4.5 still hold when considering a generalised velocity function. Thus we can apply the same types of arguments as above to extend the existing results from Section 4.4 to obtain a procedure which can construct a $\lambda$-optimal path for any piecewise constant generalised velocity function. Since a piecewise constant generalised velocity function is not necessarily regular, this has to be taken into account and slight modifications should be made to the results above. This is analogous to how a weak velocity function would be accounted for. We do not present an explicit procedure for this case, but we illustrate possible outcomes of this procedure on some simple cases in the following section.

### 5.5 Examples

As discussed in Section 5.3, it is difficult to construct all the feasibly optimal paths of the form $CSCSCSC$ (or degeneracy) with loops. In this section, we discuss the effectiveness of the procedure suggested in Section 5.4 and present different examples to illustrate that in some scenarios, the Lagrangian dual approach can be very successful while in others, it fails to produce the optimal path. Since these examples involve comparing methods which solve different problems, we will refer to the problem we are interested in solving (5.1) as the *primal problem*, while the problem solved by OPTIMALPATH in Chapter 4
as the unconstrained problem and the problem in (5.11) as the dual problem (Note that the \( \lambda \)-optimal path problem is not the same as dual problem as the optimal value of \( \lambda \) needs to be used in order for the \( \lambda \)-optimal path to be optimal for the dual problem).

By weak duality [12], we know that for any \( \lambda \geq 0 \), the value of the objective function in (5.12) is a lower bound, \( T_\lambda \), on the optimal value of the primal problem. This means that if for some \( \lambda^* \geq 0 \), we obtain a \( \lambda^* \)-optimal path \( E^* \) such that \( L(E^*) = L_{lb} \), then the primal problem is solved since \( E^* \) is a feasible path to the primal problem with \( T(E^*) = T_{\lambda^*} \). Example 5.1 below illustrates an example where this occurs.

**Example 5.1.** Consider the piecewise constant velocity function \( v \) shown in Figure 5.8(a). The optimal path \( E \) from \( p \) to \( q \) for the unconstrained problem is shown in lighter grey in Figure 5.8(c). However, this path is not feasible for the given lower bound on length \( L_{lb} \) i.e. \( L(E) < L_{lb} \). The \( \lambda \)-optimal path \( E^* \) is shown in darker grey for a particular choice of \( \lambda = \lambda^* \) such that \( L(E^*) = L_{lb} \). The generalised velocity function corresponding to \( \lambda^* \) is shown in Figure 5.8(b) with the double tangent corresponding to the pair of straight directions shown on the figure. By weak duality, we conclude that the optimal path for the dual problem corresponds to an optimal path for the primal problem, in this example. It is worth noting that in this example, an optimal solution cannot be obtained by simply adding loops to optimal paths for the unconstrained problem.

The next example illustrates a case where the dual problem approach does not give us the primal optimal path since we only consider paths of the form CSCSC (or degeneracy) without loops.

**Example 5.2.** Consider a constant velocity function \( v = 1 \) which is a special case of the primal problem where we are interested in finding the shortest path given a lower bound on length. Let \( p = (0, 0, 0), q = (\varepsilon, 0, 0) \) and \( L_{lb} = 2\pi \). It is clear that for sufficiently small \( \varepsilon > 0 \), the optimal path is simply an \( S \) path with a loop added as shown in Figure 5.9. In this case, the optimal path to the unconstrained problem with a loop added is the optimal path. The \( \lambda \)-optimal path is simply the straight line segment from \( p \) to \( q \) without a loop for \( \lambda \in (0, 1) \) and becomes an infinitely long path for \( \lambda > 1 \). When \( \lambda = 1 \), the generalised velocity function is irregular since \( c - \lambda = 0 \). In this case, any curvature-constrained path from \( p \) to \( q \) is \( \lambda \)-optimal, which is not useful for determining the primal optimal path.
Incorporating the Gradient Constraint

(a) Plot of $v(\alpha) = 1/c(\alpha)$ with the PET shown passing through the velocity points in directions $\alpha_1, \alpha_2$

(b) Plot of $1/(c(\alpha) - \lambda^*)$ with the PET shown passing through the velocity points in directions $\alpha_1^*, \alpha_2^*$

(c) Optimal CSCSC paths for the unconstrained problem and dual problem shown in lighter grey and darker grey respectively with the directions of the straight line segments annotated

Figure 5.8: Illustration of Example 5.1 where the dual optimal path is also a primal optimal path, while the unconstrained optimal path is infeasible for the primal problem.
5.5 Examples

Lastly, it is clear that the dual approach will not be able to solve the primal problem when all primal optimal paths have the form $\text{CSCSCSC}$ such as in Example 5.3 below.

**Example 5.3.** Consider the velocity function $v$ shown in Figure 5.10(a). The path $E$ from $p$ to $q$, shown in Figure 5.10(b), has length $L(E) = L_{\text{lb}}$. Since $E$ can be rearranged to form a loop together with the polygonal path $abcd$ as shown, it is clear that $E$ is an optimal path for the given velocity function. It is also clear that there is no $\text{CSCSC}$ (even with loops) path that is optimal. In this case, the dual approach will only be able to provide a lower bound on the optimal primal value since it only considers paths of the form $\text{CSCSC}$ (or degeneracy). This is still useful as it provides a performance bound on any subsequent $\text{CSCSCSC}$ (or degeneracy) path obtained via other methods.

Example 5.3 is also a simple illustration of Corollary 5.11 where the conic section is just a circle. For more complicated velocity functions, it would be difficult to construct tangential conic sections. Furthermore, it is not clear that there is a bound on the number of tangential conic sections even if we restrict ourselves to piecewise constant velocity functions. This example also illustrates that even if the velocity function is polarly convex (velocity function maps out the boundary of a convex region), we are unable to eliminate paths of the form $\text{CSCSCSC}$. This differs from Chapter 2 where without the gradient
constraint, a polarly convex velocity function guarantees the existence of an optimal path which is a Dubins path.

It is also worth pointing out that the velocity function in Example 5.3 can be easily modified to permit optimal paths with more than 3 $S$ segments by increasing the number of velocity points touching the unit circle. However, the important result is that we can always reduce such a path down to an optimal $CSCSCSC$ (or degeneracy) path with loops, by Theorem 5.8.

5.6 Conclusion

By incorporating the gradient constraint into the problem, we have developed the theory necessary to produce paths which are directly applicable to the underground mine design problem. Within a particular geological domain where the support costs can be modelled as being primarily dependent on the direction in the horizontal projection, this model incorporates the essential vehicle navigability constraints of both maximum curvature and gradient. Due to the complicated nature of gradient constraint, we are unable to present a simple procedure such as in Chapter 4 for the problem without the gradient constraint. However, we have proven the significant result that there exists an optimal path of the form $CSCSCSC$ (or degeneracy) with loops. We have also presented an approach to solving the problem by studying its dual problem and illustrated examples both where this dual approach solves the problem exactly and when it is unable to but still provides a lower bound on the optimal value of the original problem.
5.6 Conclusion

(a) Plot of $v(\alpha) = 1/c(\alpha)$ where $v(\alpha_1) = v(\alpha_2) = v(\alpha_3)$

(b) Optimal CSCSCSC path for the primal problem, $E$, shown in darker grey with the polygonal path $abcd$ formed by its 3 straight line segments shown in lighter grey

Figure 5.10: Illustration of Example 5.3 where the primal optimal path is of the form CSCSCSC.
Chapter 6
Conclusion

This thesis has developed new mathematical theory on optimal curvature-constrained paths with anisotropic cost. The problems studied in the thesis are applicable to underground mine design where the anisotropic behaviour of the ground can result in anisotropic support costs. Existing results on curvature-constrained paths have been generalised and extended. New techniques have also been developed in order to study curvature-constrained paths with anisotropic cost. The theoretical results proven have also been applied to develop an algorithm which can efficiently construct optimal paths for simple cost functions that are suitable for the application to underground mine design. The problem has also been generalised to incorporate a gradient constraint, thus accounting for the major vehicle navigability constraints required to apply the theory to underground mine declines.

6.1 Summary of Contributions

In this section, we list the new contributions of this thesis. The contributions are listed under the chapter where the result was first presented. Published and submitted papers are shown in parentheses beside the chapter heading.

6.1.1 Chapter 2 [28]

1. The optimal curvature-constrained path problem with a directional cost function is introduced (Section 2.2.3).
2. It was proven that if the velocity function is strict, then any optimal path between two given directed points is a CS-path (Section 2.3).

3. Basic properties of Euclidean length are generalised to hold for cost of paths subject to a polarly convex velocity function. These were then used to generalise Dubins’ result for shortest curvature-constrained paths. In particular, if the velocity function is strictly polarly convex, then the optimal path is a Dubins path (Section 2.4).

6.1.2 Chapter 3 [29]

1. The concept of a reversible direction of a CS-path is introduced. This is used to perform CS-path deformations which include reversible and irreversible deformations (Section 3.3).

2. Reversible directions are used to show that there exists an optimal path of the form CSCSC (or degeneracy) given any velocity function (Section 3.4).

3. An alternate proof for the case where the velocity function is strictly polarly convex is provided using reversible and irreversible deformations (Section 3.4.4).

4. The result that there exists an optimal path of the form CSCSC (or degeneracy) is shown to still hold if the cost of all C arcs is scaled up by a constant factor (Section 3.4.5).

6.1.3 Chapter 4 [26]

1. Necessary optimality conditions are derived for CSCSC (or degeneracy) paths in terms of the behaviour of the velocity function in the directions of the straight line segments or inflection points of the path (Section 4.3).

2. Sharp bounds for the different types of double tangents of a closed star convex planar curve are proven (Section 4.3.3).

3. Using the necessary optimality conditions and the result of double tangents, an \(O(n^2)\) algorithm OPTIMALPATH is presented which constructs an optimal path between given directed points for a given piecewise constant directional cost func-
6.2 Future Research

As well as answering open questions, this thesis has also brought to light many new research questions. They are listed as follows:

6.1.4 Chapter 5 [27]

1. The maximum gradient constraint is incorporated into the model and formulated as a 2-dimensional problem with a minimum length constraint (Section 5.3).
2. Length-preserving reversible deformations are introduced and used to show that there exists an optimal path of the form \( CSCSCSC \) (or degeneracy) with loops (Section 5.3).
3. It is shown that if there exists an optimal path which has length greater than the lower bound on length, then there exists an optimal path of the form \( CSCSC \) (or degeneracy) with loops (Section 5.3).
4. A necessary optimality condition is derived for \( CSCSCSC \) paths with non-degenerate straight line segments in terms of the behaviour of the velocity function relative to a particular conic section passing through the relevant points on the velocity function (Section 5.3.4).
5. Lagrangian duality is used to construct the \( \lambda \)-optimal path problem. This problem is shown to be able to provide a lower bound for the optimal value of the gradient-constrained problem, and is able to solve it exactly in some cases (Section 5.4).
6. Auxiliary reversible directions are introduced and used to show that there exists a \( \lambda \)-optimal path of the form \( CSCSC \) (or degeneracy) (Section 5.4.1).
7. Using the necessary optimality conditions from Chapter 4, a bound for the number of \( CSCSC \) paths with non-degenerate straight line segments is obtained for the \( \lambda \)-optimal path problem (Section 5.4.2).
6.2.1 Generalisation of polar convexity results

In Chapter 2, we showed that when the directional cost function has a reciprocal which is strictly polarly convex (maps out the boundary of a convex region in polar coordinates), then the optimal paths are of the same form as when the cost is isotropic i.e. the shortest path problem. One way to view this result is that the set of locally optimal paths among the set of all curvature-constrained paths between two given directed points is invariant with respect to changes in the directional cost function, as long as the reciprocal of the directional cost function remains strictly polarly convex. This raises an interesting question: What class of planar path optimisation problems (or equivalently, class planar path sets) does this statement hold for? A simpler goal would be to establish a simple set of sufficient conditions which allows us to conclude that the locally optimal paths are invariant for a given problem. This is because we can then immediately generalise any results relating to shortest paths, to the anisotropic cost version of the problem with a strictly polarly convex velocity function.

6.2.2 Improving on the planar algorithm

In Chapter 4, we presented an algorithm OPTIMAL PATH which efficiently constructs optimal paths between two given directed points, for a given piecewise constant directional cost function. The choice of restricting focus to piecewise constant functions is to suitably model underground support costs, which are discrete. It was explained that this algorithm could be modified into one which handled an arbitrary cost function using piecewise constant approximations to the function. This could be studied in more detail, as the best way to approximate a given function with a piecewise constant function is not immediately obvious. Alternatively, an entirely different algorithm could be developed which does not involve piecewise constant approximations. This could be useful for other applications where piecewise constant directional cost functions are not suitable.
6.2 Future Research

6.2.3 Developing an algorithm for the 3-dimensional problem

In Chapter 5, we proved that there must exist an optimal path of the form \( \text{CSCSCSC} \) (or degeneracy) with loops. However, we did not provide an explicit algorithm for constructing an optimal path in general. This is due to the difficulty of identifying feasible triplets of directions of straight line segments which satisfy Corollary 5.11. Furthermore, considering the degenerate forms of \( \text{CSCSCSC} \) paths further complicates the problem. Heuristics could be considered as an approach to solving the problem, especially since we provide a method of using the dual formulation in order to at least obtain a lower bound on the minimum cost of an optimal path, if not to solve the original problem.

6.2.4 Improving on the model for directional cost in 3-dimensions

Another aspect in which further theory can be developed is to consider a 3-dimensional anisotropic cost function. We assumed in Chapter 5 that the anisotropic behaviour can be modelled as a 2-dimensional property. This is not always a valid assumption, especially if the fault planes are close to horizontal. In this case, the directional behaviour should be modelled in 3-dimensions. This creates difficulties as we are unable to simply reduce the problem down to the 2-dimensional version of the problem with a minimum length constraint as we have done.

6.2.5 Heterogeneous cost functions and decline networks

The theory developed in this thesis lays the necessary foundation for incorporating heterogeneity of development cost into the model. The ground in which an underground mine is developed is typically divided into distinct geological domains with their own anisotropic features. We can model the heterogeneous behaviour of rock by considering the problem where the plane is divided into two regions, each with its own directional cost behaviour. If the path only locally passes through two geological domains, this is a valid assumption. A related problem has been studied in [57] where the velocity function is isotropic and the maximum curvature constraint varies inversely with the velocity.

This problem can then be taken to a larger scale where multiple geological domains
are considered. If the geological domains are assumed to be polytopes which partition the 3-dimensional space, then optimising a curvature and gradient-constrained path between two directed points requires deciding which geological domains to traverse. A simpler version of this problem in the plane with no curvature constraint is studied in [56]. At this level, it is also then worth considering networks of curvature and gradient-constrained paths, since an underground mine consists of a network of such paths. This would naturally lead to problems related to optimal gradient-constrained flow-dependent networks studied by Volz [66].

6.2.6 Case study

Performing a case study on an existing mine would be the first step in integrating the theory developed in this thesis into industry practice. In order to make a fair comparison between existing mine declines and optimal paths considered in this thesis, additional constraints such as stand-off from the orebody and no-go region avoidance would need to be considered. There is a wealth of literature on obstacle avoidance and Dubins paths, such as [1], [2], [3], [5], [7], [8], [9], [35]. Sufficiently accurate data on each geological domain and the corresponding directional costs would also be essential for this comparison since the theory relies on this data being available.
Appendix A

Mathematical Symbols

$\alpha$ a direction (angle) measured anti-clockwise from the horizontal axis

$\gamma$ the size of an angular deformation to a direction

$\delta$ the size of the linear displacement due to a deformation

$\rho$ type of reversible direction

$a, b$ points in the plane

c$(\alpha)$ directional-cost function

$d$ degree of a CS-path

$p, q$ directed points in the plane, with a specified heading direction

$v(\alpha)$ velocity function (reciprocal of $c(\alpha)$)

$A$ a reversible direction

$E$ a planar path

$K(\alpha)$ sign of the curvature of the velocity function in polar coordinates

$L(E)$ length of a path $E$

$O$ the origin of a function in polar coordinates

$T(E)$ cost of a path $E$ based on the given directional-cost function

$V$ velocity set corresponding to $v(\cdot)$
\( \mathcal{C} \) label for a continuous subset of the unit circle
\( \mathcal{L} \) label for a continuous subset of the unit circle no longer than \( \pi \)
\( \overline{\mathcal{C}} \) label for a continuous subset of the unit circle longer than \( \pi \)
\( \mathcal{L} \) label for a left-turning continuous subset of the unit circle
\( \mathcal{R} \) label for a right-turning continuous subset of the unit circle
\( \mathcal{V} \) planar curve corresponding to \( v(\cdot) \)
\( \mathcal{C}_{pq} \) set of all CS-paths from \( p \) to \( q \)
\( \mathcal{D}_{pq} \) set of all Dubins paths from \( p \) to \( q \)
\( \mathcal{P}_{pq} \) set of all curvature-constrained paths from \( p \) to \( q \)
\( \mathbb{R}^n \) \( n \)-dimensional Euclidean space
\( \mathbb{R}/2\pi\mathbb{Z} \) the quotient group of equivalent directions (angles)
\( \| \cdot \| \) a norm
Bibliography


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