On Lower Barrier Insurance Risk Processes

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Abstract

In this thesis we present a new model, namely the lower barrier model, based on the fundamental works in the context of ruin theory and risk process modelling, where we are concerned with the level of an insurer’s surplus for a portfolio of insurance policies. The idea is to impose a lower barrier that is above zero on the ordinary risk process, so that it prevents the process from falling below a certain level but does not prevent the ruin event. We justify the realistic application of this lower barrier model using the idea of a reinsurance arrangement.

Firstly, we impose the lower barrier on the classical risk model and study the ultimate ruin probability. We assume a fixed amount of capital to be held by the insurer, and under a number of different scenarios, we find the optimal choices of the lower barrier level that produces the minimal ultimate ruin probability. We also compare the results with the ultimate ruin probability under the classical risk model and see that the new model is effective in reducing the ultimate ruin probability. We also look at the validity of the proposed arrangement from the reinsurer’s perspective.

Having the results for the ultimate ruin probability, we then step further into the study of the deficit at ruin, the surplus before ruin, under the infinite time framework. We use numerical examples to show how the lower barrier model alters the distributions of these quantities and compare the changes in the mean and standard deviation.
Under the finite time framework, we then look at the time to ruin. We derive formulae for the probability density function and the distribution function of the time to ruin, assuming certain claim size distributions. In addition, the mean and the standard deviation of the time to ruin under the lower barrier model are calculated for illustration.

Finally, we extend the results to the Sparre Andersen model where we give further examples.
Preface

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Declaration

This is to certify that:

i The thesis comprises only my original work towards the PhD except where indicated in the Preface,

ii Due acknowledgement has been made in the text to all other material used,

iii The thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Ciyu (Jade) NIE
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Chapter 1

A review of some risk theory results

1.1 Introduction: the classical risk model

The study of ruin theory has been one of the most active research areas in actuarial science since the beginning of the 20th century. We are concerned with the level of an insurer’s surplus for a portfolio of insurance policies over an extended period of time, taking consideration of the insurer’s income, the times at which claims occur, and their amounts. In order to mathematically model and examine the behaviour of such a risk process, a real life insurance operation is simplified by assuming that the insurer starts with some non-negative amount of money, collects premiums and pays claims as they occur. Hence, our model of an insurance surplus process is determined by three components: initial surplus, premiums received and claims paid. If the insurer’s surplus falls to zero or below, we say that ruin occurs.

Much of the literature on ruin theory is based on the classical risk model due to its comparative mathematical simplicity. In the classical risk model, we denote the surplus process by \( \{U(t)\}_{t \geq 0} \). The assumption is that the insurer’s surplus starts with an initial
level $u$ and the portfolio has a constant continuous premium income at rate $c$. The aggregate claim amount up to time $t$ is denoted by $S(t)$, and hence the surplus process is represented as a stochastic process

$$U(t) = u + ct - S(t), \quad t \geq 0.$$ 

Figure 1.1.1 illustrates a realization of such a surplus process.

![Surplus process](image)

Figure 1.1.1: Surplus process

The aggregate claims process $\{S(t)\}_{t \geq 0}$ is determined by two components. The first component is the counting process for the number of claims, denoted as $\{N(t)\}_{t \geq 0}$, so that for a fixed value $t \geq 0$, the random variable $N(t)$ denotes the number of claims that occur in the fixed time interval $[0, t]$. The second component is the individual claim amounts, modelled as a sequence of independent and identically distributed (i.i.d.) random variables $\{X_i\}_{i=1}^{\infty}$, so that $X_i$ denotes the amount of the $i^{th}$ claim. Assume that $X_1$ has probability density function $p$, distribution function $P$ with $\bar{P} = 1 - P$, and that the $k^{th}$ moment of
1.2. The ultimate ruin probability

$X_1$ is $m_k$. We then have

$$S(t) = \sum_{i=1}^{N(t)} X_i.$$ 

The main assumption under the classical risk model is that \{N(t)\}_{t \geq 0} is a Poisson process with parameter $\lambda$, and therefore the aggregate claims process \{S(t)\}_{t \geq 0} is a compound Poisson process.

Clearly, the model represents a simplification of reality. Some of the more significant underlying assumptions are: claims are settled in full as soon as they occur, there is no allowance for interest on the insurer’s surplus, and there is no mention of expenses that the insurer would incur. However, this simple model is still of significant interest since it can give us some insight into the characteristics of an insurance operation.

Initially, the main research goal was the evaluation of finite and infinite time ruin probabilities. More components related to the time of ruin were examined later, for example the surplus before ruin and the deficit at ruin. Some important results and literature will now be introduced in the next few sections.

1.2 The ultimate ruin probability

Initially, researchers were interested in the probability that the insurer’s surplus falls below zero at some time in the future. We define the ultimate ruin probability as

$$\psi(u) = \Pr[U(t) < 0 \text{ for some } t > 0 \mid U(0) = u].$$

Together with the ultimate ruin probability, we also define the survival probability as $\phi(u) = 1 - \psi(u)$, which is the probability that ruin never occurs starting from initial surplus $u$. 
Chapter 1. A review of some risk theory results

When modelling the classical risk process, we usually assume that \( c > \lambda m_1 \), where \( m_1 \) is the mean individual claim amount, so that the premium income exceeds the expected aggregate claim amount per unit of time. If this condition does not hold, then \( \psi(u) = 1 \) for all \( u \geq 0 \). Sometimes we write \( c = (1 + \theta)\lambda m_1 \) for convenience, where \( \theta > 0 \) is called the premium loading factor. We now give some main results.

We firstly introduce a very famous result, namely Lundberg’s inequality (Lundberg, 1932), which provides an upper-bound for the ruin probability. We need to firstly introduce the concept of the adjustment coefficient, which gives a measure of risk for a surplus process and is denoted by \( R \). For the classical risk process, the adjustment coefficient is defined to be the unique positive root of

\[
\lambda + cr = \lambda M_X(r),
\]

where \( M_X \) is the moment generating function of the variable \( X_1 \). Equation (1.2.1) is usually called Lundberg’s equation.

**Theorem 1.2.1** (Lundberg’s inequality). Provided that the adjustment coefficient \( R \) exists, the ultimate ruin probability is bounded above by

\[
\psi(u) \leq e^{-Ru},
\]

where \( u \) is the initial surplus level.

**Proof.** See, for example, Lundberg (1932).

Lundberg’s inequality provided the first bound for the ruin probability. Later researchers have proposed several ways to generalize bounds for the ruin probability, arguing that the adjustment coefficient \( R \) does not exist for many claim size distributions such
The ultimate ruin probability

as heavy-tailed distributions. Many references can be found on this matter, for example De Vylder and Goovaerts (1984), Cai and Wu (1997), and Cai and Garrido (1999).

**Theorem 1.2.2** (Integro-differential equation). Given the initial surplus $u$, the ultimate ruin probability $\psi(u)$ satisfies the following integro-differential equation:

$$\psi'(u) = \frac{\lambda}{c} \psi(u) - \frac{\lambda}{c} \int_0^u p(x)\psi(u-x)dx - \frac{\lambda}{c} \bar{P}(u), \quad u \geq 0, \quad (1.2.2)$$

with $\psi(0) = \frac{\lambda m_1}{c} = \frac{1}{1+c}$. 

*Proof.* See, for example, Gerber (1979) or Panjer and Willmot (1992).

Another way of expressing the ultimate ruin probability is that $\psi$ satisfies a defective renewal equation. Define $P_1(x) = \frac{1}{m_1} \int_0^x \bar{P}(y)dy$, where $P_1$ is usually called the ladder-height distribution (or integrated tail distribution, or first order equilibrium distribution) of $P$. Let $\bar{P}_1(x) = 1 - P_1(x)$.

**Theorem 1.2.3** (Defective renewal equation). The ruin probability $\psi$ satisfies the following defective renewal equation:

$$\psi(u) = \psi(0) \int_0^u \psi(u-y)dp_1(y) + \psi(0)\bar{P}_1(u), \quad u \geq 0. \quad (1.2.3)$$

*Proof.* See, for example, Gerber (1979).

The ruin probability function can also be written in a compound geometric expression as follows.
**Theorem 1.2.4** (Beekman’s Convolution Formula). *The ruin probability \( \psi \) is given by the tail of a compound geometric distribution, i.e.*

\[
\psi(u) = \phi(0) \sum_{n=1}^{\infty} (\psi(0))^n \bar{P}_1^{n*}(u), \quad u \geq 0,
\]

where \( \bar{P}_1^{n*} \) is the \( n \)-fold convolution of \( \bar{P}_1 \).

*Proof.* See, for example, Feller (1971) or Beekman (1974). 

Although we can derive equations that \( \psi \) satisfies, an explicit expression for this ultimate ruin probability is often difficult to obtain. However, some special choices of individual claim size distribution do provide an explicit formula for \( \psi \), using various technique of solving intergro-differential equations and defective renewal equations, e.g., the exponential distribution (Gerber, 1979), mixed exponential distributions (Bowers et al., 1997), Erlang distributions (Dickson, 2005) and mixtures of Erlang distributions (Gerber et al., 1987). In fact, when the claim size distribution admits a rational Laplace transform, the Laplace transform of the survival function \( \phi \) can then be found and inverted.

Let \( h(y) \) be a function defined for all \( y \geq 0 \). The Laplace transform of this function \( h \), is defined as

\[
\tilde{h}(s) = \int_{0}^{\infty} e^{-sy} h(y) dy.
\]

Using equation (1.2.2), we have the following result.

**Corollary 1.2.5** (Laplace transform). *The Laplace transform of the survival probability is*

\[
\tilde{\phi}(s) = \frac{c \phi(0)}{cs - \lambda(1 - \tilde{p}(s))},
\]

*where \( \tilde{p} \) is the Laplace transform of \( p \).*
Inversion of the Laplace transform is needed to obtain $\psi$. For some particular transform functions, the inversion results are explicit and ready to use. For other general scenarios, a numerical inversion technique might be used. Some references can be found with respect to numerical inversion of Laplace transforms, such as Abate and Whitt (1995), Ahn et al. (2000) and references therein.

Cramer’s asymptotic formula is another way of looking into the features of the ruin probability besides upper and lower bounds. The most famous result in this area is the Cramér-Lundberg asymptotic formula.

**Theorem 1.2.6** (Cramér-Lundberg Asymptotic Formula).

$$\psi(u) \sim \frac{\theta m_1}{R \int_0^\infty xe^{Rx} P(x)dx} e^{-Ru}, \quad (1.2.6)$$

as $u \to \infty$, where $R$ is the adjustment coefficient.

**Proof.** See, for example, Panjer and Willmot (1992).

Approximations and numerical algorithms are some other research interests related to ruin probabilities. Panjer (1981) gives a famous recursive formula for a family of compound distributions, from which approximations to $\psi(u)$ can be calculated. Later references can be found in De Vylder and Goovaerts (1983), Dickson and Waters (1991), Panjer and Wang (1993), and Dickson et al. (1995).

In research on ruin theory, it has been argued that the probability of ruin is a very crude stability criterion when modeling and analyzing the surplus level of an insurance operation. We are not just interested in the probability of ruin, but we also want to know some other features of such a process. Should ruin occur, how serious would the deficit
be, how high is the surplus level before ruin occurs, and when does ruin occur? Under
the model of classical risk process, these properties will now be introduced.

1.3 The deficit at ruin

In this section, we will look at the amount of the insurer’s deficit at the time of ruin,
should ruin occur. We start by defining the time of ruin $T_u$, given an initial surplus $u$, as

$$T_u = \inf\{ t : U(t) < 0 | U(0) = u \},$$

with $T_u = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. Thus by the definition in Section 1.2, $\psi(u) = \Pr(T_u < \infty)$.

Gerber et al. (1987) proposed a quantitative measure $G(u, y)$, defined as

$$G(u, y) = \Pr[ T_u < \infty \text{ and } U(T_u) \geq -y ],$$

to be the probability that ruin occurs and that the insurer’s deficit at ruin is at most $y$.

Note that

$$\lim_{y \to \infty} G(u, y) = \psi(u),$$

so that

$$\frac{G(u, y)}{\psi(u)} = \Pr[|U(T_u)| \leq y | T_u < \infty].$$

Hence, for a given initial surplus $u$, $G(u, y)$ is a defective distribution with defective
density

$$g(u, y) = \frac{\partial}{\partial y} G(u, y).$$
Lemma 1.3.1. Under the classical risk model, we have

\[ g(0, y) = \frac{\lambda}{c} \bar{P}(y). \]  

(1.3.1)

Proof. See, for example, Bowers et al. (1997).

Using this lemma and by conditioning on the amount of the surplus immediately after the first time the surplus falls below its initial level, we have the following results.

Theorem 1.3.2 (Defective renewal equation for \( g(u, y) \)). The defective density function \( g(u, y) \) satisfies the defective renewal equation

\[ g(u, y) = \frac{\lambda}{c} \int_0^u g(u - x, y) \bar{P}(x) dx + \frac{\lambda}{c} \bar{P}(u + y). \]  

(1.3.2)

Corollary 1.3.3 (Laplace transform of \( g(u, y) \)). The Laplace transform of the function \( g(u, y) \), defined as

\[ \tilde{g}(s, y) = \int_0^\infty e^{-su} g(u, y) du, \]

is found to be

\[ \tilde{g}(s, y) = \frac{\lambda e^{-sy} \int_y^\infty e^{-sz} \bar{P}(x) dx}{1 - \lambda e^{-sy} \int_0^\infty e^{-sz} \bar{P}(x) dx}. \]  

(1.3.3)

Proof. See, for example, Gerber et al. (1987).

Similar to the case for the ultimate ruin probability \( \psi \), if the Laplace transform \( \tilde{g}(s, y) \) can be found, the remaining task is to invert this transform to obtain explicit solutions for \( g(u, y) \) and \( G(u, y) \). In Gerber et al. (1987), explicit solutions for \( G(u, y) \) are found for individual claim amount distributions that follow a combination of exponential distributions or a combination of gamma distributions. Some other references can also be found on approximations. For example, Dickson (1989) used a recursive method to approximate
values of $G(u, y)$ by defining a relationship between survival probabilities and the density $g(u, y)$. Dickson and Waters (1992) gave another recursive algorithm of approximation by discretizing the classical risk model.

1.4 The surplus before ruin

In this section, we will introduce the properties of the surplus immediately prior to ruin. In Section 1.3, we have introduced the notation $T_u$ to be the time of ruin given initial surplus $u$. Now, let $T_u^-$ be the time immediately prior to ruin. Denote $U(T_u^-)$ as the level of the surplus process immediately prior to payment of the claim that causes ruin. We are interested in finding the distribution of $U(T_u^-)$. The defective distribution function of $U(T_u^-)$ is defined as

$$J(u, x) = \Pr[T_u < \infty \text{ and } U(T_u^-) \leq x],$$

which represents the probability that ruin occurs from initial surplus $u$ and that the surplus immediately before ruin is at most $x$. Note that

$$\lim_{x \to \infty} J(u, x) = \psi(u).$$

The corresponding defective density function is then defined as

$$j(u, x) = \frac{\partial}{\partial x} J(u, x).$$

The distribution of the surplus immediately prior to ruin in the classical risk model was firstly considered by Gerber and Dufresne (1988), in which they looked at the joint
density of the surplus prior to ruin and the deficit at ruin when the individual claim size is a combination of exponential distributions or a combination of gamma distributions. In Dickson (1992), a method was given for finding the cumulative distribution function and the probability density function of this surplus. Explicit solutions were given when the severity of ruin function $G(u, y)$ is known.

**Theorem 1.4.1.** The defective density function of the surplus before ruin is

$$j(u, x) = \begin{cases} 
\lambda c^{-1} \bar{P}(x) \frac{\psi(u) - \psi(x)}{1 - \psi(0)}, & 0 < x \leq u, \\
\lambda c^{-1} \bar{P}(x) \frac{1 - \psi(u)}{1 - \psi(0)}, & 0 \leq u < x.
\end{cases} \tag{1.4.1}
$$

*Proof.* See, for example, Dickson (1992).

In this and previous sections, we have considered the classical risk process in an infinite time framework, i.e. the probability that the surplus process falls below zero at any time in the future, together with the associated properties at ruin. In the next sections, we will discuss the classical risk process in a finite time framework, where we are interested in the distribution of the time to ruin, the finite time ruin probability and some joint distributions of the time to ruin, the deficit at ruin and the surplus before ruin.

### 1.5 Ruin in finite time

We firstly define the finite time ruin probability $\psi(u, t)$ as

$$\psi(u, t) = \Pr[U(s) < 0 \text{ for some } s, \ 0 < s \leq t].$$
Chapter 1. A review of some risk theory results

We can study this problem by looking at the distribution of the ruin time $T_u$ since $\Pr(T_u \leq t)$ gives the probability that ruin occurs at or before time $t$, i.e. $\psi(u, t) = \Pr(T_u \leq t)$. In other words, if we know the distribution of $T_u$, we are able to compute finite time ruin probabilities. We note that the ultimate ruin probability is equivalent to $\Pr(T_u < \infty)$.

Some important results regarding the calculation of the density and moments of $T_u$ will now be introduced.

Gerber and Shiu (1998) provided a method of finding the Laplace transform of the time to ruin. Define the function $\varphi$ as

$$\varphi_\delta(u) = E[e^{-\delta T_u}I(T_u < \infty)],$$

where we can think of $\delta$ as a parameter of a Laplace transform, and $I$ is the indicator function. Note that if we interpret $\delta$ as a force of interest, this function $\varphi_\delta$ can then be seen as the expected present value of 1 payable at the time of ruin.

**Theorem 1.5.1** (Integro-differential equation for $\varphi_\delta$). Under the classical risk model, the Laplace transform of the time of ruin $\varphi_\delta$ satisfies the following integro-differential equation

$$\varphi_\delta'(u) = \frac{\lambda + \delta}{c}\varphi_\delta(u) - \frac{\lambda}{c}\int_0^u p(u-x)\varphi_\delta(x)dx - \frac{\lambda}{c}P(x). \quad (1.5.1)$$

**Proof.** See, for example, Gerber and Shiu (1998). \qed

This integro-differential equation for $\varphi_\delta(u)$ can be solved for different forms of $P$. In addition, we define the defective density function of $T_u$ to be $w_u(t)$. The function $\varphi_\delta(u)$ can then be written as

$$\varphi_\delta(u) = \int_0^\infty e^{-\delta t}w_u(t)dt.$$
If we can obtain expressions for $\varphi_\delta$, the next task is to invert this Laplace transform to identify $w_u$. Since $\psi(u, t) = \Pr(T_u \leq t)$, it follows that $w_u(t) = \frac{\partial}{\partial t} \psi(u, t)$ and hence an expression for $\psi(u, t)$ could be derived.

It is well known that an explicit solution for $\psi(u, t)$ exists when individual claims are exponentially distributed. See Seal (1978) and references therein. Drekin and Willmot (2003) inverted the Laplace transform function $\varphi_\delta$ by direct inversion and obtained an explicit expression for $w_u(t)$ when the claim sizes are exponentially distributed. Garcia (2005) provides explicit solutions for $\psi(u, t)$ when individual claims follow exponential, Erlang(2) and mixed-exponential distributions respectively, using the complex inversion formula. Dickson and Willmot (2005) find an expression for $w_u(t)$ when the claim amount distribution is a mixed Erlang distribution, also by inverting the Laplace transform.

Once again, the existence of explicit solutions for the function $w_u(t)$ depends on the choice of the individual claim distributions. When the analytical solutions do not exist, one may be interested in numerical methods in calculation and approximation. Some references can be found on this matter, for example De Vylder and Goovaerts (1988) and Dickson and Waters (1991).

In addition, Lin and Willmot (1999 and 2000) showed that when explicit solution for the ultimate ruin probability exist, explicit solutions for moments of the time to ruin can be found recursively.

Theorem 1.5.2 (Moments of $T_u$). The $k^{th}$ moment of the time to ruin, given that ruin has occurred, is

$$E[T_u^k|T_u < \infty] = \frac{\Psi_k(u)}{\psi(u)},$$  \hspace{1cm} (1.5.2)
where \( \Psi_k(u) \) satisfies a sequence of defective renewal equations

\[
\Psi_k(u) = \frac{1}{1 + \theta} \int_0^u \Psi_k(u - x) dP_1(x) + \frac{k}{c} \int_u^\infty \Psi_{k-1}(u) dx,
\]

for \( k = 1, 2, \ldots \), with \( \Psi_0(u) = \psi(u) \). Explicitly, each \( \Psi_k(u) \) is given recursively by

\[
\Psi_k(u) = \frac{k}{\lambda m_1 \theta} \left[ \int_0^u \psi(u - x) \Psi_{k-1}(u) dx + \int_u^\infty \Psi_{k-1}(x) dx - \psi(u) \int_0^\infty \Psi_{k-1}(x) dx \right].
\]

Proof. See, for example, Lin and Willmot (2000).

Albrecher and Boxma (2005) also derived the same recursive formula for the moments of the time to ruin by directly using the function \( \varphi(u) \) rather than using the results of compound geometric tails as in Lin and Willmot (2000).

### 1.6 Some joint distributions

In previous sections, we have introduced some of the main questions of interest in classical ruin theory: the deficit at ruin (Section 1.3), the surplus immediately before ruin (Section 1.4) and the time to ruin (Section 1.5), all of which have been treated separately. Gerber and Shiu (1998) opened a new chapter in classical ruin theory by introducing the discounting factor to the time of ruin. This provides a unified method of treatment to these three random variables and thereby provides a useful tool to study their joint distributions. The function they introduced has been referred to as the Gerber-Shiu function or the discounted penalty function, which we introduce now.

Consider the classical risk model as described previously, with the same definition of \( U(T_u^-) \), the surplus immediately before ruin, and \( |U(T_u)| \), the deficit at ruin. Let
1.6. Some joint distributions

\( f(x, y, t|u) \) denote the joint probability density function of \( U(T_u^-) \) (\( x \)), \( |U(T_u)| \) (\( y \)) and \( T_u \) (\( t \)). Then

\[
\int_0^\infty \int_0^\infty \int_0^\infty f(x, y, t|u) \, dt \, dx \, dy = \Pr[T_u < \infty|U(0) = u] = \psi(u).
\]

We also define

\[ f_\delta(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u) \, dt, \quad \delta \geq 0, \]

to be the discounted joint density of \( U(T_u^-) \) and \( |U(T_u)| \). In addition, define \( f_\delta(x|u) \) as

\[ f_\delta(x|u) = \int_0^\infty f_\delta(x, y|u) \, dy, \]

to be the discounted marginal probability density function of \( U(T_u^-) \). Gerber and Shiu (1998) show that

\[ f_\delta(x, y|u) = f_\delta(x|u) \frac{p(x + y)}{P(x)}. \]

Let \( w_u(y, t) \) denote the defective joint probability density function of \( |U(T_u)| \) and \( T_u \), \( v_u(x, t) \) denote the defective joint density function of \( U(T_u^-) \) and \( T_u \).

The Gerber-Shiu function is defined as

\[
\Phi(u) = E[A(U(T_u^-), |U(T_u)|)e^{-\delta T_u} I(T_u < \infty)|U(0) = u]
\]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty A(x, y) e^{-\delta t} f(x, y, t|u) \, dt \, dx \, dy \]

\[ = \int_0^\infty \int_0^\infty A(x, y) f_\delta(x, y|u) \, dx \, dy, \quad (1.6.1) \]

where \( A(x, y) \) is a non-negative function of \( x \geq 0 \) and \( y \geq 0 \), usually referred to as the \textit{penalty function}, and \( \delta \) is a discounting parameter.
Chapter 1. A review of some risk theory results

The importance of the Gerber-Shiu function is that the time to ruin is analyzed in terms of Laplace transforms which can naturally be interpreted as discounting, and the classical risk theory model is generalized by discounting with respect to the time to ruin. Hence, the analysis of the functions \( f(x, y|u) \) and \( f(x, y, t|u) \) are included within the analysis of \( \Phi(u) \). By choosing certain forms of \( A \) and certain values of \( \delta \), we are able to derive explicit forms for these joint density functions as well as the marginal densities. For example, if \( A(x_1, x_2) = I(x_1 = x, x_2 = y) \), then \( \Phi(u) \) gives the discounted joint density function of \( U(T_u^-) \) and \( |U(T_u)| \); if \( A(x_1, x_2) = x_1^n x_2^m \), then \( \Phi(u) \) gives the discounted joint moments of \( U(T_u^-) \) and \( |U(T_u)| \); if \( A(x_1, x_2) = 1 \), then \( \Phi(u) \) gives the Laplace transform of ruin time \( T_u \) with respect to \( \delta \). Notice that when \( A(x_1, x_2) = 1 \) and \( \delta = 0 \), \( \Phi(u) \) simplifies to the ruin probability \( \psi(u) \). Moreover, the Gerber-Shiu function can be interpreted differently for different modelling purposes. If we interpret \( \delta \) as a force of interest and \( A \) as some kind of penalty when ruin occurs, then \( \Phi(u) \) represents the expectation of the discounted penalty. If \( A \) is interpreted as the benefit amount of an insurance payable at the time of ruin, then \( \Phi(u) \) is the single premium for the insurance.

Gerber and Shiu (1998) showed that the Gerber-Shiu function \( \Phi(u) \) satisfies the defective renewal equation in the following theorem.

**Theorem 1.6.1** (Defective renewal equation). For \( \delta > 0 \), \( \Phi \) satisfies the following defective renewal equation,

\[
\Phi(u) = \int_0^u \Phi(u - y)g(y)dy + h(u),
\]

where

\[
g(y) = \frac{\lambda}{c} \int_0^\infty e^{-px}p(x + y)dx,
\]
and
\[ h(u) = \frac{\lambda}{c} \int_u^\infty \int_0^\infty e^{-\rho(x-u)} A(x, y)p(x + y) dy dx, \]

with \( \rho > 0 \) being the unique positive root of the equation:
\[ \ell(\xi) := \delta + \lambda - c\xi = \lambda \tilde{p}(\xi), \tag{1.6.3} \]

where \( \tilde{p}(\xi) = \int_0^\infty e^{-\xi x} p(x) dx \) is the Laplace transform of \( p \).

In Section 1.2, we have introduced the definition of the adjustment coefficient \( R \), which is calculated by equation (1.2.1), and is closely related to Lundberg’s inequality. Equation (1.6.3) is a revised version of equation (1.2.1) with inclusion of the discounting factor \( \delta \), usually referred to as Lundberg’s fundamental equation. Notice that \( \rho \) is an increasing function of \( \delta \) with \( \rho = 0 \) when \( \delta = 0 \).

Since \( \Phi(0) = h(0) \), instantly we have the following results for the special case when the initial surplus is 0:
\[ f_\delta(x|0) = \lambda c^{-1} e^{-\rho x} \tilde{P}(x), \]
\[ f_\delta(x, y|0) = \lambda c^{-1} e^{-\rho x} p(x + y), \]
\[ \varphi_\delta(0) = E[e^{-\delta T_u} I(T_u < \infty)|U(0) = 0] = \int_0^\infty g(y) dy = 1 - \frac{\delta}{c\rho}. \]

In equation (1.6.2), if \( A(x_1, x_2) = I(x_1 = x, x_2 = y) \) or \( A(x_1, x_2) = I(x_1 = x), x > 0 \), we can obtain the defective renewal equations for \( f_\delta(x, y|u) \) and \( f_\delta(x|u) \) respectively as follows.

**Corollary 1.6.2.** For non-negative \( u, x, y \) we have
\[ f_\delta(x, y|u) = \int_0^u f_\delta(x, y|u - z) g(z) dz + \lambda c^{-1} e^{-\rho(x-u)} p(x + y) I(x > u), \]
Chapter 1. A review of some risk theory results

\[ f_\delta(x|u) = \int_0^u f_\delta(x|u-z)g(z)dz + \lambda c^{-1}e^{-\rho(x-u)} \bar{P}(x)I(x > u). \]

In Section 1.4, we have introduced the result from Dickson (1992), in which he gave a formula on the probability density function of the surplus prior to ruin. By solving the above defective renewal equations, Dickson’s results can be generalized as the following theorem.

**Theorem 1.6.3.**

\[ f_\delta(x|u) = \begin{cases} 
  f_\delta(x|0) \frac{e^{\rho u} - \psi^*(u)}{1 - \psi^*(0)}, & x > u \geq 0, \\
  f_\delta(x|0) \frac{e^{\rho u} \psi^*(u - x) - \psi^*(u)}{1 - \psi^*(0)}, & 0 < x \leq u.
\end{cases} \]  

(1.6.4) with \( f_\delta(x|0) = \lambda c^{-1}e^{-\rho x} \bar{P}(x) \), and \( \psi^*(u) \) defined as

\[ \psi^*(u) = E[e^{-\delta T_u + \rho U(T_u)} I(T_u < \infty)|U(0) = u], \ u \geq 0, \]

is the extension of the definition of \( \psi(u) \) for \( \delta > 0 \).

**Proof.** See Gerber and Shiu (1998). \( \square \)

**Theorem 1.6.4.** The discounted defective joint density function of \( U(T_u^-) \) and \( |U(T_u)| \) is

\[ f_\delta(x,y|u) = \begin{cases} 
  f_\delta(x,y|0) \frac{e^{\rho u} - \psi^*(u)}{1 - \psi^*(0)}, & x > u \geq 0, \\
  f_\delta(x,y|0) \frac{e^{\rho u} \psi^*(u - x) - \psi^*(u)}{1 - \psi^*(0)}, & 0 < x \leq u.
\end{cases} \]

with \( f_\delta(x,y|0) = \lambda c^{-1}e^{-\rho x} p(x + y). \)
Since the introduction of the Gerber-Shiu function, extensive studies have used this function as a fundamental tool for analyzing marginal or joint distributions of the quantities of interest.

In Lin and Willmot (1999), a different approach is proposed for solving defective renewal equations. Under this approach, the solution of the defective renewal equation is expressed in terms of a compound geometric tail. We need to rewrite equation (1.6.2) as

$$\Phi(u) = \frac{1}{1 + \beta} \int_0^u \Phi(u - x)v(x)dx + \frac{1}{1 + \beta} H(u), \quad u \geq 0,$$

(1.6.5)

where $\beta$ is such that $(1 + \beta)^{-1} = \int_0^\infty g(y)dy = 1 - \frac{\delta}{c_0}$, $v(x) = (1 + \beta)g(x)$ is a proper density function and $H(u) = (1 + \beta)h(u)$.

In equation (1.6.2), if $A(x_1, x_2) = 1$, then $\Phi(u)$ simplifies to the function $\varphi_\delta(u)$, the Laplace transform of the ruin time $T_u$ with respect to $\delta$ as we defined in Section 1.5. The result of Gerber-Shiu functions shows that $\varphi_\delta(u)$ satisfies the following defective renewal equation:

$$\varphi_\delta(u) = \frac{1}{1 + \beta} \int_0^u \varphi_\delta(u - y)v(y)dy + \frac{1}{1 + \beta} \int_u^\infty v(y)dy.$$

We hence find that $\varphi_\delta(u)$ can also be expressed as the tail of compound geometric distribution as

$$\varphi_\delta(u) = \sum_{n=1}^{\infty} \frac{\beta}{1 + \beta} \left( \frac{1}{1 + \beta} \right)^n \bar{V}^{n*}(u), \quad u \geq 0,$$

where $\bar{V}^{n*}(u)$ is the $n$-fold convolution of the function $\bar{V}(u) = \int_u^\infty v(y)dy$ with itself. From Lin and Willmot (1999) we have the following theorem.

**Theorem 1.6.5** (Lin and Willmot (1999)). The solution of $\Phi(u)$ to equation (1.6.5) may be expressed as

$$\Phi(u) = -\frac{1}{\beta} \int_0^u H(u - x)d\varphi_\delta(x) + \frac{1}{1 + \beta} H(u),$$

(1.6.6)
or
\[
\Phi(u) = -\frac{1}{\beta} \int_0^u \varphi_\delta(u - x) dH(x) - \frac{H(0)}{\beta} \varphi_\delta(u) + \frac{1}{\beta} H(u). \tag{1.6.7}
\]

If \(H(u)\) is differentiable, \(\Phi(u)\) may be expressed as
\[
\Phi(u) = -\frac{1}{\beta} \int_0^u \varphi_\delta(u - x) H'(x) dx - \frac{H(0)}{\beta} \varphi_\delta(u) + \frac{1}{\beta} H(u), \quad u \geq 0. \tag{1.6.8}
\]


The tails of compound geometric distributions have been studied extensively, for example, recursive formulae in Panjer and Willmot (1992), and upper and lower bounds in Willmot and Lin (1997). The results provided in Lin and Willmot (1999) are of great importance since they allow for evaluation of \(\Phi(u)\) for various choices of \(A(x, y)\), and by using this, Lin and Willmot (2000) derived explicit joint and marginal moments of the time to ruin, the surplus before ruin, and the deficit at ruin.

The joint distribution of the deficit at ruin \(|U(T_u)|\) and the time to ruin \(T_u\) is another research area of great interest. Given the initial surplus \(u\), define \(W_u(y, t)\) as
\[
W_u(y, t) = \Pr[|U(T_u)| \leq y, T_u \leq t],
\]
which is a defective joint distribution function, and let
\[
w_u(y, t) = \frac{\partial^2}{\partial y \partial t} W_u(y, t)
\]
be the defective joint density.

When the individual claim sizes are exponential, it is a well known result that the
distribution of the deficit at ruin is independent of the time to ruin, due to the memoryless property of the exponential distribution. See, for example, Gerber (1979) for details of this argument. By writing the penalty function $A$ in equation (1.6.1) as $A(x, y) = e^{-sy}$, the Gerber-Shiu function then becomes the bivariate Laplace transform of the time of ruin and the deficit at ruin. Using this fact, Dickson and Drekic (2006) and Cheung et al. (2008) obtained the form of the bivariate density of $w_u(y, t)$ for certain individual claim amount distributions. However, their solutions are expressed in terms of functions that are not identified, but whose Laplace transforms are known but difficult to invert. Dickson (2008) exploits the structural forms of bivariate densities obtained in these papers and in addition, provides explicit solutions when individual claims have an Erlang(2) distribution or a mixture of two exponential distributions. The trivariate (defective) distribution of the three quantities $|U(T_u)|$, $U(T_u^-)$ and $T_u$ was discussed in Landriault and Willmot (2009), in which they have obtained a complicated expression for $f(x, y, t|u)$. 
Chapter 2

Some other risk models

2.1 The Sparre Andersen risk model

In Chapter 1, we have introduced the classical risk model where the counting process for the number of claims \( \{N(t)\}_{t \geq 0} \) is a Poisson process with parameter \( \lambda \). In other words, if we define \( N(t) \) as

\[
N(t) = \max\{n : W_1 + W_2 + \ldots + W_n \leq t\},
\]

where \( \{W_i\}_{i \geq 1} \) represent the inter-claim waiting times between the \((i-1)^{th}\) and \(i^{th}\) claim, then it follows that in the classical risk process, \( \{W_i\}_{i \geq 1} \) are i.i.d. random variables with an exponential distribution.

Instead of the Poisson process in the classical risk model, Andersen (1957) let claims occur according to a more general renewal process such that the inter-claim waiting times \( \{W_i\}_{i \geq 1} \) are a series of i.i.d. random variables with density function \( k \) and corresponding distribution function \( K \). With all other factors defined as for the classical risk model in
Chapter 1, the surplus level at time $t$ in a Sparre Andersen risk process is then

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i.$$ 

We can see that the classical risk model is a special case of the Sparre Andersen model by letting $k(t) = \lambda e^{-\lambda t}$. Additional assumptions are that $\{W_i\}_{i \geq 1}$ and $\{X_i\}_{i \geq 1}$ are independent, and $cE(W_1) > E(X_1)$ providing a positive safety loading factor. Andersen (1957) derived an integral equation for the ultimate ruin probability.

Since then, the Sparre Andersen risk model has been studied extensively using a number of different assumptions with respect to the waiting time distribution. Developments in the study of random walks and queuing theory have provided a more general framework, which has led to explicit results in the case where the inter-claim times or the claim severities have distributions related to the Erlang or phase-type distributions. We will now briefly list some important results.

### 2.1.1 Erlang(2) inter-claim times

Within the Sparre Andersen model, much of the actuarial literature focuses on the special case where the waiting time distribution is Erlang(2) with density function $k(t) = \beta^2 t e^{-\beta t}$. This type of surplus process is often referred to as an Erlang(2) risk process.

For convenience, we introduce the operator $T_r f$ for an integrable function $f$ and real $r$, defined by

$$T_r f(x) = \int_x^\infty e^{-r(u-x)} f(u) du.$$ 

Dickson (1998) showed that the ultimate survival probability $\phi(u)$ satisfies a second order integro-differential equation and hence derived the Laplace transform of $\phi$. 

2.1. The Sparre Andersen risk model

**Theorem 2.1.1.** In an Erlang(2) risk process, assume the density function of claim sizes to be \( p \) with Laplace transform \( \tilde{p} \). The ultimate survival probability \( \phi(u) \) satisfies the following second order integro-differential equation:

\[
c^2 \phi''(u) - 2\beta c \phi'(u) + \beta^2 \phi(u) = \beta^2 \int_0^u \phi(u-x)p(x)dx, \quad u \geq 0, \tag{2.1.1}
\]

and hence the Laplace transform of \( \phi \), denoted as \( \tilde{\phi}(s) = \int_0^\infty e^{-su} \phi(u)du \) can be expressed as

\[
\tilde{\phi}(s) = \frac{c^2 s \phi(0) + \beta^2 m_1 - 2\beta c}{c^2 s^2 - 2\beta cs + \beta^2 [1 - \tilde{p}(s)]}, \quad s \geq 0, \tag{2.1.2}
\]

where \( \phi(0) = \frac{2\beta c - \beta^2 m_1}{c^2 \rho} \), and \( \rho \) is the unique positive root of the equation:

\[
c^2 s^2 - 2\beta cs + \beta^2 [1 - \tilde{p}(s)] = 0. \tag{2.1.3}
\]

**Proof.** See Dickson (1998). \( \square \)

In addition, Dickson and Hipp (1998) showed that \( \phi(u) \) admits a compound geometric representation as

\[
\phi(u) = \phi(0) \sum_{n=1}^\infty [\psi(0)]^n H^n(u), \quad u \geq 0, \tag{2.1.4}
\]

where \( H(u) = \frac{\beta^2 \rho}{c^2 \rho - 2\beta c + \beta^2 m_1} T_0 T_0 p(u) \).

Later in 2001, Dickson and Hipp considered the Laplace transform of the ruin time \( T_u \) with respect to a positive parameter \( \delta \), defined as

\[
\varphi_\delta(u) = E[e^{-\delta T_u}I(T_u < \infty)].
\]
They showed that \( \varphi_\delta(u) \) satisfies the integro-differential equation

\[
c^2 \varphi''_\delta(u) - 2(\beta + \delta) c \varphi'_\delta(u) + (\beta + \delta)^2 \varphi_\delta(u) = \beta^2 \int_0^u \varphi_\delta(u - x)p(x)dx + \beta^2 \bar{P}(u).
\] (2.1.5)

Equation (2.1.5) then can be solved to find the Laplace transform of the function \( \varphi_\delta(u) \) with respect to \( u \) in the following theorem:

**Theorem 2.1.2.** The Laplace transform of \( \varphi_\delta(u) \), defined as

\[
\bar{\varphi}_\delta(s) = \int_0^\infty e^{-su} \varphi_\delta(u)du,
\]

under the Erlang(2) risk model is found to be

\[
\bar{\varphi}_\delta(s) = \frac{\beta^2 \bar{\eta}(s)}{c^2 - \beta^2 \bar{\gamma}(s)}, \quad s \geq 0,
\] (2.1.6)

where \( \bar{\gamma} \) and \( \bar{\eta} \) are the Laplace transform of functions \( \gamma(u) = T_{\rho_1} T_{\rho_2} p(u) \) and \( \eta(u) = T_{\rho_0} \gamma(u) \) respectively, while \( \rho_1 < (\beta + c)/\delta < \rho_2 \) are the only two positive roots of the generalized Lundberg equation

\[
c^2 \rho^2 - 2(\beta + \delta) c \rho + (\beta + \delta)^2 = \beta^2 \bar{p}(\rho).
\]

**Proof.** See Dickson and Hipp (2001). \( \square \)

A subsequent result is that when the initial surplus \( u = 0 \), the function \( \varphi_\delta(u) \) is reduced to

\[
\varphi_\delta(0) = \frac{\beta^2 T_{\rho_0} \bar{p}(\rho_1) - T_{\rho_0} \bar{p}(\rho_2)}{\rho_2 - \rho_1},
\]

which is a generalization of \( \phi(0) \) for \( \delta > 0 \).

Cheng and Tang (2003) extended the works in the Erlang(2) process where they in-
vestigated the moments of the surplus before ruin and the deficit at ruin, and derived
the joint density function of the two quantities. Sun and Yang (2004) also focused on
this joint distribution of the surplus before ruin and the deficit at ruin, and derived the
Laplace transform of the joint probability density function. More recently, Dickson and Li
(2010) derived expressions for the density of the time to ruin and the joint density of the
time to ruin and the deficit at ruin, assuming individual claims to follow an exponential
distribution or an Erlang(2) distribution.

The Gerber-Shiu function, defined in Chapter 1 as

\[ \Phi(u) = E[A(U(T_u^-), |U(T_u^-)|)e^{-\delta T_u}I(T_u^- < \infty)|U(0) = u], \]

is investigated in Lin (2003) by generalizing Cheng and Tang’s (2003) result. He showed
that \( \Phi \) in an Erlang (2) risk process also satisfies a second order integro-differential equation and admits a defective renewal representation.

**Theorem 2.1.3.** The Gerber-Shiu function \( \Phi \) satisfies the following integro-differential equation:

\[
\left[ (1 + \frac{\delta}{\beta}) I - \frac{c}{\beta} D \right]^2 \Phi(u) = \int_0^u \Phi(u - x)p(x)dx + \omega(u), \ u \geq 0, \quad (2.1.7)
\]

where \( I \) and \( D \) are the identity and differentiation operators, and \( \omega(u) = \int_u^\infty A(u, x - u)p(x)dx \).

Solving the equation we find that \( \Phi \) also satisfies the following defective renewal equation:

\[
\Phi(u) = \frac{\beta^2}{c^2} \int_0^u \Phi(u - x)g(x)dx + \frac{\beta^2}{c^2} h(u), \ u \geq 0 \quad (2.1.8)
\]

where \( g(x) = T_{\rho_1} T_{\rho_2} p(x) \) and \( h(u) = T_{\rho_1} T_{\rho_2} \omega(u) \).
Chapter 2. Some other risk models


2.1.2 Erlang(n) inter-claim times

By extending the Erlang(2) risk model to the more general case, we can assume the waiting times follow an Erlang(n) distribution, $n \in \mathbb{N}^+$, with density function $k(t) = \frac{\beta^n t^{n-1} e^{-\beta t}}{(n-1)!}$, $t \geq 0$. Li and Garrido (2004) studied the Gerber-Shiu function $\Phi$ under this assumption, using an approach similar to that of Gerber and Shiu (1998). They show that the function $\Phi$ satisfies an $n^{th}$ order integro-differential equation as follows.

**Theorem 2.1.4.** The Gerber-Shiu function in an Erlang(n) risk process satisfies the following equation:

$$
\sum_{i=1}^{n} \Phi^{(i)}(u) \left[ \frac{-(\beta + \delta)}{c} \right]^{n-i} \binom{n}{n-i} \left( \frac{n}{n-i} \right) - \frac{\beta}{c} \int_{0}^{u} \Phi(u-x)p(x)dx + \omega(u),
$$

where $\Phi^{(i)}(u) = \frac{d^i}{du^i}\Phi(u)$ is the $i^{th}$ order derivative of the function $\Phi$.


In the Erlang(n) risk process, the generalized Lundberg’s fundamental equation is

$$
\left( \frac{\beta + \delta}{c} - s \right)^n = \frac{\beta^n}{c^n} \tilde{p}(s).
$$

Li and Garrido (2004) proved that for $\delta > 0$, this equation has exactly $n$ roots that have a positive real part, denoted as $\rho_1, \rho_2, \ldots, \rho_n$. Moreover, if the density function $p$ is sufficiently regular, there is a negative root $-R$ such that $R > 0$. $R$ is called the generalized adjustment coefficient.
If we solve equation (2.1.9), we find that function $\Phi$ satisfies a defective renewal equation as follows.

**Theorem 2.1.5.** $\Phi(u)$ admits a defective renewal equation representation

$$
\Phi(u) = \frac{\beta^n}{\theta^n} \int_0^u \Phi(u - x)g(x)dx + \frac{\beta^n}{\theta^n} h(u), \quad u \geq 0,
$$

(2.1.10)

where $g(x) = T_\rho_1 T_\rho_2 \cdots T_\rho_n p(x)$ and $h(u) = T_\rho_1 T_\rho_2 \cdots T_\rho_n \omega(u)$.

**Proof.** See Li and Garrido (2004).

Using the Gerber-Shiu function, Li and Garrido (2004) also derived the discounted marginal density of the surplus before ruin and the Laplace transform of the time to ruin. For some results on the density of the time to ruin in the Erlang($n$) risk model, see Dickson et al. (2003), in which the individual claim amounts are assumed to be exponentially distributed. Furthermore, Li and Dickson (2006) studied the maximum surplus before ruin and related problems.

The Erlang($n$) risk model can be further extended to a generalized Erlang($n$) risk model, in which the inter-claim waiting times are assumed to be the sum of $n$ independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that the Erlang($n$) model is the special case where $\beta = \lambda_1 = \lambda_2 = \ldots = \lambda_n$. Gerber and Shiu (2005) studied the generalized Erlang($n$) model via the Gerber-Shiu function $\Phi$, and showed that $\Phi$ satisfies an integro-differential equation as follows.

**Theorem 2.1.6.** Let $I$ and $D$ be the identity operator and differentiation operator respectively. For $u \geq 0$, the Gerber-Shiu function $\Phi$ satisfies the following equation:

$$
\left\{ \prod_{j=1}^n \left[ \left( 1 + \frac{\delta}{\lambda_j} \right) I - \frac{\theta}{\lambda_j} D \right] \right\} \Phi(u) = \int_0^u \Phi(u - x)p(x)dx + \omega(u).
$$

(2.1.11)
Proof. See Gerber and Shiu (2005). \hfill \square

As in other models, this equation can be reduced to a defective renewal equation as

\[
\Phi(u) = \int_0^u \Phi(u-x)g(x)dx + h(u), \tag{2.1.12}
\]

where \(g(x) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} \tau_{\rho_1} \tau_{\rho_2} \cdots \tau_{\rho_n} p(x)\) and \(h(u) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} \tau_{\rho_1} \tau_{\rho_2} \cdots \tau_{\rho_n} \omega(u)\), while \(\rho_i\) for \(i = 1, 2, \ldots, n\) are the \(n\) roots with positive real part to the following Lundberg’s fundamental equation:

\[
\prod_{j=1}^n \left[ \left(1 + \frac{\delta}{\lambda_j}\right) - \frac{c}{\lambda_j} s \right] = \tilde{p}(s).
\]

Dickson and Drekic (2004) assumed a phase-type\((n)\) distribution for the individual claim sizes under the generalized Erlang\((n)\) model and derived expressions for the joint density function of the surplus prior to ruin and the deficit at ruin, as well as the marginal density functions of each.

### 2.1.3 General inter-claim times

Malinovskii (1998) assumed a general waiting time distribution \(k\) combined with exponentially distributed claim sizes and derived the Laplace transform of the finite time survival probability \(\phi(u, t) = 1 - \psi(u, t)\) with respect to \(t\) in the following theorem.

**Theorem 2.1.7.** Let the claim size density function be \(p(x) = \alpha e^{-\alpha x}\), and \(\tilde{k}(s)\) be the Laplace transform of the waiting times density \(k\), then

\[
\delta \int_0^\infty e^{-\delta t} \phi(u, t)dt = 1 - \rho e^{-\alpha u(1-\rho)}, \quad \delta > 0, \tag{2.1.13}
\]
where $\rho$ is the unique solution to the equation:

$$s = \tilde{k}[\delta + c\alpha(1 - s)], \quad \delta > 0.$$  


Wang and Liu (2002) extended this result to the case where the individual claim sizes are a mixture of two exponentials as the following:

**Theorem 2.1.8.** Let the claim size density function be $p(x) = q\alpha_1 e^{-\alpha_1 x} + (1 - q)\alpha_2 e^{-\alpha_2 x}$ where $0 < q < 1$, and $\tilde{k}(s)$ be the Laplace transform of the waiting times density $k$, then

$$\delta \int_0^\infty e^{-\delta t}\psi(u, t)dt = \frac{(\alpha_1 - \alpha_2)[y_1\rho_2 e^{-\rho_1 u} - y_2\rho_1 e^{-\rho_2 u}]}{(\alpha_1 - \rho_2)(\alpha_2 - \rho_1) - (\alpha_1 - \rho_1)(\alpha_2 - \rho_2)}, \quad \delta > 0, \quad (2.1.14)$$

where $\rho_1$ and $\rho_2$ are, respectively, the unique solutions of the equation

$$(\alpha_1 - \rho)(\alpha_2 - \rho) = [\alpha_1\alpha_2 - (q\alpha_1 + (1 - q)\alpha_2)\delta]\tilde{k}(\delta + c\rho), \quad \delta > 0, \quad (2.1.15)$$

in $(0, \alpha_1]$ and $[\alpha_1\alpha_2 / (q\alpha_1 + (1 - q)\alpha_2), \alpha_2]$, and $y_1 = \frac{(\alpha_1 - \rho_1)(\alpha_2 - \rho_1)}{\alpha_1\alpha_2}$, $y_2 = \frac{(\alpha_1 - \rho_2)(\alpha_2 - \rho_2)}{\alpha_1\alpha_2}$.


In both cases above, the Laplace transforms are difficult to invert except for some special waiting times distributions. Borovkov and Dickson (2008) derived a closed-form representation for the probability density function of the time to ruin, assuming exponentially distributed claim sizes and a general $k$ as follows.
Theorem 2.1.9. Assuming \( p(x) = \alpha e^{-\alpha x} \), the defective probability density function of the ruin time \( w_u(t) \) is given by

\[
w_u(t) = e^{-\lambda(u+ct)} \sum_{n=1}^{\infty} \frac{\lambda^n(u+ct)^{n-1}}{n!} \left( u + \frac{ct}{n+1} \right) k^{(n+1)\ast}(t), \tag{2.1.16}
\]

where \( k^{(n+1)\ast} \) is the \((n + 1)\)-fold convolution of \( k \).

Proof. See Borovkov and Dickson (2008). \qed

In recent years, more papers focused on the study of the joint densities with arbitrary waiting times. Dickson and Drekic (2004) looked at the joint distribution of the surplus prior to ruin and the deficit at ruin and provided general expressions. The distribution of the deficit at ruin is investigated by Drekic et al. (2004) assuming the claims sizes to be phase-type, while a formula for the joint probability that ruin occurs and the deficit at ruin exceeds a certain level is provided. Willmot (2007) and Landriault and Willmot (2008) focussed on the Gerber-Shiu discounted penalty function with general inter-claim times and a number of classes of individual claim size distributions.

2.2 The barrier models

Having introduced the Sparre Andersen model, we now return to the classical risk model and look at some other possible extensions on the risk process.

Barrier problems originated from the classical risk process, where we seek variations of the surplus process to reflect some real life scenario of an insurance portfolio. Among these, the easiest barrier problem is probably the absorbing barrier problem, where we want to find the probability that ruin occurs from initial surplus \( u \) without the surplus process reaching level \( b > u \) prior to ruin, denoted by \( \xi(u, b) \). Let \( \chi(u, b) = 1 - \xi(u, b) \)
2.2. The barrier models

denote the probability that the surplus process attains the level $b$ from initial surplus $u$ without first falling below zero. Figure 2.2.1 illustrates the two situations described.

![Graph illustrating absorption before ruin and ruin before absorption](image)

**Figure 2.2.1: Absorbing Barrier**

Dickson and Gray (1984) showed that $\xi$ and $\chi$ can be expressed in terms of the ultimate ruin probability function $\psi$ as

$$\chi(u, b) = \frac{1 - \psi(u)}{1 - \psi(b)} \quad \text{and} \quad \xi(u, b) = \frac{\psi(u) - \psi(b)}{1 - \psi(b)}.$$ 

The absorbing barrier problem is not of great interest in the actuarial literature. Instead, the results usually act as a stepping stone for another more interesting type of upper barrier model, often referred to as a dividend problem. We consider the classical risk process described in Chapter 1 but with an upper barrier $b \geq u$ ($u$ being the initial surplus). Whenever the surplus attains the level $b$, the premium income is paid to shareholders continuously as dividends until the next claim occurs, so that in this modified surplus process, the surplus never attains a level greater than $b$. Figure 2.2.2 shows a realisation of the modified surplus process.

Assume that the shareholders provide the initial surplus $u$ and pay the deficit at ruin. A question of interest is how should the level of the barrier $b$ be chosen to maximise...
the expected present value of net income to the shareholders, assuming that there is no further business after the time of ruin.

The optimal dividend strategy was initially proposed by De Finetti (1957) for a binomial model. He argued that in the traditional risk model, the surplus of an insurance company can increase without bound, which is an unrealistic assumption. He hence took the dividend payments, modelled through an upper barrier, into consideration and showed how the optimal level of the barrier can be determined. In continuous time risk models, such as the classical risk model, references to dividend strategy problems and more general barrier strategies can be found in a number of papers and books, including Bühmann (1970), Borch (1974), Segerdahl (1970) and Gerber (1972, 1973, 1979, 1981). Asmussen and Taksar (1997) discussed the optimal dividend strategy under the controlled diffusion model, where the dividend is paid out at a dynamic rate (depending on the reserve or the surplus level); Paulsen and Gjessing (1997) considered a claims process which includes a

Figure 2.2.2: Surplus process with a dividend barrier, $b$
Brownian motion element and stochastic interest on reserves driven by an independent Brownian Motion; Højgaard (2002) assumed a process with dynamic control of the premium income under a dividend barrier and looked at the optimal control strategy. In recent years, the dividend strategy problem has been considered in a wider variety of risk models. Lin et al. (2003) looked at the Gerber-Shiu function in the presence of the dividend barrier; Dickson and Waters (2004) allowed the process to continue after ruin and extended the analysis; Gerber and Shiu (2004) provided a general recursive formula to compute the moments of the present value of shareholders’ income when the surplus process in modeled by a Brownian motion with positive drift. Albrecher et al. (2005) considered a barrier strategy in a Sparre Andersen model with generalized Erlang($n$) inter-claim times; Dickson and Drekic (2006) studied the optimal dividend problem under a ruin probability constraint. Albrecher and Hartinger (2007) and Cheung et al. (2008) both looked at the moments of discounted dividends under the classical risk models with multiple thresholds.

### 2.3 Reinsurance

Reinsurance is an important mechanism by which an insurer can manage the financial risk of its operation by sharing part of its risk with a reinsurer. Traditionally there are two types of assumptions in reinsurance arrangements. The first type is the proportional reinsurance. Under this assumption, the insurer pays proportion $a$ of each claim and the reinsurer pays proportion $1 - a$. The second type is excess of loss reinsurance, where the insurer pays an amount of up to the retention level $M$ for each claim and the reinsurer pays the excess amount.

The cost of a reinsurance arrangement is the reinsurance premium, which is usually
calculated by assuming a certain principle of premium calculation. In the actuarial literature, several principles and their properties are discussed, for example the pure premium principle, the expected value principle, the variance premium principle and the standard deviation principle. Details and discussion can be found in Goovaerts et al. (1984) or Dickson (2005).

Having included a reinsurance arrangement, together with its cost, in an insurer’s surplus model, much research was concerned with the problem of determining the optimal level and/or type of reinsurance, where optimal is defined in terms of some stability criterion such as the variance of aggregate claims or the probability of ruin. Examples of earlier works on this problem include Waters (1983) who looked at the effect of both proportional and excess of loss reinsurance with different retention limits through the net adjustment coefficient, and Centeno (1986) who considered optimal mixtures of the two types of reinsurance to maximise the insurer’s net adjustment coefficient. In those earlier years, the adjustment coefficient was the main approach for evaluating the probability of ruin under reinsurance. With the development in the study of ruin probability, Dickson and Waters (1996) used numerical methods to determine the level of reinsurance that minimises the insurer’s probability of ruin in both finite and infinite time frameworks. In these papers, the level and type of reinsurance is assumed to remain constant throughout the period being considered. More recent research, for example Schmidli (2001), Hipp and Vogt (2003) and Dickson and Waters (2006) investigated dynamic reinsurance strategies so that the level of reinsurance is allowed to change. Schmidli (2001) looked at proportional reinsurance while Hipp and Vogt (2003) looked at excess of loss. In both papers, the level of reinsurance can change continuously so that mathematically both papers deal with an optimal stochastic control problem in continuous time. In Dickson and Waters (2006), rather than allowing a continuous variation in the reinsurance contract, the change in
reinsurance strategy is periodic and is assumed to be set at the start of each year.

2.4 Remarks

In these first two chapters, we have introduced some of the main results in the literature of risk theory. In Chapter 1, we have introduced the classical risk model with important results on some main quantities of interest, namely the ultimate ruin probability, the surplus before ruin, the severity of ruin, the finite time ruin probability and some joint distributions. We have seen that many explicit solutions for the functions described can be found, with different assumptions for the claim size distribution.

In Chapter 2, we firstly introduced the Sparre Andersen risk model, which is a more general model than the classical risk model, which is a special case. In Section 2.2, we have seen that based on the original risk processes, we could introduce certain mathematical conditions, for example an upper absorbing or reflecting barrier, to reflect some complex scenarios in real life insurance operations. In fact, the idea of an upper barrier model provides the main inspiration for this thesis, in which instead of introducing an upper barrier, we bring a lower barrier into the original risk process and investigate the behaviour of the surplus process under this lower barrier model. To justify the possible real life application of such a lower barrier, we use the idea of a reinsurance arrangement, which we introduced in Section 2.3.

The layout of this thesis for the next four chapters is as follows. First, in Chapter 3, we introduce the lower barrier model in detail, including the notation and assumptions. We then focus on the ultimate ruin probability under such a model and derive formulae. Our goal is then finding the optimal level of the lower barrier to minimize the ultimate ruin probability. We provide examples in three different assumptions of the claim size
distributions, namely the exponential distribution, the Erlang(2) distribution, and the mixed exponential distribution.

In Chapter 4, we look at quantities other than the ultimate ruin probability in the infinite time framework of the proposed lower barrier model, namely the deficit at ruin and the surplus prior to ruin. We derive formulae for the probability density function and the cumulative distribution function for each quantity, followed by some numerical examples, assuming an exponential, an Erlang(2) and a mixed exponential claim size distribution respectively.

In Chapter 5, we discuss the ruin related quantities under the lower barrier model in a finite time setting. We firstly give general formulae for the distribution of the time to ruin under the lower barrier model without specific assumptions for the individual claim sizes. Explicit expressions for the defective density function of the time to ruin are then derived under two assumptions of the claim size distributions, namely the exponential distribution and the Erlang(2) distribution. For each individual claim size assumption, numerical examples are given with respect to the density functions, the distribution functions and the conditional expected time to ruin. A small section investigating some joint distributions then follows.

Finally in Chapter 6, we look at the lower barrier model under the Sparre Andersen model. Specifically, we examine some risk related quantities assuming that the inter-claim times follow an Erlang(2) distribution. For the exponential claim sizes, the Erlang(2) claim sizes and the mixed exponential claim sizes, we look at the behaviour of the ultimate ruin probability and the deficit at ruin, respectively.
Chapter 3

Lower barrier model: ultimate ruin

3.1 Introduction

Based on the fundamental works of the classical risk process and its applications in insurance portfolio modelling, we made a modification to the original surplus process which will now be introduced.

The insurer we are considering has allocated a fixed amount of funds, $U$, to a portfolio of risks so that the ultimate ruin probability for this portfolio is known. Our aim is to investigate whether the insurer can reduce this ultimate ruin probability by splitting $U$ into two parts. The first of these, $u \leq U$, will be the initial surplus held for the portfolio. The second part is a reinsurance premium which we denote by $Q(u,k)$, and which equals $U − u$. The reinsurance being purchased is not a traditional type of contract. It is neither a proportional type nor an excess of loss reinsurance that has traditionally been discussed in most of actuarial literature. Moreover, it does not relate to either individual claims or aggregate claims. Rather, it relates to the amount by which the surplus process falls below a fixed level $0 \leq k \leq u$. Suppose that on the $i^{th}$ occasion that the surplus falls between 0
and \( k \) without ruin, the insurer’s surplus falls to a level \( k - r_i \) such that \( 0 < r_i < k \) the reinsurer makes an immediate payment of \( r_i \) to the insurer, restoring the insurer’s surplus to \( k \). If any claim causes the insurer’s surplus to fall from a level above \( k \) to a level below 0, the reinsurer does not make a payment and ruin for the portfolio occurs at the time of this claim.

To illustrate our modified model graphically, consider the classical risk model we have introduced in Chapter 1 where we have defined the risk process \( \{U(t)\}_{t \geq 0} \), the time to ruin \( T_u \) and the ultimate ruin probability \( \psi(u) \). The modification we have introduced is a lower barrier \( k \), where \( 0 \leq k \leq u \). Each time the surplus drops below \( k \) but not below 0, an injection of funds will immediately restore the surplus level back to \( k \), so that the surplus process continues from level \( k \) after payment of the claim that had taken the surplus below \( k \). We discuss in the next section how this injection of funds is provided.

Figure 3.1.1 shows how the insurer’s portfolio behaves through time.
3.2 The ultimate ruin probability

We now derive a formula for calculation of the ultimate ruin probability under the lower barrier model. Let $T_{u,k}$ denote the time to ruin under the modified process with initial surplus $u$ and lower barrier $k$. Now define $\psi_k(u)$ to be the ultimate ruin probability for the modified surplus process with the lower barrier at $k$. We can obtain a formula for $\psi_k(u)$, starting with the case $u = k$. By conditioning on the amount of the first drop below level $k$, we have

$$
\psi_k(k) = \int_0^k g(0,y) \psi_k(k) \, dy + \int_k^\infty g(0,y) \, dy,
$$

where $g(u,y)$ and $G(u,y)$ are defined in Section 1.3 as the density and distribution function, respectively, of the deficit at ruin in a classical risk model. Hence,

$$
\psi_k(k) = \frac{\psi(0) - G(0,k)}{1 - G(0,k)}. \tag{3.2.1}
$$

Next, we consider the more general situation when $u > k \geq 0$. Let $\phi(u) = 1 - \psi(u)$ and $\phi_k(u) = 1 - \psi_k(u)$ be the survival probabilities for the classical risk process and the modified risk process respectively. Conditioning on the amount of the first drop below level $k$, we have

$$
\phi_k(u) = \phi(u-k) + G(u-k,k) \phi_k(k),
$$

and therefore

$$
\psi_k(u) = \psi(u-k) - G(u-k,k)(1 - \psi_k(k)) \frac{1 - \psi(0)}{1 - G(0,k)}. \tag{3.2.2}
$$
For many claim size distributions we can easily find the components of $\psi_k(u)$, and hence $\psi_k(u)$ itself. See, for example, Gerber et al. (1987) or Dickson (2005).

3.3 Premium calculation for the reinsurer

Suppose that the insurer enters a reinsurance arrangement under which the reinsurer provides the funds needed to restore the surplus level to $k$ every time the surplus falls between 0 and $k$. We denote the premium required by the reinsurer as $Q(u, k)$, which is a function of the insurer’s initial surplus $u$ and the lower barrier $k$. Let the aggregate amount needed to restore the modified surplus process to $k$ up to time $t$, given initial surplus $u$, be $R_{u,k}(t)$.

In our numerical illustrations in the next section we consider premium principles for the reinsurance premium that are based on the first two moments of aggregate claims for the reinsurer, as well as a reinsurance premium based on the expected discounted claim payments by the reinsurer. We now derive formulae that can be used to calculate reinsurance premiums.

Consider first $E(R_{u,k})$ where $R_{u,k} = R_{u,k}(T_{u,k})$, i.e. the expected total claim amount for the reinsurer up to the time of ruin.

We start with the case $u = k$. Using ideas from Pafumi (1998), we can calculate $E(R_{k,k})$ as

$$E(R_{k,k}) = \int_0^k (y + E(R_{k,k})) g(0, y) \, dy$$

$$= \int_0^k y g(0, y) \, dy + E(R_{k,k}) G(0, k).$$
3.3. Premium calculation for the reinsurer

Therefore,

\[ E(R_{k,k}) = \frac{\int_0^k y \, g(0, y) \, dy}{1 - G(0, k)}. \]

When \( u > k \), we have

\[
E(R_{u,k}) = \int_0^k (y + E(R_{k,k})) \, g(u - k, y) \, dy
= \int_0^k y \, g(u - k, y) \, dy + E(R_{k,k}) \, G(u - k, k).
\] (3.3.1)

Consider next calculation of \( E(R_{u,k}^2) \). The same idea gives

\[
E(R_{k,k}^2) = \int_0^k (y^2 + E(R_{k,k}^2) + 2y E(R_{k,k})) \, g(0, y) \, dy
= \int_0^k y^2 g(0, y) \, dy + E(R_{k,k}^2) \, G(0, k) + 2E(R_{k,k}) \int_0^k \, g(0, y) \, dy,
\]

and therefore

\[
E(R_{k,k}^2) = \frac{\int_0^k y^2 g(0, y) \, dy + 2E(R_{k,k}) \int_0^k \, g(0, y) \, dy}{1 - G(0, k)}.
\]

Hence,

\[
E(R_{u,k}^2) = \int_0^k (y^2 + E(R_{k,k}^2) + 2y E(R_{k,k})) g(u - k, y) \, dy
= \int_0^k y^2 g(u - k, y) \, dy + E(R_{k,k}^2) G(u - k, k)
+ 2E(R_{k,k}) \int_0^k yg(u - k, y) \, dy.
\] (3.3.2)

We now consider the case when the reinsurer sets its premium based on the expected discounted value of the payments it will make until the time of ruin. Let \( R_{u,k}^\delta \) denote this present value at force of interest \( \delta \) per unit time. Again, we start with the case \( u = k \). In
Section 1.6 under classical risk model, we have defined function \( w_u(y, t) \) to be the joint density of the deficit at ruin and the time to ruin. Using the same argument from Pafumi (1998), \( E(R^\delta_{k,k}) \) can be calculated as

\[
E(R^\delta_{k,k}) = \int_0^\infty \int_0^k e^{-\delta t} \left( y + E(R^\delta_{k,k}) \right) w_0(y, t) \, dy \, dt
\]

\[
= \int_0^\infty \int_0^k e^{-\delta t} y w_0(y, t) \, dy \, dt
\]

\[
+ E(R^\delta_{k,k}) \int_0^\infty \int_0^k e^{-\delta t} w_0(y, t) \, dy \, dt.
\]

Let

\[
g_\delta(u, y) = \int_0^\infty e^{-\delta t} w_u(y, t) \, dt,
\]

(3.3.3)

denote the (defective) discounted probability density function of the deficit at ruin \( y \), with initial surplus \( u \) and force of interest \( \delta \). Also let \( G_\delta(u, y) \) denote the corresponding distribution function so that

\[
G_\delta(u, y) = \int_0^y g_\delta(u, x) dx.
\]

Hence \( E(R^\delta_{k,k}) \) can be simplified as

\[
E(R^\delta_{k,k}) = \frac{\int_0^\infty \int_0^k e^{-\delta t} y w_0(y, t) \, dy \, dt}{1 - \int_0^\infty \int_0^k e^{-\delta t} w_0(y, t) \, dy \, dt}
\]

\[
= \frac{\int_0^k y g_\delta(0, y) \, dy}{1 - G_\delta(0, k)}.
\]

(3.3.4)
When $u > k$ we have

$$E(R_{u,k}^\delta) = \int_0^\infty \int_0^k e^{-\delta t} (y + E(R_{k,k}^\delta)) w_{u-k}(y,t) \, dy \, dt$$

$$= \int_0^k y g_\delta(u-k,y) \, dy + E(R_{k,k}^\delta) G_\delta(u-k,k).$$

Both $w_u(y,t)$ and $g_\delta(u,y)$ can be found for certain claim size distributions (see, for example, Dickson and Drekic (2006) or Landriault and Willmot (2009)), and we will use some of these results in our subsequent examples. We note that $E(R_{u,k}^\delta) \leq E(R_{u,k})$ for all $\delta \geq 0$ and this relationship holds for all claim size distributions.

### 3.4 Minimizing the ultimate ruin probability

We now assume that the insurer holds an amount of capital, $U$, of which $u$ is allocated as the initial surplus and $Q(u,k)$ is used to buy reinsurance. Given a value of the initial surplus $u$ and therefore a value of the reinsurance premium required ($Q(u,k)$), we can find the level of the lower barrier $k$. In our examples there is a minimum value of $u$ such that the condition $u \geq k$ is satisfied. What will be of interest is whether there is a combination of $u$ and $k$ such that the ruin probability $\psi_k(u)$ is minimized. That is, we are aiming to minimize $\psi_k(u)$ subject to the constraints $U = u + Q(u,k)$ and $u \geq k$. Let $u^*$ and $k^*$ denote the optimal choices of $u$ and $k$ respectively, where by ‘optimal’ we shall always mean the $\psi_k(u)$ is minimized. The minimum ruin probability is denoted $\psi^*_k(u)$.

Solving for $u^*$ and $k^*$ is thus a constrained optimization problem. Even in the simplest case (exponential claim sizes) we are unable to find explicit solutions for $u^*$ and $k^*$. Hence, numerical techniques are required. In our examples in the next section, the values of $u^*$ and $k^*$ were found using Mathematica.
3.5 Examples

In this section, we consider three claim size distributions: exponential, Erlang(2) and a mixture of two exponentials. Under each of these claim size distributions, we investigate the optimal choice of $u$ and $k$ for given values of initial capital $U$, under different reinsurance premium scenarios. We also examine and compare the effectiveness of reinsurance in reducing the ruin probability under each circumstance.

3.5.1 Exponential claim sizes

Suppose that $P(x) = 1 - e^{-\alpha x}$. It is well known (see, for example, Dickson (2005)) that

$$
\psi(u) = \frac{\lambda}{\alpha c} e^{-(\alpha - \lambda/c)u},
$$

(3.5.1)

$$
g(u, y) = \psi(u)\alpha e^{-\alpha y},
$$

(3.5.2)

$$
G(u, y) = \psi(u)(1 - e^{-\alpha y}).
$$

(3.5.3)

Using formulae (3.3.1) and (3.3.2), we obtain expressions for $E(R_{u,k})$ and $E(R_{u,k}^2)$ as

$$
E(R_{u,k}) = \frac{\lambda}{\alpha} e^{-(\alpha - \lambda/c)(u-k)} \frac{1 - \alpha k e^{-\alpha k} - e^{-\alpha k}}{\alpha c - \lambda + \lambda e^{-\alpha k}},
$$

and

$$
E(R_{u,k}^2) = \frac{\lambda}{\alpha} \frac{1}{(\alpha c - \lambda + \lambda e^{-\alpha k})^2} e^{-(\alpha - \lambda/c)(u-k)}
$$

$$
\times (c(2 - 2e^{-\alpha k} - 2\alpha k e^{-\alpha k} - \alpha^2 k^2 e^{-\alpha k}) + \lambda ke^{-\alpha k}(2e^{-\alpha k} + \alpha k e^{-\alpha k} - 2 + k\alpha)).
$$
3.5. Examples

We remark that $E(R_{u,k})$ is a special case of $E(R_{u,k}^\delta)$ given below.

We now derive a formula for the expected present value of the reinsurance payments. From Gerber (1979) the joint probability density function $w_u(y,t)$ is

$$w_u(y,t) = \alpha e^{-\alpha y} w_u(t),$$

where $w_u(t)$ is the marginal defective density function of the time to ruin. Hence,

$$E(R_{k,k}^\delta) = \int_0^\infty e^{-\delta t} w_0(t) dt \int_0^k y \alpha e^{-\alpha y} dy \int_0^k \alpha e^{-\alpha y} dy$$

$$= \frac{\varphi_\delta(0) \alpha^{-1} [1 - e^{-\alpha k}(1 + ak)]}{1 - \varphi_\delta(0)(1 - e^{-\alpha k})},$$

where $\varphi_\delta(u) = E[e^{-\delta T_u} I(T_u < \infty)]$ is as defined in Section 1.5.

Also from equation (3.3.5), we have

$$E(R_{u,k}^\delta) = \int_0^\infty \int_0^k e^{-\delta t} (y + E(R_{k,k}^\delta)) w_u - k(t) \alpha e^{-\alpha y} dy dt$$

$$= \int_0^\infty e^{-\delta t} w_{u-k}(t) dt \left[ \int_0^k y \alpha e^{-\alpha y} dy + E(R_{k,k}^\delta) \int_0^k \alpha e^{-\alpha y} dy \right]$$

$$= \varphi_\delta(u-k) \left[ \frac{1 - e^{-\alpha k}(1 + ak)}{\alpha} + E(R_{k,k}^\delta)(1 - e^{-\alpha k}) \right].$$

We can evaluate this expression as we know from Gerber & Shiu (1998) that

$$\varphi_\delta(u) = (1 - R_\delta/\alpha) e^{-R_\delta u},$$

where

$$R_\delta = \frac{-\lambda - \delta + c\alpha + \sqrt{(c\alpha - \delta - \lambda)^2 + 4c\delta\alpha}}{2c}. $$
Setting $\delta = 0$ in the above expressions gives $E(R_{u,k})$.

We now show some numerical results under different reinsurance premium scenarios. For our numerical illustrations we set $\alpha = 1$ so that the mean and the variance of the claim size distribution are both 1. Also, we set $\lambda = 1$ and $c = 1.2$, i.e. the premium loading factor for the insurer is $\theta_1 = 0.2$.

**Example 3.5.1. Expected value principle reinsurance premium.**

Our first scenario is that $Q(u,k) = 1.6E(R_{u,k})$, i.e. the reinsurance premium is calculated using the expected value principle with loading factor $\theta_2 = 0.6$. Figure 3.5.1 illustrates how $\psi_k(u)$ changes with $u$, given that $U = 15$. For this value of $U$, $\psi(U) = 0.06840$. The optimal choice of $u$ and $k$ is found to be $u^* = 10.05$ and $k^* = 7.23$ giving $\psi_k^*(u) = 0.00226$. Some values are shown in Table 3.5.1 for different values of $U$, but we defer comment on these until after the next example.

**Example 3.5.2. Standard deviation principle reinsurance premium.**

For our second scenario we let $Q(u,k) = E(R_{u,k}) + 2 \text{St. Dev.}(R_{u,k})$, i.e. the reinsurance premium is calculated by the standard deviation principle with loading factor $\theta_3 = 2$. As a first illustration, consider the situation when the initial capital is $U = 20$, giving $\psi(U) = 0.02973$. The ruin probability $\psi_k(u)$ is plotted in Figure 3.5.2 for different values of $u$. We see in Figure 3.5.1 in Example 3.5.1 that any combination of $u$ and $k$ (such that $u + Q(u,k) = U$) provides a lower ruin probability than under the original process. In Figure 3.5.2 only certain combinations of $u$ and $k$ reduce the ruin probability compared with the original process. The minimum of $\psi_k(u)$ is obtained when $u^* = 12.50$ and $k^* = 4.28$, giving $\psi_k^*(u) = 0.01645$.

Table 3.5.1 shows the optimal values $u^*$ and $k^*$, as well as the corresponding ruin probabilities under the scenarios described in Examples 3.5.1 and 3.5.2. It also shows the
percentage reductions in the ruin probabilities, calculated as $(1 - \frac{\psi_k^*(u)}{\psi(U)}) \times 100\%$, to illustrate the effectiveness of the lower barrier system. Consider the case when the initial capital is $U = 11$, so that the ruin probability under the original process is 0.13.

When the reinsurance premium is calculated by the expected value principle with loading factor 0.6, the ruin probability is reduced by 61.04\% to 0.05 when choosing the optimal combination $u^*$ and $k^*$. When the reinsurance premium is calculated by the standard deviation principle, we find that the reinsurance premium exceeds the initial capital of $U = 11$ for any $u$ and $k$ such that $0 \leq k \leq u \leq U$. Hence the insurer would not buy reinsurance under this scenario. As $U$ increases, the insurer is able to buy reinsurance

Figure 3.5.1: $\psi_k(u)$ and $\psi(U)$, $U = 15$, $Q(u,k) = 1.6E(R_{u,k})$, Exponential claims
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Figure 3.5.2: $\psi_k(u)$ and $\psi(U)$, $U = 20$, $Q(u, k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k})$, Exponential claims

under the scenario of Example 3.5.2 and we see that the ruin probability can be reduced considerably with an appropriate choice of $u$ and $k$. Under each reinsurance premium scenario, the percentage reductions in ruin probabilities increase rapidly as $U$ increases.

Example 3.5.3. Discounted expected value principle reinsurance premium.

As our third scenario, we set $Q(u, k) = 1.6E(R_{u,k}^{0.01})$. Table 3.5.2 shows the optimal values $u^*$ and $k^*$, as well as the corresponding ruin probabilities both for this scenario and the case $\delta = 0$ from Example 3.5.1. As the introduction of discounting reduces the reinsurance premium, we can see that under this scenario the insurer is able to retain more as the initial surplus and to set a higher value for $k$ compared to our first scenario.
3.5. Examples

Therefore, the ruin probabilities are further reduced under this scenario.

Table 3.5.1: Ruin probabilities, Exponential claims

<table>
<thead>
<tr>
<th>U</th>
<th>ψ(U)</th>
<th>(Q(u, k) = 1.6E(R_{u,k}))</th>
<th>(Q(u, k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(u^<em>) (k^</em>) (\psi_k^*(u)) %</td>
<td>(u^<em>) (k^</em>) (\psi_k^*(u)) %</td>
</tr>
<tr>
<td>11</td>
<td>0.13323</td>
<td>6.83 4.01 0.05190 61.04</td>
<td>11 0 0.13323 0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.09547</td>
<td>8.25 5.43 0.01346 85.90</td>
<td>13 0 0.09547 0.00</td>
</tr>
<tr>
<td>15</td>
<td>0.06840</td>
<td>10.05 7.23 0.00226 96.70</td>
<td>15 0 0.06840 0.00</td>
</tr>
<tr>
<td>17</td>
<td>0.04901</td>
<td>12.01 9.19 0.00032 99.35</td>
<td>13.39 2.53 0.04651 5.10</td>
</tr>
<tr>
<td>19</td>
<td>0.03512</td>
<td>14.00 11.18 0.00004 99.89</td>
<td>12.59 3.69 0.02524 28.13</td>
</tr>
<tr>
<td>21</td>
<td>0.02516</td>
<td>16.00 13.18 5.9x10^{-6} 99.98</td>
<td>12.65 4.94 0.00957 61.96</td>
</tr>
</tbody>
</table>

Table 3.5.2: Ruin probabilities, Exponential claims

From the previous results, we see that the values of \(k\) can be quite high compared to the initial surplus \(u\), and such situations may not be acceptable to reinsurers. For example, in Table 3.5.1, the ultimate ruin probability before reinsurance is close to 5% when \(U = 17\). With \(k^*\) being close to 9, the insurer’s ruin probability is very small because the probability of a claim exceeding (at least) 9 is very small. The optimal choices of \(u^* = 12.01\) and \(k^* = 9.19\) may be a situation that is not acceptable to a reinsurer. To address this issue, we look at the effectiveness of the lower barrier system in reducing the ruin probability by setting \(k\) to reasonably low levels which should be acceptable to reinsurers. In Table 3.5.3, we first show the initial capital, \(U\), needed so that \(ψ(U)\) is 1%, 2%, ..., 5%. Having found the required \(U\), we calculate the initial surplus level \(u\) by setting \(k = 2\) and \(k = 3\).
with the reinsurance premium charged in Example 3.5.1. The ruin probability for each pair is then calculated.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
U & \psi(U) & k = 2 & & k = 3 & \\
\hline
16.88 & 5\% & 16.63 & 0.0352 & 29.60 & 16.32 & 0.0216 & 56.80 \\
18.22 & 4\% & 18.02 & 0.0279 & 30.25 & 17.78 & 0.0170 & 57.50 \\
19.95 & 3\% & 19.80 & 0.0208 & 30.67 & 19.62 & 0.0125 & 58.33 \\
22.38 & 2\% & 22.28 & 0.0137 & 31.50 & 22.17 & 0.0082 & 59.00 \\
26.54 & 1\% & 26.49 & 0.0068 & 32.00 & 26.43 & 0.0040 & 60.00 \\
\hline
\end{array}
\]

Table 3.5.3: Results with fixed \( k \), Exponential claims

From Table 3.5.3, when \( U = 16.88 \), the ruin probability before reinsurance is 0.05. With \( k = 3 \), we find that the optimal initial surplus for the insurer is 16.32, and that \( \psi_3(16.32) = 0.0216 \), which is a substantial reduction from 0.05.

Until now we have assumed that the insurer will try to use the capital \( U \) to purchase reinsurance to reduce the ruin probability. An alternative use of reinsurance is to release some of the insurer’s funds without changing the ruin probability. For example, suppose that \( U \) is such that \( \psi(U) = 0.05 \). The idea now is that the insurer sets the barrier level \( k \), then selects \( u \) such that \( \psi_k(u) = 0.05 \). Providing the arrangement is feasible, i.e. \( 0 \leq k \leq u \leq U \), the insurer can allocate the amount \( C_{u,k} = U - u - Q(u,k) \) to purposes other than reinsurance or the surplus process.

**Example 3.5.4. Capital release.**

Suppose that \( Q(u,k) = 1.6E(R_{u,k}) \) and consider the values of \( U \) from Table 3.5.3. Table 3.5.4 shows the amount of surplus required, \( u \), to give the same ruin probability as \( \psi(U) \) for the cases \( k = 2 \) and \( k = 3 \). It also shows the amount \( C_{u,k} \) that is released for other purposes as a result of the reinsurance arrangement.

From Table 3.5.4 we see that the amount of funds released increases as the ruin prob-
3.5. Examples

ability decreases, and that the amount released is larger in the case $k = 3$ for a given ruin probability. However, the percentage of $U$ released is decreasing. The percentage of $U$ released ranges from 11.8% to 8.6% in the case $k = 2$, and from 25.5% to 20.1% in the case $k = 3$.

### 3.5.2 Erlang(2) claim sizes

In this section, we assume that the individual claim sizes have an Erlang(2) distribution, with probability density function $p(x) = \alpha^2 x e^{-\alpha x}$. Hence, the distribution function is $P(x) = 1 - e^{-\alpha x} - \alpha x e^{-\alpha x}$. We set $\alpha = 2$ so that the mean of this distribution is 1 and variance is 0.5. Let $\lambda = 1$ and $c = 1.2$ in our numerical examples, so that the premium loading factor $\theta_1 = 0.2$.

The probability of ultimate ruin from initial surplus $u = 0$ is then $\psi(0) = \lambda m_1 / c = 5/6$. When $\delta > 0$, Gerber and Shiu (1998) show that there is a unique positive solution to Lundberg’s fundamental equation, defined by equation (1.6.3), which we denote by $\rho$. From Dickson (2005), using the Laplace transform method, we have that the ultimate ruin probability under the classical risk model as

$$\psi(u) = 0.8518e^{-\rho_1 u} - 0.0185e^{-\rho_2 u},$$

(3.5.4)
where $\rho_1 = 0.2268$ and $\rho_2 = 2.9399$ are the absolute values of the two non-zero solutions of equation (1.6.3) with $\delta = 0$. In addition, we have

\[
G(u, y) = 0.8518 e^{-\rho_1 u} (1 - e^{-2y}) - 0.5446 y e^{-\rho_1 u - 2y} \\
-0.0185 e^{-\rho_2 u} (1 - e^{-2y}) - 0.2887 y e^{-\rho_2 u - 2y}.
\]  

(3.5.5)

Using formulae (3.3.1) and (3.3.2), we can then obtain formulae for $E(R_{u,k})$ and $E(R_{u,k}^2)$ by substituting equations (3.5.4) and (3.5.5) as follows,

\[
E(R_{u,k}) \\
= 0.5621 e^{-\rho_1 (u-k)} + 0.0629 e^{-\rho_2 (u-k)} \\
+\frac{-0.6248 + 0.6250 e^{2k} - 1.2500 k - 0.8333 k^2}{0.8333 + 0.1667 e^{2k} + 0.8333 k} \\
\times (0.8518 e^{-\rho_1 (u-k)} - 0.0185 e^{-\rho_2 (u-k)}) \\
+ e^{-\rho_1 (u-k) - 2k} (-0.8518 - 0.5446 k) + e^{-\rho_2 (u-k) - 2k} (0.0185 - 0.2887 k) \\
- e^{-\rho_1 (u-k) - 2k} (0.5621 + 1.1241 k + 0.5446 k^2) \\
- e^{-\rho_2 (u-k) - 2k} (0.0629 + 0.1259 k + 0.2887 k^2),
\]

and

\[
E(R_{u,k}^2) \\
= 0.6982 e^{-\rho_1 (u-k)} + 0.1351 e^{-\rho_2 (u-k)} \\
+\frac{-0.6248 + 0.6250 e^{2k} - 1.2500 k - 0.8333 k^2}{0.8333 + 0.1667 e^{2k} + 0.8333 k} \\
\times (0.5621 e^{-\rho_1 (u-k)} + 0.0630 e^{-\rho_2 (u-k)} + e^{-\rho_1 (u-k) - 2k} (-0.5621 - 1.1241 k - 0.5446 k^2) \\
+ e^{-\rho_2 (u-k) - 2k} (-0.0630 - 0.1259 k - 0.2887 k^2)).
\]
\[ \frac{1}{(0.8333 + 0.1667e^{2k} + 0.8333k)^2} \times (0.8518e^{-\rho_1(u-k)} - 0.0185e^{-\rho_2(u-k)}) \\
+ e^{-\rho_1(u-k)-2k}(-0.8518 - 0.5446k) + e^{-\rho_2(u-k)-2k}(0.0185 - 0.2887k)) \\
\times (0.0868 + 0.9201e^{4k} + 1.0416k + 2.4304k^2 + 2.0832k^3 + 0.6944k^4 \\
+ e^{2k}(-1.0069 - 2.7082k - 2.3610k^2 - 0.1389k^3)) \\
- e^{-\rho_1(u-k)-2k}(0.6982 + 1.3964k + 1.3964k^2 + 0.5446k^3) \\
- e^{-\rho_2(u-k)-2k}(0.1351 + 0.2702k + 0.2702k^2 + 0.2887k^3). \]

We now look at the formula for $E(R_{u,k}^\delta)$. Dickson (2008) showed that the probability density function $w_u(y,t)$ is of the form

\[ w_u(y,t) = l_1(u,t)\alpha^2 ye^{-\alpha y} + l_2(u,t)\alpha e^{-\alpha y}, \]

where the functions $l_1(u,t)$ and $l_2(u,t)$ will be specified later.

Let \( \tilde{l}_1(u,\delta) = \int_0^\infty e^{-\delta t}l_1(u,t)dt \) and \( \tilde{l}_2(u,\delta) = \int_0^\infty e^{-\delta t}l_2(u,t)dt \), be the Laplace transforms of functions $l_1(u,t)$ and $l_2(u,t)$. By equation (3.3.3) we have

\[
g_\delta(u,y) = \int_0^\infty e^{-\delta t}w_u(y,t)dt \\
= \tilde{l}_1(u,\delta)\alpha^2 ye^{-\alpha y} + \tilde{l}_2(u,\delta)\alpha e^{-\alpha y}, \]

and by integration we have

\[
G_\delta(u,y) = [1 - e^{-\alpha y} - \alpha ye^{-\alpha y}]\tilde{l}_1(u,\delta) + (1 - e^{-\alpha y})\tilde{l}_2(u,\delta). \]
From Dickson (2008), we also have that

\[ w_0(y, t) = l_1(0, t)\alpha^2 ye^{-\alpha y} + l_2(0, t)\alpha e^{-\alpha y}, \]

where

\[ \tilde{l}_1(0, \delta) \overset{\text{def}}{=} \int_0^{\infty} e^{-\delta t} l_1(0, t) dt = \frac{\lambda}{c \rho + \alpha} \]

and

\[ \tilde{l}_2(0, \delta) \overset{\text{def}}{=} \int_0^{\infty} e^{-\delta t} l_2(0, t) dt = \frac{\lambda}{c(\rho + \alpha)^2}. \]

Hence,

\[ g_\delta(0, y) = \frac{\lambda}{c(\rho + \alpha)} \alpha^2 ye^{-\alpha y} + \frac{\lambda \alpha}{c(\rho + \alpha)^2} \alpha e^{-\alpha y} \]

and

\[ G_\delta(0, y) = \frac{\lambda [1 - e^{-\alpha y} - \alpha ye^{-\alpha y}]}{c(\rho + \alpha)} + \frac{\lambda \alpha (1 - e^{-\alpha y})}{c(\rho + \alpha)^2}. \]

From equation (3.3.4), we have

\[ E(R_{k,k}^\delta) = \frac{2\lambda}{c\alpha(\rho + \alpha)^2} \left[ 1 - e^{-\alpha k} - \alpha ke^{-\alpha k} - \frac{(\alpha k)^2}{2} e^{-\alpha k} \right] + \frac{\lambda}{c(\rho + \alpha)^2} \frac{1}{1 - \frac{\lambda}{c(\rho + \alpha)} [1 - e^{-\alpha k} - \alpha ke^{-\alpha k}] - \frac{\lambda \alpha}{c(\rho + \alpha)^2} (1 - e^{-\alpha k})}. \]

As described by Dickson (2008), the Laplace transform of the functions \( l_1(u, t) \) and \( l_2(u, t) \) are

\[ \tilde{l}_1(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \frac{\alpha^{p-q}}{(\rho + \alpha)^{2p-q+1}} \frac{\alpha^{p+q+1} u^{p+q} e^{-\alpha u}}{(p+q)!}, \]

and

\[ \tilde{l}_2(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \frac{\alpha^{p-q}}{(\rho + \alpha)^{2p-q+1}} \frac{\alpha^{p+q+2} u^{p+q+1} e^{-\alpha u}}{(p+q+1)!}. \]
\[ + \frac{1}{c} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \left( \frac{p}{q} \right) \left( \frac{\lambda}{c} \right)^{p+1} \frac{\alpha^{p-q+1}}{(p+\alpha)^{2p-q+2}} \frac{\alpha^{p+q+1}u^{p+q}e^{-\alpha u}}{(p+q)!}. \]

Hence, by formula (3.3.5), we have

\[ E(R_{u,k}^\delta) = \tilde{l}_1(u, \delta) \frac{2}{\alpha} \left[ 1 - e^{-\alpha k} - \alpha ke^{-\alpha k} - \frac{(\alpha k)^2}{2!} e^{-\alpha k} \right] \]
\[ + \tilde{l}_2(u, \delta) \frac{1}{\alpha} \left[ 1 - e^{-\alpha k} - \alpha ke^{-\alpha k} \right] \]
\[ + E(R_{k,k}^\delta) G_\delta(u-k,k). \]

**Example 3.5.5. Expected value principle reinsurance premium.**

In our first example under the Erlang(2) claim size assumption, we assume that \( Q(u,k) = 1.6E(R_{u,k}) \). Figure 3.5.3 illustrates changes in \( \psi_k(u) \) with changes in \( u \), given that the total capital \( U \) is 10. Under the classical risk model, the ultimate ruin probability is \( \psi(10) = 0.08818 \). By imposing the lower barrier, the optimal choice of \( u \) and \( k \) is found to be \( u^* = 6.19 \) and \( k^* = 4.23 \), such that the ultimate ruin probability is reduced to \( \psi^*_k(u) = 0.00341 \), i.e. a 96.1% reduction from the original ruin probability. For some other values of \( U \), the optimal combinations of \( u^* \) and \( k^* \) are shown in Table 3.5.5.

**Example 3.5.6. Standard deviation principle reinsurance premium.**

In this second example, instead of using the expected value principle, we use the standard deviation principle with loading factor \( \theta_3 = 2 \), i.e. \( Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}) \). We selected an initial capital of \( U = 15 \), which gives an ultimate ruin probability of \( \psi(15) = 0.02837 \) under the classical risk model. Under the lower barrier model, the optimal choice of \( u^* = 8.49 \) and \( k^* = 3.17 \) gives an optimal ruin probability of \( \psi^*_k(u) = 0.01041 \), a reduction of 63.3% from \( \psi(15) \). The patterns we see in Figure 3.5.3 and Figure 3.5.4 are quite similar to the graphs under the exponential claim sizes. With the reinsurer using the
expected value principle for the premium in Example 3.5.5, any feasible combination of $u$ and $k$ ($u \geq k$) would provide an improvement in terms of the ultimate ruin probabilities. In this example, not all pairs of $u$ and $k$ will improve the original model.

In Table 3.5.5, we have calculated the optimal $u^*$, $k^*$ and the corresponding $\psi_k^*(u)$ under the assumptions in Examples 3.5.5 and 3.5.6, using different values of $U$. The percentage reductions in ultimate ruin probabilities are also provided. Under the expected value principle for the reinsurance premium, the lower barrier model performs better than the classical model and the percentage of improvement increases as $U$ increases. The reinsurance premium becomes more expensive when it is calculated using the standard
Figure 3.5.4: $\psi_k(u)$ and $\psi(U)$, $U = 15$, $Q(u, k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k})$, Erlang(2) claims

deviation principle in Example 3.5.6. As a result, we see that when we have little capital, the reinsurance product is too expensive for the insurer and that the company is better off without purchasing any reinsurance. As $U$ increases, the insurer is then able to buy reinsurance (for example when $U = 13$). We see that by doing so, the ultimate ruin probability is considerably reduced.

**Example 3.5.7. Discounted expected value principle reinsurance premium.**

In our third scenario, we set $Q(u, k) = 1.6E(R_{0.01,0.01})$. Table 3.5.6 gives a comparison between this scenario and the special case where $\delta = 0$ as in Example 3.5.5. The optimal $u^*$ and $k^*$, together with the corresponding $\psi^*_k(u)$ are provided for each cases for the given
Chapter 3. Lower barrier model: ultimate ruin

Q(u, k) = 1.6E(Ru,k)

\[ Q(u, k) = E(R_{u,k}) + 2 \text{St.Dev.}(R_{u,k}) \]

Table 3.5.5: Ruin probabilities, Erlang(2) claims

<table>
<thead>
<tr>
<th>U</th>
<th>( \psi(U) )</th>
<th>( Q(u, k) = 1.6E(R_{u,k}) )</th>
<th>( Q(u, k) = E(R_{u,k}) + 2 \text{St.Dev.}(R_{u,k}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.17412</td>
<td>4.22 2.22 0.09918 43.04</td>
<td>7 0 0.17412 0.00</td>
</tr>
<tr>
<td>9</td>
<td>0.11063</td>
<td>5.35 3.38 0.01552 85.97</td>
<td>9 0 0.11063 0.00</td>
</tr>
<tr>
<td>11</td>
<td>0.07028</td>
<td>7.14 5.20 0.00059 99.16</td>
<td>11 0 0.07028 0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.04465</td>
<td>9.12 7.19 0.00001 99.98</td>
<td>9.04 2.09 0.03875 13.62</td>
</tr>
<tr>
<td>15</td>
<td>0.02837</td>
<td>11.12 9.19 3.3 \times 10^{-6} 99.99</td>
<td>8.49 3.17 0.01041 63.31</td>
</tr>
<tr>
<td>17</td>
<td>0.01802</td>
<td>13.12 11.19 7.25 \times 10^{-9} 100.00</td>
<td>9.57 4.80 0.00065 96.39</td>
</tr>
</tbody>
</table>

Table 3.5.6: Ruin probabilities, Erlang(2) claims

\( U \). When taking the time value of the claims payment into consideration in the reinsurance premium, the premium charged should be lower than premium charged in Example 3.5.5. Hence, we can see that both \( u^* \) and \( k^* \) are higher than the ones in the first scenario and that the ruin probabilities are further reduced.

As in the case of exponential claims, the optimal \( k^* \) in Table 3.5.5 and Table 3.5.6 can be comparatively high, considering the distribution of claim sizes has a mean of 1 and variance of 0.5. Hence, in Table 3.5.7, we chose two fixed levels of \( k \), i.e. \( k = 2 \) and \( k = 3 \). The allocated capital \( U \) is chosen so that the ultimate ruin probabilities are at 5%, 4%, \ldots, 1% level. Given \( k \) and \( U \), we can then work out the new surplus starting level \( u \), given that \( Q(u, k) = 1.6E(R_{u,k}) \), and thereby calculate the corresponding new ultimate ruin probability. In Table 3.5.7, in order to obtain a 5% ruin probability level in the
classical model, the capital needed is $U = 12.50$. When $k = 2$, the corresponding $u$ is 12.11 and that ruin probability can be reduced by 57.4% to 0.0213. As the original ruin probability level decreases, $U$ increases and so does the percentage improvement in the ruin probability. We see that this pattern also applies when $k = 3$.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\psi(U)$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$u$</td>
<td>$\psi_k(u)$</td>
</tr>
<tr>
<td>12.50</td>
<td>5%</td>
<td>12.11</td>
<td>0.0213</td>
</tr>
<tr>
<td>13.49</td>
<td>4%</td>
<td>13.18</td>
<td>0.0168</td>
</tr>
<tr>
<td>14.75</td>
<td>3%</td>
<td>14.53</td>
<td>0.0123</td>
</tr>
<tr>
<td>16.54</td>
<td>2%</td>
<td>16.39</td>
<td>0.0081</td>
</tr>
<tr>
<td>19.60</td>
<td>1%</td>
<td>19.53</td>
<td>0.0040</td>
</tr>
</tbody>
</table>

Table 3.5.7: Results with fixed $k$, Erlang(2) claims

**Example 3.5.8. Capital release.**

In our last example under Erlang(2) claim sizes, instead of keeping the capital $U$ unchanged and seeking a reduction in the ultimate ruin probability, we try to maintain the same level of ruin probability under the two models and seek for the amount of capital that could be released, if the lower barrier system were applied. We follow the same method as in Example 3.5.4. Table 3.5.8 illustrates the results and $C_{u,k}$ represents the amount of capital released in each case.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\psi(U)$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$u$</td>
<td>$C_{u,k}$</td>
</tr>
<tr>
<td>12.50</td>
<td>5%</td>
<td>8.36</td>
<td>3.23</td>
</tr>
<tr>
<td>13.49</td>
<td>4%</td>
<td>9.34</td>
<td>3.41</td>
</tr>
<tr>
<td>14.75</td>
<td>3%</td>
<td>10.61</td>
<td>3.60</td>
</tr>
<tr>
<td>16.54</td>
<td>2%</td>
<td>12.40</td>
<td>3.78</td>
</tr>
<tr>
<td>19.60</td>
<td>1%</td>
<td>15.45</td>
<td>3.96</td>
</tr>
</tbody>
</table>

Table 3.5.8: Surplus required to maintain ruin probability and $C_{u,k}$, Erlang(2) claims

In Table 3.5.8, when the ruin probability is at 5%, the capital needed is 12.50. When $k$
is set at 2, we see that the starting surplus level is at 8.36 so that the new ruin probability is also at 5%, and the amount of capital released is 3.23. However, when \( k = 3 \), any value of \( u \geq k \) would provide an ultimate ruin probability smaller than 5%, and hence the values in the table are shown as N/A.

3.5.3 Mixed exponential claim sizes

We now assume that the individual claim amount distribution is a mixture of two exponential distributions, with probability density function

\[
p(x) = \frac{1}{3} \cdot 0.5e^{-0.5x} + \frac{2}{3} \cdot 2e^{-2x}.
\]

This distribution has mean 1 and variance 1.5. Again, we let \( \lambda = 1 \) and set the premium rate as \( c = 1.2 \) in our numerical examples.

Using techniques described in Gerber et al. (1987) we find that

\[
\psi(u) = 0.7990e^{-\rho_1 u} + 0.0343e^{-\rho_2 u}, \quad (3.5.6)
\]

where \( \rho_1 = 0.1069 \) and \( \rho_2 = 1.5598 \) are the absolute values of the two non-zero solutions of Lundberg’s fundamental equation (1.6.3) when \( \delta = 0 \), and

\[
G(u, y) = (0.0752(1 - e^{-2y}) + 0.7239(1 - e^{-0.5y}))e^{-\rho_1 u}
+ (0.2026(1 - e^{-2y}) - 0.1683(1 - e^{-0.5y}))e^{-\rho_2 u}. \quad (3.5.7)
\]
Formulae for $E(R_{u,k})$ and $E(R_{u,k}^2)$ can then be obtained as

$$
E(R_{u,k}) = \frac{1.25 - e^{-0.5k}(1.1111 + 0.5556k) - e^{-2k}(0.1389 + 0.2778k)}{0.1667 + 0.2778e^{-2k} + 0.5556e^{-0.5k}} \times \left( e^{-\rho_1(u-k)}(0.7990 - 0.0752e^{-2k} - 0.7239e^{-0.5k}) + e^{-\rho_2(u-k)}(0.0343 - 0.2026e^{-2k} + 0.1683e^{-0.5k}) \right) + 0.7239e^{-\rho_1(u-k)-0.5k}(-2 + 2e^{0.5k} - k) - 0.1683e^{-\rho_2(u-k)-0.5k}(-2 + 2e^{0.5k} - k) + 0.0376e^{-\rho_1(u-k)-2k}(-1 + e^{2k} - 2k) + 0.1013e^{-\rho_2(u-k)-2k}(-1 + e^{2k} - 2k)
$$

and

$$
E(R_{u,k}^2) = \frac{2(1.25 - e^{-0.5k}(1.1111 + 0.5556k) - e^{-2k}(0.1389 + 0.2778k))}{0.1667 + 0.2778e^{-2k} + 0.5556e^{-0.5k}} \times \left( 0.7239e^{-\rho_1(u-k)-0.5k}(-2 + 2e^{0.5k} - k) - 0.1683e^{-\rho_2(u-k)-0.5k}(-2 + 2e^{0.5k} - k) + 0.0376e^{-\rho_1(u-k)-2k}(-1 + e^{2k} - 2k) + 0.1013e^{-\rho_2(u-k)-2k}(-1 + e^{2k} - 2k) \right) + \frac{1}{0.1667 + 0.2778e^{-2k} + 0.5556e^{-0.5k}} \times \left\{ (e^{-\rho_1(u-k)}(0.7990 - 0.0752e^{-2k} - 0.7239e^{-0.5k}) + e^{-\rho_2(u-k)}(0.0343 - 0.2026e^{-2k} + 0.1683e^{-0.5k})) \times \left( 2(1.25 - e^{-0.5k}(-1.1111 - 0.5556k) + e^{-2k}(-0.1389 - 0.2778k)) \right)^2 \right\} + 4.5833 + e^{-0.5k}(-4.4444 - 2.2222k - 0.5556k^2) + e^{-2k}(-0.1389 - 0.2778k - 0.27777k^2) \right\}
$$
\[ +0.7239e^{-\rho_1(u-k)-0.5k}(-8 + 8e^{0.5k} - 4k - k^2) \]
\[ -0.1683e^{-\rho_2(u-k)-0.5k}(-8 + 8e^{0.5k} - 4k - k^2) \]
\[ +0.0376e^{-\rho_1(u-k)-2k}(-1 + e^{2k} - 2k - 2k^2) \]
\[ +0.1013e^{-\rho_2(u-k)-2k}(-1 + e^{2k} - 2k - 2k^2) \]

Now consider \( E(R_{u,k}^\delta) \). Dickson and Drekic (2006) show that when claims have a mixed exponential distribution with probability density function

\[ p(x) = pae^{-\alpha x} + qbe^{-\beta x}, \]

the probability density function \( w_u(y,t) \) is of the form

\[ w_u(y,t) = \eta_1(u,t)\alpha e^{-\alpha y} + \eta_2(u,t)\beta e^{-\beta y}, \]

but they do not identify \( \eta_i(u,t) \), for \( i = 1, 2 \). Substituting this expression into equation (3.3.4), we have

\[ E(R_{u,k}^\delta) = \frac{\int_0^\infty e^{-\delta t} \eta_1(0,t) dt \int_0^k y \alpha e^{-\alpha y} dy + \int_0^\infty e^{-\delta t} \eta_2(0,t) dt \int_0^k y \beta e^{-\beta y} dy}{1 - \int_0^\infty e^{-\delta t} \eta_1(0,t) dt \int_0^k \alpha e^{-\alpha y} dy + \int_0^\infty e^{-\delta t} \eta_2(0,t) dt \int_0^k \beta e^{-\beta y} dy}. \]

An explicit formula for \( \int_0^\infty e^{-\delta t} \eta_i(u,t) dt \) is given in Dickson and Drekic (2006). They show that

\[ \int_0^\infty e^{-\delta t} \eta_i(u,t) dt = \gamma_i(\delta)e^{-\rho_{1\delta} u} + \sigma_i(\delta)e^{-\rho_{2\delta} u}, \]

for \( i = 1, 2 \), where \( \rho > 0, -\rho_{1\delta} < 0 \) and \( -\rho_{2\delta} < 0 \) are the solutions of Lundberg’s fundamental equation

\[ \lambda + \delta - ct = \frac{\lambda p\alpha}{\alpha + t} + \frac{\lambda q\beta}{\beta + t}, \]
and
\[
\begin{align*}
\gamma_1(\delta) &= \frac{(\alpha - \rho_1 \delta) (\alpha - \rho_2 \delta) (\beta - \rho_1 \delta)}{\alpha (\rho_2 - \rho_1 \delta) (\alpha - \beta)}, \\
\gamma_2(\delta) &= -\frac{(\alpha - \rho_1 \delta) (\beta - \rho_1 \delta) (\beta - \rho_2 \delta)}{\beta (\rho_2 - \rho_1 \delta) (\alpha - \beta)}, \\
\sigma_1(\delta) &= -\frac{(\alpha - \rho_1 \delta) (\alpha - \rho_2 \delta) (\beta - \rho_2 \delta)}{\alpha (\rho_2 - \rho_1 \delta) (\alpha - \beta)}, \\
\sigma_2(\delta) &= \frac{(\alpha - \rho_2 \delta) (\beta - \rho_1 \delta) (\beta - \rho_2 \delta)}{\beta (\rho_2 - \rho_1 \delta) (\alpha - \beta)}.
\end{align*}
\]

Hence,
\[
E(R_{k,k}^\delta) = \frac{[\gamma_1(\delta) + \sigma_1(\delta)] [1 - e^{-\rho_1 \delta} (1 + \alpha k)] / \alpha + [\gamma_2(\delta) + \sigma_2(\delta)] [1 - e^{-\rho_2 \delta} (1 + \beta k)] / \beta}{1 - [\gamma_1(\delta) + \sigma_1(\delta)] (1 - e^{-\alpha k}) - [\gamma_2(\delta) + \sigma_2(\delta)] (1 - e^{-\beta k})},
\]

and therefore from equation (3.3.5) we have
\[
E(R_{u,k}^\delta) = (\gamma_1(\delta)e^{-\rho_1 \delta} (u - k) + \sigma_1(\delta)e^{-\rho_2 \delta} (u - k)) [1 - e^{-\alpha k} (1 + \alpha k)] / \alpha \\
+ (\gamma_2(\delta)e^{-\rho_1 \delta} (u - k) + \sigma_2(\delta)e^{-\rho_2 \delta} (u - k)) [1 - e^{-\beta k} (1 + \beta k)] / \beta \\
+ E(R_{k,k}^\delta) ((\gamma_1(\delta)e^{-\rho_1 \delta} (u - k) + \sigma_1(\delta)e^{-\rho_2 \delta} (u - k)) (1 - e^{-\alpha k}) \\
+ (\gamma_2(\delta)e^{-\rho_1 \delta} (u - k) + \sigma_2(\delta)e^{-\rho_2 \delta} (u - k)) (1 - e^{-\beta k})).
\]

In the examples below we apply the same reinsurance scenarios as in Examples 3.5.1 to 3.5.3.

Example 3.5.9. Expected value principle reinsurance premium.

Firstly we assume that \(Q(u, k) = 1.6E(R_{u,k})\). For \(U = 15\), Figure 3.5.5 shows the ruin probabilities \(\psi_k(u)\) for feasible combinations of \(u\) and \(k\). The minimum ruin probability is \(\psi_k^*(u) = 0.10448\) when \(u^* = 10.17\) and \(k^* = 5.62\). Under the classical risk process
\( \psi(15) = 0.16088 \), so reinsurance has reduced the ruin probability by 35.6\% in this case.

Note that under this reinsurance arrangement, we obtain ruin probabilities that are less than \( \psi(15) \), regardless of the choices of \( u \) and \( k \).

![Figure 3.5.5: \( \psi_k(u) \) and \( \psi(U) \), \( U = 15 \), \( Q(u, k) = 1.6E(R_{u,k}) \), Mixed exponential claims](image)

**Example 3.5.10. Standard deviation value principle reinsurance premium.**

We now set \( Q(u, k) = E(R_{u,k}) + 2St.\,\text{Dev.}(R_{u,k}) \). For \( U = 30 \), Figure 3.5.6 shows ruin probabilities for different combinations of \( u \) and \( k \), as well as \( \psi(30) = 0.03239 \). As in Example 3.5.2, only certain choices of \( u \) and \( k \) provide a reduction in the ruin probability. The optimal choice is \( u^* = 21.37 \) and \( k^* = 6.35 \), giving \( \psi^*_k(u) = 0.02496 \).

Table 3.5.9 shows optimal values \( u^* \) and \( k^* \) for some values of \( U \) under the reinsurance
Figure 3.5.6: $\psi_k(u)$ and $\psi(U)$, $U = 30$, $Q(u, k) = E(R_{u,k}) + 2 \text{St. Dev.}(R_{u,k})$, Mixed exponential claims

premium arrangements of the previous two examples. Given these values, we are able to compare the effectiveness of reinsurance by looking at the reduction in ruin probabilities in percentage terms, as in Table 3.5.1. The percentage reduction in ruin probability increases rapidly when the reinsurance premium is calculated by the expected value principle, going from 35.06% when $U = 15$ to 98.89% when $U = 29$. When $U$ is small and the reinsurance premium is calculated by the standard deviation principle, we find that reinsurance is too expensive, and therefore the insurance company should bear all the risk itself. When $U$ is large enough, reinsurance becomes affordable and the ruin probability under reinsurance reduces from that under the original process.
Example 3.5.11. **Discounted expected value principle reinsurance premium.**

As a third example, we assume $Q(u,k) = 1.6E(R_u,k)$ with $\delta = 0.01$. Table 3.5.10 shows optimal values $u^*$ and $k^*$ for different values of $U$ under scenarios 1 and 3, together with the ruin probabilities under each choice and the percentage reduction for each combination compared to the without reinsurance situation. When discounting is introduced, the premium charged by the reinsurer is lower, so the insurer is able to arrange reinsurance with larger $k$ values, as well as retaining a greater initial surplus. Hence, the ruin probabilities are smaller under scenario 3 than under scenario 1.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\psi(U)$</th>
<th>$Q(u,k) = 1.6E(R_u,k)$</th>
<th>$Q(u,k) = E(R_u,k) + 2\text{St. Dev.}(R_u,k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$u^*$</td>
<td>$k^*$</td>
</tr>
<tr>
<td>15</td>
<td>0.16088</td>
<td>10.17</td>
<td>5.62</td>
</tr>
<tr>
<td>17</td>
<td>0.12992</td>
<td>11.30</td>
<td>6.75</td>
</tr>
<tr>
<td>19</td>
<td>0.10493</td>
<td>12.62</td>
<td>8.07</td>
</tr>
<tr>
<td>21</td>
<td>0.08474</td>
<td>14.15</td>
<td>9.60</td>
</tr>
<tr>
<td>23</td>
<td>0.06843</td>
<td>15.88</td>
<td>11.33</td>
</tr>
<tr>
<td>25</td>
<td>0.05527</td>
<td>17.74</td>
<td>13.20</td>
</tr>
<tr>
<td>27</td>
<td>0.04463</td>
<td>19.68</td>
<td>15.13</td>
</tr>
<tr>
<td>29</td>
<td>0.03604</td>
<td>21.65</td>
<td>17.11</td>
</tr>
</tbody>
</table>

Table 3.5.10: Ruin probabilities, Mixed exponential claims

We note that once again, $k^*$ can be quite high. In Table 3.5.11 we show values of $\psi_k(u)$ when $k = 2$ and $k = 3$. Compared to Table 3.5.3, the capital needed so that...
the pre-reinsurance ultimate ruin probabilities are 1%, 2%, …, 5% are much higher. This is because the claim size distribution has a larger variance compared to the exponential distribution. When \( \psi(U) = 5\% \), we found that \( U \) is 25.94. Having \( k = 3 \), the initial surplus level \( u \) is found to be 25.65 and \( \psi_3(25.65) = 0.0386 \). The table shows that reinsurance works quite effectively even when we choose smaller \( k \) rather than the optimal \( k^* \).

<table>
<thead>
<tr>
<th>( U )</th>
<th>( \psi(U) )</th>
<th>( k = 2 )</th>
<th>( u )</th>
<th>( \psi_k(u) )</th>
<th>( % )</th>
<th>( k = 3 )</th>
<th>( u )</th>
<th>( \psi_k(u) )</th>
<th>( % )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25.94</td>
<td>5%</td>
<td>25.81</td>
<td>0.0443</td>
<td>11.40</td>
<td></td>
<td></td>
<td>25.65</td>
<td>0.0386</td>
<td>22.80</td>
</tr>
<tr>
<td>28.03</td>
<td>4%</td>
<td>27.92</td>
<td>0.0353</td>
<td>11.75</td>
<td></td>
<td></td>
<td>27.79</td>
<td>0.0307</td>
<td>23.25</td>
</tr>
<tr>
<td>30.72</td>
<td>3%</td>
<td>30.64</td>
<td>0.0264</td>
<td>12.00</td>
<td></td>
<td></td>
<td>30.55</td>
<td>0.0229</td>
<td>23.67</td>
</tr>
<tr>
<td>34.51</td>
<td>2%</td>
<td>34.46</td>
<td>0.0176</td>
<td>12.00</td>
<td></td>
<td></td>
<td>34.40</td>
<td>0.0152</td>
<td>24.00</td>
</tr>
<tr>
<td>41.00</td>
<td>1%</td>
<td>40.97</td>
<td>0.0088</td>
<td>12.00</td>
<td></td>
<td></td>
<td>40.94</td>
<td>0.0075</td>
<td>25.00</td>
</tr>
</tbody>
</table>

Table 3.5.11: Results for fixed \( k \), Mixed exponential claims

Finally, we consider what happens when the ruin probability is unchanged.

**Example 3.5.12. Capital release.**

Suppose that \( Q(u,k) = 1.6 E(R_{u,k}) \) and consider the values of \( U \) from Table 3.5.11. Table 3.5.12 shows the amount of surplus required, \( u \), to give the same ruin probability as \( \psi(U) \) for the cases \( k = 2 \) and \( k = 3 \). It also shows the amount \( C_{u,k} \) that is released for other purposes as a result of the reinsurance arrangement.

As in Table 3.5.4, the amount of funds released increases as the ruin probability decreases, and the amount released is larger in the case \( k = 3 \) for a given ruin probability. The percentage of \( U \) released ranges from 3.0% to 4.3% in the case \( k = 2 \), and from 6.4% to 9.0% in the case \( k = 3 \). Although these percentages are smaller than the corresponding values in Table 3.5.4, there is still a reasonable release of funds to the insurer.
Chapter 3. Lower barrier model: ultimate ruin

<table>
<thead>
<tr>
<th>$U$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u$</td>
<td>$C_{u,k}$</td>
</tr>
<tr>
<td>25.94</td>
<td>24.67</td>
<td>1.12</td>
</tr>
<tr>
<td>28.03</td>
<td>26.76</td>
<td>1.15</td>
</tr>
<tr>
<td>30.72</td>
<td>29.45</td>
<td>1.18</td>
</tr>
<tr>
<td>34.51</td>
<td>33.25</td>
<td>1.21</td>
</tr>
<tr>
<td>41.00</td>
<td>39.73</td>
<td>1.24</td>
</tr>
</tbody>
</table>

Table 3.5.12: Surplus required to maintain ruin probability and $C_{u,k}$, Mixed exponential claims

3.6 The reinsurer’s claims

In previous sections, we have examined the effects of the lower barrier model on the insurer’s ultimate ruin probability under the classical risk model with certain individual claim size and reinsurance premium assumptions. In this section, we look at the distribution of the corresponding reinsurer’s claims under the lower barrier model.

In Section 3.3, we have defined $R_{u,k}$ to be the total claim amount for the reinsurer up to the time of ruin (of the insurer) with the insurer’s initial surplus and lower barrier being $u$ and $k$ respectively. Let

$$Z_{u,k}(x) = \Pr[R_{u,k} \leq x],$$

be the distribution function of $R_{u,k}$, and $z_{u,k}$ be the density function.

We start with the situation where $u = k$. The probability that the reinsurer has $n$ claims is

$$\Pr[N = n] = G(0, k)^n[1 - G(0, k)], \text{ for } n = 0, 1, 2, \ldots$$

Note that $\Pr[N = 0]$ is made up of $1 - \psi(0)$ and $\psi(0) - G(0, k)$, the probability of survival from 0 and the probability of ruin from 0 with a deficit at ruin that is greater than $k$. 
under the classical risk model.

Let \( A_i \) be the \( i \)th reinsurance claim amount. We then have that \( \{A_i\}_{i \geq 1} \) are i.i.d. with distribution function \( R_2 \) given by

\[
R_2(x) = \Pr[A_i \leq x] = \begin{cases} \frac{G(0, x)}{G(0, k)}, & x \leq k, \\ 1, & x > k. \end{cases}
\]

We hence have

\[
Z_{k,k}(x) = \Pr[R_{k,k} \leq x] = [1 - G(0, k)] + \sum_{n=1}^{\infty} G(0, k)^n[1 - G(0, k)]R_2^{*n}(x), \tag{3.6.1}
\]

where \( R_2^{*n} \) is the \( n \)-fold convolution of \( R_2 \) with itself.

When \( u \geq k \), the probability that the reinsurer has \( n \) claims is

\[
\Pr[N = n] = \begin{cases} 1 - G(u - k, k), & n = 0, \\ G(u - k, k)G(0, k)^{n-1}[1 - G(0, k)], & n = 1, 2, 3, \ldots \end{cases}
\]

Let \( R_1 \) be the distribution function of the first claim size. We then have

\[
R_1(x) = \begin{cases} \frac{G(u - k, x)}{G(u - k, k)}, & x \leq k, \\ 1, & x > k. \end{cases}
\]

Hence, the distribution function of \( R_{u,k} \) is

\[
Z_{u,k}(x) = 1 - G(u - k, k) + \sum_{n=1}^{\infty} G(u - k, k)G(0, k)^{n-1}[1 - G(0, k)]R_1 \ast R_2^{(n-1)*}(x), \tag{3.6.2}
\]
where $R_1 \ast R_2^{\alpha}$ is the convolution of $R_1$ and $R_2^{\alpha}$ with $R_1 \ast R_2^{0\alpha} = R_1$. In addition, let $r_1$ and $r_2$ be the derivative of $R_1$ and $R_2$ respectively. We have that the reinsurer’s claim amount is zero with probability $1 - G(0, k)$ and has density function

$$z_{u,k}(x) = \sum_{n=1}^{\infty} G(u - k, k)G(0, k)^{n-1}[1 - G(0, k)]r_1 \ast r_2^{(n-1)\alpha}(x), \ x > 0. \quad (3.6.3)$$

### 3.6.1 Exponential claim sizes

We now assume that the insurer’s individual claim sizes follow the exponential distribution with density function $p(x) = \alpha e^{-\alpha x}$. From equation (3.5.3), we have

$$R_1(x) = R_2(x) = \frac{1 - e^{-\alpha x}}{1 - e^{-\alpha k}}, \ 0 \leq x \leq k.$$

It follows that the Laplace transform of $r_1$ and $r_2$ are the same as

$$\tilde{r}_1(s) = \frac{1 - e^{-(\alpha+s)k}}{1 - e^{-\alpha k}} \frac{\alpha}{\alpha + s}.$$

From equation (3.6.3), we have that when $x > 0$, the Laplace transform of $z_{u,k}$, denoted as $\tilde{z}_{u,k}$, is

$$\tilde{z}_{u,k}(s) = \sum_{n=1}^{\infty} \psi(u - k)\psi(0)^{n-1}[1 - G(0, k)] \left(\frac{\alpha}{\alpha + s}\right)^n (1 - e^{-(\alpha+s)k})^n$$

$$= \sum_{n=1}^{\infty} \psi(u - k)\psi(0)^{n-1}[1 - G(0, k)] \sum_{m=0}^{n} \binom{n}{m} (-e^{-\alpha k})^m \left(\frac{\alpha}{\alpha + s}\right)^n e^{-(km)s}. \quad (3.6.4)$$

According to the Shift Theorem for Laplace transforms, if $I(y > a)$ is an indicator function
such that
\[
I(y > a) = \begin{cases} 
0, & y \leq a, \\
1, & y > a,
\end{cases}
\]
the Laplace transform of the function \( I(y > a)h(y - a) \) is then \( e^{-as}\tilde{h}(s) \). Hence, inverting equation (3.6.4), we have
\[
z_{u,k}(x) = \sum_{n=1}^{\infty} \psi(u - k)\psi(0)^{n-1}[1 - G(0,k)] 
\times \sum_{m=0}^{n} \binom{n}{m}(-e^{-\alpha k})^{m}I(x > km)e_{n,\alpha}(x - km), \ x > 0, (3.6.5)
\]
where \( e_{n,\beta}(t) \) is the probability density function of an Erlang(\( n \)) random variable with scale parameter \( \beta \).

**Example 3.6.1.** For numerical illustration, we set the lower barrier level \( k \) at \( k = 2 \) and assume that \( Q(u,k) = 1.6E(R_{u,k}) \). As shown in Table 3.5.3, we find the initial capital \( U \) so that the ultimate ruin probabilities under the classical risk model are at 5%, 4%, ..., 1% level. Given \( U \) and \( k \), we can work out the initial surplus level \( u \). To illustrate the reinsurer’s perspective, we calculate the probability that the reinsurer’s total claim is less than the premium received, i.e. \( \Pr[R_{u,2} \leq Q(u,2)] \).

| \( U \) | \( \psi(U) \) | \begin{tabular}{c|c|c|c|c|c} \hline
\hline
Insurer & Reinsurer & & & & \\
\hline
\hline
\( u \) & \( \psi_2(u) \) & \% reduction & \( Q(u,2) \) & \( \Pr[R_{u,2} \leq Q(u,2)] \) \\
\hline
16.88 & 5% & 16.63 & 0.0352 & 29.60 & 0.25 & 0.94207 \\
18.22 & 4% & 18.02 & 0.0279 & 30.25 & 0.20 & 0.95327 \\
19.95 & 3% & 19.80 & 0.0208 & 30.67 & 0.15 & 0.96469 \\
22.38 & 2% & 22.28 & 0.0137 & 31.50 & 0.10 & 0.97625 \\
26.54 & 1% & 26.49 & 0.0068 & 32.00 & 0.05 & 0.98803 \\
\hline
\end{tabular} |

Table 3.6.1: Reinsurer’s ultimate probability of survival, Exponential claims, \( k = 2 \)

*From Table 3.6.1, we see that when \( U \) is 16.88, the ultimate ruin probability under*
the classical risk model is 5\%. For \( k = 2 \) and under the given reinsurance premium assumption, the initial surplus level for the insurer is \( u = 16.63 \), giving the ultimate ruin probability under the lower barrier model is 0.0352. In this case, the reinsurer’s premium is 0.25, and the ultimate probability that his total claim never exceeds premium is 0.94207. As \( U \) increases, we see that the percentage reduction in the ultimate ruin probability for the insurer increases, as well as the reinsurer’s ultimate probability of survival. The results in Table 3.6.1 suggest that the reinsurance arrangement is a viable one.

3.7 Concluding remarks

We have seen from the examples of the previous section that the insurer can considerably reduce its ruin probability if it can allocate part of its capital to the surplus process and part as a reinsurance premium. However, the optimal barrier level may be sufficiently high to make a reinsurance contract unattractive to a reinsurer. Even if the insurer has to settle for a level of reinsurance that is viable to the reinsurer but sub-optimal in terms of minimizing the ruin probability, the insurer can still create a meaningful reduction in its ruin probability by setting a relatively low value for the barrier.

In our examples we have considered situations in which explicit formulae exist for the quantities of interest, such as \( \psi_k(u) \). In other situations, e.g. when the individual claim amount distribution is Pareto, explicit results do not exist. However, in such cases we can still find good approximations to quantities such as \( \psi_k(u) \) given the values of \( u \) and \( k \) using techniques described in Dickson & Waters (1991 and 1992). Unfortunately, a numerical approach is very computationally intensive to apply when we wish to find optimal values of \( u \) and \( k \), and consequently we have not tried to construct an example using this approach.
Chapter 4

Lower barrier model: infinite time

4.1 Introduction

In Chapter 3, we have proposed our modified risk model based on the classical risk model by introducing a lower barrier, and we have focused on ultimate ruin probabilities under such a modification. From the examples given, we have seen that under certain assumptions with respect to the individual claim sizes and the reinsurance premium charged, the lower barrier model can considerably reduce the insurer’s ultimate ruin probability. In this chapter, we investigate the effects of the lower barrier model on other ruin related quantities in the infinite time framework, namely the deficit at ruin and the surplus before ruin. We firstly introduce some new definitions and notation.

Recall that in Chapter 3 we have defined $T_{u,k}$ to be the time to ruin under the lower barrier model with initial surplus $u$ and lower barrier $k$. Let $G_{u,k}$ be the defective distribution function of the deficit at ruin so that

$$G_{u,k}(y) = \Pr[T_{u,k} < \infty \text{ and } |U(T_{u,k})| \leq y], \quad (4.1.1)$$
and by differentiating with respect to \( y \), we can obtain the defective density function \( g_{u,k}(y) \).

Under the lower barrier model, the surplus before ruin will always be greater than \( k \). Therefore, instead of looking at the surplus level \( U(T_{u,k}^-) \), we are interested in the quantity \( X = U(T_{u,k}^-) - k \), the surplus level in excess of \( k \). This quantity is more comparable with the surplus before ruin, \( U(T_u^-) \), under the classical risk model, since they are both distributed on \((0, \infty)\). We let \( J_{u,k} \) be the defective distribution function of \( X = U(T_{u,k}^-) - k \), such that

\[
J_{u,k}(x) = \Pr[T_{u,k} < \infty \text{ and } U(T_{u,k}^-) - k \leq x],
\]

and \( j_{u,k}(x) = \frac{d}{dx}J_{u,k}(x) \) is the defective density function.

Figure 4.1.1: Lower barrier model, \( X = U(T_{u,k}^-) - k \), \( Y = |U(T_{u,k})| \)

Figure 4.1.1 illustrates the quantities we will investigate. In the next two sections, we derive formulae for the functions \( G_{u,k}(y) \) and \( J_{u,k}(x) \) respectively, followed by some
4.2. The deficit at ruin

Examples illustrating the effects of imposing the lower barrier in the classical risk process, assuming some different claim size distributions.

4.2 The deficit at ruin

To derive a formula for $G_{u,k}(y)$, we use the results for the defective probability density function $g(u,y)$ of the deficit at ruin from the classical risk model, and the corresponding defective distribution function $G(u,y)$. Starting from the simpler case when the initial surplus $u$ is the same as $k$, and by conditioning on whether ruin happens on the first occasion the surplus drops below the barrier $k$, we have

$$G_{k,k}(y) = \int_k^{k+y} g(0,s)ds + \sum_{n=1}^{\infty} G(0,k)^n \int_k^{k+y} g(0,s)ds$$

$$= \frac{G(0,k+y) - G(0,k)}{1 - G(0,k)}.$$

By differentiating with respect to $y$, we have

$$g_{k,k}(y) = \frac{g(0,k+y)}{1 - G(0,k)}.$$

When $u > k$, using the same reasoning we obtain that

$$G_{u,k}(y) = \int_k^{k+y} g(u-k,s)ds + G(u-k,k)G_{k,k}(y)$$

$$= G(u-k,k+y) - G(u-k,k) + G(u-k,k) \frac{G(0,k+y) - G(0,k)}{1 - G(0,k)}$$

$$= G(u-k,k+y) - G(u-k,k) \frac{1 - G(0,k+y)}{1 - G(0,k)}.$$

(4.2.1)
This then gives
\[ g_{u,k}(y) = g(u-k,k+y) + \frac{G(u-k,k)}{1-G(0,k)}g(0,k+y). \]  (4.2.2)

Note that in formula (4.2.1), if we let \( y \to \infty \), then \( G_{u,k}(y) \to \psi_k(u) \) and \( G(0,k+y) \to \psi(0) \), so that equation (4.2.1) is consistent with the ultimate ruin probability \( \psi_k(u) \) in equation (3.2.2).

The unconditional \( i^{th} \) moment of the deficit at ruin under the lower barrier model can be found using the defective density function in equation (4.2.2) as
\[
E(\left|U(T_{u,k})\right|^i) = \int_0^\infty y^i g_{u,k}(y) dy \\
= \int_k^\infty (z-k)^i g(u-k,z) dz + \frac{G(u-k,k)}{1-G(0,k)} \int_k^\infty (z-k)^i g(0,z) dz \\
= \sum_{j=0}^i \binom{i}{j} (-k)^{i-j} \left[ \int_k^\infty z^j g(u-k,z) dz + \frac{G(u-k,k)}{1-G(0,k)} \int_k^\infty z^j g(0,z) dz \right].
\]

Note that this formula also applies when \( k = 0 \).

We now look at examples with three different individual claim size assumptions as in Chapter 3: the exponential distribution, the Erlang(2) distribution and a mixed-exponential distribution. All other assumptions with respect to parameters such as \( \lambda \) and \( c \) are the same as in the examples in Section 3.5. In order to make comparisons, we choose a certain level of initial capital \( U \) and the two corresponding optimal pairs of \( u^* \) and \( k^* \) that minimize the insurer’s ultimate ruin probability, assuming reinsurance charges as \( Q(u,k) = 1.6E(R_{u,k}) \) and \( Q(u,k) = E(R_{u,k}) + 2St. Dev. (R_{u,k}) \) respectively. In addition, we divide the defective distribution functions by the corresponding ultimate
ruin probabilities so that they become proper distribution functions. For example we make comparisons between functions \( G(U, y)/\psi(U) \) and \( G_{u, k^*}(y)/\psi_k^*(u) \) as conditional distribution functions.

Example 4.2.1. Exponential claim sizes.

When the individual claim sizes are exponentially distributed, we know that the deficit at ruin is independent of the initial surplus level, due to the memoryless property of the exponential distributions. Hence, when we substitute equation (3.5.3) for \( G(U, y) \) into equation (4.2.1), we find that the conditional distribution function \( G(U, y)/\psi(U) = G_{u, k}(y)/\psi_k(u) \) for any \( U, u \) and \( k \). That is, when the claim sizes are exponentially distributed, the insertion of a lower barrier does not alter the conditional distribution function of the deficit at ruin.

Example 4.2.2. Erlang(2) claim sizes.

In this second example, we assume the individual claims have an Erlang(2) distribution with \( p(x) = 4xe^{-2x} \), as in Section 3.5.2, and assume that \( \theta_1 = 0.2 \). Using equations (3.5.4), (3.5.5) and (4.2.1), we obtain that the defective distribution function of the deficit at ruin under the lower barrier model is

\[
G_{u, k}(y) = 0.8518e^{-\rho_1(u-k)}(1 - e^{-2(k+y)}) - 0.5446(k + y)e^{-\rho_2(u-k)-2(k+y)} \\
-0.0185e^{-\rho_2(u-k)}(1 - e^{-2(k+y)}) - 0.2887(k + y)e^{-\rho_2(u-k)-2(k+y)} \\
\frac{1 - 0.8333(1 - e^{-2(k+y)}) + 0.8333(k + y)e^{-2(k+y)}}{1 - 0.8333(1 - e^{-2k}) + 0.8333ke^{-2k}} \\
\times [0.8518e^{-\rho_1(u-k)}(1 - e^{-2k}) - 0.5446ke^{-2k-\rho_1(u-k)} \\
-0.0185e^{-\rho_2(u-k)}(1 - e^{-2k}) - 0.2887ke^{-2k-\rho_2(u-k)}],
\]

where \( \rho_1 = 0.2268 \) and \( \rho_2 = 2.9399 \) are the absolute values of the two non-zero solutions of Lundberg’s fundamental equation (1.6.3) with \( \delta = 0 \). We choose the initial capital as
Chapter 4. Lower barrier model: infinite time

$U = 13$. This gives an ultimate ruin probability of 0.04465 under the classical risk model when the insurer’s premium loading is $\theta_1 = 0.2$. From Table 3.5.5, we have that when the reinsurance premium is $Q(u, k) = 1.6E(R_{u,k})$, the minimum value of the ultimate ruin probability is 0.00001 when $u^* = 9.12$ and $k^* = 7.19$. If the cost of reinsurance is $Q(u, k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})$, then the optimal $u$ is $u^* = 9.04$ with $k^* = 2.09$, which gives an ultimate ruin probability of $\psi_k^*(u) = 0.03875$. Using equations (3.5.4) and (3.5.5), we plot the conditional distribution function $G(U,y)/\psi(U)$ under the classical risk model, and $G_{u^*,k^*}(y)/\psi_k^*(u)$ for the two cases under the lower barrier model in Figure 4.2.1. The conditional probability density functions $g(U,y)/\psi(U)$ and $g_{u^*,k^*}(y)/\psi_k^*(u)$ are then plotted in Figure 4.2.2.

![Figure 4.2.1: Conditional Deficit at Ruin distribution functions, Erlang(2) claims](image-url)
4.2. The deficit at ruin

From Figure 4.2.1, we can see that the introduction of a lower barrier has pushed the original conditional deficit at ruin distribution function upwards. For example, given that ruin occurs, the probability that the deficit at ruin is smaller than 1 is $G(13, 1)/\psi(13) = 0.77814$. In the first case under the lower barrier model with $u^* = 9.12$ and $k^* = 7.19$, the probability that the deficit at ruin is smaller than 1 is $G_{9.12,7.19}(1)/\psi_{7.19}(9.12) = 0.84827$. In the second case described with $u^* = 9.04$ and $k^* = 2.09$, the probability is $G_{9.04,2.09}(1)/\psi_{2.09}(9.04) = 0.82197$. Both scenarios under the lower barrier model give higher probabilities than under the classical risk model. In other words, when the claim sizes are distributed as Erlang(2), given that ruin occurs, it is more likely that the deficit
is small in the lower barrier model than in the classical risk model. Also, since the barrier level is at 2.09 in the second case, we see that the effect of the lower barrier is reduced and the conditional distribution function is closer to that of the classical risk model.

From Figure 4.2.2, we see that under the classical risk model, the conditional probability density function starts from a lower level and ends with a fatter tail, compared to the densities under the lower barrier model.

Having the conditional distribution function and conditional density function of the deficit at ruin plotted, we also calculate the conditional expected value of the deficit at ruin. Instead of a single value of $U$, we use the results obtained in Chapter 3 Table 3.5.5, in which we have provided a series of initial capital levels and the corresponding optimal $u^*$ and $k^*$ under the two reinsurance premium assumptions of expected value principle and standard deviation principle respectively. Table 4.2.1 shows the expected value of the deficit at ruin, given ruin occurs.

| $U$ | $E[|U(T_U)||T_U < \infty]$ | $Q(u, k) = 1.6E(R_{u,k})$ | $Q(u, k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})$ |
|-----|----------------------------|-----------------------------|-----------------------------------------------|
| 7   | 0.65984                    | 4.22 2.22 0.57585           | 7 0 0.65984                                  |
| 9   | 0.65984                    | 5.35 3.38 0.55624           | 9 0 0.65984                                  |
| 11  | 0.65984                    | 7.14 5.20 0.53991           | 11 0 0.65984                                 |
| 13  | 0.65984                    | 9.12 7.19 0.53029           | 9.04 2.09 0.57888                            |
| 15  | 0.65984                    | 11.12 9.19 0.52438          | 8.49 3.17 0.55901                            |
| 17  | 0.65984                    | 13.12 11.19 0.52040         | 9.57 4.80 0.54263                            |

Table 4.2.1: Conditional Expected Deficit at Ruin, Erlang(2) claims

From Table 4.2.1, we see that the conditional expected deficit at ruin under the classical risk model is 0.65984 for all initial surplus levels shown. Under the lower barrier model, this is not the case. If reinsurance premium is $Q(u, k) = 1.6E(R_{u,k})$ and the initial capital is at $U = 7$, the optimal $u^*$ is 4.22 and $k^* = 2.22$. The conditional expected deficit at ruin is then 0.57585, smaller than that under the classical risk model. When initial capital increases, the corresponding optimal $u^*$ and $k^*$ vary and we see that the
expected deficit at ruin decreases. When \( U = 17 \), we see that the optimal \( u^* = 13.12 \) and \( k^* = 11.19 \) give a conditional expected deficit at ruin of 0.52040. Under the assumption that \( Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}) \), the cost of reinsurance is much higher such that the optimal lower barrier level \( k^* \) is comparatively smaller than in the first case. However, as soon as the reinsurance is affordable and the lower barrier comes into effect, the conditional expected deficit at ruin decreases.

Next, in Table 4.2.2 we provide the standard deviation of the conditional deficit at ruin to provide further information about the random variable.

<table>
<thead>
<tr>
<th>( U )</th>
<th>St.Dev</th>
<th>( Q(u,k) = 1.6E(R_{u,k}) )</th>
<th>( u^* )</th>
<th>( k^* )</th>
<th>St.Dev</th>
<th>( Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}) )</th>
<th>( u^* )</th>
<th>( k^* )</th>
<th>St.Dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.61991</td>
<td>4.22</td>
<td>2.22</td>
<td>0.56577 &amp; 7 &amp; 0</td>
<td>0.61991</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.61991</td>
<td>5.35</td>
<td>3.38</td>
<td>0.55052 &amp; 9 &amp; 0</td>
<td>0.61991</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.61991</td>
<td>7.14</td>
<td>5.20</td>
<td>0.53695 &amp; 11 &amp; 0</td>
<td>0.61991</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.61991</td>
<td>9.12</td>
<td>7.19</td>
<td>0.52856 &amp; 9.04 &amp; 2.09</td>
<td>0.56803</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.61991</td>
<td>11.12</td>
<td>9.19</td>
<td>0.52325 &amp; 8.49 &amp; 3.17</td>
<td>0.55275</td>
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<tr>
<td>17</td>
<td>0.61991</td>
<td>13.12</td>
<td>11.19</td>
<td>0.51960 &amp; 9.57 &amp; 4.80</td>
<td>0.53927</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2.2: Conditional Standard Deviation of Deficit at Ruin, Erlang(2) claims

From Table 4.2.2, we see that the standard deviation values exhibit a similar pattern as in Table 4.2.1 with the conditional expected values. Under the classical risk model, the standard deviation remains at 0.61991. Under the first scenario of the lower barrier model, we see the standard deviation start from 0.56577 when \( U = 7 \) with optimal \( u^* \) and \( k^* \), and decreases to 0.51960 when \( U = 17 \). Under the second scenario, the highest value of \( U = 17 \) in the table gives a standard deviation of 0.53927 for the conditional deficit at ruin. Under the lower barrier model, not only the expected values are lower than those under the classical risk model, the standard deviations are also smaller.
Example 4.2.3. Mixed exponential claim sizes.

In this example, we assume the individual claim sizes to be mixed exponential, exactly as in Section 3.5.3, and assume that $\theta_1 = 0.2$. Combining equations (3.5.6), (3.5.7) and (4.2.1), we have

\[
G_{u,k}(y) = \left[0.0752(1 - e^{-2(k+y)}) + 0.7239(1 - e^{-0.5(k+y)})\right] e^{-\rho_1(u-k)} \\
+ \left[0.2026(1 - e^{-2(k+y)}) - 0.1683(1 - e^{-0.5(k+y)})\right] e^{-\rho_2(u-k)} \\
+ \frac{1 - 0.2778(1 - e^{-2(k+y)}) - 0.5556(1 - e^{-0.5(k+y)})}{1 - 0.2778(1 - e^{-2k}) - 0.5556(1 - e^{-0.5k})} \\
\times \left[(0.0752(1 - e^{-2k}) + 0.7239(1 - e^{-0.5k}))e^{-\rho_1(u-k)}ight] \\
+(0.2026(1 - e^{-2k}) - 0.1683(1 - e^{-0.5k}))e^{-\rho_2(u-k)}
\]

where $\rho_1 = 0.10685$ and $\rho_2 = 1.55982$ are the absolute values of the two non-zero solutions of Lundberg’s fundamental equation (1.6.3) with $\delta = 0$. The initial capital level is set at $U = 25$ so that under the classical risk model, the ultimate ruin probability is $\psi(25) = 0.05527$, given that the insurer’s premium loading is $0.2$. From Table 3.5.9, when $Q(u,k) = 1.6E(R_{u,k})$, the optimal $u$ is $u^* = 17.74$, with $k^* = 13.20$ giving $\psi_k^*(u) = 0.00283$. When $Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})$, the minimum ultimate ruin probability with a lower barrier is $\psi_k^*(u) = 0.05505$ with $u^* = 23.31$ and $k^* = 2.36$. In Figure 4.2.3 and Figure 4.2.4, we plot the conditional distribution function and conditional probability density function of the deficit at ruin for the classical risk model and the two cases under the lower barrier model, respectively.

Interestingly, the results under the mixed exponential claim sizes are the opposite of what we found in the Erlang(2) claims case. Compared to Figure 4.2.1 in the previ-
Figure 4.2.3: Conditional Deficit at Ruin distribution functions, Mixed exponential claims

ous example where the conditional distribution function of the deficit at ruin has been pushed up by introducing the lower barrier, in the mixed exponential case the conditional distribution function of the deficit at ruin has been pushed downwards. That is, under the mixed exponential claim size assumption, providing that ruin occurs, the deficit at ruin under the lower barrier model is more likely to be large than under the classical risk model. To see this numerically, we look at the conditional probability that the deficit at ruin is smaller than 4 given that ruin occurs. Under the classical risk model with initial surplus $U = 25$, this conditional probability is $G(25, 4)/\psi(25) = 0.87736$. Under the first reinsurance case, this probability is $G_{17.74,13.20}(4)/\psi_{13.20}(17.74) = 0.86490$ which is
lower than 0.87736 under the classical risk model. Under the second reinsurance case it is \( G_{23.31.2.36}(4)/\psi_{2.36}(23.31) = 0.86594 \), also smaller. In Figure 4.2.4 where we plot the conditional density function, we see that the lower barrier model provides a fatter tailed density for the conditional deficit at ruin than under the classical risk model. When we compare Figure 4.2.1 and Figure 4.2.3, we also find that the variation in the level of lower barrier \( k \), e.g. from \( k = 13.20 \) to \( k = 2.36 \), under the mixed exponential claims does not alter the effect on the conditional deficit at ruin much.

We now calculate the conditional expected deficit at ruin for a series of initial capital levels, as in Chapter 3 Table 3.5.9, in Table 4.2.3. In addition, in Table 4.2.4, we also
calculate the standard deviation of the deficit of ruin, with the given values of $U$, $u^*$ and $k^*$.

| $U$ | $E[[U(T_U)||T_U < \infty]]$ | $Q(u,k) = 1.6E(R_{u,k})$ | $Q(u,k) = E(R_{u,k}) + 2St.Dev.(R_{u,k})$ |
|-----|-------------------------------|--------------------------|----------------------------------|
| 15  | 1.85890                        | 10.17                     | 5.62                             | 1.99988                          | 15  | 0                           | 1.85890                          |
| 17  | 1.85890                        | 11.30                     | 6.75                             | 2.00000                          | 17  | 0                           | 1.85890                          |
| 19  | 1.85890                        | 12.62                     | 8.07                             | 2.00004                          | 19  | 0                           | 1.85890                          |
| 21  | 1.85890                        | 14.15                     | 9.60                             | 2.00009                          | 21  | 0                           | 1.85890                          |
| 23  | 1.85890                        | 15.88                     | 11.33                            | 2.00021                          | 23  | 0                           | 1.85890                          |
| 25  | 1.85890                        | 17.74                     | 13.20                            | 2.00055                          | 23.31| 2.36                        | 1.98587                          |
| 27  | 1.85890                        | 19.68                     | 15.13                            | 2.00144                          | 22.12| 4.31                        | 1.99911                          |
| 29  | 1.85890                        | 21.65                     | 17.11                            | 2.00387                          | 21.50| 5.70                        | 1.99990                          |

Table 4.2.3: Conditional Expected Deficit at Ruin, Mixed exponential claims

<table>
<thead>
<tr>
<th>$U$</th>
<th>$St.Dev$</th>
<th>$Q(u,k) = 1.6E(R_{u,k})$</th>
<th>$Q(u,k) = E(R_{u,k}) + 2St.Dev.(R_{u,k})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>1.95933</td>
<td>10.17</td>
<td>5.62</td>
</tr>
<tr>
<td>17</td>
<td>1.95933</td>
<td>11.30</td>
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<td>12.62</td>
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<tr>
<td>29</td>
<td>1.95933</td>
<td>21.65</td>
<td>17.11</td>
</tr>
</tbody>
</table>

Table 4.2.4: Conditional Standard Deviation of Deficit at Ruin, Mixed exponential claims

From Table 4.2.3, we see that the conditional expected deficit at ruin under the classical risk model is 1.85890 for all $U$ shown. The standard deviation also remains constant at 1.95933 for all $U$. Under the lower barrier model, all the expected deficit at ruin values with optimal choices of $u^*$ and $k^*$ are greater than under the classical risk models, if $k^*$ is greater than 0. The standard deviations are also greater. This is consistent with the results we saw in Figure 4.2.3 and Figure 4.2.4. When $U = 15$, the optimal $u^*$ is 10.17 with $k^* = 5.62$ under the first lower barrier case. The expected deficit at ruin is 1.99988,
with a standard deviation value at 1.99997. Instead of remaining constant as under the classical risk model, the expected deficit at ruin under the lower barrier model increases. When \( U = 29 \), \( E[|U(T_{u,k})| |T_{u,k} < \infty] \) increases to 2.00387 with optimal \( u^* \) and \( k^* \). The standard deviation values firstly increases to 2.00 when \( U \) increases to 19, and then remain constant for the rest of the \( U \) values.

### 4.3 The surplus before ruin

In this section, we look at the distribution of the surplus before ruin (in excess of \( k \)) under the lower barrier model. To derive the defective distribution function \( J_{u,k}(x) \) defined by equation (4.1.2), we need to use the defective joint density function, \( f(x, y|u) \), of the surplus before ruin \( (U(T_u^-)) \) and the deficit at ruin \( (|U(T_u)|) \) under the classical risk model. As in previous sections, we start by considering the simpler situation where the initial surplus \( u \) is the same as the lower barrier \( k \). By conditioning on whether ruin happens on the first drop below the barrier \( k \), we have

\[
J_{k,k}(x) = \int_{0}^{x} \int_{k}^{\infty} f(s, y|0)dyds + \sum_{n=1}^{\infty} G(0, k)^n \int_{0}^{x} \int_{k}^{\infty} f(s, y|0)dyds
\]

\[
= \frac{1}{1 - G(0, k)} \int_{0}^{x} \int_{k}^{\infty} f(s, y|0)dyds.
\]

We can see that

\[
\lim_{x \to \infty} J_{k,k}(x) = \frac{\psi(0) - G(0, k)}{1 - G(0, k)},
\]

which is consistent with the formula of the probability of ultimate ruin under the lower barrier model with \( u = k \).
For the general case where \( u > k \), we can see that

\[
J_{u,k}(x) = \int_0^x \int_k^\infty f(s, y|u-k)dy ds + G(u-k,k)J_{k,k}(x).
\] (4.3.1)

Recalling formula (1.4.1), we have that

\[
f(x, y|u) = \begin{cases} 
  f(x, y|0) \frac{1 - \psi(u)}{1 - \psi(0)}, & x > u \geq 0, \\
  f(x, y|0) \frac{\psi(u-x) - \psi(u)}{1 - \psi(0)}, & 0 < x \leq u.
\end{cases}
\] (4.3.2)

with \( f(x, y|0) = \lambda c^{-1} p(x + y) \). Substituting equation (4.3.2) into equation (4.3.1), we have the defective distribution function for the surplus before ruin in excess of the lower barrier \( k \) as

\[
J_{u,k}(x) = \begin{cases} 
  \frac{\lambda}{c(1-\psi(0))} \int_0^x [\psi(u-k-s) - \psi(u-k)] \tilde{P}(s+k) ds \\
  + \frac{\lambda G(u-k,k)}{c(1-G(0,k))} \int_0^x \tilde{P}(s+k) ds, & 0 < x \leq u - k, \\
  \frac{\lambda}{c(1-\psi(0))} \int_{u-k}^{u-k} [\psi(u-k-s) - \psi(u-k)] \tilde{P}(s+k) ds \\
  + \frac{\lambda[1-\psi(u-k)]}{c(1-\psi(0))} \int_{u-k}^x \tilde{P}(s+k) ds \\
  + \frac{\lambda G(u-k,k)}{c(1-G(0,k))} \int_0^x \tilde{P}(s+k) ds. & 0 \leq u - k < x.
\end{cases}
\] (4.3.3)

Differentiating this equation, we can then obtain the defective probability density function
Chapter 4. Lower barrier model: infinite time

\[ j_{u,k}(x) = \frac{d}{dx} J_{u,k}(x) \]

as

\[ j_{u,k}(x) = \begin{cases} \frac{\lambda}{c} \bar{P}(x + k) \left[ \frac{\psi(u-k-x) - \psi(u-k)}{1 - \psi(0)} + \frac{G(u-k,k)}{1 - G(0,k)} \right], & 0 < x < u - k, \\ \frac{\lambda}{c} \bar{P}(x + k) \left[ \frac{1 - \psi(u-k)}{1 - \psi(0)} + \frac{G(u-k,k)}{1 - G(0,k)} \right], & 0 < u - k < x. \end{cases} \]  

(4.3.4)

Once again, these functions are expressed in terms of functions that have been studied extensively in the literature, namely \( \psi(u) \) and \( G(u,y) \). Using these results, we now look at three examples of the surplus before ruin with different assumptions of individual claim sizes.

**Example 4.3.1. Exponential claim sizes.**

In this first example, we assume the claim sizes to be exponentially distributed with \( p(x) = e^{-x} \). The initial capital provided is assumed to be \( U = 17 \) and the insurer’s premium loading is 0.2. The ultimate ruin probability under the classical risk model is then \( \psi(17) = 0.04901 \). From Table 3.5.1, we have that under the assumption \( Q(u,k) = 1.6E(R_{u,k}) \), the optimal value of \( u \) is \( u^* = 12.01 \) with \( k^* = 9.19 \) giving a minimum ultimate ruin probability of \( \psi^*_k(u) = 0.00032 \). With \( Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}) \), \( u^* \) and \( k^* \) are found to be 13.39 and 2.53 respectively with \( \psi^*_k(u) = 0.04661 \). In Figure 4.3.1 we plot the conditional probability density functions \( j(U,x)/\psi(U) \) and \( j_{u^*,k^*}(x)/\psi^*_k(u) \) for the three given cases. In Figure 4.3.2, we plot the corresponding conditional distribution functions.

From Figure 4.3.1, we see that the lower barrier model has changed the shape of the density function of the surplus before ruin under the classical risk model. Under the classical risk model, the density function of the surplus before ruin firstly increases then decreases, whereas under the lower barrier model it decreases from 0. We already know
4.3. The surplus before ruin

that the density function of the surplus before ruin under the classical risk model has a jump at \( x = U \). Under the lower barrier model, this jump happens at \( x = u - k \). Hence, for the case where \( u^* = 12.01 \) and \( k^* = 9.19 \), we can see a clear jump at \( x = 2.82 \). For the second case under the lower barrier model, the jump happens at \( x = 10.83 \) which is barely visible in Figure 4.3.1. Using Figure 4.3.2, we can compare the probability that the surplus before ruin is below a certain level, given that ruin occurs. Under the classical risk model, the conditional probability that the surplus level before ruin \( U(T_u^-) \) is less than 2 is found to be \( J(17, 2)/\psi(17) = 0.54342 \). Under the first case of the lower barrier model, the surplus level before ruin in excess of \( k \) \( (U(T_{u,k}^-) - k) \) has a conditional probability of
Figure 4.3.2: Conditional Surplus before Ruin distribution functions, Exponential claims

\[ J_{12.01,9.19}(2) / \psi_{9.19}(12.01) = 0.81110, \] of being smaller than 2. This probability is found to be 0.78980 in the second reinsurance scenario. The results we obtain here are reasonable. Given that ruin occurs, the surplus level in excess of the barrier \( k \) is expected to be smaller than under the classical risk model where there is no barrier. However, when we compare the conditional distribution functions of the two cases under the lower barrier model, we find that although the optimal level of \( k^* \) decreases from 9.19 to 2.53, the change within the lower barrier model is small.

In addition to the conditional density function and distribution function for specific values of \( U, u \) and \( k \), we calculate the expected value of the surplus before ruin, provided
4.3. The surplus before ruin

that ruin occurs. We use the results in Chapter 3 Table 3.5.1, in which we have provided a set of values for initial capital $U$, the corresponding ultimate ruin probability under the classical risk model, the optimal choices of $u^*$ and $k^*$ under the expected value principle and the standard deviation principle for reinsurance premium, and the optimal ultimate ruin probability under the lower barrier model. We use the values obtained under the assumptions described therein, and provide the conditional expected value and standard deviation for the surplus before ruin in Table 4.3.1 and Table 4.3.2 respectively.

$Q(u,k) = 1.6$.

$E(R_{u,k}) = 1.6 E(R_{u,k}) + 2 St.Dev.(R_{u,k})$.

$E[U(T_U^\zeta < \infty)]$.

| $U$ | $E[U(T_U^\zeta ) | T_U^\zeta < \infty]$ | $Q(u,k) = 1.6 E(R_{u,k})$ | $Q(u,k) = E(R_{u,k}) + 2 St.Dev.(R_{u,k})$ |
|-----|-----------------------------------|---------------------------|-----------------------------------------------|
|     | $u^*$ | $k^*$ | $E[U(T_{u,k}^\zeta ) | T_{u,k}^\zeta < \infty]$ | $u^*$ | $k^*$ | $E[U(T_{u,k}^\zeta ) | T_{u,k}^\zeta < \infty]$ |
| 11  | 2.19989 | 6.83 | 4.01 | 1.19731 | 11 | 0 | 2.19989 |
| 13  | 2.19998 | 8.25 | 5.43 | 1.18488 | 13 | 0 | 2.19998 |
| 15  | 2.20000 | 10.05 | 7.23 | 1.18157 | 15 | 0 | 2.20000 |
| 17  | 2.20000 | 12.01 | 9.19 | 1.18100 | 13.39 | 2.53 | 1.27963 |
| 19  | 2.20000 | 14.00 | 11.18 | 1.18992 | 12.59 | 3.69 | 1.22483 |
| 21  | 2.20000 | 16.00 | 13.18 | 1.18091 | 12.65 | 4.94 | 1.20683 |

Table 4.3.1: Conditional Expected Surplus before Ruin, Exponential claims

<table>
<thead>
<tr>
<th>$U$</th>
<th>$St.Dev.$</th>
<th>$Q(u,k) = 1.6 E(R_{u,k})$</th>
<th>$Q(u,k) = E(R_{u,k}) + 2 St.Dev.(R_{u,k})$</th>
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<td>$k^*$</td>
<td>$St.Dev$</td>
</tr>
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<td>6.83</td>
<td>4.01</td>
</tr>
<tr>
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<td>8.25</td>
<td>5.43</td>
</tr>
<tr>
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<td>7.23</td>
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<td>9.19</td>
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<td>14.00</td>
<td>11.18</td>
</tr>
<tr>
<td>21</td>
<td>1.56205</td>
<td>16.00</td>
<td>13.18</td>
</tr>
</tbody>
</table>

Table 4.3.2: Conditional Standard Deviation of Surplus before Ruin, Exponential claims

From Table 4.3.1, we firstly see that with $U$ varying from 11 to 21, the conditional expected surplus before ruin under the classical risk model is around 2.20 and this value is stable for the levels of $U$ chosen. In Table 4.3.2, the standard deviation started from 1.56119 at $U = 11$ and slightly increased to 1.56205 when $U$ increased to 21. The increase
was not substantial and stabilized at 1.56205 when \( U = 19 \) and \( U = 21 \). When the
reinsurance premium is charged by the expected value principle with loading of 0.6, we
used the optimal level of \( u^* \) and \( k^* \) in the calculation processes. The expected surplus
before ruin, given that ruin occurs, is lower than under the classical risk model, and
decreases when \( U \) increases. Also note that under the lower barrier model, the expected
values are the amount in excess of the lower barrier. We see that although the lower
barrier \( k \) increases by a relatively large amount each time \( U \) increases, the conditional
expected surplus in excess of \( k \) before ruin does not alter that much. Correspondingly in
Table 4.3.2, we see that the standard deviation under the lower barrier model is much
lower than that under the classical risk model. When \( U = 11 \), the optimal \( u^* \) and \( k^* \) gives
a standard deviation of 1.14775 and unlike the classical risk model, the standard deviation
is displaying a decreasing trend as \( U \) increases. Under the reinsurance assumption that
\( Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}) \), for \( U = 11, 13, 15 \), the reinsurance premium is too
expensive. When \( U = 17 \), the optimal value \( u^* = 13.39 \), \( k^* = 2.53 \) gives a conditional
expected surplus before ruin of 1.27963, lower than 2.20 under the classical risk model. It
continues to decrease when \( U \) increases. At \( U = 17 \), the corresponding standard deviation
is 1.26182, also lower than the value of 1.56204 under the classical risk model. In addition,
we see that although the optimal level \( k^* \) is much lower in the second case than in the first
case of the lower barrier model, the conditional expected values do not differ much, which
is consistent with the results in Figure 4.3.1 and Figure 4.3.2.

Example 4.3.2. Erlang(2) claim sizes.

In this second example, the claim sizes are assumed to be distributed as Erlang(2) with
\( p(x) = 4xe^{-2x} \). The level of initial capital is chosen at \( U = 13 \). The insurer’s premium
loading and the optimal level of \( u^* \) and \( k^* \) are the same as in Example 4.2.2. The plots of
the conditional density functions and distribution functions are provided in Figure 4.3.3.
and Figure 4.3.4 respectively.

![Diagram showing conditional density of surplus before ruin](image)

**Figure 4.3.3: Conditional Density of Surplus before Ruin, Erlang(2) claims**

From Figure 4.3.3 and Figure 4.3.4, we see that the defective probability density function and distribution function of the surplus before ruin under Erlang(2) claims exhibit similar shapes and characteristics as under Exponential claims in Example 4.3.1. Under the classical risk model, the density increases initially, then decreases. Under the lower barrier model, the densities decrease from 0. Under the first case of the lower barrier model, the probability density function is discontinuous at $u^* - k^* = 1.93$. In Figure 4.3.3, we see the jump clearly. Compared with Figure 4.3.1, the jump is not as great as that is under the Exponential claims scenario. In Figure 4.3.4, we also find that the condi-
Figure 4.3.4: Conditional Surplus before Ruin distribution functions, Erlang(2) claims

ditional distribution function of the surplus before ruin is pushed up when a lower barrier is introduced. Even with a barrier level as low as 2.09, the lower barrier model still has a considerable effect on the conditional distribution function. Numerically, we can compare the conditional probability that the surplus before ruin is smaller than 1. Under the classical risk model, we found that this probability is \( J(13, 1)/\psi(13) = 0.37724 \). In the first case of the lower barrier model, it is \( J_{9.12, 7.19}(1)/\psi_{7.19}(9.12) = 0.80920 \), and in the second case, it is \( J_{9.04, 2.09}(1)/\psi_{2.09}(9.04) = 0.76591 \).

In Table 4.3.3 and Table 4.3.4 we look at the conditional expected value and standard deviation of the surplus before ruin respectively. As in the exponential claims case, under
4.3. The surplus before ruin

\[ Q(u,k) = 1.6E(R_{u,k}) \]

\[ Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k}) \]

\[ U\text{St.Dev} \]

| \( u^* \) | \( k^* \) | \( E[U(T_U^-)|T_U < \infty] \) | \( E[E(U(T_{u,k}^-)|T_{u,k} < \infty)] \) |
|---|---|---|---|
| 7 | 1.42713 | 4.22 | 2.22 | 0.67434 | 7 | 0 | 1.42713 |
| 9 | 1.42713 | 5.35 | 3.38 | 0.63592 | 9 | 0 | 1.42713 |
| 11 | 1.42713 | 7.14 | 5.20 | 0.61353 | 11 | 0 | 1.42713 |
| 13 | 1.42714 | 9.12 | 7.19 | 0.60140 | 9.04 | 2.09 | 0.68464 |
| 15 | 1.42714 | 11.12 | 9.19 | 0.59403 | 8.49 | 3.17 | 0.64189 |
| 17 | 1.42714 | 13.12 | 11.19 | 0.58906 | 9.57 | 4.80 | 0.61385 |

Table 4.3.3: Conditional Expected Surplus before Ruin, Erlang(2) claims

| \( u^* \) | \( k^* \) | \( E[U(T_U^-)|T_U < \infty] \) | \( E[E(U(T_{u,k}^-)|T_{u,k} < \infty)] \) |
|---|---|---|---|
| 7 | 0.90845 | 4.22 | 2.22 | 0.64698 | 7 | 0 | 0.90845 |
| 9 | 0.90849 | 5.35 | 3.38 | 0.61928 | 9 | 0 | 0.90849 |
| 11 | 0.90849 | 7.14 | 5.20 | 0.60181 | 11 | 0 | 0.90849 |
| 13 | 0.90850 | 9.12 | 7.19 | 0.59191 | 9.04 | 2.09 | 0.66522 |
| 15 | 0.90850 | 11.12 | 9.19 | 0.58574 | 8.49 | 3.17 | 0.63269 |
| 17 | 0.90850 | 13.12 | 11.19 | 0.58150 | 9.57 | 4.80 | 0.61342 |

Table 4.3.4: Conditional Standard Deviation of Surplus before Ruin, Erlang(2) claims

the classical risk model, \( E[U(T_U^-)|T_U < \infty] \) varies very little from 1.42713 to 1.42714 when \( U \) increases from 7 to 17. The standard deviation also varies little from 0.90845 to 0.90850. When \( k > 0 \) under the lower barrier model, both the conditional expectation and the standard deviation of the surplus before ruin are smaller than that under the classical risk model, and admit a decreasing pattern when \( U \) increases. Under the first lower barrier case, given initial capital \( U = 7 \), the optimal \( u^* = 4.22 \) and \( k^* = 2.22 \) give a conditional expectation \( E[U(T_{u^*,k^*}^-)|T_{u^*,k^*} < \infty] = 0.67434 \). The standard deviation is 0.64698. When \( U \) is at 17, the optimal \( u^* = 13.12 \) and \( k^* = 11.19 \) give a conditional expectation of 0.58906 with a standard deviation of 0.58150. For the same amount of capital provided, under the second case of the lower barrier model, the expected value is 0.61835, with \( u^* = 9.57 \) and \( k^* = 4.80 \), and the standard deviation is 0.61342.
Example 4.3.3. Mixed exponential claim sizes.

In our last example given in this section, we use the same assumption of mixed exponential claim sizes described in Example 4.2.3. Figure 4.3.5 illustrates the conditional probability density function of the surplus before ruin and Figure 4.3.6 provides the plots of the conditional distribution function. Then in Table 4.3.5, we show the conditional expected surplus before ruin for a series of values of $U$.

![Graph showing conditional density of surplus before ruin](image)

Figure 4.3.5: Conditional Density of Surplus before Ruin, Mixed exponential claims

Compared to the examples of exponential claims and Erlang(2) claims, from Figure 4.3.5 we see that when the claim sizes are mixed exponential, the defective density function of the surplus before ruin under the classical risk model has the same shape but is skewed...
4.3. The surplus before ruin

Figure 4.3.6: Conditional Surplus before Ruin distribution functions, Mixed exponential claims

more to the left. In the first case under the lower barrier model, the jump at \( u^* - k^* = 4.54 \) is slightly larger than in the case of exponential and Erlang(2) claims. We now compare the conditional probabilities that the surplus before ruin is less than 5. Under the classical risk model, given ruin occurs, the probability \( J(25, 5)/\psi(25) \) is found to be 0.67984. Under the first case of the lower barrier model, it is \( J_{17.74, 13.20}(5)/\psi_{13.20}(17.74) = 0.86881 \) and under the second case, the conditional probability is \( J_{23.31, 2.36}(5)/\psi_{2.36}(23.31) = 0.80189 \). We also see that the differences between the two cases under the lower barrier model are bigger, compared to the previous two examples.

From Table 4.3.5, the expected surplus before ruin, provided ruin occurs, increases
Chapter 4. Lower barrier model: infinite time

\[ Q(u,k) = 1.6 E(R_{u,k}) \]

\[ Q(u,k) = E(R_{u,k}) + 2 St.Dev.(R_{u,k}) \]

| \( u \) | \( E[U(T_u)|T_u < \infty] \) | \( k \) | \( Q(u,k) = 1.6 E(R_{u,k}) \) | \( Q(u,k) = E(R_{u,k}) + 2 St.Dev.(R_{u,k}) \) |
|---|---|---|---|---|
| 15 | 4.20599 | 10.17 | 5.62 | 2.53425 |
| 17 | 4.20983 | 11.30 | 6.75 | 2.49585 |
| 19 | 4.21138 | 12.62 | 8.07 | 2.47033 |
| 21 | 4.21225 | 14.15 | 9.60 | 2.45740 |
| 23 | 4.21259 | 15.88 | 11.33 | 2.45049 |
| 25 | 4.21273 | 17.74 | 13.20 | 2.44727 |
| 27 | 4.21280 | 19.68 | 15.13 | 2.44761 |
| 29 | 4.21284 | 21.65 | 17.11 | 2.44959 |

\( u^* \) | \( k^* \) | \( E[U(T_{u,k})|T_{u,k} < \infty] \) | \( u^* \) | \( k^* \) | \( E[U(T_{u,k})|T_{u,k} < \infty] \) |
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<td>21.65</td>
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<td>2.44959</td>
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Table 4.3.5: Conditional Expected Surplus before Ruin, Mixed exponential claims

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<th>( St.Dev )</th>
<th>( Q(u,k) = 1.6 E(R_{u,k}) )</th>
<th>( Q(u,k) = E(R_{u,k}) + 2 St.Dev.(R_{u,k}) )</th>
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Table 4.3.6: Conditional Standard Deviation of Surplus before Ruin, Mixed exponential claims

from 4.20599 to 4.21284 when \( U \) increases from 15 to 29. The corresponding standard deviation increases from 3.22190 to 3.25363. Although the changes in these values are not large, they are bigger compared to those we obtained in Example 4.3.1 and Example 4.3.2 under the classical risk model. Under the lower barrier model, the expected surplus before ruin again appears to be decreasing with \( U \) increasing, as well as the standard deviation. When \( U = 15 \) and \( Q(u,k) = 1.6 E(R_{u,k}) \), the optimal value of \( u^* \) is 10.17 with \( k^* = 5.62 \). The expected value \( E[U(T_{u,k})|T_{u,k} < \infty] \) is found to be 2.53425 and the standard deviation is found to be 2.53425. When \( U \) increases to 29, the optimal pair of \( u^* \) and \( k^* \) gives \( E[U(T_{u,k})|T_{u,k} < \infty] = 2.44959 \) with a standard deviation of 2.28423.
4.4 Concluding remarks

In this chapter, we examined two quantities of interest under the lower barrier model in infinite time, namely the deficit at ruin and the surplus before ruin.

In Section 4.2, we focussed on the deficit at ruin where we provided formulae for $G_{u,k}(y)$ in terms of functions defined under the classical risk model. We then gave three examples where we compared the differences between the classical risk model and the lower barrier model, assuming exponential claims, Erlang(2) claims and mixed exponential claims. We plotted the conditional distribution functions and probability density functions for each example, and calculated the conditional expected deficit at ruin as well as the standard deviations. From examples provided, we saw that how the lower barrier affects the deficit at ruin depends on the assumption of individual claim sizes.

In Section 4.3, we studied the surplus before ruin (in excess of the lower barrier $k$). Examples were also given by assuming different individual claim sizes. From the plots of the conditional density functions and distribution functions of the surplus before ruin, we saw the same pattern under the three examples. The expected surplus before ruin was then calculated. Under each scenario, provided that ruin occurs, the surplus in excess of $k$ before ruin under the lower barrier model is expected to be much smaller than the surplus before ruin under the classical risk model. It is reasonable since the surplus before ruin has to be greater than $k$ under the lower barrier model whereas under the classical risk model, the surplus before ruin does not have such a constraint.
Chapter 5

Lower barrier model: finite time

5.1 Introduction

In the previous two chapters, we have investigated the lower barrier model within an infinite time framework. In Chapter 3, we introduced the concept and the assumptions of the lower barrier model and we studied ultimate ruin probabilities. Then in Chapter 4, we derived formulae for the deficit at ruin and the surplus before ruin, under the settings of a lower barrier model. In both chapters we compared the results under both models for each quantity.

In this chapter, we focus on ruin related quantities in a finite time setting. The distribution of the time to ruin will firstly be investigated, followed by studies on the joint distribution of the time to ruin and the deficit at ruin.
5.2 On the time to ruin

Firstly, we look at the distribution of the time to ruin. As introduced in Chapter 1, the study of the probability density function of the time to ruin leads to the finite time ruin probability.

Define $W_{u,k}(t)$ to be the defective distribution function of $T_{u,k}$, the time to ruin for the process with the lower barrier, with initial surplus $u$ and lower barrier $k$, i.e.

$$W_{u,k}(t) = \Pr[T_{u,k} \leq t],$$

and define $w_{u,k}(t) = \frac{d}{dt} W_{u,k}(t)$ to be the defective probability density function. We now derive a formula for $W_{u,k}(t)$.

We start from the easier case where $u$ is the same as $k$. The probability that ruin occurs on the $n^{th}$ occasion, $n = 1, 2, 3, \ldots$, that the surplus drops below $k$ is

$$G(0,k)^{n-1}(\psi(0) - G(0,k)),$$

where $G(u, \cdot)$ is the defective distribution function of the deficit at ruin under the classical risk model with initial surplus $u$. If ruin occurs on the $n^{th}$ occasion that the surplus drops below $k$, then the ruin time is the sum of $n$ random variables. The first $n - 1$ random variables are independent and identically distributed, which we denote by $\tau$, with distribution function $D$ such that

$$D(t) = \Pr[\tau \leq t] = \Pr[T_0 \leq t \mid T_0 < \infty \text{ and } Y \leq k],$$

where $T_0$ is the time to the first drop below the initial surplus level under the classical
risk model, and $Y$ is the amount of the drop. Thus,

$$D(t) = \frac{\Pr[T_0 \leq t \text{ and } Y \leq k]}{\Pr[T_0 < \infty \text{ and } Y \leq k]}.$$  \hfill (5.2.1)

The waiting time between the $(n-1)^{th}$ and $n^{th}$ drop, denoted by $\tau_L$, has a distribution function $D_L$ given by

$$D_L(t) = \Pr[\tau_L \leq t] = \frac{\Pr[T_0 \leq t \mid T_0 < \infty \text{ and } Y > k]}{\Pr[T_0 < \infty \text{ and } Y > k]}.$$  \hfill (5.2.2)

Under the classical risk model in Chapter 1, we have defined $\psi(u,t)$ to be the probability of ruin before time $t$ from initial surplus $u$. We also defined the function $W_u(y,t)$ to be the joint distribution function of the deficit at ruin and the time to ruin given initial surplus $u$. Let $w_u(y,t) = \frac{\partial^2}{\partial y \partial t} W_u(y,t)$ be the corresponding joint density function. Using these functions, equations (5.2.1) and (5.2.2) can then be expressed as

$$D(t) = \frac{W_0(k,t)}{G(0,k)}$$ \hfill (5.2.3)

and

$$D_L(t) = \frac{\psi(0,t) - W_0(k,t)}{\psi(0) - G(0,k)}.$$ \hfill (5.2.4)

Thus

$$W_{k,k}(t) = \Pr[T_{k,k} \leq t] = \sum_{n=1}^{\infty} G(0,k)^{n-1}(\psi(0) - G(0,k))D^{(n-1)} * D_L(t),$$ \hfill (5.2.5)
where $D^{(n-1)} \ast D(t)$ represents the $(n-1)$ fold convolution of the function $D$ convoluted with the function $D_L$. Note that both $D$ and $D_L$ are proper distribution functions, so letting $t \to \infty$ we get

$$\Pr[T_{k,k} < \infty] = \sum_{n=1}^{\infty} G(0,k)^{n-1}(\psi(0) - G(0,k)) = \frac{\psi(0) - G(0,k)}{1 - G(0,k)}$$

which is the ultimate ruin probability $\psi_k(k)$ for the lower barrier model in equation (3.2.2).

We now look at the general case where $u > k$ under the lower barrier model. When $u > k$, there are two possible situations where ruin can happen. The first possible situation is that ruin happens on the first time the surplus process drops below level $k$. The second case is that ruin does not happen on the first time the surplus process drops below level $k$, and the surplus restarts from $k$ and ruin subsequently occurs. Figure 5.2.1 illustrates the two situations described.

Figure 5.2.1: Ruin in the Lower Barrier Model

In Figure 5.2.1a, ruin happens when the surplus drops below the lower barrier $k$ for
the first time. The probability of this happening is \( \psi(u - k) - G(u - k, k) \). Let \( \tau_1 \) be the waiting time for such an event to happen. Let \( D_1(t) \) be the distribution function of \( \tau_1 \) and hence

\[
D_1(t) = \Pr[\tau_1 \leq t] = \frac{\Pr[T_{u-k} \leq t \text{ and } Y > k]}{\Pr[T_{u-k} < \infty \text{ and } Y > k]} = \frac{\psi(u - k, t) - W_{u-k}(k, t)}{\psi(u - k) - G(u - k, k)}.
\]  

(5.2.6)

In Figure 5.2.1b, we show the second possible case where ruin does not occur the first time the surplus drops below \( k \). The probability of this happening is \( G(u - k, k) \). Let \( \tau_2 \) be the waiting time until the surplus first drops below \( k \) without ruin occurring, and let \( D_2(t) \) be the distribution function of \( \tau_2 \). Then,

\[
D_2(t) = \Pr[\tau_2 \leq t] = \frac{\Pr[T_{u-k} \leq t \text{ and } Y \leq k]}{\Pr[T_{u-k} < \infty \text{ and } Y \leq k]} = \frac{W_{u-k}(k, t)}{G(u - k, k)}.
\]  

(5.2.7)

Combining the two cases discussed above, we have

\[
W_{u,k}(t) = \Pr[T_{u,k} \leq t] = [\psi(u - k) - G(u - k, k)]D_1(t) + G(u - k, k)D_2 * W_{k,k}(t),
\]  

(5.2.8)

where \( D_2 * W_{k,k}(t) \) represents the convolution of the functions \( D_2 \) and \( W_{k,k} \).

For comparison, we also consider the conditional expected ruin times in the classical
risk model and the lower barrier model. Let $T_u^c$ and $T_{u,k}^c$ be the conditional time to ruin under the classical risk process with initial surplus $u$, and under the lower barrier model with initial surplus $u$ and lower barrier $k$ respectively, conditioning on the event that ruin occurs. First of all, notice that the functions $D$, $D_L$, $D_1$ and $D_2$ represented in equations (5.2.3), (5.2.4), (5.2.6), (5.2.6) are proper distribution functions and hence from equation (5.2.5), we have that

\[ E(T_{c,k}^c) = \sum_{n=1}^{\infty} G(0,k)G(0,k)^{(1-G(0,k))} \left[ (n-1)E(\tau) + E(\tau_L) \right] \]

\[ = E(\tau) \sum_{n=1}^{\infty} nG(0,k)G(0,k)^{(1-G(0,k))} + (E(\tau_L) - E(\tau)) \sum_{n=0}^{\infty} G(0,k)G(0,k)^{(1-G(0,k))} \]

\[ = \frac{G(0,k)}{1-G(0,k)} E(\tau) + E(\tau_L). \quad (5.2.9) \]

From equation (5.2.8), we have that

\[ E(T_{u,k}^c) = \frac{\psi(u-k) - G(u-k,k)}{\psi_k(u)} E(\tau_1) \]

\[ + \frac{G(u-k,k)}{\psi_k(u)} \sum_{n=1}^{\infty} G(0,k)G(0,k)^{(1-G(0,k))} \left[ E(\tau_2) + (n-1)E(\tau) + E(\tau_L) \right] \]

\[ = \frac{\psi(u-k) - G(u-k,k)}{\psi_k(u)} E(\tau_1) + \frac{G(u-k,k)\psi(0) - G(0,k)}{\psi_k(u)} \frac{1-G(0,k)}{E(T_{c,k}^c)} \left[ E(\tau_2) + E(T_{c,k}^c) \right] \]

\[ = \frac{\psi(u-k) - G(u-k,k)}{\psi_k(u)} E(\tau_1) + \frac{G(u-k,k)}{\psi_k(u)} \psi_k(k) E(\tau_2) + E(T_{c,k}^c). \quad (5.2.10) \]

From equations (5.2.6) and (5.2.7), we have the expectations of the random variables $\tau_1$
and $\tau_2$ are

$$E(\tau_1) = \frac{E(T_{u-k}) - \int_0^\infty \frac{t}{\psi(u-k)} W_{u-k}(k,t) dt}{\psi(u-k) - G(u-k,k)}$$

and

$$E(\tau_2) = \frac{\int_0^\infty t \frac{\partial}{\partial t} W_{u-k}(k,t) dt}{G(u-k,k)},$$

respectively. We therefore have,

$$E(T_{u,k}^c) = \frac{E(T_{u-k}) + G(u-k,k) \psi_k(k)}{\psi_k(u)} E(T_{k,k}^c) - \frac{\psi_k(k)}{\psi_k(u)} \int_0^\infty t \frac{\partial}{\partial t} W_{u-k}(k,t) dt.$$

(5.2.11)

Alternatively, we could use the Laplace transforms to obtain higher moments for the conditional time to ruin. From equation (5.2.5), we have that the Laplace transform of $w_{k,k}(t)$ is

$$\tilde{w}_{k,k}(s) = \sum_{n=1}^{\infty} G(0,k)^{n-1} (\psi(0) - G(0,k)) (\tilde{d}(s))^{n-1} \tilde{d}_L(s),$$

(5.2.12)

where $\tilde{d}$ and $\tilde{d}_L$ are Laplace transforms of functions $d$ and $d_L$. Also from equation (5.2.8), we have

$$\tilde{w}_{u,k}(s) = (\psi(u-k) - G(u-k,k)) \tilde{d}_1(s) + G(u-k,k) \tilde{d}_2(s) \tilde{w}_{k,k}(s),$$

(5.2.13)

where $\tilde{d}_1$ and $\tilde{d}_2$ are Laplace transforms of functions $d_1$ and $d_2$. The $i^{th}$ moment of $T_{u,k}^c$ is then

$$E(T_{u,k}^c)^i = \frac{(-1)^i \frac{d^i}{ds^i} \tilde{w}_{u,k}(s)}{\psi_k(u)} \bigg|_{s=0}.$$

(5.2.14)
From equation (5.2.8), we see that the defective distribution function of the time to
ruin under the lower barrier model is expressed in terms of functions that already have
been studied under the classical risk model. In the next two sections, we look at two
elements in which we compare the ruin time distributions between the two models, given
that individual claim sizes follow Exponential or Erlang(2) distributions. We will see in
these examples that the calculation of the final term in equation (5.2.11) does not pose a
problem.

5.2.1 Exponential claim sizes

In this section, the individual claim sizes are assumed to be exponentially distributed
with distribution function \( P(x) = 1 - e^{-\alpha x} \). This is a simpler case compared with other
claim size distributions, since most of the ruin related quantities have explicit solutions
that are ready for our use.

First of all, it is a well known result that the defective joint distribution function
\( W_u(y, t) \) under the classical risk model can be expressed as

\[
W_u(y, t) = \psi(u, t)(1 - e^{-\alpha y}),
\]

(5.2.15)
due to the memoryless property of the exponential distribution (see, for example, Gerber
(1979)). This gives

\[
W_0(k, t) = \psi(0, t)(1 - e^{-\alpha k}),
\]

and hence

\[
D(t) = \frac{\psi(0, t)}{\psi(0)},
\]
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\[ D_L(t) = \frac{\psi(0, t)(1 - (1 - e^{-ak}))}{\psi(0)(1 - (1 - e^{-ak}))} = D(t). \]

Also, from equation (3.5.3) for \( G(u, y) \), we have

\[ G(0, k)^{n-1}(\psi(0) - G(0, k)) = \psi(0)^n(1 - e^{-ak})^{n-1} e^{-ak}. \]

Hence, we can then derive the defective distribution function \( W_{k,k}(t) \) as

\[
W_{k,k}(t) = \sum_{n=1}^{\infty} \psi(0)^n(1 - e^{-ak})^{n-1} e^{-ak} D^{ns}(t) \\
= \sum_{n=1}^{\infty} (1 - e^{-ak})^{n-1} e^{-ak} \psi^{ns}(0, t), \tag{5.2.16}
\]

where \( D^{ns} \) and \( \psi^{ns} \) represent the \( n \)-fold convolution of the functions \( D \) and \( \psi \) respectively.

The defective probability density function of the time to ruin is then obtained by differentiation which gives

\[
w_{k,k}(t) = \sum_{n=1}^{\infty} (1 - e^{-ak})^{n-1} e^{-ak} w_u^{ns}(t), \tag{5.2.17}
\]

where \( w_u(t) = \frac{d}{du} \psi(u, t) \) is the defective probability density function of the time to ruin under the classical risk model with initial surplus level \( u \), and \( w_u^{ns} \) is the \( n \)-fold convolution of \( w_u \) with itself.

For the general case when \( u > k \), from equations (5.2.6), (5.2.7) and (5.2.15), we have that

\[ D_1(t) = D_2(t) = \frac{\psi(u - k, t)}{\psi(u - k)}. \]

Hence, using equations (5.2.8) and (5.2.15), and by differentiating \( W_{u,k}(t) \) with respect
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To $t$, we obtain the defective probability density function of the time to ruin as

$$w_{u,k}(t) = e^{-\alpha k}w_{u-k}(t) + \sum_{n=1}^{\infty} (1 - e^{-\alpha k})^n e^{-\alpha k} w_0^{n*} * w_{u-k}(t),$$  \hspace{1cm} (5.2.18)

where $w_0^{n*} * w_{u-k}(t)$ is the convolution of functions $w_{u,k}(t)$ and $w_0^{n*}(t)$.

From Dickson and Li (2010), we have that the defective probability density function of the time to ruin with exponential claim sizes under any assumption for inter-claim times is

$$w_u(t) = \sum_{j=1}^{\infty} w_0^{j*}(t) \frac{(\alpha u)^{j-1} e^{-\alpha u}}{\Gamma(j)}.$$  \hspace{1cm} (5.2.19)

This then gives the Laplace transform of $w_0^{n*} * w_{u-k}(t)$ in equation (5.2.18) as

$$[\tilde{w}_0(\delta)]^n \tilde{w}_{u-k}(\delta) = [\tilde{w}_0(\delta)]^n \sum_{j=1}^{\infty} [\tilde{w}_0(\delta)]^j \frac{(\alpha u)^{j-1} e^{-\alpha u}}{\Gamma(j)}$$

$$= \sum_{j=1}^{\infty} [\tilde{w}_0(\delta)]^{n+j} \frac{(\alpha u)^{j-1} e^{-\alpha u}}{\Gamma(j)},$$

which gives

$$w_0^{n*} * w_{u-k}(t) = \sum_{j=1}^{\infty} w_0^{(n+j)*}(t) \frac{(\alpha u)^{j-1} e^{-\alpha u}}{\Gamma(j)}.$$  \hspace{1cm} (5.2.20)

We now look at the $n$-fold convolution of $w_0(t)$. From Drekic and Willmot (2003),

$$w_0(t) = \frac{e^{-(\lambda + \alpha a)t}}{2 \alpha \lambda} (2\sqrt{\alpha \lambda}) I_1(2t \sqrt{\alpha \lambda}),$$

where

$$I_\nu(t) = \sum_{k=0}^{\infty} \frac{(t/2)^{2k+\nu}}{k!(k+\nu)!}.$$
is a modified Bessel function of order \( \nu \). By rearranging the function \( w_0(t) \), we have

\[
\begin{align*}
w_0(t) &= e^{-(\lambda + c\alpha)t} \sqrt{c\alpha \lambda} \sum_{k=0}^{\infty} \frac{(t\sqrt{c\alpha\lambda})^{2k+1}}{k!(k+1)!} \\
&= \sum_{k=0}^{\infty} e^{-(\lambda + c\alpha)t} \frac{1}{k!(k+1)!} \lambda^{k+1} (c\alpha)^k \\
&= \sum_{k=0}^{\infty} \frac{(\lambda + c\alpha)e^{-(\lambda + c\alpha)t} [(\lambda + c\alpha)t]^{2k}}{(2k)!} \frac{(2k)!}{k!(k+1)!} \left( \frac{\lambda}{\lambda + c\alpha} \right)^{k+1} \left( \frac{c\alpha}{\lambda + c\alpha} \right)^k e^{2k+1,\lambda+c\alpha}(t) \\
&= \sum_{k=0}^{\infty} q_k e^{2k+1,\lambda+c\alpha}(t)
\end{align*}
\]

where

\[
q_k = \frac{1}{2k+1} \binom{2k+1}{k} \left( \frac{\lambda}{\lambda + c\alpha} \right)^{k+1} \left( \frac{c\alpha}{\lambda + c\alpha} \right)^k.
\]

This suggests that \( w_0(t) \) could be a defective infinite mixture of Erlang densities, but we still need to check the property of \( \{q_k\}_{k=0}^{\infty} \). The key results we use here is the generalized binomial function \( B_t(z) \) defined as

\[
B_t(z) = \sum_{k=0}^{\infty} \binom{tk + 1}{k} \frac{z^k}{tk + 1},
\]

which according to Graham et al. (1994) has the property that

\[
B_t(z)^r = \sum_{k=0}^{\infty} \binom{tk + r}{k} \frac{z^{kr}}{tk + r}.
\]
From Graham et al. (1994), we also have that
\[
B_2(z) = \sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{z^k}{2k+1} = \frac{1 - \sqrt{1 - 4z}}{2z},
\]
(5.2.25)
and hence from equation (5.2.22) we have
\[
\sum_{k=0}^{\infty} q_k = \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k+1}{k} \left( \frac{\lambda}{\lambda + c\alpha} \right)^{k+1} \left( \frac{c\alpha}{\lambda + c\alpha} \right)^k = \frac{\lambda}{\lambda + c\alpha} B_2(z),
\]
where \( z = \frac{\lambda c\alpha}{(\lambda + c\alpha)^2} \), and therefore
\[
\sum_{k=0}^{\infty} q_k = \frac{\lambda}{\lambda + c\alpha} \frac{1 - \sqrt{1 - 4z}}{2z} = \frac{\lambda + c\alpha}{2c\alpha} \left( 1 - \frac{c\alpha - \lambda}{\lambda + c\alpha} \right) = \frac{\lambda}{\alpha c} = \psi(0),
\]
given that \( c\alpha > \lambda \). Since \( q_k \) is non-negative for \( k = 0, 1, 2, \ldots \), \( \{q_k\}_{k=0}^{\infty} \) is a defective discrete probability measure. This means that the function \( w_0(t) \) has a defective mixed Erlang structure. Hence, the Laplace transform of \( w_0(t) \) is
\[
\tilde{w}_0(s) = \sum_{k=0}^{\infty} q_k \left( \frac{\beta}{\beta + s} \right)^{2k+1},
\]
where \( \beta = \lambda + c\alpha \) is the scale parameter.
We can rewrite this as

\[
\tilde{w}_0(s) = \sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{2k+1} \left( \frac{\lambda}{\lambda + c\alpha} \right)^{k+1} \left( \frac{c\alpha}{\lambda + c\alpha} \right)^k \left( \frac{\beta}{\beta + s} \right)^{2k+1}
\]

where \( z^* = \left( \frac{\lambda}{\lambda + c\alpha} \right) \left( \frac{c\alpha}{\lambda + c\alpha} \right)^2 \). Using equation (5.2.24), we can obtain the Laplace transform of the \( n \)-fold convolution function, \( \tilde{w}_0^n(s) \), denoted as \( \tilde{w}_0^n(s) \) as

\[
\tilde{w}_0^n(s) = \left( \frac{\lambda}{\lambda + c\alpha} \right)^n \left( \frac{\beta}{\beta + s} \right)^n B_2(z^*)
\]

\[
= \left( \frac{\lambda}{\lambda + c\alpha} \right)^n \left( \frac{\beta}{\beta + s} \right)^n \sum_{k=0}^{\infty} \binom{2k+n}{k} \frac{n}{2k+n} (z^*)^k
\]

\[
= \sum_{k=0}^{\infty} \binom{2k+n}{k} \frac{n}{2k+n} \left( \frac{\lambda}{\lambda + c\alpha} \right)^{k+n} \left( \frac{c\alpha}{\lambda + c\alpha} \right)^k \left( \frac{\beta}{\beta + s} \right)^{2k+n}.
\]  

Hence, by inverting equation (5.2.27), we have

\[
w_0^n(t) = \sum_{k=0}^{\infty} \binom{2k+n}{k} \frac{n}{2k+n} \left( \frac{\lambda}{\lambda + c\alpha} \right)^{k+n} \left( \frac{c\alpha}{\lambda + c\alpha} \right)^k e^{2k+n,\lambda+c\alpha}(t)
\]

\[
= \lambda^n t^{n-1} n \sum_{m=0}^{\infty} \frac{(\alpha c\lambda t^2)^m}{m!(n+m)!}.
\]
For computational purposes, we can rewrite this as

\[
\omega_0^n(t) = \frac{\lambda^n t^{n-1} e^{-(\lambda + \alpha c)t}}{\Gamma(n)} {}_0F_1 \left( n + 1; \alpha c \lambda t^2 \right),
\]

where

\[
_pF_q(B_1, B_2, \ldots, B_p, C_1, C_2, \ldots, C_q ; Z) = \sum_{m=0}^{\infty} \frac{(B_1)_m (B_2)_m \ldots (B_p)_m}{(C_1)_m (C_2)_m \ldots (C_p)_m} \frac{Z^m}{m!}
\]

is the generalised hypergeometric function and \((a)_n = \Gamma(a + n)/\Gamma(a)\) is Pochhammer’s symbol.

Using equations (5.2.18), (5.2.20) and (5.2.28), we can then obtain the defective p.d.f. \(\omega_{u,k}(t)\). We now look at some numerical examples illustrating the effects of the lower barrier model with exponential claim sizes.

Example 5.2.1. Distribution of time to ruin

In this example, we plot the conditional density functions and distribution functions of the time to ruin under the classical risk model and the lower barrier model. The individual claim sizes are exponentially distributed with parameter \(\alpha = 1\) such that \(p(x) = e^{-x}\). We set \(U\), the initial capital level, at 17. The insurer’s premium loading is 0.2 such that \(c = 1.2\). From the results in Chapter 3, we have that the optimal \(u^* = 12.01\) and \(k^* = 9.19\) under the reinsurance assumption \(Q(u,k) = 1.6E(R_{u,k})\). When \(Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})\), the optimal level of \(u\) is \(u^* = 13.39\) with \(k^* = 2.53\). In Figure 5.2.2, we plot the conditional density function \(w_{17}(t)/\psi(17)\) under the classical risk model, together with the conditional density functions \(w_{12.01,9.19}(t)/\psi_{9.19}(12.01)\) and \(w_{13.39,2.53}(t)/\psi_{2.53}(13.39)\) under the lower barrier model. In Figure 5.2.3, we plot the corresponding conditional distribution functions.
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From Figure 5.2.2, we see that under the lower barrier model, the conditional density function of the time to ruin is more skewed to the left compared to that under the classical risk model. Under the first case of the lower barrier model, the lower barrier $k$ is at a higher level than the second case. Consequently, we see that the density of the time to ruin under the first case is more skewed to the left. In Figure 5.2.3, we see that the distribution function of the time to ruin, given that ruin occurs, is pushed upwards by introducing the lower barrier $k$. Under the classical risk model, the probability that ruin occurs before time 50, should ruin occur, is $\psi(17, 50)/\psi(17) = 0.45854$. Under the first case of the lower barrier model, this probability is $W_{12.01, 9.19}(50)/\psi_{9.19}(12.01) = 0.74501$ and under
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Figure 5.2.3: Conditional Time to Ruin distribution functions, Exponential claims

the second case it is $W_{13.39, 2.53}(50)/\psi_{2.53}(13.39) = 0.54453$.

Example 5.2.2. Conditional mean and standard deviation of time to ruin

In this example, we are interested in the changes in the mean and standard deviation of conditional time to ruin under the lower barrier model compared to the results under the classical risk model. We firstly look at the conditional expected time to ruin. Since $D(t) = D_L(t) = \psi(0, t)/\psi(0)$, which is effectively the distribution function of $T_0^c$ under the classical risk model, we hence have that

$$E(\tau) = E(\tau_L) = E(T_0^c).$$
Also, by substituting \( \psi(u,t) \) and \( G(u,y) \) into equations (5.2.6) and (5.2.7), we find that \( D_1(t) = D_2(t) = \psi(u-k,t)/\psi(u-k) \), which is the distribution function of \( T_{u-k}^c \). Therefore, the expected values \( E(\tau_1) \) and \( E(\tau_2) \) are equivalent to \( E(T_{u-k}^c) \). Inserting these results into equation (5.2.9) and (5.2.10), we obtain

\[
E(T_{k,k}^c) = \frac{E(T_0^c)}{1-G(0,k)},
\]

and

\[
E(T_{u,k}^c) = E(T_{u-k}^c) + \frac{G(u-k,k)\psi_k(k)}{\psi_k(u)} \frac{E(T_0^c)}{1-G(0,k)}.
\]

To obtain the second moment, we have

\[
\tilde{d}(s) = \tilde{d}_L(s) = \frac{\tilde{w}_0(s)}{\psi(0)},
\]

and

\[
\tilde{d}_1(s) = \tilde{d}_2(s) = \frac{\tilde{w}_{u-k}(s)}{\psi(u-k)}.
\]

When \( u = k \) under the lower barrier model, we have from equation (5.2.12)

\[
\frac{d^2}{ds^2} \tilde{w}_{k,k}(s) = \sum_{n=1}^{\infty} \frac{G(0,k)^{n-1}}{\psi(0)^n} \frac{[\psi(0) - G(0,k)]n(n-1)\tilde{w}_0(s)^{n-2} [\tilde{w}_0'(s)]^2}{n!} + \sum_{n=1}^{\infty} \frac{G(0,k)^{n-1}}{\psi(0)^n} \frac{[\psi(0) - G(0,k)]n\tilde{w}_0(s)^{n-1} \tilde{w}_0''(s)}{n!}.
\]

Since \( \tilde{w}_0(0) = \psi(0), \tilde{w}_0'(0)/\psi(0) = -E(T_0^c) \) and \( \tilde{w}_0''(0)/\psi(0) = E[(T_0^c)^2] \), we have the second moment of \( T_{k,k}^c \) as
When $u$ is greater than $k$, we have from equation (5.2.13) that

$$
\bar{w}_{u,k}(s) = [\psi(u - k) - G(u - k, k)] \frac{\bar{w}_{u-k}(s)}{\psi(u - k)} + G(u - k, k) \frac{\bar{w}_{u-k}(s)}{\psi(u - k)} \bar{w}_{k,k}(s),
$$

and therefore,

$$
\frac{d^2}{ds^2} \bar{w}_{u,k}(s) = \frac{\psi(u - k) - G(u - k, k)}{\psi(u - k)} \bar{w}_{u-k}''(s) + \frac{G(u - k, k)}{\psi(u - k)} \bar{w}_{u-k}(s) \bar{w}_{k,k}(s) + 2 \frac{G(u - k, k)}{\psi(u - k)} \bar{w}_{u-k}'(s) \bar{w}_{k,k}'(s) + \frac{G(u - k, k)}{\psi(u - k)} \bar{w}_{u-k}(s) \bar{w}_{k,k}''(s),
$$

Setting $s = 0$ and dividing by $\psi_k(u)$, we have

$$
E \left[ (T_{u,k}^c)^2 \right] = E \left[ (T_{u-k}^c)^2 \right] + 2 \frac{G(u - k, k)\psi_k(k)}{\psi_k(u)} E(T_{u,k}^c) E(T_{k,k}^c)
+ \frac{G(u - k, k)\psi_k(k)}{\psi_k(u)} E \left[ (T_{k,k}^c)^2 \right].
$$

From Gerber (1979), the conditional expected time to ruin, given that ruin occurs under the classical risk process is

$$
E[T_{u,k}^c] = \frac{c + \lambda u}{c(\alpha - \lambda)}.
$$

Now in Table 5.2.1, we provide values of mean and standard deviation of the time to ruin.
should ruin occur, with a series of initial capital level. For each $U$ given, we use the optimal level of $u^*$ and $k^*$ calculated in Chapter 3 in the illustration.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$E[T_{u^*}^c]$</th>
<th>St.Dev</th>
<th>$Q(u,k) = 1.6E(R_{u,k})$</th>
<th>$Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u^*$</td>
<td>$k^*$</td>
<td>$E[T_{u^<em>}^c,k^</em>]$</td>
<td>St.Dev</td>
</tr>
<tr>
<td>11</td>
<td>50.83</td>
<td>6.83</td>
<td>4.01</td>
<td>39.26</td>
</tr>
<tr>
<td>13</td>
<td>59.17</td>
<td>8.25</td>
<td>5.43</td>
<td>41.11</td>
</tr>
<tr>
<td>15</td>
<td>67.50</td>
<td>10.05</td>
<td>7.23</td>
<td>41.64</td>
</tr>
<tr>
<td>17</td>
<td>75.83</td>
<td>12.01</td>
<td>9.19</td>
<td>41.73</td>
</tr>
<tr>
<td>19</td>
<td>84.17</td>
<td>14.00</td>
<td>11.18</td>
<td>41.75</td>
</tr>
<tr>
<td>21</td>
<td>92.50</td>
<td>16.00</td>
<td>13.18</td>
<td>41.75</td>
</tr>
</tbody>
</table>

Table 5.2.1: Conditional Mean and Standard Deviation of Time to Ruin, Exponential claims

In Table 5.2.1, under the classical risk model, the expected time to ruin increases as $U$ increases, from 50.83 when $U = 11$, to 92.50 when $U = 21$. The standard deviation of the time to ruin also increases from 55.00 to 74.33 as $U$ increases. When the reinsurance premium is calculated by the expected value principle by $Q(u,k) = 1.6E(R_{u,k})$, we see that the mean of the conditional time to ruin is much smaller with optimal $u^*$ and $k^*$. Initially when $U = 11$, the expected time to ruin is at 39.26 with $u^* = 6.83$ and $k^* = 4.01$. This also gives a standard deviation of 60.10, which is larger than under the classical risk model. As $U$ increases, the mean and standard deviation of $T_{u^*,k^*}^c$ also increase, but the variation is much smaller compared to under the classical risk model. Note that initially, the standard deviation under the lower barrier model is larger, but when $U = 17$, we see that the standard deviation of $T_{u^*,k^*}^c$ under the lower barrier model is 63.94, smaller than 67.27 under the classical risk model. When $U = 19$, the optimal $u^* = 14.00$ with $k^* = 11.18$ gives $E[T_{u^*,k^*}^c] = 41.75$ and $\text{St.Dev}(T_{u^*,k^*}^c) = 63.96$. We also see that when $U$ increases from 19 to 21, both mean and standard deviation of the time to ruin change very little so that they remain at 41.75 and 63.96 with 2 decimal places accuracy. Under the second case of the lower barrier model where $Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})$,
initially it is optimal not to buy any reinsurance when $U$ is small. As $U$ increases and hence that the optimal $k^*$ is greater than 0, the expected time to ruin starts to decrease and becomes smaller than under the classical risk model. Again, we see that the standard deviation also increases, but with a relatively slower speed compared to under the classical risk model.

5.2.2 Erlang(2) claim sizes

In this section, we assume that the individual claim sizes have an Erlang(2) distribution, with probability density function $p(x) = \alpha^2 xe^{-\alpha x}$. Hence, the distribution function for the claim sizes is $P(x) = 1 - e^{-\alpha x} - \alpha xe^{-\alpha x}$, and the ultimate ruin probability with $u = 0$ under the classical risk model is $\psi(0) = \frac{\lambda}{c} m_1 = \frac{\alpha}{ac}$.

Gerber and Shiu (1998) show that Lundberg’s fundamental equation (1.6.3) has a unique positive solution, which we denote by $\rho$, when $\delta > 0$. When $\delta = 0$, $\rho = 0$.

From Dickson (2008), we have that the defective joint density function of the deficit at ruin and the ruin time under the classical risk model with initial surplus 0 is

$$w_0(y, t) = l_1(0, t)\alpha^2 ye^{-\alpha y} + l_2(0, t)\alpha e^{-\alpha y},$$

(5.2.31)

where

$$\tilde{l}_1(0, \delta) \overset{def}{=} \int_0^\infty e^{-\delta t} l_1(0, t) dt = \frac{\lambda}{c} \frac{1}{\rho + \alpha},$$

(5.2.32)

and

$$\tilde{l}_2(0, \delta) \overset{def}{=} \int_0^\infty e^{-\delta t} l_2(0, t) dt = \frac{\lambda}{c} \frac{\alpha}{(\rho + \alpha)^2}.$$  

(5.2.33)

Note that the functions $\tilde{l}_1(0, \delta)$ and $\tilde{l}_2(0, \delta)$ can be interpreted as Laplace transforms of the functions $l_1(0, t)$ and $l_2(0, t)$ in terms of $\delta$ respectively. Hence,
5.2. On the time to ruin

\[ W_0(k, t) = \int_0^k \int_0^t w_0(y, \tau) d\tau dy \]
\[ = \int_0^k \int_0^t \left[ l_1(0, \tau) \alpha^2 ye^{-\alpha y} + l_2(0, \tau) \alpha e^{-\alpha y} \right] d\tau dy \]
\[ = \int_0^t l_1(0, \tau) d\tau [1 - e^{-\alpha k} - \alpha e^{-\alpha k}] + \int_0^t l_2(0, \tau) d\tau [1 - e^{-\alpha k}] \]

Also, by definition, we know that

\[ \psi(0, t) = \int_0^t \int_0^\infty w_0(y, \tau) dy d\tau \]
\[ = \int_0^t \left[ l_1(0, \tau) + l_2(0, \tau) \right] d\tau, \]

giving the defective density function of the ruin time \( T_0 \) as

\[ w_0(t) = \frac{d}{dt} \psi(0, t) = l_1(0, t) + l_2(0, t). \]

Therefore, from equations (5.2.3) and (5.2.4) we have

\[ D(t) = \frac{[1 - e^{-\alpha k}] \psi(0, t) - \alpha k e^{-\alpha k} \int_0^t l_1(0, \tau) d\tau}{G(0, k)} \]

and

\[ D_L(t) = \frac{e^{-\alpha k} \psi(0, t) + \alpha k e^{-\alpha k} \int_0^t l_1(0, \tau) d\tau}{\psi(0) - G(0, k)}. \]

Letting \( d(t) = \frac{d}{dt} D(t) \) and \( d_L(t) = \frac{d}{dt} D_L(t) \), we have

\[ d(t) = \frac{(1 - e^{-\alpha k}) [l_1(0, t) + l_2(0, t)] - \alpha k e^{-\alpha k} l_1(0, t)}{G(0, k)} \]
and

\[ d_L(t) = \frac{e^{-\alpha k}[l_1(0,t) + l_2(0,t)] + \alpha ke^{-\alpha k}l_1(0,t)}{\psi(0) - G(0,k)}. \]

When we take the Laplace transform of the functions \( d \) and \( d_L \) with transform parameter \( \delta \), we have

\[ \tilde{d}(\delta) = \frac{(1 - e^{-\alpha k})[\tilde{l}_1(0,\delta) + \tilde{l}_2(0,\delta)] - \alpha ke^{-\alpha k}\tilde{l}_1(0,\delta)}{G(0,k)} \]  \hspace{1cm} (5.2.34)

and

\[ \tilde{d}_L(\delta) = \frac{e^{-\alpha k}[\tilde{l}_1(0,\delta) + \tilde{l}_2(0,\delta)] + \alpha ke^{-\alpha k}\tilde{l}_1(0,\delta)}{\psi(0) - G(0,k)}. \]  \hspace{1cm} (5.2.35)

From equation (5.2.5), by differentiation we have the defective density function of the time to ruin under the lower barrier model with \( u = k \) as

\[ w_{k,k}(t) = \frac{d}{dt} W_{k,k}(t) \]

\[ = \sum_{n=1}^{\infty} G(0,k)^{n-1}(\psi(0) - G(0,k))d^{(n-1)}\ast d_L(t). \]

Taking the Laplace transform of the function \( w_{k,k}(t) \) we have

\[ \tilde{w}_{k,k}(\delta) = \sum_{n=1}^{\infty} \left\{ (1 - e^{-\alpha k})[\tilde{l}_1(0,\delta) + \tilde{l}_2(0,\delta)] - \tilde{l}_1(0,\delta)\alpha ke^{-\alpha k} \right\}^{n-1} \]

\[ \times \left\{ e^{-\alpha k}[\tilde{l}_1(0,\delta) + \tilde{l}_2(0,\delta)] + \alpha ke^{-\alpha k}\tilde{l}_1(0,\delta) \right\}. \]

Let \( a_k = 1 - e^{-\alpha k} - \alpha ke^{-\alpha k} \) and \( b_k = 1 - e^{-\alpha k} \). Then,
\[ \tilde{w}_{k,k}(\delta) = \sum_{n=1}^{\infty} \left[ a_k \tilde{l}_1(0, \delta) + b_k \tilde{l}_2(0, \delta) \right]^{n-1} \left[ (1 - a_k) \tilde{l}_1(0, \delta) + (1 - b_k) \tilde{l}_2(0, \delta) \right] \]

\[ = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{i+1} b_k^{n-1-i} (1 - a_k) \left[ \tilde{l}_1(0, \delta) \right]^{i+1} \left[ \tilde{l}_2(0, \delta) \right]^{n-1-i} \]

\[ + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{i} b_k^{n-i} (1 - b_k) \left[ \tilde{l}_1(0, \delta) \right]^{i} \left[ \tilde{l}_2(0, \delta) \right]^{n-i} \]

\[ = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{i+1} b_k^{n-1-i} (1 - a_k) \left[ \frac{\lambda}{c(\rho + \alpha)} \right]^{i+1} \left[ \frac{\lambda \alpha}{c(\rho + \alpha)^2} \right]^{n-1-i} \]

\[ + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{i} b_k^{n-i} (1 - b_k) \left[ \frac{\lambda}{c(\rho + \alpha)} \right]^{i} \left[ \frac{\lambda \alpha}{c(\rho + \alpha)^2} \right]^{n-i} \]

\[ = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{i+1} b_k^{n-1-i} (1 - a_k) \left[ \frac{\lambda}{\alpha c} \right]^{n} \left[ \frac{\alpha}{\rho + \alpha} \right]^{2n-i-1} \]

\[ + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{i} b_k^{n-i} (1 - b_k) \left[ \frac{\lambda}{\alpha c} \right]^{n} \left[ \frac{\alpha}{\rho + \alpha} \right]^{2n-i} \].

The key result we will use in our analysis is the Laplace transform relationship given by Dickson and Willmot (2005), who showed that for two functions \(A\) and \(B\), if

\[ \tilde{A}(\rho) \overset{\text{def}}{=} \int_0^{\infty} e^{-\rho t} A(t) dt = \tilde{B}(\delta) \overset{\text{def}}{=} \int_0^{\infty} e^{-\delta t} B(t) dt, \]

then

\[ B(t) = ce^{-\lambda t} A(ct) + \int_0^{ct} \frac{x}{t} s(ct - x, t) A(x) dx, \]  

where function \(s(x, t)\) is the density of aggregate claims over \((0, t)\) so that

\[ s(x, t) = \sum_{n=1}^{\infty} e^{-\lambda x} \frac{(\lambda x)^n}{n!} p^n(x). \]
Given that the individual claim sizes follow an Erlang(2) distribution with scale parameter \( \alpha \), the \( n \)-fold convolution of \( p \) is an Erlang(2\( n \)) distribution, also with scale parameter \( \alpha \).

From equation (5.2.36), we have that inversion with respect to \( \rho \) gives the function

\[
A(x) = \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{n-i-1} b_k^{i-1} (1 - a_k) \left( \frac{\lambda}{\alpha c} \right)^n e_{2n-i-1,\alpha}(x) \\
+ \sum_{n=1}^{\infty} \frac{\lambda}{\alpha c} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^{n-i-1} b_k^{i-1} (1 - b_k) \left( \frac{\lambda}{\alpha c} \right)^n e_{2n-i,\alpha}(x).
\]

(5.2.38)

Using equation (5.2.37), we can then obtain an expression for the defective probability density function \( w_{k,k}(t) \) as

\[
w_{k,k}(t) = ce^{-\lambda t} A(ct) \\
+ \int_0^c t \left( \sum_{m=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^m}{m!} e_{2m,\alpha}(ct - x) \right) A(x) dx \\
= ce^{-\lambda t} A(ct) \\
+ \sum_{m=1}^{\infty} e^{-\lambda t} \frac{\lambda t^{m-1}}{m!} \int_0^c x e_{2m,\alpha}(ct - x) A(x) dx \\
= ce^{-\lambda t} A(ct) + \sum_{m=1}^{\infty} \frac{e_{m,\lambda}(t)}{m} \int_0^c x e_{2m,\alpha}(ct - x) A(x) dx.
\]

Since the function \( A(x) \) is an infinite sum of Erlang density functions, we look at the part

\[
\int_0^c x e_{2m,\alpha}(ct - x) e_{2n-i-1,\alpha}(x) dx \\
= \int_0^c \frac{ae^{-a(ct-x)}}{(2m-1)!} \frac{ae^{-ax}(ax)^{2n-i-2}}{(2n-i-2)!} dx \\
= e^{-ac} \alpha^{2m+2n-i-1} \int_0^c (ct - x)^{2m-1} x^{2n-i-1} (2m-1)! (2n-i-2)! dx
\]
\[ w = e^{-act}(ct)^{2n+2n-i-1} \frac{2n-i-1}{(2m+2n-i-1)!} \int_0^1 \frac{\Gamma(2m+2n-i)}{\Gamma(2m)\Gamma(2n-i)} (1-s)^{2m-1} s^{2n-i-1} ds \]

\[ = e^{-act}(ct)^{2n+2n-i-1} \frac{2n-i-1}{(2m+2n-i-1)!} \]

\[ = 2n-i-1 \frac{1}{ac} e_{2m+2n-i,ac}(t). \]

We can then write \( w_{k,k}(t) \) as

\[
\begin{align*}
  w_{k,k}(t) &= ce^{-\lambda t} A(ct) \\
  &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1-a_k) \left( \frac{\lambda}{ac} \right)^n \\
  &\times \frac{(2n-i-1)}{acm} e_{m,\lambda}(t) e_{2m+2n-i,ac}(t) \\
  &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1-b_k) \left( \frac{\lambda}{ac} \right)^n \\
  &\times \frac{(2n-i)}{acm} e_{m,\lambda}(t) e_{2m+2n-i+1,ac}(t) \\
  &= \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1-a_k) \left( \frac{\lambda}{ac} \right)^n \\
  &\times \left[ e^{-\lambda t} e_{2n-i-1,ac}(t) + \sum_{m=1}^{\infty} \frac{(2n-i-1)}{acm} e_{m,\lambda}(t) e_{2m+2n-i,ac}(t) \right] \\
  &+ \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1-b_k) \left( \frac{\lambda}{ac} \right)^n \\
  &\times \left[ e^{-\lambda t} e_{2n-i,ac}(t) + \sum_{m=1}^{\infty} \frac{(2n-i)}{acm} e_{m,\lambda}(t) e_{2m+2n-i+1,ac}(t) \right] \\
  &= \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1-a_k) \left( \frac{\lambda}{ac} \right)^n \\
  &\times e^{-\lambda t} (ac)^{2n-i-1} 2^{n-i-2} e^{-act} \sum_{m=0}^{\infty} \frac{(\lambda\alpha^2c^2t^3)^m (2n-i-1)!}{m!(2m+2n-i-1)!} \\
  &+ \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} (1-b_k) \left( \frac{\lambda}{ac} \right)^n 
\end{align*}
\]
\begin{align*}
&\times e^{-\lambda t} (\alpha c)^{2n-i} t^{2n-i-1} e^{-\alpha t} \sum_{m=0}^{\infty} \frac{(\lambda \alpha^2 c^2 t^3)^m (2n - i)!}{m!(2m + 2n - i)!} \\
&= e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k b_k^{n-i-1} (1 - a_k) \left(\frac{\lambda}{\alpha c}\right)^n e^{2n-i-1, \alpha c}(t) \\
&\times {}_0F_2\left(\frac{2n-i}{2}, \frac{2n-i+1}{2}; \frac{\lambda}{\alpha c} \right) \\
&+ e^{-\lambda t} \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \binom{n-1}{i} a_k b_k^{n-i-1} (1 - b_k) \left(\frac{\lambda}{\alpha c}\right)^n e^{2n-i, \alpha c}(t) \\
&\times {}_0F_2\left(\frac{2n-i+1}{2}, \frac{2n-i+2}{2}; \frac{\lambda^2 c^2 t^3}{4} \right), \tag{5.2.39}
\end{align*}

Refer to Dickson (2008) for more details of the simplification steps.

We now look at the case where \( u \geq k \). From Dickson (2008), we have

\begin{align*}
[w_u(y, t)] = l_1(u, t)\alpha^2 y e^{-\alpha y} + l_2(u, t)\alpha e^{-\alpha y}, \tag{5.2.40}
\end{align*}

where

\begin{align*}
l_1(u, t) &= \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left(\frac{\lambda}{\alpha c}\right)^{p+1} e^{p+q+1, \alpha}(u) e^{2p-q+1, \alpha c}(t) \\
&\times {}_0F_2\left(\frac{2p-q+2}{2}, \frac{2p-q+3}{2}; \frac{\lambda^2 c^2 t^3}{4} \right), \tag{5.2.41}
\end{align*}

and

\begin{align*}
l_2(u, t) &= \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left(\frac{\lambda}{\alpha c}\right)^{p+1} e^{p+q+2, \alpha}(u) e^{2p-q+1, \alpha c}(t) \\
&\times {}_0F_2\left(\frac{2p-q+2}{2}, \frac{2p-q+3}{2}; \frac{\lambda^2 c^2 t^3}{4} \right) \\
&+ \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left(\frac{\lambda}{\alpha c}\right)^{p+1} e^{p+q+1, \alpha}(u) e^{2p-q+2, \alpha c}(t)
\end{align*}
\[ x_0 F_2 \left( \frac{2p - q + 3}{2}, \frac{2p - q + 4}{2}; \frac{\lambda \alpha^2 c^3 t^3}{4} \right). \] (5.2.42)

Hence,

\[
W_u(k, t) = \int_0^k \int_0^t w_u(y, \tau) d\tau dy \\
= a_k \int_0^t l_1(u, \tau) d\tau + b_k \int_0^t l_2(u, \tau) d\tau
\]

and

\[
\psi(u, t) = \int_0^t \int_0^{\infty} w_u(y, \tau) dy d\tau \\
= \int_0^t [l_1(u, \tau) + l_2(u, \tau)] d\tau.
\]

From equations (5.2.6), (5.2.7) and (5.2.8), we have that

\[
W_{u,k}(t) = \psi(u - k, t) - W_{u-k}(k, t) + W_{k,k} \ast W_{u-k}(k, t),
\]

and hence, by differentiating and taking the Laplace transform we have

\[
\tilde{w}_{u,k}(\delta) = (1 - a_k) \tilde{l}_1(u - k, \delta) + (1 - b_k) \tilde{l}_2(u - k, \delta) \\
+ [a_k \tilde{l}_1(u - k, \delta) + b_k \tilde{l}_2(u - k, \delta)] \tilde{w}_{k,k}(\delta).
\] (5.2.43)

From Dickson (2008), we have the Laplace transform of the functions \(l_1(u, t)\) and \(l_2(u, t)\) as
\[\tilde{l}_1(u, \delta) = \int_0^\infty e^{-\delta t} l_1(u, t) dt = \frac{1}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \left( \begin{array}{c} p \\ q \end{array} \right) \tilde{l}_1(0, \delta)^{q+1} \tilde{l}_2(0, \delta)^{p-q} e_{p+q+1, \alpha}(u), \] 

(5.2.44)

and

\[\tilde{l}_2(u, \delta) = \int_0^\infty e^{-\delta t} l_2(u, t) dt = \frac{1}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \left( \begin{array}{c} p \\ q \end{array} \right) \tilde{l}_1(0, \delta)^{q+1} \tilde{l}_2(0, \delta)^{p-q} e_{p+q+2, \alpha}(u) \]

\[+ \frac{1}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \left( \begin{array}{c} p \\ q \end{array} \right) \tilde{l}_1(0, \delta)^{q+1} \tilde{l}_2(0, \delta)^{p-q+1} e_{p+q+1, \alpha}(u). \] 

(5.2.45)

Substituting these results and equation (5.2.36) for \(\tilde{w}_{k,k}(\delta)\) into equation (5.2.43), and by inverting the Laplace transform using the same method described before, we obtain our expression for \(w_{u,k}(t)\) as

\[w_{u,k}(t) = (1 - a_k) l_1(u - k, t) + (1 - b_k) l_2(u - k, t) \]

\[+ \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \sum_{n=1}^{n-1} \sum_{i=0}^{n-1} \left( \begin{array}{c} p \\ q \end{array} \right) \left( \begin{array}{c} n - 1 \\ i \end{array} \right) a_k^i b_k^{n-i} (1 - a_k) \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} \\
[a_k e_{p+q+1, \alpha}(u - k) + b_k e_{p+q+2, \alpha}(u - k)] \]

\[\times e_{2n+2p-i-q, \alpha c}(t) {}_2F_2 \left( \frac{2n + 2p - i - q + 1}{2}, \frac{2n + 2p - i - q + 2}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right) \]

\[+ \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^\infty \sum_{q=0}^p \sum_{n=1}^{n-1} \sum_{i=0}^{n-1} \left( \begin{array}{c} p \\ q \end{array} \right) \left( \begin{array}{c} n - 1 \\ i \end{array} \right) a_k^i b_k^{n-i} \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} \\
[(a_k + b_k - 2a_k b_k) e_{p+q+1, \alpha}(u - k) + b_k (1 - b_k) e_{p+q+2}(u - k)] \]

\[\times e_{2n+2p-i-q+1, \alpha c}(t) {}_2F_2 \left( \frac{2n + 2p - i - q + 2}{2}, \frac{2n + 2p - i - q + 3}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right) \]
5.2. On the time to ruin

\[ + \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k b_k^{n-i} (1 - b_k) \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} e_{p+q+1,\alpha}(u - k) \]

\[ \times e^{2n+2p-i-q+2,\alpha c(t)} F_2 \left( \frac{2n+2p-i-q+3}{2}, \frac{2n+2p-i-q+4}{2} ; \frac{\lambda^2 c^2 t^3}{4} \right). \]

(5.2.46)

In the next two examples, we give some numerical illustrations of the distribution of the time to ruin with Erlang(2) claim sizes with scale parameter \( \alpha = 2 \). The probability density function of \( X_1 \) is then \( p(x) = 4xe^{-2x} \), the same as in the examples given in the previous chapters. The insurer’s premium income \( c \) is 1.2 and the parameter for the Poisson process \( \lambda \) is 1, giving a premium loading of 0.2.

**Example 5.2.3. Distribution of time to ruin**

In the first example under this section, we plot the conditional density functions of the time to ruin under the classical risk model and the lower barrier model in Figure 5.2.4. Under the classical risk model, the initial capital level is set at 13, giving an ultimate ruin probability \( \psi(13) = 0.04465 \), and we plot \( w_{13}(t)/\psi(13) \). Under the first case of the lower barrier model, \( u^* = 9.12 \) and \( k^* = 7.19 \). The probability of ruin is 0.00001. Under the second case, the optimal \( u^* \) is 9.04 with \( k^* = 2.09 \), giving an ultimate ruin probability of 0.03875. For the lower barrier model, we plot the functions \( w_{u^*,k^*}(t)/\psi^*_{u^*}(u) \).

In Figure 5.2.4, we see that under the lower barrier model, the defective density of the time to ruin is more skewed to the left and the higher the lower barrier \( k \) is, the larger this effect appears. The shapes of the densities are similar to Figure 5.2.2 in the previous section. Should ruin occur, the time to ruin under the lower barrier model is more likely to be shorter than under the classical risk model. As seen from equation (5.2.46), the defective density function of the time to ruin involves infinite sums. For computational purposes, these sums are truncated to certain levels without compromising accuracy. Even
so, for large values of $t$, the computer run times are still very lengthy (several hours for a single value) and care has to be exercised in computer programs to avoid numerical underflow or overflow.

**Example 5.2.4. Conditional mean and standard deviation of time to ruin**

In this example, we calculate the conditional mean and standard deviation of the time to ruin under the classical risk model and the lower barrier model. Under the lower barrier model, we use equation (5.2.14) to find the first and second moments of $T_{u,k}^*$. 

Figure 5.2.4: Conditional Density of Time to Ruin, Erlang(2) claims
From equation (5.2.43) and by differentiating with respect to $\delta$, we have

$$
\tilde{w}_{u,k}'(\delta) = (1 - a_k)\tilde{l}_1'(u - k, \delta) + (1 - b_k)\tilde{l}_2'(u - k, \delta) \\
+ [a_k \tilde{l}_1'(u - k, \delta) + b_k \tilde{l}_2'(u - k, \delta)]\tilde{w}_{k,k}(\delta) \\
+ [a_k \tilde{l}_1(u - k, \delta) + b_k \tilde{l}_2(u - k, \delta)]\tilde{w}'_{k,k}(\delta),
$$

and that

$$
\tilde{w}_{u,k}''(\delta) = (1 - a_k)\tilde{l}_1''(u - k, \delta) + (1 - b_k)\tilde{l}_2''(u - k, \delta) \\
+ [a_k \tilde{l}_1''(u - k, \delta) + b_k \tilde{l}_2''(u - k, \delta)]\tilde{w}_{k,k}(\delta) \\
+ 2[a_k \tilde{l}_1'(u - k, \delta) + b_k \tilde{l}_2'(u - k, \delta)]\tilde{w}'_{k,k}(\delta) \\
+ [a_k \tilde{l}_1(u - k, \delta) + b_k \tilde{l}_2(u - k, \delta)]\tilde{w}''_{k,k}(\delta).
$$

Denote the unique positive solution of Lundberg’s fundamental equation as $\rho_δ$, representing a function of $\delta$. From equations (5.2.32), (5.2.33), (5.2.44) and (5.2.45), we have

$$
\tilde{l}_1(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p+q} (\rho_\delta + \alpha)^{q-1-2p} e_{p+q+1,\alpha}(u),
$$

and

$$
\tilde{l}_2(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p+q} (\rho_\delta + \alpha)^{q-1-2p} e_{p+q+2,\alpha}(u) \\
+ \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p+q+1} (\rho_\delta + \alpha)^{q-2-2p} e_{p+q+1,\alpha}(u).
$$
By differentiation, we have

\[ \tilde{l}_1'(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q}(q-1-2p)(\rho_\delta + \alpha)^{q-2-2p} \rho'_\delta e_{p+q+1,\alpha}(u), \]

and

\[ \tilde{l}_2'(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q}(q-1-2p)(\rho_\delta + \alpha)^{q-2-2p} \rho'_\delta e_{p+q+1,\alpha}(u) \]

The second derivatives are then found to be

\[ \tilde{l}_1''(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q}(q-1-2p)(q-2-2p)(\rho_\delta + \alpha)^{q-2-2p} (\rho'_\delta)^2 e_{p+q+1,\alpha}(u) \]

\[ + \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q}(q-1-2p)(\rho_\delta + \alpha)^{q-2-2p} \rho''_\delta e_{p+q+1,\alpha}(u) \]

and

\[ \tilde{l}_2''(u, \delta) = \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q}(q-1-2p)(q-2-2p)(\rho_\delta + \alpha)^{q-3-2p} (\rho'_\delta)^2 e_{p+q+2,\alpha}(u) \]

\[ + \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q}(q-1-2p)(\rho_\delta + \alpha)^{q-2-2p} \rho''_\delta e_{p+q+2,\alpha}(u) \]

\[ + \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \binom{p}{q} \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q+1}(q-2-2p)(q-3-2p)(\rho_\delta + \alpha)^{q-4-2p} (\rho'_\delta)^2 e_{p+q+1,\alpha}(u) \]
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\[ + \frac{1}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \left( \frac{p}{q} \right) \left( \frac{\lambda}{c} \right)^{p+1} \alpha^{p-q+1} (q - 2 - 2p) (\rho_{\delta} + \alpha)^{q-3-2p} \rho_{\delta}'' \ e_{p+q+1,\alpha}(u) . \]

We observe that by setting \( u = k \) in equation (5.2.43) that

\[ \tilde{w}_{k,k}(\delta) = \frac{(1 - a_k) \tilde{l}_1(0, \delta) + (1 - b_k) \tilde{l}_2(0, \delta)}{1 - a_k \tilde{l}_1(0, \delta) - b_k \tilde{l}_2(0, \delta)} , \]

from which we can obtain \( \tilde{w}_k'_{k,k}(\delta) \) and \( \tilde{w}_k''_{k,k}(\delta) \), giving everything required to find \( \tilde{w}_u'_{u,k}(\delta) \) and \( \tilde{w}_u''_{u,k}(\delta) \).

From equations (5.2.32) and (5.2.33), we have that

\[ \tilde{l}_1(0, \delta) = \frac{-\lambda \rho_{\delta}'}{c(\rho_{\delta} + \alpha)^2} , \]

and

\[ \tilde{l}_2(0, \delta) = \frac{-2\lambda \alpha \rho_{\delta}'}{c(\rho_{\delta} + \alpha)^3} . \]

In addition,

\[ \tilde{l}_1'(0, \delta) = \frac{2\lambda (\rho_{\delta}')^2}{c(\rho_{\delta} + \alpha)^3} - \frac{\lambda \rho_{\delta}''}{c(\rho_{\delta} + \alpha)^2} , \]

and

\[ \tilde{l}_2'(0, \delta) = \frac{6\lambda \alpha (\rho_{\delta}')^2}{c(\rho_{\delta} + \alpha)^4} - \frac{2\lambda \alpha \rho_{\delta}''}{c(\rho_{\delta} + \alpha)^3} . \]

We know that \( \rho_0 = 0 \). Also from Dickson (2005) we have \( \rho_0' = 1/(c - \lambda m_1) \), and

\[ \rho_0'' = \frac{-\lambda m_2}{(c - \lambda m_1)^3} . \]

We can now obtain the first and second moments of the time to ruin using equation (5.2.14).
Table 5.2.2 shows the mean and standard deviations of the time to ruin given that ruin occurs, with various levels of initial capital $U$ under the classical risk model and under the two cases of the lower barrier model with optimal $u^*$ and $k^*$. Note that the method of using the Laplace transforms to find the second moment is much more efficient than intergrating the equations $t^2l_i(0,t)$.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$E[T_{c,U}]$</th>
<th>$\text{St.Dev}$</th>
<th>$Q(u,k) = 1.6E(R_{u,k})$</th>
<th>$Q(u,k) = E(R_{u,k}) + 2\text{St.Dev.}(R_{u,k})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>33.05</td>
<td>38.03</td>
<td>4.22</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>41.57</td>
<td>42.68</td>
<td>5.35</td>
<td>9</td>
</tr>
<tr>
<td>11</td>
<td>50.10</td>
<td>46.86</td>
<td>7.14</td>
<td>11</td>
</tr>
<tr>
<td>13</td>
<td>58.62</td>
<td>50.70</td>
<td>9.12</td>
<td>9.04</td>
</tr>
<tr>
<td>15</td>
<td>67.15</td>
<td>54.27</td>
<td>11.12</td>
<td>8.49</td>
</tr>
<tr>
<td>17</td>
<td>75.68</td>
<td>57.62</td>
<td>13.12</td>
<td>9.57</td>
</tr>
</tbody>
</table>

Table 5.2.2: Conditional Mean and Standard Deviation of Time to Ruin, Erlang(2) claims

The expected time to ruin shown in Table 5.2.2 are consistent with the results in Figure 5.2.4. Under the lower barrier model, the conditional expected time to ruin is smaller than that under the classical risk model. Starting from $U = 7$, the expected time to ruin is 33.05 under the classical risk model, with a standard deviation of 38.03. As $U$ increases to 17, the expected time to ruin increases to 75.68 while the standard deviation also increased to 57.62. Under the first case of the lower barrier model, when $U = 7$, the optimal $u^*$ and $k^*$ give an expected ruin time of 27.12, much smaller than under the classical risk model. The corresponding standard deviation is 37.66, also slightly smaller than under the classical risk model. We see that $E[T^c_{u^*,k^*}]$ then increases to 30.31 when $U$ increases to 11. It then decreases slightly as $U$ increases further. For $U = 13$, $E[T^c_{u^*,k^*}] = 30.29$ and it remains at 30.29 for $U$ equals 15 and 17, at two decimal place accuracy. The standard deviation under this scenario increases to 40.93 when $U = 11$ and then remains at 40.93 at two decimal place accuracy. We see that the extent of variation is much smaller than under
the classical risk model. The observation is consistent with the example with exponential claim sizes. Under the second case of the lower barrier model, initially it is optimal not to apply the lower barrier model and therefore the results are the same as under the classical risk model. When \( U = 13 \), the lower barrier model can reduces the ultimate ruin probability, and with \( u^* = 9.04 \) and \( k^* = 2.09 \), we obtain a conditional expected time to ruin of 47.53, lower than 58.62 under the classical risk model. The standard deviation is 47.92, slightly smaller than 50.70 under the classical risk model. As \( U \) continues to increase, \( E[T_{u,k}^*] \) decreases to 42.35 when \( U = 17 \), and the standard deviation decreased slightly to 46.95.

### 5.3 Some joint densities

In Chapter 4 we discussed the deficit at ruin and the surplus prior to ruin, and in the first part of this chapter we looked at the time to ruin. We now look at some joint densities under the lower barrier model.

Let

\[
W_{u,k}(y, t) = \Pr[|U(T_{u,k})| \leq y \text{ and } T_{u,k} \leq t]
\]

be the defective joint distribution function of the deficit at ruin and the time to ruin, under the lower barrier model with initial surplus \( u \) and lower barrier \( k \). Let \( w_{u,k}(y, t) = \frac{\partial^2}{\partial y \partial t} W_{u,k}(y, t) \) be the defective joint probability density function. To derive an expression for \( W_{u,k}(y, t) \), we start from the case where \( u = k \). Conditioning on the first time that the surplus level drops below \( k \), we have
\[ W_{k,k}(y,t) = \Pr[|U(T_{k,k})| \leq y \text{ and } T_{k,k} \leq t] \]
\[ = \int_0^t \int_k^{k+y} w_0(z,\tau)dzd\tau + \int_0^t \int_0^k w_0(z,\tau)W_{k,k}(y,t-\tau)dzd\tau, \]

so that
\[ w_{k,k}(y,t) = \frac{\partial^2}{\partial y \partial t}W_{k,k}(y,t) \]
\[ = w_0(k+y,t) + \int_0^t \int_0^k w_0(z,\tau)w_{k,k}(y,t-\tau)dzd\tau. \]

By taking the Laplace transform with respect to \( t \), we have
\[ \tilde{w}_{k,k}(y,\delta) = \int_0^\infty e^{-\delta t}w_{k,k}(y,t)dt \]
\[ = \tilde{w}_0(k+y,\delta) + \tilde{w}_{k,k}(y,\delta)\int_0^k \tilde{w}_0(z,\delta)dz, \]

which gives
\[ \tilde{w}_{k,k}(y,\delta) = \frac{\tilde{w}_0(k+y,\delta)}{1 - \int_0^k \tilde{w}_0(z,\delta)dz} \]
\[ = \tilde{w}_0(k+y,\delta) \sum_{n=0}^{\infty} \left[ \int_0^k \tilde{w}_0(z,\delta)dz \right]^n. \quad (5.3.1) \]

When \( u > k \), we have
\[ W_{u,k}(y,t) = \Pr[|U(T_{u,k})| \leq y \text{ and } T_{u,k} \leq t] \]
\[ = \int_0^t \int_k^{k+y} w_{u-k}(z,\tau)dzd\tau + \int_0^t \int_0^k w_{u-k}(z,\tau)W_{k,k}(y,t-\tau)dzd\tau. \]
Therefore,

\[ w_{u,k}(y,t) = w_{u-k}(k+y,t) + \int_0^t \int_0^k w_{u-k}(z,\tau)w_{k,k}(y,t-\tau)dzd\tau. \]

The Laplace transform of \( w_{u,k}(y,t) \) with respect to \( t \) is then

\[ \tilde{w}_{u,k}(y,\delta) = \int_0^\infty e^{-\delta t}w_{u,k}(y,t)dt \]

\[ = \tilde{w}_{u-k}(k+y,\delta) + \tilde{w}_{k,k}(y,\delta) \int_0^k \tilde{w}_{u-k}(z,\delta)dz. \] (5.3.2)

We now assume two claim size distributions and derive formulae for the defective joint density functions.

### 5.3.1 Exponential claim sizes

Assume that the individual claims follow the exponential distribution with probability density function \( p(x) = \alpha e^{-\alpha x} \). Under the classical risk model, we know that \( w_u(y,t) = \alpha e^{-\alpha y}w_u(t) \). We then have the Laplace transforms

\[ \tilde{w}_u(y,\delta) = \alpha e^{-\alpha y}\tilde{w}_u(\delta) \]

and

\[ \int_0^k \tilde{w}_u(z,\delta)dz = \tilde{w}_u(\delta)(1 - e^{-\alpha k}). \]

From equations (5.2.17) and (5.3.1),

\[ \tilde{w}_{k,k}(y,\delta) = \alpha e^{-\alpha(k+y)}\tilde{w}_0(\delta) \sum_{n=0}^\infty (1 - e^{-\alpha k})^n\tilde{w}_0(\delta)^n \]

\[ = \alpha e^{-\alpha y}\tilde{w}_{k,k}(\delta), \]
which gives
\[ w_{k,k}(y, t) = \alpha e^{-\alpha y} w_{k,k}(t). \]

For \( u > k \), we have that
\[
\tilde{w}_{u,k}(y, \delta) = \alpha e^{-\alpha(k+y)} \tilde{w}_{u-k}(\delta) + \alpha e^{-\alpha y}(1 - e^{-\alpha k}) \tilde{w}_{u-\delta}(\delta) \tilde{w}_{k,k}(\delta).
\]

Since
\[
\tilde{w}_{u,k}(\delta) = e^{-\alpha k} \tilde{w}_{u-k}(\delta) + (1 - e^{-\alpha k}) \tilde{w}_{u-k}(\delta) \tilde{w}_{k,k}(\delta)
\]

from equations (5.2.17) and (5.2.18), we then have that
\[
\tilde{w}_{u,k}(y, \delta) = \alpha e^{-\alpha y} \tilde{w}_{u,k}(\delta),
\]

and hence
\[
w_{u,k}(y, t) = \alpha e^{-\alpha y} w_{u,k}(t).
\]

Note that the function \( w_{u,k}(y, t) \) under the lower barrier model is the product of the marginal densities of the deficit at ruin \( |U(T_{u,k})| \), and the time to ruin \( T_{u,k} \), which is of the same structure as \( w_u(y, t) \) under the classical risk model due to the memoryless property of the exponential distribution.

### 5.3.2 Erlang(2) claim sizes

In this section, we assume the individual claim sizes follow an Erlang(2) distribution with \( p(x) = \alpha^2 xe^{-\alpha x} \). From equation (5.2.40) we have
\[
\tilde{w}_u(y, \delta) = \tilde{l}_1(u, \delta) \alpha^2 ye^{-\alpha y} + \tilde{l}_2(u, \delta) \alpha e^{-\alpha y}
\]
and
\[ \int_0^k \tilde{w}_u(z, \delta) dz = a_k \tilde{I}_1(u, \delta) + b_k \tilde{I}_2(u, \delta), \]
where \( a_k = 1 - e^{-\alpha k} - \alpha k e^{-\alpha k} \) and \( b_k = 1 - e^{-\alpha k} \). Substituting into equation (5.3.1) we have

\[
\tilde{w}_{k,k}(y, \delta) = \sum_{n=1}^{\infty} [a_k \tilde{I}_1(0, \delta) + b_k \tilde{I}_2(0, \delta)] n^{n-1} [\tilde{I}_1(0, \delta) \alpha^2 (k + y) e^{-\alpha (k+y)} + \tilde{I}_2(0, \delta) \alpha e^{-\alpha (k+y)}] \\
= \alpha^2 (k + y) e^{-\alpha (k+y)} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} \left( \frac{\lambda}{\alpha c} \right)^n \left( \frac{\alpha}{\rho + \alpha} \right)^{2n-i-1} \\
+ \alpha e^{-\alpha (k+y)} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} \left( \frac{\lambda}{\alpha c} \right)^n \left( \frac{\alpha}{\rho + \alpha} \right)^{2n-i}.
\]

We omit the details of the simplifications since they are similar to those that appeared in equation (5.2.36). Using equation (5.3.2) and by inverting the Laplace transform \( \tilde{w}_{u,k}(y, \delta) \) we have

\[
w_{u,k}(y, t) = \alpha^2 (k + y) e^{-\alpha (k+y)} I_1(u - k, t) + \alpha e^{-\alpha (k+y)} I_2(u - k, t) \\
+ \frac{e^{-\lambda}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{p}{q} \binom{n-1}{i} a_k^i b_k^{n-1-i} \alpha^2 (k + y) e^{-\alpha (k+y)} \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} \\
[ a_k e_{p+q+1,\alpha}(u - k) + b_k e_{p+q+2,\alpha}(u - k) ] \\
\times e_{2n+2p-i-q,\alpha c}(t) {}_2F_1 \left( \frac{2n+2p-i-q+1}{2}, \frac{2n+2p-i-q+2}{2}; \lambda \alpha^2 c^2 t^3 \right) \\
+ \frac{e^{-\lambda}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{p}{q} \binom{n-1}{i} a_k^i b_k^{n-1-i} \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} \\
\left[ (a_k \alpha e^{-\alpha (k+y)} + b_k \alpha^2 (k + y) e^{-\alpha (k+y)}) e_{p+q+1,\alpha}(u - k) \\
+ b_k \alpha e^{-\alpha (k+y)} e_{p+q+2}(u - k) \right].
\]
If we rewrite equation (5.2.46) as

\[ w_{u,k}(t) = (1 - a_k) f_1(u, k, t) + (1 - b_k) f_2(u, k, t), \]

where

\[ f_1(u, k, t) = l_1(u - k, t) \]

\[ + \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=1}^{n-1} \sum_{i=0}^{p-1} \binom{p}{q} \binom{n-1}{i} a_k^i b_k^{n-i} \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} e^{p+q+1, \alpha}(u - k) \]

\[ \times e^{2n+2p-i-q+1, \alpha c}(t) \frac{\binom{2n+2p-i-q+2}{2}}{\binom{2n+2p-i-q+3}{2}} \frac{\lambda c^2 t^3}{4} \]

\[ \times e^{2n+2p-i-q+2, \alpha}(t) \frac{\binom{2n+2p-i-q+3}{2}}{\binom{2n+2p-i-q+4}{2}} \frac{\lambda c^2 t^3}{4} \]
5.4 Concluding remarks

\[
f_2(u, k, t) = l_2(u - k, t) + \frac{e^{-\lambda t}}{\alpha} \sum_{p=0}^{\infty} \sum_{q=0}^{p} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n-1}{i} a_k^i b_k^{n-1-i} \left( \frac{\lambda}{\alpha c} \right)^{n+p+1} [a_k e_{p+q+1, \alpha}(u - k) + b_k e_{p+q+2, \alpha}(u - k)]
\]

\[
\times e_{2n+2p-i-q+1, \alpha c}(t) {}_2F_2 \left( \frac{2n + 2p - i - q + 2}{2}, \frac{2n + 2p - i - q + 3}{2}; \frac{\lambda \alpha^2 c^2 t^3}{4} \right)
\]

we can then rewrite equation (5.3.3) as

\[
w_{u,k}(y, t) = f_1(u, k, t) \alpha^2 (k + y) e^{-\alpha (k+y)} + f_2(u, k, t) \alpha e^{-\alpha (k+y)}.
\]

Compared to equation (5.2.40) for the joint density function of the deficit at ruin and the time to ruin under the classical risk model, we see that under the lower barrier model, our joint density function admits a similar structure.

5.4 Concluding remarks

In this chapter, we have investigated the distribution of the time to ruin under the lower barrier model. For individual claim sizes that follow an exponential or Erlang(2) distribution, we compared the results with the distribution of the time to ruin under the classical risk model. Generally, we should expect that the time to ruin under the lower barrier model would be shorter than that under the classical risk model, should ruin occur. In addition, the variation in the expected time to ruin as \( U \) varies is smaller when a lower
barrier is imposed. We also looked at some joint densities under the lower barrier model. Under the exponential claim sizes, we found that the joint density of the deficit at ruin and the time to ruin is of the same format as under the classical risk model. When the individual claims sizes have Erlang(2) distribution, the joint densities admits a mixed Erlang structure under both models but with different weights.
Chapter 6

Lower barrier in the Sparre Andersen risk model

6.1 Introduction

In the previous chapters, we looked at the effects of introducing a lower barrier in the classical risk model. In this chapter, we examine these properties using the Sparre Andersen model instead of the classical model. The notation we use in this chapter with respect to ultimate ruin probability, the deficit at ruin, the surplus before ruin and the time to ruin follow from our introduction of these quantities in Chapter 1 and Chapter 2.

As seen in the previous chapters, the results relating to ultimate ruin, surplus before ruin and deficit at ruin in the lower barrier model are mostly based on the results for the ruin probability and deficit at ruin in the classical risk model, i.e. the formulae are expressed in terms of $\psi(u)$ and $G(u,y)$. Hence, most of the formulae we derived also apply to the Sparre Andersen model since derivations were based on the amount of the first drop of the surplus process below its initial level.
Chapter 6. Lower barrier in the Sparre Andersen risk model

6.2 Erlang(2) inter-claim times

In this section, we investigate the effects of the lower barrier model when the waiting time distribution is Erlang(2) with density function \( k(t) = \beta^2 t e^{-\beta t} \). This is probably one of the easier models under the Sparre Andersen framework. From Theorem 2.1.1, we see that the Laplace transform of the survival probability \( \phi(u) \), denoted as \( \tilde{\phi}(s) \), is expressed in terms of the Laplace transform function \( \tilde{\rho} \) of the claim size distribution as in equation (2.1.2). The probability of survival with initial surplus \( u = 0 \) is \( \phi(0) = \frac{2c-\beta^2 m_1}{c^2 \rho} \), where \( \rho \) is the unique positive root of equation (2.1.3). Also, from Dickson (1998), the Laplace transform of the \( g(0, y) \) with respect to \( y \), defined as \( \tilde{g}(0, s) = \int_0^\infty e^{-sy} g(0, y) dy \), is found via

\[
\tilde{g}(0, s) = 1 - \frac{\phi(0)}{s \tilde{\phi}(s)}, \quad (6.2.1)
\]

from which we can obtain \( g(0, y) \) by inversion. Using this result, the Laplace transform of the function \( G(u, y) \) with respect to \( u \), defined as \( \tilde{G}(s, y) = \int_0^\infty e^{-su} g(u, y) du \), is then

\[
\tilde{G}(s, y) = \frac{\int_0^\infty e^{-su} \int_u^{u+y} g(0, x) dx}{1 - \tilde{g}(0, s)}. \quad (6.2.2)
\]

If we can invert equations (2.1.2) and (6.2.2) to obtain functions \( \phi(u) \) and \( G(u, y) \), we can then calculate the ultimate ruin probability under the lower barrier model for given \( u \) and \( k \) using equation (3.2.2).

We now look at some ruin related quantities under the lower barrier model in an Erlang(2) process, assuming three individual claim sizes, as in Chapter 3, namely the exponential distribution, Erlang(2) distribution and mixed exponential distribution. For numerical illustration, we assume the distribution of the waiting time \( W_1 \) to follow an Erlang(2) distribution with scale parameter \( \beta = 2 \), so that the probability density function
is \( k(t) = 4te^{-2t} \). The assumptions with respect to the reinsurance premium follow those of Chapter 3. The expected value and the second moment for the total reinsurance claims are calculated using equations (3.3.1) and (3.3.2) respectively. We omit the numerical examples when the reinsurance premium is related to the discounted value of the total claims since the calculation involves the density functions of the time to ruin. For the following discussions, the insurer’s premium loading factor is assumed to be 0.2 such that \( cE(W_1) = 1.2E(X_1) \), meaning that the ultimate ruin probability is less than 1.

### 6.2.1 Exponential claim sizes

We firstly assume that the individual claim sizes are exponentially distributed with probability density function \( p(x) = e^{-x} \) under the Erlang(2) process. The unique positive solution of Lundberg’s fundamental equation (2.1.3) is \( \rho = 2.5511 \). It then follows that the ultimate ruin probability with initial surplus 0 is \( \psi(0) = 0.782230 \). The other non-zero solution of Lundberg’s fundamental equation is \( -R = -0.21777 \), where \( R \) is the adjustment coefficient. Theorem 2.1.1 then gives that the ultimate ruin probability with initial surplus \( u \) as

\[
\psi(u) = \psi(0)e^{-Ru}.
\]

Equations (6.2.1) and (6.2.2) give the distribution function of the deficit at ruin as

\[
G(u, y) = \psi(u)(1 - e^{-y}).
\]

Note that these functions are of the same form as equations (3.5.1) and (3.5.3) under the classical risk model, with different \( \psi(0) \) and \( R \). We now look at some numerical illustrations.
Example 6.2.1. Expected value principle reinsurance premium.

In this example, we assume that the reinsurer uses the expected value principle for premium calculation with a premium loading of 0.6, such that \( Q(u, k) = 1.6E(R_{u,k}) \). When the initial capital \( U \) is 15, under the Erlang(2) risk model, the ultimate ruin probability is \( \psi(15) = 0.02983 \). For each possible combination of \( u \) and \( k \) starting from \( u = k = 9.26 \), given \( U = 15 \), we plot the ultimate ruin probability under the lower barrier model in Figure 6.2.1. The optimal level of \( u^* \) is 11.47, with \( k^* = 9.25 \), gives an ultimate ruin probability of \( \psi^*_k(u) = 0.00021 \).

In Example 3.5.1 where the inter-claim time distribution is exponential with exponential claim sizes, we saw that \( \psi(15) = 0.06840 \) and the optimal \( u^* = 10.05 \) and \( k^* = 7.23 \), giving \( \psi^*_k(u) = 0.00226 \), a 97% reduction in ultimate ruin probability under the lower barrier model. The reduction under the Erlang(2) process higher at 99%. We also observe that the optimal \( u \) and \( k \) are both higher than in Example 3.5.1.

Example 6.2.2. Standard deviation principle reinsurance premium.

We now assume the reinsurance premium is calculated using the standard deviation principle with loading of 2, i.e. \( Q(u, k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k}) \). The initial capital is set at 20 such that the ultimate ruin probability under the Sparre Andersen model is 0.01004. The lowest possible initial surplus \( u \) under the lower barrier model is \( u = k = 7.54 \). For all possible choices of \( 7.54 \leq u \leq 20 \), we calculate the corresponding \( k \) and plot the ultimate ruin probability in Figure 6.2.2. The minimal \( \psi^*_k(u) \) is found at \( u^* = 13.21 \) and \( k^* = 7.50 \), giving \( \psi^*_k(u) = 0.00057 \), a 94% reduction. Compared to Example 3.5.2 where the inter-claim time distribution is exponential and the percentage reduction produced by the lower barrier model is 45%, the reduction in ultimate ruin probability is larger under the Erlang(2) process.

In Figure 6.2.1, we see that for all feasible combinations of \( u \) and \( k \), the ultimate ruin
Figure 6.2.1: $\psi_k(u)$ and $\psi(U)$, $U = 15$, $Q(u,k) = 1.6E(R_{u,k})$, Exponential claims

probability under the lower barrier model is lower than that under the Sparre Andersen model. Starting from $u = k = 9.26$, the ruin probability is 0.00034. It decreases at a very slow rate as $u$ increases, until it reaches minimal level. The ultimate ruin probability then increases as $u$ increases, and when $u$ approaches 15, it increases faster, approaching the same level as the ultimate ruin probability under the Sparre Andersen model. In Figure 6.2.2, we see the same pattern appears with a more visible variation in the ultimate ruin probability when $u$ changes than in Figure 6.2.1. Note the differences between Figure 6.2.2 and Figure 3.5.2 in Chapter 3. In these two figures, we have assumed the standard deviation principle for the reinsurance premium. In Figure 3.5.2, we see that only certain
Figure 6.2.2: $\psi_k(u)$ and $\psi(U)$, $U = 20$, $Q(u,k) = E(R_{u,k}) + 2 \text{St. Dev.}(R_{u,k})$, Exponential claims

combinations of $u$ and $k$ provide an ultimate ruin probability lower than that of under the classical risk model, whereas in Figure 6.2.2, all feasible $u$ and $k$ provide reductions in the ruin probability. This happens because the initial capital level we chose for Figure 6.2.2 is large enough such that the reinsurance arrangement can provide a lower barrier $k$ large enough to give a $\psi_k(u) < \psi(U)$ for all feasible values of $u$.

Instead of looking at a single value of $U$ as in Examples 6.2.1 and 6.2.2, now in Table 6.2.1, we provide the ultimate ruin probabilities for a series of values of $U$ under the Sparre Andersen model. For each given $U$, we find the optimal $u^*$ and $k^*$ under the lower barrier model, with the assumptions $Q(u,k) = 1.6E(R_{u,k})$ and $Q(u,k) = E(R_{u,k}) +$
2St.Dev.(R_{u,k}) respectively, and calculate the ruin probabilities. We also provide the percentage reduction in the ultimate ruin probability.

\[
Q(u,k) = 1.6E(R_{u,k})
\]

<table>
<thead>
<tr>
<th>U</th>
<th>\psi(U)</th>
<th>Q(u,k) = 1.6E(R_{u,k})</th>
<th>Q(u,k) = E(R_{u,k}) + 2St.Dev.(R_{u,k})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(u^*)</td>
<td>(k^*)</td>
</tr>
<tr>
<td>7</td>
<td>0.17033</td>
<td>4.84</td>
<td>2.68</td>
</tr>
<tr>
<td>9</td>
<td>0.11019</td>
<td>6.01</td>
<td>3.85</td>
</tr>
<tr>
<td>11</td>
<td>0.07128</td>
<td>7.57</td>
<td>5.41</td>
</tr>
<tr>
<td>13</td>
<td>0.04611</td>
<td>9.44</td>
<td>7.28</td>
</tr>
<tr>
<td>15</td>
<td>0.02983</td>
<td>11.41</td>
<td>9.25</td>
</tr>
<tr>
<td>17</td>
<td>0.01930</td>
<td>13.41</td>
<td>11.25</td>
</tr>
</tbody>
</table>

Table 6.2.1: Ruin probabilities, Exponential claims

In Table 6.2.1, we start from \(U = 7\) with \(\psi(7) = 0.17033\), increasing to \(U = 17\) with \(\psi(17) = 0.01930\) under the Sparre Andersen model. When \(Q(u,k) = 1.6E(R_{u,k})\), the optimal \(u^*\) is 4.84 with \(k^* = 2.68\) when \(U = 7\), giving a 27.69% reduction in the ultimate ruin probability to \(\psi^*_k(u) = 0.12316\). When \(U\) increases to 17, the percentage reduction increases to 99.84% with \(u^* = 13.41\) and \(k^* = 11.25\), giving \(\psi^*_k(u) = 0.00003\). In the case where \(Q(u,k) = E(R_{u,k}) + 2St.Dev.(R_{u,k})\), when \(U\) is at 7, 9 and 11, the reinsurance premium is too expensive such that it is optimal not to buy any reinsurance. At \(U = 13\), a reinsurance arrangement with optimal \(u^* = 11.12\) and \(k^* = 1.98\) is feasible and reduces the ultimate ruin probability slightly by 1.58% from \(\psi(13) = 0.04611\) to \(\psi^*_k(u) = 0.04538\). The percentage reduction continues to increase as \(U\) further increases, reaching 62.18% when \(U = 17\). Compare the results to Table 3.5.1, we see that the lower barrier model proves to be more effective under the Erlang(2) process than under the classical process.

A concern from Table 6.2.1 is that the optimal lower barrier \(k^*\) is relatively large for most values of \(U\), given that the mean and the standard deviation of the individual claim sizes are both 1. For example in the first case of the lower barrier model, the optimal \(u^*\) is 11.41 with \(k^* = 9.25\) when \(U = 15\). Given the assumption for the individual claim
sizes, the probability that a claim exceeds 9.25 is very small. The optimal values may not be acceptable to a reinsurer. Hence, in the next example, we investigate the effects of the lower barrier model when \( k \) is set at a comparatively low level.

**Example 6.2.3. Capital release.**

In this example, we set the lower barrier \( k = 2 \) for the lower barrier model. The assumption for the reinsurance premium is \( Q(u,k) = 1.6E(R_{u,k}) \). Firstly, we calculate the initial capital \( U \) needed so that \( \psi(U) = 1\% , 2\% , \ldots , 5\% \) under the Sparre Andersen model in Table 6.2.2. In the second part of Table 6.2.2, for each \( U \) obtained with \( k = 2 \), we calculate the corresponding values of \( u \). The ultimate ruin probability is then computed, followed by the percentage reduction compared to that under the Sparre Andersen model. In the third part of Table 6.2.2, we calculate the surplus level needed such that \( \psi_k(u) = \psi(U) \) and hence find the amount of capital released \( C_{u,k} = U - u - Q(u,k) \). The percentage is then calculated by \( C_{u,k}/U \times 100\% \).

<table>
<thead>
<tr>
<th>( U )</th>
<th>( \psi(U) )</th>
<th>( k = 2 )</th>
<th>( k = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( u )</td>
<td>( \psi_k(u) )</td>
<td>( % )</td>
</tr>
<tr>
<td>12.63</td>
<td>5%</td>
<td>12.38</td>
<td>0.03405</td>
</tr>
<tr>
<td>13.65</td>
<td>4%</td>
<td>13.46</td>
<td>0.02694</td>
</tr>
<tr>
<td>14.97</td>
<td>3%</td>
<td>14.83</td>
<td>0.01999</td>
</tr>
<tr>
<td>16.84</td>
<td>2%</td>
<td>16.74</td>
<td>0.01319</td>
</tr>
<tr>
<td>20.02</td>
<td>1%</td>
<td>19.97</td>
<td>0.00653</td>
</tr>
</tbody>
</table>

Table 6.2.2: Results with fixed \( k \), Exponential claims

The results from Table 6.2.2 show that even with a fixed \( k \) that is comparatively small, the lower barrier model is still able to produce a considerable reduction in the ultimate ruin probability. Starting from \( \psi(U) = 0.05 \), the capital required under the Sparre Andersen model is 12.63. Under the lower barrier model with \( k = 2 \), the initial surplus \( u \) is 12.38 which then gives \( \psi_2(12.38) = 0.03405 \). The reduction is 31.90\%. For \( \psi(U) = 1\% \), \( U \)
is found to be 20.02 and the ultimate ruin probability under the lower barrier model is then 0.00653, \(43.70\%\) less than 0.01. Instead of reducing the ultimate ruin probability, if we want to keep the ultimate ruin probability unchanged, we see that a smaller amount of capital is needed under the lower barrier model. For example, to have \(\psi_2(u) = 5\%\), the initial surplus should be 10.63. Including the reinsurance premium \(Q(10.63, 2)\), the total capital needed is 10.98, giving a capital release of \(C_{u,k} = 1.65\).

In addition to the ultimate ruin probability, we are also interested in the distribution of the deficit at ruin. However, due to the memoryless property of the exponential distribution, we see that the conditional distributions of the deficit at ruin are the same under the Sparre Andersen model and the lower barrier model. We hence omit further discussion here.

### 6.2.2 Erlang(2) claim sizes

In this section, the individual claim sizes are assumed to have an Erlang(2) distribution with \(p(x) = 4xe^{-2x}\). Lundberg’s fundamental equation then has a unique positive solution \(\rho = 2.4207\). The other two non-zero solutions are \(-\rho_1 = -0.33333\) and \(-\rho_2 = -2.75403\). The ultimate ruin probability \(\psi(0)\) is 0.77050. It then follows from equation (2.1.2) that

\[
\psi(u) = 0.79007e^{-\rho_1 u} - 0.01957e^{-\rho_2 u}
\]

and from equation (6.2.2) that

\[
G(u, y) = 0.79007e^{-\rho_1 u}(1 - e^{-2y}) - 0.43263e^{-\rho_1 u - 2y}y - 0.01957e^{-\rho_2 u}(1 - e^{-2y}) - 0.19573e^{-\rho_2 u - 2y}y.
\]
We now use these functions to provide some numerical illustrations.

**Example 6.2.4. Expected value principle reinsurance premium.**

As before, we firstly look at the ultimate ruin probability under the lower barrier model. We set an initial capital level $U = 8$. Under the Sparre Andersen model, we have $\psi(8) = 0.05490$. Assuming $Q(u, k) = 1.6E(R_{u,k})$, the smallest initial surplus level is $u = k = 4.25$. For $4.25 \leq u \leq 8$, the corresponding lower barrier $k$ is calculated. In Figure 6.2.3 we plot $\psi_k(u)$ for all feasible combinations of $u$ and $k$.

![Figure 6.2.3: $\psi_k(u)$ and $\psi(U)$, $U = 8$, $Q(u, k) = 1.6E(R_{u,k})$, Erlang(2) claims](image-url)
Example 6.2.5. **Standard deviation principle reinsurance premium.**

In this example, we assume $Q(u,k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k})$ and set $U = 10$. The ultimate ruin probability under the Sparre Andersen model is then 0.02819. The lowest possible $u$ is 2.70, giving $\psi_{2.7}(2.7) = 0.04716$. For $2.70 \leq u \leq 10$, we plot the corresponding ultimate ruin probabilities under the lower barrier model in Figure 6.2.4.

![Figure 6.2.4: $\psi_k(u)$ and $\psi(U)$, $U = 10$, $Q(u,k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k})$, Erlang(2) claims](image)

In Figure 6.2.3, starting from $u = 2.70$, the ultimate ruin probability decreases as $u$ increases, until it reaches a minimum of $\psi_k^*(u) = 0.00186$ at $u^* = 5.57$ and $k^* = 4.26$. It then increases until when $u = U = 8$, and hence $k = 0$, giving an ultimate ruin probability which is the same as under the Sparre Andersen model. We see that when
\[ Q(u,k) = 1.6E(R_{u,k}) \] and \( U = 8 \), any feasible \( u \) and \( k \) give an ultimate ruin probability smaller than that under the Sparre Andersen model. This is not the case in Figure 6.2.4. When \( Q(u,k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k}) \) and \( U = 10 \), we see that the lowest value of \( u \) at 2.70 yields an ultimate ruin probability of 0.04716, larger than \( \psi(10) = 0.02819 \) under the Sparre Andersen model. As \( u \) increases, the ultimate ruin probability decreases to its minimum and then increases. The optimal \( u^* \) is 6.67 with \( k^* = 2.35 \), giving \( \psi^*_k(u) = 0.01893 \) smaller than \( \psi(10) \). We see that only certain combinations of \( u \) and \( k \) provide a reduction in the ultimate ruin probability. This is because the initial capital level of 10 is not very large. In order to see the effectiveness of the lower barrier model with such a small amount of capital, one must carefully choose the balance between \( u \) and \( k \), i.e. the level of risk that the insurer bears itself and the cost of protection from the reinsurer.

In Table 6.2.3, we show the ultimate ruin probabilities under the Sparre Andersen model and the two scenarios under the lower barrier model, with a number of different values of \( U \).

<table>
<thead>
<tr>
<th>( U )</th>
<th>( \psi(U) )</th>
<th>( Q(u,k) = 1.6E(R_{u,k}) )</th>
<th>( Q(u,k) = E(R_{u,k}) + 2\text{St. Dev.}(R_{u,k}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.14923</td>
<td>( u^* = 3.32 ) ( k^* = 1.95 ) ( \psi^*_k(u) = 0.09520 ) 38.02%</td>
<td>( u^* = 5 ) ( k^* = 0 ) ( \psi^*_k(u) = 0.14923 ) 0.00%</td>
</tr>
<tr>
<td>7</td>
<td>0.07661</td>
<td>( u^* = 4.66 ) ( k^* = 3.33 ) ( \psi^*_k(u) = 0.00974 ) 87.29%</td>
<td>( u^* = 7 ) ( k^* = 0 ) ( \psi^*_k(u) = 0.07661 ) 0.00%</td>
</tr>
<tr>
<td>9</td>
<td>0.03934</td>
<td>( u^* = 6.55 ) ( k^* = 5.24 ) ( \psi^*_k(u) = 0.00301 ) 99.21%</td>
<td>( u^* = 9 ) ( k^* = 1.70 ) ( \psi^*_k(u) = 0.03634 ) 65.20%</td>
</tr>
<tr>
<td>11</td>
<td>0.02020</td>
<td>( u^* = 8.54 ) ( k^* = 7.24 ) ( \psi^*_k(u) = 7.41 	imes 10^{-6} ) 99.96%</td>
<td>( u^* = 11 ) ( k^* = 3.05 ) ( \psi^*_k(u) = 0.00031 ) 99.96%</td>
</tr>
<tr>
<td>13</td>
<td>0.01037</td>
<td>( u^* = 10.53 ) ( k^* = 9.24 ) ( \psi^*_k(u) = 1.67 	imes 10^{-7} ) 100.00%</td>
<td>( u^* = 13 ) ( k^* = 4.85 ) ( \psi^*_k(u) = 0.00031 ) 97.01%</td>
</tr>
</tbody>
</table>

Table 6.2.3: Ruin probabilities, Erlang(2) claims

From Table 6.2.3, we firstly see that although the individual claim sizes have the same mean of 1, the ultimate ruin probabilities under the Erlang(2) claims are much smaller, compared to the previous section where the individual claims are exponentially distributed. In Table 6.2.3, when \( U = 5 \), \( \psi(5) \) is 0.14923 under the Sparre Andersen
6.2. Erlang(2) inter-claim times

model. It decreases to 0.01037 when $U = 13$. In the first case of the lower barrier model where $Q(u, k) = 1.6E(R_{u,k})$, we see that the optimal $u^*$ is 3.32 with $k^* = 1.95$ when $U = 5$. The resulting ultimate ruin probability is 0.09250, a 38.02% of reduction on that under the Sparre Andersen model. The percentage reduction increases rapidly when $U$ increases from 5 to 13. In fact, when $U = 13$, the ultimate ruin probability with optimal $u^*$ and $k^*$ is very close to 0 such that the percentage reduction is seen as 100%. When $Q(u, k) = E(R_{u,k}) + 2St.Dev.(R_{u,k})$, once again we see that when $U$ is small, reinsurance is too expensive so that it is optimal for the insurer to keep all the risks itself, i.e. $k = 0$ and $u = U$. When $U$ increases to 9, the optimal $k^*$ becomes 1.70 with $u^* = 6.91$, meaning that a reinsurance arrangement is obtainable. The ultimate ruin probability decreases by 7.63% from 0.03934 under the Sparre Andersen model to 0.03634 under the lower barrier model. The percentage reduction jumps from 7.63% to 65.20% as $U$ increases from 9 to 11. When $U = 13$, the optimal $u^*$ is 8.30 and $k^*$ is 4.85, giving an ultimate ruin probability of 0.00031, which is 91.01% smaller than under the Sparre Andersen model. It shows that the effects of the lower barrier model strengthens quickly as initial capital increases.

Compare to Table 3.5.5 under the exponential inter-claim times, we again see that the lower barrier model reduces the ultimate ruin probability more effectively under this Erlang(2) process.

Just as under exponential claim sizes, one might argue that in Table 6.2.3, the optimal levels of $k$ are sometimes so large such that the reinsurance arrangements are not appealing to the reinsurer. Therefore, in the next example we keep the lower barrier $k$ at a reasonably low level and investigate the ultimate ruin probability under the lower barrier model.
Example 6.2.6. Capital release.

In this example, the lower barrier $k$ is unchanged at 2. The reinsurance premium is assumed to be calculated using the expected value principle with a loading factor of 0.6. We firstly find the capital required under the Sparre Andersen model such that the ultimate ruin probability $\psi(U)$ is at 1%, 2%, ..., 5% level. For each $U$ computed, with $k = 2$, we compute the initial surplus $u$ such that $u + Q(u, 2) = U$. The ultimate ruin probability under the lower barrier model is then produced, together with the percentage reduction compared to the Sparre Andersen model. For an alternative comparison, we find the initial surplus $u$ such that $\psi_k(u)$ is 1%, 2%, ..., 5%. With given $u$ and $k$, the total capital required under the lower barrier model is then $u + Q(u, k)$. Compared to the Sparre Andersen model, the amount of capital released is then $C_{u,k} = U - u - Q(u, k)$.

All numbers are provided in Table 6.2.4.

<table>
<thead>
<tr>
<th>$U$</th>
<th>$\psi(U)$</th>
<th>$k = 2$</th>
<th>$k = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$u$</td>
<td>$\psi_k(u)$</td>
</tr>
<tr>
<td>8.28</td>
<td>5%</td>
<td>7.90</td>
<td>0.01904</td>
</tr>
<tr>
<td>8.95</td>
<td>4%</td>
<td>8.65</td>
<td>0.01481</td>
</tr>
<tr>
<td>9.81</td>
<td>3%</td>
<td>9.60</td>
<td>0.01081</td>
</tr>
<tr>
<td>11.03</td>
<td>2%</td>
<td>10.89</td>
<td>0.00703</td>
</tr>
<tr>
<td>13.11</td>
<td>1%</td>
<td>13.04</td>
<td>0.00343</td>
</tr>
</tbody>
</table>

Table 6.2.4: Results with fixed $k$, Erlang(2) claims

In Table 6.2.4, we see that when $k$ is fixed at 2, the lower barrier model is still able to reduce the ultimate ruin probability by a meaningful amount. To have an ultimate ruin probability of 0.05 under the Sparre Andersen model, the amount of capital required is 8.28. The same amount of capital gives an ultimate ruin probability of 0.01904 under the lower barrier model, with $u = 7.90$ and $k = 2$. A 61.92% of reduction is seen. When $U = 13.11$ such that $\psi(13.11) = 0.01$, the lower barrier model reduces the ultimate ruin probability by 65.70% to $\psi_2(13.04) = 0.00343$ using the same $U$. We see that the
percentage of ruin probability reduction increases as \( U \) increases. Alternatively, we see that a smaller amount of capital is needed under the lower barrier model than under the Sparre Andersen model to provide the same ultimate ruin probability. In order to have \( \psi_k(u) = 5\% \) when \( k = 2 \), the initial surplus \( u \) is 5.00. The cost of reinsurance with the assumption that \( Q(u, k) = 1.6E(R_{u,k}) \) is then \( Q(5.00, 2) = 3.00 \). The total capital \( U \) is 8, smaller than that under the Sparre Ander model by 2.28. In addition, we see that the amount of capital released \( C_{u,k} \) increases as \( \psi(U) \) decreases and \( U \) increases.

**Example 6.2.7. Distribution of the deficit at ruin.**

In this example, instead of examining the properties of the ultimate ruin probability, we look at the changes in the deficit at ruin when a lower barrier is introduced under the Sparre Andersen model. For comparison, we set the initial capital \( U = 9 \) and from Table 6.2.3, the ultimate ruin probability is 0.03934. The optimal value of \( u^* \) under the first case of the lower barrier model is 6.55, with \( k^* = 5.24 \), giving \( \psi_{5.24}(6.55) = 0.00031 \). In the second case, \( u^* \) is 6.91 and \( k^* = 1.70 \). In Figure 6.2.5, we plot the conditional density functions of the deficit at ruin. For the Sparre Andersen model, we plot \( g(U, y) / \psi(U) \), and for lower barrier models, we plot \( g_{u,k}(y) / \psi_k(u) \), where \( g_{u,k}(y) \) is obtained from equation (4.2.2) in Chapter 4. Then in Figure 6.2.6, we plot the corresponding conditional distribution functions for each case.

In Figure 6.2.5, we see that compared to the Sparre Andersen model, under the lower barrier model the conditional density of the deficit at ruin starts from a higher value and decreases at a faster rate. The set of values was chosen because the two cases under the lower barrier model have considerably different values of the lower barrier \( k \). The first case has a much smaller gap between \( u \) and \( k \) with \( u = 6.55 \) and \( k = 5.24 \), compared to the second case where \( u = 6.91 \) and \( k = 1.70 \). As expected, when \( k \) is smaller at 1.70, the conditional density of the deficit at ruin under the lower barrier model is closer to
Chapter 6. Lower barrier in the Sparre Andersen risk model

The Sparre Andersen model than that when $k = 5.24$. From the conditional distribution functions plotted in Figure 6.2.6, we see that given that ruin occurs, the deficit of ruin under the lower barrier model has a higher probability of being small than that under the Sparre Andersen model. For example, the conditional probability that the deficit at ruin is smaller than 1 under the Sparre Andersen model is $G(9,1)/\psi(9) = 0.79056$. In the first case of the lower barrier model, this probability is $G_{6.55,5.24}(1)/\psi_{5.24}(6.55) = 0.84404$, and under the second case it is $G_{6.91,1.70}(1)/\psi_{1.70}(6.91) = 0.82035$, both larger than 0.79056.

Example 6.2.8. **Conditional expected deficit at ruin.**

Now in Table 6.2.5, we show the conditional expect value of the deficit at ruin, using
6.2. Erlang(2) inter-claim times

Figure 6.2.6: Conditional Deficit at Ruin distribution functions, Erlang(2) claims

values of \( U, u^* \) and \( k^* \) provided in Table 6.2.3.

From Table 6.2.5, we see that under the Sparre Andersen model, the expected deficit at ruin, given that ruin occurs, is 0.63690 for all \( U \). This is lower than that under the classical risk model from Table 4.2.1 where \( E[U(T_U) | T_U < \infty] \) is 0.65984 for all \( U \).

Under the lower barrier model, we see in Table 6.2.5 that \( E[U(T_{u,k}) | T_{u,k} < \infty] \) decreases with optimal values of \( u^* \) and \( k^* \). In the first case of the lower barrier model, when \( U = 5 \), the conditional expected deficit at ruin is 0.57599. It decreases to 0.52367 when \( U \) increases to 13. This decreasing pattern is also seen in the second case of the lower barrier model. For the same values of \( U \), we see that the expected deficit at ruin are larger.
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\[ Q(u,k) = 1.6E(R_{u,k}) \]

\[ \frac{Q(u,k)}{E(U(T_\infty))} = 1.6 \]

\[ \frac{Q(u,k) - 2 \text{St.Dev.}(R_{u,k})}{E(U(T_\infty))} = 1.6 \]

\[ U \]

<table>
<thead>
<tr>
<th>U</th>
<th>0.63690</th>
<th>0.63690</th>
<th>0.63690</th>
<th>0.63690</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ E[U(T_\infty)] ]</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
</tr>
<tr>
<td>[ T_\infty &lt; \infty ]</td>
<td>3.32</td>
<td>4.66</td>
<td>6.55</td>
<td>8.54</td>
</tr>
<tr>
<td>[ k^* ]</td>
<td>1.95</td>
<td>3.33</td>
<td>5.24</td>
<td>7.24</td>
</tr>
<tr>
<td>[ E[U(T_\infty)] ]</td>
<td>0.57599</td>
<td>0.55372</td>
<td>0.53810</td>
<td>0.52920</td>
</tr>
<tr>
<td>[ T_\infty &lt; \infty ]</td>
<td>5</td>
<td>7</td>
<td>6.91</td>
<td>6.85</td>
</tr>
<tr>
<td>[ k^* ]</td>
<td>0</td>
<td>0</td>
<td>1.70</td>
<td>3.05</td>
</tr>
<tr>
<td>[ E[U(T_\infty)] ]</td>
<td>0.63690</td>
<td>0.63690</td>
<td>0.58186</td>
<td>0.55790</td>
</tr>
<tr>
<td>[ T_\infty &lt; \infty ]</td>
<td>0</td>
<td>0</td>
<td>5.4048</td>
<td>5.50</td>
</tr>
</tbody>
</table>

Table 6.2.5: Conditional Expected Deficit at Ruin, Erlang(2) claims

than in Table 4.2.1 under the classical process.

6.3 Erlang(n) inter-claim times

In this section, we assume that the waiting times are i.i.d. and follow an Erlang(n) distribution with density function

\[ k(t) = \beta^nt^{n-1}e^{-\beta t}/\Gamma(n) \]

In addition, we assume that the individual claim sizes are exponentially distributed with density function

\[ p(x) = \alpha e^{-\alpha x} \]

We now find an expression for the distribution of the time to ruin under the lower barrier model, conditioning on ruin occurring.

Due to the memoryless property of the exponential distribution, we have that

\[ w_u(y,t) = w_u(t)\alpha e^{-\alpha y}, \]

and hence the defective p.d.f. of the time to ruin under the lower barrier model follows equation (5.2.18) where \( w_{k,k}(t) \) follows equation (5.2.17). From Dickson and Borovkov (2008), we have that

\[
    w_0(t) = (\beta t)^n cne^{-\alpha t - \beta t} \sum_{m=0}^{\infty} \frac{\alpha^m (ct)^m}{m!} \frac{(\beta t)^{nm}}{\Gamma(nm + n + 1)} = \sum_{m=0}^{\infty} \frac{n(nm + n + m - 1)!}{m!(nm + n)!} \frac{\alpha^m c^m \beta^{nm+n}}{\alpha c + \beta n m + n + m}
\]
6.3. Erlang($n$) inter-claim times

\[
\times \frac{(\alpha + \beta)^{nm+n+m} e^{-(\alpha+\beta)t} t^{nm+n+m-1}}{\Gamma(nm + n + m)}
\]

\[
= \sum_{m=0}^{\infty} \frac{n}{nm + n + m} \binom{nm + n + m}{m} \left( \frac{\alpha c}{\alpha c + \beta} \right)^m \left( \frac{\beta}{\alpha c + \beta} \right)^{nm+n} \epsilon_{nm+n+m, \alpha c+\beta}(t)
\]

(6.3.1)

The Laplace transform of $w_0(t)$ is then

\[
\tilde{w}_0(s) = \sum_{m=0}^{\infty} \frac{n}{nm + n + m} \binom{nm + n + m}{m} \left( \frac{\alpha c}{\alpha c + \beta} \right)^m \left( \frac{\beta}{\alpha c + \beta} \right)^{nm+n} \times \left( \frac{\alpha c + \beta}{\alpha c + \beta + s} \right)^{nm+n+m}
\]

\[
= \left( \frac{\beta}{\alpha c + \beta + s} \right)^n \sum_{m=0}^{\infty} \frac{n}{m(n+1)+n} \binom{m(n+1)+n}{m} Z^m,
\]

(6.3.2)

where

\[
Z = \frac{\alpha c \beta^n}{(\alpha c + \beta + s)^{n+1}}.
\]

We see that $\tilde{w}_0(s)$ can be written in terms of the generalised binomial function $B_l(z)$ defined in equation (5.2.23), and therefore the Laplace transform of $r$-fold convolution of $w_0(t)$ is then

\[
\tilde{w}_0^s(s) = [\tilde{w}_0(s)]^r = \left( \frac{\beta}{\beta + \alpha c + s} \right)^{nr} B_{n+1}(Z)^{nr}
\]

\[
= \beta^{nr} \sum_{m=0}^{\infty} \binom{n+1}{m} m_{nr} \frac{(\alpha c \beta)^m}{(n+1)m + nr (\alpha c + \beta + s)^{m(n+1)+nr}},
\]

which by inversion gives

\[
w_0^s(t) = \beta^{nr} t^{nr-1} n r e^{-\alpha c + \beta t} \sum_{m=0}^{\infty} \frac{(\alpha c \beta)^{m+1} (t^{m+1})^m}{m!(n(r + m))!}.
\]
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For computational purpose, we can write $w_0^*(t)$ in terms of generalised hyper-geometric function as

$$w_0^*(t) = \frac{\beta^{nr} t^{nr-1} e^{-\alpha c + \beta t}}{\Gamma(nr)} \left( r + 1, r + 2, \ldots, r + \frac{n}{n}; \frac{\alpha c \beta^n t^{n+1}}{n^n} \right).$$  \hspace{1cm} (6.3.3)

From equation (5.2.20) and equation (5.2.18), combining with equation (6.3.3) we have a readily computable formula for $w_{u,k}(t)$.

6.4 Concluding remarks

In this chapter, we have further investigated the lower barrier model under the Sparre Andersen model and more specifically the Erlang(2) risk model and a short discussion for the Erlang($n$) risk model. With two assumptions for the individual claim sizes, namely the exponential distribution and the Erlang(2) distribution, we use some numerical illustrations and discussed the effects on the ultimate ruin probability and the deficit at ruin. In general, we see that a reinsurance arrangement to insert a lower barrier in the risk process could possibly provide an effective mechanism to reduce the ultimate ruin probability, given certain assumptions for reinsurance charge, initial capital amount and the individual claim size distributions. We see that the lower barrier model is more effective in reducing the ultimate ruin probability under the Erlang(2) risk processes assumed than in Chapter 3 under the classical risk processes assumed. In addition, the lower barrier model alters the distribution of the deficit at ruin and the changes are dependent on the claim size distributions. When we assume a Erlang(2) claim size, the deficit under the lower barrier model is smaller than that under the Erlang(2) risk model. The insurer could possibly use the lower barrier model as a control mechanism for different purposes.
Chapter 7

Conclusion

In this thesis, we have proposed a modification on a traditional surplus process by introducing a lower barrier \( k \). Each time the surplus falls below \( k \) but above 0, an injection of capital is provided to bring the surplus level back to \( k \) and the risk process continues. We have assumed that the surplus injection is provided by a reinsurer with certain reinsurance premium. Under this lower barrier settings, we have investigated the changes in the ultimate ruin probability, the deficit at ruin, the surplus before ruin, the time to ruin and some joint distributions, compared to that under the two risk models, namely the classical risk model and the Sparre Andersen model.

For the ultimate ruin probability, we see that the lower barrier model can effectively reduce the ruin probability under the reinsurance premium we assumed for the same amount of initial capital provided. The distribution of the deficit at ruin is altered but the effects depend on the claim sizes assumption. The surplus before ruin (in excess of \( k \)) under the lower barrier model is smaller than that under the original risk models. The time to ruin is generally shorter, given ruin occurs.

In addition to the works presented in this thesis, there is a wide range of possible
further research opportunities based on the lower barrier framework proposed. One can possibly use the Gerber-Shiu function to study the ruin probability, the deficit at ruin and the surplus before ruin in a unified approach under the lower barrier model. For the purpose of illustration, the individual claim sizes in the examples considered in this thesis are assumed to have certain types of distributions such that moment generating functions exist. However, it is known that one of the main reasons for an insurer to purchase a reinsurance contract is to protect against the occurrence of large claims. The assumed claim sizes are relatively small compared to claim sizes from distributions for which moment generating functions do not exist, for example, the Pareto distribution. An extension of current research can be examining the lower barrier model with heavy-tailed individual claim sizes. Explicit expressions for the ruin probability, the mean and variance of the expected total claim amount covered by the reinsurer are not available, but it is possible to use a numerical approach. From published studies of reinsurance, it is well known that the optimal reinsurance strategy for an insurer to minimize its ruin probability is the excess-of-loss reinsurance under certain model assumptions and reinsurance principles. The proposed reinsurance structure provides a new reinsurance strategy for an insurer. A numerical comparison between the excess-of-loss reinsurance with the lower barrier reinsurance can be another interesting research subject.

An important point that distinguishes the reinsurance arrangement in this thesis from other studies is that the reinsurance premium is paid for from the insurer’s initial capital. In other studies (see, for example, Bowers et al. (1997) or Centeno (1986)), it is assumed that the reinsurance premium is paid from the insurer’s premium income. Our approach allows for the release of capital in a way that cannot occur if reinsurance is purchased from the insurer’s premium income.

This thesis is written in terms of a direct insurer and a reinsurer. However, as we
are considering a portfolio of risks, it may be possible to secure ‘reinsurance’ within the insurance company by passing the risk onto a different line of business within the company. It is important to appreciate that the type of reinsurance risk associated with this study is quite different to traditional reinsurance arrangements. In many such arrangements, for example excess of loss reinsurance, the insurer is seeking to limit the amount paid out on any individual claim. By contrast, under the reinsurance arrangement in this study, it is the reinsurer’s payments that are bounded above by $k$, and, depending on the method of calculating the reinsurance premium, the value of $k$ can be quite low relative to the mean individual claim amount for the insurer, although it can also be quite high. Following our numerical examples, we discuss the level of reinsurance.

Capital injections have been discussed by authors such as Pafumi (1998), Dickson and Waters (2004) and Eisenberg and Schmidli (2010). A major difference in this study is that capital injections occur before the surplus falls below zero, and the capital injections in this study do not eliminate the possibility of ruin for the insurer.
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finite and in infinite time”. ASTIN Bulletin 22, 177-190.


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