Characters of $p$-Compact Groups

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0.1 Abstract

This thesis contains a survey of homotopical group theory, integral and \(p\)-adic reflection group theory, and sets the stage to study characters of \(p\)-compact groups.

Chapter 1 motivates and introduces \(p\)-compact groups as defined by Dwyer and Wilkerson [22] in their original setting, as homotopical analogs for compact Lie groups. In particular it explores how the structure of the maximal torus and the Weyl group of a connected compact Lie group pass to the corresponding structures in a finite loop space, and that these structures then pass via \(H\mathbb{F}_p\)-localization to the defined structures on the associated \(p\)-compact group.

Chapter 2 is concerned with the connections between \(p\)-adic reflection groups and \(p\)-compact groups. It considers the topological realization of reflection groups over the \(p\)-adic integers, and goes through the construction of non-modular \(p\)-compact groups, as motivated by the spaces created by Clark and Ewing [18]. It then explores the functorial connection between these reflection groups and \(p\)-compact groups.

Chapters 3 and 4 look at studying character theory of compact connected Lie groups via equivariant \(K\)-theory and postulate that one can use \(K\)-theory with \(p\)-adic coefficients as studied by Jeanneret and Osse [32], and the stable transfer map of Bauer [10] to extend this approach to study characters of \(p\)-compact groups. In the end, we conjecture a Weyl formula for characters induced up from the maximal torus.
0.2 Declaration

This is to certify that

1. the thesis comprises only my original work towards the MPhil except where indicated,

2. due acknowledgement has been made in the text to all other material used,

3. the thesis is less than 50,000 words in length, exclusive of tables, maps, bibliographies and appendices.
0.3 Acknowledgments

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1 Homotopy Lie Groups

The study of Lie groups has been an important facet of recent mathematics, with many applications, occurring as the symmetry groups of classical geometries and gauge theories that describe the dynamics of elementary particle physics. In this chapter we will study compact Lie groups with the machinery of homotopy theory, which will lead to the definition of $p$-compact groups, as introduced by Dwyer and Wilkerson [22]. First we will highlight some key facts and properties of compact Lie groups and then we will show how they extend via homotopy theory to finite loop spaces. Then we will introduce the theory of $p$-completion and the world of $p$-compact groups.

Recall that a compact Lie group is a compact differentiable manifold $G$, together with smooth multiplication and inversion operations. These operations turn $G$ into a topological group. Every finite group can be viewed as a 0-dimensional compact Lie group with the discrete topology, but they miss out on the rich structure of the higher-dimensional groups. Additionally, at many points we will only consider connected compact Lie groups, where the only 0-dimensional example is the trivial group. The simplest higher-dimensional example is the circle group,

$$S^1 \cong \mathbb{R}/\mathbb{Z},$$

where the smooth group structure is inherited from $\mathbb{R}$.

If $G$ is a compact Lie group then a torus in $G$ is a connected abelian subgroup $T \subseteq G$ homeomorphic to some finite-dimensional standard torus,

$$T \cong S^1 \times \ldots \times S^1.$$

The rank of $T$ is the (finite) number of factors of $S^1$ in such a product decomposition. A maximal torus of $G$ is a torus in $G$ that is maximal among such subgroups, and the rank of $G$ is the rank of a maximal torus $T$. Maximal tori exist; one can see that they coincide with maximal connected abelian subgroups. It is a main theorem of Lie theory that if $G$ is a connected compact Lie group and if $T$ and $T'$ are two
maximal tori of $G$, then there exists $g \in G$ such that

$$T' = gTg^{-1},$$

i.e. $T$ and $T'$ are conjugate subgroups of $G$. Additionally, every element of $G$ is contained in some maximal torus.

Let $G$ be a compact Lie group, with $T \subseteq G$ a maximal torus of $G$. The Weyl group of $G$ is the group

$$W := N(T)/T = \{ g \in G | gTg^{-1} = T \}/T,$$

where $N(T)$ is the normalizer of the maximal torus, acting on $T$ by conjugation. Equivalently, one can define the Weyl group as the fixed point set of the left translation action of $G$ on the homogeneous space $G/T$. Since any two maximal tori are conjugate, any other choice of maximal torus $T$ leads to an isomorphic Weyl group [15, IV.§1].

Other important examples of compact Lie groups are the compact forms of the classical groups, which arise as groups of symmetries of complex, real, and quaternionic Euclidean space. Recall that a Lie group is simple if it is a connected, non-abelian Lie group such that all connected normal subgroups are trivial, and semisimple if it is a direct product of simple Lie groups.

**Example 1.** Recall that a complex $n \times n$ matrix $A$ is unitary if its conjugate transpose equals its inverse, i.e., $\bar{A}^t = A^{-1}$. The set of all $n \times n$ unitary matrices forms a compact Lie group with respect to matrix multiplication,

$$U(n) := \{ A \in GL_n(\mathbb{C}) | \bar{A}^tA = A\bar{A}^t = I \}$$

called the unitary group of rank $n$. Note that

$$U(1) \cong \{ z \in \mathbb{C}^\times : |z| = 1 \} \cong S^1$$

is the standard 1-torus. The subset of all $n \times n$ unitary matrices of determinant 1 is again a compact Lie group,

$$SU(n) := \{ A \in U(n) | det(A) = 1 \}$$
called the *special unitary group*. It is a simply connected, simple Lie group (for $n > 1$).

Let $\Delta(n) \subseteq U(n)$ be the subgroup of diagonal matrices

$$
\begin{bmatrix}
  z_1 \\
  \vdots \\
  z_n
\end{bmatrix},
$$

and let $S\Delta(n)$ be the intersection $\Delta(n) \cap SU(n)$. Note that being unitary forces each $z_r$ in $\Delta(n)$ to be on the circle of complex modulus 1, and in $S\Delta(n)$ the product of the entries $z_1...z_n$ is equal to 1. Note that $\Delta(n)$ and $S\Delta(n)$ are maximal tori for $U(n)$ and $SU(n)$, respectively [15, IV.3.1], and that $\Delta(n)$ has rank $n$, where $S\Delta(n)$ has rank $n - 1$, due to the relation $z_1...z_{n-1} = z_n^{-1}$. The Weyl groups for $U(n)$ and $SU(n)$ are both given by the symmetric group $\Sigma_n$, acting by permutations[15, IV.3.2, 3.3].

**Example 2.** The special orthogonal groups $SO(n)$ and the symplectic groups $Sp(n)$ are obtained by a similar construction as subgroups of $GL_n(\mathbb{R})$ and $GL_n(\mathbb{H})$, respectively, where $\mathbb{H}$ is the quaternions.

The notions of maximal tori and Weyl groups can be stated homotopically, via the language of classifying spaces:

### 1.1 Classifying spaces and finite loop spaces

Many of the definitions in this section date back to Rector [44], and the main approach will be the study of classifying spaces. If $G$ is a topological group (or, more generally, a topological category), then [46] there is a functorial way to associate to $G$ a topological space $BG$, such that many of the algebraic invariants of the group $G$ become topological invariants of the classifying space $BG$. We will let $B(-)$ denote such a functor from the category of topological groups to the category of (compactly generated) spaces. In particular, if $G$ is a compact Lie group, then we may apply the classifying space functor to $G$ to obtain a pointed space $BG$, called a *classifying space* for $G$.  

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Recall that a (smooth) fiber bundle $p : E \to B$, often denoted

$$\begin{array}{ccc}
F & \longrightarrow & E \\
& & \downarrow{p} \\
& & B
\end{array}$$

or $p = (E \downarrow B)$, is a (smooth) continuous surjection $p : E \to B$ that is locally trivial. That is, given a $b \in B$, there is an open neighborhood $U_b \subseteq B$ such that the inverse image $p^{-1}(U_b)$ is homeomorphic to the product space $U_b \times F$ for some fixed space $F$, called the fiber of the bundle. Thus the fibers over each point (in each connected component) of $B$ are homeomorphic to $F$. A local trivialization of $p$ is a set of pairs $\{(U_b, \phi_b)\}$, such that the set $\{U_b\}$ is an open cover of $B$ and the $\phi_b : U_b \times F \to U_b$ are all homeomorphisms that $p$ factors through, such that

$$p^{-1}(U_b) \xrightarrow{\phi_b} U_b \times F$$

commutes for each $U_b$. We call $E$ and $B$ the total space and base space of $p$, respectively.

**Definition 3.** Recall that a principal $G$-bundle is a fiber bundle $p = (E \downarrow B)$ where the fiber $F$ is equipped with a free and transitive action of the topological group $G$, such that for $x, y \in F$ there exists a unique $g \in G$ such that $g.x = y$. We will identify $F$ with $G$ under this action. If $p$ is a principal $G$ bundle with contractible total space then $p$ is a universal $G$-bundle. The term comes from the fact that if $M$ is a (paracompact) manifold with a principal $G$-bundle $q : Q \to M$, then there exists a map $f : M \to BG$ such that $q$ is isomorphic to the pullback $f^*(p)$. In this sense, the space $BG$ 'classifies' principal $G$-bundles. If $p : E \to B$ is a universal $G$-bundle then $B$ is a classifying space for $G$.

If $G$ is a topological group and $EG$ is a contractible space with a free $G$-action, then the projection onto the orbit space

$$p : EG \longrightarrow EG/G := BG$$

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is a principal $G$-bundle. Once we choose basepoints such that $p$ is basepoint-
preserving, $p$ gives a fiber sequence

$$\Omega EG \to \Omega BG \to G \to EG \to BG,$$

where $\Omega X$ denotes the space of based loops in $X$. Since $EG$ is contractible, we get a homotopy equivalence

$$\Omega BG \xrightarrow{\simeq} G.$$

The fiber bundle $p : EG \to BG$ also leads to a long exact sequence of homotopy groups

$$\ldots \to \pi_i(G) \to \pi_i(EG) \to \pi_i(BG) \to \pi_{i-1}(G) \to \ldots \to \pi_0(BG) = 0,$$

from which one can easily see that

$$\pi_i BG = \pi_{i-1} G,$$

for all $i > 0$. Additionally, if $g$ is a selfmap of $G$ that is induced by conjugation, then $B g := B(g)$ is a self-homotopy of $BG$ that is homotopic to the identity. [46]

**Remark 4.** In what follows we shall need a functorial association $G \mapsto BG$ for arbitrary topological groups $G$. There are several choices for this, but we will generally take $BG := EG/G$, where $EG := B(\ast, G, G)$ is the bar construction. For some constructions, we may need to consider the (fat) geometric realization of the nerve of the category associated to $G$. [46]

**Example 5.** Let $G$ be a group and $n \in \mathbb{Z}_{>0}$ a positive integer. Recall that a connected topological space $X$ is an *Eilenberg-MacLane space of type $K(G, n)$* if the homotopy groups of $X$ are

$$\pi_i(X) = \begin{cases} G, & i = n \\ 0, & i \neq n \end{cases}$$

If $n = 1$ or $G$ is abelian, then there exists a $CW$-complex of type $K(G, n)$, unique up to weak homotopy equivalence (the restriction on abelian groups is due to the
fact that if $Y$ is any topological space and $n > 1$ then $\pi_n(Y)$ is an abelian group). It follows that if $G$ is a discrete group and $n = 1$, then the Eilenberg-MacLane space $K(G,1)$ is a classifying space for $G$. It is worth note that if $G$ is abelian, then the inclusion of $G$ into $EG$ is a normal subgroup, and $K(G,1) = BG = EG/G$ is again an abelian group, and we can take its classifying space again, obtaining a $K(G,2)$. We are able to iterate this indefinitely, and the sequence of $K(G,n)$ forms the *Eilenberg-MacLane spectrum* of type $G$.

**Example 6.** The discrete group $Z$ acts freely on the contractible space $\mathbb{R}$ via multiplication. Thus the circle $S^1 = \mathbb{R}/Z$ is a model for $BZ = K(Z,1)$.

**Example 7.** Each sphere $S^n$ has a natural free $S^1$-action via rotation, and the limit as $n$ goes to infinity, denoted $S^\infty$, is contractible. Thus for $G = U(1) \cong S^1$, we may take $EU(1) = S^\infty$. The classifying space $BU(1)$ is thus the quotient $S^\infty/U(1) = CP^\infty$, the infinite complex projective space.

**Remark 8.** Up to homotopy, taking classifying spaces is well-behaved on products: If $L \cong \mathbb{Z}^r$ is an integral lattice, then $BL = K(\mathbb{Z}^r,1)$ is homotopy equivalent to an $r$-torus and $B^2L := B(BL)$ is homotopy equivalent to the $r$-fold product $(CP^\infty)^r$.

**Example 9.** Generalizing from $U(1)$ to $U(n)$, let $V_n(\mathbb{C}^k)$ be the Stiefel manifold, consisting of the set of orthonormal $n$-frames in $\mathbb{C}^k$. Then $V_n(\mathbb{C}^\infty)$ is a contractible space with a natural free $U(n)$-action, and we may take $EU(n) = V_n(\mathbb{C}^\infty)$. The quotient $EU(n)/U(n) = BU(n)$ can be recognized as the infinite Grassmannian $G_n(\mathbb{C}^\infty)$, the direct limit of $n$-dimensional subspaces of $\mathbb{C}^k$, as $k$ goes to infinity. The projection map sends an $n$-frame to the subspace it spans.

**Remark 10.** One sees that if $G$ is a topological group and $EG$ is a contractible free $G$-space, and if $H < G$ is a (closed) subgroup, then $EG$ is also a contractible free $H$-space and $EG \to EG/H$ is a principal $H$-bundle. Since every compact Lie group embeds in $U(n)$ for $n$ large enough (by the Peter Weyl theorem), we have just realized models for $EG$ and $BG$ for all compact Lie groups $G$. 


Let $R$ be a ring. A space $X$ is $R$-finite if the $R$-cohomology $H^*(BX; R)$ is a finitely generated $R$-module.

**Definition 11.** A finite loop space $L$ is a space homotopy equivalent to a finite CW-complex, such that $L$ is the pointed loop space $\Omega BL$ of some pointed connected space $BL$. One often refers to the space $BL$ as the ‘classifying space’ of $L$.

**Remark 12.** If $G$ is a compact Lie group, then we have already seen that there is a homotopy equivalence $G \to \Omega BG$. Since $G$ smooth and compact, it is homotopy equivalent to a finite CW-complex, thus $(G, BG, id)$ is a finite loop space. The class of finite loop spaces is much larger than the class of compact Lie groups, however, since there are no restrictions due to geometry or group structure. For example, although there is only one Lie group structure on $SU(2)$ up to isomorphism, there are uncountably many nonhomotopic spaces $Y$ such that $\Omega Y \simeq SU(2)$ [44]. Fortunately, all loop space structures agree after some localization functors are applied, which we will discuss in detail in section 1.3 below.

### 1.2 Dictionary of homotopy-theoretic group theory

We now introduce several homotopical definitions that allow us to extend the terminology of Lie groups to finite loop spaces, which we will later specialize to $p$-compact groups.

Let $L_1$ and $L_2$ be finite loop spaces.

- A **homomorphism** $f : L_1 \to L_2$ of finite loop spaces is a pointed map $Bf : BL_1 \to BL_2$. Two homomorphisms $f, f' : L_1 \to L_2$ are **conjugate** if $Bf$ and $Bf'$ are homotopic as unpointed maps.

- The **homogeneous space** $L_2/f(L_1)$ is the homotopy fiber of $Bf : BL_1 \to BL_2$. We often drop $f$ from the notation when it is understood, and call $L_2/L_1$ the homogeneous space.

- A homomorphism $Bf$ is a **monomorphism** of finite loop spaces if the homogeneous space $L_2/L_1$ is $\mathbb{Z}$-finite, and an **epimorphism** if $L_2/L_1$ is again a finite
loop space.

- A short exact sequence \( L \to M \to N \) is a fibration sequence \( BL \to BM \to BN \). The loop space \( M \) is an extension of \( N \) by \( L \).

- A subgroup \( M \) of \( L \) is a loop space \( M \), together with a monomorphism \( f : M \to L \). If \( f \) is understood, we will sometimes simply write \( M \subseteq L \) to refer to the subgroup \( M \) of \( L \). If \( M' \) is another subgroup of \( L \), then \( M' \) is contained in \( M \) up to conjugacy if the monomorphism \( M' \to L \) lifts up to conjugacy to a homomorphism \( M' \to M \).

- A subgroup \( T \) of \( L \) is a torus in \( L \) if \( T \) is a standard torus \( T = S^1 \times \ldots \times S^1 \) (\( r \) factors). The space \( BT \) is then an Eilenberg-MacLane space of type \( K(\mathbb{Z}^r, 2) \), and \( T \) is a \( K(\mathbb{Z}^r, 1) \).

- A loop space \((L, BL, e)\) is connected if \( L \) is a connected space.

**Definition 13.** Let \( L \) be a connected finite loop space. A maximal torus \( i : T \to L \) is a torus in \( L \) such that every other torus \( T' \) in \( L \) is contained in \( T \) up to conjugacy. Equivalently, \( i : T \to L \) is a maximal torus if \( T \) is a torus in \( L \) that has the same rational rank as \( L \), i.e. the rational cohomologies \( H^*(BT; \mathbb{Q}) \) and \( H^*(BL; \mathbb{Q}) \) have the same transcendence degree over \( \mathbb{Q} \).

For pointed spaces \( X \) and \( Y \), let \([X, Y]\) denote the set of homotopy classes of basepoint preserving maps from \( X \) to \( Y \).

**Definition 14.** Let \( i : T \to L \) be a maximal torus for the finite loop space \( L \). The (homotopic) Weyl monoid \( W_L \) of \( L \) is the monoid

\[
W_L := \{ w \in [BT, BT] | Bi \circ w \simeq Bi \}
\]

of homotopy classes of selfmaps of \( BT \) that make the following diagram homotopy commute:

\[
\begin{array}{ccc}
BT & \xrightarrow{w} & BT \\
\downarrow{Bi} & & \downarrow{Bi} \\
BL & \xrightarrow{Bi} & BL
\end{array}
\]
The set $W_L$ is the group of components of the Weyl space $W_T(X) := \text{Aut}_{BT}(BX)$ of self-maps of $BT$ over $BL$.

The Weyl monoid of a finite loop space plays the role of the Weyl group of a compact Lie group, as is shown in the following theorem,

**Theorem 15.** [37, 1.2] Let $L$ be a finite loop space with maximal torus $i : T \to L$. Then the following hold:

1. The Weyl monoid $W_L$ is a group under composition.

2. The representation $W_L \to \text{GL}(H^2(BT; \mathbb{Q}))$ faithfully represents $W$ as a reflection group

3. We have $(H^*BL; \mathbb{Q}) \cong H^*(BT; \mathbb{Q})^{W_L}$.

Here, if $V$ is a $k$-dimensional vector space over some field, then a subgroup $W < \text{Aut}(V)$ is a reflection group if $W$ is generated by elements that fix a $k-1$-dimensional subspace of $V$. More information on reflection groups will be given in the next chapter.

**Remark 16.** For pointed spaces $BT$ and $BG$, the set $[BT,BT]$ acts via precomposition on $[BT,BG]$. If $G$ is a connected finite loop space with maximal torus $Bi : BT \to BG$, then the Weyl group $W_G$ is the stabilizer

$$W_G = \text{Stab}(Bi)$$

under this action.

The following proposition exhibits that the homotopically defined Weyl groups are a good model for the algebraic Weyl groups of compact Lie groups.

**Proposition 17.** Let $G$ be a compact Lie group with $i : T \to G$ the inclusion of a maximal torus of $G$. The classifying space functor induces an isomorphism

$$B(-) : W \xrightarrow{\cong} W_G$$

between the Weyl group of $G$ and the homotopy Weyl group of the finite loop space $(G, BG, \text{id})$. 

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The proof of this proposition will require a lemma and some definitions from group theory.

Let $G$ be a group. Recall that an inner automorphism of $G$ is a function $f : G \to G$ such that $f(g) = xgx^{-1}$ for some fixed $x \in G$. The set of inner automorphisms of a fixed group $G$ forms a group under composition. We shall denote this group by $\text{Inn}(G)$.

Let $G, G'$ be groups. Let $\text{Rep}(G, G') = \text{Hom}(G, G')/\text{Inn}(G')$ be the set of conjugacy classes of maps from $G$ to $G'$. Specifically, we take equivalency classes with respect to the relation defined by $f_1 \simeq_c f_2 \iff$ there exists $x \in \text{Inn}(G')$ such that $f_1 = x \circ f_2$. We will abuse notation and write $f_1 = f_2 \in \text{Rep}(G, G')$ to mean that $f_1$ and $f_2$ are in the same equivalency class under $\simeq_c$.

**Remark 18.** Since all tori are abelian, the only inner automorphism is the identity. Thus each self-map of $T$ is its own conjugacy class and $\text{Rep}(T, T) \simeq \text{Hom}(T, T)$. Additionally, the group $\text{Hom}(T, T)$ acts on the set $\text{Rep}(T, G)$ via precomposition. Given a subgroup inclusion $i : T \to G$, we may define the stabilizer of $i$ with respect to this action:

$$\text{Stab}(i) := \{ \phi \in \text{Hom}(T, T) : i \circ \phi = i \in \text{Rep}(T, G) \},$$

and show that if $i$ is the inclusion of a maximal torus $T$ in a connected compact Lie group $G$, then $\text{Stab}(i)$ is isomorphic to the Weyl group of $G$.

In particular,

**Lemma 19.** Let $G$ be a connected compact Lie group with $i : T \to G$ the inclusion of the maximal torus. Let $W = N(T)/T$ be the Weyl group of $G$. Then there is an isomorphism

$$W \xrightarrow{\cong} \text{Stab}(i)$$

between the Weyl group and the stabilizer of $i$ in $\text{Hom}(T, T)$.
Proof. The Weyl group \( W \) acts on \( T \) via conjugation. Specifically, if \( w \in W \) then \( w = nT \) for some \( n \in N(T) \), and there is a representation

\[
W \rightarrow Aut(T) \\
nT \mapsto T^w \mapsto T \\
t \mapsto ntn^{-1}
\]

that faithfully represents \( W \) as a subgroup of the automorphism group \( Aut(T) \) [15, IV.2.4]. We identify \( W \) with this subgroup. Thus it is sufficient to show that \( W \) and \( Stab(i) \) are equal as subgroups of \( Aut(T) \).

Let \( w \in W \). By definition, \( w(t) = ntn^{-1} \), for some \( n \) in \( N(T) \). Since \( i(t) = t \), viewed as an element of \( G \), we have \((i \circ w)(t) = ntn^{-1}\) in \( G \). The element \( n \in N(T) \) determines an inner automorphism of \( G \), and we see that if \( t \in T \) then \((i \circ w)(t) = (n \circ i)(t)\). By definition we have \( i \circ w \simeq i \), and hence \( i \circ w = i \) in \(Rep(T,G)\). Thus \( w \) is in \( Stab(i) \).

Let \( \phi \in Stab(i) \). It is enough to show that there exists a \( w \) in \( W \) such that \( \phi \) is the inner automorphism corresponding to \( w \). Composition with \( i \) gives a map \((i \circ \phi) : T \rightarrow G\), where \((i \circ \phi)(t)\) is the element \( \phi(t) \), viewed as an element of \( G \). Modding out by conjugation in \( G \) yields a conjugacy class of maps \([i \circ \phi]\) in \(Rep(T,G)\). By assumption, \([i \circ \phi] = [i] \). Thus, \( \phi(t) = xtx^{-1} \) for some \( x \) in \( G \). By one of the main theorems of maximal tori (e.g., [15, p.166]), any two elements of the maximal torus are conjugate in \( G \) if and only if they lie in the same orbit under the action of the Weyl group. Thus there exists a \( w \) in \( W \) such that \( w(t) = \phi(t) \), and we can identify \( \phi \) with \( w \).

\[\square\]

Proof of proposition. By a result of Dwyer and Zabrodsky [20], the map \( B(-) : Rep(T,G) \rightarrow [BT,BG] \) is an isomorphism. By functoriality, the following diagram commutes,

\[
\begin{array}{ccc}
\text{Hom}(T,T) & \xrightarrow{B(-)} & [BT,BT] \\
\downarrow{i \circ -} & & \downarrow{Bio-} \\
\text{Rep}(T,G) & \xrightarrow{B(-)} & [BT,BG].
\end{array}
\]
We identify $W$ and $W_h$ as the stabilizers $W \cong \text{Stab}(i) \subseteq \text{Hom}(T, T)$ and $W_h \cong \text{Stab}(Bi) \subseteq [BT, BT]$. From here, we show that the restriction of $B(-)$ to $W$ is an isomorphism onto $W_h$, i.e., that

$$B(-)|_W : \text{Stab}(i) \xrightarrow{\cong} \text{Stab}(Bi).$$

(6)

First we must check that the image $B(\text{stab}(i)) \subseteq \text{stab}(Bi)$. Let $\phi \in \text{Stab}(i)$, i.e., $\phi : T \to T$ is a map such that $i \circ \phi = i$ in $\text{Rep}(T, G)$. Since $B(-)$ is a functor, we have $B(i \circ \phi) = Bi \circ B\phi = Bi$. Thus $B\phi$ is in $\text{Stab}(Bi)$.

The injectivity of $B(-)|_W$ we get for free, since $B(-)$ is injective. For surjectivity, let $[v]$ be a homotopy class in $\text{Stab}(Bi)$, i.e., $[Bi] \circ [v] = [Bi]$ in $[BT, BG]$. Since $B(-)$ is surjective, there exists a map $\phi$ in $\text{Hom}(T, T)$ such that $[B\phi] = [v]$. Thus $[Bi] \circ [B\phi] = [Bi]$. By functoriality of $B(-)$, we have $[B(i \circ \phi)] = [Bi] \circ [B\phi]$, and hence $[B(i \circ \phi)] = [Bi]$. Since $B(-)$ is injective, $i \circ \phi = i$, thus $\phi \in \text{Stab}(i)$. 

1.3 $\mathbb{F}_p$-localization and completion

In what follows, $p$ will denote a fixed prime, and $\mathbb{F}_p$ will denote the field of order $p$. Before we can say what a $p$-compact group is, we must first understand the theory of localizing and completing at $p$. There are several related localization and completion functors for topological spaces, but for an important class of spaces called ‘$p$-good’ spaces, these functors coincide up to homotopy. We will therefore mostly concern ourselves with the $\mathbb{F}_p$-localization of Bousfield [12], and will consider others only when they are needed to illuminate a particular construction.

**Definition 20.** Let $R$ be an abelian group. A map $f : A \to B$ is an $R$-equivalence if the induced map on $R$-homology,

$$f_* : H_*(A; R) \to H_*(B; R)$$

is an isomorphism. A space $X$ is $p$-local if any $\mathbb{F}_p$-equivalence $f : A \to B$ induces a homotopy equivalence of mapping spaces $\text{map}(B, X) \to \text{map}(A, X)$. 

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Theorem 21. [12, 3.2] There exists a functor from the homotopy category of topological spaces to itself, called the Bousfield \(F_p\)-localisation functor, denoted \((-)_p\), equipped with a natural transformation \(Id \rightarrow (-)_p\) such that if \(X\) is a space then \(\eta_X : X \rightarrow X_p\) is an \(F_p\)-equivalence and \(X_p\) is \(p\)-local.

Remark 22. There is a very nice categorical characterization of the Bousfield localisation functor \((-)_p\), in that it formally inverts \(F_p\)-equivalences. We can view localisation as a functor from the category of (pointed, compactly generated) topological spaces and continuous maps to a localised category, where we keep the same objects, but for every \(F_p\)-equivalence \(f : X \rightarrow Y\) we introduce an inverse \(f^{-1} : Y \rightarrow X\). Although working in this localised category can simplify some tasks, it is commonplace (and implicit in the definition given) to functorially project back onto the category of topological spaces, where we can again interpret each morphism as a continuous map. In general, this process changes the homotopy type of a space.

Additionally, \((-)_p\) is characterised by the following universal property:

Theorem 23. [12, 3.3] Let \(X\) be a pointed connected topological space. Let \((-)_p\) denote the Bousfield \(F_p\)-localization functor and let \(\eta_X\) denote the natural inclusion \(\eta_X : X \rightarrow X_p\). Let \(f : X \rightarrow Y\) be a map from \(X\) to some pointed connected space \(Y\).

1. If \(f\) is an \(F_p\)-equivalence, then there exists a unique homotopy class of maps \(u : Y \rightarrow X_p\) such that \(uf = \eta_X\).

2. If \(Y\) is a \(p\)-local space then there exists a unique homotopy class of maps \(u : X_p \rightarrow Y\) such that \(u\eta_X = f\).

An \(F_p\)-equivalence \(f : X \rightarrow Y\) is a localisation of \(X\) if \(Y\) is a \(p\)-local space. In light of the universal property, we will call the natural map \(\eta_X : X \rightarrow X_p\) the localisation of \(X\).

1.3.1 Nilpotent spaces and \(p\)-good spaces

[14, II, S4] Recall that a group action

\[
\phi : \pi \rightarrow Aut(G)
\]
is nilpotent if there exists a finite sequence
\[ G = G_1 \supseteq \ldots \supseteq G_j \supseteq \ldots \supseteq G_n = 1 \]
of subgroups of \( G \) such that for each \( j \)
1. \( G_j \) is closed under the action of \( \pi \),
2. the quotient \( G_j/G_{j+1} \) is a normal subgroup of \( G_j \), and
3. the induced action of \( \pi \) on \( G_j/G_{j+1} \) is trivial.

A space \( X \) is called nilpotent if the action of \( \pi_1 X \) on each \( \pi_i X \) is nilpotent. A fibration \( p : E \to B \) is called nilpotent if the homotopy fiber \( F \) is connected and the action of \( \pi_1 E \) on each \( \pi_i F \) is nilpotent. Bousfield and Kan have shown that the localisation of a nilpotent fibration \( f : E \to B \) is again a nilpotent fibration \( f_p : E_p \to B_p \), and the inclusion of the localisation of the fiber \( \text{hofib}(f)_p \to \text{hofib}(f_p) \) is a homotopy equivalence [14, II, 4.8].

A space \( X \) is \( p \)-good if \( X \) is nilpotent, or if \( \pi_1 X \) is finite, or if \( X \) is connected and \( H_1(X; \mathbb{F}_p) = 0 \). A space \( X \) is \( p \)-complete if \( X \) is \( p \)-local and \( p \)-good.

Similarly for localisations, an \( \mathbb{F}_p \)-equivalence \( f : X \to Y \) is a completion of \( X \) if \( Y \) is a \( p \)-complete space. In light of the universal property, if \( X \) is \( p \)-good, we will call the localisation \( \eta_X : X \to X_p \) the completion of \( X \).

**Corollary 24.** Let \( \eta_X : X \longrightarrow X_p \) be the completion of \( X \). If \( Y \) is a \( p \)-complete (connected, pointed) space, then \( \eta_X \) induces an isomorphism
\[ [X_p, Y] \xrightarrow{\cong} [X, Y]. \]

**Proof.** Let \( g \in [X_p, Y] \). The map \( \eta_X^* \) is precomposition with \( \eta_X \) and injectively gives a homotopy class of maps \( g \circ \eta_X \in [X, Y] \). By the second part of the universal property, if \( f \in [X, Y] \), then there exists a unique homotopy class \( u \) in \([X_p, Y]\) making the following diagram commute up to homotopy,

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\eta_X} & & \downarrow{u} \\
X_p & & \\
\end{array}
\]
hence $\eta_X^*$ is both surjective and injective.

In particular, if $U$ denotes the union of the unitary groups, then the classifying space $BU$ is a $p$-good space, so $BU_p$ is $p$-complete. Setting $Y$ to $BU_p$ in the above corollary yields

**Corollary 25.** Let $\eta_X : X \rightarrow X_p$ be the completion of $X$. Then $\eta_X$ induces an isomorphism

$$\tilde{K}(X_p; \mathbb{Z}_p) \cong \tilde{K}(X; \mathbb{Z}_p).$$

This corollary states an isomorphism between the $p$-completed $K$-theory of $X$ and the $p$-completed $K$-theory of the completion $X_p$, which we will use later in the relevant section below.

**Corollary 26.** Let $X$ and $Y$ be $p$-good spaces and let $f : X \rightarrow Y$ be an $\mathbb{F}_p$-equivalence. Then $f_p : X_p \rightarrow Y_p$ is a homotopy equivalence.

**Proof.** Since $\eta_Y$ and $f$ are $\mathbb{F}_p$-equivalences, so is their composition, $\eta_Y \circ f$. Thus $X_p$ satisfies universal property (1) with regard to $\eta_Y \circ f$ and there exists a unique homotopy class of maps $u$ such that the following diagram commutes.

\[
\begin{array}{ccc}
Y & \xrightarrow{\eta_Y} & Y_p \\
\downarrow f & & \downarrow f_p \\
X & \xrightarrow{\eta_X} & X_p \\
\downarrow f & & \downarrow u \\
Y & \xrightarrow{\eta_Y} & Y_p
\end{array}
\]

The vertical composition $f_p \circ u$ is equal to the identity $id_{Y_p}$. By a similar argument, we see that $u \circ f_p = id_{X_p}$, thus $f_p$ is a homotopy equivalence.

1.4 $p$-compact groups

With these technical details out of the way, we are prepared to state the definition of a $p$-compact group, as introduced by Dwyer and Wilkerson in [22].

**Definition 27.** A $p$-compact group $X$ is a loop space $(X, BX, e)$ such that
1. $X$ is $\mathbb{F}_p$-finite ($H^*(X; \mathbb{F}_p)$ is finitely generated over $\mathbb{F}_p$).

2. $BX$ is a pointed $p$-local space

3. $e : \Omega BX \to X$ is a homotopy equivalence.

The definitions of homomorphisms, homogeneous spaces, short exact sequences, and subgroups of finite loop spaces carry over to $p$-compact groups, but we must alter some of the other definitions to account for our $p$-localisation.

- If $X$ and $Y$ are $p$-compact groups, then a homomorphism $f : X \to Y$ is a **monomorphism** if the homogeneous space $Y/X$ is $F_p$-finite, and an **epimorphism** if $Y/X$ is again a $p$-compact group.

- A $p$-compact torus $BT$ is a $p$-compact group $(T, BT, e)$ that is homotopic to the localisation of an ordinary torus $T^r = S^1 \times \ldots \times S^1$ ($r$ factors). The space $BT$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}_p^r, 2)$. A **$p$-compact toral group** is an extension of a finite $p$-group by a $p$-compact torus.

- A **subgroup** $Y$ of $X$ is a $p$-compact group $Y$, together with a monomorphism $f : Y \to X$. If $f$ is understood, we will sometimes simply write $Y \subseteq X$ to refer to the subgroup $Y$ of $X$. If $Y'$ is another subgroup of $X$, then $Y'$ is **contained in $Y$ up to conjugacy** if the monomorphism $Y' \to X$ lifts up to conjugacy to a homomorphism $Y' \to Y$.

- The **dimension** of a $p$-compact group $X$ is the largest integer $d$ such that $H^*(X; \mathbb{F}_p) \neq 0$.

- Let $X$ be a connected $p$-compact group. A **$p$-compact torus in $X$** is a monomorphism $i : T \to X$ from a $p$-compact torus into $X$.

- A **maximal $p$-compact torus** $i : T \to X$ is a torus in $X$ such that every other torus $T'$ in $X$ is contained in $T$ up to conjugacy.
Remark 28. When there is no confusion, we will call a maximal $p$-compact torus simply a maximal torus, and we will assume it is a fibration, replacing it with a homotopy equivalent fibration if necessary.

One of the first meaningful results about $p$-compact groups is the existence of $p$-compact maximal tori:

**Proposition 29.** [22, 8.13] Let $X$ be a $p$-compact group. Then $X$ has a maximal torus $T$, unique up to conjugacy.

Let $X$ a $p$-compact group and let $i : T \to X$ be a maximal torus. Replace $Bi : BT \to BX$ with an equivalent fibration, which we shall also denote $Bi$. The Weyl space $W_T(X)$ is defined to be the space of self-maps of $BT$ over $BX$. Composition of maps gives $W_T(X)$ the structure of an associative topological monoid [22, 9.2]. Dwyer and Wilkerson have proved [22, 9.3-9.5] that any self-map of $BT$ over $BX$ is a homotopy self-equivalence, that the Weyl space is homotopically discrete, and depends on choice of maximal torus only up to (homotopical) conjugation. This leads to the definition

**Definition 30.** The Weyl group $W_T(X)$ of a $p$-compact group $X$ with maximal torus $T$ is the group given by

$$W_X := \{ w \in [BT,BT] | Bi \circ w \simeq Bi \}.$$ 

Equivalently, $W_T(X)$ is the component group of the Weyl space,

$$W_T(X) := \pi_0 W_T(X).$$

Similar to the result for finite loop spaces, Dwyer and Wilkerson showed that the Weyl groups of $p$-compact groups are faithfully represented as reflection groups:

**Theorem 31.** [22, 9.7] Suppose that $X$ is a connected $p$-compact group. Let $i : T \to X$ be a maximal torus for $X$.

1. The rank $s$ of $T$ is equal to the rank $r$ of $X$. 

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2. The action of $W_T(X)$ on $BT$ induces a map

$$W_T(X) \rightarrow \text{Aut}(H^2(BT;\mathbb{Z}_p) \otimes \mathbb{Q}) \cong GL_s(\mathbb{Q}_p)$$

which is a monomorphism whose image is a finite subgroup of $GL_s\mathbb{Q}_p$ generated by reflections.

3. The map $B_i^*: H^*(BX;\mathbb{Z}_p) \otimes \mathbb{Q} \rightarrow (H^*(BT;\mathbb{Z}_p) \otimes \mathbb{Q})^{W_T(X)}$ is an isomorphism.

A connected $p$-compact group is simple if the representation $W < GL_s\mathbb{Q}_p$ is irreducible. This definition follows the Lie case. We will study such reflection groups closely in the next chapter, but for now we will see some examples of $p$-compact groups and introduce some tools that will aid us to further understand these structures.

1.5 Example: the completion of a compact Lie group

We will now explicitly verify a fundamental example of a connected $p$-compact group: the completion of a compact Lie group. This example is integral for the theory of $p$-compact groups, but it is often stated without proof in the literature.

**Theorem 32.** Let $G$ be a connected compact Lie group.

1. For any prime $p$, the triple $(G_p, BG_p, id_p)$ is a $p$-compact group.

2. If $i : T \rightarrow G$ is a maximal torus of $G$ then $i_p : T_p \rightarrow G_p$ is a maximal torus of $G_p$.

3. The Weyl group of $G$ is isomorphic to the Weyl group of $G_p$.

We have seen that all compact Lie groups admit a finite loop space structure and that this structure preserves maximal tori and Weyl groups. We now check that $p$-completion also preserves this structure and gives rise to $p$-compact groups.

**Proof.** Let $G$ be a connected compact Lie group. We have seen that $(G, BG, e)$ is a finite loop space, and in particular, its integral cohomology $H^*(G)$ is finite dimensional over $\mathbb{Z}$ (since $G$ has the homotopy type of a finite CW-complex). By
universal coefficients $H^*(G, \mathbb{F}_p)$ is finite dimensional over $F_p$ and hence $G$ is $\mathbb{F}_p$-finite. The localization $\eta_G : G \to G_p$ is an $\mathbb{F}_p$-equivalence, thus $G_p$ is also $\mathbb{F}_p$-finite. By the basic properties of localization, the space $BG_p$ is $p$-local and the map $e_p : \Omega BGL_p \to G_p$ is an equivalence, where we may use the result that

$$\Omega(Y_p) \simeq (\Omega Y)_p$$

for simply connected $Y$ [14, VI, 6.5], proving part 1.

For part 2, the map $Bi : BT \to BG$ is equivalent to a fibration between nilpotent spaces and is hence a nilpotent fibration [14, II, 4.5]. Then, by [14, II, 4.8], the inclusion of the homotopy fiber

$$(G/T)_p \to G_p/T_p$$

is a homotopy equivalence. By the same method used in part 1, the homogeneous space $G_p/T_p$ is $\mathbb{F}_p$-finite, thus $Bi_p$ is a monomorphism. By functoriality and the fact that every $p$-compact torus is the localization of a classical torus, the maximality of $Bi$ carries over to the maximality of $Bi_p$.

For part 3, one must apply an integral Galois theory argument, as can be seen in [37, 1.8]

Remark 33. Note that a compact Lie group $G$ is not determined by $BG_p$. We are only seeing the structure at the prime $p$. For example, $BSO(2n+1)_p \simeq BSp(n)_p$ for odd primes [25].

1.6 Tools

From the definition, a $p$-compact group $(X, BX, e)$ is an $\mathbb{F}_p$-local loop space. It is not, strictly speaking, a topological group (although it is weakly equivalent to one, this equivalence does not preserve as many properties as one would like). Thus in order to study $p$-compact groups, we will need to use tools that can be formulated in terms of loop spaces.
**Definition 34.** Let $G$ be a compact Lie group, and let $X$ and $EG$ be $G$-spaces, where $EG$ is contractible and with free $G$-action. The *homotopy orbit space*, or *Borel construction* of the action of $G$ on $X$ is the quotient

$$X_{hG} := X \times_G EG := (X \times EG)/G$$

of $EG \times X$ by the free, diagonal action of $G$.

**Remark 35.** If $\tilde{EG}$ is any other contractible free $G$-space, then the quotients $X \times_G EG$ and $X \times_G \tilde{EG}$ are $G$-equivariantly homotopy-equivalent. Thus $X_{hG}$ is only well-defined up to ($G$-equivariant) homotopy.

**Remark 36.** The homotopy orbit space $X_{hG}$ is (up to homotopy) the total space of a fibration $p$ over $BG \simeq (pt)_{hG}$ with fiber $X$. To see this, let $y$ be a fixed element in $EG$. Then the fiber $p^{-1}(yG)$ is homotopic to the set of orbits $\{(x, y)G | x \in X\}$. Since the action of $G$ on $X \times EG$ is diagonal, free and we have fixed a $y$, the orbits $(x, y)G$ and $(x', y)G$ are equal if and only if $x = x'$. It follows that $p^{-1}(yG) \simeq X$, and up to homotopy, $X_{hG}$ sits in a fiber sequence

$$X \to X_{hG} \to BG.$$ 

**Definition 37.** Dually, one defines the *homotopy fixed point space* to be space of $G$-equivariant maps from $EG$ to $X$,

$$X^{hG} := map^G(EG, X),$$

and one can easily see that $X^{hG}$ is equivalent to the space of sections of the fibration $X_{hG} \to BG$ [22, 10.3].

**Example 38.** We have already seen an application of this approach in the definition of the Weyl monoid of a loop space: If $G$ is a connected compact Lie group with maximal torus $T$, then the Weyl group $W$ is the fixed point set of the left translation action of $T$ on $G/T$. The corresponding homotopy fixed point set is the topological monoid of maps $BT \to BT$ over $BG$, i.e.

$$W \cong (G/T)^{hT}.$$ 

[22, 10.6]
1.6.1 Normalizers

Let $X$ be a $p$-compact group with maximal torus $T$ and Weyl group $W$. Recall that by definition, the Weyl space $W$ acts on the classifying space $BT$. The normalizer $N(X)$ of $X$ is the loop space $(N, BN, e)$ such that the space $BN$ is the Borel construction of this action,

$$BN := (BT)_hW.$$ 

Thus up to homotopy, $BN$ sits in a fiber sequence

$$BT \to BN \to BW.$$ 

When $p$ is odd, this fibration has a section, unique up to vertical homotopy [4, 1.2]. It follows from the fiber sequence that $N$ is an extension of a $p$-compact torus by the finite group $W$, but it is rarely the case that $N$ is again a $p$-compact group. This is because the definition of a $p$-compact group requires the component group to be a finite $p$-group, but we see that $\pi_0N \cong W$ is not in general a $p$-group. To get a $p$-compact group, we restrict the action of $W$ on $BT$ to a union of components $W_p$ of $W$ corresponding to a $p$-Sylow subgroup $W_p$ of $\pi_0W = W$. Thus the $p$-normalizer $N_p(X)$ is the $p$-compact group $(N_p, BN_p, e)$ where $BN_p$ is the Borel construction

$$N_p := BT_{hW_p},$$

and up to homotopy sits in a fibration over $BW_p$ with fiber $BT$. Furthermore, [22, 9.8], the homomorphism $T \to X$ extends to

$$T \to N_p \to N \to X.$$

**Lemma 39.** [38, 3.12] If $p$ does not divide the order of $W$, then $Bj : BN \to BX$ is an $\mathbb{F}_p$-equivalence, and the localization $Bj_p : (BN)_p \to BX$ is a homotopy equivalence.
1.6.2 Centralizers

Recall that if \( f : H \to G \) is a homomorphism of compact Lie groups, then \( H \) acts on \( G \) via conjugation and \( f \). The centralizer

\[
C_G(f(H)) : \{ g \in G : \text{ if } h \in f(H) \text{ then } gh = hg \}
\]

of \( f(H) \) in \( G \) can be viewed as the fixed-point set of \( G \) with respect to this action [22, 1.3.1]. Working homotopically, if \( f : X \to Y \) is a homomorphism of loop spaces, one defines the loop space centralizer \( C_Y(f(X)) \) as the homotopy fixed point set of a conjugation action of \( X \) on \( Y \):

If \( f : X \to Y \) is a loop space homomorphism, let \( \text{Map}(X,Y)_f \) be the component of the mapping space \( \text{Map}(X,Y) \) containing \( f \). The centralizer \( C_Y(f(X)) \) of \( f \) is the loop space \( \Omega \text{Map}(BX, BY)_{Bf} \). When \( f \) is obvious from the context, we will often remove \( f \) from the notation and refer to \( C_Y(f(X)) \) simply as \( C_Y(X) \). Evaluation at the basepoint of \( BX \) yields a homomorphism \( C_Y(X) \to Y \). The map \( f \) is central if evaluation is an equivalence. A loop space \( X \) is homotopy abelian (or ‘abelian’ for short) if its identity morphism is central, i.e., if \( C_X(X) \to X \) is an equivalence.

If \( A \) is an abelian \( p \)-compact toral group, \( X \) a \( p \)-compact group with a homomorphism \( f : A \to X \), then \( f \) canonically lifts to a central homomorphism \( f' : A \to C_X(A) \) [22, 8.2]. If \( f : A \to X \) is a central homomorphism, then \( f \) naturally extends to a short exact sequence of \( p \)-compact groups \( A \to X \to X/A \).

A theorem of Dwyer and Wilkerson tells us that we may approximate centralizers of \( p \)-compact toral groups by centralizers of finite \( p \)-groups:

**Theorem 40** (D-W 6.1). Let \( X \) be a \( p \)-compact group, \( G \) a \( p \)-compact toral group, and \( f : G \to X \) a homomorphism. Then there exists a finite \( p \)-group \( K \) and a homomorphism \( K \to G \) such that the restriction map \( C_X(G) \to C_X(K) \) is an equivalence.

**Proposition 41.** [22, 5.1] Let \( f : G \to X \) be a homomorphism of a finite \( p \)-group \( G \) into a \( p \)-compact group \( X \). Then the centralizer \( C_X(f(G)) \) is a \( p \)-compact group.
Example 42. If $T \to G$ is a maximal torus and $A \to T$ is a monomorphism from a $p$-compact toral group $A$, then the natural homomorphism $C_G(A) \to G$ is of maximal rank [23, 4.3]. That is, if $A < T$ is a subtorus of the maximal torus $T < G$, then the subgroup $C_G(A) < G$ is of maximal rank.

1.6.3 Subgroups of maximal rank

For connected compact Lie groups, a subgroup of maximal rank is a subgroup that contains the maximal torus. Centralizers of subtori are important examples of subgroups of maximal rank, and this notion carries over to $p$-compact groups via the definitions:

Proposition 43. Let $G$ be a connected compact Lie group with maximal torus $T$ and Weyl group $W$. Let $\alpha \in \text{Hom}(T, S^1)$ be a nontrivial character of the maximal torus. Then $\alpha$ determines a subgroup of maximal rank with Weyl group of order 2. The inclusion of this subgroup into $G$ induces an injection of Weyl groups and determines a reflection $w_\alpha$ in the lattice $\pi_1 T$.

Proof. Let $U_\alpha$ denote the kernel of $\alpha$. It follows that $U_\alpha$ is a closed subgroup of $T$ of codimension one, and is topologically cyclic. Let $u$ be a generator of the identity component $U_\alpha^0$ of $U_\alpha$, and let

$$C_G(u) := \{ g \in G : gu = ug \}$$

be the centralizer of $u$ in $G$. Since $T \subseteq C_G(u)$, $T$ is a maximal torus for $C_G(u)$ and $C_G(u)$ is a maximal rank subgroup in $G$. By [15, IV.2.9, V.2.8], the quotient $C_G(u)/U_\alpha^0$ has maximal torus $T/U_\alpha^0 \cong S^1$. Since $C_G(u)/U_\alpha^0$ is a compact Lie group of rank 1, it follows from the classification of rank 1 compact Lie groups that $C_G(u)/U_\alpha^0$ has dimension 3 and Weyl group of order 2 [15, V.2.8, 2.9]. Also note that the Weyl group of $C_G(u)$ is isomorphic to the Weyl group of $C_G(u)/U_\alpha^0$, and that if $w_\alpha$ denotes the generator of this Weyl group and $v \in U_\alpha$, then $w_\alpha(v) = v$. Furthermore, the inclusion $C_G(u) \to G$ induces an inclusion of Weyl groups with respect to $T$. Thus $w_\alpha$ is a nontrivial automorphism of $T$ of order 2 that fixes a co-rank 1 subtorus $U_\alpha^0$, 23
and is a reflection in $\pi_1 T$.

**Definition 44.** [23, 4.1] A homomorphism $f : X \to Y$ of $p$-compact groups is of maximal rank if $f$ is a monomorphism and there exists a maximal torus $i : T_X \to X$ of $X$ such that $f \circ i : T_X \to Y$ is a maximal torus for $Y$. If $f : X \to Y$ is of maximal rank, then $X$ is said to be a subgroup of maximal rank of $Y$.

**Example 45.** [23, 4.3] If $T \to G$ is a maximal torus and $A \to T$ is a monomorphism from a $p$-compact toral group $A$, then the natural homomorphism $C_G(A) \to G$ is of maximal rank. That is, if $A < T$ is a subtorus of the maximal torus $T < G$, then the subgroup $C_G(A) < G$ is of maximal rank.

**Lemma 46.** [23, 4.4] If $H < G$ is a subgroup of maximal rank, then there is a natural monomorphism of Weyl groups $W(H) \to W(G)$, independent up to conjugation of a choice of maximal torus $T < H$.

### 1.6.4 Discrete approximations

If $S$ is a $p$-compact torus, then an extended $p$-compact torus $A$ is an extension

$$S \to A \to G,$$

of a finite group $G$ by $S$. One can construct discrete approximations to extended $p$-compact tori, by defining a $p$-discrete torus of rank $r$ to be a discrete group isomorphic to $(\mathbb{Z}/p^{\infty})^r$, and an extended $p$-discrete torus $\tilde{A}$ to be a discrete group with a normal subgroup $\tilde{S}$ of finite index such that $\tilde{S}$ is a $p$-discrete torus.

Note that [23, 3.12], if $G$ is an extended $p$-discrete torus then there is a unique normal $p$-discrete torus in $T$ in $G$ such that $G/T$ is finite; it is the maximal divisible subgroup of $G$, and we will denote it by $G_1$.

**Definition 47.** Let $G$ be an extended $p$-compact torus. A discrete approximation to $G$ is an extended $p$-discrete torus $\tilde{G}$, with a loop space homomorphism $f : \tilde{G} \to G$ such that $\tilde{G}/\tilde{G}_1 \cong \pi_0 G$ and $Bf : B\tilde{G}_1 \to BG_1$ is an $\mathbb{F}_p$-equivalence onto the identity component of $BG$. In this case, $G$ is said to be a closure of $\tilde{G}$.

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Proposition 48. [22, 6.8], [23, 3.13] If $G$ is an extended $p$-compact torus then $G$ has a discrete approximation $\tilde{G} \to G$.

Proposition 49. [22, 6.9] If $G$ is an extended $p$-discrete torus then $G$ has a functorial closure $G \to Cl(G)$, where $Cl(G) = \Omega(BG)_p$.

1.7 A note on Geometry

The structure of Lie groups as differentiable manifolds comes into play in several key areas and definitions. In particular, if $G$ is a real Lie group, then the tangent space at the identity $g := T_0G$ is the Lie algebra of $G$. Taking the differential at the identity is a functor from the category of Lie groups and smooth group homomorphisms to the category of Lie algebras and linear maps that preserve the Lie bracket. Of particular importance to the study of $G$ is the adjoint representation, [15, I.2.10]

$$Ad : G \longrightarrow Aut(\mathfrak{g})$$

$$g \mapsto d(c_g)$$

where $d(c_g)$ is the differential of the inner automorphism of $G$ determined by conjugation with $g$. Thus the adjoint representation describes the infinitesimal structure of conjugation by elements of $G$ in a neighborhood of the identity.

Example 50. If $T \cong \mathbb{R}^n / \mathbb{Z}^n$ is a torus, then the Lie algebra $\mathfrak{t}$ is isomorphic to $\mathbb{R}^n$ and there is an exponential map

$$exp : \mathfrak{t} \cong \mathbb{R}^n \longrightarrow \mathbb{R}^n / \mathbb{Z}^n \cong T,$$

which one can identify as the universal covering map, as $\mathbb{R}^n$ is contractible and the kernel $L := ker(exp) \cong \mathbb{Z}^n$ is discrete. The kernel $L$ is geometrically referred to as the integral lattice, and one may identify $L$ with the group of covering transformations of $exp$, so that $L \cong \pi_1(T)$ is the fundamental group.

Example 51. The Lie algebra of $SU(n)$, denoted $\mathfrak{su}(n)$, consists of the $n \times n$ traceless anti-Hermitian matrices, with the commutator $[X,Y] = XY - YX$ of matrices as Lie bracket.
The Lie algebra $\mathfrak{g}$ and the adjoint representation are frequently used in Lie theory. For instance, there is a one-to-one correspondence between finite-dimensional representations of $G$ and $\mathfrak{g}$, and one defines the roots of a semisimple Lie group to be the non-zero weights of the adjoint representation, and the study of root systems leads to classification of semisimple Lie groups.

Once we abstract to $p$-compact groups, however, we do not have the ability to use this geometric tool, and there is yet to be a good definition to fill the role of the Lie algebra. However, there are aspects of the Lie algebra that can carry through to homotopy theory: If $G$ is a $d$-dimensional compact Lie group, then the Lie algebra $\mathfrak{g}$ is a $d$-dimensional representation of $G$, and the one-point compactification

$$S^\mathfrak{g} := \mathfrak{g} \cup \{\infty\} \cong S^d$$

is homeomorphic to the $d$-sphere $S^d$, and is equipped with an action of $G$. Furthermore, if $G$ is abelian, then $\mathfrak{g}$ has trivial Lie product and is thus a trivial $G$-representation, and the representation sphere $S^\mathfrak{g}$ has trivial $G$-action. We will later recall some properties of $S^\mathfrak{g}$ in terms of stable homotopy theory, and see that there is a stable sphere $S^\mathfrak{g}$ for $p$-compact groups that also enjoys these properties.
2 Reflection Groups

This chapter is concerned with the $p$-adic reflection groups determined by $p$-compact groups. First, we will look at reflection groups defined as finite subgroups of $GL(V)$, where $V$ is a vector space over field $K$. We will then use the Chevalley-Shephard-Todd theorem for non-modular fields to see that the Weyl groups of connected $p$-compact groups are generated by reflections [22, 9.7]. We will go through the classification of $K$-reflection groups, for $K = \mathbb{R}, \mathbb{C}$, and, after considering embeddings of character fields, for $\mathbb{Q}_p$ [19, 47, 18], which will lead us to the Clark-Ewing construction of non-modular $p$-compact groups. We will then consider reflection groups definable over the $p$-adic integers $\mathbb{Z}_p$, and arrive at the classification of $p$-compact groups for $p$ odd, as was completed in [3]. We will then consider toric morphisms of connected $p$-compact groups and extend the classification to a functor into the category of $\mathbb{Z}_p$-reflection groups and explore some of its properties.

2.1 Finite reflection groups over a field

Let $K$ be a field and let $V$ be an $n$-dimensional vector space over $K$. An element $s \in GL(V)$ is a reflection in $V$ if $s$ fixes a co-dimension one hyperplane through the origin. That is, $s \in GL(V)$ is a reflection if the fixed point set

$\ker(1 - s) = \{v \in V : sv = v\}$

is codimension one in $V$, where $1$ is the identity transformation in $GL(V)$. A $K$-reflection group is a pair $(W, V)$, where $V$ is a finite-dimensional vector space over $K$ and

$\rho : W \rightarrow GL(V)$

is a faithful representation such that the image $\rho(W) < GL(V)$ is generated by reflections. If we say that the finite group $W$ is generated by reflections, then we have implicitly fixed a representation $\rho : W \rightarrow GL(V)$ and have identified $W$ with its image under $\rho$. We will always consider $s$ to have finite order and $V$ to be finite-dimensional.
The motivating example of a reflection group over \( \mathbb{R} \) is the Weyl group of a compact Lie group acting on the Lie algebra of its maximal torus.

**Example 52.** The simplest nontrivial example of a reflection group can be seen by letting the Weyl group \( W_{SU(2)} \cong \Sigma_2 \cong \mathbb{Z}/2 \) act on the rank-1 lattice \( \pi_1 T_{SU(2)} \cong \mathbb{Z} \). Here, the nontrivial element is a reflection in the rank-0 hyperplane \( \{0\} \).

**Example 53.** Let \( G \) be a connected compact Lie group, with maximal torus \( T \cong \mathbb{R}^n / \mathbb{Z}^n \) and Weyl group \( W \). Recall that the normalizer \( N(T) \) acts on \( T \) by conjugation, and since the action of \( T \) on itself is trivial, there is an induced action of \( W \) on \( T \),

\[
W \times T \rightarrow T
\]

\[
(nT, t) \mapsto ntn^{-1}.
\]

Moreover, this action is effective, i.e. we may view \( W \) as a subgroup of \( Aut(T) \) [15, IV.2.4]. Once one fixes an inner product on \( g \) that is invariant under the adjoint representation, one can view \( W \) as a group of orthogonal transformations on \( t \cong \mathbb{R}^n \). One generally sees that \( W \) is generated by reflections either by inspection or by considering the decomposition of \( t \) into finitely many convex regions, called Weyl chambers, corresponding to the nontrivial weight spaces of the adjoint representation [15, V.2.12]. Then one sees that \( W \) acts simply transitively on the set of Weyl chambers and that the reflections in the walls in any given Weyl chamber generate \( W \).

### 2.2 The algebra of invariants, degrees, and the Shephard-Todd theorem

If \( V \) is a vector space over the field \( K \), let \( T^r(V) \) denote the \( r \)-fold tensor power of \( V \), The *tensor algebra* \( T(V) \) of \( V \) is the direct sum

\[
T(V) := \bigoplus_{r=0}^{\infty} T^r(V).
\]

The algebra \( T(V) \) is the free associative algebra on \( V \) and the association \( V \mapsto T(V) \) is a functor from vector spaces to associative algebras. Let \( I \) denote the idea of \( T(V) \)
generated by all elements of the form $v \otimes w - w \otimes v$. The quotient $S(V) := T(V)/I$ is the symmetric algebra of $V$. It is the free commutative algebra on $V$. Since $V \cap I = \{0\}$, and $T^1(V) = V$, we may identify $V$ with its image in $S(V)$. If $v_1, ..., v_n$ is a basis for $V$, then $S(V)$ is isomorphic to the polynomial algebra on the symbols $S(V) \cong K[v_1, ..., v_n]$.

If $V^* = Hom(V, K)$ is the dual space of $V$, with dual basis the coordinate functions $X_1, ..., X_n$, then the coordinate ring of $V$ is the ring $S(V^*) \cong K[X_1, ..., X_n] =: K[V]$ which may be identified with the ring of polynomial functions on $V$.

The degree of a monomial $X_1^{m_1} ... X_n^{m_n}$ is the sum $m_1 + ... + m_n$. A polynomial $P$ is homogeneous of degree $r$ and we write $degP = r$ if $P$ is a linear combination of monomials of degree $r$. The algebra $K[V]$ is naturally graded by degree.

**Definition 54.** The group $GL(V)$ acts on $V^*$ and hence on $K[V]$; if $w \in GL(V), P \in K[V]$, then we set $wP(v) = P(w^{-1}v)$ for all $v \in V$. This is a linear action and preserves the graded algebra structure of $K[V]$. The algebra of invariants of $W < GL(V)$ is the algebra of $W$-invariant polynomials $K[V]^W := \{P \in K[V] \mid wP = P \text{ if } w \in W \}$.

If $W < GL(V)$, then a set of basic invariants for $W$ is a set of algebraically independent homogeneous polynomials that generate $K[V]^W$ as an algebra.

The following theorem was first proven case-by-case for $K = \mathbb{C}$ by Shephard and Todd, by providing a set of basic invariants for each isomorphism class of $\mathbb{C}$-reflection groups. Chevalley later gave a uniform proof for $\mathbb{R}$, that Serre showed could be applied to all non-modular reflection groups. The following statement of the theorem is due to Benson. A $K$-reflection group $(W, V)$ is non-modular if the characteristic of $K$ is coprime to $|W|$. Theorem 55 (Shephard-Todd-Chevalley). [11, 7.2.1] Let $(W, V)$ be a non-modular $K$-reflection group. Then the following are equivalent:
1. $W$ is generated by elements that act on $V$ as reflections.

2. $K[V]^W \cong K[f_1, ..., f_n]$ is a polynomial ring.


Moreover, if any of the above conditions hold, then the number of reflections in $G$ is equal to the sum $\sum_i (k_i - 1)$, where $k_i := \deg(f_i)$.

After a good deal of work by Dwyer and Wilkerson, the Chevalley-Shephard-Todd theorem is what allows us to view the Weyl groups of $p$-compact groups as being generated by reflections:

**Theorem 56.** [22, 9.7(2)] If $X$ is a connected $p$-compact group with maximal torus $T$ and Weyl group $W$, then $(W, (H^2(BT; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p))$ is a $\mathbb{Q}_p$-reflection group.

**Outline of proof.** Let $R_T$ denote $H^*(BT; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and let $R_X$ denote $H^*(BX; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Dwyer and Wilkerson used the Serre spectral sequence in the monomorphism $Bi : BT \to BX$ to show that $R_T$ is finitely generated over $R_X$. They further showed that $Bi^*$ is injective via a transfer argument. Since both $R_X$ and $R_T$ are polynomial algebras, it follows that the number of generators are equal. This, together with finite generation, implies that $R_T$ is free over $R_X$. Dwyer and Wilkerson were then able to identify the number of generators of $R_T$ over $R_X$ as $|W|$, and apply some integral Galois theory to show that $R_X$ is isomorphic to the invariants $(R_T)^W$. Then one is free to apply the Chevalley-Shephard-Todd theorem.

2.3 Reflection groups over $\mathbb{R}$

If $(W, V)$ and $(W', V')$ are two $K$-reflection groups, then $(W, L)$ and $(W', L')$ are isomorphic if we can find a $K$-linear isomorphism $\phi : L \to L'$ such that $\phi W \phi^{-1} = W'$. A $K$-reflection group $(W, V)$ is irreducible, or simple, if the representation $\rho : W \to GL(V)$ is irreducible, i.e. if $V$ is simple as a $K[W]$-module. If the characteristic of $K$ is coprime to $|W|$, then Maschke’s theorem states that the $K[W]$-module $V$ splits as a direct sum of irreducible $K[W]$-modules $V_1, ..., V_m$. Thus to classify non-modular $K$-reflection groups is enough to classify irreducibles.
Irreducible $\mathbb{R}$-reflection groups were classified by Coxeter [19], who showed that they correspond bijectively with finite Coxeter diagrams, which label symmetry groups of certain polytopes. Coxeter found four infinite families, labeled $A_n$, $BC_n$, $D_n$, and $I_2(p)$, and six exceptional cases, labeled $E_6$, $E_7$, $E_8$, $F_4$, $H_3$, and $H_4$. The Weyl groups of all the simple connected compact Lie groups appear in this classification. In particular, the symmetric group

$$A_n = \Sigma_{n+1} = W_{SU(n+1)},$$

is the Weyl group of $SU(n+1)$. 

$$BC_n = \mathbb{Z}/2 \wr \Sigma_n = W_{SO(2n+1)} = W_{Sp(n)},$$

is Hyperoctahedral group, the wreath product of the symmetric group with $\mathbb{Z}/2$, and is the Weyl group of $SO(2n+1)$ and $Sp(n)$. 

$$D_n = \mathbb{Z}/2 \wr Alt_n = W_{SO(2n)}$$

is the wreath product with the alternating group (the subgroup of the symmetric group containing permutations of order 2) and is the Weyl group of $SO(2n)$. Additionally, the Weyl group of the exceptional Lie group $G_2$ appears as the Coxeter group

$$I_2(6) = W_{G_2},$$

and the Weyl groups of the other four exceptional compact connected simple Lie groups all appear as named,

$$F_4 = W_{F_4}, E_6 = W_{E_6}, E_7 = W_{E_7}, E_8 = W_{E_8}.$$ 

Under this association, we notice that the Lie groups $SO(2n+1)$ and $Sp(n)$ correspond to the same $\mathbb{R}$-reflection group, as their Weyl groups are isomorphic. One must introduce the notion of a root system to be able to distinguish between the two families.

Also note that the Coxeter groups $H_3, H_4$ and all but finitely many of the $I_2(p)$ family don’t occur as Weyl groups. This corresponds to the crystallographic restriction theorem, as the corresponding polytopes don’t fill space. One can see that in
the $I_2(p)$ family, the Coxeter groups that correspond to polytopes that tile the plane are precisely those that occur as Weyl groups, as

\[ I_2(3) \cong A_2, I_2(4) \cong BC_2, I_2(6) \cong G_2. \]

2.4 Reflection Groups over $\mathbb{C}$ and $\mathbb{Q}_p$

Let $(W, V)$ be a $K$-reflection group and let $s \in W$ a reflection. Recall that the order of $s$, denoted $o(s)$, is the smallest integer such that $s^{o(s)} = 1$. When $R = \mathbb{Q}$ or $\mathbb{R}$, all reflections are conjugate to the matrix

\[ s = \text{diag}(-1, 1, ..., 1), \]

and thus have order 2.

Over $\mathbb{C}$, however, take for example the element

\[ s = \text{diag}(e^{2\pi i/m}, 1, ..., 1), \]

rotation in one coordinate. The element $s$ satisfies the definition of a reflection, but has order $m$. Reflections of order 2 are sometimes called ‘honest’ reflections in the literature, and reflections without restriction on order are called ‘pseudoreflections’, with the corresponding reflection groups ‘pseudoreflection groups’.

The appearance of pseudoreflections makes the structure and classification of $\mathbb{C}$-reflection groups quite different than the classification for $\mathbb{R}$.

Complex reflection groups were first classified by Shephard and Todd in 1954 [47], who found a large infinite family that varied with three parameters and 34 exceptional examples. It was not originally apparent, however, that these $\mathbb{C}$-reflection groups corresponded to any topological spaces like the connection between finite $\mathbb{R}$-reflection groups and simple connected compact Lie groups.

In their 1974 paper, Clark and Ewing showed that if $(W, V)$ is a $\mathbb{C}$-reflection group, then one could view $(W, V)$ as a $\mathbb{Q}_p$-reflection group for certain primes $p$. Additionally, they constructed spaces indexed by these reflection groups, which lead to the largest class of exotic $p$-compact groups [18]. First we will consider the passage from $\mathbb{C}$-reflection groups to $\mathbb{Q}_p$-reflection groups.
Definition 57. Let \((W, V)\) be a \(K\)-reflection group. Fixing a basis for \(V\) allows one to view \(w \in W\) as matrices in \(GL_n(V)\). The function \(\chi : W \rightarrow K\) that sends the matrix \(w \in GL_n(K)\) to its trace is called the character of (the natural representation of) \(W\) in \(K\). If \(K\) has characteristic 0 then we define the character field \(\mathbb{Q}(W) \subseteq K\) to be the field extension of \(\mathbb{Q}\) generated by the values \(\chi(w)\), for all \(w \in W\). If \(R\) is a subring of \(K\) then \((W, V)\) is definable over \(R\) if there is a basis for \(V\) such that the entries of the matrices representing the elements of \(W\) belong to \(R\).

Clark and Ewing showed that if \((W, V)\) is a \(\mathbb{C}\)-reflection group and the character field \(\mathbb{Q}(W)\) embeds into the field \(\mathbb{Q}_p\) of \(p\)-adic numbers, then \((W, V)\) is equivalent to a \(\mathbb{Q}_p\)-reflection group, and is definable over the ring \(\mathbb{Z}_p\):

Proposition 58. [18] If \(\rho : W \rightarrow GL_n(\mathbb{C})\) is a finite irreducible complex reflection group with character \(\chi\), then there is a representation \(\rho_p : W \rightarrow GL_n(\mathbb{Q}_p)\) which affords the same character \(\chi\) if and only if \(\mathbb{Q}_p\) contains a subfield isomorphic to the character field \(\mathbb{Q}(\chi)\). Moreover, \(\rho\) is definable over \(\mathbb{Z}_p\).

Clark and Ewing then calculated the various character fields for the irreducible complex reflection groups in the Shephard-Todd classification and determined for which primes these fields embed into \(\mathbb{Q}_p\), using Hensel’s lemma and the fact that the field \(\mathbb{Q}_p\) contains the element \(a^{1/2}\) if and only if \(a\) is a quadratic residue mod \(p\). They had thus obtained a classification of \(\mathbb{Q}_p\)-reflection groups. Their results closely resembled those found by Shephard and Todd, and we will describe them now. For fixed \(p\), there are 4 infinite families and up to 34 sporadic cases [18]:

The four infinite families in the classification consist of the generalized permutation matrix groups, labeled \(G(m, r, n)\). Each group depends on the three positive integers \(m, r,\) and \(n\), where \(r|m\). An \(n \times n\) matrix \(A\) is a generalized permutation matrix if \(A\) contains exactly one nonzero entry in each column and each row. To understand the matrix groups \(G(m, r, n)\), let \(A(m, r, n)\) denote the group of diagonal matrices

\[
A(m, r, n) = \{ \text{diag}(\theta_1, \theta_2, ..., \theta_n) \}
\]
where each $\theta_i$ is an $m^{th}$ root of unity, and the product $(\theta_1...\theta_n)$ is an $m/r^{th}$ root of unity. Let the symmetric group $\Sigma_n$ act on $A(m, r, n)$ by permuting indices. Then $G(m, r, n)$ is defined as the semidirect product

$$G(m, r, n) := A(m, r, n) \rtimes \Sigma_n.$$  

Clark and Ewing split the groups $G(m, r, n)$ into four infinite families, labeled $1, 2a, 2b,$ and $3$ by the different values of $p$ for which they are definable over $\mathbb{Q}_p$:

The first infinite family $G(1, 1, n), n \in \mathbb{Z}_{>0}$ is the special case of $G(m, r, n)$ where $m = 1$. These matrix groups correspond to the symmetric groups $S_{n+1}$: The symmetric groups are generated by transpositions, which are reflections of order $2$. After fixing a basis, we may view the symmetric groups as groups of permutation matrices, having exactly one $1$ in each column and each row, with all other entries $0$. These groups all have character field $\mathbb{Q}$ and can be realized as $\mathbb{Q}$-reflection groups.

The family labeled $3$ corresponds to the finite cyclic groups $\mathbb{Z}/m, m \in \mathbb{Z}_{>0}$. These groups arise as the generalized permutation matrices $G(m, 1, 1)$, which is the group of the $m$-th roots of unity. If $\zeta_m \in G(m, 1, 1)$ is a primitive $m$-th root of unity, then $\zeta_m$ is a reflection of order $m$ that generates $G(m, 1, 1)$. The character field for groups in this family is $\mathbb{Q}(\zeta_m)$.

The families labeled $2a$ and $2b$ contain the remainder of the $G(m, r, n)$, with the separation being that if $n \geq 3$ or $n = 2$ and $r < m$ then the corresponding character field is $\mathbb{Q}(\zeta_m)$, but if $n = 2$ and $m = r$ then the group $G(m, m, 2)$ has character field $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$.

### 2.4.1 Modular and non-modular $\mathbb{Z}_p$-reflection groups

In what follows, we will be invested in integral reflection groups:

**Definition 59.** Let $R$ be an integral domain with field of fractions $K$. An $R$-reflection group is a pair $(W, L)$, where $L$ is a finitely generated free $R$-module with a faithful representation

$$\rho : W \rightarrow GL(L),$$
such that the image of $W$ under $\rho$ is generated by elements $\alpha$ such that $1 - \alpha$ has rank one, viewed as a matrix over $K$.

**Example 60.** If $(W, V)$ is a $\mathbb{Q}_p$-reflection group definable over the ring $\mathbb{Z}_p$, then there is a basis for $V$ such that $(W, V)$ is a $\mathbb{Z}_p$-reflection group. There may, however, be non-isomorphic $\mathbb{Z}_p$-reflection groups that are isomorphic when viewed as $\mathbb{Q}_p$-reflection groups.

The Clark-Ewing classification serves to classify $\mathbb{Z}_p$-reflection groups up to isomorphism in $\mathbb{Q}_p$, and has been extended to classification up to isomorphism in $\mathbb{Z}_p$ by Notbohm [43].

**Definition 61.** If $W < GL_n(\mathbb{C})$ is a complex reflection group such that $\chi(W)$ embeds into $\mathbb{Q}_p$ and $p$ divides the order of $W$, then $p$ is called a *modular prime* for $W$, and $W$, viewed over $\mathbb{Q}_p$ or $\mathbb{Z}_p$, is a *modular reflection group*. Otherwise, $p$ is a *non-modular prime* for $W$, and $W$ is a *non-modular reflection group*.

**Remark 62.** Although the families 3 and 2a in the Clark-Ewing classification appear to have the same character fields, and thus can be realized over the field $\mathbb{Q}_p$ for the same values of $p$, the order of the group $G(m, r, n)$ is $n!m^n/r$, but when we specialize to family 3 the order of $G(m, 1, 1)$ is simply $m$. This affects which primes are modular and which are non-modular, and is the reason for the splitting off of family 3.

### 2.5 Exotic $p$-compact groups

A $p$-compact group $BX$ is *exotic* if $BX$ is not homotopy-equivalent to the completion of a compact Lie group. The whole point of studying $p$-compact groups is that exotic $p$-compact groups exist; there are structures localized at a specific primes $p$ that behave in many ways like Lie groups, but do not correspond to any particular Lie group. These exotic $p$-compact groups fall into two categories: the non-modular groups, which we attribute to Clark-Ewing, and the modular groups studied by Agaudé and others.
In 1974, well before the definition of $p$-compact groups, Clark and Ewing constructed a large class of spaces with polynomial mod-$p$ cohomology, the completions of which are $p$-compact groups. By the nature of their construction, they are some of the most easily understood $p$-compact groups.

**Definition 63.** A connected $p$-compact group $X$ with maximal torus $T$ and Weyl group $W$ is **non-modular** if $p$ does not divide the order of $W$. If $p$ divides $|W|$ then $X$ is said to be **modular**. It follows that $X$ is (non-)modular if and only if the $\mathbb{Q}_p$-reflection group $(W, \pi_2 BT)$ [22, 9.7(2)] is (non-)modular. A connected $p$-compact group $X$ is **polynomial** if its $\mathbb{Z}_p$-cohomology $H^*(BX; \mathbb{Z}_p)$ is a polynomial algebra.

**Proposition 64** (Clark-Ewing construction). Let $(W, L)$ be a $\mathbb{Z}_p$-reflection group such that $p$ is odd and does not divide $|W|$. Then there exists a non-modular, polynomial $p$-compact group $X$ with maximal torus $T_X$ and Weyl group $W_X$ such that $W_X \cong W$ and $\pi_2 BT_X \cong L$.

**Proof.** Let $W$ be a finite subgroup of $GL_n(\mathbb{Z}_p)$. Then there is a natural action of $W$ on $BT := B^2 L = K((\mathbb{Z}_p)^n, 2)$, which is a $p$-compact torus of rank $n$. If we let $BN := BT_\text{hW}$ be the homotopy orbit space of this action, then up to homotopy $BN$ sits in a fibration

$$BT \to BN \to BW$$

where $BW = K(W, 1)$, the classifying space of the discrete group $W$. Applying the Leray-Serre spectral sequence to this fibration allows us to calculate the mod-$p$ cohomology of $BN$ via

$$E_2^{r,s} = H^r(BW; H^s(BT; \mathbb{Z}_p)) \implies H^{r+s}(BN; \mathbb{Z}_p),$$

where the $E_2$-term is cohomology with respect to the system of local coefficients given by the action of $W$ on the $\mathbb{Z}_p$-module $H^*(BT; \mathbb{Z}_p)$. We may also write the $E_2$-term as the group cohomology of $W$

$$E_2^{r,s} = H^r(W; H^s(BT; \mathbb{Z}_p)).$$
It is a classical result that if $M$ is a $W$-module, then the selfmap of $H^n(W; M)$ that is multiplication by $|W|$ is the 0 map, thus $|W| \cdot E_2^{r,s} = 0$ [16, XII.2.5]. However, since $E_2^{r,s}$ is also a $\mathbb{Z}_p$-module for $r > 0$ and the Jacobson radical of $\mathbb{Z}_p$ is $< p >$, all torsion in $E_2^{r,s}$ is $p$-torsion. Thus if we assume that $|W|$ is prime to $p$, then it follows that $E_2^{r,s} = 0$ for $r > 0$, and the spectral sequence collapses at the line $r = 0$. Thus

$$E_\infty = E_2 = H^0(W; H^*(BT; \mathbb{Z}_p)) = \text{Ext}_{\mathbb{Z}_p[W]}^0(\mathbb{Z}_p, H^*(BT; \mathbb{Z}_p)) = H^*(BT; \mathbb{Z}_p)^W.$$\[16, XII.2.5]\]

Furthermore, since $E_\infty$ is a free $\mathbb{Z}_p$-module, the spectral sequence converges as a $\mathbb{Z}_p$-module and

$$H^*(BN; \mathbb{Z}_p) \cong H^*(BT; \mathbb{Z}_p)^W$$

as modules over $\mathbb{Z}_p$.

By the Chevalley-Shephard-Todd theorem, the reduction mod-$p$ $H^*(BN; \mathbb{Z}_p) \otimes \mathbb{F}_p = (H(BT; \mathbb{Z}_p) \otimes \mathbb{F}_p)^W$ is a polynomial algebra precisely when $W$ is a $\mathbb{F}_p$-reflection group, thus if $W$ is a $\mathbb{F}_p$-reflection group, with order prime to $p$, then $H^*(BN; \mathbb{Z}_p) \otimes \mathbb{F}_p$ is polynomial on generators of even degree. One completes the construction by forming the localization

$$BX := BN_p,$$

and seeing that $BX$ is an $\mathbb{F}_p$-local space with polynomial mod-$p$ cohomology, and by the Eilenberg-Moore spectral sequence in the path-loop fibration, $X := \Omega BX$ is $\mathbb{F}_p$-finite and hence a $p$-compact group. Thus $BX$ is a polynomial, non-modular $p$-compact group.

\[\Box\]

**Example 65** (Sullivan spheres). The Clark-Ewing $p$-compact groups corresponding to the rank-1 finite cyclic $\mathbb{Q}_p$-reflection groups in family 3 are called the ‘Sullivan spheres.’ They are the first exotic $p$-compact groups to be constructed, and the method of their construction was generalized by Clark and Ewing to create the other non-modular $p$-compact groups. In particular, if $p$ is an odd prime not dividing positive integer $l$, then $\mathbb{Z}/l$ is a subgroup of the $p$-adic units $\mathbb{Z}_p^\times$ by Hensel’s lemma. The corresponding Sullivan sphere is then the loop space of the completion of the Borel construction of the natural action of $\mathbb{Z}/l$ on the $p$-compact 1-torus $BT =$

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$K(\mathbb{Z}_p, 2)$. It is a ‘sphere’ in the sense that it is homotopic to the completion of the $2l - 1$-sphere,

$$X \simeq (S^{2l-1})_p.$$ 

The Sullivan spheres satisfy

$$H^*(BX; \mathbb{F}_p) = \mathbb{F}_p[t]^{\mathbb{Z}/l} = \mathbb{F}_p[t'^l],$$

where the degree of $(t'^l)$ is $2l$, and

$$H^*(X'; \mathbb{F}_p) = \Lambda[x],$$

with $\text{deg}(x) = 2l - 1$. An interesting fact about the Sullivan spheres is that the homogeneous space

$$X/T = \text{hofib}(Bi : BT \to BX) \simeq (\mathbb{C}P^{l-1})_p$$

is homotopy equivalent to the completion of a manifold, a property that is enjoyed by $p$-compact groups of Lie type, but not by general exotic $p$-compact groups. Note that when $l = 2$, the corresponding Sullivan sphere is equivalent to the completion $(S^3)_p \simeq SU(2)_p$.

**Remark 66.** The Clark-Ewing construction realizes many examples of $p$-compact groups, but fundamentally requires that $p$ not divide $|W|$. To realize the remaining $\mathbb{Z}_p$-reflection groups, one must generalize the homotopy orbit space construction to a homotopy colimit, inspired by the work of Jackowski and McClure in [31].

### 2.6 Classification and functoriality

We begin by stating the classification for $p$-compact groups, where $p$ is an odd prime.

**Theorem 67.** [3, 1.1] Let $p$ be an odd prime. The assignment that sends each connected $p$-compact group $X$ to the associated $\mathbb{Z}_p$-reflection group $(W_X, \pi_2 BT_X)$ defines a bijection between the set of isomorphism classes of connected $p$-compact groups and the set of isomorphism classes of finite $\mathbb{Z}_p$-reflection groups.
The goal of this section is to expand on the comment in [41] that one can choose morphisms such that this association becomes a functor between a category of p-compact groups and a category of \( \mathbb{Z}_p \)-reflection groups, and to explore the properties of this functor.

2.6.1 Maps between reflection groups

The theory of \( R \)-reflection groups naturally extends to a category: If we have two reflection groups \((W_1, L_1)\) and \((W_2, L_2)\), we can define a morphism between them as a pair \((\alpha, \theta)\), where \(\alpha : W_1 \rightarrow W_2\) is a group homomorphism, and \(\theta : L_1 \rightarrow L_2\) is an \(\alpha\)-equivariant \(R\)-module homomorphism, i.e., if \(w \in W_1, v \in L_1\), then \(\theta(w \ast t) = \alpha(w) \ast \theta(v)\).

Definition 68 (The Category \( R\)-Refl). [41, 4.1] For \( R = \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{F}_p \), let \( R\)-Refl be the category with

- objects: pairs \((W, L)\), where \(L\) is a finitely generated free \(R\)-module and \(W\) a finite reflection subgroup of \(\text{Aut}(L)\), and
- morphisms: pairs \((\alpha, \theta) : (W_1, L_1) \rightarrow (W_2, L_2)\), where \(\alpha : W_1 \rightarrow W_2\) is a group homomorphism and \(\theta : L_1 \rightarrow L_2\) is an \(\alpha\)-equivariant \(R\)-module homomorphism.

We can now use the machinery of category theory to define substructures in \( R\)-Refl. In particular, a morphism \((\alpha, \theta)\) is a monomorphism if \((\alpha, \theta)\) is left-cancellative, which is the case when both \(\alpha\) and \(\theta\) are monomorphisms in their respective categories. A subgroup is an equivalence class of monomorphisms, under the relation of common factorization. Thus two monomorphisms determine the same subgroup if they factor through each other.

Example 69. Let \((W, L)\) be the \(p\)-adic reflection group \((\mathbb{Z}/m, \mathbb{Z}_p)\), where \(m|p - 1\). The sub-reflection groups of \((\mathbb{Z}/m, \mathbb{Z}_p)\) are pairs \((W', L')\) such that \(L'\) is a sublattice of \(\mathbb{Z}_p\) and \(W'\) is a subgroup of \(\text{GL}(L')\) generated by reflections. In particular, the order of \(W'\) is always a divisor of \(m\), and if \(r \in \mathbb{Z}_{>0}\) such that \(r|m\), then there is
exactly one sub-reflection group

$$(Z/r, \mathbb{Z}_p) \to (Z/m, \mathbb{Z}_p),$$

As abstract groups generated by reflections, these subreflection groups correspond to

$$G(m, r, 1) \cong G(m/r, 1, 1) < G(m, 1, 1).$$

**Remark 70.** While every subreflection group $(W, L) < (W', L')$ corresponds to a subgroup $W < W'$, not every abstract embedding $W < W'$ of groups generated by reflections corresponds to a subreflection group, as we must consider the corresponding representations on the lattices. [34, 8.12]

### 2.6.2 Toric morphisms and functoriality

Normally, one considers the morphisms in the category of $p$-compact groups to consist of general homotopy classes of unbased maps between $p$-compact groups. Unfortunately, not every map induces a group homomorphism between the corresponding Weyl groups, which one immediately sees is a necessary condition to induce a morphism in the category $\mathbb{Z}_p\text{-refl}$ of $\mathbb{Z}_p$-reflection groups. However [40, 42], we have a good handle on the class of maps that do extend as we would like:

**Definition 71.** [40] Let $X$ and $Y$ be $p$-compact groups with maximal tori $B_i_X : T_X \to BX$ and $B_i_Y : T_Y \to BY$, and $Y$ connected. A $p$-toric morphism $f : X \to Y$ is a loop space homomorphism such that the centralizer of the composition

$$C_Y(T_X) := C_Y(Bf \circ B_i_X) \to Y$$

is a maximal torus for $Y$.

To motivate this definition, let $f : X \to Y$ be a homomorphism of $p$-compact groups with maximal tori $T_X$ and $T_Y$. By [22, 8.11], $f$ lifts to map of maximal tori
$T(f) : T_X \to T_Y$, i.e. the space $Y/T_Y^{T_X}$ of lifts is non-empty. Thus

$$
\begin{array}{c}
T_X \xrightarrow{T(f)} T_Y \\
\downarrow \quad \downarrow \\
X \xrightarrow{f} Y
\end{array}
$$

commutes, up to conjugacy, and $T(f)$ is unique up to left action by $W_Y$. It follows that if $w \in W(X)$, then there exists a $v \in W(Y)$ such that $T(f)w = vT(f)$. In general, $v$ is not uniquely determined by $w$, however. Let

$$(W_Y)^{T(f)} := \{w \in W_Y | w \circ T(f) = T(f)\}$$

be the stabilizer subgroup at $T(f)$. If $(W_Y)^{T(f)}$ is trivial, then $v$ is uniquely determined by $w$, and one can check that the association $w \mapsto v$ determines a homomorphism $\alpha(f) : W_X \to W_Y$. It follows that $\theta(f) := \pi_2 T(f)$ is $\alpha(f)$-equivariant, and $(\alpha(f), \theta(f))$ is a morphism of the associated reflection groups.

Thus having trivial isotropy group $(W_Y)^{T(f)}$ is a necessary and sufficient condition for $f$ to induce a morphism. Fortunately [42, 3.2], one can identify $(W_Y)^{T(f)}$ as the Weyl group of the centralizer $C_Y(T_X)$ of $T_X$ in $Y$, and requiring $(W_Y)^{T(f)}$ to be trivial amounts to requiring $C_Y(T_X)$ to be a $p$-compact torus. Since $C_Y(T_X)$ is also a subgroup of maximal rank, the $p$-compact group $C_Y(T_X)$ is a maximal torus for $Y$. Thus we have arrived at definition 70.

**Example 72.** [40, 2.12] Subgroups of maximal rank are $p$-toric: Recall that a subgroup of maximal rank is a monomorphism $f : X \to Y$ of $p$-compact groups such that there exists a maximal torus $i : T_X \to X$ of $X$ such that $f \circ i : T_X \to Y$ is a maximal torus for $Y$ [23, 4.1]. It follows that if $Y$ is connected, then the natural map $T_X \to C_Y(T_X)$ is an equivalence [20], $C_Y(T_X)$ is a maximal torus for $Y$, and $f$ is a $p$-toric morphism [22, 9.1].

Thus if we let $\mathbf{Pcg}$ denote the category with

- objects: isomorphism classes of connected $p$-compact groups for a fixed odd prime $p$
• morphisms: conjugacy classes of $p$-toric morphisms

Then the association $X \mapsto (W_X, L_X)$ is a functor from $\text{Pcg}$ to $\text{Z}_p\text{-Refl}$. We would like to understand how close this functor is to an equivalence of categories.

2.6.3 Properties

The traditional statement of the classification implies that the functor $(W, L)$ is essentially surjective, i.e. every isomorphism class of $\text{Z}_p$-reflection groups. Thus given an isomorphism class $(W, L)$ of $\text{Z}_p$-reflection groups, there is a unique isomorphism class of $p$-compact groups $X$ such that $(W_X, L_X) = (W, L)$.

We now consider fullness. We would like to be able to realize arbitrary morphisms between $\text{Z}_p$-reflection groups as coming from toric morphisms of the corresponding $p$-compact groups. For now, we are able to say that

**Theorem 73 (Fullness).** Let $p$ be an odd prime, and let $(W, L)$ and $(W', L')$ be two non-modular $\text{Z}_p$-reflection groups (def. 61), and let $(\alpha, \theta) : (W, L) \to (W', L')$ be a homomorphism between them (sect. 2.6.1) such that $(W, L)$ is a parabolic subgroup of $(W', L')$ (def 76). Then $(\alpha, \theta)$ is realizable by a toric morphism (def 71) between the $p$-compact groups $X$ and $X'$ given by the Clark-Ewing construction (Prop 64).

Thus the assignment that sends each connected $p$-compact group $X$ with Weyl group $W_X$ and maximal torus $T_X$ to the $\text{Z}_p$-reflection group $(W_X, \pi_2 BT_X)$ is a full, essentially surjective functor from the category of non-modular $p$-compact groups and toric morphisms to the category of non-modular $\text{Z}_p$-reflection groups and inclusions of parabolic subgroups. We can concretely construct the morphisms when restricting to this case, though the result may be able to extend more generally.

**Lemma 74.** If $\theta : L_1 \to L_2$ is a $\text{Z}_p$-module homomorphism between finitely generated free $\text{Z}_p$-modules, then $\theta$ can be realized by a loop space homomorphism $f : T_1 \to T_2$ between $p$-compact tori. Moreover, if $\theta$ is injective, then $f$ is a monomorphism of $p$-compact groups.
Proof. Let \( r_i \) denote the rank of \( L_i \cong \mathbb{Z}_p^{r_i} \). Then \( L_i \cong \pi_2 BT_i \) for some \( p \)-compact torus

\[
BT_i \cong K((\mathbb{Z}_p^{r_i}), 2) \cong B^2 L_i.
\]

The map \( BT_1 \to BT_2 \) follows from the functoriality of Eilenberg-MacLane spaces. It remains to see that the mod \( p \) cohomology of \( L_2 \) is finitely generated over the mod \( p \) cohomology of \( L_1 \).

Let \( X \) be a connected \( p \)-compact group with maximal torus \( T \) of rank \( r \) and Weyl group \( W \). Let \( N \) be the normalizer of \( T \) with a discrete approximation \( \tilde{N} \to N \). Let \( \tilde{T} \) be the maximal divisible subgroup of \( \tilde{N} \), so that \( \tilde{T} \to T \) is a discrete approximation for \( T \).

Note that \( Aut(\pi_2 BT) \cong Aut(\tilde{T}) \), and under this isomorphism, \( H^3(W; \pi_2 BT) \cong H^2(W; \tilde{T}) \), giving a bijection between fibrations

\[
BT \to BN \to BW
\]
classified by elements of \( H^3(W; \pi_2 BT) \) and group extensions

\[
\tilde{T} \to \tilde{N} \to W
\]
classified by \( H^2(W; \tilde{T}) \) [39, 2.4]. Under this association, the action of \( W \) on \( BT \) by monodromy corresponds to the action of \( W \) on \( \tilde{T} \) by conjugation, after taking classifying spaces and fiberwise \( \mathbb{F}_p \)-completion [23, 7.2]. Additionally, if \( BF : BN_1 \to BN_2 \) is a morphism of extended \( p \)-compact tori, then [23, 3.13] \( BF \) induces a morphism

\[
\begin{array}{c}
\tilde{T}_1 \\
\downarrow f_1
\end{array}
\begin{array}{c}
\tilde{N}_1 \\
\downarrow f
\end{array}
\begin{array}{c}
W_1 \\
\downarrow \pi_0(f)
\end{array}
\begin{array}{c}
\tilde{T}_2 \\
\downarrow f_2
\end{array}
\begin{array}{c}
\tilde{N}_2 \\
\downarrow
\end{array}
\begin{array}{c}
W_2
\end{array}
\]

d of group extensions, and one may pass from a morphism of group extensions to a fiber map by taking classifying spaces and fiberwise completing.

Proposition 75. If \( (\alpha, \theta) : (W_1, L_1) \to (W_2, L_2) \) is a morphism of simple \( \mathbb{Z}_p \)-reflection groups such that \( p > 2 \) does not divide the order of \( W_i, i = 1, 2 \), then \( (\alpha, \theta) \) is realizable as a morphism \( f : X_1 \to X_2 \) of \( p \)-compact groups.
Proof. Let $X_1$ and $X_2$ be $p$-compact groups that correspond under the classification [26] to the Weyl groups $W_1$ and $W_2$, with maximal tori $T_1$ and $T_2$, respectively. Recall that the normalizer of the maximal torus is defined as the Borel construction $BN := (BT)_{hW}$ of the Weyl group acting on the maximal torus, and up to homotopy sits in a fiber sequence

$$BT \to BN \to BW.$$ 

When $p$ is odd, this fibration has a section, unique up to vertical homotopy [4, 1.2]. If $\tilde{N}$ and $\tilde{T}$ denote discrete approximations to $N$ and $T$, then this splitting implies that $\tilde{N} \cong \tilde{T} \times W$. The maps $\tilde{\theta} := \mathbb{Z}/p^\infty \otimes \theta$ and $\alpha$ determine a map on discrete normalizers

$$(\alpha, \tilde{\theta}) : \tilde{N}_1 \to \tilde{N}_2$$

which passes to a loop space homomorphism after taking classifying spaces and fiberwise $p$-completing.

Thus up to homotopy, we have the diagram of fiber sequences,

$$\begin{array}{ccc}
BT_1 & \longrightarrow & BN_1 \longrightarrow BW_1 \\
\downarrow B^{2\theta} & & \downarrow f(\alpha,\theta) \downarrow B\alpha \\
BT_2 & \longrightarrow & BN_2 \longrightarrow BW_2
\end{array}$$

where $f(\alpha,\theta) := B(\alpha,\tilde{\theta})_p$. It remains to show that $f(\alpha,\theta)$ extends to a map $BX_1 \to BX_2$.

Since the $BX_i$ are assumed to be Clark-Ewing $p$-compact groups, the inclusions $Bj_i : BN_i \to BX_i$ are $\mathbb{F}_p$-equivalences and factor as

$$Bj_i : BN_i \longrightarrow (BN_i)_p \overset{\cong}{\longrightarrow} BX_i,$$

where the first map is $p$-completion, and the second map is an isomorphism of $p$-compact groups [38, 3.12]. Thus we have realized $(\alpha, \theta)$ as a map

$$f(\alpha, \theta)_p : BX_1 \longrightarrow BX_2.$$

$\square$
It remains to check that the morphism obtained from this proposition is toric. We will now show that if \((\alpha, \theta)\) corresponds to the inclusion of a parabolic subgroup, then we can realize \((\alpha, \theta)\) as a subgroup of maximal rank, which is toric and hence a morphism in our category.

**Definition 76.** Let \((W, L)\) be an R-reflection group, and let \(V \subseteq L\) be a subset. The *parabolic subgroup* \(W_V\) is the pointwise stabilizer of \(V\) in \(W\),

\[
W_V := \{ w \in W | \forall v \in V, wv = v \}
\]

Over \(\mathbb{C}\), Steinberg’s fixed point theorem [48, 1.5] gives the result that parabolic subgroups corresponding to arbitrary subsets are again \(\mathbb{C}\)-reflection groups, generated by the reflections in the hyperplanes that contain the subset.

For odd primes \(p\), we have the following result by Nakajima, which we have specialized here for non-modular reflection groups, using [43, 3.6, 3.7]

**Lemma 77** (Nakajima’s lemma). [3, 7.1] let \((W, L)\) be a non-modular \(\mathbb{Z}_p\)-reflection group, \(p\) odd. Let \(V\) be a non-trivial subset of \(L \otimes_{\mathbb{Z}_p} \mathbb{F}_p\), and let \(W_V\) denote the pointwise stabilizer of \(V\) in \(W\). Then \((W_V, L)\) is again a non-modular \(\mathbb{Z}_p\)-reflection group.

In general, for \(p\)-compact groups, we can consider subgroups of maximal rank corresponding to subgroups of the \(p\)-discrete maximal torus:

Recall that for \(p\) odd, an element \(w \in W\) is a reflection if and only if the fixed point set \(F(w)\) of the action of \(w\) on \(\pi_2 BT\) is isomorphic to \((\mathbb{Z}_p)^{r-1}\), if and only if the fixed point set \(\tilde{F}(w)\) of the action of \(w\) on \(\tilde{T} = \pi_2 BT \otimes \mathbb{Z}/p^\infty\) is isomorphic to \((\mathbb{Z}/p^\infty)^{r-1}\).

**Theorem 78.** [23, 7.6] Let \(X\) be a connected \(p\)-compact group, \(p\) odd, with maximal torus \(T\) and Weyl group \(W\). Suppose that \(A \subseteq \tilde{T}\) is a subgroup. Let \(W_A\) denote the Weyl group of the centralizer \(C_X(A)\), and let \(W_{A1}\) denote the Weyl group of the centralizer of the identity component \(C_X(A)_1\) of \(C_X(A)\). Then

1. The group \(W_A\) is the parabolic subgroup containing the elements of \(W\) that pointwise fix \(A\).
2. The group $W_{A_1}$ is the subgroup of $W_A$ generated by the reflections contained in $W_A$.

3. If $w \in W$ is a reflection with reflecting hyperplane $F(w)$, then $w \in W_A$ if and only if $A \subseteq F(w)$.

It follows that, since the identity component $C_X(A)_1$ is a connected $p$-compact group, its Weyl group $W_{A_1}$ is a $\mathbb{Z}_p$-reflection group, generated by the reflections in the hyperplanes that contain $A$. In particular, if $C_X(A)$ is connected, then $W_A$ is a reflection group. This is the case, for example, if $A$ is a $p$-discrete torus [23, 7.8].

**Proposition 79.** Let $(W_X, L_X)$ be the non-modular $\mathbb{Z}_p$-reflection group corresponding to the Clark-Ewing $p$-compact group $X$, and let $\alpha_U : W_U \to W_X$ be the inclusion of a parabolic subgroup for some $U \subseteq L_X \otimes \mathbb{F}_p$. Then $(\alpha_U, id)$ can be realized as a subgroup of maximal rank in $X$.

**Proof.** If $s \in W_X$, let $F(s) := \{v \in L_X|sv = v\}$ denote the reflecting hyperplane of $s$ and let $T_s := B^2F(s) \to T_X$ denote the fixed-point subtorus corresponding to the inclusion $F(s) \to F(1)$.

Recall that Nakajima’s lemma tells us that $W_U$ is generated by the reflections $s \in W_X$ such that the reflecting hyperplanes $F(s) \supseteq U$.

Since the pointwise stabilizer of a subset also pointwise fixes the span of that subset, we have $W_U = W_{<U>}$, where $<U>$ is the subspace of $L_X$ spanned by $U$. If $U$ contains a regular element, an element not contained in any reflecting hyperplane, then only the identity fixes all of $U$ and $W_U = W_{L_X}$ is trivial. In this case, $(0, id)$ corresponds to the inclusion of the maximal torus $BT \to BX$.

Thus without loss of generality, we may assume $U$ is in the lattice of intersecting hyperplanes of $L_X$, i.e.

$$U = F(s_1) \cap \ldots \cap F(s_l) = F(s_1 \ldots s_l)$$

for some finite product of reflections $s_1 \ldots s_l$. Then the fixed point torus $T_{s_1 \ldots s_l}$ yields a monomorphism $T_{s_1 \ldots s_l} \to T_X$ of a $p$-compact subtorus and applying [23, section 4],
the centralizer $C_X(T_{s_1...s_l})$ is a subgroup of maximal rank, and

$$T_X \rightarrow C_X(T_{s_1...s_l}) \rightarrow X$$

induces inclusions

$$1 \rightarrow W_U \rightarrow W_X$$

of Weyl groups.

\[\square\]

**Theorem 80.** [3, 11.5] Let $G$ and $G'$ be two compact connected Lie groups and $p$ an odd prime. Then the following conditions are equivalent:

1. $(W_G, L_G \otimes \mathbb{Z}_p)$ and $(W_{G'}, L_{G'} \otimes \mathbb{Z}_p)$ are isomorphic.
2. $(BG)_p \simeq (BG')_p$

Note that this contains the result that, at odd primes, $BSO(2n+1)_p$ is homotopy equivalent to $BSp(n)_p$.  

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3 Characters of $p$-compact Groups

Here we motivate the definition of a character of a $p$-compact group $X$ as an element in the $K$-theory of $BX$ with $\mathbb{Z}_p$-coefficients. First we recall how characters of connected compact Lie groups can be described in terms of a restriction map in equivariant $K$-theory, then we look at non-equivariant $K$-theory of classifying spaces via Atiyah-Segal completion [8]. Finally we outline construction of characters on $K(BG; \mathbb{Z}_p)$, introduced by Adams in [1] and apply this to $p$-compact groups, using results about $K(BX; \mathbb{Z}_p)$ obtained by Jeanneret and Osse in [32].

3.1 Classical setup

If $V$ is a complex representation of the group $G$ then the character of $V$ is the map that sends $g \in G$ to the trace of its action on $V$,

$$\chi_V : G \rightarrow \mathbb{C}$$

$$g \mapsto \text{Tr}(g|V).$$

The representation $V$ is determined by $\chi_V$ up to isomorphism, and under the operations of direct sum and tensor product of representations, the association $V \mapsto \chi_V$ determines a ring map

$$\text{char} : R(G) \rightarrow Cl(G)$$

$$[V] \mapsto \chi_V,$$

where the representation ring $R(G)$ is the (Grothendieck group of) isomorphism classes of complex linear representation of $G$ and $Cl(G)$ is the ring of continuous functions on the space of conjugacy classes of $G$, with pointwise addition and multiplication. Furthermore, the map $\text{char}$ is injective, thus we may think of $R(G)$ as a subring of $Cl(G)$.

Given a group homomorphism $\alpha : H \rightarrow G$, we may consider restricting $G$-representations along $\alpha$ to obtain $H$-representations, and if $G/H$ is a complex variety
we may induce $H$-representations up along $\alpha$ to obtain $G$-representations. These lead to ring maps between $R(G)$ and $R(H)$.

If $G$ is a compact connected Lie group with maximal torus $T$ and corresponding Weyl group $W$, then we know a good deal about its conjugacy classes. In particular, for maximal torus $T$ and corresponding Weyl group $W$, we have the commuting diagram,

\[
\begin{array}{ccc}
R(G) & \xrightarrow{\text{res}} & R(T)^W \\
\downarrow \text{char} & & \downarrow \\
Cl(G) & \cong & C^0(T)^W
\end{array}
\]

where $R(T)^W$ is the subring of the representation ring of the torus that is invariant under the action of Weyl group. The top map $\text{res}$ turns out to be an isomorphism ([15, VI, 2.1]), and we can identify the images of $\text{res}$ and $\text{char}$ and view the restriction as sending a representation to its character, giving us

\[
\text{char} : R(G) \xrightarrow{\cong} R(T)^W.
\]

Let $G$ be a compact Lie group and $X$ a compact $G$-space. Let $K_G(X)$ denote the (Grothendieck group of) isomorphism classes of $G$-equivariant vector bundles over $X$. Note that equivariant vector bundles over the one-point space are simply $G$-equivariant vector spaces, also known as complex linear representations of $G$. Thus the representation ring $R(G)$ is isomorphic to the equivariant $K$-theory of a point,

\[
R(G) \cong K_G(pt).
\]

For general spaces $X$, $K_G(X)$ is a module over $K_G(pt)$.

If $\phi : H \to G$ is a subgroup inclusion of connected compact Lie groups, then restriction along $\phi$ gives a homomorphism of rings

\[
\text{res}|_{\phi} : K_G(X) \to K_H(X),
\]

that sends each $G$-equivariant vector bundle over $X$ to itself, viewed as an $H$-equivariant vector bundle. Now that we’ve identified $R(G)$ with $K_G(pt)$, elements in $K_G(pt)$ will be called (virtual) representations. If $\rho \in K_G(pt)$ is a representation,
then the character of $\rho$ is its image under the restriction

$$K_G(pt) \rightarrow K_T(pt)^W.$$  

3.2 Atiyah-Segal completion

Let $G$ be a finite group. Atiyah [5] constructed a homomorphism of graded rings

$$\alpha : R(G) \rightarrow K(BG)$$

from the representation ring of $G$ to the $K$-theory of the classifying space $BG$.

Let us recall that if $R$ is a commutative ring with a given ideal $I$, and if $K$ is an $R$-module, then $K$ can be given the $I$-adic topology, where the submodules $I^n \cdot K$ form a basis of the neighborhoods of the identity. The Hausdorff completion with respect to this topology can be identified with the inverse limit

$$K_I \cong \lim_{\rightarrow} K/I^n \cdot K.$$  

If we let $I_G$ denote the augmentation ideal of $G$, by definition the kernel of the homomorphism

$$\epsilon : R(G) \rightarrow \mathbb{Z}$$

that sends each virtual representation to its dimension, then Atiyah showed that $\alpha$ passes via $I_G$-adic completion to an isomorphism of topological rings,

$$\hat{\alpha} : R(G)_{I_G} \xrightarrow{\cong} K(BG).$$

Atiyah and Segal extend this result to all compact Lie groups and all (complex) $G$-spaces $X$ via [8, 4.2]. They showed that if $K_G(X)$ is finitely generated over $R(G)$ then there exists a homomorphism

$$\alpha : K_G(X) \rightarrow K(X_{hG}),$$

where $X_{hG}$ denotes the Borel construction of $G$ acting on $X$, such that $\alpha$ induces an isomorphism of the $I_G$-adic completion,

$$K_G(X)_{I_G} \cong K(X_{hG}).$$
3.3 Adams characters

Let $K^*(-; \mathbb{Z}_p)$ denote representable $p$-completed complex $K$-theory. For our purposes, we can treat this as the inverse limit

$$K^*(-; \mathbb{Z}_p) := K^*_p(-) := \lim K^*(-; \mathbb{Z}/p^k),$$

where $K^*(-; \mathbb{Z}/p^k)$ denotes $K$-theory with coefficients in the finite field $\mathbb{Z}/p^k$. Recall that Adams [2] defined the underlying spectrum of such a theory as the spectrum $K\mathbb{Z}/p^k := K \wedge M(\mathbb{Z}/p^k)$, where $K$ is the 2-periodic $BU$-spectrum of complex $K$-theory and $M(\mathbb{Z}/p^k)$ is a Moore spectrum of type $\mathbb{Z}/p^k$. It follows that $K^*_p(-)$ is a 2-periodic theory, with $K^0_p$ represented by the $p$-completed classifying space of the infinite unitary group $\mathbb{Z}_p \times BU_p$ and $K^1_p$ represented by $U_p$. Thus the homotopy groups are

$$\pi_n(K\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & n = \text{even} \\ * & n = \text{odd}. \end{cases}$$

The properties of the functor $K_p(-)$ are outlined in greater detail in appendix B. By a result of Adams,

**Proposition 81.** [1, 2.3] If $G$ is a connected compact Lie group (or with $\pi_0G$ a $p$-group), then elements of the $p$-completed $K$-theory $K(BG; \mathbb{Z}_p)$ can be interpreted as class functions

$$\tilde{G} \rightarrow \mathbb{Z}_p \otimes \mathbb{C},$$

where $\tilde{G}$ is the set

$$\tilde{G} := \{ g \in G \mid g^{p^k} = 1, k \in \mathbb{Z}_{\geq 0} \}$$

of elements of $p$-power order in $G$.

Since the localization $G_p$ is a $p$-compact group and localization induces an isomorphism (Cor. 25),

$$K(BG; \mathbb{Z}_p) \cong K(BG_p; \mathbb{Z}_p),$$

elements of $K(BG_p; \mathbb{Z}_p)$ can be interpreted as class functions on $\tilde{G}$. This motivates the definition,
Definition 82. If $BX$ is a $p$-compact group, then a \textit{(p-complete) character} of $BX$ is an element $\rho \in K(BX; \mathbb{Z}_p)$.

Example 83. [1, 2.1] Assume that $G$ is a finite $p$-group. Let $\alpha$ denote the $p$-adic extension of Atiyah-Segal completion. Then

$$\alpha : \mathbb{Z}_p \otimes R(G) \longrightarrow K(BG; \mathbb{Z}_p)$$

is an isomorphism, and the coefficient homomorphism map on reduced $K$-theory,

$$\tilde{K}(BG) \longrightarrow \tilde{K}(BG; \mathbb{Z}_p),$$

is an isomorphism (moreover, a homeomorphism, where the filtration topology on $K(BG; \mathbb{Z}_p)$ coincides with the $p$-adic topology).

The idea is that if $G$ is a connected $p$-compact group and $X$ is a $G$-space, then we replace the equivariant $K$-theory $K_G(X)$ with the non-equivariant, $p$-completed $K$-theory of $X_{hG}$,

$$K_G(X) := K_p(X_{hG}),$$

and use this setup to study character theory for $p$-compact groups.

For $p$-compact groups, Jeanneret and Osse have the following results:

Theorem 84. [32, Main Theorem] Let $X$ a connected $p$-compact group, with maximal torus $i : T \to X$ and Weyl group $W$. then

$$K^*(BX; \mathbb{Z}_p) \xrightarrow{\text{Bi}*} K^*(BT; \mathbb{Z}_p)^W$$

is a ring isomorphism.

Theorem 85. [32, 1.2] Let $X$ be a simply-connected $p$-compact group.

1. The $K$-theory of $X$ is an exterior algebra

$$K^*(X; \mathbb{Z}_p) \cong \Lambda_{\mathbb{Z}_p}[\eta_1, \ldots, \eta_r],$$

where the generators $\eta_1, \ldots, \eta_r$ are in $K^1(X; \mathbb{Z}_p)$. 

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2. The $K$-theory of $BX$ is a power series ring

\[ K^*(BX; \mathbb{Z}_p) \cong \mathbb{Z}_p[[\xi_1, \ldots, \xi_r]], \]

where the generators $\xi_1, \ldots, \xi_r$ are in $K^0(BK; \mathbb{Z}_p)$. 
4 Conjectured Weyl Formula

The goal of this chapter is to find a formula for characters of $BX$ that are induced from characters of the maximal torus $BT$. We develop a program that uses the transfer map of Bauer [10] and a conjectured form of localization theory to state a Weyl formula for induced characters of $p$-compact groups.

4.1 Induction in the classical setup

This section follows Chriss and Ginzburg [17], see also [27]. In terms of $K$-theory, induction provides a ring map in the opposite direction to that given by restriction; if $\phi : H \to G$ is an inclusion of subgroups, then we have

$$\text{ind}|_{\phi} : K_H(pt) \to K_G(pt).$$

To understand $\text{ind}|_{\phi}$, first note that if $V$ is an $H$-representation, then $H$ acts diagonally on the space $G$-space $V \times G$, and identifying the $H$-orbits provides a $G$-equivariant vector bundle $V \times_H G$ over $G/H$. In general, there is a change of groups isomorphism,

$$K_H(X) \to K_G(G \times_H X),$$

$$V \mapsto G \times_H V,$$

and $\text{ind}|_H^G$ breaks up as

$$K_H(pt) \xrightarrow{=} K_G(G/H) \xrightarrow{=} K_G(pt),$$

where the first map is the change of groups isomorphism in the special case where $X = pt$, and the second map is the transfer associated to the unique map $\pi : G/H \to pt$ that sends all of $G/H$ to a point. We will define this transfer in the next section. For now, we will cite the following proposition:

**Proposition 86.** [17, 6.1.7] Let $T$ be a maximal torus for connected compact Lie
group $G$. Then the following diagram commutes,

\[
\begin{array}{c}
K_T(pt) \xrightarrow{\cong} K_G(G/T) \xrightarrow{\pi_1} K_G(pt) \\
\downarrow \text{res}_G^T \downarrow \quad \downarrow \text{res}_G^T \\
K_T(G/T) \xrightarrow{\tilde{\pi}_1} K_T(pt)
\end{array}
\]

where $\tilde{\pi}_1$ is the $T$-equivariant transfer associated to $\pi : G/T \to pt$.

It follows that if we are interested in calculating the character of an induced representation, we may first restrict to $T$ and then calculate the effect of the $T$-equivariant transfer.

### 4.2 Transfer maps

Let $T$ be a torus, and let $f : X \to Y$ be an inclusion of $T$-manifolds. Since $K$-theory is a contravariant functor, we get an induced map $K_T^*(Y) \to K_T^*(X)$. We want to construct a map in the other direction, $f_! : K_T^*(X) \to K_T^*(Y)$. We can add a disjoint basepoint and pass to Spanier-Whitehead duals, which gets a map

\[ Df_+ : DY_+ \longrightarrow DX_+, \]

where $DX_+ = map(X_+, S)$ and $DY_+ = map(Y_+, S)$ are the functional duals of maps into the sphere spectrum $S$. Applying this to the collapsing map of the flag variety $\pi : G/T \to pt$, we get a map

\[ D\pi_+ : S^0 \longrightarrow D(G/T)_+, \]

where we have identified $D(\text{point})_+ \simeq S^0$ as the 0-sphere. On $T$-equivariant $K$-theory, $D\pi_+$ has the right target, $K_T(pt)$, but we must precompose it with something to obtain a map out of $K_T(G/T)$. This can be accomplished via Atiyah duality and a Thom isomorphism:

If $\xi$ is a real vector bundle over $X$, then let $X^\xi$ denote the Thom space of $\xi$, which is the quotient

\[ X^\xi := D(\xi)/S(\xi) \]
of the unit disk bundle $D(\xi)$ of $\xi$ by the unit sphere bundle $S(\xi)$ of $\xi$. If we let $\xi + n$ denote the direct sum of $\xi$ and the trivial $n$-bundle over $X$, then $X^{\xi+n}$ is equivalent to the $n$-th suspension of $X^\xi$. In this manner, one extends the definition of the Thom space to all virtual bundles.

**Theorem 87** (Atiyah Duality). 
[6, 3.3] Let $X$ be a compact differentiable manifold with tangent bundle $TX$. Then the Spanier-Whitehead dual of $X_+$ is $X^{-TX}$,

$$DX_+ \simeq X^{-TX}.$$ 

The last ingredient we need is a Thom isomorphism,

$$\Theta : K_T(X) \xrightarrow{\cong} K_T(X^{-TX}),$$

which one can find in most algebraic topology texts, the $K$-theoretic version due to Atiyah as a special case of equivariant Bott periodicity ([9, 4.7]).

Let $\pi : G/T \to pt$ be the collapsing map, and let $\tau$ be the tangent bundle of $G/T$. Then the composition: Thom isomorphism, Atiyah duality, Spanier-Whitehead duality leads to the definition

$$K_T(G/T) \xrightarrow{\theta} \tilde{K}_T(G/T^{-\tau}) \xrightarrow{\cong} \tilde{K}_T(D(G/T_+)) \xrightarrow{(D\pi_+)^*} \tilde{K}_T(S^0) \cong K_T(pt)$$

of the transfer map $\pi_!$. We can calculate the effect of this map via fixed point theory:

**4.3 Fixed point theory**

This section follows Atiyah and Segal [7]. Let $T$ be a connected, topologically cyclic compact Lie group and $X$ a complex $T$-manifold. If $Y$ is a space, then $\pi_Y$ denotes the unique map from $Y$ to the one-point space. Note that the fixed-point set $X^T$ is a submanifold of $X$, and let $\beta$ be an index for the connected components $X^T_\beta$ of $X^T$. Let $N_\beta$ denote the normal bundle of the inclusion $i_\beta : X^T_\beta \to X^T$.

**Proposition 88.** Let $a \in K_T(X)$. Then $\pi_!^X(a)$ is the element in $K_T(pt)$ given by

$$\pi_!^X(a) = \sum_\beta \frac{\pi_!^{X^T_\beta}}{\pi_!^{X^T_\beta}} \left( \frac{a|_{X^T_\beta}}{\pi_K(N_\beta)} \right),$$

of the unit disk bundle $D(\xi)$ of $\xi$ by the unit sphere bundle $S(\xi)$ of $\xi$. If we let $\xi + n$ denote the direct sum of $\xi$ and the trivial $n$-bundle over $X$, then $X^{\xi+n}$ is equivalent to the $n$-th suspension of $X^\xi$. In this manner, one extends the definition of the Thom space to all virtual bundles.

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$$\pi_!^X(a) = \sum_\beta \frac{\pi_!^{X^T_\beta}}{\pi_!^{X^T_\beta}} \left( \frac{a|_{X^T_\beta}}{\pi_K(N_\beta)} \right),$$

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where $e_K$ denotes the $K$-theoretic Euler class.

As in [7], one may set up the commutative diagram

$$
\begin{array}{c}
K_T(X) \xrightarrow{\pi!} K_T(pt) \\
\downarrow i^* \downarrow \leftarrow \downarrow \\
K_T(X^T) \xrightarrow{i^*\downarrow=(\pi\downarrow)^=r_{\pi(N_i)}} K_T(X^T)
\end{array}
$$

and calculate the result of the top arrow by following along the bottom edge. One first localizes at the multiplicative set

$$S = \{e(\mathbb{C}_\lambda)\}_{\lambda \neq 0},$$

as the maps $i^*$, $i^!$, and $i^*i^!$ become isomorphisms after localization. Then one realizes that the localization

$$K_T(pt) \longrightarrow K_T(pt)[S^{-1}]$$

is an injection, and that the formula given by (prop. 88) (which a priori is a formula in $K_T(pt)[S^{-1}]$) holds in $K_T(pt)$.

We now apply this formula to the flag variety $X = G/T$ and the line bundle associated to the irreducible $T$-representation indexed by $\lambda \in Hom(T,U(1))$ (viewed as a $G$-representation)

$$a = G \times_T \mathbb{C}_\lambda.$$

We are left with a formula that calculates the character of the induced representation $ind^G_T\mathbb{C}_\lambda$:

Note that a coset $gT$ is fixed under the $T$-action if and only if $g$ is in the normalizer of $T$, so the fixed-point set $(G/T)^T$ is precisely the (discrete) Weyl group $W$. For $w \in W$, the normal bundle $N_w$ of the inclusion $i : w \rightarrow G/T$ is simply the tangent at the identity of $G/T$, but with the action shifted by $w$,

$$N_w = w_*(\mathfrak{g}/\mathfrak{t}).$$

As a vector space, $N_w$ is isomorphic to the sum

$$N_w \cong \bigoplus_{\alpha \in \Delta_-} \mathbb{C}_{w(\alpha)};$$

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where $\mathcal{R}_-$ is a set of negative roots [15, V, 4.4]. Hence the Euler class $e_K(N_w)$ is equal to the tensor product

$$e_K(N_w) \cong \bigotimes_{\alpha \in \mathcal{R}_-} (C - C_{w(\alpha)}).$$

Similarly, the restriction of $(G \times T \mathcal{C}_\lambda)$ to each $w$-component becomes

$$(G \times T \mathcal{C}_\lambda)|_w \cong \mathcal{C}_{w(\lambda)}.$$

Putting all of these results into the fixed-point formula (prop. 88) leads to the Weyl character formula:

**Theorem 89** (Weyl character formula). [17, 6.6.5] Let $G$ be a connected compact Lie group with maximal torus $T$, and let $\mathcal{C}_\lambda$ be an irreducible $T$-representation indexed by $\lambda \in \text{Hom}(T, U(1))$. Let $\mathcal{R}_+$ be a set of positive roots [15, V, 4.4]. Then the character of the induced representation of $\mathcal{C}_\lambda$ is the element of $K_T(\text{pt})$ given by

$$\text{char}(\text{ind}|_T \mathcal{C}_\lambda) = \sum_{w \in W} \frac{\mathcal{C}_{w(\lambda)}}{\prod_{\alpha \in \mathcal{R}_+} (C - C_{w(\alpha)}^{-1}).}$$

To summarize, the information of this formula can be described by the commutative diagram:

$$\begin{array}{ccc}
K_T(\text{pt}) & \xrightarrow{\cong} & K_G(G/T) \\
\downarrow \text{res}|_T & & \downarrow \text{res}|_T \\
K_T(G/T) & \xrightarrow{\pi_1} & K_T(\text{pt}) \\
\downarrow i^* & & \downarrow i_1 \\
K_T((G/T)^T) & \xrightarrow{i^* n = e_{K(N_i)}} & K_T((G/T)^T) \\
\end{array}$$

4.4 Stably dualizable $H\mathbb{F}_p$-local topological groups

Bauer [10] has constructed a transfer for $p$-compact groups that we would like to make use of. However, until now, we have viewed $p$-compact groups as loop spaces and not topological groups. In order to apply Bauer’s transfer, we must change our
setting slightly to consider $p$-compact group actions on spectra, which is much more easily understood in terms of topological spaces rather than loop spaces. Luckily, the switch is not that hard to make, as the space of based loops in a 1-connected pointed space is homotopy equivalent to a topological group, as is highlighted in the following proposition:

**Proposition 90.** Let $(X, BX, e)$ be a connected $p$-compact group. Then,

1. There is a weak homotopy equivalence between $X$ and a topological group $G$.
2. The classifying space $BG$ of $G$ is homotopy equivalent to $BX$.

**Proof.** This follows from a direct application of [21, 1.2]. One method to construct such a $G$ is to take the geometric realization of Kan’s loop group functor. 

Furthermore, [10, 3.6] these groups can be chosen in such a way that monomorphisms of $p$-compact groups correspond to honest subgroup inclusions of topological groups. In order to obtain a transfer $\pi_1$, we will have to view $p$-compact groups in the setting of stable homotopy theory:

### 4.4.1 $G$-spectra

Let $G$ be a topological group corresponding to a connected $p$-compact group via (prop. 90). Let $\mathcal{M}_S$ be the (bicomplete, bitensored) closed symmetric monoidal category of $S$-modules from [24]. We will write $- \wedge -$ for the symmetric monoidal pairing, and $S$ for its two-sided unit object, the sphere spectrum. We will call objects of $\mathcal{M}_S$ ‘spectra’ and leave the extra structure implicit. The mapping spectrum $F(X, Y)$ serves as the internal function object, i.e., the association

$$ Y \mapsto F(X, Y) $$

is right adjoint to the association

$$ Y \mapsto X \wedge Y \cong Y \wedge X. $$

More precisely, if $X, Y,$ and $Z$ are spectra, then

$$ F(X \wedge Y, Z) \cong F(Y, F(X, Z)). $$
We will write
\[ DX := F(X, S) \]
for the functional dual. For an (unbased) topological space \( Z \), let \( Z_+ \) denote its union with a disjoint basepoint. Then the (reduced) suspension spectrum
\[ S[Z] := \Sigma^\infty Z_+ \]
is a spectrum, and given another spectrum \( X \), we will take \( X \wedge Z_+ \) to mean \( X \wedge S[Z] \) and \( F(Z_+, X) \) to mean \( F(S[Z], X) \).

Recall that if \( E \) is a spectrum and \( n \in \mathbb{Z} \geq 0 \) then (for example, [2]) we define the homotopy groups of \( E \) to be
\[ \pi_n E := [S^n, E], \]
and that \( E \) determines reduced generalized homology and cohomology theories on spectra: if \( X \) is a spectrum then
\[ E_n(X) := \pi_n(E \wedge X) \]
and
\[ E^n(X) := [\Sigma^{-n} X, E]. \]

Let \( H\mathbb{F}_p \) denote the Eilenberg-MacLane spectrum. If \( X \) is a spectrum then the reduced \( H\mathbb{F}_p \)-homology of \( X \) is the graded abelian group
\[ H_*(X; \mathbb{F}_p) := H\mathbb{F}_p_*(X) = \pi_*(H\mathbb{F}_p \wedge X). \]

Similarly to (def. 20), a map \( f : X \to Y \) of spectra is an \( H\mathbb{F}_p \)-equivalence if the induced homomorphism \( f_* : H_*(X; \mathbb{F}_p) \to H_*(Y; \mathbb{F}_p) \) is an isomorphism, and a spectrum \( Z \) is \( H\mathbb{F}_p \)-local if for each \( H\mathbb{F}_p \)-equivalence \( f : X \to Y \), the induced homomorphism \( f^# : [Y, Z]_* \to [X, Z]_* \) is an isomorphism. Let \( \mathcal{M}_{S,p} \) denote the full subcategory of \( \mathcal{M}_S \) of \( H\mathbb{F}_p \)-local spectra. There is a Bousfield localization functor
\[ L_{H\mathbb{F}_p} : \mathcal{M}_S \to \mathcal{M}_{S,p} \]
[13], [24, Ch. VIII], and a natural transformation 1 → \(L_{HF_p}\) such that for each spectrum \(X\), there is an \(HF_p\)-equivalence natural in \(X\),

\[\eta_X : X \to L_{HF_p}X,\]

where \(L_{HF_p}X\) is \(HF_p\)-local. Let \(\mathcal{D}_S = \mathcal{M}_S\) be the homotopy category of \(M_S\), and let \(\mathcal{D}_{S,p} = \mathcal{M}_{S,p}\) denote the homotopy category of \(M_{S,p}\). They are both stable homotopy categories in the sense of [28, 1.2.2]. Let \(L_p\) denote the induced \(HF_p\)-localization functor \(L_p : \mathcal{D}_S \to \mathcal{D}_{S,p}\). It is left adjoint to the forgetful functor \(\mathcal{D}_{S,p} \to \mathcal{D}_S\). The local category \(M_{S,p}\) is again a bicomplete, bitensored closed symmetric monoidal category, where we must apply \(L_{HF_p}\) to each construction formed in \(M_S\). Thus the symmetric monoidal pairing takes \(X\) and \(Y\) to \(L_{HF_p}(X \land Y)\), with unit object the \(p\)-local sphere spectrum \(L_{HF_p}S\). It is worth noting that if \(Y\) is local, then the internal function object \(F(X, Y)\) is already local and is thus equivalent to its localization.

In the absence of a good description of the representation theory of \(G\), we have followed Bauer and Rognes and chosen to work in a trivial \(G\)-universe. Indexing on trivial representations is equivalent to indexing on \(Z\), thus a (left) \(G\)-spectrum is a spectrum \(X\), together with a (left) \(G\)-action on every space \(E_n\), for \(n \in Z\), such that the (adjoint) structure maps \(E_n \to \Omega E_{n+1}\) are \(G\)-equivariant homeomorphisms. Morphisms of \(G\)-spectra consist of \(G\)-equivariant maps, and weak equivalences are morphisms that are weak equivalences of underlying spectra. It is worth saying that the \(G\)-action on a smash product is the diagonal action, whereas the \(G\)-action on a function object is given by conjugation of functions.

### 4.4.2 Dualizable objects

Following [35, III.1], in any closed symmetric monoidal category \(\mathcal{C}\) there are canonical natural maps \(\rho : X \to DX, \nu : F(X, Y) \land Z \to F(X, Y \land Z)\), and \(\land : F(X, Y) \land F(Z, W) \to F(X \land Z, Y \land W)\). For localized spectra, Hovey and Strickland then defined [29, 1.5] a spectrum \(X\) to be \(HF_p\)-locally dualizable or simply
dualizable if the canonical map

\[ \nu : DX \wedge X \to F(X, X) \]

is an equivalence in \( \mathcal{M}_{\mathbb{S}, p} \).

**Remark 91.** Lewis et al. then show [35, III.1.2, 1.3] that \( \rho \) is an equivalence if \( X \) is dualizable, \( \nu \) is an equivalence if \( X \) or \( Z \) is dualizable, and \( \wedge \) is an equivalence if \( X \) and \( Z \) are dualizable, or if \( X \) is dualizable and \( Y = S \). It follows that \( F(X, Y) \) and \( X \wedge Y \) are dualizable when \( X \) and \( Y \) are both dualizable.

**Definition 92.** A topological group \( G \) is \( H\mathbb{F}_p \)-locally stably dualizable if the spectrum \( S[G] \) is locally dualizable in \( \mathcal{M}_{\mathbb{S}, p} \), thus if

\[ \nu : DG_+ \wedge S[G] \to F(S[G], S[G]) \]

is an \( H\mathbb{F}_p \)-equivalence.

**Example 93.** [45, 2.3.4],[29, 8.6] A topological group \( G \) is \( H\mathbb{F}_p \)-locally stably dualizable if and only if \( H_*(G, \mathbb{F}_p) \) is a finitely generated \( H_*(pt, \mathbb{F}_p) \)-module, i.e. if \( G \) is \( \mathbb{F}_p \)-finite. Thus if \( G \) is a \( p \)-compact group, then \( G \) is homotopy equivalent to an \( H\mathbb{F}_p \)-locally stably dualizable topological group.

4.5 Duality and a transfer

Let \( G \) be a topological group. Following Rognes [45, 2.3], we will assume all spectra are localized from now on, and write

\[ S[G] := S \wedge G_+ := L_{H\mathbb{F}_p} \Sigma^\infty G_+ \]

for the \((H\mathbb{F}_p\text{-localization of the})\) unreduced suspension spectrum on \( G \), and

\[ DG_+ := F(S[G], S) := F(G_+, L_{H\mathbb{F}_p} \Sigma^\infty S^0) \]

for its (localized) functional dual. We will suppose that \( G \) is cofibrantly based and has the homotopy type of a based \( CW \)-complex. Let \( EG = B(*, G, G) \) be the usual free, contractible right \( G \)-space obtained from the bar construction. If \( X \) is
a spectrum that $G$ acts on from the right, then one defines the $G$-homotopy fixed points of $X$ to be the mapping space of $G$-equivariant maps from $EG_+$ to $X$,

$$X^{hG} := F(EG_+, X)^G.$$ 

If $Y$ is a spectrum that $G$ acts on from the left, then one defines the $G$-homotopy orbits of $Y$ to be the reduced Borel construction

$$Y_{hG} := EG_+ \wedge_G Y.$$ 

**Definition 94.** [33, 10, 45] Let $G$ be a topological group. The group multiplication provides $S[G]$ with mutually commuting left and right $G$-actions. The dualizing spectrum $S^g$ of $G$ is the $G$-homotopy fixed point spectrum

$$S^g := S[G]^{hG} = F(EG_+, S[G])^G$$

of $S[G]$, formed with respect to the right action of $G$. The left action on $S[G]$ induces a left $G$-action on $S^g$.

**Remark 95.** In the non-localized setting, Klein shows [33, 10.1] that if $G$ is a connected compact Lie group, then the dualizing spectrum $S[G]^{hG}$ of $G$ is indeed equivalent (as a spectrum with left $G$-action) to the suspension spectrum of the one-point compactification $S^g$ of the adjoint representation $g$. This motivates the notation of the previous definition.

**Theorem 96.** [10, 1.3]

1. Let $G$ be a $p$-compact group of $\mathbb{F}_p$-homological dimension $d$. Then the homotopy fixed-point spectrum

$$S^g := S[G]^{hG}$$

is a $H\mathbb{Z}_p$-local $d$-dimensional sphere with a stable $G$-action. Moreover, if $G$ is the localization of a compact Lie group $G$ with Lie algebra $\mathfrak{g}$, then $S^g$ is equivalent to the localization of the one-point compactification of $\mathfrak{g}$ with the adjoint action.
2. For every monomorphism $H < G$ of $p$-compact groups, there is a duality

$$S^g \to G_+ \wedge_H S^h \simeq D(G/H_+) \wedge S^g$$

which is an isomorphism in $H_d(-; \mathbb{F}_p)$.

Bauer went on to define the adjoint Thom spectrum for a $p$-compact group $G$ as the homotopy orbit space

$$BG^g := (S^g)_{hG}$$

and verified that the association $G \mapsto BG^g$ was well-behaved:

**Theorem 97.** [10, 1.4] There is a contravariant functor $t$ from the category of connected $p$-compact groups and monomorphisms to the stable category such that:

1. The spectrum $t(G) := BG^g$ is $\mathbb{F}_p$-local and connective, and $H^*(BG^g; \mathbb{Z}_p)$ is a free module over $H^*(BG; \mathbb{Z}_p)$ on a Thom class in dimension $d = \dim(G)$.

2. If $i : H \to G$ is a monomorphism of connected $p$-compact groups, then the functor $t$ makes the following diagram commutative:

$$
\begin{array}{ccc}
S^g & \to & G_+ \wedge_H S^h \\
\downarrow & & \downarrow \\
BG^g & \to & BH^h
\end{array}
$$

3. The composition $t \circ L_\pi$, defined on the category of compact Lie groups and monomorphisms, is equivalent to the functor $L_\pi \circ (-)_!$.

Furthermore, if $H$ is a torus $T$ of rank $r$, then the commutator map of $S^t$ is homotopic to the identity, and the action of $T$ on $S^t$ is homotopy trivial [45, 2.5.4]. Thus

$$G_+ \wedge_T S^t \simeq S[G/T] \wedge S^t.$$  

Thus $G_+ \wedge_T S^t \simeq S[G/T] \wedge S^t$.

For convenience, let

$$G/T^t := S[G/T] \wedge S^t.$$  

For $T < G$, Bauer’s transfer gives us a map

$$\tilde{i} : S^g \to D[G/T] \wedge S^g \simeq G/T^t,$$
which will be our replacement for $D\pi_+$. One may take $G$-homotopy orbits and obtain a map

$$t = \tilde{t}_{hG} : BG^g \rightarrow BT^4,$$

or one may take $T$-homotopy orbits and obtain a map

$$\tilde{t}_{hT} : EG_+ \wedge_T S^g \rightarrow EG_+ \wedge_T (G/T^4).$$

In terms of $K$-theory, this gives us the commutative diagram

$$
\begin{array}{ccc}
K_p(BT^4) & \xrightarrow{\tilde{t}_{hG}} & K_p(BG^g) \\
| & res & | \\
K_p(EG_+ \wedge_T G/T^4) & \xrightarrow{\tilde{t}_{hT}} & K_p(EG_+ \wedge_T S^g),
\end{array}
$$

where $res$ is the map that restricts the $G$-homotopy orbits to $T$-homotopy orbits.

### 4.6 Conjectural Weyl character formula

We end this chapter by outlining an approach that may possibly be extended to yield a Weyl character formula for $p$-compact groups. The shape of this formula would very strongly resemble the classical Weyl character formula for compact Lie groups. Namely, if $G$ is a connected $p$-compact group with maximal torus $T$ and Weyl group $W$, and if $x \in K_p(BT)$ is a character of $T$, then the induced character of $x$ is an element of $K_p(BG)$, conjecturally given by

$$\sum_{w \in W} x|_w e_{i_w},$$

where $e_{i_w}$ is a suitable Euler class. However, more work must be done to verify that such a formula holds. Of note, one must take care that the Euler classes $e_{i_w}$ are invertible elements in $K_p(BG)$, which would require a currently unformulated localization theory.
4.6.1 The picture thus far

As usual, let $G$ be a connected $p$-compact group with maximal torus $T$ and Weyl group $W$. Let $\mathcal{W} = (G/T)^{hT}$ denote the Weyl space of $G$, and let

$$i : \mathcal{W} \to G/T$$

be the inclusion of the homotopy fixed points. Recall that $\mathcal{W}$ is homotopically equivalent to a discrete space [22, 8.10], so that $i_+ : \mathcal{W}_+ \to G/T_+$ has Spanier-Whitehead dual of the form

$$Di_+ : D(G/T_+) \to \mathcal{W}_+,$$

and that if $X$ is any space, then

$$\mathcal{W}_+ \wedge X \simeq \bigvee_W X.$$

Recall the definitions of the dualizing spectra $S^g$ and $S^t$ from definition 94, and let $i^*$ and $i_!$ be the maps in $K$-theory with $p$-adic coefficients induced by

$$EG_+ \wedge_T (i_+ \wedge S^t) : EG_+ \wedge_T (\mathcal{W}_+ \wedge S^t) \to EG_+ \wedge_T (G/T_+ \wedge S^t)$$

and

$$EG_+ \wedge_T (Di_+ \wedge S^g) : EG_+ \wedge_T (D(G/T_+) \wedge S^g) \to EG_+ \wedge_T (\mathcal{W}_+ \wedge S^g),$$

respectively. Also let

$$BT^g := EG_+ \wedge_T S^g$$

denote the homotopy orbit spectrum of $S^g$ (defined in theorem 97), restricted to $T$. Lastly, note that

$$K_p \left( \bigvee_W BT^t \right) \cong \bigoplus_W K_p(BT^t)$$

and

$$K_p \left( \bigvee_W BT^g \right) \cong \bigoplus_W K_p(BT^g).$$

With these tools we are able to define all of the solid maps in the following commutative diagram. A full understanding of the dotted composite maps and a localization argument would yield a Weyl character formula for $p$-compact groups.
Conjecture 98. 1. The following diagram commutes and is well-defined

\[
\begin{array}{ccc}
K_p(BT) & \overset{\theta}{\cong} & K_p(BT^G) \\
\downarrow_{\text{res}} & & \downarrow_{\text{res}} \\
K_p(EG_+ \wedge_T (G/T^G)) & \overset{i_{BG}}{\longrightarrow} & K_p(BG^T) \\
\downarrow_{\text{left}} & & \downarrow_{\text{left}} \\
\bigoplus_W K_p(BT^w) & & \bigoplus_W K_p(BT^w) \\
\downarrow_{\theta} & & \downarrow_{\theta} \\
\bigoplus_W K_p(BT) & \overset{\theta}{\cong} & \bigoplus_W K_p(BT) \\
\end{array}
\]

Any map labeled by \(\theta\) is a Thom isomorphism, and the ones not labeled with a \(\cong\) symbol depend on the conjecture (con. 100) that the Thom isomorphism holds in the nonabelian case.

2. The left dotted composite map

\[\text{left} : K_p(BT) \longrightarrow \bigoplus_W K_p(BT)\]

is, on the component labeled by \(w\), the map that sends \(x\) to itself, acted on by \(w\).

3. The bottom dotted composite map

\[\text{bottom} : \bigoplus_W K_p(BT) \longrightarrow \bigoplus_W K_p(BT)\]

is, on the component labeled by \(w\), multiplication by a particular element \(e_i\) \(w \in K_p(BT)\) that depends on the inclusion \(i : W \rightarrow G/T\) and deserves the name ‘Euler class’.

4. The right dotted composite map

\[\text{right} : \bigoplus_W K_p(BT) \longrightarrow K_p(BT)\]

is the summation map \(\Sigma_{w \in W}(-)\), and its precomposition with \(\text{left}\) and \(\text{bottom}^{-1}\) has image in the invariants \(K_p(BT)^W\).
5. There is a multiplicative set $S \subseteq K_p(BT)$ such that after localizing at $S$, the map \textit{bottom} becomes an isomorphism and we can traverse it in the wrong direction, dividing by the ‘Euler classes’.

\textbf{Corollary 99.} Let $G, T, W$, and $e_{i_w}$ as above, and let $x \in K_p(BT)$ be a character of $T$. Then the induced character of $x$ is an element of $K_p(BG)$ given by

$$\text{char}(\text{ind}|x) = \sum_{w \in W} \frac{x|_w}{e_{i_w}}.$$ 

We now outline the current state of our research on the relevant Thom isomorphisms and give partial results and motivations for the descriptions of the composite arrows \textit{left} and \textit{bottom}.

\textbf{4.6.2 Thom isomorphism}

Bauer has shown \cite[4.1]{ref10} that if $G$ is a $p$-compact group of $\mathbb{F}_p$-cohomological dimension $d$, that mod-$p$, the cohomology of the adjoint Thom spectrum $BG^a$ is a free module over the cohomology of $BG$ on one generator in dimension $d$, via a spectral sequence argument.

Multiplication by this generator is an analog for the Thom isomorphism

$$H^*(BG; \mathbb{Z}_p) \cong H^*(BG^a; \mathbb{Z}_p).$$

Atiyah extended the Thom isomorphism to equivariant $K$-theory in \cite[4.7]{ref9}. For our $p$-complete setup, the desired result would mirror Bauer’s,

\textbf{Conjecture 100.} If $G$ is a $d$-dimensional connected $p$-compact group, then the $p$-completed $K$-theory $K^*_p(BG^a)$ is a free module over $K^*_p(BG)$ on one generator in degree $d$.

We can quickly prove the special case of an abelian $p$-compact group,

\textbf{Proposition 101.} Let $T$ be a $p$-compact torus or rank $r$. Then the $p$-completed $K$-theory $K^*_p(BT^d)$ is a free module over $K^*_p(BT)$ on one generator in degree $r$. 

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Proof. Since $T$ is homotopy abelian, the action of $T$ on $S[T]$ factors through the contractible space $ET$ and is thus homotopy trivial. [45, 2.5.4]. Thus the dualizing spectrum $S^t$ has a homotopy trivial left action of $T$ and the adjoint Thom spectrum $BT^t := (S^t)_{hT} \cong BT \wedge S^r$ and the result is immediate. 

We would like to extend this result to obtain the Thom isomorphism for general nonabelian $p$-compact groups using the Leray-Serre-Atiyah-Hirzebruch spectral sequence. We would then be able to prove part 1 of conjecture 98.

4.6.3 Identification of left composition

To illuminate the left-hand dotted composite map

$$\text{left} : K_p(BT) \rightarrow \bigoplus_W K_p(BT)$$

of conjecture 98, we provide the following definitions.

Definition 102. If $G$ is a $p$-compact group then a $G$-space $X$ is a fibration over $BG$ with homotopy fiber $X$,

$$X \xrightarrow{i} X_{hG} \xrightarrow{p_X} BG.$$ 

We will denote the total space of this fibration by $X_{hG}$, which is an abuse of notation, as $X_{hG}$ implies that $G$ is already a group. However, $X_{hG}$ is homotopy equivalent to the Borel construction $X_{hG}'$ for the topological group $G'$ obtained from (prop. 90), which motivates the notation.

The fixed point set of the action of $G$ on $X$ is the homotopy fixed point set $X^{hG}$, which is equivalent to the space of sections of $X_{hG} \to BG$, [22, 10.4]

$$X^{hG} = \{BG \xrightarrow{s} X_{hG}| p_X \circ s = id_{BG}\}.$$ 

Two actions are equivalent if their defining fibrations are fiber homotopy equivalent. An equivariant map $X \to Y$ is a map of spaces together with an extension to a map of homotopy orbit spaces $X_{hG} \to Y_{hG}$ [22, 10.8]. A trivial $G$-space $X$ is a trivial fiber bundle over $BG$ with homotopy fiber $X$. 

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Example 103. Recall that if \( f : H \rightarrow G \) is a subgroup, then the homogeneous space \( G/H \) is the homotopy fiber of \( f \) and we have a homotopy equivalence

\[
(G/H)_{hG} \simeq BH.
\]

Example 104. If \( H \) and \( K \) are conjugate subgroups of a \( p \)-compact group \( G \), then there are \( G \)-maps \( f : (G/H)_{hG} \rightarrow (G/K)_{hG} \) and \( g : (G/K)_{hG} \rightarrow (G/H)_{hG} \), such that \( p_K \circ f \simeq p_H \) and \( p_H \circ g \simeq p_K \).

Remark 105. If \( X \) is a \( G \) space and \( f : H \rightarrow G \) is a \( p \)-compact group homomorphism, then the homotopy pullback along \( Bf \) exhibits \( X \) as an \( H \)-space,

\[
X_{hH} = X_{hG} \times_{Bf} BH = \{(x, \sigma, h) \in X_{hG} \times BG^I \times BH | \sigma(0) = x, \sigma(1) = h\},
\]

where \( \sigma \) is a path from \( p_X(x) \) to \( Bf(h) \). This will be our interpretation of restriction.

Example 106. If \( f : T \rightarrow G \) is a maximal torus, then the homogeneous space \( G/T \) is a \( G \)-space. In our current \( p \)-compact setup, we may view \( G/T \) as a \( T \)-space as it is the homotopy pullback of \( Bf \) along itself,

\[
(G/T)_{hT} = Bf^*(BG) =: BT \times_{BG} BT.
\]

We wish to \( p \)-compactly model the inclusion \((G/T)^T \rightarrow G/T\) of the \( T \)-fixed points into the flag manifold. With our current definitions, the inclusion is a map

\[
i_{hT} : ((G/T)^hT)_{hT} \rightarrow (G/T)_{hT}
\]

of fibrations over \( BT \). Recall that we can identify \((G/T)^hT\) as the Weyl space \([22, 10.8]\)

\[
W = \{w \in Aut(BT) | Bi \circ w \simeq Bi\},
\]

and that the total space of the fixed-point set is the trivial bundle

\[
W_{hT} = W \times BT.
\]

We now define the map

\[
eval, pr : W \times BT \rightarrow BT \times_{BG} BT
\]
up to homotopy as the map that sends the pair \((w, t)\) to the element \((w(t), \sigma, t)\), where \(\sigma\) is a path from \(Bi(w(t))\) to \(Bi(t)\) guaranteed by the definition of \(W\). This makes the following diagram homotopy commute:

\[
\begin{array}{ccc}
W & \to & G/T \\
\downarrow & & \downarrow \\
W \times BT & \xrightarrow{ev,pr} & BT \times BG \\
\downarrow & & \downarrow \\
BT & \to & BG
\end{array}
\]

4.6.4 Identification of the bottom composition and beyond

We would like to identify the bottom composite map

\[
\text{bottom} : \bigoplus_{W} K_p(BT) \to \bigoplus_{W} K_p(BT)
\]

of conjecture 98 as, on the component labeled by \(w\), multiplication by an element \(e_{i_w} \in K_p(BT)\). Further, each \(e_{i_w}\) should be a restriction of some Euler class \(e_i\), where \(i\) is the inclusion \(W \to G/T\). We would like to proceed as follows: Let \(i_w\) be the inclusion of the connected component of \(W\) labeled by \(w\) into \(G/T\), and note that the \(i_w\) factor through \(i\). If we take

\[
e_i := i^*i_! \theta(1),
\]

and the Thom isomorphism holds (conj. 100), then \(e_i = i^*i_!(u)\) and \(\theta(x) = u \cdot x\), where \(u\) is the universal element given by the Thom isomorphism. Thus the composition

\[
i^*i_! \theta(x) = i^*i_!(u) \cdot x = e_i \cdot x
\]

would be multiplication by an ‘Euler class’, and the formula would fall mostly in place. The next step would be to assert that the right composite map

\[
\text{right} : \bigoplus_{W} K_p(BT) \to K_p(BT)^W
\]

of conjecture 98 is the summation map

\[
\Sigma_{w \in W} (-).
\]
Then, one would only need to prove the last part of conjecture 98, that we may invert the $e_{iu}$ and write the formula as stated.

Answers to these final questions would give us a better understanding of the enigmatic structures that are $p$-compact groups, and would show that a form of Weyl’s formula holds at the localized level. We hope that the approaches and motivations outlined here might one day be fully realized and extended to continue to develop the representation theory of $p$-compact groups.
5 Appendix A: The Stable Category

Here we will recall Lewis and May’s construction of the stable category, as introduced in [35].

If $V$ is a finite-dimensional vector space, then $S^V$ denotes the one-point compactification of $V$, and if $X$ is a based space, let

$$\Sigma^V X := S^V \wedge X$$

denote the smash product of $S^V$ and $X$, and

$$\Omega^V X := F(S^V, X)$$

the space of based maps from $S^V$ to $X$. If $W \subseteq V$ then let $W - V$ denote the orthogonal complement of $V$ in $W$.

**Remark 107.** If $V = \mathbb{R}^n$, for some $n \in \mathbb{Z}_{\geq 0}$, then $S^V \cong S^n$ and we will denote $\Sigma^V(-)$ and $\Omega^V(-)$ by $\Sigma^n(-)$ and $\Omega^n X$, respectively, and note that these are in fact the standard $n$-th iterated suspension and loop space functors.

A *universe* $U$ is a countably infinite-dimensional real inner product space $U \cong \mathbb{R}^\infty$. An *indexing subspace* $V \subsetneq U$ is a finite dimensional subspace of $U$.

**Definition 108.** Let $U \cong \mathbb{R}^\infty$ be a fixed universe. A *prespectrum* $E$ *indexed on* $U$ is a collection of based spaces $\{EV\}$, one for each indexing subspace $V \subseteq U$, with (adjoint) structure maps

$$\sigma_{V,W} : EV \xrightarrow{\cong} \Omega^{W-V} EW$$

whenever $V \subseteq W$, satisfying the appropriate compatibility relations. If $E$ and $E'$ are prespectra indexed on $U$ then a *map of prespectra* $f : E \to E'$ is a collection of
based maps $f_V : EV \to E'V$ such that the following diagram commutes:

\[
\begin{array}{ccc}
EV & \xrightarrow{f_V} & E'V \\
\sigma_{V,W} \downarrow & & \downarrow \sigma'_{V,W} \\
\Omega^{W-V}EW & \xrightarrow{\Omega^{W-V}f_W} & \Omega^{W-V}E'W.
\end{array}
\]

A prespectrum $E$ indexed on $U$ is a \textit{spectrum} if the structure maps $\{\sigma_{V,W}\}$ are homeomorphisms.

Spectra and prespectra indexed on a universe $U$ define categories $SU$ and $PU$, respectively. The forgetful functor $SU \to PU$ has a left adjoint ‘spectrification’ functor $L$, which is described in detail in [35, App].

Functors on prespectra that do not preserve spectra are extended to spectra by composing with $L$. For instance, for a based space $X$ and a prespectrum $E$, $E \wedge X$ is the prespectrum specified by $(E \wedge X)(V) = EV \wedge X$. If $E$ is a spectrum, however, the structure maps for this prespectrum level smash product are not homeomorphisms, thus we take $E \wedge X$ to mean the spectrum $L(E \wedge X)$.

A morphism $f$ in $SU$ is a \textit{weak equivalence} if $f_V$ is a weak equivalence of spaces for each finite-dimensional $V \subseteq U$. If we let $I_+$ denote the unit interval union a disjoint basepoint $[0,1] \sqcup \{\ast\}$, then a \textit{homotopy} of maps between spectra $E$ and $E'$ in $SU$ is a map

\[E \wedge I_+ \longrightarrow E'.\]

One can then state the relevant homotopy lifting and extension properties and classify maps of spectra as fibrations and cofibrations just as in the category of spaces. It follows that $SU$ has (co)fiber sequences that behave exactly as in the category of spaces.

The homotopy category $hSU$ is obtained in the normal fashion from $SU$ by taking the same objects, but considering homotopy classes of maps of spectra as morphisms. One then creates \textit{the stable homotopy category $\bar{h}SU$} by adjoining formal inverses to the weak equivalences. This process can be made rigorous by CW approximation.
5.1 $S$-modules

This section briefly introduces the closed symmetric monoidal category $\mathcal{M}_S$ of $S$-modules, as developed in [24]. [24, I, section 2] Let $U$ and $U'$ be two universes, and let $\text{map}(U, U')$ denote the set of linear isometries $U$ to $U'$, given the function space topology. Let $A$ be an unbased space with a structure map $\alpha : A \to \text{map}(U, U')$ and let $E$ be a spectrum indexed on $U$. The ‘twisted half-smash product’ (which depends on $\alpha$ and not just $A$), is denoted $A \ltimes E$ is a spectrum indexed on $U'$.

[24, I, 3.1] Fix a universe $U \cong \mathbb{R}^\infty$. Let $U^j$ denote the direct sum of $j$ copies of $U$ and let $\mathcal{L}(j)$ be the space of linear transformations from $U^j$ to $U$. Note that the space $\mathcal{L}(0)$ is the point $i : \{0\} \to U$, and $\mathcal{L}(1)$ contains the identity map $1 = \text{id}_U : U \to U$. There is an obvious left action of the symmetric group $\Sigma_j$ on $U^j$ by permutations, and it induces a free right action of $\Sigma_j$ on the space $\mathcal{L}(j)$, which one notes is contractible. The spaces $\mathcal{L}(j)$ form an operad, called the ‘linear isometries operad,’ with structural maps

$$\gamma : \mathcal{L}(k) \times \mathcal{L}(j_1) \times \ldots \times \mathcal{L}(j_k) \to \mathcal{L}(j_1 + \ldots + j + k)$$

where

$$\gamma(g; f_1, \ldots, f_k) := g \circ (f_1 \oplus \ldots \oplus f_k).$$

Points $f \in \mathcal{L}(j)$ give inclusions $\{f\} \to \mathcal{L}(j)$. There is a corresponding twisted half-smash product $f_*$ that sends spectra indexed on $U^j$ to spectra indexed on $U$. We think of $\mathcal{L}(j) \ltimes (E_1 \wedge \ldots \wedge E_j)$ as a canonical $j$-fold internal smash product.

[24, I, 4.1] Let $\mathbb{L}$ denote the monad in the category of spectra indexed on $U$ specified by $\mathbb{L}E := \mathcal{L}(1) \ltimes E$. The product

$$\gamma : \mathcal{L}(1) \times \mathcal{L}(1) \to \mathcal{L}(1)$$

induces the product

$$\mu : \mathbb{L}\mathbb{L}E \cong (\mathcal{L}(1) \times \mathcal{L}(1)) \ltimes E \to \mathcal{L}(1) \ltimes E = \mathbb{L}E,$$

and the inclusion of the identity $\{1\} \to \mathcal{L}(1)$ induces the unit

$$\eta : E \cong \{1\} \ltimes E \to \mathcal{L}(1) \ltimes E = \mathbb{L}E.$$
**Definition 109.** [24, I,4.2] An \(\mathcal{L}\)-spectrum is an \(\mathcal{L}\)-algebra \(M\), that is, a spectrum \(M\) together with an action \(\xi : \mathcal{L}M \to M\) by the monad \(\mathcal{L}\). Explicitly, the following diagrams are required to commute:

\[
\begin{array}{ccc}
\mathcal{L}\mathcal{L}M & \xrightarrow{\mu} & \mathcal{L}M \\
\downarrow{\mathcal{L}\xi} & & \downarrow{\xi} \\
\mathcal{L}M & \xrightarrow{\xi} & M
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
M & \xrightarrow{\eta} & \mathcal{L}M \\
\downarrow{=} & & \downarrow{\xi} \\
M & & M.
\end{array}
\]

A map \(f : M \to N\) of \(\mathcal{L}\)-spectra is a map of spectra such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{L}M & \xrightarrow{Lf} & \mathcal{L}N \\
\downarrow{\xi_M} & & \downarrow{\xi_N} \\
M & \xrightarrow{f} & N.
\end{array}
\]

**Example 110.** [24, I,4.5] If \(X\) is a based space, then \(X\) may be viewed as a spectrum indexed on \(\{0\}\) and the reduced suspension spectrum \(\Sigma^\infty X \cong S \wedge X\) is equivalent to \(\mathcal{L}(0) \ltimes X\), associated to the inclusion \(\{0\} \to U\). Thus \(\Sigma^\infty X\) is naturally an \(\mathcal{L}\)-spectrum. In particular, the sphere spectrum \(S\) is an \(\mathcal{L}\)-spectrum.

They then go on to define the smash product \(\wedge_\mathcal{L}\) of \(\mathcal{L}\)-spectra, but note that the ‘unit’ map

\[
\lambda : S \wedge_\mathcal{L} M \longrightarrow M
\]

is not always an isomorphism. They thus define an \(S\)-module to be an \(\mathcal{L}\)-spectrum \(M\) such that the unit \(\lambda\) is an isomorphism. We will follow their notation and let \(\mathcal{M}_S\) denote the full subcategory of the category of \(\mathcal{L}\)-spectra whose objects are \(S\)-modules.[24, II, 1.1]

**Example 111.** [24, II,1.2] If \(X\) and \(Y\) are based spaces, then the reduced suspension spectra \(\Sigma^\infty X\), \(\Sigma^\infty Y\), and

\[
\Sigma^\infty X \wedge_\mathcal{L} \Sigma^\infty Y \cong \Sigma^\infty (X \wedge Y)
\]

are \(S\)-modules. Furthermore, if \(M\) is an \(S\)-module and \(N\) is an \(\mathcal{L}\)-spectrum, then \(M \wedge N\) is an \(S\)-module.

**Theorem 112.** The category \(\mathcal{M}_S\) is closed symmetric monoidal under \(\wedge_\mathcal{L}\).
5.2 Equivariance

Let $G$ be a topological group. One can easily extend the category $SU$ of spectra indexed on $U$ to a category $G SU$ of $G$-spectra indexed on $U$ by letting $G$ act on the universe $U$. [35]

**Definition 113.** A $G$-universe $U$ is a countably infinite dimensional real inner product space that $G$ acts on via linear isometries. A $G$-universe is complete if it contains a copy of every irreducible representation of $G$, and trivial if it contains only trivial $G$-representations. The category $G SU$ is called a genuine equivariant category if $U$ is complete, and a naive equivariant category if $U$ is trivial.

Additionally, the constructions in [24] allow one to naturally define a $G$-action on an $S$-module $M$ by defining an action on the underlying spectrum.

As there is yet to be a formulation of a genuine equivariant category for $p$-compact groups, we work in a trivial universe $U$. Thus if $V \subset U$ is an indexing subspace then $V \cong \mathbb{R}^n$ for some $n$, and is a trivial representation. Given the notation from before, it is sufficient to index our spectra on $\mathbb{Z}_{\geq 0}$. 


6 Appendix B: $p$-completed $K$-theory

Recall that for a connected paracompact space $X$, one defines the topological complex $K$-theory to be the Grothendieck group of $\text{Vect}(X)$,

$$K(X) = (\text{Vect}(X))^{gp} \cong [X, \mathbb{Z} \times BU],$$

where $BU$ is the direct limit of $BU(n)$ as $n$ tends to infinity. One extends this definition to arbitrary infinite complexes and spectra by letting $K$ denote the spectrum that has $\mathbb{Z} \times BU$ in every even dimension and $U \simeq \Omega BU$ in every odd dimension, and defining

$$K^n(X) := [\Sigma^{-n}X, K].$$

6.1 Moore spectra and putting coefficients into spectra

Let $G$ be an abelian group. A Moore spectrum of type $G$ is a spectrum $MG$ such that the 0-th ordinary homology $H_0(MG) = G$ and all other ordinary homology groups $H_i(MG) = 0$.

Let $R = \bigoplus_i \mathbb{Z}$ be a free abelian group. We may construct a Moore spectrum of type $R$ via $MR := \bigvee_i S^0$. Surely, the spectrum $MR$ satisfies $H_0(MR) = R$ and $H_i(MR) = 0$ for all $i \neq 0$. Now let $F$ be an arbitrary abelian group. For any homomorphism

$$\alpha : R \longrightarrow F,$$

we may construct the map

$$M\alpha : \bigvee_i S^0 \longrightarrow MF$$

componentwise as the unique map that induces $\alpha$. Given a free resolution

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$$

of $G$, we define the Moore spectrum $MG$ to be the cofiber

$$MR \xrightarrow{M\alpha} MF \longrightarrow MG$$

Moore spectra are the stable homotopy analogues of Moore spaces. In fact,
Example 114. The suspension spectrum of a Moore space of type \((G,n)\) (after desuspending \(n\) times) is a Moore spectrum of type \(G\), i.e., we may take \(MG = \Sigma^{-n}(\Sigma^\infty_+(M(G,n)))\), such that

\[ MG_k = S^k M(G, n) \simeq M(G, k + n) \]

Since Moore spectra (and spaces) are unique up to homotopy, this construction characterizes all Moore spectra.

Let \(MG\) be a Moore spectrum of type \(G\). A spectrum \(E\) with coefficients in \(G\) is a spectrum \(EG := E \wedge MG\).

Example 115. \(HG = H\mathbb{Z} \wedge MG\) is the Eilenberg-MacLane spectrum with coefficients in \(G\). It classifies ordinary cohomology \(H^*(X;G) = HG^*(X) = [\Sigma^{-n}X,HG]\).

Definition 116. Let \(p\) be a fixed prime number, and let \(X\) be a spectrum. The \(n\)-th \(K\)-theory of \(X\) with coefficients in \(\mathbb{Z}_p\) is denoted by \(K\mathbb{Z}_p^n(-)\), and is defined

\[ K\mathbb{Z}_p^n(X) := [\Sigma^{-n}X,K\mathbb{Z}_p] \]

where \(K\mathbb{Z}_p\) is \(K \wedge M\mathbb{Z}_p\).

Unfortunately, this definition is not very illuminating, because \(M\mathbb{Z}_p\) is a very complicated spectrum. However, since \(\mathbb{Z}_p\) is the inverse limit of the rings \(\mathbb{Z}/p^k\) over increasing \(k\), it is possible to construct the more simple spectra \(M\mathbb{Z}/p^k\), and then relate these back to \(M\mathbb{Z}_p\). For these Moore spectra we may take the desuspension of Moore spaces of type \((\mathbb{Z}/p^k,1)\), which are much easier to understand; the 1-sphere \(S^1\) has homology \(H_1(S^1) = \mathbb{Z}\), and it is a basic construction of Algebraic Topology that attaching a 2-cell via a map of degree \(p^k\) gets us the desired homology \(H_1(M(\mathbb{Z}/p^k,1)) = \mathbb{Z}/p^k\). Now we take the suspension spectrum and shift the terms down by 1 to get a Moore spectrum of type \(\mathbb{Z}/p^k\).

Remark 117. The association \(G \mapsto MG\) is not a functor on arbitrary abelian groups. However, we may view \(M(-)\) as a functor from the full subcategory of 2-torsion free abelian groups to the homotopy category of CW spectra. Luckily, once we smash everything with the \(K\) spectrum, we can define a functor \(Ab \to \mathcal{S}\) [30]
6.2 Description as an inverse limit

In the rest of this appendix we will verify that the inverse limit of the $K$-theories $\lim_k K\mathbb{Z}/p^k(\cdot)$ is also a cohomology theory, and that it agrees with $K\mathbb{Z}_p(\cdot)$ on arbitrary spaces and spectra.

**Lemma 118.** $\lim (K\mathbb{Z}/p^k)$ defines a generalized cohomology theory on pairs, i.e., it satisfies the following axioms:

1. **Homotopy:** If $f: (X, A) \to (Y, B)$ is a homotopy equivalence, then $f$ induces an isomorphism
   $$f^* : \lim (K\mathbb{Z}/p^k)^n(Y, B) \cong \lim (K\mathbb{Z}/p^k)^n(X, A)$$

2. **Excision:** For $U, A$ such that the closure of $U$ is contained in interior of $A$, excision of $U$ induces an isomorphism
   $$\lim (K\mathbb{Z}/p^k)^n(X, A) \cong \lim (K\mathbb{Z}/p^k)^n(X - U, A - U).$$

3. **Additivity:** If $X = \bigsqcup \alpha X_\alpha$, then $\lim (K\mathbb{Z}/p^k)^n(X) \cong \bigoplus \alpha \lim (K\mathbb{Z}/p^k)^n(X)$

4. **Exactness:** The maps $i : A \to X$, and $j : X \to (X, A)$ induce a long exact sequence:
   $$\cdots \to \lim (K\mathbb{Z}/p^k)^{n-1}(A) \xrightarrow{\delta} \lim (K\mathbb{Z}/p^k)^n(X, A) \xrightarrow{j^*} \lim (K\mathbb{Z}/p^k)^n(X) \xrightarrow{\iota} \cdots$$

   **Outline of proof:** The first three axioms are straightforward, but it will require some work to show that the inverse limit preserves the long exact sequence of a pair. First we will recall some definitions and facts about towers of abelian groups, then we will show that the system $\{K\mathbb{Z}/p^k(X)\}$ is Mittag-Leffler, and from there we will prove the exactness axiom.

Let $I$ be a partially ordered set. An **inverse system** of Abelian groups is a list $\{G_n\}_{n \in I}$ of abelian groups, with homomorphisms $g^n_m : G_m \to G_n$ whenever $m \geq n$. These maps satisfy $g^a_a = id$ for all $a$ in $I$, and $g^b_c g^a_b = g^a_c$ for all $a \geq b \geq c$ in $I$. We will usually be concerned with systems indexed by $I = \mathbb{Z}_{\geq 0}$, the nonnegative integers, and in this case, it is sufficient to give maps $g_n := g^{n+1}_n$ for all $n$ in $\mathbb{Z}_{\geq 0}$. 
Let \( \{G_n\} \) be an inverse system of Abelian groups. The inverse limit \( \lim_n \{G_n\} \) is defined to be the subgroup of the product \( \prod_n G_n \) consisting of lists \( \{x_n\} \) satisfying \( x_a = g^b_a(x_b) \) whenever \( a \leq b \). The limit is equipped with natural projections onto each factor, and is characterized by a universal property.

A map \( \theta : \{G_n\} \rightarrow \{H_n\} \) of inverse systems over the same indexing set \( I \) is a list \( \{\theta_n\} \) containing homomorphisms \( \theta_n : G_n \rightarrow H_n \) for each \( n \in I \), such that each \( \theta_n \) commutes with the system maps, i.e. \( \theta_n g^m_n = h^m_n \theta_m \) for all \( m \geq n \). A map \( \{\theta_n\} : \{G_n\} \rightarrow \{H_n\} \) between inverse systems is a pro-isomorphism if for every \( n \) in \( I \), there exists \( m \geq n \) and a map \( H_m \rightarrow G_n \) such that the following diagram commutes.

\[
\begin{array}{ccc}
G_m & \xrightarrow{\theta_m} & H_m \\
\downarrow{g^m_m} & & \downarrow{h^m_m} \\
G_n & \xrightarrow{\theta_n} & H_n
\end{array}
\]

Remark 119. A pro-isomorphism \( \{\theta_n\} : \{G_n\} \rightarrow \{H_n\} \) induces isomorphisms [14, III, 2.6]

\[
\lim \{G_n\} \cong \lim \{H_n\} \quad \text{and} \quad \lim^1 \{G_n\} \cong \lim^1 \{H_n\},
\]

and a morphism \( \{\theta_n\} : \{G_n\} \rightarrow \{H_n\} \) is a pro-isomorphism if and only if the tower of pointed sets \( \{\ker \theta_n\} \) and \( \{\coker \theta_n\} \) are pro-isomorphic to the trivial tower [14, III, 2.2].

Let \( \{G_n\} \) be an inverse system of abelian groups indexed by \( \mathbb{Z}_{\geq 0} \). We say that \( \{G_n\} \) is Mittag-Leffler if for each \( n \), there exists \( m \) such that \( \text{Im}(g^p_n) = \text{Im}(g^m_n) \) for all \( p \geq m \). In other words, for fixed \( n \), the image of \( g^p_n \) in \( G_n \) stabilizes as \( p \rightarrow \infty \).

We shall list some consequences of the above definitions for Mittag-Leffler towers in the following lemma.

**Lemma 120 (ML Lemma).** Let \( \{G_n\} \) be a tower of abelian groups.

1. \( \{G_n\} \) is Mittag-Leffler if and only if \( \{G_n\} \) is pro-isomorphic to a tower of surjections. [Shipley, 2.1]
2. If each $G_n$ is finite, then $\{G_n\}$ is Mittag-Leffler. [Shipley, 2.1]

3. If $\{F_a\}$ and $\{H_a\}$ are Mittag-Leffler, and

$$0 \to \{F_a\} \to \{G_a\} \to \{H_a\} \to 0$$

is an exact sequence of towers of abelian groups, then $\{G_a\}$ is Mittag-Leffler.

[Atiyah, 3.7, citing Grothendieck]

4. If $\theta : \{G_a\} \to \{H_a\}$ is a map of systems with $\{G_a\}$ Mittag-Leffler and each $\theta_a$ a surjection, then $\{H_a\}$ is Mittag-Leffler.

5. If $\{F_a\}$ is Mittag-Leffler, then $\lim^{-1}\{F_a\} = 0$, i.e. if

$$0 \to \{F_a\} \to \{G_a\} \to \{H_a\} \to 0$$

is exact and $\{F_a\}$ is Mittag-Leffler, then

$$0 \to \varprojlim F_a \to \varprojlim G_a \to \varprojlim H_a \to 0$$

is an exact sequence of abelian groups. [Atiyah, 3.8, citing Grothendieck]

6. If

$$\ldots \to \{A_1\} \to \{A_2\} \to \{A_3\} \to \ldots$$

is a long exact sequence of inverse systems and each $\{A_i\}$ is Mittag-Leffler, then

$$\ldots \to \varprojlim \{A_1\} \to \varprojlim \{A_2\} \to \varprojlim \{A_3\} \to \ldots$$

is a long exact sequence.

**proof of (6):** We may break the long exact sequence up into overlapping (diagonal) short exact sequences,
If each \( \{ A_i \} \) is Mittag-Leffler, then property (4) from the ML Lemma implies that each \( \{ C_{i+1} \} \) is also Mittag-Leffler. Thus we may apply property (5) of the ML lemma to each diagonal short exact sequence, obtaining overlapping short exact sequences on limits. These limits then fit into the long exact sequence

\[
\ldots \rightarrow \lim \{ A_1 \} \rightarrow \lim \{ A_2 \} \rightarrow \lim \{ A_3 \} \rightarrow \ldots
\]
as required.

**Lemma 121.** For each finite CW-complex \( X \), the system \( \{ K\mathbb{Z}/p^k(X) \} \) is Mittag-Leffler.

**Proof.** First, we must be explicit about the system maps \( f_k : [X, K\mathbb{Z}/p^{k+1}]_n \rightarrow [X, K\mathbb{Z}/p^k]_n \). Let \( r_k : \mathbb{Z}/p^{k+1} \rightarrow \mathbb{Z}/p^k \) denote reduction modulo \( p^k \). Our maps \( f_k \) are the maps induced by \( r_k \) on the functor \( G \mapsto -\rightarrow KG^n(X) \).

For each \( k \), \( r_k \) induces maps of universal coefficient sequences,

\[
\begin{align*}
0 \rightarrow K^n(X) \otimes \mathbb{Z}/p^{k+1} & \rightarrow (K\mathbb{Z}/p^{k+1})^n(X) \rightarrow \text{Tor}(K^{n+1}(X), \mathbb{Z}/p^{k+1}) \rightarrow 0 \\
0 \rightarrow K^n(X) \otimes \mathbb{Z}/p^k & \rightarrow (K\mathbb{Z}/p^k)^n(X) \rightarrow \text{Tor}(K^{n+1}(X), \mathbb{Z}/p^k) \rightarrow 0
\end{align*}
\]
Here, the left vertical map \( g_k \) is the surjection \( \text{id} \otimes r_k \). The right vertical map \( h_k \) takes elements of the \( p^{k+1} \)-torsion of \( K^{n+1}(X) \) to elements of \( p^k \)-torsion, via multiplication by \( p \). To see this, we form resolutions of \( \mathbb{Z}/p^{k+1} \) and \( \mathbb{Z}/p^k \) which fit into the following diagram.

\[
\begin{array}{c}
0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow 0 \\
| & | & |
0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow 0
\end{array}
\]

Tensoring with \( K^{n+1}(X) \) and taking homology yields the stated map

\[ \ker(p^{k+1}) \rightarrow \ker(p^k). \]

The left inverse system is a tower of surjections, and is thus Mittag-Leffler by property (1) from the ML lemma. For finite \( X \), \( K(X) \) is finitely generated, and by the classification of finitely generated abelian groups, \( K(X) \) has finite torsion. Thus the
right system is a tower of finite groups, and is Mittag-Leffler by property (2). Since both the left and right systems are Mittag-Leffler, property (3) of the ML lemma states that the middle system, \{K\mathbb{Z}/p^k(X)\}, is also Mittag-Leffler.

**Corollary 122.** The maps \(i : A \to X\), and \(j : X \to (X, A)\) induce a long exact sequence:

\[
\vdots \longrightarrow \lim_{\to} (K\mathbb{Z}/p^k)^{n-1}(A) \overset{\delta^*}{\longrightarrow} \lim_{\to} (K\mathbb{Z}/p^k)^n(X, A) \overset{j^*}{\longrightarrow} \lim_{\to} (K\mathbb{Z}/p^k)^n(X) \overset{i^*}{\longrightarrow} \vdots
\]

**Proof.** Since each \(K\mathbb{Z}/p^k\) is a generalized cohomology theory, we may (with vertical maps \(f_k\) as above) construct a long exact sequence of inverse systems that contains, in the \(k\)-th row, the long exact sequence for \(K\mathbb{Z}/p^k\),

\[
\vdots \longrightarrow (K\mathbb{Z}/p^k)^{n-1}(A) \overset{\delta^*}{\longrightarrow} (K\mathbb{Z}/p^k)^n(X, A) \overset{j^*}{\longrightarrow} (K\mathbb{Z}/p^k)^n(X) \overset{i^*}{\longrightarrow} \vdots
\]

By the previous lemma, each inverse system is Mittag-Leffler, and by property (6) of the ML lemma, the corresponding sequence of limits

\[
\vdots \longrightarrow \lim_{\to} (K\mathbb{Z}/p^k)^{n-1}(A) \overset{\delta^*}{\longrightarrow} \lim_{\to} (K\mathbb{Z}/p^k)^n(X, A) \overset{j^*}{\longrightarrow} \lim_{\to} (K\mathbb{Z}/p^k)^n(X) \overset{i^*}{\longrightarrow} \vdots
\]

is exact, finishing the proof. \(\square\)

For the sake of completeness, I have included the following lemma that is the key to proving basic results in this section.

**Lemma 123.** If \(f : X \to Y\) induces an isomorphism

\[f^* : (K\mathbb{Z}/p^k)^n(Y) \to (K\mathbb{Z}/p^k)^n(X)\]

for each \(k \geq 1\), then \(f\) induces a pro-isomorphism of towers

\[\{(K\mathbb{Z}/p^k)^n(Y)\} \longrightarrow \{(K\mathbb{Z}/p^k)^n(X)\}\]

**Proof.** The following diagram commutes,

\[
\begin{array}{ccc}
[Y, K\mathbb{Z}/p^{k+1}] & \xrightarrow{f^*} & [X, K\mathbb{Z}/p^{k+1}] \\
\downarrow r_{k^*} & & \downarrow r_{k^*} \\
[Y, K\mathbb{Z}/p^k] & \xrightarrow{f^*} & [X, K\mathbb{Z}/p^k]
\end{array}
\]

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and admits a diagonal map \((f^*)^{-1} \circ r_{k*} \simeq r_{k*} \circ (f^*)^{-1}\), where \(r_k : \mathbb{Z}/p^{k+1} \to \mathbb{Z}/p^k\) is reduction modulo \(p^k\).

\[\text{\Box}\]

\section{Comparing the theories}

\begin{lemma}
\[K\mathbb{Z}_p(S^n) = \varprojlim K\mathbb{Z}/p(S^n)\]
\end{lemma}

\begin{proof}
For an abelian group \(G\), we have
\[KG(S^n) := [S^n, KG] := \pi_n(KG).\]

Thus, in order to calculate \(\varprojlim K\mathbb{Z}/p^k(-)\) on spheres, it is sufficient to calculate the homotopy groups \(\pi_n(K\mathbb{Z}/p^k)\) and take the inverse limit.

Adams proves [1, p. 201] that the homotopy groups of the \(KG\) spectrum are given by the exact sequence
\[0 \to \pi_n(K) \otimes G \to \pi_n(KG) \to Tor(\pi_{n-1}(K), G) \to 0.\]

Here we recall that the homotopy groups of \(K\), the \(BU\)-spectrum, are
\[\pi_n(K) = \begin{cases} 
\mathbb{Z} & (n = \text{even}) \\
0 & (n = \text{odd})
\end{cases}\]

and that \(Tor(\mathbb{Z}, A) = 0\) for any abelian group \(A\), since \(\mathbb{Z}\) has a free resolution length of zero over \(\mathbb{Z}\). Similarly, \(Tor(0, A) = 0\), as we may take the trivial resolution of 0 and its cohomology is trivial.

Plugging in the appropriate groups, we see that \(\pi_n(KG)\) is 2-periodic, with
\[\pi_0(KG) = \mathbb{Z} \otimes G, \quad \text{and} \quad \pi_1(KG) = 0 \otimes G = 0.\]

For \(G = \mathbb{Z}/p^r\), we have
\[\pi_n(K\mathbb{Z}/p^r) = \begin{cases} 
\mathbb{Z}/p^r & (n = \text{even}) \\
0 & (n = \text{odd}),
\end{cases}\]

which implies that \(\varprojlim K\mathbb{Z}/p^k(S^{2n}) = 0\) and \(\varprojlim K\mathbb{Z}/p^k(S^{2n+1}) = \mathbb{Z}_p\), the \(p\)-adic integers.
For $G = \mathbb{Z}_{\hat{p}}$, we have

$$\pi_n(K\mathbb{Z}_{\hat{p}}) = \begin{cases} \mathbb{Z}_{\hat{p}} & (n = \text{even}) \\ 0 & (n = \text{odd}) \end{cases}.$$

Thus $K\mathbb{Z}_{\hat{p}}(S^{2n}) = 0$ and $K\mathbb{Z}_{\hat{p}}(S^{2n+1}) = \mathbb{Z}_{\hat{p}}$. □

Once one has a map of cohomology theories that induces isomorphisms on spheres, one can show that $\lim\limits_{\alpha} K\mathbb{Z}/p(-)$ and $K\mathbb{Z}_p(-)$ agree on arbitrary finite CW-complexes by the standard arguments of cell induction and the 5-lemma. One then finishes the proof that $K\mathbb{Z}_p(-) \cong \lim\limits_{\alpha} K\mathbb{Z}/p^k(-)$ on infinite complexes by a phantom argument: Huber and Meier show [30, p.239] that if $G$ is an abelian group and $Y$ is an infinite CW-complex, then

$$(KG)^n(Y) \cong \lim\limits_{\alpha} (KG)^n(Y_\alpha),$$

Where $\{Y_\alpha\}$ denotes the system of finite subcomplexes of $Y$, directed by inclusion. Thus one can construct maps that yield the following chain of isomorphisms, giving the result.

$$K\mathbb{Z}_p(Y) \cong \lim\limits_{\alpha} K\mathbb{Z}_p(Y_\alpha)$$

$$\cong \lim\limits_{\alpha} (\lim\limits_k K\mathbb{Z}/p^k(Y_\alpha))$$

$$\cong \lim\limits_k (\lim\limits_{\alpha} K\mathbb{Z}/p^k(Y_\alpha))$$

$$\cong \lim\limits_k K\mathbb{Z}/p^k(Y)$$
Many of the structures we consider are modules over the \( p \)-adic integers \( \mathbb{Z}_p \), and in this section, we recall some basic properties of \( \mathbb{Z}_p \) and \( p \)-adic modules.

A \( p \)-adic integer is a formal series

\[
\sum_{i \geq 0} a_i p^i,
\]

where the integer coefficients \( a_i \) satisfy \( 0 \leq a_i \leq p - 1 \). Two \( p \)-adic integers \( a \) and \( b \) are the same if and only if they have the same coefficients. Thus we may view \( \mathbb{Z}_p \) as the inverse limit

\[
\mathbb{Z}_p \cong \lim_{\rightarrow} \mathbb{Z}/p^k
\]

over \( k \in \mathbb{Z}_{>0} \).

There is a natural embedding of the integers \( \tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_p \), given by taking expansion in base \( p \). Note that these integers will end in infinitely many zeros, and that \( \tau \) is not surjective, as the set \( \mathbb{Z}_p \) is uncountable (and non-finitely generated).

For calculations, it makes sense to write the series \( \sum_{i \geq 0} a_i p^i \) as \( ...a_3a_2a_1a_0 \), as if we have written out an infinitely large integer in base \( p \) and are looking at its first digits. From this setup, we may add, subtract, and multiply \( p \)-adic numbers componentwise just as we would with integers base \( p \), using the system of carries. These operations extend the usual ones on base \( p \) expansions of nonnegative integers, and the series

\[
1 = \ldots 001
\]

serves as the multiplicative unit, with

\[
-1 = \ldots (p - 1)(p - 1)(p - 1)
\]

its additive inverse.

Additive inverses for arbitrary elements are defined: if \( a = \sum_{i \geq 0} a_i p^i \), then

\[
-a = \sigma(a) + 1 := \sum_{i \geq 0} (p - 1 - a_i)p^i + 1,
\]
and addition and multiplication distribute in the usual sense, making \( \mathbb{Z}_p \) into a ring. Since there are no zero divisors, \( \mathbb{Z}_p \) is an integral domain, and one can also see that it is a principal ideal domain, thus all ideals are generated by a single element.

Dividing \( p \)-adic integers is not always possible, however, as the element \( p \) has no inverse. More precisely, \( p \) is the unique (up to a unit) irreducible element of \( \mathbb{Z}_p \), which is equivalent to the statement that the ideal \( < p > \) is the unique maximal ideal. Thus \( \mathbb{Z}_p \) is a discrete valuation ring, and the ideal \( < p > \) is the Jacobson radical, the set containing elements that annihilate all simple \( \mathbb{Z}_p \)-modules. Thus if \( M \) is a \( \mathbb{Z}_p \)-module, then all torsion in \( M \) is \( p \)-torsion. It follows that if \( M \) is a finitely generated module over \( \mathbb{Z}_p \), then \( M \) is of the form \( \mathbb{Z}_p^a \oplus C \), where \( C \) is a finite \( p \)-group.

**Lemma 125** (Hensel’s Lemma). Let \( f(x) \) be a polynomial with coefficients in \( \mathbb{Z} \) or \( \mathbb{Z}_p \). Let \( k, m \in \mathbb{Z}_{>0} \) with \( m \leq k \). If \( r \in \mathbb{Z} \) satisfies \( f(r) = 0(\text{mod} p^k) \) and \( f'(r) \neq 0(\text{mod} p) \) then there exists \( s \in \mathbb{Z} \) such that \( f(s) = 0(\text{mod} p^{k+m}) \) and \( r = s(\text{mod} p^k) \).

**Corollary 126.** Let \( p \) be a prime, with \( m \) dividing \( p-1 \). Then the \( p \)-adic integers \( \mathbb{Z}_p \) contain a cyclic group isomorphic to the group of \( m \)-th roots of unity.

**Proof.** The polynomial \( f(x) = x^m - 1 \) satisfies the criteria for Hensel’s lemma for \( m = k = 1 \) (by way of Fermat’s little theorem), and we may lift any solution of \( f(x) \mod p \) to a solution in \( \mathbb{Z}_p \), uniquely determined by its (nonzero) residue mod \( p \). This method gives us at least \( m \) solutions, and there can be no more, since \( \mathbb{Z}_p \) is an integral domain. \( \Box \)

Let \( \mathbb{Z}/p^\infty \) denote the discrete group containing the \( p^k \)-th roots of unity for all \( k \in \mathbb{Z}_{>0} \). It can be viewed as the direct limit of the \( p \)-groups \( \mathbb{Z}/p^k \), or as the quotient \( \mathbb{Q}_p / \mathbb{Z}_p \). The group \( \mathbb{Z}/p^\infty \) is divisible, and thus tensoring with \( \mathbb{Z}/p^\infty \) kills all torsion. Note that \( Aut(\mathbb{Z}/p^\infty) \cong \mathbb{Z}_p \)

**Definition 127.** A \( \mathbb{Z}_p \)-lattice is a \( \mathbb{Z}_p \)-module \( L \) such that \( L \cong \mathbb{Z}_p^r \) for some finite \( r \in \mathbb{Z}_{>0} \). A \( \mathbb{Z}_p \)-torus is a \( \mathbb{Z}_p \)-module \( L \cong (\mathbb{Z}/p^\infty)^r \) for some finite \( r \in \mathbb{Z}_{>0} \).
**Example 128.** Examples of lattices that arise in the objects we study are the modules $H_2(BT; \mathbb{Z}_p)$, $H^2(BT; \mathbb{Z}_p)$ and $\pi_2(BT)$ for $p$-compact tori $T$. Also note that $H^2(BT; \mathbb{Z}_p)$ is the dual over $\mathbb{Z}_p$ of $\pi_2 BT = (\mathbb{Z}_p)^r$.

**Remark 129.** There is a pair of adjoint functors from the category $\textbf{Ab}$ of abelian groups to itself that interchanges $\mathbb{Z}_p$-modules and $\mathbb{Z}_p$-lattices. In particular, let $\tilde{T}$ and $L$ be given by

$$
\tilde{T} = (\mathbb{Z}/p^\infty \otimes -) \quad \text{and} \quad L = \text{Hom}(\mathbb{Z}/p^\infty, -).
$$

Then, the left adjoint functor $\tilde{T}$ is right exact, vanishes on finite $\mathbb{Z}_p$-modules, and turns $\mathbb{Z}_p$-lattices into $\mathbb{Z}_p$-tori. Its left derived functor $\tilde{T}_1 = Tor(\mathbb{Z}/p^\infty, -)$ preserves finite $\mathbb{Z}_p$-modules and vanishes on $\mathbb{Z}_p$-lattices. The right adjoint functor $L$ is left exact, vanishes on finite $\mathbb{Z}_p$-modules, turns $\mathbb{Z}_p$-tori into $\mathbb{Z}_p$-lattices, and has right derived functor $L_1 = \text{Ext}(\mathbb{Z}/p^\infty, -)$, which preserves finite $\mathbb{Z}_p$-modules and vanishes on $\mathbb{Z}_p$-tori.
8 Appendix D: \( \lambda \)-Rings

Example 130. [15, II, §7] Let \( G \) be a connected compact Lie group and let \( V \) and \( W \) be complex \( G \)-modules. Then the direct sum \( V \oplus W \) and the tensor product \( V \otimes W \) are again complex \( G \)-modules, and \( V \) is isomorphic to \( W \) if there exists a \( G \)-equivariant invertible linear map between them. Let \( R^+(G) \) be the set of isomorphism classes of complex \( G \)-modules, with composition laws given by \( \oplus \) and \( \otimes \). Then \( R^+(G) \) is a commutative ring, except that there are no additive inverses. One uses the Grothendieck construction to complete this additive semigroup into a group by introducing formal inverses, and obtains a commutative ring \( R(G) \), the complex representation ring of \( G \). Additionally, \( R(G) \) is a \( \lambda \)-ring:

Let \( V \in R^+(G) \) be a \( G \)-module. Recall that one may form the exterior algebra \( \Lambda(V) \) as the quotient of the tensor algebra by the ideal \( I \) generated by all elements of the form \( v \otimes v \). The \( k \)th exterior power of \( V \), denoted \( \Lambda^k(V) \) is the submodule of \( \Lambda(V) \) spanned by elements of the form \( v_1 \otimes ... \otimes v_k \) (mod \( I \)), with \( \Lambda(V) \) graded as

\[
\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus ...
\]

Note that if \( W \) is another \( G \)-module then

\[
\Lambda^k(V \oplus W) = \Sigma_{i+j=k} \Lambda^i(V) \otimes \Lambda^j(W),
\]

and that \( \Lambda^0(V) = \mathbb{C} \) with the trivial action is the unit of \( R(G) \). We may form the total exterior power as the formal power series with coefficients in \( R(G) \),

\[
\lambda_t(V) := 1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + ...,
\]

and express the previous relations simply as

\[
\lambda_t(V \oplus W) = \lambda_t(V) \cdot \lambda_t(W).
\]

Hence \( V \mapsto \lambda_t(V) \) is a homomorphism from the additive semigroup \( R^+(G) \) into the multiplicative group

\[
1 + tR(G)[[t]]
\]
of formal power series over $R(G)$ with constant term 1. The universal property of
the Grothendieck construction induces a homomorphism from the ring $R(G)$,

$$\lambda_t : R(G) \longrightarrow 1 + tR(G)[[t]].$$

For $x \in R(G)$ we set

$$\lambda_t(x) = \lambda^0(x) + \lambda^1(x)t + \lambda^2(x)t^2 + \ldots,$$

which gives us maps $\lambda^t : R(G) \to R(G)$ such that

$$\lambda^0(x) = 1, \quad \lambda^1(x) = x, \quad \lambda^k(x + y) = \sum \lambda^l(x) \cdot \lambda^{k-l}(y),$$

exhibiting a $\lambda$-ring structure on $R(G)$. 
9 Appendix E: The Dualizing Spectrum of a Topological Group

We have included Klein’s proof of the role of the motivating example of the dualizing spectrum of a compact Lie group:

**Theorem 131.** [33, 10.1] Let $G$ be a compact Lie group. Then there is a $G$-equivariant weak equivalence

$$S[G]^{hG} \simeq_G S^g,$$

where the homotopy fixed points on the left is taking with respect to the standard right action of $G$ on its suspension spectrum $S[G]$, and $S^g$ denotes the suspension spectrum of the one-point compactification of the adjoint representation of $G$.

**Proof.** Let $S^g$ denote the one-point compactification of the lie algebra $g$. We give $S^g$ the $G \times G$ action where $G \times 1$ acts trivially and $1 \times G$ acts via the adjoint representation $Ad : G \to GL(g)$. Let $G \times G$ act on $G_+$ by $(g, h) \ast x = gxh^{-1}$, and let the based function space $map(G_+, S^g)$ have the $G \times G$ action defined by conjugation of functions, i.e., $(g, h) \ast \phi(y) = Ad_G(h)(\phi(g^{-1}yh))$.

Let $\epsilon > 0$ such that $exp : g \to G$ is an embedding on $D(\epsilon)$, the disk of radius $\epsilon$. Identify $S^{Ad_G}$ with $D(\epsilon)/\delta D(\epsilon)$. Define $log : G \to S^{Ad_G}$ such that $log(x) = z$ if $exp(z) = x$ and $z$ has norm $\geq \epsilon$, and $\infty$ otherwise. Then, the map

$$\alpha : G_+ \to map(G_+, S^{Ad_G})$$

that sends $x$ to the map that sends $y$ to $log(x^{-1}y)$ is $(G \times G)$-equivariant, with adjunction map

$$\hat{\alpha} : G_+ \wedge G_+ \to S^{Ad_G}.$$

The map $\hat{\alpha}$ is a Spanier-Whitehead duality: (using the trivialization of the tangent bundle of $G$ given by left translation, identify $map(G_+, S^{Ad_G})$ with the space of sections of the fiberwise one-point compactification of the tangent bundle. Thus $\alpha$ is the tangential version of Atiyah duality). In the stable category, this yields a
$(G \times G)$-equivariant weak equivalence of spectra,

$$\Sigma^{\infty}G_+ \simeq_{G \times G} \text{map}(G_+, S^{Ad_G}).$$

Now if we take homotopy fixed points with respect to the action of $G = G \times 1 \subseteq G \times G$, we see that

$$D_G := (\Sigma^{\infty}G_+)^{h(G \times 1)} \simeq_G \text{map}^{G \times 1}(G_+, S^{Ad_G}) = S^{Ad_G}.$$

\[ \square \]

10 Appendix F: Complex Projective Space

The completion of Complex projective space is the only homogeneous space of a $p$-compact group that has the benefit of being the completion of an actual manifold. In particular, the homogeneous space of the Sullivan sphere $S^{2m-1}_p$ is homotopy equivalent to $\mathbb{C}P^{m-1}$. It would be interesting to apply Bauer’s transfer in this case and reach some results. For now, this section just contains basic information about $\mathbb{C}P^n$ and its Atiyah dual.

**Definition 132.** The complex projective space $\mathbb{C}P^n$ is the space of complex one-dimensional subspaces of $\mathbb{C}^{n+1}$. Equivalently, it is the quotient of $\mathbb{C}^{n+1}\setminus\{0\}$ by the equivalence relation that identifies all complex multiples of each point:

$$\mathbb{C}P^n = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}\setminus\{0\}\}/\{(z_0, z_1, \ldots, z_n) \sim [cz_0, cz_1, \ldots, cz_n], c \in \mathbb{C}^\times \}.$$

**Remark 133.** (specific bundles) The quotient from the definition of $\mathbb{C}P^n$ is the base space of the **tautological line bundle**, $\mathbb{C}^{n+1}\setminus\{0\} \xrightarrow{\pi} \mathbb{C}P^n$.

We denote this bundle by $L$. Specifically, $L$ is the sub-bundle of the product bundle $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ over $\mathbb{C}P^n$ given by pairs of the form $(x, v)$, where $v$ is a point on the complex line $x$. The bundle projection sends $(x, v) \mapsto x$, collapsing each line in $\mathbb{C}^{n+1}$ to a point. Thus each fiber $L_x$ is isomorphic to $\mathbb{C}$, with the multiplicative group $\mathbb{C}^\times$.
acting as the structure group. The associated principal $\mathbb{C}^\times$-bundle will be denoted $M$. Let $L^* = \text{Hom}_\mathbb{C}(L, \mathbb{C})$ denote the dual bundle of $L$. If we forget the complex structure, then we may view $L^*$ as a 2-dimensional real vector bundle, which we will denote by $\xi$. Over each point $x \in \mathbb{C}P^n$, the principal bundle $M$ determines an exact sequence of vector spaces $S_x \to Q_x \to T_x$, where $T_x$ is the tangent space of $\mathbb{C}P^n$ at $x$, $Q_x$ is a field of tangent vectors of $\mathbb{C}^{n+1}\setminus\{0\}$ invariant under $\mathbb{C}^\times$, and $S_x$ is the subspace of fields tangent to the fiber $\pi^{-1}(x)$. For $y$ in $\pi^{-1}(x)$, there is a canonical isomorphism

$$V \otimes_{\mathbb{C}} L^*_x \xrightarrow{\theta} Q_x$$

$$(v \otimes \phi)(y) \mapsto v\phi(y)$$

which induces a canonical isomorphism $L \otimes_{\mathbb{C}} L^* \to S_x$.

**Remark 134.** (relation to Hopf fibration) Since multiplication by a nonzero complex scalar $c = Re^{i\theta}$ can be uniquely decomposed into a dilation by $R$ and a rotation by $\theta$, this quotient may be broken into two pieces,

$$\mathbb{C}^{n+1}\setminus\{0\} \to S^{2n+1} \to \mathbb{C}P^n,$$

where we first project to the unit sphere $S^{2n+1} \subseteq \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$, and then mod out by the natural action of $S^1 \cong U(1)$, via the Hopf fibration

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n.$$

**Remark 135.** (recursive definition, inclusion maps) Let $H$ be a fixed hyperplane through the origin in $\mathbb{C}^{n+1}$. Under the above projection to $\mathbb{C}P^n$, $H$ is taken to a subspace homeomorphic to $\mathbb{C}P^{n-1}$. The complement of $H$ in $\mathbb{C}P^n$ is homeomorphic to $\mathbb{C}^n$. Thus if $m \leq n$, there is a natural inclusion of projective spaces $\mathbb{C}P^m \to \mathbb{C}P^n$. Thus we may build $\mathbb{C}P^n$ inductively by adding $2n$-cells:

$$\mathbb{C}P^n = \mathbb{C}P^{n-1} \cup e^{2n}$$

If we view the $2n$-cell as the open unit ball in $\mathbb{C}^n$, then the attaching map is the Hopf fibration in the boundary.
Example 136.  

- $n = 0$: We consider the unit circle $U(1) \subseteq \mathbb{C}$, and mod out by the action of $U(1)$ on itself by multiplication. Thus $\mathbb{C}P^0 \cong *$ is the one-point space.

- $n = 1$: We consider the 3-sphere $S^3 \subseteq \mathbb{C}^2$. The Hopf fibration $U(1) \cong S^1 \rightarrow S^3 \rightarrow \mathbb{C}P^1 \cong S^2$ gives an action of $U(1)$

10.1 Duality

The goal of this section is to identify the Spanier-Whitehead dual $D(\mathbb{C}P^n_+)$ of $\mathbb{C}P^n$ with a disjoint basepoint.

Theorem 137. [6, 3.3] Let $\alpha$ be a real vector bundle over paracompact base $X$. Then the Spanier-Whitehead dual of the Thom complex $X^\alpha$ is $X^{-\alpha-\tau}$, where $\tau$ is the (real) tangent bundle of $X$.

Recall that the Thom spectrum $X^0 \cong X_+$, so

$$D(\mathbb{C}P^n_+) \cong (\mathbb{C}P^n)^{0-\tau} = (\mathbb{C}P^n)^{-\tau}.$$  

By [6, 4.5], or [36, §14], if $\tau_n$ is the tangent bundle of $\mathbb{C}P^n$, then

$$\tau_n \oplus 1 \cong (n + 1)L^*$$

as complex vector bundles. Thus

$$D(\mathbb{C}P^n_+) \cong (\mathbb{C}P^n)^{-(n+1)L^*\oplus 1} \cong S^2 \wedge (\mathbb{C}P^n)^{-(n+1)L},$$

where we have used that if $\alpha$ is a complex vector bundle over $X$, then the Thom spectrum $X^\alpha$ is to be interpreted as the Thom spectrum of the underlying real vector bundle, $\alpha_\mathbb{R}$, and the identification $L^*_\mathbb{R} \cong L_\mathbb{R}$. Furthermore, if $\alpha$ is a real vector bundle over $X$, with $\alpha \oplus \beta = n$, then we shall interpret $X^{-\alpha}$ as the desuspension $S^{-n} \wedge X^\beta$. 

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11 Appendix G: A Note on Root Data and the Prime $p = 2$

Just as the classification of simple compact Lie groups via $\mathbb{R}$-reflection groups carries the ambiguity between groups of type $B$ and $C$, for the prime $p = 2$, the classification of 2-compact groups carries an ambiguity between the completions of $Sp(n)$ and $SO(2n + 1)$. In both of these settings, the classification can be made more complete by considering root data in addition to the underlying reflection groups, and we are able to obtain a complete classification. This thesis, however, is mainly focused on the simple case of $p$-compact groups for $p$-odd, where there is a one-to-one correspondence between $\mathbb{Z}_p$-reflection groups and $\mathbb{Z}_p$-root data. Additionally, many constructions for spaces simplify; the torsion elements in $\mathbb{Z}_2$ often require different proof strategies.

For completeness, the following definition of a root datum is from [3].

A $\mathbb{Z}_p$-root datum is a triple $(W, L, \{\mathbb{Z}_p b_\sigma\})$, where $(W, L)$ is a finite $\mathbb{Z}_p$-reflection group, and $\{\mathbb{Z}_p b_\sigma\}$ is a collection of rank one submodules of $L$, indexed by the set of reflections $\sigma \in W$, such that

- $\text{im}(1 - \sigma) \subseteq \mathbb{Z}_p b_\sigma$
- if $w \in W$ then $w(\mathbb{Z}_p b_\sigma) = \mathbb{Z}_p b_{\sigma w}^{-1}$

The set $\{\mathbb{Z}_p b_\sigma\}$ amounts to choosing a cyclic $\mathbb{Z}_p$-submodule of $H^1(<\sigma>; L)$ for each conjugacy class of reflections $\sigma$. For $p$ odd, if $\sigma$ is a reflection in $(W, L)$, then the order of $\sigma$ divides $p - 1$, and is hence prime to $p$. And there is a unique cyclic submodule, giving the stated correspondence.

An isomorphism of $\mathbb{Z}_p$-root data $\varphi : D \rightarrow D'$ is a $\mathbb{Z}_p$-module isomorphism $\varphi : L \rightarrow L'$ such that

- $\varphi W \varphi^{-1} = W'$ as subgroups of $\text{Aut}(L')$
- $\varphi(\mathbb{Z}_p b_\sigma) = \mathbb{Z}_p b'_{\varphi \sigma \varphi^{-1}}$ for every reflection $\sigma \in W$

The statement of the classification then is that there is a one-to-one correspondence between the sets of $p$-compact groups up to isomorphism and $\mathbb{Z}_p$ root data up
to isomorphism. [3]

References


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