Convective methods of pumping and drag reduction

James D. Woodcock
BE (Hons.) BSc (Hons.)

Submitted in total fulfillment of the requirements of the degree of Doctor of Philosophy

September 2013

Department of Mechanical Engineering
&
Department of Mathematics and Statistics

The University of Melbourne
Abstract

It is the convection of the velocity field by itself that renders many fluid mechanics problems mathematically challenging, and produces complicated, and often non-intuitive flow phenomena. Pumping and drag reduction are effectively related concepts, in that they both involve increasing the volume flux of the fluid. In this work, we consider three different methods of pumping and drag reduction, all of which result, partially or entirely, from the effect of convection.

The first of these methods is the drag reduction obtained by the addition of elastic polymers to a turbulently flowing liquid. This effect is not well understood, and a complete physical explanation of the phenomenon remains to be made. However, it is clear that the polymer has the capacity to transport energy and momentum within the fluid, and energy may also dissipate within the polymer itself. In this work, it is proved that the addition of elastic polymers to a turbulent flow cannot reduce the drag to a level below that of the equivalent laminar flow. This proof can also be applied to similar methods of drag reduction, such as the presence of surfactant micelles within a turbulently flowing liquid and the presence of sand particles within high winds and water droplets within cyclones.

The second method is known as “transpiration”, and consists of a dynamic regime of blowing and suction at the wall of the pipe or channel which imparts no net volume flux upon the flow. Using a perturbation analysis, the pumping effect of transpiration has been quantified in this work. It is shown that this pumping results from convection, and relies on the presence of large velocity gradients within the flow.

The third method consists of oscillating waves in the wall of the pipe or chan-
nel. This has particular relevance to the valveless impedance pump, which consists of a thin tube, one section of which is elastic and is subjected to rhythmic pinching at some point offset from its centre. This pinching induces oscillating waves within the wall of the tube, which in turn induce a flow. The flow induced by small-amplitude oscillations, in the wall, has been derived through a perturbation analysis. In this way, we are able to separate the effect of convection from the more readily intuitive dragging effect that the wave has upon the fluid, and thereby quantify the importance of convection in such systems. It is found that even within small tubes, the effect of convection remains generally of the same order of magnitude as the dragging effect, and that no effective model of the valveless impedance pump could safely neglect the effect of convection.
Declaration

This is to certify that

1. The thesis comprises only my original work towards the PhD except where indicated in the Preface.

2. Due acknowledgment has been made in the text to all other material used.

3. The thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

James Woodcock
Preface

The work presented in this thesis is my own. The only exceptions are the figures within the Introduction, which have been reproduced from previous works. The authors of those works have been appropriately cited.
Acknowledgments

I would like to thank my two supervisors, Professors Ivan Marusic and John Sader, for their guidance during my PhD candidature. It is generally agreed that the most important part of a PhD is choosing the right supervisors, and without their insight and direction, the process of completing this degree would have been considerably more difficult.

In addition, thanks must go to the university and its departments of Mechanical Engineering and Mathematics & Statistics, for finding the necessary funding to send me to the American Physical Society Division of Fluid Dynamics meeting in San Diego 2012. This was an invaluable experience which not only provided significant insight into the most important current research in fluid dynamics, but also allowed this thesis to be as up to date as possible.

I am also grateful to the university for providing me with a Melbourne Research Scholarship during my time as a PhD student.
Publications


## Contents

Abstract .......................................................... i  
Declaration ......................................................... iii  
Preface ............................................................... iv  
Acknowledgments .................................................... v  
Publications ........................................................ vi  

### List of Figures

#### 1 Introduction to drag reduction
1.1 Turbulent flows and their topology .......................... 2  
1.1.1 Isotropic turbulence ...................................... 3  
1.1.2 Wall-flows .................................................. 5  
1.2 Passive methods of drag reduction .......................... 8  
1.2.1 Riblets ..................................................... 8  
1.2.2 Polymer drag reduction and related methods .......... 11  
1.3 Active methods of drag reduction and pumping .......... 20  
1.3.1 Spanwise oscillations of the wall ....................... 20  
1.3.2 Transpiration ............................................... 21  
1.3.3 Small-amplitude peristalsis ............................. 27  
1.3.4 The valveless impedance pump ......................... 29  

#### 2 On the maximum drag reduction due to added polymers in Poiseuille flow
2.1 Equations of channel flow .................................... 34
### Contents

2.1.1 Energy equations ........................................... 37
2.2 Work done by the polymer upon the flow .................. 38
2.3 Volume-flux comparison between laminar and turbulent flows .......... 39
2.4 Minimising drag ............................................ 42
2.5 Pipe flow .................................................. 42
2.6 Conclusions ............................................... 45

3 Induced flow due to blowing and suction flow control: An analysis of transpiration 48
  3.1 Equations of channel flow .................................. 49
     3.1.1 Boundary conditions .................................. 50
     3.1.2 The scaled equations .................................. 52
     3.1.3 Averaging ............................................. 54
     3.1.4 Streamfunction ....................................... 55
  3.2 Perturbation analysis ...................................... 55
     3.2.1 Leading order flow .................................... 56
     3.2.2 First order correction to the flow ....................... 59
     3.2.3 First order velocity field ............................... 66
     3.2.4 Boundary conditions of long wavelength ............... 67
     3.2.5 Boundary conditions of short wavelength ............. 70
     3.2.6 Dependence on the frequency of oscillation .......... 71
     3.2.7 Generalised boundary conditions ......................... 73
  3.3 Energy considerations ..................................... 75
     3.3.1 Energy input ......................................... 76
  3.4 Boundary conditions of large amplitude .................... 79
  3.5 Discussion ............................................... 85
     3.5.1 Scales of motion ..................................... 85
     3.5.2 The induced bulk flow ................................ 87
     3.5.3 Generalising the boundary conditions ................. 89
  3.6 Conclusions ............................................... 89

4 Flows driven by oscillating walls: On the physics behind valveless pumping 92
  4.1 Equations of pipe flow .................................... 93
     4.1.1 Boundary conditions .................................. 94
List of Figures

1.1 Results of drag reduction experiments conducted by Virk (1975). ........................................ 12
1.2 More results of drag reduction experiments conducted by Virk (1975). ........................................ 13
1.3 Drag reduction produced by the presence of rod-shaped surfactant micelles within the flow, reproduced from Warholic et al. (1999b). ........................................ 18
1.4 Experimental valveless impedance pump, reproduced from Hickerson (2005). ................................. 30

2.1 Diagram of the channel domain. ........................................ 35
2.2 Diagram of the pipe domain. ........................................ 43

3.1 Instantaneous realisations of various cases of the family of boundary conditions defined by equation (3.1.12). ........................................ 51
3.2 Diagram of the channel domain, in scaled coordinates. ........................................ 53
3.3 Plot of instantaneous streamlines of $u_0(x,t)$, the leading order approximation of the flow. ........................................ 59
3.4 Log-log plots of $\langle u_1 \rangle/\beta$ against the Stokes number, $\beta = \rho h^2 \omega/\mu$, for several values of the ratio of the channel height to the boundary condition wavelength, $\eta = h/\lambda$. ........................................ 63
3.5 Log-log plots of $\langle u_1 \rangle/\beta$ against $\eta (= h/\lambda)$ for several values of $\beta (= \rho h^2 \omega/\mu)$. ........................................ 64
3.6 Contour plots of $\langle u_1 \rangle/\beta$ against $\beta (= \rho h^2 \omega/\mu)$ and $\eta (= h/\lambda)$. ........................................ 65
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.7</td>
<td>Plots of instantaneous streamlines of $u_1(x,t)/\beta$, the first order correction of the flow for $\eta = 2$ and several values of $\beta$.</td>
<td>68</td>
</tr>
<tr>
<td>3.8</td>
<td>Log-log plots of $\mathcal{W}_0$, the leading order component of the rate at which energy is being supplied to the flow, against $\beta$ and $\eta$.</td>
<td>77</td>
</tr>
<tr>
<td>3.9</td>
<td>Log-log plots of the leading order component of the efficiency of the pumping, against $\beta$ and $\eta$.</td>
<td>78</td>
</tr>
<tr>
<td>4.1</td>
<td>Diagram of the pipe domain in the absence of oscillations, in scaled coordinates.</td>
<td>94</td>
</tr>
<tr>
<td>4.2</td>
<td>Plots of $\langle u_1 \rangle_{BC}$, the first order bulk flow due to wall-entrainment, against $\beta (= \rho h^2 \omega / \mu)$ and $\eta (= h / \lambda)$.</td>
<td>107</td>
</tr>
<tr>
<td>4.3</td>
<td>Plots of $\langle u_1 \rangle_{conv}$, the first order bulk flow due to convection, against $\beta (= \rho h^2 \omega / \mu)$ and $\eta (= h / \lambda)$.</td>
<td>108</td>
</tr>
<tr>
<td>4.4</td>
<td>Plots of $\langle u_1 \rangle$, the total first order bulk flow, against $\beta$ and $\eta$.</td>
<td>109</td>
</tr>
<tr>
<td>4.5</td>
<td>Plots of the asymptotic behaviour of the total bulk flow, and its convective and wall-entrainment components, in the limit as $\eta \rightarrow 0$.</td>
<td>112</td>
</tr>
<tr>
<td>4.6</td>
<td>Plots of $\langle \epsilon_0 \rangle$, the leading order of the average rate of dissipation within the flow, against $\beta$ and $\eta$.</td>
<td>118</td>
</tr>
<tr>
<td>4.7</td>
<td>Diagram of the channel domain, in scaled coordinates.</td>
<td>120</td>
</tr>
</tbody>
</table>
Introduction to drag reduction

The behaviour of flowing fluids is often counterintuitive, and even unpredictable. There are several potential causes of such behaviour, some of which result from the nature of the fluid itself, such as its compressibility or non-Newtonian nature. Others result from the nature of the system, such as any temperature gradients which may be imposed upon the fluid. In this work, we focus on effects which are inherent to the flow field itself. Specifically, we focus on the effects of convection within fluids that are both incompressible and Newtonian.

Convection is the transport of a substance by a flowing fluid. Where the word is used in this work, it refers specifically to the transport of the velocity field by itself\footnote{It is common to refer to the convective transfer of heat or particles within a flow, or to mixing due to the convective transfer of a scalar. None of these phenomena will be discussed further within this thesis.}, a process made physically and mathematically complicated by the fact that the velocity field is perpetually altering itself. This non-linear behaviour renders it impossible to exactly derive the behaviour of most convective systems. This includes systems subjected to any of the three varieties of drag reduction and pumping that are the main focus of this study.

Drag is the resistance experienced by a flowing fluid. All fluids that flow around surfaces or through pipes, tubes, or channels will experience some amount of drag. This is because the dissipation of the fluid’s kinetic energy into heat will occur wherever there are velocity gradients within the fluid. Since the velocity of the fluid directly adjacent to the surface will conform to the velocity of the surface...
itself, avoiding velocity gradients in such near-wall regions is not possible, and hence some amount of drag must exist.

Numerous methods exist for reducing the drag experienced by the fluid, and no doubt many more will be developed in the future. They can be broadly classified into two groups: active methods of drag reduction, which provide their own energy to the flow, and passive methods, which can only function by altering the nature of the flow itself.

### 1.1 Turbulent flows and their topology

Many active methods of drag reduction, and all of the passive methods that are considered here, function by altering the nature of turbulent flows. These methods therefore cannot be explained without some understanding of the nature of turbulent wall-flows, and the coherent structures that form within them. This is the subject to this section. This section is not intended to be a complete discussion of the known behaviours and characteristics of turbulent flows, as such a discussion would likely be longer than this thesis. For more general information about turbulent flows, see Davidson (2004).

As early as the 16th century, Leonardo da Vinci observed that there existed two distinct varieties of flows. Slower flows would exhibit simple behaviours, forming reproducible flow fields. However, as the rate of the flow was increased, so did its complexity. This culminated in a flow phenomenon he named “la turbolenza”, in which the flow field consists of a sum of numerous eddying motions, and behaves in a seemingly random manner.

It would be a further three centuries before turbulence would again be subjected to rigorous scientific analysis. In 1883, Osborne Reynolds conducted a series of experiments on flows, in which he verified that there existed two distinct varieties of flow, which he labelled “direct” and “sinuous”. His “sinuous” flow was what we now call turbulence, and its counterpart, “direct” flow, is now known as laminar flow.

The motion of slow-moving fluids are dominated by their viscosity. In other words, the effect of convection within such flows is dominated by that of diffusion. Diffusion is a mathematically linear phenomenon, which causes slow-moving flows to be simple, predictable and mathematically tractable.
As the momentum of the flow is increased, so too is the significance of convection. Convection is a mathematically non-linear phenomenon, which has complicated, and often unpredictable and counter-intuitive effects. With every further increase in momentum comes a further increase in complexity and unpredictability. Once the momentum is sufficiently great, the formerly laminar flow becomes unstable, and may be tripped into turbulence by a small disturbance of the flow field.

The now turbulent flow is no longer, simple, predictable or mathematically tractable. It has become chaotic\(^2\), and it therefore impossible to produce any accurate simulation of a specific flow field, since any minute deviation in the specified initial conditions from the physical system being simulated will produce markedly different results. (This is leaving aside the great computational expense involved in most simulations of turbulent flows.) It is nonetheless possible to produce simulations of individual realisations of turbulent flows, which are likely to have similar properties to that physical flow. However, the computational cost of the simulations increases with the Reynolds number\(^3\), and as a result it becomes impractical to simulate turbulent flows once the relative importance of momentum becomes too great (Pope, 2000).

Even if two turbulently flowing fluids are identical, possessing the same properties, such as temperature, density and viscosity, their natures will depend heavily upon the geometry through and around which the fluid is flowing.

### 1.1.1 Isotropic turbulence

Isotropy denotes the independence of a quantity from changes in orientation. Isotropic turbulent flows are those whose statistically-averaged properties remain constant regardless of the orientation from which the flow is viewed.

In reality, perfectly isotropic turbulence is a physical impossibility, as the presence of any solid surfaces will impose an orientation-dependence upon the flow.

\(^2\) A chaotic system is one in which small changes to the initial conditions will result in large and often unpredictable changes in output. The behaviours of such systems are generally impossible to anticipate, and may appear to be random.

\(^3\) The Reynolds number is the ratio of the flow’s inertial forces to its viscous forces. It is effectively a measure of the relative importance of convection within the flow. Its precise mathematical definition will depend upon the type of flow in question.
Moreover, the perpetuation of turbulence requires that the flow should have non-zero vorticity. And therefore, since vorticity is generated at the wall and transported from there to the rest of the flow, isotropic turbulence can only be an abstract ideal. Turbulence will however tend to approach isotropy within wakes and away from walls (Davidson, 2004).

Richardson (1922) argued that momentum (and hence kinetic energy) is input to a turbulent flow at the same length scale\(^4\) of whatever surface is providing the vorticity the sustains the turbulence\(^5\). At this scale, diffusion (and thus dissipation) is negligible, and energy and momentum are generally transferred to smaller scales by the effect of convection. From these smaller scales, convection transfers energy and momentum to yet smaller scales, until it reaches scales sufficiently small that viscosity takes effect, and hence the energy is dissipated as heat. This process has been named the Richardson cascade.

Kolmogorov (1941\(a,b,c\)) refined the Richardson cascade for isotropic turbulence. He argued that the statistical properties of the smaller scales depend only upon the viscosity and the rate at which energy is passed down the cascade from the larger scales. As a result, the behaviour of the small scales becomes self-similar. That is, the flow appears the same regardless of which scale it is viewed from, so long as that scale is sufficiently small\(^6\).

The mathematical representation of Kolmogorov’s result is instructive. If we

---

\(^4\)It is not particularly important how the term “length scale” is defined here. One possible definition is that if the turbulent flow field were divided into a series of eddies, the average radius of each eddy is its length scale. For isotropic turbulence, another possible length scale presents itself if the velocity is divided up into Fourier modes. In which case, the length of each Fourier mode is its length scale. See Davidson (2004).

\(^5\)For example, if the turbulence is produced by flowing the fluid through a wire mesh, the length scale of the mesh would be the spacing between wires.

\(^6\)This very important result was made more remarkable by the fact that it was published in, and disseminated from, the Soviet Union during the Second World War. During the war, the USSR’s Academy of Sciences had been evacuated from Moscow to the city of Kazan at the foothills of the Ural mountains. The academy nonetheless produced English language translations of Kolmogorov’s now seminal papers. These translations were then used as ballast by Allied supply ships, during their return journeys, after resuppling the Soviet Union via an Arctic route. In this way, Kolmogorov’s ideas on the scale-similarity we disseminated to the west (Barenblatt, 2001). Many of Kolmogorov’s other contributions were not to be widely known in the West until after the break up of the Soviet Union (Davidson \textit{et al.}, 2011)
1.1. Turbulent flows and their topology

define a velocity difference function, \( \Delta u(r) \):

\[
\Delta u(r, t) \overset{\text{def}}{=} u(x + r, t) + u(x, t),
\]

where \( u(x, t) \) is the velocity at the position \( x \) and time \( t \), and \( r \) is some spacial displacement from \( x \), Kolmogorov found the counterintuitive result that (for \( r \) sufficiently small),

\[
\Delta u(\lambda r, t) = \lambda^{1/3} \Delta u(r, t).
\]

However, Batchelor & Townsend (1949) found that at the smallest scales, the flow field is not self-similar, but instead exhibits spatially dependent and intermittent features. Nonetheless, at scales small enough to be independent of the method by which the turbulence has been produced, yet large enough to experience negligible dissipation, the flow is in fact scale-similar, as Kolmogorov predicted. For a thorough discussion of scale-similarity, intermittency, and methods of accounting for it, see Frisch (1996).

1.1.2 Wall-flows

Turbulent flows behave markedly differently in the region adjacent to a solid surface. In this near-wall region, the no-slip boundary condition\(^7\) necessitates that the flow will be slow, or at least slow enough that diffusive effects remain relevant at higher scales. Moreover, the flow could never be isotropic in the near-wall region, because of the presence of a wall. Accordingly, the scale-similarity that applies to isotropic flows does not apply to wall-bounded turbulent flows.

The greater complexity of wall-bounded turbulent flows has made progress toward a theoretical understanding of such flows significantly more complicated than it has been for isotropic turbulence.

Wall-bounded turbulence consists of a near-wall “inner region”, in which viscosity is important, and an outer region, in which it is not. Again, dimensional analysis has proven useful in discerning the behaviour of turbulent flows near to walls. By assuming that there exists a region sufficiently distant from the wall that viscosity has negligible direct upon the mean flow, while also being not close

\(^7\)The no-slip boundary condition simply states that the velocity of fluid at a solid surface will equal the velocity of that surface.
1.1. Turbulent flows and their topology

enough to the edge of the boundary layer to be affected by the outer flow, it is possible to derive a theoretical relation between the scaled mean flow, \( U^+ \), and the scaled distance from the wall, \( z^+ \), in this region:

\[
U^+ = \frac{1}{\kappa} \log e z^+ + C. \tag{1.1.3}
\]

Here, \( \kappa \) is known as Von Kármán’s constant\(^9\) and \( C \) is a constant of integration. We will not recount the derivation of the above equation here. However a simple derivation has been provided by Nickels et al. (2007), and a more complete description of turbulent boundary layers can be found in Rotta (1962). This result is known as the log law, or variously the law of the wall, and has been found to be in good agreement with experimental data (Nickels et al., 2007). Of course, given the assumptions that have been made in deriving the log law, it cannot be expected to apply either too near to the wall, or too far from it.\(^10\)

Observing that the average rate of dissipation at a point within the log region is proportional to its distance from the wall, Townsend (1976) noted that this implied that if the flow were divided into a series of eddies\(^11\), the diameters of these eddies should be proportional to the distance of their centre from the wall. This implies that the dominant discernible persistent flow patterns (or “coherent structures”) that exist within the near-wall flow will extend to the wall itself. This reasoning was further extended into what is known as the attached eddy hypothesis, which states that the majority of the flow’s kinetic energy may be described by a random superposition of such wall-bounded eddies.

\(^8\)The superscript “+” denotes scaling with respect to the shear stress at the wall, \( \tau_w \). The non-dimensionalised mean velocity, \( U^+ \), differs from its dimensional counterpart, \( U \), via \( U^+ = U/\sqrt{\tau_w \rho} \) (where \( \rho \) is again the fluid’s density). The non-dimensionalised distance from the wall, \( z^+ \), differs from its dimensional counterpart, \( z \), via \( z^+ = z \sqrt{\tau_w \rho/\nu} \) (where \( \nu \) is the fluid’s kinematic viscosity). While this is the conventional scaling employed within the turbulence literature, alternative scalings will be used in the bulk of this thesis. The use of a capitalised \( U \) to denote the mean velocity is also conventional, but will also not be used within the bulk of this thesis.

\(^9\)Despite its name, it has been argued that Von Kármán’s constant is in fact a variable. Von Kármán’s constant remains an empirical observation and has not, to the author’s knowledge, been derived from first principles. It is generally found to have a value of \( \kappa \approx 0.41 \) (Marusic et al., 2010).

\(^10\)And not least because \( U^+ \) diverges as \( z^+ \) approaches 0 or \( \infty \), according to (1.1.3).

\(^11\)We leave aside here the thorny issue of what exactly constitutes an eddy.
Townsend, however, did not specify the nature of the wall-bounded eddies. Subsequently, Head & Bandyopadhyay (1981) found that the near-wall region was populated by persistent hairpin-shaped vortices. These “hairpin vortices” are attached to the wall and protrude into the main flow, where their protruding portion is subjected to a faster mean flow than the rest of the vortex. The hairpin therefore aligns itself with the direction of the mean flow, at an average angle of $45^\circ$ to the wall (Jiménez, 2012). The nature, and physical importance, of these hairpin vortices is an area of continuing study. There is also no formal consensus that complete hairpin vortices even exist within turbulent wall flows (Schoppa & Hussain, 2002).

Perry & Chong (1982) argued that these hairpin vortices are a candidate for Townsend’s attached eddies, and that a turbulent wall-flow may be modelled by a superposition of such eddies. This model was later refined by Perry & Marusic (1995) and Marusic (2001).

Another prominent type of coherent structure that has been found to be present within wall-flows are long streaks of either high or low velocity. These streaks take the form of vortices they are far longer than than they are wide. These vortices generally, but imperfectly, align with the streamwise direction, and are accordingly known as quasi-streamwise vortices (Adrian & Marusic, 2012). The low-speed streaks emanate from closer to the wall, where the mean velocity is lower, while the high-speed streaks conversely emanate from further from the wall (Panton, 2001).

The interaction of these quasi-streamwise vortices with the near-wall eddies is an intrinsic part of the turbulence regeneration cycle, by which vorticity is transferred from the wall to the outer flow (Roa et al., 1971; Wark & Nagib, 1991). (The presence of a non-negligible vorticity, throughout the flow, is necessary for the flow to be turbulent. Vorticity cannot be generated within the flow, and must therefore be transferred to the outer flow from the wall. Accordingly, anything which disrupts this transfer of vorticity from the wall to the outer flow will have the effect of reducing the turbulence intensity, and thereby reducing the drag.) For a thorough review of quasi-streamwise vortices, and their importance to the turbulence regeneration cycle see Marusic et al. (2010).

Hairpin vortices are also variously known as horseshoe vortices and Λ vortices.
1.2 Passive methods of drag reduction

The volume flux of a laminar Poiseuille flow, in a channel or pipe, will always be greater than the volume flux of the equivalent turbulent flow\footnote{Within this work, two flows are considered ‘equivalent’ if they are driven by the same average pressure gradient.} (Thomas, 1942). However, various passive methods have been developed by which the volume flux of a turbulent flow can be increased, without the addition of extra energy or momentum to the flow.

These methods are often very beneficial, when employed deliberately, as they can often produce quite significant drag reduction at very little cost. Conversely, they can often increase the destructive power of high winds, such as within cyclones and sandstorms, whose drag is counter intuitively reduced by the addition of water droplets or sand particles to the turbulently flowing air. Such methods of drag reduction will be summarised shortly.

Passive methods of drag reduction function by altering the very nature of the flow, changing its topology, and its appearance at every scale. Laminar flows are generally energy efficient. Their flow fields orient themselves such that they do not experience high degrees of drag, at least when compared to the equivalent turbulent flows. It it therefore unlikely that any passive method of drag reduction should be capable of significantly reducing the drag experienced by a laminar flow.\footnote{It should be noted here that for many methods of drag reduction, sublaminar drag is not only unlikely, but impossible. In §2 this is proven to be the case for methods such as polymer drag reduction, which do not alter the flow domain. However, methods such as adding riblets to the surface do alter the flow domain, and so are not covered by this proof. Moreover, the proof assumes a pipe or channel flow, in which any laminar flow will obey Stokes’ equation, and hence cannot be assumed to apply to flows in all possible domains.} For that reason, passive drag reduction is generally confined to turbulent flows, whose complicated, energy inefficient natures render them a candidate for such drag reduction methods.

1.2.1 Riblets

One very practical method of drag reduction involves the addition of riblets to the wall around which a fluid flows. This has been shown to reduce the drag
1.2. Passive methods of drag reduction

experienced by a turbulent fluid by up to 10%. For a recent review riblet drag reduction, see García-Mayoral & Jiménez (2011).

Riblets are small protrusions along a wall into the flow. Walsh (1980) studied the effects of such riblets on turbulent flows and found that V-shaped riblets could, if aligned with the direction of the flow and correctly spaced, produce significant drag reduction. Like many methods of reducing the drag of turbulent flows, the effect of riblets is counter intuitive. Any roughening of the surface would naively be expected to increase the friction, and thereby to increase the drag. However, because of the complicated nature and behaviours of turbulent flows, its effect is the opposite.

It has been shown that practical use of riblets may be made in the design of boats (Alving & Freeberg, 1995), airfoils (Lee & Jang, 2005) and airplanes (Viswanath, 2002), and even briefly in low-drag swimming suits, before their use in competition was officially banned in 2009. However, the earliest known use of riblets is in the skin of sharks. Some fast-swimming species of sharks have developed a rough skin, the texture of which acts as drag-reducing riblets (Dean & Bhushan, 2010).

Mechanisms of drag reduction caused by riblets

The effect that riblets have upon a turbulent flow, and the mechanisms by which they induce drag reduction, are an open field of study. It has been postulated that riblets induce drag reduction by impeding the migration of long vortices that form along the walls (Choi, 1989; Dean & Bhushan, 2010). In so doing, they reduce the occurrence of collisions between the vorticies. These collisions then cause vortices to be forced from the near-wall region to the outer flow, thereby transferring their vorticity to the outer flow. It is therefore argued that reducing the incidence of these collisions will reduce the rate at which vorticity is transferred from the near-wall region to the outer boundary layer. This, in turn, reduces the turbulence intensity of the flow. However, the existence of these postulated streamwise vortices in turbulent flows over smooth walls has, to the author’s knowledge, never been definitively demonstrated.

15These should not be confused with the quasi-streamwise vortices that were described in §1.1.2. The vortices described here are attached to the wall and do not constitute regions of high and low streamwise velocity.
The spacing of the riblets is vitally important. If they are too close together, they would fail to impede the spanwise migration of the long vortices, as the vortices will instead inhabit the space above the riblets’ tips. The riblets’ sole contribution will then be to increase the surface area of the wall, and thereby increase the drag. Conversely, if the riblets are too widely spaced, they would allow the vortices too much room to migrate, and would therefore not prevent the bursting events. Their only effect would therefore be to increase the surface area, and thereby the drag. For a thorough experimental study of riblet spacing and its effects, see Bechert et al. (1997a).

Conversely however, Frohnapfel et al. (2007) studied the effects of smaller than conventional riblets on turbulent flows. Their experiments showed that such riblets could significantly reduce the drag experienced by such flows.

On the other hand, the effect of riblets will be reasonably tolerant of small changes in orientation. In order to completely negate the effect of riblets, they must be at an angle of 30° to the mean flow (Gaudet, 1987). With any further increase in this angle, the effect of the riblets will be to increase the drag (Choi & Hamid, 1991). This is because the presence of the riblets then causes the boundary layer to become locally separated.

Existing studies of flows over ribbed surfaces

Experimental studies of flows over rough surfaces date back to the 1850s, and the work of Hagen (1854) and Darcy (1857), who found that a rough surface would noticeably alter the nature of the flow. They reported that the pressure loss within turbulent pipe flows will become independent of the viscosity once the pipe’s surface becomes sufficiently rough. For a review of turbulent flows over rough surfaces, see Jiménez (2004).

Riblets are a special category of surface roughness, being coherent shapes, rather than the typical random diffusive topology. Walsh & Lindemann (1984) studied the effect of riblets of varying shapes and spacings. Using oil channels, Bechert et al. (1997a) conducted extensive studies on riblets, using semi-circular, blade-shaped and trapezoidal-shaped riblets.

The riblet-like effects of the skins of various marine animals have also been tested. Brusse et al. (1993) studied the drag reduction produced by replicas of shark skin, as well as hairy surfaces and riblets with adjustable geometries. Bechert
et al. (1997b) and (2000) studied the effects of various biological surfaces, including replica sharkskin and hairy surfaces. Itoh et al. (2006) studied flows over seal fur, finding that it is capable of producing a drag reduction of up to 12%.

1.2.2 Polymer drag reduction and related methods

It has been well demonstrated that adding elastic polymers to a turbulent liquid will dramatically reduce the drag on the fluid, and significantly increase the flowrate (Toms, 1948). See White & Mungal (2008) for a recent review. It has been argued that the polymers reduce drag by transporting momentum within the fluid (Min et al., 2003), as well as by countering vorticity and eddying motions and thereby reducing fluctuations (Luchik & Tiederman, 1988; Kim et al., 2007).

Experimental results show that at critical values of polymer concentration and polymer relaxation time, the drag experienced by the fluid will be reduced, and that this effect can require only very low concentrations of the polymer (Virk et al., 1967; Sreenivasan & White, 2000). In one particular example, a concentration in the order of 10ppm of a certain long-chain polymer has been found to be capable of reducing drag by up to 80% (Virk, 1975). A graph demonstrating this effect is reproduced in figure 1.1.

However, a limit appears to exist to the drag reduction that an added polymer may produce. This limit is known as Virk’s asymptote (Virk, 1971). It, however, remains a purely empirical result, and there is no theoretical proof that it constitutes the maximum allowable drag reduction. The universality of this maximum drag reduction can be seen in figure 1.2.

Recently, it has been found that the presence of such long-chain elastic polymers within a turbulent flow so fundamentally alters the nature of the flow that it must be considered to constitute a new state of turbulence (Hof et al., 2012; Dubief et al., 2012). This new state, named “elasto-inertial turbulence”, is a chaotic flow, whose dynamics are the combined result of both inertial and elastic effects. The relative contributions of which depend upon the Reynolds number. It is found also that flows containing elastic polymers will transition to elasto-inertial turbu-

---

16This is an example of an important scientific discovery being made entirely by accident. The experiments conducted by B. A. Toms were originally intended to study the degradation of polymers within a turbulently flowing liquid (Toms, 1977).
1.2. Passive methods of drag reduction

Figure 1.1: Results of drag reduction experiments conducted by Virk (1975). Here $Q$ refers to the volume flux of fluid through the pipe, while $T_w$ refers to the corresponding shear stress. The solvent used was distilled water, while the polymer solution contained 300 parts per million by weight of polyethylene-oxide, molecular weight $5.7 \times 10^5$. Here it can clearly be seen that while the addition of polymers will not appreciably reduce the drag of slower flows, once the volume flux has become sufficiently large, the presence of polymers will significantly reduce the drag experienced by the fluid.
1.2. Passive methods of drag reduction

Figure 1.2: More results of drag reduction experiments conducted by Virk (1975). Here, $Re$ denotes the Reynolds number, while $f$ denotes the Fanning friction factor. Two different polymers were used at different concentrations. These are polyethylene-oxide (PEO) and polyacrylamide (PAM). The solvent was again distilled water. It can clearly be seen that polymer solutions behave like their pure solvents at low Reynolds number, and that while they may obey the Poiseuille and Prandtl-Karman Laws at low Reynolds numbers, once the Reynolds number is sufficiently high, their behaviour becomes markedly different. The maximum drag drag reduction produced can be seen to be independent of the polymer concentration, and even the species of elastic polymer used.
1.2. Passive methods of drag reduction

Turbulent flows are computationally expensive to simulate. This is because turbulent flows consist of multiple scales of motion, the smallest of which are just larger than the random fluctuations we call heat (Davidson, 2004), and each of these scales must be accounted for within an accurate simulation. For a discussion of techniques for simulating turbulent flows, see Pope (2000).

When a turbulent flow is also subject to a body force due to the presence of an elastic polymer, this computational cost is increased significantly. It is for this reason that computational simulations of turbulent flow containing polymers are a relatively recent field of study (Sureshkumar, Beris & Handler, 1997).

Polymer molecules have the ability to function like microscopic springs, storing energy as they are forcibly elongated by the flow field, and then releasing it again as the local velocity gradients decrease. The methods generally used to model the effect of the polymer treat it as a simple spring, as if the polymer consisted of two beads connected by an elastic bond. For a thorough review of constitutive equations for modelling polymer flows, see Bird & Wiest (1995).

The simplest commonly used such model is known as Oldroyd-B. It assumes
that the polymer behaves like a simple linear spring, obeying Hooke’s law regardless of how far it has been stretched. The implication being that the polymer may be stretched to an infinite length. As a result of being a linear model, Oldroyd-B fails to effectively capture the true behaviour of polymer solutions, which is often non-linear (Quinzani et al., 1990).

A more complicated non-linear spring model has been developed, known as the finitely extensible nonlinear elastic (FENE) model. This model only allows the polymer molecules stretch to a finite length, with the force required to further elongate the polymer increasing as the molecule is stretched, becoming infinite in the limit as the polymer’s length approaches its maximum. The great downside of the FENE model is that it is particularly expensive to implement due to fact that the average elongation of polymers must be determined at every point within the flow (Jin & Collins, 2007).

Because of the computational expense involved in using the FENE model, a simplified alternative has been produced. This has been named the FENE-P model (where P stands for Peterlin). This is a simplification of the FENE model in which the extension of each individual polymer molecule is replaced by its average (Zhou & Akhavan, 2003). While this approximation significantly simplifies the equation, it does fail to capture the correct physics in certain circumstances. Specifically, it has been shown that the predictions of the FENE-P model differ from those of FENE in flows in which the average polymer elongation is time-dependent, and in shear flows (van den Brule, 1993; Herrchen & Öttinger, 1997). More importantly, the results of the FENE-P model have also been shown to differ from experimental results (Tirtaatmadja & Sridhar, 1995; Verhoef et al., 1999).

However, all of these models are decidedly imperfect. They treat a carbon chain polymer of the order of $10^5$ monomers as if it contained a single elastic bond. Furthermore, they do not account for interactions between polymer molecules, which have been shown to play an important part in certain polymeric flows (Kalashnikov, 1994). Due to the complicated nature of polymer molecules and their interactions, it is likely that any attempt to model such flows will always face a difficult trade off between physical realism and computational cost.
1.2. Passive methods of drag reduction

The current work

In §2, we prove that the force caused by the presence of the polymer will be incapable of raising the bulk flowrate of a turbulent Poiseuille flow to beyond that of the equivalent unforced\textsuperscript{17} laminar flow. We prove this without reference to any constitutive equation for the polymer force.

We also prove that the polymer force, when acting upon a laminar Poiseuille flow, will decrease the fluid’s bulk flowrate.

A caveat must be made about this proof: The presence of elastic polymers within a Newtonian liquid generally causes the solution to become significantly shear-thinning. However, if the polymer is sufficiently dilute, then the fluid’s viscosity will be only negligibly affected by the polymer’s presence, and the fluid will remain effectively Newtonian. Such fluids are also known as Boger fluids, and their advantage over other polymer-containing fluids is that they allow the effects of the polymer’s elasticity to be considered separately from their shear-thinning effects (James, 2009).

Within the proof in §2, it has been assumed that the fluid’s density and viscosity are not significantly affected by the presence of the polymer. However, it must be noted that the presence of the polymer increases the viscosity of the solution beyond that of the pure solvent, and that a stress applied to the solution will merely act to reduce this increase. Thus, regardless of the applied stress, the solution’s viscosity should remain greater than that of the pure Newtonian solvent.

It should therefore be reasonable to assume that if the addition of the polymer cannot produce sublaminar drag when the increase in viscosity is neglected, then neither can it when the increase in viscosity is not neglected. This follows from the fact that a higher viscosity will impede flow.

As has been discussed previously within this section, the polymer may also affect the fluid in a more fundamental way, by either delaying or facilitating the onset of turbulence.

\textsuperscript{17}Throughout this work, the term ‘unforced’ is used to describe a flow which is not subject to any body forces.
1.2. Passive methods of drag reduction

Other forms of passive drag reduction also exist, to which the proof presented herein (and its caveat) will apply. The presence of aggregates of surfactant molecules, known as micelles, within a turbulently flowing liquid has been known to be capable of producing a similar degree of drag reduction to the presence of elastic polymers (Warholic et al., 1999b). For a complete description and review of surfactant drag reduction, see Li et al. (2012). A similar phenomenon is also believed to occur within sandstorms (Gore & Crowe, 1989) and cyclones (Barenblatt et al., 2005). Under certain circumstances, strong turbulent winds suspending sand particles or water droplets can encounter significantly less drag than the equivalent flows of pure turbulent air.

The example of surfactant drag reduction is instructive, particularly because it is capable of producing roughly the same degree of drag reduction as the addition of elastic polymers, as can be seen in figure 1.3. This is despite the significant physical differences between rod-shaped surfactant micelles and elastic polymers.

Surfactants are molecules which possess both a polar end (usually an ion) and a non-polar end (such as a hydrocarbon chain). When placed within a polar liquid, such as water, the polar end will orient itself in order to maximise contact with the liquid, while the non-polar end will orient itself in order to avoid contact with the liquid. (Conversely, when placed in a non-polar liquid, such as oil, the non-polar end will be drawn into contact with liquid, while the polar end will be repulsed.)

For this reason, surfactants tend to migrate to the surface of the liquid, where one end can point downward into the liquid, while the other points upward into the air. However, the number of molecules that can be accommodated at this interface will be limited. Therefore, once the surfactant concentration exceeds a certain point, any additional molecules must remain entirely within the solution.

Micelles are structures which surfactants form within the liquid. In a polar liquid, they consist of generally spherical arrangements in which the surfactant’s non-polar ends face inwards, attracted to each other by weak Van der Waals forces, while their polar ends face outwards (with the directions reversed in the case of a non-polar liquid). At higher surfactant concentrations, rod-shaped micelles will form, in addition to the more common spheres (Zana, 2005). It is the presence of the rod-shaped structures that has been shown to cause drag reduction.

Elastic polymers are stretchable molecules, capable of storing significant quan-
Figure 1.3: Drag reduction produced via the presence of rod-shaped surfactant micelles within the flow, reproduced from Warholic et al. (1999b). These experiments were performed using aqueous solutions of Ethoquad T/13-50. Here $y^+$ denotes the distance from the wall, while $U^+$ denotes the average streamwise velocity at that point. It can be seen that surfactant drag reduction is capable of producing a similar degree of drag reduction to the addition of elastic polymers, whose maximum drag reduction (Virk’s asymptote) is denoted by a straight line.
1.2. Passive methods of drag reduction

tities of energy and transporting it within the flow. Micelles, on the other hand, are temporary structures, spontaneously forming and spontaneously disbanding. Furthermore, if they are significantly stretched, the modest Van der Waals forces that bind the micelles to each other could easily be overcome, causing the micelle to immediately disband.

That similar degrees of drag reduction can be produced by two such different additives suggests that the physical explanation for this maximum achievable drag reduction may not be due to the unique nature, and exact effects, of polymer additives (or those of surfactants), but may instead be a more general property of flows subjected to body forces.

The work of Warholic et al. (1999b) offers some illumination on the nature of such flows at maximum drag reduction. Once maximum drag reduction has been reached in their surfactant-containing flows, the Reynolds stress has been found to be everywhere zero. In other words, if we take an average of the flow (for example, averaged over time), and then were to define any deviations from that average as “fluctuations”, then they have found that at maximum drag reduction, the fluctuations draw no energy directly from the mean flow. Instead, the fluctuations must draw energy from the micelles, which in turn draw energy from the mean flow.

Warholic, Massah & Hanratty (1999a) found a similar result in their experiments on liquids containing elastic polymers at maximum drag reduction. They find that although the Reynolds stress does not quite become universally zero in such flows, it does become very small at all points within the pipe. This again suggests an inherent similarity between polymer drag reduction and its surfactant counterpart.

A number of authors have performed computational simulations on polymer-containing flows at maximum drag reduction. Min, Choi & Yoo (2003a) performed simulations using the Oldroyd-B model. Their results show that the Reynolds stress decreases as the drag is reduced.

More recently, simulations performed by Xi & Graham (2012), using the non-linear FENE-P model for the polymer-liquid interactions. They find that at maximum drag reduction, the Reynolds stress is reduced to zero, except in a small region adjacent to the wall. This is in general agreement with the experiments of Warholic et al. (1999a), and the slight discrepancy between the experimental and
computational results could alternatively be explained by the imperfect nature of the FENE-P model or the inevitable experimental uncertainty.

1.3 Active methods of drag reduction and pumping

As with passive methods of drag reduction, active methods of drag reduction may function by altering the nature of a turbulent flow. However, because active methods are capable of providing their own impetus to the flow, they may also induce drag reduction by adding their own impetus to the flow.

Active drag reduction methods may be further divided into two categories: open-loop and closed-loop. Closed-loop methods involve observing the flow and adjusting the imposed conditions accordingly. Open-loop methods, on the other hand, impose pre-specified conditions upon the flow.

Moreover, because active methods do not necessarily rely on altering the nature of turbulent flows, they may also be capable of reducing the drag of laminar fluids. Some are even able to induce a bulk flow in an otherwise stagnant fluid, and can therefore equivalently be considered methods of pumping.

Conversely, however, the only methods of pumping we shall consider here are those which could equivalently be considered methods of drag reduction. In other words, we consider only pumping methods that may act upon already flowing fluids. We therefore ignore numerous conventional methods of pumping, such as centrifugal pumps, gear pumps, piston pumps and conventional peristaltic pumps. For a complete discussion of pumping mechanisms, see Karassik et al. (2007).

1.3.1 Spanwise oscillations of the wall

In 1992, Jung, Mangiavacchi & Akhavan demonstrated that the skin-friction drag of a turbulent flow can be significantly reduced by oscillating the wall in a spanwise direction. In their direct numerical simulations, they found that the drag experienced by a turbulent flow can be reduced by up to 40%. This effect was later confirmed experimentally by Laadhari, Skandaji & Morel (1994). For a recent review of the effect of spanwise oscillations of the wall, see Quadrio (2011).

The early studies by Jung et al. (1992) and Akhavan et al. (1993) of the effect of spanwise oscillations on turbulent flows were motivated by a similar effect
1.3. Active methods of drag reduction and pumping

seen in analogous systems subjected to a transverse pressure gradient or cross flow (Bradshaw & Pontikos, 1985; Driver & Hebbar, 1987; Moin et al., 1990).

This phenomenon was extended to longitudinal oscillations within pipe flows by Choi & Graham (1998), whose experiments demonstrated that such oscillations were capable of reducing the friction factor of a turbulent fluid by 25%.

Mechanisms of drag reduction caused by spanwise oscillations

No complete explanation yet exists for the mechanisms that result in drag reduction in turbulent flows over spanwise oscillating walls. It has been shown, however, to be connected to the evolution of coherent structures (these are described in §1.1.2).

Dhanak & Si (1999) found that spanwise wall-oscillations disrupt the quasi-streamwise vortices that form within turbulent wall-flows, which have an important role in sustaining turbulence. In this way, it is proposed that they are able to induce drag reduction.

Ricco et al. (2012) studied the effect of spanwise oscillations on a turbulent flow via direct numerical simulations. From this, they were able to also determine energy and enstropy balances.

Touber & Leschziner (2012) also studied spanwise oscillations via direct numerical simulations. They were able not only to produce statistical data from the simulated flows, but also to investigate the coherent structures that form along the walls, and to visualise the effect of spanwise oscillations on such structures. They find that the unsteady cross flow generated by the oscillating wall, causes significant spanwise distortions of the vorticies that form near to the walls. This disrupts the cycle of sweeping and ejection events that sustain the turbulence by transferring vorticity from the near-wall region to the main body of the flow.

This concurs with the earlier experimental results of Choi & Clayton (2001), who found that the spanwise oscillation of the wall reduces the incidence of bursting events within turbulent flows.

1.3.2 Transpiration

A novel method of active drag reduction has previously been discovered to affect both laminar and turbulent flows (Choi, Moin & Kim, 1994). This method,
1.3. Active methods of drag reduction and pumping

now known as ‘transpiration’, involves subjecting the flow to a non-zero and non-constant wall-normal velocity at the flow’s surface. This boundary condition consists of both blowing and suction at the surface, and imparts no net volume flux upon the flow. This effect has so far been investigated via numerical and analytical methods, and to the author’s knowledge its existence has thus far not been verified experimentally.

Transpiration can be enacted in either a closed-loop configuration or an open-loop configuration. Closed-loop transpiration\(^{18}\) involves observing the flow, in order to subject it to targeted bursts from the wall, whose purpose is to disrupt the coherent structures that form along the wall within turbulent flows.

**Studies of open-loop transpiration**

It had previously been suspected that transpiration should be incapable of producing a sustainable sublaminar drag within a Poiseuille flow (Bewley, 2001).\(^ {19}\) Indeed, initial attempts to produce simulations of flows with minimised drag resulted in only transient periods of sublaminar drag (Cortelezzi \textit{et al.}, 1998; Aamo \textit{et al.}, 2003).

However, more recently Min \textit{et al.} (2006) demonstrated that it is possible, within a laminar Poiseuille flow, for transpiration to produce a sublaminar drag which is persistent and sustainable. The boundary conditions they employed were of a travelling sine wave at the top wall of the channel, matched with a sine wave of equal magnitude and opposite sign at the bottom wall, which moved counter to the overall direction of the flow. (This arrangement has been referred to as “varicose mode”.) These boundary conditions had been chosen as a result of analysing an expression for the drag acting on the fluid that was developed by Fukagata, Iwamoto & Kasagni (2002), and which related the drag to a weighted integral of the Reynolds stress.

\(^{18}\)It should be noted that the term “closed-loop transpiration” is not generally used in the relevant literature. Instead, its practitioners tend to refer to it by the less specific descriptor, “active flow control”. The term “transpiration” is more commonly used to describe open-loop transpiration. We will use the term “transpiration” to alternatively refer to either the closed and open loop configurations in this section, because the published theoretical results can apply to either.

\(^{19}\)A Poiseuille flow is a flow that is driven by an imposed pressure gradient, and by ‘sublaminar’ drag, we mean that the drag is less than would be experienced by a laminar Poiseuille flow in the absence of transpiration.
Specifically, Min et al. (2006) demonstrated numerically that sustained sub-laminar drag could be produced within low Reynolds number laminar flows, and produced simulations of turbulent flows in which transpiration resulted in significant sustained drag reduction.

Min et al. conclude their paper with the observation that transpiration would be physically difficult to implement. They suggest instead that “a moving surface with wavy motion would produce a similar effect, since wavy walls with small amplitudes can be approximated by surface blowing and suction.” In other words, that the effect of transpiration may be approximated by that of peristalsis, provided that the maximum amplitude of the deformation is small. By solving the Navier-Stokes equation both through numerical simulations and a perturbation analysis, Hœpffner & Fukagata (2009) compared such peristalsis-driven flows to those driven by transpiration. They report that despite the apparent similarity between transpiration and peristalsis (they are both driven by a non-zero wall-normal velocity at the walls), their effects are noticeably different. Specifically, they report that the bulk flow induced by peristalsis generally moves in the same direction as the variation in the boundary conditions, while the flow induced by transpiration moves counter to the boundary condition (although the results of their perturbation analysis suggest that it would be possible, under certain conditions, for small-amplitude peristalsis to induce flow in the opposite direction to the variation in the boundary condition). The nature of flows through tubes induced by such rapid oscillations in the tube’s radius have been further studied by Whitaker et al. (2010b), and extended to flows through elliptical tubes by Whittaker et al. (2010a). The flow induced by such oscillating walls is the subject of §4.

By determining the Reynolds shear stress numerically, Mamori, Fukagata & Hœpffner (2010) investigated the effect of transpiration upon Poiseuille flows in a channel. They considered the combination of a travelling sine wave at one wall in conjunction with an equal and opposite travelling sine wave at the other, which had also previously been used by Min et al. They also considered a system in which a travelling sine wave at one wall is combined with an identical sine wave at the other (an arrangement known as “sinuous mode”). They found that the former arrangement produces significantly greater drag reduction than the latter.

Lee, Min & Kim (2008) studied the stability of Poiseuille flows subjected to transpiration. They found that if the boundary conditions consist of upstream trav-
1.3. Active methods of drag reduction and pumping

...elling waves, there will be a destabilising effect upon the flow once the amplitude of the boundary condition reaches 1.5% of the centreline velocity. Conversely, if the boundary conditions consist of downstream travelling waves, they will have the opposite effect, potentially stabilising an otherwise unstable Poiseuille flow. They found that this stabilising effect was achieved when the phase speed (the speed at which the sine wave boundary condition travels along the wall) exceeded the centerline velocity. They also report that although upstream travelling wave boundary conditions decrease the drag acting on the fluid, downstream travelling waves have the opposite effect, resulting in increased drag.

Moarref & Jovanović (2010) also studied the effect of both upstream and downstream travelling waves upon Poiseuille flows in a channel. Using a perturbation analysis for boundary conditions of small amplitude, they similarly showed that the drag-reducing upstream travelling waves potentially destabilise the flow, while the drag-increasing downstream travelling waves can render stability to otherwise unstable flows. They also derived the rate at which transpiration imparts energy to the flow, and thereby determined the energy efficiency of the resulting drag reduction. The theoretical predictions resulting from their analysis have been verified by a series of direct numerical simulations presented in a companion paper by Lieu, Moarref & Jovanović (2010).

**Studies of closed-loop transpiration**

The concept of closed-loop transpiration was pioneered by Choi et al. (1994).\(^{20}\) Using direct numerical simulations, they tested the effect of targeted blowing and suction in order to reduce the velocity gradients in the near-wall region within turbulent flows. To this end, where a large velocity is detected against the wall, it is negated by a commensurate blowing at the wall. Conversely, any large velocities directed away from the wall are countered by the application of a localised sucking at the wall. In this way, the closed-loop transpiration reduces the velocity gradients within the near-wall region, and thereby locally reduces the effect of convection.

Using direct numerical simulations, Chung & Talha (2011) studied the opti-

\(^{20}\)Although they did not use the term “transpiration”, instead referring to the method as “active turbulence control”.
1.3. Active methods of drag reduction and pumping

mal arrangement for closed-loop transpiration. They considered both the optimal placement of detectors within the flow, and the optimal amplitude of the imposed blowing and suction.

For a recent review of the use of microsensors in closed-loop turbulence control, see Kasagi et al. (2009).

Theoretical studies

Bewley (2009) proved that the power cost of producing sublaminar drag via transpiration in a Poiseuille flow through a channel for an incompressible Newtonian fluid must necessarily be greater than the power saved due to that drag reduction. This result can be seen as a natural extension of the principle of minimum dissipation (Helmholtz, 1868; Batchelor, 1967), which states that the velocity field of a Stokes flow will orient itself such that the total rate of dissipation within the flow will be minimised. (While it is true that a laminar Poiseuille flow within a channel will not necessarily be a Stokes flow, since the magnitude of the convective term in such systems is rendered zero by their geometry, they nonetheless obey Stokes equation. They therefore will orient themselves in the same manner as a Stokes flow, and hence the principle of minimum dissipation should be expected to apply.)

Fukagata et al. (2009) derived the lower bound of the net driving power for a flow through a duct with arbitrary cross-section. They showed that “the lowest net power required to drive an incompressible constant mass-flux flow in a periodic duct having arbitrary constant-shape cross-section, when controlled via a distribution of zero-net mass-flux blowing/suction over the no-slip channel walls or via any body forces, is exactly that of the Stokes flow.”

Marusic, Joseph & Mahesh (2007) derived the conditions under which transpiration will produce sublaminar drag. Their formula relates the rate at which the transpiration provides energy to the flow to the rate at which the flow’s energy dissipates. We shall make use of their formula subsequently in §3.3. It is the ability of active forms of drag reduction, such as transpiration, to impart energy upon the flow that enables them to produce sublaminar drag.

Transpiration can also be viewed as an example of a variety of flow phenomena known as “acoustic streaming” (Riley, 2001). These consist of a fluid that is subjected to a fluctuating force or boundary condition, which induces a change in
the fluid’s overall bulk flow. Despite its misleading name, these phenomena are not limited to flows subjected to acoustic vibrations, and neither are they limited to compressible fluids, but include all flows subjected to oscillating body forces and boundary conditions.

The current study

In §3, we have derived explicit asymptotic formulae for the behaviour of flows induced by transpiration. We consider an overall streamwise flow that is induced by transpiration alone, as was reported by Höpffner & Fukagata (2009). The boundary conditions employed are those in which the wall-normal velocities are travelling sine waves. The waves defining the velocities at either wall are of equal wavelength and frequency, but may differ in magnitude and phase. This is a generalisation of the boundary conditions employed in their investigations by Min et al. (2006) and Höpffner & Fukagata (2009). Using a perturbation analysis, the asymptotic behaviour of such flows has been derived. The bulk flow induced by transpiration is derived in §3.2, along with the optimal arrangement, wavelength and frequency of the boundary conditions for maximising the bulk flow. The asymptotic behaviour of the bulk flow is detailed in the conclusion. The energy imparted to the flow is derived in §3.3.

We also examine, in §3.4, the behaviour of such flows in which the magnitude of the transpiration is considerably greater than the speed at which the boundary condition moves down the channel. We prove that the bulk flow induced by such boundary conditions depends only upon the speed at which the boundary condition moves down the channel. Our analysis here applies to two-dimensional systems whose boundary conditions are the same as those employed in the aforementioned perturbation analysis, and can only be readily extended to a limited set of functionally similar two-dimensional boundary conditions. These represent only a small subset of all possible periodic functions that could define the transpiration boundary conditions. The extension of this proof to generalised boundary conditions, as well as to three dimensional flows, is posed as a conjecture.
1.3.3 Small-amplitude peristalsis

The passage of travelling waves oscillations along a wall can have a significant effect upon the fluid flowing around it. The nature of this effect will depend upon the direction, amplitude and frequency of the oscillations, as well as whether the prevailing flow is laminar or turbulent. This phenomenon can be considered a form of peristalsis. It differs from conventional peristalsis (which is a subject beyond the scope of this thesis) in the amplitude of the imposed wall deformations. Under conventional peristalsis, a tube would be highly deformed, even so much as to be entirely occluded. Here, we instead consider only small-amplitude travelling wave oscillations, a method which can be applied to already flowing fluids.

Within both turbulent and laminar flows, such travelling wave oscillations of the wall are capable of inducing drag reduction. It is possible to reduce the drag of a laminar flow, or to pump an otherwise stationary flow, by the imposition of oscillating waves along the wall in the direction of the flow (Höpffner & Fukagata, 2009). This was demonstrated using direct numerical simulations. More recently, however, Urano et al. (2012) demonstrated experimentally that downstream travelling wave oscillations can reduce the drag experienced by a turbulent flow.

This pumping effect of oscillating walls may not be their greatest potential drag reducing effect. Nakanishi, Mamori & Fukagata (2012) found that a downstream travelling wave is capable of relaminarising an otherwise turbulent flow. This compounds the previously mentioned tendency of such downstream travelling waves to induce drag reduction by providing their own impetus to the flow. This compares favourably with transpiration, described in §1.3.2, in which the downstream travelling waves stabilise the flow but increase the drag, and the upstream travelling waves decrease the drag but destabilise the flow. Similarly, the use of riblets and elastic polymers, described in §1.2.1 and §1.2.2 respectively, will potentially destabilise a laminar flow.

The general problem of flows driven by such oscillating walls of their containers has been studied by a number of authors. Whittaker et al. (2010b) derived asymptotic solutions for the pressure and flow fields resulting from imposed oscillations of the tube wall for a prescribed steady flux or pressure difference along the tube. They also evaluated the energy budget, and derived the conditions under which there is no net energy transfer from the flow to the wall. These results were
then extended to flows through elliptical tubes by Whittaker et al. (2010a).

Another interesting use of oscillating surfaces is in what’s known as a “flying carpet”. Here, a thin sheet of elastic material is able to levitate and propel itself via the propagation of travelling waves along the sheet. The sheet’s propulsion results from the driving of air behind the sheet, and it is therefore another example of a flow driven by an oscillating surface. See Argentina, Skotheim & Mahadevan (2007) for a mathematical model of the flying carpet, and Jafferis, Stone & Sturm (2011) for an experimental study.

Comparison to transpiration

It may not be immediately apparent how transpiration relates to flows induced by oscillating surface waves. However, if the amplitude of these waves is sufficiently small, there is an inherent similarity between the two pumping mechanisms, since the motion of waving walls will induce velocities in the near-wall region which are normal to the time-averaged location of the wall. In other words, for oscillations of small amplitude, oscillating surface waves should induce qualitatively similar flow fields to transpiration (although, as we shall show, these induced flow fields do show significant differences). Observing this similarity between these two pumping mechanisms, Min et al. (2006) acknowledge that transpiration would be difficult to investigate experimentally, and, in concluding their paper, propose instead that “a moving surface with wavy motion would produce a similar effect, since wavy walls with small amplitudes can be approximated by surface blowing and suction.”

However, subsequently Hœpffner & Fukagata (2009) investigated the effects of both pumping mechanisms in isolation. They demonstrated that flows driven by waving walls are significantly different from flows driven by transpiration. Specifically, they report that while transpiration induces flows in the direction counter to the motion of the travelling wave boundary condition, the bulk flow that is induced by waving walls is generally in the same direction as that of the boundary condition. However, using a perturbation analysis, they also found that it would be possible, under certain conditions, for small-amplitude oscillations of waving walls to induce a bulk flow in the direction counter to that of the surface wave. The relationship between transpiration and surface wave pumping is investigated in §4.
The current study

In §4 we consider the flow through tubes that is induced by small-amplitude oscillations in the tube’s wall. This we study with particular reference to the valveless impedance pump, which we shall describe in the next section.

Using a perturbation analysis, we derive expressions for the flow induced by such small-amplitude oscillations. We find that convection plays an unexpectedly large role within such flows\textsuperscript{21}. It is also found that contrary to the predictions of Min et al. (2006), flows driven by small-amplitude oscillations do not behave similarly to those driven by transpiration. This confirms and extends the results reported by Höpffner & Fukagata (2009). By considering the physics at work within such flows, we argue that it would also be impossible for larger-amplitude oscillations to produce flows similar to transpiration.

1.3.4 The valveless impedance pump

In 1954, Gerhart Liebau proposed a model for a valveless pumping mechanism in which a fluid is induced to flow through a tube via the periodic pinching at one location of the tube wall. This pinching causes travelling waves to propagate along the tube, which in turn induce motion within the fluid. If the waves are more readily able to propagate in one direction than the other, then this may result in an overall bulk flow being induced within the tube.

These pumps are constructed by connecting a length of elastic tubing, at both ends, to another inelastic tube. Any travelling waves induced within the elastic section will then be reflected upon reaching the boundary with the inelastic section. The necessary asymmetry is achieved by situating the point at which the waves are generated away from the centre of the elastic section. In this way, a flow is generated due to the mismatch in impedance between the different sections of the tube. A photo of an experimental valveless impedance pump is reproduced in figure 1.4.

It has been found that as the frequency at which the tube is periodically pinched is increased, the direction of the induced flow may be reversed. This flow reversal

\textsuperscript{21}As shall be discussed in §1.3.4, it is commonly assumed, in modelling the valveless impedance pump that convection has a negligible effect within tubes with oscillating walls.
Figure 1.4: Experimental valveless impedance pump, reproduced from Hickerson (2005). The labels denote: \textbf{a} - elastic section, \textbf{b} - pressure ports, \textbf{c} - flow measurement location, \textbf{d} - reservoirs, \textbf{f} - secondary reservoir. Absent from this photo is a pinching mechanism, which would be placed upon section \textbf{a}, at a point offset from the centre.
1.3. Active methods of drag reduction and pumping

was observed in experimental studies of the valveless impedance pump by Hickerson et al. (2005), as well as in studies of the impedance pump at resonance by Hickerson & Gharib (2006).

This does not appear to be the earliest use of the valveless impedance pumping mechanism. In the early stages of their embryonic development, the hearts of invertebrates have been found to be valveless tubes, which drive a flow via the propagation of small amplitude oscillations along their walls (Forouhar et al., 2006). This would appear to be a very similar mechanism to the valveless impedance pump. It is inferred from this that the hearts of the earliest vertebrates most likely functioned via a similar mechanism (Cartwright, Piro & Tuval, 2009).

Furthermore, it has been found that the more rudimentary circulatory systems of some invertebrates are also driven by a tubular heart, which operates via a similar mechanism (Wenning & Meyer, 2007).

Inspired by the structure of the embryonic heart, Loumes, Avrahami & Gharib (2008) developed a form of valveless impedance pump in which the tube is constructed of two separate layers, the outer layer being less elastic than the inner layer. They find that this “multilayer” valveless impedance pump is able to amplify the elastic waves that drive the flow.

Experimental studies of the valveless impedance pump have been undertaken by several authors. Accurate mathematical models of the impedance pump, however, have so far proved elusive.

Existing models of the valveless impedance pump

Thomann (1978) developed a simple one dimensional model for the valveless impedance pump. This model applied to a flow through a torus, and made various assumptions about the pressure and the volume of fluid displaced by the pinching of the tube. The system was further simplified by linearising the governing equations.

A different model was proposed by Moser et al. (1998), consisting of two distensible reservoirs, that are connected in a loop by two tubes, one of which has a wider diameter than the other. This model was based upon an electrical circuit. It does, however, fail to display the characteristic behaviour of the valveless impedance pump.

Jung & Peskin (2001) performed computational simulations of the valveless
impedance pump using the immersed boundary method. Their simulations were performed for a two dimensional viscous, incompressible fluid of constant density. They found that the flow rate induced, and the direction in which it flows, will depend on the frequency of the periodic pinching of the tube, and the location at which it is pinched.

Ottesen (2003) produced a one dimensional model of the pump, consisting of a torus constructed from two separate lengths of tube, which are of differing elasticities. It was found that the outcome of their model resembled the experimental behaviour of the pump. It is also reported that the magnitude, and the direction, of the induced flow will depend upon the frequency of oscillation. Increasing the elasticity of the tube was found to generally increase the induced flow. They also performed experiments to validate their model.

Borzì & Propst (2003) developed a different model, which consisted of a single tube connecting two reservoirs. This system they simulated using a one dimensional model. Their analysis showed that such a pump could produce a net pressure head.

Another one dimensional model has been developed by Manopoulos, Mathioulakis & Tsangaris (2006). This model takes into account the hydraulic losses at the compressed segment of the tube. They report that the addition of this quantity improves the accuracy of the model, particularly when the tube is highly contracted at the point of compression.

In addition to their experimental studies, Hickerson & Gharib (2006) also developed a one dimensional wave model for the pump.

A model based on the acoustic pressure field within the flow was developed by Huang, Wen & Jiao (2010). In this model, they seek to draw a parallel between the effect of the impedance pump and a series of pumping mechanisms collectively known as acoustic streaming (Riley, 2001).

Simulations of the valveless impedance pump at resonance were carried out by Avrahami & Gharib (2008). They also determined the rate at which energy is imparted to the fluid by the pumping mechanism. They report that the pump is most effective when operated at resonance. Unlike some other authors, they did not find that the flow reverses direction at higher frequency.

Bringley et al. (2008) produced a simple model consisting of a torus divided into four sections: the section that is pinched by the pumping mechanism, the two
1.3. Active methods of drag reduction and pumping

elastic regions to either side of it, and the inelastic region. They also performed experimental measurements upon the pump, and compared the results to their predictions. Their model does not include the same resonance behaviour as earlier authors, and neither does it show a flow reversal at high frequency.

The current study

In §4, we derive explicit expressions for the asymptotic behaviour of flows of an incompressible Newtonian fluid induced by small amplitude variations in the radius of a pipe. The radius of the pipe varies as a travelling sine wave. The asymptotic behaviour of the flow has been derived via a perturbation analysis, the method of which is described in §4.2. The work done by such a pumping mechanism is derived in §4.3. The asymptotic behaviour of the flow and the work are detailed in the conclusion. The derivation is repeated for flows through channels in §4.4. Because the boundary conditions at either wall of the channel may be defined independently, we allow the oscillations at either wall to differ in phase and amplitude.

From this we can explore the physics that govern the behaviour of the valveless impedance pump. We consider the physical mechanisms that may induce a bulk flow in tubes whose walls are subject to small amplitude oscillations, and thereby discern the nature of some of the physical mechanisms that would be involved in the impedance pump.

We also compare the nature of such flows driven by oscillating walls to flows driven by transpiration. We show that there does not appear to exist any regime in which transpiration and oscillating walls are physically equivalent pumping mechanisms, and hence there do not appear to be any circumstances under which oscillating walls would be a feasible experimental substitute for transpiration.
On the maximum drag reduction due to added polymers in Poiseuille flow

The addition of elastic polymers to turbulent liquids is known to produce significant drag reduction. In this chapter, we prove that the drag in pipe and channel flows of an unforced laminar fluid constitutes a lower bound for the drag of a fluid containing dilute elastic polymers. Further, the addition of elastic polymers to laminar fluids invariably increases drag. This proof does not rely on the adoption of a particular constitutive equation for the polymer force, and would also be applicable to other similar methods of drag reduction, which are also achieved by the addition of certain particles to a flow. Examples of such methods include the addition of surfactants to a flowing liquid and the presence of sand particles within sandstorms and water droplets within cyclones.

2.1 Equations of channel flow

The proof is presented in §2.3. It is an extension of work by Busse (1970) and Howard (1972), who considered the minimum drag for the flow of an unforced fluid in a channel. The derivation follows a similar path to that presented by Marusic et al. (2007) for a Poiseuille flow subject to blowing and suction flow control. This section contains the mathematical preliminaries, which we make use of in §2.3 and in an alternative proof presented in §2.4.
2.1. Equations of channel flow

We consider an incompressible fluid driven by a constant pressure gradient and subject to a force per unit volume \( f \) caused by the presence of the polymer. Within this work, time and velocity have been normalised using \( \nu \), the kinematic viscosity of the fluid, and \( d \), the height of the channel. The pressure and polymer force have also been normalised by the fluid’s density \( \rho \). The resulting Navier-Stokes and continuity equations are written

\[
\begin{align*}
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} &= -\nabla p + \mathbf{e}_x P + \nabla^2 \mathbf{V} + \mathbf{f}, \quad (2.1.1) \\
\nabla \cdot \mathbf{V} &= 0, \quad (2.1.2)
\end{align*}
\]

where \( \mathbf{V} \) denotes the normalised velocity. The pressure has been split between a constant pressure gradient \( P \), caused by an imposed pressure gradient, and a variable pressure function \( p \), such that the total pressure at any point is given by

\[
p_{\text{total}}(x, y, z, t) = p(x, y, z, t) - P x. \quad (2.1.3)
\]

Hence the \( x \)-direction is the direction of the imposed pressure gradient, and is
therefore also the streamwise direction. The $y$-direction is defined as the spanwise direction, and the $z$-direction is defined as the wall-normal direction.

The system has been defined as a channel of infinite length and width, and unit height. A diagram of the channel is shown in figure 2.1. For the purposes of the subsequent derivations, we define the channel as having length and width $L$, and we consider the limiting case as $L \to \infty$. The Cartesian coordinate system has been used, and the domain is therefore

$$-\infty < x, y < \infty, \quad -\frac{1}{2} \leq z \leq \frac{1}{2}.$$  

(The subsequent derivations were also performed for a pipe flow, the results of which can be found in §2.5.) For all quantities, wall-parallel averages are denoted by an overbar and are defined via

$$\bar{F}(z, t) \overset{\text{def}}{=} \lim_{L \to \infty} \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} F(x, y, z, t) \, dx \, dy. \quad (2.1.4)$$

It should be noted that for a statistically steady state flow, a wall-parallel average will be equivalent to a temporal average. This fact will be used subsequently. In this work, a ‘statistically steady-state’ flow refers to a fully-developed flow, whose bulk (spacially averaged) characteristics are independent of time. This will not necessarily be a flow in which the local velocity at any point is independent of time.

An average over the entire channel is denoted by angled brackets and is defined via

$$\langle F(t) \rangle \overset{\text{def}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{F}(z, t) \, dz. \quad (2.1.5)$$

The Reynolds number, based on the channel height and the bulk velocity, is $Re_B = \langle \bar{V}_z \rangle$. We also decompose the velocity, pressure and polymer force into wall-parallel averaged components (referred to from this point on as ‘mean’ components) and fluctuating components,

$$V = \bar{V} + u, \quad p = \bar{p} + p', \quad f = \bar{f} + f',$$

where the mean of each fluctuation is zero: $\bar{u} = \bar{f}' = 0, \quad \bar{p}' = 0$. We assume the flow is subject to a no-slip boundary condition, and so we have,

$$\bar{V} = u = \mathbf{0}, \quad \text{at } z = \pm \frac{1}{2}. \quad (2.1.6)$$
2.1. Equations of channel flow

Since the $x$-direction is the direction of the constant pressure gradient $P$, and since there exists no driving force which should sustain a bulk flow in the $y$ or $z$ directions, the velocity functions may be written as

\[ \bar{V}(z, t) = (\bar{V}_x, 0, 0), \quad u(x, y, z, t) = (u, v, w). \]

2.1.1 Energy equations

We derive energy equations relating to the mean and fluctuating components of a statistically steady-state flow. These will be used within subsequent equations.

To do this we first decompose all terms within the Navier-Stokes equation into their mean and fluctuating components, and then making use of equation (2.1.2) we have

\[ \frac{\partial \bar{V}}{\partial t} + \frac{\partial u}{\partial t} + \nabla \cdot (\bar{V}u + u\bar{V} + uu) = -e_x \frac{\partial \bar{p}}{\partial z} - \nabla p' + e_x P + \frac{\partial^2 \bar{V}}{\partial z^2} + \nabla^2 u + \bar{f} + f'. \]  

(2.1.7)

The wall-parallel average of equation (2.1.7) is

\[ \frac{\partial \bar{V}}{\partial t} + \frac{\partial (\bar{u}w)}{\partial z} = -e_x \frac{\partial \bar{p}}{\partial z} + e_x P + \frac{\partial^2 \bar{V}}{\partial z^2} + \bar{f}. \]  

(2.1.8)

The evolution equation for the fluctuations in velocity is therefore the difference (2.1.7) – (2.1.8), which is

\[ \frac{\partial u}{\partial t} + \nabla \cdot (\bar{V}u + u\bar{V} + uu - \bar{u}w) = -\nabla p' + \nabla^2 u + f'. \]  

(2.1.9)

The energy equation for the mean flow can be found by taking $\langle \bar{V} \cdot (2.1.8) \rangle$, which gives

\[ \frac{1}{2} \frac{d}{dt} \langle |\bar{V}|^2 \rangle - \left\langle \bar{u}w \frac{\partial \bar{V}_x}{\partial z} \right\rangle = P \langle \bar{V}_x \rangle - \left\langle \left| \frac{\partial \bar{V}_x}{\partial z} \right|^2 \right\rangle + \langle \nabla \cdot \bar{f} \rangle. \]  

(2.1.10)

Similarly, the energy equation for the fluctuations within the flow can be found by taking $\langle u \cdot (2.1.9) \rangle$, which gives

\[ \frac{1}{2} \frac{d}{dt} \langle |u|^2 \rangle + \left\langle \bar{u}w \frac{\partial \bar{V}_x}{\partial z} \right\rangle = -\langle |\nabla u|^2 \rangle + \langle u \cdot f' \rangle. \]  

(2.1.11)
The overall energy equation is then simply the sum of equations (2.1.12) and (2.1.11), which is

$$\frac{1}{2} \frac{d}{dt} \langle |V|^2 \rangle = P\langle \bar{V}_x \rangle - \left\langle \left( \frac{\partial \bar{V}_x}{\partial z} \right)^2 \right\rangle - \langle |\nabla u|^2 \rangle + \langle V \cdot f \rangle. \quad (2.1.12)$$

If the flow is steady state, then \( \frac{d}{dt} \langle |\tilde{V}|^2 \rangle = \frac{d}{dt} \langle |u|^2 \rangle = 0 \), and the above equations may be simplified to

$$-\left\langle \bar{u} w d\bar{V}_x \frac{d}{dz} \right\rangle = P\langle \bar{V}_x \rangle - \left\langle \left( \frac{d\bar{V}_x}{dz} \right)^2 \right\rangle + \langle \bar{V} \cdot \bar{f} \rangle, \quad (2.1.13)$$

$$\left\langle \bar{u} w \frac{d\bar{V}_x}{dz} \right\rangle = -\langle |\nabla u|^2 \rangle + \langle u \cdot f' \rangle, \quad (2.1.14)$$

$$0 = P\langle \bar{V}_x \rangle - \left\langle \left( \frac{d\bar{V}_x}{dz} \right)^2 \right\rangle - \langle |\nabla u|^2 \rangle + \langle V \cdot f \rangle. \quad (2.1.15)$$

The term \( \langle V \cdot f \rangle \) in equation (2.1.15) represents the rate at which the polymer does work upon the fluid per unit of volume, throughout the channel. If negative, \( \langle V \cdot f \rangle \) represents the rate at which the polymer extracts energy from the flow.

### 2.2 Work done by the polymer upon the flow

Within the previous section, we showed that \( \langle V \cdot f \rangle \), which represents the overall rate at which the polymer does work upon the flow per unit of volume, is crucial to determining the bulk flow rate of the fluid, \( \langle \bar{V}_x \rangle \). Within this section, we show that \( \langle V \cdot f \rangle < 0 \).

The exact physical causes of drag reduction due to elastic polymers are open to debate. As has been mentioned within the Introduction, §1.2.2, they are believed to involve fundamentally altering the turbulent fluid’s flow profile by transporting momentum within the fluid, and by countering vorticity and eddying motions. However, none of these mechanisms involves a net transfer of energy from the polymer to the fluid.

The polymer draws energy from the flow, and may return energy to the flow. The energy is stored meanwhile as elastic energy within the polymer. Where the
term $V \cdot f$ is positive, the polymer is imparting energy to the flow; and where it is negative, the polymer is drawing energy from the flow. The polymer has no source of energy apart from the flow, and therefore the overall work done by the polymer upon the flow, $\langle V \cdot f \rangle$, cannot be positive.

There are two paths by which the presence of the polymer may affect the flow’s total energy: The first is the transport of energy into and out of the channel as elastic energy due to the stretching of the polymer molecules, and the second is the dissipation of elastic energy from within the polymer molecules. However, in a statistically steady-state flow, the average elongation of a polymer molecule entering the channel will equal the average elongation exiting the channel. Hence there will in fact be no net transport of elastic energy into, or out of, the channel.

Thus, since the overall work done by the polymer upon the flow must be equal to the rate of dissipation of elastic energy within the polymer molecules, we may say that,

$$\langle V \cdot f \rangle = -\langle \epsilon_p \rangle \leq 0, \quad (2.2.1)$$

where $\epsilon_p(x, y, z, t)$ denotes the rate at which elastic energy from within the polymer molecules is dissipating at a point within the flow. For a discussion of the magnitude of this dissipation rate see Ptasinski et al. (2003).

### 2.3 Volume-flux comparison between laminar and turbulent flows

The flow profile for an unforced laminar fluid is given by

$$U_l = \frac{P}{2} \left( \frac{1}{4} - z^2 \right), \quad (2.3.1)$$

and its bulk flowrate will be

$$\langle U_l \rangle = \frac{P}{12}. \quad (2.3.2)$$

We now consider a turbulent flow subjected to a polymer force. To this end, we redefine the polymer force in terms of a stress tensor

$$\nabla \cdot \tau = f. \quad (2.3.3)$$
This tensor refers to the stress experienced by the fluid due to the polymer force. For a statistically steady state flow, the component of equation (2.1.8) in the $x$-dimension may now be written as

\[
\frac{d}{dz} \left[ \bar{uw} - P_z - \frac{d\bar{V}_x}{dz} - \bar{\tau}_{xz} \right] = 0,
\]

where $\bar{\tau}_{xz}$ is one component of the mean of the stress tensor, and is defined by

\[
\frac{d\bar{\tau}_{xz}}{dz} = \bar{f}_x.
\]

We can obtain an expression for $P$ by integrating equation (2.3.4), which results in

\[
P_z = \bar{uw} - \langle \bar{uw} \rangle - \bar{\tau}_{xz} + \langle \bar{\tau}_{xz} \rangle - \frac{d\bar{V}_x}{dz}.
\]

The bulk flowrate can therefore be found by taking $\langle z \cdot (2.3.6) \rangle$ and rearranging. By doing so, we obtain

\[
\langle \bar{V}_x \rangle = \frac{P}{12} - \langle z \bar{uw} \rangle + \langle z \bar{f}_x \rangle.
\]

The $\langle z \bar{uw} \rangle$ term in the above equation can be evaluated by taking $\langle \bar{uw} \cdot (2.3.6) \rangle$ and substituting the energy equation for the fluctuations (2.1.14). The $\langle z \bar{\tau}_{xz} \rangle$ term can be evaluated similarly by taking $\langle \bar{\tau}_{xz} \cdot (2.3.6) \rangle$, and integrating one of the resulting terms by parts, taking into account the no-slip boundary condition and equation (2.3.5). In so doing, we obtain a new equation for $\langle \bar{V}_x \rangle$ after substituting the results into equation (2.3.7). By then comparing this result to equation (2.3.2), we obtain the following relation between the flowrates of a turbulent fluid containing polymers and an unforced laminar flow.

\[
\langle U_l - \bar{V}_x \rangle = \frac{1}{P} \left[ \langle (\bar{uw} - \langle \bar{uw} \rangle - \bar{\tau}_{xz} + \langle \bar{\tau}_{xz} \rangle)^2 \rangle + \langle |\nabla \bar{u}|^2 \rangle - \langle \bar{V} \cdot \bar{f} \rangle \right].
\]
since the required concentration of polymers is very low. Hence, the term \( \langle |\nabla u|^2 \rangle \) dominates the above equation, since the dissipation due to fluctuations is known to be significantly greater than that due to the mean flow within a turbulent fluid (Pope, 2000). Any significant drag reduction will therefore be achieved by reducing \( \langle |\nabla u|^2 \rangle \).

It is now clear that for a polymer force to raise the bulk flowrate of the fluid to greater than or equal to that of a laminar Poiseuille flow, the following would need to hold:

\[
\langle \mathbf{V} \cdot \mathbf{f} \rangle \geq \langle (u\overline{w} - \langle u\overline{w} \rangle - \overline{\tau}_{xz} + \langle \overline{\tau}_{xz} \rangle)^2 \rangle + \langle |\nabla u|^2 \rangle. \tag{2.3.9}
\]

The question of whether or not the presence of a polymer force may produce sublaminar drag therefore becomes a question of the sign and magnitude of \( \langle \mathbf{V} \cdot \mathbf{f} \rangle \). And so, because of equation (2.2.1), we conclude that polymer forces cannot produce sublaminar drag in turbulent fluids.

Furthermore, by removing all of the fluctuating terms from equation (2.3.8), we obtain a relation between the bulk flowrate of a laminar Poiseuille flow subject to a polymer force to that of the equivalent unforced laminar flow. It is clear by inspection that if the polymer force is anywhere non-zero, then the bulk flowrate will be reduced. We may infer from this that while such polymer forces are known to be capable of causing drag reduction within turbulent fluids, they will invariably increase the drag when acting upon a laminar flow.

An alternative methodology, which could have been employed in this proof, has been presented by Bewley & Aamo (2004), who employed it in reference to drag reduction for a Poiseuille flow subject to blowing and suction flow control at the walls. The difference between their methodology, and that employed here is that they have explicitly considered the magnitude of the drag at the wall. To rederive this result via their methodology would simply be a matter of removing all terms in Bewley (2009) which relate to the blowing and suction method of drag reduction, and adding a body force term, as we have done here, to account for the effect of the polymer.
2.4 Minimising drag

There is an alternative way to prove the result given in §2.3: By adding $P\langle \bar{V}_x \rangle$ to both sides of equation (2.1.15), and substituting equation (2.2.1), we obtain

$$\langle \bar{V}_x \rangle = \lim_{L \to \infty} \frac{1}{L^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int \left( 2 \bar{V}_x - \frac{1}{P} \left| \frac{d\bar{V}_x}{dz} \right|^2 - \frac{1}{P} |\nabla u|^2 \right) dx
dy
dz - \langle \epsilon_p \rangle. \tag{2.4.1}$$

While there is no unique solution to the above equation, it is possible, using the calculus of variations, to find the functions for $\bar{V}_x$ and $u$ which maximise the value of the integral. To do so, we must first set $\langle \epsilon_p \rangle$ to zero. This is equivalent to assuming that the rate of dissipation of elastic energy within the polymer is negligible. We can, however, do this without loss of generality, since a non-zero $\langle \epsilon_p \rangle$ term would only ever reduce the value of $\langle \bar{V}_x \rangle$.

The derivation is omitted here, but the functions which maximise $\langle \bar{V}_x \rangle$ are

$$\bar{V}_x \big|_{\text{max}(\bar{V}_x)} = \frac{P}{2} \left( \frac{1}{4} - z^2 \right), \quad u \big|_{\text{max}(\bar{V}_x)} = 0, \quad (2.4.2)$$

which is the profile of an unforced laminar flow, $U_l(z)$. This implies that of all the physically allowable flow profiles of a Poiseuille fluid, the greatest bulk flowrate is obtained when flow profile is exactly that of an unforced laminar flow.

Therefore, since equation (2.3.6) indicates that the presence of a polymer force will cause a deviation to the profile of a laminar flow away from (2.4.2), we can conclude that adding such elastic polymers to a laminar flow will only result in increased drag.

2.5 Pipe flow

In this section, we extend these results to the common case of flow through a pipe. To do so, we use the cylindrical polar coordinates $(r, \theta, x)$, where $x$ is the streamwise distance from the beginning of the pipe and $r$ is the radial distance from the pipe’s center. A diagram of the system is shown in figure 2.2. The system’s domain is
Figure 2.2: Diagram of the pipe domain. We consider an infinite pipe in which $L \to \infty$. 
The wall-parallel averages are again denoted by an overbar, and are now defined via
\[ \bar{F}(r, t) \overset{\text{def}}{=} \lim_{L \to \infty} \frac{1}{2\pi L} \int_{-L/2}^{L/2} \int_{0}^{2\pi} F(r, \theta, x, t) \, d\theta \, dx, \] (2.5.1)
and again the overall average is denoted by angled brackets. It is now defined via
\[ \langle F(t) \rangle \overset{\text{def}}{=} 2 \int_{0}^{1} \bar{F}(r, t) r \, dr. \] (2.5.2)

Again the Reynolds number, this time given by the pipe’s radius and the bulk velocity, is \( Re_B = \langle \bar{V}_x \rangle \). The pressure, velocity and polymer force are decomposed into mean and fluctuating components as before, this time with
\[ V_x = \bar{V}_x + u; \quad V_{\theta} = v; \quad V_r = w. \]

The derivation follows a similar path to the channel flow case, so much of the detail will be omitted here. Beginning with the Navier-Stokes (2.1.1) and continuity (2.1.2) equations as before, we obtain the energy equations for a statistically steady-state flow via an entirely analogous path. The energy equations thus obtained are identical to those of channel flow, equations (2.1.13)-(2.1.15), except that the wall-normal direction \( z \) is here replaced by its pipe flow analogue \( r \).

We then proceed along the path described within §2.3, in this way we derive the following comparison between the flowrates of a turbulent fluid containing the polymer and an unforced laminar fluid:
\[ \langle U_l - \bar{V}_x \rangle = \frac{1}{\bar{P}} \left[ \langle (\bar{\omega}_w - \bar{r}_{xx})^2 \rangle + \langle |\nabla \bar{u}|^2 \rangle + \langle \epsilon_p \rangle \right], \] (2.5.3)
which, for reasons already given, must be positive if \( \bar{\omega}_w \) is anywhere non-zero, or the polymer is present.

We may also prove this same result via the calculus of variations, as described in §2.3. Since the energy equations are identical in form to those in the channel flow case, we perform the calculus of variations upon an integral whose integrand is identical to that in equation (2.4.1), with the Cartesian coordinates used in the channel flow case replaced by their cylindrical polar equivalents for the pipe flow.
2.6. Conclusions

In so doing, we find that the flow profile which maximises the flowrate of a fluid in a pipe is,

\[ \bar{V}_x \bigg|_{\max(\bar{V}_x)} = \frac{P}{4} (1 - r^2), \quad u \bigg|_{\max(\bar{V}_x)} = 0, \quad (2.5.4) \]

which is the flow profile for an unforced laminar flow within a pipe.

2.6 Conclusions

The results show that the flowrate of an unforced laminar fluid constitutes an upper bound for the flowrate of a fluid containing dilute elastic polymers. This result can also potentially be applied to flows subject to drag reduction achieved by the addition of other types of particles to the fluid. Examples include the addition of surfactant molecules to a flowing liquid, and the addition of water droplets or sand particles to a flowing gas.

However, the additives which produce the drag reduction may also alter the fluid’s density and viscosity, as is the case for elastic polymers. If their presence has affected the fluid’s density or viscosity, then it is the flowrate of a laminar fluid with the same density and viscosity as the solution (rather than the density and viscosity of the pure solvent) which constitutes an upper bound on the mixture’s bulk flowrate.

Furthermore, if the resulting mixture produces a fluid which is either compressible or non-Newtonian, then this proof may not apply. However, as we have argued within §1.2.2 of the Introduction, the minimum viscosity of a polymer solution will be greater than the viscosity of the pure Newtonian solvent.

An increased viscosity will reduce the flowrate. We can thus conclude that since the addition of the polymer cannot produce sublaminar drag when the thickening effect of the polymer is neglected, it will neither be capable of producing sublaminar drag in real flows in which the thickening effect of the polymer will often be significant.

By considering the overall energy equation (2.1.15) and the drag minimising procedure used within §2.4, we see that for a body force (or boundary force) to produce sublaminar drag would require an energy input that is greater than the combination of the rate of dissipation due to turbulent fluctuations, \( \langle |\nabla u|^2 \rangle \), the dissipation within the polymer molecules, \( \langle \epsilon_p \rangle \), and the effect of the deviation of
2.6. Conclusions

the flow profile, $\bar{V}_x(z)$, away from its laminar equivalent, $U_l(z)$, caused by the Reynolds stress and the body force.

This concurs with Bewley’s recent result (2009), showing that the power saved through sublaminar drag reduction produced by flow control must be less than the power cost to produce that drag reduction.

This method has not proved capable of deriving or approximating Virk’s asymptote from first principles. The reason for this is that Virk’s asymptote applies strictly to fully developed turbulent flow, for which the flow profile has also yet to be derived from first principles. Hence we are only able to compare turbulent flows with drag reduction to laminar flow.

For this reason, we are only able to conclude that the drag is minimised by removing the fluctuations. However, by its definition, fully developed turbulent flow contains significant random fluctuations. At maximum drag reduction, these fluctuations will only have been reduced by the action of the polymer. Therefore, to derive Virk’s asymptote would require the ability to quantify the amount by which these fluctuations may be reduced. This is something we cannot do without a theoretical basis for the nature of the fluctuations, and an analytical closure for the Navier-Stokes equation.

If the assumption of fully developed turbulent flow allowed some further assumptions to be made about the nature of the velocity vectors $\bar{V}$ and $u$, then it may prove possible to derive Virk’s asymptote.

It is notable that the maximum drag reduction produced by the presence of surfactant micelles within the fluid is very similar to that produced by the presence of the polymer. In fact, the maximum drag reduction produced by the micelles slightly exceeds Virk’s asymptote. It is also notable that when a flow containing a surfactant reaches maximum drag reduction, the Reynolds stress is everywhere zero (Warholic et al., 1999b).\footnote{The Reynolds stress is given by $\overline{uu}$. It first appears within the second term on left hand side of equation (2.1.8).} The same is not true for a flow containing a polymer at the maximum drag reduction (Ptasinski et al., 2001).

The relevance of this is that the Reynolds stress represents the effect of the fluctuations upon the mean flow. It is through the Reynolds stress that the fluctuations draw energy from the mean flow. This transfer of energy is represented by the first term on the left hand side of equations (2.1.13) and (2.1.14).

The Reynolds stress is given by $\overline{uu}$. It first appears within the second term on left hand side of equation (2.1.8).
2.6. Conclusions

Any method of drag reduction, which does not act by directly imparting energy upon the mean flow, will function by altering the Reynolds stress. We have shown that no method of drag reduction which is due to the presence of a body force will be capable of producing sublaminar drag, unless its overall action imparts energy upon the flow. It follows that the net effect of a non-zero Reynolds stress in such flows must be to extract energy from the mean flow. Such methods of drag reduction therefore work by minimising the effect of the Reynolds stress upon the mean flow.

A flow with zero (or negligible) Reynolds stress therefore constitutes a natural upper bound for any such methods of drag reduction. Within such a flow, the fluctuations can only extract energy from the mean flow indirectly, through the body force. Such flows could then only differ in the extent to which the body force affects the mean flow.
Chapter 3

Induced flow due to blowing and suction flow control: An analysis of transpiration

It has previously been demonstrated that the drag experienced by a Poiseuille flow in a channel can be reduced by subjecting the flow to a dynamic regime of blowing and suction at the walls of the channel (also known as ‘transpiration’). Furthermore, it has been found to be possible to induce a ‘bulk flow’, or steady motion through the channel, via transpiration alone.

In this work, we derive explicit asymptotic expressions for the induced bulk flow via a perturbation analysis. From this we gain insight into the physical mechanisms at work within the flow. The boundary conditions used are of travelling sine waves at either wall, which may differ in amplitude and phase. Here it is demonstrated that the induced bulk flow results from the effect of convection.

We find that the most effective arrangement for inducing a bulk flow is that in which the boundary conditions at either wall are equal in magnitude and opposite in sign. We also show that for the bulk flow induced to be non-negligible, the wavelength of the boundary condition should be comparable to, or greater than, the height of the channel. Moreover, we derive the optimal frequency of oscillation, for maximising the induced bulk flow, under such boundary conditions. The asymptotic behaviour of the bulk flow is detailed within the conclusion.
3.1 Equations of channel flow

It is found, under certain caveats, that if the amplitude of the boundary condition is too great, the bulk flow induced will become dependent only upon the speed at which the boundary condition travels along the walls of the channel. We propose the conjecture that for all similar flows, if the magnitude of the transpiration is sufficiently great, the bulk flow will depend only upon the speed of the boundary condition.

3.1 Equations of channel flow

We consider the steady-state flow of an incompressible Newtonian fluid through a channel of infinite length and width and finite height. The flow is driven by the boundary conditions at the base and top of the channel that are non-zero in only the wall-normal, or $z$-direction, and vary spatially only in the streamwise, or $x$-direction.

We assume that the flow remains two-dimensional and laminar. The spanwise, or $y$-direction is therefore not included in our derivation. We consider only boundary conditions in which the wall-normal velocity can be represented as a single sine wave travelling backwards in the $x$-direction.

The Navier-Stokes and continuity equations are given by

$$\rho \left( \frac{\partial}{\partial t} \hat{u} + \hat{u} \cdot \nabla \hat{u} \right) = -\nabla \hat{p} + \mu \nabla^2 \hat{u}, \quad (3.1.1)$$

$$\hat{\nabla} \cdot \hat{u} = 0, \quad (3.1.2)$$

where $\rho$ and $\mu$ represent the density and the dynamic viscosity of the fluid respectively. These equations will subsequently be non-dimensionalised via the properties of the boundary conditions. Throughout this chapter, quantities expressed in terms of dimensional units, such as the velocity, $\hat{u}(\hat{x}, \hat{t})$, and pressure, $\hat{p}(\hat{x}, \hat{t})$, above, are differentiated from their non-dimensionalised counterparts, $u(x, t)$ and $p(x, t)$, by the presence of a circumflex. The position vector is denoted by $\hat{x}$, and is defined such that $\hat{x} = (\hat{x}, \hat{z})$.

The height of the infinitely long channel is $h$, and hence the domain of the flow is given by

$$-\infty < \hat{x} < \infty, \quad 0 \leq \hat{z} \leq h, \quad (3.1.3)$$
3.1. Equations of channel flow

3.1.1 Boundary conditions

The no-slip boundary condition applies to the streamwise flow. For the wall-normal velocity, we consider a family of possible boundary conditions, which consist of a single-modal travelling sine wave at either wall. The sine wave at the top wall has an amplitude of $A$, while its counterpart at the bottom wall has an amplitude of $\gamma A$. Hence $\gamma$ is simply a dimensionless ratio of the two amplitudes. The sine waves travel backwards along the wall in the streamwise direction, and are of equal wavelength, $\lambda$, and temporal frequency, $\omega$ (henceforth referred to simply as the ‘frequency’). They may however differ in phase by some quantity $\phi$. The wall-normal velocities at the boundaries are therefore given by

$$\hat{w} = \begin{cases} 
A \sin \left( \frac{\hat{x}}{\lambda} + \omega \hat{t} \right), & \text{at } \hat{z} = h \\
\gamma A \sin \left( \frac{\hat{x}}{\lambda} + \omega \hat{t} - \phi \right), & \text{at } \hat{z} = 0 
\end{cases}$$

(3.1.4)

The parameters $\gamma$ and $\phi$ may take the following values:

$$0 \leq \gamma \leq 1, \quad 0 \leq \phi < 2\pi.$$  

We analyse and compare the resulting flows from four different arrangements of boundary conditions. The first case, which we shall call the “mixed” boundary condition, involves transpiration only on the top wall of the channel, while the no-slip boundary condition applies to $w$ at the bottom wall. The second involves identical boundary conditions at the top and bottom walls, and is referred to here as the in-phase boundary condition. The third involves a travelling sine wave at the top wall and an equivalent cosine wave at the bottom wall, and is referred to here as the out-of-phase boundary condition. The fourth involves boundary conditions that are equal in magnitude but opposite in sign at either wall, and is referred to here as the antiphase boundary condition. (It was under this arrangement that Min et al. (2006) demonstrated sustainable sublaminar drag.) Diagrams of an instantaneous realisation of each of these cases can be seen in figure 3.1, and the values of $\gamma$ and $\phi$ corresponding to each case are given in table 3.1.
3.1. Equations of channel flow

Figure 3.1: Instantaneous realisations of various cases of the family of boundary conditions defined by equation (3.1.12). **Top left:** ‘Mixed’ boundary conditions. **Top right:** In-phase boundary conditions. **Bottom left:** Out-of-phase boundary conditions. **Bottom right:** Antiphase boundary conditions.

Table 3.1: Values of $\gamma$ (the ratio of boundary condition amplitudes) and $\phi$ (the phase difference) for the four varieties of boundary conditions.

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>$\gamma$</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mixed</td>
<td>0</td>
<td>$-\pi$</td>
</tr>
<tr>
<td>In-phase</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Out-of-phase</td>
<td>1</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>Antiphase</td>
<td>1</td>
<td>$\pi$</td>
</tr>
</tbody>
</table>
3.1. Equations of channel flow

3.1.2 The scaled equations

The system contains two natural length scales: $h$, the height of the channel, and $\lambda$, the wavelength of the sinusoidal waves defining the boundary conditions. The other parameters that have been used to non-dimensionalise the quantities of the Navier-Stokes equation are the maximum amplitude (velocity) of the boundary condition, $A$, the frequency of the boundary condition, $\omega$, and the density, $\rho$, and dynamic viscosity, $\mu$ of the fluid.

Using these quantities, the variables in the Navier-Stokes equation are scaled according to

\[ \hat{x} = hx, \quad \hat{u} = Au, \quad \hat{p} = \frac{\mu A}{h}p, \quad \hat{t} = \frac{1}{\omega}t. \]  

(3.1.5)

Scaling the Navier-Stokes (3.1.1) and continuity (3.1.2) equations in this way leads to

\[ \beta \left( \frac{\partial}{\partial t} u + \alpha u \cdot \nabla u \right) = -\nabla p + \nabla^2 u, \]  

(3.1.6)

\[ \nabla \cdot u = 0. \]  

(3.1.7)

There are two dimensionless numbers in the scaled Navier-Stokes equation, denoted $\alpha$ and $\beta$. This is due to the presence of two separate timescales, $h/A$ and $1/\omega$, within the system. The former constitutes the convective timescale, while the latter constitutes the diffusive timescale. The parameter $\beta$ is referred to as the Stokes number, and relates to the rate of diffusion of vorticity within the flow. The parameter $\alpha$ is effectively a measure of the relative importance of convection within the system. The values of these parameters are given by

\[ \alpha = \frac{A}{h} \omega, \quad \beta = \frac{\rho h^2 \omega}{\mu}. \]  

(3.1.8)

If we define a Reynolds number based upon the amplitude of the boundary condition by

\[ Re = \frac{\rho h A}{\mu}, \]  

(3.1.9)

then we may also express $\alpha$ as a ratio of dimensionless numbers:

\[ \alpha = \frac{Re}{\beta}. \]  

(3.1.10)
Figure 3.2: Diagram of the channel domain, in scaled coordinates. We consider an infinite channel in which $L \to \infty$. Because of the inherent symmetry of the flow, we may neglect the spanwise (or $y$) dimension.

The scaled domain of the flow, equivalent to (3.1.3), is given by

$$-\infty < x < \infty, \quad 0 \leq z \leq 1.$$  \hfill (3.1.11)

A diagram of the domain of the flow, in scaled variables, is shown in figure 3.2.

In scaled variables, the boundary conditions defined in equation (3.1.4) become

$$w = \begin{cases} \sin(\eta x + t), & \text{at } z = 1 \\ \gamma \sin(\eta x + t - \phi), & \text{at } z = 0 \end{cases}$$  \hfill (3.1.12)

A new dimensionless parameter, $\eta$, appears in these dimensionless boundary conditions. It represents the ratio of the height of the channel to the wavelength of the sinusoidal waves defining the boundary conditions, and is formally defined

$$\eta = \frac{h}{\lambda}.$$  \hfill (3.1.13)
Since the no-slip boundary condition applies to the flow in the streamwise direction, we have
\[ u = 0, \quad \text{at } z = 0, \ 1. \tag{3.1.14} \]

### 3.1.3 Averaging

For all quantities, wall-parallel averages are denoted by an overbar and are defined
\[ \bar{F}(z, t) \overset{\text{def}}{=} \lim_{L \to \infty} \frac{1}{L} \int_{-L/2}^{L/2} F(x, t) \, dx, \tag{3.1.15} \]

A non-zero wall-parallel average of the streamwise velocity, \( \bar{u}(z, t) \), is referred to here as a ‘translational’ flow. This concept is introduced here as a mathematical convenience. Use will be made of it in §3.3. Fluctuations of quantities (by which we mean here any deviation in the value of that quantity from its wall-parallel average) are denoted by a prime and are defined
\[ F'(x, t) \overset{\text{def}}{=} F(x, t) - \bar{F}(z, t). \tag{3.1.16} \]

An average over the entire channel is denoted by angled brackets and is defined
\[ \langle F(t) \rangle \overset{\text{def}}{=} \int_{0}^{1} F(z, t) \, dz. \tag{3.1.17} \]

In dimensional variables, the wall-parallel averages of quantities are mathematically defined in an identical manner to their equivalents in scaled variables. The average over the entire channel, however, is defined differently, due to its dependence upon \( h \), the height of the channel. In terms of dimensional variables, equation (3.1.17) is equivalent to
\[ \langle \hat{F}(\hat{t}) \rangle \overset{\text{def}}{=} \frac{1}{h} \int_{0}^{h} \hat{F}(\hat{\hat{z}}, \hat{t}) \, d\hat{\hat{z}}. \tag{3.1.18} \]

For the non-dimensionalised velocity, \( \langle \hat{u} \rangle \) can denote either the average streamwise velocity within the channel, or the volume flux through the channel (henceforth referred to as the ‘bulk flow’). However, for the dimensionalised velocity, \( \langle \hat{u} \rangle \) denotes exclusively the average velocity, and must be multiplied by \( h \) to obtain the bulk flow.
3.1.4 Streamfunction

Because the flow is two dimensional, we can express the velocity in terms of a streamfunction $\Psi(x,t)$. In this formulation, the steamwise, $u$, and wall-normal, $w$, components of the velocity vector are given by

$$u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x}. \quad (3.1.19)$$

By taking the curl of the Navier-Stokes equation, and substituting the streamfunction for the fluid’s velocity, we obtain the evolution equation for $\Psi(x,t)$:

$$\beta \frac{\partial}{\partial t} \nabla^2 \Psi + \alpha \beta \left( \frac{\partial \Psi}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial}{\partial z} \right) \nabla^2 \Psi = \nabla^4 \Psi. \quad (3.1.20)$$

3.2 Perturbation analysis

Within this section, we describe the methodology we have used to determine the properties of the flow. We analyse flows driven by transpiration through a perturbation analysis for small values of the perturbation parameter $\alpha$. The derivation and the results are presented within this section.

The method involves expanding the velocity, streamfunction and pressure in terms of $\alpha$, as follows:

$$u = u_0 + \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \ldots \quad (3.2.1a)$$

$$\Psi = \Psi_0 + \alpha \Psi_1 + \alpha^2 \Psi_2 + \alpha^3 \Psi_3 + \ldots \quad (3.2.1b)$$

$$p = p_0 + \alpha p_1 + \alpha^2 p_2 + \alpha^3 p_3 + \ldots \quad (3.2.1c)$$

Substituting equation (3.2.1) into (3.1.20), and equating orders of $\alpha$, we obtain

$$\nabla^4 \Psi_0 - \beta \frac{\partial}{\partial t} \nabla^2 \Psi_0 = 0, \quad (3.2.2a)$$

$$\nabla^4 \Psi_1 - \beta \frac{\partial}{\partial t} \nabla^2 \Psi_1 = \beta \left( \frac{\partial \Psi_0}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \Psi_0}{\partial x} \frac{\partial}{\partial z} \right) \nabla^2 \Psi_0, \quad (3.2.2b)$$

$$\nabla^4 \Psi_2 - \beta \frac{\partial}{\partial t} \nabla^2 \Psi_2 = \beta \left( \frac{\partial \Psi_0}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \Psi_0}{\partial x} \frac{\partial}{\partial z} \right) \nabla^2 \Psi_1$$

$$+ \beta \left( \frac{\partial \Psi_1}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \Psi_1}{\partial x} \frac{\partial}{\partial z} \right) \nabla^2 \Psi_0, \quad (3.2.2c)$$

$$\ldots$$
We can see from equation (3.2.2a) that the leading order term, $\Psi_0$, represents an unsteady Stokes flow. For all bar the leading order term our boundary conditions are

$$\frac{\partial \Psi_n}{\partial x} = \frac{\partial \Psi_n}{\partial z} = 0, \quad \text{at } z = 0, 1, \quad \text{for } n > 0 \tag{3.2.3}$$

while the boundary conditions given within equations (3.1.12) and (3.1.14) apply to the leading order term, $\Psi_0$. This implies that the leading order boundary conditions are

$$\frac{\partial \Psi_0}{\partial z} = 0, \quad \text{at } z = 0, 1 \tag{3.2.4}$$

$$\frac{\partial \Psi_0}{\partial x} = \begin{cases} \sin(\eta x + t), & \text{at } z = 1 \\ \gamma \sin(\eta x + t - \phi), & \text{at } z = 0 \end{cases} \tag{3.2.5}$$

### 3.2.1 Leading order flow

The leading order term $u_0(x, t)$ represents a Stokes flow subject to transpiration boundary conditions. In order to find a closed-form solution for this leading order flow, we introduce the following assumed form, which is based upon the boundary condition at the top wall:

$$\Psi_0 = \xi_{0,1}(z) e^{i(\eta x + t)} + \xi_{0,-1}(z) e^{-i(\eta x + t)}, \tag{3.2.6}$$

where $\xi_{0,1}(z)$ and $\xi_{0,-1}(z)$ are functions to be determined. By substituting the above into equation (3.2.2a), then equating coefficients of $e^{i(\eta x + t)}$ and $e^{-i(\eta x + t)}$, we obtain the following differential equations:

$$\frac{d^4 \xi_{0,1}}{dz^4} - (2\eta^2 + \beta i) \frac{d^2 \xi_{0,1}}{dz^2} + \eta^2 (\eta^2 + \beta i) \xi_{0,1} = 0, \tag{3.2.7}$$

$$\frac{d^4 \xi_{0,-1}}{dz^4} - (2\eta^2 - \beta i) \frac{d^2 \xi_{0,-1}}{dz^2} + \eta^2 (\eta^2 - \beta i) \xi_{0,-1} = 0. \tag{3.2.8}$$

From the boundary conditions acting upon $\Psi_0(x, t)$, we can infer the boundary conditions for $\xi_{0,1}(z)$ and $\xi_{0,-1}(z)$. From (3.2.5), we obtain

$$\xi_{0,\pm 1} = \begin{cases} \frac{1}{2\eta}, & \text{at } z = 1 \\ \frac{\gamma}{2\eta} e^{\pm i\phi}, & \text{at } z = 0 \end{cases} \tag{3.2.9}$$
3.2. Perturbation analysis

while from (3.2.4), we obtain

\[
\frac{d\xi_{0,1}}{dz} = \frac{d\xi_{0,-1}}{dz} = 0. \quad \text{at } z = 0, 1 \tag{3.2.10}
\]

From the above equations and boundary conditions, we can derive closed form solutions,

\[
\xi_{0,1} = c_1 e^{\eta z} + c_2 e^{-\eta z} + c_3 e^{Hz} + c_4 e^{-Hz}, \tag{3.2.11}
\]

\[
\xi_{0,-1} = c^*_1 e^{\eta z} + c^*_2 e^{-\eta z} + c^*_3 e^{H^*_z} + c^*_4 e^{-H^*_z}, \tag{3.2.12}
\]

where

\[
H \equiv \sqrt{\eta^2 + \beta i}, \quad H^* \equiv \sqrt{\eta^2 - \beta i},
\]

and the various constant coefficients \(c_1, c^*_1, \ldots\) etc. are dependent upon the choice of boundary conditions (in dimensional units). They are given by

\[
c_1 = [8\eta \left( \sinh H \left( 2\eta^2 + i\beta \right) \sinh \eta + 2H\eta - 2H\eta \cosh H \cosh \eta \right)]^{-1}
\times \left\{ e^{-H-\eta-i\phi} \left[ \gamma(\eta(\eta - H) + i\beta) + e^{2H}\gamma(-\eta(H + \eta) - i\beta) 
+ (\eta(\eta - H) + i\beta)e^{2H+\eta+i\phi} - e^{2H+i\phi}(\eta(H + \eta) + i\beta) + 2H\gamma e^{H+\eta} 
+ 2H\eta e^{H+i\phi} \right] \right\}, \tag{3.2.13a}
\]

\[
c_2 = [8\eta \left( \sinh H \left( 2\eta^2 + i\beta \right) \sinh \eta + 2H\eta - 2H\eta \cosh H \cosh \eta \right)]^{-1}
\times \left\{ e^{-H-i\phi} \left[ -2e^{H+\eta} \left( H\eta \cosh H - \sinh H \left( \eta^2 + i\beta \right) \right) 
+ e^{2H+i\phi}(\eta(\eta - H) + i\beta) + e^{2H+i\phi}(\eta(H + \eta) - i\beta) + 2e^H H\gamma \eta 
+ 2H\eta e^{H+i\phi} \right] \right\}, \tag{3.2.13b}
\]

\[
c_3 = [8 \left( 2H\eta(\cosh H \cosh \eta - 1) - \sinh H \left( 2\eta^2 + i\beta \right) \sinh \eta \right)]^{-1}
\times \left\{ e^{-H-\eta-i\phi} \left[ -2H\gamma e^{H+\eta} + \gamma(H - \eta) + \gamma e^{2\eta}(H + \eta) 
- 2He^{\eta+i\phi} + (H - \eta)e^{H+2\eta+i\phi} + (H + \eta)e^{H+i\phi} \right] \right\}, \tag{3.2.13c}
\]
3.2. Perturbation analysis

We can now express \( \Psi_0(x, t) \) in a closed form. To do so, we substitute equations (3.2.11) and (3.2.12) for \( \xi_{0,1} \) and \( \xi_{0,-1} \) into equation (3.2.6). Doing so, we arrive at

\[
\Psi_0 = \left( c_1 e^{i \eta z} + c_2 e^{-i \eta z} + c_3 e^{H z} + c_4 e^{-H z} \right) e^{i(\eta_0 x + t)}
+ \left( c_1^* e^{i \eta z} + c_2^* e^{-i \eta z} + c_3^* e^{H^* z} + c_4^* e^{-H^* z} \right) e^{-i(\eta_0 x + t)}. \tag{3.2.14}
\]
Despite containing several complex terms, the above function for $\Psi_0$ is in fact real for all applicable values of the parameters $\beta$, $\eta$, $\gamma$ and $\phi$. A vector plot of an instantaneous realisation of the leading order flow, from an antiphase boundary condition, can be seen in figure 3.3. Regardless of the values of $\beta$ and $\eta$, all such leading order vector plots have the same general appearance.

From this result, we can easily determine the pressure to leading order. To do so, we substitute $u_0$ (or $\partial \Psi_0 / \partial z$) into the Stokes equation. Then, by assuming that the pressure exhibits the same streamwise scales of motion as the velocity field, we find that

$$p_0(x,t) = \beta \left( c_2 e^{-\eta z} - c_1 e^{\eta z} \right) e^{i(\eta x + t)} + \beta \left( c_2^* e^{-\eta z} - c_1^* e^{\eta z} \right) e^{-i(\eta x + t)}. \quad (3.2.15)$$

### 3.2.2 First order correction to the flow

Here we derive the first order correction to the bulk flow, $\langle u_1 \rangle$. In order to determine the first order correction to the streamfunction, $\Psi_1$, we first substitute equation (3.2.14) into equation (3.2.2b). This produces an equation of the form

$$\nabla^4 \Psi_1 - \beta \frac{\partial}{\partial t} \nabla^2 \Psi_1 = Z_{1,0}(z) + Z_{1,2}(z) e^{2i(\eta x + t)} + Z_{1,-2}(z) e^{-2i(\eta x + t)}, \quad (3.2.16)$$
where \( Z_{1,0}, Z_{1,2} \) and \( Z_{1,-2} \) are known functions, which depend upon \( \xi_{0,1} \) and \( \xi_{0,-1} \). From this we can infer that \( \Psi_1 \) takes the form

\[
\Psi_1 = \xi_{1,0}(z) + \xi_{1,2}(z)e^{2i(\eta z + t)} + \xi_{1,-2}(z)e^{-2i(\eta z + t)},
\]

(3.2.17)

where \( \xi_{1,0}, \xi_{1,2} \) and \( \xi_{1,-2} \) are functions to be determined. The functions \( \xi_{1,2}(z) \) and \( \xi_{1,-2}(z) \) can be determined by substituting the above into equation (3.2.16) and equating coefficients of the exponentials. This results in the following,

\[
\frac{d^4 \xi_{1,\pm2}}{dz^4} - 2(4\eta^2 \pm \beta i) \frac{d^2 \xi_{1,\pm2}}{dz^2} + 8\eta^2(2\eta^2 \pm \beta i) \xi_{1,\pm2} = Z_{1,\pm2},
\]

(3.2.18)

with the boundary conditions,

\[
\xi_{1,\pm2}(0) = \xi_{1,\pm2}(1) = \frac{d\xi_{1,\pm2}}{dz}(0) = \frac{d\xi_{1,\pm2}}{dz}(1) = 0.
\]

(3.2.19)

It is not possible to determine the translational flow term, \( \xi_{1,0}(z) \), via an analogous method, since it would lead to a fourth order differential equation with only two boundary conditions. Instead, we consider the Navier-Stokes equation expanded in \( \alpha \). The first order components of the expansion in the streamwise direction are given by

\[
\beta \frac{\partial}{\partial t} u_1 + \beta u_0 \cdot \nabla u_0 = -\frac{\partial}{\partial x} p_1 + \nabla^2 u_1.
\]

(3.2.20)

From equation (3.2.17) we can infer that the streamwise velocity must take the following form,

\[
u_1 = U_{1,0}(z) + U_{1,2}(z)e^{2i(\eta z + t)} + U_{1,-2}(z)e^{-2i(\eta z + t)},
\]

(3.2.21)

where \( U_{1,0}, U_{1,2} \) and \( U_{1,-2} \) are functions to be determined. By substituting the above into equation (3.2.20), and then averaging the result over the streamwise direction, we obtain

\[
\frac{d^2}{dz^2} \overline{u_1} = \frac{d^2}{dz^2} U_{1,0} = \beta u_0 \cdot \nabla u_0 \equiv \beta \left( \frac{\partial \Psi_0}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \Psi_0}{\partial x} \frac{\partial}{\partial z} \right) \frac{\partial \Psi_0}{\partial z}.
\]

(3.2.22)

By substituting equation (3.2.14) into the above, we obtain

\[
\frac{d^2}{dz^2} \overline{u_1} = -2\beta \eta \left[ c_3 c_1^* e^{(\eta + H)z} + c_4 c_1^* e^{(\eta - H)z} + c_3 c_2^* e^{-2(\eta + H)z} + c_4 c_2^* e^{-2(\eta + H)z}
\right.
\]

\[
+ c_1 c_3^* e^{(\eta + H^*)z} + c_1 c_4^* e^{(\eta - H^*)z} + c_2 c_3^* e^{(-\eta + H^*)z} + c_2 c_4^* e^{(-\eta + H^*)z}
\]

\[
+ 2c_3 c_3^* e^{(H + H^*)z} + 2c_3 c_4^* e^{(H - H^*)z} + 2c_4 c_3^* e^{(-H + H^*)z}
\]

\[
+ 2c_4 c_4^* e^{-(H + H^*)z} \right].
\]

(3.2.23)
By solving the above equation, subject to the no-slip boundary condition, we can determine $\bar{u}_1$. Its value is given by

$$
\bar{u}_1 = \beta^2 \eta \left\{ \frac{c_3 e_1^*}{(\eta + H)^2} \left[ 1 - e^{(\eta + H)z} + z \left( e^{(\eta + H)} - 1 \right) \right] \\
+ \frac{c_4 e_1^*}{(\eta - H)^2} \left[ 1 - e^{(\eta - H)z} + z \left( e^{(\eta - H)} - 1 \right) \right] \\
+ \frac{c_3 e_2^*}{(\eta - H)^2} \left[ 1 - e^{(-\eta + H)z} + z \left( e^{(-\eta + H)} - 1 \right) \right] \\
+ \frac{c_4 e_2^*}{(\eta + H)^2} \left[ 1 - e^{(-\eta - H)z} + z \left( e^{(-\eta - H)} - 1 \right) \right] \\
+ \frac{c_1 e_3^*}{(\eta + H^*)^2} \left[ 1 - e^{(\eta + H^*)z} + z \left( e^{(\eta + H^*)} - 1 \right) \right] \\
+ \frac{c_1 e_4^*}{(\eta - H^*)^2} \left[ 1 - e^{(\eta - H^*)z} + z \left( e^{(\eta - H^*)} - 1 \right) \right] \\
+ \frac{c_2 e_3^*}{(\eta - H^*)^2} \left[ 1 - e^{(-\eta + H^*)z} + z \left( e^{(-\eta + H^*)} - 1 \right) \right] \\
+ \frac{c_2 e_4^*}{(\eta + H^*)^2} \left[ 1 - e^{(-\eta - H^*)z} + z \left( e^{(-\eta - H^*)} - 1 \right) \right] \\
+ \frac{2c_3 e_3^*}{(H + H^*)^2} \left[ 1 - e^{(H + H^*)z} + z \left( e^{(H + H^*)} - 1 \right) \right] \\
+ \frac{2c_3 e_4^*}{(H - H^*)^2} \left[ 1 - e^{(H - H^*)z} + z \left( e^{(H - H^*)} - 1 \right) \right] \\
+ \frac{2c_4 e_3^*}{(H - H^*)^2} \left[ 1 - e^{(-H + H^*)z} + z \left( e^{(-H + H^*)} - 1 \right) \right] \\
+ \frac{2c_4 e_4^*}{(H + H^*)^2} \left[ 1 - e^{(-H - H^*)z} + z \left( e^{(-H - H^*)} - 1 \right) \right] \right\}. \quad (3.2.24)
$$
In order to determine the corresponding first order approximation of the overall average \( \langle u_1 \rangle \), we integrate \( \bar{u}_1 \) with respect to \( z \) from 0 to 1. This results in

\[
\langle u_1 \rangle = \frac{\beta^2 \eta}{2} \left\{ \frac{c_3 c_1^*}{(\eta + H)^3} \left[ 2 + \eta + H + e^{\eta + H} (\eta + H - 2) \right] \\
+ \frac{c_4 c_1^*}{(\eta - H)^3} \left[ 2 + \eta - H + e^{\eta - H} (\eta - H - 2) \right] \\
- \frac{c_3 c_2^*}{(\eta - H)^3} \left[ 2 - \eta + H + e^{-\eta + H} (\eta + H - 2) \right] \\
- \frac{c_4 c_2^*}{(\eta + H)^3} \left[ 2 - \eta - H - e^{-\eta - H} (\eta + H + 2) \right] \\
+ \frac{c_1 c_3^*}{(\eta + H^*)^3} \left[ 2 + \eta + H^* + e^{H + H^*} (\eta + H^* - 2) \right] \\
+ \frac{c_1 c_4^*}{(\eta - H^*)^3} \left[ 2 + \eta - H^* + e^{H - H^*} (\eta - H^* - 2) \right] \\
- \frac{c_2 c_3^*}{(\eta - H^*)^3} \left[ 2 - \eta + H^* + e^{-\eta + H^*} (\eta + H^* - 2) \right] \\
- \frac{c_2 c_4^*}{(\eta + H^*)^3} \left[ 2 - \eta - H^* - e^{H^*} (\eta + H^* - 2) \right] \\
+ \frac{c_3 c_3^*}{(H + H^*)^3} \left[ 2 + H + H^* + e^{H + H^*} (H + H^* - 2) \right] \right\} .
\]

Plots of \( \langle u_1 \rangle / \beta \) can be seen in figures 3.4, 3.5 & 3.6. Here we have plotted \( \langle u_1 \rangle / \beta \), rather than \( \langle u_1 \rangle \), since we are most interested in determining the bulk flow induced for a particular amplitude of the boundary condition (that is, for a particular \( Re \)). From the definition of the convection parameter \( \alpha \), given in equation (3.1.10), we can see that an expansion of the velocity in \( \alpha \) must be divided by \( \beta \) to become the equivalent expansion in \( Re \).

If we were to plot \( \langle u_1 \rangle \), we would find that it increases monotonically with the Stokes number, \( \beta \), potentially giving the misleading impression that increasing \( \beta \)
3.2. Perturbation analysis

Figure 3.4: Log-log plots of $\langle u_1 \rangle / \beta$ against the Stokes number, $\beta = \rho h^2 \omega / \mu$, for several values of the ratio of the channel height to the boundary condition wavelength, $\eta = h / \lambda$. **Top left:** ‘Mixed’ boundary conditions. **Top right:** In-phase boundary conditions. **Bottom left:** Out-of-phase boundary conditions. **Bottom right:** Antiphase boundary conditions.
3.2. Perturbation analysis

Figure 3.5: Log-log plots of $\langle u_1 \rangle / \beta$ against $\eta$ ($= h/\lambda$) for several values of $\beta$ ($= \rho h^2 \omega / \mu$). **Top left:** ‘Mixed’ boundary conditions. **Top right:** In-phase boundary conditions. **Bottom left:** Out-of-phase boundary conditions. **Bottom right:** Antiphase boundary conditions.
3.2. Perturbation analysis

![Contour plots](figure)

Figure 3.6: Contour plots of $\langle u_1 \rangle/\beta$ against $\beta$ ($= \rho h^2 \omega/\mu$) and $\eta$ ($= h/\lambda$). **Top left:** ‘Mixed’ boundary conditions. **Top right:** In-phase boundary conditions. **Bottom left:** Out-of-phase boundary conditions. **Bottom right:** Antiphase boundary conditions.
3.2. Perturbation analysis

invariably increases the induced bulk flow. This is, however, an illusion caused by the fact that increasing $\beta$ while holding $\alpha$ constant implies commensurately increasing $Re$.

We are able to approximate the bulk flow induced via transpiration as

$$\langle u \rangle = \alpha \langle u_1 \rangle + O(\alpha^3). \quad (3.2.26)$$

Note that the error term in the above equation is of order $\alpha^3$. It can easily be verified that the next order term in the velocity expansion, $u_2(x,t)$, contains only swirling, rather than translational flow, by substituting equations (3.2.6) and (3.2.17) into (3.2.2c). (In their analysis of flows driven solely by transpiration, Min et al. (2006) employed values of $\alpha$ ranging from $\alpha = 0.0025$ to $\alpha = 0.075$.)

Equation (3.2.26) appears to imply that the bulk flow increases linearly with the amplitude of the boundary condition, $A$. However, it is important to note that because the velocity has been scaled via $A$, as defined in equation (3.1.5), the dimensional velocity will in fact be proportional to $A^2$.

3.2.3 First order velocity field

In order that we should be able to produce an approximation of the entire velocity field to first order, we now seek to determine the functions $\xi_{1,2}(z)$ and $\xi_{1,-2}(z)$ from (3.2.17). We begin with the derivation of $\xi_{1,2}$. By substituting equation (3.2.14) into equation (3.2.2b), and equating coefficients of $e^{2i(\eta x+t)}$ in the result to that in equation (3.2.16), we find that

$$Z_{1,2} = \beta^2 \eta \left[ c_1 c_3 (H - \eta) e^{(\eta + H)z} - c_1 c_4 (\eta + H) e^{(\eta - H)z} 
+ c_2 c_3 (\eta + H) e^{(H - \eta)z} + c_2 c_4 (\eta - H) e^{-(\eta + H)z} \right]. \quad (3.2.27)$$

We can now determine $\xi_{1,2}$ by substituting the above into equation (3.2.18) and solving for the no-slip boundary conditions given within equation (3.2.19). This results in

$$\xi_{1,2} = k_1 e^{(\eta + H)z} + k_2 e^{(\eta - H)z} + k_3 e^{(H - \eta)z} + k_4 e^{-(\eta + H)z} 
+ k_5 e^{2\eta z} + k_6 e^{-2\eta z} + k_7 e^{\sqrt{4\eta^2 + 2\beta^2}z} + k_8 e^{-\sqrt{4\eta^2 + 2\beta^2}z}, \quad (3.2.28)$$
3.2. Perturbation analysis

where the constants $k_1, k_2, \ldots$, etc. depend only upon the values of $\beta$, $\eta$, $\gamma$ and $\phi$. Because the explicit forms of several of these constants are very cumbersome, and moreover physically unrevealing, we do not reproduce them here. We now similarly seek to derive $\xi_{1,-2}$. We again substitute equation (3.2.14) into equation (3.2.2b), this time equating the resulting coefficient of $e^{-2i(\eta x + t)}$ to that in equation (3.2.16). This gives

$$ Z_{1,-2} = \beta^2 \eta \left[ c_1^* c_3^* (\eta - H^*) e^{(\eta + H^*)z} - c_1^* c_4^* (\eta + H^*) e^{(\eta - H^*)z} ight. $$

$$ + \left. c_2^* c_3^* (\eta + H^*) e^{(H^* - \eta)z} + c_2^* c_4^* (\eta - H^*) e^{-(\eta + H^*)z} \right]. \quad (3.2.29) $$

We can now determine $\xi_{1,-2}$ via an analogous method to that used above to determine $\xi_{1,2}$. This yields

$$ \xi_{1,-2} = k_1^* e^{(\eta + H^*)z} + k_2^* e^{(\eta - H^*)z} + k_3^* e^{(H^* - \eta)z} + k_4^* e^{-(\eta + H^*)z} $$

$$ + k_5^* e^{2\eta z} + k_6^* e^{-2\eta z} + k_7^* e^{\sqrt{4\eta^2 - 2\beta iz}} + k_8^* e^{-\sqrt{4\eta^2 - 2\beta iz}}, \quad (3.2.30) $$

where similarly the constants $k_1^*, k_2^*, \ldots$, etc. depend on the values of $\beta$, $\eta$, $\gamma$ and $\phi$, and are too cumbersome to be worth reproducing here. Plots of the first order velocity field can be seen in figure 3.7.

3.2.4 Boundary conditions of long wavelength

In all results bar the in-phase case, it can be clearly seen that $\langle u_1 \rangle$ is maximised by minimising $\eta$ (the ratio of the channel height to the wavelength of the boundary condition). The asymptotic behaviour of $\langle u_1 \rangle$ in the small $\eta$ limit is given by

$$ \langle u_1 \rangle \sim \frac{\gamma^2 - 2\gamma \cos \phi + 1}{\eta} f_1(\beta), \quad (3.2.31) $$
Figure 3.7: Plots of instantaneous streamlines of $u_1(x, t)/\beta$, the first order correction of the flow for $\eta = 2$ and several values of $\beta$, using an antiphase boundary condition ($\gamma = 1, \phi = \pi$).
where,

\[
f_1(\beta) \equiv \left[ (10-\beta) \sinh\left(\frac{\sqrt{i\beta}}{2}\right) \cosh\left(\frac{\sqrt{i\beta}}{2}\right) + \cosh\left(\frac{\sqrt{-i\beta}}{2}\right) \times \beta \left( (\beta - 10i) \sinh\left(\frac{\sqrt{i\beta}}{2}\right) + 10i\sqrt{i\beta} \cosh\left(\frac{\sqrt{i\beta}}{2}\right) \right) \right]
\]

\[
\times \sinh\left(\frac{\sqrt{-i\beta}}{2}\right) \sinh\left(\frac{\sqrt{i\beta}}{2}\right) / \left[ 2 \left( -\sqrt{i\beta} \sinh\left(\sqrt{i\beta}\right) + 2 \sqrt{-i\beta} - 2 \sqrt{-i\beta} \sinh\left(\sqrt{-i\beta}\right) \right) \left( -2\sqrt{-i\beta} + i\beta \sinh\left(\sqrt{-i\beta}\right) + 2\sqrt{-i\beta} \cosh\left(\sqrt{-i\beta}\right) \right) \right].
\]

(3.2.32)

The above equation clearly demonstrates that the antiphase boundary condition induces the greatest bulk flow for \( \eta < 1 \). This concurs with the findings of Mamori et al. (2010), who reported that the antiphase boundary condition produced significantly greater drag reduction than the in-phase case, for an upstream travelling wave acting upon a laminar Poiseuille flow.

By inspection of figures 3.5 and 3.6, it can be verified that equation (3.2.31) is a good approximation, in the ‘mixed’, out-of-phase and antiphase cases, for values of \( \eta \) less than unity (in other words, all systems in which the wavelength of the boundary condition is greater than the height of the channel). In fact, it is a good approximation, at such values of \( \eta \), for all boundary conditions, except for those that are similar to the in-phase case (i.e. those in which \( 2\gamma \cos \phi - \gamma^2 \approx 1 \)). If we further consider the limiting cases in which the Stokes number, \( \beta \), approaches zero or infinity, we find that

\[
\langle u_1 \rangle \sim \begin{cases} 
\frac{1 + \gamma^2 - 2\gamma \cos \phi \beta^2}{5040 \eta}, & \eta \to 0, \ \beta \to 0 \\
\frac{1 + \gamma^2 - 2\gamma \cos \phi \sqrt{\beta}}{2\sqrt{2} \eta}, & \eta \to 0, \ \beta \to \infty
\end{cases}
\]

(3.2.33)

From equation (3.2.31) it can be determined that the optimal value of \( \beta \) (i.e. that which maximises \( \langle u_1 \rangle / \beta \)) is given by

\[
\beta_{\text{max}} \approx 107.
\]

(3.2.34)
3.2. Perturbation analysis

This of course implies that there exists an optimal frequency, beyond which any further increase to the frequency will result in a reduced bulk flow. This concurs with previous findings by Min et al. (2006), Hœpffner & Fukagata (2009), Mamori et al. (2010) and Moarref & Jovanović (2010).

It might be expected that the ‘mixed’ boundary condition should always induce the least bulk flow, since it involves transpiration acting at only one wall. However, while it does invariably induce less bulk flow than the out-of-phase and antiphase cases, it can be clearly seen that for \( \eta < 1 \), it is by far the in-phase case that is the least effective in inducing a bulk flow. The reason for this is that to reduce \( \eta \) is equivalent to reducing the height of the channel (it follows from the definition of \( \eta \) that this is also equivalent to increasing the wavelength of the boundary conditions). Because the velocity gradients at either wall are defined to be equal in the in-phase case, a short channel contains very little space for velocity gradients to develop, within the fluid. Where the velocity gradients are small, the effect of convection will also be small.

3.2.5 Boundary conditions of short wavelength

The flow is qualitatively different at large \( \eta \), where the height of the channel is significantly greater than the wavelength of the boundary condition. If we consider the limiting case in which \( \eta \to \infty \), it is clear, by inspection of equation (3.2.13), that in this limit, all of \( c_1, c_1^*, \ldots, \) etc. are zero for all varieties of boundary conditions. From this, we may infer that for all boundary conditions,

\[
\lim_{\eta \to \infty} \langle u \rangle = 0. \tag{3.2.35}
\]

This is an intuitive result, since the greater the relative height of the channel to the wavelength of the boundary condition, the more likely that at any point within the flow, the effects of the adjacent peaks and troughs of the boundary condition will combine, resulting in a flow containing only small velocity gradients.

Similarly, as \( \eta \) increases, the dependence of \( \langle u_1 \rangle \) upon the phase difference, \( \phi \), generally decreases. By inspection, it is clear that for values of \( \eta \) greater than around 10, the bulk flow is effectively independent of \( \phi \). This is due to the fact that as the height of the channel increases, the effect of the boundary condition at one wall upon the flow near to the opposite wall decreases. As a result, the effect
of the phase difference becomes negligible in the large \( \eta \) limit. The asymptotic behaviour of the bulk flow in this large \( \eta \) limit is given by

\[
\langle u_1 \rangle \sim \begin{cases} 
\frac{3(1 + \gamma^2) \beta^2}{64 \eta^3}, & \eta \to \infty, \frac{\beta}{\eta^2} \to 0 \\
\frac{1 + \gamma^2}{4\sqrt{2}} \sqrt{\beta}, & \eta \to \infty, \frac{\beta}{\eta^2} \to \infty
\end{cases}
\]  

(3.2.36)

### 3.2.6 Dependence on the frequency of oscillation

The Stokes number, \( \beta \), defined in equation (3.1.8), is proportional to the frequency of oscillation of the boundary condition. Hence, by considering the effect of \( \beta \) upon \( \langle u_1 \rangle / \beta \), we can see the effect of the frequency upon the bulk flow.

Plots of the first order correction to the flow, \( \langle u_1 \rangle / \beta \), for \( \eta = 2 \), can be seen in figure 3.7. There it can be clearly seen that for \( \beta < 1 \), the first order correction to the flow tends to consist largely of swirling, rather than translational motion, while for \( \beta > 1 \), the first order flow is primarily translational. In fact, even for very large values of \( \beta \), the swirling motions within the first order correction are negligible in comparison to the streamwise translational motion. This is despite the fact that, as can be seen in figure 3.4, the translational motion is itself minute at such values of \( \beta \).

In fact, a notable result is found in the limit as \( \beta \to \infty \), with \( Re \) held constant. This corresponds to a system in which the boundary condition is travelling very rapidly along the wall of the channel. In this limit, the streamfunction asymptotes to

\[
\Psi \sim \frac{\operatorname{csch} \eta}{\eta} [ \sinh(\eta z) \cos(\eta x + t) + \gamma \sinh(\eta(1 - z)) \cos(\eta x + t - \phi) ], \text{ as } \beta \to \infty.
\]

(3.2.37)

Note that the above equation contains the total streamfunction, \( \Psi(x, t) \), rather than merely its leading order term, \( \Psi_0(x, t) \). This conclusive result follows from the fact that in this limit, the constants \( c_3, c'_3, c_4 \) and \( c'_4 \) are zero. In fact, they approach zero sufficiently rapidly that the right hand side of equation (3.2.23) is zero. By inspection of equation (3.2.25), it is clear that this implies that

\[
\lim_{\beta \to \infty} \langle u \rangle = 0, \quad Re = \text{constant}
\]

(3.2.38)
They also approach zero sufficiently rapidly that
\[ \Psi_1(x, t) = 0. \quad (3.2.39) \]

It is clear from equation (3.2.2c) and the perturbation methodology that in this limit
\[ \Psi_n(x, t) = 0, \quad \text{for all } n > 0. \quad (3.2.40) \]

In other words, as \( \beta \) dominates \( Re \), the effect of convection becomes negligible, and the entire flow will be of the same streamwise scale of motion as the boundary condition. (In this sense, the flow at large \( \beta \) resembles the flow at small \( Re \).

The most notable aspect of equation (3.2.37) is that it does not satisfy the no-slip boundary condition (3.1.14). It may appear a paradoxical result that the solution to the Navier-Stokes and continuity equations, subject to transpiration, asymptotes toward a velocity field which does not satisfy its own boundary conditions. This result can be explained, however, by analogy with the motion of a fluid of low viscosity adjacent to an oscillating body (Stokes, 1851). In such flows, a thin vorticity-containing boundary layer is known to form, the thickness of which decreases as the frequency of oscillation increases.

Similarly, in the case of transpiration, an irrotational region forms within the centre of the channel at high \( \beta \). The reason for this is that as \( \beta \) increases, the vorticity generated by the adjacent peaks and troughs of the boundary condition begins to cancel away from the wall. This results in vorticity being confined to thin regions near to the walls. As \( \beta \) increases further, these vorticity-containing boundary layers decrease in width, approaching an infinitesimal width in the limit as \( \beta \to \infty \). (Indeed, by inspection, it is clear that the flow represented by equation (3.2.37) is in fact entirely irrotational.) It is for this reason that the limiting behaviour of the flow as \( \beta \to \infty \) does not satisfy its own boundary conditions.

In the case of high \( Re \) flows however, no such irrotational region forms. This is because the transpiration boundary condition causes vorticity to be convected away from the wall towards the centre of the channel. The system is therefore not reliant upon diffusion to spread vorticity from the near-wall region to the centre of the channel at high \( Re \). We discuss the case of flows at high \( Re \) further in §3.4.
3.2.7 Generalised boundary conditions

It may appear that the family of boundary conditions that have been defined by equation (3.1.12) constitute a significant restriction upon the analysis presented in this work. However, as we demonstrate in this section, these results can be readily extended to any channel flow for which the functions defining $w$ at the walls can be expressed as convergent Fourier series in $\eta x + t$.

In order to explain how these results can be extended, it is necessary to formally define what we call streamwise scales of motion. Since the flow field is periodic in the $x$-direction, and with respect to time, the entire flow field can be expressed as a Fourier series in $\eta x + t$. Each component of that Fourier series will be referred to as a streamwise scale of motion (these could alternatively be called Fourier modes in $\eta x + t$). For example, equation (3.2.6) indicates that $\Psi_0(x, t)$ contains just one streamwise scale of motion, which is equal to that of the boundary condition, while equation (3.2.16) indicates that $\Psi_1(x, t)$ contains two scales of motion. One of these scales has half the wavelength of the boundary condition, and the other is translational, by which we mean that it is not periodic in $\eta x + t$.

Fundamental to the method of extending these results is the observation that it is only the convection of each scale of motion at leading order by itself which can produce a translational flow at first order. Only at second order or higher may two different streamwise scales of motion produce a translational flow. The method of extending these results to more general boundary conditions consists of expanding the boundary conditions in a Fourier series in order to re-express them as a sum of functions, each of which satisfy equation (3.1.12), with appropriate substitutions for $\beta$, $\eta$, $\gamma$ and $\phi$.

Hence, if we define a new function $U(\beta, \eta, \gamma, \phi)$ to be the first order bulk flow for a specific set of the flow parameters, so that

$$U(\beta', \eta', \gamma', \phi') \overset{\text{def}}{=} \langle u_1 \rangle; \quad \beta = \beta', \; \eta = \eta', \; \gamma = \gamma', \; \phi = \phi',$$

it follows that if the boundary condition can be expressed as

$$w = \begin{cases} \sum_{n=0}^{\infty} A_n \sin n(\eta x + t), & \text{at } z = 1 \\ \sum_{n=0}^{\infty} A_n \gamma_n \sin n(\eta x + t - \phi_n), & \text{at } z = 0 \end{cases}$$

(3.2.42)
3.2. Perturbation analysis

where each $A_n$ are constants, then the bulk flow at first order can be expressed as

$$\langle u_1 \rangle = \sum_{n=0}^{\infty} \frac{A_n}{n} U(n\beta, n\eta, \gamma_n, \phi_n).$$

(3.2.43)

For flows at low $\eta$, it should significantly reduce the complexity of the resulting function to define $U$ in terms of the low $\eta$ asymptote given in equation (3.2.31).

Note that there is a factor of $1/n$ in the above equation, which results from equation (3.2.26), and the dependence of $\alpha$ upon $\beta$. In scaling the Navier-Stokes equation, the characteristic velocity (equivalent to $A$ in this study) should be chosen to be the highest of the amplitudes of the scales of motion that are present within the boundary conditions. In that way, it can be guaranteed that for all $n$, we will have $0 \leq A_n \leq 1$. The uncertainty involved in this extended derivation will be either $O(\alpha^2)$ or $O(\alpha^3)$, depending on the scales of motion present within the boundary conditions.

The summation in equation (3.2.43) should converge, assuming that the Fourier series that defines the boundary conditions converges. This is because $\langle u_1 \rangle$ approaches zero for high values of $\beta$ and $\eta$. However, particularly at low values of $\beta$ and $\eta$, the summation in (3.2.43) may contain a large number of non-negligible terms, and may therefore converge only very slowly.

While only a subset of all possible periodic boundary conditions can be expressed in the form of equation (3.2.42), this nonetheless suggests a practical method by which the bulk flow induced by any periodic wall-normal boundary conditions may be determined: This method involves first solving for a system whose boundary conditions take the form of the combination of a sine wave and a cosine wave at both walls. For example:

$$w = \begin{cases} 
\sin(\eta x + t) + \gamma \cos(\eta x + t), & \text{at } z = 1 \\
\delta \sin(\eta x + t) + \epsilon \cos(\eta x + t), & \text{at } z = 0
\end{cases}$$

(3.2.44)

where $\gamma$, $\delta$ and $\epsilon$ are constants between $-1$ and 1. Then, $\langle u_1 \rangle$ could be determined for such a system via an analogous derivation to that by which $\langle u_1 \rangle$ has been determined in this work. It would be found that the leading order streamfunction, $\Psi_0(x, t)$, would still take the form of equation (3.2.14), with the exception that the constants $c_1$, $c_1^*$, $c_2$, etc. would take different values from those they have here,
3.3. Energy considerations

and would depend upon the parameters $\gamma$, $\delta$ and $\epsilon$. Having rederived the values of $c_1$, $c_1^*$, etc. in this way, the remainder of the derivation would proceed as before, and $\langle u_1 \rangle$ would again be given by equation (3.2.25).

Because any periodic function may be expanded as a Fourier series, it follows that if we were to define a new function $U(\beta, \eta, \gamma, \delta, \epsilon)$ in an equivalent manner to (3.2.41), and expand the boundary conditions in an analogous way to (3.2.42), the bulk flow could then be represented by the equivalent of equation (3.2.43).

3.3 Energy considerations

Marusic et al. (2007) considered the effect of transpiration in conjunction with an externally applied pressure gradient across the channel. They derived the conditions required for transpiration to produce sublaminar drag, in the presence of such an applied pressure gradient. Their formula relates the rate at which the transpiration imparts energy upon the flow to the rate at which energy is dissipated within the flow. They derived the rule that the drag will be sublaminar if and only if

$$ W > \left( |\nabla u'|^2 \right) + Re \left( \left( \overline{u'w'} - \langle u'w' \rangle \right)^2 \right), $$

(3.3.1)

where $W$ represents the rate at which energy is being imparted upon the flow via transpiration. It is given by

$$ W = \frac{1}{2} Re \left. (w')^3 + w'p' \right|_{z=0} - \frac{1}{2} Re \left. (w')^3 + w'p' \right|_{z=1}. $$

(3.3.2)

The $\frac{1}{2} (w')^3$ term represents the overall rate at which energy is input (or removed) as kinetic energy, while the $w'p'$ term represents the overall rate at which energy is transferred by flowing against (or with) a local pressure gradient. This notably excludes any energy that is imparted due to an overall cross-flow within the channel. The first term on the right hand side of (3.3.1) represents the dissipation due to fluctuations within the flow (i.e. not counting dissipation due to the wall-parallel averaged flow). The $\overline{u'w'}$ term represents the Reynolds shear stress (which is the stress acting upon the wall-parallel averaged flow due to the presence of the fluctuations).

There are factors of $Re$ in equations (3.3.1) and (3.3.2) above, which are absent from the equivalent equations in the work by Marusic et al. These result from the
3.3. Energy considerations

fact that a different scaling that has been employed in this work. The scaling used in this section is the same as that in §3.2.

It must be stressed, however, that equation (3.3.1) applies strictly to those flows in which the applied pressure gradient across the channel is non-zero. The systems we have considered in this work all have no such applied pressure gradients. The equivalent equation for a flow in the absence of an applied pressure gradient or an overall cross-flow can be found by removing all of the terms relating to time derivatives, pressure gradients or a cross-flow from equations (2.10) and (3.5) of Marusic et al. (2007), and combining the results. This leads to

$$W = \left\langle |\nabla u'|^2 \right\rangle + Re \left\langle \left( w'w' - \langle u'w' \rangle \right)^2 \right\rangle.$$  (3.3.3)

The above equation applies to all such flows, regardless of whether or not the transpiration has induced a bulk flow. The reason for this discrepancy between pressure-driven and non pressure-driven flows is that if a pressure gradient is applied in the streamwise direction of the flow, then the bulk flow will be removing energy from the system due to the pressure difference between the inlet and outlet of the channel. This removal of energy must be accounted for within the flow’s overall energy balance, and therefore flows that are subject to an applied pressure gradient are fundamentally different from flows in which the inlet and outlet are at equal pressure.

It should be noted also that for all of the flows investigated herein, the overall kinetic transfer, $\frac{1}{2} (w')^3$, at either wall will be zero. The driving energy of the flow therefore derives solely from the $w'p'$ term.

3.3.1 Energy input

We can now determine the rate at which transpiration imparts energy upon the flow by substituting equation (3.2.15), for the leading order of the pressure, into (3.3.2). This results in

$$W = \frac{\beta}{2i} \left[ \gamma (c_1 - c_2) e^{i\phi} + \gamma (c_2 - c_1^*) e^{-i\phi} + (c_1^* e^{-\eta} - c_2^* e^{-\eta}) + c_2 e^{-\eta} - c_1 e^\eta \right] \mathcal{W}_0 + O \left( \alpha^2 \right).$$  (3.3.4)
3.3. Energy considerations

Figure 3.8: Log-log plots of $W_0$, the leading order component of the rate at which energy is being supplied to the flow, against $\beta$ and $\eta$. **Left:** Plots for $\beta = 100$. **Right:** Plots for $\eta = 1$.

The first term on the right hand side is the leading order contribution to $W$, and will henceforth be denoted by $W_0$. That there is no first order correction to $W$ is clear from the streamwise scales of motion that are present at first order. Plots of $W_0$ against $\beta$ and $\eta$ can be found in figure 3.8.

For values of $\eta$ less than 1 (i.e. for systems in which the wavelength of the boundary condition is greater than the height of the channel), $W_0$ increases as $\eta$ decreases in all bar the in-phase case. This is to be expected, since the induced bulk flow also increases as $\eta$ decreases, at such values of $\eta$. The asymptotic behaviour of $W_0$, in the small $\eta$ limit, is given by

$$W_0 \sim \frac{\gamma^2 - 2\gamma \cos \phi + 1}{\eta^2} f_2(\beta), \quad (3.3.5)$$

where,

$$f_2(\beta) \equiv \beta^2 \left( \sinh \left( \frac{\sqrt{-i\beta}}{2} \right) \cosh \left( \frac{\sqrt{i\beta}}{2} \right) + i \sinh \left( \frac{\sqrt{i\beta}}{2} \right) \cosh \left( \frac{\sqrt{-i\beta}}{2} \right) \right)$$

$$\sqrt{2} \left( \sqrt{i\beta} \cosh \left( \frac{\sqrt{i\beta}}{2} \right) - 2 \sinh \left( \frac{\sqrt{i\beta}}{2} \right) \right)$$

$$\times \left( 2(-1)^{1/4} \sqrt{\beta} \sinh \left( \frac{1}{2} (-1)^{3/4} \sqrt{\beta} \right) + \beta \cosh \left( \frac{\sqrt{-i\beta}}{2} \right) \right). \quad (3.3.6)$$
3.3. Energy considerations

![Log-log plots of the leading order component of the efficiency of the pumping, against β and η. Left: Plots for β = 100. Right: Plots for η = 1.](image)

Figure 3.9: Log-log plots of the leading order component of the efficiency of the pumping, against β and η. **Left:** Plots for β = 100. **Right:** Plots for η = 1.

If we compare the above to equation (3.2.31), for the asymptotic behaviour of \( \langle u_1 \rangle \) in the low η limit, we find that although the bulk flow is maximised by minimising η, the energy cost of producing that flow increases at an even greater rate. We therefore have a diminishing return, in terms of induced bulk flow, for the energy input, as we increase the wavelength of the boundary condition. Moarref & Jovanović (2010), in their studies of Poiseuille flows subjected to transpiration, also found that transpiration became less energy efficient at longer wavelength, despite the fact that the drag-reduction induced by transpiration increases with the wavelength. They also similarly found that the energy cost increases with the frequency of the boundary condition (or equivalently with β), despite the fact that the induced bulk flow is negligible at large β.

Also noteworthy is the fact that at high η and at high β, \( W_0 \) is large. This is despite the fact that at such values of η and β, the bulk flow induced is negligible. This has also been found to be true for Poiseuille flows subjected to transpiration, in the work by Moarref & Jovanović (2010).

The efficiency of transpiration as a pumping mechanism can be defined as

\[
Efficiency = \frac{\langle u \rangle}{W} = \frac{\langle u_1 \rangle}{W_0} + O(\alpha^2).
\]

The leading order of the efficiency can be seen in figure 3.9. Notably, higher
3.4 Boundary conditions of large amplitude

In this section, we consider flows in which the maximum amplitude of the wall-normal velocity at the boundaries is very high in comparison to the speed at which the boundary condition moves along the wall of the channel. Specifically, we consider the mathematical limit as \( \lambda \omega/A \to 0 \), under the assumption that the flow remains two-dimensional. These amount to flows in which the effect of the time derivative is negligible in comparison to that of convection. Although transpiration at large amplitude has been found to be very energy inefficient, and therefore of little practical use (Moarref & Jovanović, 2010), it is considered here from a theoretical perspective for completeness.

We show that, in the limit as \( \lambda \omega/A \to 0 \), the bulk flow induced will become independent of \( A \), depending instead only upon \( \lambda \omega \) (the speed of the boundary condition).

In this section, the Navier-Stokes equation is written

\[
Re_{\lambda \omega} \frac{\partial}{\partial t} \mathbf{u} + Re_A \mathbf{u} \cdot \nabla \mathbf{u} = -Re_A \nabla p + \nabla^2 \mathbf{u}. \tag{3.4.1}
\]

The continuity equation (3.1.7) applies as before. A different scaling have been employed here from that used in the preceding sections. The reason for the change of scaling is because we will subsequently make use of a change of the inertial reference frame from which the flow is observed, and it is therefore preferable that the scaling should be independent of the reference frame.

Here we have used as our length scale the wavelength of the boundary condition, \( \lambda \), rather than the height of the channel, \( h \). In this section, the variables within the Navier-Stokes equation are scaled according to

\[
\dot{x} = \lambda \dot{x}, \quad \dot{\mathbf{u}} = A \dot{\mathbf{u}}, \quad \dot{p} = \rho A^2 \dot{p}, \quad \dot{t} = \frac{1}{\omega} t. \tag{3.4.2}
\]

Equation (3.4.1) contains two different Reynolds numbers, \( Re_A \) and \( Re_{\lambda \omega} \). Here \( Re_A \) is based upon the amplitude of the boundary condition, while \( Re_{\lambda \omega} \) is based
3.4. Boundary conditions of large amplitude

upon the speed at which the boundary condition moves along the channel. They have been defined

\[ \text{Re}_A = \frac{\rho \lambda A}{\mu}, \quad \text{Re}_{\gamma \omega} = \frac{\rho \lambda^2 \omega}{\mu}. \]  

(3.4.3)

Notice that \( \text{Re}_A \) is analogous to the Reynolds number \( Re \), that has been used in the preceding sections, while \( \text{Re}_{\gamma \omega} \) is analogous to \( \beta/\eta \).

We consider again the same set of possible boundary conditions as before, those defined by equations (3.1.12) and (3.1.14). When scaled according to (3.4.2), these boundary conditions become

\[ u = 0, \quad z = 0, \quad \eta \]  

(3.4.4)

\[ w = \begin{cases} \sin(x + t), & \text{at } z = \eta \\ \gamma \sin(x + t - \phi), & \text{at } z = 0 \end{cases} \]  

(3.4.5)

Here we introduce a new inertial reference frame defined such that the observer is travelling along the channel at the same speed and in the same direction as the boundary condition. If \( X \) denotes the streamwise position in this new reference frame, then it is given by

\[ X = x + t. \]  

(3.4.6)

A new position vector \( \mathbf{X} \) is introduced for use in this new reference frame. It is defined by

\[ \mathbf{X} \overset{\text{def}}{=} (X, z) \]  

(3.4.7)

The velocity in this new frame will be denoted by \( \mathbf{U}(\mathbf{X}, t) \). It has streamwise component \( U \), and wall-normal component \( W \). The boundary conditions in this new frame are given by

\[ U = \frac{\text{Re}_{\gamma \omega}}{\text{Re}_A}, \quad z = 0, \quad \eta \]  

(3.4.8)

\[ W = \begin{cases} \sin X, & \text{at } z = \eta \\ \gamma \sin(X - \phi), & \text{at } z = 0 \end{cases} \]  

(3.4.9)

We shall denote the pressure within this new frame by \( P \). It is clear that the relationship between the bulk flows in two frames will be simply

\[ \langle U \rangle - \langle u \rangle = \frac{\text{Re}_{\gamma \omega}}{\text{Re}_A} \equiv \frac{\lambda \omega}{A}. \]  

(3.4.10)
3.4. Boundary conditions of large amplitude

We convert the Navier-Stokes equation to this new frame by substituting equation (3.4.6) into (3.4.1). This results in

\[ Re_{\lambda\omega} \frac{\partial}{\partial X} \mathbf{U} + Re_A \mathbf{U} \cdot \vec{\nabla} \mathbf{U} = -Re_A \vec{\nabla} P + \vec{\nabla}^2 \mathbf{U}, \tag{3.4.11} \]

where \( \vec{\nabla} \) is the gradient vector in the new frame, and is defined by

\[ \vec{\nabla} \overset{\text{def}}{=} \left( \frac{\partial}{\partial X}, \frac{\partial}{\partial z} \right). \tag{3.4.12} \]

It is therefore clear that if \( Re_A \) is sufficiently greater than \( Re_{\lambda\omega} \), the effect of convection will outweigh the effect of the time derivative in the Navier-Stokes equation. Moreover, we can see in equation (3.4.11) that the time derivative will be negligible if

\[ Re_A \gg Re_{\lambda\omega}. \tag{3.4.13a} \]

The above equation is in fact a comparison of velocity scales, specifying that the maximum amplitude of the boundary condition should be much greater than the speed at which it travels along the wall of the channel. Indeed, in expressing the above dimensionless numbers in terms of the quantities through they have been defined in equation (3.4.3), the above inequality becomes

\[ A \gg \lambda\omega. \tag{3.4.13b} \]

We therefore expand \( u(x,t) \) and \( U(X,t) \) as follows:

\[ u = u_0 + \frac{\lambda\omega}{A} u_1 + \left( \frac{\lambda\omega}{A} \right)^2 u_2 + \ldots \tag{3.4.14a} \]

\[ U = U_0 + \frac{\lambda\omega}{A} U_1 + \left( \frac{\lambda\omega}{A} \right)^2 U_2 + \ldots \tag{3.4.14b} \]

We expand the pressure in the same manner. By substituting the above into the Navier-Stokes equation (3.4.1), and equating coefficients of orders of \( \lambda\omega/A \), we obtain

\[ U_0 \cdot \vec{\nabla} U_0 = -\vec{\nabla} P_0 + \frac{1}{Re_A} \vec{\nabla}^2 U_0. \tag{3.4.15} \]

The viscous term survives, regardless of the magnitude of \( Re_A \), due to the necessity of having a region of low streamwise velocity adjacent to either wall, in order
to satisfy the no-slip boundary condition. The boundary conditions acting upon $U_0(X,t)$ are

$$U_0 = 0, \quad z = 0, \eta$$

(3.4.16)

$$W_0 = \begin{cases} \sin X, & \text{at } z = \eta \\ \gamma \sin(X - \phi), & \text{at } z = 0 \end{cases}$$

(3.4.17)

Clearly, therefore, the system in the limit as $\lambda \omega / A \to 0$ is mathematically equivalent to a flow in which the boundary condition is stationary. If we consider a system in which the boundary condition is stationary, then due to the symmetry of such a system, there could be no bulk flow, since there would be no preferential direction in which it could flow. This implies simply that

$$\langle U_0 \rangle = 0.$$  

(3.4.18)

The magnitude of the bulk flow, in the new frame, will therefore be of the same order as the second term in the expansion of $U(X,t)$ given in (3.4.14b). We therefore conclude that

$$\langle U \rangle \leq O \left( \frac{\lambda \omega}{A} \right), \quad \text{as } \frac{\lambda \omega}{A} \to 0$$

(3.4.19)

From the relationship between the bulk flows in the two frames, given in equation (3.4.10), we can therefore see that

$$\langle u \rangle \leq O \left( \frac{\lambda \omega}{A} \right), \quad \text{as } \frac{\lambda \omega}{A} \to 0$$

(3.4.20)

The factor of $A$ in the denominator above results from the velocity scaling, as defined in equation (3.4.2). In terms of the dimensional velocity, $\hat{v}$, this result becomes

$$\langle \hat{u} \rangle \leq O \left( \frac{\lambda \omega}{A} \right), \quad \text{as } \frac{\lambda \omega}{A} \to 0$$

(3.4.21)

This proof is reliant upon there being a single unique solution to the Navier-Stokes equation, since otherwise there would be no need for the flow to have a preferential direction, and hence the symmetry argument used here would not hold. While there exists no accepted proof that the full three-dimensional Navier-Stokes equation exhibits a unique solution, the uniqueness of solutions for two-dimensional flows, for initial boundary valued problems on bounded domains, has previously
been shown by Ladyzhenskaya (1958, 1963). (That Ladyzhenskaya’s proof applies to problems defined on bounded domains does not prevent it from being applicable here, since, although the channel is of infinite length, it is periodic in the streamwise direction. By exploiting the periodicity of the system, therefore, the domain can be split into an infinite number of identical sub-domains, each of finite streamwise length.)

The importance of this proof, from a practical perspective, is that it shows that to increase \( A \) will not necessarily result in an increased bulk flow. We do not know whether there is in fact a non-zero bulk flow in this limit, or its direction. We have proved this for the family of boundary conditions that are defined by equation (3.1.12). However, since the proof relies upon the inherent symmetry of a sine wave, it cannot be easily extended to a more generalised set of boundary conditions. By inspection, there are two types of boundary conditions to which this proof can readily be extended: The first is those in which the function defining the boundary condition at the bottom wall, \( w^- (x, z, t) \), is related to its counterpart at the top wall, \( w^+ (x, z, t) \), via

\[
    w^+ (x, z, t) = -w^- (-x + \phi, z, t). \tag{3.4.22}
\]

Notice that the parameter \( \phi \) has been included above to indicate that the two functions may be out of phase to some degree. The second case is those in which \( w = 0 \) at one wall and the boundary condition at the other wall is symmetric in the streamwise direction. In both cases, the boundary conditions must be independent of the spanwise location, in order that the flow should remain two dimensional.

One possible explanation for the diminishing increase in the bulk flow with \( Re_A / Re_{\lambda \omega} \) follows from the effect of convection upon the scales of motion present within the flow. As has been shown in §3.2, if the convection parameter \( \alpha \) is small, only the larger scales of motion will be non-negligible. However, as \( \alpha \) increases, so too will the relative magnitude of the smaller scales of motion. These smaller scales of motion subsist by drawing energy and momentum from the scales above them. The quantity \( Re_A / Re_{\lambda \omega} \) defines the relative importance of convection within the flow, and hence is analogous to \( \alpha \), within the present scaling. It follows, therefore, that at high \( Re_A / Re_{\lambda \omega} \), the momentum that is imparted to the flow via transpiration at the walls will be readily convected down to the smaller scales of motion.
As has been demonstrated in this work, the effect of convection can also be to produce motions of higher scale from motions of lower scale. However, the results presented herein also demonstrate that the larger the scale of motion of the boundary condition (in other words, the lower the value of $\theta$), the greater will be its general tendency to transfer its momentum to a bulk flow. It is likely to be generally true that the smaller the scale of motion, the lesser the tendency for convection to transfer its momentum to a bulk flow. Hence, the increasing prevalence of the smaller scales of motion at high $\text{Re}_A/\text{Re}_\lambda\omega$ could potentially explain this result.

Notably, in equation (3.4.21), there is no dependence of the dimensional average streamwise velocity, $\langle \hat{u} \rangle$, upon the density or viscosity of the fluid. This can be explained by the fact at high $\text{Re}_A$, the effect of viscosity, upon the larger scales of motion present within $U_0(X, t)$, becomes negligible. This is clear by inspection of equation (3.4.15). As a result, at high $\text{Re}_A$, energy is primarily drawn from the larger scales of motion by convection, rather than dissipation.

It is important to stress the limitations of the above proof. The proof relies upon the assumption that the flow remains two dimensional. However, the stability of Poiseuille flows subjected to transpiration has been studied by Lee et al. (2008), who showed that an upstream travelling wave boundary condition will reduce the stability of the flow. The same was reported by Moarref & Jovanović (2010), and in simulations by Lieu et al. (2010). From this, we can safely infer that at sufficiently high $\text{Re}_A$, the flow becomes unstable, and will turn turbulent. Once the flow becomes turbulent, this proof no longer holds, since turbulent flows are three dimensional. It is also possible that the flow could begin to exhibit three-dimensional laminar behaviour at some sub-turbulent value of $\text{Re}_A$.

Regardless of the shape of the function defining the wall-normal boundary condition, if the magnitude of that boundary condition is much greater than the speed at which it moves down the wall of the channel, then the system will effectively be governed by equation (3.4.15) and the effectively stationary boundary conditions given in equations (3.4.16) and (3.4.17). It seems likely, therefore, that such flows will, as the amplitude further increases, also begin to exhibit zero or negligible increases to their bulk flows. Equally, it may also be that non-symmetric (and also spanwise-dependent) boundary conditions will similarly find that their
bulk flows will not increase with the magnitudes of their boundary conditions, beyond a certain point. We have therefore posed the following conjecture in the Journal of Fluid Mechanics (Woodcock, Sader & Marusic, 2012):

*For a flow that is driven by a zero-net mass-flux blowing and suction over the no-slip channel walls (also known as transpiration), regardless of the shape of the function that defines the boundary condition, if the magnitude of that transpiration is sufficiently great, and sufficiently greater than the speed at which the boundary condition moves along the wall of the channel, the bulk flow will become dependent only upon the speed at which the boundary condition moves along the wall of the channel.*

If it could be shown that the Navier-Stokes equation exhibited a unique solution in three dimensions, then this proof could be extended to three dimensional flows driven by transpiration (provided that their boundary conditions were appropriately symmetric in the streamwise direction). Conversely, if the above conjecture could be shown to be incorrect for some such symmetric boundary conditions acting upon a three dimensional flow, then it must follow that the Navier-Stokes equation does not in fact exhibit a unique solution in three dimensions. This we can infer from the previously stated fact that equation (3.4.15), subject to symmetric boundary conditions such as (3.4.16) and (3.4.17), could only induce a bulk flow if the Navier-Stokes equation were able to exhibit more than one solution.

### 3.5 Discussion

#### 3.5.1 Scales of motion

The perturbation expansion presented within §3.2 demonstrates the effect of convection upon the scales of motion present within the flow. In a Stokes flow, in which the effect of convection is absent or negligible, any separate streamwise scales of motion present will simply superimpose upon one another, without altering each other or giving rise to new scales of motion. (The exact meaning,
3.5. Discussion

in this work, of the term “scale of motion” has been defined in the beginning of §3.2.7.)

In inertial flows, however, convection acts to transfer energy and momentum from one scale of motion to another. In this example, the wavelength of the boundary condition forms a natural streamwise scale for the motion of the fluid within the channel. Naturally then, since the leading order term in the perturbation expansion is an unsteady Stokes flow, the motion at leading order is of the scale of the boundary condition. At first order however, two new streamwise scales arise due to the effect of convection: a series of eddies with a lengthscale half that of the boundary condition, and a translational flow, independent of the streamwise location.

If we were to extend our perturbation analysis (in the manner of Moarref & Jovanović (2010)) to determine the second order components of the flow, we would find that there exists at second order a scale of motion one third the length of that of the boundary condition. As we pursue the analysis further, we would find that at the \( n \)-th order, the expansion introduces a scale of motion \( 1/(n + 1) \) times the length of the scale of the boundary condition. Hence we can see that energy and momentum is drawn by each lesser scale of motion from the scales above it.

However, the translational flow is of a higher streamwise scale of motion than that of the boundary condition. The question of whether transpiration will be capable of inducing a non-zero bulk flow therefore becomes a question of whether the motion of the fluid may, under the influence of convection, provide energy and momentum to the translational flow from the lower scales of streamwise motion.

The analysis presented herein demonstrates that an inertial flow driven by transpiration will indeed experience such a transference of energy and momentum upwards in scale, to feed the translational motion of the fluid. Not only that, but an analysis of the higher order terms within the perturbation expansion reveals that all scales of motion are drawing some of their energy from lower scales. This general tendency of energy to transfer downwards in scale is to be expected, since any upward transfer of energy requires that the scales of motion within the flow should combine in a coherent manner.
3.5.2 The induced bulk flow

The exact nature of the induced streamwise flow varies considerably with the geometry of the channel and the boundary conditions. However, certain general statements can be made about it: for example, the induced flow is invariably in what we have consequently referred to here as the streamwise direction, the direction counter to the motion of the boundary condition. This rule has been reported previously by Hœpffner & Fukagata (2009).

It might be expected that the in-phase boundary condition would produce a greater bulk flow than the ‘mixed’ boundary condition, since the in-phase boundary condition contains, in effect, twice the active flow-control. However, as has been discussed in section §3.2.2, this is only true for \( \eta > 10 \), or equivalently when the height of the channel is significantly greater than the wavelength of the boundary condition.

More revealing perhaps is the fact that for \( \eta < 1 \) (where the wavelength of the boundary condition is longer than the height of the channel), the ‘mixed’ boundary condition induces a significantly higher bulk flow than the in-phase boundary condition. This can be attributed to the fact that the in-phase boundary condition sets an equal velocity at corresponding points on both walls, and thus reduces the velocity gradients within the flow, thereby decreasing the effect of convection. This will naturally be most noticeable at small \( \eta \), where the ‘mixed’, out-of-phase or antiphase boundary conditions produce significant pressure and velocity gradients at the two walls, which are completely absent when an in-phase boundary condition is applied. This concurs with the findings of Mamori et al. (2010), who reported that the antiphase boundary condition produced significantly greater drag reduction for a Poiseuille channel flow than the in-phase boundary condition.

The justification for considering the out-of-phase mode separately from the antiphase mode is that the translational motion at first order results from the interaction of components of the leading order flow that are out of phase by \( \pi/2 \). When ‘mixed’, in-phase or antiphase boundary conditions are employed, any leading order flow that is out-of-phase with the boundary conditions by \( \pi/2 \) (that is any flow whose streamfunction can be expressed as a multiple of \( \sin(\eta x + t) \)) will result from the presence of the time derivative term within the Navier-Stokes equation. Thus the advantage of the out-of-phase boundary conditions is that since the flows at the two bound-
aries are out of phase by $\pi/2$, the presence of these two phases of the flow will not be dependent upon the time derivative. It follows that the out-of-phase mode could reasonably be expected to produce the greatest overall flow for sufficiently small values of the Stokes number, $\beta$.

However, the results show that the out-of-phase mode only generally induces a greater bulk flow than the antiphase flow for $\beta < 1$ and $1 < \eta < 10$, and even in these cases, this difference is not significant. Moreover, at these values of $\beta$ the bulk flow induced by the transpiration is so small as to render these cases of little interest. For $\eta < 1$, the antiphase boundary condition produces the greatest bulk flow. We conclude, therefore, that the greater velocity gradients produced within the flow via the antiphase boundary conditions outweighs the reduced dependence upon the time derivative that is created by the out-of-phase boundary condition.

The antiphase mode produces the greatest bulk flow because it produces the greatest velocity gradients within the channel, and therefore should be the most affected by convection. This analysis suggests that an antiphase boundary condition should be considered as a flow-control regime for producing drag-reduction, with a small $\eta$ and a value for $\beta$ of around 100.

The energy input required to support the system has been derived in §3.3. Notably, for all bar the in-phase case, the energy cost is proportional to $\eta^{-2}$ for $\eta < 1$. This contrasts with the induced bulk flow, which has been shown to be proportional to $\eta^{-1}$ in this limit. This tells us that although it is possible to increase the induced bulk flow by increasing the wavelength of the boundary condition, by doing so an even greater increase in the power cost of the system will be incurred. A boundary condition of long wavelength will therefore not be the most energy efficient method of driving a flow via transpiration. This is in agreement with the analysis of drag reduction with Poiseuille flows by Moarref & Jovanović (2010), who similarly found that for boundary conditions of sufficiently long wavelength, the energy cost increased, with increasing wavelength, faster than the resulting drag reduction.

This analysis, as well as that of Hœpffner & Fukagata (2009) and Moarref & Jovanović (2010), demonstrates that at small $A$ (the amplitude of the boundary condition), the induced bulk flow will increase parabolically with $A$. (Here we are referring to the dimensional velocity $\hat{u}$, rather than the non-dimensionalised...
3.6. Conclusions

The asymptotic behaviour of the bulk flow at small $A/h\omega$ is detailed in §3.6.

At higher values of $A$, however, the bulk flow will not increase parabolically with $A$. In §3.3, it has been shown that if the amplitude of the boundary condition is sufficiently greater than the speed at which it travels along the channel wall, then the effect upon the bulk flow of increasing $A$ will diminish.

Furthermore, it has been shown in §3.2 that for boundary conditions of small amplitude there exists an optimal frequency for the boundary condition, at which the bulk flow will be maximised. Beyond this frequency, any increase will have the effect of reducing the bulk flow. In fact, the bulk flow will be reduced to zero in the limit as the frequency approaches infinity. This has been explained via an analogy with the motion of fluid adjacent to a shuffling body.

3.5.3 Generalising the boundary conditions

So far, we have only considered a very narrow range of the possible boundary conditions that could be used to induce a bulk flow via transpiration. Specifically, we have considered those boundary conditions that can be expressed as single-modal travelling waves, in which boundary conditions are of equal wavelength and frequency. However, in §3.2.7 we have demonstrated a method by which this perturbation analysis could be readily extended to investigate all periodic boundary conditions. This method relies upon the fact that if a boundary condition may be expressed as a sum of two or more travelling sinusoidal waves, then the first order approximation of the overall translational flow will be the sum of the first order translational flows that each of those component travelling waves would induce separately as a boundary condition.

3.6 Conclusions

We have analysed the behaviour of fluids that are subjected to a non-zero wall-normal velocity at the surfaces of the channel. This arrangement, known also as ‘transpiration’, allows no net volume flux to be imparted upon the flow through
the channel walls, and does not require the boundary conditions to remain constant over time.

We have shown, though a perturbation analysis, that the effect of convection is capable of inducing a bulk flow in a fluid that is driven by transpiration. The set of boundary conditions considered were those in which the wall-normal velocities at either wall of the channel can be expressed as travelling sine waves. The waves at either wall may differ in their magnitude, and may be out of phase, but they are of equal wavelength and frequency. The induced bulk flow is found to move in the direction counter to the motion of the boundary condition, which concurs with the results of reported numerical simulations, and perturbation analyses, of systems driven by transpiration.

The asymptotic behaviour of \( \langle \hat{u} \rangle \), the average streamwise velocity (in dimensional units) within the channel, is detailed here. Again, \( A \), denotes the maximum amplitude of the boundary condition, \( \lambda \) denotes its wavelength, and \( \omega \) its temporal frequency of oscillation, \( h \) denotes the height of the channel, and \( \nu \) the kinematic viscosity of the fluid (\( \nu \equiv \mu / \rho \)). The ratio of the amplitude of the boundary conditions at either wall is denoted by \( \gamma \), and the degree to which they are out of phase by \( \phi \). Note that in order to obtain the bulk flow from \( \langle \hat{u} \rangle \), it must be multiplied by \( h \).

The magnitude of the average streamwise velocity in the limit as \( A/h \omega \to 0 \) is given in equations (3.2.33) and (3.2.36). In dimensional units, the average velocity is given by

\[
\langle \hat{u} \rangle \sim \begin{cases}
\frac{1 + \gamma^2 - 2 \gamma \cos \phi h^2 \omega A^2 \lambda}{5040} \frac{h^2 \omega}{\nu^2}, & h \lambda \to 0, \frac{h^2 \omega}{\nu} \to 0 \\
\frac{1 + \gamma^2 - 2 \gamma \cos \phi A^2 \lambda}{2\sqrt{2}} \frac{h}{h \sqrt{\nu \omega}}, & h \lambda \to 0, \frac{h^2 \omega}{\nu} \to \infty \\
\frac{3(1 + \gamma^2) \omega A^2 \lambda^3}{64} \frac{h}{\nu^2}, & h \lambda \to \infty, \frac{\lambda^2 \omega}{\nu} \to 0 \\
\frac{1 + \gamma^2 A^2}{4\sqrt{2}} \frac{h}{h \sqrt{\nu \omega}}, & h \lambda \to \infty, \frac{\lambda^2 \omega}{\nu} \to \infty 
\end{cases}
\]  
(3.6.1)

In this small \( A/h \omega \) limit, it has been found that the maximum flow is obtained by minimising \( h/\lambda \), while maintaining an optimal frequency \( \omega_{\text{max}} \), which is given by

\[
\omega_{\text{max}} \approx 100 \frac{\nu}{h^2}.
\]  
(3.6.2)
3.6. Conclusions

Using the optimal frequency, as well as a wavelength that is longer than the height of the channel, we find that the average flow will be approximately

\[ \langle \hat{u} \rangle \approx \frac{\lambda}{h} (1 + \gamma^2 - 2\gamma \cos \phi), \quad \lambda > h, \quad \omega = \omega_{\text{max}}. \tag{3.6.3} \]

Decidedly different behaviour is found when the amplitude of the boundary condition, \( A \), is much greater than the speed at which it moves along the wall of the channel, \( \lambda \omega \). In the limit as \( \lambda \omega / A \to 0 \). Here, assuming that the flow remains two dimensional, the average velocity becomes

\[ \langle \hat{u} \rangle \leq O(\lambda \omega). \tag{3.6.4} \]

This implies that \( \langle \hat{u} \rangle \) will not remain proportional to \( A^2 \) for all values of \( A \). Moreover that as \( A \) is increased, \( \partial \langle \hat{u} \rangle / \partial A \) will at some point diminish, approaching zero in the limit as \( \lambda \omega / A \to 0 \). It remains to be determined whether there in fact exists a bulk flow in this limit, and the direction of the flow.

This has been proved only for two-dimensional flows resulting from a limited set of the possible transpiration boundary conditions. These include the boundary conditions for which the perturbation analysis was performed. The extension of this result to three-dimensional flows and generalised boundary conditions has been posed as a conjecture.
Flows driven by oscillating walls: On the physics behind valveless pumping

The valveless impedance pump consists of a thin tube, one section of which is elastic and is subjected to rhythmic pinching at some point offset from its centre. This induces travelling waves to propagate back and forth along an elastic section of a tube, which in turn induces a flow within the tube. In this chapter we investigate the physics behind impedance pumping. To this end, we derive the asymptotic behaviour of flows induced by a simple travelling sine wave along the surface of a tube, in the limit of small-amplitude of wall deformation. We find that there are two mechanisms by which the wave may induce a bulk flow within the tube: the first of which we call ‘wall-entrainment’ and consists of the wave dragging the fluid along with it, and the second is the effect of convection. We find that these two effects drive the fluid in opposite directions, and that both effects must be accurately accounted for within any reasonable model of the pump. The asymptotic behaviour of the bulk flow has been detailed in the conclusions, and these derivations have been subsequently repeated for flows within a channel in §4.4.

Furthermore, it has been proposed that the action of oscillating walls, such as in an impedance pump, should, if of sufficiently small amplitude, be similar to another pumping mechanism known as transpiration (which involves subjecting the flow to a non-zero and non-constant wall-normal velocity at the flow’s sur-
4.1. Equations of pipe flow

While this has previously been shown to be untrue in some cases, we seek to determine whether there exist any circumstances in which an oscillating wall would act similarly to transpiration. We show that there are no circumstances under which a system driven by oscillating walls behaves like the equivalent system driven by transpiration, and therefore there are no circumstances under which a flow driven by oscillating walls, such as in an impedance pump, would be a feasible experimental substitute for transpiration.

4.1 Equations of pipe flow

We consider the steady-state flow of an incompressible Newtonian fluid through a pipe of infinite length and finite radius. The flow is driven by the sinusoidal oscillation of the tube’s radius. The ends of the tube are not assumed to be clamped, so that the oscillations continue to propagate to the ends. The no-slip boundary condition applies at the tube’s wall.

Because of the inherent symmetry of the system, we may simplify the derivation by assuming the flow to be axisymmetric. The azimuthal direction, $\hat{\theta}$, is therefore not included in our derivation. We consider only boundary conditions in which the variation in the pipe’s radius can be represented as a single sine wave travelling in the $x$-direction.

The Navier-Stokes and continuity equations may be written as

\[
\rho \left( \frac{\partial}{\partial t} \hat{u} + \hat{u} \cdot \nabla \hat{u} \right) = -\nabla \hat{p} + \mu \nabla^2 \hat{u}, \tag{4.1.1}
\]

\[
\nabla \cdot \hat{u} = 0, \tag{4.1.2}
\]

where $\rho$ and $\mu$ represent the density and the shear viscosity of the fluid respectively. These equations will subsequently be non-dimensionalised via the properties of the boundary conditions. Throughout this chapter, quantities expressed in terms of dimensional units, such as the velocity, $\hat{u}(\hat{x}, \hat{t})$, and pressure, $\hat{p}(\hat{x}, \hat{t})$, above, are differentiated from their non-dimensionalised counterparts, $u(x, t)$ and $p(x, t)$, by the presence of a circumflex. The position vector is denoted by $\hat{x}$, and is defined such that $\hat{x} = (\hat{x}, \hat{r})$. 
4.1. Equations of pipe flow

4.1. Boundary conditions

The no-slip boundary condition applies at the wall of the tube. The radius of the tube varies sinusoidally with time and with the streamwise location along the tube. This variation takes the form of a sine wave that travels forwards in the \( x \) direction, along the wall of the channel. The average radius of the tube is \( h \). A diagram of the tube, in the absence of oscillations, is shown in figure 4.1. The amplitude of the oscillations is \( A \). The frequency of the oscillations is \( \omega \) and their wavelength is \( \lambda \). The radius of the tube, \( \hat{R}(\hat{x}, \hat{t}) \), is therefore given by

\[
\hat{R}(\hat{x}, \hat{t}) = h - A \sin \left( \frac{\hat{x}}{\lambda} - \omega \hat{t} \right),
\]

and hence the domain of the flow is given by

\[
-\infty < \hat{x} < \infty, \quad 0 \leq \hat{r} \leq \hat{R}(\hat{x}, \hat{t}).
\]
4.1. Equations of pipe flow

The radial velocity at the wall of the tube is the time derivative of \( \hat{R}(\hat{x}, \hat{t}) \), and hence the boundary conditions are

\[
\hat{u}(\hat{R}, \hat{t}) = 0, \tag{4.1.5}
\]
\[
\hat{w}(\hat{R}, \hat{t}) = A\omega \cos \left( \frac{\hat{x}}{\lambda} - \omega \hat{t} \right). \tag{4.1.6}
\]

4.1.2 The scaled equations

The system contains three natural length scales: \( h \), the average radius of the tube, which is the radius of the pipe in the absence of deformation; \( A \), the amplitude of oscillations of tube’s radius; and \( \lambda \), the wavelength of the oscillations. The other quantities that have been used to non-dimensionalise the quantities of the Navier-Stokes equation are the frequency, \( \omega \), of the oscillations of the boundary condition, and the density, \( \rho \), and shear viscosity, \( \mu \), of the fluid.

Using these quantities, the variables in the Navier-Stokes equations are scaled according to

\[
\hat{r} = h r, \quad \hat{u} = A\omega u, \quad \hat{\rho} = \frac{\mu A\omega}{\rho h} p, \quad \hat{t} = \frac{1}{\omega} t. \tag{4.1.7}
\]

Scaling the Navier-Stokes equation, (4.1.1) and continuity equation (4.1.2) in this way leads to

\[
\beta \left( \frac{\partial}{\partial \hat{t}} u + \alpha u \cdot \nabla u \right) = -\nabla p + \nabla^2 u, \tag{4.1.8}
\]
\[
\nabla \cdot u = 0. \tag{4.1.9}
\]

There are two dimensionless numbers in the above equation, denoted \( \alpha \) and \( \beta \). The parameter \( \beta \) is referred to as the Stokes number, and relates to the rate of diffusion of vorticity within the flow. The parameter \( \alpha \) is the non-dimensionalised amplitude of the deformations, and is a measure of the relative importance of convection within the system. The values of these parameters are given by

\[
\alpha = \frac{A}{h}, \quad \beta = \frac{\rho h^2 \omega}{\mu}, \tag{4.1.10}
\]

where \( \rho \) denotes the density of the fluid. It is immediately clear that

\[0 \leq \alpha < 1.\]
If we define a Reynolds number based upon the amplitude of the boundary condition by

\[ Re = \frac{\rho h A \omega}{\mu} , \]

then we may also express \( \alpha \) as a ratio of dimensionless numbers:

\[ \alpha = \frac{Re}{\beta} . \]

There is a third length scale present, which is the wavelength of the boundary condition, \( \lambda \). We shall subsequently make use of a further parameter \( \eta \), which is a ratio of length scales and is defined

\[ \eta = \frac{h}{\lambda} . \] (4.1.11)

In scaled variables, the radius of the pipe, \( R(z, t) \), equivalent to (4.1.3), is given by

\[ R(x, t) = 1 - \alpha \sin(\eta x - t) . \] (4.1.12)

The scaled domain of the flow is given by

\[ 0 \leq r \leq R(x, t), \quad -\infty < x < \infty . \]

The velocity of the flow at the wall of the tube, in scaled variables, equivalent to (4.1.5) and (4.1.6), is given by

\[ u(R, t) = 0, \]

\[ w(R, t) = \cos(\eta x - t) . \] (4.1.13) (4.1.14)

For all quantities, an average over the entire pipe is denoted by angled brackets. We define such average quantities via

\[ \langle F(t) \rangle \overset{\text{def}}{=} \lim_{L \to \infty} \frac{2}{L} \int_{-L/2}^{L/2} \int_0^{R(z,t)} F(r, t) r \, dr \, dx . \] (4.1.15)

It is important to note that for the non-dimensionalised velocity, \( \langle u \rangle \) denotes the average streamwise velocity within the channel, and must be multiplied by \( \pi \) to obtain the bulk flow. However, for the dimensionalised velocity, \( \langle \hat{u} \rangle \) must be multiplied by \( \pi h \) to obtain the bulk flow.
4.2 Perturbation analysis

In this section, we describe the methodology we have used to derive the properties of the flow. We consider flows that are driven by small sinusoidal deviations in the radius of the tube. We analyse these flows using a perturbation analysis for small values of the parameter $\alpha$, which amount to flows in which the amplitude of the deviation of the boundary is small compared to the radius of the tube.

The derivation of this perturbation analysis, along with its results, is presented in this section. Our derivation involves expanding the velocity and pressure in terms of $\alpha$ as follows:

\[
\begin{align*}
\mathbf{u} &= u_0 + \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \ldots \quad (4.2.1a) \\
p &= p_0 + \alpha p_1 + \alpha^2 p_2 + \alpha^3 p_3 + \ldots \quad (4.2.1b)
\end{align*}
\]

By substituting equation (4.2.1) into (4.1.8), and equating orders of $\alpha$, we obtain

\[
\begin{align*}
\beta \frac{\partial}{\partial t} u_0 &= -\nabla p_0 + \nabla^2 u_0, \quad (4.2.2a) \\
\beta \frac{\partial}{\partial t} u_1 + \beta u_0 \cdot \nabla u_0 &= -\nabla p_1 + \nabla^2 u_1, \quad (4.2.2b) \\
\beta \frac{\partial}{\partial t} u_2 + \beta u_0 \cdot \nabla u_1 + \beta u_1 \cdot \nabla u_0 &= -\nabla p_2 + \nabla^2 u_2, \quad (4.2.2c) \\
&\quad \ldots
\end{align*}
\]

Of course, the continuity equation (4.1.9) implies that,

\[
\nabla \cdot \mathbf{u}_n = 0, \quad \text{for all } n. \quad (4.2.3)
\]

To determine the boundary conditions acting upon each of $\mathbf{u}_n$, we first expand the velocity at the wall in the manner shown in equation (4.2.1a), taking into account equations (4.1.13) and (4.1.14). This gives

\[
\mathbf{u}_0(R, x, t) + \alpha \mathbf{u}_1(R, x, t) + \alpha^2 \mathbf{u}_2(R, x, t) + \cdots = (0, \cos(\eta x - t)), \quad (4.2.4)
\]

where $R$ has been defined in equation (4.1.12). We then expand $\mathbf{u}(R, x, t)$ in a
4.2. Perturbation analysis

Taylor series around $\alpha = 0$. This results in

$$u(R, x, t) = u(R, x, t)|_{\alpha=0} + \alpha \frac{\partial u}{\partial \alpha} u(R, x, t)|_{\alpha=0} + \frac{\alpha^2}{2!} \left( \frac{\partial R}{\partial \alpha} \right)^2 \frac{\partial^2 u}{\partial R^2} u(R, x, t)|_{\alpha=0} + \ldots$$

$$= u(1, x, t) - \alpha \sin(\eta x - t) \frac{\partial u(1, x, t)}{\partial r} + \frac{\alpha^2}{2!} \sin^2(\eta x - t) \frac{\partial^2 u(1, x, t)}{\partial r^2} + \ldots$$

(4.2.5)

By substituting equation (4.2.5) into (4.2.4), and equating orders of $\alpha$, we obtain the following boundary conditions for the velocity:

$$u_0(1, x, t) = (0, \cos(\eta x - t)), \quad (4.2.6a)$$

$$u_1(1, x, t) = \sin(\eta x - t) \frac{\partial}{\partial r} u_0(1, x, t), \quad (4.2.6b)$$

$$u_2(1, x, t) = \sin(\eta x - t) \frac{\partial}{\partial r} u_1(1, x, t) - \frac{\sin^2(\eta x - t)}{2} \frac{\partial^2}{\partial r^2} u_0(1, x, t), \quad (4.2.6c)$$

$$\ldots$$

$$u_n(1, x, t) = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j!} \sin^j(\eta x - t) \frac{\partial^j}{\partial r^j} u_{n-j}(1, x, t), \quad n > 0. \quad (4.2.6d)$$

4.2.1 Leading order flow

The leading order of the flow, whose velocity is given by $u_0(r, t)$, is the limiting behaviour of the flow as $\alpha \to 0$. It is noticeable that, in this limit, the flow is equivalent to an analogous limit in flows driven by transpiration, as discussed in §3.2.1.

In order to find a closed form solution for $u_0(r, t)$, we introduce the following assumed forms for $u_0(r, t)$, $w_0(r, t)$ and $p_0(r, t)$:

$$u_0 = U_{0,1}(r)e^{i(\eta z - t)} + U_{0,-1}(r)e^{-i(\eta z - t)}, \quad (4.2.7)$$

$$w_0 = W_{0,1}(r)e^{i(\eta z - t)} + W_{0,-1}(r)e^{-i(\eta z - t)}, \quad (4.2.8)$$

$$p_0 = P_{0,1}(r)e^{i(\eta z - t)} + P_{0,-1}(r)e^{-i(\eta z - t)}, \quad (4.2.9)$$
4.2. Perturbation analysis

where $U_{0,1}(r), W_{0,1}(r)$, etc. are functions to be determined. By taking the gradient of the unsteady Stokes equation (4.2.2a), we find that

$$\nabla^2 p_0 = 0,$$

(4.2.10)

which leads to

$$P_{0,\pm 1}(r) = \mathcal{P}_{0,\pm 1} I_0(\eta r),$$

(4.2.11)

where $\mathcal{P}_{0,1}$ and $\mathcal{P}_{0,-1}$ are constants to be determined. The function $I_n(x)$ is a modified Bessel function. In deriving the above, we have made use of the fact that the pressure must be finite at $r = 0$. We now substitute the pressure, as represented by equations (4.2.9) and (4.2.11), back into the unsteady Stokes equation (4.2.2a). We then make use of the assumed form for $u_0(R,t)$ given in (4.2.7). By equating coefficients of the complex exponentials in the resulting equation we arrive at

$$r^2 U''_{0,\pm 1} + r U'_{0,\pm 1} - (\eta^2 \mp \beta i) r^2 U_{0,\pm 1} = \pm i \eta \mathcal{P}_{0,\pm 1} r^2 I_0(\eta r),$$

(4.2.12)

$$r^2 W''_{0,\pm 1} + r W'_{0,\pm 1} - [(\eta^2 \mp \beta i) r^2 + 1] W_{0,\pm 1} = \eta \mathcal{P}_{0,\pm 1} r^2 I_1(\eta r).$$

(4.2.13)

From equation (4.2.6a), we find that the boundary conditions affecting $U_{0,1}(r)$, $U_{0,-1}(r), W_{0,1}(r)$ and $W_{0,1}(r)$ at the pipe’s wall are

$$U_{0,\pm 1}(1) = 0,$$

(4.2.14)

$$W_{0,\pm 1}(1) = \frac{1}{2}.$$  

(4.2.15)

while a second implicit boundary boundary condition is that both are finite at $r = 0$. From these equations and boundary conditions, we obtain the following:

$$U_{0,1} = \frac{\eta \mathcal{P}_{0,1}}{\beta} \left( I_0(\eta r) - \frac{I_0(\eta)}{I_0(\sqrt{\eta^2 - \beta i})} I_0\left( \sqrt{\eta^2 - \beta i} r \right) \right),$$

(4.2.16)

$$U_{0,-1} = \frac{\eta \mathcal{P}_{0,-1}}{\beta} \left( I_0(\eta r) - \frac{I_0(\eta)}{I_0(\sqrt{\eta^2 + \beta i})} I_0\left( \sqrt{\eta^2 + \beta i} r \right) \right),$$

(4.2.17)
4.2. Perturbation analysis

\[ W_{0,1} = \frac{1}{2} \frac{I_1(\sqrt{\eta^2 - \beta ir})}{I_1(\sqrt{\eta^2 - \beta i})} + \frac{i\eta P_{0,1}}{\beta} \left( \frac{I_1(\eta)}{I_1(\sqrt{\eta^2 - \beta i})} I_1(\sqrt{\eta^2 - \beta ir}) - I_1(\eta r) \right), \quad (4.2.18) \]

\[ W_{0,-1} = \frac{1}{2} \frac{I_1(\sqrt{\eta^2 + \beta ir})}{I_1(\sqrt{\eta^2 + \beta i})} + \frac{i\eta P_{0,-1}}{\beta} \left( I_1(\eta r) - \frac{I_1(\eta)}{I_1(\sqrt{\eta^2 + \beta i})} I_1(\sqrt{\eta^2 + \beta ir}) \right). \quad (4.2.19) \]

In order to determine \( P_{0,1} \) and \( P_{0,-1} \), we express the continuity equation (4.2.3) in terms of \( U_{0,\pm1}(r) \) and \( W_{0,\pm1}(r) \). This results in

\[ W'_{0,\pm1} + \frac{1}{r} W_{0,\pm1} \pm i\eta U_{0,\pm1} = 0. \quad (4.2.20) \]

The pressure can now be determined, to leading order. First, we determine \( P_{0,1} \) and \( P_{0,-1} \) by substituting (4.2.16)-(4.2.19) into the above. Then, by substituting the results into (4.2.11), and these results in turn into (4.2.9), we obtain the leading order of the pressure:

\[ p_0(r, t) = \]

\[ \frac{i\beta I_0(\eta)}{\eta} \left[ 2I_1(\eta) - \eta \frac{I_0(\eta)}{I_1(\sqrt{\eta^2 - \beta i})} F_1(2; \frac{1}{4} (\eta^2 - \beta i)) \right]^{-1} e^{i(\eta x - t)} \]

\[ - \frac{i\beta I_0(\eta)}{\eta} \left[ 2I_1(\eta) - \eta \frac{I_0(\eta)}{I_1(\sqrt{\eta^2 + \beta i})} F_1(2; \frac{1}{4} (\eta^2 + \beta i)) \right]^{-1} e^{-i(\eta x - t)}. \quad (4.2.21) \]

The term \( _pF_q \) above represents the hypergeometric function, which is defined

\[ _pF_q(a_1, a_2, \ldots, a_p; b_1, \ldots, b_q; x) \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \ldots (a_p)_n x^n}{(b_1)_n \ldots (b_q)_n n!}. \quad (4.2.22) \]
where \((x)_n\) is Pochhammer’s symbol, which is given by

\[
(x)_n \equiv \frac{\Gamma(x + n)}{\Gamma(a)}.
\]

The leading order of the velocity can now be determined by substituting the above back into equations (4.2.16)-(4.2.19). This results in

\[
U_{0,1} = \frac{i}{2I_0(\sqrt{\eta^2 - \beta i}) I_1(\eta)} \left[ I_0\left(\sqrt{\eta^2 - \beta i}\right) I_0(\eta + \eta I_0(\eta) F_1(2; \frac{1}{4}(\eta^2 - \beta i)) - I_0\left(\sqrt{\eta^2 - \beta i}\right) I_0(\eta)ight)
\]

\[
U_{0,-1} = \frac{i}{2I_0(\sqrt{\eta^2 + \beta i}) I_1(\eta)} \left[ I_0\left(\sqrt{\eta^2 + \beta i}\right) I_0(\eta + \eta I_0(\eta) F_1(2; \frac{1}{4}(\eta^2 + \beta i)) - I_0\left(\sqrt{\eta^2 + \beta i}\right) I_0(\eta)ight)
\]

\[
W_{0,1} = \frac{4I_0\left(\sqrt{\eta^2 - \beta i}\right) I_1(\eta fr) - r \eta I_0(\eta) F_1(2; \frac{1}{4}(\eta^2 - \beta i))}{8I_0(\sqrt{\eta^2 - \beta i}) I_1(\eta) - 2 \eta I_0(\eta) F_1(2; \frac{1}{4}(\eta^2 - \beta i))}
\]

\[
W_{0,-1} = \frac{4I_0\left(\sqrt{\eta^2 + \beta i}\right) I_1(\eta fr) - r \eta I_0(\eta) F_1(2; \frac{1}{4}(\eta^2 + \beta i))}{8I_0(\sqrt{\eta^2 + \beta i}) I_1(\eta) - 2 \eta I_0(\eta) F_1(2; \frac{1}{4}(\eta^2 + \beta i))}
\]

The leading order of the velocity can now be found by substituting the above back into equations (4.2.7) and (4.2.8). This results in

\[
u_0 = \frac{i\sqrt{\eta^2 - \beta i}}{2\sqrt{\eta^2 - \beta i} I_0(\sqrt{\eta^2 - \beta i}) I_1(\eta) - 2 \eta I_0(\eta) I_1 \left(\sqrt{\eta^2 - \beta i}\right)} e^{i(\eta z - t)}
\]

\[
+ \frac{i\sqrt{\eta^2 + \beta i}}{2\sqrt{\eta^2 + \beta i} I_0(\sqrt{\eta^2 + \beta i}) I_1(\eta) - 2 \eta I_0(\eta) I_1 \left(\sqrt{\eta^2 + \beta i}\right)} e^{-i(\eta z - t)}
\]

\[(4.2.28)\]
4.2. Perturbation analysis

\[ w_0 = \frac{\sqrt{\eta^2 - \beta i I_0 (\sqrt{\eta^2 - \beta i})} I_1 (\eta r) - \eta I_0 (\eta) I_1 (\sqrt{\eta^2 - \beta i r})}{2 \sqrt{\eta^2 - \beta i I_0 (\sqrt{\eta^2 - \beta i})} I_1 (\eta) - 2 \eta I_0 (\eta) I_1 (\sqrt{\eta^2 - \beta i})} e^{i (\eta z - t)} + \frac{\sqrt{\eta^2 + \beta i I_0 (\sqrt{\eta^2 + \beta i})} I_1 (\eta) - \eta I_0 (\eta) I_1 (\sqrt{\eta^2 + \beta i})}{2 \sqrt{\eta^2 + \beta i I_0 (\sqrt{\eta^2 + \beta i})} I_1 (\eta) - 2 \eta I_0 (\eta) I_1 (\sqrt{\eta^2 + \beta i})} e^{-i (\eta z - t)}. \] (4.2.29)

Despite containing several complex and imaginary terms, the above functions are in fact everywhere real for all applicable values of \( \eta \) and \( \beta \). The function \( I_n (x) \) is a modified Bessel function. Note that leading order flow is purely oscillatory, and contains no bulk flow in either direction.

The average rate of dissipation within the flow, \( \langle \epsilon \rangle \), can now be derived to leading order according to the formula given in equation (4.3.6). Substituting (4.2.7) and (4.2.15) into (4.3.6), for the velocity, and (4.2.21) for the pressure, we obtain the following for the leading order of the dissipation:

\[ \langle \epsilon_0 \rangle = \frac{i \beta I_0 (\eta)}{2 \eta} \left\{ \left[ 2 I_1 (\eta) - \frac{\eta I_0 (\eta) \text{}_0 F_1 \left( ; 2; \frac{1}{4} \left( \eta^2 + i \beta \right) \right)}{I_0 \left( \sqrt{\eta^2 + i \beta} \right)} \right]^{-1} + \left[ \frac{\eta I_0 (\eta) \text{}_0 F_1 \left( ; 2; \frac{1}{4} \left( \eta^2 - i \beta \right) \right)}{I_0 \left( \sqrt{\eta^2 - i \beta} \right)} - 2 I_1 (\eta) \right]^{-1} \right\}. \] (4.2.30)

4.2.2 First order correction to the flow

In order to find a closed form solution for the first order correction to the velocity, \( u_1 (R, t) \), we introduce the following assumed forms for \( u_1 (R, t) \), \( w_1 (R, t) \) and \( p_1 (R, t) \):

\[ u_1 = U_{1,0} (r) + U_{1,2} (r) e^{2i (\eta z - t)} + U_{1,-2} (r) e^{-2i (\eta z - t)}, \] (4.2.31)
\[ w_1 = W_{1,2} (r) e^{2i (\eta z - t)} + W_{1,-2} (r) e^{-2i (\eta z - t)}, \] (4.2.32)
\[ p_1 = P_{1,2} (r) e^{2i (\eta z - t)} + P_{1,-2} (r) e^{-2i (\eta z - t)}, \] (4.2.33)
4.2. Perturbation analysis

where \( U_{1,0}(r) \), \( W_{1,2}(r) \), etc. are functions to be determined. It’s clear that the first order correction to the bulk flow is given by

\[
\langle u_1 \rangle = 2 \int_0^1 r U_{1,0}(r) dr.
\]  
(4.2.34)

We substitute the above assumed forms into the first order correction to the Navier-Stokes equation (4.2.2b), along with the leading order velocity given by (4.2.28) and (4.2.29). By equating coefficients of the complex exponentials, we find the following:

\[
U''_{1,0} + \frac{1}{r} U'_i = \\
\left\{ 4 \left[ \eta I_1(\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right) - \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 - i\beta} \right) \right] \right. \\
\times \left[ \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 + i\beta} \right) - \sqrt{\eta^2 + i\beta} I_0(\eta) I_1 \left( \sqrt{\eta^2 + i\beta} \right) \right]^{-1} \\
\times \beta^2 I_0(\eta) \left[ \sqrt{\eta^2 + i\beta} I_1(r\eta) I_0 \left( \sqrt{\eta^2 + i\beta} \right) I_1 \left( r \sqrt{\eta^2 - i\beta} \right) \right. \\
+ \left. \left( \sqrt{\eta^2 - i\beta} I_1(r\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right) - 2 \eta I_0(\eta) I_1 \left( r \sqrt{\eta^2 - i\beta} \right) \right) \right. \\
\times \left. I_1 \left( r \sqrt{\eta^2 + i\beta} \right) \right\}.
\]  
(4.2.35)

The function \( U_{1,0}(r) \) is subject to the implicit boundary condition that it is finite at \( r = 0 \). The boundary conditions for \( U_{1,0}(r) \), \( U_{1,2}(r) \), and \( U_{1,-2}(r) \) can be found by substituting equation (4.2.29) into (4.2.6b), and equating coefficients of the complex exponentials. This results in

\[
U_{1,0}(1) = \frac{1}{2} \left( \frac{\eta I_1(\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right) - \sqrt{\eta^2 - i\beta} I_0(\eta) I_1 \left( \sqrt{\eta^2 - i\beta} \right)}{\eta I_0(\eta) _0 F_1 \left( 2; \frac{1}{4} \left( \eta^2 - i\beta \right) \right) - 2 I_1(\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right)} \\
+ \frac{\eta I_1(\eta) I_0 \left( \sqrt{\eta^2 + i\beta} \right) - \sqrt{\eta^2 + i\beta} I_0(\eta) I_1 \left( \sqrt{\eta^2 + i\beta} \right)}{\eta I_0(\eta) _0 F_1 \left( 2; \frac{1}{4} \left( \eta^2 + i\beta \right) \right) - 2 I_1(\eta) I_0 \left( \sqrt{\eta^2 + i\beta} \right)} \right).
\]  
(4.2.36)

In §B of the appendix, we derive an expression for the product of two modified Bessel functions as a power series. This is given in equation (B.6). Substituting
4.2. Perturbation analysis

this into equation (4.2.35) and using the above boundary conditions, we can obtain

\[ U_{1,0}(r) = U_{1,0}(1) + \left\{ 4 \left[ \sqrt{\eta^2 - i\beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right) - \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 - i\beta} \right) \right] \right. \]

\[ \times \left[ I_0(\eta) I_1 \left( \sqrt{\eta^2 + i\beta} \right) - \sqrt{\eta^2 + i\beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 + i\beta} \right) \right] \right\}^{-1} \]

\[ \times \beta^2 I_0(\eta) \eta (\eta^4 + \beta^2) \sum_{k=1}^{\infty} \frac{\eta^{2k} - 1}{2^{2k} k^2 (k-2)! (k-1)!} \left[ I_0 \left( \sqrt{\eta^2 + i\beta} \right) \right. \]

\[ \times 2 F_1 \left( 2 - k, 1 - k; 2; \frac{\eta^2 - i\beta}{\eta^2} \right) \eta^{2k-4} + I_0 \left( \sqrt{\eta^2 - i\beta} \right) \]

\[ \times 2 F_1 \left( 2 - k, 1 - k; 2; \frac{\eta^2 + i\beta}{\eta^2} \right) \eta^{2k-4} - 2 I_0(\eta) \]

\[ \left. \left[ 2 F_1 \left( 2 - k, 1 - k; 2; \frac{(\eta^2 + i\beta)^2}{\eta^4 + \beta^2} \right) \left( \eta^2 - i\beta \right)^{k-2} \right] \right\} \right\}^{(4.2.37)} \]

It is clear that the first term on the right hand side of the above equation represents the effect of the non-zero boundary condition, while the rest denotes the flow that would exist were the no-slip boundary condition to apply at first order (in other words, that which is due to convection). We can determine the first order correction to the bulk flow by substituting the above into equation (4.2.34). Hence, it is clear that the first order correction to the bulk flow can be separated into contributions due to the non-zero boundary conditions, \( \langle u_1 \rangle_{BC} \), and due to convection, \( \langle u_1 \rangle_{conv} \), so that the total first order bulk flow is given by

\[ \langle u_1 \rangle = \langle u_1 \rangle_{conv}^{<0} + \langle u_1 \rangle_{BC}^{>0}. \] (4.2.38)

Splitting the bulk flow in this way is possible because there are two physical mechanisms by which the oscillating walls induce a bulk flow. Convection induces a component of the bulk flow that is invariably in the direction counter to the motion of the travelling wave along the wall of the tube. On the other hand, there is a tendency for the travelling wave at the wall to effectively drag fluid along with it. We refer to this effect here as “wall-entrainment”. (The portion of the bulk flow that is induced by wall-entrainment has been denoted by \( \langle u_1 \rangle_{BC} \), because the subscript ‘BC’ is intended to indicate that the term derives from the presence of
4.2. Perturbation analysis

Unsurprisingly, wall-entrainment invariably induces a flow in the same direction as the oscillations of the wall. The contribution due to wall-entrainment is simply given by

$$\langle u_1 \rangle_{BC} = \frac{1}{2} \left( \eta \sqrt{\eta^2 - i \beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 - i \beta} \right) - (\eta^2 - i \beta) I_0(\eta) I_1 \left( \sqrt{\eta^2 - i \beta} \right) \right) - \frac{\eta \sqrt{\eta^2 + i \beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 + i \beta} \right)}{2 \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 + i \beta} \right) - 2 \sqrt{\eta^2 - i \beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 - i \beta} \right)} \right).$$

(4.2.39)

and the contribution due to convection is

$$\langle u_1 \rangle_{conv} =$$

$$- \left\{ 4 \left[ \sqrt{\eta^2 - i \beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 - i \beta} \right) - \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 - i \beta} \right) \right] \times \right.$$

$$\left[ \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 + i \beta} \right) - \sqrt{\eta^2 + i \beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 + i \beta} \right) \right] \bigg\}^{-1}$$

$$\times \beta^2 I_0(\eta) \eta \sqrt{\eta^4 + \beta^2} \sum_{k=1}^{\infty} \frac{k/(k+1)}{2^{2k} k^2 (k-2)! (k-1)!} \left[ I_0 \left( \sqrt{\eta^2 + i \beta} \right) \right]$$

$$\times 2 F_1 \left( 2 - k, 1 - k; 2; \frac{\eta^2 - i \beta}{\eta^2} \right) \eta^{2k-4} + I_0 \left( \sqrt{\eta^2 - i \beta} \right)$$

$$\times 2 F_1 \left( 2 - k, 1 - k; 2; \frac{\eta^2 + i \beta}{\eta^2} \right) \eta^{2k-4} - 2 I_0(\eta)$$

$$\times 2 F_1 \left( 2 - k, 1 - k; 2; \frac{(\eta^2 + i \beta)^2}{\eta^4 + \beta^2} \right) \left( \eta^2 - i \beta \right)^{k-2} \right\}.$$

(4.2.40)

An alternative method for deriving $\langle u_1 \rangle_{conv}$ is through the use of a Greens function. The derivation of such a Greens function for solving (4.2.35) is given in §A of the appendix. By solving (4.2.35) through the use of this Greens function, we derive the following alternative expression for $\langle u_1 \rangle_{conv}$:

$$\langle u_1 \rangle_{conv} = \int_{0}^{1} (r^3 - r) f_1(\eta, \beta) \, dr,$$

(4.2.41)
where
\[
f_1 = \left\{ \begin{array}{l}
\beta^2 I_0(\eta) \left[ \sqrt{\eta^2 + i\beta} I_1(r\eta) I_0 \left( \sqrt{\eta^2 + i\beta} \right) I_1 \left( r \sqrt{\eta^2 - i\beta} \right) \\
+ \left( \sqrt{\eta^2 - i\beta} I_1(r\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right) \\
- 2\eta I_0(\eta) I_1 \left( r \sqrt{\eta^2 - i\beta} \right) I_1 \left( r \sqrt{\eta^2 + i\beta} \right) \right) \right] \\
\times \left\{ 2 \left[ 4 \left( \sqrt{\eta^2 - i\beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 - i\beta} \right) - \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 - i\beta} \right) \right) \\
\times \left( \eta I_0(\eta) I_1 \left( \sqrt{\eta^2 + i\beta} \right) - \sqrt{\eta^2 + i\beta} I_1(\eta) I_0 \left( \sqrt{\eta^2 + i\beta} \right) \right) \right]^{-1} \right\}.
\end{array} \right.
\]

The asymptotic behaviour of \( \langle u_1 \rangle_{BC} \), \( \langle u_1 \rangle_{conv} \) and \( \langle u_1 \rangle \) are detailed in table 4.1. Note that in the low \( \eta \) and high \( \beta \) case, as well as in the high \( \eta \) and high \( \beta/\eta^2 \) case, \( \langle u_1 \rangle_{BC} \) and \( \langle u_1 \rangle_{conv} \) asymptote towards values that are equal in magnitude and opposite in sign. As a result, the asymptotic behaviour of \( \langle u_1 \rangle \), in this limit, is due to a higher order term within \( \langle u_1 \rangle \).

A notable feature of the component of the first order flow that results from wall-entrainment is that it appears to violate the no-slip boundary condition. While the effect of wall-entrainment is to drag the fluid along with the motion of the travelling wave at the wall, the no-slip boundary condition must nonetheless apply. The reason why it appears not to in this analysis is that there exists a thin boundary layer in which the velocity is small. The width of this boundary layer decreases as \( \alpha \) is reduced, becoming infinitesimal in the limit as \( \alpha \to 0 \). In our perturbation analysis, this manifests as a non-zero streamwise boundary condition at first order.

### 4.2.3 Boundary conditions of long wavelength

It is immediately apparent that \( \langle u_1 \rangle_{conv} \) is maximised by minimising \( \eta \) (the ratio of the average radius of the tube to the wavelength of the oscillations of the tube wall). In other words, the effect of convection is greatest when the wavelength of the oscillations of the wall of the tube is long. The equivalent is shown to be true of transpiration within channels in §3.2.4. The other contribution to the first
4.2. Perturbation analysis

Figure 4.2: Plots of $\langle u_1 \rangle_{BC}$, the first order bulk flow due to wall-entrainment, against $\beta (= \rho h^2 \omega / \mu)$ and $\eta (= h / \lambda)$. 
Figure 4.3: Plots of $\langle u_1 \rangle_{\text{conv}}$, the first order bulk flow due to convection, against $\beta (= \rho h^2 \omega / \mu)$ and $\eta (= h / \lambda)$.
4.2. Perturbation analysis

\begin{align*}
\langle u_1 \rangle &= h = 1 \\
\langle u_1 \rangle &= h = 0.1 \\
\langle u_1 \rangle &= h = 10 \\
\langle u_1 \rangle &= h = 100 \\
0.01 &< 1 < 100 < 10 < 4 < 10 < 6 \\
\beta = 1 \\
\beta = 10^8 \\
\eta = 0.1 \\
\eta = 10 \\
\eta = 100 \\
\end{align*}

Figure 4.4: Plots of $\langle u_1 \rangle$, the total first order bulk flow, against $\beta (= \rho h^2 \omega / \mu)$ and $\eta (= h/\lambda)$. Here we can see that $\langle u_1 \rangle$ has only a small dependence upon $\beta$. In fact, the graph in the centre shows only two very disparate values of $\beta$, in order to demonstrate how small this dependence is. Since $\langle u_1 \rangle$ decreases monotonically with increasing $\beta$, all intermediate values of $\beta$ will produce curves somewhere between these two.
4.2. Perturbation analysis

Table 4.1: Asymptotic behaviours of the first order correction to the bulk flow, in terms of the Stokes number, $\beta = \rho h^2 \omega / \mu$, and the ratio of the average radius of the tube to the wavelength of the deformations of the boundary conditions, $\eta = h / \lambda$. Here $\langle u_1 \rangle_{BC}$ and $\langle u_1 \rangle_{conv}$ refer to the contributions due to the boundary conditions and convection respectively, while $\langle u_1 \rangle$ refers to the total of the first order correction to the bulk flow.

<table>
<thead>
<tr>
<th>$\eta \to 0$, $\beta \to 0$</th>
<th>$\langle u_1 \rangle_{BC}$</th>
<th>$\langle u_1 \rangle_{conv}$</th>
<th>$\langle u_1 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta \to 0$, $\beta \to \infty$</td>
<td>$\frac{1}{\sqrt{2}} \frac{\sqrt{\beta}}{\eta}$</td>
<td>$\frac{1}{\sqrt{2}} \frac{\sqrt{\beta}}{\eta}$</td>
<td>$\frac{1}{\eta}$</td>
</tr>
<tr>
<td>$\eta \to \infty$, $\frac{\beta}{\eta^2} \to 0$</td>
<td>$\frac{1}{2} \eta$</td>
<td>$-\frac{3}{32} \eta \left( \frac{\beta}{\eta^2} \right)^2$</td>
<td>$\frac{1}{2 \eta}$</td>
</tr>
<tr>
<td>$\eta \to \infty$, $\frac{\beta}{\eta^2} \to \infty$</td>
<td>$\frac{1}{2 \sqrt{2}} \frac{\sqrt{\beta}}{\eta^2}$</td>
<td>$-\frac{1}{2 \sqrt{2}} \frac{\sqrt{\beta}}{\eta^2}$</td>
<td>$\frac{1}{4 \eta}$</td>
</tr>
</tbody>
</table>

order bulk blow, $\langle u_1 \rangle_{BC}$, which results from wall-entrainment, also increases with decreasing $\eta$.

In fact, both the convective and wall-entrainment components of the bulk flow exhibit a simple inversely proportional dependence upon $\eta$ in the low $\eta$ limit. It is possible to express $\langle u_1 \rangle$ as a closed form function in the limit as $\eta \to 0$. To this end, we first recognise that in this limit,

$$\langle u_1 \rangle \propto \frac{1}{\eta}, \quad (4.2.42)$$

as well as that

$$\int_0^1 (r^3 - r) I_1 (\eta^2 r) I_1 \left( \sqrt{\eta^2 + i \beta r} \right) dr = i \frac{\eta}{\beta} I_3 \left( \sqrt{i \beta} \right) + O \left( \eta^2 \right), \quad (4.2.43)$$

$$\int_0^1 (r^3 - r) I_1 (\eta^2 r) I_1 \left( \sqrt{\eta^2 - i \beta r} \right) dr = \frac{\eta}{\beta} I_3 \left( \sqrt{i \beta} \right) + O \left( \eta^2 \right), \quad (4.2.44)$$
\[
\int_0^1 (r^3 - r) I_1 \left( \sqrt{\eta^2 + i\beta r} \right) I_1 \left( \sqrt{\eta^2 - i\beta r} \right) \, dr \\
= \int_0^1 (r^3 - r) I_1 \left( \sqrt{i\beta r} \right) I_1 \left( \sqrt{-i\beta r} \right) \, dr + O(\eta^2), \quad (4.2.45)
\]

where \( J_n(x) \) represents a Bessel function. These allow us to express the limiting behaviour of (4.2.41) with just one pair of modified Bessel functions within the integral. We can further simplify equation (4.2.45) above through the use of (B.6), which allows one to express the product of the two modified Bessel functions as an infinite sum. The integral can then be exactly evaluated, and the result can be re-expressed as a closed form function.

In this way, we derive the following asymptotic behaviour in the small \( \eta \) limit

\[
\langle u_1 \rangle \sim \frac{1}{\eta} \left( \frac{-i\beta I_1(\sqrt{-i\beta})}{2I_2(\sqrt{-i\beta})} \right) + \frac{1}{\eta} f_2(\beta).
\]

(4.2.46)

The first term on the right hand side above results from the non-zero boundary conditions, and is therefore due to the effect of wall-entrainment. The second term is due to convection. The function \( f_2(\beta) \) is given by

\[
f_2(\beta) = -\left\{ 2 \left[ \sqrt{\beta} \text{ber}_0 \left( \sqrt{\beta} \right)^2 + \sqrt{\beta} \text{bei}_0 \left( \sqrt{\beta} \right)^2 + \sqrt{2} \left( \text{ber}_0 \left( \sqrt{\beta} \right) \right. \\
- \left. \text{bei}_1 \left( \sqrt{\beta} \right) \right] + \text{bei}_0 \left( \sqrt{\beta} \right) \left( \text{ber}_1 \left( \sqrt{\beta} \right) \right) \\
+ \text{bei}_1 \left( \sqrt{\beta} \right) \right\} + \sqrt{\beta} \left[ \left( \sqrt{-i\beta} I_0 \left( \sqrt{-i\beta} \right) I_3 \left( \sqrt{-i\beta} \right) \right) + \sqrt{i\beta} I_0 \left( \sqrt{-i\beta} \right) I_3 \left( \sqrt{i\beta} \right) \right] / 2 \sqrt{\beta} I_2 \left( \sqrt{-i\beta} \right) I_2 \left( \sqrt{i\beta} \right),
\]

(4.2.47)

where \( \text{ber}_n(x) \) and \( \text{bei}_n(x) \) are the Kelvin functions, which are defined respectively as the real and imaginary components of

\[
e^{n\pi i} J_n \left( xe^{-\pi i/4} \right).
\]

(4.2.48)

As can clearly be seen in figures 4.2 & 4.3, the above asymptotic forms are decent approximations for \( \langle u_1 \rangle_{BC} \) and \( \langle u_1 \rangle_{conv} \) for \( \eta < 1 \). Plots of the bulk flow in
Figure 4.5: Plots of the asymptotic behaviour of the total bulk flow, and its convective and wall-entrainment components, in the limit as \( \eta \to 0 \). These are reasonable approximations for the bulk flow for \( \eta < 1 \). Here we can clearly see that while the effect of wall-entrainment dominates at low \( \beta \), the effect of convection becomes comparatively more important as \( \beta \) is increased.

What is immediately clear in figure 4.5 is that despite resulting from the combined effects of two different physical mechanisms, \( \eta \langle u_1 \rangle \) behaves in a very simple manner in the small \( \eta \) limit. In fact it decreases monotonically with \( \beta \), never exceeding 4 and always greater than 3.

### 4.2.4 Boundary conditions of short wavelength

The behaviour of the flow is different at large \( \eta \), where the average radius of the tube is significantly greater than the wavelength of the oscillations at the wall. The bulk flow due to convection, \( \langle u_1 \rangle_{\text{conv}} \), diminishes in the large \( \eta \) limit. This is because a large \( \eta \) is equivalent to having wall oscillations of small wavelength. When this wavelength is small, the effects of the adjacent peaks and troughs, within the flow, will combine. The effect is therefore similar to that of high \( \beta \),
mentioned in §4.2.3. Naturally, a similar phenomenon occurs in flows driven by transpiration, which has been discussed in detail in §3.2.5.

However, there is again a counteracting effect if $\beta$ is sufficiently high. In this case, it occurs in the large $\beta/\eta^2 \to \infty$ limit. The effect is again due to the fact that the radial velocity at the wall is proportional to the oscillating frequency of the wall.

The contribution to the bulk flow that is due to wall-entrainment, $(u_1)_{BC}$, increases monotonically with decreasing $\eta$, once $\eta$ has exceeded a critical value.

### 4.2.5 Bulk flow at high $\eta$ and low $\beta/\eta^2$

In this section, we derive the exact bulk flow, at any $\alpha$, in the limits as $\eta \to \infty$ and $\beta/\eta^2 \to 0$. As has been shown, in this limit, all swirling motions, and their subsequent effects upon higher order terms, are negligible. The other fact that simplifies derivations in this limit is that the streaming motion that is induced by the the deformation of the boundary is a plug flow. That is, the magnitude of this streaming velocity (which is the non-swirling component of the velocity) is independent of the radial location.

The implication of this is that since there are negligible velocity gradients within such flows at high $\eta$, the effect of convection upon higher orders will also be negligible. Therefore, at any order, the only component flow that will need to be determined will be the streaming flow induced by the deformation of the boundary. This in turn requires only knowledge of the leading order flow field to determine, since all other swirling components of the flow will be comparatively negligible. In light of this, equation (4.2.6d), for the boundary conditions acting upon higher order components of the flow, becomes

$$u_n(1, z, t) = \frac{(-1)^{n+1}}{n!} \sin^n(\eta z - t) \frac{\partial^n}{\partial r^n} u_0(1, z, t), \quad \text{for all } n > 0. \quad (4.2.49)$$

Following the example of equations (4.2.8) and (4.2.32), the steamwise velocity at any odd numbered order, $n$, can be split into

$$u_n(R, t) = U_{n,0}(r) + U_{n,2}(r)e^{2i(\eta z - t)} + U_{n,-2}(r)e^{-2i(\eta z - t)} + \cdots + U_{n,n+1}(r)e^{(n+1)i(\eta z - t)} + U_{n,-(n+1)}(r)e^{-(n+1)i(\eta z - t)}, \quad n \text{ odd.} \quad (4.2.50)$$
Again for odd \( n \), we use as our expression of the \( n \)th derivative of a modified Bessel function:

\[
\frac{\partial^n}{\partial r^n} I_0(\alpha r) = \frac{\alpha^n}{2^{n-1}} \sum_{m=0}^{(n-1)/2} \binom{n}{m} I_{n-2m}(\alpha r), \quad n \text{ odd. (4.2.51)}
\]

Therefore, substituting the above into equation (4.2.49), and substituting the resulting function into (4.2.1a), we obtain the following result for the bulk velocity at high \( \eta \):

\[
\langle u \rangle =
\sum_{n \text{ odd}} \frac{\alpha^n}{2^{2n-1}n!} \left( \binom{n}{(n-1)/2} \sum_{m=0}^{(n-1)/2} \binom{n}{m} \right)
\times \left[ \frac{\eta^n I_0(\sqrt{\eta^2 - \beta i}) I_{n-2m}(\eta) - (\eta^2 - \beta i)^{n/2} I_0(\eta) I_{n-2m}(\sqrt{\eta^2 - \beta i})}{2I_0(\sqrt{\eta^2 - \beta i}) I_1(\eta) - \eta I_0(\eta) \, _0F_1(2; \frac{1}{4} (\eta^2 - \beta i))} \right]
\times \left[ \frac{\eta^n I_0(\sqrt{\eta^2 + \beta i}) I_{n-2m}(\eta) - (\eta^2 + \beta i)^{n/2} I_0(\eta) I_{n-2m}(\sqrt{\eta^2 + \beta i})}{2I_0(\sqrt{\eta^2 + \beta i}) I_1(\eta) - \eta I_0(\eta) \, _0F_1(2; \frac{1}{4} (\eta^2 + \beta i))} \right].
\]

\[
(4.2.52)
\]

It is also worth noting that in the high \( \eta \) limit,

\[
u(R, t) = u(1, z, t) = \langle u \rangle,
\]

since the flow is effectively a plug flow in this limit.

### 4.2.6 Generalised boundary conditions

It is a simple procedure to extend these results to any pipe flow driven by periodic small-amplitude oscillations in the radius of the pipe. All flows that have been considered so far are driven by oscillations that take the form of a single sine wave. Since any periodic function may be decomposed into a Fourier series, any periodic oscillations may be decomposed into a sum of sine and cosine waves, the arguments of each of which will be multiples of either \( \eta x - t \) or \( \eta x + t \) (indicating
downstream travelling waves and upstream travelling waves respectively), these can each be referred to as Fourier modes.

Fundamental to this method of generalising the result is the fact that each Fourier mode of the leading order flow may only contribute to the first order bulk flow by interacting with itself. In other words, it is only the convection of each individual Fourier mode of the leading order flow by itself that can contribute to $\langle u_1 \rangle_{\text{conv}}$. Similarly, it is only the interaction of one Fourier mode of the oscillation of the wall with the same Fourier mode of the leading order flow adjacent to the wall that contributes to $\langle u_1 \rangle_{\text{BC}}$. Only at second, or higher, order may two different Fourier modes interact to produce a bulk flow.

Hence we introduce a new function $U(\beta, \eta, \kappa)$ to be the bulk flow induced at a specified Stokes number, $\beta$, driven by an oscillating wall, whose radius is given by

$$R(x, t) = 1 - \alpha \left[ \sin(\eta x - t) + \kappa \cos(\eta x - t) \right].$$  (4.2.54)

This new function therefore relates to $\langle u_1 \rangle$ via

$$U(\beta', \eta', 0) \overset{\text{def}}{=} \langle u_1 \rangle; \quad \beta = \beta', \eta = \eta', \kappa = 0.$$  (4.2.55)

By an entirely analogous derivation to that detailed in §4.2, it can be shown that the above boundary condition produces a bulk flow of

$$U(\beta, \eta, \kappa) = (1 + \kappa^2) U(\beta, \eta, 0).$$  (4.2.56)

Reversing the direction of the travelling wave at the wall (or equivalently reversing the sign of $\beta$) naturally reverses the direction of the induced bulk flow, so that

$$U(-\beta, \eta, \kappa) = -U(\beta, \eta, \kappa).$$  (4.2.57)

If we then express the deformation of the wall as

$$R(x, t) = 1 - \sum_{n=1}^{\infty} \left\{ A_n \left[ \sin n(\eta x - t) + \kappa_n \cos n(\eta x - t) \right] \ight.$$

$$- B_n \left[ \sin n(\eta x + t) + \kappa_n \cos n(\eta x + t) \right] \right\},$$  (4.2.58)

then the bulk flow at first order can be written as

$$\langle u_1 \rangle = \sum_{n=1}^{\infty} \left[ A_n (1 + \kappa_n^2) U(n\beta, n\eta, 0) - B_n (1 + \kappa_n^2) U(n\beta, n\eta, 0) \right].$$  (4.2.59)
4.3 Energy considerations

There are two mechanisms by which the oscillation of the wall may impart energy to the flow: The first is the transfer of momentum from the moving wall to the fluid. The second is due to the motion of the wall against a local pressure gradient. In a steady state system, the rate at which energy is imparted to the flow at the walls must be equal to the rate at which energy is dissipated within the flow.

This energy balance can be derived from the Navier-Stokes equation: First we take the dot product of the velocity, \( \mathbf{u}(x, t) \), with the Navier-Stokes equation (4.1.8). Then we average each term in the resulting equation over the entire flow field, using the average as defined in (4.1.15). After rearranging a few terms via integration by parts, we arrive at

\[
\beta \frac{\partial}{\partial t} \langle u^2 \rangle = \mathcal{W} - \langle \epsilon \rangle.
\]

The term on the left hand side above represents the average kinetic energy of the entire system. On the right hand side, \( \mathcal{W} \) represents the rate at which energy is imparted to the fluid at the walls and \( \langle \epsilon \rangle \) represents the average rate at which energy is dissipated within the flow. These terms are equal to the following.

\[
\mathcal{W} = - \lim_{L \to \infty} \frac{2}{L} \int_{-L/2}^{L/2} \left[ w p + \frac{\alpha \beta}{2} w^3 \right]_{r=R(z,t)} dx,
\]

and

\[
\langle \epsilon \rangle = \langle |\nabla \mathbf{u}|^2 \rangle.
\]

The \((\alpha \beta/2) w^3\) term in (4.3.2) above represents the transfer of momentum from the wall to the fluid, while the \(wp\) term represents the energy imparted by the motion of the wall against a local pressure gradient. Either of these terms may, of course, be negative or zero. In steady state systems, the total energy of the system is a constant, and hence the time derivative term in (4.3.1) will be zero. It therefore follows that for steady state systems,

\[
\langle \epsilon \rangle = \mathcal{W}.
\]

Since this present work investigates exclusively steady state systems, \( \mathcal{W} \) and \( \langle \epsilon \rangle \) will hereafter be treated as equivalent terms. We therefore choose to refer from
4.3. Energy considerations

this point onwards to $\langle \epsilon \rangle$, with the implication that for these systems it may represent equivalently either the dissipation rate or the rate of energy transfer to the fluid.

We can derive $\langle \epsilon \rangle$ in the small $\alpha$ limit through a similar method to that by which the bulk flow in this limit as been derived. We first expand $\langle \epsilon \rangle$ in $\alpha$ as follows:

$$\langle \epsilon \rangle = \langle \epsilon_0 \rangle + \alpha \langle \epsilon_1 \rangle + \alpha^2 \langle \epsilon_2 \rangle + \ldots$$  \hspace{1cm} (4.3.5)

The pressure and velocity are again expanded in $\alpha$ as in (4.2.1). The location of the boundary is also expanded in $\alpha$ in the same manner as before, which is defined in (4.2.5). In this way we find that the leading order component of the dissipation is

$$\langle \epsilon_0 \rangle = - \lim_{L \to \infty} \frac{2}{L} \int_{-L/2}^{L/2} [w_0 p_0]_{r=1}. \hspace{1cm} (4.3.6)$$

Hence, at leading order, the energy imparted to the flow derives entirely from the motion of the wall against a local pressure gradient. This is derived in §4.2.1, and the value $\langle \epsilon_0 \rangle$ is given in equation (4.2.30). The limiting behaviour of $\langle \epsilon_0 \rangle$ is

$$\langle \epsilon_0 \rangle \sim \begin{cases} 
\frac{8}{\eta^2}, & \eta \to 0, \beta \to 0 \\
\sqrt{2} \frac{\sqrt{\beta}}{\eta^2}, & \eta \to 0, \beta \to \infty \\
\eta, & \eta \to \infty, \frac{\beta}{\eta^2} \to 0 \\
\frac{1}{2\sqrt{2}} \eta \sqrt{\frac{\beta}{\eta^2}}, & \eta \to \infty, \frac{\beta}{\eta^2} \to \infty
\end{cases} \hspace{1cm} (4.3.7)$$

If we compare the asymptotic behaviour of $\langle \epsilon_0 \rangle$, found in table 4.6, to that of $\langle u_1 \rangle$, found in table 4.1, we gain some insight into the most energy efficient arrangement of the pump. At low $\eta$, the bulk flow increases as $\eta^{-1}$. However, in this limit, the dissipation increases as $\eta^{-2}$. We therefore have a diminishing return, in terms of increases bulk flow, for the energy input, as we decrease $\eta$.

This contrasts with the high $\eta$ limit, in which dissipation increases only as $\eta$ when $\beta/\eta^2$ is small, and becomes independent of $\eta$ when $\beta/\eta^2$ is large. As a result, pumping is more energy efficient at high $\eta$. 
Figure 4.6: Plots of $\langle \epsilon_0 \rangle$, the leading order of the average rate of dissipation within the flow, against $\beta$ and $\eta$. 
We can now determine the efficiency of this pumping method, which we define as the induced bulk flow divided by the energy input required to produce it, so that

\[
\text{Efficiency} = \frac{\langle u \rangle}{\langle \epsilon \rangle} = \frac{\langle u_1 \rangle}{\langle \epsilon_0 \rangle} \alpha + O(\alpha^2).
\] (4.3.8)

The limiting behaviour of the first order component of the efficiency is given by

\[
\text{Efficiency}_1 \sim \begin{cases} 
\frac{1}{2} \eta, & \eta \to 0, \beta \to 0 \\
\frac{3}{\sqrt{2}} \frac{\eta}{\sqrt{\beta}}, & \eta \to 0, \beta \to \infty \\
\frac{1}{2}, & \eta \to \infty, \frac{\beta}{\eta^2} \to 0 \\
\frac{1}{\sqrt{2}} \sqrt{\frac{\eta^2}{\beta}}, & \eta \to \infty, \frac{\beta}{\eta^2} \to \infty
\end{cases}
\] (4.3.9)

It can be seen above that it is only the \( \eta \to \infty, \beta/\eta^2 \to 0 \) case in which energy cost does not outstrip the induced bulk flow as we approach the limit.

## 4.4 Channel flows

In this section, we derive the asymptotic behaviour of flows within channels driven by oscillating walls. We consider a channel of infinite length and width and finite height. The system and its parameters are that same within this section, as they have been throughout this work, with the exception that \( h \) represents here the height of the channel. A diagram of the channel, in dimensional coordinates, is given in figure 4.7.

The parameters, \( \beta, \eta, Re \) and \( \alpha \), defined in §4.1, also apply within this section. Because in a channel flow the boundary conditions at either wall may be defined independently, the oscillations at either wall are permitted to be out of phase by a degree \( \phi \), and to differ in amplitude by a ratio \( \gamma \), which is taken to be between 0 and 1. Here, \( u \) is taken to represent the velocity in the streamwise direction, while
Figure 4.7: Diagram of the channel domain, in scaled coordinates. We consider an infinite channel in which $L \to \infty$. Because of the inherent symmetry of the flow, we may neglect the spanwise (or $y$) dimension.
4.4. Channel flows

\( w \) represents the wall-normal velocity. Here, \( x \) represents the streamwise location, while \( z \) represents the wall-normal location.

The location of the upper wall is denoted by \( Z^+(x, t) \), in non-dimensional units. It is given by

\[
Z^+(x, t) = 1 + \alpha \cos(\eta x - t). \tag{4.4.1}
\]

The location of the lower wall is denoted by \( Z^-(x, t) \), and it is given by

\[
Z^-(x, t) = \alpha \gamma \cos(\eta x - t - \phi). \tag{4.4.2}
\]

This is very similar to the system driven by transpiration, that has been detailed in §3. There are three important mathematical differences between the system studied in §3 and the system we consider here: Here the boundary wave travels in the forward direction, whereas in §3 it travels backwards; the velocities at the walls are equivalent, but the locations of the walls differ; and the physical meanings of the dimensionless parameters \( \beta, Re \) and \( \alpha \) are not the same.

Mathematically, however, the two systems are very similar. A system driven by transpiration would be equivalent to one driven by oscillating walls were it not for the effect of wall-entrainment. By expanding the velocity in \( \alpha \) in the same manner as before, we may determine the leading order of the velocity, and the component of the first order correction that is due to convection, by following the derivation in §3, remembering always to place a negative sign in front of the bulk flow, to account for the fact that the boundary wave travels in the opposite direction in that chapter.

The boundary conditions can be determined by expanding the velocities at either wall, \( u(x, Z^+, t) \) and \( u(x, Z^-, t) \), using a Taylor series around \( \alpha = 0 \), in a similar manner to (4.2.5). Doing so, we find that the boundary conditions at leading order are given by

\[
u_0(x, 1, t) = (0, \sin(\eta x - t)), \tag{4.4.3}
\]

\[
u_0(x, 0, t) = (0, \gamma \sin(\eta x - t - \phi)). \tag{4.4.4}
\]

The boundary conditions conditions that apply to the first order velocity are given by

\[
u_1(x, 1, t) = -\cos(\eta x - t) \frac{\partial}{\partial z} \nu_0(x, 1, t), \tag{4.4.5}
\]

\[
u_1(x, 0, t) = -\gamma \cos(\eta x - t - \phi) \frac{\partial}{\partial z} \nu_0(x, 0, t). \tag{4.4.6}
\]
Table 4.2: Asymptotic behaviours of the first order correction to the bulk flow in a channel, in terms of the Stokes number, $\beta = \rho h^2 \omega / \mu$, the ratio of the average radius of the tube to the wavelength of the deformations of the boundary conditions, $\eta = h / \lambda$, the phase difference between the oscillations at either wall, $\phi$, and the ratio of amplitudes of the oscillations at either wall, $\gamma$. Here $\langle u_1 \rangle_{BC}$ and $\langle u_1 \rangle_{conv}$ refer to the contributions due to wall-entrainment and convection respectively, while $\langle u_1 \rangle$ refers to the total of the first order correction to the bulk flow.

<table>
<thead>
<tr>
<th>Condition</th>
<th>$\langle u_1 \rangle_{BC}$</th>
<th>$\langle u_1 \rangle_{conv}$</th>
<th>$\langle u_1 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta \to 0$, $\beta \to 0$</td>
<td>$\frac{3(1+\gamma^2-2\gamma \cos \phi)}{2 \eta}$</td>
<td>$\frac{1+\gamma^2-2\gamma \cos \phi}{5040 \eta}$</td>
<td>$\frac{3(1+\gamma^2-2\gamma \cos \phi)}{2 \eta}$</td>
</tr>
<tr>
<td>$\eta \to 0$, $\beta \to \infty$</td>
<td>$\frac{1+\gamma^2-2\gamma \cos \phi}{4 \sqrt{2} \eta}$</td>
<td>$\frac{1+\gamma^2-2\gamma \cos \phi}{4 \sqrt{2} \eta}$</td>
<td>$\frac{5(1+\gamma^2-2\gamma \cos \phi)}{4 \eta}$</td>
</tr>
<tr>
<td>$\eta \to \infty$, $\frac{\beta}{\eta^2} \to 0$</td>
<td>$\frac{1+\gamma^2}{4 \eta}$</td>
<td>$\frac{3(1+\gamma^2)}{64 \eta^2} \left(\frac{\beta}{\eta^2}\right)^2$</td>
<td>$\frac{1+\gamma^2}{4 \eta}$</td>
</tr>
<tr>
<td>$\eta \to \infty$, $\frac{\beta}{\eta^2} \to \infty$</td>
<td>$\frac{1+\gamma^2}{4 \sqrt{2} \eta^2} \sqrt{\frac{\beta}{\eta^2}}$</td>
<td>$\frac{1+\gamma^2}{4 \sqrt{2} \eta^2} \sqrt{\frac{\beta}{\eta^2}}$</td>
<td>$\frac{1+\gamma^2}{8 \eta}$</td>
</tr>
</tbody>
</table>
4.4. Channel flows

The derivation of the leading order velocity is very similar to the equivalent derivation in §3.2.1, since the only mathematical difference between the two is that the boundary wave propagates in the opposite direction in this case, and so will not be reproduced here. By then substituting the leading order velocity into equations (4.4.5) and (4.4.6) above, we obtain the boundary conditions applying to the first order correction. Using these, we may now derive the bulk flow via an entirely analogous derivation to §3.2.1.

We can thereby determine the component of the bulk flow that is due to wall-entrainment, which results mathematically from the non-zero boundary conditions. It is given by

\[
\langle u_1 \rangle_{BC} = - \frac{1}{4} \left[ \eta^2 c_1 \left( \gamma e^{i\phi} + e^{\eta} \right) + \eta^2 c_2 \left( \gamma e^{i\phi} + e^{-\eta} \right) + \left( \eta^2 + \beta i \right) c_3 \left( \gamma e^{i\phi} + e^{\sqrt{\eta^2 + \beta^2}} \right) + \left( \eta^2 + \beta i \right) c_4 \left( \gamma e^{i\phi} + e^{-\sqrt{\eta^2 + \beta^2}} \right) + \eta^2 c_1^* \left( \gamma e^{-i\phi} + e^{\eta} \right) + \eta^2 c_2^* \left( \gamma e^{-i\phi} + e^{-\eta} \right) + \left( \eta^2 - \beta i \right) c_3^* \left( \gamma e^{-i\phi} + e^{\sqrt{\eta^2 - \beta^2}} \right) + \left( \eta^2 - \beta i \right) c_4^* \left( \gamma e^{-i\phi} + e^{-\sqrt{\eta^2 - \beta^2}} \right) \right],
\]

(4.4.7)

where the constants \( c_1, c_1^*, c_2, \) etc. are defined in equation (3.2.13). The leading order bulk flow due to convection, \( \langle u_1 \rangle_{conv} \), is equal in magnitude, and opposite in sign, to the bulk flow due to transpiration, which is given in equation (3.2.25).

The asymptotic behaviours of \( \langle u_1 \rangle_{BC} \) and \( \langle u_1 \rangle_{conv} \), as well as the total bulk flow, \( \langle u_1 \rangle \), are detailed in table 4.2. The asymptotic behaviour of the dimensional velocity \( \langle \hat{u} \rangle \), and its two constituent components, \( \langle \hat{u} \rangle_{BC} \) and \( \langle \hat{u} \rangle_{conv} \), are detailed in table 4.3.

It can be seen that the behaviours of channel flows are qualitatively similar to the equivalent flows within a tube. The assumption that the tube must be axisymmetric clearly results in a system that is physically similar to a channel flow with \( \phi = \pi \) and \( \gamma = 1 \).

Tube flows will generally induce similar bulk flows to their equivalent channel flows for most values of the parameters \( \eta, \beta, \gamma \) and \( \phi \). The exception to this rule
is found in cases in which \( \eta < 1, \phi \approx 0 \) and \( \gamma \approx 1 \). Here, because the oscillations of either wall are equal in magnitude and opposite in sign, the effect is merely to shuffle the fluid back and forth. The reason for this is that to reduce \( \eta \) is effectively to reduce the height of the channel. Because the oscillations are defined to be equal in this case, a short channel contains very little space for a velocity gradient to develop within the fluid. Since the velocity gradients are small, the effect of convection will also be small.

4.4.1 Energy considerations

In non-dimensional units, the rate at which the oscillating walls impart energy to the flow (or equivalently the rate of dissipation), \( \langle \epsilon \rangle \), is, to leading order, mathematically identical to the equivalent rate in a system driven by transpiration. We therefore do not need to rederive \( \langle \epsilon_0 \rangle \) here, since it has been derived for transpiration in §3.3. From this we may derive the following asymptotic behaviours for the rate of dissipation within the flow, in dimensional units, in the limit as \( A/h \to 0 \):

\[
\langle \epsilon \rangle \sim \begin{cases} 
6(1 + \gamma^2 - 2\gamma \cos \phi) \frac{\rho \lambda^2 \omega^3 A^2}{h^2}, & h \to 0, \ h^2 \omega \nu \to 0 \\
\frac{1 + \gamma^2 - 2\gamma \cos \phi}{\sqrt{2}} \frac{\rho \lambda^3 \omega^{7/2} A^2}{h \sqrt{\nu}}, & h \to 0, \ h^2 \omega \nu \to \infty \\
\frac{(1 + \gamma^2) \rho h \omega^3 A^2}{\lambda}, & h \to \infty, \ \frac{\lambda^2 \omega}{\nu} \to 0 \\
\frac{1 + \gamma^2 \rho \lambda \omega^{7/2} A^2}{2\sqrt{2} \nu}, & h \to \infty, \ \frac{\lambda^2 \omega}{\nu} \to \infty 
\end{cases}
\] (4.4.8)

4.5 Discussion

The perturbation expansion presented within §4.2 demonstrates the two mechanisms by which a travelling wave propagating along the wall of a tube may induce a bulk flow. These two mechanisms are convection and the dragging of fluid by the travelling wave. This dragging effect we refer to here as “wall-entrainment”. The effects of these two mechanisms become disentangled in the limit as \( \alpha \to 0 \), and may therefore be considered separately. For any finite value of \( \alpha \), however, this separation should not be assumed to hold.
Table 4.3: Asymptotic behaviours of the bulk flow in a channel, in terms of dimensional parameters. These parameters are: $h$, the average radius of the tube, which is the radius of the pipe in the absence of deformation; $A$, the amplitude of oscillations of tube’s radius; and $\lambda$, the wavelength of the oscillations. The other quantities that have been used to non-dimensionalise the quantities of the Navier-Stokes equation are the frequency, $\omega$, of the oscillations of the boundary condition, $\phi$, the phase difference between the waves at either wall, $\gamma$ the ratio of the amplitudes of the waves at either wall, and the density, $\rho$ and kinematic viscosity, $\nu (= \mu/\rho)$, of the fluid. Here $\langle \hat{w} \rangle_{BC}$ and $\langle \hat{w} \rangle_{conv}$ refer to the contributions due to the boundary conditions and convection respectively, while $\langle \hat{u} \rangle$ refers to the total of the first order correction to the bulk flow. The average rate of dissipation per unit of volume, within the fluid, is given by $\langle \hat{\epsilon} \rangle$. These apply strictly in the limit as $A/h \rightarrow 0$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda$</th>
<th>$\frac{h^2 \omega}{\nu}$</th>
<th>$\langle \hat{u} \rangle_{BC}$</th>
<th>$\langle \hat{u} \rangle_{conv}$</th>
<th>$\langle \hat{u} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{h}{\lambda} \rightarrow 0$,</td>
<td>$\frac{h^2 \omega}{\nu} \rightarrow 0$</td>
<td>$\frac{3(1+\gamma^2-2\gamma \cos \phi)}{2} \frac{\lambda \omega A^2}{h^2}$</td>
<td>$\frac{1+\gamma^2-2\gamma \cos \phi}{5040} \frac{h^2 \lambda \omega^3 A^2}{\nu^2}$</td>
<td>$\frac{3(1+\gamma^2-2\gamma \cos \phi)}{2} \frac{\lambda \omega A^2}{h^2}$</td>
<td></td>
</tr>
<tr>
<td>$h \rightarrow 0$,</td>
<td>$\frac{h^2 \omega}{\nu} \rightarrow \infty$</td>
<td>$\frac{1+\gamma^2-2\gamma \cos \phi}{4\sqrt{2}} \frac{\lambda \omega^{3/2} A^2}{h \sqrt{\nu}}$</td>
<td>$\frac{1+\gamma^2-2\gamma \cos \phi}{4\sqrt{2}} \frac{\lambda \omega^{3/2} A^2}{h \sqrt{\nu}}$</td>
<td>$\frac{5(1+\gamma^2-2\gamma \cos \phi)}{4} \frac{\lambda \omega A^2}{h^2}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{h}{\lambda} \rightarrow \infty$,</td>
<td>$\frac{\lambda^2 \omega}{\nu} \rightarrow 0$</td>
<td>$\frac{1+\gamma^2 \omega A^2}{4}$</td>
<td>$\frac{3(1+\gamma^2) \lambda^3 \omega^3 A^2}{64} \frac{\nu^2}{\lambda}$</td>
<td>$\frac{1+\gamma^2 \omega A^2}{4}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{h}{\lambda} \rightarrow \infty$,</td>
<td>$\frac{\lambda^2 \omega}{\nu} \rightarrow \infty$</td>
<td>$\frac{1+\gamma^2 \omega^{3/2} A^2}{4\sqrt{2}} \frac{\lambda}{\sqrt{\nu}}$</td>
<td>$\frac{1+\gamma^2 \omega^{3/2} A^2}{4\sqrt{2}} \frac{\lambda}{\sqrt{\nu}}$</td>
<td>$\frac{1+\gamma^2 \omega A^2}{8}$</td>
<td></td>
</tr>
</tbody>
</table>
Neither can it necessarily be assumed that the nature of the effect upon the flow of either convection or wall-entrainment will be as they are here for all values of $\alpha$. While it would be quite unexpected if wall-entrainment were ever to induce a flow in the direction opposite to oscillation of the wall, it is perhaps more readily conceivable that the effect of convection might, in certain circumstances, induce a flow in the same direction as the wall’s oscillations.

4.5.1 The valveless impedance pump

The purpose of this work is to explore the underlying physics of what is a complicated flow phenomenon. A greater understanding of the physics underlying the pump should lead to an understanding of which features must be accounted for within any viable model of the pump, and which aspects of the flow may safely be neglected.

Because the bulk flow induced will increase with the amplitude of the deformation of the wall, it is often desirable to run the pump with large deformations of large amplitude (Loumes et al., 2008). However, much of the underlying physics behind the pump should be identifiable at small amplitude. Any peristaltic effects, however, should be absent at small amplitude, because here there is negligible direct displacement of the fluid.

Within the elastic section of the valveless impedance pump, a series of waves will propagate in either direction. These may have varying amplitudes, frequencies and wavelengths, and may therefore have quite different effects upon the bulk flow. The results of this work suggest that the effects of each of these waves upon the bulk flow will vary according to the frequency, wavelength and direction of the wave. Their effect may also vary significantly with the amplitude of the wave.

It conceivable that some of the waves that propagate along the wall will make far more significant contributions to the bulk flow than others. In this case, the pump models could be simplified by neglecting the effects of the less significant waves.

A number of authors have reported that as the frequency of the pump is increased (i.e. the pinching becomes more rapid), a flow reversal will occur (Jung & Peskin, 2001; Ottesen, 2003; Hickerson et al., 2005). However, Avrahami & Gharib (2008) have reported that in their simulations no such flow reversals took
place. This flow reversal behaviour is notably absent from the present model.

There are, however, several important differences between this model and the physical pump: Here the wave has been allowed to propagate along the entire length of the tube, whereas in a valveless impedance pump the waves may propagate only along a portion of the tube. Here also only a single sine wave has been allowed to propagate at once, whereas the physical pump will experience a series of travelling sine waves at the wall, and the action of the pump is likely to depend upon which sine waves are present, and their relative magnitudes. Furthermore, this present analysis neglects the possible effects of any fluid-structure interactions within the system. Finally, as has been mentioned, this analysis has been undertaken for tubes whose deformations are of small amplitude, whereas most impedance pumps operate with relatively large amplitude deformations. Any of these differences could account for the discrepancies between these results and the action of the physical pump.

### 4.5.2 Comparison to transpiration

The action of an oscillating wall, such as in an impedance pump, is shown to share physical characteristics with another method of drag reduction known as transpiration, which consists of zero-net flux blowing and suction at the wall. The reason for this is that both systems produce a bulk flow via the action of convection. The great difference between the two methods is the effect of wall-entrainment exists only on the case of flows driven by oscillating walls.

This strongly suggests that flows subjected to oscillating wall are likely to display other characteristics of flows driven by transpiration. It is likely that, much like transpiration, the oscillating walls can effect the stability of flows. It has been found that in transpiration it is downstream travelling waves at the walls that stabilise a Poiseuille flow, while upstream travelling waves destabilise the flow (Lee et al., 2008). (This is unfortunate, since it implies that it is the drag-reducing waves that destabilise the flows, while the stabilising waves are drag increasing.) It is quite possible, therefore, that an oscillating wall with a downstream-travelling wave will have the dual effect of reducing the drag along with stabilising the flow.

In addition, the increased thermal mixing within fluids subjected to transpiration (Hasegawa & Kasagi, 2011) strongly suggests the action of oscillating walls
should also have the effect of increasing thermal mixing.

One question we have sought to answer in this work is whether a flow driven by an oscillating wall can ever be used to approximate a flow driven by transpiration. The reason for this is that transpiration is a difficult phenomenon to investigate experimentally, and realising this, Min et al. (2006) proposed that transpiration could be approximated by the action of an oscillating wall, provided that the amplitude of deformation of the wall is sufficiently small. This has been elaborated upon further in §1.3.3 of the Introduction. This has been shown to be untrue in several cases by Heepfner & Fukagata (2009), and we generalise this result here by showing that there exists no case in which the action of an oscillating wall effectively approximates transpiration.

In order for a flow driven by oscillating walls to be quantitatively similar to that driven by transpiration, it would be necessary for the effect of convection to be significantly greater than the effect of wall-entrainment. This is for the simple reason that the effect of wall-entrainment is absent from flows driven by transpiration. However, it is clear in table 4.1 that there is no asymptotic regime in which $\langle u_1 \rangle_{\text{conv}}$ dominates $\langle u_1 \rangle_{BC}$, and hence no regime in which the effect of convection dominates that of wall-entrainment, in the limit of small amplitude of oscillation.

Neither should it be expected that there might be a regime at higher $\alpha$ at which the system more closely resembles one driven by transpiration. The reason for this is that $\langle u_1 \rangle_{BC}$ and $\langle u_1 \rangle_{\text{conv}}$ are only effectively decoupled in the limit as $\alpha \to 0$. If we were to include higher order terms in our analysis, or somehow solve for a system with a finite $\alpha$, we would find that the effect of convection would be inseparable from that of wall-entrainment, and hence the system would be unlikely to ever resemble one driven by transpiration.

The energy efficiency of the two methods of pumping are also vastly different. It can be seen by comparing the results contained in §3.3 to those in §4.3 that oscillating walls constitute a significantly more efficient pumping mechanism than transpiration at low $\beta$. This is because wall entrainment, which is the dominant mechanism by which a bulk flow is induced in a flow driven by oscillating walls, is a more energy efficient mechanism than transpiration. Conversely, at high $\beta$ (or high $\beta/\eta^2$ in the high $\eta$ case), it is transpiration that is the most efficient pumping method. This is because under this arrangement, oscillating walls produce two
competing effects, wall-entrainment and convection, who effects are of similar magnitude. Transpiration will therefore be more efficient, since it entails only one of these effects.

Moreover, there does remain a fundamental difference between transpiration and flows induced by wavy walls: in transpiration the wall normal velocity at the wall is defined separately from the oscillation of the boundary condition. Conversely, in flows driven by waving walls, the two are inextricably linked. This is due to the fact that here the wall normal velocity at the wall is proportional to $\omega$, the frequency of the wall’s oscillation. It is for this reason that while $\langle u_1 \rangle_{\text{conv}}$ increases monotonically with $\beta$ (and therefore with $\omega$) in flows driven by waving walls, in flows driven by transpiration the equivalent bulk flow has been shown to decrease with $\beta$, once $\beta$ exceeds a certain value. It is necessary, therefore, to be cautious when inferring the effect of convection within flows driven by waving walls from the behaviour of flows driven by transpiration, and vice-versa.

The reason why the bulk flow diminishes in transpiration driven flows in the large $\beta$ limit is that at sufficiently high oscillation frequency, the effects of adjacent peaks and troughs will combine and cancel away from the wall. This results in only small velocity gradients existing within the flow. Where the velocity gradients are small, the effect of convection will be small, and hence the effect of convection will be small at large $\eta$. This effect has been described more thoroughly in §3.2.6.

This is also true in flows driven by oscillating walls. However, in this case there is also a counteracting effect. Here again the effects of the adjacent peaks and troughs of the wall oscillations combine and cancel away from the wall, resulting in small velocity gradients and negligible convection. This effect will in fact be enhanced by the high frequency of oscillation. Nonetheless, this is overshadowed by the direct effect that the high frequency of oscillation has upon the radial velocity at the wall: Since the maximum amplitude of the velocity at the wall, as defined in (4.1.5), is proportional to the frequency of oscillation of the wall, the ultimate driving force of convection also increases as the frequency (and therefore $\beta$) increases. This effect is more significant than the aforementioned cancelling effect of the adjacent peaks and troughs.
4.6 Conclusions

We have analysed the behaviour of flows driven by sinusoidal travelling oscillations of the walls of their tube. Despite the superficial similarity to peristalsis, this is in fact an entirely different pumping mechanism. The vital difference between the two is that while peristalsis involves large deformations of the tube’s wall, resulting in flow by the direct displacement of the fluid, in these systems, the deformation of the tube can be very small. In fact, this study formally considers the behaviour of such flows in the limit as the amplitude of the deformation of the tube approaches zero. It is found that there exist two mechanisms by which this method of pumping which may induce a bulk flow.

The first of these mechanisms is the dragging of the fluid adjacent to the wall by the oscillating wall. This has been referred to in this work as wall-entrainment. The second method is via the effect of convection. This induces a flow counter to the direction via which the wave propagates along the wall, and is therefore counter to the bulk flow induced via wall-entrainment. We find the effect of wall-entrainment invariably dominates that of convection, and therefore the induced flow will invariably be in the direction in which the travelling wave propagates. However, it is also found that both of these effects must be accurately modelled, in order to produce an effective model of the valveless impedance pump.

When in non-dimensionalised units, the bulk flow induced by convection is denoted $\langle u \rangle_{\text{conv}}$, and the bulk flow induced by wall-entrainment is denoted $\langle u \rangle_{\text{BC}}$. The total flow is the sum of these two and is denoted $\langle u \rangle$. (These dimensionless quantities are differentiated here from their dimensional counterparts, $\langle \hat{u} \rangle_{\text{BC}}$, $\langle \hat{u} \rangle_{\text{conv}}$ and $\langle \hat{u} \rangle$, by the addition of a circumflex.) In the limit as the amplitude of the oscillation of the wall becomes very small, the flow due to convection and the flow due to wall-entrainment become separate quantities, and may be expressed independently.

The asymptotic behaviour of the bulk flow is detailed in table 4.1, in non-dimensionalised units. They are also given in dimensional units in table 4.4. Also in table 4.4 is the asymptotic behaviour of the average rate of dissipation within the flow, $\langle \bar{\epsilon} \rangle$, which is equivalent to the energy input at the wall required to drive the flow. This is also given in non-dimensional units in (4.3.7).
Table 4.4: Asymptotic behaviours of the bulk flow and the average dissipation, in terms of dimensional parameters. These parameters are: $h$, the average radius of the tube, which is the radius of the pipe in the absence of deformation; $A$, the amplitude of oscillations of tube’s radius; and $\lambda$, the wavelength of the oscillations. The other quantities that have been used to non-dimensionalise the quantities of the Navier-Stokes equation are the frequency, $\omega$, of the oscillations of the boundary condition, and the density, $\rho$ and kinematic viscosity, $\nu$ ($= \mu/\rho$), of the fluid. Here $\langle \hat{u} \rangle_{BC}$ and $\langle \hat{u} \rangle_{conv}$ refer to the contributions due to the boundary conditions and convection respectively, while $\langle \hat{u} \rangle$ refers to the total of the first order correction to the bulk flow. The average rate of dissipation per unit of volume, within the fluid, is given by $\langle \hat{\epsilon} \rangle$. These apply strictly in the limit as $A/h \to 0$.

<table>
<thead>
<tr>
<th>Condition</th>
<th>$\langle \hat{u} \rangle_{BC}$</th>
<th>$\langle \hat{u} \rangle_{conv}$</th>
<th>$\langle \hat{u} \rangle$</th>
<th>$\langle \hat{\epsilon} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h \to 0$, $h^2/\omega \to 0$</td>
<td>$\frac{4}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{4}{h^2} \lambda \omega A^2$</td>
<td>$\frac{8}{h^2} \rho \lambda^2 \omega^2 A^2$</td>
</tr>
<tr>
<td>$\lambda \to \infty$, $h \to 0$, $h^2/\omega \to \infty$</td>
<td>$\sqrt{2} \frac{h \sqrt{\nu}}{h \sqrt{\nu}}$</td>
<td>$\sqrt{2} \frac{h \sqrt{\nu}}{h \sqrt{\nu}}$</td>
<td>$\frac{2 \lambda}{\sqrt{\nu}}$</td>
<td>$\frac{2 \lambda}{\sqrt{\nu}}$</td>
</tr>
<tr>
<td>$h \to \infty$, $\lambda^2 \omega \to 0$, $\nu \to \infty$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
</tr>
<tr>
<td>$h \to \infty$, $\lambda^2 \omega \to \infty$, $\nu \to \infty$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
<td>$\frac{1}{h^2} \lambda \omega A^2$</td>
</tr>
<tr>
<td>$\lambda \to \infty$, $\nu \to \infty$</td>
<td>$\frac{2 \sqrt{2}}{\sqrt{\nu}}$</td>
<td>$\frac{2 \sqrt{2}}{\sqrt{\nu}}$</td>
<td>$\frac{2 \sqrt{2}}{\sqrt{\nu}}$</td>
<td>$\frac{2 \sqrt{2}}{\sqrt{\nu}}$</td>
</tr>
</tbody>
</table>
The effects that inertial forces can have within fluid flows are often counter-intuitive and unpredictable. Only when the flow’s inertia is sufficiently large will the flow field convect itself to a significant degree. Beginning with the original discovery that there exist two distinct states of flow: laminar and turbulent (the latter being due to the destabilising and energy-transferring effects of convection), many interesting flow phenomena have been found to result from the effect of convection. These include the drag reduction mechanisms that are the subjects of this thesis.

The focus of this thesis is on methods of drag reduction and pumping in which convection plays an important part. The first of these is the addition of long-chain elastic polymers to a turbulently flowing liquid. Even the addition of small quantities of such polymers has previously been shown to very significantly reduce the drag experienced by the flow. In §2, it has been proved that the drag experienced by the equivalent laminar flow constitutes a theoretical lower bound on the drag experienced by a turbulent flow containing such elastic polymers. In other words, the addition of polymers to a turbulently flowing liquid cannot reduce the drag experienced by a turbulent flow to the same magnitude as the drag experienced by the equivalent laminar flow. Furthermore, it is shown the addition of such polymers to a laminar flow will instead increase the drag.

In other words, polymer drag reduction affects only turbulent flows. Since the addition of polymers to a flow does not impart energy or momentum to the fluid, the resulting drag reduction must therefore be explained as a modification of the
turbulent flow itself. Subsequently to the publication of the findings presented in §2 (Woodcock, Sader & Marusic, 2010), it has been shown that the addition of such long-chain elastic polymers to a turbulently flowing liquid causes the flow to take on a new state, distinct from conventional turbulence (Hof et al., 2012; Dubief et al., 2012). This state, dubbed elasto-inertial turbulence, is a chaotic flow, whose behaviour is characterised by a combination of inertial and elastic effects.

The second method of drag reduction considered within this thesis is known as transpiration, and consists of a dynamic regime of blowing and suction at the wall, which imparts no net volume flux to the fluid. Transpiration can equivalently be considered a form of pumping, since it is capable of inducing an average flow within an otherwise stagnant fluid. In §3, the flow resulting from transpiration in a channel has been derived using a perturbation analysis. This allows us not only to model and visualise the resulting flow, but also to investigate the physical mechanisms at work within the flow.

Here it has been shown that the average flow induced via transpiration results from the effect of convection. It is convection that transfers energy and momentum from one scale of motion to another. And accordingly, it is convection that transfers momentum from the swirling motions that transpiration inputs to the flow at the wall to the streaming motion that flows in a single direction.

The third method that has been studied here is the imposition of a travelling wave oscillation along the wall of the fluid’s container. This has been studied with particular reference to the valveless impedance pump, which is a method of pumping particularly useful for flows within small tubes.

The valveless impedance pump consists of a short elastic tube, which is connected at both ends to inelastic tubes. Any travelling waves induced within the elastic section will propagate within it and rebound once they reach the inelastic sections. A rhythmic pinching mechanism within the elastic section (offset from the centre), thereby induces waves to propagate along the tube’s wall, which in turn induce a flow within the tube. Using the simple model of a travelling wave, the common assumption that convection is negligible within the valveless impedance pump has been investigated.

Using a perturbation analysis, it has been shown that the effect of convection is not in fact negligible within such flows. In fact, it is of the same order of
magnitude as the other flow-inducing mechanism present (which is the simple dragging effect of the fluid by the travelling wave at the wall).

This work is not, however, intended to constitute a practical model of the valveless impedance pump, one which could be used to quantitatively predict the output of the pump. The present purpose is to determine which physical effects must be taken into account in order to accurately model the pump, and which can be safely neglected. It would therefore be valuable to produce a quantitative model of the valveless impedance pump which accounts for the effect of convection. It must be noted, however, that it would likely prove to be difficult to account for convection within such a model, since its effect is considerably non-linear, and as a result it can be computationally expensive to model.
Bibliography


RICHARDSON, L. F. 1922 *Weather prediction by numerical analysis*. Cambridge University Press.


Appendix

Greens functions

In this section, we derive a Greens function for use in solving equations of the form
\[ f''(r) + \frac{1}{r}f'(r) = g(r), \quad 0 \leq r \leq 1, \]  
(A.1)
where \( f(r) \) is everywhere finite, and
\[ f(1) = 0. \]  
(A.2)
This Greens function is then further extended in order to determine \( \langle f \rangle \), the average of \( f(r) \) over the entire flow domain.

Equations of the form (A.1) result from functions, defined on the unit cylinder, which are independent of the streamwise and azimuthal directions. In order to derive a Greens function for the above equation, we temporarily restore the hitherto neglected azimuthal dimension, \( \hat{\theta} \). We then seek to determine a Greens function for \( f(r) \) on the unit circle. This function we call \( G(r|\theta, \theta') \), and is defined such that
\[ f(r) = \int_0^1 \int_0^{2\pi} G(r|\theta, \theta') g(\theta') d\theta' dr'. \]  
(A.3)
Using the method of images, we can derive the following:
\[ G(r|\theta, \theta') = \frac{1}{2\pi} \log \left( \frac{r^2 + (r')^2 - 2rr' \cos \theta'}{r^2(r')^2 + 1 - 2rr' \cos \theta'} \right). \]  
(A.4)
By substituting the above into (A.3) and integrating over \( \theta' \), we find that
\[ f(r) = \int_0^1 r' \log \left( \frac{r^2 + (r')^2 + |r^2 - (r')^2|}{2} \right) g(\theta') dr'. \]  
(A.5)
If we also want to represent, $\langle f \rangle$, as an integral containing $g(r)$, we first recognise that

$$\langle f \rangle = 2 \int_0^1 r f(r) \, dr. \quad (A.6)$$

By substituting (A.5) into the above and integrating over $r$, we obtain

$$\langle f \rangle = \frac{1}{2} \int_0^1 \left((r')^3 - r'\right) g(r') \, dr'. \quad (A.7)$$
Appendix B

Products of Bessel functions

In this section, we derive an expression for the product of two modified Bessel functions, of which we have made use at several points within §4.2. A similar derivation can be found in Watson (1995).

We begin with the definition of a modified Bessel function as a power series:

$$I_m(\ar) = \sum_{j=0}^{\infty} \frac{1}{j!(j+m)!} \left(\frac{\ar}{2}\right)^{2j+m}.$$  \hspace{1cm} (B.1)

This leads to the following power series expression for the product of two modified Bessel functions:

$$I_m(\ar)I_n(br) = \sum_{j=0}^{\infty} \left(\frac{r}{2}\right)^{2j+m+n} a^{2j+m} b^n \sum_{k=0}^{j} \frac{(b^2/a^2)^k}{(j-k)!(j+m-k)!k!(k+n)!}.$$  \hspace{1cm} (B.2)

Since this expression involves two infinite summations, it can be computationally expensive to evaluate. We therefore seek to reduce the number of summations.

Using the fact that Pochhammer symbol, which is defined in equation (4.2.23), can be expressed as

$$(-x)_n = (-1)^n \frac{\Gamma(x + 1)}{\Gamma(x - n + 1)},$$  \hspace{1cm} (B.3)

151
we may rewrite equation (B.2) as

\[
I_m(ar)I_n(br) = \sum_{j=0}^{\infty} \left( \frac{r}{2} \right)^{2j+m+n} \frac{a^{2j+m}b^n}{j!(j+m)!n!} \sum_{k=0}^{j} \frac{(-j)_k(-j-m)_k(b^2/a^2)^k}{(n+1)_k k!}.
\]

(B.4)

It is clear from (B.3) that \((-x)_n = 0\), if \(x\) and \(n\) are both positive integers with \(n > x\). Therefore an alternative expression for the hypergeometric function \(2F_1\), which is defined in equation (4.2.22), is (Abramowitz & Stegun, 1965)

\[
2F_1(-j, -\xi; \gamma) = \sum_{k=0}^{j} \frac{(-j)_k(-\xi)_k \gamma^k}{(\chi)_k k!},
\]

(B.5)

where \(\xi\) is a positive integer. Using this expression, we can rewrite (B.4) as single summation:

\[
I_m(ar)I_n(br) = \sum_{j=0}^{\infty} 2F_1 \left( -j, -j - m; n + 1; \frac{b^2}{a^2} \right) \frac{a^{2j+m}b^n}{j!(j+m)!n!} \left\{ \frac{r}{2} \right\}^{2j+m+n}.
\]

(B.6)