Eynard-Orantin Theory of the A-Polynomial

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Abstract

This thesis studies the Eynard-Orantin invariants of an important knot invariant: the $A$-polynomial. In particular, period integrals of the Eynard-Orantin invariants are studied. First, formulae for the period integrals are derived in the case of a general elliptic curve, and relations to the Gauss-Manin connection and integrable systems are shown. These constructions are then elaborated to the case of the $A$-polynomial of a knot. In the specific case of the figure eight knot, explicit formula are given.
Declaration

This is to certify that

1. This thesis comprises only my original work towards the PhD except where indicated in the Introduction.

2. Due acknowledgement has been made in the text to all other material used.

3. This thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Callum Sleigh
I would like to especially acknowledge the enormous amount of time and effort that Associate Professor Paul Norbury has generously given to assist me in completing this thesis. His incredible facility with an enormous range of mathematics, and his obvious passion for the subject have been a constant source of inspiration for me. I would like to also acknowledge the help of my supervisory panel: Nora Ganter and Arun Ram. This thesis would not have been possible without the support of Olivia Nikkel and my family in New Zealand and Australia.
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Introduction

0.0.1 Quantum and Classical Topology

In the mid 1980s, Vaughan Jones revolutionised low-dimensional topology with the discovery of a new knot invariant: the Jones polynomial. This invariant was powerful enough to resolve several outstanding questions in topology, and it was constructed using a seemingly unrelated area of mathematics; the theory of operator algebras.

In 1988, Edward Witten vastly extended this theory by giving a physical interpretation of the Jones polynomial using a three dimensional topological quantum field theory (TQFT), whose action is determined by the Chern-Simons invariant of a connection. Witten’s work showed that gauge theory could be used to understand the Jones polynomial, and extend Jones’s results to give invariants of arbitrary three manifolds at a physical level of rigour. Witten’s theory also incorporated the various generalisations of the Jones polynomial, such as the ‘coloured Jones polynomial’, and the HOMFLYPT polynomial.

Subsequently, Reshetikhin and Turaev gave a mathematically rigorous construction of Witten’s quantum field theory, using the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$ and the fact that an arbitrary three manifold can be expressed as surgery on a link.

The outstanding question now is: how do the new ‘quantum’ invariants - based on representation theory and quantum field theory - relate to ‘classical’ invariants from low-
dimensional topology and geometry? In the case of hyperbolic knots, an important example of a classical invariant is the Riemannian volume of the knot complement, with the hyperbolic metric.

The earliest such proposal is the Volume Conjecture of Murakami, (which is outlined, for example, in the review paper ([Mu])). One of the main motivations for the material in this thesis is a more recent - and vastly more general - proposal for such a relationship given in the physics literature by Dijkgraaf, Fuji and Manabe ([DF], ([DFM]).

On the side of ‘classical’ geometry is a knot invariant called the $A$-polynomial. The $A$ polynomial was discovered by Culler and Shalen in 1994; it defines a one-dimensional algebraic variety parametrising certain representations of the fundamental group of the knot complement into $SL(2, \mathbb{C})$. Many important classical invariants are encoded in this algebraic variety: for example, a result of Yoshida ([Y] Theorem 2) gives that the volume and Chern-Simons invariant of the knot complement can be calculated as a certain period integral on this algebraic curve.

On the ‘quantum’ side is the coloured Jones polynomial of a knot, arising in the Witten-Reshetikhin-Turaev topological quantum field theory.

The precise statement of the Dijkgraaf-Manabe-Fuji conjecture is quite complex; roughly speaking, the conjecture is that certain recursion relations satisfied by the coloured Jones polynomial determine a differential equation for period integrals on the zero locus of the $A$-polynomial.

Thus the purely combinatorial/algebraic information that defines the coloured Jones polynomial (coming from the representation theory of $U_q(sl(2, \mathbb{C}))$) determines period integrals on a variety that is defined solely by the topology of the knot complement, and is closely related to the knot complement’s hyperbolic structure.

The original conjecture of Dijkgraaf, Manabe and Fuji involved inherent ambiguities which needed to be resolved on a case by case basis; in a recent preprint ([E]), Eynard and Borot have shown how to fix these ambiguities.
The Dijkgraaf-Manabe-Fuji conjecture is remarkable because the period integrals that arise are examples of an infinite family of new invariants attached to an algebraic curve, called Eynard-Orantin invariants. The Eynard-Orantin invariants are an infinite family of multidifferentials, indexed by two integers \((g,n)\), which were discovered in a totally different context (the study of random matrices). These invariants have many exceptional properties that are discussed in chapter 1. Perhaps the most surprising is that the Eynard-Orantin invariants are closely related to intersection theory on the moduli space of curves and Gromov-Witten invariants. This aspect of the Eynard-Orantin theory will not be discussed here, but provides another important motivation for the study of these objects.

Regardless of the motivation, the Dijkgraaf-Fuji-Manabe conjecture arose in physics, and the problem remains of interpreting it from a mathematical point of view. What do period integrals on the zero locus of the \(A\)-polynomial calculate?

In this thesis, some answers to these question will be given, focusing mainly on the case of period integrals over closed chains, although the original work of Dijkgraaf-Manabe-Fuji also made use of period integrals over intervals. The approach is inspired by Witten’s physical interpretation of the Witten-Reshetikhin-Turaev TQFT: Using the identification between flat connections and representations of the fundamental group, the \(A\)-polynomial will be approached from the point of view of gauge theory and symplectic geometry of the space of connections. This way of thinking of the \(A\)-polynomial was pioneered by Gukov in the physical literature ([Gu]), and will necessitate considering connections with noncompact structure group (usually \(SL(2,\mathbb{C})\)).

The novel point of view put forth in this thesis is that the period integrals of Eynard-Orantin invariants should really come from varying the \(A\)-polynomial to a family of related algebraic curves. The Eynard-Orantin invariants of the \(A\)-polynomial will then be related to the variation of parallel transport of the Chern-Simons line bundle over these families. The original definition of the \(A\)-polynomial does not have any moduli, so it is a nontrivial
problem to try and find a candidate family of curves that will vary the $A$-polynomial in the correct way. An original contribution of this thesis is to explicitly find a variation of the $A$-polynomial of the figure eight knot which gives rise to Eynard-Orantin invariants. This is the content of Proposition (4.4) and Theorem (4.1).

Another original contribution of this thesis is to explicitly relate the families of curves that yield Eynard-Orantin invariants to the families of curves that appear in the study of integrable systems. This relationship is discussed in section (2.1.1), and then, for the case of the $A$-polynomial, in section (??). In particular, this thesis will propose a relationship between Eynard-Orantin theory of the $A$-polynomial and the Hitchin system. This is the content of Theorem (4.5).

0.0.2 Chapter Description

1. In chapter 1, the basics of Eynard-Orantin theory are introduced, using the motivating example of random matrices. In order to describe the Eynard-Orantin theory, various standard facts of Riemann surface theory are also summarised.

In the original papers of Eynard and Orantin, reference is made to the ‘Rauch variational formula’ which gives the variation of the $(0, 2)$ Eynard-Orantin invariant in terms of certain period integrals of the $(0, 3)$ invariant. It is not clear to what extent these formula are completely rigorous (and for which curves they apply to), so, in section (1.3), a self-contained proof of the Rauch variational formula is given, following the classical work of Rauch. Careful formulation of the necessary conditions are given, for use in subsequent chapters.

2. In chapter 2 the Eynard-Orantin invariants of genus one curves are studied. Firstly, standard objects in the study of genus one curves, such as elliptic functions and elliptic modular forms are described. Then, it is shown that the Rauch variational formula from chapter 1 can be used to give a geometric interpretation of the pe-
period integrals of the (0, 3) Eynard-Orantin invariant: they are the ‘derivatives’ of the lower order invariants as the spectral curve is varied with respect to certain parameters. Here the ‘derivative’ is with respect to the Gauss-Manin connection, and explicit formulae for the correct variation are presented in section (2.1.1). For elliptic curves, this gives a rigorous verification of a formula in the original work of Eynard and Orantin. Next, these results are used to prove some useful new expressions for the period integrals of Eynard-Orantin invariants on an elliptic curve in Theorem (2.2).

In section 2.2, it is shown that it is possible to give expressions for these integrals in terms of quasi-modular forms. In fact, these are quasi-modular forms for a special congruence subgroup $\Gamma(2)$ of $SL(2, \mathbb{Z})$. In Theorem (2.3) we prove an original result that all period integrals of higher order Eynard-Orantin invariants are rational functions in two simple $\Gamma(2)$ quasi-modular forms: the level 4 and level 2 congruence Eisenstein series.

3. Chapter 3 introduces the necessary background for discussing the $A$-polynomial of a knot from the point of gauge theory. Consider a tubular neighbourhood $N_K$ of a knot $K \subset S^3$. The boundary of this tubular neighbourhood is a two-torus $T^2$. The essential idea is that the $A$-polynomial describes the variety of flat $SL(2, \mathbb{C})$-connections on this torus $T^2$ that extend to the bounding three manifold $S^3 \setminus N_K$. Thus it is necessary to introduce the space of flat $SL(2, \mathbb{C})$-connections (denoted $\mathcal{M}_{T^2}$) on a smooth two-dimensional torus $T^2$, and discuss its properties. Since $SL(2, \mathbb{C})$ is an algebraic group, it is shown that the space of flat $SL(2, \mathbb{C})$ connections has the structure of an algebraic variety, which is discussed in section 3.3.1.

The space of flat connections also has more structure, as was noticed by Atiyah and Bott; their observation was that this space can also be endowed with a symplectic structure, using the symplectic structure on the space of all connections $A$ together with the theory of symplectic reduction. The zero locus of the $A$-polynomial defines
a holomorphic Lagrangian subvariety of this symplectic space. This point of view is natural in the physical approach using gauge theory ([Gu]).

Another important geometric concept with applications to the Eynard-Orantin theory is the Chern-Simons functional. This is a function on the space of all connections $\mathcal{A}$ that depends on a trivialisation of the $\text{SL}(2, \mathbb{C})$ bundle. Because of the dependence on trivialisations, it is more convenient ([F2]), ([RSW]) to think of the Chern-Simons functional as a section of a line bundle over $\mathcal{A}$, called the Chern-Simons line bundle. This line bundle then reduces to a line bundle over $\mathcal{M}_{T^2}$, which is closely related to the symplectic structure on $\mathcal{M}_{T^2}$. Specifically, the logarithm of the parallel transport of the Chern-Simons line bundle is given by the integral of the Liouville one-form of the symplectic form $\omega$.

This link between the Chern-Simons line bundle and the symplectic structure of $\mathcal{M}_{T^2}$ is the starting point of a geometric interpretation of the Eynard-Orantin theory: the first term in the Eynard-Orantin recursion is a one-form $\omega^{(0,1)}$, and the physical literature suggests ([DF]), ([DFM]), ([E]) that the one-form $\omega^{(0,1)}$ should be closed and cohomologous to the Liouville one-form of $\omega$. This then gives that integrals of $\omega^{(0,1)}$ around closed cycles give the logarithm of parallel transport of the Chern-Simons line bundle.

4. In Chapter 4, the definition of the $A$-polynomial is given and its relation to the space $\mathcal{M}_{T^2}$ is described. Some simple examples of the computation of various $A$-polynomials are given.

To study the Eynard-Orantin theory using the ideas of the previous chapters, it is necessary to find a good expression for the cohomology class of the Liouville one-form of $\omega$. This is not so easy, so in section 4.2.1 we use some technical manipulations (of the ‘Gelfand-Leray’ form) to get a nice expression for a one-form representing this class. Once this is done, we can give a good definition of what it means to vary the $A$-polynomial, as determined by Eynard-Orantin theory, and proceed onto
the main results of Proposition (4.4) and Theorem (4.1), using the formalism of the Gauss-Manin connection. These results parallel those of section (2.1.1), now in the more complicated setting of the $A$-polynomial.

In section ?? we show that this variation determines an algebraically integrable system, just like the elliptic case in section (2.1.1). This is related to the Hitchin system in section (4.2.2).
Chapter 1

Eynard-Orantin Theory

1.1 Hermitian Matrix Models

One approach to Eynard-Orantin theory is via the study of random matrices. Their methods can be seen as the continuation of a line of research into the combinatoric structure of the large $N$ distribution of eigenvalues of random matrices that started with Wigner’s observation relating moments of the semicircle distribution to Catalan numbers ([AGZ]). The new ingredients which Eynard and Orantin introduce are the extensive use of complex analysis and Riemann surfaces to study the combinatorics.

Most of the properties of the recursion that we will be using (see the review papers ([EO1], [EO2]) and the references below) generalise important features of random Hermitian matrices. As such, the definitions and motivations for some features of the Eynard-Orantin recursion can remain rather opaque without knowledge of the precedent in random matrices. Here, we provide an overview of the study of random matrices, focusing on the simplest case of a Hermitian one-matrix model.
1.1.1 Definitions

Let $\mathcal{H}_N$ be the space of $N \times N$ Hermitian matrices, along with its natural Haar measure $d\mu_N$ which can be written,

$$d\mu_N(M) = 2^{\frac{N(N-1)}{2}} \prod_{i=1}^{N} dM_{ii} \prod_{i \leq j} d\text{Re}M_{ij} d\text{Im}M_{ij}.$$  

Let $V(x) \in \mathbb{R}[x]$ be a polynomial of degree $d$ of the form,

$$\frac{x^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} x^j,$$

where the coefficients $t_j$ will be called times.

For any matrix $M$, let $V(M)$ denote the polynomial $V$ evaluated at $M$ in the obvious way. The **Hermitian one-matrix model** is the following integral,

$$Z(t_3, t_4, \ldots, t_d) := \int_{\mathcal{H}_N} e^{-g \text{tr} V(M)} d\mu_N(M).$$ (1.1)

The Haar measure $d\mu_N$ is conjugation invariant, so it is natural to consider the expectation of conjugation invariant functions with respect to this measure. For any conjugation invariant integrable function, $f : \mathcal{H}_N \to \mathbb{R}$ define the symbol,

$$\langle f \rangle := \int_{\mathcal{H}_N} e^{-g \text{tr} V(M)} f(M) d\mu_N(M).$$

and for the Gaussian potential $V(x) = \frac{x^2}{2}$ define the **Gaussian integral**,

$$\langle f \rangle_G := \int_{\mathcal{H}_N} e^{-g \frac{1}{2} M^2} f(M) d\mu_N(M).$$

Important examples are the **kth-moments**, $\langle \text{Tr} M^k \rangle_G = \sum_{i_1, i_2, \ldots, i_k} \langle M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_k i_1} \rangle_G.$  
Because $\text{Tr} M^k = \sum_{i=1}^{N} \lambda_i^k$, where $\lambda_i \in \sigma(M)$, the kth-moments encode the statistical properties of the spectrum of a random Hermitian matrix.

Note that, in much of the physical literature, the integral above is considered ‘formally’, in the sense of Feynman path integrals. What this means in practice is that analytic
issues such as whether the summation and integration can be commuted are ignored, and the ‘integral’ is considered as a convenient notation for a certain formal power series of Gaussian integrals. Specifically, the integral,

$$\int_\mathcal{H}_N e^{-g_s (\text{tr} M^2 - \sum_{j=3}^d \frac{t_j}{j} \text{tr} M^j)} d\mu_N(M),$$

will stand for the formal power series of Gaussian integrals,

$$\sum_{k=0}^\infty \frac{1}{k!} \left\langle \left( g_s \sum_{j=3}^d \frac{t_j}{j} \text{tr} M^j \right)^k \right\rangle_G.$$

In the case of a single Hermitian matrix with polynomial potential, the analytic issues necessary to verify that the ‘formal’ expansion agrees with the rigorous asymptotic expansion have been dealt with by various authors, such as in ([Gui]).

A random matrix is a $N \times N$ hermitian matrix whose entries are random variables (ie. each complex entry $z = x + iy$ has $x$ and $y$ real random variables) with the probability distribution given above. In this case, the study of random matrices is the same thing as studying Hermitian one-matrix models. Of course there are other interesting probability distributions to study on the coordinates of a Hermitian matrix, see ([AGZ]).

The subject originated with the revelation that the spectra of random matrices have explicit distributions as $N$ approaches infinity: this is the so-called universality property of the eigenvalues of a random Hermitian matrix.

### 1.1.2 The Combinatorics of Random Spectra

To understand the behaviour of the spectrum at large $N$, it is useful to examine the moments $\langle \text{tr} M^k \rangle_G$ using the special properties of Gaussian integrals.

**Proposition 1.1** (Wick’s Theorem for Matrices). If $M_{ij}$ denotes the entry in the $i$th row and $j$th column of an Hermitian matrix, then, for even $k$, 

$$\langle M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_k j_k} \rangle_G = \sum_{\text{pairings}} \prod_{\text{pairs}(k,l)} \langle M_{i_{k,l} j_{k,l}} \rangle_G,$$

3
and for odd $k$,
\[
\langle M_{i_1j_1} M_{i_2j_2} \ldots M_{i_kj_k} \rangle_G = 0.
\]

This proposition is well known, see, for example section 2.2 of ([Z]).

The righthand side of Wick’s theorem consists of known moments by the following important property of Gaussian integrals (recall that $g_s$ is the constant in the exponential of the original integral),
\[
\langle M_{ij} M_{kl} \rangle_G = g_s \delta_{il} \delta_{jk}.
\]

(1.3)

A simple example is $\langle TrM^4 \rangle_G = \sum_{i,j,k,l} \langle M_{ij} M_{jk} M_{kl} M_{li} \rangle_G$, which can be expanded, using Wick’s Theorem, into,
\[
\sum_{i,j,k,l} \left( \langle M_{ij} M_{jk} \rangle_G \langle M_{kl} M_{li} \rangle_G + \langle M_{ij} M_{kl} \rangle_G \langle M_{jk} M_{li} \rangle_G + \langle M_{ij} M_{li} \rangle_G \langle M_{jk} M_{kl} \rangle_G \right).
\]

Notice that (1.3) implies that most of these terms will vanish. The easiest way to get rid of the redundancy is to organise this information with a simple graphical calculus. The original integrand $TrM^4$ is a product of 4 terms: visualise this as a four-valent vertex, with thickened edges,

![Diagram of a four-valent vertex with thickened edges]

It is convenient to choose an orientation of the plane and pick a marked edge of the four-valent vertex. The marked edge goes to the far left, and all the edges are required to point upwards, as in our diagram.

Each pairing from Wick’s theorem can be thought of as a pairing of thickened half-edges, and the number of ‘free’ variables is given by the number of faces of the resulting ribbon.
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graph. For example, the following is such a pairing:

\[
\langle M_{ij} M_{li} \rangle \langle M_{kl} M_{ti} \rangle,
\]

which represents \( \langle M_{ij} M_{li} \rangle \langle M_{jk} M_{kl} \rangle \), and

\[
\langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle,
\]

represents the pairing \( \langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle \).

The equation (1.3) implies that the pairing \( \langle M_{ij} M_{jk} \rangle \langle M_{kl} M_{li} \rangle_G \) is non zero if and only if \( i = k \) while \( j, l \) are free, while \( \langle M_{ij} M_{kl} \rangle_G \langle M_{jk} M_{li} \rangle_G \neq 0 \) implies that \( i = l = k = j \), so there is only one free variable. This gives,

\[
< Tr M^4 >_G = \sum_{i,j,l=1}^{N} \langle M_{ij} M_{li} \rangle_G \langle M_{il} M_{li} \rangle_G + \sum_{i=1}^{N} \langle M_{ii} M_{ii} \rangle_G \langle M_{ii} M_{ii} \rangle_G
\]

\[+ \sum_{i,j,k} \langle M_{ij} M_{ji} \rangle_G \langle M_{jk} M_{kj} \rangle_G \]

\[= g_s^2 N^3 + g_s^2 N + g_s^2 N^3. \]

This is the basis of the link between random matrices and enumerative combinatorics; if we use \( \Gamma_{0,0,0,1} \) to denote the set of graphs that have one 4-valent vertex
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1.1

(and $\Gamma_{n_1,n_2,n_3,...}$ for the set of graphs with $n_i$, $i$-valent vertices), and let $E(\gamma)$, $V(\gamma)$ be the number of edges and vertices of a graph, then we can rewrite the expression above suggestively as,

$$\sum_{\gamma \in \Gamma_{0,0,0,1}} g_s^{E(\gamma)} N^{F(\gamma)},$$

(even though it is clear that $E(\gamma) = 2$ for all $\gamma \in \Gamma_{0,1}$). Putting $g_s = \frac{t}{N}$, and using the definition of the Euler characteristic gives,

$$\sum_{\gamma \in \Gamma_{0,0,0,1}} t^{E(\gamma)} N^{\chi(\gamma)-V(\gamma)}.$$

A similar calculation shows that expressions of the form,

$$\langle (TrM)^{k_1} (TrM)^{k_2} \ldots (TrM)^{k_n} \rangle_G,$$

give the number of ribbon graphs involving all pairings of $k_i$, $i_i$-valent graphs. Note that this expression will be zero if the total $\sum_i k_i$ is not even (from the condition in Wick’s Theorem). Just as for the case of $\langle TrM^4 \rangle$, the result can be written as polynomial in $t$, if we substitute $g_s = \frac{t}{N},$

$$\sum t^{E(\gamma)} N^{\chi(\gamma)-V(\gamma)},$$

where the sum is over the appropriate graphs.

The combinatoric interpretation associated to the calculation of expectation values such as $\langle TrM^{2k} \rangle_G$ can be used to give recursive formulae for these quantities. For any such expectation value, use $\langle TrM^{2k} \rangle_G^0$ to denote coefficient of $N^1 = N^{2-V}$ in the polynomial $\langle TrM^{2k} \rangle_G$. This is the contribution from genus 0 (or ‘planar’) graphs For example,

$$\langle TrM^4 \rangle_G^0 = 2 t^2,$$

because the contribution from the pairing $\langle M_{ij} M_{kl} \rangle_G \langle M_{jk} M_{li} \rangle_G$ gives a graph with Euler characteristic 0, so that it can only be embedded in a genus one surface (recall that $g_s$ has been set to $\frac{t}{N}$).
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Examine the diagram that gives the calculation of $\langle TrM^{2k} \rangle^0_G$: the far left edge is connected to another edge by a loop, and since the diagram is planar, every other edge is either inside or outside this loop. For example, with $k = 4$ a pairing might look like,

By cutting the diagram along the edge attached to the far left vertex, the diagram is reduced to two disconnected planar diagrams with $2l_1$, $2l_2$ vertices each, and $2l_1 + 2l_2 = 2k - 2$. Thus the following recursion holds,

$$\langle TrM^{2k} \rangle^0_G = \sum_{2l_1+2l_2=2k-2} \langle TrM^{2l_1} \rangle^0_G \langle TrM^{2l_2} \rangle^0_G,$$

(1.4)

with the initial condition that $\langle TrM^0 \rangle^0_G = 1$. This is closely related to the famous recursion for the Catalan numbers $C_k$,

$$1, 1, 2, 5, 14, 42, 132, \ldots,$$

and if we set $t = 1$, then we get that $\langle TrM^{2k} \rangle^0_G (1) = C_k$. Indeed, it is well known that counting the number of non-crossing pairings of $k$ legs is one of the numerous counting problems that leads to the Catalan numbers.

Now examine the correlators $\langle TrM^k TrM^3 \rangle^0_G$ (recall that Wick’s theorem means that this will automatically be zero if $k+3$ is odd). The combinatoric interpretation of this quantity
is to count the number of connected pairings between a 3-valent vertex and a $k$-valent vertex (where the vertices may pair with themselves). For example,

Examine the marked leg of the $k$-valent vertex, (which is at the far left of the $k$-valent vertex). This will either be attached to a 3 valent vertex or attached by a loop to a unique other leg of the $k$-valent vertex. In the latter case, all the legs of the diagram will either be inside or outside the loop, as above; so they are partitioned into a collection of $j$ legs and $k - 2 - j$ legs.

Otherwise, the marked leg is attached to a 3 valent vertex, and this figure is homeomorphic to a single $k + 1$ valent vertex. This gives the following recursion,

$$
\langle Tr^k Tr^3 \rangle_0^0 t^{1/2} = \sum_{j=0}^{k-2} \left( \langle Tr^j \rangle_G^0 \langle Tr^{k-2-j} Tr^3 \rangle_0^0 \right) + \langle Tr^{k+1} \rangle_0^0 \quad (1.5)
$$

Similar reasoning yields recursive formula for planar diagrams comprised of a base of arbitrary valence and attaching diagrams of valence $j$, $3 \leq j \leq d$, as well as pairing the base with itself. Recursions of these sort were originally discovered by Tutte ([Tu]).
1.1.3 Solving the Recursion Relation

Because the $k$th moments satisfy the recursions described above, their generating functions will then satisfy corresponding identities, that can be analysed using the tools of complex analysis. This will give a powerful tool for calculating the $k$th moments, since the graphical method becomes increasingly complex as the number of vertices grows.

The simplest example is to examine the moments $\langle TrM^{2k}\rangle_0^G$ (recall that (1.3) implies that the odd moments vanish). Assemble them into a generating function (omitting the dependence on $t$ for simplicity),

$$W^{(0,1)}(x) = \sum_{k=0}^{\infty} \langle TrM^{2k}\rangle_0^G \frac{x^k}{x^{k+1}}.$$  

The recursion (1.4) ensures that,

$$(W^{(0,1)}(x))^2 = W^{(0,1)}(x) - \frac{1}{x},$$

and thus it is possible to solve and find the closed form for $W^{(0,1)}(x)$,

$$W^{(0,1)}(x) = \frac{x + \sqrt{x^2 + 4}}{2x},$$  

(1.6)

which gives a simple method of computing the higher terms $\langle TrM^{2k}\rangle_0^G$.

By construction, the coefficients of the generating function $W^{(0,1)}(x)$ store interesting combinatoric information. From a physicist’s point of view, we would also like to calculate more complicated moments such as,

$$\langle (TrM^{n_1})^{k_1}(TrM^{n_2})^{k_2} \ldots (TrM^{n_l})^{k_l}\rangle_0^G,$$

(1.7)

where $g$ means that only connected diagrams on genus $g$ surfaces are counted.

In the formal calculations of physicists, these moments arise from expanding out the exponential as in (1.2), and considering the corresponding formal series of Gaussian integrals. The goal is to find a closed expression for these terms.

The idea of Eynard and Orantin theory is to consider a more invariant approach: we will
promote the generating function $W^{(0,1)}(x)$ to a meromorphic differential $W^{(0,1)}(x)dx$, and look at the residues,

$$\text{Res}_{x=\infty} x^k W^{(0,1)}(x),$$

which will give the required coefficients. The advantage of this reformulation of the problem is that the full power of Riemann surface theory can be brought into play; we can think of $W^{(0,1)}(x)dx$ as a differential on the Riemann surface $\mathbb{C}P^1$. Eynard and Orantin developed an algorithm to find quantities such as (1.7), extensively using the geometry of Riemann surfaces, as well as the recursions of Tutte type such as (1.4). Their algorithm is based on a Riemann surface determined by the particular counting problem. For example, in the case of (1.6), the Riemann surface is the double branched cover of the $x$-plane given by,

$$y(x) = \frac{x + \sqrt{x^2 + 4}}{2x},$$

so that $W^{(0,1)}(x)dx = ydx$ on this Riemann surface.

For Hermitian matrix models (and several other types of Matrix models), there is a canonical way to extract a Riemann surface $C$ with two meromorphic functions $x, y$ from the matrix model data, inspired by the process we just outlined in the case of the Gaussian matrix model. The Eynard-Orantin recursion successfully finds generating functions for all moments of this model, using complex analysis. Subsequently it was realised that if $(C, x, y)$ is an arbitrary triple of this form, then it is possible to apply the Eynard-Orantin algorithm, and that this process often outputs geometrically interesting quantities. This algorithm will now be described.

### 1.2 Eynard-Orantin Topological Recursion

A triple $(C, x, y)$, where $C$ is a compact Riemann surface, and $x, y$ are two functions that are holomorphic on an open subset of $C$ will be called a *spectral curve*. Recall that the
function field of meromorphic functions on $C$ is denoted $K(C)$.

A simple example of a spectral curve is given by an element $p \in \mathbb{C}[x, y]$ with no singular points: the compactification of the zero locus of $p(x, y)$ in $\mathbb{C}^2$ is a compact Riemann surface $C$, with the meromorphic functions $x$ and $y$ in $K(C)$. Notice that since two functions $x, y \in K(C)$ are specified, if $\Sigma \subset C$ is the locus of poles of $x, y$ then the particular affine immersion of $C/\Sigma$ in $\mathbb{C}^2$ is part of the data. Two different immersions of the same Riemann surface in $\mathbb{C}^2$ give different spectral curves.

Another important example is when the spectral curve defined by a Laurent polynomial $p(x, y) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Here the Riemann surface is given by the compactification of the zero locus of $p(x, y)$, and the two functions are $\text{Log}(x), \text{Log}(y)$. Note that it is not possible to choose a consistent branch of the Logarithm of these two functions, they can only be defined on local open sets. A spectral curve of this sort will be called a non-meromorphic spectral curve.

Recall that a multidifferential is a tensor product of meromorphic 1-forms on $\mathbb{C}^n$, for some $n \in \mathbb{N}$. A bidifferential is a multidifferential for $n = 2$. The Eynard-Orantin topological recursion gives an algorithm for producing an infinite family $\omega^{(g,n)}(z_1, \ldots, z_n)$ of multidifferentials on $\mathbb{C}^n$, for $g \geq 0, n \geq 1$, from the data of a spectral curve. In the original case, coming from matrix models, these invariants gave the generating functions for moments such as (1.7); for general spectral curves, the interpretation of these invariants is an open question (it is conjectured, that these invariants are related to intersection theory on the moduli space of curves: see the discussion in ([EO2])).

### 1.2.1 Riemann Surfaces

Let $(C, x, y)$ be a spectral curve. From now on, assume that the meromorphic form $dx$ has a finite number of zeroes, which will be labelled $a_i$ (this notation will hold henceforth, for any spectral curve). Assume also that all the zeroes $dx(a_i)$ are simple, and that $dy(a_i) \neq 0$
for all $a_i$. This assumption means, roughly, that the algebraic curve defined by $x$ and $y$, ‘looks like’ $y = x^2$ near the zeroes of $dx$. In a small neighbourhood of a point $a_i$ the curve is a simple double branched cover of the $x$-plane.

As a consequence of this condition there is an involution in a small neighbourhood $U_i$ of each $a_i$. This will be notated $z \mapsto \z, \forall z \in U_i$, and is called the branch involution. The notation $z \mapsto \z$ does not specify which branch point is being considered, but context will always make this clear. Because of the local structure of the curve near $a_i$, the branch involution has the property that $x(\z) = x(z)$. This branch involution may turn out to be global (such as in the case of hyperelliptic algebraic curves), but that is not generally true.

For example, let $C = \mathbb{CP}^1$ and define the spectral curve $(C, x, y)$ using the functions $x(z) = z + \frac{1}{z}, y(z) = z$, where $z$ is the usual coordinate on $\mathbb{CP}^1$. These functions satisfy the equation,

$$xy = y^2 + 1,$$

The zeroes of $dx$ are at $\pm 1$, and the branch involution is the global map $z \mapsto \frac{1}{z}, \forall z \in \mathbb{CP}^1$. Notice that this involution fixes the zeroes of $dx$, as it should. We can see the structure of the branch involution on the real graph,
A contrasting example is the ‘folium of Descartes’, given by $C = \mathbb{CP}^1$, and the two functions,

$$x(z) = \frac{3z}{1 + z^3}, \quad y(z) = \frac{3z^2}{1 + z^3},$$

that satisfy the equation,

$$x^3 + y^3 - 3xy = 0,$$

with real graph,

![Graph of the folium of Descartes](image)

The zeroes of $dx$ are, $\frac{1}{24\pi}(-1 - \sqrt{3})$, $\frac{1}{24\pi}$, and $\frac{1}{24\pi}(-1 + \sqrt{3})$. However, while the involution,

$$z \mapsto \frac{-z^2 - \sqrt{4z + z^3}}{2z},$$

fixes $\frac{1}{24\pi}(-1 - \sqrt{3})$ and $\frac{1}{24\pi}(-1 + \sqrt{3})$, it does not fix the remaining zero of $dx$. Near $\frac{1}{24\pi}$, one must use the branch involution,

$$z \mapsto \frac{-z^2 + \sqrt{4z + z^3}}{2z}.$$

It is important to notice that the recursion below always concerns the zeroes of $dx$, and does not include poles of $dx$; even though there may be poles of $dx$ where the curve locally behaves like a double branched cover in the manner described above. In this sense, the Eynard-Orantin theory does not notice information about the curve ‘at infinity’.

This means the usual terminology ‘branch points’, which occurs sometimes in the literature
on Eynard-Orantin invariants is slightly inaccurate, and it is better to simply use the terminology ‘zeroes of $dx$‘.

### 1.2.2 Cycles and Period Matrices

For any compact, orientable Riemann surface $C$ of genus $g$ the classical representation of $C$ as a polygon with identified faces gives a natural choice of basis for $H_1(C,\mathbb{Z})$. The interior of this polygon will be called the *fundamental domain*. For $1 \leq i, j \leq g$, it is customary to write this basis as $\{A_i, B_j\}$ with the defining properties,

$$A_i \cdot A_j = 0, \quad B_i \cdot B_j = 0, \quad A_i \cdot B_j = \delta_{ij}. \quad (1.8)$$

These cycles will be called the $A$ and $B$ cycles, respectively.

The Riemann-Roch theorem gives that the dimension of $H^0(C, K)$ is $g$. Once a choice of basis for $H_1(C, \mathbb{Z})$ has been made that satisfies (1.8), there is a canonical basis $\{z_i\}$ of $H^0(C, K)$, with the property that,

$$\oint_{a_j} z_i = \delta_{ij}, \quad 1 \leq i, j \leq g.$$

Define the $g \times g$ *period matrix* $\tau$ by,

$$\tau_{ij} := \oint_{b_j} z_i.$$

The period matrix has the symmetry $\tau_{ij} = \tau_{ji}$ and $\text{Im} \tau > 0$ (this will be shown below as a consequence of the Riemann bilinear relations).

### 1.2.3 Kernel Functions and Canonical Forms

In complex analysis, ‘kernel functions’ are a convenient way of describing many operations. The fundamental example, on $\mathbb{CP}^1$, is the ‘Cauchy kernel’,

$$\frac{dw}{(w - z)^2},$$
1.2 Eynard-Orantin Theory

which has the property that, for any function \( f(z) \) with \( f \) analytic at \( z \),

\[
f'(z) = \text{Res}_{w=z} f(w)dw / (w-z)^2.
\]

To generalise this concept, introduce the *Bergmann kernel*,

\[
B(z_1, z_2),
\]
as the unique symmetric, bidifferential satisfying the following properties:

1. In each argument, \( B(z_1, z_2) \) has a double pole at \( z_1 = z_2 \) and no other pole.

2. For \( z \) a local parameter on \( C \),

\[
B(z_1, z_2) \sim \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{holomorphic terms}.
\]

3. \( \forall \, 1 \leq i \leq g, \oint B(z_1, z_2) = 0. \)

The Bergmann kernel has the property that, for any meromorphic function \( f(z) \),

\[
df(z) = \text{Res}_{q=z} B(z, q)f(z),
\]

so it can be thought of as a natural generalisation of the Cauchy kernel. In the case when \( C \) is genus \( 0 \), there are no \( A \)-cycles, so the Cauchy kernel (multiplied by \( dz \)) is the Bergmann kernel,

\[
B(z, w) = \frac{dzdw}{(z-w)^2}.
\]

For a general, genus \( g \) curve, there is a formula for the Bergmann kernel:

\[
B(z_1, z_2) = dz_1 dz_2 \ln(\theta(u(z_1) - u(z_2) - c, \tau)),
\]

where \( u(z) \) is the Abel map, \( c \) is an odd characteristic, \( \tau \) is the period matrix \( \{\tau_{i,j}\} \), and \( \theta \) is the Riemann theta function. Although this formula is general, in practice it is usually easier to try and determine a formula for the Bergmann kernel which is easier to calculate.
with. This will be done for genus one curves in chapter 2.

A canonical form closely associated to the Bergmann kernel is the classical normalised differential of the third kind, \(dS_{xy}(z)\), where \(x, y\) are points on \(C\). This one-form is uniquely determined by the requirements that its only singularities are simple poles at \(x\) and \(y\), and,

\[
Res_{z=x} dS_{xy}(z) = 1 = -Res_{z=y} dS_{xy}(z), \quad \text{and,} \quad \oint_{A_i} dS_{xy}(z) = 0.
\]

For example, for the Riemann surface \(\mathbb{CP}^1\), the differential of the third kind is,

\[
dS_{xy}(z) = \frac{2(y-x)dz}{(z-y)(z-x)}.
\]

For any spectral curve \((C, x, y)\), the recursion kernel \(K(z_0, z)\) is defined, for \(z_0 \in C\), and \(z\) close to a zero \(a_i\) of \(dx\) by,

\[
K^i(z_0, z) := \frac{-1}{2} \int_{z'=z} B(z_0, z') \frac{dy(z) - y(z)}{dx(z)}.
\]

This is a meromorphic 1-form in \(z_0\) with simple poles at \(z_0 = z\) and \(z_0 = \overline{z}\). Notice that, because of the use of the branch involution, for the variable \(z\), the recursion kernel is only locally defined near the zeroes of \(dx\). This is why we have include the superscript \(i\).

For example, for the genus 0 curve above with \((\mathbb{CP}^1, z + 1/z, z)\), the recursion kernel is,

\[
\frac{z^3dz_0}{(z^2 - 1)(z - z_0)(zz_0 - 1)}dz.
\]

1.2.4 The Riemann Bilinear Relation

Let \(\omega_1\) be a holomorphic one-form on \(C\), and \(\omega_2\) be a meromorphic one-form on \(C\), with \(f \in K(C)\) a primitive \(f(z) = \int_0^z \omega_1\) whose path of integration lies entirely in the fundamental domain. Let \(\Gamma\) be the boundary of the fundamental domain; an important classical result in the theory of Riemann surfaces is,

\[
\int_{\Gamma} f \omega_2 = \frac{1}{2\pi i} \sum_{k=0}^{g} \left( \oint_{A_k} \omega_1 \oint_{B_k} \omega_2 - \oint_{B_k} \omega_1 \oint_{A_k} \omega_2 \right). \tag{1.9}
\]
This is called the \textit{Riemann bilinear relation}, it dates back to the nineteenth century, and a proof can be found in any text on Riemann surface theory (for example, Section 5.3 of ([Gr])). Notice that setting $\omega_1 = z_i$ and $\omega_2 = z_j$ gives the symmetry of the period matrix $\tau_{ij} = \tau_{ji}$ alluded to earlier.

Setting $\omega_2(z) = dS_{xy}(z)$, and $\omega_1(z) = z_i$ in (1.9) gives (using the fact that, for all $1 \leq i \leq g$, $\oint A_i dS_{xy} = 0$),

$$\int_{\Gamma} f dS_{xy} = f(y) - f(x) = \frac{1}{2\pi i} \oint_{B_i} dS_{xy}.$$  

Differentiating both sides with respect to $y$ gives,

$$z_i(y) = \frac{1}{2\pi i} \oint_{z \in B_i} B(y, z). \quad (1.10)$$

So that,

$$\tau_{ij} = \frac{1}{2\pi i} \oint_{B_j} \oint_{B_i} B(y, z).$$

\subsection{1.2.5 Eynard-Orantin Invariants}

The \textit{Eynard-Orantin invariants} of $(C,x,y)$ are defined recursively by,

$$\omega^{(0,1)}(z) = -y(z)dx(z),$$

$$\omega^{(0,2)}(z_0, z_1) = B(z_0, z_1),$$

and for $2g - 2 + n \geq 0$, and $J = \{z_1, \ldots, z_n\}$:

$$\omega^{(g,n+1)}(z_0, J) = \sum_{i} \text{Res}_{z=a_i} K(z_0, z) \left[ \omega^{(g-1,n+2)}(z, z, J) + \sum_{h=0}^{g} \sum_{I \subseteq J} \omega^{(h,1+|I|)}(z, I) \omega^{(g-h,1+n-|I|)}(z, J/I) \right], \quad (1.11)$$

where the symbol $\sum$ means that terms with $(h, I) = (0, \emptyset)$, and $(g, J)$ are excluded.

The original definition in (EO) also defines an infinite family of complex numbers $F_g$ for $g \geq 2$ by,

$$F_g = \omega_{0}^g = \frac{1}{2 - 2g} \sum_{z=a_i} \text{Res}_{z=a_i} \Phi(z) \omega_{1}^g(z),$$
where $\Phi(z)$ is any primitive of the one-form $ydx$. The terms $F_g$ do not depend on the choice of primitive. The main point of interest in this thesis will not be the numbers $F_g$, but the multidifferentials $\omega^{(g,n)}$, which will be called ‘Eynard-Orantin invariants’. Inspired by the Dijkgraaf-Fuji-Manabe conjecture, the period integrals of the Eynard-Orantin invariants will actually be our main focus.

Remark 1.1. Part of the interest generated by the Eynard-Orantin invariants is an extraordinary conjecture, originally due to the physicists Bouchard, Klemm, Marino and Pasquitti ([BKMP]), that relates the Eynard-Orantin theory to the study of (local) Gromov-Witten invariants of certain non-compact Calabi-Yau varieties.

1.2.6 Properties of the Eynard-Orantin Invariants

The Eynard-Orantin invariants have several remarkable properties, such as the following three theorems. These theorems were all originally proved in ([EO1]) for the case when the spectral curve is a triple $(C,x,y)$ where $x,y$ are meromorphic functions on $C$. However, careful inspection of the proofs show that they also apply to arbitrary spectral curves.

Proposition 1.2. The $(0,3)$ Eynard-Orantin invariant can be calculated using the simpler expression,

$$\omega^{(0,3)}(p_1,p_2,p_3) = \sum_{a_i} \text{Res}_{p=a_i} \frac{B(p,p_1)B(p,p_2)B(p,p_3)}{dx(p)dy(p)}.$$  

This is easy to show, using the original definition (1.11). The reason this is useful is that it is an expression without the recursion kernel $K^i$. It is often the recursion kernel that is the hardest to calculate. We reiterate also that the functions $\{x,y\}$ may only be locally defined around the zeroes of $dx$.

Theorem 1.1 (Eynard, Orantin). For all $(g,n) \neq (1,0)$ the Eynard-Orantin invariants $\omega^{(g,n)}(p_1,\ldots,p_n)$ have poles (in any $p_i$) only at the branch points.
Theorem 1.2 (Eynard, Orantin). For any $A$-cycle, and any $\omega^{(g,n)}$ with $(g,n) \neq (1,0)$,
\[ \oint_{p_k \in A_i} \omega^{(g,n)}(p_1, \ldots, p_n) = 0, \]
for all $1 \leq i \leq g$ and $1 \leq k \leq n$.

Theorem 1.3 (Eynard, Orantin). For all $(g,n) \neq (0,1)$, $1 \leq k \leq n$, and all branch points $a_i$,
\[ \text{Res}_{p_k = a_i} \omega^{(g,n)}(p_1, \ldots, p_n) = 0. \]

The $\omega^{(g,n)}$ are meromorphic forms in each coordinate with poles at the branch points $\{a_1, \ldots, a_k\}$, and the vanishing residue means that, in each argument, they define elements of $H^1(C \setminus \{a_1, \ldots, a_k\}, \mathbb{C})$ (recall that the defining properties of the Bergmann kernel means that this applies also to the case $\omega^{(0,2)}$). In fact, more is true:

Lemma 1.1. In each coordinate, the Eynard-Orantin invariant defines an element of $H^1(C, \mathbb{C})$.

Proof. It is clear, using Stoke’s theorem, that the residue map can be defined on cohomology. Let $P$ be the union of the branch points (this is a finite set), the following sequence ([CMP], Problem 1.1.10) is exact:
\[ 0 \rightarrow H^1(C, \mathbb{C}) \rightarrow H^1(C \setminus P, \mathbb{C}) \xrightarrow{\text{res}} H^0(P, \mathbb{C}) \rightarrow H^0(C, \mathbb{C}) \rightarrow 0, \]
Here, elements of $H^0(P, \mathbb{C})$ are thought of as linear functionals on the vector space spanned by points of $P$, and the usual residue at $p \in P$ is just the restriction of ‘res’ to $p \in P$.

Using Theorem (1.3), in each argument, the Eynard-Orantin forms are in $Ker(\text{res}) \subset H^1(C/P, \mathbb{C})$, and they can be extended to $H^1(C, \mathbb{C})$. \qed

Remark 1.2. This is simply the classical idea of ‘differentials of the second kind’ giving cohomology classes. Notice that the holomorphic differentials are not sufficient to span the first cohomology group of $C$, since $\dim H^1(C, \mathbb{C}) = 2g$ and the space of holomorphic differentials has dimension $g$.  

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1.3 Eynard-Orantin Theory and Geometry

As discussed above, Eynard-Orantin theory arose out of the idea of translating a combinatoric problem (enumerating ribbons graphs of a given genus) into complex geometry, and then using standard tools from that field. After abstracting this process from its combinatoric origins, it is natural to ask: is there a geometric meaning to the Eynard-Orantin theory?

Our focus will be on the simplest case: the ‘planar’ Eynard-Orantin invariants of the form $\omega^{(0,n)}$. Here we will argue that the way to understand these invariants is via the idea of ‘differentiating’ a parametrised family of cohomology classes, using the Gauss-Manin connection. The parametrised family of cohomology classes is very special, and comes from a certain variation of the spectral curve $C$ which we call (following Eynard) a ‘filling fractions variation’.

The idea that the planar Eynard-Orantin invariants can be thought of in this way is implicit in the original work of Eynard and Orantin (\cite{EO1}), although not stated in this way. Hopefully the discussion below will make the theory easier to understand for geometers. Once the idea of viewing the (planar) Eynard-Orantin theory via the Gauss-Manin connection has been made rigorous, it can be used to understand what the Eynard-Orantin theory of the $A$-polynomial should be calculating, in chapter 4.

1.3.1 The Rauch Variational Formula

In our context, parametrised families of cohomology classes will arise in the following way. Let $\pi : \mathcal{X} \to \mathcal{E}$ be a holomorphic fibration, with the fibres $\pi^{-1}(\epsilon) = C_\epsilon$ diffeomorphic to a smooth surface of genus $g$. Choose a base point $0 \in \mathcal{E}$, and let $C_0 = C$, so that the family $C_\epsilon$ is thought of as a variation of $C_0$. Let $\Omega$ be a one-form on $\mathcal{X}$ such that $\Omega_\epsilon = \Omega|_{C_\epsilon}$ is closed for all $\epsilon \in \mathcal{E}$. Now suppose that there is a local trivialisation $T : \pi^{-1}(U) \cong C_0 \times U$,
for some $U \subset \mathcal{E}$. The trivialisation identifies $C_\epsilon$ and $C_0$ (as smooth manifolds), so we can consider $(\Omega_\epsilon)$ as a family of differential forms $\omega_\epsilon$ on $C_0$ that vary smoothly with $\epsilon \in \mathcal{E}$. The ‘derivative’ of this family at the point $\epsilon = 0$ will give us the derivative of the cohomology class $[\omega_0]$.

Now let’s define (following, for example, ([K]), or ([V]) Chapter 9 and the references therein) what it means to differentiate a family of one-forms $\omega_b$ on $C_0$, where $\omega_0$ is closed. Choose a function $x \in K(C)$, so that any one-form $\omega_b$ can be written as $f_b(x)dx$, for some meromorphic function $f_b(x)$, that depends on $\epsilon \in B$. In general, the one-form $\partial_\epsilon(f_\epsilon(x))dx$ depends on which function $x$ was chosen. However, it can be shown that the one-form $\partial_\epsilon(f_\epsilon(x))dx$ has vanishing residue (and so defines a cohomology class, using Lemma (1.1)), and this cohomology class does not depend on the function $x$. That is, the two one-forms $\partial_\epsilon f_\epsilon(y)dy$ and $\partial_\epsilon f_\epsilon(x)dx$ differ by an exact one-form. This gives the covariant derivative of the cohomology class $[\omega_0]$ with respect to $\epsilon \in \mathcal{E}$ at $\epsilon = 0$.

This can be thought of as a covariant derivative at $\epsilon = 0$ on sections of the trivial vector bundle $\mathcal{E} \times H^1(C_\epsilon, \mathbb{C})$ over $\mathcal{E}$. The corresponding connection is the ‘Gauss-Manin connection’.

In summary, a variation of Riemann surfaces $C = C_0$, (with a one-form $\Omega$ on the total space) will give us families of one-forms, that we will then differentiate using the definition above.

In order to do this in detail, we will now discuss a simple way to vary a Riemann surface $C$, which is essentially due to Riemann himself. Think of $C$ as a smooth two-manifold: one way to determine a complex structure on $C$ is to express $C$ as a smooth branched cover of $\mathbb{C}P^1$ via some smooth map $x : C \to \mathbb{C}P^1$. If $a_i \in C$ are the branch points, and $a_i = x(a_i)$, then the complex structure is determined by letting $x$ be a chart away from the branch points, and setting $t^n = x - a_i$ near a branch point $a_i$ (with the corresponding adjustments for branching at infinity). This complex structure on $C$ is unique, and $x$ will now be a meromorphic function.
Now, alter the position of the branch points slightly so that $C$ is a smooth branched cover of $\mathbb{CP}^1$ with branch points at $a_i + \epsilon$. This gives a slightly deformed complex structure on $C$, and it is these sorts of variations that we will be considering. Notice that, from this heuristic description, it is clear that the resulting variation $C_\epsilon$ of $C_0$ is 'locally trivial' as a smooth fibration: if $\epsilon$ is close to zero, there is obviously a diffeomorphism (given by smoothly pulling the branch point back to $a_i$) from $C_\epsilon$ to $C_0$. Let’s discuss this in some more detail.

Let $C_0$ be a compact Riemann surface with $x : C_0 \to \mathbb{CP}^1$ a branched cover, with $n$ branch points $a_i \in X$, of degree $m_i$, $1 \leq i \leq n$. Let $t_i : V_i \to \mathbb{C}$ be local charts at each branch point, with $t_i(a_i) = 0$.

Define the family of new Riemann surfaces $C_\epsilon$, by changing all the coordinate charts $t_i|_{V_i}$ to $s_{\epsilon,i}$, where $s_{0,i} = t_i$. Of course, this change of structure might not define a Riemann surface if the new coordinate charts don’t have holomorphic transition functions. We will assume the existence of a family $U_{i,\epsilon} \subset V$ where, $\epsilon$ sufficiently small, the substitution $t_i|_{U_i} \mapsto s_{\epsilon,i}$ defines a Riemann surface. Also we will stipulate that the image under $t$ of $\partial U_i$ is a loop around 0 of radius $r(\epsilon)$, and $r(\epsilon)$ is a nondecreasing function with $r(0) = 0$.

Denote the first derivative of $s_{\epsilon}(t)$ with respect to $\epsilon$ evaluated at $\epsilon = 0$ by $\delta s$. It will also be required that $s_{\epsilon}(t) - t = \epsilon \delta s + \epsilon^2 f(t)$, for some holomorphic $f(t)$.

**Definition 1.1.** Let $C$ be a Riemann surface with a fixed branched covering $x : C \to \mathbb{CP}^1$. A family of Riemann surfaces $C_\epsilon$ defined as above will be called a variation of the branched cover $x$.

For any deformation of a branched cover, with corresponding family $U_{i,\epsilon}$ of open sets in $C_\epsilon$, we will denote $U_\epsilon = \bigcup U_{i,\epsilon}$, and $\gamma_i,\epsilon = \partial U_{i,\epsilon}$. Also denote $\gamma_\epsilon = \partial U_\epsilon$. Recall that the $x$ projection of $\gamma_i$ is a loop whose radius decreases as $\epsilon$ approaches 0.

The simplest example is to take $C_0$ to be the compactification of the zero locus of $y^2 - x$, and the branched covering to be the function $x : C_0 \to \mathbb{CP}^1$. The only branch point
is at \((0, 0)\), and a local coordinate near the branch point (which is in fact also a global coordinate) is given by the function \(y : C_0 \to \mathbb{C}P^1\). Define the family of Riemann surfaces \(C_\epsilon\) to be the projective zero locus of \(\tilde{y}^2 - y^2 - \epsilon\), the family of surfaces \(C_\epsilon\) is a variation of the branched cover \(x\).

Another example is to define \(C_0\) to be the zero locus of \(y^2 - \sigma(x)\) where \(\sigma(x) = \prod_{i=1}^{n} (x-a_i)\), and to define \(C_\epsilon\) by the zero locus of \(\tilde{y}^2 - \sigma(x) - \epsilon\). The function \(y\) is, again, a local coordinate around any of the branch points \((a_i, 0)\), and the family \(C_\epsilon\) is a variation of the branched cover \((C_0, x)\).

Notice that in the example above, one also start with local coordinates \(t_i\) such that \(t_i^2 = x-a_i\), and deform each of these individually to, for example, \(s_\epsilon\), where \(s_\epsilon^2 + \epsilon s_\epsilon = x-a_i\).

The two examples above are atypical in the sense that each of the local deformations is controlled by the deformation of one global coordinate.

As we outlined above, given a locally trivial family \(C_\epsilon\) of Riemann surfaces we can pull back a one-form \(\Omega\) on the total space of the variation to a family of one-forms \(\omega_\epsilon\) on \(C_0\).

**Theorem 1.4** (Rauch). Let \((C, x)\) be a branched cover with branch points \(a_i \in C\). Let \(C_\epsilon\) be a variation of the branch cover \((C_0, x)\). If \(B(p_1, p_2)\) is the Bergmann kernel of \(C_0\), with respect to some canonical dissection \(\Gamma\), \(B_\epsilon(p_1, p_2)\) is a Bergmann kernel of \(C_\epsilon\) with respect to \(\Gamma\), then the variation,

\[
B_\epsilon(p_1, p_2) - B(p_1, p_2),
\]

is defined on \(C_0/U_\epsilon \cong C_\epsilon/U_\epsilon\), and the derivative is given by,

\[
\frac{\partial}{\partial \epsilon} B_\epsilon(p_1, p_2)|_{\epsilon=0} := \delta B(p_1, p_2) = \sum_{i=1}^{n} \frac{\text{Res} \delta s(p) B(p_1, p) B(p_2, p)}{dt}. \tag{1.12}
\]

**Remark 1.3.** In terms of the Gauss-Manin connection, since both sides of (1.12) define cohomology classes in each of the arguments (the Bergmann kernel has vanishing residue), we have,

\[
\nabla_{\frac{\partial}{\partial \epsilon}} [B_\epsilon(p_1, p_2)] = \left[ \sum_{i=1}^{n} \frac{\text{Res} \delta s(p) B(p_1, p) B(p_2, p)}{dt} \right],
\]
where the notation \([\cdot]\) is used to mean the cohomology class defined in each argument.

Proof. The proof follows the classical case of ([R]), with some adjustments to use the modern language of differential forms.

Let \(dS_{xy}\) and \(dS_{\epsilon,xy}\) be the fundamental differentials of the third kind on \(C\) and \(C_{\epsilon}\), with respect to dissections \(\Gamma\) and \(\Gamma_{\epsilon}\). Let \(S_{xy}\) and \(S_{\epsilon,xy}\) be arbitrary primitives of \(dS_{xy}\) and \(dS_{\epsilon,xy}\).

Consider the function,
\[
\delta_{\epsilon}S_{xy}(z) := i^*S_{\epsilon,xy}(z) - S_{xy}(z),
\]
on \(C/U_{\epsilon}\). This function is analytic on \(C/U_{\epsilon}\); choose two points \(z, w \notin U_{\epsilon}\), and use the Riemann bilinear relation for a surface with boundary \(\gamma_{\epsilon}\), and the defining properties of the fundamental third kind differential, to get,
\[
\int_{\gamma_{\epsilon} \cup \Gamma} \delta_{\epsilon}S_{xy}dS_{zw} = \int_{\gamma_{\epsilon}} \delta_{\epsilon}S_{xy}dS_{zw} + \sum_{i=1}^{g} \left( \oint_{A_i} \delta_{\epsilon}dS_{xy} \oint_{B_i} dS_{zw} - \oint_{A_i} dS_{zw} \oint_{B_i} \delta_{\epsilon}dS_{xy} \right)
\]
\[
= \int_{\gamma_{\epsilon}} \delta_{\epsilon}S_{xy}dS_{zw} + \sum_{i=1}^{g} \left( \oint_{A_i} (i^*dS_{\epsilon,xy} - S_{xy}) \oint_{B_i} dS_{zw} \right)
\]
\[
= \int_{\gamma_{\epsilon}} \delta_{\epsilon}S_{xy}dS_{zw}.
\]
Because \(\delta_{\epsilon}S_{xy}\) is holomorphic on \(C/U_{\epsilon}\),
\[
\delta_{\epsilon}S_{xy}(z) - \delta_{\epsilon}S_{xy}(w) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon} \cup \Gamma} \delta_{\epsilon}S_{xy}(u)dS_{zw}(u) = \int_{\gamma_{\epsilon}} \delta_{\epsilon}S_{xy}(u)dS_{zw}(u).
\]
Furthermore, \(dS_{zw}(u)\) is holomorphic on \(\gamma_{\epsilon}\), so there is the further reduction,
\[
\delta_{\epsilon}S_{xy}(z) - \delta_{\epsilon}S_{xy}(w) = \frac{1}{2\pi i} \int_{\gamma_{\epsilon}} dS_{zw}(u)\delta_{\epsilon}S_{xy}(u)
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_{\epsilon}} dS_{zw}(u)i^*S_{\epsilon,xy}(u) - \frac{1}{2\pi i} \int_{\gamma_{\epsilon}} dS_{zw}(u)S_{xy}(u)
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_{\epsilon}} dS_{zw}(u)i^*S_{\epsilon,xy}(u).
\]
Differentiation with respect to \( w \) yields,

\[
\delta_x dS_{xy}(z) = \frac{1}{2\pi i} \int_{\gamma_x} B(u, z) i^\tau S_{\epsilon,xy}(u).
\]

To calculate this integral, use the coordinates \( t_i \) so that,

\[
\delta_\epsilon dS_{xy}(z) = \frac{1}{2\pi i} \sum_i \int_{t_i(\gamma_{\epsilon,i})} B(t_i, z) S_{\epsilon,xy}(s(t_i)).
\]

Notice that the related integral \( \int_{t_i(\gamma_{\epsilon,i})} B(t_i, z) S_{\epsilon,xy}(s(t_i)) = 0 \) because the integrand is holomorphic around \( \gamma_{\epsilon,i} \) and rewrite the above as,

\[
\delta_\epsilon dS_{xy}(z) = \frac{1}{2\pi i} \sum_i \int_{t_i(\gamma_{\epsilon,i})} B(t_i, z) (S_{\epsilon,xy}(s(t_i)) - S_{\epsilon,xy}(t_i)).
\]

Now, since \( s_\epsilon(t) - t = \epsilon \delta s + \epsilon^2 f(t) \),

\[
\delta_\epsilon dS_{xy}(z) = \frac{\epsilon}{2\pi i} \sum_i \int_{t_i(\gamma_{\epsilon,i})} B(t_i, z) S'_{\epsilon,xy}(t) \delta s + o(\epsilon).
\]

The equation above means that, for some analytic \( f(z), g(z) \),

\[
dS_{\epsilon,xy}(z) - dS_{xy}(z) = f(z) \epsilon + g(z) \epsilon^2,
\]

so,

\[
S_{\epsilon,xy}(z) - S_{xy}(z) = F(z) \epsilon + G(z) \epsilon^2,
\]

for some analytic \( F(z), G(z) \). This gives,

\[
\epsilon \int_{t_i(\gamma_{\epsilon,i})} S'_{\epsilon,xy} \delta s(t) B(z, t) = \epsilon \int_{t_i(\gamma_{\epsilon,i})} S_{xy} \delta s(t) B(z, t) + o(\epsilon)
\]

So \( \delta dS_{xy}(z) = \lim_{\epsilon \to 0} \frac{\delta_x dS_{xy}(z)}{\epsilon} = \frac{1}{2\pi i} \sum_i \int_{t_i(\gamma_{\epsilon,i})} B(t_i, z) S'_{xy}(t) \delta s \). Differentiation with respect to \( x \) and the residue theorem gives the result.
Chapter 2

Eynard-Orantin Theory of Elliptic Curves

The Eynard-Orantin invariants are defined for spectral curves of arbitrary genus. However, calculation of the Eynard-Orantin invariants for higher genus curves is difficult, mainly because one needs a good parametrisation to express the recursion kernel and calculate its Laurent series at the branch points. The difficulty of calculating the Eynard-Orantin invariants is determined not only by the genus of the fixed Riemann surface, but the complexity of the two functions that define the spectral curve. These functions determine things like the number of branch points, and the numerator of the recursion kernel. Because of the complications that can develop at higher genus, we will focus on the case when the spectral curve is genus one. This is the simplest situation that still allows us to still demonstrate the use of the variational formula from Chapter 1 (since genus zero curves have no homology). Apart from applying these results to certain $A$-polynomials in chapter 4, the genus one case is of interest in the various other applications of Eynard-Orantin theory, and is relatively uninvestigated: most of the mathematical results on
Eynard-Orantin theory (reviewed in [EO2]) only apply to genus zero spectral curves.

## 2.1 Elliptic Curves and Elliptic Functions

To fix notation (and point out some non-standard phenomenon that are peculiar to the Eynard-Orantin theory) we will now introduce some conventions for elliptic curves and elliptic functions. There are many sources for this material, such as ([WW]), ([MM]), ([L]).

Firstly, a rank 2 lattice $\Lambda \in \mathbb{C}$ gives a compact Riemann surface $E_\Lambda$ of genus one by taking the quotient $E_\Lambda := \mathbb{C}/\Lambda$. This Riemann surface has a distinguished point, given by the equivalence class of the identity element of $\Lambda$. Let $\mathcal{H}$ denote the upper-half plane of complex numbers with strictly positive imaginary part. Choosing generators $\omega_1, \omega_2$ for $\Lambda$ with $\omega_2 \omega_1 \in \mathcal{H}$, gives the identification $\Lambda := \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2$. If $\omega'_1$ and $\omega'_2$ are another set of generators, and $\omega'_1, \omega'_2$ are related to $\omega_1, \omega_2$ by an $SL(2, \mathbb{Z})$ transformation, then the resulting Riemann surface is isomorphic.

**Remark 2.1.** Note that these conventions differ from some standard sources, eg. ([WW]).

Conversely, let $E$ be a compact, genus 1 Riemann surface $E$ with a marked point $o$; applying the Riemann-Roch theorem to the case of a trivial line bundle $L \to E$ yields,

$$h^0(E, K) = 1,$$

so there is a global non-vanishing holomorphic 1-form, denoted $dz$.

As discussed in section 1.2.2, the group $H_1(E, \mathbb{Z})$ has a pair of generators $\{A, B\}$ such that $A \cdot B = 1$ and $A \cdot A = B \cdot B = 0$. Define the lattice $\Lambda := \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2$ by,

$$\omega_1 := \int_A dz, \quad \omega_2 := \int_B dz,$$
with the convention that $\omega_2/\omega_1 \in \mathcal{H}$. It is a short exercise to show that the mapping $\Phi : E \to \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$, given by,

$$\Phi(x) := \int_0^x dz \mod \Lambda,$$

is an isomorphism of Riemann surfaces.

Therefore, genus one Riemann surfaces will be identified with rank two lattices in $\mathbb{C}$, and a genus one Riemann surface with a choice of basis for $H_1(C, \mathbb{Z})$ will be identified with a rank two lattice with a choice of generators $\{\omega_1, \omega_2\}$ (the convention will always hold that $\omega_2/\omega_1 \in \mathcal{H}$).

Given such a lattice $\Lambda$, define the Weierstrass $\wp$ function,

$$\wp(z) := \frac{1}{z^2} + \sum_{(m,n)\in \Lambda, (m,n)\neq 0} \left( \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right), \quad (2.1)$$

Since this function is invariant under translation by an element of the lattice and is uniformly convergent on compact sets in the complement of the lattice, it is a meromorphic function on the genus one Riemann surface $E_{\Lambda}$ (an ‘elliptic function’). It has a double pole at the point $z = 0$, and satisfies,

$$\wp'(z) = 4\wp^3(z) - g_2\wp(z) - g_3 = (\wp(z) - a_1)(\wp(z) - a_2)(\wp(z) - a_3), \quad (2.2)$$

for constants $a_i$, which gives a realisation of $E_{\Lambda}$ as an affine curve in $\mathbb{C}^2$. The Riemann surface has a (holomorphic) group structure given by the natural additive group structure of $\mathbb{C}/\Lambda$; this is mapped to the affine curve using the identity,

$$\wp(z + w) = \frac{1}{4} \left( \frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2 - \wp(z) - \wp(w), \forall z, w \in \mathbb{C}. \quad (2.3)$$

Define $\omega_3 := \omega_1 + \omega_2$. The images of the half-lattice points $\omega_i/2$ have the property that they map to the roots of (2.2), so that, $a_i = \wp(\omega_i/2)$, for all $1 \leq i \leq 3$. The roots also satisfy the equation $a_1 + a_2 + a_3 = 0$. Using the equation (2.2), it also follows that the image of $\omega_i/2$
under $\wp'(z)$ is 0.

Given a choice of lattice $\Lambda$, the function $Z_{\Lambda}(z)$ on $\mathbb{C}$ defined by the equation,

$$\frac{dZ_{\Lambda}(z)}{dz} = -\wp(z),$$

with the condition,

$$\lim_{z \to 0} (Z_{\Lambda}(z) - z^{-1}) = 0,$$

is called the Weierstrass Zeta Function; this is not an elliptic function.

The periods of the Weierstrass Zeta function are defined by,

$$\eta_i(\Lambda) := Z_{\Lambda}(z + \omega_i) - Z_{\Lambda}(z),$$

for any $z \in \mathbb{C}$ and $1 \leq i \leq 3$. The periods satisfy,

$$Z_{\Lambda}(z + n_1\omega_1 + n_2\omega_2) = Z_{\Lambda}(z) + n_1\eta_1(\Lambda) + n_2\eta_2(\Lambda), \quad n_1, n_2 \in \mathbb{Z}. \quad (2.4)$$

So that $\eta_3(\Lambda) = \eta_1(\Lambda) + \eta_2(\Lambda)$. Moreover, (recall that our convention is that $\frac{\omega_2}{\omega_1} \in \mathcal{H}$),

the periods satisfy the Legendre relation,

$$\eta_1(\Lambda)\omega_2 - \eta_2(\Lambda)\omega_1 = 2\pi i. \quad (2.5)$$

### 2.1.1 The Gauss-Manin Connection

Now we will discuss the geometry of the Eynard-Orantin theory of a genus one curve. Let $C_0$ be the nonsingular elliptic curve, defined by the double branched cover of the $x$-plane given by,

$$y^2 = 4x^3 - g_2x - g_3 = 4 \prod_{i=1}^{3} (x - a_i).$$

Now define the family $C_{\epsilon}$ by the branched covers,

$$y_{\epsilon}^2 = 4 \prod_{i=1}^{3} (x - a_i) + \epsilon.$$

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Notice that $C_\epsilon$ is a variation as described in section (1.3) of chapter 1: locally near each branch point $a_i$, the function $y_0$ is a coordinate, and we have changed this to $y_\epsilon$. This variation simultaneously alters all the branch points.

The family $C_\epsilon$ can be thought of via a fibration $\pi : C^2 \to B$, where $\pi(x, y) = y^2 - 4x^3 + g_2x + g_3$. Consider the one-form $\Omega = ydx$ on $C^2$. On each curve $C_\epsilon$, $\omega_\epsilon := \Omega_\epsilon|C_\epsilon = y_\epsilon dx$ is of the form,

$$y'(z)^2dz,$$

if we use the usual uniformising coordinate $z$. Thus the one-form $y_\epsilon dx$ has zero residue at it’s only pole $z = 0$, which means that it defines a cohomology class $[y_\epsilon dx] \in H^1(C_\epsilon, \mathbb{C})$.

We would like to apply the Gauss-Manin connection to this cohomology class, at $\epsilon = 0$ (or indeed, at any $\epsilon$). Recall from chapter 1 section (1.3) that the effect of the Gauss-Manin connection on a cohomology class can be calculated in a given coordinate patch simply by differentiating with respect to $\epsilon$ (and the resulting cohomology class is independent of the coordinate chosen).

Differentiate with respect to $\epsilon$, we get,

$$\nabla \frac{\partial}{\partial \epsilon} [y_\epsilon dx] = \left[ \frac{dx}{2y} \right] = 2[dz].$$

So that the derivative of $[ydx]$ is a holomorphic form at all $\epsilon$.

Recall from section (1.10) of chapter 1 that the canonical holomorphic one-forms on a Riemann surface are related to period integrals of the Bergmann kernel by,

$$\oint_{B_j} B(\cdot, p) = 2\pi i \delta_j(p).$$

In fact, in the genus one case this can be verified by direct calculation, because (2.5) gives that,

$$\oint_B B(\cdot, p)\omega_1 = (\eta_2 \omega_1 + \eta_1 \omega_2)dz(p) = 2\pi idz(p),$$

and the canonical holomorphic one-form is $\frac{dz}{\omega_1}$, where $z$ is the uniformising coordinate.

This gives that,

$$\nabla \frac{\partial}{\partial \epsilon} [y_\epsilon dx] = \left[ \oint_{z \in B} B(z, \cdot) \right].$$
So, in the language of Eynard-Orantin theory,
\[ \nabla_{\partial} [\omega^{(0,1)}] = \{ \oint_{z \in B} \omega^{(0,2)} \}. \]

**Definition 2.1.** A family of genus one curves \( y^2_\epsilon = \sigma(x) \) in \( \mathbb{C}^2 \) with the property that \( \nabla_{\partial} [y_\epsilon dx] \) is holomorphic is called a variation by filling fractions of the zero locus of \( y^2_0 = \sigma(x) \).

This terminology is inspired by the original case of matrix models. The variations by fillings fractions play a special role in Eynard-Orantin theory.

Using Theorem (1.4), and the fact that, \( \partial_{\epsilon} y_{\epsilon} \Big|_{\epsilon=0} = \oint_{z \in B} B(\cdot, p) \), we get, \( \nabla_{\partial} [\omega^{(0,2)}] = [\oint_{z \in B} \omega^{(0,3)}] \) (one also has to check that there are no residues at infinity). Here, we have also used Lemma (1.1) to give that \( \omega^{(0,3)} \) defines a cohomology class, in each of its arguments. This result can be seen as the ‘base case’ of the following theorem proven by Eynard and Orantin in ([EO1], section 5) for meromorphic spectral curves,

**Theorem 2.1** (Eynard, Orantin). For all \((g,n) \neq (0,2)\), and \( C_\epsilon \) the family of genus one curves defined above, then,
\[ \nabla_{\partial} [\omega^{(g,n)}(p_1, \ldots, p_n)] = [\oint_{B} \omega^{(g,n+1)}(\cdot, p_1, \ldots, p_n)]. \]

**Remark 2.2.** The theory discussed in section 5 of ([EO1]) covers a much wider situation than the genus one variation of filling fractions. However, in the more general situation, it is not clear to what extent the proofs are mathematically rigorous: the questionable step is the use in ([EO1]) of Rauch’s variational formula to go from \( \omega^{(0,2)} \) to \( \omega^{(0,3)} \).

This is one reason for focusing on the elliptic case: here, for filling fraction variations of elliptic curves we can rigorously show the \((0,2)\) case, using the argument above based on
on Theorem (1.4). Once this step is verified the formula (5-10) in ([EO1]) follows, and the remaining arguments to show the higher \((0,n)\) cases only involve standard complex analysis (and the combinatorics of the graphical calculus outlined in [EO1]).

**Remark 2.3.** Note that Lemma (1.1) is used to show that \(\omega^{(g,n)}\) determines a cohomology class.

The importance of this theorem is that it gives the correct framework to try and generalise the Eynard-Orantin theory to the case of the \(A\)-polynomial, via the idea of the variation of filling fractions.

The variation of filling fractions also gives natural links to the geometry of integrable systems. Recall that we have a family of compact Riemann surfaces \(C_\epsilon\) varying over a subset of the \(\epsilon\)-plane (to avoid complications with global monodromy). On \(C_0\) we have the family \([y_\epsilon, dx]\) of cohomology classes, given by pulling back \([y_\epsilon, dx]\) to \(C_0\) via a local trivialisation of \(\pi : C^2 \rightarrow B\). If we use the Abel-Jacobi map on \([y_\epsilon, dx]\) (evaluating on \(A\) and \(B\) cycles) we end up with a flow in the Jacobian \(J(C_0)\) of \(C_0\). If the filling fractions variation is constant, then this flow is linear. In fact, if we identify \(J(C_0)\) with the lattice \(\mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2\), the flow is a straight line in the direction of \(\omega_2\). This aspect of Eynard-Orantin theory will be discussed in chapter 4.

The idea of thinking of Eynard-Orantin theory via the covariant derivatives of cohomology classes is also powerful computationally. For even the simplest spectral curves, the Eynard-Orantin invariants are enormously difficult to compute, and the complexity of the computation grows rapidly as the integers \((g,n)\) indexing the invariants grow. Even for the planar invariants \((0,n)\), for a simple genus one spectral curve \((C, \wp(z), \wp'(z))\), to calculate an invariant like \(\omega^{(0,1000)}\) is impossible, even using a computer.

To calculate period integrals such as \(\oint_B \omega^{(g,n)}\), it seems that calculating the full Eynard-Orantin invariant, and then calculating its integrals might not be the most efficient algorithm. In fact, the following theorem shows a simple way to calculate the period integrals.
of \((0, n)\) Eynard-Orantin invariants (‘planar’ Eynard-Orantin invariants), that circumvents the whole recursion and works for extremely high values of \(n\).

Henceforth for any cycle \(\Gamma \subset C\), and a multidifferential \(\omega\) of degree \(n\), use the notation \(\int_{\Gamma}^{k} \omega\) for the multidifferential of degree \(n - k\) given by integrating the first \(k\) arguments of \(\omega\) over \(\Gamma\). For example \(\int_{\Gamma}^{3} B \omega(0, 3) = \int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)\).

**Theorem 2.2.** If \(\omega^{(g,n)}\) are the Eynard-Orantin invariants of a genus 1 spectral curve \((C, \wp(z), \wp'(z))\), then the equation,

\[
\int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 4)(\cdot, \cdot, \cdot, \cdot, p) = \beta \int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, p),
\]

holds in \(H^1(C, \mathbb{C})\), for some \(\beta \in \mathbb{C}\). Furthermore, the equation,

\[
\int_{\Gamma}^{4+n} B \omega(0, 4+n) = \beta^n \int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)
\]

holds for the planar Eynard-Orantin invariants.

**Proof.** First recall that Lemma (1.1) gives that, in each coordinate, the Eynard-Orantin invariants do define an element of \(H^1(C, \mathbb{C})\). Let \(\langle \cdot, \cdot \rangle\) be the pairing between homology and cohomology. Since \(\langle \nabla_{\frac{\partial}{\partial x}} B [ydx], A \rangle = 1\), and \(\langle \int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, p) \rangle = 0\) the classes \(\nabla_{\frac{\partial}{\partial x}} [ydx]\) and \(\int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, p)\) are not scalar multiples.

Since the dimension of \(H^1(C, \mathbb{C})\) is 2, the cohomology class \(\int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, \cdot)\) can be written as a linear combination of \(\nabla_{\frac{\partial}{\partial x}} [ydx]\) and \(\int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, \cdot)\):

\[
\int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 4)(\cdot, \cdot, \cdot, \cdot, p) = \alpha [\delta ydx] + \beta [\int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, \cdot, p)].
\]

In fact, since \(\langle \nabla_{\frac{\partial}{\partial x}} [ydx], A \rangle = 1\), and,

\[
\langle \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, p), A \rangle = \langle \int_{\Gamma} B \int_{\Gamma} B \omega(0, 4)(\cdot, \cdot, \cdot, \cdot), A \rangle = 0,
\]

then \(\alpha = 0\). Since the Gauss-Manin connection is a linear operator, Theorem (2.1) means that,

\[
\langle \nabla_{\frac{\partial}{\partial x}} \int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 4)(\cdot, \cdot, \cdot, \cdot, p), B \rangle = \beta^n < \int_{\Gamma} B \int_{\Gamma} B \int_{\Gamma} B \omega(0, 3)(\cdot, \cdot, \cdot, p), B >.
\]
and the left hand side is equal to $f_B^{4+n} \omega^{(0,4+n)}$.

**Remark 2.4.** The scalar $\beta$ in Theorem (2.13) can be computed by finding the ratio of $f_B f_B f_B f_B \omega^{(0,3)}(\cdot, \cdot, \cdot, \cdot)$ and $f_B f_B f_B f_B \omega^{(0,4)}(\cdot, \cdot, \cdot, \cdot)$.

### 2.2 Modular Forms and Formulae for Eynard-Orantin Theory

In this section expressions for the Eynard-Orantin invariants of elliptic curves are derived in terms of classical objects such as modular forms and elliptic functions.

The group $SL(2, \mathbb{Z})$ acts on the upper half plane $\mathcal{H} \subset \mathbb{C}$ by linear fractional transformations. A *modular form* of weight $k$ is a holomorphic function $f$ on $\mathcal{H}$ such that,

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z),$$

for all $\gamma \in SL(2, \mathbb{Z})$, satisfying certain growth conditions at infinity. To define these conditions, notice that any function on $\mathcal{H}$ satisfying these transformation laws is periodic of period 1, since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$, so that a modular form will have a Fourier expansion in $q = \exp(2\pi i z)$. We will require that the Fourier expansion has no negative powers of $q$.

**Definition 2.2.** For any rank two lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$, and $k \geq 2$ an even integer the function,

$$G_k(\Lambda) = \sum_{(n,m) \in \mathbb{Z}^2 \setminus (n,m) \neq 0} \frac{1}{(n\omega_1 + m\omega_2)^k},$$

is the level $k$ Eisenstein series for $\Lambda$.

We will mainly be interested in Eisenstein series for the normalised lattices of the form $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$, where $\tau \in \mathcal{H}$. In this case, it is easy to show that the level $k$ Eisenstein
series is a modular form in \( \tau \), when \( k > 2 \). The level 2 Eisenstein series are not modular forms, they are the simplest examples of what are known as ‘quasimodular’ forms (see Don Zagier’s chapter in ([BGZ])).

For the normalised lattice \( \Lambda_{\tau} \), the Eisenstein series are related to the period of the Weierstrass Zeta function by the following identity in ([L], chapter 18):

\[
\eta_1(\Lambda_{\tau}) = G_2(\tau). \tag{2.6}
\]

Using the Legendre relation (2.5) it then follows that

\[
\eta_2(\Lambda_{\tau}) = G_2(\tau)\tau + 2\pi i. \tag{2.7}
\]

Let \( N \) be a positive integer. The \textit{principal congruence subgroup of level} \( N \) is,

\[
\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{N} \right\}.
\]

**Definition 2.3.** A subgroup \( \Gamma \) of \( SL(2, \mathbb{Z}) \) is a congruence subgroup of level \( N \) if \( \Gamma(N) \subset \Gamma \) for some \( N \in \mathbb{Z}^+ \).

Modular forms were defined as functions that transform in a certain manner under the action of the full modular group \( SL(2, \mathbb{Z}) \). It is possible to extend this definition to define modular forms for congruence subgroups in the obvious manner. Such modular forms will be called \textit{congruence modular forms} of weight \( k \).

The main congruence subgroup used below will be \( \Gamma(2) \), and the most important example of a congruence modular form will be the analogues of the Eisenstein series for the congruence subgroups.

**Definition 2.4.** Let \( \bar{a} = (a_1, a_2) \) be a vector in \( (\mathbb{Z}/2\mathbb{Z}) \), \( \Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C} \) a rank two lattice, and \( k \) a positive integer, then,

\[
G_k^\bar{a}(\tau) := \sum_{\substack{(n,m) \equiv \bar{a} \\ (n,m) \neq 0}} \frac{1}{(n\omega_1 + m\omega_2)^k}
\]
2.2 Eynard-Orantin Theory of Elliptic Curves

is the level $k \Gamma(2)$-Eisenstein series for $\Lambda$.

Remark 2.5. It is more usual in this definition to only work with the normalised lattice $\Lambda_\tau$.

Like the standard Eisenstein series, it is easy to show that the $\Gamma(2)$-Eisenstein series for $\Lambda_\tau$ are congruence modular forms in the variable $\tau$, for $k > 2$.

2.2.1 Recursion Kernel

To study the Eynard-Orantin theory of an elliptic curve, the first step is to calculate the recursion kernel $K(z_0, z)$. To do this, it is necessary to determine the branch involution, and find the Bergmann kernel. On an elliptic curve $C$ in Weierstrass form,

$$y^2 = 4x^3 - g_2x - g_3 = (x - a_1)(x - a_2)(x - a_3) = \sigma(x)$$

the zeroes of $dx$ are at the half lattice points $(a_i, 0)$, so the branch involution for the spectral curve $(C, x, y)$ is the global map $(x, y) \mapsto (x, -y)$. Notice that it is more usual to think of an elliptic curve as a compact projective variety in $\mathbb{CP}^2$, and consider a fourth branch point at infinity which corresponds to the pole of $dx = \wp'(z)dz$ at $z = 0$. This fourth branch point is not considered in the Eynard-Orantin recursion.

Typically, we will be applying these considerations to the more general case of a genus one spectral curve defined by a polynomial $p(x, l) \in \mathbb{C}[x, l]$ that is quadratic in $l$, so that,

$$l(z) = \frac{b(x) + \sqrt{\sigma(x)}}{c(x)},$$

with $b(x), c(x) \in \mathbb{C}[x]$. Since $\sqrt{\sigma(z)} = -\sqrt{\sigma(z)}$, the branch involution for such a spectral curve is the (unique) involution that sends,

$$\left( x, \frac{b(x) + \sqrt{\sigma(x)}}{c(x)} \right) \mapsto \left( x, \frac{b(x) - \sqrt{\sigma(x)}}{b(x)} \right). \quad (2.8)$$
If we think of $l$ as defining a meromorphic function on the curve $y^2 = \sigma(x)$, then the involution that sends $(x, y) \mapsto (x, -y)$ induces the involution (2.8).

Using the usual uniformising coordinate $z$, with $\wp(z) = x$, the branch points are at $z = \frac{\omega_i}{2}$. Near a branch point, the branch involution has the form $\tau := -z + \omega_i$.

The next step is to calculate the Bergmann kernel. The bidifferential,

$$\wp(z_1 - z_2) dz_1 dz_2,$$

has the correct singularities (a double pole at $z_1 = z_2$), and it is symmetric in $z_i$. To adjust the differential to have zero $A$-cycle periods, use the Weierstrass zeta function above, so that,

$$\oint_A \wp(z_1 - z_2) dz_1 dz_2 = (-\zeta(\omega_1 - z_2) + \zeta(-z_2)) dz_2 = (\zeta(z_2 - \omega_1) - \zeta(z_2)) dz_2 = ((\zeta(z_2 + \omega_1 - 2\omega_1) - \zeta(z_2)) dz_2 = (\zeta(z_2 + \omega_1) - 2\eta_1 - \zeta(z_2)) dz_2 = -\eta_1 dz_2,$$

(recall that by definition, $\eta_1$ is the constant $\zeta(z + \omega_1) - \zeta(z)$). Notice that the $z_2$ dependence only occurs in the one-form $dz_2$. This gives the Bergmann kernel with zero $A$-cycle integrals,

$$B(z_1, z_2) = \left(\wp(z_1 - z_2) + \frac{\eta_1}{\omega_1}\right) dz_1 dz_2.$$

Now one gets that the numerator of the recursion kernel at the branch point $\omega_i$ looks like,

$$\zeta(z_0 - z) dz_0 - \zeta(z_0 + z - \omega_i) dz_0 + \left(\frac{\eta_1}{\omega_1}\right) (2z - \omega_i).$$

Notice that $\zeta(z)$ is not an elliptic function, so the recursion kernel is not globally defined on the curve (unlike the simple genus zero examples in chapter 1). This gives the recursion
kernel for the spectral curve \((C, x, l)\) to be,

\[
\frac{\zeta(z_0 - z)dz_0 - \zeta(z_0 + z - \omega_i)dz_0 + \left(\frac{m}{\omega_1}\right)(2z - \omega_i)}{\wp'(z)dl'(z)dz}.
\]

If, on the other we want to look at the non-meromorphic spectral curve \((C, \log(x), \log(l))\), the main factor of the denominator of the recursion kernel is,

\[
\log(l(z)) - \log(l(\bar{z})),
\]

for some choice of branch of the logarithm. This can be simplified using the identity ([M]),

\[
\log \left( \frac{b(x) + \sqrt{\sigma(x)}}{c(x)} \right) = \frac{1}{2} \log \left( \frac{b(x)^2 - \sigma(x)}{c(x)^2} \right) + \tanh^{-1} \left( \frac{\sqrt{\sigma(x)}}{b(x)} \right),
\]

to get,

\[
\log(l(z)) - \log(l(\bar{z})) = 2 \tanh^{-1} \left( \frac{\sqrt{\sigma(x)}}{b(x)} \right).
\]

So that the recursion kernel is,

\[
\frac{\left(\zeta(z_0 - z)dz_0 - \zeta(z_0 + z - \omega_i)dz_0 + \left(\frac{m}{\omega_1}\right)(2z - \omega_i)\right)\wp(z)}{2 \tanh^{-1} \left( \frac{\sqrt{\sigma(x)}}{b(x)} \right) \wp'(z)dz}.
\]

(2.10)

It is useful to look closer at the series for the recursion kernel at each branch point. The series for the numerator of the recursion kernel (2.9) at the branch point \(\omega_i\) is,

\[
2 \sum_{k=0}^{\infty} \wp^{2k} \left( z \right) \left( z - \frac{\omega_i}{2} \right)^{2k+1} = \sum_{k=0}^{\infty} a_{2k+1,i}(z_0) \left( z - \frac{\omega_i}{2} \right)^{2k+1}
\]

Denote the series for the recursion kernel at a branch point by,

\[
-4K^i(z_0, z) = \sum_{k=0}^{\infty} a_{2k+1,i}(z_0) \left( z - \frac{\omega_i}{2} \right)^{2k+1} = \sum_{k=-1}^{\infty} K^i_k(z_0) \left( z - \frac{\omega_i}{2} \right)^k,
\]

so that the terms \(K^i_k(z_0)\) are polynomials in \(a_{2l+1,i}(z_0)\) with coefficients that are rational functions of the \(\{b_{j,i}\}\).
2.2.2 Taylor Series for the Weierstrass Function

As a short digression, let us briefly examine the Taylor series for the Weierstrass $\wp$ function at the zeroes of $dx = \wp'(z)dz$, since these terms are the fundamental ingredients of the Eynard-Orantin calculation: we will discuss how the integral of an arbitrary Eynard-Orantin invariant can be written in terms of the coefficients of this Taylor series below.

There are several ways of viewing the coefficients of this Taylor series; to emphasise the ‘global’ aspect, and have expressions that are valid throughout the moduli space we would like to write everything in terms of the ratio $\tau = \frac{\omega_2}{\omega_1}$. It turns out that the coefficients of the Taylor series of $\wp(z)$ at the branch points are closely related to congruence modular forms in the parameter $\tau$.

As a preliminary observation, notice that a direct calculation using the definition of the Weierstrass $\wp$-function yields that, for any constant $c \in \mathbb{C}$ with $c \neq 0$,

$$\wp_{c\Lambda}(cz) = c^{-2} \wp_{\Lambda}(z). \quad (2.11)$$

Now we will give expressions for the Taylor series of the Weierstrass function in terms of $\tau$.

**Proposition 2.1.** For the normalised lattice $\mathbb{Z} \oplus \mathbb{Z} \tau \subset \mathbb{C}$, the functions $\wp^{(n)}(\omega_i/2)$ are modular forms in the variable $\tau$. In particular,

$$\wp^{(n)}\left(\frac{a_1 + a_2 \tau}{2}\right) = (-1)^n(n + 1)! 2^{n+2} G_{n+2}^{(a_1, a_2)}(\tau)$$

**Proof.** The function $\wp(z)$ differentiated $n$ times and evaluated at the point $\frac{a_1 + a_2 \tau}{2}$ becomes,

$$\wp^{(n)}\left(\frac{a_1 + a_2 \tau}{2}\right) = (-1)^n(n + 1)! \sum_{(n,m) \in \mathbb{Z}^2} \frac{1}{(\frac{a_1}{2} - n + (\frac{a_2}{2} - m)\tau)^{n+2}},$$

and the sum on the right is equal to,

$$2^{n+2} \sum_{(n,m) \equiv (a_1, a_2) \text{ Mod 2}} \frac{1}{(n + m\tau)^{n+2}},$$

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which gives the required congruence modular form.

**Remark 2.6.** This proposition also includes the case when \( n = 0 \), although in that case the corresponding level \( n \) Eisenstein series is only a quasimodular form.

As any lattice \( \Lambda \) can be scaled to a normalised lattice \( \Lambda \frac{\omega_2}{\omega_1} = \Lambda \tau \) by multiplication with \( \omega_1^{-1} \), the equation (2.11) gives that the Weierstrass \( \wp \) function for any lattice \( \Lambda \) satisfies,

\[
\wp^{(n)}_\tau \left( \frac{1}{\omega_1} z \right) = \omega_1^{n-2} \wp^{(n)}_\Lambda(z),
\]

and this means that,

\[
\wp^{(n)}_\Lambda \left( \frac{\omega_i}{2} \right) = \omega_1^{2-n} \wp^{(n)}_\tau \left( \frac{a_1 + a_2 \tau}{2} \right),
\] (2.12)

for \( (a_1, a_2) \in (\mathbb{Z}/2\mathbb{Z})^2 \) corresponding to \( \frac{\omega_i}{2} \) via \( \omega_i = a_1 \omega_1 + a_2 \omega_2 \). This means that proposition (2.1) can be used to calculate the series expansions for Weierstrass functions of arbitrary lattices \( \Lambda \).

However, if one does not want to express the calculation as a function on the Teichmüller space \( \mathcal{H} \), but to focus on a particular spectral curve, it is more useful to express the coefficients of the Taylor series in terms of the roots \( \sigma_i \) of \( \sigma(x) \). This can be done by noticing that the terms in the Taylor series for the Weierstrass function also satisfy a useful recursion. Let,

\[
\wp(z) = \sum_{n=0}^{\infty} b_{2n+1}(z - \omega_i/2)^{2n},
\]

(recall that the Weierstrass \( \wp \)-function is even and is holomorphic at the half-lattice points \( \frac{\omega}{2} \)).

Since the Weierstrass \( \wp(z) \) function satisfies,

\[
\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2},
\]

the following recursion holds for \( n \geq 1 \) (omit the \( i \) for ease of reading):

\[
b_{2n+2} = \frac{3}{(n + 1)(2n + 1)} \sum_{k=0}^{n} b_{2k} b_{2n-2k},
\] (2.13)
with the initial conditions,
\[ b_0 = e_i, \quad 2b_2 = 6e_i^2 - \frac{g_2}{2}. \]

There is also a third way to express the terms in the Taylor series for \( \wp(z) \) is in terms of theta functions, or rather, the theta Nullwerte.

**Proposition 2.2.** Any of the coefficients \( a_{2k+1}(z_0) \) of the Taylor series of \( \wp(z) \) at the half-lattice points \( \frac{\omega_i}{2} \) can be expressed in terms of the theta Nullwerte \( \theta_i(0) \) and the derivatives \( \theta_i^n(0) \).

**Proof.** First use the classical identity (eg. ([WW]) or ([MM]), Section 3.3),
\[ \wp(z) = e_1 + \left[ \frac{\theta_1'(0)\theta_2(z)}{\theta_1(z)\theta_2(0)} \right]^2 \]
(2.14)
to express \( \wp^n(\frac{\omega_i}{2}) \) as a rational function in \( \theta_i^n(\frac{\omega_i}{2}) \). Then use the transformation laws for changing the argument of a theta function by a half-lattice point. \( \square \)

### 2.2.3 Period Integrals and Modular Forms

We will introduce this section by examining the (0,3) Eynard-Orantin invariant: the (0,3) case is the simplest nontrivial example of the Eynard-Orantin recursion. It also plays a special role in the theory, due to its occurrence in the Rauch variational formula.

From the definition of the recursion,
\[ \omega^{(0,3)}(z_0, z_1, z_2) = \sum_{i}^{3} Res_{z=a_i} K(z_0, z) [B(z_1, z)B(z_2, \bar{z}) + B(z_1, \bar{z})B(z_2, z)], \]

and since, in the variable \( z \), the Bergman kernel is generically holomorphic at \( a_i \), this becomes,
\[ \omega^{(0,3)}(z_0, z_1, z_2) = 8 \sum_{i=1}^{3} K_{-1}(z_0)B(z_1, a_i)B(z_2, a_i) \]
\[ \omega^{(0,3)}(z_0, z_1, z_2) = 8 \sum_{i=1}^{3} K_{-1}(z_0) \left( \wp(z_1 - \frac{\omega_i}{2}) + \eta_1 \right) \left( \wp(z_2 - \frac{\omega_i}{2}) + \eta_1 \right) \]
To express the periods of the Zeta function \( \eta_1(\Lambda) \) in terms of \( \tau \), it is useful to use (2.11), and the definition of \( \eta_1(\Lambda) \) to relate this to the periods of the Zeta function for the normalised lattice \( \Lambda_\tau \). This is done by comparing primitives of (2.11) to get, for any non-zero constant \( c \in \mathbb{C} \),

\[
cZ_{c\Lambda}(cz) = Z_\Lambda(z) + k,
\]

for a constant \( k \). So the defining equation for the periods \( \eta_1(\Lambda) \) gives,

\[
\eta_1(\Lambda_\tau) = \eta_1\left(\frac{1}{\omega_1}\Lambda\right) = Z_{\frac{1}{\omega_1}\Lambda}\left(\frac{z}{\omega_1} + \frac{\omega_i}{\omega_1}\right) - Z_{\frac{1}{\omega_1}\Lambda}\left(\frac{z}{\omega_1}\right)
= \omega_1\eta_1(\Lambda).
\]

This means that (2.6) implies,

\[
\eta_1(\Lambda) = \frac{1}{\omega_1} G_2(\tau),
\]

so we can rewrite the (0, 3) Eynard-Orantin invariant as,

\[
\omega^{(0,3)}(z_0, z_1, z_2) = 8 \sum_{i=1}^{3} K_{-1}(z_0) \left( \varphi(z_1 - \frac{\omega_i}{2}) + \frac{G_2(\tau)}{\omega_1^2} \right) dz_1 \left( \varphi(z_2 - \frac{\omega_i}{2}) + \frac{G_2(\tau)}{\omega_1^2} \right) dz_2,
\]

so that \( \omega^{(0,3)}(z_0, z_1, z_2) \) is equal to,

\[
8 \sum_{i=1}^{3} \frac{\left( \varphi(z_0 - \frac{\omega_i}{2}) + \frac{G_2(\tau)}{\omega_1^2} \right) dz_0 \left( \varphi(z_1 - \frac{\omega_i}{2}) + \frac{G_2(\tau)}{\omega_1^2} \right) dz_1 \left( \varphi(z_2 - \frac{\omega_i}{2}) + \frac{G_2(\tau)}{\omega_1^2} \right) dz_2}{b_{2,i}}.
\]

Now we would like to calculate the period integrals,

\[
\oint_B \oint_B \oint_B \omega^{(0,3)}(z_0, z_1, z_2),
\]

so it is necessary to calculate \( \oint_B \varphi(z - \frac{\omega_i}{2}) \).

Using the defining properties of \( Z(z) \) (2.4), and the fact that \( Z(z) \) is odd:

\[
\int_0^{\omega_1} \varphi(z - \frac{\omega_1}{2}) = \int_0^{\omega_2} \varphi(z - \frac{\omega_2}{2}) = \int_0^{\omega_3} \varphi(z - \frac{\omega_3}{2}) = -\eta_2.
\]

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This gives that,
\[ \oint_B \oint_B \oint_B \omega(0, 3)(z_0, z_1, z_2) = 8 \left( -\eta_2 + \frac{G_2(\tau)}{\omega_1} \right)^3 \left( \sum_{i=1}^{3} \frac{1}{b_{2,i}} \right), \]
and the identity (2.6) with (2.15) yields the compact formula,
\[ \oint_B \oint_B \oint_B \omega(0, 3)(z_0, z_1, z_2) = -8 \left( \frac{2\pi i}{\omega_1} \right)^3 \left( \sum_{i=1}^{3} \frac{1}{b_{2,i}} \right). \]

Notice that the terms \( b_{2,i} \) are rational functions of \( G_a^2(\tau) \) and \( G_4(\tau) \). So that this expression for the \( B \)-cycle integrals of \( \omega(0, 3) \) is a rational function in \( \{ G_a^2(\tau), G_4(\tau) \} \). This fact holds more generally.

**Theorem 2.3.** For all stable \((g, n)\), with \( \bar{a} \) any vector in \((\mathbb{Z}/2)^2\), the integral,
\[ \oint_B \ldots \oint_B \omega^{(g, n)}(z_0, \ldots, z_n), \]
is a rational function of \( G_a^2(\tau) \) and \( G_4(\tau) \).

**Proof.** The terms \( K_i^l(z_0) \) in the Laurent series for the recursion kernel are polynomials in \( \{ a_0^l(z_0), a_1^l(z_0), \ldots, a_l^l(z_0) \} \) with coefficients in \( \mathbb{C}(b_{i,1}, b_{i,2}, \ldots, b_{i,l}) \). By uniformising using \( x = \wp(z) \) and \( y = \wp'(z) \), the terms \( b_{i,j} \) in the Taylor series for the denominator of the recursion kernel are rational functions in \( \wp^2(\omega_i^2) \). Using (2.13), and the fact that \( g_2 = 60G_4(\tau) \), any expression of this form can be written as a polynomial in \( G_a^2(\tau) \) and \( G_4(\tau) \), so the terms \( K_i^l(z_0) \) are polynomials in \( a_i^l(z_0) \) with coefficients in \( \mathbb{C}(G_a^2(\tau), G_4(\tau)) \).

By induction on the recursive definition, every \( \omega^{(g, n)}(z_0, \ldots, z_n) \) is a polynomial in the terms \( K_i^l(z_0) \) and three kinds of terms: derivatives of \( B(z, z_2) \), \( B(z, z_2) \), and \( B(z, \bar{z}) \), evaluated at \( z = \omega_i^2 \). Using that \( \bar{z} = -z + \omega_i \), these derivatives of the Bergmann kernel will either be of the form \( \wp(m)(\omega_i^2) \), \( \wp(m)(\omega_i^2 - z_2)dz_2 \), or \( \wp(m)(-\omega_i^2 - z_2)dz_2 \), for some \( m \in \mathbb{Z} \).

Again using (2.13), we can write these derivatives as polynomials in \( \wp(z_2 - \omega_i^2)dz_2 \), and
\[ g_2 = 60G_4(\tau), \] and (2.16) gives that the \( B \)-cycle integrals of \( \wp(z - \frac{\omega_i}{2}) \) are \( \eta_i \), so that (2.7) and (2.12) gives the result.

It would be interesting to see if there is a higher genus generalisation of this result, incorporating Siegel modular forms.
Chapter 3

Character Varieties and the
Chern-Simons Line Bundle

This chapter revolves around the space of flat connections on a surface \( \Sigma \) modulo the action of the group of bundle automorphisms. This space can be equipped with a natural line-bundle, which is defined using the Chern-Simons functional given below. The main reference used for the Chern-Simons functional is ([F2]), as well as the lecture notes ([AH]). The goal is to focus on the case when \( \Sigma \) has genus one, with a view to relating this to the \( A \)-polynomial. In the genus one case, the symplectic structure and the line bundle can be made explicit, and will be related to Eynard-Orantin theory in subsequent chapters.

3.1 The Chern-Simons functional

For this section, let \( M \) be a compact, oriented 3-manifold. Let \( K \) be a compact, connected, simply connected Lie group with Lie algebra \( \mathfrak{k} \), so that \( G := K^\mathbb{C} \) is possibly noncompact. Recall that a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) in \( S^2 \mathfrak{k}^* \) is called \( Ad\)\text{-}invariant if \( \langle Ad_g X, Ad_g Y \rangle = \langle X, Y \rangle \)
\langle X, Y \rangle \) for all \( X, Y \in \mathfrak{k} \), and \( g \in K \). For such an Ad-invariant form \( \langle \cdot, \cdot \rangle \), define the multilinear form \( \omega \) on \( \mathfrak{k} \) by,

\[
\omega[X,Y,Z] := \langle X, [Y,Z] \rangle, \quad \forall X,Y,Z \in \mathfrak{k}.
\]

It follows from the fact that the structure constants of a Lie algebra are antisymmetric that the form \( \omega \) is actually alternating; since a left-invariant differential form in \( \Omega^\bullet(K) \) is determined by it’s values on \( T_eK = \mathfrak{k} \), it follows that \( \omega \) defines a left-invariant differential form in \( \Omega^3(K) \), and the Jacobi identity gives that \( \omega \) is closed.

Thus, \( \omega \) defines an element of \( H^3(K, \mathbb{R}) \). In Chern-Simons theory, it will useful to find an Ad-invariant form \( \langle \cdot, \cdot \rangle \) so that the resulting three-form \( \omega \) is actually an element of \( H^3(K, \mathbb{Z}) \subset H^3(K, \mathbb{R}) \).

In fact, the case we will be interested in is when \( K = SU(2) \). In this case, the form \( \omega \) induces a holomorphic form on \( SU(2) \mathbb{C} \cong SL(2, \mathbb{C}) \), which we will also denote by \( \omega \). Since \( SL(2, \mathbb{C}) \) is contractible onto \( SU(2) \), the condition that \( \omega \) gives an element of \( H^3(SU(2), \mathbb{Z}) \subset H^3(SU(2), \mathbb{R}) \) is equivalent to the condition that \( \omega \) gives an element of \( H^3(SL(2, \mathbb{C}), \mathbb{Z}) \subset H^3(SL(2, \mathbb{C}), \mathbb{C}) \). This is why we use the same notation for \( \omega \) in both cases. In subsequent sections, it is \( SL(2, \mathbb{C}) \) that will be the focus of our interest.

An immediate candidate for an Ad-invariant element of \( S^2\mathfrak{k}^* \) is the Killing form. Recall that the Killing form in \( K(X,Y) := kTr(XY) \), for an integer \( k \), which is a normalisation constant. We must determine \( k \) by stipulating that \( [\omega] \) define an integral class in \( H^3(SL(2, \mathbb{C}), \mathbb{R}) \).

To this end, recall that the Maurer-Cartan form is the unique element \( \theta \) in \( \Omega^1(K, \mathfrak{k}) \) that sends a vector field \( X \) to it’s left-invariant extension, so that \( l_g^*\theta = \theta \), where \( l_g \) is left multiplication with \( g \). This gives that the form \( \omega \) can be written \( \omega(X,Y,Z) = \langle \theta(X), [\theta(Y), \theta(Z)] \rangle \), for any \( X,Y,Z \in \Gamma(TG) \).

For any Lie algebra-valued one-forms \( \eta_i \) on the manifold \( M \), one can extend the wedge product to a map \( \Omega^1(M) \otimes \mathfrak{k} \times \Omega^1(M) \otimes \mathfrak{k} \to \Omega^2(M) \otimes (\mathfrak{k} \otimes \mathfrak{k}) \) in the standard fashion. We will use the usual notation \( [\eta_i \wedge \eta_j] \) for the composition of this extended wedge product
with the element \([\cdot, \cdot] \in \text{Hom}(\mathfrak{k} \times \mathfrak{k}, \mathfrak{k})\), so that \([\eta_1, \eta_2] \in \Omega^1(M) \otimes \mathfrak{k}\). Similarly for the notation \([\eta_i, \eta_j]\), with the Ad-invariant form \(\langle \cdot, \cdot \rangle\). With this notation fixed, it follows that \(\omega = \langle \theta, [\theta, \theta] \rangle\).

Using the basis \(\{u_1, u_2, u_3\}\) of \(\mathfrak{su}(2)\) given by,
\[
\begin{align*}
    u_1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\
    u_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
    u_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{align*}
\]
Let \(u_i^*\) be the unique left-invariant one-forms dual to the left-invariant vector fields \(u_i\). Since \(\theta(u_i) = u_i\) the Maurer-Cartan form can be written \(\theta = \sum_{i=1}^3 u_i \otimes u_i^*\) and since,
\[
[\theta, \theta] = 2[u_1, u_2] \otimes u_1^* \wedge u_2^* + 2[u_2, u_3] \otimes u_2^* \wedge u_3^* + 2[u_1, u_3] \otimes u_1^* \wedge u_3^*,
\]
it follows that,
\[
\langle \theta, [\theta \wedge \theta] \rangle = 2k (\langle u_1, [u_2, u_3] \rangle - \langle u_2, [u_1, u_3] \rangle + \langle u_3, [u_1, u_2] \rangle) u_1^* \wedge u_2^* \wedge u_3^* = -24ku_1^* \wedge u_2^* \wedge u_3^*.
\]

If \(g\) is the ambient Euclidean metric on \(SU(2) \cong S^3 \subset \mathbb{R}^4\), the Haar measure \(u_1^* \wedge u_2^* \wedge u_3^*\) is \(Vol_g\). Since \(\int_{S^3} Vol_g = 2\pi^2\), it follows that, if \(K(X, Y) = ktr(XY)\), then,
\[
\int_{SU(2)} \omega = k(-48)\pi^2,
\]
and this allows us to choose appropriate \(k\).

Even more than the case of a compact manifold \(M\), the case of a manifold \(M\) with two-dimensional boundary \(\partial M = \Sigma\) will be of interest. In this case, we will want to be even more specific about the normalisation of the Ad-invariant form \(\omega\); not only must it define an integral class, but we will choose \(k = -\frac{1}{8\pi^2}\), so that,
\[
[\omega] = 6[SU(2)],
\]
where \([SU(2)]\) is the fundamental class. This choice will have consequences for the following functional:
Definition 3.1. If $\mathcal{M}$ is a three-manifold with boundary $\partial \mathcal{M} = \Sigma$, for $\Sigma$ a smooth, orientable two-manifold, $g \in C^\infty(\Sigma, G)$, and $\omega$ is the form defined above, then the Wess-Zumino-Witten functional is,

$$W_\Sigma(g) := -\int_\mathcal{M} \frac{1}{6} g^* \omega.$$  \hspace{1cm} (3.1)

For our purposes, the Wess-Zumino-Witten functional will be used when $g \in C^\infty(\Sigma, G)$ is the expression for a $G$-bundle automorphism, in some trivialisation. The point of choosing $k$ as above, is that the Wess-Zumino-Witten functional will then be an integer (the degree of $g \in C^\infty(\Sigma, G)$).

The Wess-Zumino-Witten functional will mainly be of interest because of its relation to the behaviour of another functional: the Chern-Simons functional.

To define the Chern-Simons functional, let $P \to \mathcal{M}$ be a principle $G$-bundle, and $\langle \cdot, \cdot \rangle$ the Killing form normalised with level $k = \frac{-1}{8\pi^2}$ as above. Denote the space of $G$-connections on $P$ by $\mathcal{A}_{\mathcal{M}}$ (the subscript $\mathcal{M}$ will be omitted, if the context makes it clear what the base manifold is). By fixing a ‘base’ connection $A \in \Omega^1(P, \mathfrak{g})$, any other connection $B$ can be written $B = A + a$ for $a \in \Omega^1(\mathcal{M}, \mathfrak{g})$. This gives the space $\mathcal{A}$ an affine structure (modelled on $\Omega^1(\mathcal{M}, \mathfrak{g})$), and since an affine space is isomorphic to its tangent space, the tangent space $T_{A} \mathcal{A}_{\mathcal{M}}$ is $\Omega^1(\mathcal{M}, \mathfrak{g})$.

The group of bundle automorphisms (which is customary to call the ‘gauge group’) will be denoted $\mathcal{G}$. The action of an element $v \in \mathcal{G}$ on $P$ will be denoted $p \cdot v$, for $p \in P$. By pulling back the action on $P$, the gauge group can be made to act on $\mathcal{A}_{\mathcal{M}}$ as well. This action will also be denoted $A \cdot v$ for $A \in \mathcal{A}_{\mathcal{M}}$. Given a local trivialisation $s : U \subset \mathcal{M} \to P$ a connection $A$ on $P$ can be pulled back to an element $a \in \Omega^1(U, \mathfrak{g})$, and in this trivialisation and element $v \in \mathcal{G}$ can be written as an element $g_s(v) \in C^\infty(\mathcal{M}, G)$ using the defining rule $v \circ c = s \cdot g_s(v)$. In this trivialisation, the action of the gauge group is given by,

$$a \cdot g = gag^{-1} - (dg)g^{-1}.$$  \hspace{1cm}

Identifying elements of $G$ with constant elements of $\mathcal{G}$, it is easy to see that every connec-
tion $A$ has at least the centre $Z(G)$ in it’s stabiliser, since, for $g \in Z(G) \subset \mathcal{G}$,

$$A \cdot g = gAg^{-1} - (dg)g^{-1} = gAg^{-1} = A.$$ 

A connection whose stabiliser only consists of $Z(G)$ will be called irreducible, while a connection with larger stabiliser will be called reducible.

If the bundle is trivialisable, so that there is a global section $s : M \to P$, then one can alternatively think of the pullback $s^*A = a$ by fixing the trivial connection $A_0$ on the trivial bundle $P$ (which is simply the exterior derivative) so that, $A = A_0 + a$.

**Definition 3.2.** Let $s : M \to P$ be a trivialisation of $P$, and $A \in \Omega^1(P; \mathfrak{g})$ a connection on $P$. Then define,

$$\alpha(A) := \langle A \wedge F_A \rangle - \frac{1}{6} \langle A \wedge [A \wedge A] \rangle = \langle A \wedge dA \rangle + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle.$$ 

and the Chern-Simons action is,

$$cs_s(A) = \int_M s^*\alpha(A).$$ 

The Chern-Simons action is also called the Chern-Simons invariant of $A$. Note the dependence of this functional on the trivialisation $s$. Up until now, we have been assuming that a trivialisation exists. It turns out this is justified by the following theorem of topology:

**Theorem 3.1.** If $G$ is a simply connected Lie group, then every principle $G$-bundle over a three dimensional manifold is trivialisable.

See ([F2]) for an outline of the proof.

So, for our case of a principle $SL(2, \mathbb{C})$-bundle there exists a section $s \in \Gamma(P)$. Using the trivialisation $s$ to express a connection $A$ as an element of $\Omega^1(M, \mathfrak{g})$, the expression for
the Chern-Simons action, with this identification becomes,

\[
cs_s(A) = -\frac{1}{4\pi^2} \int_M Tr(a \wedge da + \frac{2}{3} a \wedge a \wedge a).
\]

Here, the notation \( a \wedge a \) for \( a \in \Omega^1(M, g) \) is similar to that used above, in that it means the composition of wedge product on forms with the matrix multiplication on elements of \( g \) is taken. To get the coefficient of the second summand above, use the fact that, for matrix groups, \([a \wedge a] = 2a \wedge a\) (this can be verified explicitly by writing the element \( a \in \Omega^1(M, g) \) in indices \( a = a_{ij}^e \omega^j \) and performing the multiplication). Note that, if the structure group of \( P \) is a complex Lie group, the Chern-Simons invariant is complex valued.

An important example of the Chern-Simons invariant is to consider a Riemannian metric on an oriented three-manifold \( M \), and the corresponding frame bundle \( F(M) \). This is an \( SO(3) \)-bundle with a corresponding Levi-Civita connection. After choosing an orthonormal frame, which gives a section of \( F(M) \), one can define the Chern-Simons invariant of the Levi-Civita connection. This was the original motivation for studying this invariant: the Chern-Simons invariant of the Levi-Civita connection is an obstruction to the existence of conformal immersions of \( M \).

The Chern-Simons invariant depends on the choice of trivialisation \( s : M \to P \), and we would like to investigate how it depends on \( s \). This is equivalent to the question of how the Chern-Simons functional changes under the action of the gauge group on \( A \), since, if \( t : M \to P \) is another trivialisation, then there exists \( v \in G \) so that \( t = s \cdot v \), and,

\[
cs_{s \cdot v}(A) = cs_s(A \cdot v), \forall g \in G.
\]

Using the trivialisation \( s \), any element \( v \) of the gauge group determines a map \( g_s(v) \in C^\infty(M, G) \). It can be shown ([F2]) that,

\[
cs_s(A \cdot v) = cs_s(A) - \int_M \frac{1}{6} \langle g_s^* \theta \wedge [g_s^* \theta \wedge g_s^* \theta] \rangle,
\]

(3.2)

(the dependence of the righthand side on \( v \) is suppressed for clarity).

This is a slightly subtle point; it is important to understand that there are two ways of
looking at the same process. One can fix a trivialisation and then examine the action of
the gauge group on connections, or one can alter the trivialisation by the action of the
gauge group, to give a new local expression of any given connection: these two processes
are equivalent.
Using the choice of \( k \) above, it follows that the integral in (3.2) is an integer so that,

\[
cs_{s,g}(A) - cs_s(A \cdot g) \in \mathbb{Z}.
\]

Let \( \mathbb{Z} \) act on \( \mathbb{C} \) via \( n \cdot (a + bi) = (a + n + bi) \), then the Chern-Simons functional defines a
map \( cs_s : A/\mathcal{G} \to \mathbb{C}/\mathbb{Z} \). Notice that the gauge-indeterminacy occurs only in the real part.

It will be important to see how the Chern-Simons functional behaves under the action
of \( \mathcal{G} \) when \( M \) has non-empty boundary. The following important proposition is proven
in several places, for example ([F2]) and ([AH]). It shows how the Wess-Zumino-Witten
functional on a two-dimensional boundary \( \Sigma \) is related to the transformation of the Chern-Simons functional on the bounding three manifold, under the action of the gauge group.

**Proposition 3.1.** If \( M \) is a 3-manifold with boundary \( \partial M = \Sigma \), \( A \) is a connection
on \( P \to M \), \( s : M \to P \) is a trivialisation, \( a = s^* A \in \Omega^1(M,\mathfrak{g}) \), and \( g \in \mathcal{G} \) with
\( g_s \in C^\infty(M,\mathcal{G}) \), then,

\[
cs_s(A \cdot g) = cs_s(A) + \int_{\Sigma} \langle Ad_{g_s^{-1}}a \wedge g_s^* \theta \rangle + W_\Sigma(g_s).
\]

**Remark 3.1.** It is sometimes customary (especially in the physics literature) to omit
the dependence on the trivialisation, and write this formula solely in terms of elements
of \( \Omega^1(M;\mathfrak{g}) \) and elements of \( C^\infty(M,\mathcal{G}) \), with the assumption that a certain trivialisation
has been fixed.
3.2 The Chern-Simons Line Bundle

For $M$ with $\partial M = \Sigma$ as before, we would like to consider the exponential of the Chern-Simons action varying over the space of connections $A_M$. However, due to the dependence of the Chern-Simons functional on a trivialisation, and the nontrivial change under the action of the gauge group, it is more convenient to view $\exp(2\pi i cs_s(A))$ as a section of a line bundle. This construction is originally due (in the case where the structure group of $P$ is compact) to ([RSW]).

Actually, since the data for the construction of this line bundle will be essentially determined by the difference $cs_s(A \cdot g) - cs_s(A)$, and proposition (3.1) means that this only depends on the restriction of $A$ to $\Sigma$, a line bundle over the space $A_\Sigma$ will be constructed; and this will be the focus of interest.

Define the line bundle as follows: for any connection $A \in A_\Sigma$, let $\tilde{A}$ be an extension of $A$ to a connection on the bounding three-manifold $M$, with $s^*\tilde{A} = a$. Such an extension exists, because $\pi_1(SL(2, \mathbb{C})) = \pi_2(SL(2, \mathbb{C})) = 0$. From proposition (3.1), the expression,

$$c_\Sigma(a, g_s) := \exp \left( 2\pi i \left( \int_\Sigma (Ad_{g_s^{-1}} a \wedge g_s^* \theta) + W_\Sigma(g_s) \right) \right),$$

is the ratio $\exp(2\pi i cs_s(\tilde{A} \cdot v)) / \exp(2\pi i cs_s(\tilde{A}))$. It can be shown that the term $c_\Sigma(a, g_s)$ does not depend on the choice of extension (the argument in ([RSW]) is still valid for $SL(2, \mathbb{C})$).

Define $L := A_\Sigma \times \mathbb{C}$, and, for each trivialisation $s : \Sigma \to P$ define the trivialisation $\phi_s$ of $L$ by,

$$\phi_s(A, z) \to (A, z \exp(2\pi i cs_\Sigma(a, s))).$$

For $t = v \cdot s$, the transition function, $\psi_{ts}(A) := \psi_t \circ \psi_s^{-1}$, is of the form,

$$\psi_{ts}(A) := \frac{\exp(2\pi i (cs_{v,s}(A)))}{\exp(2\pi i (cs_s(A)))} = \frac{\exp(2\pi i cs_s(v \cdot A))}{\exp(2\pi i cs_s(A))} = c_\Sigma(s^* A, g_s(v)).$$

To define the structure of a line bundle on $L$, the transition function $\phi_{ts} = c_\Sigma(a, g_s)$ must satisfy the cocycle identity, shown in ([RSW]),
Proposition 3.2. Let $s, t, u$ be three trivialisations $s, t, u : \Sigma \to P$, with $t = v \cdot s$, $u = \nu \cdot t$, for $v, \nu \in G$. The transition functions, $\phi_{ts}(A) = c_{\Sigma}(s^*A, g_s(v))$, $\phi_{ut} = c_{\Sigma}(t^*A, g_t(\nu))$, and $\phi_{us} = c_{\Sigma}(s^*A, (g_s(\nu\nu))$, satisfy,

$$\phi_{ut} \circ \phi_{ts} = \phi_{us}.$$ 

The line bundle $L$ is called the Chern-Simons line bundle over $A_{\Sigma}$. Notice that the transition functions are $U(1)$ valued, so that $L$ is an Hermitian line bundle. We will want to put a unitary connection on this line bundle, whose curvature is proportional to the symplectic form $\omega \in \Omega^2(A_{\Sigma})$. Such a line bundle is called a pre-quantum line bundle because of its role in the theory of geometric quantization. An equivalent way of phrasing this condition is that $c_1(L) \sim [\omega]$, since a connection on $L$ determines $c_1(L)$ by Chern-Weil theory.

The Lie algebra of $U(1)$ is $i\mathbb{R}$, so we can construct a connection on $L$ by writing down a family of $i\mathbb{R}$-valued one-forms, for each trivialisation $s$ of $L$, that transform correctly when the trivialisation is changed. So, consider the family $\Theta_s \in \Omega_1(A_{\Sigma}, i\mathbb{R})$,

$$(\Theta_s)_A(\eta) := 2\pi i \int_\Sigma (s^*A \wedge \eta), \ A \in A_{\Sigma}, \ \eta \in T_{A_{\Sigma}}.$$  

(3.3)

It can be shown that these connection forms patch together to define a connection $\Theta$ on $L$. Since $U(1)$ is abelian, the term $\Theta \wedge \Theta$ vanishes, so the curvature of $\Theta$ is simply $d\Theta$. Thus, the curvature is given by,

$$d\Theta_A(\eta_1, \eta_2) = 2\pi i \omega(\eta_1, \eta_2), \ \eta_i \in T_{A_{\Sigma}}.$$ 

The curvature form of a connection is always closed, so this gives another proof that $\omega$ is a symplectic form; it also implies that the first Chern class of $L$ is $2\pi^2[\omega]$.

3.2.1 Symplectic Structure On the Space of Connections

The space of connections $A_{\Sigma}$ has more structure; it was an important observation of ([AB]) in the case of compact structure group $G$, that $A_{\Sigma}$ can profitably be studied from
the point of view of symplectic geometry and moment maps. These ideas can also be
transferred to the case of noncompact structure group, such as $SL(2, \mathbb{C})$ (as in [G]), and
will be used to give a symplectic structure on the space of flat connections modulo the
action of the gauge group.

**Lemma 3.1.** For two elements $\eta_1, \eta_2 \in \Omega_1(\Sigma, \mathfrak{g})$, consider $\eta_i$ as tangent vectors in the
vector space $T_A \mathcal{A}_\Sigma$. If $\langle \cdot, \cdot \rangle$ is an Ad-invariant form on $\mathfrak{g}$, the two-form,

$$
\omega(\eta_1, \eta_2) := \int_{\Sigma} \langle \eta_1 \wedge \eta_2 \rangle,
$$

is a symplectic form. The action of the gauge group on $\mathcal{A}_\Sigma$ preserves $\omega$.

Proof: The definition of $\omega$ does not depend on the point $A$ at which $\omega$ is evaluated,
so it is invariant under translations of the affine space $\mathcal{A}_\Sigma$, and hence closed. It is simple
to check explicitly that the action of $\mathcal{G}$ preserves $\omega$. $\square$.

When a symplectic manifold has a Hamiltonian action of a group $H$, it is endowed with
a huge degree of symmetry, since each element $h \in H$ gives a symmetry of the manifold.
At the infinitesimal level, each direction of the action, given by an element of the Lie
algebra of $H$ generates a flow that conserves the symplectic form. It is natural to look
for ‘conserved quantities’ for any of these symmetries, by the analogy with the symplectic
spaces that arise in classical dynamics, such as the phase space of a particle in $\mathbb{R}^3$ with
$SO(3)$ acting by rotations. The formalism of a moment map makes this precise.

**Definition 3.3.** If $(M, \omega)$ is a symplectic manifold with a Lie group $H$ acting freely on
$M$, each element $X \in \mathfrak{h}$ defines a vector field in $\Gamma(TM)$ by pushing forward the group
action from $e \in H$. A function,

$$
\mu : M \to \mathfrak{h}^*,
$$

can be paired with such a vector field $X$ to form a function $\mu(X)$, and $\omega$ can be used to
define the corresponding hamiltonian vector field $Y_{\mu(X)} \in \Gamma(TM)$. Such a function $\mu$
is called a moment map if $Y_{\mu(X)} = X$. 56

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This means that a moment map gives a conserved quantity $\mu(X)$ for any direction of the action of $H$ on $M$.

Returning to the symplectic space $A_\Sigma$, another important observation of ([AB]) was that the curvature $F_A$ can be thought of as a moment map for the symplectic action of the gauge group $\mathcal{G}$. The Lie algebra of $\mathcal{G}$ can naturally be identified with $\Omega^0(\Sigma, \mathfrak{g})$ by differentiating a path of gauge transformations starting at the identity. By differentiating the action,

$$g(A) = A - (\nabla_A g) g^{-1},$$

one sees that the derivative is the map,

$$-d_A : \Omega^0(\mathfrak{g}) \to \Omega^1(\mathfrak{g}),$$

thought of as a map from $T_e \mathcal{G}$ to $T_A A$.

Now, any $\eta \in \Omega^2(\Sigma, \mathfrak{g})$ can be paired with $f \in \Omega^0(\Sigma, \mathfrak{g})$ by,

$$\eta(f) := \int_\Sigma Tr(\eta \wedge f),$$

so that $\Omega^2(\Sigma, \mathfrak{g}) = \Omega^0(\Sigma, \mathfrak{g})^*$. The latter vector space is $\text{Lie}(\mathcal{G})$, so, $F_A : A \to \Omega^2(\Sigma, \mathfrak{g})$, is a candidate for a moment map for the action of $\mathcal{G}$ on $\mathcal{A}$.

In order to verify that curvature is a moment map, it is necessary to verify that, for any element $X \in \Omega^0(M; \mathfrak{g})$, the Hamiltonian vector field generated by $F_A(X)$ is equal to the image of $X$ in $T_A A$ (the image under the map $-d_A : \Omega^0(\mathfrak{g}) \to \Omega^1(\mathfrak{g})$ from above). That is, the one-form $dF_A(X)$ on $\mathcal{A}$ must have the property that,

$$-\int_\Sigma Tr(a \wedge d_A X) = d(F_A(X))(a), \quad (3.4)$$

for all $a \in T_A A$: this follows by direct calculation.

Since moment maps give conserved quantities for the orbits of group actions, it is natural use these to try and understand quotients $M/H$. In the simplest case, when $H$ acts freely or with finite stabiliser on $M$, then the space $\mu^{-1}(0)$ is a smooth submanifold of $M$, and the symplectic quotient $\mu^{-1}(0)/H$ has an induced symplectic form.
The discussion above regarding symplectic quotients implies that, restricted to irreducible connections, the space of flat connections $F^{-1}(0)$ modulo the action of $G$ is a symplectic manifold. Denote this space $\mathcal{M}_\Sigma^*$, and use $\mathcal{M}_\Sigma$ to denote the space of all flat connections modulo $G$, including reducible connections. The theory of symplectic reduction also applies to $\mathcal{M}_\Sigma = F^{-1}(0)/G$, except that this space might not be a manifold, and appropriate modifications of the notion of symplectic form need to be made.

For general surfaces $\Sigma$, these issues will not be addressed here, except to say that the space $M_\Sigma$ does have have an induced symplectic structure on its smooth strata. The case of interest to the topic of this thesis is when $\Sigma$ is the two-torus $T^2$, and the space $\mathcal{M}_{T^2}$ will be discussed explicitly below: it is a symplectic orbifold, which means that the symplectic form is a closed non-degenerate orbifold two-form.

It will be important to consider how the Chern-Simons line bundle $\mathcal{L} \to \mathcal{A}_\Sigma$ behaves with respect to symplectic reduction. Recall that $\mathcal{L}$ has the special property that $c_1(\mathcal{L}) \sim [\omega]$; such a line bundle ‘behaves well’ with respect to symplectic reduction in a sense captured by the following proposition (which is discussed in many places, for example ([DK] 6.5)).

**Proposition 3.3.** Suppose $(M, \omega)$ is a symplectic manifold with $H$ acting on $M$ with an induced moment map $\mu$. If $\mathcal{L}' \to M$ is a complex line bundle over $M$ with a unitary connection $\Theta$ that has curvature form $2\pi i \omega$, the moment map for the action of $G$ lifts to an action of the the Lie algebra $\mathfrak{h}$ on $\mathcal{L}'$, that covers the action on $M$, and preserves the $U(1)$ action. This induces a line bundle $\mathcal{L} \to \mu^{-1}(0)/H$ over the symplectic quotient with an induced unitary connection $\Theta|_{\mu^{-1}(0)/H}$.

In applying this proposition to the space of connections and the Chern-Simons line bundle, the fact that the space $F^{-1}(0)/G$ may not be a smooth manifold at all points must be taken into account. This can be done by either restricting to the space of irreducible connections, to study the induced bundle over $\mathcal{M}_\Sigma^*$, or applying the theory to a broader category of spaces (such as orbifolds). The latter option will be taken with respect to
studying $\mathcal{M}_{T^2}$.

Lastly, it is helpful to see an expression for parallel transport via the connection $\Theta$. In our application to Eynard-Orantin theory, varying the holonomy of $\Theta$ around closed cycles will play an important role.

First, we will discuss parallel transport over the space $\mathcal{A}$, and then discuss how this behaves under the symplectic quotient. Let $\gamma : [0, 1] \to \mathcal{A}$ be a smooth path with $\gamma(t) = A_t$. We will fix a trivialisation of $\mathcal{L}$ by choosing a trivialisation $s : M \to P$. In this trivialisation, sections of $\mathcal{L}$ are functions from $\mathcal{A}$ to $\mathbb{C}$ that transform correctly with respect to the transition functions.

A horizontal lift of the path $\gamma(t) = A_t$ is then a section $c : [0, 1] \to \gamma^*(\mathcal{L})$ of the trivial pullback bundle $\gamma^*(\mathcal{L})$, with,

$$ dc = -\gamma^* \Theta s c, $$

so that,

$$ dc(d\frac{dp}{dt}) = c'(t) = -\Theta s(\gamma^* d\frac{dp}{d\lambda})c(t), \forall t \in [0, 1], $$

and,

$$ c(t) = K \exp \left( -\int_0^t \Theta s(\gamma^* d\frac{dp}{d\lambda}) d\lambda \right) = K \exp \left( -2\pi i \int_0^t \int_{\Sigma} \left( s^* A_\lambda \wedge \gamma^* d\frac{dp}{d\lambda} \right) d\lambda \right), \tag{3.5} $$

for all $t \in [0, 1]$, and where $K \in \mathbb{C}$ determines the particular lift. If $\gamma(0) = \gamma(1)$, then the holonomy around $\gamma$ is,

$$ \exp \left( -2\pi i \int_0^1 \int_{\Sigma} \left( s^* A_\lambda \wedge \gamma^* d\frac{dp}{d\lambda} \right) d\lambda \right). $$

This expression can be related back to the Chern-Simons functional, by considering the family of connections $s^* A_\lambda$ as a family of elements of $\Omega^1(\Sigma, g)$, written in local coordinates $(x, y)$ on $\Sigma$ as,

$$ s^* A_\lambda = \alpha(\lambda) dx + \beta(\lambda) dy $$

for $\alpha(\lambda), \beta(\lambda) \in g$. Now consider the three manifold $\Sigma \times [0, 1]$. The tangent bundle of $\Sigma \times [0, 1]$ canonically splits, so that every vector field $Z \in \Gamma(T(\Sigma \times [0, 1]))$ can be split.
into a vector field \( X \in \Gamma(T(\Sigma)) \) and a vector \( Y \in \Gamma(T([0, 1])) \). Let \( Q \) be the bundle over \( \Sigma \times [0, 1] \) that is isomorphic to \( P \) on each slice \( P|_{\{\text{pt.}\}} \). Define the connection \( \tilde{A} \) on \( Q \) by,

\[
\tilde{A}(X) = A(X), \quad \text{and} \quad \tilde{A}(Y) = 0.
\]

So, in local coordinates \((x, y, t)\), with the trivialisation induced by \( s \), \( s^*\tilde{A} = \alpha(\lambda)dx + \beta(\lambda)dy + 0dt \). Now, calculate that, for any connection \( B \) on \( Q \), with \( B = adx + bdy + cdt \),

\[
B \wedge B \wedge B = a[b, c]dx \wedge dy \wedge dt + c[a, b]dt \wedge dx \wedge dy + b[a, c]dy \wedge dx \wedge dt,
\]

so that, since \( \tilde{A} \) has zero \( dt \) component, \( \tilde{A} \wedge \tilde{A} \wedge \tilde{A} = 0 \). Furthermore,

\[
\tilde{A} \wedge d\tilde{A} = \beta(\lambda)\alpha'(\lambda)dy \wedge dt \wedge dx + \alpha(\lambda)\beta'(\lambda)dx \wedge dt \wedge dy
\]

\[
= -(A \wedge \hat{A}) \wedge dt.
\]

This gives that the holonomy calculated by (3.5) along a path in \( A \) is \( \exp(-2\pi i cs_s(\tilde{A})) \). If we want to understand the holonomy of paths in the symplectic quotient \( M_{\Sigma} \), use that sections of \( L \to M_{\Sigma} \) are \( \mathcal{G} \)-invariant sections of \( L \to A \), and the connection \( \Theta \) restricts to the desired connection on \( L \to A \). This means that we can unambiguously calculate holonomy using \( \Theta \) on \( L \to M_{T^2} \).

### 3.3 Torus Boundary

In the previous section, the Chern-Simons line bundle over the space of flat connections was discussed in general terms, for an arbitrary two dimensional boundary. In the case when the boundary is genus one, many of the above constructions, such as the Chern-Simons line bundle and its connection, can be given more explicitly, without using the machinery of symplectic reduction. This was first considered by ([KK]).

Our motivation for focusing on the case of genus one boundary is that this will be related
3.3 Character Varieties and the Chern-Simons Line Bundle

to the $A$-polynomial in Section 4.1.

Denote a smooth, genus one surface by $T^2$. Choose generators for $\pi_1(T^2)$ given by a meridian and longitude cycle so that $\pi_1(T^2) = \langle M, L | ML = LM \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. By taking the holonomy, the space of flat connections $\mathcal{M}_{T^2}$ can be identified with the space of all representations $\rho : \mathbb{Z} \oplus \mathbb{Z} \to SL(2, \mathbb{C})$ modulo conjugation. Since $SL(2, \mathbb{C})$ is an algebraic group, this space turns out to be an algebraic variety.

For our purposes, the variety $Hom(\pi_1(T), SL(2, \mathbb{C}))$ modulo conjugation is too badly behaved (it can be non-Hausdorff, for example), and so a related variety, the variety of characters of representations is more useful to work with. These varieties will now be discussed.

3.3.1 Representation Varieties and Character Varieties

For any finitely generated group $G$ (this is not to be confused with the structure group of the bundle $P \to T^2$, which is $SL(2, \mathbb{C})$), so that $G$ has generators $\{g_1, g_2, \ldots, g_n\}$. The set,

$$Rep(G) := \{ \rho : G \to SL(2, \mathbb{C}) \mid \rho \in Hom(G, SL(2, \mathbb{C})) \},$$

will be called the $SL(2, \mathbb{C})$-representation variety of $G$. Identifying each element of $SL(2, \mathbb{C})$ with a length 4 row vector, the set $Hom(G, SL(2, \mathbb{C}))$ can be thought of as a subset of $\mathbb{C}^{4n}$ by mapping $\rho \in Hom(G, SL(2, \mathbb{C}))$ to the vector $(\rho(g_1), \rho(g_2), \ldots, \rho(g_n)) \in \mathbb{C}^{4n}$. Since matrix multiplication and the defining equations for $SL(2, \mathbb{C})$ are polynomial operations, the resulting subset of $\mathbb{C}^{4n}$ is an affine variety.

The character of a representation $\rho : G \to SL(2, \mathbb{C})$ is the function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) := tr(\rho(g))$; the character only depends on the conjugacy class of $g$. For each $g \in G$, a character $\chi_\rho$ gives a function $\tau_g$ on $Rep(G)$, defined by $\tau_g(\rho) = \chi_\rho(g)$. In order to prove that the set of characters is also a variety, it is necessary to first prove some Lemmas about the functions $\tau_g$. Define $\mathbb{T}$ to be the ring generated by all functions $\tau_g$. 
The following Lemma appears in ([CS]) 1.4.1.

**Lemma 3.2.** $\mathcal{T}$ is finitely generated.

Fix a finite set $\{\gamma_1, \ldots, \gamma_m\} \in G$ such that $\tau_{\gamma_1}, \ldots, \tau_{\gamma_m}$ generate $\mathcal{T}$ as in Lemma 3.2. Define the function $t : \text{Rep}(G) \to \mathbb{C}^m$ by $t(\rho) = (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_m}(\rho))$, and define $\chi(G) := t(\text{Rep}(G))$. The set $\chi(G)$ is a closed algebraic subset of $\mathbb{C}^n$, with coordinate ring $\mathcal{T}_\mathbb{C} := \mathcal{T} \otimes \mathbb{C}$.

For example, with $G$ being the fundamental group of $T^2$ with generators as above, the representation variety is the subset of $\mathbb{C}^8 = \{(a, b, c, d, e, f, g, h), \ a, b, c, d, e, f, g, h \in \mathbb{C}\}$ cut out by,

\[
\begin{align*}
ae + bg &= ea + fc \\
af + bh &= eb + fd \\
\text{cd} + dg &= ga + hc \\
cf + dh &= gb + hd \\
ad - bc &= 1 \\
ch - fg &= 1,
\end{align*}
\]

and the coordinate ring of the character variety is generated by $\tau_M(a, b, c, d, e, f, g, h) = a + d$, $\tau_L(a, b, c, d, e, f, g, h) = e + h$, and $\tau_{ML}(a, b, c, d, e, f, g, h) = ae + bf + cf + dh$.

To understand the relationship between the space $\mathcal{M}_{T^2}$ and $\chi(T^2)$, the following Lemma from ([CS]) is useful,

**Lemma 3.3** (Culler, Shalen 1.5.2). If $\rho$ and $\rho'$ are representations of $G$ in $\text{SL}(2, \mathbb{C})$ with $\chi_\rho = \chi_{\rho'}$, and if $\rho$ is irreducible, then $\rho$ and $\rho'$ are conjugate.

Thus, for irreducible flat connections, there is a bijection between $\mathcal{M}_{T^2}$ and points in the character variety. Furthermore, if any two distinct representation $\rho$ and $\rho'$ are not
conjugate, but have the same character, then they are in $\mathcal{M}_T^2/\mathcal{M}^*_T$. The possibility of the existence of such non-conjugate elements with the same character means that the space $\chi(T^2)$ can be ‘coarser’ than $\mathcal{M}_T^2$.

However, even for reducible flat connections, it can be shown the Chern-Simons invariant (with respect to some trivialisation of $P$) only depends on the character of the connection: this motivates working with $\chi(T^2)$ instead of $\mathcal{M}_T^2$.

To describe the variety $\chi(T^2)$, first notice that, because $\pi_1(T^2)$ is abelian, any representation of $\pi_1(T^2)$ can be simultaneously diagonalised. This means that any element of the space $\mathcal{M}_T^2$ is conjugate to two representations with diagonal matrices:

$$\rho(M) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(L) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix},$$

and,

$$\rho'(M) = \begin{pmatrix} m^{-1} & 0 \\ 0 & m \end{pmatrix}, \quad \rho'(L) = \begin{pmatrix} l^{-1} & 0 \\ 0 & l \end{pmatrix}.$$

Let $\Delta \subset \text{Rep}(T^2)$ be the set of representations with diagonal matrices; the space $\mathcal{M}_T^2$ is the quotient of the space $\Delta$ by the action that identifies the pairs $\rho$ and $\rho'$ from above.

Now observe that the map $t : \text{Rep}(T^2) \to \chi(T^2)$ is still surjective when restricted to $\Delta$.

To see this, first notice that, for any $\rho \in \text{Rep}(T^2)$, $\rho(M)$ can be conjugated into the form $\rho(M) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}$, and so, since $M$ and $L$ commute, $\rho(L)$ will be of the form:

$$\rho(L) = \begin{pmatrix} l & * \\ 0 & l^{-1} \end{pmatrix}.$$ So, on any representation $\rho \in \text{Rep}(T^2)$,

$$t(\rho) = (m + m^{-1}, l + l^{-1}, ml + m^{-1}l^{-1}).$$

For any such triple in $\chi(T^2)$, there is a diagonal representation,

$$\rho'(M) = \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho'(L) = \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}.$$
in $\Delta$ such that $t(\rho') = t(\rho)$, so $t|_{\Delta}$ is surjective.

Let $\mathbb{Z}_2$ act on $\mathbb{C}^* \times \mathbb{C}^*$ by sending $(m, l) \in \mathbb{C}^* \times \mathbb{C}^*$ to $(m^{-1}, l^{-1})$, and let $\sigma$ be the map that takes a diagonal matrix to it’s top left coordinate. The above considerations give that the bottom arrow of the following commutative diagram is an isomorphism,

$$
\begin{array}{ccc}
\Delta & \xrightarrow{\sigma} & \mathbb{C}^* \times \mathbb{C}^* \\
\downarrow t & & \downarrow \pi \\
\chi(T^2) & \longrightarrow & (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2
\end{array}
$$

The algebraic variety $\chi(T^2) \cong (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$ is singular at the four points $(\pm1, \pm1)$.

### 3.3.2 The Chern-Simons Line Bundle for the two-torus

An explicit description of the Chern-Simons line bundle on $T^2$ was given by Kirk and Klassen; this section follows ([KK]) closely.

From the previous discussion, away from the four singularities, the space $\chi(T^2)$ has coordinates $(m, l) \in \mathbb{C}^* \times \mathbb{C}^*$. Given such a pair, it will be useful to find a canonical expression for the corresponding flat connection, and hence to get a more explicit expression for the symplectic form.

The idea is to find some good coordinates on a flat covering space of $\chi(T^2)$, then construct the Chern-Simons line bundle in those coordinates, making sure everything is invariant under the deck transformations of the covering.

Firstly, identify $T^2$ with $S^1 \times S^1$ and use $2\pi$-periodic coordinates $(x, y)$; the tangent space to $\mathcal{A}_{T^2}$ is $\Omega^1(T; \mathfrak{sl}(2, \mathbb{C}))$ so that elements of this space can be written,

$$A = \alpha_1(x, y)dx + \alpha_2(x, y)dy, \alpha_i \in C^\infty(T^2, \mathfrak{sl}(2, \mathbb{C})).$$

In these coordinates, for $\eta_1 = \alpha_1(x, y)dx + \alpha_2(x, y)dy$, and $\eta_2 = \beta_1(x, y)dx + \beta_2(x, y)dy$ the symplectic form on $\mathcal{A}_{T^2}$ is,

$$\omega(\eta_1, \eta_2) = k \int_{T^2} (Tr(\alpha_1\beta_2) - Tr(\alpha_2\beta_1)) \, dx \wedge dy.$$
Lemma 3.4. Fix a trivialisation $s$ to identify $A$ with $\Omega^1(T^2; \mathfrak{sl}(2, \mathbb{C}))$. Denote the connection $A_s = \frac{1}{2\pi}(\theta_1 dx + \theta_2 dy)$ where $\theta_i$ are constant and diagonal $\mathfrak{sl}(2, \mathbb{C})$ matrices. The connection $A$ is flat, and if $\gamma_L(t) = (0, 2\pi t)$, $\gamma_M(t) = (2\pi t, 0)$, are loops around the longitude and meridian, respectively, then $A$ has holonomy $\exp(\theta_1)$ around $\gamma_L$ and $\exp(\theta_2)$ around $\gamma_M$.

Proof. By direct calculation $dA_s + A_s \wedge A_s = [\theta_1, \theta_2] \otimes dx \wedge dy = 0$.

For the loop $\gamma_L$, the equations for finding a flat section $\sigma(t)$ around $\gamma_L$ are,

$$d\sigma(t) = \frac{1}{2\pi}\theta_1 \sigma(t),$$

where $\theta_1$ has been pulled back to $[0, 2\pi]$. By the usual theory of linear O.D.Es, a solution is $\exp(\frac{1}{2\pi}\theta_1 t)$, and the holonomy is given by observing,

$$\exp(\frac{1}{2\pi}\theta_1(t + 2\pi)) = \exp(\frac{1}{2\pi}\theta_1 t) \exp(\theta_1),$$

and similarly for the loop $\gamma_M$. $\Box$

Choose a point $[A]$ in $\mathcal{M}_{T^2}$ and consider it’s image under the map from $\mathcal{M}_{T^2}$ to $\chi(T^2)$. Locally, the identification $\chi(T^2) \cong (\mathbb{C} \times \mathbb{C})/\mathbb{Z}_2$ means that there are coordinates on $\chi(T^2)$ provided by $\mathbb{C}^* \times \mathbb{C}^*$, so we get a point $(m, l)$. Kirk and Klassen showed that $[A]$ can be represented by a connection of the form given by Lemma (3.4).

This means that $[A]$ can be represented by the flat connection with constant matrices,

$$A = \theta_M dx + \theta_L dy,$$

with,

$$\theta_M = \begin{pmatrix} \log(m) & 0 \\ 0 & \log(m)^{-1} \end{pmatrix}, \quad \theta_L = \begin{pmatrix} \log(l) & 0 \\ 0 & \log(l)^{-1} \end{pmatrix}.$$}

This gives us a good candidate for the coordinates on our covering space: the idea is to use $u = \text{Log}(m)$ and $v = \text{Log}(l)$, for some branch of Log. In these coordinates, the
Atiyah-Bott symplectic form on $\mathcal{A}_T^2$ descends to the form on flat connections that can be written,
\[
\omega = -\frac{1}{4\pi^2}du \wedge dv.
\]
We would like to discuss this construction in a more invariant fashion.

To this end, define the covering map $\pi : \mathbb{C}^2 \to \chi(T^2)$, by sending the pair $(u,v)$ to the character of the representation,
\[
\rho(M) = \begin{pmatrix}
\exp(2\pi i u) & 0 \\
0 & \exp(-2\pi i u)
\end{pmatrix}, \quad \rho(L) = \begin{pmatrix}
\exp(2\pi i v) & 0 \\
0 & \exp(-2\pi i v)
\end{pmatrix}.
\]
This map is an algebraic covering map with covering group $H = (\mathbb{Z} \oplus \mathbb{Z}) \rtimes \mathbb{Z}_2$. Give $H$ the presentation,
\[
<x, y, b | [x, y] = bxbx = byby = b^2 = 1>,
\]
and let $H$ act on the trivial $U(1)$-bundle $\mathbb{C}^2 \times U(1)$, by,
\[
x(u, v, z) = (u + 1, v, z \exp(2\pi i v))
\]
\[
y(u, v, z) = (u, v + 1, z \exp(-2\pi i u))
\]
\[
b(u, v, z) = (-u, -v, z).
\]
We get,

**Theorem 3.2** (Kirk, Klassen). *The quotient bundle $\mathbb{C}^2 \times U(1)$ under the action of $H$ is isomorphic to the Chern-Simons line bundle over $\chi(T^2)$.*

**Remark 3.2.** *One has to show that the Chern-Simons line bundle is well-defined after quotienting from $\mathcal{M}_T^2$ to $\chi(T^2)$, but this follows because the Chern-Simons invariant only depends on the character of a representation in $\text{Hom}(\pi_1(T^2), SL(2, \mathbb{C}))$.***

The action of $H$ restricted to $\mathbb{C}^2$ gives the quotient $\mathbb{C}^2/H$ an orbifold structure.

**Lemma 3.5.** *The symplectic form $\omega = -\frac{1}{4\pi^2}du \wedge dv$ defines an orbifold two-form on $\chi(T^2)$.***
Proof. One only has to show that $\omega \in \Omega^2(\mathbb{C}^2)$ is invariant under the action of $H$ on $\mathbb{C}^2$. This follows since $x_\ast = y_\ast = Id$ and $b_\ast = -Id$ so that, for any $X, Y \in \Gamma(T^*\mathbb{C}^2)$,

$$b^*\omega(X, Y) = \omega(b_\ast X, b_\ast Y) = \omega(-X, -Y) = \omega(X, Y).$$

\[ \Box \]

Using the coordinates $(u, v)$ on the covering space $\mathbb{C}^2$, the connection one-form $\Theta^s$ on $\mathcal{L} \to \mathcal{M}_{T^2}$ is of the form,

$$\Theta^s(u, v) = -\frac{i}{\pi}(udv - vdu).$$

This gives that parallel transport on along a path $c : [0, 1] \to \mathcal{L}$ is given by finding lifts of the form (3.5). So let denote the path $c(t) = (u(t), v(t), z(t))$ so that a horizontal lift of $c(t)$ is given by,

$$\exp \left( \frac{i}{\pi} \int \left( u(t) \frac{dv}{dt} - v(t) \frac{du}{dt} \right) dt \right),$$

and the total transport along an interval is given by,

$$\tilde{c}(1)\tilde{c}(0)^{-1} = \exp \left( -8\pi i \int_0^1 (u \frac{dv}{dt} - v \frac{du}{dt}) dt \right).$$
Chapter 4

Integrable Systems and the $A$-polynomial

In this chapter, we will discuss a family of Lagrangian subvarieties of $\mathcal{M}_{T^2}$, determined by a knot $K \subset S^3$. Then we will outline the construction of an integrable system associated to this family.

4.1 The $A$-Polynomial

One reason the variety $\chi(T^2)$ is of interest is because certain subvarieties of $\chi(T^2)$ can give invariants of knots and three manifolds. Let $K$ be a knot in $S^3$. Take a small tubular neighbourhood of $K$ with boundary diffeomorphic to a torus $T^2$. The complement of this neighbourhood is a three-manifold $X_K$ with $\partial X_K = T^2$.

There is a natural inclusion map $i : \pi_1(T^2) \to \pi_1(X_K)$ given by letting a loop on the surface of the torus extend into $X_K$. Call the image of this map the *peripheral subgroup* of $\pi_1(X_K)$. 
If the knot $K$ is a two-bridge knot, then it is possible to be more explicit about this inclusion map. Two bridge knots are classified by a pair of integers $(p, q)$ with $gcd(p, q) = 1$. In this case the fundamental group of $X_K$ (the so-called knot group) has a presentation, 

$$\langle x, y \mid xw = wy \rangle, \quad w = y^{e_1}x^{e_2} \cdots x^{e_{q-1}}, \quad \text{where } e_i = (-1)^{\frac{ip}{q}}.$$ 

This presentation is well known, see standard references such as ([Ro], [BZ]).

If $\pi_1(T^2)$ has the presentation with generators $M, L$ as in the previous chapter, the inclusion map is simply,

$$M \mapsto x$$

$$L \mapsto x^n w\bar{w},$$

where $n$ is chosen so that the exponent sum of $L$ is zero, and $\bar{w}$ is the word in $x$ and $y$ obtained by reversing the order of $w$.

For example, the figure eight knot has $w = yx^{-1}y^{-1}x$, and the peripheral subgroup of the figure eight knot complement is generated by $L = yx^{-1}y^{-1}xy^{-1}x^{-1}y$, and $M = x$.

By applying the $\text{Hom}(-, SL(2, \mathbb{C}))$ functor to the inclusion map $i : \pi_1(T^2) \hookrightarrow \pi_1(X_K)$, we get a map $r : \text{Rep}(X_K) \rightarrow \text{Rep}(T^2)$, (since $\text{Hom}$ is contravariant).

To define the $A$-polynomial, notice that the map $r$ induces $r : \chi(X_K) \rightarrow \chi(T^2)$ (which will be also be denoted $r$). The $A$-polynomial defines the image of $r$ in $\chi(T)$. Thinking of this in the language of gauge theory, the zero locus of the $A$-polynomial is the subvariety of $SL(2, \mathbb{C})$-connections on $T^2$ that also can be extended to connections on the bounding knot complement $X_K$.

More specifically, recall the diagram (3.6) in the previous chapter. Let $Y$ be the union of the irreducible components $Y'$ of the character variety $\chi(X_K)$ such that $r(Y')$ is a one-dimensional variety. Any such component lies in the image of $t_\Delta$, so define the deformation variety, $D_K$ to be,

$$D_K := \sigma(\bigcup_{Y' \in Y} t_\Delta^{-1}r(Y')).$$
Up to multiplication by non-zero constants, there is a unique polynomial in $l$ and $m$ that vanishes exactly on $D_K$: this is the $A$-polynomial $A_K(m, l)$ of the knot. The $A$-polynomial of a knot was first defined in ([CGLS]).

Notice that, from this definition, the zero locus of the $A$-polynomial can have multiple components. Indeed, a moment’s thought yields that $(l-1)$ will always be a factor of the $A$-polynomial, so its zero locus will always have at least two connected components. The component $(l-1) = 0$ will be called the Abelian component.

Recall that a hyperbolic knot $K$ is a knot whose complement $X_K$ admits a hyperbolic metric. Let $\mathbb{H}^3$ denote hyperbolic three-space. If $X_K$ admits a hyperbolic metric, then it is isometrically isomorphic to $\pi_1(X_K) \backslash \mathbb{H}^3$, where $\pi_1(X_K)$ acts on $\mathbb{H}^3$ by isometries. The group of isometries of $\mathbb{H}^3$ is $PSL(2, \mathbb{C}) \cong Conf^+(S^2)$ (each isometry can uniquely be specified by a conformal action on the hyperbolic boundary $S^2$).

Thus, for a hyperbolic knot, there is a representation $\rho : \pi_1(X_K) \to PSL(2, \mathbb{C})$ determined by the geometry, and it can be shown that this representation is unique, up to conjugation. This conjugacy class will be called the geometric representation.

**Proposition 4.1** (Thurston). Let $X$ be a hyperbolic three-manifold. The geometric representation $\rho : \pi_1(X) \to PSL(2, \mathbb{C})$ can be lifted to a representation into $SL(2, \mathbb{C})$.

A proof of this is in section three of [CS]. We will abuse language slightly and use the term geometric representation for this lift into $SL(2, \mathbb{C})$. The perennial example will be the figure eight knot. This is a hyperbolic knot; in fact, Thurston gave a famous and very explicit construction of the hyperbolic structure via glueing hyperbolic tetrahedra, see ([T]). For this knot (using the generators $\{x, y\}$ for $\pi_1(X_K)$ as above), the geometric representation is,

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}(1 + i\sqrt{3}) & 1 \end{pmatrix}.$$ 

Henceforth, the connected component of the zero locus of the $A$-polynomial containing
the geometric representation of $\pi_1(X_K)$ will be called the geometric component.

Going back to the definition of the $A$-polynomial also shows that the zero locus of the $A$-polynomial is invariant under the action of $\mathbb{Z}_2$ on $\mathbb{C}^* \times \mathbb{C}^*$. This is basically because we used $\chi(T^2)$ in the definition, rather than $\mathbb{C}^* \times \mathbb{C}^*$. More formally:

**Proposition 4.2.** For the $A$-polynomial of a knot the equality $A(m, l) = A(m^{-1}, l^{-1})$ holds, up to powers of $l$ and $m$.

Another important property satisfied by the $A$-polynomial that we will use is Proposition 2.9 of the original paper ([CGLS]):

**Proposition 4.3.** The $A$-polynomial of a knot only has even powers of $m$.

**Remark 4.1.** One can actually define the $A$-polynomial for a knot in any compact three manifold, and the previous proposition applies to all knots in homology spheres.

### 4.1.1 Example Calculations

According to the original paper ([CS]), the $A$-polynomial of a two-bridge knot is - in principle - calculable. The idea is to use the fact that if $\rho(M)$ is in upper triangular form, then the character of $\rho$ is determined entirely by the upper left coordinates of $\rho(M)$ and $\rho(L)$.

To use this observation, notice that, for a representation $\rho \in \text{Rep}(X_K)$, any two elements $\rho(x), \rho(y) \in SL(2, \mathbb{C})$ are conjugate, and hence have the same trace. Lemma 7 of ([Ri]) implies that $\rho$ can be conjugated into the following form,

$$
\rho(x) = \begin{pmatrix} m & 1 \\ 0 & m^{-1} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} m & 0 \\ s & m^{-1} \end{pmatrix}.
$$

Once $s$ has been determined (as a function of $m$) by imposing the conditions that $\rho$ has to be a representation, then $\rho(M) = \rho(x)$ and $\rho(L) = \rho(x^n w \bar{w})$ can be determined.
4.1 Integrable Systems and the A-polynomial

As mentioned above, \( \rho(L) \) will also be upper triangular, since \( \rho(L) \) commutes with \( \rho(M) \). To find the A polynomial, simply examine the resulting conditions on the upper left entries of \( \rho(M) \) and \( \rho(L) \). Specifically, let \( p(m, s) \) be the upper right entry of \( \rho(wx) - \rho(xw) \), and \( q(m, s) \) be the upper left entry of \( \rho(L) \). Then the deformation variety is given by the projection onto the \((m, l)\) plane of the curve defined by \( p = 0 \) and \( q = l \).

For example, for the figure eight knot group, assuming \( \rho(x) \) and \( \rho(y) \) are in the triangular form above,

\[
\rho(xw) - \rho(yw) = \begin{pmatrix}
0 & -1 + m^4(-1 + s) + m^2(3 - 3s + s^2) \\
-s(-1 + m^4(-1 + s) + s + m^2(3 - 3s + s^2)) & m^2
\end{pmatrix},
\]

and \( \rho \) is a representation if,

\[
-1 + m^4(-1 + s) + s + m^2(3 - 3s + s^2)
\]

or \( s = 0 \). In the latter case, we get \( \rho(L) = \text{Id} \), so this component of the A-polynomial is simply \((l - 1) = 0\). In the former case, solving to find \( s \) as a function of \( m \) gives the representation with \( \rho(x) \) as above and,

\[
\rho(y) = \begin{pmatrix}
m & 0 \\
-1 + 3m^2 - m^4 + \sqrt{1 - 2m^2 - m^4 - 2m^6 + m^8} & m^{-1}
\end{pmatrix}.
\]

Now that \( \rho(y) \) is known, we can explicitly calculate the matrix for \( \rho(L) \) to get,

\[
\begin{pmatrix}
p(m) & \frac{(1 + m^2) \sqrt{1 - 2m^2 - m^4 - 2m^6 + m^8}}{m^4} & 0 \\
0 & p(m)^{-1}
\end{pmatrix},
\]

for \( p(m) \) the multivalued function,

\[
p(m) = \frac{1 - m^2 - 2m^4 - m^6 + m^8 - (1 - m^4) \sqrt{1 - 2m^2 - m^4 - 2m^6 + m^8}}{2m^4}.
\]

The other generator of the peripheral subgroup given by the meridian cycle is simply \( \rho(x) = \rho(M) \). Examining the upper left entry \( p(m) \) of \( \rho(L) \) gives that the A-polynomial of the figure eight knot is,

\[
(l - 1)(m^4l^2 - l(1 - m^2 - 2m^4 - m^6 + m^8) + m^4).
\]

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Another simple example is the knot group of the trefoil,

$$\langle x, y|xyx = yxy \rangle.$$  

So that, for the $\rho$ given above,

$$\rho(xw) - \rho(yw) = \begin{pmatrix} 0 & -1 + m^{-2} + m^2 + s \\ (1 - m^{-2} - m^2)s - s^2 & 0 \end{pmatrix}.$$  

Which means $\rho$ is a representation if,

$$s = 1 - m^{-2} - m^2,$$

or $s = 0$. To investigate the former case, use that the longitude holonomy corresponds to $L = x^{-4}yxxy$ while the meridian is $m = x$, so,

$$\rho(L) = \begin{pmatrix} -m^{-6} & (m^{-5} + m^{-3} + m^{-1} + m + m^3 + m^5) \\ 0 & -m^6 \end{pmatrix}.$$  

Examining the upper left entry of $\rho(L)$ gives the non-Abelian component of the $A$-polynomial to be,

$$l = -m^{-6}.$$  

For applications to the Eynard-Orantin theory, it will be important to determine which $A$-polynomials have low genus components, and to have information about their singularities and normalisations. A first step in this direction is in the Appendix to ([E]) where computer experiments give a survey of the genus of various $A$-polynomials with up to 10 crossings.

An infinite class of knots for which the $A$-polynomials can be determined recursively is the class of ‘twist’ knots, where formulae for the $n$th twist knot were determined by ([HIS]). By calculating the genus and singularities of a sequence of twist knots, the following conjecture suggests itself.
Conjecture 4.1. The normalisation of the $n$-twist knot is a compact Riemann surface of genus $2n - 2$. The $n$-th twist knot has ordinary singularities at $\{(-1,1), (1,-1)\}$, and for $n \geq 3$ at $\{(0,0), (0,1)\}$ (so these last two do not lie on the $A$-polynomial). The $n$-th twist knot also has 2 more complicated singularities at infinity.

In particular, this conjecture would imply that the figure eight knot is the only twist knot with genus one $A$-polynomial.

### 4.2 Deformations of the $A$-polynomial and Integrable Systems

Now, we will study how the Chern-Simons line bundle restricts to the zero locus of the $A$-polynomial. Define $\mathcal{X}$ to be the subvariety of $\mathbb{C}^* \times \mathbb{C}^*$ defined by the geometric component of the $A$ polynomial.

Lemma 4.1. The variety $\mathcal{X}$ is a Lagrangian subvariety of the symplectic orbifold $\chi(T^2)$. The restriction of the Chern-Simons line bundle to the nonsingular component of $\mathcal{X}$ is flat, and the restriction of the connection one-form $\Theta|_{\mathcal{X}}$ is closed.

Remark 4.2. Recall that the connection one-form $\Theta$ is defined in section 3.2.

Proof. Since $\chi(T^2) \cong (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$, for the purpose of computations, it is useful to think of the zero locus of the $A$-polynomial as a $\mathbb{Z}_2$-invariant subvariety of $\mathbb{C}^* \times \mathbb{C}^*$. Let $(m,l)$ be coordinates on $\mathbb{C}^* \times \mathbb{C}^*$, the symplectic form $\omega$ is written in these coordinates as $\omega = k4\pi^2 \frac{dm}{m} \wedge \frac{dl}{l}$. Since there is a polynomial relation between $m$ and $l$ on the zero locus of the $A$-polynomial, $\omega$ vanishes on this subvariety, and so the curvature of $\mathcal{L}|_A$ is zero. Since $d\Theta = F_\Theta$, the connection one-form $\Theta$ is closed on the restriction to the zero locus of $A(m,l)$. 

\[\square\]
As mentioned in the proof of Lemma (4.1), the definition of the $A$-polynomial gives that the affine variety defined by its zero locus $X$ can be thought of as $\mathbb{Z}_2$-invariant subvariety of $\mathbb{C}^* \times \mathbb{C}^*$. This is because the definition really involves the Lagrangian subvariety inside $\chi(T^2)$ consisting of flat connections that extend onto the bounding three manifold; the object to consider should actually be the quotient $X/\mathbb{Z}_2$.

For example, the geometric component of the $A$-polynomial of the figure 8 knot defines the algebraic curve,

$$m^4l^2 - l(1 - m^2 - 2m^4 - m^6 + m^8) + m^4 = 0. \quad (4.1)$$

In the original work of Dijkgraaf, Fuji and Manabe ([DFM]), the curve (4.1) was replaced with,

$$x^2l^2 - l(1 - x - 2x^2 - x^3 + x^4) + x^2,$$

where $m^2 = x$ (this $x$ is not to be confused with the element $x \in \pi_1(X_K)$ that was introduced in the previous section). Using Proposition (4.3), this is a substitution that can be applied to the $A$-polynomial of any knot, and so, to make contact with the work of Dijkgraaf, Fuji and Manabe, we will use this substitution in our study of the Eynard-Orantin theory of the $A$-polynomial.

Now, recall that the $\mathbb{Z}_2$ action on $\mathbb{C}^* \times \mathbb{C}^*$ is $(x, l) \mapsto (x^{-1}, l^{-1})$, and this curve is singular at the $\mathbb{Z}_2$ fixed points $\{(1, -1), (-1, 1)\}$. This curve is birational to the nonsingular genus 1 curve,

$$y^2 = x^4 - 2x^3 - x^2 - 2x + 1, \quad (4.2)$$

via the birational transformation that sends $x$ to $x$ and,

$$y(x, l) = \frac{(l - l^{-1})x^2}{(x^2 - 1)}.$$

This birational map sends the involution $(x, l) \mapsto (x, l^{-1})$ on (4.1) to the involution $(x, y) \mapsto (x, -y)$ on (4.2). Now we will quotient the curve (4.2) by the involution
(x, y) \mapsto (x^{-1}, -y) to get \mathcal{X}/\mathbb{Z}_2. Let s := \frac{y^2}{x^2}, and t := \frac{x + x^{-1}}{2}, yields the quotient curve given by the zero locus of,

\[ s = 4t^2 - 4t - 3, \]

in the (s, t)-plane. This is a nonsingular rational curve.

In chapter 1 and chapter 2, we showed that certain aspects of the (planar) Eynard-Orantin theory can be understood via the idea of covariant derivatives of special cohomology classes via the Gauss-Manin connection. To apply this idea to the A-polynomial, we will now introduce a family $X_\epsilon$ of Lagrangian subvarieties of $\mathbb{C}^* \times \mathbb{C}^*$. These are defined by the zero locus of certain polynomials $A_\epsilon (m, l)$. We will require that this family is a deformation of the original A-polynomial, in the sense that $A_0 (m, l) = A(m, l)$.

Using similar reasoning to Lemma (4.1), it follows that the restriction of the connection one-form $\Theta$ to $X_\epsilon$ is closed, so gives a class $[\Theta|_{X_\epsilon}] \in H^1(X_\epsilon, \mathbb{C})$. The family $[\Theta|_{X_\epsilon}]$ of cohomology classes will be differentiated by the Gauss-Manin connection: if the family $X_\epsilon$ is constructed correctly, then the derivatives of $[\Theta|_{X_\epsilon}]$ will be holomorphic forms, so that $X_\epsilon$ is a variation by filling fractions of $X_0$. This will yield planar Eynard-Orantin invariants (actually this idea must be adjusted slightly to give non-meromorphic Eynard-Orantin invariants, this is detailed below).

The family of one-forms $\Theta|_{X_\epsilon}$ coming from the global one-form $\Theta$ on $\chi(T^2)$ is the correct replacement for the family of one forms $ydx|_{C}$ coming from the global one-form $ydx$ on $\mathbb{C}^2$ that we discussed in Chapter (2.1.1). This is because $ydx$ is a Liouville one-form for the symplectic form $dy \wedge dx$, and the same holds true for $\Theta$ with respect to the symplectic form $\omega$ on $\chi(T^2)$.

One obstacle to developing this construction is that the form $\Theta$ is actually defined in the infinite-dimensional space of all connections. To do calculations, we would like to work in a finite-dimensional space. It is actually easier to work with $\mathbb{Z}_2$-invariant objects in the space $\mathbb{C}^* \times \mathbb{C}^*$ than the space $\chi(T^2) \cong \mathcal{M}_{g,2}^* \cong (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2$. However, this makes it difficult to deal with the one-form $\Theta$: unlike the symplectic form $\omega$, we do not have a
good expression for this one-form in the coordinates \((m, l) \in \mathbb{C}^* \times \mathbb{C}^*\).

To understand why, go back to the covering space \(\mathbb{C}^2\) with coordinates \((u, v)\) as in the previous section. Here, \(\omega = du \wedge dv\), which is \(H\)-invariant. It is easy to represent \(\Theta\) by the one form \(udv - vdu\). However, unlike \(\omega\), the one-form \(udv - vdu\) does not descend to a form on \(\chi(T^2)\) or \(\mathbb{C}^* \times \mathbb{C}^*\); it is not invariant under the action of the group \(H\). In the physics literature, this difficulty is manifested by the use of the 'local' one-form \(udv = log(l) \frac{dm}{m}\) to represent \([\Theta]\) on various open sets of \(\mathcal{X}_0\), because of the dependence on a choice of branch of the Logarithm.

Rather than differentiate classes defined by the ill-defined form \(udv\) (representing \(\Theta\)), we will show how period integrals of \(\Theta\) can be related to integrals of \(\omega\) over smooth two-chains in \(H_2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z})\). The advantage of this is that the two-form has an explicit (\(\mathbb{Z}_2\)-equivariant) expression in terms of the coordinates \((m, l) \in \mathbb{C}^* \times \mathbb{C}^*\).

### 4.2.1 Covariant derivatives of the cohomology class \([\Theta]\)

Recall that we defined the coordinate \(x\) by \(m^2 = x\). For convenience, we will use this coordinate on \(\mathbb{C}^* \times \mathbb{C}^*\) so that the form \(\omega\) becomes \(\frac{1}{2} \frac{dx}{x} \wedge \frac{dl}{l}\); this does not alter any calculations.

The \(\mathbb{Z}_2\)-equivariant form \(\omega = \frac{dz}{z} \wedge \frac{dl}{l}\) is closed, and integrating over the two-chain \(S^1 \times S^1 \subset \mathbb{C}^* \times \mathbb{C}^*\) where \(S^1 = \{e^{i\theta} \in \mathbb{C}\text{ s.t. } 0 \leq \theta \leq 2\pi\}\) gives that \([\omega]\) is nontrivial in \(H^2(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{C})\).

Now we will use a map, called the Leray co-boundary map, from \(H_1(\mathcal{X}_\epsilon, \mathbb{Z})\) to \(H_2((\mathbb{C}^* \times \mathbb{C}^*) - \mathcal{X}_\epsilon, \mathbb{Z})\). This map is defined as follows: take a tubular neighbourhood of \(\mathcal{X}_\epsilon\), the boundary of this tubular neighbourhood gives a fibre bundle with fibre a circle over \(\mathcal{X}_\epsilon\).

For any smooth cycle \(\gamma \in H_1(\mathcal{X}_\epsilon, \mathbb{Z})\), take the preimage of the fibre bundle projection of \(\gamma\) to get a two-chain \(\Gamma \subset (\mathbb{C}^* \times \mathbb{C}^*) - \mathcal{X}_\epsilon\). It can be shown ([A] section 10) that, if \(\gamma\) is a cycle, then \(\Gamma\) is a cycle, and the homology class of \(\Gamma\) in \(H_2(\mathbb{C}^* \times \mathbb{C}^* - \mathcal{X}_\epsilon, \mathbb{Z})\) only depends on the homology class of \(\gamma\). The cycle \(\Gamma\) is the image of \(\gamma\) under the Leray co-boundary
map.

To apply this to our situation, use the following Lemmata:

**Lemma 4.2.** [Arnold] If $F : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}$ determines a hypersurface $X$ in $\mathbb{C}^* \times \mathbb{C}^*$ and $\Gamma$ is the image of the cycle $\gamma \in H_1(X, \mathbb{Z})$ under the Leray coboundary map, then,

$$\oint_{\gamma} \eta = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\eta}{F} \wedge dF,$$

for any one-form $\eta$ on $\mathbb{C}^* \times \mathbb{C}^*$.

**Lemma 4.3.** Let $A_\epsilon(m,l)$ be a polynomial with $dA_\epsilon \neq 0$ except at a (necessarily finite) set of points $\{p_i\} \subset \mathbb{C}^* \times \mathbb{C}^*$. At any point outside of $\{p_i\}$, there exists a one-form $\Phi_\epsilon$ with the property that $\omega = dA_\epsilon \wedge \Phi_\epsilon$. If $X_\epsilon$ is the variety defined by the zero locus of $A_\epsilon$, the one-form $\Phi_\epsilon$ is holomorphic on $X_\epsilon/\{p_i\}$ and the restriction of $\Phi_\epsilon$ to $X_\epsilon/\{p_i\}$ is unique.

**Proof.** To show existence of $\Phi_\epsilon$ away from $\{p_i\}$, use the fact that $A_\epsilon$ gives a coordinate at in $\mathbb{C}^* \times \mathbb{C}^*/\{p_i\}$.

To show the next part of the Lemma, we will construct $\Phi_\epsilon|_{X_\epsilon}$ explicitly. The required one-form must have the expression $\Phi_\epsilon = \Phi_x^\epsilon dx + \Phi_l^\epsilon dl$, for some functions $\Phi_x^\epsilon$, $\Phi_l^\epsilon$. It also must satisfy the equation,

$$(\partial_x A_\epsilon)\Phi_l^\epsilon - (\partial_l A_\epsilon)\Phi_x^\epsilon = \frac{1}{xl},$$

at all points in $\mathbb{C}^* \times \mathbb{C}^*/\{p_i\}$. On $X_\epsilon$ the equation $dx = -\frac{\partial A_\epsilon}{\partial x} dl$ holds, so that, away from the zeroes of $\partial_x A_\epsilon$, $\Phi_\epsilon$ must be of the form,

$$\Phi_\epsilon = \left(\Phi_x^\epsilon \left( -\frac{\partial_l A_\epsilon}{\partial_x A_\epsilon} \right) + \Phi_l^\epsilon \right) dl = \frac{dl}{(\partial_x A_\epsilon)xl}.$$  

On $X_\epsilon$, the form $\Phi_\epsilon$ is given by $\frac{dl}{\partial_x A_\epsilon x^l}$ away from the zeroes of $\partial_x A_\epsilon$, and by $-\frac{dx}{\partial_l A_\epsilon x^l}$ away from the zeroes of $\partial_l A_\epsilon$. These agree on the overlap, and are holomorphic on $X_\epsilon/\{p_i\}$.  

**Remark 4.3.** The one-form $\Phi_\epsilon$ is an example of a ‘Gelfand-Leray’ form.
Now, if \( \gamma \in H_1(X_\epsilon, \mathbb{Z}) \) is trivial in \( H_1(\mathbb{C}^* \times \mathbb{C}^*, \mathbb{Z}) \), so that there exists a two-cycle \( \Sigma \) with \( \partial \Sigma = \gamma \) and \( \Gamma \) is the image of \( \gamma \) under the Leray coboundary map, then (using Lemma 4.2 and Lemma 4.3), we have,

\[
\int_\Sigma \omega = \oint_\gamma \Theta = \oint_\gamma A_\epsilon \wedge \Phi = -\frac{1}{2\pi i} \int_\Gamma \omega.
\]

So, for such \( \gamma \), we can use the form \( A_\epsilon \wedge \Phi \) to calculate period integrals of \( \Theta \).

If \( \gamma \) is not of this form, we will still use \( A_\epsilon \wedge \Phi \) to as a representative to calculate with, since it is still connected with the symplectic form \( \omega \) via the Gelfand-Leray map, and is a Liouville form for \( \omega \) on \( X_\epsilon \).

This allows us to define the correct analogue of the notion of a filling fraction variation (in the definition below, recall that the function field of a curve \( C \) is denoted \( K(C) \)).

**Definition 4.1.** A logarithmic filling fractions variation of the polynomial \( A_0 \) is a family of polynomials \( \pi : \mathbb{C}[x, y] \to \mathcal{E} \), with \( \pi^{-1}(\epsilon) \) denoted \( A_\epsilon \) and the algebraic curve defined by the zero locus of \( A_\epsilon \) denoted \( X_\epsilon \) with the following properties:

- The genus of \( X_\epsilon \) is equal to the dimension of the base \( \mathcal{E} \),
- \( \nabla_{\frac{\partial}{\partial \epsilon}} [A_\epsilon \wedge \Phi] \in H^0(X_\epsilon, K(X_\epsilon)) \) for all \( 1 \leq i \leq \dim(\mathcal{E}) \),
- The map,
  \[
  T_\epsilon \mathcal{E} \to H^0(\pi^{-1}(\epsilon), K(X_\epsilon)),
  \]
  given by \( X \mapsto \nabla_X [A_\epsilon \wedge \Phi] \), is an isomorphism of vector spaces.

**Remark 4.4.** The name comes from the fact that this is designed to be the analogue of the situation in the physics literature where the ‘logarithmic’ one-form \( u \, dv \) is varied to give a holomorphic form. The terminology ‘Seiberg-Witten’ differential is also used for a differential on a family of curves with the properties of \( A_\epsilon \wedge \Phi \) in the above definition.

**Remark 4.5.** The isomorphism from the base to the space of holomorphic differentials on the fibre is local; globally, there may be monodromy if \( \mathcal{E} \) is not simply connected.
Our main example of such a variation will be the deformation of the geometric component of the $A$-polynomial of the figure eight knot, given in (4.1). Define the polynomial family,

$$A_{\epsilon}(x, l) = x^2 l^2 - l(1 - (1 + \epsilon_1)x - (2 + \epsilon_2)x^2 - (1 + \epsilon_3)x^4 + x^4) + x^2,$$

(4.3)

so that $A_0(x, l)$ is the $A$-polynomial of the figure eight knot.

**Proposition 4.4.** The family $\pi : X_\epsilon \to \mathcal{E} = \{(\epsilon_1, \epsilon_2, \epsilon_3)\}$ defines a logarithmic filling fractions variation of $X_0$.

**Proof.** First notice that $X_\epsilon$ is generically nonsingular, since the discriminant only has multiple roots at the zeroes of $65536\epsilon^2(-16 + \epsilon^2)(225 + 136\epsilon + 16\epsilon^2)^2$, which are at \{-25 \frac{1}{4}, -4, -\frac{9}{4}, 0, 4\}. Away from these points, the curve $X_\epsilon$ is a genus three hyperelliptic compact Riemann surface, that can be realised as a double branched cover of the $x$-plane via the equation $y^2 = \sigma(x)$, where,

$$\sigma(x) = -4x^4 + (1 - (1 + \epsilon_1)x - (2 + \epsilon_2)x^2 - (1 + \epsilon_3)x^3 + x^4)^2.$$  

(4.4)

Four of the branch points come together in two pairs when $\epsilon = 0$, which defines the singular genus one curve $X_0$. It is well known that a basis for $H^0(X_\epsilon, K)$ is given by \{\frac{dx}{y}, \frac{x dx}{y}, \frac{x^2 dx}{y}\}. Now, using the fact, on $X_\epsilon$,

$$l = \frac{b(x) + y}{c(x)},$$

where $A_{\epsilon}(x, l) = l^2 \frac{c(x)}{2} - lb(x) + \frac{c(x)}{2}$, and $\partial_l A_{\epsilon} = lc(x) - b(x) = y$, we get,

$$\frac{\partial A_{\epsilon_1}}{\partial \epsilon_1} \wedge \Phi = \frac{dx}{y}, \quad \frac{\partial A_{\epsilon_2}}{\partial \epsilon_2} \wedge \Phi = \frac{x dx}{y}, \quad \frac{\partial A_{\epsilon_3}}{\partial \epsilon_3} \wedge \Phi = \frac{x^2 dx}{y}.$$

Now, we can see how the logarithmic filling fractions of the $A$-polynomial of the figure eight knot relates to Eynard-Orantin theory.

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Theorem 4.1. If $\omega^{(0,3)}$ is the $(0, 3)$ Eynard-Orantin invariant for the non-meromorphic spectral curve $X_\varepsilon$ with the functions $\text{Log}(l), \text{Log}(x)$, then,

$$\nabla_{\frac{\partial}{\partial \varepsilon_i}} \left[ \omega^{(0,2)}(p_1, p_2) \right] = \left[ \oint_{B_i} \omega^{(0,3)}(\cdot, p_1, p_2) \right],$$

for all $1 \leq i \leq 3$.

Proof. The function,

$$l_\varepsilon = \frac{b_\varepsilon(x) + y_\varepsilon(x)}{c(x)},$$

can be used as a local coordinate around each branch point, as in the discussion of the Rauch variational formula. Since $y_\varepsilon(x) = \sqrt{b_\varepsilon(x)^2 - c(x)}$, and varying in the $\varepsilon_i$ direction sends $b_\varepsilon(x) \mapsto b_\varepsilon(x) + \varepsilon_i x^i$, it follows that,

$$y_{\varepsilon_i}(x) = \sqrt{y_0^2 + 2\varepsilon_i b_0(x) x^i + \varepsilon_i^2 x^i}.$$ 

Which gives the derivative,

$$\delta_{\varepsilon_i} y = \frac{b_0(x) x^i}{y_0(x)}.$$ 

So that,

$$\delta_{\varepsilon_i} l_\varepsilon|_{\varepsilon_i=0} = \frac{\delta b_\varepsilon}{c(x)} + \frac{b_0(x) x^i}{c(x) y_0(x)}.$$ 

Using Theorem (1.4), we get that,

$$\nabla_{\frac{\partial}{\partial \varepsilon_i}} \left[ \omega^{(0,2)}(p_1, p_2) \right] = \left[ \sum_{i=1}^{8} \text{Res}_{p=a_i} \frac{\delta_{\varepsilon_i} l_\varepsilon B(p, p_1) B(p, p_2)}{dl} \right].$$ 

Now calculate $\omega^{(0,3)}$, using Proposition (1.2),

$$\omega^{(0,3)}(p_1, p_2, p_3) = \sum_{i=1}^{8} \text{Res}_{p=a_i} \frac{x l B(\cdot, p_1) B(\cdot, p_2) B(\cdot, p_3)}{dx dl}.$$ 

Notice that the residue at a branch point $a_j$ of the first summand of $\delta_{\varepsilon_i} l$ is zero at a branch point (both the numerator and the denominator as holomorphic there). Also notice that, since $y(a_j) = 0$ the the residue of the second summand is,

$$l(a_j) a_j \left[ \oint_{B_{a_j}} \frac{B(a_j, \cdot)}{dx(a_i)} \right],$$ 

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(this has again used the fact the a basis for $H^0(X, K)$ is $\{dx, x^2d\bar{z}, \bar{z}\}$, and that, $\oint_{B_i} B(p, \cdot) = 3^i(p)$). This gives the result.

We will use the following definition, from ([D]) and ([F1]).

**Definition 4.2.** A complex integrable system is a holomorphic proper Lagrangian map $\pi : A \to B$ from a $2n$-dimensional complex manifold $A$ with a holomorphic symplectic form $\omega \in \Omega^{(2,0)}(A, \mathbb{C})$ to an $n$-dimensional complex manifold $B$. An algebraically integrable system must satisfy the additional requirement that the fibres should be Abelian varieties.

This is a complex-geometric formulation of the classical notion of action-angle coordinates. The fibres $\pi^{-1}(b)$ are tori with ‘angle’ coordinates, and the coordinates on the base are ‘action’ coordinates. Think of the base as parametrising momenta: for fixed momenta the system evolves linearly by wrapping around the torus fibre. In mechanics, this is the law of conservation of momentum. See, for example ([Au]).

For an algebraically integrable system, we can lift any function on the base $H$ to get a one form $d\pi^*H$ on the total space. Using the symplectic form, this one-form can be identified with a vector field $X_H$ on the total space, tangent to the fibres of $\pi$. Any two pairs of such vector fields $X_{H_1}, X_{H_2}$ Poisson-commute: thus the concept of algebraically integrable system reproduces the classical notion of an integrable system as ‘a $2n$-dimensional symplectic manifold with $n$ Poisson-commuting functions’.

We have been considering families of curves $X_\epsilon$, given by filling fractions variations of a fixed curve $X_0$. This gives a family of Jacobians $J(X_\epsilon)$ suggestive of the structure of an algebraically integrable system. We must point out, however, that a given family of Abelian varieties does not, in general determine an algebraically integrable system. This is the importance of the idea of filling fractions variations (both logarithmic and non-logarithmic) that was defined in the previous chapters.

**Theorem 4.2.** The family of Jacobians $J(X_\epsilon)$ corresponding to the family of curves $X_\epsilon$,
comes from a logarithmic filling fractions variation if and only if it gives rise to an algebraically integrable system.

**Proof.** First, suppose that the family $J(X_e)$ is an algebraically integrable system. This means that there is a holomorphic symplectic form $\omega$ on the total space $|J(X_e)|$. On a contractible open subset of the base $E$, the two-form $\omega$ is exact, so that $\omega = d\theta$. The divisor associated to $\theta|_e$ gives a divisor in $J(X_e)$. Using the Abel-Jacobi isomorphism $u : Pic(X_e) \to J(X_e)$, $\theta|_e$ gives a divisor $u^*(\theta|_e) \in Pic(X_e)$. Taking a generic section of this line bundle gives a differential $\lambda_e$ on $X_e$. This can clearly be done in a smooth fashion, since each $u^*(\theta|_e)$ arose from the differential on the total space given by $\omega$. Holomorphicity and the fact that the fibres are Lagrangian gives that the map from $T_E$ to $H^0(X_e, K)$ given by covariant differentiation using the Gauss-Manin connection is an isomorphism.

Conversely, if we start with a logarithmic filling fractions variation $X_e$ over a base $E$, we can use the Abel-Jacobi isomorphism to send the element of the Picard group defined by $A_e \wedge \Phi$ to a differential on $J(X_e)$ and hence to a differential on the total space $|J(X_e)|$. The exterior derivative of this differential gives the holomorphic symplectic form $\omega$ that we need.

**Remark 4.6.** A similar construction appears in ([Ho]).

### 4.2.2 The Hitchin System and Eynard-Orantin Theory

We have showed how the logarithmic filling fractions variation that naturally arises from Eynard-Orantin theory gives rise to an algebraically integrable system. This algebraically integrable system has Jacobians of a family of curves for Lagrangian fibres, and it is natural to ask what is the relation to one of the most famous algebraically integrable systems: the Hitchin System, originally introduced in ([Hi2]).

To understand the Hitchin system, we need to first discuss Hitchin’s correspondence be-
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tween line bundles on a curve and objects called Higgs bundles.

This correspondence goes as follows. Let \( \pi : S \to C \) be a degree \( n \) covering of a Riemann surface \( C \) by a Riemann surface \( S \). If \( K \) is the canonical bundle of \( S \), and \( V \) is a \( SL(2, \mathbb{C}) \)-bundle (we can also consider other algebraic groups) over \( C \), a holomorphic section \( \phi \in H^0(C; End(V) \otimes K) \) is called a Higgs bundle. If \( C \) is genus 0, then we replace the definition with \( \phi \in H^0(C; End(V) \otimes \mathcal{O}(n)) \).

**Theorem 4.3** (Hitchin). Let \( C \) be a Riemann surface. There is a one-to-one correspondence between Higgs fields on \( C \) and pairs \( (\pi, S, L) \), where \( \pi : S \to C \) is a branched covering, \( S \) is a non-singular Riemann surface and \( L \to S \) is a line bundle.

Here is an outline of the proof. First, fix a line bundle \( L \) on \( S \). The direct image \( \pi_* L \) is the sheaf of sections of a vector bundle of rank \( n \). Away from branch points, this sheaf clearly has stalk \( \bigoplus_{y \in \pi^{-1} L} L_y \), so is locally free. Hitchin showed that this sheaf can be extended to give a locally free sheaf on all of \( C \), and hence a vector bundle \( V \) of rank \( n \). A section \( \lambda \in H^0(S, \pi^* K) \) acts by multiplication on \( V \) and so the line bundle \( L \) has induced a Higgs field \( \Phi \in H^0(C; End(V) \otimes K) \).

Conversely, given a Higgs bundle \( \phi \in H^0(C; End(V) \otimes K) \), we would like to construct a covering \( S \) with a line bundle \( L \to S \). Since eigenvalues don’t depend on a choice of basis, the polynomial \( det(x - \phi) \) is well defined, for \( x \in C \). The characteristic polynomial defines \( S \) by its zero locus in the total space of \( K \) (which will be generically nonsingular). Generically, the eigenvalues will be distinct, so the eigenspaces for each eigenvalue on \( S \) define a line bundle \( L \).

This construction shows that the set of Higgs bundle on \( C \) with fixed characteristic polynomial can be thought of as the space of line bundles on \( S \) of degree,

\[
-m(m - 1)(g - 1) - deg V^*,
\]

which comes from applying the Grothendieck-Riemann-Roch theorem to the direct image \( \pi_* L \).
So the space of Higgs fields of fixed degree with a given spectral curve $S$ (under the correspondence of Theorem 4.3) can be thought of as the Jacobian of $S$. As we vary the coefficients of the characteristic polynomial, we get a family of Abelian varieties fibering the space of Higgs bundles. The moduli space of Higgs bundles is a symplectic manifold (away from certain bad points), so it is natural to now ask if we get an algebraically integrable system. In order for this to work, the dimensions must match up: the genus of the spectral curve $S$ (which gives the dimension of the Jacobian $J(S)$) must be equal to the number of algebraically independent coefficients of a characteristic polynomial that determines $S$. Hitchin showed that this ‘numerical miracle’ does occur; both the base and the fibre of this integrable system are of dimension,

$$1 - m^2(1 - g),$$

and the moduli space of Higgs bundles is an algebraically integrable system. With some alterations (such as considering Prym varieties rather than Jacobians), this correspondence also applies to other reductive algebraic groups.

**Theorem 4.4** (Hitchin). *The moduli space of Higgs bundles on a curve $C$ is an algebraically integrable system, whose Lagrangian fibres are the Jacobians of a family of algebraic curves.*

In the main body of this thesis we have argued that the underlying structure to the Eynard-Orantin theory is the concept of the variation by filling fractions and the associated algebraically integrable system (on the family of Jacobians). By combining Theorems (4.4) and (4.2) we see that the Hitchin system is a natural source of such structures. That is, the Hitchin system gives a family of curves with differentials that have holomorphic covariant derivative. Therefore, if one starts with a curve $S$ equipped with a line bundle (or, equivalently a meromorphic differential with the same divisor), and a branched covering $\pi : S \to C$ of another Riemann surface then an answer to the question “where is a family of curves that
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varies S and gives rise to Eynard-Orantin theory?" is “the Hitchin system on C”.
In most situations where Eynard-Orantin theory arises, the natural branched cover to use
is the cover of \( \mathbb{CP}^1 \) given by the function \( x \), and the divisor comes from the divisor of
\( \omega^{(0,1)} = ydx \). In the main body of thesis, we have explored explicit constructions of the
filling fractions variation, the Hitchin system provides a more abstract approach, where
it is harder to actually calculate.

4.2.3 The Hitchin System and the A-polynomial

Now we will apply these ideas to the A-polynomial. In this situation we have a (degree
0) line bundle on the zero locus of the A polynomial, but I shall argue that the correct
branched cover is not necessarily the ‘obvious’ one given by considering the covering of
the m plane.
First of all, rather than consider the zero-locus \( X_\epsilon \) as a degree \( n \) cover of the m-plane
(where \( n \) is the degree of the A-polynomial in \( l \)), we will consider it as a degree 2 cover of
a Riemann surface \( C \) that covers the m-plane.
To see how this works, recall from the construction in section (4.1.1) that determining
the matrix \( \rho(L) \) for a representation \( \rho : \pi_1(T^2) \rightarrow SL(2, \mathbb{C}) \) that extends to \( X_K \), means
solving to find the function \( s(m) \) in the lower left hand corner of the matrix \( \rho(y) \). For the
case of the figure eight knot, the function \( s(m) \) is quadratic in \( m \),
\[
m^4l^2 - l(1 - m^2 - 2m^4 - m^6 + m^8) + m^4.
\]
While, in general \( s(m) \) will be a higher degree polynomial, for example,
\[
s^4m^4 + s^3(2m^6 - 5m^4 + 2m^2) + s^2(m^8 - 6m^6 + 13m^4 - 6m^2 + 1)
+ s(-m^8 + 7m^6 - 14m^4 + 7m^2 - 1) - 2m^6 + 5m^4 - 2m^2,
\]
for the knot 5_2 in Rolfsen’s table.
This means that \( s(m) \) will be an algebraic function of \( m \), and then, \( \rho(L) = \rho(x^n w \bar{w}) \) will
have multivalued algebraic functions of $m$ as its entries. These multivalued functions will be single valued on a unique Riemann surface $C$ that covers the $m$-plane, so that, for polynomials $b(m)$, $\Delta(m)$, $a(m)$, and algebraic functions $\phi_i$ of $m$,

$$l(m) = \frac{b(\phi_1(m)) + \sqrt{\Delta(\phi_2(m))}}{a(\phi_3(m))},$$  \hspace{1cm} (4.5)

is a double-valued function on $C$. The case of the figure eight knot that we studied before is exceptional, here the covering $C$ of the $m$-plane is trivial, and $C$ is just $\mathbb{CP}^1$. This is because, for the figure eight knot, $s(m)$ is quadratic.

Thus it may be that the correct algebraically integrable system comes from the Hitchin system on $C$, which would mean considering $SL(2, \mathbb{C})$ Higgs bundles (a more satisfactory situation than having to deal with Higgs bundles of arbitrary rank). The fact that in the case of the figure 8 knot $C$ is just the $m$-plane $\mathbb{CP}^1$ may be a misleading coincidence due to the simplicity of the figure 8 knot.

In any case, using Theorems 4.4 and 4.2 again we get:

**Theorem 4.5.** The Hitchin system on $C$ gives rise to a filling fraction variation of the curve defined by (4.5).
Bibliography


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