Managed DC Power Reticulation Systems

Anthony B. Morton
B.E.(Hons.)/B.Sc., MEngSc(Melb)

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Department of Electrical and Electronic Engineering
School of Electrical Engineering and Computer Science
University of Melbourne
Victoria 3010, Australia
Declaration

The material presented in this thesis has not been previously published or submitted for any other degree or examination. All work is original and has been performed by the author except where due reference or acknowledgement has been made in the text.

This thesis is less than 100,000 words in length, exclusive of appendices, tables and bibliography.

Anthony B. Morton

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Abstract

Electric power engineering, as it applies to low-voltage power reticulation in buildings and industrial sites, is ripe for a ‘paradigm shift’ to bring it properly into the Electronic Age. The conventional alternating-current approach, now over a hundred years old, is increasingly unsatisfactory from the point of view of plant and appliance requirements. Alternative approaches can deliver substantial cost savings, higher efficiencies, power quality improvements, and greater safety.

Power reticulation systems in the future can be expected to differ from present systems in two key respects. The first is a greatly increased role for direct current; the second is the augmentation of the power system with a wide range of ‘management’ technologies. Combining these two trends, which can already be observed today, leads to consideration of ‘managed DC’ power reticulation systems, operating from AC bulk supply mains via AC–DC converters.

This thesis presents arguments for the use of such systems in future, and investigates a broad range of novel technical challenges that arise. With electricity management in place, new opportunities arise to save energy by minimising the losses in the network, and to enhance system protection by anticipating and preventing overload conditions. A managed DC system will also need a sophisticated control scheme to ensure system stability and high power quality. An important aspect of such a system is the design of input converters using state-of-the-art technology to source bulk DC supply while minimising AC-side harmonic distortion.

We present in this thesis algorithms for loss minimisation and overload prevention in managed DC networks, new models and switching schemes for switch-mode AC–DC converters, and novel approaches to the dynamics and control of DC power systems. The research programme set out in this thesis aims to obtain the highest possible performance from the proposed managed DC system in all its electrical aspects.
## Contents

Declaration iii  

Acknowledgements v  

Abstract vii  

### I Introduction 1

1 Background and Motivation 3  
1.1 Origins: DC and AC 5  
1.1.1 Battle of the Systems 6  
1.1.2 AC Consensus 7  
1.1.3 The Electronic Revolution 8  
1.2 A New Case for DC 9  
1.2.1 Cost Savings 10  
1.2.2 Power Factor 15  
1.2.3 Variable-Speed Drives 17  
1.2.4 Harmonic Mitigation 20  
1.2.5 Alternative Energy Sources 23  
1.2.6 Electrical Appliances 25  
1.2.7 Safety 27  
1.3 The Problems with DC 30  
1.3.1 The Switching Problem 30  
1.3.2 Load Incompatibility 32  
1.3.3 Lack of Transformability 33  
1.3.4 Corrosion of Cables 33  
1.4 Management of Electricity 34
1.4.1 Plain Old Electricity Supply ........................................ 34
1.4.2 Intelligent Buildings .............................................. 35
1.4.3 Energy Management Systems ...................................... 36
1.4.4 ‘Soft’ Overload Prevention ....................................... 39
1.4.5 SCADA and Industrial Automation .............................. 39
1.4.6 Power Quality .................................................... 40

1.5 Applications: The Managed DC Bus ............................ 41
1.5.1 Initial Design ..................................................... 42
1.5.2 Discretionary Current ........................................... 43
1.5.3 Other Features ................................................... 45

2 The Problems .......................................................... 47
2.1 Static Problems ....................................................... 47
2.2 Dynamic Problems .................................................... 49
2.3 Structure of the Thesis ............................................... 50
2.4 Contributions of the Thesis ......................................... 52

II Statics ......................................................................... 55

3 Managed Distribution Networks: Mathematical Formulation .... 57
3.1 Circuit Topology ........................................................ 57
3.2 Managed Networks: Demand and Discretion .................... 59
3.3 The Discretionary Map ............................................... 61
  3.3.1 Definitions ......................................................... 61
  3.3.2 Geometry of the discretionary map ............................ 62
  3.3.3 Affine Discretionary Maps ...................................... 62
3.4 The Sensitivity Matrix: Two Formulations ....................... 64
  3.4.1 Incidence Matrix Formulation ................................ 65
  3.4.2 Loop Matrix Formulation ...................................... 67
  3.4.3 Relationships Between the Two Formulations ............... 70
3.5 The Voltage-Drop Matrix ............................................. 72
3.6 The Demand Sensitivity Matrix ..................................... 75
  3.6.1 Affine Discretionary Maps ...................................... 76
  3.6.2 General Discretionary Maps .................................... 77
4 Loss Minimisation by Discretionary Control 87
4.1 Loss Calculations and Problem Statement ................................. 88
4.2 Solving the Loss Minimisation Problem in a DC Network ............... 90
  4.2.1 Restatement in Standard Form ........................................... 90
  4.2.2 Kuhn-Tucker Conditions and Active Constraints .................... 92
  4.2.3 Incorporating Non-Discretionary Nodes ................................. 93
4.3 Solution by Least Squares Methods .......................................... 95
4.4 Examples ................................................................. 96
  4.4.1 Two-Node Network ....................................................... 96
  4.4.2 Three-Node Network ..................................................... 98

5 Loss Minimisation by Network Reconfiguration 103
5.1 The Network Reconfiguration Problem ....................................... 103
5.2 Elementary Tree Transformations ............................................. 105
5.3 Application to Loss Minimisation ............................................. 107
5.4 Transformation of Sensitivity Matrix ....................................... 108
5.5 An Algorithm for Topological Loss Optimisation .......................... 110
5.6 An Example ............................................................... 112

6 Overload Prevention 115
6.1 Choice of Cables .......................................................... 116
6.2 Operational Measures ....................................................... 117
  6.2.1 Linear Discretionary Map ............................................... 118
  6.2.2 General Discretionary Map .............................................. 119
  6.2.3 Minimal Loss Discretionary Map ....................................... 119
6.3 Example: Overload Prevention with a Linear Discretionary Map ...... 120

III Dynamics 125

7 Models Large and Small 127
7.1 Introduction ................................................................................. 127
7.2 Extension of Topological Analysis: the ‘Large’ Model .............. 129
  7.2.1 Cable Network Dynamics ......................................................... 129
  7.2.2 External Circuit Modelling ....................................................... 130
  7.2.3 A Complete System Model ..................................................... 131
  7.2.4 Control Action ...................................................................... 133
7.3 The ‘Small’ Model ..................................................................... 133
  7.3.1 Analysis in LC Units .............................................................. 134
  7.3.2 Small Model with Source Resistance ................................... 135
7.4 Generalised Synchronous Switching Model ............................ 136
  7.4.1 Topology and Operating States ................................................ 137
  7.4.2 Discrete-Time Circuit Model ................................................... 138
  7.4.3 Practical Constraints .............................................................. 139
7.5 PWM Switching Model ............................................................. 141
  7.5.1 System Model for Voltage-Driven Converter ....................... 142
  7.5.2 Current-Driven PWM Converters ........................................... 146
7.6 Generalised Asynchronous Switching Model ........................... 149
  7.6.1 Generalised Space Vector Formalism ...................................... 149
  7.6.2 A Generalised State-Space Model ......................................... 151
  7.6.3 Control Design Issues ............................................................ 163

8 Advanced Switching Strategies for PWM Converters ................... 167
  8.1 Realisation of (Almost) Arbitrary Switching Sequences .......... 167
  8.2 Voltage-Driven Converters Revisited ........................................ 170
  8.3 An Enhanced Current-Driven Converter .................................. 172
    8.3.1 Sixfold CDC Switching Strategy .......................................... 174
    8.3.2 VDC-Like Switching Strategy ................................................. 178
    8.3.3 Twelvefold CDC Switching Strategy ...................................... 181
  8.4 Simulation Results ................................................................. 185
    8.4.1 Sixfold CDC Scheme ............................................................... 185
    8.4.2 Twelvefold CDC Scheme ....................................................... 186

9 AC-DC Converters in the Steady State ........................................ 195
  9.1 The Small Model with Thyristor Converter .............................. 195
9.1.1 Analysis of DC Ripple ........................................ 196
9.1.2 Attenuation of DC Ripple ................................... 198
9.2 Voltage-Driven PWM Converters: the Boost Characteristic .... 200
  9.2.1 No-Load Input Current ..................................... 202
  9.2.2 Steady-State Characteristic with Ohmic Load ............. 203
  9.2.3 Regeneration, Voltage Reversal and Other Load Types ... 206
9.3 Current-Driven PWM Converters: the Buck Characteristic .... 208
  9.3.1 Equilibrium Conditions for the Sixfold CDC ............. 208
  9.3.2 Equilibrium Characteristic with a Constant-Current Load .. 209
  9.3.3 The Buck Characteristic .................................... 212
  9.3.4 Relation to Synchronising Control ....................... 215

10 Synchronising Control of Switch-Mode AC-DC Converters ...... 217
  10.1 Control of Voltage-Driven Converters ...................... 218
    10.1.1 Problem Formulation .................................... 218
    10.1.2 Evolution of Models and Control ....................... 220
    10.1.3 Optimal Control Approach ............................. 223
    10.1.4 Energy-Based Approach .................................. 227
    10.1.5 Simulation Results ..................................... 235
  10.2 Control of Current-Driven Converters ...................... 239
    10.2.1 Problem Formulation ..................................... 241
    10.2.2 Quasilinear State Feedback Design ..................... 243
    10.2.3 A Two-Gain Controller .................................. 247
    10.2.4 Simulation Results ..................................... 252

11 Regulatory Control of Converter-Fed DC Networks .............. 255
  11.1 Problem Formulation ........................................ 256
    11.1.1 The General Problem ..................................... 256
    11.1.2 Simplifications and Special Cases ....................... 258
  11.2 Bifurcation Phenomena in LC Circuits with Nonlinear Loads ... 261
    11.2.1 Normal Form of the LC Circuit Equations ............. 261
    11.2.2 Bifurcations ........................................... 263
    11.2.3 Analysis of the Hopf Bifurcation ....................... 265
    11.2.4 Example: the Constant-Power Load ..................... 269
11.3 The Regulatory Control Problem ........................................ 281
11.3.1 Preliminaries ..................................................... 281
11.3.2 Problem Formulation ............................................. 282
11.4 LTIBU Regulator Problems ........................................... 284
11.4.1 Approach I: Linear-Quadratic (LQ) Control .................. 285
11.4.2 Approach II: Saturated Deadbeat (SD) Control ............ 286
11.4.3 Approach III: $H_\infty$ Control ................................. 293
11.5 Control Design for the Small Model ............................... 295
11.5.1 A Hybrid Solution ................................................. 295
11.5.2 Estimation of Damping Coefficients .......................... 296
11.5.3 Robustness Issues ................................................. 298
11.6 Control of Multiple Converters .................................... 300
11.7 Simulation Results ..................................................... 302

IV Conclusion 315

12 Conclusions and Recommendations 317
12.1 Concluding Remarks ................................................ 317
12.2 Directions for Further Work ....................................... 320

V Appendices 323

A Graph Theory and Electrical Networks 325
A.1 Graph Definitions .................................................. 325
A.2 Graph Algebra ...................................................... 326
A.3 Electrical Networks ................................................ 328

B A Geometric Approach to Loss Minimisation and Other Minimum-
Distance Problems 331
B.1 Preliminaries ........................................................ 331
B.2 Outward Projections for an Arbitrary Convex Polytope ....... 334
B.3 Computing the constraint directions $\phi^{(k)}$ .................. 347
B.4 Elements of an Algorithm ......................................... 350

C The Number of Spanning Trees in a Graph 353
E Polyphase Signals: Definition and Properties 363
   E.1 Definitions .................................................. 363
   E.2 Elementary Properties ..................................... 364
   E.3 The Park Transform for Three-Phase Signals .......... 365
      E.3.1 Stationary Park Transform .......................... 365
      E.3.2 Synchronous Park Transform ....................... 367

F The Asynchronous Thyristor Switching Model 369
   F.1 Equivalent Circuits ....................................... 370
   F.2 Dynamics in the Single Conduction Mode .............. 372
   F.3 Dynamics in the Commutation Mode ..................... 374
   F.4 An Asynchronous Discrete-Time Model ................. 377

Bibliography 379
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>A variable-speed AC drive system</td>
<td>18</td>
</tr>
<tr>
<td>1.2</td>
<td>Multiple VSDs supplied from a common DC bus</td>
<td>19</td>
</tr>
<tr>
<td>1.3</td>
<td>Electronic appliance power supply input stage</td>
<td>26</td>
</tr>
<tr>
<td>1.4</td>
<td>A bank of computers supplied by a UPS</td>
<td>28</td>
</tr>
<tr>
<td>1.5</td>
<td>Computer bank with uninterruptible DC bus supply</td>
<td>28</td>
</tr>
<tr>
<td>3.1</td>
<td>A general distribution network</td>
<td>58</td>
</tr>
<tr>
<td>3.2</td>
<td>Geometry of the discretionary map (d(r))</td>
<td>63</td>
</tr>
<tr>
<td>4.1</td>
<td>Discretionary loss minimisation for two-node network</td>
<td>97</td>
</tr>
<tr>
<td>4.2</td>
<td>A three-node distribution network</td>
<td>99</td>
</tr>
<tr>
<td>4.3</td>
<td>Minimum distance to a polygon</td>
<td>100</td>
</tr>
<tr>
<td>5.1</td>
<td>An elementary tree transformation</td>
<td>108</td>
</tr>
<tr>
<td>6.1</td>
<td>A single-ring cable network</td>
<td>120</td>
</tr>
<tr>
<td>7.1</td>
<td>Power circuit for Small Model</td>
<td>133</td>
</tr>
<tr>
<td>7.2</td>
<td>A two-port LC network</td>
<td>134</td>
</tr>
<tr>
<td>7.3</td>
<td>PWM voltage-driven AC-DC converter</td>
<td>142</td>
</tr>
<tr>
<td>7.4</td>
<td>A symmetrical PWM waveform</td>
<td>144</td>
</tr>
<tr>
<td>7.5</td>
<td>Space vectors for voltage-driven PWM converter</td>
<td>146</td>
</tr>
<tr>
<td>7.6</td>
<td>Current-driven PWM converter</td>
<td>147</td>
</tr>
<tr>
<td>7.7</td>
<td>Generalised space vectors for three-phase converter</td>
<td>150</td>
</tr>
<tr>
<td>7.8</td>
<td>Generic converter circuit for GAS model</td>
<td>152</td>
</tr>
<tr>
<td>7.9</td>
<td>Construction of coupling network</td>
<td>154</td>
</tr>
<tr>
<td>8.1</td>
<td>A versatile pulsewidth modulator (one per device)</td>
<td>169</td>
</tr>
<tr>
<td>8.2</td>
<td>Space vectors for conventional CDC (Park interpretation)</td>
<td>174</td>
</tr>
</tbody>
</table>
8.3 Calculation of DC offset for CDC switching strategy .......... 177
8.4 Phase 0 current and capacitor voltage for conventional CDC ...... 186
8.5 Power spectrum of phase 0 current for conventional CDC .......... 187
8.6 DC bus voltage and link current for conventional CDC .......... 187
8.7 AC and DC currents for GAS model simulation ................. 188
8.8 AC and DC currents for GAS model simulation ................. 188
8.9 AC capacitor voltages for GAS model simulation ............... 189
8.10 AC capacitor voltages for GAS model simulation .............. 189
8.11 DC output voltage for GAS model simulation ............... 190
8.12 DC output voltage for GAS model simulation .......... 190
8.13 Phase 0 current and capacitor voltage for hybrid CDC ........ 192
8.14 Frequency spectrum of phase 0 current for hybrid CDC ........ 192
8.15 DC bus voltage for hybrid CDC .......... 193
8.16 DC link current for hybrid CDC .......... 193

9.1 The Boost Characteristic ........................................... 206
9.2 Contour plot of $\beta(D, \psi)$ with $\Lambda = 10$. (Contour spacing = 0.04.) ... 213
9.3 Contour plot of $\eta(D, \psi)$ with $\Lambda = 10$. (Contour spacing = 0.01.) ... 214

10.1 Phase portrait of (10.24) with $K_P/C = K_I/C = 1$ ............... 230
10.2 Phase portrait of (10.24) with $K_P = 0, K_I/C = 1$ ............... 231
10.3 DC voltage response for VDC synchronising controllers ........ 236
10.4 AC current response for VDC synchronising controllers .......... 237
10.5 Response of $V^*$ for passivity-based control .................. 237
10.6 DC voltage response for PCFF with regenerative load .......... 238
10.7 AC current response for PCFF with regenerative load .......... 238
10.8 DC voltage response for PCFF with varying load types .......... 239
10.9 AC current response for PCFF with varying load types .......... 240
10.10 Root loci for polynomial (10.46), $\mu \leq \mu^*$ ............... 248
10.11 Root loci for polynomial (10.46), $\mu \geq \mu^*$ ............... 249
10.12 Voltage transient response of two-gain CDC synchronising control ... 252

11.1 'Real' constant-power characteristic $F(V, P)$ ............... 271
11.2 Limit cycle behaviour in systems with 'real' constant-power loads ... 274
11.3 Phase portrait with resistive and subcritical constant-power load ... 275

xviii
11.4 Phase portrait with resistive and supercritical constant-power load . . 275
11.5 Phase portrait with resistive and ideal constant-power load . . . . . 276
11.6 Graphical determination of equilibrium when $R > 0$ . . . . . . . . 277
11.7 Root locus of Small Model with SD control, $\zeta_0 = 0.5$ . . . . . . . 299
11.8 Root locus of Small Model with SD control, $\zeta_0 = 0$ . . . . . . . 299
11.9 Root locus of Small Model with SD control, $\zeta_0 = -0.5$ . . . . . 300
11.10 Deadbeat voltage response with $\zeta = 0$, $z_0 = 2$ . . . . . . . 304
11.11 Deadbeat current response with $\zeta = 0$, $z_0 = 2$ . . . . . . . 304
11.12 Voltage response with 100A constant-current load, $z_0 = 2$ . . . . 305
11.13 Current response with 100A constant-current load, $z_0 = 2$ . . . . 305
11.14 Load estimation with 100A constant-current load . . . . . . . . . 306
11.15 Voltage response with 100A ohmic load, $z_0 = 2$ . . . . . . . . . 307
11.16 Current response with 100A ohmic load, $z_0 = 2$ . . . . . . . . . 307
11.17 Load estimation with 100A ohmic load . . . . . . . . . . . . . . . 308
11.18 Voltage response with 100A constant-power load, $z_0 = 0.2$ . . . 309
11.19 Current response with 100A constant-power load, $z_0 = 0.2$ . . . 309
11.20 Load estimation with 100A constant-power load . . . . . . . . . 310
11.21 Estimation error for Figure 11.20 . . . . . . . . . . . . . . . . . . . 310
11.22 Voltage response with ohmic and switched constant-power load . . 311
11.23 Current response with ohmic and switched constant-power load . . . 312
11.24 Load estimation with ohmic and switched constant-power load . . . 312
11.25 Estimation error for Figure 11.24 . . . . . . . . . . . . . . . . . . . 313

C.1 Graph of nullity $\nu = 2$ with two overlapping loops . . . . . . . . 355
C.2 Topologically distinct graphs with nullity $\nu = 3$ . . . . . . . . . 356
C.3 Loop graphs with nullity $\nu = 3$ . . . . . . . . . . . . . . . . . . . 357

F.1 ‘Simple’ bridge configuration . . . . . . . . . . . . . . . . . . . . . 370
F.2 Equivalent circuit for single conduction . . . . . . . . . . . . . . . 371
F.3 Equivalent circuit for commutation . . . . . . . . . . . . . . . . . . 371
List of Tables

1.1 Optimal design variables and cost by installation type ........... 14
1.2 $\alpha$ coefficient by skin effect and power factor ............... 14
5.1 The 33-bus test system ............................................. 113
7.1 Definition of LC units .............................................. 135
7.2 Equivalence classes of switch states .............................. 159
7.3 Coupling coefficients for current-driven GAS model ............ 160
7.4 Coupling coefficients for voltage-driven GAS model ............ 161
8.1 Mnemonic coding for modulator command $M$ .................... 169
D.1 The DHO family of functions ...................................... 360
Part I

Introduction
Chapter 1

Background and Motivation

A silent revolution is underway in electric power engineering. It is part of the wider revolution in electrical engineering wrought by the rise of electronic technology, but its effects on the power discipline are less visible than in other areas such as telecommunications and computer systems. Even as computers, mobile phones and CD players replace older technologies in our homes and workplaces, we continue to take for granted the electricity that supplies them.

It is somewhat ironic that virtually all our hi-tech equipment, most of it developed in the last two or three decades by engineers working in entirely new disciplines, is powered by technology that in its fundamentals has changed little in the past hundred years. The theory and practice that underlies alternating current, three-phase transmission, busbars, fuses, circuit breakers and the other components of conventional power reticulation systems, was laid down by engineers such as Edison and Tesla in the latter half of the nineteenth century and has barely changed since; meanwhile, the ways we use electricity have altered beyond recognition.

Of course, there is nothing necessarily wrong with this state of affairs. Generally speaking, it makes good sense not to alter existing, well-tested practice unless there are clear problems in need of fixing, or significant advantages to be had from change. It is our opinion nonetheless that electric power engineering, as it applies to low-voltage power reticulation in buildings and industrial sites, is ripe for a ‘paradigm shift’ that would bring it properly into the Electronic Age.

We offer the following reasons to support this contention:

1. Despite the common belief that all is well with electrical technology, significant problems are emerging as a direct consequence of the revolution in electrical
applications. Most significant among these is harmonic distortion in supply mains, owing to the high proportion of nonlinear loads on today’s power systems. This problem is inherent in conventional power systems, and solutions within the conventional paradigm do not come easily or cheaply.

2. The conventional approach to power reticulation, like all good engineering standards, is based on established good practice: it became the standard because there were sound reasons for preferring it over the alternatives. However, the standard is now a hundred years old and the underlying reasons for it are showing their age, as is explained below. At no time since World War II has the electrical engineering community seriously reassessed the standard approach in the light of changing circumstances.

3. Alternative approaches to power reticulation offer advantages over the conventional approach, whether in cost to the customer, cost to the utility, safety, or energy efficiency. Electric utilities and consumers alike are being disadvantaged as a result of entrenched attitudes.

Given the specific developments in technology driving the need for new ways of thinking, we believe there are two elements particularly likely to figure in the reticulation systems of the 21st century.

The first element is the increased use of direct current. There is now an unstoppable trend toward use of direct current where power is utilised, in computers, electronics and motor drive systems. Even in high-voltage transmission systems DC has been used and recognised for many years, though AC retains its role in voltage stepping and system protection. At the low-voltage reticulation level, much can be gained simply by improving the match between the type of current supplied and demanded.

The second element is the sophisticated management and control of electricity supply and demand. Computers and microprocessor systems, with their superlative flexibility and scalability, have proved their worth in countless applications. In something as basic and essential as electricity reticulation, software has the potential to radically improve operating efficiency, power quality and other objectives.

In the following sections we elaborate on these points in more detail. Sections 1.1, 1.2 and 1.3 respectively explain the historical context for DC power systems,
the advantages of DC and the drawbacks of DC. Section 1.4 briefly sets out the state of the art in electricity management technologies, as a guide to the managed electricity networks of the future. Finally, Section 1.5 describes the Managed DC Bus, a prototype reticulation system based on these principles. This provides a concrete motivation for the theoretical investigations that comprise the thesis proper.

1.1 Origins: DC and AC

Direct current, or DC, is as old as the notion of electric current itself. The first observations of electric current—the controlled flow of electrons in a conductor, carrying energy from one point to another in a circuit—were of direct current obtained from electrochemical cells or ‘voltaic piles’. For the first three decades after Volta’s discovery of electric current in 1800, DC batteries remained the only source of electricity.

The next major development came with Faraday’s discovery of electromagnetic induction in 1830, which paved the way for mechanical generation of electricity. It did not take the experimenters long to discover that a rotating electrical machine, in the absence of a commutator, produced a current that was approximately sinusoidal. (Imperfections in the design caused—and still causes—the current to deviate from a true sine wave.) This new type of current, reversing in sign with every half-turn of the machine shaft, was dubbed alternating current, or AC. It was recognised that, although the current by itself averaged to zero, positive power flow was still possible due to the alternating voltage. From that point on, AC took its place alongside DC as a potential means for transmitting electrical energy.

DC nonetheless had the head start. By 1870 Gramme had devised a machine capable of generating a steady, reliable DC current from a mechanical input [2], and the following decade saw the emergence of the first public arc-lighting systems driven by electricity. Most of these were DC systems, which could take advantage of battery storage and the possibility of connecting machines in parallel to build up larger systems. AC machines, meanwhile, were plagued by high running costs and imperfections in the voltage waveforms, which created doubts about parallel operation.
1.1.1 Battle of the Systems

Thomas Edison, by all accounts, was highly influential in winning public acceptance of electricity for lighting and other applications from the 1880s onward. Though his invention of the incandescent lamp in 1879 was in all likelihood anticipated by others [9, pp.113–4], Edison’s real achievement was inventing a complete supply system to complement his lamps, including central generators, cables, fittings, switches, fuses, meters and 110V DC distribution. The Edison System was designed to match in as many respects as possible the well-established system for gas lighting, in order to better penetrate the market. Its debut in 1882 made it very nearly the first electrical undertaking to provide domestic supply to the general public.

Though Edison deserves the credit for much of the low-voltage reticulation technology on which we still rely over a hundred years later, the pioneer of high-voltage AC transmission and distribution as we know it was Sebastian Ziani de Ferranti. In 1885 Ferranti, the Chief Engineer of the Grosvenor Gallery Company in London, unveiled a scheme to supply power to central London from a huge power station on the Thames in Deptford. AC power would be transmitted through underground cables at an unprecedented 10kV, to be stepped down to 2400V by transformers at the London end. Like Edison, Ferranti was following the lead of the gas companies, which had recently centralised their networks to improve operating efficiency. Though the Deptford scheme was overly ambitious for its time, being plagued with difficulties from the outset, it must nonetheless be regarded as the forerunner of today’s centralised AC power systems.

Edison and Ferranti represented two rival paradigms in the fledgling electricity industry. The Edison System, along with many others built on similar principles, was characterised by small-scale generation and low-voltage DC distribution; Ferranti’s system on the other hand relied on large-scale centralised generation, high-voltage AC transmission and transformers to step down voltage at the load. The former systems had the advantage of battery storage and were perceived as being both simpler and safer; the latter on the other hand could claim higher efficiency and economies of scale.

Supporters of the rival approaches soon crystallised into warring factions, as the Battle of the Systems gathered force. Any city in Europe or America seeking to establish a public electricity supply would be courted by companies representing both
sides. The ‘battle for the city of Frankfurt’ was typical: while Ganz and Co. were persuading the city to follow the example of nearby Cologne and adopt AC, Siemens was putting the case for DC.

Though the battle can be portrayed as simply AC versus DC, issues covering the whole approach to power system design were being decided: large-scale versus localised generation, high versus low transmission voltages, and so on. Perhaps surprisingly, the key argument now used to support AC over DC in low-voltage reticulation—namely, the relative ease of interrupting AC current—appears not to have played any significant role in this debate.

1.1.2 AC Consensus

The development which was to prove the key deciding factor in the Battle of the Systems was the invention of the polyphase AC induction motor and synchronous machine by Nikola Tesla in 1888. To this day they remain the most efficient, cost-effective, reliable and versatile devices for the mechanical generation and utilisation of electricity.

The machines were rapidly commercialised by George Westinghouse as part of his centralised AC supply systems, and industrial applications for the induction motor rapidly emerged. After experimenting with two supply phases at a 90-degree separation, Westinghouse settled on a three-phase system at 120 degrees phase separation, which had the advantage of constant total power flow under balanced conditions. Within twenty years, three-phase AC had become the preferred option for electricity generation and transmission in most places throughout the world.

The following decades saw the rise of industry-wide standards governing electric power systems. Domestic voltages were standardised to between 110V and 240V for single-phase AC circuits, and frequencies standardised to 50Hz in Europe and 60Hz in America. Despite the trend towards making AC standard throughout the industry, DC systems persisted in many cities and towns (including Melbourne’s CBD) through the first half of the twentieth century. As three-phase AC became the norm for generation and transmission, these DC systems came to be supplied from rotary converter units in substations, then from mercury-arc rectifiers. Almost all were eventually phased out in favour of AC.

Two specific applications in which DC continued to dominate were the electro-
chemical industry and electric traction systems. But in the area of general purpose supply, there was and is a perception that all the fundamental problems of design had been solved. The focus of electric power engineering shifted from the pioneering of new approaches to electricity supply, to the gradual improvement of existing approaches: larger and more efficient generators, higher transmission voltages, increasing system scale, better circuit breakers, faster fuses and more accurate metering. The locus of innovation in electrical engineering had shifted, from the delivery of electricity to its applications.

1.1.3 The Electronic Revolution

The development of semiconductor devices and electronics in the postwar era transformed the entire field of electrical engineering. Prior to World War II, most electrical plant could be classified as either resistive elements or motors, all amenable to description via linear circuit theory. The transistor changed all this. Nowadays, a large proportion of appliances are primarily electronic in nature—particularly if one counts equipment that makes use of so-called ‘power electronics’.

Power electronics sits at the interface between power engineering and the newer disciplines of electronic system design and digital control. Power electronic technology now makes it possible to convert single-phase or polyphase AC to DC, to convert DC to lower or higher voltages, to synthesise special-purpose power waveforms such as square waves, and to convert DC to AC at any desired voltage and frequency. In each case the electronic devices allow rapid and fine control over the output voltage (and frequency, where the output is AC).

However, in sophisticated power electronics it is generally more convenient to work with a DC input than with an AC input. Variable-speed drives for AC machines, for example, typically use two conversion stages: the AC input is converted to DC, then the DC is fed to a variable-voltage variable-frequency (VVVF) inverter. Were such an appliance to be provided with a DC input directly, the design could be much simplified and the cost reduced. Ironically, although the desired output is AC, an AC input acts as a handicap. When placed on an AC supply, such equipment is the main contributor to one of the power industry’s greatest contemporary problems: harmonic distortion.
1.2 A New Case for DC

The above discussion establishes an historical precedent for the use of DC in general purpose power reticulation, but also draws attention to the reasons why DC was superseded by AC. AC was deemed superior primarily for two reasons:

1. the superior reliability, efficiency and cost-effectiveness of AC machines; and

2. the ability to transform AC to high voltages for efficient long-distance transmission, and back down to low voltages for reticulation.

The rise of power electronics has rendered both these reasons obsolete—not because they are any less true than in the past, but because their truth has declined and will continue to decline in relevance.

In the past, it was taken for granted that converting between AC and DC was inefficient, unreliable and expensive, and therefore to be avoided. So, rather than use DC in those parts of the system where it proved advantageous and AC elsewhere, it was preferable to use AC or DC throughout, and AC won out on this basis.

Nothing of the sort can be said today. We live in the Age of Electronics, and in electrical technology, electronics has been applied primarily to the task of AC–DC, DC–AC and DC–DC conversion. So, while the three-phase squirrel-cage AC induction machine is still considered the best general-purpose motor for medium-to-high-power applications, engineers have seen the advantages of variable-speed drives, the principal component of which is an inverter driven from a DC input. The AC–DC front end required for contemporary variable-speed drives is a mere adjunct, the sole purpose of which is to connect the inverter with an AC main.

Large-scale power systems will probably always be largely AC-based, because of the superlative efficiency of transformers. Power electronics has, however, greatly increased the flexibility engineers have to convert between AC and DC, to the extent where it is now economic to transmit electricity over long distances on high-voltage DC (HVDC) lines, even though this requires costly conversion equipment at each end of the line. Similarly, it may be economic for a building to accept an AC mains input but reticulate DC internally, if there are compelling reasons for it to do so. Technology has advanced to the point where we may recognise that the form of electricity most appropriate for one part of a power system may not be the most appropriate for other parts.
The type of system here advocated is a DC reticulation system on the scale of a commercial building, industrial site or residential subdivision, that accepts a conventional low-voltage three-phase AC feed (typically 415V in Australia), converted to DC using a small number of AC–DC bridge converters, and possibly supplemented with other generation sources. The following pages provide arguments in support of this approach.

1.2.1 Cost Savings

Cost is the paramount consideration when designing an electrical installation. Two kinds of cost must be accounted for: the upfront or capital cost of installing the system, and the ongoing cost of running the system. To allow these costs to be compared, it is standard practice to annualise the capital costs, converting the upfront figure into an annual figure which also takes into account financing, depreciation, maintenance and other such costs. Henceforth we shall assume that all capital costs are annualised.

To minimise system costs, the engineer has a degree of freedom in choosing the values of certain design variables, of which the most important for our purposes are:

1. the cable size, expressed as the cross-sectional area $A$ of a single cable;

2. the RMS voltage level $V_{\text{rms}}$ (taken as phase-to-neutral for a polyphase installation); and

3. the installation type: single-phase AC, three-phase AC, or DC.

The values chosen for these variables must be consistent with the design requirements, in particular with:

1. the real power rating $P$ for the installation; and

2. the net AC power factor $pf$ of the load, if an AC installation is being considered.

We model the various capital and operating costs of an electrical installation by splitting the costs into three components as follows:

**Copper cost** This is the annualised upfront cost of cable material, and has the form

$$C_1 = K_1 A \quad (\text{$/\text{cable-m / year}$}) \quad (1.1)$$
where \( K_1 \) is a constant that takes into account the cost of material, the rate of depreciation and other relevant factors.

**Voltage-related cost** This term models costs associated with the level of voltage, including the cost of insulation material, operational risk costs, and maintenance. It has the form

\[
C_2 = K_2 V_p \quad (\text{\$} / \text{cable-m / year}) \tag{1.2}
\]

where \( V_p \) is the peak instantaneous voltage to earth.

**Cost of losses** This accounts for the cost of power lost in heating the wires, and is equal to

\[
C_3 = K_3 \frac{\sigma I_{\text{rms}}^2}{A} \quad (\text{\$} / \text{cable-m / year}) \tag{1.3}
\]

where \( I_{\text{rms}} \) is the RMS current averaged over a year of operation, and the factor \( \sigma \) accounts for skin effect.

These cost expressions depend on the introduced quantities \( V_p \) and \( I_{\text{rms}} \), which must be related back to the fundamental design variables. For a given type of installation, there is a dimensionless factor \( \gamma \) relating the peak and RMS voltages:

\[
V_p = \gamma V_{\text{rms}}. \tag{1.4}
\]

(For example, in a single-phase AC system we have \( \gamma = \sqrt{2} \).) The RMS current and voltage are related via the power rating \( P \):

\[
P = \eta V_{\text{rms}} I_{\text{rms}} \tag{1.5}
\]

where \( \eta \) is another dimensionless constant related to the installation type. For later convenience, we define

\[
\kappa = \left( \frac{\sigma}{\eta^2} \right)^{\frac{2}{3}}. \tag{1.6}
\]

Summing these components and multiplying by the number of cables per circuit \( n_C \) (which may vary with the type of installation), we obtain an expression for the annual cost of an electrical installation, per metre of installed power circuit:

\[
C_T = n_C \left( K_1 A + \gamma K_2 V_{\text{rms}} + \kappa^4 K_3 \frac{P^2}{A V_{\text{rms}}^2} \right). \tag{1.7}
\]
(Although we could have multiplied this expression by the total length of installed circuits to obtain the total annual cost, this only complicates matters unnecessarily since we take length as a fixed requirement rather than a design variable.)

The above expression is a simplification, in that we are considering the current and voltage to be uniform throughout the installation. A more thorough analysis would consider each individual circuit, choosing an optimal cable size for each while optimising the voltage across the entire installation. It would also take account of voltage drop if cable length were significant. Our analysis, however, is applicable to single circuits whose cable routes are not unduly long, and to multiple circuits if the load is evenly balanced between them. In the latter case \( P \) is the real power rating of a single circuit rather than of the entire installation.

Given an installation type, we may optimise the cost (1.7) with respect to the cable size \( A \) and voltage \( V_{\text{rms}} \). We then compare the optimal costs to determine the overall cheapest installation, assuming that manufacturing and labour costs are the same across the various installation types.

Differentiating with respect to \( A \) we obtain

\[
\frac{\partial C_T}{\partial A} = n_C \left( K_1 - \kappa^4 K_3 \frac{P^2}{A^2 V_{\text{rms}}^2} \right)
\]

which vanishes when

\[
AV_{\text{rms}} = \kappa^2 P \left( \frac{K_3}{K_1} \right)^{\frac{1}{2}}. \tag{1.8}
\]

Differentiating with respect to \( V_{\text{rms}} \) we obtain

\[
\frac{\partial C_T}{\partial V_{\text{rms}}} = n_C \left( \gamma K_2 - 2 \kappa^4 K_3 \frac{P^2}{AV_{\text{rms}}^3} \right)
\]

which vanishes when

\[
AV_{\text{rms}}^3 = \frac{2\kappa^4 P^2 K_3}{\gamma K_2}. \tag{1.9}
\]

Evaluating the second derivatives, one finds after a routine calculation that \( \frac{\partial^2 C_T}{\partial A^2} \) and \( \frac{\partial^2 C_T}{\partial V_{\text{rms}}^2} \) are both positive, while the Hessian determinant is

\[
|H| = 8 \left( \frac{\kappa^4 K_3 P^2}{A^2 V_{\text{rms}}^3} \right)^2 > 0
\]

indicating that the solution identified by (1.8) and (1.9) is a global minimum, as required. (This is also apparent intuitively, based on the uniqueness of the extremum and the asymptotic behaviour of \( C_T \).)
Dividing (1.9) by (1.8), and taking the square root, we obtain the optimal voltage as

\[ V_{\text{rms}}^* = \kappa \sqrt{\frac{2P}{\gamma} \left( \frac{K_1K_3}{K_2^2} \right) ^{\frac{1}{2}}}. \]  

(1.10)

Dividing this into (1.8) we obtain the optimal cable size as

\[ A^* = \kappa \sqrt{\frac{\gamma P}{2} \left( \frac{K_2^2K_3}{K_1^3} \right) ^{\frac{1}{2}}}. \]  

(1.11)

Finally, substituting for \( A \) and \( V_{\text{rms}} \) in (1.7) we obtain

\[ C_T^* = n_C \kappa \sqrt{8\gamma P(K_1K_2^2K_3)^{1/4}} \]  

(1.12)
of which \( C_2 \) contributes half and \( C_1 \) and \( C_3 \) a quarter each, regardless of the relative magnitudes of \( K_1, K_2 \) and \( K_3 \).

The absolute values of \( V_{\text{rms}}^* \), \( A^* \) and \( C_T^* \) certainly will depend on the power rating and on the cost parameters \( K_i \), as well as on the factors \( \gamma \) and \( \kappa \). However, if we fix all parameters other than the installation type, then the ratios between the optimal values will depend only on the specific values of \( \gamma \), \( \kappa \) and \( n_C \) pertaining to each type of installation. We may therefore compare the relative optimal cable size, voltage and cost for various types of installation, without regard for the size of the installation or the cost parameters (provided the model assumptions are met).

Table 1.1 compares the optimal cable size, voltage and cost for three types of installation: single-phase AC, three-phase AC and DC. For clarity of presentation we have defined the dimensionless quantity

\[ \alpha = \frac{\sigma_{\Omega}^{1/4}}{\sqrt{pf}} \]  

(1.13)

where \( \sigma_{\Omega} > 1 \) is the change in resistivity due to skin effect at AC frequency \( \Omega \), and \( pf \) is the AC power factor of the load. The quantity \( \alpha \) obtains its minimum value 1 at unity power factor with no skin effect, and becomes greater than 1 as power factor decreases or skin effect increases. Table 1.2 tabulates \( \alpha \) for typical values of skin effect and power factor.

As an example of the cost savings attainable, consider a conventional 415V three-phase AC installation with a load power factor of 0.8 and a 5% rise in cable resistance from its DC value due to skin effect. We shall assume that 415V line-to-line, equivalently 240V line-to-neutral, is in fact the optimal voltage level for this installation. The \( \alpha \) value for this installation is 1.132.
<table>
<thead>
<tr>
<th></th>
<th>AC, 1ϕ</th>
<th>AC, 3ϕ</th>
<th>DC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cables per circuit, $n_C$</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Peak voltage factor, $\gamma$</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
<td>$1/2^*$</td>
</tr>
<tr>
<td>Power constant, $\eta$</td>
<td>$pf$</td>
<td>$3pf$</td>
<td>1</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$\alpha^\dagger$</td>
<td>$\alpha/\sqrt{3}$</td>
<td>1</td>
</tr>
<tr>
<td>Optimal cable size $A^\ddagger$</td>
<td>$1.682\alpha^\ddagger$</td>
<td>$0.971\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>Optimal RMS voltage $V_{rms}$</td>
<td>$0.595\alpha$</td>
<td>$0.343\alpha^\ddagger$</td>
<td>1</td>
</tr>
<tr>
<td>Minimum annual cost $C_T$</td>
<td>$1.682\alpha$</td>
<td>$1.456\alpha$</td>
<td>1</td>
</tr>
</tbody>
</table>

$^*$ Achieved with bipolar operation at $\pm V_{rms}/2$.
$^\dagger$ See text for definition of $\alpha$.
$^\ddagger$ Cross section of a single cable.
$^\ddagger$ As proportion of optimal DC value.
$^\ddagger$ Line-to-neutral voltage. Line-to-line voltage is same as single-phase optimal voltage.

Table 1.1: Optimal design variables and cost by installation type

<table>
<thead>
<tr>
<th>Power factor</th>
<th>Change in resistivity due to skin effect</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>1.00</td>
<td>1.000</td>
</tr>
<tr>
<td>0.95</td>
<td>1.026</td>
</tr>
<tr>
<td>0.90</td>
<td>1.054</td>
</tr>
<tr>
<td>0.85</td>
<td>1.085</td>
</tr>
<tr>
<td>0.80</td>
<td>1.118</td>
</tr>
<tr>
<td>0.75</td>
<td>1.155</td>
</tr>
<tr>
<td>0.70</td>
<td>1.195</td>
</tr>
</tbody>
</table>

Table 1.2: $\alpha$ coefficient by skin effect and power factor
If one were to build an equivalent DC installation, Table 1.1 asserts that one should use a DC voltage of \( \frac{240}{(0.343)(1.132)} \), or around 600V, distributed using ±300V conductors so that the voltage to earth is only half the system voltage. The cables should be \((0.971)(1.132) = 1.1\) times thinner than the AC cables; in practice the same cables could be used. Note, however, that this represents a one-third saving in copper, since only two cables are required for each circuit instead of three. Last but not least, one calculates the cost saving for the DC installation as \( 1/(1.456)(1.132) = 0.607 \), or around 39% over the AC installation. Even if the AC power factor were unity and skin effect were negligible, the cost saving would still be over 30%.

This cost saving disregards the cost of items such as mains conversion equipment that would be required for a mains-fed DC installation. Although cost savings in the order of 30–40% leave plenty of room for additional costs to be absorbed, the following sections show that to a large extent, these costs need not be additional to the cost of a conventional AC installation with typical loads.

### 1.2.2 Power Factor

An unstated assumption in the above cost analysis is that DC loads always operate at unity power factor, irrespective of the power factor under AC operation. This is a compelling theoretical argument in favour of DC, and we therefore proceed to justify the assumption rigorously.

We define *power factor* as the constant \( \eta \) in equation (1.5) for the active power flow in a (single-phase) circuit. The active power \( P \) in the steady state is defined, for any periodic voltage and current waveforms \( v(t) \) and \( i(t) \), as the time average of the product \( p = vi \) over one period. Since various factors of installation cost depend on the quantities \( V_{\text{rms}} \) and \( I_{\text{rms}} \) independently of each other, and not directly on the real power requirement \( P \), the power factor \( \eta \) provides a measure of the ‘capital efficiency’ of an installation.

A convenient framework for the analysis of generalised power waveforms involves the representation of steady-state periodic voltages and currents as elements of the space \( L_2[T_1,T_2] \) of time functions \( f(t) \) defined on \([T_1,T_2]\) for which the root-mean-square norm

\[
\|f\|_2 = \left( \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f^2(t) \, dt \right)^{\frac{1}{2}}
\]
is finite. Define the inner product of \( f, g \in \mathcal{L}_2[T_1, T_2] \) as

\[
\langle f, g \rangle_2 = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f(t) g(t) \, dt
\]

so that \( \|f\|_2^2 = \langle f, f \rangle_2 \). Every \( f \in \mathcal{L}_2[T_1, T_2] \) has a Fourier series representation

\[
f(t) = f_0 + \sum_{k=1}^{\infty} \left( f_k^C \cos(k\omega t) + f_k^S \sin(k\omega t) \right)
\]

where \( \omega = 2\pi/(T_2 - T_1) \), \( f_0 = \langle f, 1 \rangle_2 \), \( f_k^C = 2 \langle f, \cos k\omega t \rangle_2 \) and \( f_k^S = 2 \langle f, \sin k\omega t \rangle_2 \).

If \( v, i \in \mathcal{L}_2[T_1, T_2] \) represent one period of the steady-state voltage and current, respectively, in a power circuit, then the active power is \( P = \langle v, i \rangle_2 \) and the power factor (defined for nonzero \( v, i \))

\[
\eta = \frac{\langle v, i \rangle_2}{\|v\|_2 \|i\|_2}.
\]

By the Cauchy-Schwarz inequality we have \( \eta \leq 1 \) for all \( v \) and \( i \), with \( \eta = 1 \) if and only if \( i = \gamma v \) with \( \gamma \) a scalar.

Within this framework, the load on the circuit in the steady state is represented as a system \( G : v \mapsto i \) with the impressed voltage as input and load current as output. For brevity, we refer to \( G \) as a system on \( \mathcal{L}_2[T_1, T_2] \). Given a load \( G \), we may consider the problem of finding all voltage waveforms \( v^* \) such that \( G \) operates at unity power factor. This amounts to an eigenvector problem for the system \( G \): that of finding all functions \( v^* \in \mathcal{L}_2[T_1, T_2] \) such that

\[
Gv^* = \gamma v^*
\]

with \( \gamma \) a scalar. Alternatively, we may fix \( v^* \) and consider the class of all loads \( G \) that satisfy (1.16). The latter is the ‘power factor correction’ problem as commonly understood.

A system that satisfies (1.16) for all \( v^* \) is termed a scalar system, and corresponds to a load that obeys Ohm’s Law with conductance \( \gamma \). It follows that resistive loads always operate at unity power factor in the steady state, regardless of the impressed voltage.

Ideally, one would like to choose a voltage waveform \( v^* \) that makes the class of systems \( G \) satisfying (1.16) as large as possible. DC is a prime candidate for such a waveform, as the following result shows:

**Theorem 1.1** Let \( G : v \mapsto i \) be a stable linear system on \( \mathcal{L}_2[T_1, T_2] \), and let \( v(t) = V_0 \) for \( t \in [T_1, T_2] \), with \( V_0 \) a real constant. Then \( Gv = \gamma v \), where \( \gamma \) is the DC gain of \( G \).
In physical terms, Theorem 1.1 states that DC guarantees unity-power-factor operation in the steady state for any stable linear load. (The stability requirement is necessary to ensure a steady state exists.) One can also state a converse result, namely that there are certain categories of stable linear load for which DC is the only waveform that gives unity-power-factor operation.

**Theorem 1.2** Let \( \mathcal{G} : v \mapsto i \) be the series-inductive load modelled by

\[
L \frac{di}{dt} + Ri = v
\]

with \( v, i \in \mathcal{L}_2[T_1, T_2] \). If \( \mathcal{G}v = \gamma v \) for some scalar \( \gamma \), then \( v \) is constant on \([T_1, T_2] \).

This particular example is important, since all power reticulation systems exhibit series inductance. In [52], this result is extended to give necessary and sufficient conditions for a solution to (1.16) when \( \mathcal{G} \) is a general linear (time-invariant) system on \( \mathcal{L}_2[T_1, T_2] \). It is concluded that for ‘almost all’ linear loads other than pure conductances, there is no periodic waveform other than DC that guarantees unity-power-factor operation.

No such general conclusion can be drawn for nonlinear loads. However, explicit calculation of the power factor under DC operation with \( v = V_0 \) gives

\[
\eta_{DC} = \frac{V_0 \langle i, 1 \rangle_2}{V_0 \|i\|_2} = \frac{i_0}{\|i\|_2} \tag{1.17}
\]

so that the power factor may be calculated for any load as the ratio between average and RMS load current, assuming a constant DC voltage input. (In fact, DC obtained from rectifiers and switch-mode converters contains a small ripple component which leads to power factor reductions, but in practice this ripple may be made as small as desired.)

Deviations from unity power factor under DC operation are analogous to the ‘distortion factor’ under AC operation (see below), but are much rarer owing to the relative ease of designing power electronic circuits to draw a constant input current, in contrast to a sinusoidal input current.

### 1.2.3 Variable-Speed Drives

The variable-speed AC drive (henceforth VSD) is the most popular application of power electronic technology, accounting for roughly two-thirds of the total electrical
Figure 1.1: A variable-speed AC drive system

load in the USA alone [8]. In conjunction with sophisticated control techniques such as vector control, VSDs permit the operation of AC induction and synchronous motors with high efficiency and performance, and consequently a thriving VSD industry has emerged over the last two decades. VSDs are used extensively in industrial applications and also ‘behind the scenes’ in high-power commercial applications such as lifts and air-conditioning plant.

The components of a typical VSD system are depicted in Figure 1.1. Standard practice is to use a two-stage converter, producing variable-frequency AC from fixed-frequency AC with an intermediate DC link. In theory a cycloconverter could accomplish the same task, but the two-stage approach is almost universally preferred owing to the relative simplicity of the control task and the more economical use of switching devices.

The AC–DC conversion stage is one of the following:

• a diode rectifier, permitting single-quadrant operation only but at low cost;

• a thyristor rectifier, permitting reversible DC voltage but not reversible DC current; or

• a switch-mode converter, permitting reversible current though at a relatively high cost.

The latter approach is an emerging technology (discussed in more detail in Part III) whose use at present is justified only where the load creates a significant amount of regenerative energy or where power quality is of paramount concern. Under the two more traditional, lower-cost approaches, any regenerative energy from the load is diverted to a braking resistor placed across the DC bus.

The DC–AC conversion stage is accomplished by a force-commutated inverter. For control of AC motors, stator voltage must be varied in proportion to line frequency in order to maintain the air gap flux at constant magnitude. Accordingly, the
inverter must be capable of variable-voltage as well as variable-frequency sinusoidal output. Typically a switch-mode inverter, with devices such as insulated-gate bipolar transistors (IGBTs), and reversible input current, is used.

In a typical installation, a large number of these VSD systems will be connected in parallel on a common AC reticulation system. Figure 1.2 depicts an alternative configuration, in which the individual DC–AC stages of the VSDs share a common DC bus, supplied by a single AC–DC converter (or more than one, in a larger installation). This alternative configuration offers a number of immediate advantages.

- The ability to share regenerative energy among all the drives instead of dissipating the energy in braking resistors, without the need for relatively expensive switch-mode rectifiers.

- Economies of scale in the rectifiers. In general, a single device with current rating $N \times I$ will cost less than $N$ devices each with current rating $I$. The saving may justify the use of a switch-mode converter at the system input to improve power quality in the AC supply, even where switch-mode converters on the individual VSDs were not justified.

- The cost savings inherent in substituting a DC reticulation system for the AC reticulation system, as outlined above.

- The ability to connect battery storage without the need for an additional inverter.
Of course, each of these advantages is amplified as the scope of the DC reticulation system increases beyond the VSD systems to encompass (ideally) all the electrical load in an installation. The issue of non-VSD loads is addressed under ‘Appliance Design’ below.

1.2.4 Harmonic Mitigation

‘Harmonic distortion’ refers to distortion of AC voltage and current waveforms from the pure sinusoids expected in linear circuits. In the Fourier decomposition (1.14) of a power signal, the numbers \( f_k^C \) and \( f_k^S \) are termed the ‘coefficients of the \( k \)th harmonic’. Harmonic distortion is present whenever \( f_k^C \neq 0 \) or \( f_k^S \neq 0 \) for some \( k \neq 1 \).

Traditionally, harmonic distortion was recognised as arising from magnetic nonlinearities in transformers and machines. In recent times, however, this relatively minor source of distortion has been eclipsed by switching circuits at the load end of the system. As more and more traditional applications yield to the lure of sophisticated power-electronic controllers, and wholly new applications dominated by electronics emerge, this source can only continue to grow in magnitude and importance.

Harmonic distortion poses problems to suppliers and users of electricity for a variety of reasons, not least among which is its adverse effect on power factor. If voltage is purely sinusoidal, and

\[
I_k = \frac{1}{\sqrt{2}} \sqrt{(i_k^C)^2 + (i_k^S)^2}
\]

denotes the RMS magnitude of the \( k \)th harmonic of current, then the power factor is given by

\[
\eta = \frac{I_1}{\|i\|_2} \cdot \cos \phi_1 \quad (1.18)
\]

where \( \phi_1 \) denotes the phase difference between the voltage and the fundamental (first harmonic) of current. (If voltage harmonics are present, (1.18) must be modified to account for interaction between higher-order voltage and current harmonics. (1.18) is nonetheless useful in practice, as voltage harmonics are usually insignificant by comparison with current harmonics.)

The first term in (1.18) is commonly referred to as ‘distortion factor’ and the second as ‘displacement factor’. The distortion factor is unity in the absence of harmonic
distortion, but decreases rapidly in its presence. A common measure of distortion, the Total Harmonic Distortion (THD), is closely related to distortion factor:

$$\text{THD} = \frac{1}{I_1} \sqrt{\sum_{k \neq 1} I_k^2} = \sqrt{\frac{1}{DF^2} - 1}.$$ 

THD may be defined for either current or voltage.

As a consequence of the decrease in power factor, the presence of significant harmonics leads to overheating of conductors and direct-on-line motors. Other problems arising from harmonic distortion include:

**Nonstandard phase sequences.** The harmonic components of three-phase voltage and current can have a phase sequence different to that of the fundamental. Harmonics of order $3k - 1$ (where $k$ is an integer) have a negative phase sequence relative to the fundamental, which leads to pulsating torques and consequent mechanical stress in AC motors. Triplen harmonics (those of order $3k$) have a zero phase sequence, which means that triplen harmonic currents add at the neutral point and can cause overloading of neutral conductors not designed to carry such currents.

**Stress on power-factor-correction capacitors.** Since the reactance of a capacitor decreases with frequency, any capacitor can become a ‘harmonic sink’, drawing significant currents at the harmonic frequencies. A similar problem arises with charging currents on transmission lines, for the same reason.

**Voltage notching.** Current harmonics at the load induce voltage harmonics in the supply, due to the frequency-dependent voltage drop in series inductance elements. As inductive reactance increases with frequency, high-order harmonics can add to produce ‘notches’ and other undesirable distortions in the voltage waveform. These voltage harmonics contribute to appliance failures resulting from ‘poor power quality’, most notably in equipment sensitive to zero-crossings (clocks, AC controllers).

**Failure of protection devices.** The change in wave shape caused by harmonics can lead to nuisance tripping of circuit breakers and specialised protection relays, or blowing of fuses.
The problem is given an ironic twist by the fact that electronic circuits are not only greater *producers* of harmonics than traditional loads, but are also a good deal more *susceptible* to harmonic distortion.

Most industrialised nations now define power quality standards which stipulate, among other things, upper limits to voltage and/or current harmonics and corresponding levels of immunity for appliances. The paradigm for these standards at the time of writing is the IEC 61000 series, expected to be adopted over time as an international standard. As industry awareness of power quality issues increases, appliance manufacturers are likely to face increasing pressure to build filtering and other compliance equipment into individual appliances, thereby adding to costs.

Power quality in a DC context is both easier to define and easier to maintain. Electronics accounts for most of the nonlinear load on power systems, and the vast majority of electronic systems require a DC input, which they obtain from AC mains via a low-cost, line-commutated bridge rectifier. These rectifiers are the source of most low-order harmonic distortion (frequencies up to 1kHz), while switch-mode circuits on the DC side produce higher-order harmonics only (at or above the switching frequency). It is the former that are the source of most distortion-related power quality problems, owing to the lowpass characteristic of typical transmission systems.

Given the nature of electronic loads, one is led to expect fewer power quality problems with DC reticulation than with AC reticulation, provided the DC system voltage is sufficiently well controlled and sufficient capacitance is provided near switching circuits to absorb high-frequency currents. One significant problem remains: namely, the harmonic distortion in the AC mains produced at the input to the system. Here the concentration of harmonic sources makes possible a solution strategy superior to that possible with conventional AC reticulation.

Strategies for reducing harmonic distortion are of two general types:

**Source-level solutions**: strategies aimed at reducing the harmonics *produced* by a particular load, such as finer control or advanced switching techniques; and

**System-level solutions**: strategies aimed at limiting the *propagation* of harmonics produced, usually through filtering.

Each strategy has characteristic advantages and disadvantages. Both are costly, although relative cost depends on the scale of the problem. Filtering, though perhaps
The most common solution strategy, is fraught with difficulties including:

- absorption of harmonic currents originating from sources other than one’s own, and consequent overloading;

- resonance with other elements of the power system;

- detuning of passive filters with time (though active elements can compensate for this); and

- escalation in cost with the number of harmonic components to be filtered.

In a conventional AC system, comprising a large number of small, independent sources of harmonics, the only strategy generally open to the system designer is a system-level solution. The alternative source-level solution generally imposes unreasonable costs on appliance manufacturers. It is largely this characteristic of conventional systems that accounts for the popularity of filters despite their disadvantages.

With a mains-fed DC reticulation system the number of harmonic sources seen at the mains is greatly reduced. There may be only a single converter, although most installations could be expected to use two or more converters for added reliability. With this approach a source-level solution to harmonic distortion (for example, a switch-mode rectifier with active lowpass filter) becomes possible, even desirable on the basis that prevention is better than cure. This strategy avoids many of the problems of filters and can be justified by economies of scale.

1.2.5 Alternative Energy Sources

By an alternative energy source we mean any plant on a reticulation system capable of acting as an energy source independent of the AC main, at call and on an essentially continuous basis (thereby excluding regenerative drives and ordinary capacitors). Some sources (batteries, diesel generators) are intended primarily as backup in the event of mains failure, while others (photovoltaics, wind generators) are used for their inherent value as a renewable or ‘free’ energy source.

While our proposed ‘generic’ reticulation system is mains-fed, this does not preclude its use as a stand-alone system supplied entirely through these alternative sources. Whether mains-fed or not, a DC system has inherent advantages over AC
systems in ease of connection of alternative energy sources. We list below a number of energy sources, along with the issues involved in connecting to an AC or DC system.

**Photovoltaics.** Solar cells, owing to their nature as semiconductor junctions, produce a DC output voltage and unidirectional output current. Typical cell voltages are in the order of 1V; higher voltages are obtained by connecting cells in series. Photovoltaics can be connected to a DC system either directly or through a DC-DC converter. Connection to an AC system requires a sinusoidal inverter with mains synchronisation; in practice the inverter accounts for a significant portion of the cost of grid-connected PV installations.

**Wind Generators.** A typical wind generator comprises a wind-driven turbine coupled to an AC synchronous or induction machine. Its mechanical and electrical characteristics depend on the method of speed control, to which there are three basic approaches.

1. *Fixed-speed, direct drive:* control the speed of the turbine to give a constant synchronous speed for the AC machine.

2. *Fixed-speed, indirect drive:* place a gearbox on the shaft and allow the turbine speed to vary while maintaining constant synchronous speed as before.

3. *Variable-speed:* use no gearbox and allow the generator speed to vary with the turbine speed.

The first two approaches permit direct AC connection but are suboptimal from a whole-system point of view: the first because the turbine is unable to track the maximum power operating point as wind speed varies, and the second because the gearbox adds to cost while subtracting from efficiency. The third approach is mechanically superior, but complicates the connection to an AC grid, due to the variable-frequency output; essentially a VSD running in reverse is required. The cost of this makes a gearbox look attractive by comparison. On the other hand, DC connection requires only that a controlled rectifier be placed on the output, and the third option then, appropriately, emerges as the most attractive. This same argument applies to any generator whose optimal speed varies with environmental conditions, such as run-of-river hydro or tidal generators.

24
Fuel Cells. Fuel cells work on electrochemical principles and, therefore, produce a DC output voltage and unidirectional output current. The same comments regarding AC and DC connection apply as for photovoltaics.

Batteries. Batteries have a long and distinguished history as DC energy storage devices. The availability of batteries was a key argument used in support of DC power systems in the earliest days of electric power. Today it is common to find battery-driven ‘ancillary’ power systems for such applications as building security or power substation protection, while automotive electrics are probably the most widespread instance of battery-fed DC power systems. On mains-fed reticulation systems, batteries find their primary application in static uninterruptible power supplies (UPS), which resemble VSDs in overall structure in that they comprise a DC link through which power is fed from supply to load. As with VSDs, a static UPS can connect directly to a DC reticulation system without a rectifier.

Diesel Generators and Turbines. Of all the alternative energy sources we consider, diesel generators and (steam or gas) turbines alone are better suited to AC than to DC connection, as their optimal speed is fixed, allowing efficient generation of fixed-frequency AC. A generator for DC operation would comprise either a DC machine or an AC machine with output rectifier, giving increased capital cost and slightly lower efficiency than an AC generator. This minor disadvantage can be averted in a mains-fed DC system by connecting the generator on the AC side of the mains rectifier(s).

1.2.6 Electrical Appliances

People are often surprised to learn that many common household appliances, designed for AC connection, will run without modification on a DC input. Others will run after minor modifications, such as bypassing of input transformers or connection of additional series impedance. Just as surprisingly, redesigning the appliance for DC-only connection would in many cases simplify the design and improve reliability.

Appliances can be divided into several categories based on their suitability for AC or DC connection. It should be noted, however, that many appliances (in particular, those with embedded microprocessors) fall into multiple categories by virtue of being
built up from diverse subsystems. Some heavy-duty appliances may also contain input transformers which will need to be bypassed.

**Resistive elements.** Suitable for AC or DC connection. Examples include incandescent lamps, heaters, cookers, toasters and kettles.

**Universal motors.** Suitable for AC or DC connection, but may run at higher speed on DC due to lower input impedance. Examples include vacuum cleaners, food processors, steam pumps in irons, hair dryers, power drills and most high-speed motors.

**Electronics.** Suitable for AC or DC connection, but more ‘naturally’ suited to DC connection. Examples include computers, compact fluorescent lights (if electronically ballasted), DC motors, AC motors with VSDs, hi-fi systems, TV sets, telephones, and embedded processors in many other appliances.

**AC motors without VSDs.** Require AC connection, or inverter. Examples include older ‘whitegoods’: refrigerators, washing machines, driers and pumps.

**Miscellaneous.** Require AC connection due to inherent design features. Examples include lamps with dimmer switches, fluorescent tubes with choke ballasts, and special-purpose transformers.

The remark about electronic appliances requires further elaboration. Figure 1.3 depicts the input stage of the power supply for a typical household electronic appliance. The purpose of this input stage is to provide a reasonable approximation to a DC supply, for input to a switch-mode DC-DC converter. Resistance $R$ is often included as a protection measure, to limit the input current during power surges. The diode bridge with output capacitor is the most common rectifier topology, although the current drawn contains a substantial third harmonic component. (In future the input stage may have to incorporate an harmonic filter.) A large electrolytic capacitor
$C$ is required to smooth the DC-side voltage. This capacitor suffers significant stress whenever the power is switched on or off, and in practice is among the least reliable components in the whole appliance.

If the same appliance were redesigned for DC-only connection, this input stage would be greatly simplified, as the requisite DC input is already present. The resistance $R$ may or may not be required, depending on the stringency of the protection standard adopted for the DC system. The capacitance $C$ would still be present to smooth out residual voltage fluctuations and noise from the switch-mode converter, but would not need to be as large if the DC supply is of reasonable quality. It may indeed be possible to substitute a ceramic capacitor, thereby greatly improving the reliability of the power supply unit.

Even now, then, a large proportion of our appliances can be accommodated on a DC reticulation system, and in many cases this has the potential to simplify appliance design. The argument becomes particularly compelling, however, when one examines future trends. The overwhelming trend is for the number of appliances in the ‘Electronics’ category to increase, at the expense of the other categories. Computers, embedded processors, and other electronic gadgets continue to proliferate. Compact fluorescent lamps are slowly displacing incandescent bulbs. Washing machines with intelligent drives are replacing older, single-speed models. In line with this trend, we can expect ever greater problems with harmonic distortion, arising from the increasing proportion of electrical loads more suited to DC than AC supply.

The inherent logic of a common DC reticulation system becomes even more compelling when power is fed to appliances such as computers through a static uninterruptible power supply (UPS). Figure 1.4 depicts the stages of the power circuit in this situation. Incoming AC power is converted to DC and back to AC, before winding up as DC inside the computer. Figure 1.5 shows a more rational DC bus approach to this application.

1.2.7 Safety

Understanding of the physiological effects of electric current has progressed somewhat since purposely-electrocuted dogs and horses were presented by the Edison Company as evidence of the dangers of alternating current [45]. It is now understood that a primary cause of death by electrocution is ventricular fibrillation (a disordered
Figure 1.4: A bank of computers supplied by a UPS

Figure 1.5: Computer bank with uninterruptible DC bus supply
contraction of the heart muscle), and that the risk of fibrillation for a given amount of body current is greater for AC than for DC, increasing with frequency.

The relevant issues are covered by Australian Standard AS3859, “Effects of current passing through the human body”, which is reproduced from the international standard IEC 479. This document includes the following statement [73, Chapter 3]:

Accidents with direct current are much less frequent than would be expected from the number of d.c. applications, and fatal accidents occur only under very unfavourable conditions, for example in mines. This is partly due to the fact that the let-go of parts gripped is less difficult and that for shock durations longer than the period of the cardiac cycle the threshold of ventricular fibrillation remains considerably higher than for alternating current.

The main differences between the effects of a.c. and d.c. on the human body result from the fact that excitatory actions of the current ... are linked to the changes of the current magnitude especially when making and breaking the current. To produce the same excitatory effects the magnitude of direct current flow of constant strength is two to four times greater than that of alternating current.

The standard also notes that, whereas a person suffering AC current flow greater than about 10mA is generally unable to let go of the conductor, let-go is possible at DC currents up to approximately 300mA [73, Chapter 2]. These observations seem quite compatible with the following excerpt from the Reminiscences of R.E.Crompton (a DC advocate during the Battle of the Systems), an account which might shock many people today [9, p.219]:

The Prince of Wales, afterwards Edward VII, and Princess Alexandra visited our exhibition more than once, and I personally had the honour of conducting her Royal Highness round, to point out and explain to her the electrical phenomena. I assured her that she could touch bare conductors carrying energy of 200 volts pressure without danger. She made the experiment, and agreed with me that such a pressure was quite safe to use.
Bipolar distribution further reduces the risk of death or injury for a given DC system voltage, as alluded to in the discussion on costs.

1.3 The Problems with DC

Direct current is not without its disadvantages, even at the low-voltage reticulation level. Here we list the main concerns and suggest solutions.

1.3.1 The Switching Problem

When a switch is opened on an electrical circuit, current does not cease to flow instantly. Initially there is a sudden decrease in current flow, which causes a large voltage to build up over series inductive elements in the circuit. This ‘back EMF’ appears as a large negative voltage across the air gap in the switch, sufficient to ionise the air and strike an arc. Once the arc is struck the air gap voltage reverts to a small value, and current flow is maintained in the circuit until the arc is extinguished.

In an AC circuit, the supply voltage drops to zero within 10 milliseconds, and the consequent drop in current extinguishes the arc. Depending on the level of current, the arc may re-strike after the zero crossing, as the air has not completely deionised. After about three cycles, however, enough of the arc energy will have dissipated to eliminate further arcing. High-performance circuit breakers employ arc suppression strategies, such as breaking the arc into smaller arcs, or ‘blowing out’ the arc by lengthening the discharge path. These strategies can extinguish arcs even before the first zero crossing [67], but add substantially to the cost of a circuit breaker.

In DC circuits, by comparison, the supply voltage does not drop, and an arc can sustain itself indefinitely. For this reason, DC currents are more difficult to interrupt than AC currents.

This problem, as we have noted, received surprisingly little attention in the original Battle of the Systems. This may be due to the fact that arcing is more severe at high voltage levels, whereas the early DC systems operated at relatively safe domestic voltages. Users of Edison’s Electric Light, and DC customers in many Australian cities through the first half of the 20th century, switched their lamps on and off using ‘snap-action’ switches [9], apparently without ill effect considering that such systems were marketed on safety grounds.
Today, however, the switching problem is seen as a significant drawback with DC from the point of view of protection. Mechanical switches and contactors rated for a given level of DC current are significantly larger and more expensive than for an equivalent level of AC current.

There are a number of strategies for avoiding this problem without incurring significant extra cost. First, it must be recognised that a switch performs two related but logically distinct functions:

1. Circuit interruption: interrupting a current flow of sufficiently high magnitude in sufficiently short time.

2. Mechanical isolation: maintaining a physical separation between two parts of a circuit, so that resumption of current flow is prevented.

These functions together define the notion of a protection device as commonly understood, and both functions are essential to this purpose. However, there is no reason why these two functions must be provided by the same device.

The second of these functions can be provided without any difficulty. Assuming that we have some method for interrupting the current, any isolating mechanism whatsoever will serve this purpose. In particular, an AC contactor of suitably high current rating will provide the requisite isolation capability.

What we really seek, then, is a method for interrupting a DC current flow, without necessarily providing mechanical isolation, whose cost is small by comparison with an AC contactor. For some decades now there has existed a device admirably suited to this task: the transistor. The price of semiconductors is now sufficiently low that an IGBT or similar device can be used in conjunction with an AC contactor and control circuitry to form a DC switching module that is cost-competitive with its AC equivalent. A solid-state circuit breaker design, on the lines that we are proposing, is presented for example in [76].

A second strategy for circuit interruption is to mimic the effect of the zero crossings, by dropping the DC system voltage to zero just long enough to extinguish an arc in the AC contactor. The resulting disturbance in the system voltage is of very short duration (on the same time scale as the sinusoidal fluctuation in AC quantities) and should not therefore have adverse effects on other loads in the system. This second approach is suitable for use as a backup strategy in the event of device failure under
the first approach, and is particularly appropriate for switching under fault conditions. In the latter event, mains rectifiers supplying the system can be programmed to a negative voltage, to draw fault energy out of the system.

Using either of these strategies, interruption of current is expected to occur more rapidly than on an AC system with standard switchgear. Simulations presented in [52, p.31,p.200] demonstrate this in the case of the second strategy; here the interruption time is less than 10 milliseconds. By comparison, the interruption time for ‘instantaneous tripping’ in a typical AC circuit breaker is between 8 and 60 milliseconds [30, p.9]. For the first, transistor-based strategy, comparable or better results can be expected given the capabilities of today’s semiconductors [8].

1.3.2 Load Incompatibility

As mentioned in Section 1.2.6, there is a (shrinking) category of appliances and plant that rely on alternating current for their operation. Direct-on-line AC motors are the most prominent, but the category also includes such common items as inductive fluorescent lamps, dimmers, and diesel generators.

One should not exaggerate the difficulty thus posed. All other things being equal, it is most sensible to select the distribution system that comes closest to matching the characteristics of the majority of loads. We have argued that in typical installations, the majority of loads are of the kind that either use DC internally or are indifferent between AC and DC. Of course, all other things are not equal: many factors count in favour of DC (as explained above), while others (notably the cost differential between inverters and diode rectifiers) count in favour of AC. On balance, it still appears preferable to accommodate AC-dependent loads within a DC system rather than forgo the benefits of DC purely for their sake.

Strategies for accommodating such loads within DC systems include:

Substitution of DC-compatible alternatives: In most cases, alternatives to AC-dependent appliances can be procured at similar cost. Examples include electronic ballasts in place of choke ballasts for fluorescent lamps, and DC switchers in place of dimmers.

Connection via inverter: A special case of this is the introduction of VSDs for AC motors, which often yields efficiency benefits. In other cases, a group of smaller
AC-dependent appliances may be fed from a single inverter, thereby achieving economies of scale.

**Connection on AC side of mains-fed installation:** This can be attractive particularly where plant is located close to the source of mains supply. Generators provide obvious candidates for this approach.

### 1.3.3 Lack of Transformability

To a large extent, efficient high-power DC transmission and distribution systems are restricted to a single voltage level. This is because there is no high-power DC–DC converter, particularly at utility voltage levels, that can match the efficiency of the AC transformer. Large power systems, however, demand a multiplicity of voltage levels. While site-level reticulation demands low voltages (less than 1000V) for reasons of safety and economy, long-distance transmission demands high voltages to minimise losses. Although high-voltage DC transmission lines are increasingly common, the transformation to lower voltage levels and distribution in local areas is still accomplished with AC.

It is for this reason that we propose DC only at the reticulation level, where the voltage has already been transformed to below 1000V, and there is no need for a multiplicity of system voltages. At this low voltage level, efficient and inexpensive DC–DC converters are available to provide the extra-low voltages required by particular appliances. As explained in Section 1.2.6, it is already standard practice for appliances to generate their extra-low-voltage supplies from a DC–DC converter rather than from a (necessarily bulky) AC transformer.

Bipolar operation of the DC bus opens up the additional possibility (with the addition of a balanced neutral) for low-power appliances to be connected on only one side of the bus, thus drawing only half the full DC voltage. This may be viewed as the DC analogue of a single-phase connection on a three-phase AC system.

### 1.3.4 Corrosion of Cables

Problems occur when DC cables are buried underground, due to the constant electric field experienced by these cables. This field sets up electrolysis reactions with the surrounding earth, leading to corrosion of cable material in conjunction with electrical
Where an installation includes underground cables, these need to be appropriately sheathed to mitigate the effects of electrolysis.

1.4 Management of Electricity

1.4.1 Plain Old Electricity Supply

The concept of ‘managed’ electricity ranges over a broad spectrum of issues. One way to draw attention to the issues involved, drawing on an analogy with telecommunications, is to contrast managed electricity with its traditional counterpart, the ‘plain old electricity supply’.

Plain old electricity supply comes down a wire, through a meter, a fuse and a switch or two, into an appliance or plant. Current flows while the switch is on, and stops when the switch is turned off. Occasionally the consumer overloads a circuit, and the fuse blows and is replaced, with no power available in the meantime. Slightly more sophisticated protection devices, such as thermal overloads and automatic circuit breakers, are used with motors and other high-power equipment. Nonetheless, as long as the power is on it is more or less taken for granted.

Plain old electricity is billed by the kilowatt-hour. Very large users may also be charged a fee based on peak demand, recognising the cost of the dedicated infrastructure required to supply them with electricity. But by and large, the supplier of plain old electricity is primarily concerned with selling kilowatt-hours. Consumers of plain old electricity, for their part, are not overly concerned with power quality; one kilowatt-hour is much like another, and they all come fairly cheaply. The occasional dip or surge is tolerated.

Attitudes like this are now very much in the past, particularly among industrial and commercial users of electricity. Generally, the attitude taken by users toward the electricity they consume is a good deal more exacting, whether in terms of power quality, cost, or efficiency.

Over time, a whole host of electricity ‘management’ technologies has emerged in an effort to bring the plain old electricity supply more in line with heightened consumer expectations. Like a data modem operating on a low-bandwidth telephone line, such technologies come largely in the form of ‘add-ons’ to an existing power
system designed for simpler times and lower expectations. We argue that the time is ripe to develop an entirely new approach to power distribution that incorporates power-management philosophy from the start. An integrated power distribution and management system could both fulfil current expectations better than an add-on approach, and serve as an ‘enabler’ for future developments. In the following pages we develop this argument, briefly surveying some of the technologies that currently exist and suggesting improvements that could result from an integrated system approach.

1.4.2 Intelligent Buildings

The drive toward better management of electricity receives much of its impetus from the ‘intelligent buildings’ movement. This rapidly emerging approach to building services, also known as facility management or total property management, seeks to integrate the functions now provided by a burgeoning array of systems. These include lighting, environmental control, computers, networking, telecommunications, security, fire services and even office layout. Integration of these systems serves a number of purposes: it increases the efficiency and effectiveness of the building services themselves, eases the task of building managers, and helps provide a better working and living environment for the occupants.

Advocates of intelligent buildings vary in their degree of ambition. One particularly strong advocate provides this vision statement:

When the occupant comes into work and passes his [sic] badge or access card through the reader, the intelligent building looks him up in the database and automatically turns on the lights in his work area, supplies power to his computer, and activates the HVAC system to increase the amount of ventilation going into his work area. The temperature in his work area remains constant throughout the day …

External communications from his computer work through seamless interfaces to other networks, both inside and outside the organisation … And at the end of the day, as he is working late, the building senses his presence and his lights and temperature remain constant. When he leaves, the [Building Automation System] senses him passing through the security system, and the BAS shuts down his area. [53]
While some accept this as a vision of a better life, to others the image is of being imprisoned within a technological Leviathan. One is reminded of the modernist architect Le Corbusier, who saw buildings and cities as vast machines for living in [17, 16]. The present author would certainly question whether the imposition by remote control of so much structure is warranted simply to avoid the need for flicking a few switches. On a more pragmatic level, it is widely observed that the more ‘clever’ technology becomes at trying to ‘second-guess’ the user’s desired action, the greater the scope for user frustration (see [39]).

Having noted these reservations, there is much of value in the idea of an ‘intelligent’ building when conceived more narrowly as the integration and optimisation of ‘utility’ services such as power supply, lighting, HVAC (heating, ventilation and air-conditioning), mechanical plant, and communications. Electricity being a common denominator in all such schemes, it is appropriate that power reticulation provide the ‘backbone’ for an integrated system. Important components of intelligent buildings include Energy Management Systems (EMS), Lighting Management Systems (LMS) and environmental controllers, all of which are natural adjuncts to the electric power system. Collectively, these make up the Building Automation System (BAS) together with high-level coordination and user interface tools.

Still in the future are ‘intelligent’ appliances capable of communicating with each other, with the EMS and with other components of the BAS. The recently developed CEBus (for Consumer Electronics Bus) standard [15] provides a ready-made standard platform for appliances wishing to communicate over domestic mains without the need for additional cabling. While the present CEBus standard is oriented toward conventional AC mains, one may envisage an alternative physical layer specification allowing CEBus to be implemented on a DC reticulation system. The field of intelligent appliances is still in its infancy.

### 1.4.3 Energy Management Systems

Energy management systems, or EMS, are used to control a consumer’s level of electricity demand. The term may be used to describe any control action aimed at saving energy, such as a lighting controller that dims the lights in proportion to the amount of outside light available. Such systems are highly application-specific and are best implemented at the level of individual load circuits or a local group of such circuits,
We shall take an EMS to be a high-level control system that manages the general demand for electricity. Such a system may take into account, where appropriate and in a generic manner, the ability of individual circuits to manage their own power demand, but should not concern itself with the details of application-specific measures. High-level systems of this sort are also known as demand limiters.

While an EMS has clear advantages in energy efficiency, the demand limiting function is particularly beneficial for large consumers who receive supply on a ‘demand tariff’ which charges for maximum peak demand as well as for total consumption. Typically, demand is calculated as the average over a 30-minute period, and the maximum value for any one 30-minute period in a month determines the demand charge, specified in dollars per kW.

A typical EMS controls the 30-minute average demand by shedding non-essential loads when the demand exceeds the desired level, and reinstating these loads when demand falls below this level. At any time, the load shed is the least amount compatible with maintaining the average demand below the desired maximum. The exact manner in which load shedding takes place is specified by the user, and most EMSs permit a high level of detail in this specification. EMSs are now in common use and have proved their ability to conserve energy and thereby deliver cost savings to the consumer.

The principles underlying an EMS are exemplified by a typical system, developed by Edwards et al. for use in Queensland hospitals [22]. In addition to shedding loads to control maximum demand, this system allows loads to be automatically switched on or off on a time-of-day basis for greater energy conservation. At any given time, each load in the building is assigned a priority: unconditionally on, unconditionally off, or sheddable at any one of eight priority levels. Loads at priority level 1 are shed first, and those at priority level 8 last. The time-of-day control feature allows the operator a high degree of flexibility in configuring the system. Loads can be designated on, off, or sheddable according to the time of day and/or the day of the week, and the software includes a module that calculates sunrise and sunset times to provide time references for loads such as lighting.

This system is typical of most EMSs in that its action on the power system consists entirely in switching load circuits on and off. The task of the EMS, albeit
an important and complex one, lies entirely in determining which loads should be switched on and which switched off at any given time. Conceiving an EMS purely as a switching device makes it easier to add an EMS to an existing system. Nonetheless, the framing of control issues exclusively in terms of switching is, as mentioned above, a characteristic of plain old electricity supply.

As long as switching is the only form of control considered, loads must be considered as falling into one of just three categories, from a demand management perspective [22]:

1. essential loads: those which are required to be on at all times, such as security, computers and production systems;

2. sheddable loads: those which are normally on but can be switched off if necessary, such as non-essential lighting and air conditioning; and

3. reschedulable loads: those which can be switched off at times of peak demand, as long as they are permitted to operate at another time instead, such as hot water, ice production and other energy storage systems.

This classification is appropriate for loads that exercise no control over their power input. Many loads these days are controllable, however. Heating and air-conditioning units, for example, will respond to a user setpoint, which in most cases is continuously variable. Many energy storage systems can be set to run at half power, rather than being switched off entirely if there is insufficient energy to run them at full power.

These considerations form the basis of a more flexible approach to energy management, in which the above categories of sheddable and reschedulable load emerge as particular special cases. The key to this approach is that the EMS have partial or full control over not only the on-off switch, but also the operating setpoint. The EMS then determines not only whether a given load should be on or off, but also how much power it should draw.

To be more precise, let each load have associated with it a ‘level of service factor’ between 0 and 1, where 1 represents full power and 0 indicates the load is switched off. Sheddable loads, by the above definition, have a level of service that alternates between 0 and 1. Reschedulable loads are, in addition, constrained such that the integral of the level of service over a given time period lies within set bounds (which will depend also on demand for the stored energy). Under the more flexible approach,
the level of service becomes a continuous variable rather than a discrete quantity. The
EMS may then choose among a wide range of level-of-service profiles, all of which
give the same total power demand.

A key advantage of this strategy is that a wider range of loads can be brought
into the scope of the EMS. Many loads are of the sort where a certain minimum level
of service is essential, while a level of service above the minimum is optional. In the
above classification, such loads would be classified as essential and placed outside the
scope of the EMS; in a more flexible approach, the minimum level of service would
be specified and the EMS given discretion above that level.

Other factors may be brought to bear on EMS decisions. Given the choice between
winding back a pump on the ground floor or a heater on the 10th floor, all else being
equal, the EMS may choose the heater on the grounds that the power losses in the
distribution system are thereby reduced. With continuously-variable levels of service
it becomes possible to select an optimum profile that minimises the system losses for
a given total demand. Such a strategy is the subject of Chapter 4.

1.4.4 ‘Soft’ Overload Prevention

Related to energy management systems is the idea of preventing system overloads by
monitoring the current drawn by a load circuit and switching off the load if the current
falls outside preset limits. This technique may be referred to [3] as ‘soft’ protection,
in contrast with the ‘hard’ protection provided by fuses or equivalent devices.

Aside from its avoidance of blown fuses, this technique has potential advantages
as an energy management measure. An EMS, capable of communicating with the
embedded control system in a piece of equipment, may negotiate demand limits with
the equipment and subsequently monitor the current to ensure the limits are adhered
to. The demand limits would be renegotiated whenever the operating environment
changes.

1.4.5 SCADA and Industrial Automation

A recent development in the area of large-scale transmission and distribution system
control, mirroring the emergence of computer-aided facilities management in build-
ings, is the Supervisory Control And Data Acquisition (SCADA) system. In the
SCADA-equipped power system, relevant state variables at individual substations
(such as voltages, currents, frequency, phase shift, temperature, and the status of circuit breakers and other plant) are sampled regularly and transmitted in digital form to regional control centres. Transmission is accomplished by a Remote Terminal Unit (RTU) at each substation. System operators can, in turn, issue instructions which are relayed as control signals to RTUs and cause equipment to operate automatically.

Numerous examples of SCADA installations can be found in the literature. One prominent recent example is the ongoing project to upgrade power system control in the London Underground [36, 11]. This is a DC traction system fed by 11kV and 22kV AC distribution networks. Part of the motivation for this project is the desire to operate each Underground line as a separate business unit with its own control function; SCADA was seen as providing the necessary flexibility. A useful feature of this SCADA system, as with many others, is the easy transfer of control authority from one centre to another.

The logical next step is to automate some of the functions at the operator level, and here the parallel with intelligent buildings is particularly evident. The addition of EMS functions to the SCADA system shows particular promise, and has been adopted within a number of European railway systems; the Swiss case is typical [62]. By the nature of these systems, the EMS acts more as a resource scheduler than as a demand manager, but railway systems are in any case rather special in their needs.

1.4.6 Power Quality

A wide range of power supply issues are today bundled together under the heading of ‘power quality’. These include continuity of supply, voltage dips and surges, harmonic voltage distortion and electromagnetic interference (EMI) in voltages and currents, and in general any conceivable deviation of the supply voltage waveform from the ideal.

In AC systems, there will always be a tradeoff between energy efficiency and power quality. Plant and process efficiency is increased through the use of power electronics, but the resulting harmonic distortion degrades the quality of supply. Power quality thus presents the greatest contemporary challenge for the power system industry. The solution, as echoed by most contemporary experts on power quality, is to improve the match between the characteristics of the electricity supply and the characteristics of the consumer’s installation.
However, an impediment to an efficient solution is presented by the cultural split that currently exists within electric power engineering. Traditional power system engineers, on the one hand, tend not to concern themselves with small-scale electrical applications except in aggregate, as a source of power quality problems. On the other hand, power electronics practitioners with a focus on applications tend to take the AC mains supply for granted. Thus, the electrical revolution mentioned at the outset has so far largely stopped at the point where the appliance meets the point of supply.

For this reason, most approaches to improving power quality have relied on re-designing electrical loads to suit the existing AC system. Usually, this entails adding an harmonic filter to an AC–DC input stage that is itself superfluous to the operation of the plant or appliance. If the AC supply itself were called into question, a radical but superior solution could emerge: that of redesigning the supply system to suit the loads, just as Tesla originally designed three-phase distribution to suit his AC machines. Power system science would thereby be brought up to date with half a century of new developments in electrical applications.

Already, some electricity distributors are experimenting with the idea of tailoring their supply to the individual consumer. These initiatives go by the collective name of ‘custom power’ or ‘premium power’ [32], and involve the use of innovative power conditioning equipment such as dynamic voltage restorers (DVRs) or static transfer switches (which transfer supply from one feeder to another without interruption). The importance of these projects is less in their specifics than in the general idea they espouse, that the resolution of power quality issues rests with the supply system, not in the end-uses of electricity.

1.5 Applications: The Managed DC Bus

The twin notions of DC reticulation and managed electricity converge in the Managed DC Bus, a prototype power supply system developed by Boral Elevators and later by Otis Engineering Centre Australia. This project involved the present author as principal control engineer; a more detailed exposition can be found in [52].
1.5.1 Initial Design

The core ideas of the Managed DC Bus are contained in a US patent by its inventor, Gregory Eckersley [21]. The concept arose from the consideration of lift installations in commercial buildings, which typically involve the operation of several variable-speed AC drives from a common supply bus. As explained in Section 1.2.3 and by Eckersley [20], the use of a DC bus in place of an AC bus for such an installation offers advantages in cost, energy efficiency and protection.

The system voltage was chosen as 600V, close to that obtained from a three-phase diode rectifier operating from a standard 415V main. Similarly, the 300V unipolar supply corresponds roughly to a rectified 240V single-phase supply. Thus, all existing single-phase loads with diode rectifier front ends can be connected on such a system without any modification being required. Third-party VSDs may be operated from the 600V supply, connected at the internal DC bus terminals. As discussed in Section 1.2.1, a 600V DC supply is the optimal equivalent of a 415V three-phase AC supply from an economic perspective.

The need for a coordinated response to faults to prevent sustained arcing led naturally to the incorporation of ‘management’ elements into this DC-based system. In the Managed DC Bus as originally conceived, the AC–DC supply source(s) and the various loads each have an associated control processor, and all control processors are linked by a communication network. When a load circuit wishes to draw current, its control processor sends a request over the network, specifying minimum and maximum limits for the current to be drawn. A central demand controller responds by granting the request, granting it with more stringent limits, or refusing it. The current limits may be renegotiated at any time by sending a new request.

When a request to draw current is granted, the approved limits are transmitted back to the control processor. The current actually drawn by the load is monitored, and if an excursion outside the limits is detected, the control processor integrates the ‘let-through’ $(I - I_L)^2\,dt$, where $I_L$ is the limit current. If the current drops back within the limits before a preset threshold is reached, the let-through calculation is reset and normal operation continues. Otherwise, a fault is signalled when the let-through exceeds the threshold. The controller thus mimics to some extent the behaviour of a fuse, providing ‘soft overload protection’ as discussed in Section 1.4.4.

Upon the signalling of a fault, a master control processor initiates a ‘bus trip’.
This is a momentary interruption in the DC supply from the input rectifier, sufficient to reduce the DC voltage and fault current to zero and thereby extinguish any arcs. Should the fault persist, the trip action is repeated, and ultimately the faulty load circuit is disconnected permanently by opening a contactor.

1.5.2 Discretionary Current

During the prototype development phase, the initial design was refined in several respects. A key innovation was the concept of discretionary current. This refers to the ability of a supply or load circuit to control its current in response to externally determined setpoints. In a conventional distribution system with one point of supply, the supply current is determined from the sum of the various load currents; if there is more than one supply, other factors must be brought to bear on the allocation of current between supplies. Taking this idea to a higher level of abstraction, any load with a controllable setpoint may be thought of as a ‘virtual supply’, where an increase in the virtual supply current corresponds to a reduction in load current brought about by a change in the setpoint. A higher-level controller may be given ‘discretion’ over this virtual supply current, and the higher-level controller may then be used to implement a flexible energy management system of the sort proposed in Section 1.4.3.

In the revised Managed DC Bus specification, there is no logical distinction between supply and load circuits; all are referred to as external circuits. The external circuit current is divided into two components: the demand component, subject to the request mechanism described earlier, and the discretionary component. For supply units, the discretionary current is the conventional input current, while for load units it is the virtual supply current. The discretionary current is thus always positive; its maximum value for a given external circuit is the discretionary rating for that circuit. An external circuit’s discretionary rating defaults to zero, and can be modified at any time by the circuit’s control processor sending a message to the demand controller. Any external circuit with a nonzero discretionary rating will receive periodic setpoint commands from the regulatory control system, and is expected to control its discretionary current to track this setpoint.

The sum of all external circuit discretionary ratings is the capacity of the system, and sets a limit on the aggregate of all demand requests. The principal task of the
demand controller is to manage demand requests so that aggregate demand never exceeds the system capacity. For this purpose every request for demand current specifies a priority level, and higher-priority requests are permitted to override lower-priority requests in the event of a capacity shortage. The external circuit requesting current at the lower priority is sent a ‘load shed’ message, which has the effect of forcing the demand current limits at that priority level to zero. Load shedding may also potentially occur when an external circuit signals a reduction in its discretionary rating, leading to a reduction in system capacity. In either case, however, an adjustable load unit can avoid the possibility of abrupt shedding either by increasing its priority (as in a conventional EMS) or by placing all or part of its current under the discretionary regime.

A simple example will serve to illustrate the demand management implications of discretionary current. Suppose an installation contains three external circuits as follows:

- a 100A supply unit;

- Load 1, a low-priority adjustable load unit with a full-load rating of 60A and a minimal operating current of 10A; and

- Load 2, a high-priority uncontrollable load whose current varies over long time scales between zero and 100A.

A conventional system cannot instruct Load 1 to change its setpoint, and Load 1 itself has no information about demand elsewhere on the system, so cannot autonomously change its setpoint in response. Typically Load 1 would draw the full 60 amps, and would be shed by the EMS whenever the current to Load 2 exceeded 40 amps, or 40% of full load.

In a Managed DC Bus situation, Load 1 might instead nominate a discretionary rating of, say, 50 amps. Suppose that the aggregate demand is shared among all discretionary external circuits in proportion to their discretionary ratings; since the supply unit has a discretionary rating of 100A, this means that two-thirds of the demand will be allocated to the supply and one-third to Load 1 as a virtual supply. If Load 2 is drawing no current, the aggregate demand is 60A from Load 1 and the actual Load 1 current is 40A. (In this example the supply capacity is unusually low;
in a more typical installation the current in Load 1 in the absence of Load 2 would be much closer to the full 60A.)

As the current in Load 2 increases, the supply current increases and the current in Load 1 decreases proportionately. Only when Load 2 is drawing 90A, or 90% of full load, is the virtual supply current from Load 1 equal to the nominated maximum of 50A. (The aggregate demand is 90 + 60 = 150A, and one-third of this is 50A.) Thus, it is only when Load 2 exceeds 90A that Load 1 must be instructed to shed the last 10A of low-priority current. (As a side effect Load 1 must also shed its virtual supply, reducing the system capacity to 100A, all of which goes to Load 2.)

This example shows that for loads which can provide an adequate level of service with reduced current, the discretionary current concept provides superior energy management capabilities in the presence of large but unpredictable load variations, such as those arising from lift machines.

1.5.3 Other Features

The mechanism for switching external circuits and interrupting fault currents was refined as part of the (now suspended) commercialisation of the Managed DC Bus. Rather than initiate a bus trip whenever switching a DC current is necessary, it was thought preferable to incorporate some inexpensive power semiconductors in the individual switching modules and use the bus trip mechanism only as a backup measure.

In a commercial Managed DC Bus, all routine switching would be accomplished at the local circuit level using ‘active arc suppression’ technology. As discussed in Section 1.3.1, a semiconductor switch placed in series with a low-cost mechanical isolator performs the function of a standard AC circuit breaker without the risk of arcing on DC. In the event of device failure, or fault currents in excess of the switch rating, arcing in the isolator is prevented using the bus trip mechanism.

The protection function is assisted by the use of precharging circuits in conjunction with switches. Before the main switch is closed to bring the circuit into operation, an auxiliary switch closes on to a series resistive precharging circuit, the main purpose of which is to limit the inrush current to capacitive loads. The charging current can be monitored to detect the presence of a short-circuit fault in the load, and switching on of the load can be aborted in this case without the need to interrupt the full
short-circuit current.

In summary, the Managed DC Bus provides a practical instance of the ‘new paradigm’ in electric power reticulation characterised by the use of DC and the increased management of electricity. The Managed DC Bus incorporates both energy management and soft overload prevention in its basic design; its use of distributed control processors with a common communication network and operator interface is reminiscent of SCADA systems; and a regulatory control system operating on the level of the power electronic supply sources ensures high DC-side power quality. The Managed DC Bus may be seen as an enabling technology for intelligent buildings and for superior energy efficiency technologies based on power electronics and computer control.

The aim of this thesis is to abstract from the Managed DC Bus to a generic managed DC power reticulation system and to find solutions to the various technical challenges posed by all such systems.
Chapter 2

The Problems

The previous chapter set out, in broad terms, the concept of managed DC electricity as an idea worthy of further investigation. The purpose of this chapter is to motivate the specific technical challenges which proceed from this concept, and which are explored in the main body of the thesis.

The problems investigated fall into two broad categories. First, there are those which relate to the operation of the power system in steady state, ignoring all transient effects; these we refer to as static problems. Second, there are problems relating to the system’s transient behaviour during the transition from one steady state to another; these are termed dynamic problems.

In the following pages we summarise each in turn and foreshadow the results to be obtained. References to relevant literature will be found in the main text.

2.1 Static Problems

The steady state behaviour of power systems is determined by the network topology and impedances, and the steady-state load requirements. Problems that arise in steady-state operation are of two main types: the optimisation of certain aspects of network performance, and the avoidance of certain adverse operating conditions (‘faults’).

In this thesis we are concerned with the behaviour of the network, not with the performance of specific loads; moreover, we are concerned with the specific problems that arise in (managed) DC networks as distinct from AC networks. Thus the issue of network reliability, for example, does not concern us here as the essential differences
between AC and managed DC networks do not impact on reliability, and so any result concerning the reliability of AC networks may be applied directly to DC networks without modification.

The steady-state behaviour of any electrical network is given by Ohm's and Kirchhoff's Laws in conjunction with the steady-state characteristics of the supply and load circuits. In regard to steady-state analysis, therefore, the only essential difference between (nonmanaged) AC and DC networks is that between complex and real numbers. The problems with which we are concerned thus relate mainly to the managed nature of the networks under consideration. Our paradigm for electricity management is the ‘discretionary current’ framework used by the Managed DC Bus and outlined in Section 1.5.2.

In our treatment of static problems we begin by developing a topological sensitivity analysis for generic impedance networks in the steady state. The theory developed is applicable to any network in which the external circuits can be modelled as current sinks, and is therefore well suited to our management framework, which uses current as the unit of supply and demand. The analysis is then specialised to managed DC networks by introducing the concept of a ‘discretionary map’, which expresses the quantitative relationship between discretionary and demand components of current. This framework for managed DC network analysis is subsequently applied to the problems of performance optimisation and fault avoidance.

The most commonly used measures of distribution network performance that are electrical in nature (and therefore relevant to managed networks) are power loss, voltage balance and load balance. The three measures are linked to some degree; power loss, for example, can be thought of as a weighted sum of cable voltage drops, with each voltage drop term being weighted by the cable current. To a large extent, then, measures that reduce network power loss will also tend to improve the other performance measures, whether by reducing voltage drops or by equalising the cable currents. Our work therefore focusses on loss minimisation.

In a managed DC network, the opportunity exists to minimise losses by an efficient allocation of discretionary currents. Of all the ways of sharing the total discretionary requirement among external circuits, some will give rise to greater network losses than others; ideally, the bulk of the task should go to those external circuits nearest the points of greatest demand. This provides us with our first major technical challenge.
For a given supply and demand profile, network losses may also be reduced by more traditional means, such as by reconfiguration of the network. As well as providing a framework for analysis of managed DC networks, the generic sensitivity analysis provides a fresh approach to the well-known ‘reconfiguration problem’ for AC distribution networks. As an adjunct to the work on managed DC networks, we provide a new algorithm for the solution of this more traditional problem.

In regard to fault avoidance, the principal steady-state problem is that of preventing cable overload conditions. Here we explore two aspects of the problem: design rules for the selection of cables in a new installation, and operational measures to anticipate overloads and take preventative action. In regard to the latter, the demand request mechanism provides the opportunity to screen requests and ensure that no network overloads will result from them. We accordingly provide (elements of) an algorithm to determine whether a request for current should be denied on the grounds of overload prevention.

### 2.2 Dynamic Problems

The dynamics of DC reticulation systems open up a range of largely unexplored problems. In contrast to AC power system dynamics, which rests on the decomposition of power systems into machines and passive elements, DC system dynamics rests heavily on an understanding of switching power converters. While the behaviour of converters in a stand-alone situation has been (in most cases) extensively analysed in the literature, the interaction of one or more converters with nonlinear loads on a common DC bus is not well studied.

The starting point for dynamic analysis of DC systems is a comprehensive model, encompassing the dynamics of all converters and load modules. A chapter of the thesis is devoted to this modelling task. With the model in place, dynamic problems such as transient stability and suppression of voltage excursions can be formulated as problems in the control of dynamical systems. In a dynamic context, ‘management’ of electricity is neither more nor less than the implementation of efficient solutions to these control problems.

A substantial part of the thesis is devoted to the modelling and control of switch-mode AC–DC converters, with particular emphasis on the ‘current-driven’ variety
(with a series inductive DC link). This emphasis comes about because, inevitably, a large part of the control task for DC systems rests with the input converters, and because the current-driven switch-mode converter offers certain advantages as a bulk supply source. The material in this thesis is intended to help fill what we perceive as significant gaps in the understanding of such converters.

For a DC system fed from one or more switch-mode AC–DC converters, the control problem can be divided into three quasi-independent parts. The first two relate to low-level switching control and to control of the AC input current for individual converters; these problems are already well understood, although some original contributions are made here in regard to current-driven converters. The third subproblem is essentially to stabilise the voltages and currents in the DC network as a whole, with controllable sources in place of the converters, and this problem is poorly understood. We accordingly provide a rigorous formulation of this control problem, and demonstrate that the need for control arises from the inherent instability of the system. We also describe some approaches to control design in simplified cases, leaving the problem in its full complexity as a topic for future research.

2.3 Structure of the Thesis

The main body of the thesis falls into two parts, the first devoted to static problems (Chapters 3 through 6) and the second to dynamic problems (Chapters 7 through 11). The thesis is supplemented with a series of appendices, which have two functions: to provide background material not commonly available in contemporary textbooks, and to present original work that is peripheral to the main development.

Chapter 1 and this chapter together form Part I, the introduction. Part II, Statics, commences with a chapter setting out definitions and preliminary results for the steady-state analysis of managed DC networks. Readers wishing to skip the technical detail of Chapter 3 should proceed directly to the summary, Section 3.8, which concludes the chapter. Appendix A provides some standard results in graph and network theory that are essential to this work.

Subsequent chapters in Part II are devoted to particular static problems and their solution. The minimisation of network losses by allocation of discretionary currents is the subject of Chapter 4. Here we formulate the problem rigorously as a quadratic
optimisation problem with linear constraints, soluble by least-squares methods. The chapter concludes with an example of optimal discretionary allocation. Appendix B considers an alternative geometric formulation of the problem which offers some additional insight into the structure of the solution and is relevant to the overload prevention problem discussed later.

Chapter 5 is devoted to the reconfiguration problem for generic networks. Given a network with redundant links (‘tie lines’), the problem is to find from among all radial subnetworks (spanning trees) a configuration with minimal network loss. The restriction to radial networks is made for operational reasons in AC distribution systems, and is of questionable relevance in DC reticulation. Nonetheless, this chapter is included as a straightforward application of the steady-state theory of Chapter 3 and as an important contribution in its own right.

The practicality of the algorithm presented in this chapter leads to an open problem in graph theory, namely the existence of a topological formula for the number of spanning trees in a graph. Some comments in regard to this problem are made in Appendix C.

In Chapter 6, the steady-state theory is applied to the prevention of overload conditions. Section 6.1 briefly discusses the selection of cables in new installations with known external circuit ratings, based on the global current sensitivity analysis. The remainder of the chapter is devoted to the effect of a change in external circuit demand on the network cable currents, which leads to a procedure for determining whether to accept or reject a demand request on the basis of overload prevention. The procedure turns out to be relatively straightforward when discretionary currents are allocated in proportion to discretionary ratings, and a simple example is provided at the conclusion of the chapter. If on the other hand the currents are allocated so as to minimise losses, as suggested in Chapter 4, the procedure is much more complicated, and requires further work to specify properly.

Part III, *Dynamics*, commences with a chapter setting out various dynamical models for DC power systems and the converters that feed them. Converter models are developed in order of complexity, from simple controlled sources to detailed piecewise linear switching models. Appendices D and E provide supplementary technical material on discrete harmonic oscillators and polyphase signals, respectively. Throughout this part of the thesis the emphasis is on switch-mode converters. However, Appendix
F provides an additional model capable of capturing the detailed switching behaviour of thyristor converters.

Section 7.6.3 presents the three-tiered approach to control on which the remainder of Part III is structured. Chapter 8 discusses the lowest of these tiers, which we label switching control. This chapter develops a novel approach to the control of current-driven converters based on the extensive model development in Chapter 7.

Chapter 9 presents a detailed steady-state analysis of AC–DC thyristor and switch-mode converters. The thyristor converter analysis leads to a design rule for the DC-side inductor and capacitor, while the analyses for switch-mode converters lead to restrictions on the steady-state voltage and current.

Chapter 10 discusses the synchronising control of switch-mode AC–DC converters: the simultaneous control of AC input current and DC output voltage. For voltage-driven converters this amounts to a review and comparison of existing techniques, but for current-driven converters a new method is proposed and tested in simulations.

Chapter 11, which concludes Part III, is devoted to the regulatory control of the DC power system; this is the control problem with all converters treated as controllable current or voltage sources. The loads on such a system are frequently nonlinear, and we present some results on the qualitative open-loop dynamics which demonstrate the need for closed-loop control. The control problem is analysed in detail for the simplified scenario where the DC system is reduced to a single node with a single converter. Brief comments are made on the generalisation to multiple converters.

A brief concluding chapter summarises the main results of the thesis and suggests directions for further work.

## 2.4 Contributions of the Thesis

The work described in this thesis spans a broad range of subject areas within the wider discipline of electric power engineering. It was therefore thought appropriate to review the relevant literature on a chapter-by-chapter basis, rather than attempt a monolithic review of several unrelated bodies of research.

A brief note on the idea of DC reticulation per se is in order. Setting aside the Battle of the Systems 100 years ago, alternatives to AC for residential and commercial
reticulation have been considered by a handful of authors, notably Ferreira [23] and Lee, Lee and Lin [41]. The approaches of these authors differ somewhat from that proposed here, and because of their complexity would not give rise to the same cost benefits. DC reticulation systems also exist for special-purpose installations, such as spacecraft [72]. Our approach proceeds from the work of Eckersley [21, 20] and the discretionary current concept as developed in previous work by the author [52, 51].

The following original contributions arise from the work presented in this thesis. Numbers in parentheses refer to relevant sections of the thesis.

1. A cost model for small scale electrical installations, and a demonstration that DC distribution can generate cost savings of between 30% and 40% over AC distribution for new installations, exclusive of conversion equipment. (1.2.1)

2. The formulation of power factor correction as an eigenvector problem for a dynamical system. (1.2.2)

3. A steady-state theory of managed DC distribution networks, using discretionary current as the basis for energy management. (3.3)

4. A method for allocating discretionary currents in a managed DC distribution network so as to minimise network losses. (4)

5. A characterisation of the complement of a convex polytope in $\mathbb{R}^N$, applicable to the problem of finding the point on the polytope closest to an arbitrary point in $\mathbb{R}^N$. (B.2)

6. An efficient ‘brute force’ algorithm for solving the network reconfiguration problem in a distribution network having few tie lines. (5)

7. A design rule for the selection of cables in a new electrical installation, based on sensitivity analysis of the network topology. (6.1)

8. The formulation of the overload prevention problem in managed DC distribution networks, and an algorithm for determining the effect of a change in demand on the cable currents in a network with a linear discretionary map. (6.2)

9. A linear time-varying model for switch-mode AC–DC converters, giving the approximate dynamical behaviour for all admissible switch states of the converter. (7.6)
10. Three novel switching control schemes for current-driven switch-mode AC–DC converters, allowing ‘optimal’ implementation of space vector modulation for these converters. (8.3)

11. A steady-state analysis for switch-mode converters, yielding constraints on the DC voltage and current for unity-power-factor operation with particular load characteristics. (9.2, 9.3)

12. A synchronising controller design for current-driven AC–DC converters, with good robustness properties and a time response comparable to the period of the natural supply-side LC oscillations. (10.2)

13. Bifurcation theorems for LC circuits, applicable to nonlinear DC power systems with one current-driven external circuit, and predicting the existence of Hopf bifurcations in such circuits. (11.2)

14. The formulation of the LTIBU regulator problem and the application of modified LQ and deadbeat control techniques to such problems. (11.4)

15. A polynomial-time algorithm for the solution of UBLEs of codimension 1. (11.4.2)

16. A regulatory control design for the single-node, single-converter DC power system, based on a saturated deadbeat control scheme and proven in simulations. (11.5)
Part II

Statics
Chapter 3

Managed Distribution Networks: Mathematical Formulation

To commence the investigation of the steady-state operation of managed DC reticulation systems, we present a mathematical model. The ‘management’ concepts introduced in Chapter 1 are here made rigorous. Key definitions, basic to the work in subsequent chapters are also introduced here. Readers more interested in the applications may wish to skip to Section 3.8, which summarises the main concepts.

3.1 Circuit Topology

Our mathematical formulation starts by considering the interconnections of the distribution network itself. The framework adopted is that of classical network theory as developed, for example, in [69] and [48].

We denote by $N$ the number of nodes in the network, a node being any junction in the power circuit. Throughout this thesis, the term external circuit refers to any power source or load connected at a node, or where appropriate, to composite entities formed by parallel combinations of sources and loads. While every external circuit may be associated with a particular node, not every node will necessarily have external circuits connected.

The network has an associated directed graph $\mathcal{N}$ (see Appendix A), whose vertices represent the nodes of the network and whose arcs represent connections. As well as the $N$ vertices corresponding to distribution nodes, labelled $\overline{1}$ through $\overline{N}$ in the sequel, $\mathcal{N}$ contains an additional vertex $\overline{0}$ representing the network ground, to which all arcs
corresponding to external circuits are connected. In the usual manner we associate with each arc $j$ a current $i_j$ and a voltage drop $v_j$, and with each vertex $k$ a potential $u_k$. The zero reference of potential is the network ground, giving $v_\bar{0} = 0$.

Thus far, we have made no formal distinction between those branches which correspond to external circuits, and those which make up the distribution feeders. This distinction is fundamental to our analysis, and we shall therefore view $\mathcal{N}$ as composed from two subgraphs: $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$, as indicated in Figure 3.1. $\mathcal{C}$, the cable network, is a graph on the nodes $\bar{1}$ through $\bar{N}$ and comprises just those arcs (termed cable segments) which connect these nodes. These make up the distribution network proper, and we denote their number by $C$. $\mathcal{E}$, the external network, shall be taken to comprise $N$ arcs, one connected from each of the nodes $\bar{k}$ to the ground vertex $\bar{0}$. The currents $j_k$ in each of these $N$ arcs, or external circuits, satisfy

$$\sum_{k=1}^{N} j_k = 0.$$  \hspace{1cm} (3.1)

In physical terms, $j_k$ is ‘the external circuit current out of node $k$’. If there is no external circuit connected at $\bar{k}$ in the physical network, we set $j_k = 0$. The voltage drops on the arcs in $\mathcal{E}$ may be identified with the node potentials $u_k$.

We assume throughout that the cable network $\mathcal{C}$ is connected, remembering that the ground vertex $\bar{0}$ is not included. It follows that $C \geq N - 1$, and that $\mathcal{C}$ has rank $N - 1$. Since the labelling of nodes is arbitrary, we designate $\bar{N}$ as the datum node,
and denote by $\mathbf{A}$ the corresponding reduced incidence matrix of $\mathcal{C}$ (see Appendix A), with $N - 1$ rows and $C$ columns.

The nullity of $\mathcal{C}$, equal to the number of independent loops, is denoted $\nu$, and satisfies the identity

$$
\nu = C - N + 1 \geq 0.
$$

Note in particular that $\nu = 0$ for a radial (or tree) network, while $\nu = 1$ for a network with a single ring main.

For the steady-state analysis, we model each cable segment $i$ by a series resistance $z_i$. Though series inductance is also present in general, it may be ignored in the DC steady state. Of course, our formulation may also be applied to AC networks by defining a complex impedance $z_i = r_i + j x_i$.

The external circuits in $\mathcal{E}$ corresponding to nodes $\overline{1}$ through $\overline{N - 1}$ are modelled in the steady state by current sinks with values $j_k$, $1 \leq k \leq N - 1$. The $N$th external circuit, on the other hand, is modelled using a voltage source for reasons that will shortly become apparent. This source is taken to have a fixed voltage $u_N = V_0$.

The graph $\mathcal{N}$ for the complete network has rank $N$, and its nullity is $C$ regardless of the value of $\nu$. An external circuit profile for $\mathcal{N}$ is a specification of the sink currents $j_k$ (hence also $j_N$ by (3.1)) and the source voltage $V_0$. Since the cable network contains no sources, an external circuit profile is a source assignment according to the definition in Appendix A.

Consider the spanning tree of $\mathcal{N}$ formed by taking any spanning tree in $\mathcal{C}$ and adjoining the arc from $\overline{N}$ to $\overline{1}$. Since this tree contains all the voltage sources and no current sources, $\mathcal{N}$ is shown to be well-posed by Theorem A.1. It follows that $\mathcal{N}$ has a unique solution for any external circuit profile. This is the reason for introducing the voltage source $V_0$; had all arcs in $\mathcal{E}$ been represented by current sinks, the resulting network would not have been well-posed.

### 3.2 Managed Networks: Demand and Discretion

In Section 1.5 we introduced the concepts of demand and discretion in connection with the Managed DC Bus. For the purpose of analysis, these will be taken to define the more general concept of a ‘managed’ network.
By a managed distribution network, then, we mean one in which the current $j_k$ flowing from node $\overline{k}$ of the power bus into external circuit $k$ is considered to comprise two components:

- the demand component $r_k$, which is under the control of the external circuit, but subject to authorisation from the control system; and
- the discretionary component $d_k$, which is under the control of the bus through current setpoints transmitted periodically to the external circuit controller.

The demand component of the current is sensed positive flowing from the bus to the external circuit, while the discretionary component is sensed positive flowing from the external circuit to the bus. (This reflects the fact that in a conventional system, the supply currents play the role of discretionary currents while the load currents constitute the demand.) Under this sign convention we have

$$j_k = r_k - d_k.$$  \hfill (3.3)

The demand component $r_k$ may in principle take any value, subject to authorised limits. The discretionary component $d_k$ is constrained to be positive, and is not permitted to exceed the discretionary rating $D_k$ nominated by the external circuit:

$$0 \leq d_k \leq D_k, \quad 1 \leq k \leq N.$$ \hfill (3.4)

The canonical example of a discretionary external circuit is a power source, with $D_k$ equal to the maximum output current. Load circuits can, however, be given discretionary capability by combining discretion with demand. In a circuit with no discretion ($D_k = 0$), the current is simply $r_k$. With discretion and a given level of demand, the current may vary between a minimum of $r_k - D_k$ and a maximum of $r_k$. Accordingly, a load circuit which can vary its current on demand between two values $I_1$ and $I_2$ (with $I_1 < I_2$) could fix its demand component as $r_k = I_2$ and request a discretionary rating $D_k = I_2 - I_1$.

If there are $N$ external circuits, then the total demand on the bus is $\sum_{k=1}^{N} r_k$. In order to achieve overall current balance in the steady state, the discretionary setpoints must be chosen so that their total is equal to this figure. We must therefore have at all times

$$\sum_{k=1}^{N} d_k = \sum_{k=1}^{N} r_k.$$ \hfill (3.5)
There is a certain sense in which the concept of ‘management’ elaborated here is specific to DC systems, where circuit quantities have just one degree of freedom. In AC systems, quantities such as $d_k$ and $r_k$ would have to be defined as complex numbers, and the limits on demand and discretion would need to be quantified in a reasonably flexible manner as two-dimensional regions in the complex plane, taking into account magnitude and power-factor constraints. This increased complexity limits the practical application of the ‘demand and discretion’ concept to AC systems, although the concept remains quite valid theoretically. (Indeed, the ‘capability curve’ familiar to operators of synchronous generators [66] is readily interpreted, after rescaling of the axes, as a portrait of the machine’s discretionary capacity.)

### 3.3 The Discretionary Map

#### 3.3.1 Definitions

A *demand profile* is a specification of the demand component $r_k$ for each external circuit, $1 \leq k \leq N$. The demand profile may be represented as a vector $\mathbf{r} \in \mathbb{R}^N$. Given a particular demand profile, it falls on the control system to choose a set of (steady-state) discretionary components $d_k$ for each external circuit, which we may call the *discretionary profile* and represent by a vector $\mathbf{d} \in \mathbb{R}^N$. The discretionary profile must satisfy the constraints (3.4) and (3.5) but may vary freely within these constraints.

We shall assume that the choice of discretionary profile is determined purely from the demand profile at any given time, giving what may be called a ‘deterministic’ steady-state control scheme. This notion may be formalised, giving what we call the *discretionary map*.

Recall that a vector $\mathbf{a} \in \mathbb{R}^N$ is *nonnegative* if $a_i \geq 0$ for $1 \leq i \leq N$. The *balance* of a vector $\mathbf{a} \in \mathbb{R}^N$, denoted $\text{bal} \, \mathbf{a}$, is defined as the algebraic sum of its components:

$$\text{bal} \, \mathbf{a} = \sum_{i=1}^{N} a_i. \quad (3.6)$$

The vector $\mathbf{a}$ is *balanced* if $\text{bal} \, \mathbf{a} = 0$. If we define $\mathbf{j}'$ to be the vector of all external circuit currents $j_k$, then (3.1) simply asserts that $\mathbf{j}'$ is balanced.

Geometrically speaking, the set of all vectors $\mathbf{x}$ such that $\text{bal} \, \mathbf{x} = b$ is a hyperplane in Euclidean $N$-space, orthogonal to the vector $\mathbf{1}_N$ having all unit entries, and passing

61
through the point \(b_1 N\). In particular, the set of all balanced vectors in \(\mathbb{R}^N\) is an 
\((N - 1)\)-dimensional subspace (the orthogonal complement of span \(1_N\)) called the
*balanced hyperplane* of \(\mathbb{R}^N\).

The following definition formalises the concept of a (deterministic) discretionary allocation scheme.

**Definition 3.1** Let \(D \in \mathbb{R}^N\) be nonnegative. A map \(d : \Xi_D \to \mathbb{R}^N\) defined on the
set \(\Xi_D\) of all vectors \(r \in \mathbb{R}^N\) having \(0 \leq \text{bal} \, r \leq \text{bal} \, D\) is a discretionary map with
capacity \(D\) if the following conditions hold for all \(r \in \Xi_D\):

1. both \(d(r)\) and \(D - d(r)\) are nonnegative; and
2. \(\text{bal} \, d(r) = \text{bal} \, r\) (\(d\) preserves balance).

Note that we use an italic \(d\) to denote the discretionary map itself and a bold \(d\) to
denote its image, the discretionary profile. The notation \(d_k\) will occasionally be used
to denote a component of \(d\), as well as the discretionary current at an external circuit;
where a distinction is necessary, the context should indicate to which entity we refer.

### 3.3.2 Geometry of the discretionary map

The discretionary map has an elegant geometric interpretation. In the space \(\mathbb{R}^N\),
\(\Xi_D\) is a region bounded by two parallel hyperplanes, one of these being the balanced
hyperplane. The set of all admissible points \(d(r)\), according to Condition 1, is a
rectangular parallelepiped aligned with the coordinate axes and defined by the two
opposite corners \(0\) and \(D\); we refer to this set as the *discretionary orthotope* \(\Gamma_D\). It
is easily established that \(\Gamma_D\) lies entirely within \(\Xi_D\).

Since the map \(d\) must preserve balance it follows that, given \(r\), the feasible region
for \(d(r)\) is just the intersection of \(\Gamma_D\) with the hyperplane \(\{x : \text{bal} \, x = \text{bal} \, r\}\). Again,
one may readily verify that when \(r\) is balanced or \(\text{bal} \, r = \text{bal} \, D\), this feasible region
reduces to a single point.

Figure 3.2 illustrates this geometric interpretation in the three-dimensional case.

### 3.3.3 Affine Discretionary Maps

A straightforward way to choose the discretionary profile for any given demand pro-
file is simply to share the total discretionary current among the external circuits in
proportion to their discretionary ratings. This gives the following discretionary map:

\[ d_k(r) = \frac{D_k}{\sum_{i=1}^{N} D_i} \sum_{i=1}^{N} r_i. \]  \hspace{1cm} (3.7)

This map is, trivially, a linear function of the demand profile \( r \). The following theorem shows that it is, in fact, the only affine function of \( r \) that satisfies the conditions for a discretionary map, for a given fixed capacity \( D \).

**Theorem 3.1** An \( N \)-dimensional discretionary map \( d \) with capacity \( D \) is affine if and only if it is the map (3.7).

**Proof.** Assume that \( d \) is affine, so that \( d(r) \) is given for all \( r \in \Xi_D \) by a rule of the form

\[ d(r) = Ar + b \]  \hspace{1cm} (3.8)

where \( A \) is a constant matrix and \( b \in \mathbb{R}^N \) a constant vector.

We prove first that \( b = 0 \). Set \( r = 0 \), which is certainly a vector in \( \Xi_D \). We require that \( d(0) \) be nonnegative and that \( \text{bal} \, d(0) = \text{bal} \, 0 = 0 \), which implies that \( d(0) = b = 0 \). Thus, any affine discretionary map must be linear.

Now choose any \( k, 1 \leq k \leq N \) and set \( r = (\text{bal} \, D)e_k \) (where \( e_k \) is the vector whose \( k \)-th component is 1 and whose other components are 0). From (3.8) with \( b = 0 \) we
obtain \( d(r) = (\text{bal} D)a_k \), where \( a_k \) is the \( k \)th column of \( A \). In this case, however, it is evident that there is only one way to choose the components \( d_i \) of \( d(r) \) so that 
\[
\text{bal} \ d = \text{bal} \ r = \text{bal} \ D \quad \text{and} \quad d_i \leq D_i \quad \text{for each} \ i, \text{and that is to set} \ d_i = D_i \ \text{for all} \ i.
\]
It follows that \( a_{ik} = D_i/\text{bal} \ D \) for each \( i \), and since the choice of \( k \) was arbitrary, this suffices to determine all elements of \( A \).

Substituting the values obtained for \( a_{ik} \) into (3.8), we obtain
\[
d_i(r) = \sum_{k=1}^{N} \frac{D_i}{\text{bal} \ D} r_k = D_i \frac{\text{bal} \ r}{\text{bal} \ D}
\]
which is equivalent to (3.7).

Conversely, it is easily seen that the map (3.7) is affine and satisfies all the properties of a discretionary map, for any \( D \) and \( r \in \Xi_D \). √

If \( d_k \) is chosen according to (3.7), then the total current \( j_k \) leaving the bus at the \( k \)th external circuit is
\[
j_k = r_k - \lambda D_k
\]
where
\[
\lambda = \frac{\text{bal} \ r}{\text{bal} \ D}
\]
is the load factor, equal to the total demand divided by the total available discretion.

With a given fixed demand \( r_k \) and discretionary rating \( D_k \), the maximum current drawn by an external circuit is in principle \( r_k \). With an affine discretionary map, however, the maximum is not \( r_k \) but rather \( r_k (1 - D_k/\text{bal} \ D) \), and occurs when this is the only circuit demanding current. The maximum becomes equal to \( r_k \) in the ‘infinite bus’ limit, as the circuit’s share in the total discretionary capacity tends to zero.

### 3.4 The Sensitivity Matrix: Two Formulations

The sensitivity matrix is defined for any distribution network, and relates the currents in the cable network \( C \) to those in the external network \( \mathcal{E} \). The equation expressing the relationship is
\[
i = Sj
\]
(3.11)
where vector \( \mathbf{i} = [i_1 \ i_2 \ldots \ i_C]^T \) represents the \( C \) cable currents \( i_j \), and vector \( \mathbf{j} = [j_1 \ j_2 \ldots \ j_{N-1}]^T \) comprises the first \( N-1 \) external circuit currents \( j_k \). (The current \( j_N \) is omitted as its value is determined from \( \mathbf{j} \) by (3.1).)

As will be seen below, a relationship of the form (3.11) holds for any network, assuming only that the elements of \( \mathbf{C} \) are linear impedances. (3.11) implies that the cable currents do not depend explicitly on the voltage \( V_0 \). Intuitively, this follows from the fact that \( \mathbf{C} \) omits the vertex \( \overline{U} \); its mathematical justification will be obtained below.

The entries \( s_{jk} \) of the sensitivity matrix \( \mathbf{S} \) are dimensionless quantities, expressing the sensitivity of particular cable currents \( i_j \) to changes in particular external circuit currents \( j_k \). In general, they will depend on the impedances of the cable network. We give here two methods for deriving \( \mathbf{S} \), based on the classical node and mesh transformation techniques.

### 3.4.1 Incidence Matrix Formulation

For the network \( \mathcal{N} \) of Figure 3.1, we may write the KCL equations as

\[
\begin{bmatrix}
\mathbf{A} & -\mathbf{I}_{N-1} & 0 \\
0 & \mathbf{1}_{N-1}^T & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{i} \\
\mathbf{j} \\
\mathbf{j}_N
\end{bmatrix} = 0
\]

(3.12)

where \( \mathbf{1}_{N-1} \) denotes an \((N-1)\)-vector with unit entries. (3.12) simply combines (3.1) with the \( N-1 \) linearly independent equations

\[
\mathbf{A}\mathbf{i} = \mathbf{j}
\]

(3.13)

obtained by applying Kirchhoff's Current Law at vertices \( \overline{U} \) through \( \overline{N-1} \).

It is well known that (3.12) can be used to define a node transformation expressing the \( N+C \) voltage drops as linear combinations of \( N \) voltages \( \mathbf{v}_p \), obtained as differences in potential between particular pairs of vertices in \( \mathcal{N} \). In what follows we shall partition \( \mathbf{v}_p \) as \([\mathbf{v}_p' \ \mathbf{v}_p'']^T\) in conformance with the row partitioning of (3.12); it follows that \( \mathbf{v}_p'' \) is a scalar. If we denote by \( \mathbf{u} \) the \((N-1)\)-vector of external network potentials excluding \( V_0 \), and by \( \mathbf{v} \) the \( C \)-vector of cable segment voltage drops, we
\[
\begin{bmatrix}
\mathbf{v} \\
\mathbf{u} \\
V_0
\end{bmatrix}
= \begin{bmatrix}
A^T & 0 \\
-I_{N-1} & 1_{N-1} \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}'_p \\
\mathbf{v}''_p
\end{bmatrix}.
\] (3.14)

It follows immediately from (3.14) that \( \mathbf{v}''_p = V_0 \) and that

\[
\mathbf{v} = A^T \mathbf{v}'_p.
\] (3.15)

Note that this expression for \( \mathbf{v} \) is independent of \( V_0 \); this follows from the form of the equations, and specifically from the fact that none of the arcs of \( C \) connect with the ground vertex \( \overline{0} \). For this reason, \( V_0 \) will not appear in the final expression for \( \mathbf{S} \).

From the second set of equations (3.14), we obtain

\[
u_k = -v'_p + V_0
\]
or equivalently,

\[
v'_p = V_0 - u_k.
\] (3.16)

In other words, the fundamental voltages \( \mathbf{v}'_p \) are simply the potential differences between \( \overline{N} \) and the remaining \( N - 1 \) nodes of the distribution network. (3.16) will later prove useful in obtaining the network voltage profile.

Thus far, the analysis has made no assumptions about the cable models. We now use the cable impedances to relate the cable voltage drops back to the cable currents \( \mathbf{i} \), setting \( y_k = z_k^{-1} \) and \( \mathbf{Y} = \text{diag}\{y_1, y_2, \ldots, y_C\} \) to give

\[
\mathbf{i} = \mathbf{Y} \mathbf{v}.
\] (3.17)

Combining (3.13), (3.17) and (3.15) gives

\[
AYA^T \mathbf{v}'_p = \mathbf{j},
\] (3.18)

a set of \( N - 1 \) equations which, assuming all \( y_k \) are positive, can always be solved for \( \mathbf{v}'_p \). Combining this solution with the equation \( \mathbf{i} = \mathbf{Y}A^T \mathbf{v}'_p \) expressing \( \mathbf{i} \) in terms of \( \mathbf{v}'_p \) we finally obtain our first expression for the sensitivity matrix \( \mathbf{S} \) which we state as a theorem:
Theorem 3.2 Let $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$ be a distribution network in which $\mathcal{C}$ is connected, and let $\mathbf{A}$ be the reduced incidence matrix of $\mathcal{C}$. Let $\mathbf{Y}$ be the matrix of cable admittances corresponding to the arcs of $\mathcal{C}$, and let $\mathbf{i}$ denote the corresponding cable currents. Then

$$i = S \mathbf{j}$$

where $\mathbf{j}$ is the $(N-1)$-vector of external circuit currents, excluding the current at the datum node, and

$$S = \mathbf{Y} \mathbf{A}^T (\mathbf{A} \mathbf{Y} \mathbf{A}^T)^{-1}. \quad (3.19)$$

Thus, one may calculate the sensitivity matrix directly from the reduced incidence matrix $\mathbf{A}$ of the cable network $\mathcal{C}$ and from the admittances $y_k$ of the $C$ cable segments. Calculation of $S$ by this method, however, requires the inversion of an $(N-1) \times (N-1)$ matrix, which is computationally severe for large $N$. Thus, while (3.19) will prove theoretically useful, $S$ is usually better calculated by means of a different expression, to be derived below.

Corollary 3.1 In Theorem 3.2, suppose $\mathcal{C}$ is a tree, or equivalently that $\nu = 0$. Then $\mathbf{A}$ is nonsingular and

$$S = \mathbf{A}^{-1}. \quad (3.20)$$

It follows that the sensitivity matrix is purely topological, and may be calculated even if the cable admittances are not known.

Note that even if $\nu \neq 0$, it necessarily follows from (3.11) and (3.13) and the rank condition on $\mathbf{A}$ that

$$\mathbf{A} \mathbf{S} = \mathbf{I}_{N-1}, \quad (3.21)$$

in other words, $S$ is a right inverse of $A$.

3.4.2 Loop Matrix Formulation

An alternative expression for $S$ is obtained by considering the KVL equations for the network $\mathcal{N}$, which take the following form (assuming $\nu > 0$):

$$\begin{bmatrix}
\mathbf{B}_1 & \mathbf{I}_{N-1} & -\mathbf{1}_{N-1} \\
\mathbf{B}_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{v} \\
\mathbf{u} \\
\mathbf{V}_0
\end{bmatrix} = \mathbf{0}. \quad (3.22)$$
Here, \( \mathbf{v}, \mathbf{u} \) and \( \mathbf{1} \) are as defined above, while the matrices \( \mathbf{B}_1 \) and \( \mathbf{B}_2 \) are obtained as follows. Choose any tree \( \mathcal{T} \) in \( \mathcal{C} \), and let \( e_N \) denote the arc bearing the voltage source \( V_0 \). It is easy to see that \( \mathcal{T}' = \mathcal{T} \cup e_N \) is a tree in \( \mathcal{N} \); its cotree is formed from the remaining \( N - 1 \) external circuits and from \( \nu \) arcs in \( \mathcal{C} \). The total number of cotree arcs accordingly is \( N - 1 + \nu = C \), the nullity of \( \mathcal{N} \). Of the fundamental loops defined by \( \mathcal{T}' \), precisely \( \nu \) involve only arcs in \( \mathcal{C} \), and are in fact identical to the \( \nu \) fundamental loops defined by \( \mathcal{T} \) in \( \mathcal{C} \). We define \( \mathbf{B}_2 \) to be the \( \nu \times C \) matrix corresponding to these loops in \( \mathcal{C} \), giving

\[
\mathbf{B}_2 \mathbf{v} = \mathbf{0}. \tag{3.23}
\]

The remaining \( N - 1 \) fundamental loops each include the arc \( e_N \) and one other external circuit. If we take these loops to be directed as the external circuit arcs, we obtain (3.22), where \( \mathbf{B}_1 \) is the \( (N - 1) \times C \) matrix whose \( k \)th row represents the path from \( \mathcal{N} \) to \( \overline{k} \) in \( \mathcal{T} \). We shall subsequently refer to \( \mathbf{B}_1 \) as the path-set matrix and to \( \mathbf{B}_2 \) as the ring-main matrix.

By analogy with the previous derivation, we obtain the following mesh transformation expressing all currents as linear combinations of \( C \) independent mesh currents \( \mathbf{i}_m \), partitioned as \([i''_m, i'_m]^T\) in conformance with the row partitioning of (3.22):

\[
\begin{bmatrix}
\mathbf{i} \\
\mathbf{j} \\
\dot{j}_N
\end{bmatrix} =
\begin{bmatrix}
\mathbf{B}_1^T & \mathbf{B}_2^T \\
\mathbf{I}_{N-1} & 0 \\
-\mathbf{I}_{N-1}^T & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{i}''_m \\
\mathbf{i}'_m
\end{bmatrix}. \tag{3.24}
\]

From (3.24) we obtain \( \mathbf{i}''_m = \mathbf{j} \) and thence

\[
\mathbf{i} = \mathbf{B}_1^T \mathbf{j} + \mathbf{B}_2^T \mathbf{i}'_m \tag{3.25}
\]

where now \( \mathbf{i}'_m \) are \( \nu \) independent mesh currents corresponding to the fundamental loops of \( \mathcal{T} \) in \( \mathcal{C} \). Defining \( \mathbf{Z} = \text{diag}\{z_1, z_2, \ldots, z_C\} = \mathbf{Y}^{-1} \) gives

\[
\mathbf{v} = \mathbf{Z} \mathbf{i} \tag{3.26}
\]

whence, premultiplying (3.25) by \( \mathbf{B}_2 \mathbf{Z} \) and using (3.23), we obtain

\[
\mathbf{B}_2 \mathbf{Z} \mathbf{B}_2^T \mathbf{i}'_m = -\mathbf{B}_2 \mathbf{Z} \mathbf{B}_1^T \mathbf{j}.
\]

Again we may assume that this system of \( \nu \) equations is soluble for \( \mathbf{i}'_m \). Substitution of this solution into (3.25) gives:

68
Theorem 3.3 Let $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$ be a distribution network, and let $\mathcal{T}$ be any tree in $\mathcal{C}$. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be, respectively, the path-set matrix and ring-main matrix of $\mathcal{C}$ with respect to $\mathcal{T}$. Let $\mathbf{Z}$ be the matrix of cable impedances corresponding to the arcs of $\mathcal{C}$, and let $\mathbf{i}$ denote the corresponding cable currents. Then $\mathbf{i} = \mathbf{Sj}$ where $\mathbf{j}$ is the $(N - 1)$-vector of external circuit currents, excluding the current at the datum node, and

$$
\mathbf{S} = \mathbf{B}_1^T - \mathbf{B}_2^T (\mathbf{B}_2 \mathbf{ZB}_2^T)^{-1} \mathbf{B}_2 \mathbf{ZB}_1^T. \tag{3.27}
$$

(3.27) remains valid if for $\mathbf{B}_2$ we substitute any matrix defined by $\nu$ independent loops in $\mathcal{C}$.

Proof. Only the final statement remains to be proved. Let $\mathbf{B'}_2$ be any matrix formed by $\nu$ independent loops of $\mathcal{C}$; this matrix is therefore of full rank $\nu$. By a standard result of graph theory (see [48]), we have

$$
\mathbf{B'}_2 = \mathbf{DB}_2 \tag{3.28}
$$

where $\mathbf{B}_2$ is a fundamental loop matrix (which we take to be the ring-main matrix) and $\mathbf{D}$ is a nonsingular $\nu \times \nu$ matrix. Substituting $\mathbf{B'}_2$ for $\mathbf{B}_2$ in (3.27) and using (3.28) we obtain

$$
\mathbf{S} = \mathbf{B}_1^T - \mathbf{B}_2^T \mathbf{D}^T (\mathbf{DAD}^T)^{-1} \mathbf{DB}_2 \mathbf{ZB}_1^T \tag{3.29}
$$

where $\mathbf{A} = \mathbf{B}_2 \mathbf{ZB}_2^T$. By assumption, $\mathbf{A}$ is a nonsingular $\nu \times \nu$ matrix, and accordingly we may write

$$(\mathbf{DAD}^T)^{-1} = \mathbf{D}^{-T} \mathbf{A}^{-1} \mathbf{D}^{-1}$$

and with this substitution, it is readily verified that (3.29) reduces to (3.27). √

Although (3.27) appears at first glance more complicated than (3.19) obtained earlier, it is often simpler to calculate since the matrix to be inverted is of dimension $\nu$, which is in most practical situations much smaller than $N$. In particular, if $\nu = 0$ the contribution from the mesh currents $\mathbf{i}_m'$ vanishes and we have:

Corollary 3.2 If $\mathcal{C}$ is a tree, the sensitivity matrix of Theorem 3.3 is given by

$$
\mathbf{S} = \mathbf{B}_1^T. \tag{3.30}
$$
We note in passing that the loop matrix formulation allows for the possibility of short-circuit branches (for which \( z_k = 0 \) but \( y_k \) is undefined) in addition to strictly positive impedances, provided there is no loop formed by such branches. (This ensures that there is at least one cotree consisting entirely of strictly positive impedances, which is sufficient to ensure that \( \det \mathbf{B}_2 \mathbf{Z} \mathbf{B}_2^T \) is nonzero; see [69].)

### 3.4.3 Relationships Between the Two Formulations

Comparing (3.20) and (3.30) one sees immediately that the incidence and path-set matrices are related by \( \mathbf{A}^{-1} = \mathbf{B}_1^T \) when \( \nu = 0 \). This is a restatement of a well-known result in algebraic graph theory, concerning the inverse of the incidence matrix for a tree. (See for example [6, result 5e].) A useful generalisation of this result follows from (3.21) and the identity

\[
\mathbf{A} \mathbf{B}_2^T = 0
\]  

which holds because \( \mathbf{A} \) and \( \mathbf{B}_2 \) are incidence and loop matrices respectively of the same graph \( \mathcal{C} \). Substituting Equation (3.27) for \( \mathbf{S} \) into (3.21) and using (3.31), we immediately obtain

\[
\mathbf{A} \mathbf{S} = \mathbf{A} \mathbf{B}_1^T = \mathbf{I}_{N-1}.
\]  

We find then that \( \mathbf{B}_1^T \) is always a right inverse for \( \mathbf{A} \), regardless of the nullity of \( \mathcal{C} \). (Similarly, \( \mathbf{B}_1 \) is a left inverse for \( \mathbf{A}^T \).)

Consider now the matrix

\[
\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}
\]  

which is a square matrix having \( \mathcal{C} \) rows and columns. It is obtained from the complete voltage-constraint matrix of \( \mathcal{N} \) (see (3.22)) by selecting those columns corresponding to the \( \mathcal{C} \) arcs of the cable network \( \mathcal{C} \). It is not hard to see that these arcs form a cotree of \( \mathcal{N} \), and hence that \( \mathbf{B} \) is not only square but nonsingular. \( \mathbf{B}^T \) therefore possesses an inverse, the first \( N - 1 \) rows of which are given by \( \mathbf{A} \) owing to (3.32) and (3.31). The question of the remaining rows has a simple solution, at least when \( \mathbf{B}_2 \) is the ring-main matrix:

**Theorem 3.4** Let \( \mathcal{C} \) be any connected directed graph on \( N \) vertices with \( \mathcal{C} \) arcs, of nullity \( \nu \), with reduced incidence matrix \( \mathbf{A} \). Let \( \mathcal{T} \) be any tree of \( \mathcal{C} \), and denote by
\( \{k_1, k_2, \ldots, k_v\} \) the arc indices of the cotree. Let \( B_1 \) be the matrix of paths in \( T \) from the datum node to the remaining \( N - 1 \) nodes (the path-set matrix), and let \( B_2 \) be the fundamental loop matrix with respect to \( T \) (the ring-main matrix), in which the loops are directed similarly to the arcs \( k_i \). Let \( B \) be defined as in (3.33).

Now let \( A \) be augmented by the addition of \( v \) rows \( \{e_{k_1}, e_{k_2}, \ldots, e_{k_v}\} \) (where \( e_i \) denotes the \( i \)-th elementary row vector in \( \mathbb{R}^C \) ), to form a new matrix \( A_+ \). Then \( B \) and \( A_+ \) are nonsingular and

\[
A_+B^T = B^T A_+ = I_C. \tag{3.34}
\]

**Proof.** We first give an alternative proof of the identity (3.32) that uses only the graph-theoretic properties of \( A \) and \( B_1 \) and does not assume the existence of the sensitivity matrix \( S \). Let \( C = AB_1^T \). Then the elements of \( C \) are given as

\[
c_{ij} = \sum_{r=1}^{C} a_{ir}b_{jr} \tag{3.35}
\]

where the \( a_{ir} \) represent the connections to \( \overline{i} \), and the \( b_{jr} \) represent the path \( P_j \) directed from the datum node to \( \overline{j} \). Consequently, the nonzero terms in (3.35) correspond to those arcs in \( P_j \) which also intersect \( \overline{i} \). Now, if \( i = j \) there is precisely one such arc \( p \), and in this case the sign conventions ensure that \( a_{ip} \) and \( b_{ip} \) have the same sign, giving \( c_{ii} = 1 \) for all \( i \). If \( i \neq j \), then either \( P_j \) does not intersect \( \overline{i} \), in which case \( c_{ij} = 0 \) immediately, or else there are two arcs of \( P_j \) incident with \( \overline{i} \). Let these arcs be denoted \( p \) and \( q \), in order of traversal in \( P_j \). As with the case \( i = j \), the sign conventions ensure that \( a_{ip} \) and \( b_{jp} \) have similar signs; it may also be easily verified that \( a_{iq} \) and \( b_{jq} \) have opposite signs. Combining these observations, we obtain

\[
c_{ij} = a_{ip}b_{jp} + a_{iq}b_{jq} = 1 - 1 = 0.
\]

The desired result \( C = AB_1^T = I_{N-1} \) follows.

Since \( AB_2^T = 0 \) by a standard result, it remains to examine the last \( v \) rows of the product \( A_+B^T \). Denote by \( c_r \) the \((N - 1 + r)\)th row of this product, equal to

\[
c_r = e_{k_r}B^T = b_{k_r}^T,
\]

that is, to the transpose of the column of \( B \) corresponding to the \( r \)-th arc of the cotree. Now, the first \( N - 1 \) entries of this column are zero, since the arc \( k_r \), not belonging to
Moreover, \( T \), a fortiori does not belong to any of the paths \( P_j \) comprising \( B_1 \). Likewise, all but one of the remaining \( \nu \) entries are zero, since the arc in question appears in only one of the fundamental loops. The unique nonzero entry appears by definition in position \((N - 1 + r)\), and is equal to 1 as we assume that the fundamental loops are directed similarly to the cotree arcs. It follows then that \( c_r = e_{N-1+r} \) and hence \( A_+B^T = I_C \). Since \( A_+ \) and \( B \) are both square matrices it follows that they are nonsingular and are inverses of one another, hence the desired result.

Let \( E \) denote the \( \nu \) rows added to \( A \) to form \( A_+ \) in Theorem 3.4. It follows from the above proof that \( E \) satisfies the identities \( EB_1^T = 0 \) and \( EB_2^T = I_\nu \). Now suppose that in Theorem 3.4 we substitute for \( B_2 \) any loop matrix \( B'_2 \) of \( C \) having rank \( \nu \). Then the product is found as

\[
A_+B^T = \begin{bmatrix} I_{N-1} & 0 \\ 0 & D^T \end{bmatrix} \tag{3.36}
\]

where \( D \) is obtained from (3.28). Indeed, the effect of substituting \( B'_2 \) is to postmultiply the \( \nu \) rightmost columns of the product \( A_+B^T \) by \( D^T \). Let us now denote by \( D^{-T} \) the inverse of \( D^T \), and form

\[
A'_+ = \begin{bmatrix} A \\ D^{-T}E \end{bmatrix} \tag{3.37}
\]

A straightforward calculation then establishes that

\[
A'_+B^T = I_C
\]

and combining this result with some simple properties of the structure of \( E \) gives

**Theorem 3.5** If in Theorem 3.4 any full-rank loop matrix \( B'_2 = DB_2 \) of \( C \) is substituted for the ring-main matrix \( B_2 \), the inverse of \( B^T \) is given by the matrix \( A'_+ \) in (3.37). The matrix \( D^{-T}E \) figuring in the definition of \( A'_+ \) has precisely \( \nu \) nonzero columns, corresponding to the arc indices of the cotree by which \( B_2 \) is defined. If \( B'_2 \) is a fundamental loop matrix, then \( A'_+ \) reduces to the form of Theorem 3.4.

### 3.5 The Voltage-Drop Matrix

A straightforward extension of the results of the previous section allows us to calculate the system voltage profile from the external circuit currents \( j \) (excluding the dependent
current $j_N$) and the voltage $V_0$ imposed at the datum node. (If the distribution system has only one energy source, it is most convenient to identify the datum node with the source, in which case $V_0$ is the source voltage, $j$ the load currents, and $j_N$ the source current.)

From the network analysis in terms of fundamental node-pair voltages, it was found that the voltage transformation induced by the cable network incidence matrix $A$ gave node-pair voltages equal to the voltage drops between the datum node $\bar{N}$ and the remaining $N - 1$ nodes (Equation (3.16)). Again, this has a convenient interpretation in networks with a single source. In terms of $A$ and the cable admittance matrix $Y$, the solution of (3.18) is

$$v'_p = (AYA^T)^{-1}j.$$  \hspace{1cm} (3.38)

Combining with (3.16), we see that the $N - 1$ node potentials $u$ are given by

$$u = 1_{N-1}V_0 - (AYA^T)^{-1}j$$  \hspace{1cm} (3.39)

where, again, $1_{N-1}$ denotes an $(N - 1)$-vector with unit entries.

An alternative expression for the node potentials is obtained from (3.22):

$$u = 1_{N-1}V_0 - B_1v$$  \hspace{1cm} (3.40)

which gives, upon comparison with (3.16), the identity

$$v'_p = B_1v.$$  \hspace{1cm} (3.41)

(This may also be verified using (3.15) and the identity $B_1A^T = I_{N-1}$.)

Making the substitution $v = Zi = ZSj$ in (3.41) we immediately obtain

$$v'_p = B_1ZSj.$$ 

At this point, substituting Equation (3.19) for $S$ immediately yields (3.38) by virtue of the identity $B_1A^T = I_{N-1}$. However, if we instead substitute Equation (3.27) for $S$, we obtain

$$v'_p = (B_1ZB_1^T - B_1ZB_2^T(B_2ZB_2^T)^{-1}B_2ZB_1^T)j,$$

which, though more complicated an expression, usually involves the inversion of a much smaller matrix in practice.

We collect the above results into the next theorem:
Theorem 3.6 Let $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$ be a distribution network. Let $A$, $B_1$, $B_2$, $Y$, $Z$ and $j$ have their usual definitions. If the potential at the datum node is $V_0$, then the potentials at the remaining $N - 1$ external circuits are given by

$$u = 1_{N-1}V_0 - \Sigma j$$

(3.42)

where

$$\Sigma = (AYA^T)^{-1} = B_1ZB_1^T - B_1ZB_2^T(B_2ZB_2^T)^{-1}B_2ZB_1^T.$$  

(3.43)

The nonsingular matrix $\Sigma$ is referred to as the voltage-drop matrix for the network $\mathcal{N}$. It is seen from (3.42) that a general element $\Sigma_{ik}$ of the voltage-drop matrix $\Sigma$ measures the contribution of the $k$th external circuit current $j_k$ to the voltage drop at node $i$, relative to the voltage $V_0$ at the datum node $\overline{N}$. In fact the definition of $\Sigma$ entails that $\Sigma_{ik}$ has the dimensions of impedance.

In a DC network, the matrix $\Sigma$ is both Hermitian and (if there are no short-circuit branches) positive definite. For an AC network, neither statement is true in general; as $\Sigma$ is complex symmetric only. In fact one easily finds that $\Sigma^* = (AY^*A^T)^{-1}$, so that $\Sigma$ is Hermitian if and only if the diagonal matrix $Y$ is Hermitian, which can be the case only if $Y$ is real. We can, however, recover positive definiteness by taking the real part of $\Sigma$.

Lemma 3.1 Let $\Sigma$ be the voltage-drop matrix for a distribution network with arbitrary complex admittances $Y$ having positive resistive components. Then $\Re(\Sigma)$ is positive definite.

Proof. By assumption, $\Re(Y)$ is a diagonal matrix all of whose diagonal entries are positive; hence, $\Re(Y)$ is positive definite. Since in addition $A$ is of full rank and has at least as many columns as rows, it follows that $A\Re(Y)A^T$ is positive definite. Now consider

$$\Re(AYA^T) = \frac{1}{2}(AYA^T + A\bar{Y}A^*).$$

Since $A$ is real this is equal to

$$\frac{1}{2}(AYA^T + A\bar{Y}A^T) = A\Re(Y)A^T,$$

74
and hence \( \text{Re}(AYA^T) \) is positive definite. Let \( \Gamma = AYA^T = \Gamma^T \), then

\[
\text{Re}(\Sigma) = \text{Re}(\Gamma^{-1}) = \frac{1}{2}(\Gamma^{-1} + \tilde{\Gamma}^{-1}) = \frac{1}{2}\Gamma^{-1}(\Gamma + \tilde{\Gamma})\tilde{\Gamma}^{-1} = \Gamma^{-1}\text{Re}(\Gamma)(\Gamma^{-1})^*.
\]

Now \( \text{Re}(\Gamma) \) is positive definite, and \( \Gamma \) (hence \( \Gamma^{-1} \)) is of full rank. It follows that \( \text{Re}(\Sigma) \) is positive definite, proving the Lemma. \( \sqrt{\text{.}} \)

When \( \mathcal{C} \) contains pure imaginary impedances, or short-circuit branches, it is evident from the physical context that \( \text{Re}(\Sigma) \) is positive semidefinite. For pure imaginary impedances the formal proof is analogous to that given above, while for short-circuit branches it relies (for both AC and DC networks) on the second, more complicated, expression for \( \Sigma \) and will not be given here.

### 3.6 The Demand Sensitivity Matrix

The demand sensitivity matrix \( \mathbf{T} \) plays a similar logical role to the sensitivity matrix \( \mathbf{S} \), except that it describes the sensitivity of cable currents to the demand \( r_k \) at an external circuit rather than the net current \( j_k \). Whereas the sensitivity matrix is determined purely by the network topology and impedances, the demand sensitivity matrix is defined relative to a specific discretionary map \( d \). If \( d \) is nonlinear, it will also depend on the initial demand profile \( r_0 \). The relationship expressing the demand sensitivity for a particular network \( \mathcal{N} \) is then in general

\[
\Delta i \approx T(d, r_0) \Delta r.
\]  \hspace{1cm} (3.44)

The following notational convention will prove convenient in the sequel.

**Definition 3.2** Let \( \mathcal{V} \) be a vector space and let \( a \in \mathcal{V}^N \) be an ordered \( N \)-tuple with elements drawn from \( \mathcal{V} \). The diminished tuple \( a_b \in \mathcal{V}^{N-1} \) is the \((N-1)\)-tuple formed from the first \( N - 1 \) elements of \( a \) taken in order.

**Examples 3.1**

1. Let \( \mathcal{V} = \mathbb{C} \). Then for any \( N \)-vector \( a \in \mathbb{C}^N \), the diminished vector \( a_b \in \mathbb{C}^{N-1} \) is the \((N-1)\)-vector formed by removing the last element from \( a \).
2. Let $\mathcal{V} = \mathbb{C}^{M^*}$, a space of row vectors. Then for any $N \times M$ matrix $A$, the diminished matrix $A_b$ is the $(N - 1) \times M$ matrix formed by removing the last row from $A$.

Note that with this convention in mind we may write

$$j = r_v - d_v.$$  \hspace{1cm} (3.45)

### 3.6.1 Affine Discretionary Maps

When $d$ is affine, Theorem 3.1 asserts that it has the general form given by (3.7). Let $\Delta r_k$ denote the amount by which the demand at external circuit $k$ changes at a given instant. Then the change in current in a given cable segment $j$ may be found by combining (3.11) and (3.3) and taking differences:

$$\Delta i_j = \sum_{k=1}^{N-1} s_{jk} (\Delta r_k - \Delta d_k(r))$$  \hspace{1cm} (3.46)

where $s_{jk}$ denotes an element of the sensitivity matrix $S$ and $\Delta d_k(r)$ is shorthand for the difference $d_k(r + \Delta r) - d_k(r)$. Since the map $d$ is affine, this latter quantity has a precise expression in terms of $\Delta r$:

$$\Delta d_k(r) = \nabla d_k \cdot \Delta r = \sum_{i=1}^{N} \frac{D_k}{\text{bal}D} \Delta r_i = \frac{D_k}{\text{bal}D} \text{bal} \Delta r.$$  \hspace{1cm} (3.47)

The net result may be expressed as follows.

**Theorem 3.7** Let a managed DC distribution network with $N$ external circuits and sensitivity matrix $S$ have its discretionary currents determined by an affine discretionary map $d$ with capacity $D$. Then the change $\Delta i$ in steady-state cable currents $i_j$ resulting from a change $\Delta r$ in steady-state external circuit demand is given by

$$\Delta i = T \Delta r$$

where $T$, the demand sensitivity matrix, may be calculated as

$$T = S \left( I_N - \frac{\partial d}{\partial r} \right).$$  \hspace{1cm} (3.48)

$\partial d/\partial r$ denotes the Jacobian of $d$. An explicit formula for $T$ in terms of $D$ is

$$T = S \left( I_N - \frac{1}{\text{bal}D} D I_N^T \right),$$  \hspace{1cm} (3.49)

where $I_N^T$ denotes a row vector containing $N$ unit entries.
The following result, for a change at one external circuit only, follows immediately. For convenience we shall adopt the convention that \( s_{jN} = 0 \) for all \( j \).

**Corollary 3.3** Let a managed DC distribution network have \( N \) external circuits, sensitivity matrix \( S \) and an affine discretionary map \( d \) with capacity \( D \). The change \( \Delta i_j \) in the steady-state cable current \( i_j \) resulting from a change \( \Delta r_k \) in the steady-state demand at external circuit \( k \) is

\[
\Delta i_j = (s_{jk} - C_j) \Delta r_k \tag{3.50}
\]

where

\[
C_j = \frac{\sum_{i=1}^{N-1} s_{ji}D_i}{\text{bal } D}. \tag{3.51}
\]

The numbers \( C_j \) are in one-to-one correspondence with the cable segments and are referred to as the cable sensitivity coefficients.

### 3.6.2 General Discretionary Maps

Equation (3.46) still holds in the case where the discretionary map \( d \) is nonlinear. However, in this case the differences \( \Delta d_k(r) \) are not so readily calculated, and do not depend on \( \Delta r \) alone. If \( \Delta r \) is sufficiently small it is possible to approximate \( \Delta d_k(r) \) by expanding \( d_k \) in a Taylor series about \( r \):

\[
d_k(r + \Delta r) = d_k(r) + \nabla d_k(r) \cdot \Delta r + \frac{1}{2} \Delta r^T H d_k(r) \Delta r + \ldots \tag{3.52}
\]

where \( \nabla d_k(r) \) is the gradient of \( d_k \) and \( H d_k(r) \) the Hessian of \( d_k \), both evaluated at \( r \).

If the above approximation is carried out to first order only, the form of the equations is similar to that for the affine case, and will be adequate for our task provided the quadratic terms in (3.52) are small by comparison with the largest linear term. In sensitivity analyses of nonlinear systems it is customary to work only to first order. We therefore define, for a given initial demand profile \( r_0 \), the demand sensitivity matrix \( T(d, r_0) \) by

\[
T(d, r_0) = S \left( I_N - \frac{\partial d}{\partial r} \bigg|_{r_0} \right), \tag{3.53}
\]

where the Jacobian \( \partial d/\partial r \) is evaluated at \( r_0 \).
Theorem 3.8 Let a managed DC distribution network with N external circuits and sensitivity matrix $S$ have its discretionary currents determined by a discretionary map $d$. For sufficiently small changes in demand $\Delta r$ about an initial demand profile $r_0$, the change $\Delta i$ in cable currents may be approximated as

$$\Delta i \approx T(d, r_0) \Delta r$$

(3.54)

where $T(d, r_0)$ is given by (3.53).

Irrespective of the magnitude of $\Delta r$, the right hand side of (3.54) is an upper bound for $\Delta i$ if $d$ is convex, and a lower bound for $\Delta i$ if $d$ is concave.

Proof. If $d$ is convex, it follows by definition that, for all $\Delta r$ and $k$,

$$\Delta d_k(r) \geq \nabla d_k(r_0) \cdot \Delta r.$$

Combining this with (3.46) it follows that

$$\Delta i_j \leq \sum_{k=1}^{N-1} s_{jk} \left( \Delta r_k - \sum_{i=1}^{N} \frac{\partial d_i}{\partial r_k} \bigg|_{r_0} \Delta r_i \right)$$

and hence, by definition of $T$,

$$\Delta i \leq T(d, r_0) \Delta r$$

which is the desired result. A similar argument establishes the lower bound in the case of concave $d$. √

Corollary 3.4 Let a managed DC distribution network have $N$ external circuits, sensitivity matrix $S$ and discretionary map $d$, and consider a change in demand $\Delta r_k$ at the $k$th external circuit from a given initial demand profile $r_0$. Suppose there exists $R > 0$ such that for all $1 \leq i \leq N$, $0 < \rho \leq R$ and $0 < \epsilon < 1$ we have

$$\frac{\partial^2 d_i}{\partial r_k^2} \bigg|_{r_0} \rho^2 \ll \frac{\partial d_i}{\partial r_k} \bigg|_{r_0} \rho.$$  

(3.55)

Then for all $\Delta r_k \leq R$ and $1 \leq j \leq C$ the change $\Delta i_j$ in the $j$th cable current is well approximated as

$$\Delta i_j \approx \left( s_{jk} - \sum_{i=1}^{N-1} s_{ji} \frac{\partial d_i}{\partial r_k} \bigg|_{r_0} \right) \Delta r_k.$$  

(3.56)

Irrespective of the magnitude of $\Delta r_k$, the right hand side of (3.56) is an upper (lower) bound for $\Delta i_j$ if $d$ is convex (concave) with respect to $r_k$. 

78
The condition (3.55) provides an explicit test (using Lagrange’s form for the remainder of a Taylor series) of the validity of the linear approximation for changes in demand at a single external circuit. A similar condition may be derived in the more general case, but we refrain from stating it here. Greater accuracy may be obtained by including higher-order terms in the approximation (3.52), but this of course complicates the calculation of $\Delta i$.

### 3.7 Sensitivity Matrices and Voltage-Drop Matrices for Sparse Networks

The results of the previous sections apply to all distribution networks whose cable network is connected. In practice one often encounters simple network structures with only a small number of interconnections. In graph-theoretic terms, such networks have low nullity.

Specific practical examples come easily to hand. Conventional power distribution systems are usually constrained to operate as radial networks, so that their nullity is zero and the structure is that of a tree. It is envisaged that practical implementations of the Managed DC Bus will be constrained such that the number of cable segments does not exceed the number of bus nodes. Since the number of cable segments in a connected network is at least $N - 1$, it follows that the nullity of a Managed DC Bus cable network is restricted to the values 0 and 1; thus, there can be at most one ring main in a Managed DC Bus installation. For most purposes this is not an unduly harsh restriction, but in any case the theory does not depend on it.

In this section we investigate the special cases $\nu = 0$ and $\nu = 1$ in more detail, with a view to obtaining useful results applicable to many practical networks.

#### 3.7.1 Tree Networks ($\nu = 0$)

In this case we already know from Corollaries 3.1 and 3.2 that

$$S = A^{-1} = B_1^T,$$

so that the sensitivity matrix relating the cable currents to the external circuit currents may be calculated either directly from the path-set matrix or by inversion of the reduced incidence matrix, without any knowledge of the cable impedances.
It is evident also that in a tree network, for a given labelling of the nodes and
cable segments and a given choice of orientations, the path-set matrix $B_1$ is unique,
since there is only one possible choice of tree $T$ from which to define it. Common
sense requires that this be the case, since $S$ is by definition unique up to relabelling
of the nodes and cable segments and reorientation of the cable segments.

In a tree network, the voltage-drop matrix $\Sigma$ also has a greatly simplified form:

$$\Sigma = (AYA^T)^{-1} = A^{-T}ZAZ^{-1} = B_1ZB_1^T.$$ \hspace{1cm} (3.57)

The calculation of $\Sigma$ is aided by the following theorem:

**Theorem 3.9** Let $N = C \cup E$ be a tree network, and let $P_k$ denote the path from $\overline{N}$
to $\overline{K}$ in the tree $C$, for $1 \leq k \leq N - 1$. If $\Sigma$ is the voltage-drop matrix of $N$, then

1. $\Sigma_{ii}$ is the total series impedance of path $P_i$; and

2. $\Sigma_{ij} = \Sigma_{ji}$ is the total series impedance of the path $P_{ij}$, where $P_{ij} = P_i \cap P_j$.

**Proof.** From Equation (3.57) we obtain

$$\Sigma_{ij} = \sum_{k=1}^{C} Z_k b_{ik} b_{jk},$$

where $b_{ik}$ denotes an element of $B_1$ in the usual manner. If $i = j$ we obtain a sum
of terms of the form $Z_k (b_{ik})^2$, which is precisely the total impedance of the path $P_i$, 
by definition of $B_1$. If $i \neq j$, then because $P_i$ and $P_j$ share the same origin and
are defined with respect to the same tree, the only nonzero terms comprising $\Sigma_{ij}$ are
those corresponding to cable segments in $P_{ij}$, and in all such terms we have $b_{ik} = b_{jk}$.

The stated result follows. \checkmark

### 3.7.2 Single-Ring Networks ($\nu = 1$)

Analysis of networks containing a single ring main is greatly simplified by the fact that
the ring-main matrix $B_2$ reduces to a row vector. Choose an arbitrary orientation for
the ring main, and let the vector $\gamma$ be defined such that $\gamma_k = 1$ if cable segment $k$
is part of the ring main and similarly oriented, $-1$ if it is oppositely oriented and $0$
if it is not in the ring main. Then we may write $B_2 = \gamma^T$, and $B_2 Z B_2^T = \gamma^T Z \gamma$ is
a scalar, equal to the total series impedance $z_{\text{loop}}$ of the ring main. Accordingly, the sensitivity matrix calculated according to Equation (3.27) reduces to

$$S = B_1^T - z_{\text{loop}}^{-1} \gamma \gamma^T Z B_1^T. \tag{3.58}$$

Working from right to left in this expression, we observe that:

- $ZB_1^T$ is the same as $B_1^T$, with the $k$th row multiplied by $z_k$ for all $k$. This simply means that each element of $B_1^T$ is weighted by the impedance of the corresponding cable segment.

- $\gamma^T ZB_1^T$ is a row vector $\zeta^T$ with $N - 1$ entries. The $k$th entry $\zeta_k$ is equal to the scalar product of $\gamma$ and the $k$th column of $ZB_1^T$; thus, it consists of sums and differences of cable impedances. More precisely, the impedances which appear in $\zeta_k$ are those of the ring-main cable segments traversed on the path from $\mathcal{N}$ to $\mathcal{T}_k$ in the tree $\mathcal{T}$ by which $B_1$ is defined. The sign of a given impedance $z_j$ appearing in $\zeta_k$ is given by the relative orientation of the ring main and the path $\mathcal{P}_k$.

- $\gamma \gamma^T ZB_1^T = \gamma \zeta^T$ is a unity-rank matrix $C$, of similar dimensions to $B_1^T$, whose elements are sums and differences of cable impedances. Rows corresponding to zero entries of $\gamma$ (that is, to cable segments not in the ring main) consist entirely of zeros; other rows are equal either to $\zeta^T$ or to $-\zeta^T$, depending on the orientation of the corresponding cable segment relative to the ring main.

- Finally, $S$ is given by

$$S = B_1^T - z_{\text{loop}}^{-1} C \tag{3.59}$$

with $C$ as above. Accordingly, $S$ has the general form $z_{\text{loop}}^{-1} S'$ where the entries of $S'$ are sums and differences of cable impedances. It follows that the entries of $S$ are dimensionless, as one would expect.

Referring back to the derivation of $S$ by the loop matrix approach, we see that the scalar quantity $z_{\text{loop}}^{-1} \zeta^T j$ is the mesh current $\dot{v}_m$ corresponding to the single ring main. Equation (3.25) in this context simply states that, if one of the ring-main segments were to be removed to form a tree network, the current in each remaining ring-main segment would change by an amount equal to the current in the segment removed, while currents in other cable segments would be unaffected.
In the light of the above comments, a canonical form for $S$ can be determined quite simply by an appropriate choice of cable orientations and $T$.

**Theorem 3.10** Let $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$ be a single-ring distribution network with $N$ nodes. For $1 \leq j \leq N$, let $\rho(j)$ denote the node in the ring main closest to $j$, under the usual definition of distance in a graph.

Let the reference tree $T$ and the orientations of arcs in $\mathcal{C}$ satisfy the following conventions:

1. the cable segments forming the ring main are given a uniform orientation, corresponding to the orientation of the ring main itself;

2. the cable segments not forming the ring main are oriented toward the node further from $\mathcal{N}$; and

3. the segment removed from $\mathcal{C}$ to form $T$ is the (unique) segment in the ring main directed toward $\rho(N)$.

Let $\hat{z}_j$ denote the total series impedance on the directed path in the ring main from $\rho(N)$ to $\rho(j)$, and let $\hat{z}'_j = z_{\text{loop}} - \hat{z}_j$ denote the impedance of the complementary path from $\rho(j)$ to $\rho(N)$. In the case $\rho(j) = \rho(N)$, we set $\hat{z}_j = 0$ and $\hat{z}'_j = z_{\text{loop}}$.

Then the elements of the sensitivity matrix $S$ are given as follows:

1. if segment $i$ belongs neither to the ring main nor to the path $P_j$ from $\mathcal{N}$ to $j$, then $s_{ij} = 0$;

2. if segment $i$ belongs to the path $P_j$ but not to the ring main, then $s_{ij} = 1$;

3. if segment $i$ belongs to the ring main and is part of the directed path from $\rho(N)$ to $\rho(j)$, then $s_{ij} = \hat{z}'_j / z_{\text{loop}}$;

4. finally, if segment $i$ belongs to the ring main and is not on the directed path from $\rho(N)$ to $\rho(j)$, then $s_{ij} = -\hat{z}_j / z_{\text{loop}}$.

**Proof.** Convention 1 of the Theorem ensures that all elements of $\gamma$ are zeros and ones; together with convention 3 this ensures that each path $P_k$ in $T$ is uniform with the same orientation as the ring main, hence that all impedances appearing in the matrix $C$ appear with a positive sign. It follows that if $i$ is a ring-main segment then $c_{ij} = \hat{z}_j$ as defined above, while if $i$ is not a ring-main segment then $c_{ij} = 0$ for all $j$. 

82
These conventions also ensure that all elements of \( B_1 \) are zeros and ones; equivalently, that the \((ij)\)-th element of \( B_1^\top \) is equal to 1 if cable segment \( i \) belongs to the directed path \( P_j \) from \( N \) to \( j \), and zero otherwise.

The conclusions of the Theorem follow in a straightforward manner from these observations and (3.59). √

**Corollary 3.5** In any single-ring network with arbitrary orientations, all elements \( s_{ij} \) of the sensitivity matrix satisfy \(|s_{ij}| \leq 1\). If in addition the ring main contains no short-circuit elements, then \( |s_{ij}| < 1 \) whenever \( i \) is a ring-main segment.

**Proof.** These bounds on \( s_{ij} \) are certainly satisfied by the canonical form of \( S \) as determined by Theorem 3.10. Indeed, we have \( 0 \leq \hat{z}_j \leq z_{\text{loop}} \) and \( 0 \leq \hat{z}'_j \leq z_{\text{loop}} \) for all \( j \), which ensures that \(|s_{ij}| \leq 1\) for all \( i \) and \( j \), including the case where \( i \) is a ring-main segment. If in addition all ring-main impedances are strictly positive, then we also have \( \hat{z}_j < z_{\text{loop}} \) and \( \hat{z}'_j < z_{\text{loop}} \), except in the case where \( \rho(j) = \rho(N) \), in which case \( \hat{z}_j = 0 \) and case 3 of the Theorem never arises, so the value of \( \hat{z}'_j \) is irrelevant.

Now, it is evident from the definition of the sensitivity matrix that the values of its elements must be independent of the choice of \( T \), and also that the only effect of an arbitrary change in cable segment orientations is to change the sign of certain rows. Since the statement to be proved is independent of sign, it follows that we may relax the assumptions leading to the canonical form of \( S \), and conclude that the statement holds for any single-ring network. √

### 3.8 Chapter Summary

A distribution network is a directed graph \( \mathcal{N} = \mathcal{C} \cup \mathcal{E} \) comprising a cable network \( \mathcal{C} \) with \( N \) nodes and \( C \) branches, and an external network \( \mathcal{E} \) of \( N \) branches representing supply and load modules. The nullity of \( \mathcal{C} \) is \( \nu = C - N + 1 \) and equals the number of independent loops or ‘ring mains’ in \( \mathcal{C} \).

For the steady-state analysis, we represent each branch of \( \mathcal{C} \) by a resistance \( r_j \), which in the AC case becomes an impedance \( z_j = r_j + jx_j \). \( Z \) is the diagonal matrix of branch impedances \( z_j \), and \( Y = Z^{-1} \) the matrix of branch admittances. The \( N \)th node is arbitrarily denoted a voltage reference or ‘datum node’; the first \( N - 1 \) branches of \( \mathcal{E} \) are represented by current sinks \( j_k \), while the \( N \)th branch is a voltage

83
source $V_0$. $A$ denotes the node-to-branch incidence matrix of $C$ with the $N$th row removed; it has $C$ columns and $N - 1$ linearly independent rows.

A managed distribution network is a network in which the external circuit currents $j_k$ are divided into discretionary and demand components, $(-d_k)$ and $r_k$ respectively. In a managed DC network, the discretionary rating $D_k$ specifies the maximum value of $d_k$ for each external circuit; the minimum is zero by convention. Managed AC networks are beyond the scope of this thesis.

A discretionary map with capacity $D$ is a map $d : \Xi \rightarrow \mathbb{R}^N$ assigning a feasible discretionary profile $d$ to each demand profile $r$ such that $0 \leq \text{bal } r \leq D$. There is only one affine discretionary map with a given capacity $D$; this is the linear map $d_k = D_k \text{ bal } r / \text{ bal } D$.

Let $j$ denote the vector of sink currents $j_1$ through $j_{N-1}$ in $C$, and let $i$ be the vector of cable currents $i_1$ through $i_C$ in $C$. We then have $i = Sj$ where the sensitivity matrix $S$ is given by either of two expressions:

$$S = YA^T(AYA^T)^{-1} = B_1^T - B_2^T(B_2ZB_2^T)^{-1}B_2ZB_1^T$$

where $B_1$ and $B_2$ are the path-set and ring-main matrices defined in Section 3.4.2. If $C$ is a tree, then $S = A^{-1} = B_1^T$.

Similarly, let $u$ denote the vector of node potentials $u_1$ through $u_{N-1}$. (We have $u_N = V_0$ by definition.) Then $u = 1_{N-1}V_0 - \Sigma j$ where the voltage-drop matrix $\Sigma$ again has two equivalent expressions:

$$\Sigma = (AYA^T)^{-1} = B_1ZB_1^T - B_1ZB_2^T(B_2ZB_2^T)^{-1}B_2ZB_1^T.$$  

$\Sigma$ is positive definite in a DC network with positive cable impedances. For an AC network $\Sigma$ is only complex symmetric, but Re ($\Sigma$) is positive definite provided all cable impedances have positive real part. If $C$ is a tree, then $\Sigma$ has a simple definition in terms of the impedances of certain paths in $C$; see Section 3.7.1.

The demand sensitivity matrix $T$ expresses the sensitivity of the cable currents $i$ to the demand profile $r$. If the discretionary map $d$ is affine, then $\Delta i = T\Delta r$ where

$$T = S\left(I_N - \frac{1}{\text{bal } D} D I_N^T\right).$$

For a change in demand at one external circuit only, we have $\Delta i_j = (s_{jk} - C_j)\Delta r_k$ where the coefficients $C_j$ are the same for each external circuit $k$. 

84
With a general discretionary map, we can define demand sensitivity matrices only locally. For changes in demand about an initial value \( r_0 \) we have \( \Delta i \approx T(d, r_0) \Delta r \) where

\[
T(d, r_0) = S \left( I_N - \frac{\partial d}{\partial r} \bigg|_{r_0} \right).
\]

This is an upper bound for \( \Delta i \) if \( d \) is convex, and a lower bound if \( d \) is concave.
Chapter 4

Loss Minimisation by Discretionary Control

Section 3.3 explored the framework for choosing the steady-state discretionary currents $d$ in a managed distribution network. That section gave an indication of the breadth of choice possible when external circuits take advantage of these management functions, but did not attempt to compare discretionary profiles or maps on the basis of any performance criteria. Here, we examine one such performance criterion: the minimisation of the steady-state losses in the cable network.

The chapter is organised as follows. In Section 4.1 we derive expressions for the network losses and state in formal terms the problem to be solved. Section 4.2 shows how this may be restated as a standard quadratic optimisation problem, taking into account the constraints on the discretionary map, and describes the application of the Kuhn-Tucker theorem to the problem. A convenient approach to the numerical solution of this optimisation problem is sketched in Section 4.3, and applied to concrete examples in Section 4.4.

The reader will note that this chapter is concerned chiefly with practicalities, proceeding quite directly from a problem to its solution. The problem we state nonetheless possesses an elegant geometry, which is explored at some length in Appendix B. While this approach does not yield a simple solution to the problem as do the least-squares methods of Section 4.3, it does provide important geometrical insights relating to the overload prevention scheme of Chapter 6.
4.1 Loss Calculations and Problem Statement

Let \( \mathcal{N} = \mathcal{C} \cup \mathcal{E} \) be a distribution network, and let \( P_{\text{loss}} \) denote the total loss in the cable network \( \mathcal{C} \), equal to the sum of the real power dissipated in each of the \( C \) cable segments. In keeping with the spirit of the previous chapter, we work in full generality, restricting our attention to DC networks only when the discretionary framework is explicitly introduced.

An expression for the loss, in terms of the external circuit currents, may be derived in two ways. The first is to recognise that \( P_{\text{loss}} \) is the net real power delivered from the external network \( \mathcal{E} \) to the cable network \( \mathcal{C} \); equivalently,

\[
P_{\text{loss}} = \text{Re} (-j_N^* V_0 - j^* u) = \text{Re} (j^* 1_{N-1} V_0 - j^* u) = \text{Re} (j^* \Sigma j),
\]

where the voltage equation (3.42) has been used at the final step. Alternatively, one may sum the cable losses directly and write

\[
P_{\text{loss}} = \text{Re} (i^* z) = \text{Re} (j^* S^* Z S j),
\]

and a quick calculation will verify that in fact \( S^* Z S = \Sigma \), so the two derivations are equivalent. We therefore have

**Theorem 4.1** The total loss in the cable network \( \mathcal{C} \) of a distribution network is

\[
P_{\text{loss}} = \text{Re} (j^* \Sigma j) = j^* \text{Re} (\Sigma) j, \tag{4.1}
\]

where \( \Sigma \) is the voltage-drop matrix for the network.

The operation of taking the real part of \( \Sigma \) is necessitated in this theorem (for AC networks only) by the fact that \( \Sigma \) is symmetric, but not Hermitian. In DC networks, all quantities are real and the loss is given as \( P_{\text{loss}} = j^T \Sigma j \) where \( \Sigma \) is a real symmetric matrix. If \( \mathcal{C} \) contains no lossless branches then \( \text{Re} (\Sigma) \) is positive definite; otherwise it is positive semidefinite.

Equation (4.1) involves only the first \( N-1 \) external circuit currents; it follows that the demand \( r_N \) and discretionary setpoint \( d_N \) for the \( N \)th external circuit will not figure explicitly in the loss calculation. The constraints on the setpoints \( d_1 \) through \( d_{N-1} \) do however depend on the conditions at the \( N \)th external circuit, in the sense that

88
$d_N$ is determined from the remaining setpoints through the requirement to preserve balance, but must nevertheless lie in the interval $[0, D_N]$.

Making use of the ‘diminished vector’ convention introduced in Section 3.6, we may write

$$P_{\text{loss}} = \text{Re} \left( (r_b^* - d_b^*) \Sigma (r_b - d_b) \right)$$

$$= \text{Re} \left( d_b^* \Sigma d_b - r_b^* \Sigma d_b - d_b^* \Sigma r_b + r_b^* \Sigma r_b \right).$$ \hspace{1cm} (4.2)

This quadratic equation for $P_{\text{loss}}$, and its minimisation with respect to $d$, may be viewed in a number of ways.

Looking at (4.2) purely algebraically, we notice that the last term $r_b^* \Sigma r_b$ is constant for any given steady-state demand profile $r$, so may be omitted if the minimisation is with respect to $d$ alone (as is the case here). The two cross-product terms have the same real part, and combine accordingly. We therefore arrive at the following criterion:

**Minimal Loss Criterion (1)** The loss in a managed distribution network with voltage-drop matrix $\Sigma$ is said to be minimised if the cost function

$$J_\Sigma(d, r) = d_b^* \text{Re} (\Sigma) d_b - 2 \text{Re} (r_b^* \text{Re} (\Sigma) d_b)$$ \hspace{1cm} (4.3)

is minimal with respect to $d_b$.

(4.2) also admits a geometric interpretation if we use the positive definiteness of $\text{Re} (\Sigma)$ to define a norm on $\mathbb{C}^{N-1}$.

**Definition 4.1** Let $\Sigma$ be the voltage-drop matrix of a distribution network with $N$ nodes. Given two vectors $a, b \in \mathbb{C}^{N-1}$ we define their $\Sigma$-product as

$$(a, b)_\Sigma = b' \text{Re} (\Sigma) a.$$ \hspace{1cm} (4.4)

This is an inner product on $\mathbb{C}^{N-1}$, and induces a norm (the $\Sigma$-norm) defined for any vector $a \in \mathbb{C}^{N-1}$ as

$$\|a\|_\Sigma = \sqrt{a' \text{Re} (\Sigma) a}.$$ \hspace{1cm} (4.5)

For any $a, b \in \mathbb{C}^{N-1}$ the $\Sigma$-distance between $a$ and $b$ is defined from the $\Sigma$-norm as

$$d_\Sigma(a, b) = \|a - b\|_\Sigma = \sqrt{(a - b)^* \text{Re} (\Sigma) (a - b)}.$$ \hspace{1cm} (4.6)
Definition 4.1 has two trivial but useful consequences:

1. If \( \mathbf{j} \) is the vector of the first \( N - 1 \) external circuit currents, then the loss in the cable network is \( P_{\text{loss}} = \|\mathbf{j}\|^2_{\Sigma} \).

2. If \( \mathbf{r}_s \) and \( \mathbf{d}_s \) are the diminished demand and discretion vectors respectively, then the loss is \( P_{\text{loss}} = d_{\Sigma}^2(\mathbf{r}_s, \mathbf{d}_s) = d_{\Sigma}^2(\mathbf{d}_s, \mathbf{r}_s) \).

This motivates the following restatement of the Minimal Loss Criterion:

**Minimal Loss Criterion (2)** The loss in a managed distribution network with voltage-drop matrix \( \Sigma \) is said to be minimised if the discretionary setpoints \( \mathbf{d} \) are chosen to minimise the \( \Sigma \)-distance between \( \mathbf{d}_s \) and the diminished demand profile \( \mathbf{r}_s \).

It is readily seen that the two statements of the Minimal Loss Criterion are equivalent. We may accordingly state the fundamental problem to be solved, at least in the DC case, as follows:

**Loss Minimisation Problem for Managed DC Networks** Given \( N, \mathbf{D} \in \mathbb{R}^N \) nonnegative, and a real voltage-drop matrix \( \Sigma \), find a discretionary map \( d : \Xi_{\mathbf{D}} \to \mathbb{R}^N \) with capacity \( \mathbf{D} \) such that the Minimal Loss Criterion is satisfied for all \( \mathbf{r} \in \Xi_{\mathbf{D}} \) and \( \mathbf{d} = d(\mathbf{r}) \), subject to the intrinsic constraints on \( d \).

### 4.2 Solving the Loss Minimisation Problem in a DC Network

#### 4.2.1 Restatement in Standard Form

It is readily seen that the Loss Minimisation Problem is a quadratic (hence convex) optimisation problem with linear constraints, amenable to solution by a number of standard techniques. When applying these techniques it is convenient to state the problem in the ‘standard form’ (see [44]):

Given \( \Sigma \in \mathbb{R}^{(N-1) \times (N-1)}, \mathbf{D} \in \mathbb{R}^N, \mathbf{r} \in \mathbb{R}^N \)

minimise \( d_{\Sigma}(\mathbf{d}_s, \mathbf{r}_s) \) \hspace{1cm} (4.7)

subject to \( G(\mathbf{d}) \leq 0, H(\mathbf{d}) = 0 \)
Where $\Sigma$ is real positive definite, $D$ is nonnegative, $r$ satisfies $0 \leq \text{bal} \, r \leq \text{bal} \, D$, and $G: \mathbb{R}^N \rightarrow \mathbb{R}^{2N}$ and $H: \mathbb{R}^N \rightarrow \mathbb{R}$ are mappings defined by

\[ G_i(d) = -d_i \]
\[ G_{N+i}(d) = d_i - D_i \]
\[ H(d) = \text{bal} \, d - \text{bal} \, r \] (4.8)

where $i$ runs from 1 to $N$. (Here we make the implicit assumption that $D_k > 0$ for all $k$; Section 4.2.3 discusses the case where some of the $D_k$ are zero.)

Note that if $C$ contains lossless branches, making $\Sigma$ only positive semidefinite rather than positive definite, we may obtain an equivalent distribution network $N'$ with $N' < N$ nodes and a positive definite voltage-drop matrix by contracting the lossless branches in $C$ and combining the associated external circuits.

As stated, the problem involves $2N$ inequality constraints and one equality constraint, with a quadratic objective function defined on $N - 1$ out of the $N$ variables. The problem may be analysed most simply as that of determining the minimum distance from $r$ to the convex set defined by (4.8), but in order to do so we must reformulate the problem in the $(N - 1)$-dimensional space of diminished vectors with the inner product defined in (4.4).

**Definition 4.2** Given a vector $r \in \mathbb{R}^N$ and a real positive definite $(N - 1) \times (N - 1)$ matrix $\Sigma$, let $\Lambda_{\Sigma, r}$ be the Hilbert space obtained from $\mathbb{R}^{N-1}$ with the $\Sigma$-product (4.4) as inner product. With each element $x \in \Lambda_{\Sigma, r}$ we associate a fictitious $N$th coordinate, or auxiliary coordinate, defined by

\[ x_N = \text{bal} \, r - \text{bal} \, x. \] (4.9)

Note that this definition of auxiliary coordinate introduces no ambiguities, even when $x = r$, as in this case the auxiliary coordinate and the $N$th coordinate of $r$ are identical. Definition 4.2 ensures that if $d_0 \in \Lambda_{\Sigma, r}$, then the vector $d$ defined from $d_0$ together with the auxiliary coordinate $d_N$ satisfies the equality constraint $H$ of the Loss Minimisation Problem. We may therefore regard this problem as a minimum-distance problem in $\Lambda_{\Sigma, r}$ subject only to the inequality constraints $G \leq 0$. 

91
4.2.2 Kuhn-Tucker Conditions and Active Constraints

For this problem, the Kuhn-Tucker theorem (see, for example, [44] or [40]) asserts that, if \( \hat{\mathbf{d}} = \left[ \hat{d}_i^T \hat{d}_N \right]^T \) is the optimal solution vector, then there exist nonnegative vectors \( \lambda \in \mathbb{R}^N \) and \( \mu \in \mathbb{R}^N \) such that

\[
\lambda^T \hat{\mathbf{d}} + \mu^T (\mathbf{D} - \hat{\mathbf{d}}) = 0 \tag{4.10}
\]

and

\[
\Sigma(\hat{\mathbf{d}}_i - \mathbf{r}_i) - \lambda_i + \mu_i + 1_{N-1}(\lambda_N - \mu_N) = 0. \tag{4.11}
\]

Given \( \mathbf{r} \in \Xi_{\mathbf{D}} \), there will be anywhere between 0 and \( N \) constraints active at the optimal solution \( \hat{\mathbf{d}} \). (The constraints associated with \( \lambda_i \) and \( \mu_i \) certainly cannot be active simultaneously.) If it is known \textit{a priori} which constraints are active at \( \hat{\mathbf{d}} \), then of the \( 3N \) unknowns \( \hat{d}_i, \lambda_i, \mu_i, 1 \leq i \leq N \), precisely \( 2N \) are fixed by this knowledge together with (4.10). Equation (4.11), together with the definition of \( \hat{d}_N \) from \( \hat{\mathbf{d}} \), then provides \( N \) linear equations connecting the remaining \( N \) unknowns.

It is known from well-established theory [44] that the numbers \( \lambda_i \) and \( \mu_i \) provide a measure of the discrepancy between the constrained optimum \( \hat{\mathbf{d}} \) and the unconstrained optimum, which in this case is simply \( \mathbf{r} \). This idea leads to the following result connecting these numbers with the network loss.

**Lemma 4.1** Let \( \hat{\mathbf{d}}, \lambda \) and \( \mu \) be the Kuhn-Tucker solution to the Loss Minimisation Problem for a managed distribution network \( \mathcal{N} \) with voltage-drop matrix \( \Sigma \) and demand profile \( \mathbf{r} \). If the discretionary currents in \( \mathcal{N} \) are set to \( \hat{\mathbf{d}} \), then the loss in the cable network \( \mathcal{C} \) is given as

\[
P_{\text{loss}} = \sum_{i=1}^{N} (-r_i)\lambda_i + \sum_{i=1}^{N} (r_i - D_i)\mu_i. \tag{4.12}
\]

**Proof.** Equating \( \mathbf{d} \) with \( \hat{\mathbf{d}} \) and premultiplying (4.11) by \( (\mathbf{d}_b - \mathbf{r}_b)^* \) we obtain

\[
P_{\text{loss}} &= d_S^2(\mathbf{d}_b, \mathbf{r}_b) = (\mathbf{d}_b - \mathbf{r}_b)^* \Sigma (\mathbf{d}_b - \mathbf{r}_b) \\
&= (\mathbf{d}_b - \mathbf{r}_b)^* (\lambda_b - \mu_b - 1_{N-1}(\lambda_N - \mu_N)) \\
&= \sum_{i=1}^{N-1} (-r_i)\lambda_i + \sum_{i=1}^{N-1} (D_i - r_i)\mu_i - \text{bal}(\mathbf{d}_b - \mathbf{r}_b)(\lambda_N - \mu_N) \\
&= \sum_{i=1}^{N-1} (-r_i)\lambda_i + \sum_{i=1}^{N-1} (r_i - D_i)\mu_i + (d_N - r_N)(\lambda_N - \mu_N)
\]

92
which, taking into account the restrictions on \( d_N \), \( \lambda_N \) and \( \mu_N \) imposed by (4.10), is equivalent to (4.12). \( \checkmark \)

The above theory reduces our problem to one of determining, for a given \( \mathbf{r} \), exactly which constraints are active at \( \mathbf{d} \). Appendix B explores a geometric approach to this question.

### 4.2.3 Incorporating Non-Discretionary Nodes

The application of the Kuhn-Tucker theorem to the minimisation problem (4.7) technically requires that the set defined by the inequality constraints \( G \leq 0 \) possesses an interior point. This is equivalent to requiring that \( D_k > 0 \) for all \( 1 \leq k \leq N \). In practice, however, one does not expect all external circuits to have a positive discretionary rating; some nodes of \( \mathcal{C} \) may not even have external circuits connected. In such cases it is desirable to reduce the number of dimensions to the problem, both to restore the Kuhn-Tucker conditions and to simplify the solution procedure.

The reduction of the Loss Minimisation Problem to a lower number of dimensions proceeds in one of two ways, depending whether or not the datum node \( \overline{N} \) has a positive discretionary rating \( D_N \).

**Case 1:** \( D_N > 0 \)

The case where \( D_N > 0 \) will be treated first, as the geometry of the problem is essentially unaffected (apart from the dimensionality of \( \mathbf{A} \mathbf{r} \)).

Let \( N' < N \) denote the number of external circuits with positive discretionary ratings. The case \( N' = 1 \) (where \( \overline{N} \) is the only discretionary node) is trivial, as there is then at most one feasible choice of \( \mathbf{d} \). We therefore suppose without loss of generality that \( N' \geq 2 \), and that the nodes are labelled such that \( D_k > 0 \) for \( 1 \leq k \leq N' - 1 \), and \( D_k = 0 \) for \( N' \leq k \leq N - 1 \). This gives rise to the following partitioning of the voltage-drop matrix \( \mathbf{\Sigma} \) and the vectors \( \mathbf{r} \) and \( \mathbf{d} \):

\[
\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{12}^T & \mathbf{\Sigma}_{22} \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}^{(1)} \\ 0 \\ d_N \end{bmatrix} \quad \mathbf{r} = \begin{bmatrix} \mathbf{r}^{(1)} \\ \mathbf{r}^{(2)} \end{bmatrix}
\]

(4.13)

where \( \mathbf{d}^{(1)} \) and \( \mathbf{r}^{(1)} \) are \( (N' - 1) \)-vectors, \( \mathbf{\Sigma}_{11} \) is a positive definite \( (N' - 1) \times (N' - 1) \)
matrix, and so on. We then define

\[ r_0 = \Sigma_{11}^{-1} \Sigma_{12} r^{(2)}. \]  \hspace{1cm} (4.14)

Subject to the above partitioning and (4.14) we have

\[ d_{\Sigma}^2(d, r) = (d^{(1)} - r^{(1)})^* \Sigma_{11} (d^{(1)} - r^{(1)}) - (d^{(1)} - r^{(1)})^* \Sigma_{12} r^{(2)} 
- r^{(2)*} \Sigma_{12}^T (d^{(1)} - r^{(1)}) + r^{(2)*} \Sigma_{22} r^{(2)} 
= (d^{(1)} - r^{(1)})^* \Sigma_{11} (d^{(1)} - r^{(1)}) - (d^{(1)} - r^{(1)})^* \Sigma_{11} r_0 
- r_0^* \Sigma_{11}(d^{(1)} - r^{(1)}) - r_0^* \Sigma_{11} r_0 + r^{(2)*} \Sigma_{22} r^{(2)}. \]

The latter two terms in this last expression are constant with respect to d and may therefore be disregarded. There follows

**Theorem 4.2** Suppose that there exists \( N' \), \( 1 < N' < N \), such that \( D_k > 0 \) for \( 1 \leq k \leq N' - 1 \) and \( D_k = 0 \) for \( N' \leq k \leq N - 1 \). Suppose also that \( \Sigma \), d and r are partitioned as in (4.13). Then the problem of choosing d to minimise \( d_{\Sigma}(d, r) \) is equivalent to choosing \( d' \) to minimise \( d_{\Sigma'}(d', r) \) where \( \Sigma' = \Sigma_{11} \) and

\[ d' = \begin{bmatrix} d^{(1)} \\ d_N \end{bmatrix}, \quad r' = \begin{bmatrix} r^{(1)} + \Sigma_{11}^{-1} \Sigma_{12} r^{(2)} \\ r_N \end{bmatrix}. \]  \hspace{1cm} (4.15)

This problem may be formulated in an \((N' - 1)\)-dimensional Hilbert space \( \Lambda_{\Sigma', r'} \) analogous to the \((N - 1)\)-dimensional space \( \Lambda_{\Sigma, r} \) of the original problem.

**Case 2: \( D_N = 0 \)**

In this case we may define a partitioning of the nodes as above, into those for which \( D_k > 0 \) and those for which \( D_k = 0 \), and construct \( \Lambda_{\Sigma', r'} \) using Theorem 4.2. \( N' \) now counts the number of discretionary nodes, plus one.

The geometry of this problem is slightly altered by the fact that the datum node has no discretion. Specifically, the \( N \)th pair of inequality constraints (those acting on the auxiliary component in \( \Lambda_{\Sigma, r} \) or \( \Lambda_{\Sigma', r'} \)) is replaced by the single equality constraint

\[ \text{bal } d_k = \text{bal } r = \text{bal } r_5 + r_N. \]  \hspace{1cm} (4.16)

This constraint locates the feasible set on a \((N' - 1)\)-dimensional hyperplane intersecting the \((N' - 1)\)-dimensional orthotope defined by the remaining inequality
The Kuhn-Tucker conditions for this modified problem may be stated as follows: if \( \hat{d} \) is the optimal solution \((N' - 1)\)-vector giving the discretionary currents for the first \( N' - 1 \) external circuits, then there exist nonnegative vectors \( \lambda \in \mathbb{R}^{N' - 1} \) and \( \mu \in \mathbb{R}^{N' - 1} \) and a real number \( \beta \) such that

\[
\lambda^T \hat{d} + \mu^T (D^{(1)} - \hat{d}) = 0 \tag{4.17}
\]

and

\[
\Sigma' (\hat{d} - r'_i) - \lambda + \mu - \beta 1_{N'-1} = 0. \tag{4.18}
\]

These equations involve \( 3N' - 2 \) unknowns, these being the elements of \( \hat{d} \), \( \lambda \) and \( \mu \) together with \( \beta \). Again, if it is known which of the various inequality constraints are active at \( \hat{d}, \) (4.17) gives known values to \( 2(N' - 1) \) of these unknowns. Equations (4.18), together with the balance-preserving condition (4.16), then determine the values of the remaining \( N' \) unknowns.

It may be easily checked that Lemma 4.1 still holds, with the addition of a term \( r_N/\beta \) to the loss expression. This term arises because, if \( r_N \neq 0 \), \( r'_i \) cannot be a feasible point for \( \hat{d} \) even if it satisfies all the inequality constraints.

### 4.3 Solution by Least Squares Methods

An efficient iterative method for obtaining a solution to our problem is the Least Distance Programming (LDP) algorithm of Lawson and Hanson [40], based on a least-squares approach. To apply this approach, we rewrite the constraints (4.8) in terms of \( j \) as \( G(j) \leq 0 \) where

\[
\begin{align*}
G_i(j) &= j_i - r_i \\
G_N(j) &= - \text{bal } j - r_N \\
G_{N+1}(j) &= - j_i + r_i - D_i \\
G_{2N}(j) &= \text{bal } j + r_N - D_N.
\end{align*}
\]  

These constraints are linear in \( j \) and may be expressed in the form

\[
Lj + \eta \leq 0
\]
where $L \in \mathbb{R}^{2N \times (N-1)}$ and $\eta \in \mathbb{R}^{2N}$. Now, choose any matrix $R$ such that $R^T R = \Sigma$ (for example, the unique positive definite square root of $\Sigma$), and set
\[ x = Rj \in \mathbb{R}^{N-1}. \]  
(4.20)

The problem then becomes one of minimising $\|x\|$ (the norm now being Euclidean) subject to constraints of the form
\[ \Gamma x \geq \eta \]  
(4.21)

where
\[ \Gamma = -LR^{-1}. \]  
(4.22)

This is precisely the LDP problem considered in [40, Chapter 23]. Lawson and Hanson provide an algorithm that solves this problem typically in $O(N)$ steps of the basic least-squares procedure. (If the required number of steps exceeds $3(N-1)$, the algorithm terminates with an approximate solution.) Given the minimising vector $\hat{x}$, the required discretionary setpoints $\hat{d}$ may be calculated as
\[ \hat{d}_b = r_b - R^{-1}\hat{x} \]  
(4.23)

\[ \hat{d}_N = \text{bal}\ r - \text{bal}\ \hat{d}_b. \]

Note that, owing to the definition of $\Sigma$, the matrix $R^{-1}$ appearing above can be related to the node-admittance matrix $G = AYA^T$, via the identity
\[ R^{-1}(R^{-1})^T = G. \]  
(4.24)

In particular, if $C$ is a tree (and hence $A$ a square matrix) we may set $R^{-1} = AY^{1/2}$, where $Y^{1/2}$ is the diagonal matrix whose entries are the positive square roots of the cable conductances. For a general network, we may choose $R^{-1}$ using the Cholesky decomposition of $G$ [26, p.89].

### 4.4 Examples

#### 4.4.1 Two-Node Network

The simplest nontrivial instance of the Loss Minimisation Problem, which will nicely illustrate the underlying geometry, is the generic two-node network with discretionary ratings $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}^T$, demand $r = \begin{bmatrix} r_1 & r_2 \end{bmatrix}^T$ and a single cable segment.
In this network the reduced incidence matrix $A$ is a scalar: 1 or $-1$ depending on the direction chosen for the cable current $i$. If the cable segment has impedance $z$ (admittance $y$) then the (scalar) voltage-drop matrix is $\Sigma = (AYA^T)^{-1} = y^{-1} = z$ and the network loss is $P_{\text{loss}} = z(r_1 - d_1)^2$. The loss is thus minimised by minimising the absolute difference $|r_1 - d_1|$, as required by the second form of the Minimal Loss Criterion.

Geometrically, the demand and discretionary profiles can be represented in the plane $\mathbb{R}^2$ as shown in Figure 4.1. The discretionary orthotope $\Gamma_D$ is the (closed) rectangle with corners $0$ and $D$, and the feasible demand set $\Xi_D$ is the strip bounded by the thick lines $r_1 + r_2 = 0$ and $r_1 + r_2 = \text{bal } D$. Given a demand profile $r$, the allowable discretionary profiles $d$ are those on the line $L_r : \text{bal } d = \text{bal } r$, which runs parallel to the boundaries of $\Xi_D$.

The discretionary profile that minimises the network loss is the nearest point to
\( \mathbf{r} \) in the set \( \Gamma_{\mathcal{D}} \cap L_r \). More formally, working in the one-dimensional space \( \mathbf{A}_{\mathcal{E}r} \), we evaluate the inequality constraints

\[
\begin{align*}
d_1 & \geq 0 \\
d_1 & \leq D_1 \\
d_1 & \leq \text{bal } \mathbf{r} \\
d_1 & \geq \text{bal } \mathbf{r} - D_2
\end{align*}
\]

to determine the projection of \( \Gamma_{\mathcal{D}} \cap L_r \) onto the \( d_1 \)-axis. We then find \( d_1 \) in this set nearest to \( r_1 \), and assign \( d_2 \) using (4.9) as \( d_2 = \text{bal } \mathbf{r} - d_1 \). This induces a discretionary map \( \mathbf{d} \) with the five piecewise-linear domains indicated in Figure 4.1. If \( \mathbf{r} \) is inside \( \Gamma_{\mathcal{D}} \) then \( \mathbf{d} = \mathbf{r} \); otherwise the choice of \( \mathbf{d} \) is governed by one of the four inequality constraints and the equality constraint \( \text{bal } \mathbf{d} = \text{bal } \mathbf{r} \).

To make this example more concrete, suppose the two-node network corresponds to an installation with two supply sources, rated at \( D_1 = 40 \text{A} \) and \( D_2 = 80 \text{A} \), and separated by some distance. In close proximity to each of these supply sources are load units whose demand varies widely.

Figure 4.1 illustrates the case where load 1, near the 40A supply, demands 60A while load 2 demands 20A. The current \( i \) in the distribution cable, hence the network loss, is minimised if the 40A supply runs at full power and the 80A supply generates 40A, 20A of which travels across the cable to load 1. Under a simple load-sharing arrangement, where the 40A supply generated one-third and the 80A supply two-thirds of the total load requirement, the cable current in this case would be 33A and the losses nearly three times higher.

### 4.4.2 Three-Node Network

The above example was kept deliberately simple in order to illustrate the geometry of the problem. To describe the working of the least-squares solution of Section 4.3, we turn to the more complex example of a generic three-node network having three cable segments arranged in a ring topology. The cable network is depicted in Figure 4.2.

The reduced incidence matrix for this cable network is

\[
\mathbf{A} = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix}
\]
and the voltage-drop matrix (with cable admittances \( y_k \)) is

\[
\Sigma = \frac{1}{y_1 y_2 + y_1 y_3 + y_2 y_3} \begin{bmatrix}
y_1 + y_3 & y_3 \\
y_3 & y_2 + y_3
\end{bmatrix}.
\]

Note that if \( y_3 = 0 \), indicating the absence of the cable connecting nodes 1 and 2, \( \Sigma \) becomes diagonal and the loss can be written simply as \( z_2(r_1 - d_1)^2 + z_1(r_2 - d_2)^2 \).

In general, however, the loss takes the form

\[
P_{\text{loss}} = \frac{(r_1 - d_1)^2}{y_2 + y_1 \oplus y_3} + \frac{(r_2 - d_2)^2}{y_1 + y_2 \oplus y_3} + \frac{2(r_1 - d_1)(r_2 - d_2)}{y_1 + y_2 + y_1 y_2 / y_3}
\]

where \( y_a \oplus y_b \) denotes the admittance of the series combination of \( y_a \) and \( y_b \).

Given \( \mathbf{D} \) and \( \mathbf{r} \), the intersection of the three-dimensional orthotope \( \Gamma_{\mathbf{D}} \) with the plane \( \text{bal } \mathbf{d} = \text{bal } \mathbf{r} \) is a single point (when \( \text{bal } \mathbf{r} = 0 \) or \( \text{bal } \mathbf{r} = \text{bal } \mathbf{D} \)) or a polygon with between \( N = 3 \) and \( 2N = 6 \) sides. Each of these sides is determined by one of the six inequality constraints

\[
\begin{align*}
d_1 & \geq 0 \\
d_1 & \leq D_1 \\
d_2 & \geq 0 \\
d_2 & \leq D_2 \\
d_1 + d_2 & \leq \text{bal } \mathbf{r} \\
d_1 + d_2 & \geq \text{bal } \mathbf{r} - D_3.
\end{align*}
\]

Given this polygon and the metric induced by \( \Sigma \), the problem of minimising \( P_{\text{loss}} \) may be formulated as a minimum-distance problem in two dimensions: that of finding
Figure 4.3: Minimum distance to a polygon

\[ \mathbf{d}_b = \begin{bmatrix} d_1 & d_2 \end{bmatrix}^T \] within this polygon minimising the \( \Sigma \)-distance to \( \mathbf{r}_s = \begin{bmatrix} r_1 & r_2 \end{bmatrix}^T \).

(See Figure 4.3.)

Let us now work the solution to this problem by the least-squares method of Section 4.3. The constraints (4.19) have the form \( \mathbf{L}\mathbf{j} + \eta \leq 0 \) where

\[
\mathbf{L} = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
-1 & -1 \\
1 & 0 \\
0 & -1 \\
1 & 1
\end{bmatrix}, \quad \eta = \begin{bmatrix}
-r_1 \\
-r_2 \\
r_3 \\
-r_3 \\
r_1 - D_1 \\
r_2 - D_2 \\
r_3 - D_3
\end{bmatrix}.
\]

The other quantity we require is the transformation matrix \( \mathbf{R}^{-1} \), which is obtained from the node-admittance matrix

\[
\mathbf{G} = \mathbf{AYA}^T = \begin{bmatrix}
y_2 + y_3 & -y_3 \\
-y_3 & y_1 + y_3
\end{bmatrix}
\]

by square root or Cholesky decomposition.

For purposes of illustration, suppose all cable admittances are equal: \( y_1 = y_2 = y_3 = y \). In this case the node-admittance matrix is

\[
\mathbf{G} = y \begin{bmatrix}
2 & -1 \\
-1 & 2
\end{bmatrix}
\]
and its square root is

$$G^{1/2} = \sqrt{y} \begin{bmatrix} \frac{\alpha}{\alpha} & -\beta \\ -\beta & \frac{\alpha}{\alpha} \end{bmatrix} \quad \alpha = \sqrt{1 + \frac{\sqrt{3}}{2}} \quad \beta = \sqrt{1 - \frac{\sqrt{3}}{2}}.$$  

This square root is a candidate for $R^{-1}$, and so we obtain the transformed constraint matrix as

$$\Gamma = -L R^{-1} = \sqrt{y} \begin{bmatrix} -\alpha & \beta \\ \beta & -\alpha \\ \alpha - \beta & \alpha - \beta \\ \alpha & -\beta \\ -\beta & \alpha \\ \beta - \alpha & \beta - \alpha \end{bmatrix} = \sqrt{y} \begin{bmatrix} -1.366 & 0.366 \\ 0.366 & -1.366 \\ 1 & 1 \\ 1.366 & -0.366 \\ -0.366 & 1.366 \\ -1 & -1 \end{bmatrix}.$$ 

Note that we have $\alpha - \beta = 1$, as is easily demonstrated by calculating $(\alpha - \beta)^2$.

Given a demand profile $r$ and capacity $D$, we calculate $\eta$ as above, then find a vector $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ of minimum (Euclidean) norm satisfying the constraints $\Gamma x \geq \eta$. Knowing the optimal $x$, we calculate $d$ by (4.23):

$$
\begin{align*}
d_1 &= r_1 - \sqrt{y}(\alpha x_1 - \beta x_2) \\
d_2 &= r_2 - \sqrt{y}(\alpha x_2 - \beta x_1) \\
d_3 &= r_1 + r_2 + r_3 - d_1 - d_2 \\
&= r_3 + \sqrt{y}(x_1 + x_2).
\end{align*}
$$

Notice that this solution algorithm does not require calculation of $\Sigma$ or $R$, and that $\Gamma$ need only be calculated once.
Chapter 5

Loss Minimisation by Network Reconfiguration

The previous chapter explored the question of minimising the steady-state losses in a distribution network by exploiting the ‘discretionary’ capabilities of external circuits, for a given fixed network topology. Here, we explore a more traditional approach, to which the theoretical apparatus of Chapter 3 is equally well-suited: the minimisation of steady-state losses with a given fixed external circuit profile, by altering the network topology. The relevance of this approach goes beyond the DC reticulation systems to which this thesis is devoted, to high-voltage AC distribution networks.

5.1 The Network Reconfiguration Problem

A substantial literature has been devoted over the past 20 years to the so-called ‘reconfiguration problem’ for radial power distribution networks. The problem may be stated as follows:

Network Reconfiguration Problem Given a load profile for a distribution network with a number of redundant branches (tie lines), find a radial configuration for the network which minimises the network losses.

In the language of Chapter 3, we are given a distribution network \( \mathcal{N} = \mathcal{C} \cup \mathcal{E} \) with nullity \( \nu > 0 \). The network is always operated in a radial configuration, meaning that at any given time some \( \nu \) cable segments are open-circuited, leaving a spanning tree \( \mathcal{T} \subset \mathcal{C} \) as the effective cable network. We seek a tree \( \mathcal{T} \) which minimises the real power losses \( P_{\text{loss}} \) in the cable network.
A variety of approaches to this problem are surveyed by Sarfi et al. in [64]. This survey commences by stating that “[t]he generalised reconfiguration problem presents a considerable computational burden for a distribution system of even moderate proportions.” This assumed computational burden follows from the observation that “[t]he nonlinear nature of the distribution system necessitates that at each iteration of an optimisation algorithm a load flow operation be performed to determine a new system operating point.” If this is correct, it follows that a direct or exhaustive solution is infeasible, so that a practical solution must employ some heuristic search method, possibly guided by a simplified optimisation procedure. This indeed is the approach taken by all the methods surveyed, as well as more recent approaches such as that of Haque and Khan [29].

We suggest that this presumption of infeasibility can be challenged, if one is prepared to model the network loads using a constant-current characteristic rather than the constant-power characteristic assumed in load-flow studies. With this assumption, the feeder currents can be obtained from the load currents by a simple equation involving the sensitivity matrix $S$ of the network, which for tree networks can be written down directly by inspection of the network topology. Knowing the feeder currents and the (fixed) cable resistances, the network losses are easily calculated without the use of computationally intensive load-flow algorithms.

The use of a constant-current load model can be justified most simply on the basis that it is no less ‘arbitrary’ than a constant-power model. In reality, the load on a power system exhibits a combination of ohmic, constant-current and constant-power characteristics, as well as switch-mode circuits that approximate one or other of these characteristics. The use of constant-power models in load-flow studies is justified on the basis of conservatism: given that these types of load have more severe effects on stability and voltage drop than the others, it makes sense to assume a worst-case scenario. If, on the other hand, one is concerned purely with comparing the network losses across different configurations, there is no ‘worst case’: the question is purely whether the inevitable modelling error is sufficiently large to throw one’s conclusions into doubt. Particularly when the anticipated voltage drop is small, a constant-power assumption provides no greater guarantee of a favourable response to this question than a constant-current assumption.
Given an initial radial configuration or spanning tree, all other trees can be generated, without repetition, by means of a recursive branch-exchange algorithm. Such a branch exchange or ‘elementary tree transformation’ involves a ‘semi-sparse’ perturbation of the sensitivity matrix for a radial network. These two facts form the basis of a highly efficient ‘brute force’ algorithm for the solution of the network reconfiguration problem.

5.2 Elementary Tree Transformations

Let $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$ be a distribution network, with $N$ nodes $\mathcal{K}$, $1 \leq k \leq N$ representing substations or buses and $C$ arcs $e_j$, $1 \leq j \leq C$ representing cable segments in $\mathcal{C}$. Recall that the nullity of $\mathcal{C}$ is $\nu = C - N + 1$, equal to the number of independent loops. If the network is radial, with no tie lines, then $\nu = 0$.

Given a (spanning) tree in $\mathcal{C}$, the simplest way to obtain a different tree (if $\nu > 0$) is by exchanging one cable segment for another in $\mathcal{C}$. The following definition formalises this notion.

**Definition 5.1** An elementary tree transformation in $\mathcal{C}$ is a transformation from a tree $\mathcal{T}_1 \subset \mathcal{C}$ to a tree $\mathcal{T}_2 \subset \mathcal{C}$ accomplished by removing one arc $e_i$ from $\mathcal{T}_1$ and adding an arc $e_j \in \mathcal{C} \setminus \mathcal{T}_1$. We abbreviate this operation by writing it in the form

$$\mathcal{T}_2 = T_{i,j}(\mathcal{T}_1).$$

(5.1)

Define the distance $d(\mathcal{T}_1, \mathcal{T}_2)$ between two trees $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{C}$ as the number of arcs in $\mathcal{T}_1$ but not in $\mathcal{T}_2$. We also denote by $K_i(\mathcal{T})$ the fundamental cut set defined by $e_i$ with respect to $\mathcal{T}$; that is, the set of all arcs in $\mathcal{C}$ which connect the two components into which $\mathcal{T}$ is divided by the removal of $e_i$.

We then have the following results, the proofs of which can be found in [48]:

- Let $\mathcal{T} \subset \mathcal{C}$ be a tree. Then $T_{i,j}(\mathcal{T})$ is a tree if and only if $e_i \in \mathcal{T}$ and $e_j \in K_i(\mathcal{T})$.
- Trees $\mathcal{T}_1$ and $\mathcal{T}_2$ in $\mathcal{C}$ are related by an elementary tree transformation in $\mathcal{C}$ if and only if $d(\mathcal{T}_1, \mathcal{T}_2) = 1$.
- Let $\mathcal{T}_1, \mathcal{T}_2$ be any trees in $\mathcal{C}$. If $d(\mathcal{T}_1, \mathcal{T}_2) = k$, then $\mathcal{T}_2$ can be obtained from $\mathcal{T}_1$ through a sequence of exactly $k$ elementary tree transformations.

105
Because the maximum possible distance between any two trees in \( \mathcal{C} \) is \( \nu \), it follows from the last of these statements that, given any tree \( \mathcal{T}_0 \subset \mathcal{C} \), we can generate any tree \( \mathcal{T} \subset \mathcal{C} \) by applying at most \( \nu \) elementary tree transformations. This idea forms the basis of a simple and reasonably efficient algorithm for generating all the trees in a graph from an initial tree \( \mathcal{T}_0 \). The basic procedure is as follows:

1. Find an initial tree \( \mathcal{T}_0 \), having \( N - 1 \) arcs.

2. For each arc \( e_i \in \mathcal{T}_0 \), find the cut set \( K_i(\mathcal{T}_0) \) and substitute the arcs of \( K_i(\mathcal{T}_0) \) one by one (if any) for \( e_i \) to obtain a new set of trees. When this is done for every arc of \( \mathcal{T}_0 \), one obtains all trees of distance 1 from \( \mathcal{T}_0 \).

3. Repeat the above process (subject to conditions detailed below) for each tree of distance 1 from \( \mathcal{T}_0 \), hence obtaining the trees of distance 2 from \( \mathcal{T}_0 \).

4. Repeat until one has all trees of distance up to \( \nu \) or \( N - 1 \) (whichever is the lesser) from \( \mathcal{T}_0 \). One then has all the trees in the graph.

To this basic procedure, of course, must be added some checks to ensure no back-tracking occurs. The following conditions are sufficient [48] to ensure that each tree in the graph is generated once and once only by the above procedure:

1. The arcs forming the tree \( \mathcal{T}_0 \) are ordered such that every initial subsequence forms a connected subgraph.

2. Arcs are substituted in strict ascending order as they appear in the arc sequence comprising \( \mathcal{T}_0 \). (In other words, if \( \mathcal{T}_0 = \{e_1, e_2, e_3\} \) and \( e_i \) is substituted for \( e_2 \) to form a new tree \( \mathcal{T}' \), then neither \( e_1 \) nor \( e_i \) may be substituted to generate a new tree from \( \mathcal{T}' \).

3. When generating new trees from a tree \( \mathcal{T} \) by substituting an arc \( e_i \), a new arc is substituted if and only if it belongs both to the set \( K_i(\mathcal{T}) \) and to the set \( K_i(\mathcal{T}_0) \). (Note that \( e_i \in \mathcal{T}_0 \) by Condition (2) above.)

Let \( \tau(\mathcal{C}) \) denote the total number of spanning trees in \( \mathcal{C} \), often called the complexity or tree-number. The execution time for the above algorithm, due to Mayeda and Seshu, is \( O(NC) \times \tau(\mathcal{C}) \), and the storage space required is \( O(N + C) \). Smith [71] outlines a number of alternative algorithms for generation of spanning trees, the
most recent of which have execution time \( O(1) \times \tau(C) \). (This is referred to as ‘constant amortized time’, the ‘amortized time’ being the execution time divided by the number of trees generated.) Some of these algorithms generate trees using successive elementary tree transformations, others do not. The more efficient algorithms, understandably, are very difficult to describe and we shall not attempt to describe them here. Suffice it to say, however, that any of these algorithms, provided it operates by means of elementary transformations, can be substituted for the Mayeda-Seshu algorithm in our application.

5.3 Application to Loss Minimisation

Suppose we have an AC or DC distribution network which we wish to operate in a radial manner (that is, as a tree network) but which nevertheless contains a number of ‘tie lines’ that allow the network topology to be reconfigured in various ways so as to minimise the cable losses. The problem is to find, from amongst all spanning trees of \( C \), a tree \( T \) such that the tree network \( T \) has minimal cable losses.

In a radial distribution network with a single point of supply, it is convenient to identify this supply point with the datum node \( \bar{N} \). As before, we denote by \( \mathbf{j} \) the vector of currents (assumed constant) leaving the network at each node \( \mathbf{T} \) through \( \bar{N} - 1 \), excluding the datum node. The vector of cable currents is denoted \( \mathbf{i} \), and we have \( \mathbf{i} = \mathbf{S} \mathbf{j} \) where \( \mathbf{S} \) is the sensitivity matrix. If \( C \) is a tree then

\[
\mathbf{S} = \mathbf{A}^{-1} = \mathbf{B}_1^T
\]

where \( \mathbf{A} \) is the reduced incidence matrix, and \( \mathbf{B}_1 \) is the path-set matrix defined in Section 3.4.2.

To determine the network losses, we first calculate \( \mathbf{i} = \mathbf{S} \mathbf{j} \) and then the losses as the sum \( \sum z_j i_j^2 \). Our brute-force approach involves generating all the spanning trees of \( C \) and comparing the above sum for each. Though simple, this method can be practicable as long as the process of recalculating \( \mathbf{S} \), \( \mathbf{i} \) and the losses at each step is not too complex. That this is so is established in the following section.
Figure 5.1: An elementary tree transformation

5.4 Transformation of Sensitivity Matrix

Let $\mathcal{T} \subset \mathcal{C}$ be a tree with sensitivity matrix $\mathbf{S}$, and let $\mathcal{T}' = T_{q,r}(\mathcal{T})$ be an elementary tree transformation involving arcs $e_q, e_r \in \mathcal{C}$. We denote by $\mathbf{S}'$ the sensitivity matrix of $\mathcal{T}'$, and seek to understand the relationship between $\mathbf{S}$ and $\mathbf{S}'$. This information allows us to relate the cable currents and hence the losses in the two subtrees $\mathcal{T}$ and $\mathcal{T}'$.

Consider $\mathcal{C}' = \mathcal{T} \cup \mathcal{T}'$, a subgraph of $\mathcal{C}$ of nullity 1. Then $\{e_q, e_r\}$ is a cut set of $\mathcal{C}'$ partitioning it into two components $\mathcal{C}_1, \mathcal{C}_2$ such that $\mathcal{C}' = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{e_q, e_r\}$. We take $\mathcal{C}_1$ to be the component containing the datum node $\mathcal{N}$. (See Figure 5.1.)

Let $\overline{e_q}$ and $\overline{e_r}$ be the nodes incident with $e_q$, where we take $\overline{e_q} \in \mathcal{C}_1$ and $\overline{e_r} \in \mathcal{C}_2$. Similarly, let $\overline{e_r}$, $\overline{e_r}$ be the nodes incident with $e_r$. To take account of the orientation of $e_r$ we define $\sigma_r$ to be 1 if $e_r$ is directed from $\overline{e_r}$ to $\overline{e_r}$, and $-1$ otherwise.

Now, $\mathbf{S} = \mathbf{B}_1^T$ has a row corresponding to every arc of $\mathcal{T}$ and a column corresponding to each node of $\mathcal{C}$ other than the datum node $\mathcal{N}$. To simplify matters here we augment $\mathbf{S}$ by a zero column corresponding to the datum node, so that $s_{jn} = 0$ for all $j$ by our convention. One row of $\mathbf{S}$ will correspond to the arc $e_q$ being substituted; let $m$ be the index of this row.

Our first observation regarding the transition from $\mathbf{S}$ to $\mathbf{S}'$ is that the $j$th row of $\mathbf{S}$ is unchanged if the corresponding arc is not in the unique loop $L$ of $\mathcal{C}'$. For, if $\mathcal{P}_k$ is the path from $\mathcal{N}$ to some node $\mathcal{K}$, then replacing $e_q$ by $e_r$ does not affect that part of $\mathcal{P}_k$ that does not lie in the loop $L$. It follows that, if the external circuits have constant-current characteristics, the cable current $i_j$ is also unchanged. An arc lies in $L$ if and only if it belongs to exactly one of the paths $\mathcal{P}_{r_1}$ and $\mathcal{P}_{r_2}$ in $\mathcal{T}$; if it belongs to both then it has the same orientation with respect to both. Accordingly,

$$s'_{jk} = s_{jk} \quad \text{if} \quad s_{jr_1} - s_{jr_2} = 0.$$  \hfill (5.2)
We observe also that the kth column of $S$ is unchanged if the node $\bar{k}$ lies in $C_1$. For in this case, the path $P_k$ from $\bar{N}$ to $\bar{k}$ in $T$ must lie entirely within $C_1$, so that no part of this path is affected by the transformation. Equivalently, the path $P_k$ in $T$ does not include $e_q$, giving

$$s'_{jk} = s_{jk} \text{ if } s_{qk} = 0. \quad (5.3)$$

We may therefore restrict our attention to rows of $S$ corresponding to arcs in $L$, and to elements of these rows corresponding to nodes in $C_2$. Consider first the row $m$ corresponding to $e_q$ itself. In $S'$ this will be replaced by a row corresponding to $e_r$; it is apparent that $e_r$ in $T'$ feeds precisely those nodes (in $C_2$) that are fed by $e_q$ in $T$. Thus the elements of row $m$ are unchanged except in sign. A sign change occurs if the cut set $\{e_q, e_r\}$ is not uniform, alternatively if $\sigma_r$ (defined above) and $s_{mr_2}$ differ in sign. It follows that

$$s'_{mk} = \sigma_r s_{mr_2} s_{mk} \quad (5.4)$$

for all $k < N$, and (if constant-current models are assumed) that

$$i'_r = \sigma_r s_{mr_2} i_q.$$ 

Now suppose $j$ and $k$ are indices corresponding (with some abuse of notation) to an arc $e_j \in L$ and a node $\bar{k} \in C_2$. Let $P_k$ be the path from $\bar{N}$ to $\bar{k}$ in $T$, and let $P'_k$ be the corresponding path in $T'$. Then $P_k \cap L$ and $P'_k \cap L$ (viewed as arc-sets) form a partition of $L$. It follows that

$$s'_{jk} = 0 \text{ if } s_{jk} \neq 0. \quad (5.5)$$

If $s_{jk} = 0$ then there are two cases to consider, according as $e_j \in C_1$ or $e_j \in C_2$. In the former case $P'_k$ must include $P_{r_1}$ ($= P'_{r_1}$), giving $s'_{jk} = s_{jr_1}$. In the latter case on the other hand $P_k$ passes through $\bar{r}_2$, but in the opposite sense to $P_{r_2}$; thus $s'_{jk} = -s_{jr_2}$. It turns out in fact that we may write

$$s'_{jk} = s_{jr_1} - s_{jr_2} \text{ if } s_{jk} = 0 \quad (5.6)$$

without regard for whether $e_j \in C_1$ or $e_j \in C_2$. For we already know that $P_{r_1}$ lies entirely within $C_1$, hence that $s_{jr_1} = 0$ if $e_j \in C_2$. It is also readily seen that $P_k \cap C_1 = P_{q_1}$ for all $\bar{k} \in C_2$, hence that if $e_j \in C_1$ and $s_{jk} = 0$, then $s_{jl} = 0$ for all $\bar{l} \in C_2$, and in particular $s_{jr_2} = 0$. 

109
Equations (5.2) through (5.6) completely describe the effect of an elementary tree transformation on the sensitivity matrix for a tree network. For convenience we collect these into a theorem.

**Theorem 5.1** Let $\mathcal{N} = \mathcal{C} \cup \mathcal{E}$ be a distribution network, let $\mathcal{T} \subset \mathcal{C}$ be a tree and let $\mathcal{T}' = T_{q,r}(\mathcal{T})$ be an elementary tree transformation involving arcs $e_q, e_r \in \mathcal{C}$. Let $\mathbf{S}$ and $\mathbf{S}'$ be the sensitivity matrices of the tree networks $\mathcal{T} \cup \mathcal{E}$ and $\mathcal{T}' \cup \mathcal{E}$ respectively, in which the $m$th row corresponds to the arc $e_q$, respectively $e_r$, and other rows correspond to like arcs. Let $\overline{r_1} \in \mathcal{C}_1$, $\overline{r_2} \in \mathcal{C}_2$ be the nodes incident with $e_r$, and let $\sigma_r = 1$ if $e_r$ is directed from $\overline{r_1}$ to $\overline{r_2}$ and $-1$ otherwise. Then for all $j$, $k$

1. $s'_{jk} = s_{jk}$ if $s_{jr_1} - s_{jr_2} = 0$ or $s_{mk} = 0$.

2. $s'_{mk} = \sigma_r s_{mr_1} s_{mk}$.

3. If $s_{jr_1} - s_{jr_2} \neq 0$ and $s_{mk} \neq 0$, with $j \neq m$, then $s'_{jk} = 0$ if $s_{jk} \neq 0$, otherwise $s'_{jk} = s_{jr_1} - s_{jr_2}$.

### 5.5 An Algorithm for Topological Loss Optimisation

Theorem 5.1 forms the basis for an exhaustive yet efficient algorithm to solve the network reconfiguration problem. The algorithm takes as input the reduced incidence matrix $\mathbf{A}$ of the distribution network with tie lines included, the resistances $r_j$ of each cable segment, and the external circuit current $j_k$ for the first $N - 1$ nodes. (Since the sensitivity matrix and hence the cable currents in a tree network do not depend on the cable impedances, it is not necessary to specify the reactances of the cables as input to the algorithm.) It begins by determining an initial spanning tree $\mathcal{T}_0$ for the given network, and the sensitivity matrix $\mathbf{S}_0$ for the corresponding tree network (obtained by inverting the appropriate submatrix of $\mathbf{A}$). The initial cable currents and losses are also calculated at this point.

Proceeding from these initial conditions, the algorithm generates all the spanning trees for the network, using an algorithm such as that of Section 5.2, and at each step modifies the sensitivity matrix according to Theorem 5.1. The sensitivity matrix is modified row by row; whenever a row of the matrix changes, the corresponding cable
current is recalculated and the losses perturbed by the difference between the squares of the old and new currents, multiplied by the appropriate cable resistance:

\[ P'_{\text{loss}} = P_{\text{loss}} + r_j (|i_j'|^2 - |i_j|^2). \]

In the case of the row corresponding to the arc being substituted, the magnitude of the current is unchanged, and the loss need only be altered to take account of the physical cable substitution:

\[ P'_{\text{loss}} = P_{\text{loss}} + (r_r - r_q)|i_m|^2. \]

After each step the loss is compared with the minimum loss obtained so far, and if not higher, the current configuration is recorded. In those cases where the minimal loss configuration is not unique, the algorithm is capable of reporting all configurations having the same minimal loss.

The algorithm as presented here will solve the network reconfiguration problem for a DC, single-phase AC, or balanced three-phase AC network, though it takes no account of operational limits, such as maximum current or power-transfer limits on individual cable segments, or node voltage constraints. It is not in principle difficult to incorporate checks for such limits into the algorithm, and to reject any particular configuration that fails to satisfy these checks. The algorithm could also be modified to handle unbalanced three-phase networks simply by treating each current quantity as a vector containing either the line currents or the magnitudes of their symmetrical components. A symmetrical-component representation would be necessary if the cable resistances depended significantly on phase sequence, but this dependence is generally much less for resistance than for reactance quantities.

In assuming constant-current characteristics for the external circuits, our approach follows that of Liu et al [43], who recognised that in this case the problem could be formulated purely in terms of currents. The graphical approach we use is more akin to the original study of Merlin and Back [49], and particularly to the work of Civanlar et al [14] whose ‘branch exchanges’ correspond to our elementary tree transformations. The recognition that the problem is essentially combinatorial in nature, and that a generic switching operation may be represented as a series of branch exchanges, is due to Baran and Wu [4].

The computational burden of the algorithm is significant, but not prohibitively so for practical networks where the number of tie lines (the nullity) is small by compari-
son with the number of buses. Quantitative estimates of complexity require estimates of the tree-number $\tau(C)$ for arbitrary graphs $C$ based on topological properties, an open problem explored in Appendix C. Unlike other algorithms proposed in the literature, this algorithm guarantees a minimal-loss solution, and as indicated above is easily modified to take into account phase imbalance and operational constraints.

5.6 An Example

As an example of the algorithm outlined above, we choose the 33-bus system of Baran and Wu [4]. In our terminology, this system comprises a cable network $C$ having $N = 33$ nodes (or ‘buses’), $C = 37$ cable segments, and nullity $\nu = 5$. This network is usually taken to model an AC distribution network having one energy input at bus 0 and specified power demand at buses 1 through 32.

To fit in our framework, we relabel the supply bus 0 as bus 33, and take it as the datum node. The cable network topology and resistances are given in Table 5.1 below. The 32 ‘load’ circuits are assumed to have constant currents whose real and imaginary parts are provided in Table 5.1, next to the data for the first 32 cable segments. (These are derived respectively from the quoted values of real and reactive power in [4, Table 1]. The power quantities were used directly, as the unit of current is of no importance to the algorithm.) Cable segments 33 through 37 are the tie lines. We ran the algorithm both on the usual AC network and on a DC network having the same topology and the same RMS current magnitude at each node.

A MATLAB implementation of the minimal-topology algorithm took around two hours on a multi-user UNIX workstation to process the 50,751 spanning trees for the network of Table 5.1. As the subnetwork comprising branches 1 through 32 is connected and therefore a tree network, it was automatically selected as the starting tree $T_0$. For the AC network with external currents specified as in Table 5.1, it was found that the following elementary tree transformations gave a network with minimal losses of 20.413 MW, as compared with 28.266 MW for the initial network $T_0$. 
<table>
<thead>
<tr>
<th>Seg.</th>
<th>Src.</th>
<th>Rec.</th>
<th>$r_j$ ($\Omega$)</th>
<th>Rec. bus $j_k$ (A)</th>
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Table 5.1: The 33-bus test system
Replace branch      With branch
                       
    7     33
    9     35
   14     34
   32     36

This corresponds to closing four out of the five tie lines, and opening four branches of the original network to maintain the radial structure. The result may be compared with those of Baran and Wu [4] and Haque and Khan [29], though an exact comparison is not possible owing to the different assumptions on the external circuit models. The aforementioned studies assume constant-power characteristics, in keeping with the load-flow approach, whereas we assume constant current; one might therefore expect slightly different results to be obtained due to the effect of voltage drop.
Chapter 6

Overload Prevention

Conventional distribution circuit cable protection schemes employ fuses and relays to interrupt current when a thermal overload condition is detected. While effective as protection schemes, such methods lack the ability to prevent overload situations before they occur, and the resulting inconvenience is significant.

More recently, as discussed in Chapter 1, it has become common to install energy management systems on these same circuits, for the purpose of limiting cumulative electricity demand to prescribed levels. Since the purpose of traditional cable protection schemes, namely that of limiting accumulated thermal energy in power circuit cables, bears some similarity to that of the newer energy management systems, it would appear that there is scope for bridging the gap between the two. What is envisaged is a demand control system which would supplement (though not replace) a conventional fuse or relay-based protection scheme by using the functions of the energy management system for the purpose of preventing cable overloads as well as controlling demand.

There are two aspects to the prevention of thermal overloads in conventional power systems: the choice of cables during the design phase to permit continuous current flow within certain limits; and operational measures to ensure that cumulative current flow does not exceed the thermal capacity of the cables. We now seek to apply the topological approach to both these issues.
### 6.1 Choice of Cables

In both conventional and managed distribution systems, it is important that the bus bars and/or cables be sized to provide a current-carrying capacity appropriate to the design task. In particular it is important to know, given some characterisation of the supply and demand requirements for the external circuits, the maximum current expected to flow in each cable segment. The topological sensitivity analysis is readily applicable to this problem.

In order to characterise the maximal current requirements of external circuits, we define \( R_k \) for \( 1 \leq k \leq N \) to be the maximum current that external circuit \( k \) can be expected to demand from the bus, and \( S_k \) to be the maximum current that it can be expected to supply to the bus. Thus, we anticipate that under all foreseeable loading conditions the external circuit currents \( j \) are constrained by the relations

\[
-S_k \leq j_k \leq R_k \quad R_k, S_k \geq 0 \quad 1 \leq k \leq N - 1.
\] (6.1)

(We assume that similar bounds for the current \( j_N \) are given implicitly by these constraints together with the fundamental equation 3.1.) The required continuous current rating for cable segment \( j \) is then given by the maximum of \( |i_j| \) subject to Equation (3.11) and the constraints (6.1). The following theorem provides the result we need.

**Theorem 6.1** Suppose the external circuit currents \( j \) in a distribution network satisfy the constraints (6.1), and that \( S \) is the sensitivity matrix of the network. Then the current \( i_j \) in cable segment \( j \) satisfies 

\[
-I_{j,\text{min}} \leq i_j \leq I_{j,\text{max}} \quad \text{where}
\]

\[
I_{j,\text{max}} = \sum_{s_{jk} > 0} |s_{jk}| R_k + \sum_{s_{jk} < 0} |s_{jk}| S_k,
\]

\[
I_{j,\text{min}} = \sum_{s_{jk} > 0} |s_{jk}| S_k + \sum_{s_{jk} < 0} |s_{jk}| R_k.
\] (6.2)

It follows that \( \max_j |i_j| = \max \{I_{j,\text{max}}, I_{j,\text{min}}\} \).

**Proof.** From Equation 3.11 we have

\[
i_j = \sum_{k=1}^{N-1} s_{jk} j_k = \sum_{s_{jk} j_k > 0} |s_{jk}||j_k| - \sum_{s_{jk} j_k < 0} |s_{jk}||j_k|.
\]

In order to make the sum on the right hand side of this equation as positive as possible, subject to the constraints, it is apparent that we should set \( j_k = R_k \) whenever
$s_{jk} > 0$, and $j_k = -S_k$ otherwise, thus ensuring that $s_{jk,j_k}$ is nonnegative for all $k$.

The resulting value of $i_j$ is precisely $I_{j,\text{max}}$ as stated in the Theorem. Similarly, to make the sum as negative as possible, we should set $j_k = -S_k$ whenever $s_{jk} > 0$ and $j_k = R_k$ otherwise, giving the value $I_{j,\text{min}}$ as stated above. Since both $I_{j,\text{max}}$ and $I_{j,\text{min}}$ as defined are nonnegative, the result follows. √

**Corollary 6.1** If $R_k = S_k$ for all $k$ in the constraints (6.1), then for all $j$ we have

$$\max_j |i_j| = \sum_{k=1}^{N-1} |s_{jk}|R_k. \quad (6.3)$$

Theorem 6.1 forms the basis of a simple design rule that can be used to select bus cables during the installation design phase. External circuit cables, needless to say, would be chosen to have a continuous current rating equal to at least $\max\{R_k, S_k\}$.

For tree networks of course the problem is trivial, since it is always possible to choose cable orientations such that $S$ consists entirely of zeros and ones, the unit entries $s_{jk} = 1$ corresponding to nodes $k$ that are ‘downstream’ from cable segment $j$. It follows that the maximum of $|i_j|$ is the sum of $R_k$ for all downstream nodes, or of $S_k$ for all downstream nodes, whichever is the greater. (The datum node is excluded from these calculations by virtue of being the ‘root’ of the tree.)

### 6.2 Operational Measures

In a managed distribution network, it is possible to prevent overload conditions before they occur by using the request mechanism to limit authorised current in accordance with the thermal capacity of the distribution network. Although it is possible to make better use of short-term thermal capacity through a time-limited request mechanism, we discuss here only the simplest approach: to authorise current requests on an unlimited time basis, and limit the steady-state current authorised on each external circuit in such a way that all steady-state currents in the distribution network $N$ remain within the continuous current ratings of the conductors.

An overload prevention scheme of this sort must take into account the manner in which changes in demand affect the cable currents via the discretionary current distribution. Accordingly, the demand sensitivity matrix defined in Section 3.6 will
figure prominently in the following discussion, which is restricted to managed DC networks.

### 6.2.1 Linear Discretionary Map

When a linear discretionary map is used, the overload prevention scheme is particularly simple. Recall from Section 3.6.1 that in this case the change in a cable current \( i_j \) resulting from a change in demand \( r_k \) at some external circuit is given by

\[
\Delta i_j = (s_{jk} - C_j) \Delta r_k
\]

where

\[
C_j = \frac{\sum_{i=1}^{N-1} s_{ji} D_i}{\text{bal} \ D_i}
\]

is the \( j \)th cable sensitivity coefficient.

The proposed scheme would function along these lines:

1. As part of the system design process, one specifies the bus topology and the type and length of all cable segments. From this information the elements \( s_{jk} \) of the sensitivity matrix are calculated and stored.

2. The cable sensitivity coefficients \( C_j \) are calculated online from the matrix elements and discretionary ratings.

3. When an external circuit requests a change in its demand levels, the prospective increase or decrease of current in each cable segment is determined. On this basis, taking into account the measured cable currents and the continuous ratings of the cables, it can be determined whether the request should be authorised, partially authorised, or refused.

4. If a change in discretion occurs, the coefficients \( C_j \) are recalculated. Loads will be shed or additional capacity dispatched if necessary to bring the load factor \( \lambda \) below unity. It may also be necessary to shed further loads in order to bring the cable currents below their continuous rating under these new conditions. Generally, this additional load shedding would proceed in priority order; it would also be possible to identify the external circuits to which particular cable currents are most sensitive, by calculating and comparing \( s_{jk} - C_j \) for various values of \( k \).
6.2.2 General Discretionary Map

Section 3.6.2 details the sensitivity calculations for a general discretionary map \( d \). In this case, the demand sensitivity matrix \( T \) is valid only locally. From the point of view of cable overload prevention, the results of that Section suggest the desirability of a convex discretionary map, in order that the linear approximation provide at least an upper bound for the change in cable currents.

It is expected that the major practical difficulty in implementing a cable overload prevention scheme based on these results for a nonlinear discretionary map would be the need to calculate the elements of \( T \) by evaluating the partial derivatives of the discretionary map for each demand profile \( r \) (or at least a representative set of demand profiles). If the discretionary setpoints are assigned by some sort of optimisation algorithm, it may not even be possible to obtain values for these partial derivatives. In such situations it may be necessary for the system to ‘learn’ the map \( d \) and estimate the derivatives at \( r \) from nearby values of \( d \); this would necessitate ‘relearning’ \( d \) whenever the capacity \( D \) changes. Whether such an approach is satisfactory in practice remains to be determined.

6.2.3 Minimal Loss Discretionary Map

In the case where the discretionary map is determined using the loss minimisation algorithm of Chapter 4, the map is not in general linear, but it is piecewise linear, the breakpoints being defined by the bounding hyperplanes of the constraint domains. The various piecewise linear domains of the Kuhn-Tucker solution are characterised by the particular set of constraints active within each, and the boundaries between them occur at the points where a constraint goes from being active to being inactive, and vice versa.

As explained in Chapter 4 and Appendix B, the Loss Minimisation Problem may be formulated as one of determining the minimum distance from a given point to a convex polytope within the Hilbert space \( \Lambda_{\Sigma,r} \). Each of the \( 2N \) constraints defines a hyperplane whose normal vector \( \mathbf{n}_k \) has a fairly simple expression in terms of the cable admittances and network topology. In Section B.4 we provide straightforward conditions under which the hyperplane in fact determines a facet of the polytope in question. Having determined the facets of the polytope, it is a relatively simple
matter (in principle) to determine the boundaries between the various constraint regimes. Here the theory developed in Appendix B is of key importance.

Within any one domain enclosed by these boundary hyperplanes, the discretionary map may be treated as though it were linear (or rather affine). Thus, a precise demand sensitivity matrix $T$ may be derived, just as in the case of the linear discretionary map, and is applicable so long as the demand profile $r$ does not cross a domain boundary. Once a boundary is crossed the demand sensitivity matrix must be replaced with the matrix applicable to the new domain.

Naturally, in real-life situations where the number of discretionary external circuits is large, the process of identifying to which domain a given demand profile belongs is computationally severe. It is hoped that the theory of Appendix B will lead to heuristic methods for simplifying this task; this is a subject for future work.

### 6.3 Example: Overload Prevention with a Linear Discretionary Map

As a simple example of the ideas explored in the previous section, and some of the theorems of Chapter 3, consider the six-node single-ring distribution network whose cable network is depicted in Figure 6.1. The orientations of this network have been chosen to satisfy the conventions of Theorem 3.10. In particular, the ring main has been given a uniform anticlockwise orientation as drawn, while the remaining cable segments have been directed away from the datum node $\bar{6}$. (Since in fact the datum node is part of the ring main in this example, it suffices that the non-ring-main
segments form a ‘uniform rooted forest’ with the root nodes being ring-main nodes.)

We may accordingly use Theorem 3.10 to write down the sensitivity matrix \( \mathbf{S} \) directly by inspection of Figure 6.1 as follows:

\[
\mathbf{S} = \begin{bmatrix}
-\zeta_1 & \xi_2 & \xi_2 & \xi_2 & \xi_2 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-\zeta_1 & -\zeta_2 & -\zeta_2 & -\zeta_2 & -\zeta_2 \\
\xi_1 & \xi_2 & \xi_2 & \xi_2 & \xi_2 \\
\end{bmatrix} \tag{6.4}
\]

where

\[
\zeta_1 = \frac{\hat{z}_1}{z_{\text{loop}}} = \frac{z_6}{z_1 + z_5 + z_6} \\
\zeta_2 = \frac{\hat{z}_2}{z_{\text{loop}}} = \frac{z_1}{z_1 + z_5 + z_6} \\
\xi_1 = \frac{\hat{z}_1'}{z_{\text{loop}}} = \frac{z_1 + z_5}{z_1 + z_5 + z_6} = 1 - \zeta_1 \\
\xi_2 = \frac{\hat{z}_2'}{z_{\text{loop}}} = \frac{z_5}{z_1 + z_5 + z_6} = 1 - \zeta_2.
\]

This matrix can also be constructed directly using Equation (3.59) as the difference \( \mathbf{B}_1^T - (1/z_{\text{loop}})\mathbf{C} \), where \( \mathbf{B}_1 \) is the path-set matrix

\[
\mathbf{B}_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}, \tag{6.5}
\]

and \( \mathbf{C} \) the unity-rank matrix calculated as \( \mathbf{C} = \gamma \zeta^T \), where

\[
\gamma^T = \mathbf{B}_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix} \tag{6.6}
\]

and

\[
\zeta^T = \begin{bmatrix}
\hat{z}_1 & \hat{z}_2 & \hat{z}_3 & \hat{z}_4 & \hat{z}_5 \\
\end{bmatrix} = \begin{bmatrix}
z_6 & z_1 + z_6 & z_1 + z_6 & z_1 + z_6 & z_1 + z_6 \\
\end{bmatrix}. \tag{6.7}
\]

From (6.5) and (6.6) one may construct the \( N \times N \) matrix \( \mathbf{B} \) defined in (3.33)—or
rather its transpose—as

\[
B^T = \begin{bmatrix} B_1^T & \gamma \end{bmatrix} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 
\end{bmatrix},
\]

(6.8)

It is easily verified that this matrix is nonsingular and that its inverse is

\[
A_+ = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix},
\]

(6.9)

the first five rows of which are precisely the reduced incidence matrix of the cable network (the row corresponding to \( \bar{6} \) having been removed), while the last row is \( e_5^T \), corresponding to the single cotree branch \( i_5 \) removed from the network to form the tree from which \( B_1 \) was defined. Thus Theorem 3.4 is seen to hold.

Suppose now that the external circuits having discretion are those at \( \bar{1}, \bar{4}, \bar{5} \) and \( \bar{6} \). If the discretionary map is linear, the demand sensitivity matrix \( T \) may be calculated
\[ T = S \left( I_6 - \frac{1}{\text{bal } D} D I_6^T \right) \]
\[
= \frac{1}{D_1 + D_4 + D_5 + D_6} \begin{bmatrix}
-\xi_1 & \xi_2 & \xi_2 & \xi_2 & \xi_2 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-\xi_1 & -\xi_2 & -\xi_2 & -\xi_2 & -\xi_2 \\
\xi_1 & \xi_2 & \xi_2 & \xi_2 & \xi_2
\end{bmatrix}
\times \begin{bmatrix}
D_4 + D_5 + D_6 & -D_1 & -D_1 & -D_1 & -D_1 \\
0 & \text{bal } D & 0 & 0 & 0 \\
0 & 0 & \text{bal } D & 0 & 0 \\
-D_4 & -D_4 & -D_4 & D_1 + D_5 + D_6 & -D_4 & -D_4 \\
-D_6 & -D_5 & -D_5 & -D_5 & D_1 + D_4 + D_6 & -D_5
\end{bmatrix}.
\]

Defining \( C_j \) for each cable segment \( j \) as in Corollary 3.3 one readily verifies that
\( t_{jk} = s_{jk} - C_j \). For example, when calculating the effects of demand changes on the
cable current \( i_1 \), one has
\[
C_1 = \frac{-\xi_1 D_1 + \xi_2 (D_4 + D_5)}{D_1 + D_4 + D_5 + D_6} = \frac{-z_6 D_1 + z_5 (D_4 + D_5)}{(z_1 + z_5 + z_6)(D_1 + D_4 + D_5 + D_6)},
\]
a quantity depending only on the ring-main impedances and the discretionary ratings
\( D_i \). The corresponding sensitivity to a change in demand at, say, \( \mathcal{I} \) is
\[
t_{i_1} = s_{i_1} - C_1 = \frac{z_6 D_1 + z_5 (D_1 + D_6)}{(z_1 + z_5 + z_6)(D_1 + D_4 + D_5 + D_6}).
\]

It is apparent also that, given an arbitrary demand change \( \Delta \mathbf{d} \), the resulting change
in the cable current \( i_j \) is
\[
\Delta i_j = \sum_{k=1}^{N-1} s_{jk} \Delta d_k - C_j \text{ bal } \Delta \mathbf{d}
\]
(6.10)

which implies in particular that if the change in demand is balanced, the effect on the
discretionary components of current may be neglected. This is true because, with a
linear discretionary map, the steady-state discretionary setpoint at any given external
circuit depends only on the balance of the demand profile and not on the distribution
of demand between external circuits.
Part III

Dynamics
Chapter 7

Models Large and Small

7.1 Introduction

This section of the thesis is devoted to the dynamic, or transient, problems complementary to the steady-state problems discussed earlier. Given a desirable DC steady state, and an initial state differing from this, we seek to know how the control mechanisms in a managed DC reticulation system can make the transition from one state to the other in the ‘best’ manner.

Previous studies [52] have demonstrated the need for some sort of fine control over the DC bus voltage. Impedances in DC power systems are generally much lower than in AC systems, and the effect of transient disturbances correspondingly more severe. Voltage transients in uncontrolled DC circuits are characterised by large relative magnitudes and very short time scales. These observations militate against the simplest solution, which is to deny that a problem exists and allow the circuit to regulate itself, as has traditionally been possible with plain old electricity supply.

In a typical managed DC installation, it is expected that mains-fed AC-DC converters will be the primary energy source. Therefore, for most applications, the general transient problem reduces to that of controlling the switching in one or more such converters connected in parallel. The performance criteria used to decide on the ‘best’ transient behaviour may then incorporate one or more of the following:

1. DC bus voltage settling time.

2. Deviation of DC bus voltage from its desired value (integral-squared-error or ISE criterion).
3. Settling time or ISE deviation of converter output current. (Note that the output current is \( r_k - d_k \) where \( d_k \) is a rapidly-varying setpoint determined by the bus control system and \( r_k \) is a slowly-varying offset determined by the converter itself or a higher-level energy management system.)

4. AC input power factor and/or harmonic distortion.

In this chapter we develop dynamical models for DC networks fed by converters. Our starting point is the topological analysis of Chapter 3, which is extended in Section 7.2 to account for the dynamic behaviour of network elements and external circuits. The ‘Large Model’ that results is suitable for computer simulation of networks with known topology and external circuit behaviour. For analytical purposes it is often convenient to use the greatly simplified ‘Small Model’ of Section 7.3, in which the network is collapsed to a single node.

For control on fast time scales, the Large or Small Model must be elaborated to account for the switching behaviour of the supply converter(s). The Generalised Synchronous Switching (GSS) model of Section 7.4 considers the simplest case, that of a bridge converter whose switch state changes at regular discrete time intervals.

This model is inadequate, however, to describe switch-mode converters which are pulsewidth modulated to allow switching to occur at arbitrary times. After reviewing some standard approaches to PWM switching in Section 7.5, we are in a position to combine PWM with the generalised approach to switch states taken in the GSS model. The result is the Generalised Asynchronous Switching (GAS) model, described in Section 7.6. The GAS model allows us to investigate the behaviour of voltage-driven or current-driven switch-mode converters with completely arbitrary switching sequences. It will be applied in the next chapter to the design of novel switching schemes for current-driven converters.

As stated at the outset, the emphasis in the present work is on control of switch-mode converters, which are the prime candidates for supply sources to future DC reticulation systems. Appendix F provides another model which describes the switching behaviour in thyristor converters. Called the Asynchronous Thyristor Switching (ATS) model, it can be regarded as an asynchronous discrete-time model in which the control is accomplished through the switching times themselves and the system evolves autonomously (and linearly) between switching times. The ATS model is, however, not used in the remainder of the thesis.

128
7.2 Extension of Topological Analysis: the ‘Large’ Model

In this section we outline the extension of the theory of Chapter 3 to the dynamics of managed distribution networks. This extension entails some minor redefinitions. For notational convenience in Chapters 3 through 6, we defined $\mathbf{j}$ and $\mathbf{u}$ as $(N-1)$-vectors, excluding the external circuit current and potential at the datum node $\overline{N}$. Here we relabel the potential as $\mathbf{v}$, to distinguish it from the control input $\mathbf{u}$. From this point onward, $\mathbf{j}$ and $\mathbf{v}$ are the $N$-vectors comprising all $N$ external circuit currents and node potentials respectively. Similarly, $\mathbf{A}$ is the full incidence matrix of the cable network $\mathcal{C}$, with $N$ rows and $C$ columns.

### 7.2.1 Cable Network Dynamics

The principal modifications required to the static theory are the addition of inductive and capacitive elements to the network model. Specifically, each segment in the cable network $\mathcal{C}$ is modelled as a short transmission line with resistance $r_j$ and series inductance $\lambda_j$, $1 \leq j \leq C$. Thus (where cable segment $e_j$ is directed from node $\overline{p}$ to node $\overline{q}$) we obtain $C$ equations of the form

$$\lambda_j \frac{di_j}{dt} + r_j i_j = v_p - v_q$$

which combine into the matrix equation

$$\mathbf{A} \frac{d\mathbf{i}}{dt} = -\mathbf{A}^T \mathbf{v} - \mathbf{Ri}. \quad (7.1)$$

Here $\mathbf{A}$ and $\mathbf{R}$ are matrices having the numbers $\lambda_j$ and $r_j$, respectively, on the diagonal. Inductive coupling between cable segments is easily accounted for in (7.1), if necessary, by introducing additional elements in $\mathbf{A}$.

Capacitance is distributed throughout the DC system to provide voltage support and stability. We assume that at each node $\overline{k}$ there is connected a positive capacitance $C_k$ so that

$$C_k \frac{dv_{\overline{k}}}{dt} = \sum_j a_{kj} i_j - j_k$$

where $j_k$ is the external circuit current leaving $\overline{k}$. In matrix form this becomes

$$\mathbf{C} \frac{d\mathbf{v}}{dt} = \mathbf{Ai} - \mathbf{j} \quad (7.2)$$

129
where $C$ is a diagonal matrix of capacitances.

Together, (7.1) and (7.2) describe the dynamics of the $N$-port cable network $\mathcal{C}$, with $N + C$ state variables $(\mathbf{v}, \mathbf{i})$ and $N$ system inputs, the port currents $\mathbf{j}$. If desired, these equations may be cast into the alternative flux-charge representation by introducing the equivalent state variables $\phi = \mathbf{A} \mathbf{i}$ and $\mathbf{q} = \mathbf{C} \mathbf{v}$ to give

$$
\dot{\phi} = (-\mathbf{A}^T \mathbf{C}^{-1}) \mathbf{q} - (\mathbf{R} \mathbf{A}^{-1}) \phi
$$

$$
\dot{\mathbf{q}} = (\mathbf{A} \mathbf{A}^{-1}) \phi - \mathbf{j}.
$$

### 7.2.2 External Circuit Modelling

For analysis and simulation purposes, the model above is extended to incorporate simplified models of the external circuits. These have two broad classifications, reflecting those commonly used for inverter circuits. Voltage-driven circuits are modelled using a (possibly time-dependent) functional relationship between voltage and current, thus:

$$
\mathbf{j} = \sigma f(v, t).
$$

(7.3)

The coefficient $\sigma$, called the sign of the circuit, is either $+1$ or $-1$ and allows us to adopt different sign conventions for $f$ in supply and load circuits. Circuits which may be modelled by (7.3) include resistive elements, constant-power loads and clamping circuits, as well as simple current sources and sinks.

Current-driven circuits are modelled using a functional characteristic in series with an inductance $L$, thus:

$$
\sigma L \frac{dj}{dt} + h(j, t) = v.
$$

(7.4)

Again, the sign $\sigma$ is chosen according to whether $j$ is a sink current ($\sigma = 1$) or source current ($\sigma = -1$). Suitable candidates for $h$ include resistive elements and voltage sources. In order to keep our whole-system model tractable, we restrict the external circuit models to the two forms (7.3) and (7.4).

In general, a number of external circuits with differing characteristics will be connected at a single DC bus node. Multiple voltage-driven circuits are easily represented by a composite function $f$ equal to the sum of the component characteristics. Parallel combination of $m$ current-driven circuits is also possible in the special case where the
characteristics to be combined have the form

\[ h_i(j_i, t) = V_i(t) + \sigma_i \rho(t) L_i j_i, \quad 1 \leq i \leq m. \] (7.5)

Here either \( V_i \) or \( \rho \) may be zero. Writing the equation for the \( i \)th circuit as

\[ \sigma_i \frac{dj_i}{dt} + \frac{V_i(t)}{L_i} + \sigma_i \rho(t) j_i = \frac{v}{L_i} \]

and setting \( j = \sigma \sum_i \sigma_i j_i \) (with arbitrary sign \( \sigma \)), we obtain upon defining \( L = (\sum_i (1/L_i))^{-1} \)

\[ \sigma L \left( \frac{dj}{dt} + \rho(t) j \right) + \sum_{i=1}^{m} \frac{L_i}{L_i} V_i(t) = v. \] (7.6)

Physically, (7.6) expresses the parallel combination of \( m \) voltage sources with series inductive impedances, each having the same time constant \( L_i/R_i = 1/\rho \). Note that this is also a current-driven circuit, and further has the form (7.5) when the \( i \) subscripts in the latter are dropped.

### 7.2.3 A Complete System Model

Without loss of generality, we shall consider as a single circuit two or more linear current-driven circuits with the same time constant connected at the same node. It then follows that every current-driven external circuit increases the order of the entire system by one. Suppose there are \( E \) such circuits; we then define \( E \) additional state variables \( \mathbf{\iota} \) representing the inductor currents in these circuits. The connection of these \( E \) circuits at the \( N \) ports of \( \mathcal{C} \) is represented by an \( N \times E \) matrix \( \mathbf{E} \) of zeroes and ones, defined by the equation

\[ \mathbf{j} = \sigma_v f(\mathbf{v}, t) + \mathbf{E} \sigma_c \mathbf{\iota}. \] (7.7)

Here the function \( f \) (with \( f_k = f_k(\mathbf{v}_k, t) \) for all \( k \)) represents the voltage-driven part of the external network \( \mathcal{E} \), and \( \mathbf{E} \sigma_c \mathbf{\iota} \) the current-driven part. The diagonal matrices \( \sigma_v \) and \( \sigma_c \) (with diagonal entries \( +1, -1 \)) allow the \( f_k \), respectively \( \mathbf{\iota}_k \), to be defined as either sink or source currents.

Following (7.4), we obtain for the external circuit dynamics

\[ \sigma_c L \frac{d\mathbf{\iota}}{dt} = \mathbf{E}^T \mathbf{v} - h(\mathbf{\iota}, t) \] (7.8)
where again, the matrix \( \mathbf{L} \) allows for inductive coupling between external circuits. Finally, equation (7.2) must be modified to make the dynamics of \( \mathbf{j} \) explicit:

\[
\mathbf{C} \frac{d\mathbf{v}}{dt} = \mathbf{A} \mathbf{i} - \mathbf{E} \sigma_\nu \mathbf{u} - \sigma_v f(\mathbf{v}, t). \tag{7.9}
\]

Equations (7.1), (7.9) and (7.8) combine to produce a nonlinear system of order \( N + C + E \). This can be expressed as a perturbed linear system if we write

\[
f(\mathbf{v}, t) = \mathbf{I}_0(t) + \sigma_v \mathbf{G}(t)\mathbf{v} + F_0(\mathbf{v}, t) \tag{7.10}
\]

\[
h(\mathbf{u}, t) = \mathbf{V}_0(t) + \sigma_\nu \mathbf{P}(t)\mathbf{u} + H_0(\mathbf{u}, t) \tag{7.11}
\]

with \( F_0(0, t) \equiv H_0(0, t) \equiv \mathcal{D}_1F_0(0, t) \equiv \mathcal{D}_1H_0(0, t) \equiv 0 \). (We use the Dieudonné notation for partial derivatives, whereby \( \mathcal{D}_k f \) denotes the derivative of \( f \) with respect to its \( k \)th argument.) Note that the linear terms have been constructed so as to maintain a passive sign convention for \( \mathbf{G} \) and \( \mathbf{P} \). Set \( \mathbf{x} = \begin{bmatrix} \mathbf{v}^T & \mathbf{i}^T & \mathbf{u}^T \end{bmatrix}^T \), and let \( \mathbf{w} = \begin{bmatrix} \mathbf{I}_0^T & \mathbf{V}_0^T \end{bmatrix}^T \) be the independent input to the system. Then

\[
\dot{\mathbf{x}} = \Phi(t)\mathbf{x} + \phi_0(\mathbf{x}, t) + \Gamma \mathbf{w} \tag{7.12}
\]

where

\[
\Phi(t) = \begin{bmatrix}
-C^{-1}\mathbf{G} & C^{-1}\mathbf{A} & -C^{-1}\mathbf{E} \sigma_v \\
-C^{-1}\mathbf{A}^T & -\Lambda^{-1}\mathbf{R} & 0 \\
L^{-1} \sigma_\nu \mathbf{E}^T & 0 & -L^{-1} \mathbf{P}
\end{bmatrix}
\]

\[
\phi_0(\mathbf{x}, t) = \begin{bmatrix}
-C^{-1}\sigma_v F_0(\mathbf{v}, t) \\
0 \\
-L^{-1} \sigma_\nu H_0(\mathbf{u}, t)
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
-C^{-1}\sigma_v & 0 \\
0 & 0 \\
0 & -L^{-1} \sigma_c
\end{bmatrix}.
\]

An equivalent flux-charge representation is easily derived.

The Large Model developed here is suitable for computer simulation of DC reticulation systems whose topology and external circuits have been completely specified. When topology is specified only in very general terms one may, by analogy with large AC power systems, attempt an ‘area-based’ study in which a large number of nodes are collapsed into a smaller number, reflecting the dominant eigenvalues of \( \Phi \). When the emphasis is on investigating the behaviour at a single external circuit, one may go still further and condense the entire cable network to a single node, at which are connected all external circuits. This is the approach taken in the supply control models developed below.
Figure 7.1: Power circuit for Small Model

7.2.4 Control Action

In the system as modelled above, the control action will usually be expressed through the source values $I_0$ and $V_0$. For external circuits with discretionary capability, these correspond to the output of discretionary controllers programmed to track a current setpoint $d_k - r_k$. When there is no discretion, the source values may be taken as extraneous inputs or disturbances.

Of course, external circuit current is not the only variable of interest; voltage is at least as important. In [52] it is suggested that system voltage control be accomplished by adding to each discretionary setpoint an ‘error current’ determined from the voltage error and its derivative:

$$d_k = d_k^* + K_k (V_R - v_k) - K_{Dk} \frac{dv_k}{dt}$$

(7.13)

where $d_k^*$ is the steady-state setpoint, $V_R$ is the voltage setpoint, and $K_k$, $K_{Dk}$ are tunable gains. (In [52], network topology is not taken into account and the error current simply added to the total net demand bal r before the discretionary profile is determined. The two strategies are equivalent provided the system voltage profile is fairly uniform.)

In Chapter 11 we explore a number of alternative approaches to voltage control.

7.3 The ‘Small’ Model

The simplest of the control models we denote as the Small Model. In this model the system is represented as a single node with capacitance $C$ and two external circuits: a voltage-fed ‘load’ circuit (with current $I_L$) and a current-fed ‘supply’ circuit (with current $I_S$). The supply circuit consists of an inductance $L$ in series with a continuously variable voltage source $V_s$. Figure 7.1 depicts the power circuit.
The behaviour of the Small Model is extensively analysed in [52]. Its significance here is less as a self-contained model for the system than as a basic framework of which the more complicated models below are extensions.

### 7.3.1 Analysis in LC Units

The power circuit of Figure 7.1 can be regarded as an instance of the two-port $LC$ configuration depicted in Figure 7.2. For this particular configuration, the various circuit quantities have a convenient natural representation in dimensionless units, which we shall call *LC units*.

The dynamics of the LC network are governed by the two equations

$$ V_1 - L \dot{I}_1 = V_2 $$
$$ I_1 - C \dot{V}_2 = I_2 $$

where a dot denotes differentiation with respect to time $t$. These combine to form a second-order system with a natural resonant frequency $1/\sqrt{LC}$. The base quantities which shall serve as normalisation factors for the circuit quantities are:

1. The characteristic frequency: $\omega = \frac{1}{\sqrt{LC}}$.

2. The characteristic impedance: $Z_C = \sqrt{\frac{L}{C}} = \omega L = \frac{1}{\omega C}$.

3. The nominal voltage, $V_R$ (arbitrarily chosen).

Table 7.1 defines the LC units for time, voltage, current, impedance and power. Note that absolute voltage quantities, in addition to being scaled, are also shifted so that the nominal voltage becomes zero in LC units. This is done for convenience in control studies, where it is customary to locate the desired equilibrium at the origin.
<table>
<thead>
<tr>
<th>Quantity</th>
<th>Standard symbol</th>
<th>LC definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$t$</td>
<td>$\tau = \omega t$</td>
</tr>
<tr>
<td>Voltage</td>
<td>$V$</td>
<td>$v = \frac{V}{Z_C} - 1$</td>
</tr>
<tr>
<td>Current</td>
<td>$I$</td>
<td>$i = \frac{V_R}{Z_C}$</td>
</tr>
<tr>
<td>Impedance</td>
<td>$Z$</td>
<td>$z = \frac{Z_C}{Z_C}$</td>
</tr>
<tr>
<td>Power</td>
<td>$P$</td>
<td>$p = \frac{P}{P_C}$ where $P_C = \frac{V_R^2}{Z_C}$</td>
</tr>
</tbody>
</table>

Table 7.1: Definition of LC units

of the state space. Thus, zero volts is $-1$ in LC units, while the power equation is $p = (1 + v)i$. Voltage drops and line-to-line voltages, being differences of absolute voltages, are not shifted in this manner, although it will be necessary on occasion to introduce a $-1$ term when a line-to-line voltage needs to be regarded as an absolute voltage in its own right. With these provisos, Ohm’s and Kirchhoff’s laws remain unchanged.

Derivatives with respect to the scaled time variable $\tau$ are denoted with a prime symbol:

$$x' = \frac{dx}{d\tau} = \omega^{-1}x.$$  \hspace{1cm} (7.14)

Recast in LC units, the network equations become simply

$$v_1 - v'_1 = v_2$$
$$i_1 - i'_1 = i_2$$

which combine as

$$v'_2 + v_2 = v_1 - v'_1.$$ 

The resulting second order system may be described using the state variables $v_2$ and $i_1$, or alternatively $v'_2$ and $v'_2$.

### 7.3.2 Small Model with Source Resistance

The Small Model may be extended by adding a resistance $R$ in series with the inductance $L$. This element accounts for a number of effects including switching losses, line
losses between the supply source and the bus, the resistance of the inductor \( L \) itself, and voltage drop due to the effect of AC impedance on converter switching. With state-of-the-art components, typical values for \( R \) will not be large and will depend on the size of the converter, so that at full loading one may expect a voltage drop due to \( R \) of no more than about 5% of rated DC voltage.

For a thyristor converter rated at a few hundred amps, operating on a 415V AC main, a reasonable value is \( R = 50 \text{mΩ} \). Previous simulation studies [52] have shown that series resistance of this magnitude does not significantly affect the behaviour of the Small Model from a control perspective. We shall therefore neglect source resistance in control studies based on the Small Model power circuit.

With source resistance taken into account, the DC voltage gain from \( V_s \) to \( V \) is slightly less than unity (assuming the net current flow is from supply to load). In addition, if the load has a resistive characteristic then frequency response expressions require slight modifications, both in the DC gain and in the natural frequency \( \omega_0 \). We shall not pursue this issue in detail, but simply underline the need to allow a margin for error in all design calculations.

### 7.4 Generalised Synchronous Switching Model

The Generalised Synchronous Switching (GSS) Model was motivated by the need to include the switching behaviour of the input converter in the DC system model, while at the same time keeping the model amenable to standard controller design techniques. It was also desired to leave open the possibility of including ‘nonstandard’ switching sequences within the scope of the model; thus, the devices in a thyristor converter might be switched out of standard phase sequence. Neither of these criteria apply (for example) to the ATS model in Appendix F, which was an early forerunner to the GSS model.

The GSS model accordingly makes no prior assumptions about the nature of the switching control, other than that it operates at regular, arbitrarily short intervals \( nT \). We start from the most general picture of the switched circuit, define the control objective, and only then introduce constraints relating to the nature of the devices or the times at which any given device may be switched on or off.

A model developed according to these principles, as well as providing a tool for the
design of optimal controllers, allows us to answer questions of the form: is strategy $X$ the best strategy for achieving objective $Y$ with constraints $Z$? Here $X$ could be phase control, or a quantised approximation to PWM (though true PWM requires a still more general model to be developed below).

### 7.4.1 Topology and Operating States

For simplicity, we restrict our attention to simple bridge converters, with an odd number $p$ of input phases having instantaneous voltages $V_k$, $0 \leq k \leq p - 1$ given by

$$V_k = \sqrt{2}V_{AC} \cos \left( \Omega t - \frac{2\pi k}{p} \right). \quad (7.15)$$

The $p$ supplies are connected through $p$ semiconductor devices to the common cathode (CK); another set of $p$ devices connects the supplies to the common anode (CA). The 'DC' output is taken between CK and CA. (See Figure F.1.)

The circuit on the DC side may be represented using either the Large Model or the Small Model. If the Large Model is used, the converter becomes a voltage-driven or (with a DC-side inductance) a current-driven external circuit, and the converter output voltage, respectively current, becomes a state variable in the Large Model. However, for ease of presentation we shall henceforth embed our converter models within the Small Model power circuit. Thus the power circuit on the DC side is as for the Small Model, with series inductance between converter and DC bus. We assume a time-varying linear load $i_L = i_0(\tau) + \zeta v(\tau)$ (in LC units).

By switching the semiconductor devices, any combination of the $p$ supply phases can (in principle) be brought together at the CK or CA terminals. We define a *switch state* as a pair $(Q, R)$ with $Q, R \in \{0, 1\}^p$. $p$-vectors $Q$ and $R$ indicate the state of the devices on the CK and CA respectively. The switch state is a *conduction state* or *c-state* if $Q \neq 0$ and $R \neq 0$, and an *open state* otherwise. Phase $k$ is said to be *open* if $Q_k = R_k = 0$ and *shorted* if $Q_k = R_k = 1$. A c-state with no shorted phases is a *driving state* or *d-state*, otherwise it is a *free state* (in the sense of ‘freewheeling’).

We define $|Q| = \sum Q_k$ and $|R| = \sum R_k$.

In the ideal unconstrained situation, the set of possible controls is the set of all c-states. For simplicity, we may consider c-states in which only one device in each half is conducting: $|Q| = |R| = 1$. We refer to these as the *simple* c-states; such states may also be identified with a pair $(q, r)$, $q, r \in \mathbb{Z}_p$. With the converter in simple
c-state \((q, r)\), and defining \(\sigma_{qr} = \text{sgn}(q - r)\), the instantaneous output voltage is

\[
V_S = V_q - V_r = V_{qr} \cos \left( \Omega t - \pi \frac{q + r + \sigma_{qr} p/2}{p} \right), \tag{7.16}
\]

\[
V_{qr} = 2\sqrt{2}V_{AC} \sin \frac{\pi |q - r|}{p}. \tag{7.17}
\]

A simple d-state is \textit{maximal} if \(V_{qr}\) is maximal, which occurs when \(|q - r| = (p+1)/2\). There are precisely \(2p\) maximal d-states, with instantaneous output voltage

\[
V_S = \sqrt{2}V_L \cos \left( \Omega t - \left( k + \frac{1}{2} \right) \frac{\pi}{p} \right) \tag{7.17}
\]

for \(0 \leq k < 2p\) where \(V_L\), the maximal line-to-line supply voltage, is given by

\[
V_L = 2V_{AC} \cos \frac{\pi}{2p}. \tag{7.18}
\]

At any given time instant, the largest and the smallest of the possible DC-side voltages will be found among the maximal d-states. In terms of the GSS model, standard phase control in a thyristor converter is no more or less than switching between the maximal d-states in order of increasing \(k\), with the switching times delayed according to the parameter \(\alpha\).

### 7.4.2 Discrete-Time Circuit Model

We work in LC units with the usual definitions of \(\omega\) and \(Z\), and once again assume the load to be slowly-varying, with \(\zeta' \ll 1\) and \(i'_0 \ll 1\). The dynamics are

\[
v'' + 2\zeta(\tau)v' + v = v_S(\tau)
\]

which is a DHO (see Appendix D) with forcing function \(v_S = V_S/V_R - 1\).

Now, choose a sampling time \(T \ll 2\pi/\Omega\), and set \(\tau_0 = \omega T\). We may then regard \(v_S\) as approximately constant over one sample interval, and work with the zero-order-hold equivalent discrete-time system. Define

\[
C = C_{\zeta}(\tau_0), \quad S = S_{\zeta}(\tau_0),
\]

\[
\alpha = C + \zeta S, \quad \beta = C - \zeta S,
\]

\[
\gamma = e^{-\zeta \tau_0}, \quad \mu = 1 - \gamma \alpha
\]

\(C_{\zeta}(\tau_0)\) and \(S_{\zeta}(\tau_0)\) are defined in Appendix D) and set

\[
v_n = \begin{bmatrix} v(n\tau_0) \\ v'(n\tau_0) \end{bmatrix}, \quad u_n = v_S(n\tau_0).
\]

138
One then derives the approximate discrete system

\[ \mathbf{v}_{n+1} = \begin{bmatrix} \alpha & S \\ -S & \beta \end{bmatrix} \mathbf{v}_n + \begin{bmatrix} \mu \\ \gamma S \end{bmatrix} u_n. \]  

(7.19)

Note that in general this is a time-varying system, as the damping coefficient \( \zeta \) varies with time. Nonetheless, in what follows we shall frequently make the simplifying assumption that \( \zeta \) is constant or slowly-varying, so that the system is effectively time-invariant.

Let \( \theta \) be a vector of physical control parameters (such as \( p, q \) for control based on simple \( c \)-states). We assert that there exist functions \( \rho(\theta) \) and \( \phi(\theta) \) such that, for fixed \( \theta \)

\[ v_S(\tau) = \rho(\theta) \cos(\sigma \tau - \phi(\theta)) - 1 \]  

(7.20)

where \( \sigma = \Omega/\omega \). Now, assume that the sample period divides evenly into the AC supply period: \( 2\pi/\Omega = MT, M \in \mathbb{Z} \). Assuming \( M \) is sufficiently large, the control input to our model is

\[ u_n = \rho(\theta) \cos 2\pi \Gamma_n(\theta_n) - 1, \]  

\[ \Gamma_n(\theta_n) = \frac{n}{M} - \frac{\phi(\theta_n)}{2\pi}. \]  

(7.21)

Our control strategy then has two components. First, we use linear theory to find a desired control input \( u_n \) to the model (7.19). Given \( u_n \), we then find \( \theta_n \) to minimise

\[ |\rho(\theta) \cos 2\pi \Gamma_n(\theta) - 1 - u_n|. \]  

(7.22)

This approach can be expected to give reasonable performance, provided we incorporate the key limitations implicit in (7.21) as constraints on our linearised design.

### 7.4.3 Practical Constraints

Two salient features of (7.21) that will need to be taken into account in the linear control design are:

1. \( u_n \) is bounded: regardless of the choice of switching state, we necessarily have

\[ |1 + u_n| \leq V_L/V_R. \]

2. \( u_n \) is quantised: at any given time there is only a finite number of choices for \( \theta_n \), hence we must be prepared to tolerate a certain level of ‘quantisation error’ in the actuator.
There are also more subtle constraints arising from the nature of the devices used in the converter. It was suggested above that at each sample it was possible to choose our control input from among all c-states (or at least all simple c-states). For this simplified assumption to work, we require that our converter have *forced instantaneous commutation*. This is not the case for thyristor converters, in which the following additional constraints operate.

1. **Natural commutation**: a device will not start conducting current unless it is forward biased. Usually, this means its instantaneous phase voltage must be greater than that of other active devices.

2. **Finite commutation time**: a device will cease conducting only when the current reaches zero under natural commutation. The pattern of conduction passes through an intermediate state in which two sets of devices conduct simultaneously.

Both of these factors affect the degree of quantisation in the control, and hence the level of quantisation error. In addition, the second will impact on the control bounds.

**Natural Commutation Constraints**

Consider a transition from state \((j, r)\) to state \((k, r)\) at time \(t_0\). Under natural commutation, this transition will occur only if

\[
V_k(t) > V_j(t) \quad t_0 \leq t \leq t_0 + t_C
\]

where \(t_C\) is the commutation time required. Substituting for \(V_j\) and \(V_k\), we require

\[
\sin \frac{\pi(k - j)}{p} \sin \left( \frac{\Omega t - \pi(j + k)}{p} \right) > 0
\]

(7.23)

for \(\theta_0 \leq \Omega t \leq \theta_0 + \theta_C\) (where \(\theta_0 = \Omega t_0, \theta_C = \Omega t_C\)). In LC units, we write \(\sigma \tau\) in place of \(\Omega t\); all else remains the same. The inequality reverses for transitions of the form \((q, j) \rightarrow (q, k)\).

(7.23) defines a switching ‘window’ of width \(\pi - \theta_C\) in which the transition is permitted. For a given \(j\) one has a set of overlapping windows corresponding to the \(p - 1\) possible values of \(k\). Due to this overlapping effect, over any one AC cycle there will be intervals of width \(\pi/p\) during which the number of simple c-state alternatives one may choose from (including the present state) is precisely \(q\) on the CK and \(p - q\)
on the CA, for any \( q \) between 0 and \( p \). This results in a time-dependent quantisation error.

**Intermediate States**

Even if the control strategy is restricted to simple c-states, finite commutation time entails that the converter will pass through intermediate c-states that are not simple. As the converter may spend significant time in these intermediate states, they must be provided for in a practical model.

As the analysis in Appendix F shows, for generic DC circuit behaviour the commutation time in a thyristor converter depends in a complicated way on timing and current levels, and is usually not controllable. This adds a degree of difficulty to the proposed switching strategy. In principle, the more stringent bounds arising from intermediate states may be taken into account in the control design, although the best way of doing this is an open question. Usually, the entire issue is avoided by using devices such as IGBTs or GTOs with turn-off capability.

### 7.5 PWM Switching Model

Aside from thyristor converters, pulse-width modulation (PWM) is the most popular technique for AC-DC as well as force-commutated DC-AC conversion. A PWM (switch-mode) converter uses devices with turn-off capability, such as IGBTs or GTOs, which are switched at high frequency (typically a few kHz) with a varying duty ratio. The switching process produces substantial high-frequency voltage and current harmonics, requiring the connection of lowpass filters on input and output. PWM converters offer numerous advantages over thyristor converters, notably bidirectional current flow, high AC power factor, and negligible low-frequency harmonics [50, 78, 7]. Their chief disadvantages are in the higher cost and lower power-handling capability of the switching devices, though as semiconductor technology continues to evolve this will probably count for less in the future.

The PWM rectifier originated as an offshoot of the more popular PWM inverter, so its theory owes much to inverter concepts. To a large extent this is warranted, as the bidirectional power capability allows for an almost completely symmetric treatment. Thus, in the theory of DC-AC conversion one speaks of voltage-source and current-
source inverters. This same terminology is used when similar forms of circuit are employed for AC-DC conversion, and one speaks of voltage-source or current-source converters [79, 7]. These terms, though standard, are apt to confuse those who approach the subject without a background in inverter theory, since in the case of an AC-DC converter the ‘source’ is really a sink. Accordingly, some practitioners refer instead to ‘current-type’ [65] or ‘voltage-link’ [37] converters. For our part, we shall use the terms ‘voltage-driven’ and ‘current-driven’, which is consistent with the classification set out in Section 7.2.2.

The primary objective with PWM converters, as commonly understood, is to optimise the AC-side characteristics, in particular power factor and harmonics. Accordingly, analysis of such converters includes extensive manipulation of polyphase AC quantities, the formalism of which is presented in Appendix E.

### 7.5.1 System Model for Voltage-Driven Converter

The most studied form of PWM converter is the voltage-driven topology, in which the converter output feeds directly onto the DC bus with no intervening inductor as shown in Figure 7.3. Each of the three AC phases is connected by a ‘positive switch’ to the CK and by a ‘negative switch’ to the CA. The AC supply is assumed to have series inductance $L_S$ and may also have series resistance $R_S$ (not shown).

The switches themselves are normally IGBTs with antiparallel diodes [7]. This device combination permits current to flow in either direction while the switch is closed, but can only block voltage in one direction when open. Such a limitation does not concern us as long as the DC bus voltage is greater than the peak-to-peak AC voltage. The steady-state analysis of Chapter 9 shows that such ‘boosting’ behaviour
is in fact characteristic of voltage-driven PWM converters, or VDCs for short.

The converter switching occurs in such a manner that at any time instant, one switch on each phase is open and the other closed; this together with the AC line inductance guarantees a continuous input current on each phase. The voltage-driven configuration permits the simultaneous opening of all three positive (or negative) switches as part of the switching cycle, as the DC bus capacitance provides for continuity of output current in the short term.

The PWM strategy itself has two essentially equivalent representations, in terms of either duty ratios or space vectors. The framework we adopt is that of Wu et al [78, 79]. We begin by associating a switching function \( d_k^* \) with the \( k \)th input phase, where \( k \in \{0, 1, 2\} \). This has the value 1 when the positive switch is on (and the negative switch off) and 0 when the positive switch is off (and the negative switch on). Let \( E_k \) be the \( k \)th phase voltage at the generator, \( I_k \) the \( k \)th phase current, \( V_d \) and \( I_d \) the output voltage and current respectively, and \( V_- \) the voltage between the negative DC busbar and the supply neutral.

Applying Kirchhoff’s voltage law to each phase in turn for the cases \( d_k^* = 1 \) and \( d_k^* = 0 \) we obtain

\[
L_S \dot{I}_k = E_k - R_S I_k - d_k^* V_d - V_-.
\]  

(7.24)

Assuming that \( \mathbf{E} \) is balanced we have \( \sum E_k = \sum I_k = 0 \), and summing (7.24) over \( k \) then gives

\[
V_- = -\frac{V_d}{3} \sum_{k=1}^{3} d_k^*.
\]  

(7.25)

Substituting back into (7.24) then yields, in vector form

\[
L_S \dot{\mathbf{I}} = \mathbf{E} - R_S \mathbf{I} - \left( \hat{\mathbf{d}}^* - \frac{\text{bal} \hat{\mathbf{d}}^*}{3} \right) V_d.
\]  

(7.26)

Similarly, applying Kirchoff’s current law at the positive busbar provides an equation for the output voltage dynamics:

\[
C \dot{V}_d = \hat{\mathbf{d}}^T \mathbf{I} - I_d
\]  

(7.27)

(\text{where } I_d \text{ may depend implicitly on } V_d \text{ via the load characteristic). Equations (7.26) and (7.27) together provide a complete description of the system dynamics.}

We next assume that over a single sample period \( T \), the switching function \( d_k^* \) has the symmetric form depicted in Figure 7.4. The width of the ‘on’ pulse is governed
by the duty ratio \( d_k = T_{on}/T \), which over any sample period takes a constant (or very nearly constant) value between 0 and 1. This strategy ensures that, provided \( d_k \) does not remain at the fixed value 0 or 1 over multiple sample periods, every device is switched at a constant frequency \( F_s = 1/T \) (with two switchings per cycle). In practice, this PWM technique is implemented as an analogue circuit which compares the signal \( d_k \) with a triangle wave of period \( T \), as explained in [78].

As consequences of this modulation technique, we assert the following:

1. When \( \mathbf{d} \) is constant, the only harmonics produced are at multiples of the switching frequency \( F_s \); in particular, there are no low-frequency harmonics present.

2. The low-frequency behaviour of the converter is obtained by substituting the duty ratios \( \mathbf{d} \) for the switching functions \( \mathbf{d}^* \) in the model equations (7.26) and (7.27).

3. The PWM strategy is ‘optimal’ in that it minimises the deviations of the actual input currents from their low-frequency values.

In [79] the first two assertions are proved by means of Fourier analysis. The second assertion may also be demonstrated more directly by a simple averaging argument. Let the system above be represented as \( \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{E}, I_d, d_k^*, t) \), where \( \mathbf{x} = \begin{bmatrix} \mathbf{I}^T & V_d \end{bmatrix}^T \).

Now introduce a ‘fine’ time variable \( \tau = t/\epsilon \), where \( \epsilon \) is a small parameter, and consider the (equivalent) augmented system

\[
\frac{d\mathbf{x}}{d\tau} = \epsilon f(\mathbf{x}, \mathbf{E}, I_d, d_k^*, t)
\]

\[
\frac{d\tau}{d\tau} = \epsilon.
\]
Augmenting the system in this manner allows us to treat \( t \) and \( \tau \) as independent variables. Now, since everything on the right hand side is multiplied by the small parameter \( \epsilon \), we may obtain a low-frequency approximation by averaging the right hand side over the fine time variable \( \tau \). In doing so, we replace the switching functions \( d_k^* \) with their slowly-varying averages \( d_k \), and notionally replace the system variables \( x, E, I_d \) with their lowpass equivalents. Finally, we ‘collapse’ the system back into its original form, varying with the coarse time \( t \). For a rigorous justification of this approach, see [63].

The third assertion above rests on the alternative system representation by means of space vectors. Under this approach, a three-phase signal \( \mathbf{X} \) is represented by its (stationary) Park transform

\[
\ddot{X} = \frac{\sqrt{2}}{3} (X_0 + \alpha X_1 + \alpha^2 X_2).
\]  

(7.28)

For a balanced signal, \( \ddot{X} \) is a vector rotating anticlockwise, whose value at \( t = 0 \) is the phasor \( \ddot{X}_0 \). (See Appendix E for further details.) The components of \( \ddot{X} \) are of relevance in generalised machine theory [59, 60, 66] and, by extension, in field-oriented control of motor drive circuits [61]. In rectifier circuits, the Park transform is merely a mathematical device without much greater significance.

With each of the eight switching states \( \mathbf{d}^* \) of the converter, we may associate a voltage ‘space vector’ \( V_d \). This is just the Park transform of the three-phase signal \( \mathbf{d}^* V_d \), where the converter is treated as an inverter driven from a constant voltage source \( V_d \). There are two zero vectors, corresponding to \( \mathbf{d}^* = [0 \ 0 \ 0]^T \) and \( \mathbf{d}^* = [1 \ 1 \ 1]^T \); the remaining six nonzero vectors form a hexagon in the complex plane as shown in Figure 7.5. The hexagon is inscribed in a circle of radius \( \frac{2}{3} V_d \).

By switching the converter, we synthesise a series of space vectors \( V \). The time average \( \bar{V} \) of these space vectors is associated with (the Park transform of) the fundamental three-phase input current, while the difference \( V - \bar{V} \) is associated with the current harmonics. For each sample period, the desired input current is determined and an appropriate value of \( \bar{V} \) chosen based on the AC-side dynamics:

\[
\bar{V} = \bar{E} - R_s \bar{I} - L_s \bar{\tilde{I}}
\]  

(7.29)

where \( \bar{\tilde{I}} \) denotes the Park transform of \( \tilde{I} \). Integrating (7.29) over one sample period gives \( \tilde{I} \) as a function of \( V \) and \( \bar{E} \); one may then rearrange this expression to obtain a choice of \( V \) (or rather \( \bar{V} \)) in terms of the desired input current [61].
According to [75], the current harmonics are minimised if the switching sequence to achieve a given $\tilde{V}$ includes only those states adjacent to it in the diagram; the two zero states, and the two states bounding the sextant in which $\tilde{V}$ lies. These can always be switched in such a manner that only one converter leg switches in each state transition; we shall call this a Gray code sequence (VDC sense). The required average vector is synthesised as a convex combination of the four states, with the zero-state time being divided evenly between the two zero states.

In [78], it is shown that these two PWM strategies are equivalent: symmetrical PWM with arbitrary duty ratios $d$ produces a switching pattern corresponding to a Gray code sequence of adjacent space vectors. We may therefore adopt either formalism as the basis of the control framework; the task of the controller is to determine either a set of duty ratios, or an average space vector. The two are indeed related via the Park transform: the average space vector is essentially $\bar{d}V_a$, the Park transform of the duty ratios.

### 7.5.2 Current-Driven PWM Converters

The chief alternative to the standard voltage-driven configuration for PWM converters, though less common, is the current-driven configuration shown in Figure 7.6. Unlike the voltage-driven converter (henceforth VDC), the devices in a current-driven converter (or CDC) must be capable of withstanding both positive and negative voltages when open. However, provided the DC-side current is unidirectional, it will suffice to use a single controllable switching device, without a parallel diode. In practice, the gate turn-off (GTO) thyristor is preferred in CDCs owing to its higher
power capability, though IGBTs could also be used in principle. One then has a two-quadrant converter with reversible DC voltage, analogous to a thyristor converter. The DC-side capacitor $C$ may be omitted, depending on the application; for our purposes we shall assume it is present.

CDCs have attracted less interest than their voltage-driven counterparts. One could say this is a self-reinforcing trend, as the better understanding of VDCs leads to a presumption in favour of their use. Given the evolution of PWM rectifiers from inverters, one may also point to the narrow applicability of current-source inverters as partly explaining the lack of interest in CDCs. There are some inherent drawbacks with CDCs, notably the added cost of the DC-side inductor and AC-side filter capacitors, and the inability to reverse the current flow while maintaining a positive DC voltage. Thus, CDCs tend to be used only where a VDC would be inappropriate, chiefly in DC motor drives (themselves declining in popularity).

CDCs are nonetheless attractive when viewed from the perspective of mains-fed DC power systems. A key concern with bulk supply systems is the limitation of charging, inrush and fault currents. AC systems rely on series inductance to reduce the fault level (a measure of short-circuit apparent power flow); in DC systems, series inductance has the effect of limiting the rise time for current transients without (ideally) contributing to voltage drop. For this reason the current-driven configuration, with its DC-side inductance, is worthy of consideration.

At this time there is no agreement on a preferred PWM strategy for CDCs, analogous to fixed-frequency or space-vector PWM for VDCs. A common theoretical treatment is lacking; instead, a diverse range of techniques are described in the literature [35, 65, 47]. One promising line of argument is, however, suggested by Kolar
et al [37], who posit a duality relation connecting current-driven and voltage-driven converter circuits. This duality (or rather quasi-duality) has long been acknowledged in a vague and intuitive sense, but is made properly rigorous in Kolar’s treatment.

**Duality of Bridge Converters**

Any electrical network with a planar underlying graph has a dual network, obtained by taking the planar dual of the graph, preserving voltage and current orientations by a right-hand-rule or similar convention, and exchanging voltage and current in the network equations. The duality operation interchanges nodes and meshes, voltage and current sources, inductors and capacitors, series and parallel connections, and three-phase star and delta connections. Whatever is true of voltage in the original network becomes true of current in the dual network, and vice versa.

The notional duality between three-phase VDCs and CDCs stems from this relationship between networks. The transformation must be handled with care, however, as bridge converter circuits with more than two legs are nonplanar. (Graphs of such circuits contain a subdivision of the complete bipartite graph $K_{3,3}$.) It is nonetheless true that in a three-phase converter, opening at least one switch produces a planar graph, which therefore has a dual. Kolar et al [37] deduce the dual networks for each of the eight standard switching states of a VDC, and show by construction that these dual networks may collectively be mapped onto the CDC. The relationship between the two networks under this ‘piecemeal’ transformation is referred to as *quasi-dual* to distinguish it from strict duality.

Under quasi-duality, each of the six nontrivial switching states for the VDC maps uniquely onto a switching state for the CDC. The six corresponding states are in fact just the maximal d-states, in the terminology of Section 7.4.1. For the two open states of the VDC, the mapping is indeterminate; any of the three simple free states of the CDC is dual to any open state of the VDC. If the VDC is viewed as a circuit with three double-throw switches connected to the AC inputs, the CDC may be viewed as containing two triple-throw switches connected to the DC bus rails.

Translation of PWM signals from the VDC to the CDC may accordingly be carried out in a natural manner, with one proviso. In the VDC, the optimal switching occurs in a Gray code sequence including the two open states. In the CDC, we have three free states to choose from, and these are “incorporated sensibly into the switching
state sequence in such a way that the current generation is performed with a minimum number of switchings” [37]. Given the switching sequence in any sextant of the space vector diagram, there are two free states that can occur at the end of the sequence while preserving the Gray code property (in a suitably reinterpreted form). If it is presumed that the next state vector will be either in the same sextant or in the next one in an anticlockwise direction (as is normally the case), then we may choose one of these two states in such a way that the Gray code property is preserved in both cases.

In Chapter 8 we present PWM schemes for CDCs based on a generalised space vector calculus, in a similar spirit to Kolar et al., although we do not make explicit use of circuit duality relationships. The framework in which these schemes are developed is the GAS model, to which we now turn.

### 7.6 Generalised Asynchronous Switching Model

The Generalised Asynchronous Switching (or GAS) model is an extension of the GSS model, with its methodological emphasis on considering the broadest possible range of switching sequences, to incorporate PWM control strategies. An important property of PWM is that the switching times of individual devices vary (in principle) along a continuum. The control law itself still operates in discrete time at regular sample intervals; the difference is that it determines not a single switching state, but a sequence of switching states: an average space vector.

#### 7.6.1 Generalised Space Vector Formalism

We base our model on the standard three-phase bridge topology; the absence of low-frequency harmonics makes consideration of more than three input phases unnecessary. The switching devices may be either single controlled switches (IGBTs, GTOs or like devices) or controlled switches with antiparallel diodes, depending whether the converter is designed for reversible current or reversible voltage. (We leave open the possibility of reversible voltage as it can provide a quick and efficient way of drawing fault energy out of the DC system.)

The terminology of switch states we adopt is the same as in Section 7.4.1. In a three-phase context it is convenient to represent an arbitrary switch state as a 3-
Figure 7.7: Generalised space vectors for three-phase converter: (a) Park transform interpretation, (b) phasor interpretation

vector whose components represent the connections of the three input phases: + for a CK connection, – for a CA connection, 1 for a shorted phase and 0 for an open phase.

Considered in full generality there are 64 switch states for the converter, of which 15 are open states and 37 free states. This leaves 12 d-states, six of which are simple and the others ‘compound’ states. These 12 states may be arranged in a space vector diagram as shown in Figure 7.7(a). The space vectors may be viewed in any of the following equivalent senses:

1. As the (stationary) Park transform of the switching function \( \mathbf{d}^* \), with \( d^*_k = \frac{1}{2} \) for an open phase. (We may also use the alternative representation \( \hat{\mathbf{d}}^* = \mathbf{d}^* - \frac{1}{2} \mathbf{1} \), as the two have the same Park transform neglecting the zero-sequence component.)

2. As the Park transform of the three-phase input voltage, assuming a fixed DC output voltage of \( \pm V_d/2 \) relative to the AC neutral, and zero volts on any open phase.

3. As the Park transform of the three-phase input current (with qualifications), assuming a fixed DC output current \( I_d \) and balanced AC impedances.

The last of these three interpretations is strictly correct only for the six simple d-states. When two or more phases are connected on the same side of the converter, the DC current is not shared evenly between them (as the diagram implicitly assumes), but instead commutes from one to the other. Accordingly, the vector representing the Park transform of the currents is not static but rotates at a rate determined by the
AC source inductance and the difference in voltage between the commuting phases. This effect is taken up in Section 7.6.3 and studied in more detail in Chapter 8.

Figure 7.7(a) and its interpretation assumes that the DC side of the converter may be regarded as an infinite bus, a fundamental assumption underlying the usual VDC and CDC theory. If one assumes instead that the infinite bus lies on the AC side (a not unreasonable assumption given a high-fault-level AC main), the appropriate picture is that of Figure 7.7(b). This is obtained from the standard picture by reflecting in the real axis and interchanging the sizes of the two hexagons (shown in dotted outline). The space vectors are now viewed in one of the following senses:

1. As a phasor representation of the instantaneous output voltage, assuming a balanced three-phase input voltage \( V \) with balanced series impedances and with \( V_0 \) as the phase reference.

2. As a phasor representation of the instantaneous output current, assuming a balanced three-phase input current \( I \) with \( I_0 \) as phase reference.

Whatever interpretation of the state vectors we adopt, it is apparent that the control in the steady state amounts to tracking an average space vector reference signal that rotates anticlockwise relative to Figure 7.7(a), or clockwise relative to Figure 7.7(b). (In the phasor interpretation, we desire that the phase of the output be constant in time, but the diagram itself is attached to a reference frame rotating anticlockwise.)

We note that the six compound states correspond to those of the standard VDC, and the six simple states to those of the standard CDC as well as the standard thyristor converter. One may restrict the switching control to use only one or the other set of states, but such an a priori limitation should be justified. As the simple states include an open phase, their use implies a discontinuous input current, necessitating the connection of AC-side capacitors. (Such capacitors may be required in any case to filter out the switching noise, if the line reactance alone is insufficient.)

### 7.6.2 A Generalised State-Space Model

In the case of the standard VDC, the space-vector formalism is complemented by an explicit behavioural model for the time-averaged circuit quantities, based on duty
Figure 7.8: Generic converter circuit for GAS model

ratios. Here we provide a generalisation of that model to encompass both VDCs and
CDCs, under the full range of switch states subject only to well-posedness.

**Preliminary Model Equations**

The most general form of the converter circuit we study is shown in Figure 7.8. Not
all elements of this circuit will be present in particular cases; in particular, VDC
circuits usually omit the input capacitor \( C_S \) and the DC inductor \( L \). Conventional
CDC circuits include the inductor \( L \) but may omit the capacitor \( C \). As the number
of state variables in an electrical circuit depends on the number of energy-storage
elements, omission of such elements will affect the order of the model.

The AC inductance \( L_S \) and resistance \( R_S \) (possibly zero) are assumed to be com-
mon to all such models, contributing the vector equation

\[
L_S \dot{\mathbf{I}} = \mathbf{E} - \mathbf{V} - R_S \mathbf{I}
\]  

(7.30)

with the three state variables \( \mathbf{I} \). Let \( \mathbf{I} \) denote the input current to the converter. If a
nonzero capacitance \( C_S \) is connected at the input, we have a second vector equation

\[
C_S \dot{\mathbf{V}} = \mathbf{I} - \mathbf{I}'
\]  

(7.31)

which introduces three additional state variables \( \mathbf{V} \). If no capacitance is connected
we write

\[
\mathbf{I} = \mathbf{I}'
\]

which leads to an algebraic relation between input voltage and current.
For the DC side, we let \( V' \) denote the converter output voltage, \( I_d \) the output current, and \( V_d \) the DC bus voltage. As in the Small Model the load is assumed voltage-driven, with the characteristic \( I = f(V_d, t) \). If a nonzero DC link inductance \( L \) is connected we may write

\[
L\dot{I}_d = V' - V_d, \tag{7.32}
\]

thereby introducing the state variable \( I_d \). If there is no link inductance we have simply

\[
V_d = V'
\]

which, analogous to the case \( C_S = 0 \), expresses an implicit algebraic dependence between \( V_d \) and \( I_d \). Finally, there is the contribution of the DC bus capacitor \( C \) and the load:

\[
CV_d = I_d - f(V_d, t). \tag{7.33}
\]

For our purposes it is not necessary to consider the case where no DC bus capacitor exists, although this may be readily accomplished by setting \( I_d = f(V_d, t) \).

We identify three special cases of this generic model framework. The first is the standard VDC, in which both the capacitor \( C_S \) and the inductor \( L \) are omitted; here we obtain a fourth-order model, incorporating equations (7.30) and (7.33) together with the identities \( I = I' \) and \( V_d = V' \). Under appropriate constraints on the switch state of the converter, this reduces to the model of Section 7.5.1. The second case is the VDC with input capacitance, which omits only the DC inductor \( L \); this gives a seventh-order model including equations (7.30), (7.31), (7.33) and the identity \( V_d = V' \). The third special case is the CDC with LC output configuration, which includes all the circuit elements of Figure 7.8; this leads to an eighth-order model with the state variables \( I, V, I_d \) and \( V_d \).

**The Coupling Network**

The remaining task is to establish the relationship among the eight converter variables \( V, I, V' \) and \( I_d \). In a generic converter, precisely four of these are state variables: \( V \) and \( I_d \) for the CDC, \( I' \) and \( V' \) for the standard VDC, and \( V \) and \( V' \) for the VDC with input capacitance.

A dynamic model of the converter switching devices, including parasitic elements and snubbers, is beyond the scope of our investigation. We instead opt for a static
model in which each device is approximated by an open circuit when off, and by a small resistance \( R_D \) when on. Depending on the specific devices used, there may be positivity constraints on the resistor current or on the open-circuit voltage.

Let \( S \) be any switch state. The coupling network for state \( S \) is a linear resistive multiport network obtained by substitution of open circuits and resistors for the switching devices in a bridge converter according to the state \( S \). Its construction is suggested in Figure 7.9. The network has a ‘three-phase input port’ with voltages \( \mathbf{V} \) and (incoming) currents \( \mathbf{I}' \). It has in addition a single ‘output’ port with potentials \( V_+ \) and \( V_- \), output voltage \( V' = V_+ - V_- \), and output current \( I_d \).

The input port variables are not independent, satisfying the identities \( \text{bal} \mathbf{V} = \text{bal} \mathbf{I}' = 0 \). Accordingly, the port may be represented using just two independent voltages \( V_a \) and \( V_b \) and currents \( I_a \) and \( I_b \). One of many convenient choices is \( V_a = V_0 - V_1 \) and \( V_b = V_0 - V_2 \). We then have

\[
\begin{bmatrix}
V_a \\
V_b
\end{bmatrix} = \mathbf{T}_1 \mathbf{V},
\begin{bmatrix}
I_a \\
I_b
\end{bmatrix} = \mathbf{T}_1^T \begin{bmatrix}
I_a \\
I_b
\end{bmatrix},
\begin{bmatrix}
V_a \\
V_b
\end{bmatrix} = \mathbf{T}_2 \begin{bmatrix}
V_a \\
V_b
\end{bmatrix},
\begin{bmatrix}
I_a \\
I_b
\end{bmatrix} = \mathbf{T}_3 \mathbf{I}'
\]

where

\[
\mathbf{T}_1 = \begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix},
\mathbf{T}_2 = \frac{1}{3} \begin{bmatrix}
1 & -2 & 1 \\
1 & 1 & -2
\end{bmatrix},
\mathbf{T}_3 = \begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}.
\]

The network may then be solved as a conventional three-port under this ‘collapsed’ representation.

From the coupling network one obtains three independent linear equations relating the variables \( \mathbf{x} = [ V_a \ I_a \ V_b \ I_b \ V' \ I_d ]^T \), expressible as

\[
\mathbf{A} \mathbf{x} = 0
\]

(7.34)
where $A \in \mathbb{R}^{3 \times 6}$ has full row rank. The application of these equations depends on the ability to obtain three of these variables as an explicit function of the remaining three. In particular, for a CDC model we require an expression of the form

$$
\begin{bmatrix}
I_a \\
I_b \\
V'
\end{bmatrix} =
\begin{bmatrix}
R_D^{-1} \mathbf{G} & \eta \\
\eta^T & -\gamma R_D
\end{bmatrix}
\begin{bmatrix}
V_a \\
V_b \\
I_d
\end{bmatrix}
$$

(7.35)

while the standard VDC model (for example) requires the inverse:

$$
\begin{bmatrix}
V_a \\
V_b \\
I_d
\end{bmatrix} =
\begin{bmatrix}
R_D \mathbf{H} & \zeta \\
\zeta^T & -\delta R_D^{-1}
\end{bmatrix}
\begin{bmatrix}
I_a \\
I_b \\
V'
\end{bmatrix}
$$

(7.36)

Note that we have attributed a certain symmetry to these expressions: this is a simple consequence of reciprocity for linear networks. The condition for well-posedness is different in the two cases (7.35) and (7.36); in each case we require the nonsingularity of the submatrix of $A$ corresponding to the ‘nonstate’ variables appearing on the left hand side.

We may use the network theory of Appendix A to state the exact topological criteria for admissibility of switch states.

**Theorem 7.1**

1. A switch state $S$ is admissible for the CDC, providing coupling network equations of the form (7.35), if and only if each DC output terminal is connected by a closed switch to at least one AC input terminal.

2. A switch state $S$ is admissible for the standard VDC, providing coupling network equations of the form (7.36), if and only if each AC input terminal is connected by a closed switch to at least one DC output terminal.

3. A switch state $S$ is admissible for the VDC with input capacitance if and only if at least one switch is closed.

**Proof.**

1. A switch state yields consistent equations of the form (7.35) if and only if its coupling network is well-posed when voltage sources are connected across the AC input terminals and a current source connected across the DC output. By Theorem A.1, such a network must have a spanning tree that includes the voltage sources and excludes the current source. Equivalently, the coupling
network must have a spanning forest rooted at the AC input terminals. This is readily seen to correspond to the condition stated in the theorem.

2. In this case, we require well-posedness with current sources connected at the AC input and a voltage source at the DC output. Reasoning as above, the requirement is for a spanning forest rooted at the DC output terminals, giving the condition stated.

3. Since the relevant state variables in this model are all voltage variables, we require a spanning tree that includes voltage sources connected across the input and output terminals. For such a tree to exist it suffices that at least one input terminal be connected to an output terminal.

\( \sqrt{ } \)

It is already well known that for a state to be admissible for the CDC, it is necessary that it not be an open state; similarly, for the (standard) VDC it is necessary that it have no open phases. Theorem 7.1, however, shows that these conditions are also sufficient for admissibility.

Of course, this admissibility criterion is contingent on the device model we have adopted, which permits the short-circuiting of capacitors via the bridge devices but does not permit the opening of inductive circuits. These strictures we regard as appropriate; as simulations will show, the capacitor discharge currents under normal operation are not excessive.

Note that there is a fourth class of model we could have considered, namely the CDC without input capacitance. However, the well-posedness condition for this class of model would require a path to exist between any pair of terminals in the coupling network. This condition severely restricts the range of possible switch states (in particular, it excludes all driving states), but is necessary if the inductor currents are to be continuous across arbitrary state transitions.

**CDC and VDC Models for a Constant Switch State**

Before giving explicit formulae for the coupling network solutions, we state the forms of the complete system models that result.

Assuming a state is admissible for the CDC, the solution (7.35) of its coupling
network leads to
\[
\begin{bmatrix}
I' \\
V'
\end{bmatrix} = \begin{bmatrix}
R_D^{-1} \Gamma & \alpha \\
\alpha^T & -\gamma R_D
\end{bmatrix} \begin{bmatrix}
V \\
I_d
\end{bmatrix}
\] (7.37)

with \( \Gamma = T_1^T G T \) and \( \alpha = T_1^T \eta \). Substitution for \( I' \) in (7.31) and \( V' \) in (7.32) yields the eighth-order model, restated here for completeness:

\[
L_S \dot{I} = E - V - R_s I \\
C_S \dot{V} = I - R_D^{-1} \Gamma V - \alpha I_d \\
L I_d = \alpha^T V - V_d - \gamma R_D I_d \\
C \dot{V}_d = I_d - f(V_d, t).
\] (7.38)

The solution (7.36) for a VDC-admissible state leads to
\[
\begin{bmatrix}
V \\
I_d
\end{bmatrix} = \begin{bmatrix}
R_D \Psi & \beta_1 \\
\beta_2^T & -\delta R_D^{-1}
\end{bmatrix} \begin{bmatrix}
I' \\
V'
\end{bmatrix}
\] (7.39)

with \( \Psi = T_2^T H T_3 \), \( \beta_1 = T_2^T \zeta \) and \( \beta_2 = T_3^T \zeta \). The fourth-order VDC model results upon substituting for \( V \) in (7.30) and \( I_d \) in (7.33), and equating \( I' = I \), \( V' = V_d \):

\[
L_S \dot{I} = E - (R_s I + R_D \Psi) I - \beta_1 V_d \\
C \dot{V}_d = \beta_2^T \Psi - \delta R_D^{-1} V_d - f(V_d, t).
\] (7.40)

Similarly, the seventh-order model for the VDC with input capacitance takes the form

\[
L_S \dot{I} = E - V - R_s I \\
C_S \dot{V} = I - R_D^{-1} (\Phi V - \mu V_d) \\
C \dot{V}_d = R_D^{-1} (\mu^T V - \epsilon V_d) - f(V_d, t)
\] (7.41)

for some matrix \( \Phi \), vectors \( \lambda, \mu \) and scalar \( \epsilon \). By rearrangement of (7.35) we find that, for any CDC-admissible state

\[
\Phi = \Gamma + \gamma^{-1} \alpha \alpha^T, \quad \mu = \gamma^{-1} \alpha, \quad \epsilon = \gamma^{-1}.
\] (7.42)

We have thus derived a coherent behavioural model for a generic converter, assuming a switch state given a priori. However, it is often important to know the behaviour of certain auxiliary variables such as the individual device currents, voltage drops across devices, and the potentials \( V_+ \) and \( V_- \), which are not explicitly
included in the model. Provided the appropriate condition in Theorem 7.1 is satisfied, the theory of Appendix A guarantees that all such quantities have an expression in terms of the state variables, given by the solution to the coupling network.

The coupling coefficients appearing in the models (7.38), (7.40) and (7.41) are not unique due to the degeneracy \( \text{bal} \mathbf{V} = \text{bal} \mathbf{I} = 0 \). In each case, adding a multiple of the vector \( \mathbf{1}^T = [1 \ 1 \ 1] \) to the coefficients of \( \mathbf{V} \) (\( \mathbf{I} \)) in each equation gives a model with identical dynamics. In particular, since \( \mathbf{T}_2 \) and \( \mathbf{T}_3 \) differ by a matrix all of whose entries are identical, we may take \( \beta_2 = \beta_1 \) in (7.40) without changing the model.

We note that the GAS model (7.38) for a CDC may be incorporated into the Large Model of Section 7.2 simply by including (7.38) in the equations of the Large Model and treating the current \( I_d \) as one element of the vector \( \mathbf{I} \) of external circuit driving currents. The model (7.40) (or (7.41)) for the VDC is accommodated by allowing the coupling matrix \( \mathbf{E} \) to vary; here it is the AC currents \( \mathbf{I} \) (or voltages \( \mathbf{V} \)) which act like elements of \( \mathbf{I} \).

**Calculation of Coupling Coefficients**

When surveying the full range of coupling networks it is convenient to deal with equivalence classes of networks rather than with each one individually. Two switch states, or coupling networks, are called equivalent if one may be obtained from the other by permuting the AC input terminals and/or exchanging the DC output terminals. This equivalence relation partitions the set of all switch states into 13 equivalence classes, listed in Table 7.2 together with a representative and the number of states in each class.

Tables 7.3 and 7.4 give, for each of these canonical representatives subject to admissibility, the model coefficients as follows.

1. The parameters \( \mathbf{\alpha}, \mathbf{\Gamma} \) and \( \gamma \) required for (7.38).

2. Coefficients for calculating the DC bus midpoint voltage \( V_N \) (relative to AC neutral) in the CDC model, by the formula

\[
V_N = \mathbf{\alpha}_0^T \mathbf{V} - \gamma_0 R_D I_d.
\]  

(7.43)

3. The parameters \( \beta_1, \mathbf{\Psi} \) and \( \delta \) required for (7.40). (Where a state is inadmissible, these are left blank in Table 7.4.)
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Table 7.2: Equivalence classes of switch states
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Table 7.3: Coupling coefficients for current-driven GAS model
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Table 7.4: Coupling coefficients for voltage-driven GAS model
The parameters $\mu$, $\Phi$ and $\epsilon$ required for (7.41).

The coefficients for equivalent states are easily derived, by permutation of vector and matrix elements and/or changes of sign where appropriate.

We note once again that the values for the matrices $\Gamma$, $\Psi$ and $\Phi$ given in these tables are not unique, but rather represent equivalence classes of matrices under transformations of the form

$$A \mapsto A + \lambda 1^T$$  \hspace{1cm} (7.44)

where $\lambda$ is any vector in $\mathbb{R}^3$. Nonetheless, an important property of these matrices, which is invariant under transformations of the form (7.44) is that they are column quasi-stochastic: their column sums are equal, though not necessarily to 1. By appropriate choice of $\lambda$ in (7.44), any such matrix can be made doubly quasi-stochastic with any desired row (and column) sum. In Table 7.3, the $\Gamma$ matrices are presented in this form, with row and column sums equal to zero.

For the matrices in Table 7.4, we have for the most part adopted an arbitrary presentation with the diagonal elements equal to unity. Again, the transformation (7.44) allows the diagonal elements to be arbitrarily assigned if one does not care about the value of the column sum. The quantity obtained by subtracting the column sum from the trace (sum of diagonal elements) is invariant under the transformation (7.44).

The coefficients for the CDC model given in Table 7.3, and others besides, may be obtained by simple formulae. As in Section 7.4, let $(Q,R)$ with $Q,R \in \{0,1\}^3$ denote the switch state. Set

$$\gamma_+ = \frac{1}{|Q|}, \quad \gamma_- = \frac{1}{|R|}, \quad \alpha_+ = \gamma_+ Q, \quad \alpha_- = \gamma_- R.$$  

$\alpha_+$ and $\alpha_-$ may be thought of as normalisations of $Q$ and $R$. The following formulae then hold for the output terminal potentials $V_+$ and $V_-.$

$$V_+ = \alpha_+^T V - \gamma_+ R_D I_d, \quad V_- = \alpha_-^T V + \gamma_- R_D I_d.$$  \hspace{1cm} (7.45)

As a consequence of (7.45) we obtain

$$\alpha = \alpha_+ - \alpha_-, \quad \gamma = \gamma_+ + \gamma_-, \quad \alpha_0 = \frac{\alpha_+ + \alpha_-}{2}, \quad \gamma_0 = \frac{\gamma_+ - \gamma_-}{2}.$$  \hspace{1cm} (7.46)

For the matrix $\Gamma$, define matrices $\Gamma_k \in \mathbb{R}^{k \times k}, k = 1,2,3$ by

$$\Gamma_k = I_k - \frac{1}{k} J_k$$  \hspace{1cm} (7.47)

162
where \( I_k \) is the \( k \times k \) identity matrix, and \( J_k \) is a \( k \times k \) matrix of all ones. (Note that \( \Gamma_1 = 0 \) by this definition.) Now let \( A \) be a \( m \times m \) matrix and \( \sigma \in \mathbb{R}^n \), \( n \geq m \) be a vector with \( m \) nonzero entries. We define the matrix embedding \( A[\sigma] \) as the \( n \times n \) matrix obtained from \( A \) by inserting rows and columns of zeros corresponding to the zero elements of \( \sigma \). With this notation we have

\[
\Gamma = \Gamma_{|Q|}[Q] + \Gamma_{|R|}[R].
\]

(7.48)

We can also obtain formulae for the individual device currents, useful when simulating real (unidirectional) devices. Let \( I_+ \) denote the currents entering the CK from the three input phases, and let \( I_- \) denote the currents leaving the CA. Then

\[
I_+ = R_D^{-1} \Gamma_+ V + \alpha_+ I_d, \quad I_- = -R_D^{-1} \Gamma_- V + \alpha_- I_d
\]

(7.49)

where \( \alpha_+ \), \( \alpha_- \) are as above, and

\[
\Gamma_+ = \Gamma_{|Q|}[Q], \quad \Gamma_- = \Gamma_{|R|}[R].
\]

7.6.3 Control Design Issues

Section 7.6.2 presented model equations for a fixed switch state, whereas in practice the switch state changes at a rapid frequency. The switching control framework we adopt is a synchronous discrete-time approach, whereby we fix a control period \( T \) (as in the GSS model) and at each ‘sample time’ \( kT \), \( k \in \mathbb{Z} \), program the converter with a given switching sequence. Formally, a switching sequence may be viewed as an arbitrary-length sequence of tuples \( (\tau_k, Q_k, R_k) \), \( 1 \leq k \leq L \), where \( (Q_k, R_k) \) denotes a switch state, \( 0 \leq \tau_k \leq 1 \) denotes the proportion of the period \( T \) spent in that state (giving \( \sum_k \tau_k = 1 \)), and \( L \geq 1 \) is the number of states in the sequence. The states are taken to be programmed in the order specified; multiple occurrences of switch states are permitted.

As with the PWM model (which forms a special case), analysis of the GAS model works on the assumption that the system behaviour on time scales longer than the switching period is adequately described by averaging the (time-varying) model equations over one period. If \( \theta \) represents a generic parameter (for example, \( \alpha \) in the CDC model), then the averaging process involves replacing each occurrence of \( \theta \) in the model equations with the averaged value

\[
\bar{\theta} = \sum_{k=1}^{L} \tau_k \theta_k
\]

(7.50)
where $\theta_k$ is the value of $\theta$ for the switch state $(Q_k, R_k)$.

Depending on the precise form of circuit used, the vector $\alpha$, $\beta$, or $\mu$ (as appropriate) may be treated analogously to the duty ratio vector $d$ in the usual PWM model. Accordingly it may be related, via the Park transform, to the space vector formalism of Section 7.6.1. In each case, taking the Park transform of the appropriate vector for the 12 d-states (where admissible) yields the vector diagram of Figure 7.7(a), up to a constant.

If the AC-AC interaction matrix ($\Gamma$, $\Psi$ or $\Phi$) is diagonal or zero, then the corresponding space vector is static. In this case the dependent input on any one phase ($I_k$ for a converter with input capacitance, otherwise $V_k$) is determined as a linear combination of a DC variable and an AC variable for phase $k$ alone. If the latter are constant (or slowly varying), then so is the former. If on the other hand the interaction matrix contains off-diagonal terms, then in addition to the ‘static’ component of current one obtains a ‘circulating’ component which flows between phases of the supply. Suppose there are two coupled phases $q$ and $r$; one may then write

$$I_q = I_{qs} + I_C, \quad I_r = I_{rs} - I_C$$

where $I_{qs}, I_{rs}$ are slowly-varying, while $I_C$ satisfies, in the absence of significant source resistance

$$\left| \frac{dI_C}{dt} \right| = \frac{1}{L_S} |(E_q - E_r) - (V_q - V_r)|. \quad (7.51)$$

Small values of $L_S$ thus give rise to rapid variations in the currents not accounted for by space vector calculus. But even if these variations can be kept reasonably small on the time scale $T$, $I_C$ may yet have an initial value far from zero, so that the Park transform of the dependent input differs substantially from the value given by the space vector diagram. Another complication is that, if unidirectional devices are used, the circulating components may attempt to force a device current to be negative, in which case the device fails to turn on and the switch state itself differs from that which was programmed. Space vectors corresponding to switch states in which AC-AC coupling occurs are termed variable; space vector calculus on its own does not predict the behaviour of switching strategies employing such states.

Setting to one side the difficulties posed by variable space vectors, the theory of Section 7.6.1 entails that, given a space vector command, the converter synthesises an output voltage whose average over the control period is determined from the
command vector according to Figure 7.7(b). More precisely, by selecting a command vector having magnitude $V$ and phase $\phi$, measured anticlockwise from the positive real axis in Figure 7.7(b), the output voltage produced is

$$V_S = \sqrt{2}V \cos(\phi + \phi_0^V)$$

(7.52)

where $\phi_0^V$ is the phase of $V_0$, the input voltage on phase 0. We thereby obtain, in principle, a continuum of possible DC supply voltages $V_S$, limited only by the hexagonal boundary of Figure 7.7(b) (the inscribed circle of which has radius $3|V|/2$, giving $|V_S| < 3|V|/\sqrt{2}$).

This somewhat idealised observation permits us in effect to decouple the DC-side and AC-side control tasks. From a DC-side perspective, we may regard the converter as a DC voltage source, programmable at discrete intervals $nT$ with voltages $V_S$ which are limited in magnitude but otherwise arbitrary. From an AC-side perspective, the load on the converter is a DC voltage or current sink whose state is slowly-varying, so that a space vector command determines the AC-side behaviour according to Figure 7.7(a). In the absence of zero-sequence components (entailing that AC quantities are determined by their two-dimensional Park transforms), a rotating space vector command signal simultaneously produces a sinusoidal AC input and a constant DC output (after filtering of switching harmonics).

We therefore propose the following three-step approach to control.

1. DC voltage control setpoints $V_S$ are generated by a DC-side controller which operates on DC variables only. For a CDC supplying a DC bus, this controller can be based on the Large Model, the Small Model or the GSS model, which operates explicitly in discrete time.

2. The setpoints $V_S$ are converted to space vector commands $(V, \phi)$ which (in the steady state at least) satisfy long-timescale AC control objectives, such as maximising the input power factor.

3. The commands $(V, \phi)$ are converted to switching sequences $(\tau_k, Q_k, R_k)$ which satisfy short-timescale control objectives, such as minimising harmonic distortion and device stress.

The first of these we term *regulatory control*, its purpose being to regulate DC voltages and currents. The second we term *synchronising control*, as its purpose is (approx-
imately) to synchronise the AC voltage and current. The third is *switching control*, which falls firmly within the discipline of power electronics.

This control framework, once again, depends on a number of idealising assumptions; in particular, on the validity of averaging the model, and on the approximation of variable space vectors by static space vectors. Ultimately, these assumptions will need to be justified by simulation studies and real-world testing.

New techniques for switching control based on the GAS model are the subject of Chapter 8. Regulatory and synchronising control are discussed in subsequent chapters.
Chapter 8

Advanced Switching Strategies for PWM Converters

In this chapter we outline new approaches to the switching control of PWM converters, with particular emphasis on current-driven converters to which space vector techniques have not been as readily applied. The design framework is that of the GAS model, described in Section 7.6.

The chapter is organised as follows. Section 8.1 is devoted to explaining how switching sequences containing arbitrary state transitions may be realised in practice, when such sequences clearly cannot be synthesised by conventional pulsewidth modulators. Section 8.2 looks at the general question of designing converter circuits with bidirectional current flow, rederives the voltage-driven converter by considering the device constraints, and concludes that it is unlikely the performance of the VDC can be improved by permitting a wider range of switch states. Section 8.3 similarly looks at the design of unidirectional converters, rederives the current-driven converter, and describes new approaches to switching control for the CDC. Section 8.4 presents simulation results to verify the operation of the proposed switching schemes.

8.1 Realisation of (Almost) Arbitrary Switching Sequences

As alluded to in Section 7.5.1, switching in the standard VDC is controlled by three duty ratios, which are compared with a triangle-wave carrier. The three switching functions generated are fed to the three positive devices, and their complements to the
three negative devices. While simple, this modulator restricts the possible switching sequences to those of the form

\[ [- - -] \rightarrow S_1 \rightarrow S_2 \rightarrow [+++] \rightarrow S_2 \rightarrow S_1 \rightarrow [- - -] \ldots \]

where \( S_1 \) and \( S_2 \) are adjacent VDC states, and the state timings are completely symmetric around the central \([+++]\) state. Nonetheless, an important positive feature of this modulator is that, since the triangle-wave comparisons are done by analogue circuitry, it permits continuously variable sequence timings \( \tau_k \).

The GAS model is based on the presumption that there be no arbitrary restriction on the switching sequence that may be programmed into the converter over the period \( T \). We do however wish to retain the continuous variability of transition times within this sequence. This immediately raises the question whether these two aims are simultaneously realisable, and what combination of analogue and digital circuitry would serve to realise these aims.

Evidently, we cannot realise these aims if we permit the switching sequences to be arbitrarily complex. However, a practical realisation is possible provided we place an upper limit on the number of times any particular device may change state over the period \( T \). Indeed, the switching schemes we present later in this chapter can all be realised on the basis that no device switches more than twice per period, and so a realisation based on this assumption will suffice for our purposes.

Figure 8.1 depicts an elementary pulsewidth modulator capable of synthesising an arbitrary switching sequence in which no device switches more than twice in the fixed period \( T \). The circuit of Figure 8.1, based on two comparators, two logic gates and an 8-way multiplexer, is replicated for each of the six devices; the twin sawtooth carriers \( C_1 \) and \( C_2 \) are common to all. The sequence for any one device is programmed at the start of the period through the two analogue duty ratios \( D_1 \) and \( D_2 \) and the 3-bit digital command signal \( M \), which selects the multiplexer channel. Depending on the value of \( M \), either or both of the duty ratio signals may be ignored. If an off–on transition is commanded, then \( D_1 \) controls the time between 0 and \( T \) at which this transition occurs; similarly if an on–off transition is commanded, then \( D_2 \) controls the time at which this occurs. Both signals are used if a double transition is requested.

As there are only six transition types, and hence six inputs to the multiplexer, two values of \( M \) are redundant. The assignment of channels to specific values of \( M \) is entirely arbitrary, but Table 8.1 suggests a useful mnemonic coding.
Figure 8.1: A versatile pulsewidth modulator (one per device)

<table>
<thead>
<tr>
<th>Octal value</th>
<th>Binary value</th>
<th>Transition type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
<td>off</td>
</tr>
<tr>
<td>1</td>
<td>001</td>
<td>off–on</td>
</tr>
<tr>
<td>2</td>
<td>010</td>
<td>off–on–off</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
<td>off–on (redundant)</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>on–off</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>on–off–on</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>on–off (redundant)</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>on</td>
</tr>
</tbody>
</table>

Table 8.1: Mnemonic coding for modulator command $M$
As a simple application of this modulator, we show how the standard VDC scheme may be implemented in two different ways. In the first method, we use a switching period $T$ equal to one period of the VDC sequence. The three positive devices are programmed with a fixed off–on–off command, while the three negative devices receive a fixed on–off–on command. Given $d_k$, the VDC duty ratio for phase $k$, we programme the positive switch on leg $k$ with $D1 = D2 = (1 + d_k)/2$, and the negative switch with $D1 = D2 = (1 - d_k)/2$. Setting $D1$ and $D2$ to the same value gives a symmetric pulse around $T/2$ of the requisite width.

The second method uses a switching period equal to half a period of the VDC sequence; each device now switches just once per period. In even periods, we programme the positive devices with an off–on command and the negative devices with an on–off command; in odd periods we do the reverse. Symmetrical pulses are achieved by programming the positive switch on leg $k$ with $D1 = D2 = d_k$ and the negative switch with $D1 = D2 = 1 - d_k$. Although one of the duty ratio signals will be ignored in each case, programming both simultaneously allows us to disregard the alternation between $D1$ and $D2$.

In the proposed modulator it is essential that changes in the command $M$ be triggered to coincide precisely with the discontinuities in the sawtooth carriers, or spurious pulses may result. In practice $M$ would be latched and the latch triggered by the falling edge of $C1$ or the rising edge of $C2$. If logic hazards still present a problem, a modified design is possible, in which the carriers $C1$ and $C2$ are twin triangle waves with period $2T$. In even periods the circuit behaviour is as described above; in odd periods there is an effective transposition between $D1$ and $D2$, and between off–on and on–off commands, which is easily handled by the controller. The description of the second method for VDC switching is actually simpler with this modified design; not surprisingly, as it corresponds more closely with the modulator used in a standard VDC.

### 8.2 Voltage-Driven Converters Revisited

Approaching the control problem from the most general perspective, in the spirit of the GAS model, we may now ask: in what way do the operational requirements of the converter dictate the circuit conditions on the DC side, and the switching strategy for
the converter? Suppose that we desire a reversible DC current, and seek to achieve this in the usual manner, using unidirectional switching devices with antiparallel diodes. We then require that the voltage on each of the DC bus rails, relative to the AC neutral point, be greater than the maximum AC voltage. Further, the DC voltage needs to be relatively ‘stiff’ to ensure that the turning on of a switch does not cause the DC voltage to drop to the AC level.

Thus, the use of antiparallel diodes essentially dictates that a large capacitor or voltage source be placed across the output terminals, guaranteeing a slowly-varying DC voltage which remains above a certain threshold value. In short, we require a voltage-driven converter.

In the standard PWM regime for a converter of this kind, eight switch states are used: the six compound d-states, and the two zero states [+++] and [----]. Guided by Figure 7.7, we could in principle modify the switching control to include the six simple d-states as well. Given an average space vector, we first identify the 30° segment in which it lies, and the switch states $S_1$, $S_2$ at the corresponding vertices of the dodecagon. Without loss of generality we assume the quantity $|Q| - |R|$ is strictly greater for $S_2$ than for $S_1$. We then form the switching sequence as

$$[----] \rightarrow S_1 \rightarrow S_2 \rightarrow [+++] \rightarrow S_2 \rightarrow S_1 \rightarrow [----] \ldots$$

which shares the desirable properties of the standard VDC sequence, namely that the average space vector attains its desired value in each half-period of the sequence, and that each device switches just once in that same time period.

The key advantage of this twelfold switching scheme is a substantial reduction in switching harmonics compared with the usual sixfold scheme, due to the fact that the deviation of the actual space vector from its average value is approximately halved. This advantage, however, must be balanced against a number of technical drawbacks of the twelfold scheme.

Firstly, the simple d-states are inadmissible for the standard VDC, so their use will require the connection of AC-side capacitance, which is not necessary under the sixfold scheme. The simple d-states also give rise to variable space vectors for the VDC, as is seen by inspection of Table 7.4; by comparison, the space vectors for the compound d-states are static. This raises the possibility of distorted input currents and other non-ideal phenomena in the twelfold scheme.

A third drawback is the appearance of nonzero DC offsets. We use the term DC
offset to denote the difference in potential between the DC bus midpoint and the AC neutral; ideally this should be as close to zero as possible. If $\mathbf{d}$ denotes the average of $\mathbf{d}^*$ over one control period (whose Park transform is the average space vector), then from Section 7.5.1 we can ensure the average DC offset over a switching period is zero, provided $\text{bal} \mathbf{d} = \frac{3}{2}$. This can be achieved provided the average space vector $\bar{\mathbf{d}}$ lies within the smaller hexagon in Figure 7.7(a).

If the circuit conditions could ensure that the voltage on any open phase were the same as at the DC bus midpoint (through a biasing circuit, for example), then incorporating the simple d-states would still give a zero DC offset. It is more likely, however, that with both switches on one leg of the converter in blocking state and negligible current flow, the input voltage on that phase will simply follow the potential $E_k$ at the supply. Under these conditions we obtain that the DC offset is $-E_k/2$, where $k$ is the open phase, and we cannot then expect to zero the DC offset over the period $T$. The best we can hope for is that the DC offset averages to zero over a longer time scale, and indeed this will occur in the steady state owing to symmetry.

In practice, we consider it likely that these drawbacks (requirement for input capacitance, variable space vectors, and oscillating DC offset), while by no means fatal, will nonetheless outweigh any advantages of the twelfold scheme.

We conclude therefore that a switch-mode converter using antiparallel diodes for reversible current flow must have the DC-side characteristics of the standard VDC, and that the optimal switching control for such a converter involves restricting the switch states to those of the standard VDC. The requirement for reversible current can of course be achieved by means other than antiparallel diodes, but would entail greater cost. We conclude therefore that the standard VDC with sixfold switching control is uniquely well-suited to the task of AC-DC conversion with reversible current, unless an inductive DC link is of paramount importance.

### 8.3 An Enhanced Current-Driven Converter

We turn now to the alternative, a converter with reversible voltage but unidirectional current. Such a converter might be envisioned as the primary supply source for a DC power system, with a VDC as a smaller secondary source, since net regeneration would never be likely to exceed a small fraction of the system power rating. Among
other things, the task of raising the DC voltage to its rated value from zero at system startup would fall to the primary converter.

A converter with these characteristics can be realised with stand-alone unidirectional switching devices such as IGBTs or GTOs. The only constraint such devices impose on the DC-side circuit conditions is that a continuous, unidirectional current flow be maintained. This is most naturally accomplished with a DC link inductance, giving a current-driven converter topology. While other configurations are possible, we shall henceforth assume that the converter output is connected via inductance \( L \) to a node of the DC reticulation system, at which is connected a capacitance \( C \). If the system is modelled with a single node, then the DC-side circuit is equivalent to that of the Small Model. As with the thyristor converter, the \( LC \) combination acts as a passive lowpass filter to remove the high-frequency harmonics from the DC bus voltage.

We now require a switching strategy that synthesises any given average space vector by switching devices in a specified sequence over a control period \( T \). As with any switching control, the strategy chosen should fulfil the following 'optimality criteria', based on those stated in [75].

**Balance criterion:** The disparity in switching frequency, averaged over the period \( T \), should be minimal across the six devices; ideally, each device should be switched the same number of times as every other device over the period \( T \). This ensures both that devices are not overly stressed by being switched too often, and that no low-frequency AC current harmonics are produced due to devices being switched too infrequently.

**Economy criterion:** Subject to balance, the total number of device switchings required to bring the average space vector to its desired value should be minimal. This ensures, by the argument of [75], that the AC current harmonics are minimised for a given (average) switching frequency.

In formulating the switching strategy we should also consider the issue of DC offset, as this has implications for protective earthing systems. DC offset is a more tractable issue than under the VDC assumptions, as we may now consistently treat the AC supply as the voltage bus. DC offsets will be calculated in terms of the input voltage \( V \), which for averaging purposes will be approximately equal to \( E \), apart from a small
phase shift. (A justification for this assertion will come in Section 9.3.)

We shall discuss three broad types of switching scheme: a sixfold CDC scheme similar to that in [37], an alternative sixfold scheme using compound d-states as in a VDC, and a twelvefold ‘hybrid’ scheme.

8.3.1 Sixfold CDC Switching Strategy

In this strategy we adhere to ‘standard’ CDC conventions (mirroring those of thyristor converters), using only the six simple d-states and the three simple free states. Space vectors are synthesised as in [37]. Adopting the Park transform interpretation for reference, and extracting the relevant space vectors from Figure 7.7(a), we obtain the picture shown in Figure 8.2. Here we have labelled the switch states with the \((q,r)\) notation of Section 7.4.1; this notation extends to the free states also. Two such states are considered adjacent if they have the same value of either \(q\) or \(r\); a Gray code sequence (\(CDC\) sense) is a sequence of states adjacent by this definition.

Switching Scheme

For each sextant of Figure 8.2 in which the average space vector may lie, we identify two nontrivial ‘fundamental states’ on adjacent vertices of the hexagon. The desired average vector may be synthesised as a convex combination of these fundamental states, together with free states, switched in a Gray code sequence to fulfil the economy criterion. In particular, we have the following:

**Lemma 8.1 (CDC Switching Principle)** Let \((q,q)\) and \((r,r)\) be any two distinct free states, and let \(S_1, S_2\) be any two states adjacent in Figure 8.2. Then there is a
unique 4-step Gray code sequence (CDC sense) starting with \((q, q)\), ending with \((r, r)\) and having \(S_1\) and \(S_2\) as intermediate states. In this sequence the devices on legs \(q\) and \(r\) each switch once, one device on the remaining leg switches twice, and the other remains off throughout.

Lemma 8.1 is easily verified by construction. Simply take any two adjacent states, write down the eight possible Gray code sequences having those states as intermediates and free states as endpoints, and observe that the endpoints of these sequences include all six ordered pairs of distinct free states. (There are in addition two sequences that start and end with the same free state, which we shall not be using.)

So long as the average space vector remains within one sextant, the balance criterion is best met by cycling through the three free states, giving a periodic sequence of nine switch states. (Compare with the standard scheme for VDCs, where there are only two zero vectors and a periodic sequence comprises six states.) As an example, for switching in the sextant between \(+30^\circ\) and \(-30^\circ\) in Figure 8.2, with fundamental states \((0, 1)\) and \((0, 2)\), one could use the sequence

\[
(0, 0) \rightarrow (0, 2) \rightarrow (0, 1) \rightarrow (1, 1) \rightarrow (0, 1) \rightarrow \\
(0, 2) \rightarrow (2, 2) \rightarrow (0, 2) \rightarrow (0, 1) \rightarrow (0, 0) \ldots
\]

in which we cycle through the free states \((0, 0)\), \((1, 1)\) and \((2, 2)\) in that order. (If we chose a different ordering for the free states, we would obtain a cyclic permutation of the sequence above, or the equivalent time-reversed sequence.) For realisation purposes, we take the switching period as the time between two free states in this sequence, so that a full cycle of the sequence actually represents three switching periods.

In this nine-step approach to space vector synthesis, three out of the six devices are switched four times per cycle, and the other three only twice per cycle. By the uniqueness part of Lemma 8.1 this partitioning of the six devices into ‘fast’ and ‘slow’ sets is determined by the fundamental states, and there is no alternative switching sequence that reduces this disparity in switching frequency. Nevertheless, in the steady state at least, the switching frequency averaged over a period of the AC supply is the same for all six devices, notwithstanding the fact that each device will need to work harder in some parts of the cycle than in others. (If the average frequency is \(F\), then any given device switches at frequency \(\frac{4}{3}F\) half the time and at \(\frac{2}{3}F\) half the time.)
We consider first the conditions under which the average DC offset may be made equal to zero over the control period $T$.

Let $V$ be the voltage at the converter input terminals, which we assume to be balanced. While the converter is in free state $(q,q)$, both DC output terminals are at potential $V_q$, hence the DC offset is also $V_q$. In state $(q,r)$, with the third phase (say phase $s$) open, the DC offset is

$$\frac{V_q + V_r}{2} = -\frac{V_s}{2}.$$  

Consider now our nine-step switching sequence, with fundamental states $S_1$ and $S_2$. Let $p$ be the phase common to $S_1$ and $S_2$, with $q$ and $r$ the other two phases. Let $\tau_1$ and $\tau_2$ be the time spent in states $S_1$ and $S_2$ respectively, as a proportion of the switching period, and similarly let $\phi_m$ be the time spent in free state $(m,m)$. Setting $\tau_3 = \phi_0 + \phi_1 + \phi_2$, we must have

$$\tau_1 + \tau_2 + \tau_3 = 1.$$  

If the switching frequency is sufficiently high, we may assume $V$ is constant over the switching period. The DC offset, averaged over the switching period, is then given as

$$V_N = -\frac{1}{2}(\tau_1 V_r + \tau_2 V_q) + \phi_0 V_0 + \phi_1 V_1 + \phi_2 V_2.$$  

Now, the ratios $\tau_k$ are determined by the average space vector to be synthesised, but we have some discretion in selecting the $\phi_m$. To obtain a zero DC offset, we should choose the $\phi_m$ so that

$$\phi_0 V_0 + \phi_1 V_1 + \phi_2 V_2 = \frac{1}{2}(\tau_1 V_r + \tau_2 V_q)$$  

and

$$\phi_0 + \phi_1 + \phi_2 = \tau_3.$$  

This problem is interpreted geometrically in Figure 8.3. (We have taken $q = 0$, $r = 1$ without loss of generality.) The left hand side of (8.2) is a phasor inside the triangle

$$\text{conv} \{\tau_3 V_0, \tau_3 V_1, \tau_3 V_2\}$$

while the right hand side is located on the line segment

$$\text{conv} \left\{ \frac{1 - \tau_3}{2} V_q, \frac{1 - \tau_3}{2} V_r \right\}.$$
Figure 8.3: Calculation of DC offset for CDC switching strategy

It is evident that if \( \tau_3 \geq \frac{1}{3} \) in Figure 8.3, then there is a valid choice of \( \phi_m \) giving a zero DC offset, while if \( \tau_3 < \frac{1}{3} \) there is no such choice available.

It follows that we can ensure a zero average DC offset if and only if the average space vector falls within a hexagon two-thirds the size of the bounding hexagon in Figure 8.2. The largest circle contained within this set corresponds to a DC voltage equal to \( \sqrt{2}|V| \), the peak AC input voltage. (Without this constraint, the largest achievable DC voltage is \( 3|V|/\sqrt{2} \).)

If we cannot zero the average DC offset on the time scale \( T \), the ‘second best’ alternative is to zero the average DC offset over a longer time scale. In the sinusoidal steady state the average space vector describes a circular trajectory at the AC line frequency, and the symmetry of the system then ensures that the DC offset averages to zero over a period no greater than one cycle of the AC supply. This will be the case for any arbitrary (fixed) values of the \( \phi_m \); for simplicity we may take them to be equal, in which case (8.1) reduces to

\[
V_N = -\frac{1}{2}(\tau_1 V_r + \tau_2 V_q). \tag{8.3}
\]

Recall that in the steady state the average space vector rotates anticlockwise through
the diagram of Figure 8.2, describing a circle of radius \( V \) (as a proportion of the side length of the hexagon). Consider one 60° segment of this rotation, from one pure state to another. We designate \( S_1 \) as the initial state, so that \( \tau_1 \) decreases monotonically from \( V \) to 0 while \( \tau_2 \) increases from 0 to \( V \). The voltages \( V_q \) and \( V_r \) satisfy

\[
V_q = \alpha V_r
\]

where \( \alpha \) is the phase constant, expressing an anticlockwise rotation of 120 degrees. As a phasor quantity, \( V_N \) thus describes a 120° anticlockwise rotation relative to \( V_r \). In the same time period, \( V_r \) itself describes a 60° anticlockwise rotation; thus, relative to a constant phase reference, \( V_N \) rotates a full 180 degrees as the space vector traverses a sextant of Figure 8.2.

It follows that in the steady state, \( V_N \) oscillates sinusoidally with a frequency equal to three times the AC supply frequency. The magnitude of this oscillation depends on the phase displacement between the space vector reference and the supply voltage (a quantity studied in Chapter 9), but its maximum value is \( V|V|/2 \) RMS volts.

### 8.3.2 VDC-Like Switching Strategy

As a complementary strategy to the conventional CDC scheme, we may formulate a scheme using the six compound d-states, along with free states chosen according to the economy principle. Since the compound d-states give variable space vectors for the CDC, the behaviour predicted here is an idealisation only. However, if the free states chosen for the strategy contain no open phases, we might thereby hope to reduce the requirement for input capacitance (though we can never eliminate it completely).

In this strategy, as in the hybrid strategy below, we consider ‘nonsimple’ free states, in which more than two devices conduct. In such a state the instantaneous DC voltage becomes zero, but the unidirectional device configuration ensures that all possible current paths include the DC bus. Assuming equal AC impedances, the potential at both output terminals (hence the DC offset) is equal to the arithmetic mean of \( E_k \) over all bridge legs \( k \) having at least one device conducting. (We assume the switching period is sufficiently short that no device current drops to zero before the next state is triggered, although this possibility will need to be accounted for in a fuller analysis.)
Switching Scheme

In each of the six standard VDC d-states, three devices conduct; one on each leg. Any one of these states may be brought to one of seven free states, by switching on one or more of the three nonconducting devices. These free states similarly have the property that at least one device conducts on each converter leg. There are 19 such free states; 12 with one shorted phase, 6 with two shorted phases and one with three shorted phases.

Again, to synthesise a given average space vector we form a sequence of switch states from the two ‘fundamental states’ (adjacent vertices of the hexagon of Figure 7.5) together with free states, keeping in mind the balance and economy criteria. Suppose for the sake of argument that the two fundamental states are \([-–+]\) and \([-++\]). In a VDC, we would use a switching sequence of the form

\[
[-––] \rightarrow [-–+] \rightarrow [-++] \rightarrow [+++] \rightarrow [-++] \rightarrow [-–+] \rightarrow [-––] \ldots
\]

in which each device is switched exactly twice per control period.

Our equivalents of the open states \([-––]\) and \([+++]\) will include shorted phases. To minimise overall switching activity, it is desirable that one phase only be shorted; furthermore, if we require that every device switches at least once (thereby ensuring no low-frequency input harmonics), it is not hard to see that the unique substitution in this case is

\[
[-––] \mapsto [1––] \\
[++] \mapsto [++1]
\]

as any other substitution entails that at least one device remains conducting throughout the cycle. With this modification we obtain

\[
[1––] \rightarrow [-–+] \rightarrow [-++] \rightarrow [+1] \rightarrow [-++] \rightarrow [-–+] \rightarrow [1––] \ldots
\]

This sequence is unbalanced, as two devices are switched twice as often as the remaining four. A more radical modification gives the alternative candidate sequence

\[
[1––] \rightarrow [-–+] \rightarrow [-++] \rightarrow [+1] \rightarrow [1––] \ldots \quad (8.4)
\]

in which we have abandoned the symmetry property of the VDC sequence, but satisfied the balance criterion in its strongest form: each device in this sequence is switched
twice in one period. It is accordingly suggested that (8.4) and its reversal (if required) form the basis for a VDC-like switching scheme for CDCs.

Given any pair of fundamental states, we may form a switching sequence by analogy with (8.4). Moving from one sextant of the space vector diagram to another still presents a problem, however. In this case we can employ appropriate transitional sequences, although we cannot necessarily avoid a one-off disparity in switching frequencies.

Consider once again the example above with fundamental states [–+–] and [–++], whose sequence ends in the free state [1––]. If the next space vector is in the adjacent sextant with fundamental states [–++ ] and [–––], then we may proceed directly with the switching sequence for that sextant, as it shares the endpoint [1––]. Constant switching frequency is maintained in this special case. Suppose on the other hand that the next space vector is in the sextant with fundamental states [––+] and [+-+]. One might then programme a transitional sequence such as

\[ [1––] \rightarrow [–+–] \rightarrow [+-+] \rightarrow [+++] \rightarrow [-1–] \]

and follow thereon with the sequence

\[ [-1–] \rightarrow [–+–] \rightarrow [+-+] \rightarrow [+++] \rightarrow [-1–] \]

appropriate to that sextant. In this case, however, the transitional sequence requires one device to switch three times, not twice, during a control period; this may place undue stress on the devices when transitions occur on a regular basis. It is also not realisable by the pulsewidth modulator proposed in Section 8.1. It may therefore be desirable to instead programme the truncated sequence

\[ [1––] \rightarrow [–+–] \rightarrow [+-+] \rightarrow [+++] \]

which prepares the converter to follow the equivalent reversed sequence

\[ [+++] \rightarrow [+-+] \rightarrow [–+–] \rightarrow [-1–] \rightarrow [+++] \ldots \]

In the truncated sequence, four of the devices switch only once; their effective switching frequency during this period is only half the usual frequency, and subharmonics below the switching frequency may result. In practice, the choice between these two alternative strategies will depend on the operating frequency relative to the device capabilities.

180
DC Offset

As all the free states used in this scheme have at least one device conducting on each leg, the DC offset is zero in any free state. The average DC offset over one cycle is then determined as a linear combination of the DC offsets in the two fundamental states. This can be compared to a VDC in which the zero-state time is divided evenly between the two open states, as in [75], whereupon the contribution of open states to the DC offset is zero.

To determine the DC offset in the fundamental states (remembering that the driving voltage is now on the AC side), we divide the six VDC d-states into two sets of three: those with one positive switch, and those with two positive switches. In a state with one positive switch, letting $p$ denote the phase of the positive switch and $q, r$ the phases of the negative switches, the DC offset is

$$\frac{1}{2} \left( V_p + \frac{V_q + V_r}{2} \right) = \frac{V_p}{4}.$$  

In a state with two positive switches, similar reasoning gives the DC offset as $V_n/4$, where $n$ is the phase of the negative switch. Now, any pair of fundamental states includes a state with one positive switch and a state with one negative switch. If $\tau_1$ is the proportion of the switching period spend in the former state, and $\tau_2$ in the latter, the average DC offset for the cycle is

$$V_N = \frac{1}{4}(\tau_1 V_p + \tau_2 V_n).$$

This equation has the same form as (8.3) for the sixfold CDC strategy. Reasoning in a similar manner, we find that as the average space vector follows a circular trajectory at the AC supply frequency $\Omega$, the DC offset follows a circular trajectory at frequency $3\Omega$.

8.3.3 Twelvefold CDC Switching Strategy

We now consider the most general switching scheme possible, in which there are no a priori restrictions on the d-states we may use. We accordingly turn our attention to the vector diagrams of Figure 7.7, which are divided into twelve equal regions rather than six. For any one of these regions we may identify two fundamental states as before, but now one of these states will have an open phase while the other will not. As these states have only a $30^\circ$ phase separation, we may expect a reduction in the
magnitude of switching harmonics relative to previous schemes; however, six of the twelve space vectors are variable, resulting in low-frequency harmonic distortion.

**Switching Scheme**

Under the twelvefold or ‘hybrid’ scheme, we regard two switch states as adjacent if they differ only in the state of one switch; again, any two states adjacent in Figure 7.7 are also adjacent in this sense. A sequence of states adjacent by this definition is a *Gray code sequence (hybrid sense)*. We define the *complement* \( \tilde{S} \) of any switch state \( S \) as the state obtained by opening all closed switches and closing all open switches. It follows that in any six-step Gray code sequence (hybrid sense) starting at an arbitrary state \( S \) and ending at \( \tilde{S} \), every device switches exactly once. Such a sequence is therefore optimal according to our balance and economy criteria.

This property is exploited in the standard VDC, where we take \( S = [-] \) and \( \tilde{S} = [++] \) (or vice versa). The sequences taking \( S \) to \( \tilde{S} \) are three-step Gray code sequences in the VDC sense, but may be taken as six-step sequences in the hybrid sense through the insertion of ‘blanking’ states, as in

\[
[-] \rightarrow [-0] \rightarrow [-+] \rightarrow [-0+] \rightarrow [++] \rightarrow [0++] \rightarrow [++].
\]

In theory, transitions involving these blanking states are presumed to occur simultaneously. (In reality a finite ‘blanking time’ is necessary in the VDC to avoid shoot-through.)

Recall that in the CDC switching scheme we restricted our attention to simple states, while in the VDC-like switching scheme the restriction was to states having no open phases. Lifting both restrictions permits consideration of free states whose complements are also free states. Here either \( S \) or \( \tilde{S} \) must be nonsimple, and both must include open phases. The following result, analogous to Lemma 8.1, then forms the basis for our switching scheme.

**Lemma 8.2 (Hybrid Switching Principle)** Let \( S_0 \) be a free state whose complement \( \tilde{S}_0 \) is a free state, and let \( S_1, S_2 \) be any two states adjacent in Figure 7.7. Then there is a 6-step Gray code sequence (hybrid sense) starting with \( S_0 \), ending with \( \tilde{S}_0 \) and having \( S_1 \) and \( S_2 \), plus free states, as intermediate states.

**Proof.** Since \( S_1 \) and \( S_2 \) are adjacent, they have five switch positions in common. As for the sixth switch there are two possibilities: its state in \( S_1 \) (say) matches that in
either \( S_0 \) or \( \tilde{S}_0 \). Without loss of generality we may assume that its state in \( S_1 \) matches that in \( S_0 \) (and hence its state in \( S_2 \) matches that in \( \tilde{S}_0 \)). Let \( k \) be the number of switch positions in which \( S_0 \) and \( S_1 \) differ.

By symmetry it remains only to show that the \( k \) switchings required to change from \( S_0 \) to \( S_1 \) can be accomplished while maintaining at least one shorted phase at all but the final step. (We may then repeat the argument for \( \tilde{S}_0 \) and \( S_2 \).) Now, this is trivial as long as there is a shorted phase \( p \) in \( S_0 \) which has one device conducting in \( S_1 \); just ensure that the complementary device on phase \( p \) is the last to be switched. Since \( S_1 \) has no shorted phases, this leaves only the case where \( S_1 \) has an open phase \( p \), which coincides with the only shorted phase in \( S_0 \). But then the single switch whose state changes in the transition \( S_1 \to S_2 \) must be on this same phase \( p \); by hypothesis its state (off) in \( S_1 \) matches its state in \( S_0 \), yielding a contradiction. \( \sqrt{ } \)

According to Lemma 8.2 we may base a fixed-frequency hybrid switching scheme on any arbitrary state \( S_0 \) having at least one open and one shorted phase. Space vectors are synthesised using timed Gray code transitions from \( S_0 \) to \( \tilde{S}_0 \) and vice versa, analogous to the standard VDC scheme. The additional free states occurring in the Gray code sequences may be treated as ‘virtual’ states, with multiple transitions occurring simultaneously. The result is a periodic switching sequence of the form

\[
S_0 \to S_1 \to S_2 \to \tilde{S}_0 \to S_2 \to S_1 \to S_0 \ldots
\]

in which every device switches once per half-cycle. For realisation purposes, we may take the switching period as a half-cycle of this sequence, so the state at the end of a period alternates between \( S_0 \) and \( \tilde{S}_0 \).

**DC Offset**

From previous work, we know the DC offset is \(-V_p/2\) for a state with one open phase \( p \), and \(V_q/4\) for a state with no open phases but one positive (or one negative) phase \( q \). Of the fundamental states \( S_1 \) and \( S_2 \) used for space vector synthesis, one will fall into each of these categories; furthermore, the phases \( p \) and \( q \) will necessarily be distinct.

The DC offset in states \( S_0 \) and \( \tilde{S}_0 \) will depend on the state characteristics. If \( S_0 \) includes precisely one shorted phase and one open phase, then both \( S_0 \) and \( \tilde{S}_0 \) will have two phases conducting, and the DC offsets will be \(-V_i/2\) and \(-V_j/2\), for some \( i \) and \( j \). If on the other hand \( S_0 \) has one shorted phase \( k \) and two open phases, then \( \tilde{S}_0 \)
has two shorted phases and one open phase \( k \) and the DC offsets are respectively \( V_k \) and \(-V_k/2\).

By varying the proportion of the zero-state time spent in the two states \( S_0 \) and \( S_0' \), we may vary these states’ contribution to the DC offset along a line segment in the space vector diagram. The contribution from the fundamental states lies along another line segment (with endpoints proportional to \(-V_p/2 \) and \( V_q/4\)); the disposition of the endpoints of these line segments entails that they are not collinear, and so meet in at most one point.

It follows that we cannot zero the DC offset over a single switching cycle. Nonetheless, as with the two sixfold schemes, under steady-state operation the contribution from the fundamental states will average to zero over a fraction of the AC supply period. It is also not difficult to arrange things so that the contribution from \( S_0 \) and \( S_0' \) averaged over one period is zero; to do this, we make \( S_0 \) a simple free state and time the sequence so that the converter always spends twice as much time in \( S_0' \) as in \( S_0 \).

**Phase Transposition**

The state \( S_0 \) is necessarily asymmetric over the three input phases. We have seen that this does not constitute a problem for the switching scheme, nor does it raise problems with DC offset or with the fundamental component of the input currents (as the space-vector approach effectively guarantees balanced currents in the steady state). The asymmetry could, nonetheless, have implications for the current harmonics or for the non-ideal characteristics present in a practical converter. We therefore outline a ‘transposition strategy’ which ensures the three phases are treated symmetrically, on a long-term average at least.

The following lemma generalises our Hybrid Switching Principle to the case where we desire a final state other than the complement \( S_0' \) of the initial state.

**Lemma 8.3** Let \( S \) be a free state whose complement is a free state, and let \( S' \neq S \) be the image of \( S \) under a phase rotation, such that at least one open phase in \( S \) does not coincide with a shorted phase in \( S' \). Let \( S_1, S_2 \) be two adjacent \( d \)-states. Then there is a 6-step Gray code sequence (hybrid sense) starting with \( S \), ending with \( S' \) and having \( S_1 \) and \( S_2 \), plus free states, as intermediate states. The states \( S \) and \( S' \) differ in four switching positions, which switch once in any such sequence; of the
Proof. By relabelling if necessary, we assume that $S_2$ follows $S_1$ in the Gray code sequence taking $S$ to $\bar{S}$. That $S$ and $S'$ differ in four switching positions is readily established by enumerating the possible choices of $S$. It follows that $S'$ differs from $\bar{S}$ in two switching positions $\sigma_1$ and $\sigma_2$; by hypothesis there is a shorted phase $p$ in $\bar{S}$ that does not coincide with any shorted phase in $S'$, whereby at least one of these switches (say $\sigma_1$) must be on phase $p$. As $S_2$ has no shorted phase, it further follows that $\sigma_1$ must change state between $S_2$ and $\bar{S}$. Let $S''$ denote the state obtained from $\bar{S}$ by switching $\sigma_1$; by the above we may arrange the switching sequence so that $S''$ is the penultimate state. Then by switching $\sigma_2$, we obtain a six-step Gray code sequence concluding in $S'$, in which $\sigma_2$ switches twice and $\sigma_1$ not at all. √

By Lemma 8.3 we may choose an appropriate $S_0$ and periodically ‘transpose’ phases by rotating the phases in $S_0$. In practice the technical requirement on $S$ and $S'$ is not difficult to satisfy; in particular, it is satisfied by any simple free state $S$ under either a clockwise or anticlockwise phase rotation.

The strategy is reminiscent of phase transposition in long power transmission lines. It does, nonetheless, involve a slight relaxation of the fixed-frequency principle. If a ‘phase change’ occurs once in every $M$ sample periods, then in every $3M$ sample periods three devices will be switched $3M + 1$ times, and the other three $3M - 1$ times. For any given device, the switching frequency will nonetheless average out to $F$ over $6M$ periods.

### 8.4 Simulation Results

All simulations were carried out on a 10kHz IGBT converter with $L_\text{S} = 1\text{mH}$, $C_\text{S} = 80\mu\text{F}$, $R_\text{D} = 1\text{m}\Omega$, $L = 8\text{mH}$ and $C = 1000\mu\text{F}$. A 50Hz three-phase supply with $|E| = 240\text{V}$ was used. The nominal DC bus voltage was set at 500V and a 100A constant-current load placed on the DC bus.

#### 8.4.1 Sixfold CDC Scheme

Figure 8.4 shows the steady-state phase 0 current and voltage over three AC cycles, for a converter using the sixfold CDC switching scheme of Section 8.3.1. The open-loop
Figure 8.4: Phase 0 current and capacitor voltage for conventional CDC

PWM control used for these simulations gives unity power factor only approximately, so the phase shift between current and voltage is very nearly but not exactly zero. That low-frequency harmonics are virtually absent is confirmed by plotting the power spectrum of the phase 0 input current (Figure 8.5). Attenuation of the dominant 11th harmonic is around –47dB.

Figure 8.6 shows the DC bus voltage $V_d$ and the DC link current $I_d$ over the same period. Again, the 500 volts used in the simulation is a nominal value only, as feedback control has not been implemented at this stage.

### 8.4.2 Twelvefold CDC Scheme

The instantaneous behaviour of the GAS model was simulated to study the detailed working of the CDC, and in particular the effect of variable space vectors in the twelvefold switching scheme of Section 8.3.3. Figures 8.7 through 8.12 show typical simulations for the periodic steady state in a CDC under twelvefold switching control: a ‘short’ simulation carried out over two control periods (0.2ms) and a ‘long’ simulation over twenty periods (2ms, or one-tenth of an AC cycle). The ‘short’ simulation corresponds to the time-slice between 0.8ms and 1ms in the longer simula-
Figure 8.5: Power spectrum of phase 0 current for conventional CDC

Figure 8.6: DC bus voltage and link current for conventional CDC
Figure 8.7: AC and DC currents for GAS model simulation

Figure 8.8: AC and DC currents for GAS model simulation
Figure 8.9: AC capacitor voltages for GAS model simulation

Figure 8.10: AC capacitor voltages for GAS model simulation
Figure 8.11: DC output voltage for GAS model simulation

Figure 8.12: DC output voltage for GAS model simulation
tion, where the space vector reference is approximately equidistant between the two fundamental vectors \([+0-]\) and \([+-]\). Dotted lines in the ‘short’ simulation mark the transitions between switch states in the sequence

\[
[100] \rightarrow [+0-] \rightarrow [+--] \rightarrow [011] \rightarrow [+--] \rightarrow [+0-] \rightarrow [100]
\]

representing the twelvefold synthesis of a space vector with phase \(\psi = \pi/12\).

Although the waveforms in these simulations represent a periodic steady state, a significant low-frequency oscillation can be discerned. While a proper theoretical study of these oscillations has not been carried out, it would appear that these arise from AC–AC coupling in switch states such as \([+-]\). The precise dynamical consequences of this coupling are most readily discerned from the voltage plots. In the state \([+0-]\), there is a positive net current into the capacitor on phase 2, and so the voltage \(V_2\) increases; \(V_1\) on the other hand decreases as there is a net current out of the capacitor on phase 1. Meanwhile, the states \([+-]\) and \([011]\) cannot be sustained unless \(V_1 = V_2\); otherwise, the state \([+-]\) (for example) reverts to \([+-0]\) or \([+0-]\) according to which of \(V_1\) or \(V_2\) is greater. The tendency then is for \(V_1\) and \(V_2\) to equalise, after which the converter is able to go into the compound state \([+-]\). This effect is postulated as the root cause of the low-frequency oscillation.

Figure 8.13 shows, for the twelvefold scheme, the steady-state phase 0 current and capacitor voltage over three AC cycles. The spectrum of the low-frequency harmonics is shown in Figure 8.14. An important (and as yet unexplained) feature of this scheme is that although harmonics of order 5, 7, 11 and 13 are produced, there are no higher-order harmonics present (apart from those at the switching frequency). These limited harmonics can be readily removed by an appropriate filter (see for example [68]). As things stand, the total current harmonic distortion is 16.6%; by comparison, the current distortion in a six-pulse diode rectifier (as used in almost all commercial VSDs) is 19% even after removal of the fifth and seventh harmonics.

Figures 8.15 and 8.16 show the DC bus voltage and link current over the same period for the hybrid converter. The variation in the bus voltage is around 1% of its DC value, and that in the link current around 10%. As Figure 8.12 shows, there is a substantial ripple component in the converter output voltage which must be filtered by the link inductance \(L\). The next chapter considers the size requirement on \(L\) for filtering the voltage ripple from thyristor converters; the extension to the CDC with twelvefold switching is a subject for future work.
Figure 8.13: Phase 0 current and capacitor voltage for hybrid CDC

Figure 8.14: Frequency spectrum of phase 0 current for hybrid CDC
Figure 8.15: DC bus voltage for hybrid CDC

Figure 8.16: DC link current for hybrid CDC
Chapter 9

AC-DC Converters in the Steady State

The subject of this chapter is the steady-state behaviour of AC-DC converters. Section 9.1 investigates the effect of DC voltage ripple induced by thyristor converters, and provides a design rule for choosing $L$ and $C$ so that this ripple component is sufficiently attenuated. Sections 9.2 and 9.3 are concerned with the steady-state operating characteristics of PWM converters, and specifically with the constraints on operation if the input power factor is to be maximised. The standard VDC is analysed in Section 9.2, and the CDC with sixfold switching in Section 9.3. The latter analysis forms the basis for the design of feedback synchronising control, carried out in the next chapter.

9.1 The Small Model with Thyristor Converter

Given the present state of the art, it is likely that DC reticulation systems in the near future will use thyristor bridge converters as their primary power source. These offer a good approximation to a continuously-controllable source at time scales greater than the switching period, as well as the capability to invert the output voltage for rapid dissipation of DC bus fault energy. Though much finer control is possible with switch-mode converters and IGBTs, for the present thyristors remain superior in cost and power-handling capability.

The chief drawback of thyristor converters, apart from AC-side harmonic distortion, is the ripple component in the output voltage. If $p$ is the number of AC phases
at the input to the bridge (assumed odd), the converter output voltage will fluctuate uncontrollably at a frequency of $2p$ ‘pulses’ per AC cycle. The magnitude of the fluctuations may be reduced by increasing $p$ with a phase-splitting transformer, or by connecting multiple bridges in series with phase-shifted inputs. (Two bridges in series, with a phase shift of $\pi/2p$ at the input to the second bridge, produce an output waveform with $4p$ pulses.)

If the values of converter link inductance $L$ or bus capacitance $C$ are poorly chosen, resonance may occur at the ripple frequency, so that the voltage ripple is amplified between the converter output and the DC bus proper. Conversely, if some care is taken in the choice of these values, the $LC$ network will act as a lowpass filter, attenuating the ripple to within reasonable tolerances.

### 9.1.1 Analysis of DC Ripple

For the purpose of analysing ripple attenuation, the converter output may be approximated by the sum of its DC component and its first harmonic, found by Fourier analysis. Let $V_{AC}$ be the RMS AC line-to-neutral input voltage; the line-to-line voltage tracked by the converter is then

$$V_L = 2V_{AC} \cos \frac{\pi}{2p}. $$

(Note that for $p = 3$, $V_L = \sqrt{3}V_{AC}$; the ratio $V_L/V_{AC}$ tends toward 2 as $p$ tends to infinity, as one would expect.) Let $\Omega = 2\pi f$ be the frequency of the AC supply and $T = 2\pi/\Omega$ its period. The period of the output voltage signal is then $T/2p$ and its fundamental frequency $\omega = 2p\Omega$. With a phase delay angle $\alpha$ and an appropriate choice of time origin, a single period of the output voltage $v(t)$ appears as $\sqrt{2}V_L \cos \Omega t$ between angular times $\Omega t = \pm \pi/2p + \alpha$. Thus, using the notation of (1.14)

$$v_0 = \frac{2p}{T} \int_{\Omega t = -\pi/2p + \alpha}^{\Omega t = \pi/2p + \alpha} \sqrt{2}V_L \cos(\Omega t) \, dt $$

$$= \frac{p}{\pi} \int_{-\pi/2p + \alpha}^{\pi/2p + \alpha} \sqrt{2}V_L \cos \theta \, d\theta \quad (\theta = \Omega t) $$

$$= \sqrt{2} \frac{p}{\pi} V_L (\sin(\pi/2p + \alpha) - \sin(-\pi/2p + \alpha)) $$

$$= 2\sqrt{2} \frac{p}{\pi} V_L \sin \frac{\pi}{2p} \cos \alpha. $$

Thus, following standard theory we have $v_0 = V_{\text{max}} \cos \alpha$ where

$$V_{\text{max}} = 2\sqrt{2} \frac{p}{\pi} \sin \frac{\pi}{2p} V_L = 2\sqrt{2} \frac{p}{\pi} \sin \frac{\pi}{p} V_{AC}. $$

(9.1)
For the Fourier cosine coefficients we obtain

\[
 v_k^C = \frac{2p}{T} \int_{\Omega=-\pi/2p+\alpha}^{\Omega=\pi/2p+\alpha} \sqrt{2}V_L \cos(\Omega t) \cos(2pk\Omega t) \, dt \\
 = \frac{2p}{\pi} \int_{-\pi/2p+\alpha}^{\pi/2p+\alpha} \sqrt{2}V_L \cos \theta \cos(2pk\theta) \, d\theta \\
 = 2\sqrt{2}V_L \cos \theta \left( \frac{1}{2pk + 1} \sin \left( \frac{(2pk + 1)\pi}{2p} \right) \cos((2pk + 1)\alpha) \\
 + \frac{1}{2pk - 1} \sin \left( \frac{(2pk - 1)\pi}{2p} \right) \cos((2pk - 1)\alpha) \right)
\]

Recognising that

\[
 \sin \left( \frac{(2pk + 1)\pi}{2p} \right) = \sin \left( k\pi + \frac{\pi}{2p} \right) = (-1)^k \sin \frac{\pi}{2p} = -\sin \left( \frac{(2pk - 1)\pi}{2p} \right)
\]

and normalising by \( V_{\text{max}} \) using (9.1) we obtain

\[
 \frac{v_k^C}{V_{\text{max}}} = (-1)^k \left( \frac{\cos((2pk + 1)\alpha)}{2pk + 1} - \frac{\cos((2pk - 1)\alpha)}{2pk - 1} \right). \tag{9.2}
\]

Similarly for the sine coefficients

\[
 v_k^S = \frac{2p}{\pi} \int_{-\pi/2p+\alpha}^{\pi/2p+\alpha} \sqrt{2}V_L \cos \theta \sin(2pk\theta) \, d\theta \\
 = 2\sqrt{2}V_L \sin \theta \left( \frac{1}{2pk + 1} \sin \left( \frac{(2pk + 1)\pi}{2p} \right) \sin((2pk + 1)\alpha) \\
 + \frac{1}{2pk - 1} \sin \left( \frac{(2pk - 1)\pi}{2p} \right) \sin((2pk - 1)\alpha) \right)
\]

and

\[
 \frac{v_k^S}{V_{\text{max}}} = (-1)^k \left( \frac{\sin((2pk + 1)\alpha)}{2pk + 1} - \frac{\sin((2pk - 1)\alpha)}{2pk - 1} \right). \tag{9.3}
\]

Letting \( V_k \) denote the RMS magnitude of the \( k \)th voltage harmonic, summing the squares of (9.2) and (9.3) we find after some routine calculation that

\[
 \left( \frac{V_k}{V_{\text{max}}} \right)^2 = \frac{1}{(2pk + 1)^2} + \frac{1}{(2pk - 1)^2} - \frac{2 \cos 2\alpha}{4p^2k^2 - 1}
\]

which may be rewritten as

\[
 \frac{V_k}{V_{\text{max}}} = \frac{2}{4p^2k^2 - 1} \sqrt{1 + (4p^2k^2 - 1) \sin^2 \alpha}. \tag{9.4}
\]

This equation is of use in a number of respects. Firstly, it allows us to quantify the reduction in DC-side ripple obtained by increasing the number of phases on the input to the converter. This is sensitive to the operating value of \( \alpha \); if \( \sin^2 \alpha \) is significant
the reduction in ripple follows a simple inverse law: \( V_k \sim 1/p \). However, for \( \alpha \) small \((\sin^2 \alpha \ll 1)\), this gives way to an inverse-square law: \( V_k \sim 1/p^2 \). A similar rule governs the decrease in harmonic magnitude with increasing harmonic order \( k \); thus, convergence of the Fourier series is fastest when \( \alpha = 0 \) and slowest when \( \alpha = \pi/2 \). On the other hand, one sees that the ripple magnitude is greatest when \( \alpha = \pi/2 \) (when the output most resembles a sawtooth wave), and is least when \( \alpha = 0 \). Symmetry entails that the harmonics for \( \alpha \) and \( \pi - \alpha \) are identical, apart from a change of sign in the cosine coefficients.

Setting \( k = 1 \), we obtain the magnitude of the fundamental component; it is this component that will determine the filtering requirement. Since we assume that \( \alpha \) will vary freely as a result of control action, we obtain the ‘worst case’ value by setting \( \alpha = \pi/2 \):

\[
V_{1,\text{max}} = \frac{4p}{4p^2 - 1} V_{\text{max}} \approx \frac{1}{p} V_{\text{max}}. \tag{9.5}
\]

Note that this is a peak value, not an RMS value.

### 9.1.2 Attenuation of DC Ripple

Consider now the circuit of Figure 7.1, in which

\[
V_s = V_{\text{max}} \cos \alpha + V_{1,\text{max}} \cos(2p\Omega t).
\]

For present purposes we shall assume the load circuit is (at least approximately) linear and time-invariant, so that we may write

\[
V \approx V_{\text{max}} \cos \alpha + V_r \cos(2p\Omega t + \phi).
\]

(The DC value of voltage will be the same, as \( L \) is transparent in the DC steady state.) Our objective is to bring \( V_r \) within set limits, say \( V_r < \epsilon V_R \) where \( V_R \) is the rated DC voltage.

Following standard theory, the circuit dynamics may be written as

\[
\ddot{V} + 2\zeta \omega_0 \dot{V} + \omega_0^2 V = \omega_0^2 V_s \tag{9.6}
\]

where \( \omega_0 = 1/\sqrt{LC} \) is the natural frequency, and \( \zeta > 0 \) is an unspecified damping coefficient related to the load impedance. (Specifically, if \( g \) is the load conductance in \( LC \) units, then \( \zeta = g/2 \).)
Now consider a sinusoidal input \( V_S = |V_S| \cos \omega t \) and let \( V = |V| \cos(\omega t + \phi) \) denote the response. Setting \( d/dt \equiv j \omega \) and taking the magnitude of the frequency-domain transfer function \( V/V_S \) we obtain

\[
\frac{|V|}{|V_S|} = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_0}\right)^4 + 2(2\zeta^2 - 1) \left(\frac{\omega}{\omega_0}\right)^2 + 1}}. \tag{9.7}
\]

When \( \zeta < 1/\sqrt{2} \), the expression under the square-root sign attains a minimum when \( \omega = \sqrt{1 - 2\zeta^2}\omega_0 \); this, then, is the frequency at which maximum amplification occurs, and for any \( \zeta \) is no greater than \( \omega_0 \). (When \( \zeta \geq 1/\sqrt{2} \) this is a true lowpass filter, with maximum amplification at \( \omega = 0 \).)

Now set \( |V_S| = V_{1,\max} \approx V_{\max}/p \) and define \( \rho = (\omega/\omega_0)^2 \). Then, to obtain \( |V| \leq \epsilon V_R \) we require, for arbitrary \( \zeta \)

\[
\rho^2 + 2(2\zeta^2 - 1)\rho + 1 \geq \left(\frac{V_{\max}}{\epsilon p V_R}\right)^2.
\]

For positive \( \rho \), the above inequality is certainly true for all \( \zeta \) if it is true for \( \zeta \to 0 \); accordingly, we let \( \zeta = 0 \) on the left hand side and obtain

\[
\rho = \left(\frac{\omega}{\omega_0}\right)^2 \geq 1 + \frac{V_{\max}}{\epsilon p V_R}
\]

or in terms of physical parameters, noting that \( \omega = 2p\Omega \)

\[
LC \geq \frac{1}{(2p\Omega)^2} \left(1 + \frac{2\sqrt{2}V_{AC}}{\pi \epsilon V_R \sin \frac{\pi}{p}}\right). \tag{9.8}
\]

(9.8) may be regarded as a design rule governing the choice of link inductance \( L \) for thyristor converters and/or the equivalent node capacitance \( C \). As an example, let a 600V DC bus be fed from a 50Hz AC supply with \( V_{AC} = 240V \), by an 18-pulse converter (\( p = 9 \)), and let \( \epsilon = 0.02 \) (so that we tolerate at most a \( \pm 2\% \) fluctuation in DC bus voltage due to converter ripple). Then (9.8) gives \( LC \geq 0.224 \times 10^{-6} \). If \( C \) is given \( a \ priori \) as 1mF, say, then \( L \) should be at least 0.2mH to ensure sufficient ripple attenuation.

Given that the \( LC \) network acts as a second-order lowpass filter in the frequency range of interest, harmonics at frequencies greater than \( 2p\Omega \) will be further attenuated, the gain decreasing at a rate of \( 40\text{dB/decade} \). The contribution of these higher harmonics to \( V \) will accordingly be negligible, so we are justified in ignoring them in the above analysis.
Dynamical considerations will of course dictate additional conditions on the values of $L$ and $C$ for control performance and stability. For example, if a discrete-time regulatory controller with sample period $T$ is used, the Nyquist criterion requires $\sqrt{LC} > T/\pi$. However, the rule (9.8) does provide a necessary and sufficient condition for adequate steady-state performance once stability is assured.

### 9.2 Voltage-Driven PWM Converters: the Boost Characteristic

Here we consider the steady-state operation of a standard voltage-driven PWM converter, with sinusoidal inputs and constant DC outputs. The modelling framework for this section is the PWM model of Section 7.5; we work with the duty ratio formalism, rather than the space vector formalism. Our calculations are similar in form to the steady-state analysis of Wu et al in [79], generalising and extending the results found there.

For control based on duty ratios, one usually chooses $d$ so that $\text{bal} d = 3/2$; this ensures that $V_0 = -V_d/2$, ensuring bipolar DC operation with respect to the supply neutral. One may then write $d = \frac{1}{2}I + \hat{d}$, with $\text{bal} \hat{d} = 0$. In terms of $\hat{d}$, the system equations are

\[
L_s \dot{I} = E - R_s I - \hat{d}V_d \tag{9.9}
\]

\[
CV_d = \hat{d}^T I - I_d. \tag{9.10}
\]

For operation in the steady state, we desire that the DC voltage have the constant value $V_R$, and that $I$ be a balanced signal. We take $E$ as the phase reference, with $\phi_0^E = 0$ and $\phi_0^I \equiv \phi$, the power factor angle at the generator. (Note that positive $\phi$ is a leading power factor.) Since $I$ is assumed sinusoidal, so is $\dot{I}$ and hence also $\hat{d}$ by (9.9). We may therefore assume that

\[
\hat{d}_k = \frac{D}{2} \cos \left( \Omega t + \psi - \frac{2\pi k}{3} \right) \tag{9.11}
\]

for some $D \in [0, 1]$, $\psi \in (-\pi, \pi]$. This gives the RMS magnitude $|\hat{d}| = D/(2\sqrt{2})$. A quantity that will figure prominently is the peak-to-peak input voltage

\[
V_P = 2\sqrt{2}|E|. \tag{9.12}
\]

200
Setting \( \dot{V}_d = 0 \) in (9.10), we find that in the steady state

\[
D \cos(\phi - \psi) = \frac{2\sqrt{2} I_d}{3 |I|},
\]

(9.13)

For the AC side, multiplying (9.9) through by \( I^T/3|I| \) and invoking the CPF property on each term (see Appendix E) gives

\[
0 = |E| \cos \phi - R_S |I| - \frac{D}{2\sqrt{2}} V_d \cos(\phi - \psi)
\]

(9.14)

which in view of (9.13) simply expresses the overall power balance. A similar equation may be obtained by multiplying (9.9) through by \( E^T/3|E| \), giving

\[
\Omega L_S |I| \sin \phi = |E| - R_S |I| \cos \phi - \frac{D}{2\sqrt{2}} V_d \cos \psi.
\]

(9.15)

Now, eliminating \( |E| \) between (9.14) and (9.15) gives, after some manipulation

\[
\frac{|I|}{V_d} = \frac{D \sin \psi}{2\sqrt{2}(\Omega L_S \cos \phi - R_S \sin \phi)}
\]

(9.16)

which expresses the input current magnitude as a function of the steady-state control parameters \( D, \psi \), and the power factor angle \( \phi \).

Usually, one requires that the converter operate at ‘unity power factor’. What is really desired is that the input current magnitude \( |I| \) be minimal for a given load requirement. It is a not quite trivial fact that this is equivalent to ensuring that \( E \) and \( I \) are in phase; a proper justification of this fact follows.

The load requirement in our case is for a fixed DC voltage \( V_R \), a fixed DC current \( f(V_R) \), hence a fixed DC power \( P \). If \( R_S = 0 \) then \( P = 3|E||I| \cos \phi \), and so if \( |E| \) is fixed the current \( |I| \) is minimised for fixed \( P \) by setting \( \phi = 0 \) or \( \pi \) (depending on the sign of \( P \)). If \( R_S \neq 0 \), on the other hand, the input power must also supply the loss in \( R_S \), and this in turn depends on \( |I| \). However, by differentiating the identity

\[
3|E||I|(\phi) \cos \phi = 3R_S |I|^2(\phi) + P
\]

implicitly with respect to \( \phi \) and setting \( d|I|/d\phi = 0 \), we arrive at \( \sin \phi = 0 \).

Suppose we are given \( |E|, \Omega L_S, R_S \) and the load characteristic \( I_d = f(V_d) \). Then, given values for the control variables \( D \) and \( \psi \), the three equations (9.13) through (9.15) together with \( f \) suffice to determine the remaining variables \( \phi, |I|, V_d \) and \( I_d \).

We may then fix values for \( D \) and \( \psi \) by imposing two constraints: that \( V_d = V_R \), and that \( \sin \phi = 0 \) (ensuring that \( |I| \) is minimal).
9.2.1 No-Load Input Current

We solve first for the no-load behaviour, which \textit{a fortiori} does not require knowledge of any load characteristics. At no load (9.13) gives \( \cos(\phi - \psi) = 0 \), while by (9.14) we have \( \cos \phi = R_S |I|/|E| \). Accordingly, only certain values of \( \psi \) will produce an equilibrium. Power factor is also less relevant in the no-load case, as we can minimise \( |I| \) by setting \( I = 0 \) regardless of \( \phi \).

If \( R_S = 0 \), then \( \cos \phi = \sin \psi = 0 \) and so by (9.15)

\[
|I| = \frac{1}{\Omega L_S} \left| E \right| \pm \frac{D}{2\sqrt{2}} V_d
\]

with the sign chosen according to whether \( \psi = 0 \) or \( \psi = \pi \). If \( R_S > 0 \), then we have

\[
\sin \phi = \epsilon_1 \sqrt{|E|^2 - R_S^2 |I|^2} \quad \cos \psi = \epsilon_2 \sqrt{|E|^2 - R_S^2 |I|^2}
\]

where \( |\epsilon_1| = |\epsilon_2| = 1 \). Substituting in (9.15) and squaring yields the following quadratic equation for \( |I| \):

\[
((\Omega L_S)^2 + R_S^2) |I|^2 + \epsilon_1 \epsilon_2 \Omega L_S \frac{D}{\sqrt{2}} V_d |I| + \frac{D^2}{8} V_d^2 - |E|^2 = 0 \quad \text{(9.17)}
\]

for which the two admissible (positive) solutions are

\[
|I| = \frac{1}{(\Omega L_S)^2 + R_S^2} \left( \Omega L_S \frac{D}{2\sqrt{2}} V_d \pm \sqrt{(\Omega L_S)^2 |E|^2 - R_S^2 \left( \frac{D^2}{8} V_d^2 - |E|^2 \right)} \right). \quad \text{(9.18)}
\]

As a special case, we obtain the result already found for \( R_S = 0 \). Note that in all cases, \( |I| \) tends toward zero as \( D|V_d| \) tends toward \( V_P \).

If \( D|V_d| > V_P \), then the absolute value signs may be omitted from (9.18), and the admissible solutions are then obtained by setting \( \epsilon_1 \epsilon_2 = -1 \) in (9.17). Examination of the alternatives shows that of eight putative values for \( \psi \), only the four in the range \([0, \pi]\) are admissible. We then have \( \sin \psi = \cos \phi = R_S |I|/|E| \) and \( \cos \psi = -\sin \phi = \pm \sqrt{|E|^2 - R_S^2 |I|^2} / |E| \), where \( |I| \) is either of the two values (9.18). Choices of \( \psi \) in the first quadrant correspond to a lagging power factor, and in the second quadrant to a leading power factor.

If \( D|V_d| < V_P \), then the smaller of the two solutions obtains when \( \epsilon_1 \epsilon_2 = 1 \) in (9.17). This places \( \psi \) in the range \([-\pi, 0]\), whence \( \sin \psi = -R_S |I|_\cdot/|E| \) where \( |I|_\cdot \) is the alternative with the negative sign in (9.18).

For a concrete example, consider the no-load operation of a converter with \( |E| = 200 \text{V}, V_R = 600 \text{V}, R_S = 0.01 \Omega \) and \( \Omega L_S = 0.05 \Omega \). To obtain an average input
current close to zero we should choose $D$ close to $V_P/V_R \approx 0.943$. Suppose that in fact $D = 0.95$; then the smaller of the two values given by (9.18) is $|I| = 30.5\text{A}$, obtained by setting $\psi = 0.087^\circ$.

It is seen that the input current magnitude is highly sensitive to $D$; if not properly controlled, the converter can appear to the supply as a short circuit. In practice closed-loop synchronising control is necessary to select $D$ and locate the correct equilibrium value of $\psi$. Note that for this current-minimising strategy to work, the nominal DC voltage $V_R$ must be no less than $V_P$. A similar conclusion will be drawn from the loaded equilibrium analysis below.

### 9.2.2 Steady-State Characteristic with Ohmic Load

In the presence of a DC load, we must augment the steady-state equations for the converter with a description of the load. We provide a detailed treatment here for the case of an ohmic load, $I_d = GV_d$. From (9.13) we now obtain

$$
\frac{|I|}{V_d} = \frac{2\sqrt{2}}{3} \frac{G}{D \cos(\phi - \psi)}
$$

(9.19)

which equated with (9.16) gives, after rearranging

$$
\tan \phi = \frac{\frac{8}{3}G\Omega L_S - D^2 \sin \psi \cos \psi}{\frac{8}{3}GR_S + D^2 \sin^2 \psi}.
$$

Note that if $G = 0$ we obtain $\tan \phi = -\cot \psi$, consistent with the no-load analysis above.

Defining the dimensionless parameters $\rho = R_S/(\Omega L_S)$ and $\Gamma = \frac{8}{3}\Omega L_S G$, the above equation becomes

$$
\tan \phi = \frac{\frac{\Gamma}{\rho} - D^2 \sin \psi \cos \psi}{\rho \Gamma + D^2 \sin^2 \psi}.
$$

(9.20)

In order to fix $V_d$, we rewrite (9.14) as

$$
\frac{|E|}{V_d} - \frac{D}{2\sqrt{2}} \cos \psi = \frac{|I|}{V_d}(\Omega L_S \sin \phi + R_S \cos \phi)
$$

and substitute for $|I|/V_d$ from (9.16) to obtain

$$
\frac{V_P}{V_d} = D \cos \psi + D \sin \psi \frac{\Omega L_S \sin \phi + R_S \cos \phi}{\Omega L_S \cos \phi - R_S \sin \phi}
$$

$$
= D \cos \psi + D \sin \psi \frac{\tan \phi + \rho}{1 - \rho \tan \phi}.
$$

(9.21)
provided \( \cos \phi \neq 0 \). We may now substitute (9.20) for \( \tan \phi \) to obtain, after clearing fractions

\[
\frac{V_d}{V_p} = \frac{D(\sin \psi + \rho \cos \psi)}{(1 + \rho^2)\Gamma + \rho D^2}. \tag{9.22}
\]

Since \(|I|\) is positive, it will suffice to have an equation for \(|I|^2\) rather than \(|I|\). We first note that, from (9.16)

\[
(2\sqrt{2} \Omega L_s)^2 \left( \frac{|I|}{V_d} \right)^2 = \frac{D^2 \sin^2 \psi}{\cos^2 \phi(1 - \rho \tan \phi)^2} = \frac{D^2 \sin^2 \psi (1 + \tan^2 \phi)}{1 - 2 \rho \tan \phi + \rho^2 \tan^2 \phi}
\]

(again assuming \( \cos \phi \neq 0 \)), and substituting for \( \tan \phi \) gives

\[
(2\sqrt{2} \Omega L_s)^2 \left( \frac{|I|}{V_d} \right)^2 = \frac{(1 + \rho^2)\Gamma^2 - 2\Gamma D^2 \sin \psi (\cos \psi - \rho \sin \psi) + D^4 \sin^2 \psi}{D^2 (\sin^2 \psi + \rho \cos \psi)^2}.
\]

Finally, multiplying by \((V_d/V_p)^2\) from (9.22) we obtain

\[
\left( \frac{\Omega L_s}{|E|} \right)^2 |I|^2 = \frac{(1 + \rho^2)\Gamma^2 - 2\Gamma D^2 \sin \psi (\cos \psi - \rho \sin \psi) + D^4 \sin^2 \psi}{((1 + \rho^2)\Gamma + \rho D^2)^2}. \tag{9.23}
\]

Physically, (9.23) is the square of the converter input admittance, normalised by the reactance \( \Omega L_s \). Note that as \( \Gamma \to 0 \) with \( D \sin \psi \neq 0 \), this admittance tends toward \( 1/\rho \), characteristic of a short circuit as alluded to above.

Equations (9.20), (9.22) and (9.23) together with the load characteristic \( I_d = GV_d \) provide a complete description of the steady-state behaviour with an ohmic load, once the normalised conductance \( \Gamma \), normalised input resistance \( \rho \) and peak-to-peak input voltage \( V_p \) are specified. To determine which values of \( D \) and \( \psi \) are appropriate, we impose the constraints \( V_d = V_R \) and \( \sin \phi = 0 \). In view of (9.20), the latter becomes

\[
D^2 \sin \psi \cos \psi = \Gamma. \tag{9.24}
\]

We could also have obtained this result by explicitly minimising (9.23) subject to the constraint \( V_d = V_R \), using (9.22) for \( V_d \). However, the calculations involved are cumbersome; the simple power factor argument outlined above provides a much simpler path to the same result. For the output voltage, we substitute \( \Gamma \) from (9.24) in (9.22) to obtain

\[
\frac{V_R}{V_p} = \frac{\sin \psi + \rho \cos \psi}{D((1 + \rho^2) \sin \psi \cos \psi + \rho)}.
\]

In the ideal case \( \rho = 0 \), provided \( \sin \psi \neq 0 \) we obtain

\[
V_d = \frac{V_p}{D \cos \psi}. \tag{9.25}
\]
At this point we may draw two elementary conclusions. First, from (9.24) it is
evident that phase synchronism between $E$ and $I$ cannot be achieved if $|\Gamma| > 1/2$,
equivalently if $|I_d| > \frac{4}{10}V_R/(\Omega L_s)$. Second, (9.25) entails that for optimal operation,
the DC voltage should be set higher than $V_P$, the peak-to-peak AC voltage. For
this reason, the voltage-driven PWM rectifier is sometimes referred to as a ‘boost’
converter.

The specific values of $V_R$ at which the converter is capable of operating in this
optimal fashion depend on the loading $\Gamma$. To see this, notice that we have $D \cos \psi =
V_P/V_R$ from (9.25), and also $D \sin \psi = V_R \Gamma / V_P$ by substituting for $D \cos \psi$ in (9.24).
Accordingly we have

$$D^2 = \left( \frac{V_P}{V_R} \right)^2 + \left( \frac{V_R \Gamma}{V_P} \right)^2 \quad \text{and} \quad \tan \psi = \left( \frac{V_R}{V_P} \right)^2 \Gamma.$$ 

The equation for $\tan \psi$ does not introduce any restrictions, but that for $D^2$ is a
different matter, as $D$ can be no greater than 1.

Let us define the boost coefficient

$$\beta = \frac{V_R}{V_P} > 0$$ \hspace{1cm} (9.26)

whereupon we have

$$D^2 = \frac{1}{\beta^2} + \Gamma^2 \beta^2.$$ \hspace{1cm} (9.27)

When $\Gamma = 0$ we obtain $\beta = 1/D$, implying that any operating voltage greater than
$V_P$ is acceptable provided we set $D = V_P/V_R$. This is entirely consistent with the
no-load analysis above.

To obtain $\beta$ as a function of $D$ and $\Gamma$ in the general case, we may rewrite (9.27)
as a quadratic equation in $\beta^2$, the solutions to which are

$$\beta^2 = \frac{D^2 \pm \sqrt{D^4 - 4 \Gamma^2}}{2 \Gamma^2}.$$ 

For any fixed $\Gamma > 0$ there is a minimum value for $D$, $D = \sqrt{2}\Gamma$, at which $\beta$ has
the unique value $1/\sqrt{\Gamma}$. As $D$ increases beyond this value, the plausible values for $\beta$
follow two branches, one strictly increasing, the other strictly decreasing. Both the
maximum and minimum values for $\beta$ are therefore attained when $D = 1$, and for any
given $\Gamma$ are

$$\beta_{\min} = \sqrt{\frac{1 - \sqrt{1 - 4 \Gamma^2}}{2 \Gamma^2}} \quad \text{and} \quad \beta_{\max} = \sqrt{\frac{1 + \sqrt{1 - 4 \Gamma^2}}{2 \Gamma^2}}.$$ \hspace{1cm} (9.28)
Figure 9.1: The Boost Characteristic: Optimal operating voltages for voltage-driven PWM converter

The plausible values of $\beta$ for varying $|\Gamma|$ are depicted in Figure 9.1. Note that when $|\Gamma| = 1/2$ we have $\beta_{\min} = \beta_{\max} = \sqrt{2}$, so that the only plausible operating voltage at this extremal value of $\Gamma$ is $\sqrt{2}V_P = 4|E|$. On the other hand, as $\Gamma$ approaches zero $\beta_{\max}$ increases (monotonically) without limit, in keeping with the limiting no-load case. Similarly, $\beta_{\min}$ decreases monotonically to its limiting value 1 at $\Gamma = 0$. We therefore find that the choice of $V_R$ is increasingly constrained as the load rating increases, with $V_R = 4|E|$ the only value compatible with all values of $|\Gamma|$ from 0 to 1/2.

### 9.2.3 Regeneration, Voltage Reversal and Other Load Types

The above analysis remains perfectly valid if we allow the DC current $I_d$ (hence $\Gamma$) to take on negative values, reflecting the bidirectional current flow property of the converter. However, the ohmic characteristic we have assumed is probably less realistic for regenerative loads than for normal rectifier loads.

The system equations, in the abstract, also permit negative values for the DC voltage setpoint $V_R$; an inverted boost characteristic is readily obtained by substituting
\[ -\beta \] for \( \beta \) in the preceding analysis. Inversion of the DC voltage is, however, not possible with the standard converter circuit, due to the unipolar blocking characteristic of the switching devices. One would instead require devices combining bidirectional on-state current with bidirectional off-state voltage, such as antiparallel IGBTs with a common gate circuit. This capability obviously comes with an added cost.

The above observation highlights an important distinction between switch-mode rectifiers and inverters, which seems to attract very little attention. A switch-mode inverter is known as a ‘four-quadrant’ conversion device, meaning that both its output current and output voltage are reversible relative to the input. When viewed as a rectifier, the same circuit is not a four-quadrant device, but merely a two-quadrant device. Given the relative symmetry of the two cases one is tempted to regard the rectifier as a four-quadrant device also, and the symmetry of the model equations (as well as some textbook presentations; see [§14.7, §3.5, §18.4]) can reinforce this belief. It is nonetheless fallacious.

As we noted above, the operating characteristic we have developed here applies only to the case of an ohmic load. Loads with constant-current, constant-power or other characteristics will necessarily yield different feasible operating voltages as a function of the converter loading, although the ‘voltage boost’ property is preserved in broad terms.

For example, with a constant-current load (9.13) no longer yields an explicit dependence between \( |I| \) and \( V_d \) of the form (9.19). The equation analogous to (9.20) is

\[
\tan \phi = \frac{\Lambda - V_d D^2 \sin \psi \cos \psi}{\rho \Lambda + V_d D^2 \sin^2 \psi}
\]  

(9.29)

where we now define \( \Lambda = \frac{8}{3} \Omega L S I_d \), with the dimension of voltage. (As with \( \Gamma \) above, \( \Lambda \) may be positive or negative.) Equation (9.21) for \( V_d \) remains valid regardless of the load type, and substituting (9.29) gives

\[
\rho V_d D^2 - V_P D (\sin \psi + \rho \cos \psi) + (1 + \rho^2) \Lambda = 0.
\]  

(9.30)

If \( \rho = 0 \), it follows that \( D \sin \psi = \Lambda / V_P \) is completely determined at equilibrium, regardless of the DC voltage. If \( \rho > 0 \) then given \( \psi \), \( D \) is in general constrained to lie in a specific range; this contrasts with the ohmic case where equilibria exist for all values of \( (D, \psi) \). One may also discern a threshold value for \( \Lambda \) above which no equilibrium is possible.
9.3 Current-Driven PWM Converters: the Buck Characteristic

We turn now to the current-driven converter, and attempt to derive similar steady-state characteristics. For simplicity, we undertake a detailed analysis only for the sixfold CDC switching strategy described in Section 8.3.1.

9.3.1 Equilibrium Conditions for the Sixfold CDC

For all states used in the sixfold CDC strategy, the AC-AC coupling matrix $\Gamma$ is zero, and the DC-side voltage drop due to $\gamma$ is $2RDI_d$, which we assume is negligible. The model equations (7.38) for the CDC, on timescales longer than the switching period, accordingly read

$$L_S \dot{I} = E - V - R_S I$$  \hspace{1cm} (9.31)
$$C_S \dot{V} = I - \bar{\alpha} I_d$$  \hspace{1cm} (9.32)
$$L \dot{I}_d = \bar{\alpha}^T V - V_d$$  \hspace{1cm} (9.33)
$$CV_d = I_d - f(V_d, t)$$  \hspace{1cm} (9.34)

where $\bar{\alpha}$ denotes the moving average of the coupling coefficients $\alpha$ over the control period $T$. The Park transform of $\bar{\alpha}$ is the average space vector (up to a scaling factor), and accordingly the switching control allows us to synthesise any $\bar{\alpha}$ having bal $\bar{\alpha} = 0$ and $\|\bar{\alpha}\|_\infty \leq 1$. (The $L_\infty$ norm $\|a\|_\infty$ is the supremum of $|a_k|$; equivalently, we require $|\bar{\alpha}_k| \leq 1$ for all $k$.)

The connection with space vectors entails that in the steady state $\bar{\alpha}$ should be a balanced three-phase signal. We can also derive this fact from (9.32), taking $I_d$ to be constant and $V$ and $I$ balanced. By analogy with the VDC analysis above, we write

$$\bar{\alpha} = D \cos \left( \Omega t + \psi - \frac{2\pi k}{3} \right)$$  \hspace{1cm} (9.35)

for some $D \in [0, 1]$, $\psi \in (-\pi, \pi]$. Note also that (9.32) and (9.33) are formally dual to the VDC equations with $R_S = 0$; however, in order to relate the steady-state quantities to the fixed driving voltage $|E|$, it is necessary in addition to use (9.31), thereby adding a degree of complexity.

Fixing the generator $E$ as the phase reference, we set

$$\phi_0^E = 0, \quad \phi_0^I = \phi, \quad \phi_0^V = \delta.$$
Proceeding initially as in the VDC analysis, we set \( \dot{I}_d = 0 \) in (9.33) to obtain

\[
D \cos(\delta - \psi) = \frac{\sqrt{2}}{3} \frac{V_d}{|V|}. \tag{9.36}
\]

Meanwhile, multiplying (9.32) through in turn by \( V/3|V| \) and by \( I/3|I| \) gives

\[
0 = |I| \cos(\delta - \phi) - \frac{D}{\sqrt{2}} I_d \cos(\delta - \psi) \tag{9.37}
\]

\[
\Omega C_S |V| \sin(\delta - \phi) = |I| - \frac{D}{\sqrt{2}} I_d \cos(\psi - \phi). \tag{9.38}
\]

A similar pair of equations is obtained by multiplying (9.31) through by \( I/3|I| \) and \( \dot{I}/3|I| \) respectively:

\[
0 = |E| \cos \phi - |V| \cos(\delta - \phi) - R_S |I| \tag{9.39}
\]

\[
\Omega L_S |I| = |E| \sin \phi - |V| \sin(\delta - \phi). \tag{9.40}
\]

Lastly, for completeness, we include the equilibrium of (9.34) which simply expresses the load characteristic:

\[
I_d = f(V_d, t). \tag{9.41}
\]

The six equations (9.36) through (9.41) are independent and thus suffice to determine the six unknowns \( |I|, \phi, |V|, \delta, V_d \) and \( I_d \) given the load characteristic \( f \) and the inputs \( |E|, D \) and \( \psi \). As with the VDC, the choice of \( D \) and \( \psi \) gives us two degrees of freedom to ensure that \( V_d = V_R \) (the desired DC voltage) while simultaneously minimising \( |I| \).

Among the consequences of these equations are the power conservation laws

\[
V_d I_d = 3|V||I| \cos(\delta - \phi) = 3|E||I| \cos \phi - 3R_S |I|^2 \tag{9.42}
\]

which may also be written down by inspection.

### 9.3.2 Equilibrium Characteristic with a Constant-Current Load

The equilibrium analysis is most straightforward in the case of a constant-current load. Suppose then that the DC current \( I_d = I_L \) is a constant independent of \( V_d \). Multiplying (9.38) through by \( \cos(\delta - \phi) \) and using (9.37), we obtain

\[
|V| \cos(\delta - \phi) = \frac{D}{\sqrt{2} \Omega C_S} \frac{I_L}{\sin(\psi - \phi)}. \tag{9.42}
\]
We define the quantities

$$E_P = \sqrt{2}|E|, \quad \Lambda = \frac{I_L}{\Omega C_S E_P}. \quad (9.43)$$

(Thus, \(\Lambda\) is a dimensionless measure of the load.) We then have

$$\frac{|V|}{|E|} \cos(\delta - \phi) = \Lambda D \sin(\psi - \phi). \quad (9.44)$$

We now seek a similar identity involving \(|I|/|E|\). Notice that the power conservation law (9.42) implies

$$\frac{|V|}{I_L} \cos(\delta - \phi) = \frac{V_d}{3|I|}. \quad (9.44)$$

Accordingly, dividing (9.44) by \(I_L\) and substituting gives

$$\frac{V_d}{3|E||I|} = \frac{D \sin(\psi - \phi)}{\Omega C_S E_P}$$

or

$$|I| = \frac{\sqrt{2}}{3} \frac{\Omega C_S}{D \sin(\psi - \phi)} V_d. \quad (9.45)$$

We define the *buck factor*

$$\beta = \frac{2 V_d}{3 E_P} \quad (9.45)$$

which measures the attenuation or ‘bucking’ in DC voltage relative to the theoretical steady-state maximum \(3 E_P/2\). We then have

$$\frac{|I|}{|E|} = \beta \frac{\Omega C_S}{D \sin(\psi - \phi)}. \quad (9.46)$$

Now, divide (9.39) by \(|E|\) and use the identities (9.44) and (9.46) to eliminate \(|I|, |V|\) and \(\delta\). The resulting equation is

$$\cos \phi = \Lambda D \sin(\psi - \phi) - R_S \beta \frac{\Omega C_S}{D \sin(\psi - \phi)}. \quad (9.47)$$

For \(R_S > 0\) this is essentially a quadratic equation in \(\cos \phi\) and \(\sin \phi\), which yields \(\phi\) as a function of \(D, \psi,\) the load \(\Lambda,\) the buck factor \(\beta,\) and the dimensionless quantity \(\rho = \Omega C_S R_S\). Having made this general observation, we henceforth assume that \(R_S\) is negligible.

With \(R_S = 0\) (9.47) reduces to

$$\cos \phi = \Lambda D (\sin \psi \cos \phi - \cos \psi \sin \phi)$$

210
\[ \tan \phi = \frac{\Lambda D \sin \psi - 1}{\Lambda D \cos \psi}, \quad (9.48) \]

an expression inviting comparison with (9.20) for a VDC with ohmic load. As consequences of (9.48) we obtain

\[
\begin{align*}
\cos \phi &= \frac{\Lambda D \cos \psi}{\sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi + 1}} \quad & \sin \phi &= \frac{\Lambda D \sin \psi - 1}{\sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi + 1}} \\
\cos(\psi - \phi) &= \frac{\Lambda D - \sin \psi}{\sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi + 1}} \quad & \sin(\psi - \phi) &= \frac{\cos \psi}{\sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi + 1}}.
\end{align*} \quad (9.49)
\]

Having isolated \( \phi \), we now proceed to eliminate the phase angles \( \phi \) and \( \delta \) from the equations. First, we use (9.44) and (9.36) to write

\[ \Lambda D^2 \sin(\psi - \phi) \cos(\delta - \phi) = \beta \cos(\delta - \phi). \]

We then use the identity

\[ \cos(\delta - \psi) = \cos(\delta - \phi) \cos(\psi - \phi) + \sin(\delta - \phi) \sin(\psi - \phi) \]

and replace all occurrences of \( \cos(\psi - \phi) \), \( \sin(\psi - \phi) \) using (9.49). Gathering together the terms in \( \cos(\delta - \phi) \) and \( \sin(\delta - \phi) \) we obtain (after much algebra)

\[ \tan(\delta - \phi) = \frac{\beta(\Lambda^2 D^2 - 2\Lambda D \sin \psi + 1) - \Lambda D^2 \cos \psi(\Lambda D - \sin \psi)}{\Lambda D^2 \cos^2 \psi}. \quad (9.50) \]

Next, we call on the hitherto unused equation (9.40), which we write as

\[ |V| \sin(\delta - \phi) = |E| \sin \phi - \Omega L_s |I| \]

and divide by (9.39), which with \( R_S = 0 \) reads

\[ |V| \cos(\delta - \phi) = |E| \cos \phi. \]

The result is a second equation for \( \tan(\delta - \phi) \):

\[ \tan(\delta - \phi) = \tan \phi - \frac{\Omega L_s |I|}{\cos \phi |E|}. \quad (9.51) \]

We now normalise the input current by introducing the dimensionless quantity

\[ \eta = \frac{\Omega L_s |I|}{|E|} \]

and substitute for \( \tan \phi \) and \( \cos \phi \) to obtain

\[ \tan(\delta - \phi) = \frac{\Lambda D \sin \psi - 1 - \eta \sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi + 1}}{\Lambda D \cos \psi}. \quad (9.52) \]
We are now in a position to eliminate $\delta$ by equating (9.50) and (9.52). This yields the following relation between $\eta$ and $\beta$:

$$
\eta + \frac{\Lambda}{\cos \phi} \beta = \frac{\Lambda^2 D^2 - 1}{\sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi} + 1}.
$$

(9.53)

Now, $\eta$ and $\beta$ are also related via (9.46). If we define $\sigma$ to be the ratio of the AC supply frequency to the natural frequency of the LC input network

$$
\sigma = \frac{\Omega}{\omega_s} = \Omega \sqrt{L_S C_S}
$$

then (9.46) can be made to read

$$
\eta = \beta \frac{\sigma^2}{D \sin (\psi - \phi)} = \sigma^2 \frac{\Lambda}{\cos \phi} \beta.
$$

(9.54)

Combining (9.53) and (9.54) gives, finally, explicit formulae for the normalised current and buck factor as a function of $D$, $\psi$, and the normalised load. These are as follows:

$$
\eta = \frac{\sigma^2}{1 + \sigma^2 \sqrt{\Lambda^2 D^2 - 2\Lambda D \sin \psi} + 1}
$$

(9.55)

$$
\beta = \frac{1}{1 + \sigma^2 \Lambda^2 D^2 - 2\Lambda D \sin \psi + 1}. \quad (9.56)
$$

### 9.3.3 The Buck Characteristic

We now have a description of the steady-state behaviour of the CDC with sixfold switching control, $R_S = 0$, and a constant-current load $I_L = \Lambda (\Omega C_S E_P)$. Equations (9.55) and (9.56) give the input current and output voltage respectively, while (9.49) gives the power factor angle. A few observations are in order before we proceed.

- Typical values of $\Lambda$ are large. With $\Lambda = 1$, the load current is of the same order of magnitude as the current in the input capacitor $C_S$.

- Setting $D = 0$ or $\psi = \pm \pi/2$ zeroes the output voltage. Setting $D = 1/\Lambda$ also zeroes the voltage unless $\psi = \pi/2$, which corresponds to a singularity in the characteristic. There are no other singularities besides this one.

- When $\Lambda D > 1$, voltage inversion is achieved by setting $|\psi| > \pi/2$. The reverse applies when $\Lambda D < 1$.

- The maximum achievable buck factor is not quite 1 but rather $1/(1 + \sigma^2)$, achieved when $D = 1$ and $\sin \psi = 2\Lambda/(\Lambda^2 + 1)$.
Figure 9.2: Contour plot of $\beta(D, \psi)$ with $\Lambda = 10$. (Contour spacing = 0.04.)

- $|\beta| \sim D|\cos \psi|$ both in the large load limit $\Lambda D \gg 1$ and the small load limit $\Lambda D \ll 1$.

Figures 9.2 and 9.3 provide contour plots of $\beta$ and $\eta$ respectively as functions of $D$ and $\psi$, for $\Lambda = 10$. In Figure 9.2 we can see the loci of constant $\beta$ tend toward straight lines as $D$ becomes large, as expected under the large load approximation. Notice also the $\beta = 0$ contour at $D = 1/\Lambda = 0.1$, and the singularity at $D = 0.1$, $\psi = \pi/2$.

We now seek the unity-power-factor operating point $(D, \psi)$ for a given DC voltage $V_R$. Setting $\sin \phi = 0$ in (9.49) gives

$$\Lambda D \sin \psi = 1. \quad (9.57)$$

(When $R_S \neq 0$, a quadratic equation for $D \sin \psi$ may be deduced from (9.47).) Equation (9.57) gives two values for $\psi$, corresponding to positive and negative output voltage, and we have

$$\eta = \frac{\sigma^2}{1 + \sigma^2} \sqrt{\Lambda^2 D^2 - 1}$$

$$|\beta| = \frac{1}{1 + \sigma^2} \frac{\sqrt{\Lambda^2 D^2 - 1}}{\Lambda}.$$
Figure 9.3: Contour plot of $\eta(D, \psi)$ with $\Lambda = 10$. (Contour spacing = 0.01.)

Note that for a given $\Lambda$, the highest buck factor attainable at unity power factor is

$$\beta_{\text{max}} = \frac{1}{1 + \sigma^2} \sqrt{\Lambda^2 - 1}. \frac{\Lambda}{\Lambda}.$$  

This becomes equal to 1 in the large load limit, when $|\beta| \sim D$. Rearranging this expression, we obtain a lower limit on $\Lambda$ for operation at a given output voltage:

$$\Lambda_{\text{min}} = \frac{1}{\sqrt{1 - (1 + \sigma^2)^2 \beta^2}}.$$  

This limit becomes more stringent as $\beta$ approaches 1. If $\Lambda < \Lambda_{\text{min}}$, unity-power-factor operation must be abandoned.

At unity power factor, the magnitude of the input current may be found directly from the power balance as

$$|I| = \frac{|V_R| I_L}{3|E|}$$

while the phase shift $\delta$, from (9.51), is given by

$$\tan \delta = -\frac{\eta}{\cos \phi} = -\frac{P}{S}, \quad -\frac{\pi}{2} < \delta < \frac{\pi}{2} \quad (9.58)$$

where $P$ is the DC power, and $S = 3|E|^2/(\Omega L_S)$ is the fault level of the AC supply. Note that $\delta \to 0$ as $P \to 0$ or as $L_S \to 0$; there is an obvious analogy here with the
power angle in AC power system control theory. Its main use to us is in determining
the magnitude of the steady-state input voltage \(|V|\):

\[
\frac{|V|}{|E|} = \frac{\cos \phi}{\cos(\delta - \phi)} = \frac{1}{\cos \delta} = \frac{\sqrt{P^2 + S^2}}{S}.
\]  

(9.59)

Provided that the converter power rating is a small fraction of the fault level, as is
the norm, we may assume that \(|V| \approx |E|\) and \(\delta \approx 0\). This assumption was used
in the DC offset calculations of Chapter 8; in general (9.58) and (9.59) hold for any
series-inductive circuit having unity power factor at the source.

9.3.4 Relation to Synchronising Control

The above analysis was carried out in a rotating coordinate frame, with \(E_0\) as the
phase reference. The parameter \(\psi\) measures the phase displacement between \(\bar{\delta}_k\) and
\(E_k\), so that constant \(\psi\) entails that \(\bar{\alpha}\) rotates anticlockwise at the supply frequency.

Recall that the Park transform of a balanced three-phase signal \(X\) is a complex
number whose magnitude and phase are identical to that of \(X_0\). Suppose that we
have measured the phase of \(E_0\) as \(\phi^E_0\) at time \(t_0\), and let \((V, \Phi)\) be the magnitude and
phase of the average space vector, interpreted as the Park transform of \(\bar{\alpha}\). In order
to achieve steady-state behaviour with given values of \(D\) and \(\psi\), it is necessary to set

\[
V = D, \quad \Phi = \psi + \phi^E_0 + \Omega(t - t_0).
\]  

(9.60)

If our measurement is of \(\phi^V_0\) rather than \(\phi^E_0\), there will be a small adjustment for \(\delta\),
which may be estimated from the power flow and fault level according to (9.58).

This observation is entirely consistent with Equation (7.52) for the DC voltage
command, provided one first ‘translates’ from the Park transform interpretation to
the output phasor interpretation of the space vector. If \((V, \Phi)\) is the space vector on
the former interpretation, then \((V', \Phi')\) is the space vector on the latter interpretation,
where

\[
V' = \frac{3}{2}|V|V, \quad \Phi' = -\Phi.
\]  

(9.61)

After substituting for \(V'\) and \(\Phi'\), and noting that \(\phi^V_0 - \phi^E_0 = \delta\), Equation (7.52) reads

\[
V_s = \frac{3}{\sqrt{2}}|V|D \cos(\delta - \psi)
\]

which is precisely Equation (9.36) with \(V_s\) and \(V_d\) identified (appropriately, keeping
in mind that our remarks refer only to steady-state operation).

215
Chapter 10

Synchronising Control of
Switch-Mode AC-DC Converters

Synchronising control, as defined in Section 7.6.3, is the selection of control inputs (space vectors, in the case of switch-mode converters) so as to simultaneously track a constant output voltage setpoint and a sinusoidal input current or voltage reference signal. It is intermediate between regulatory control, which deals with the selection of output voltage setpoints and is discussed in the next chapter, and switching control, which deals with the mapping from control inputs to instantaneous switching states—the subject of Chapter 8.

As before, we divide switch-mode converters into voltage-driven and current-driven types. Section 10.1 deals with the conceptually simpler voltage-driven converter, for which a number of synchronising control techniques have been described in the literature. After reviewing these approaches and discussing their advantages and shortcomings, we describe an approach to design based on energy functions, which provides an alternative justification for the more popular techniques. We also describe a new approach to VDC control based on an existing approach for DC-DC converters. In Section 10.2 elements of these design methodologies are applied to the current-driven converter, where the synchronising control is more easily decoupled from the regulatory control but presents a more difficult problem on its own. Here, a conceptually simple design based purely on linear theory is found to share many of the features of the VDC controllers.
Because in a VDC there is no series inductance to decouple the converter output and DC bus voltages, the regulatory control problem for VDCs is subsumed within the synchronising control problem. However, as we shall see, most strategies for synchronising control of VDCs tend to notionally decouple the AC current control and DC voltage control, designing the former under the assumption that the DC voltage is constant (as in an inverter) and then placing this within a higher-level DC voltage loop.

As it happens, the structure of the nonlinearities in the VDC model make the input currents easier to control than the output voltage, for reasons that will become apparent in Section 10.1.4. On the basis of our investigations, we suggest that the best way to satisfy both AC-side and DC-side control objectives is indeed through a two-tiered controller structure, though one that avoids the notional treatment of DC voltage as constant. The two tiers may be identified with regulatory control and synchronising control, in keeping with previous discussion, but in the VDC case the role of the regulatory control is to provide current, not voltage, setpoints.

In Section 10.1.1 we formulate the synchronising control problem for VDCs and describe the models used in subsequent analysis. Section 10.1.2 outlines the evolution of these models in the research literature and describes PCFF control, which we adopt as our control ‘benchmark’. Subsequent sections describe some newer approaches, including one of our own based on work of Ortega et al. [58]. Finally, the controllers are subjected to simulation studies in Section 10.1.5.

**10.1.1 Problem Formulation**

Our starting point is the time-averaged VDC GAS model (7.40). Assuming the usual switching control (the use of which was defended in Chapter 8) and no input capacitance, this has the form originally presented in Section 7.5:

\[
L_S \dot{I} = E - R_S I - \dot{d}V_d
\]

\[
CV_d = \dot{d}^T I - f(V_d, t).
\]

Here the resistance \( R_S \) includes any device resistance as well as the AC source resistance.
We assume for the present that all system variables \((E, I, V_d \text{ and } I_d = f(V_d, t))\) are measurable, and that the parameter values \(L_S, R_S\) and \(C\) are known. The continuous-time control task is then as follows: given measurements of the system variables at times \(t \geq t_0\), find an admissible control \(d(t), t \geq t_0\), which ensures that

1. \(V_d(t) \in [(1 - \epsilon_1)V_R, (1 + \epsilon_1)V_R]\) for \(t - t_0 \geq T_1\), for some nominal DC voltage \(V_R\), tolerance \(\epsilon_1\) and convergence time \(T_1\);

2. \(I_k(t) \in [I^*_k(t) - \epsilon_2|\mathbf{I}|, I^*_k(t) + \epsilon_2|\mathbf{I}|]\) for \(t - t_0 \geq T_2, k \in \{0, 1, 2\}\), for some tolerance \(\epsilon_2\) and convergence time \(T_2\), where \(\mathbf{I}\) is a balanced three-phase signal having \(|\mathbf{I}|\) minimal with regard to the DC load requirements.

Since the system variables and parameters are not known exactly, we require the control to be robust against modelling errors and disturbances. The convergence times \(T_1\) and \(T_2\) give the maximum duration of any transient excursion outside the respective tolerance limits for ‘occasional’ disturbances such as load switching. In practice, there are also constraints on the maximum values of \(V_d\) and \(I_k\) which apply at all times, including transients.

We now make use of the Park transform (see Appendix E) to simplify the model. The Park transform accomplishes two things: first, it eliminates the redundancy among the state variables owing to the (assumed) absence of zero-sequence current; and second, it simplifies the analysis by replacing 120 \(^\circ\) phase separations with 90 \(^\circ\) phase separations.

Without loss of generality we choose the time origin so that \(\phi_0^E = 0\). Applying the stationary Park transform (E.3) to the variables \(E, I\) and \(d\), and neglecting the zero-sequence variable \(I_z\) (assumed identically zero), we obtain

\[
\begin{align*}
L_S \dot{i}_a &= |E| \cos(\Omega t) - R_S i_a - d_a V_d \\
L_S \dot{i}_b &= |E| \sin(\Omega t) - R_S i_b - d_b V_d \\
C \dot{V}_d &= 3(d_a i_a + d_b i_b) - f(V_d, t).
\end{align*}
\]

(10.2)

The reference signal \(\mathbf{I}^*\) can similarly be written in Park components, as reference signals for \(I_a\) and \(I_b\) respectively:

\[
\begin{align*}
I^*_a(t) &= |\mathbf{I}^*| \cos(\Omega t + \phi^*) \\
I^*_b(t) &= |\mathbf{I}^*| \sin(\Omega t + \phi^*)
\end{align*}
\]

219
where $\phi^*$ is the relative phase shift of $E$ and $I$ required to minimise $|I|$.

In place of the stationary Park transform one may use the synchronous Park transform (E.7), and thereby obtain time-invariant equations with constant setpoints:

\[
L_S I_a' = |E| - R_S I_a + \Omega L_S I_\beta - d_a V_d \\
L_S I_\beta' = -R_S I_\beta - \Omega L_S I_a - d_\beta V_d \\
CV_d = 3(d_a I_a + d_\beta I_\beta) - f(V_d, t)
\]

with

\[
I^*_a = |\Gamma| \cos \phi^*, \quad I^*_\beta = |\Gamma| \sin \phi^*
\]

and tolerance bounds

\[
I_a(t) \in \left[(\cos \phi - \epsilon_2)|\Gamma|, (\cos \phi + \epsilon_2)|\Gamma|\right], \quad t - t_0 \geq T_2 \\
I_\beta(t) \in \left[(\sin \phi - \epsilon_2)|\Gamma|, (\sin \phi + \epsilon_2)|\Gamma|\right], \quad t - t_0 \geq T_2.
\]

Additional complexity is introduced into this form of the model, however, by the cross-coupling between $I_a$ and $I_\beta$, which comes about through the identity (E.12) for the synchronous Park transform of a derivative. In some situations, therefore, it may be advisable to use the stationary Park model despite its time-varying nature.

### 10.1.2 Evolution of Models and Control

The model (10.1) was originally derived by Wu, Dewan and Slemon, henceforth WDS [78], who proposed a fixed-frequency discrete-time controller based on this model. By averaging the converter dynamics to focus on the low-frequency behaviour, WDS were able to draw a clear distinction between what we call synchronising control and switching control.

Previous approaches to closed-loop current control had used analogue comparators operating in continuous time; the most popular was hysteresis current control (HCC), essentially a set of three bang-bang controllers operating on individual converter legs. A key drawback of such controllers, as noted by WDS and others, is the variability in switching frequency, which rises as DC load increases. With some clever modifications (such as the insertion of zero states to simultaneously reduce the current error on all three phases) the switching frequency can be reduced, but the essential variability

220
remains. (See [10] for a survey of HCC and other ‘classical’ approaches, as applied to voltage-source inverters.)

The design principle employed by the WDS controller, called predicted-current fixed-frequency control or PCFF, is the mimicking of linear dynamics by the ‘inversion’ of a nonlinear model. Let \( \mathbf{I} \) be a balanced three-phase signal, and \( T \) be the controller sample period. Set

\[
\hat{\mathbf{d}} = \frac{1}{V_d} \left( \mathbf{E} - \left( R_s - \frac{L_s}{T} \right) \mathbf{I} - \frac{L_s}{T} \mathbf{I}^* \right).
\]

(10.4)

Upon substitution into the model (10.1), the current dynamics are rendered into the form

\[
T \frac{d\mathbf{I}}{dt} + \mathbf{I} = \mathbf{I}^*.
\]

(10.5)

This is a simple first-order equation, which ensures according to WDS that the input current \( \mathbf{I} \) tracks the signal \( \mathbf{I}^* \), “delayed by one sample period \( T \)” [sic]. On the assumption that \( |\mathbf{I}| \) is minimised when \( \mathbf{E} \) and \( \mathbf{I} \) are in phase, we need now only set

\[
\mathbf{I}^* = \mathcal{R}_\theta \cdot \frac{I^*}{|\mathbf{E}|} \mathbf{E}
\]

(10.6)

where \( I^* \) is the desired RMS AC input current, and \( \mathcal{R}_\theta \) is a phase-shift operator that applies the leading phase shift \( \theta = \Omega T \) to compensate for the (approximate) time delay from (10.5).

It remains to determine the AC current setpoint \( I^* \). For a DC current setpoint \( I^*_d \), \( I^* \) may be found by solving the power balance, applicable when \( E \) and \( \mathbf{I} \) are in phase:

\[
3R_s(I^*)^2 - 3|\mathbf{E}|I^* + V_R I^*_d = 0.
\]

(10.7)

In PCFF control, \( I^*_d \) is the output of a feedforward-feedback regulator driven by the DC voltage error and current \( I_d \). The specific form of the regulator is not specified in [78]; if we assume a PI regulator, then we have (in the continuous-time approximation)

\[
I^*_d = I_d + \left( K_P + K_I \int \right) (V_R - V_d).
\]

(10.8)

The use of a regulator of this form has a straightforward justification. Under the assumption that the converter output current tracks \( I^*_d \), the current difference \( I^*_d - I_d \) is just the capacitor current \( I_c = CV_d \). With \( V_d \) as the controlled variable and \( I_c \)
as the effective control input, the controlled system is an integrator. (The regulator (10.8) may be compared with that of (7.13), derived independently in [52].) Analysis of
the voltage control loop is deferred to a later section.

An inherent drawback with controllers based on model inversion is their sensitivity
to the model parameters. Since the control relies on subtracting or dividing out
nonlinearities and coupling terms in the natural dynamics, a poor estimation of these
terms can cause problems. WDS nonetheless make a case for the robustness of their
control design based on the negative feedback inherent in (10.4) and (10.8), and this
robustness appears to be borne out in simulations (Section 10.1.5). We conclude that
PCFF control is an appropriate benchmark against which alternative approaches may
be evaluated.

In a second paper [79], the same authors studied the behaviour of general control
schemes in which \( \hat{d} \) is assumed only to be a balanced three-phase signal, in which
case it can be represented by a modulation index \( m \) and phase shift \( \psi \) such that

\[
\hat{d}_i = \frac{m}{2} \cos \left( \Omega t - \psi - \frac{2\pi j}{3} \right).
\]

(In Chapter 9 we made use of a variant of this representation to carry out the steady-
state analysis.) This generic scheme WDS referred to as phase and amplitude control,
or PAC. The synchronous Park transform (or “rotating frame of reference”) was
then introduced to render the system time-invariant. Much subsequent work on
synchronising control, including our own, has drawn on this basic framework.

A further refinement was made by Blasko and Kaura [7], who sought to bypass the
complexity of the steady-state and small-signal models in [79], and produce models
amenable to control-theoretic approaches. Our statement above of the converter
system model in the stationary and synchronous Park transform domains is essentially
that of [7]. These models eschew the phase-amplitude representation in favour of a
representation by the Park components \( d_a \) and \( d_b \), or \( d_\alpha \) and \( d_\beta \). (The latter are
related to the phase-amplitude representation as rectangular coordinates are to polar
coordinates.)

The control suggested in [7] uses the synchronous Park model (10.3). Separate PI
regulators are used for the \( \alpha \) and \( \beta \) components, driven from the current error signals
\( I^*_\alpha - I_\alpha \) and \( I^*_\beta - I_\beta \) and producing PWM control voltages (equivalent to \( dV_d \)). The
inductive cross-coupling is removed by feedforward compensation; again, a form of
model inversion. Blasko and Kaura note that “the only work which is performed by
the regulators during normal operation is to correct for the errors in the parameters and for the above compensations.” Just as with PCFF control, the bulk of the control signal derives from feedforward terms.

Indeed, the Blasko-Kaura control can be likened to PCFF control in a number of respects. As in PCFF control, the AC current is controlled to be in phase with the supply voltage (thus \( I_\beta^* \) is set to zero). The setpoint \( I_\alpha^* \) plays the same role as \( I^* \) in PCFF control, and likewise is obtained from a PI regulator driven by the DC voltage error (although Blasko and Kaura appear to regard the DC current as an extraneous disturbance rather than as an explicit feedforward term in the control). Equation (10.4) for the duty ratios in PCFF control is equivalent to a pair of simple proportional regulators acting on the \( \alpha \) and \( \beta \) components, with gain \( L_s/T \), feedforward compensation for the voltage \( E \) and resistive voltage drop, but no compensation for the inherent cross-coupling due to \( L_s \). As a result, the proportional regulator has to do some work in PCFF control that is not necessary in the Blasko-Kaura control. Blasko and Kaura allow also for integral action on the AC current error. Seen in this way, the Blasko-Kaura control is a modified and extended version of PCFF control.

The explicit Blasko-Kaura equivalent of (10.4) is

\[
\begin{align*}
d_\alpha V_d &= |E| + \Omega L_s I_\beta - \left( K_{P\alpha} + K_{I\alpha} \int \right) (I_\alpha^* - I_\alpha) \quad (10.9) \\
d_\beta V_d &= -\Omega L_s I_\alpha - \left( K_{P\beta} + K_{I\beta} \int \right) (I_\beta^* - I_\beta). \quad (10.10)
\end{align*}
\]

Note that if \( R_s \) is known we could add an extra feedforward term to compensate for that also, as was done in (10.4), and thereby speed up the response further. However, as \( R_s \) is usually small and uncertain there is probably little merit in including such a term.

### 10.1.3 Optimal Control Approach

An approach to synchronising control that moves away from the classical model inversion and PI regulator design methods is the nonlinear optimal controller of Choi and Sul [12]. Their work is significant, not only because it treats the complete model (instead of just subtracting away the ‘awkward’ terms and hoping a PI regulator can compensate for errors), but also because of the explicit consideration given to the problem of large transients, and the differing time scales inherent in the control objectives.
PCFF control (in its original and Blasko-Kaura variants), like the hysteresis current control that preceded it, is predominantly concerned with tracking a sinusoidal current reference, in phase with the voltage $E$. It does this by effectively decoupling the two Park components of the current and then forcing the ‘quadrature’ component $I_\beta$ to be as small as possible. However, the requirement for $E$ and $I$ to be in phase is only a medium-term objective; it is motivated by concern over the power factor, which is a meaningless notion during short-term transients. By permitting $I_\beta$ to take on nonzero values during transients, it may be possible, by exploiting the inductive cross-coupling, to bring both $I_a$ and $I_\beta$ close to their desired values in less time.

For reasons that will become apparent, the Choi-Sul control starts not from the synchronous Park model but from the stationary Park model (10.2). One translates the desired (time-varying) operating point to the origin by defining new variables

$$x_a = I_a - I_a^* = I_a - I^* \cos(\Omega t)$$

$$x_b = I_b - I_b^* = I_b - I^* \sin(\Omega t).$$

In terms of these variables the AC current component of the model, neglecting the source resistance and assuming a slowly-varying current setpoint $I^*$, becomes

$$\dot{x}_a = \frac{|E|}{L_s} \cos(\Omega t) + \Omega I^* \sin(\Omega t) - \frac{1}{L_s} d_a V_d$$

$$\dot{x}_b = \frac{|E|}{L_s} \sin(\Omega t) - \Omega I^* \cos(\Omega t) - \frac{1}{L_s} d_b V_d.$$  (10.11)

(10.12)

The problem now becomes one of forcing $x_a$ and $x_b$ to zero in minimum time, given the constraint

$$d_a^2 + d_b^2 \leq d_{\text{max}}^2 = \frac{1}{8},$$

(10.13)

(In [12] this is presented as a constraint on the voltage space vector, with components $d_a V_d$ and $d_b V_d$. The latter is not quite correct, as larger values of $V_d$ permit larger space vectors independently of the constraint (10.13).)

Based on the intuitive fact that $\dot{x}_a$ ($\dot{x}_b$) can be made arbitrarily large by making $|d_a|$ ($|d_b|$) arbitrarily large, Choi and Sul treat the inequality constraint as an equality constraint, making the substitution

$$d_b = \sqrt{d_{\text{max}}^2 - d_a^2}.$$  (10.14)

The optimal control problem is then one of minimising the cost function

$$J = \int_{t_0}^{t_f} 1 \, dt$$

224
for the system (10.11), (10.12), (10.14) with control variable \(d_a\) and specified boundary conditions

\[
x_a(t_0) = I_a(t_0) - I^* \cos(\Omega t_0) \\
x_b(t_0) = I_b(t_0) - I^* \sin(\Omega t_0) \\
x_a(t_f) = x_b(t_f) = 0.
\]

From classical optimal control theory [1, 42], the Hamiltonian for this problem has the form

\[
H = 1 + \dot{x}_a p_a + \dot{x}_b p_b
\]

where \(p_a\), \(p_b\) are costate variables, and the optimal control satisfies

\[
\begin{align*}
\dot{p}_a &= -\frac{\partial H}{\partial x_a} = 0 \\
\dot{p}_b &= -\frac{\partial H}{\partial x_b} = 0 \\
0 &= \frac{\partial H}{\partial d_a} = -\frac{V_d}{L_S} \left( p_a - \frac{d_a}{\sqrt{d_a^2 - d_b^2}} p_b \right).
\end{align*}
\]

We accordingly find that the costate variables \(p_a\) and \(p_b\) are constant for the optimal trajectory, while the control has the form

\[
d_a = d_{\text{max}} \frac{p_a}{\sqrt{p_a^2 + p_b^2}}, \quad d_b = d_{\text{max}} \frac{p_b}{\sqrt{p_a^2 + p_b^2}}.
\]

That \(p_a\) and \(p_b\) are constant derives from the fact that the system equations (10.11), (10.12) contain no explicit dependence on the variables \(x_a\) and \(x_b\). This is a consequence of the use of the stationary Park model. Importantly, (10.15) tells us the optimal \(d_a\) and \(d_b\) are constant; this corresponds to a constant (not rotating) duty ratio vector \(\hat{d}\) in the original model (10.1). We thereby arrive at a generalisation of the old HCC strategy of programming a zero vector when appropriate to simultaneously reduce the current error on all three phases.

Let \(\theta_0 = \Omega t_0\) be the phase of \(E\) at time \(t_0\). Given the constant control input, we may integrate (10.11) and (10.12) to give, assuming \(V_d\) is slowly varying

\[
\begin{align*}
x_a(t) &\approx x_a(t_0) + \frac{|E|}{\Omega L_S} (\sin(\Omega t) - \sin \theta_0) - I^*(\cos(\Omega t) - \cos \theta_0) - \frac{1}{L_S} d_a V_d(t - t_0) \\
x_b(t) &\approx x_b(t_0) - \frac{|E|}{\Omega L_S} (\cos(\Omega t) - \cos \theta_0) - I^*(\sin(\Omega t) - \sin \theta_0) - \frac{1}{L_S} d_b V_d(t - t_0).
\end{align*}
\]
We still do not know the values of \( d_a \) and \( d_b \). However, in order to obtain \( x_a(t_0 + \tau) = x_b(t_0 + \tau) = 0 \) for some arbitrary \( \tau > 0 \), we would need to set \( d_a \) and \( d_b \) such that

\[
\frac{1}{L_S} V_a d_a \tau = x_a(t_0) + \frac{|E|}{\Omega L_S} (\sin(\theta_0 + \Omega \tau) - \sin \theta_0) - I^*(\cos(\theta_0 + \Omega \tau) - \cos \theta_0)
\]

\[
= x_a(t_0) + f_a(\tau)
\]

(10.16)

\[
\frac{1}{L_S} V_b d_b \tau = x_b(t_0) - \frac{|E|}{\Omega L_S} (\cos(\theta_0 + \Omega \tau) - \cos \theta_0) - I^*(\sin(\theta_0 + \Omega \tau) - \sin \theta_0)
\]

\[
= x_b(t_0) + f_b(\tau).
\]

(10.17)

At the same time, we have the constraint

\[
d_a^2 + d_b^2 = d_{\text{max}}^2 = \frac{1}{8}.
\]

So, squaring the above equations and adding, we obtain

\[
\frac{1}{8} \left( \frac{V_a}{L_S} \right)^2 \tau^2 = (x_a(t_0) + f_a(\tau))^2 + (x_b(t_0) + f_b(\tau))^2.
\]

Now, a little algebra shows that

\[
f_a^2(\tau) + f_b^2(\tau) = 2 \left( \left( \frac{|E|}{\Omega L_S} \right)^2 + (I^*)^2 \right) (1 - \cos(\Omega \tau)).
\]

It follows then that \( \tau \) satisfies the equation

\[
F(\tau) = \frac{x_a^2(t_0) + x_b^2(t_0)}{2} + x_a(t_0) f_a(\tau) + x_b(t_0) f_b(\tau)
\]

\[
+ \left( \left( \frac{|E|}{\Omega L_S} \right)^2 + (I^*)^2 \right) (1 - \cos(\Omega \tau)) - \left( \frac{V_a}{4L_S} \right)^2 \tau^2 = 0.
\]

(10.18)

The Choi-Sul control at some time \( t_0 \) consists of finding \( \tau^* \), the smallest positive root of \( F(\tau) \), then setting \( d_a \) and \( d_b \) based on \( \tau^* \), using (10.16) and (10.17). The actual duty ratios are then found by inverse Park transform:

\[
\dot{d}_0 = \sqrt{2} d_a, \quad \dot{d}_1 = \sqrt{2} \left( -\frac{1}{2} d_a + \frac{\sqrt{3}}{2} d_b \right), \quad \dot{d}_2 = \sqrt{2} \left( -\frac{1}{2} d_a - \frac{\sqrt{3}}{2} d_b \right).
\]

(10.19)

In practice, the above controller is implemented in discrete-time, with the optimal control reassessed and updated at regular sample intervals \( T \). While not explicitly stated in [12], it is clear that \( \tau^* \) should be bounded below by \( T \) in order to prevent overshoots. Also, the minimum-time trajectory is not necessarily monotonic in \( I \), so that \( I \) may temporarily exceed the converter current rating. Choi and Sul acknowledge this, and provide a ‘current-bounding algorithm’ to avoid overloading. Their strategy is to calculate the prospective current at the next sample period, using a first-order
approximation to (10.2) with the optimal values of \( d_a \) and \( d_b \). If the net AC current 
\[ \sqrt{I_a^2(t_0 + T) + I_b^2(t_0 + T)} \]
exceeds a preset threshold \( I_{\text{max}} \), an alternative method is 
used to set \( d_a \) and \( d_b \). The values chosen are those which satisfy the simultaneous 
equations

\[
I_a^2(t_0 + T) + I_b^2(t_0 + T) = I_{\text{max}}^2 \\
\quad \quad d_a^2 + d_b^2 = d_{\text{max}}^2.
\]

Of the two solutions to this system, the one giving the lesser current error is selected.

Taken on its own, the Choi-Sul controller has one chief drawback. Like all 
minimum-time controllers operating in continuous time, this is a bang-bang control 
law (in the sense that \( d_a^2 + d_b^2 \) is always at its maximum value). In the steady-state 
this produces a ‘chattering’ control whose stability is not guaranteed, although Choi 
and Sul provide simulations and experiments in [12] which suggest stable operation. 
While chattering noise cannot be discerned in the simulation plots, it is certainly 
evident in the experimental plots.

A better strategy would be to use the Choi-Sul control only during well-defined 
transient events, in which the AC current error exceeds a preset threshold, and to 
revert to a feedback control such as PCFF when the error falls below the threshold. 
The question then arises, of course, whether the gain in performance is worth the 
added complexity posed by two controllers over one, particularly given the inherent 
complexity of solving transcendental equations such as (10.18). (This equation also 
appears from experiments to be vulnerable to numerical instability.) Experimental 
results in [12] suggest an approximate halving in response time over PI regulators 
with feedforward compensation.

### 10.1.4 Energy-Based Approach

An important recent development in nonlinear control is the passivity-based or energy-
shaping approach, the basic principles of which were set out by Takegaki and Arimoto 
[74] and developed by Ortega and others [57, 34]. The approach was applied initially 
to robot control, but has subsequently been extended by Ortega, Loria, Nicklasson and 
Sira-Ramírez [58] to general electromechanical systems described by Euler-Lagrange 
equations. Of the many electrical applications described in the literature, the most 
relevant in this context is to the switch-mode DC-DC boost converter [70, 58], which
can be thought of as a two-dimensional analogue of the VDC. We present the AC-DC version of this controller, after showing how PCFF control can be recast in energy terms.

**PCFF as an Energy-Shaping Control**

PCFF control, as elaborated by Blasko and Kaura, can be rederived from an elementary application of energy-based design principles. Consider the synchronous Park model (10.3), neglect source resistance, and let $I^*$ be the setpoint for the AC current as before. Assuming once again that our objective is to zero the quadrature component $I_\beta$, we define new state variables

$$I_e = I_a - I^*$$
$$V_e = V_a - V_R$$

in terms of which we obtain a model with the desired equilibrium at the origin of the state space:

$$L_S \dot{I}_e = |E| + \Omega L_S I_\beta - d_a(V_R + V_e)$$
$$L_S \dot{I}_\beta = -\Omega L_S (I^* + I_e) - d_\beta(V_R + V_e)$$
$$C V_e = 3d_a(I^* + I_e) + 3d_\beta I_\beta - f(V_R + V_e, t).$$

(10.20)

For this system we can define the ‘energy function’

$$J = \frac{3L_S}{2} I_e^2 + \frac{3L_S}{2} I_\beta^2 + \frac{C}{2} V_e^2 \dot{=} J_e + J_\beta + J_V.$$  

This is trivially a positive definite function of the state; if its derivative can be made negative definite, this will be sufficient to stabilise the system at the origin. The control variables at our disposal are the duty ratios $d_a$ and $d_\beta$, and also the current setpoint $I^*$. This gives us three degrees of freedom, with which we may independently ‘passivise’ the three energy components:

$$\dot{J}_e = -2\gamma_1 J_e, \quad \dot{J}_\beta = -2\gamma_2 J_\beta, \quad \dot{J}_V = -2\gamma_3 J_V$$

where the $\gamma_i$ are arbitrary positive real numbers. So for $J_e$, for example, we first observe that $\dot{J}_e = 3L_S I_e \dot{I}_e$, then substitute for $\dot{I}_e$ from (10.20) and equate the result to $-3\gamma_1 L_S I_e^2$. Since we are imposing an exponential shape on the energy functions, we may expect the result will be akin to feedback linearisation, although the latter is not the primary intent of our design.
Consider first the subsystem formed by the currents $I_\epsilon$ and $I_\beta$, treating $V_d = V_R + V_\epsilon$ as a slow variable and $I^*$ as an independently-chosen setpoint. The control law that achieves the desired exponential shaping of $J_\epsilon$ and $J_\beta$ through $d_\alpha$ and $d_\beta$ is

\[
d_\alpha = \frac{1}{V_d}(|E| + L_S(\gamma_1 I_\epsilon + \Omega I_\beta)) \tag{10.21}
\]
\[
d_\beta = \frac{1}{V_d}L_S(\gamma_2 I_\beta - \Omega(I^* + I_\epsilon)). \tag{10.22}
\]

This is identical to the Blasko-Kaura control (10.9)–(10.10) apart from omitting the integral action. The latter is included purely to compensate for steady-state error due to parameter variation, an issue not taken into account in our simple energy-shaping design.

When we consider the full system we obtain, in addition to (10.21)–(10.22), a formula for $I^*$ which assigns the required dynamics to the DC-side energy $J_V$. The equation to be satisfied, in terms of the measurable quantity $I_\alpha$, is

\[
3d_\alpha V_d I_\alpha + 3d_\beta V_d I_\beta - V_d I_d + C\gamma_3 V_d V_\epsilon = 0
\]

where we have multiplied through by $V_d = V_R + V_\epsilon$ for convenience. If we now assume that the current control operates instantaneously, so that $I_\alpha = I^*$, $I_\beta = 0$, and $d_\alpha V_d$ is equal to its steady-state value $|E|$, we obtain

\[
I^* = \frac{V_d}{3|E|}(I_d - \gamma_3 CV_\epsilon) \tag{10.23}
\]

which is (almost) the usual PCFF regulator, with proportional gain $\gamma_3 C$ and no integral action. Once again, the proportional action obtained here from energy-shaping considerations could be supplemented with integral action on $V_\epsilon$ if modelling and measurement errors shift the voltage equilibrium too far away from $V_R$.

(10.23) differs slightly from the usual PCFF voltage controller in that the scaling factor involves $V_d$ rather than $V_R$. The underlying reason is the fact that equations (10.21)–(10.22) for the duty ratios are scaled by the factor $1/V_d$, and this factor carries through to the voltage dynamics. If we include integral action in the voltage regulator and scale by $V_R$, the approximate closed-loop behaviour takes the form

\[
CV_\epsilon = -\frac{V_R}{V_R + V_\epsilon} \left(K_P + K_I \int V_\epsilon \right)
\]

Defining $x = V_\epsilon/V_R$, the above is equivalent to

\[
(1 + x)\ddot{x} + \dot{x}^2 + \frac{K_P}{C} \dot{x} + \frac{K_I}{C} x = 0 \tag{10.24}
\]
with a unique equilibrium at $x = 0$. By comparison, if we scale by $V_d$ and thereby cancel the nonlinear term, the approximate voltage dynamics are linear:

$$\ddot{x} + \frac{K_P}{C} \dot{x} + \frac{K_I}{C} x = 0.$$ 

In practice one cannot expect to exactly cancel a nonlinearity, and so (10.24) merits closer attention.

When $K_P = K_I = 0$, the equilibrium in both equations has a double zero eigenvalue; in other words it is a Takens-Bogdanov point. This is the most structurally unstable equilibrium that can occur in a second-order system with two parameters [38]; the parameter space includes families of saddle-node bifurcations (as $K_I$ passes through zero for nonzero $K_P$) and Hopf bifurcations (as $K_P$ passes through zero for nonzero $K_I$). The implication for (10.24) is that, while the origin is locally stable for all positive gain values, it may fail to be globally stable.

The Takens-Bogdanov point in (10.24) is degenerate (having vanishing Hessian coefficients), but numerical plots of the phase portrait confirm the absence of global stability. Figure 10.1 shows a ‘generic’ phase portrait with nonzero $K_P$ and $K_I$, while Figure 10.2 shows the phase portrait at the degenerate Hopf bifurcation with $K_P = 0$.

In the latter case the system becomes a nonlinear oscillator, thus failing to possess
an attractor at all. We conclude that the energy-shaping control law (10.23) is likely to have better stability properties than the PCFF law, which calculates a desired AC current without regard for the current value of DC voltage. Simulations show, however, that PCFF control in its (still quite large) region of stability has slightly superior performance to (10.23).

Having rederived a version of PCFF control, it appears highly unlikely that any further gains in performance can be made using this approach. Recall that the current control was derived on the assumption that $I^*$, which determines $I_c$, is slowly-varying. Hence, any attempt to control $V_c$ on a similar timescale to $I_c$ will jeopardise the current control. One might instead seek a controller which chooses $d_\alpha$ or $d_\beta$ on the basis of $J_V$, and makes $I_\alpha$ or $I_\beta$ the ‘slow’ variable. However, such attempts always lead to controllers that are degenerate (failing to produce two independent equations for $d_\alpha$ and $d_\beta$) or ill-conditioned (possessing singularities at the desired operating point). Stabilising controllers capable of outperforming PCFF control, if indeed such controllers exist, will require an entirely new approach.
The VDC as an Euler-Lagrange System

Ortega et al. [58] provide a generalised framework for the design of passivity-based controllers for mechanical and electrical systems. The control system is assumed to have a representation in the so-called Euler-Lagrange form

\[
\frac{d}{dt} \frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})}{\partial \mathbf{q}} = -\frac{\partial \mathcal{F}(\mathbf{u}, \dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} + \mathbf{M}\mathbf{u}
\]

(10.25)

where the ‘generalised coordinates’ \( \mathbf{q} \) correspond to state variables, \( \mathbf{u} \) is the control, \( \mathbf{M} \) is a real matrix, \( \mathcal{L} \) is the Lagrangian and \( \mathcal{F} \) the Rayleigh dissipation function. Lagrange’s equation (10.25) is the cornerstone of analytical mechanics (see for example [25]); the Lagrangian is usually expressed as

\[
\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{T}(\dot{\mathbf{q}}) - \mathcal{V}(\mathbf{q})
\]

where \( \mathcal{T} \) is a positive definite ‘kinetic energy’ and \( \mathcal{V} \) the ‘potential’. (The terminology is somewhat stretched in electrical systems, but \( \mathcal{T} \) and \( \mathcal{V} \) retain their meaning as energy functions.) The function \( \mathcal{F} \), representing external forces in a mechanical system, accounts for dissipative terms in the system equations. Together \( \mathcal{T}, \mathcal{V}, \mathcal{F} \) and \( \mathbf{M} \) fully characterise the system in question.

As one example of passivity-based controller design, the authors consider a DC-DC boost converter with resistive load [58, §II], a time-averaged model for which is

\[
\begin{align*}
L\dot{I} &= E - dV \\
C\dot{V} &= dI - \frac{V}{R}
\end{align*}
\]

(10.26)

where \( E \) and \( I \) are the source voltage and current respectively, \( d \in [0, 1] \) is the duty ratio, \( V \) is the output voltage and \( R \) the load resistance. To obtain the Euler-Lagrange representation, we define generalised coordinates \( q_1 \) and \( q_2 \) such that \( \dot{q}_1 = I \) and \( q_2 = CV \). Thus, \( q_1 \) and \( q_2 \) are the electric charges on the inductor and capacitor respectively. We then take \( \mathcal{T} \) as the stored inductive energy and \( \mathcal{V} \) as the capacitive energy, giving the Euler-Lagrange parameters

\[
\mathcal{T}(\dot{\mathbf{q}}) = \frac{L}{2}q_1^2, \quad \mathcal{V}(\mathbf{q}) = \frac{1}{2C}q_2^2, \quad \mathcal{F}(\mathbf{q}, \mathbf{u}) = \frac{R}{2}(dq_1 - \dot{q}_2)^2.
\]

(10.27)

It is observed that the system (10.26) is nonlinear and nonminimum phase with respect to the voltage \( V \), making control of \( V \) a challenging problem. It is then noted
that due to the overall power balance, it is possible to control \( V \) indirectly through \( I \). The control is chosen so as to preserve the Euler-Lagrange structure (10.27) but replace the generalised coordinates with ‘error’ coordinates \( I_e, V_e \) about a transient setpoint \((I_d, V_d)\). An extra dissipation term \( R_1 I_e^2 \) is added to \( F \) in the closed-loop system to ensure asymptotic stability, as there is no damping term corresponding to \( I \) in the original system. A controller that achieves the desired closed-loop dynamics is

\[
\begin{align*}
L \dot{I}_d &= E - dV_d + R_1 (I - I_d) \\
CV_d &= dI_d - \frac{V_d}{R}.
\end{align*}
\]

Finally, to ensure the desired steady state voltage \( V_R \), the controller variable \( I_d \) is fixed as \( V_R^2 / RE \), giving the final controller equations

\[
\begin{align*}
d &= \frac{1}{V_d} \left( E + R_1 \left( I - \frac{V_R^2}{RE} \right) \right) \\
CV_d &= \frac{V_R^2}{RE} d - \frac{1}{R} V_d.
\end{align*}
\]

The problem solved here by Ortega et al. is directly analogous to the VDC control problem with a resistive load; indeed, apart from an additional inductive variable the only new feature is a sinusoidal input. We may therefore carry out the design exactly as above, starting from the stationary Park model of the converter. (The synchronous model does not permit an Euler-Lagrange representation, owing to the cross-coupling.) With a resistive load, the model is

\[
\begin{align*}
L_S \dot{I}_a &= |E| \cos \Omega t - d_a V_d \\
L_S \dot{I}_b &= |E| \sin \Omega t - d_b V_d \\
CV_d &= 3d_a I_a + 3d_b I_b - \frac{1}{R} V_d.
\end{align*}
\]

Define the generalised coordinates by \( \dot{q}_1 = I_a, \dot{q}_2 = I_b \) and \( q_3 = CV_d \). We then have the Euler-Lagrange parameters

\[
\begin{align*}
T(\dot{q}) &= \frac{3L_S}{2} (\dot{q}_1^2 + \dot{q}_2^2), \quad V(q) = \frac{1}{2C} q_3^2, \quad F(q, u) = \frac{R}{2} (3d_a \dot{q}_1 + 3d_b \dot{q}_2 - \dot{q}_3)^2.
\end{align*}
\]

To this Euler-Lagrange structure with error variables in place of \( q \), we add dissipation terms \( R_1 I_{ae}^2 \) and \( R_2 I_{be}^2 \) to ensure asymptotic stability of the currents. We make one
further alteration: in case the natural dynamics of the load prove too slow, we add a
dissipation term \( V_c^2/R_3 \).

Let the transient setpoint be denoted \((I_a^*, I_b^*, V^*)\). The controller is then given
implicitly by the equations

\[
\begin{align*}
L_S I_a^* &= |E| \cos \Omega t - d_a V^* + R_1 (I_a - I_a^*) \\
L_S I_b^* &= |E| \sin \Omega t - d_b V^* + R_2 (I_b - I_b^*) \\
CV^* &= 3d_a I_a^* + 3d_b I_b^* - \frac{1}{R} V^* + \left( \frac{1}{R_3} - \frac{1}{R} \right) (V_d - V^*).
\end{align*}
\]

(10.31)

Finally, we fix \( I_a^* = I^* \cos \Omega t, I_b^* = I^* \sin \Omega t \), with \( I^* = V_R^2/3R|E| \), to give

\[
\begin{align*}
d_a &= \frac{1}{V^*} (|E| \cos \Omega t + R_1 I_a - I^*(R_1 \cos \Omega t - \Omega L_S \sin \Omega t)) \\
d_b &= \frac{1}{V^*} (|E| \sin \Omega t + R_2 I_b - I^*(R_2 \sin \Omega t + \Omega L_S \cos \Omega t)) \\
CV^* &= 3I^* (d_a \cos \Omega t + d_b \sin \Omega t) + \frac{1}{R_3} (V_d - V^*) - \frac{1}{R} V_d.
\end{align*}
\]

(10.32)

The controller we obtain is once again reminiscent of PCFF control; for the sake
of comparison, the PCFF current control in the stationary Park domain with \( \gamma_1 = \gamma_2 = \gamma \) is

\[
\begin{align*}
d_a &= \frac{1}{V_d} ((|E| - \gamma L_S I^*) \cos \Omega t + \gamma L_S I_a + \Omega L_S I_b) \\
d_b &= \frac{1}{V_d} ((|E| - \gamma L_S I^*) \sin \Omega t + \gamma L_S I_b - \Omega L_S I_a).
\end{align*}
\]

(10.33)

Interestingly, although the controller was designed for resistive loads only, it will also
work with more general load characteristics. For the general case, we reinterpret \( V_d/R \)
as \( I_d \), the DC current, which can be directly measured. We thereby also obviate the
need to estimate the load resistance. (Note that the introduction of the \( 1/R_3 \) term
eliminated the term \( V^*/R \) in the controller, making a resistance estimate unnecessary
here also.) While the controller in its original form is very sensitive to \( R \), the revised
form using \( I_d \) is quite robust.

In general, however, the voltage response time of the controller (10.32) is inferior
to that of PCFF control. One sees why by inspecting the dynamics of \( V^* \). When \( R_3 \)
is sufficiently large, \( V_d \) tracks \( V^* \) quite closely. The quantity \( d_a \cos \Omega t + d_b \sin \Omega t \) for
an approximately sinusoidal input current is the space vector magnitude \( D \), which is
roughly equal to \( |E|/V_d \). Upon substituting \( I^* = V_R I_d/3|E| \) we obtain

\[
CV_e \approx CV^* \approx \left( \frac{V_R}{V_d} - 1 \right) I_d = \frac{I_d}{V_d} V_e.
\]

234
Thus, if we define a notional resistance $R_d = V_d/I_d$, the DC voltage will converge exponentially at a rate given by the time constant $R_dC$. The greater the DC load the faster the convergence, (largely) irrespective of the load characteristic. (In general $R_d$ varies with voltage, making this only approximately true. The convergence is actually slightly faster for constant-power loads than for resistive or constant-current loads.)

10.1.5 Simulation Results

Simulations were undertaken to compare the various VDC control strategies. In each case the VDC was operated from a three-phase 50Hz AC supply with a phase voltage of 150V RMS and 1mH source inductance. The DC bus capacitance was chosen as 1000μF and the controller sampling frequency as 10kHz (giving $T = 100μs$). Each of the control strategies described above was tuned to approximate time-optimal performance, and a ‘benchmark’ simulation run in which a step change in DC output voltage from 600V to 500V concided with a step change in load from zero to 100A constant-current.

For the original PCFF control, the sole tuning parameters are the gains $K_P$ and $K_I$ in (10.8). After a number of trials, the gain $K_P$ was set equal to 1; the integral action was omitted entirely as it was found not to improve performance. The Blasko-Kaura and energy-based approaches to PCFF control yield the same controller structure, which can be likened to that of WDS if the $\alpha$ and $\beta$-axis gains are both set equal to $L_S/T$ and integral action omitted in the current regulators. When this is done, the observed response in our benchmark simulation is identical to that of the original PCFF control. Furthermore, this response could not be markedly improved either by adjusting the proportional gains or by introducing integral action.

For the passivity-based control of Section 10.1.4, the tunable parameters are the notional resistances $R_1$, $R_2$ and $R_3$. Again, this controller can be likened to the energy-based PCFF control if we set $R_1 = \gamma_1 L_S$, $R_2 = \gamma_2 L_S$ and $R_3 = 1/\gamma_3 C$. These values were used in our benchmark simulation; variation around these values gave no further performance improvement. Simulation of the Choi-Sul Hamiltonian controller in Section 10.1.3 was also attempted, but this controller failed to produce a stable steady state owing to numerical difficulties. The transient performance of this controller is very similar to that of the optimally tuned PCFF controllers, but the latter have the advantage of relative simplicity.

235
The benchmark performance of the PCFF controllers is contrasted with that of the passivity-based control in Figures 10.3 and 10.4. These results point to a trade-off between voltage and current settling time. While the current settling time for the passivity-based control is extremely short, the voltage settling time is hampered by the relatively slow convergence rate of the voltage reference signal $V^*$, shown in Figure 10.5. Theoretically, the time constant of this response will be $CV_d/I_d = 5\text{ms}$; for the exponential part of the response, the observed time constant is quite close to this value.

The passivity-based control, in the form stated here, is unfortunately not suitable for regenerative loads as the convergence of $V^*$ necessitates that $V_d/I_d$ be positive. PCFF control on the other hand will support negative output currents without modification, as is demonstrated by WDS in their original paper [78]. Figures 10.6 and 10.7 show the DC voltage and AC current responses for the same step change in voltage, with a step change in load current from $100\text{A}$ to $-100\text{A}$. Even after half an AC cycle, the requisite $180^\circ$ phase shift in AC currents has occurred. There is, however, a substantial overshoot both in the DC voltage and in the phase 2 current, as a consequence of the high gain settings. Reduced gains result in a lower overshoot,
Figure 10.4: AC current response for VDC synchronising controllers

Figure 10.5: Response of $V^*$ for passivity-based control
Figure 10.6: DC voltage response for PCFF with regenerative load

Figure 10.7: AC current response for PCFF with regenerative load
at the expense of settling time.

Simulations were also carried out to explore the effect of different load types. Figures 10.8 and 10.9 compare the DC voltage and phase 0 current responses under energy-based PCFF control for ohmic, constant-current and constant-power loads each with a nominal current (current drawn at rated voltage) equal to 100A. The results here are in line with intuition; for a given level of nominal load current, ohmic loads improve the controller performance slightly, while constant-power loads have the opposite effect. (Under regeneration conditions the reverse is in fact true, with constant-power loads giving the best performance. This effect is clarified theoretically in the following chapter.)

### 10.2 Control of Current-Driven Converters

Superficially, the synchronising control problem for the CDC (at least with sixfold CDC switching control) closely resembles that for the VDC. Indeed, if one looks merely at that part of the dynamics mediated by the converter itself, the equations are formally identical (apart from insignificant resistive terms). However, on closer
Figure 10.9: AC current response for PCFF with varying load types

investigation this is seen to be a very different control problem, one that is in some respects more difficult to solve and in some respects simpler.

In the CDC model, the equations into which the control variables enter directly are those relating AC voltage to DC current. For our purposes however, regarding the CDC as a mains supply source for a DC power system, these are not the variables we wish to control. Our control objectives relate to DC voltage (to be stabilised at some nominal value) and AC current (to be made sinusoidal with minimal magnitude); these variables are separated by an integrator relationship from the others, while the directly controlled variables are only of secondary importance.

The indirect nature of the DC voltage control fits nicely with our overall methodology. In the CDC synchronising control, we seek only to achieve a certain output voltage $V_S$ at the converter; the regulatory control may then be designed purely with reference to the Small Model, with $V_S$ as its control input. By contrast with the other DC-side variables, the output voltage is a direct function of the CDC space vector, so can be controlled on a fast timescale. This however requires rapid stabilisation of the AC variables, the main dynamical task of synchronising control.

As with the section on VDC control, we begin in Section 10.2.1 by formulating the
synchronising control problem, stating models and objectives. Techniques developed for the VDC are not directly applicable to this problem but nonetheless hint at a promising ‘quasilinear’ approach, which we describe in Section 10.2.2, and apply in Section 10.2.3 to the design of a relatively simple controller. Simulations presented in Section 10.2.4 verify that our control design achieves both robust stability and fast convergence to the desired equilibrium.

10.2.1 Problem Formulation

For the CDC, the model on which we base the control problem will depend on the choice of switching control. To keep things simple in the present work we opt for the sixfold CDC switching scheme (Section 8.3.1). While it involves cyclic disparities in switching frequency and greater high-frequency harmonics compared with the twelvefold scheme, it involves only fixed space vectors and thereby simplifies the model greatly. In particular, the matrix $\Gamma$ is zero and the switching-loss coefficient $\gamma$ equal to 2 for all states (see Section 7.6.2). The time-averaged GAS model (7.38) for the CDC, ignoring the small voltage drop due to $R_D$, is then

$$
L_s \dot{\mathbf{I}} = \mathbf{E} - R_s \mathbf{I} - \mathbf{V} \\
C_s \dot{\mathbf{V}} = \mathbf{I} - \bar{\alpha} I_d \\
L \dot{I}_d = \alpha^T \mathbf{V} - V_d \\
C \dot{V}_d = I_d - f(V_d, t).
$$

As noted above, an important auxiliary variable is the converter output voltage $V_o$:

$$
V_o = \bar{\alpha}^T \mathbf{V}.
$$

We assume that the input variables $\mathbf{I}$ and $\mathbf{V}$, and the output current $I_d$, are measurable. In order to properly select the equilibrium we also need to know the effective generated voltage $\mathbf{E}$, which cannot be measured locally. As suggested in Section 9.3.4, one can estimate the magnitude of $\mathbf{E}$ and its phase shift relative to $\mathbf{V}$ if the fault level at the input is known to sufficient accuracy. Otherwise, one may resort to the high-fault-level approximation $\mathbf{E} = \mathbf{V}$, which yields an adequate though possibly suboptimal controller. We shall design the controller assuming known values for the parameters $L_S$ and $C_S$, while noting that the controller we obtain should be
robust with regard to parameter uncertainty as well as measurement and modelling errors.

The control task, analogous to that for the VDC, is as follows. Given measurements of the system variables at times \( t \geq t_0 \), find an admissible control \( \alpha(t), t \geq t_0 \), which ensures that

1. \( V_o(t) \in [(1 - \epsilon_1)V_S, (1 + \epsilon_1)V_S] \) for \( t - t_0 \geq T_1 \), for some voltage command \( V_S \), tolerance \( \epsilon_1 \) and convergence time \( T_1 \);

2. \( I_k(t) \in [I_k^*(t) - \epsilon_2|\mathbf{I}^*|, I_k^*(t) + \epsilon_2|\mathbf{I}^*|] \) for \( t - t_0 \geq T_2 \), \( k \in \{0, 1, 2\} \), for some tolerance \( \epsilon_2 \) and convergence time \( T_2 \), where \( \mathbf{I}^* \) is a balanced three-phase signal having \( |\mathbf{I}^*| \) minimal with regard to the DC load requirements.

In practice, these objectives are supplemented by constraints on the maximum values of \( I_d \), \( I_k \) and \( V_k \), applicable at all times.

Again we choose the time origin so that \( \phi_0^E = 0 \), and apply the Park transform to simplify the analysis. We omit the equations for the DC-side dynamics as they are readily inferred from the expression for \( V_o \). In the stationary Park domain we obtain

\[
\begin{align*}
L_S \dot{I}_a &= |E| \cos(\Omega t) - R_S I_a - V_a \\
L_S \dot{I}_b &= |E| \sin(\Omega t) - R_S I_b - V_b \\
C_S \dot{V}_a &= I_a - \alpha_a I_d \\
C_S \dot{V}_b &= I_b - \alpha_b I_d \\
V_o &= 3\alpha_a V_a + 3\alpha_b V_b 
\end{align*}
\]  

with current reference

\[
\begin{align*}
I^*_a(t) &= |\mathbf{I}^*| \cos(\Omega t + \phi^*) \\
I^*_b(t) &= |\mathbf{I}^*| \sin(\Omega t + \phi^*).
\end{align*}
\]

Similarly, in the synchronous Park domain one obtains the time-invariant system

\[
\begin{align*}
L_S \dot{I}_a &= |E| - R_S I_a + \Omega L_S I_\beta - V_a \\
L_S \dot{I}_\beta &= -R_S I_\beta - \Omega L_S I_a - V_\beta \\
C_S \dot{V}_a &= I_a + \Omega C_S V_\beta - \alpha_a I_d \\
C_S \dot{V}_\beta &= I_\beta - \Omega C_S V_a - \alpha_\beta I_d \\
V_o &= 3\alpha_a V_a + 3\alpha_\beta V_\beta
\end{align*}
\]

242
with the constant current reference

\[ I_a^* = |\Gamma| \cos \phi^*, \quad I_\beta^* = |\Gamma| \sin \phi^* \]

and tolerance bounds as for the VDC.

### 10.2.2 Quasilinear State Feedback Design

For the control design we focus on the synchronous Park model (10.35). Once again, we assume the input current \( I \) is minimal when it is in phase with \( E \). Then \( I_\beta^* = 0 \) and

\[ I_a^* = |\Gamma| \approx \frac{V_S I_d}{3|E|} \quad (10.36) \]

assuming slowly-varying \( I_d \) and negligible \( R_S \). (If \( R_S \) is known, a more accurate setpoint is found by solving a quadratic equation similar to (10.7).) From these values and the equilibrium conditions of the model, we obtain corresponding setpoints for \( V_\alpha \) and \( V_\beta \):

\[ V_\alpha^* = |E| - R_S I_a^* \]
\[ V_\beta^* = -\Omega L_S I_a^*. \quad (10.37) \]

(If the approximation \( |E| = |V| \) is used, one should instead set \( V_\alpha^* = |V| \) and \( V_\beta^* = 0 \).) This in turn implies that the control inputs \( \alpha_a \) and \( \alpha_\beta \) must have the equilibrium values

\[ \alpha_a^* = (1 - \Omega^2 L_S C_S) \frac{I_a^*}{I_d} \]
\[ \alpha_\beta^* = -\Omega C_S \frac{|E| - R_S I_a^*}{I_d}. \quad (10.38) \]

As a consistency check, we may calculate

\[ 3\alpha_a^* V_\alpha^* + 3\alpha_\beta^* V_\beta^* = \frac{1}{I_d} (3|E|I_a^* - 3R_S(I_a^*)^2) \]

which ensures that \( V_o = V_S \) in the steady state as desired, just as long as \( I_a^* \) is set correctly according to the power balance.

Our task then is to design a controller which assures a sufficiently rapid convergence of the system (10.35) to the equilibrium defined by (10.36) and (10.37). This is still a nonlinear problem, as the current \( I_d \) is obviously affected by changes in the
input. For a guide to how to proceed, we might look back at the VDC designs and attempt to draw parallels. One promising approach, which we pursue here, lies in recognising that all the successful VDC current controllers are ‘quasilinear’. In other words, they treat the DC variable multiplying the input as a slowly-varying parameter rather than a plant variable, ignoring its dynamical interaction with the AC side until the time comes to wrap the DC-side control loop around the controlled AC system.

By analogy with this approach we shall tentatively proceed on the assumption that, as long as the AC-side dynamics are sufficiently fast, the feedback correlation between $I_d$ and $I$ can be ignored. This effectively reduces (10.35) to a linear time-invariant system, for which all manner of design tools are available. A further advantage, as will be seen, is that the controller tuning is sensitive only to the parameters $L_s, C_s$ and (to a lesser degree) $R_s$, which are essentially fixed for all time, unlike the DC load which can vary.

For a quasilinear design, we define error variables around our desired equilibrium as

$$I_\epsilon = I_\alpha - I^*_\alpha, \quad I_\delta = I_\beta - I^*_\beta, \quad V_\epsilon = V_\alpha - V^*_\alpha, \quad V_\delta = V_\beta - V^*_\beta$$

and set $x = [I_\epsilon, I_\delta, V_\epsilon, V_\delta]^T$. The ‘pseudo-LTI’ system equations are

$$\dot{x} = Ax + Bu + e$$

where $u = [\alpha_\alpha, \alpha_\beta]^T$ is the control input, and

$$A = \begin{bmatrix} -\frac{1}{L_s} & 0 & -\frac{1}{L_s} & 0 \\ -\frac{1}{L_s} & 0 & 0 & -\frac{1}{L_s} \\ \frac{1}{C_s} & 0 & 0 & \Omega \\ 0 & \frac{1}{C_s} & -\Omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{I_d}{C_s} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e = \begin{bmatrix} \frac{1}{L_s}(|\mathbf{E}| - R_s I^*_\alpha - V^*_\alpha) + \Omega I^*_\beta \\ -\frac{1}{L_s}(R_s I^*_\beta + V^*_\beta) - \Omega I^*_\alpha \\ \frac{1}{C_s} I^*_\alpha + \Omega V^*_\beta \\ \frac{1}{C_s} I^*_\beta - \Omega V^*_\alpha \end{bmatrix} = \frac{1}{C_s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ (1 - \sigma^2) I^*_\alpha \\ -\Omega C_s (|\mathbf{E}| - R_s I^*_\alpha) \end{bmatrix}.$$

As in Chapter 9 we define $\omega_s = 1/\sqrt{L_s C_s}$ as the natural frequency of the LC input configuration, and $\sigma = \Omega \sqrt{L_s C_s}$ as the ratio of $\Omega$ to $\omega_s$. 

244
Since \( \mathbf{I} \) and \( \mathbf{V} \) are measurable, we may carry out a simple state feedback pole-placement design. First, we transform the system (10.39) into its controllable canonical form by a change of variables \( \mathbf{x} = \mathbf{T}\mathbf{x} \). The computation of \( \mathbf{T} \) is a textbook problem; see [55, p.672ff] for an exhaustive treatment. The matrix that accomplishes the transformation in this case is

\[
\mathbf{T} = \omega_\delta^2 \mathbf{I}_d \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-R_S & -L_S & \Omega L_S & 0 \\
-\Omega L_S & 0 & -R_S & -L_S \\
\end{bmatrix}
\]

and the canonical variables may be calculated from the physical variables as

\[
\dot{\mathbf{x}} = \mathbf{T}^{-1}\mathbf{x} = \frac{C_S}{I_d} \begin{bmatrix}
L_S I_e \\
-V_e - R_S I_e + \Omega L_S I_\delta \\
L_S I_\delta \\
-V_\delta - R_S I_\delta - \Omega L_S I_e \\
\end{bmatrix}.
\]  
(10.40)

In canonical form, the system equations are

\[
\dot{\mathbf{x}} = \mathbf{\hat{A}} \mathbf{x} + \mathbf{B} \mathbf{u} + \mathbf{T}^{-1} \mathbf{e}
\]

where, setting \( \kappa = R_S/L_S \)

\[
\mathbf{\hat{A}} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-(\omega_\delta^2 - \Omega^2) & -\kappa & \kappa \Omega & 2\Omega \\
0 & 0 & \kappa \Omega & 1 \\
-\kappa \Omega & -2\Omega & -(\omega_\delta^2 - \Omega^2) & -\kappa \\
\end{bmatrix}
\]  
(10.41)

\[
\mathbf{\hat{B}} = \mathbf{T}^{-1} \mathbf{B} = \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
1 \\
\end{bmatrix}.
\]

From \( \mathbf{\hat{A}} \) one obtains the open-loop characteristic polynomial as

\[
P_0(s) = (s^2 + \kappa s + \omega_\delta^2 - \Omega^2)^2 + (2s + \kappa)^2 \Omega^2
\]

which with \( R_S = 0 \) (hence \( \kappa = 0 \)) factorises as

\[
P_0(s) = (s^2 + (\omega_S + \Omega)^2)(s^2 + (\omega_S - \Omega)^2)
\]

245
Thus, in the absence of feedback the CDC is a resonant system, with two pairs of poles on or near the imaginary axis. This resonant property, along with the double-integrator structure of the input-output map implicit in the model, make the controller design very difficult with anything other than full state feedback.

Now, let the control law have the form

$$ u = u_0 - F \dot{x} = \begin{bmatrix} \alpha_{a0} \\ \alpha_{b0} \end{bmatrix} - \begin{bmatrix} \zeta_{a\alpha} & \zeta_{a \beta} & \zeta_{a \alpha} & \zeta_{a \beta} \\ \zeta_{b \alpha} & \zeta_{b \beta} & \zeta_{b \alpha} & \zeta_{b \beta} \end{bmatrix} \dot{x} \tag{10.42} $$

giving the closed-loop system

$$ \dot{x} = (\hat{A} - \hat{B}F)\dot{x} + T^{-1}(Bu_0 + e) $$

with

$$ \hat{A} - \hat{B}F = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -\omega_S^2 - \Omega^2 & \zeta_{a \alpha} & -\kappa - \zeta_{a \alpha} & \zeta_{a \beta} & \zeta_{a \beta} \\ 0 & \zeta_{b \alpha} & 2\Omega - \zeta_{a \beta} & \zeta_{b \beta} & \zeta_{b \beta} \end{bmatrix} $$

The constant term $u_0$ in the control is included (formally) to compensate for the constant term $e$ in the plant equations and (physically) to ensure the correct value for $u$ at equilibrium. One easily verifies that setting $u_0 = [\alpha_{a}^* \alpha_{b}^*]^T$ from (10.38) eliminates the term $Bu_0 + e$, rendering the closed-loop dynamics homogeneous.

To simplify the equations we take $R_S = 0$ in the following analysis. This involves no real loss of generality, as the effect of a nonzero $R_S$ is the same as that of an additive perturbation in $\zeta_{a \alpha}$, $\zeta_{a \beta}$, $\zeta_{b \alpha}$ and $\zeta_{b \beta}$. From (10.43) with $\kappa = 0$, the characteristic polynomial is

$$ P(s) = s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 $$

with

$$ a_3 = \zeta_{a \alpha} + \zeta_{b \beta} $$

$$ a_2 = 2(\omega_S^2 + \Omega^2) + \zeta_{a \alpha} \zeta_{b \beta} + \zeta_{a \alpha} + \zeta_{b \beta} + 2(\zeta_{b \alpha} - \zeta_{a \beta})\Omega - \zeta_{b \alpha} \zeta_{a \beta} $$

$$ a_1 = (\zeta_{a \alpha} + \zeta_{b \beta})(\omega_S^2 + \Omega^2) + \zeta_{a \alpha} \zeta_{b \beta} + \zeta_{b \alpha} \zeta_{a \beta} + 2(\zeta_{b \alpha} - \zeta_{a \beta})\Omega - \zeta_{b \alpha} \zeta_{a \beta} - \zeta_{a \beta} \zeta_{a \beta} $$

$$ a_0 = (\omega_S^2 - \Omega^2)^2 + (\zeta_{a \alpha} + \zeta_{b \beta})(\omega_S^2 - \Omega^2) + \zeta_{a \alpha} \zeta_{b \beta} - \zeta_{b \alpha} \zeta_{a \beta}. $$

246
In principle, by choosing the elements of \( \mathbf{F} \) we may assign any values to these coefficients, and thereby place the closed-loop poles anywhere we like (with the usual conjugacy restriction). Furthermore, since there are eight tunable parameters and only four coefficients, we actually have four degrees of freedom in assigning the poles, in addition to the choice of the poles themselves.

Given all this design freedom, we need some idea of what constitutes a ‘nice’ controller. Our chief trade-off is between minimising the real part of the dominant closed-loop pole(s), and minimising the gains themselves. We could if we wished state these criteria in the form of a nonlinear optimisation problem over \( \mathbf{F} \) (the mapping from gains to dominant poles is continuous), but this approach is not fruitful in practice. In the following section we provide a ‘sufficient’ design based on engineering intuition and simple root-locus analysis.

### 10.2.3 A Two-Gain Controller

Our starting point is the observation that, of the eight gains in (10.42), the most important for shifting the poles leftward are \( \zeta_{aa} \) and \( \zeta_{\beta \beta} \). The coefficient \( a_3 \) in the characteristic polynomial is equal to the negative sum of the four poles; if their real part is not to exceed \( -\sigma \), then \( a_3 \) must be at least \( 4\sigma \). But from (10.44) \( a_3 \) is just \( \zeta_{aa} + \zeta_{\beta \beta} \), with a possible small contribution from \( \kappa \). Intuitively, we should hope to obtain a \( \sigma \) value of the same order of magnitude as \( \omega_S \); as a minimum, then, we must assign values of this order to \( \zeta_{aa} \) and/or \( \zeta_{\beta \beta} \).

Next, invoking Ockham’s Razor, we investigate whether a satisfactory design can be had in which \( \zeta_{aa} \) and \( \zeta_{\beta \beta} \) are the only nonzero gains. Our tool here is root-locus analysis. We note first that \( \zeta_{aa} \) and \( \zeta_{\beta \beta} \) enter the equations (10.44) either as the sum \( \zeta_{aa} + \zeta_{\beta \beta} \) or as the product \( \zeta_{aa} \zeta_{\beta \beta} \). This leads us to introduce the new gain variables \( K, \mu \) defined by

\[
\begin{align*}
\zeta_{aa} + \zeta_{\beta \beta} &= K, \\
\zeta_{aa} \zeta_{\beta \beta} &= \mu K.
\end{align*}
\]

In terms of these variables, the characteristic polynomial is

\[
P(s) = P_0(s) + K(s^3 + \mu s^2 + (\omega_S^2 - \Omega^2)s). \quad (10.46)
\]

In this form the roots of \( P(s) \) are amenable to study by standard root locus techniques.
Recall [56] that the root locus is a complex-plane plot of the roots of

\[ P(s) = P_0(s) + K P_1(s) \]  

(10.47)
as \( K \) increases from zero to infinity. Because the theory was developed in the context of a unity-feedback system with open-loop transfer function \( P_1/P_0 \), the roots of \( P_0 \) are referred to as the ‘open-loop poles’ and those of \( P_1 \) as the ‘open-loop zeros’. Nonetheless, the theory is applicable to any polynomial with a gain that enters affinely, even though the standard terminology may cease to be entirely meaningful.

From (10.46) we obtain a root locus with three open-loop zeros, one fixed at the origin and the others dependent on the value of \( \mu \). The critical value of \( \mu \) giving a double zero is

\[ \mu_0 = 2 \sqrt{\frac{\omega_N^2}{\Omega^2}}. \]

For \( \mu < \mu_0 \) we obtain two complex zeros, and for \( \mu \geq \mu_0 \) two real zeros. These zeros repel the root loci for low values of \( K \), and attract them for high values of \( K \).

Figures 10.10 and 10.11 depict the evolution of the root loci for several representative values of \( \mu \), with \( \omega_N = 3500 \) rad/s and \( \Omega = 100\pi \) rad/s. (For reasons of space
and clarity, only the top half of the root locus diagram is depicted; the bottom half is a simple mirror reflection.)

From these diagrams it is evident that \( \mu \) should be neither too small nor too large. Owing to the presence of the zero at the origin, performance will start to degrade as soon as any locus ‘breaks in’ to the real axis. For small values of \( \mu \), the loci describe ovals centred on the two complex zeros, and break-in occurs only after these loci have ‘collided’ at point \( C \). (Three such points, for varying \( \mu \), are shown in Figure 10.10).

The collision occurs (at gain \( K^* \), say) when \( P(s) \) has the form

\[
P(s) = (s^2 + bs + c)^2 = s^4 + 2bs^3 + (b^2 + 2c)s^2 + 2bcs + c^2.
\]

Equating coefficients with (10.46) we obtain \( b = K^*/2 \) and \( c = \omega_3^2 - \Omega^2 \), whereupon solving for the coefficient of \( s^2 \) gives

\[
K^* = 2(\mu + \sqrt{\mu^2 + 4\Omega^2}) \tag{10.48}
\]

or slightly more than \( 4\mu \). For \( \mu \) fixed, the minimal dominant poles occur at these same collision points, and have real part equal to \( -K^*/4 \). Evidently this improves (decreases) as \( \mu \) increases.
Eventually $\mu$ passes through a critical value $\mu^*$ where the ovals collide with the real axis. At $\mu = \mu^*$ (10.46) has a quadruple root for some $K$; equating coefficients between (10.46) and $(s + \sigma)^4$ we obtain $\sigma = \sqrt{\omega_S^2 - \Omega^2}$, $K = 4\sigma$ and

$$
\mu^* = \frac{\omega_S^2 - 2\Omega^2}{\sqrt{\omega_S^2 - \Omega^2}}. \quad (10.49)
$$

Regarding the overall topology of the root loci $\mu^*$ is of greater significance than $\mu_0$, which is why we have taken it as the transition point between Figures 10.10 and 10.11.

As $\mu$ increases beyond $\mu^*$ (Figure 10.11) the minimal dominant poles shift rapidly to the right, and so further improvement is not possible. Note that equation (10.48) remains valid, but the collision points are now break-in points on the real axis. The location of the minimal dominant poles is the larger (real) root of

$$
 s^2 + (\mu + \sqrt{\mu^2 + 4\Omega^2})s + (\omega_S^2 - \Omega^2) = 0.
$$

For design purposes it is helpful to know the value of $K$ at the break-in point $B$ for $\mu < \mu^*$. The breakpoints are the stationary points of $K(s) = -P_0(s)/P_1(s)$ for $s$ real, and are accordingly the real negative roots of the sixth-degree polynomial

$$
P_B(s) = P_0(s)P_1(s) - P_1'(s)P_0(s)
= s^6 + 2\mu s^5 + (\omega_S^2 - 5\Omega^2)s^4 - (\omega_S^2 - 5\Omega^2)\omega_0^2 s^2 - 2\mu \omega_0^4 s - \omega_0^6
$$

with $\omega_0 = \sqrt{\omega_S^2 - \Omega^2}$. This polynomial factorises into three quadratic factors as

$$
P_B(s) = (s^2 - \omega_0^2)(s^2 + b_1 s + \omega_0^2)(s^2 + b_2 s + \omega_0^2) \quad (10.50)
$$

where

$$
\omega_0 = \sqrt{\omega_S^2 - \Omega^2}
$$

$$
b_1 = \mu + \sqrt{\mu^2 + 4\Omega^2}
$$

$$
b_2 = \mu - \sqrt{\mu^2 + 4\Omega^2}.
$$

The following observations are readily checked, and provide analytical confirmation of our empirical inferences from Figures 10.10 and 10.11.

1. The first factor in (10.50) has the one fixed negative root $s = -\omega_0$, plus a positive root that we ignore. $-\omega_0$ is a break-in point for $\mu < \mu^*$ and a breakout point for $\mu^* < \mu < \mu_0$. For $\mu > \mu_0$, $-\omega_0$ is not in the root locus.

250
2. When \( \omega_s > \sqrt{3} \Omega \), the third factor in (10.50) never has real roots. Thus, the number of breakpoints (generically) is either one or three.

3. The transition from one to three breakpoints occurs at \( \mu = \mu^* \). For all \( \mu > \mu^* \) the roots of the second factor in (10.50) are break-in points, occurring at the gain \( K^* \) calculated earlier.

The value of the gain \( K^{**} \) at the point \( B = -\omega_0 \) may now be calculated as

\[
K^{**} = \frac{-P_0(-\omega_0)}{P_1(-\omega_0)} = \frac{4\omega_0^2}{\mu_0 - \mu}.
\] (10.51)

For our example with \( \omega_s = 3500 \) and \( \Omega = 100\pi \), \( \mu^* \) is around 3458; we accordingly choose \( \mu = 3000 \) to obtain reasonable performance while allowing some margin for error. From a robustness point of view, a sensible choice for \( K \) is a point on the locus between the collision point \( C_3 \) and the break-in point \( B \). In this neighbourhood the real parts of the poles are least sensitive to variations in \( K \). A suitable \( K \) is accordingly an intermediate value between \( K^* = 12130 \) (from (10.48)) and \( K^{**} = 12337 \) (from (10.51)).

Given \( \mu \) and \( K \), the design values for \( \zeta_{\alpha\alpha} \) and \( \zeta_{\beta\beta} \) are found by solving (10.45). To obtain a solution we must impose the constraint

\[
K \geq 4\mu
\]

which, by construction, is satisfied in our example. The gains are then found as \( \zeta_{\alpha\alpha} = \Delta K \), \( \zeta_{\beta\beta} = \mu/\Delta \) (or vice versa), where

\[
\Delta = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{4\mu}{K}} \right).
\] (10.52)

Either the positive or negative solution for \( \Delta \) may be used. Taking the positive solution and \( K = 12150 \) (say), we obtain \( \zeta_{\alpha\alpha} = 6667 \) and \( \zeta_{\beta\beta} = 5400 \). (Lest these values appear overly large, recall that in the controller they are ‘normalised’ by multiplying by \( C_s \).)

It is certainly possible to improve on this somewhat parsimonious design. A promising next step is to shift the third open-loop zero away from the origin by incorporating the pair \( (\xi_{\alpha\alpha}, \xi_{\beta\beta}) \) into the design. Nonetheless, as seen in the simulations to follow, the two-gain design already achieves impressive performance even in the presence of errors.
Figure 10.12: Voltage transient response of two-gain CDC synchronising control

10.2.4 Simulation Results

The two-gain quasilinear controller of Section 10.2.3 was simulated for a CDC with the same characteristics as in Chapter 8, except that the resistance $R_D$ was omitted and source resistance $R_S = 50\text{m}\Omega$ added. The synchronising control was implemented assuming zero source resistance, to investigate the magnitude of the resulting error. Because there is as yet no regulatory control, we place a resistive load on the DC bus to ensure adequate damping. The nominal DC load current is 72 amps, corresponding to a nominal AC current at unity power factor (and assuming no source resistance) of 50 amps.

Figure 10.12 plots the transient response of the converter output voltage $V_S$ under five separate test conditions:

1. The base case, with all parameters other than resistance known exactly.

2. Base case with zero source resistance.

3. Error of 100% in the value of $L_S$ used in the controller design.

4. Error of 100% in the value of $C_S$ used in the controller design.
5. High-fault-level approximation $|E| = |V|$ used in the controller design. In this case we make $V_c$ identically zero, and set $V_\beta^* = 0$.

Under each of these conditions apart from (2), we observe steady-state error of roughly 5% due to the effect of source resistance. Slightly greater error results when $L_S$ or $C_S$ are poorly estimated, but the effect is not pronounced. From (5) we observe that the high-fault-level approximation does not significantly impair the voltage performance, apart from a small amount of chattering which will be filtered by the DC bus circuit. Note that for this example, the input power is roughly 7% of the AC fault level.
Chapter 11

Regulatory Control of Converter-Fed DC Networks

This chapter is devoted to the top-level dynamic control problem for DC distribution networks. In essence, the task of regulatory control is to drive the DC network to a specified steady state on a sufficiently rapid time scale, and ensure the robust stability of that state. Control action is via the system’s external circuits, specifically those which possess ‘dynamic discretion’ (as opposed to ‘steady-state discretion’, the topic of Chapters 3 through 6). The canonical example of an external circuit with dynamic discretion is a converter attached to a supply source, though load modules may also possess dynamic discretion.

In Section 11.1 we formulate the regulatory control problem in full generality. Much of the remainder of this chapter is devoted to the simplest instance of this problem, namely a single AC-DC converter with an uncontrolled load. In this instance, the managed DC network is functionally identical to a conventional power electronic system. Accordingly, much of the work of this chapter is directly applicable to these systems also.

In the single-supply, single-load case, the power circuit from a regulatory control perspective reduces to a form of the Small Model. Section 11.2 studies the natural dynamics of the Small Model with a general nonlinear load, and deduces the presence of Hopf bifurcations, with implications for open-loop stability. After stating general bifurcation theorems, the important example of a constant-power load is discussed in some detail. Sections 11.3 through 11.5 then look at the regulatory control design for the Small Model, analysing it as a special case of a more general ‘LTIBU regulator’
Having dealt satisfactorily with the control problem in the simplest instance, we return in Section 11.6 to the general case, and suggest approaches to the control design with multiple control points. Finally, Section 11.7 presents some simulation results to verify the operation of the proposed controllers.

11.1 Problem Formulation

11.1.1 The General Problem

Large Model

To grasp the general problem we return to the Large Model of Section 7.2. This has the form

\[ \dot{x} = \Phi(t)x + \phi_0(x, t) + \Gamma w \]

where the components of the state \( x \) are the node voltages \( v \), the cable currents \( i \) and the currents \( \iota \) in current-driven external circuits. The control action enters through the external circuit characteristics

\[
\begin{align*}
    f(v, t) &= I_0(t) + \sigma_v G(t)v + F_0(v, t) \\
    h(\iota, t) &= V_0(t) + \sigma_{i} P(t)\iota + H_0(\iota, t).
\end{align*}
\]

where the \( N + E \) independent terms \( I_0, V_0 \) make up the exogenous input \( w \). We consider here only the case where the control enters linearly, so that

\[ w = w_0 + B_0 u \]

where \( w_0 \) represents uncontrollable sources and \( u \in \mathbb{R}^M \) the control variables. The general control system model is

\[ \dot{x} = \Phi(t)x + \phi_0(x, t) + Bu(t) + \Gamma w_0(t) \]  \hspace{1cm} (11.1)

where \( B = \Gamma B_0 \).

Dynamic Discretion

An external circuit corresponding to a nonzero row in \( B_0 \) is said to have *dynamic discretion*. This should not be confused with discretion in the sense of ‘discretionary
map’, which is a steady-state concept only. For voltage-driven circuits, dynamic
discretion takes the form of a current setpoint, which becomes equal to the usual
discretionary current in the steady state. For current-driven circuits, on the other
hand, dynamic discretion appears as a voltage setpoint, and its value at equilibrium
does not determine the steady-state discretionary current.

Electrical circuits cannot supply or absorb infinite energy, and so there will be
some intrinsic limits on the voltage and current setpoints, analogous to the steady-
state discretionary ratings \( D \). Accordingly, with each component \( u_k \) of \( u \) we associate
bounds \( u_k^- \) and \( u_k^+ \), such that \([u_k^-, u_k^+]\) is the operating region permitted for \( u_k \). For
example, if \( u_k \) represents the output \( V_S \) of a current-driven switch-mode AC–DC
converter, we have \( u^- = -3V_{AC}/\sqrt{2} \), \( u^+ = 3V_{AC}/\sqrt{2} \) where \( V_{AC} \) is the RMS AC-side
voltage. A control \( u \) is feasible if \( u_k^- \leq u \leq u_k^+ \) for all \( k \).

**Admissible Equilibria and Problem Statement**

An equilibrium \( \tilde{x} = \begin{bmatrix} \tilde{v}^T & \tilde{i}^T & \tilde{v}^T \end{bmatrix}^T \) of (11.1) is admissible if

1. \((1 - \epsilon)V_R \leq \tilde{v}_k \leq (1 + \epsilon)V_R \) for \( 1 \leq k \leq N \), with \( V_R \) and \( \epsilon \) given, and

2. There exists \( r \in \mathbb{R}^N \) such that \( \tilde{i} = S(r, -d_0(r)) \) and \( E\tilde{i} + f(\tilde{v}, t) = r - d(r) \).

Here \( d : \mathbb{R}^N \rightarrow \mathbb{R}^N \) is the discretionary map and \( S \) the sensitivity matrix.

Broadly speaking, \( \tilde{x} \) is an admissible equilibrium if the voltage profile is acceptable
and the currents are compatible with some arbitrary demand profile \( r \).

Given an arbitrary state \( x \in \mathbb{R}^{N+C+E} \), a nominal voltage \( V_R \) and a nominal
‘current rating’ \( I_R \), define a normalised state \( \hat{x} \in \mathbb{R}^{N+C+E} \) by \( \hat{v} = v/V_R \), \( \hat{i} = i/I_R \)
and \( \hat{r} = r/I_R \). We let \( B_\delta(\hat{x}) \) denote the closed ball of radius \( \delta \) centred on \( \hat{x} \). \( x \) is said
to be near equilibrium with tolerance \( \delta \) if \( x \in B_\delta(\hat{x}) \) for some admissible equilibrium
\( \hat{x} \).

Our general regulatory control problem may now be stated as follows: given

- the system (11.1),
- an initial state \( x_0 \) at time \( T_0 \),
- bounds \( u^- \) and \( u^+ \) on the control, and
- constants \( V_R, I_R, \epsilon, \delta \) and \( T_1 \);
find a feasible control $u(t), \ t \geq T_0$ such that $x(t)$ is near equilibrium with tolerance $\delta$ for $t \geq T_0 + T_1$. Note that this entails a maximum steady-state relative voltage error equal to $\epsilon + \delta$.

### 11.1.2 Simplifications and Special Cases

#### The Single-Node Model

If the cable network impedances in (11.1) are sufficiently small, we may treat the cable network as a single node with potential $V$. Defining $\tilde{x} = \begin{bmatrix} V & \iota^T \end{bmatrix}^T$ as the reduced state, we have

$$\dot{\tilde{x}} = \tilde{\Phi}(t) \tilde{x} + \tilde{\phi}_0(\tilde{x}, t) + \tilde{\Gamma} \tilde{w}$$  \hspace{1cm} (11.2)

where

$$\tilde{\Phi}(t) = \begin{bmatrix} -G/C & -C^{-1}e \\ L^{-1}e^T & -L^{-1}P \end{bmatrix} \quad \tilde{\phi}_0(\tilde{x}, t) = \begin{bmatrix} -(\sigma/C)\tilde{F}_0(V, t) \\ -L^{-1}\sigma_c H_0(\iota, t) \end{bmatrix}$$

$$\tilde{\Gamma} = \begin{bmatrix} -\sigma/C & 0 \\ 0 & -L^{-1}\sigma_c \end{bmatrix} \quad \tilde{w} = \begin{bmatrix} \sigma \sum_{i=1}^N \sigma_{vi} J_{0i}(t) \\ V_0^T(t) \end{bmatrix}$$

$$C = \text{tr} C \quad G = \text{tr} G \quad \tilde{F}_0(V, t) = \sigma \sum_{i=1}^N \sigma_{vi} F_{0i}(V, t).$$

Here there is at most one control input corresponding to voltage-driven external circuits, whose sign $\sigma$ may be chosen arbitrarily. Other inputs correspond to current-driven circuits. $e$ is the row vector formed from the diagonal elements of $\sigma_c$.

While the inductances $L$ in (11.2) are in general different from one another, we can nonetheless represent (11.2) in dimensionless LC units, using the capacitance $C$ and an arbitrary inductance $L_0$ for normalisation. Because the voltage variable in LC units is translated, we expand the voltage-driven characteristic $f$ around the nominal voltage $V_R$, giving (in LC units)

$$f(v, \tau) = i_0(\tau) + \sigma g(\tau)v + f_0(v, \tau)$$  \hspace{1cm} (11.3)

where (with $t = \tau / \omega$)

$$i_0 = \frac{Z_C}{V_R} (I_0(t) + \sigma V_R G(t) + F_0(V_R, t)) \quad g = Z_C (G(t) + \sigma D_1 F_0(V_R, t)).$$

258
Reusing \( \nu \) to denote its LC equivalent (with some notational abuse) and \( \rho \) as the LC equivalent of \( P \), the scaled equations are

\[
\begin{align*}
v' &= -g v - e \nu - \sigma f_0(v, \tau) - \sigma i_0 \\
\nu' &= L_0 L^{-1} (e^T v - \rho \nu - \sigma_e h_0(\nu, \tau) - \sigma_e v_0).
\end{align*}
\]

Note that in the second equation \( e^T v - \sigma_e v_0 \) is equivalent to \( \sigma_e (v_1 - v_0) \).

**The Small Model**

In many cases the single-node model (11.2) can be further simplified, at least approximately. Recalling Section 7.2 once again, we know that if \( H_0 \equiv 0 \) and \( L^{-1} P = \rho I \) for some \( \rho \) (including zero), then the currents \( \nu \) can also be combined into a single state variable, with a corresponding input variable equal to a signed sum over \( V_0 \). In this case we obtain a model with just two state variables, \( V \) and \( I \), and the equations

\[
\dot{x} = \begin{bmatrix} -G/C & -\sigma_I/C \\ \sigma_I/L & -R/L \end{bmatrix} x + \begin{bmatrix} -(\sigma_V/C) F_0(V, t) \\ -(\sigma_I/L) H_0(I, t) \end{bmatrix} w
\]

\[
= \begin{bmatrix} -\sigma_V/C & 0 \\ 0 & -\sigma_I/L \end{bmatrix} w
\]

(11.4)

where \( x = \begin{bmatrix} V & I \end{bmatrix}^T \), \( w = \begin{bmatrix} I_0 & V_0 \end{bmatrix}^T \), and \( R = \rho L \). These equations are also valid in the case where there is only one current-driven external circuit, and for this reason we have retained the nonlinear term \( H_0(I, t) \). In LC units, with the components of \( f \) redefined as in (11.3), the model is

\[
\begin{bmatrix} v' \\ i' \end{bmatrix} = \begin{bmatrix} -g & -\sigma_I \\ \sigma_I & -r \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} -\sigma_V f_0(v, \tau) \\ -\sigma_I h_0(i, \tau) \end{bmatrix} + \begin{bmatrix} -\sigma_V & 0 \\ 0 & -\sigma_I \end{bmatrix} \begin{bmatrix} i_0 \\ v_0 \end{bmatrix}.
\]

(11.5)

It is convenient to restate this model in 'standard form', with state variables \( x_1 = v \) and \( x_2 = v' \). This entails a redefinition of the components of \( h \), analogously to those of \( f \) above, as the inductor current \( i \) is no longer zero at \( x = 0 \). Indeed, in terms of the state \( x \) we have

\[
i(x) = -\sigma_I x_2 - \sigma_I \sigma_V f(x_1, \tau)
\]

so that \( i(0) = -\sigma_I \sigma_V i_0(\tau) \). (Note that in the case where a current-driven source supplies a voltage-driven load, or vice versa, we have \( -\sigma_I \sigma_V = 1 \).) Denoting the characteristic in (11.5) by

\[
h(i, \tau) = \tilde{v}_0(\tau) + \sigma_I \tilde{r}(\tau) i + \tilde{h}_0(i, \tau)
\]

259
we define
\[ h(i(\mathbf{x}), \tau) = v_0(\tau) + \sigma_I r(\tau)(i(\mathbf{x}) - i(0)) + h_0(i(\mathbf{x}), \tau) \quad (11.6) \]

where
\[
\begin{align*}
v_0(\tau) &= h(i(0), \tau) \\
&= \tilde{v}_0(\tau) - \sigma_V \tilde{r}(\tau) i_0(\tau) + \tilde{h}_0(-\sigma_I \sigma_V i_0(\tau), \tau) \\
r(\tau) &= \sigma_V D_1 h(i(0), \tau) \\
&= \tilde{r}(\tau) + \sigma_I D_1 \tilde{h}_0(-\sigma_I \sigma_V i_0(\tau), \tau)
\end{align*}
\]

thereby ensuring that \( h_0(i(0), \tau) \equiv 0 \) and \( h'_0(i(0), \tau) \equiv 0 \) (hence \( \nabla_x h_0(i(0), \tau) \equiv 0 \) by the chain rule).

With these redefinitions, we obtain the standard-form model
\[
\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -(1 + \mathcal{R} \cdot g(\tau)) & -(r(\tau) + g(\tau)) \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \phi_0(\mathbf{x}, \tau) \end{bmatrix} + \begin{bmatrix} 0 \\ w(\tau) \end{bmatrix} \quad (11.7)
\]

where
\[
\phi_0(\mathbf{x}, \tau) = h_0(i(\mathbf{x}), \tau) - \sigma_V (D_1 f_0(x_1, \tau)) x_2 - \sigma_V \mathcal{R} \cdot f_0(x_1, \tau) \\
w(\tau) = v_0(\tau) - \sigma_V i'_0(\tau) \\
i(\mathbf{x}) = -\sigma_I x_2 - \sigma_I \sigma_V f(x_1, \tau).
\]

For brevity, we have defined the ‘voltage-drop operator’
\[
\mathcal{R} = r(\tau) + \frac{\partial}{\partial \tau}
\]

which vanishes on constant signals when \( r \equiv 0 \).

It is important to note that the system (11.7) is completely described by the tuple \((f(v, \tau), h(i, \tau), \sigma_V, \sigma_I)\), or equivalently by the tuple
\[(i_0(\tau), g(\tau), f_0(v, \tau), v_0(\tau), r(\tau), h_0(i, \tau), \sigma_V, \sigma_I).\]

These are uniquely determined from the original system (11.4) with the characteristic functions \( F(V, t) \) and \( H(I, t) \), via the circuit parameters \( L, C \) and \( V_R \). To recap: the functions \( f \) and \( h \) are simple LC transforms of \( F \) and \( H \), while the derived parameters \((i_0, g, f_0, v_0, r, h_0)\) are found by expanding \( f \) and \( h \) to second order about \( v = 0 \) and \( i = -\sigma_I \sigma_V f(0) \), respectively.
When \( r \equiv h_0 \equiv 0 \), we recover the Small Model of Chapter 7, with the source voltage \( v_0 \) as control variable and the generalised ‘load’ characteristic \( f(v, \tau) \). While this model was developed initially for the case where a current-driven source feeds a voltage-driven load, our general control framework allows for the possibility that \( f \) includes the output of one or more voltage-driven AC–DC or DC–DC converters, providing an additional control variable.

### 11.2 Bifurcation Phenomena in LC Circuits with Nonlinear Loads

The model (11.7) is quite general in form; it describes the dynamics of any electrical circuit in which the only significant energy-storage elements are an inductor and capacitor arranged in the two-port configuration of Figure 7.2. In this section we seek a qualitative understanding of the dynamics of (11.7) when the external-circuit parameters \( i_0, g, f_0, v_0, r, h_0 \) are constant (or slowly-varying) in time. Because the case \( r = 0 \) is applicable to many real-world situations and simplifies the equations, we single it out for special attention.

In Section 11.2.1 we derive ‘normal forms’ for the system (11.7) in the cases \( r = 0 \) and \( r \neq 0 \). The two cases prove to be formally similar provided \( rg > -1 \). We then show in Section 11.2.2 that the qualitative dynamics of the case \( r = 0 \) boils down to the analysis of Hopf bifurcations, while that of the case \( r \neq 0 \) involves a Takens-Bogdanov point when \( r \) and \( g \) are permitted to vary freely over \( \mathbb{R} \). Section 11.2.3 carries out a detailed analysis of the Hopf bifurcations and provides the key results. Finally, in Section 11.2.4 we apply these results to the most important nonlinear phenomenon in DC power systems, the constant-power load. We conclude that in virtually all realistic scenarios involving constant-power loads, the voltage dynamics undergoes a subcritical Hopf bifurcation as the load power approaches a certain critical value.

#### 11.2.1 Normal Form of the LC Circuit Equations

If \( r = 0 \) then the system has the form

\[
\dot{x}' = \begin{bmatrix} 0 & 1 \\ -1 & -g \end{bmatrix} x + \begin{bmatrix} 0 \\ \phi_0(x) \end{bmatrix} + \begin{bmatrix} 0 \\ v_0 \end{bmatrix}
\]

(11.8)
with $\phi_0 = h_0(i(\mathbf{x})) - \sigma_V f'_0(x_1)x_2$. This system has an equilibrium at $\mathbf{x} = \begin{bmatrix} v^* & 0 \end{bmatrix}^T$ where

$$v^* - h_0(-\sigma_I \sigma_V f(v^*)) = v_0.$$ 

This equilibrium is unique provided

$$-\sigma_I \sigma_V h'_0(-\sigma_I \sigma_V f(v)) \cdot f'(v) < 1$$

for all voltages $v$. Since the nominal voltage $V_R$ used in the definition of the LC units is arbitrary, we may without loss of generality assume $V_R = V_0$, in which case $v_0 = 0$ and hence $v^* = 0$ by definition of $h_0$. System (11.8), omitting the constant term, then provides an explicit linearisation at the equilibrium $\mathbf{x} = \mathbf{0}$.

If $r \neq 0$ and $rg > -1$, define the quantity

$$\gamma = \frac{r + g}{\sqrt{1 + rg}}.$$

Then we may use a linear change of variables and a time scaling

$$y_1 = \left(\sqrt{1 + rg}\right)x_1 + \gamma x_2 \quad y_2 = x_2 \quad \tau_1 = \left(\sqrt{1 + rg}\right)\tau$$

to bring (11.7) into the form

$$\mathbf{y}' = \begin{bmatrix} -\gamma & 1 \\ -1 & 0 \end{bmatrix} \mathbf{y} + \frac{1}{\sqrt{1 + rg}} \begin{bmatrix} \gamma \\ 1 \end{bmatrix} (\phi_0(\mathbf{y}) + v_0) \quad (11.9)$$

with

$$\phi_0(\mathbf{y}) = h_0(i(\mathbf{x}(\mathbf{y}))) - \sigma_V (y_2D_1 + r)f_0(x_1(\mathbf{y}))$$

$$\mathbf{x}(\mathbf{y}) = \begin{bmatrix} \frac{y_1 - \gamma y_2}{\sqrt{1 + rg}} \\ y_2 \end{bmatrix}^T.$$

The equilibrium of this system is $\mathbf{y} = \begin{bmatrix} (\sqrt{1 + rg})v^* & 0 \end{bmatrix}^T$ where

$$(1 + rg)v^* + \sigma_V r f_0(v^*) - h_0(-\sigma_I \sigma_V f(v^*)) = v_0.$$ 

Again, by placing suitable bounds on the derivatives of $f_0$ and $h_0$ we may ensure that $v^*$ is unique. By appropriate choice of $V_R$ we may assume $v_0 = 0$ without loss of generality, and thence $v^* = 0$ by definition of $f_0$ and $h_0$. (11.8), like (11.8) with $r = 0$, then expresses the linearisation of the Small Model (11.7) around its equilibrium.

If either $r$ or $g$ are negative and sufficiently large, then the conditions on $f_0$ and $h_0$ for a unique equilibrium become more stringent as $rg$ approaches $-1$. In the case
\( rg \leq -1 \) (which may occur under extreme conditions in realistic scenarios; see Section 11.2.4) the system (11.7) cannot be brought into the form (11.9). Depending on the specific form of \( f_0 \) and \( h_0 \), it may also cease to have a unique equilibrium. Scenarios in which \( rg \lesssim -1 \) must be analysed on a case-by-case basis, although some general features are discussed in the next section.

### 11.2.2 Bifurcations

A *bifurcation* in a parameter-dependent system is defined by Kuznetsov [38] as “the appearance of a topologically nonequivalent phase portrait under variation of parameters.” In general the parameter space of a dynamical system may be partitioned into domains of structural stability, with the boundaries corresponding to certain kinds of bifurcation. Bifurcations may be either local, involving equilibria, or global, involving properties of entire trajectories. The local variety are relatively easy to detect, occurring just when an equilibrium point of the system becomes nonhyperbolic (marginally stable).

In both (11.8) and (11.9) one recognises a canonical form for one such local bifurcation, the Hopf bifurcation [38, Ch.3]. Indeed, in each case there is a unique equilibrium at the origin, and a single parameter \((g \text{ or } \gamma)\) which determines the eigenvalues at that equilibrium. The two eigenvalues are located on the imaginary axis, at points \( \pm j \), if and only if this parameter is zero.

In a generic Hopf bifurcation, a stable equilibrium becomes unstable as the parameter passes through the critical value where the eigenvalues cross the imaginary axis. In addition, one of the following phenomena occurs:

1. a stable limit cycle bifurcates from the critical equilibrium, and grows outward as the equilibrium becomes increasingly unstable (*supercritical Hopf bifurcation*); or

2. an unstable limit cycle bifurcates from the critical equilibrium, and grows outward as the equilibrium becomes increasingly stable (*subcritical Hopf bifurcation*).

While similar in description, the two possibilities have very different implications for overall system stability, the former implying global stability for all parameter values (albeit with steady-state oscillations), and the latter implying a shrinking domain
of attraction that vanishes at the critical parameter value. The supercritical and
subcritical cases are distinguished by examining the quadratic and cubic nonlinearities
in the dynamics.

Provided the equilibrium is unique, the Hopf bifurcation is the only type of lo-
cal bifurcation that can occur in systems (11.8) and (11.9). Also, the only global
bifurcations that can occur are those involving the limit cycles that arise from Hopf
bifurcations. Homoclinic and heteroclinic orbit bifurcations cannot occur, as such bi-
furcations in a two-dimensional system imply the existence of more than one equilib-
rium. We can therefore obtain a complete qualitative picture of the system dynamics
by analysing the Hopf bifurcations, and the multipliers of the limit cycles arising from
these bifurcations. (The latter is by far the more difficult and is not attempted here.)

If on the other hand we consider (11.7) with arbitrarily large negative values of $r$
and/or $g$, the picture is more complicated. In particular, if $r = 1$ and $g = -1$ (or vice
versa), the linearised dynamics has a double zero eigenvalue. Any equilibrium (we no
longer assume uniqueness) having these values for $r$ and $g$ yields a Takens-Bogdanov
point for the system.

A generic Takens-Bogdanov point is the intersection of bifurcation curves in pa-
parameter space corresponding to saddle-node bifurcations, Hopf bifurcations and ho-
mclinic orbit bifurcations. In order for this to be true it must, however, satisfy
certain nondegeneracy conditions specified, for example, in [38, §8.4]. Checking these
conditions requires detailed knowledge of the system, including the number and location
of equilibria and the structure of nonlinearities. For this reason we do not seek
general results for the case $r g \lesssim -1$, returning to it only in the context of a specific
example in Section 11.2.4. We may nonetheless deduce the existence of the following
critical phenomena:

- Saddle-node bifurcations (real eigenvalue passing through zero) when $g = -1/r$, $|r| \neq 1$.

- Hopf bifurcations (those of the system (11.9)) when $g = -r$, $|r| < 1$.

- Neutral saddles (real eigenvalues of same magnitude but opposite sign) when
  $g = -r$, $|r| > 1$.  

264
11.2.3 Analysis of the Hopf Bifurcation

To analyse the bifurcations of our LC circuit model, we make the circuit characteristics \( f(v, \mu) \) and \( h(i, \mu) \) depend on a parameter \( \mu \in \mathbb{R} \). The normalisation to LC units is chosen so that \( x = 0 \) is an equilibrium for all \( \mu \) in a domain of interest. In general, this requires that the normalisation itself be dependent on \( \mu \).

As noted above, for either of the systems (11.8) or (11.9), the origin is an equilibrium if \( v_0 = 0 \). Recall that \( v_0 \) is the constant term in the Taylor expansion of \( h \) around \( i(0) = -\sigma_I \sigma_V f(0, \mu) \). If \( F(V, \mu) \) and \( H(I, \mu) \) are the circuit characteristics in the original system (11.4), the condition \( v_0 = 0 \) implies

\[
H(-\sigma_I \sigma_V F(V_R, \mu), \mu) = V_R. \tag{11.10}
\]

It turns out that any equilibrium voltage for the original system (11.4) must satisfy (11.10).

Suppose (11.10) is satisfied for some pair \( (V_R^*, \mu^*) \); it then determines \( V_R \) as an implicit function of \( \mu \) in a neighbourhood of \( (V_R^*, \mu^*) \) provided

\[
-\sigma_I \sigma_V D_1 F(V_R^*, \mu^*) D_1 H(-\sigma_I \sigma_V F(V_R^*, \mu^*), \mu^*) \neq 1. \tag{11.11}
\]

It follows that there exists an open domain \( D \) containing \( \mu^* \) such that \( V_R(\mu) \) is well-defined by (11.10) for \( \mu \in D \). This gives a \( \mu \)-dependent transformation to LC units, such that when the transformed system is put in the form (11.7) with state variables \( x_1 = v, x_2 = v' \) and induced characteristics \( f(x_1, \mu) \) and \( h(i(x), \mu) \), the latter satisfies \( h(i(0), \mu) = 0 \) for all \( \mu \in D \).

To determine the type of the Hopf bifurcations, we make use of a formula of Guckenheimer and Holmes [28, p.152]. Let \( \dot{x} = F(x, \mu) \) be a parameter-dependent second-order dynamical system, which at \( \mu = \mu_0 \) has the form

\[
\dot{x} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} x + \begin{bmatrix} \xi(x) \\ \eta(x) \end{bmatrix} \tag{11.12}
\]

with \( \xi(0) \equiv \eta(0) \equiv \xi'(0) \equiv \eta'(0) \equiv 0 \). This system has an equilibrium at the origin with a pair of imaginary eigenvalues \( \pm j\omega \). Set

\[
a = \frac{1}{16}(\xi_{111} + \xi_{122} + \eta_{112} + \eta_{222})
+ \frac{1}{16\omega}(\xi_{12}(\xi_{11} + \xi_{22}) - \eta_{12}(\eta_{11} + \eta_{22}) - \xi_{11}\eta_{11} + \xi_{22}\eta_{22}) \tag{11.13}
\]

265
where $\xi_{11}$, $\eta_{12}$ and so on represent second and third derivatives of $\xi$ and $\eta$ with respect to $x_1$ and $x_2$, evaluated at $x = 0$.

Let $\kappa(\mu)$ denote the real part of the equilibrium eigenvalues as a function of $\mu$, in a neighbourhood of $\mu_0$. A nondegenerate Hopf bifurcation occurs if $\kappa'(\mu_0) \neq 0$ (the transversality condition) and $a \neq 0$. Furthermore, the limit cycles bifurcating from the equilibrium at $\mu = \mu_0$ are stable if $a < 0$ (supercritical case) and unstable if $a > 0$ (subcritical case).

Now, our canonical forms (11.8) and (11.9) differ from the reference system (11.12) in the sign of the linear terms. However, the transformation $(x_1, x_2) \mapsto (x_1, -x_2)$ shows that the formula (11.13) is formally correct for negative as well as positive $\omega$.

**Case 1: $r = 0$**

We assume first that $r \equiv \sigma_I D_1 h(i, \mu) \equiv 0$ at equilibrium, reducing the system to the normal form (11.8). Letting $\mu_0$ denote any value of $\mu \in D$ for which $g(\mu) = 0$, we make the assumption $g'(\mu_0) \neq 0$, thereby ensuring the transversality condition is satisfied. We then have $\omega = -1$, $\xi \equiv 0$ and

$$\eta(x) = \phi_0(x, \mu_0) = h_0(i(x), \mu_0) - \sigma_V(D_1 f_0(x_1, \mu_0)) x_2.$$ 

The algebra required in differentiating $\eta(x)$ is simplified by noting that the only nonzero derivatives of $i(x)$ are

$$D_1^k i(x_1, x_2) = -\sigma_I \sigma_V D_1^k f(x_1, \mu_0)$$

$$D_2 i(x_1, x_2) = -\sigma_I.$$ 

Evaluating the derivatives at $x_1 = x_2 = 0$, we note that $h_0'(i(0), \mu_0) = 0$ by definition, that $D_1 f(0, \mu_0) = \sigma_V g(\mu_0) = 0$, and that all higher derivatives of $f$ are identical to those of $f_0$, again by definition. Applying all these facts we arrive, after a long but routine calculation, at

$$a = -\frac{1}{16} \left( \sigma_V D_1^3 f_0(0, \mu_0) + \sigma_I D_1^3 h_0(-\sigma_I \sigma_V i_0, \mu_0) \right). \quad (11.14)$$

Thus, the supercritical or subcritical nature of the Hopf bifurcations in this system depends purely on the relative values of the third derivatives of $f_0$ and $h_0$ (equivalently, $f$ and $h$) at the bifurcation point.

This result provides the following theorem.
Theorem 11.1 Let an LC circuit with arbitrary parameter \( \mu \in \mathbb{R} \) comprise the parallel combination of three one-port elements at one node with voltage \( V \):

- a capacitor \( C \);
- a current-driven circuit, with inductance \( L \), sign \( \sigma_I \) and characteristic \( H(I, \mu) \) of class \( C^3 \); and
- a voltage-driven circuit, with sign \( \sigma_V \) and characteristic \( F(V, \mu) \) of class \( C^3 \).

Let \( D \subset \mathbb{R} \) be a domain on which the relation

\[
H(-\sigma_I \sigma_V F(V_R, \mu), \mu) - V_R = 0
\]

defines an implicit function \( V_R(\mu) \neq 0 \), and let \( h(i, \mu) \) and \( f(v, \mu) \) denote the representation of \( H \) and \( F \) in LC units with parameters \( L, C \) and \( V_R(\mu) \). Suppose that for all \( \mu \in D \)

\[
\mathcal{D}_i h(-\sigma_I \sigma_V f(0, \mu), \mu) \equiv 0.
\]

Define

\[
\begin{align*}
\kappa(\mu) &= \sigma_V \mathcal{D}_i f(0, \mu) \\
\alpha(\mu) &= \sigma_V \mathcal{D}_i^3 f(0, \mu) + \sigma_I \mathcal{D}_i^3 h(-\sigma_I \sigma_V f(0, \mu), \mu).
\end{align*}
\]

Then:

1. For all \( \mu \in D \), the voltage dynamics possesses an equilibrium at \( V = V_R(\mu) \).
   The equilibrium is stable if \( \kappa(\mu) > 0 \) and unstable if \( \kappa(\mu) < 0 \).

2. At values of \( \mu \) for which \( \kappa(\mu) = 0 \) and \( \kappa'(\mu) \neq 0 \), the equilibrium at \( V = V_R(\mu) \) undergoes a Hopf bifurcation. This is supercritical if \( \alpha(\mu) > 0 \), subcritical if \( \alpha(\mu) < 0 \) and degenerate if \( \alpha(\mu) = 0 \).

Case 2: \( r \neq 0, \; rg > -1 \)

In the general case, provided \( r(\mu)g(\mu) > -1 \) at the equilibrium \( x = 0 \) we obtain the normal form (11.9). However, the condition \( rg > -1 \) at equilibrium is equivalent to the ‘less than’ part of the necessary condition (11.11) for a well-defined equilibrium voltage \( V_R(\mu) \). It follows that if \( D \) is the domain of definition for \( V_R(\mu) \), the condition
\( rg > -1 \) is automatically satisfied for \( \mu \in D \) provided it is satisfied for some \( \mu \in D \) and \( r(\mu), g(\mu) \) are continuous.

Again, we take \( \mu_0 \) to be any value of \( \mu \) giving \( \gamma(\mu) = 0 \) and \( \gamma'(\mu) \neq 0 \); obvious candidates for \( \mu \) include \( \mu = g \) with \( \mu_0 = -r \), or \( \mu = r \) with \( \mu_0 = -g \). When \( \gamma = 0 \) in (11.9) we obtain \( \omega = -1, \xi \equiv 0 \) and

\[
\eta(y) = \frac{1}{\sqrt{1 + rg}} (h_0(i(x(y)), \mu_0) - \sigma_V D_1 f_0(x_1(y), \mu_0) y_2 - \sigma_V r f_0(x_1(y), \mu_0)).
\]

Because the mapping \( x(y) \) with \( \gamma = 0 \) is a simple scaling \( (x_1 = y_1/\sqrt{1 + rg}, x_2 = y_2) \), the calculation of the second and third derivatives of \( \eta \) goes through almost exactly as in Case 1, with two modifications:

1. Every derivative with respect to \( y_1 \) introduces an additional multiplicative factor \( 1/\sqrt{1 + rg} \).

2. Derivatives of the form \( D_k^r \eta(y) \) include an extra term \( -\sigma_V r D_k f_0 \), which is not present when \( r = 0 \). This does not affect derivatives involving \( y_2 \).

The main differences appear when evaluating these derivatives at \( y = 0 \); this is because \( D_1 f(0, \mu_0) = \sigma_V g(\mu_0) \) is no longer zero as it was in Case 1. However, the fact that \( r(\mu_0) + g(\mu_0) = 0 \) leads to cancellation of several terms in the formula for \( a \). The final result is

\[
a = -\frac{1}{16(1 + rg)^{3/2}} \left( \sigma_V D_3^3 f_0(0, \mu_0) + \sigma_I D_3^3 h_0(-\sigma_I \sigma_V i_0, \mu_0) \right.
\]

\[
- \left. \frac{1}{1 + rg} \left( r(D_1^2 f_0(0, \mu_0))^2 + g(D_1^2 h_0(-\sigma_I \sigma_V i_0, \mu_0))^2 \right) \right). \tag{11.15}
\]

In addition to the third derivatives of \( f \) and \( h \), (11.15) involves the squares of the second derivatives, multiplied by \( r \) and \( g \) respectively. As one would expect, this formula reduces to (11.14) when we make \( r = 0 \) (hence also \( g = 0 \)).

The generalisation of Theorem 11.1 to nonzero \( r \) follows.

**Theorem 11.2** Let an LC circuit with arbitrary parameter \( \mu \in \mathbb{R} \) comprise the parallel combination of three one-port elements at one node with voltage \( V \):

- a capacitor \( C \);

- a current-driven circuit, with inductance \( L \), sign \( \sigma_I \) and characteristic \( H(I, \mu) \) of class \( C^3 \); and

268
Let $D \subset \mathbb{R}$ be a domain on which the relation

$$H(-\sigma_I \sigma_V F(V_R, \mu), \mu) - V_R = 0$$

defines an implicit function $V_R(\mu) \neq 0$, and let $h(i, \mu)$ and $f(v, \mu)$ denote the representation of $H$ and $F$ in LC units with parameters $L$, $C$ and $V_R(\mu)$. Define

$$g(\mu) = \sigma_V D_1 f(0, \mu)$$
$$r(\mu) = \sigma_I D_1 h(-\sigma_I \sigma_V f(0, \mu), \mu)$$
$$\kappa(\mu) = g(\mu) + r(\mu)$$
$$\alpha(\mu) = \sigma_V D_1^3 f(0, \mu) + \sigma_I D_1^3 h(-\sigma_I \sigma_V f(0, \mu), \mu)$$
$$\frac{1}{1 + r(\mu)g(\mu)} \left( r(\mu)(D_1^2 f(0, \mu))^2 + g(\mu)(D_1 h(-\sigma_I \sigma_V f(0, \mu), \mu))^2 \right)$$

and suppose there exists some $\mu^* \in D$ with $1 + r(\mu^*)g(\mu^*) > 0$. Then:

1. For all $\mu \in D$, the voltage dynamics possesses an equilibrium at $V = V_R(\mu)$, at which $1 + r(\mu)g(\mu) > 0$. The equilibrium is stable if $\kappa(\mu) > 0$ and unstable if $\kappa(\mu) < 0$.

2. At values of $\mu$ for which $\kappa(\mu) = 0$ and $\kappa'(\mu) \neq 0$, the equilibrium at $V = V_R(\mu)$ undergoes a Hopf bifurcation. This is supercritical if $\alpha(\mu) > 0$, subcritical if $\alpha(\mu) < 0$ and degenerate if $\alpha(\mu) = 0$.

11.2.4 Example: the Constant-Power Load

Ideal Constant-Power Elements

The ‘ideal’ constant-power load is a circuit element having sign $\sigma = 1$ and the characteristic relation $VI = P$, where $P > 0$ is the power level parameter. It may be regarded either as a voltage-driven circuit with $F(V, P) = P/V$, or as a current-driven circuit, in series with an inductor and with $H(I, P) = P/I$. In LC units, the characteristics are $f(v, p) = p/(1 + v)$ and $h(i, p) = p/i - 1$ respectively, with $p = P/P_0$, $P_0 = V_R^2/Z$.

The constant-power load is the most basic nonlinear element in behavioural models of power systems in general and DC power systems in particular. In a DC context,
any switch-mode DC–DC or DC–AC converter that regulates its output voltage and current will display a constant-power characteristic. Constant-power loads are important also because their incremental resistance is negative, leading to dynamic instability.

If we reverse the sign of the constant-power load, or set $P$ negative, we obtain a constant-power source. The negative-$P$ case is appropriate for modelling regenerative drives. The case with $\sigma = -1$ is rarer; it would correspond, for example, to a switch-mode converter attached to a stand-alone power source which regulated the supply-side voltage and current but not the bus-side voltage and current. It is unlikely that a circuit intended to supply energy to the system would ever be operated in such a manner (as opposed to a regenerating drive, which supplies energy only incidentally).

**Real Constant-Power Elements**

When undertaking global analyses of systems with constant-power loads, the ideal characteristic needs to be modified. For the sake of discussion, consider the voltage-driven characteristic $F(V, P) = P/V$. This characteristic possesses a singularity at $V = 0$, in the vicinity of which the current becomes arbitrarily large; a real constant-power load, on the other hand, would be expected to cease operating during an undervoltage condition. The real-world behaviour can be more closely approximated, and the singularity avoided, by means of the following ‘real’ characteristic:

$$F(V, P) = \exp \left( - \left( \frac{V_C}{V} \right)^4 \right) \frac{P}{V}. \quad (11.16)$$

This is a function defined for all $V \in \mathbb{R}$, of class $C^\infty$. It has the value zero at $V = 0$, while for sufficiently large $V$ (as determined by $V_C$) it approximates the ideal characteristic $P/V$. Indeed, if we set

$$V_C = V_R \left( \log \frac{1}{1 - \epsilon} \right)^{1/4},$$

then the relative error between (11.16) and $P/V$ is no greater than $\epsilon$ for all $V \geq V_R$.

Figure 11.1 depicts the general shape of $F(V, P)$ for $V \geq 0$; for negative $V$ we have $F(V, P) = -F(-V, P)$. Notice that the function is relatively ‘flat’ in a neighbourhood of zero, rising rapidly in the vicinity of $V_C$ before converging on the ideal characteristic. The use of the fourth power is purely arbitrary; higher powers give a sharper transition from zero to the ideal characteristic, but the exponent should be even to ensure $F(V, P)$ behaves properly for negative $V$.  

270
Figure 11.1: ‘Real’ constant-power characteristic $F(V, P)$

The maximum of $F$ with respect to $V$ occurs at $V_m = \sqrt{2}V_C$, and its value is $F(V_m, P) = e^{-1/4}P/V_m \approx 0.7788P/V_m$. Based on these observations a convenient choice for $V_C$ is $V_R/(2\sqrt{2})$, in which case $V_m = V_R/2$ and the maximum current is about 1.55 times the nominal current $P/V_R$. The relative error from the ideal characteristic at $V_R$ is $\epsilon \approx 1.55\%$.

In LC units, the characteristic (11.16) becomes

$$f(v, p) = \exp \left( - \left( \frac{\nu}{1 + v} \right)^4 \right) \frac{p}{1 + v}$$

where $\nu = V_C/V_R$. The corresponding characteristics for current-driven elements are

$$H(I, P) = \exp \left( - \left( \frac{I_C}{I} \right)^4 \right) \frac{P}{I}$$

$$h(i, p) = \exp \left( - \left( \frac{\nu}{i} \right)^4 \right) \frac{p}{i} - 1$$

with $\nu = ZI_C/V_R$. Again, a convenient choice is $\nu = 1/2\sqrt{2}$.

The use of these ‘real’ characteristics yields more meaningful results for global system behaviour. As will be seen below, for local phenomena such as bifurcations of an equilibrium near $V_R$, use of the ideal characteristic may be more appropriate.
In the following paragraphs we explore the dynamics of some typical LC circuit configurations with constant-power loads, which provide simple examples of the application of Theorems 11.1 and 11.2.

**Pure Voltage Source with Constant-Power Load**

The simplest LC configuration with a constant-power load has $H(I, \mu) = V_S$ (a pure voltage source with series inductance), $F(V, \mu) = \Psi(V)\mu/V$, $\sigma_I = -1$, $\sigma_V = 1$. Here, as in all subsequent examples, we have taken the parameter $\mu$ as the power level $P$. For brevity, we have defined

$$
\Psi(V) = \exp \left( - \left( \frac{V_C}{V} \right)^4 \right)
$$

where $V_C$ is assumed sufficiently small to make $\Psi(V) \approx 1$ for $V \sim V_S$.

Since $H$ is constant, the equation (11.10) defining $V_R$ is trivial, giving $V_R = V_S$ for all $\mu \in \mathbb{R}$. Accordingly, the transformation to LC units is independent of $\mu$ and gives

$$
f(v, \mu) = \psi(v) \frac{\mu}{P_0} \frac{1}{1 + v} \quad h(i, \mu) = 0
$$

where $\psi(v)$ denotes the LC equivalent of $\Psi(V)$. Since $\mathcal{D}_1 h$ is identically zero we use Theorem 1, obtaining

$$
\kappa(\mu) = \mathcal{D}_1 f(0, \mu) = -\frac{\mu}{P_0} e^{\nu^4} (1 - 4\nu^4) \approx -\frac{\mu}{P_0}
$$

$$
\alpha(\mu) = \mathcal{D}_3 f(0, \mu) = -6 \frac{\mu}{P_0} e^{\nu^4} \left( 1 - 34\nu^4 + 48\nu^8 - \frac{32}{3} \nu^{12} \right) \approx -6 \frac{\mu}{P_0}.
$$

One notices here that higher derivatives of $f$ grow increasingly sensitive to deviations from the ideal constant-power characteristic. If $\nu = 1/2\sqrt{2}$, for example, the quantity $34\nu^4$ is actually quite significant, though still less than 1. We should not read too much into this, however, as the factor $\Psi$ is simply a mathematical fiction introduced to suppress the singularity while retaining smoothness of $f$. This suggests that it is actually more appropriate, when examining local behaviour, to ignore $\Psi$ and just use the ideal characteristic. This approach will be taken in all subsequent examples.

The conclusion in the case of a pure voltage source with a constant-power load is that for all power levels $P \in \mathbb{R}$ there is a unique equilibrium at the source voltage $V_S$. The equilibrium is stable if and only if $P$ is negative (that is, the load is regenerating). When $P = 0$ the equilibrium is nonhyperbolic and would give rise to a Hopf
bifurcation, but since \( f \) is identically zero we have merely a linear oscillator. Even if we failed to notice this simple fact we would find \( \alpha(0) = 0 \), implying a degeneracy. Thus, in this simplest of cases we obtain no useful information about the existence of limit cycles.

**Pure Voltage Source with Generalised Load**

Generic loads on DC power systems are usually regarded as combinations of three characteristics: ohmic, constant-current, and constant-power [24, 52]. This results in a voltage-driven characteristic of the form

\[
F(V, G, J, P) = J + GV + \Psi(V)\frac{P}{V}
\]

with three parameters \( G \geq 0 \), \( J \), \( P \) representing the ohmic, constant-current and constant-power components respectively. For the bifurcation analysis we take \( G, J \) as constant and set \( \mu = P \).

Again we take \( H = V_S \), giving a constant and unique equilibrium for all \( \mu \) at \( V_R = V_S \). Transforming to LC units we obtain \( h = 0 \) and

\[
f(v, \mu) = j + g(1 + v) + \psi(v)\frac{\mu}{P_0} \frac{1}{1 + v}.
\]

Disregarding \( \psi \) and applying Theorem 1 we obtain

\[
\kappa(\mu) = g - \frac{\mu}{P_0},
\]

\[
\alpha(\mu) = -6 \frac{\mu}{P_0}.
\]

In this case we discover that the equilibrium is stable if and only if \( p < g \), equivalently if

\[
P < GV_S^2.
\]

This is equivalent to the nominal current \( P/V_S \) drawn by constant-power loads being less than the nominal current \( GV_S \) drawn by ohmic loads. The constant-current component of the load has no effect on stability. When \( p = g \), then provided \( g > 0 \) we observe a subcritical Hopf bifurcation, implying a shrinking domain of attraction as \( P \) approaches its critical value. If \( g = 0 \), we obtain the degenerate case already studied.

Our analysis confirms the well-known fact that constant-power loads make power systems unstable (although the converse applies for regenerating loads), and also
Figure 11.2: Limit cycle behaviour in systems with ‘real’ constant-power loads reveals a worse fact: even when the equilibrium is locally stable it can fail to be globally stable. Since there are no other equilibria in this system, any trajectory that is not attracted to the origin must go to infinity, or to a stable limit cycle arising from a global bifurcation. Given that the energy input to the system is finite, a trajectory can only go to infinity in two ways: \(|V| \to \infty, I \to 0\) (impossible, as a large \(|V|\) minimises the effect of the nonlinearity and renders the system passive), or \(|I| \to \infty, V \to 0\). The latter implies \(|\dot{V}| \to \infty\)—a genuine ‘voltage collapse’ event. However, it too cannot occur if the system is passive in a neighbourhood of \(V = 0\), as is the case with the ‘real’ constant-power characteristic.

We conclude that in systems with real constant-power loads with sufficiently large \(P\), the dynamics must produce a stable limit cycle. This stable cycle must, further, be associated with a bifurcation of the unstable limit cycle arising from the subcritical Hopf bifurcation. The picture we obtain is that of Figure 11.2. For large \(P\) there is a stable limit cycle, which contracts as \(P\) decreases. In the subcritical region there is also an unstable limit cycle, which expands as \(P\) decreases. At some value \(P^\ast < GV^2_R\), the stable and unstable cycles collide and disappear in a ‘saddle-node bifurcation of cycles’ (see [38, §5.3]).

We can observe this global behaviour by plotting phase portraits. Figure 11.3 depicts a phase portrait with \(g = 1, j = 0\) and \(p = 0.9\), using the ‘real’ constant-power characteristic with \(\nu = 1/2\sqrt{2}\). Both the unstable and stable limit cycles are plainly evident. Figure 11.4 illustrates the same system after the Hopf bifurcation; all trajectories are now attracted toward the stable limit cycle. It is in the production of phase portraits such as these that the realistic yet smooth characteristic (11.16) really proves its worth.

For the sake of comparison, Figure 11.5 depicts the phase portrait for the system
\[ x' = y \]
\[ y' = -x - (g - p \exp(-1/(64 (1 + x)^4)(1 + x)^2)) y \]

Figure 11.3: Phase portrait with resistive and subcritical constant-power load

\[ x' = y \]
\[ y' = -x - (g - p \exp(-1/(64 (1 + x)^4)(1 + x)^2)) y \]

Figure 11.4: Phase portrait with resistive and supercritical constant-power load
Figure 11.5: Phase portrait with resistive and ideal constant-power load

with parameters as in Figure 11.3 but using the ideal constant-power characteristic. This characteristic fails to be passive in a neighbourhood of \( V = 0 \), and thus admits the voltage collapse seen in the unbounded trajectories at the bottom of the figure. As \( P \) is reduced the unstable limit cycle does not bifurcate, but persists all the way down to \( P = 0 \).

**Resistive Voltage Source with Generalised Load**

We now introduce a nontrivial \( H \), setting \( H(I, \mu) = V_S - RI \) \((R > 0)\) to investigate the effect of source resistance. \( F(V, \mu) \) is kept the same as in our previous example. The relation defining \( V_R \) is now

\[
H(F(V_R, \mu), \mu) - V_R = V_S - R \left( J + GV_R + \Psi(V_R) \frac{\mu}{V_R} \right) - V_R = 0.
\]

This is a transcendental equation, owing to the presence of \( \Psi(V_R) \). It is helpful to write it in the form

\[
-(1 + RG)V_R(V_R - V_\sigma) = \mu R \Psi(V_R)
\]

where

\[
V_\sigma = \frac{V_S - RJ}{1 + RG}
\]
Figure 11.6: Graphical determination of equilibrium when $R > 0$

is the equilibrium voltage when $\mu = 0$. Writing the equation in this form suggests
the graphical solution of Figure 11.6. This figure suggests that under some fairly
general conditions, there is just one solution for $V_R$, aside from the fictitious solution
$V_R = 0$. To keep matters concrete, we replace $\Psi$ with a step function that changes
from 0 to 1 at the critical voltage $V_C$, and further assume that $V_C/2 < V_C < V_\sigma$.
(This corresponds to a ‘hard shutoff’ of constant-power loads at $V = V_C$.) We then
obtain the unique equilibrium

$$ V_R(\mu) = \begin{cases} \frac{1}{2} \left( V_\sigma + \sqrt{V_\sigma^2 - \frac{4R}{1+RG} \mu} \right) & \mu \leq \mu^* \\ V_C & \mu \geq \mu^* \end{cases} $$

where

$$ \mu^* = \left( \frac{1}{R} + G \right) V_C (V_\sigma - V_C). $$

For $\mu \geq \mu^*$ the equilibrium is located on the discontinuity in the load characteristic,
where ‘chattering’ behaviour may be expected. Accordingly we restrict the bifurcation
analysis to the domain $D = (-\infty, \mu^*)$ where the equations are smooth. Observe that
$V_R(\mu)$ is a decreasing function on $D$.  

277
Transforming to LC units, we obtain

\[
f(v, \mu) = \frac{Z}{V_R(\mu)} F((1 + v)V_R(\mu), \mu) = \frac{ZJ}{V_R(\mu)} + ZG(1 + v) + \frac{Z\mu}{V_R^2(\mu)} \frac{1}{1 + v}
\]

\[
g(i, \mu) = \frac{1}{V_R(\mu)} \left( V_S - R \frac{V_R(\mu)}{Z} i \right) - 1 = \frac{V_S}{V_R(\mu)} - 1 - \frac{R}{Z} i.
\]

and applying Theorem 11.2 calculate

\[
g(\mu) = D_1 f(0, \mu) = ZG - \frac{Z\mu}{V_R^2(\mu)}
\]

\[
r(\mu) = -D_1 h(f(0, \mu), \mu) = \frac{R}{Z}
\]

\[
\kappa(\mu) = r(\mu) + g(\mu)
\]

\[
1 + r(\mu)g(\mu) = 1 + RG - \frac{R\mu}{V_R^2(\mu)}
\]

\[
\alpha(\mu) = D_1^2 f(0, \mu) - \frac{r}{1 + rg}(D_1 f(0, \mu))^2
\]

\[
= -\frac{6Z\mu}{V_R^2(\mu)} \left( 1 + \frac{2}{3} \frac{R\mu}{(1 + RG)V_R^2(\mu) - R\mu} \right).
\]

Observing that \(1 + rg > 0\) when \(\mu = 0 \in D\), we infer that \(1 + rg > 0\) for all \(\mu \in D\). \(\kappa(\mu)\) is a decreasing function of \(\mu\), and hence the equilibrium at \(V_R(\mu)\) is stable for \(\mu \in D\) if and only if \(\mu < \mu_0\), where \(\mu_0 > 0\) is defined by

\[
\mu_0 = \left( G + \frac{RC}{L} \right) V_R^2(\mu_0).
\]

(11.18)

The equilibrium is stable for all \(\mu \in D\) if \(\mu_0 > \mu^*\), or equivalently if

\[
G + \frac{RC}{L} = \frac{\mu_0}{V_R^2(\mu_0)} > \frac{\mu^*}{V_C^2} = \left( \frac{1}{R} + G \right) \left( \frac{V_\sigma}{V_C} - 1 \right)
\]

or

\[
V_C > \frac{1 + RG}{1 + 2RG + R^2C/L} V_\sigma = \frac{V_S - RJ}{1 + 2RG + R^2C/L}.
\]

For small values of \(R\), this requires that the cutoff voltage \(V_C\) be set very close to \(V_S\), making the operation of constant-power loads sensitive to small voltage fluctuations. We conclude that open-loop instability in this system is practically unavoidable, for a sufficiently large ratio of constant-power load to ohmic load.

The Hopf bifurcation occurs at \(\mu = \mu_0\) and is subcritical provided

\[
\left( \frac{R}{Z} \right)^2 < 3 + 2RG,
\]

a condition satisfied for practical values of \(R\) in a power system context. (Even in a ‘small’ system with \(V_S = 600\text{V}\) and a current rating of 600A, \(R\) should be no greater

278
than 0.01Ω to give a 1% voltage drop at rated current. To obtain a characteristic impedance of similar magnitude, it would be necessary to take \( C \) very large and \( L \) very small—say \( C = 0.1F \) and \( L = 10\mu H. \)

Qualitatively, then, the introduction of source resistance yields a similar conclusion to the zero-resistance case: a threshold value for \( P \), given by (11.18), above which the unique equilibrium loses stability, and below and near which the domain of attraction is bounded by an unstable limit cycle. Replacing the step function with the globally smooth \( \Psi(V) \), we can expect to observe a bifurcation of the limit cycle analogous to that in the zero-source-resistance case, yielding a stable limit cycle for supercritical values of \( P \).

**Pure Current Source with Generalised Inductive Load**

To exemplify the effect of nonlinearities in \( H \), we exchange the voltage-driven and current-driven components and consider a current source \( F(V, \mu) = I_S \) feeding the generalised current-driven load

\[
H(I, \mu) = V_0 + RI + \Psi(I) \frac{\mu}{I}
\]

representing the series combination of a counter-EMF source, resistance \( R \geq 0 \), constant-power load and inductance \( L \). Note that we now take \( \sigma_V = -1 \) and \( \sigma_I = 1 \).

The equilibrium equation gives

\[
V_R(\mu) = V_0 + R I_S + \Psi(I_S) \frac{\mu}{I_S}.
\]

For the transformation to LC units, define

\[
\tilde{v}_0 = \frac{V_0}{Z I_S} \quad \tilde{r} = \frac{R}{Z} \quad \tilde{\mu} = \frac{\mu}{Z I_S^2} \quad v_* = \frac{V_R}{Z I_S} = \tilde{v}_0 + \tilde{r} + \Psi(I_S) \tilde{\mu}.
\]

Then, provided \( v_* \neq 0 \)

\[
f(v, \tilde{\mu}) = \frac{1}{v_*} \quad h(i, \tilde{\mu}) = \frac{\tilde{v}_0}{v_*} - 1 + \tilde{r} i + \Psi(v_* I_S) \frac{\tilde{\mu}}{v_*^2}.
\]

Taking \( \Psi(I) \equiv 1 \) for \( I \sim I_S \) and applying Theorem 11.2, we obtain \( g(\tilde{\mu}) = 0 \) and

\[
\kappa(\tilde{\mu}) = r(\tilde{\mu}) = D_1 h \left( \frac{1}{v_*}, \tilde{\mu} \right) = \tilde{r} - \tilde{\mu}
\]

\[
\alpha(\tilde{\mu}) = D_3 h \left( \frac{1}{v_*}, \tilde{\mu} \right) = -6 \tilde{\mu} v_*^2 = -6(\tilde{v}_0 + \tilde{r} + \tilde{\mu})^2.
\]
Since \( V_R \) is a linear function of \( \mu \), there is one value \( \mu^* \) giving \( V_R = 0 \), for which the LC representation is undefined (the load appearing as a short circuit at equilibrium). Accordingly, we must treat the dynamics separately on the domains \( \mu > \mu^* \) (where the equilibrium voltage is positive) and \( \mu < \mu^* \) (where it is negative). Assuming \( I_S > 0 \) and \( V_0 > 0 \) we obtain \( \mu^* < 0 \), and can therefore conclude that equilibrium is reached at a positive bus voltage as long as the constant-power load is consuming energy.

The equilibrium at \( V_R(\mu) \) is stable if and only if \( \tilde{\mu} < \tilde{\tau} \), equivalently if

\[
P < Rl_S^2.
\]

Assuming nonzero \( R \) and \( \mu^* < 0 \), there is a subcritical Hopf bifurcation at the transition to instability. However, if \( \mu^* > 0 \) is sufficiently large, we observe a new phenomenon: a supercritical Hopf bifurcation of an equilibrium at negative voltage. This occurs when \( \delta_0 < -2\tilde{\tau} \), making \( \kappa = 0 \) and \( \alpha > 0 \) at \( \tilde{\mu} = \tilde{\tau} \). Note that this means the source \( V_0 \) is generating rather than absorbing energy. This supercritical bifurcation is unlikely to be of more than intellectual interest.

As a generalisation of the above we may introduce resistance into the current source and set \( F(V, \mu) = I_S - GV \) with \( G > 0 \). This results in a quadratic-transcendental equation for \( V_R \) similar in form to that obtained in Section 11.2.4. By analogy with the results of that section, we expect that while the introduction of source resistance complicates the analysis, it should not yield any qualitatively different conclusions.

**Conclusions**

We have analysed a variety of ‘generic’ LC circuits involving constant-power elements, and established in most cases that as the power level \( P \) of the constant-power load passes through a critical threshold \( P_0 \), the voltage dynamics undergoes a subcritical Hopf bifurcation. This holds over a wide range of scenarios, whenever the system exhibits dissipation as all real systems must. It remains true for all ‘generalised’ loads comprising independent sources and resistive elements in addition to constant-power elements, whether voltage-driven or current-driven with series inductance. In the most common situation, where such a load is fed by a power source having a small internal resistance, the critical threshold is close to the point where the same power is demanded by constant-power and ohmic load components.
The global dynamics proves to be more subtle. If a ‘real’ constant-power load is assumed, energy considerations dictate that there be a finite attractor for all finite $P$. In practice, the necessary attractor comes about through a saddle-node bifurcation of the unstable Hopf cycle, at some value $P^* < P_0$ which may be established experimentally. Thus, the equilibrium is globally stable for $P < P^*$, while for $P > P^*$ the voltage may oscillate. Conversely, if one uses an ideal constant-power load then the state trajectories are unbounded for all $x_0$ when $P > P_0$, and for sufficiently large $x_0$ when $0 < P < P_0$. In this case one observes ‘voltage collapse’, with $V \to 0$ and $\dot{V} \to -\infty$.

For constant-power loads that can undergo regeneration, a converse result applies: regeneration actually improves the stability of the system. It is also possible in some extreme cases for the Hopf bifurcation to be rendered supercritical; for example, when the characteristic impedance is very small, when the constant-power characteristic deviates significantly from its ideal shape at the equilibrium voltage, or when a current-driven constant-power load is placed in series with a constant-voltage generator. Such cases are generally undesirable in practice for reasons other than dynamic stability.

### 11.3 The Regulatory Control Problem

#### 11.3.1 Preliminaries

From the previous section it should be evident that simply programming the energy sources in a DC power system to the desired steady-state values is insufficient for stable operation. Even if stability of the desired equilibrium could be assured, the natural dynamics may yield an insufficient convergence rate or settling time. Some form of dynamic control is needed, and the design of such a control is the subject of this section.

For the present we focus on the single-input problem, with a single current-driven energy source and a voltage-driven load. For reasons already discussed we neglect source resistance, and model the energy input as a pure controllable voltage source. As the controller will likely be implemented in digital form, we shall work explicitly in discrete time, this being feasible with a plant as simple as the Small Model.

We make one further simplification. In order that we might have the widest
possible range of design tools at our disposal, we work with the linearised plant rather than the exact nonlinear system. Assuming that voltage excursions do not take the system too far from equilibrium, the error involved will be comparable to the inherent uncertainty in load measurements. It will thus be necessary to design the control with a certain degree of robustness, to account for uncertainties in the load as well as modelling errors arising from the linearisation.

In the second part of this section we reformulate the regulatory control problem for the Small Model in discrete time and state the control objectives applying to this formulation. This is an instance of a more general class of problem, which we call the \textit{LTIBU regulator problem}. Section 11.4 formalises the LTIBU problem and explores various strategies for its solution, building on standard approaches from linear control theory. A novel feature of LTIBU problems is the underdetermination of the initial condition, which can be exploited to improve transient performance.

After the discussion of the general LTIBU problem, we return to the Small Model in Section 11.5 and propose a hybrid adaptive controller, using a combination of linear-quadratic and deadbeat control techniques. We also discuss the design of a reliable parameter estimator, which is required by the adaptive strategy.

### 11.3.2 Problem Formulation

Consider the Small Model (11.7) with a current-driven controllable voltage source as input and a voltage-driven load $f(v, t)$. (Thus, we take $\sigma_I = -1$ and $\sigma_V = 1$.) Suppose that the load has the generic linear characteristic

$$f(v, t) = i_0(t) + 2\zeta(t)v.$$  \hfill (11.19)

The quantity $\zeta$ corresponds to the ‘damping coefficient’ in second-order linear system theory [56]. If $|v| \ll 1$, a constant-power load $p/(1 + v)$ can be represented approximately by the modified terms

$$i_0' = i_0 + p \quad \zeta' = \zeta - p/2.$$  

We assume that $i_0$ and $\zeta$ (and $p$ if present) are slowly-varying in time and that $\zeta$ can be estimated.

With the load characteristic (11.19), the system equations are

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_s.$$  \hfill (11.20)

282
Assuming that the current \( i_0 \) is compatible with the system’s steady-state requirements, the origin \( x = 0 \) is an admissible equilibrium. For control purposes, we convert the system into its discrete-time representation with sampling period \( T (\tau_0 \) in LC units), just as we did in Chapter 7 when using a similar system as the basis for the GSS model. The discrete-time system equations are those of (7.19):

\[
\mathbf{v}_{n+1} = \gamma \begin{bmatrix} \alpha & S \\ -S & \beta \end{bmatrix} \mathbf{v}_n + \begin{bmatrix} \mu \\ \gamma S \end{bmatrix} u_n
\]

with \( \mathbf{v}_n = \begin{bmatrix} v(n\tau_0) & v'(n\tau_0) \end{bmatrix}^T \) and \( u_n = v_S(n\tau_0) \). (Refer to Section 7.4.2 for other definitions.)

In anticipation of our controller employing integral action, we introduce a variable \( z \) representing the (discrete-time) integral of the voltage \( v \):

\[
z_{n+1} = z_n + v_n.
\]

A stable \( z \) then implies that \( v \) is stable at zero. Defining a new state

\[
\mathbf{x}_n = \begin{bmatrix} v(n\tau_0) & v'(n\tau_0) & z_n \end{bmatrix}^T
\]

we obtain the third-order system

\[
\mathbf{x}_{n+1} = \mathbf{A} \mathbf{x}_n + \mathbf{B} u_n
\]

(11.21)

with

\[
\mathbf{A} = \begin{bmatrix} \gamma \alpha & \gamma S & 0 \\ -\gamma S & \gamma \beta & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mu \\ \gamma S \\ 0 \end{bmatrix}.
\]

The entries of \( \mathbf{A} \) and \( \mathbf{B} \) depend on \( \zeta \), and so will vary with time. However, under our assumption that \( \zeta \) is slowly-varying, we shall treat the system as time-invariant.

The system (11.21) will form the basis of our control design. Our objective, in keeping with the general statement of Section 11.1.1, is to find a discrete sequence of feasible controls \( \{u_n\} \), \( n \geq 0 \) which takes any given initial state \( \mathbf{v}_0 \) of the plant to a neighbourhood of the origin of radius \( \delta \) and keeps it there after \( N_0 \) sample periods, for \( N_0 \) and \( \delta \) fixed.
11.4 LTIBU Regulator Problems

The control problem described in the previous section may be treated as an instance of a more general class of problems. The \textit{(discrete) LTIBU regulator problem} takes as its starting point the linear time-invariant (LTI) system

\[
x_{n+1} = Ax_n + Bu_n
\]

partitioned as

\[
\begin{bmatrix}
v_{n+1} \\
z_{n+1}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
v_n \\
z_n
\end{bmatrix}
+ 
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u_n
\] (11.22)

where \( v \in \mathbb{R}^{N_1} \) represents the plant state and \( z \in \mathbb{R}^{N_2} \) the state of a dynamic controller (such as a PI regulator). Starting from some initial plant state \( v_0 \), the combined state \( x \) is to be stabilised at the origin via the control input \( u \in \mathbb{R}^M \).

The LTIBU problem is characterised by two features:

1. The control \( u \) is \textit{bounded} elementwise by the constant vectors \( u^- \), \( u^+ \in \mathbb{R}^M \).

   The bounds will in general be asymmetric.

2. The initial state \( x_0 \) is \textit{underdetermined}. \( v_0 \), being the initial state of the plant, is given in advance, while \( z_0 \) is not. Since \( z \) is determined in part from \( v \) and must be driven to zero by the control, the choice of \( z_0 \) will affect the control performance.

Both these aspects are of importance in the specific instance of controlling (11.21). Here the bounds on \( u \) are \( u^- = -(v_L + 1) \) and \( u^+ = v_L - 1 \), for some value \( v_L = 3V_{AC}/\sqrt{2}V_R \). Typically \( v_L \) is only slightly greater than 1, making the constraint much more severe on the positive side than on the negative side. Also, since \( z \) is the integral of \( v \), problems can arise if \( z \) is not initialised appropriately and \( v_0 \) is far from equilibrium, as at startup when \( v_0 = -1 \). If we set \( z_0 = 0 \) in this instance, then \( z_1 = -1 \) and a significant voltage overshoot is required in order to bring \( z \) back to zero.

There is now a substantial literature on LTI problems with bounded inputs, starting from the work of Desoer and Wing [19, 18, 77]. By comparison, virtually no attention has been given to the underdetermination aspect, even though this feature is present in any PID regulator. It will therefore be of use to obtain techniques for solving general LTIBU problems.
A simple strategy for driving $\mathbf{x}$ to the origin is the state feedback law

$$u_n = S(-F \mathbf{x}_n)$$

where $S$ is a saturation function. When $-F \mathbf{x}_n$ is feasible, this transforms the system evolution into the pure exponential form

$$\mathbf{x}_n = (A - BF)^n \mathbf{x}_0.$$ 

The controllers we derive will all be of this general form, although the feedback matrix $F$ may be time-varying.

11.4.1 Approach I: Linear-Quadratic (LQ) Control

In LQ control, $u$ is calculated to minimise the quadratic performance index

$$J = \frac{1}{2} \mathbf{x}_N^T S \mathbf{x}_N + \frac{1}{2} \sum_{n=0}^{N-1} (\mathbf{x}_n^T Q \mathbf{x}_n + u_n^T R u_n).$$

with $S$, $Q$ and $R$ positive definite. The solution (see [42, §2.2]) is a time-varying feedback $u_n = -F_n \mathbf{x}_n$. To obtain this solution one solves the Riccati equation

$$S_n = A^T (S_{n+1} - S_{n+1} B (B^T S_{n+1} B + R)^{-1} B^T S_{n+1}) A + Q$$

which is a backward recurrence for $0 \leq n < N$, with the final condition $S_N = S$. The feedback matrix is then given as

$$F_n = (B^T S_{n+1} B + R)^{-1} B^T S_{n+1} A$$

for $0 \leq n < N$. The calculation is easily modified to take into account time-varying system matrices or performance indices, provided these are known $N$ steps in advance. For a time-invariant system, $F$ becomes a static feedback in the limit $N \to \infty$, obtained from the fixed point of the Riccati equation.

The LQ regulator is derived on the basis of a fixed initial state $\mathbf{x}_0$ and a free final state $\mathbf{x}_N$. If we were to fix the final state as $\mathbf{x}_N = 0$ we would obtain an open-loop control, which does not have the desired robustness properties. We therefore adhere to the usual practice, leaving $\mathbf{x}_N$ free but choosing the weighting matrices appropriately so that $\mathbf{x}_N$ is made as small as we require.

The initial state, on the other hand, is only partially fixed in a LTIBU problem. However, if we carry out the optimisation assuming $\mathbf{v}_0$ fixed but $\mathbf{z}_0$ free, this does
not alter the usual Riccati solution; it simply asserts that $z_0$ must have a certain relationship to the (independently derived) costate variables in order to minimise the performance index. Rather than studying this relationship in detail, we shall calculate the optimal $z_0$ by a more direct method. From the solution of the Riccati equation one obtains a positive definite matrix $S_0$ such that the performance index, minimised over all variables other than $z_0$, is

$$J = x_0^T S_0 x_0.$$ 

Let $S_0$ be partitioned conformably with $x$ into ‘plant’ and ‘controller’ components, thus:

$$S_0 = \begin{bmatrix} S_{11} & S_{21}^T \\ S_{21} & S_{22} \end{bmatrix}.$$ 

Then given $v_0$, the value of $z_0$ which minimises $J$ is

$$z_0^* = -S_{22}^{-1}S_{21}v_0.$$ (11.23)

The LQ method is attractive for our application, as it prescribes the optimal initial condition $z_0^*$ through a straightforward adjunct to the Riccati solution, incorporates time-varying systems in a natural manner, and guarantees robust stability in the static limit for time-invariant systems. Unfortunately, LQ control is not readily modified to take input bounds into account, particularly when the bounds are asymmetric as in our case. For control of the Small Model, even a heavy quadratic weighting on $u$ is insufficiently ‘hard-limiting’ for our purposes, tending as it does to penalise large negative values of $u$ (which are perfectly admissible) while allowing moderately large positive values (which are not).

### 11.4.2 Approach II: Saturated Deadbeat (SD) Control

#### SD and Minimum-Time Control

In deadbeat control, the feedback $F$ is chosen so that $A - BF$ is a nilpotent matrix. If $u_k^* = -Fx_k$ is feasible for $0 \leq k < N$, where $N = N_1 + N_2$ is the order of the system, then the deadbeat response results, with $x$ driven to zero in at most $N$ time steps. If there is more than one control input ($M > 1$), the deadbeat response can be achieved for an arbitrary initial state with an unbounded control in less than $N$
steps [55, p.681]. Here, however, we shall treat only the case of one control input \( u \in [u^-, u^+] \). We shall also assume that \((A, B)\) is complete-state controllable, that \( A \in \mathbb{R}^{N \times N} \) is nonsingular, and that \( 0 \in [u^-, u^+] \).

When the set of feasible inputs is bounded, so is the set of initial states \( x_0 \in \mathbb{R}^N \) which can be controlled to zero in any finite number of steps. For all \( m \geq 0 \), let \( \Omega_m \subset \mathbb{R}^N \) denote the set of initial states that are feasibly controllable to the origin in no more than \( m \) steps (where \( \Omega_0 = \{0\} \)). Since we assume \( u = 0 \) is a feasible control, it follows that \( \Omega_m \subseteq \Omega_{m+1} \) for all \( m \). We may then define \( \Omega^*_m = \Omega_m \setminus \Omega_{m-1} \) as the set of initial states controllable to the origin in \( m \) steps and no fewer. Note that \( \Omega^*_i \), \( 0 \leq i \leq m \) forms a partition of \( \Omega_m \).

The sets \( \Omega_m \) and \( \Omega^*_m \) were defined (with different notation) by Desoer and Wing in their seminal papers on discrete minimum-time control with bounded inputs [19, 18]. It can be easily established by induction that \( x \in \Omega_m \) with \( m > 0 \) if and only if it can be written in the form

\[
x = \sum_{k=1}^{m} r_k u_{k-1}
\]  

where \( r_k = -A^{-k}B \) and \( u_{k-1} \in [u^-, u^+] \) for all \( k \). (This result can also be found in [18], although the latter made the assumption of symmetrical input constraints.) It follows immediately that the \( \Omega_m \) are convex sets, and that for \( m \leq N \) the set \( \Omega_m \) is \( m \)-dimensional. (The first \( N \) vectors of the sequence \( \{r_k\} \) are homeomorphic to the columns of the controllability matrix, and thus linearly independent by assumption.)

The following result provides the key to our saturated deadbeat control strategy.

**Theorem 11.3** Let \((A, B)\) be a single-input discrete LTI system that is complete-
state controllable, with \( A \in \mathbb{R}^{N \times N} \) nonsingular, \( B \in \mathbb{R}^{N \times 1} \), and suppose the input \( u \in \mathbb{R} \) is constrained to the interval \([u^-, u^+]\) with \( u^- \leq 0 \) and \( u^+ \geq 0 \). There exists \( F \in \mathbb{R}^{1 \times N} \) such that \( A - BF \) is nilpotent, that is, has all zero eigenvalues. The saturated deadbeat (SD) control is \( u_n = \mathcal{S}(-Fx_n) \) where \( \mathcal{S} \) is the saturation function limiting \( u \) to \([u^-, u^+]\).

Suppose \( x \in \Omega^*_m \) with \( m \leq N \); that is, there is a feasible control taking \( x \) to zero in precisely \( m \) steps. Then this feasible control is unique, and is identical to both the SD control and to the ordinary (unsaturated) deadbeat control \( u_n = -Fx_n \). The set \( \Omega_N = \bigcup_{m=0}^{N} \Omega^*_m \) may be characterised as the set of all initial states \( x \) for which SD control and deadbeat control are equivalent.
Proof. The existence of $\mathbf{F}$ is a standard consequence of complete-state controllability, which allows the eigenvalues of $\mathbf{A} - \mathbf{BF}$ to be arbitrarily assigned [55]. Now suppose $\mathbf{x} \in \Omega_n^m$ with $m \leq N$; it follows that $\mathbf{x}$ has a representation $\sum_{k=1}^{N} r_k u_{k-1}$ with $u_{k-1} \in [u^-, u^+]$ for all $k$. (If $m < N$ we may take $u_{k-1} = 0$ for $k > m$.) This is expressible as a matrix equation

$$\mathbf{x} = -\mathbf{A}^{-N} \mathbf{C} \mathbf{P} \mathbf{u} \equiv \mathbf{Ru}$$

where $\mathbf{u}$ represents the control sequence, $\mathbf{C}$ is the controllability matrix, and $\mathbf{P}$ is the exchange (‘antidiagonal’) matrix of order $N$. The uniqueness of $\mathbf{u}$ now follows from the nonsingularity of the matrix $\mathbf{R} = -\mathbf{A}^{-N} \mathbf{C} \mathbf{P}$ whose columns are the $r_k$.

To show that $\mathbf{u}$ is actually the SD control, we construct the inverse of $\mathbf{R}$. Our starting point is the identity

$$\mathbf{A}^{N-k}(\mathbf{A} - \mathbf{BF})^k = \mathbf{A}^{N-(k-1)}(\mathbf{A} - \mathbf{BF})^{k-1} - \mathbf{A}^{N-k} \mathbf{BF}(\mathbf{A} - \mathbf{BF})^{k-1}.$$ 

A repeated application of this identity starting from $(\mathbf{A} - \mathbf{BF})^N$ (the case $k = N$) yields the identity

$$(\mathbf{A} - \mathbf{BF})^N = \mathbf{A}^{N} - \sum_{k=1}^{N} \mathbf{A}^{N-k} \mathbf{BF}(\mathbf{A} - \mathbf{BF})^{k-1}.$$ 

Now suppose $\mathbf{F}$ is a deadbeat gain matrix, so $(\mathbf{A} - \mathbf{BF})^N = \mathbf{0}$. Premultiplying the above by $\mathbf{A}^{-N}$ we obtain

$$\mathbf{I} = \sum_{k=1}^{N} \mathbf{A}^{-k} \mathbf{BF}(\mathbf{A} - \mathbf{BF})^{k-1} = \sum_{k=1}^{N} r_k \theta_k^T$$

where $r_k$ is as above and $\theta_k^T = -\mathbf{F}(\mathbf{A} - \mathbf{BF})^{k-1}$. This last sum is no more than the matrix product $\mathbf{R}\mathbf{\Theta}$, where $\mathbf{\Theta}$ is the matrix whose rows are the $\theta^T_k$. It follows that $\mathbf{\Theta}$ is the inverse of $\mathbf{R}$, and accordingly that

$$u_{k-1} = \theta_k^T \mathbf{x} = -\mathbf{F}(\mathbf{A} - \mathbf{BF})^{k-1} \mathbf{x}.$$ 

This equation is precisely that for deadbeat control in the absence of saturation. Since the $u_{k-1}$ are $a$ priori feasible, we infer that SD control and unsaturated deadbeat control are equivalent on the set $\Omega_N$ (but on no larger set, by definition of $\Omega_N$), and that either gives the unique minimum-time control. √

Theorem 11.3 states that if there is any feasible control that takes $\mathbf{x}_0$ to zero in at most $N$ steps, then (saturated) deadbeat control will accomplish this. As a corollary
to the proof we obtain a variant of Ackermann’s formula [55, p.663] for the deadbeat gain matrix $F$ given $A$ and $B$ (and hence $R$):

$$F = -\theta_1^T = -e_1^T R^{-1}. \quad (11.25)$$

(Elements of the proof of Ackermann’s formula in [55] were used in our proof to obtain the formula for $\Theta_1$.)

Note that Theorem 11.3 is silent on whether SD control is still a minimum-time control when $x_0 \not\in \Omega_N$. This is in fact true for one-dimensional systems, but in general this is false: even with $N = 2$ one can easily construct counterexamples for which there is a feasible control sequence bringing $x_0$ to the origin in fewer steps than SD control. The construction of a true global minimum-time control is complicated, and is dealt with in detail in [18] (for the single-input case) and [77] (for the multiple-input case).

Rather than attempt a global minimum-time control, we adopt another approach: to use the underdetermination of $x_0$ in the LTIBU problem to enlarge the set of SD-controllable initial states. By choosing $z_0$ where possible to make $[v_0 \ z_0]^T \in \Omega_N$, the set of $v_0$ amenable to SD control can be made quite large.

**UBLEs and Their Solution**

An initial state $x_0$ is in $\Omega_N$ if and only if $x_0 = Ru$ for a feasible input sequence $u$. Let us partition this equation as

$$\begin{bmatrix} v_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} u. \quad (11.26)$$

This makes our task explicit, the main part of which may be stated as follows:

**Underdetermined Bounded Linear Equations (UBLE)** Given $v_0 \in \mathbb{R}^{N_1}$ and $R_1 \in \mathbb{R}^{N_1 \times N}$, with $N_1 < N$, find $u \in \mathbb{R}^N$ such that $u_k \in [u^+, u^-]$ for all $k$ and

$$R_1 u = v_0 \quad (11.26)$$

or conclude that no such $u$ exists.

Given a solution $u$ to the UBLE (11.26), it is a simple matter to calculate $z_0 = R_2 u$. We shall refer to this as the *adjoint equation* of the UBLE, with *codimension* $N_2 = N - N_1$.
Note that from the LTIBU system structure (11.22) the columns of $\mathbf{R}_1$ are determined from the plant matrices $(\mathbf{A}_{11}, \mathbf{B}_1)$ as

$$r_{1k} = A_{11}^{-k} \mathbf{B}_1.$$ (11.27)

In particular, if the plant on its own is complete-state controllable then $\mathbf{R}_1$ is of maximum rank. This assumption will prove useful when solving (11.26).

The convexity of the constraints on $\mathbf{u}$ implies that the solution set of the UBLE (where it exists) is convex, and with it the adjoint solution set of initial states $\mathbf{z}_0$. We shall provide an explicit solution for the case $N_2 = 1$. In this case, the $\mathbf{z}_0$ set is a closed interval, and its endpoints are obtainable as solutions of a straightforward linear program (LP): we just minimise, respectively maximise, the function $\mathbf{R}_2 \mathbf{u}$ subject to the equality constraints $\mathbf{R}_1 \mathbf{u} = \mathbf{v}_0$ and the convex inequalities $\mathbf{u} \geq u^- \mathbf{1}_N$, $\mathbf{u} \leq u^+ \mathbf{1}_N$.

Our solution of the LP problem follows the approach in standard texts, for example [54, Ch.11] or [80, §3.11.1]. We begin by putting the problem in standard form with equality and nonnegativity constraints only. Defining the slack variables

$$t_k = u_k - u^- \quad s_k = u^+ - u_k$$

we obtain $t_k + s_k = u^+ - u^-$ (a constant) for all $k$, and

$$\mathbf{R}_1 \mathbf{u} = \mathbf{R}_1 \mathbf{t} + u^- (\mathbf{R}_1 \mathbf{1}_N) = \mathbf{v}_0$$

$$\mathbf{R}_2 \mathbf{u} = \mathbf{R}_2 \mathbf{t} + u^- (\mathbf{R}_2 \mathbf{1}_N).$$

Define $\rho_1 = \mathbf{R}_1 \mathbf{1}_N \in \mathbb{R}^{N_1}$ as the vector of row sums of $\mathbf{R}_1$, and $\rho_2 = \text{bal} \mathbf{R}_2$. In standard form, with variables $\mathbf{t}, \mathbf{s} \in \mathbb{R}^N$, the problem is to extremise $\mathbf{R}_2 \mathbf{t} + u^- \rho_2$ (or just $\mathbf{R}_2 \mathbf{t}$) subject to

$$\begin{bmatrix} \mathbf{R}_1 & 0 \\ \mathbf{I}_N & \mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0 - u^- \rho_1 \\ (u^+ - u^-) \mathbf{1}_N \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{s} \end{bmatrix} \geq \mathbf{0}.\]$$

The constraint set for this problem is a (finite) convex polytope, and accordingly both the required extremal solutions lie at extreme points of this polytope. Because the problem is of codimension 1 (the codimension is unchanged by introducing slack
variables), the extreme points are just those for which precisely one variable \( t_k \) or \( s_k \) is zero, and there are just \( 2N \) such points. It is therefore an easy matter to search through these \( 2N \) basic solutions to find those which are feasible (nonnegative) and extremise \( R_2 t \). The remainder of this section is devoted to an efficient method for calculating these basic solutions.

Let \( y^{(n)} \) denote (in some order to be determined later) the nonzero variables in the \( n \)th basic solution, where \( 1 \leq n \leq 2N \). If the equality constraints in the standard form are denoted \( Cy = d \), then the \( n \)th basic solution is found by solving

\[
C^{(n)} y^{(n)} = d
\]

where \( C^{(n)} \) denotes \( C \) with the \( n \)th column removed and the others rearranged in conformance with \( y^{(n)} \). We assume that all the \( C^{(n)} \) are nonsingular.

For the first basic solution, we remove the first column of \( C \) to obtain

\[
C^{(1)} = \begin{bmatrix}
0 \\
R_{11} & \vdots & 0 \\
0 \\
0 & \cdots & 1 & 0 & \cdots & 0 \\
0 \\
I_{N-1} & \vdots & I_{N-1} \\
0
\end{bmatrix}
\]

where \( R_{11} \) denotes \( R_1 \) with its first column removed. Its inverse is

\[
(C^{(1)})^{-1} = \begin{bmatrix}
0 \\
R_{11}^{-1} & \vdots & 0 \\
0 \\
0 & \cdots & 1 & 0 & \cdots & 0 \\
0 \\
-R_{11}^{-1} & \vdots & I_{N-1} \\
0
\end{bmatrix}
\]

Denote the \( n \)th column of \( C \) by \( c_n \), and the \( n \)th row of \( (C^{(1)})^{-1} \) by \( \gamma_n^T \). We set

\[
C^{(n)} = C^{(1)} + (c_1 - c_n)c_{n-1}^T
\]

thereby substituting \( t_1 \) for the variable to be omitted in \( y^{(n)} \). We then obtain for \( n > 1 \), using a well-known formula for the inverse of a unity-rank-modified matrix
\[ y^{(n)} = (C^{(1)} + (c_1 - c_n)e_{n-1}^T)^{-1}d \]
\[ = y^{(1)} - \frac{(C^{(1)})^{-1} y_{n-1}^{(1)}}{1 + \gamma_{n-1}^*(c_1 - c_n)} \]
\[ = y^{(1)} - ((C^{(1)})^{-1} c_1 - e_{n-1}) \frac{y_{n-1}^{(1)}}{\gamma_{n-1}^* c_1} \]

the last line eventuating because \((C^{(1)})^{-1} c_n = e_{n-1}\) for \(n > 1\), by definition of \(C^{(1)}\).

Now, define \(r_{11}\) as the first column of \(R_1\) (removed in the formation of \(C^{(1)}\)) and set

\[ \sigma \doteq (C^{(1)})^{-1} c_1 = \begin{bmatrix} \bar{R}_{11}^{-1} r_{11} \\ 1 \\ -R_{11}^{-1} r_{11} \end{bmatrix} \tag{11.28} \]

thereby also obtaining \(\gamma_{n-1}^T c_1 = \sigma_{n-1}\). We then obtain, for \(2 \leq n \leq 2N\)

\[ y^{(n)} = y^{(1)} - (\sigma - e_{n-1}) \frac{y_{n-1}^{(1)}}{\sigma_{n-1}} \tag{11.29} \]

so that, once \(\sigma\) and \(y^{(1)} = (C^{(1)})^{-1} d\) have been calculated, all other basic solutions can be obtained almost immediately.

Note that we require \(\sigma\) to have all nonzero entries if we are to obtain a meaningful solution. Geometrically, this entails that when \(r_{11}\) is written as a linear combination of the remaining columns in \(R_1\), none of the coefficients are zero; the vectors are in ‘general position’. This is equivalent to requiring that all \(N_1 \times N_1\) submatrices of \(R_1\) are nonsingular, as is required for a nondegenerate solution of the LP problem.

**Selection of Optimal \(z_0\)**

As a result of solving the UBLE posed in the previous section, we either obtain a convex set of \(z_0\) for which \(x_0 \in \Omega_N\), or discover that there are no feasible solutions, in which case an alternative approach is required. Assuming the UBLE has a solution, we are free to choose any value from the solution set; the specific choice will need to depend on criteria other than time considerations.

One possibility is to choose \(z_0\) to minimise the LQ-like performance index

\[ J = \frac{1}{2} \sum_{n=0}^{N-1} (x_n^T Q x_n + R u_n^2) \tag{11.30} \]
where \( x_n \) and \( u = \{ u_n \} \) are taken as functions of \( z_0 \). The implicit relationship is given by \( u = \Theta x_0 \), where \( \Theta \) is the matrix obtained in the proof of Theorem 11.3. Let the columns of \( \Theta \) be partitioned as \( [\Theta_1 \ \Theta_2] \) conformably with the rows of \( R \); then

\[
u = \Theta_1 v_0 + \Theta_2 z_0\]

and

\[
x_n = \Psi_1^{(n)} v_0 + \Psi_2^{(n)} z_0\]

where \( [\Psi_1^{(n)} \ \Psi_2^{(n)}] \) is the corresponding partition of \((A - BF)^n\). Upon substituting these expressions into (11.30), this becomes a (static) quadratic optimisation problem over a convex set.

A brief calculation shows that the solution to the un\textit{constrained} optimisation problem is

\[
z_0^* = Q_{zz}^{-1} Q_{vz} v_0 \quad (11.31)\]

where

\[
Q_{zz} = \sum_{n=0}^{N-1} (\Psi_2^{(n)})^T Q \Psi_2^{(n)} + R \Theta_2^T \Theta_2
\]

\[
Q_{vz} = \sum_{n=0}^{N-1} (\Psi_1^{(n)})^T Q \Psi_2^{(n)} + R \Theta_1^T \Theta_2.
\]

If \( z_0^* \) is not in the adjoint solution set of the UBLE, then the optimal solution is the point in this set closest to \( z_0^* \), in a suitably-defined metric.

When \( N_2 = 1 \) things are particularly simple: the solution set is an interval \([z_{\text{min}}, z_{\text{max}}]\), and \( z_0^* \) is a linear combination of the plant variables \( v_0 \). If \( z_0^* \not\in [z_{\text{min}}, z_{\text{max}}] \), one simply chooses the nearest endpoint.

### 11.4.3 Approach III: \( \mathcal{H}_\infty \) Control

Both the approaches so far proposed for the LTIBU regulator problem (LQ and SD) are sensitive to the plant parameters \( A_{11} \) and \( B_1 \). If these are not known precisely, one must question how much model error can be tolerated before the closed-loop performance degrades to unacceptable levels and (in extremely unfavourable cases) becomes unstable. The LQ and SD approaches both promise a certain degree of robustness, in that the closed-loop eigenvalues are placed well within the unit circle. However,
this is no guarantee of true robustness unless the eigenvalue locations themselves are sufficiently insensitive to modelling errors.

Is there any controller that promises a degree of robustness much higher than that of LQ and SD control? The answer, for linear systems, is provided by $\mathcal{H}_\infty$ design theory. In $\mathcal{H}_\infty$ control [27], the system model is taken as

$$x_{n+1} = A_0 x_n + B_0 u_n + w_n$$

where $(A_0, B_0)$ is the assumed model, and the quantity $w_n$ represents modelling error. If the actual system is $(A_0 + A'(n), B_0 + B'(n))$, then

$$w_n = A'(n)x_n + B'(n)u_n.$$ 

The control design is based on seeking the smallest $\gamma > 0$ such that

$$J_\gamma = \sum_{n=1}^{N}(x_n^T x_n + u_n^T u_n - \gamma^2 w_n^T w_n) \leq -\epsilon \|w\|_2^2$$

for all bounded real signals $w$ and $\epsilon > 0$. As with LQ control, the $\mathcal{H}_\infty$ design produces a time-varying feedback $u_n = F_n x_n$ for $0 \leq n \leq N$, with $N$ chosen arbitrarily.

The minimal $\gamma$ obtained from the $\mathcal{H}_\infty$ design provides a measure of the robustness of the resulting controller. For example, if $B$ is known exactly so that $w_n = A'(n)x_n$, we obtain

$$0 \geq J_\gamma \geq \sum_{n=1}^{N} \left( (1 - \gamma^2 \|A'(n)\|_2^2) x_n^T x_n + u_n^T u_n \right)$$

where $\|A'(n)\|_2$ is the spectral norm for matrices, equal to the largest singular value. It follows that if

$$\|A'(n)\|_2 \leq \frac{1}{\gamma}$$

for all $n$, the $\mathcal{H}_\infty$ control is stabilising. More generally, $\gamma$ sets an upper bound (in some sense) on the modelling error that can be tolerated by any linear controller. $\mathcal{H}_\infty$ control says nothing about nonlinear controllers, including adaptive controllers (which recalculate their gains on the basis of state estimates).

The $\mathcal{H}_\infty$ design problem is technically complicated, and for all but the most trivial problems must be solved by numerical computation. The question whether $\mathcal{H}_\infty$ control provides sufficient robustness must therefore be assessed on a case-by-case basis.
11.5 Control Design for the Small Model

11.5.1 A Hybrid Solution

The regulatory control problem for the Small Model (11.21) is an instance of the
LTIBU regulator problem with $N_1 = 2$, $N_2 = 1$ and $M = 1$. The plant matrices $A_{11}$
and $B_1$ both depend on the load damping coefficient $\zeta$, which can vary over a wide
range and take both positive and negative values in a typical system. An attempt at
$H_\infty$ control design yields a negative conclusion: even neglecting the variation in $B$,
the variation in the elements of $A$ due to changes in $\zeta$ is simply too great for a fixed
linear controller to achieve global stability.

We therefore discard the $H_\infty$ approach in favour of an adaptive strategy. Our
strategy will be based on certainty equivalence: given an estimate $\hat{\zeta}$ of the unknown
parameter $\zeta$, we carry out the control design on the assumption that $\zeta = \hat{\zeta}$. Provided
the estimation error is sufficiently small, such a controller should achieve the desired
objective. The strategy accordingly falls into two parts: an estimation component
(discussed in the following section) and a control component.

Our preferred control strategy is SD control, as it provides a minimum-time control
over a wide range of initial plant states $v_0$ and is easily tailored to the asymmetric
control bounds. Given successive estimates of the damping coefficient $\zeta$, we calculate
the system matrices $A$ and $B$ for (11.21) and choose the gain vector $F$ using (11.25).
The control law takes the form

$$u_n = -Fx_n, \quad F = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$$

which, due to the construction of the state $x$, is equivalent to PID control on the
voltage $v$. (In practice $v$ can be measured directly, while the derivative $v'$ may be
obtained from the capacitor current.) Explicit formulae for the gains are

$$f_1 = \frac{1 + 2C \gamma - \gamma^2}{1 - 2C \gamma + \gamma^2} - \frac{\gamma(C - \zeta S - \gamma)}{(1 - 2C \gamma + \gamma^2)^2}$$

$$f_2 = \gamma \left( \frac{2(1 + \zeta CS - C^2) + (C - \zeta S) \gamma}{S(1 - 2C \gamma + \gamma^2)} - \frac{(1 - 2\zeta^2)S + 2\zeta(C - \gamma)}{(1 - 2C \gamma + \gamma^2)^2} \right)$$

$$f_3 = \frac{1}{1 - 2C \gamma + \gamma^2}$$

where $\gamma = e^{-\zeta \tau_0}$, $C = C_\zeta(\tau_0)$ and $S = S_\zeta(\tau_0)$.

One may verify that provided $0 < \tau_0 < \pi$, the denominators in (11.32) do not
vanish for any value of $\zeta$. We conjecture that this condition on $\tau_0$ is sufficient for the

295
complete-state controllability of \((A, B)\) and also for the nondegeneracy of \(R_1\) in the sense of the LP problem in Section 11.4.2. (One can easily verify these properties for any specific values of \(\tau_0\) and \(\zeta\).) Since \(\det A = \gamma^2\) is always positive we conclude that the SD control is always well-defined provided \(\tau_0 < \pi\).

The integrator \(z\) is ‘reset’ by the method of Section 11.4.2 under either of the following conditions:

1. the previous integrator state is meaningless (as when the controller first comes online, or when it reenters ‘SD mode’ as described below); \textit{or}

2. the voltage \(v\) is outside the tolerance limits, \textit{and} the state vector \(x\) is larger in magnitude than at the previous iteration, \textit{and} at least \(N\) iterations have elapsed since the last reset.

As foreshadowed in Section 11.4.2, the use of SD control necessitates a fallback strategy for the case where \(v_0\) is outside the SD-controllable set. LQ control is one possible strategy, but does not choose \(u\) sufficiently well, owing to the asymmetric constraints on \(u\). An alternative is to effectively substitute \(N = 1\) into the LQ problem and solve for \(u_0\) as a function of \(v_0\). If necessary, this is repeated at the next time step, until \(v\) is brought within the SD-controllable set. The resulting feedback is

\[
u_k = -\frac{1}{R + B^T_1S_{B_1}B^T_1SA_{11}v_k}
\]

where the weighting matrix \(S\) includes plant variables only. The integrator is redundant and is not used in this strategy, which would otherwise call for it to be reinitialised at every time step. When a solution for the UBLE is eventually obtained, the integrator is reset to the new value \(z_0\) and we revert to SD control.

### 11.5.2 Estimation of Damping Coefficients

To complete the adaptive strategy, we require a method for estimating the equivalent load damping coefficient \(\zeta\). For this task, we assume that measurements of the bus voltage \(v\) and the load current \(i_L = f(v, t)\) are available. If necessary \(i_L\) may be obtained indirectly as \(i - v'\), where \(i\) is the inductor current and \(v'\) the capacitor current.

Since the control problem is formulated with a linear load, an obvious approach
is to identify the parameters \( i_0, \zeta \) in the model

\[
i_L = i_0 + 2\zeta v.
\]

An alternative approach is to go back to the nonlinear generalised load model

\[
i_L = i_0 + g(1 + v) + \frac{p}{1 + v},
\]

identify the parameters \( i_0, g \) and \( p \), and then set

\[
\zeta = \frac{\dot{\theta} - \dot{\hat{\theta}}}{2}.
\]

In general, identification problems perform better if the putative model is well-matched to the actual underlying model. We shall therefore follow the second approach, in a slightly modified form: setting \( p_L = (1 + v)i_L \), we can model the \( p_L - v \) relationship as a simple quadratic law

\[
p_L = g(1 + v)^2 + i_0(1 + v) + p.
\]

This model is amenable to estimation by a standard recursive least-squares (RLS) procedure [46, §3.3]. We first express the model in standard form as follows:

\[
p_L = \theta^T \phi \quad \theta = \begin{bmatrix} g \\ i_0 \\ p \end{bmatrix} \quad \phi = \begin{bmatrix} (1 + v)^2 \\ 1 + v \\ 1 \end{bmatrix} \tag{11.34}
\]

where \( \theta \) represents the parameters to be estimated, and \( \phi \) a measurable regressor vector. Given an initial estimate \( \hat{\theta}_0 \) and a matrix \( \mathbf{P}_0 \in \mathbb{R}^{3 \times 3} \) (see below), the RLS estimator based on successive estimates \( p_L^{(n)} \) with \( n \geq 1 \) takes the form

\[
\dot{\hat{\theta}}_n = \hat{\theta}_{n-1} + \mathbf{K}_n \left( p_L^{(n)} - \hat{\theta}_n^T \phi_n \right)
\]

\[
\mathbf{K}_n = \frac{\mathbf{P}_{n-1} \phi_n}{\lambda + \phi_n^T \mathbf{P}_{n-1} \phi_n}
\]

\[
\mathbf{P}_n = \lambda^{-1}(\mathbf{I} - \mathbf{K}_n \phi_n^T)\mathbf{P}_{n-1}.
\]

The first of these equations updates the estimate \( \hat{\theta} \) based on the estimation error, with the time-varying gain \( \mathbf{K}_n \). The last equation is a recurrence for the matrix \( \mathbf{P} \), whose inverse (the ‘information matrix’) plays a fundamental role in least-squares theory. It is best to choose a large \( \mathbf{P}_0 \), as the estimates produced by RLS are biased toward \( \hat{\theta}_0 \) by an amount proportional to \( \mathbf{P}_0^{-1} \).
RLS estimation is based on the assumption that $\theta$ is constant. Since in practice $\theta$ varies (slowly) with time, we provide for a ‘forgetting factor’ $\lambda$, which is set equal to 1 in the standard RLS estimator. Setting $\lambda$ to a value slightly less than 1 allows the estimation procedure to notionally discard some of its earlier information.

11.5.3 Robustness Issues

With sufficiently good estimates $\hat{\zeta}$ provided to the adaptive SD control, stability of the closed-loop system is assured. However, in practice problems are posed by estimation error, by rapid load variations and by load characteristics not conforming to the standard model. It is therefore still important to study the performance of SD control when the actual coefficient $\zeta$ differs from the estimate $\hat{\zeta}$ used in the SD design.

The following conjecture offers the hope that a certain degree of uncertainty can be accommodated through robustness of the SD control alone.

**Conjecture 11.1** Suppose the SD switching control is designed for optimal performance at $\zeta = \zeta_0$. Then, provided $\zeta_0$ is sufficiently large, there exists $\epsilon > 0$ such that the closed-loop system is stable for all $\zeta > \zeta_0 - \epsilon$.

Experiments suggest this is true, but a formal proof is difficult. Figures 11.7 through 11.9 show ‘root loci’ of the eigenvalues of $A(\zeta) - B(\zeta)F(\zeta_0)$ as $\zeta$ varies in regular increments. The values of $\zeta_0$ used to generate the loci are 0.5, 0 and $-0.5$ respectively. Note that since the elements of $A$ and $B$ depend nonlinearly on $\zeta$, these loci do not follow the rules of ordinary affine root loci of the form (10.47).

The figures show that the ‘sufficiently large’ condition in our conjecture is necessary: when $\zeta_0$ is small or negative, there is an interval of $\zeta > \zeta_0$ in which one closed-loop eigenvalue is observed to become unstable. For this reason (as well as on performance grounds) it is not feasible to simply design for a small $\zeta_0$ and let that design suffice for all $\zeta > \zeta_0$. One also sees that as $\zeta_0$ becomes increasingly negative, the number $\epsilon$ expressing the stability margin for $\zeta < \zeta_0$ shrinks rapidly. These simple plots underscore the necessity for an effective adaptation strategy, with an estimator up to the task.

298
Figure 11.7: Root locus of Small Model with SD control, $\zeta_0 = 0.5$

Figure 11.8: Root locus of Small Model with SD control, $\zeta_0 = 0$
11.6 Control of Multiple Converters

Apart from the statement of the general regulatory control problem at the outset, this chapter has been concerned entirely with the simplest case of this problem: the Small Model with one current-driven source and a voltage-driven load. Two aspects of the general problem accordingly have not been considered: the multiplicity of control inputs, and the increase in order as more nodes are added to the model.

A first step from the scalar problem considered to this point is to allow for a voltage-driven circuit with dynamic discretion, in addition to the current-driven source. This voltage-driven circuit could be a VDC acting as an additional supply source, or a discretionary load. If we suppose, as in the Small Model, that the current-driven source has negligible series resistance, then the standard-form model (11.7) takes the form

\[
\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & -g \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \phi_0(\mathbf{x}, \tau) \end{bmatrix} + \begin{bmatrix} 0 \\ v_0(\tau) - \sigma v'_0(\tau) \end{bmatrix}.
\]

Thus, the autonomous current \(i_0\) from the voltage-driven part of the circuit enters the model only in the first derivative. Suppose that \(u_1 = v_0\) is a control input as
before, and that

\[ i_0(\tau) = i_q(\tau) + i_q^* + i_{\infty} \]

where \( i_q \) is a controllable current (or sum of controllable currents), \( i_q^* \) is the desired steady-state value (ensuring that \( i_q = 0 \) at equilibrium) and \( i_{\infty} \) accounts for constant-current loads with no dynamic discretion. Now, let \( i_q \) be controlled by the process

\[ i_q' = u_2(\tau) \]

where \( u_2 \) is a second control input. We may then augment the system state as

\[ \mathbf{v} = \begin{bmatrix} x^T & i_q \end{bmatrix}^T \]

and write

\[ \mathbf{v}' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -g & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{v} + \begin{bmatrix} 0 \\ \phi_0(\mathbf{v}, \tau) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -\sigma_v \end{bmatrix} \mathbf{u} \]

where \( \mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T \). If the nonlinearity \( \phi_0 \) is negligible and the variation in \( g \) and \( i_{\infty} \) slow, this is just a two-input LTI system, whose discrete-time equivalent is

\[ \mathbf{v}_{n+1} = \begin{bmatrix} \gamma \alpha & \gamma S & 0 \\ -\gamma S & \gamma / \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}_n + \begin{bmatrix} \mu & -\sigma_v \mu \\ \gamma S & -\sigma_v \gamma S \\ 0 & \tau_0 \end{bmatrix} \mathbf{u}_n \quad (11.35) \]

taking \( \zeta = g/2 \) in the customary manner.

Just as we did above, we may augment the third-order plant (11.35) with an integrator \( z \) and formulate the resulting system as an LTIBU problem, this time with two bounded inputs and one undetermined initial condition. In the case of \( u_2 \), the bounds are those applying to the rate of change of the controlled current. The full state feedback will include terms in \( i_q \); depending on the design circumstances, it may be desirable to pursue a decentralised design in which the gain from \( i_q \) to \( u_1 \) is forced to zero. Similarly it may be desirable to use two integrators, one integrating the voltage as before and another integrating the current \( i_q \). The first would contribute to both \( u_1 \) and \( u_2 \), and the second to \( u_2 \) only.

The present framework permits a limited extension to arbitrary numbers of discretionary external circuits, provided the single-node approximation remains valid. Recall that several voltage sources with series inductance and negligible series resistance, when connected at the same node, can be regarded as a single source, by
virtue of their common (infinite) time constant. Thus we may regard \( u_1 \) in general as a linear combination of discretionary voltages, and for a given \( u_1 \) assign these voltages by means of an auxiliary algorithm (which would attend among other things to the sharing of current among individual sources). Similarly, \( i_q \) is in general a sum of discretionary currents, and \( u_2 \) a sum of respective control inputs.

When the single-node approximation breaks down, we must return to the Large Model and consider systems of arbitrary order and more complicated structure. Owing to the greater complexity, it is unlikely that a generic design procedure can be given for the Large Model such as we have provided for the Small Model. Much further work is required, both to analyse the LTIBU regulator problem with multiple inputs and to adjudicate when the single-node approximation is and is not appropriate.

### 11.7 Simulation Results

Simulations were undertaken to test the proposed regulatory control and estimation scheme. The CDC with parameters as in Section 8.4 and 10.2.4 is used as the supply source. For the purposes of regulatory control this CDC is modelled as a programmable voltage source with series inductance \( L = 8 \text{mH} \) and capacitance \( C = 1000 \mu \text{F} \) on the DC bus. The nominal voltage \( V_R \) is once again taken as 500V.

The load estimator of Section 11.5.2 is run at the converter sampling frequency, namely 10kHz. The time scale appropriate for the saturated deadbeat control, on the other hand, is much slower owing to the tradeoff between response time and deadbeat gains (and SD-controllability as a consequence). As the bandwidth of the system under consideration is essentially the natural LC frequency \( \omega = 1/\sqrt{LC} = 354 \text{rad s}^{-1} \approx 56 \text{Hz} \), any controller frequency from around 120Hz upwards will satisfy the Nyquist criterion. For these simulations we have chosen a frequency of 250Hz, which keeps the deadbeat gains favourably small while ensuring a sizeable Nyquist stability margin. The sampling period is \( T_0 = 4 \text{ms} \) (five samples per AC cycle) and the LC sampling period is \( \tau_0 = \omega T_0 = \sqrt{2} \), substantially less than \( \pi \).

Consider first the case \( \zeta = 0 \), corresponding to no load, constant-current load only, or evenly balanced ohmic and constant-power load. The system matrices in this
case are

\[
A = \begin{bmatrix}
0.1559 & 0.9878 & 0 \\
-0.9878 & 0.1559 & 0 \\
1 & 0 & 1
\end{bmatrix} \quad B = \begin{bmatrix}
0.8441 \\
0.9878 \\
0
\end{bmatrix}
\]

and the matrix \( R \) defined in Section 11.4.2 is

\[
R = \begin{bmatrix}
0.8441 & 1.1073 & -0.4987 \\
-0.9878 & 0.6797 & 1.1998 \\
-0.8441 & -1.9514 & -1.4527
\end{bmatrix}
\]

Inverting \( R \) gives the deadbeat gains:

\[
F = -e_1^T R^{-1} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} = \begin{bmatrix}
0.4809 & 0.9172 & 0.5924
\end{bmatrix}
\]

Suppose now that the system starts in ‘shutdown’ condition (zero volts), giving

\[
v_0 = \begin{bmatrix}
-1 \\
0
\end{bmatrix}
\]

The input constraints are \(-v_L - 1 \leq u \leq v_L - 1\) where

\[
v_L = \frac{3V_{AC}}{\sqrt{2}V_R} = 1.0182.
\]

These constraints together with \( R \) and \( v_0 \) define an UBLE of the form (11.26), which may be solved by the method of Section 11.4.2 to give the adjoint solution set

\[
z_0 \in [z_{\min}, z_{\max}] = [1.4692, 4.2189].
\]

Initialising the integrator with any value from this set will produce a deadbeat response respecting the control bounds. (Note on the other hand that the default choice \( z_0 = 0 \) is not in this set.)

Making the arbitrary choice \( z_0 = 2 \) gives the ‘ideal’ deadbeat response shown in Figures 11.10 and 11.11. In this simulation we have not yet incorporated the load estimator into the control. For the purpose of comparison with later simulations, we have also included a 100A constant-current load; any other constant-current load, as well as no load, would give a similar response.

Figures 11.12 and 11.13 show the effect of closing the control loop around the load estimator. The estimator is now permitted to adjust the working value of \( \zeta \), and hence the gains \( F \), according to the formula \( \hat{\zeta} = (\hat{\zeta} - \hat{\zeta})/2 \). The concurrent time
Figure 11.10: Deadbeat voltage response with $\zeta = 0, \omega_0 = 2$

Figure 11.11: Deadbeat current response with $\zeta = 0, \omega_0 = 2$
Figure 11.12: Voltage response with 100A constant-current load, $z_0 = 2$

Figure 11.13: Current response with 100A constant-current load, $z_0 = 2$
Figure 11.14: Load estimation with 100A constant-current load

evolution of the estimates \( \hat{g}, \hat{i}_0 \) and \( \hat{p} \) (starting from zero) is shown in Figure 11.14. Note that the actual value of the load current in LC units is

\[
\hat{i}_0 = \frac{100}{500} \sqrt{\frac{L}{C}} = 0.5657.
\]

We now carry out the same experiment, but with an ohmic load having the same nominal current of 100A at rated voltage. Because the nominal currents are the same, the numerical value of \( g \) in LC units is the same as \( i_0 \) above (0.5657). However, the value of \( \zeta \) is now \( g/2 = 0.2828 \), and this affects the feasible set for \( z_0 \). Repeating the above analysis with this value of \( \zeta \) gives the result

\[
z_0 \in [1.3822, 3.0796]
\]

so that \( z_0 = 2 \) still gives an SD-controllable initial state. The response, again with the estimator in the control loop, is shown in Figures 11.15 and 11.16. Figure 11.17 shows the evolving parameter estimates.

The obvious next step is to run the same simulation with a constant-power load having a nominal current of 100A. However, such a load could not possibly operate at zero volts, so instead we start the simulation with the voltage at 90% of its rated
Figure 11.15: Voltage response with 100A ohmic load, $z_0 = 2$

Figure 11.16: Current response with 100A ohmic load, $z_0 = 2$
Figure 11.17: Load estimation with 100A ohmic load

value: $v_0 = \begin{bmatrix} -0.1 & 0 \end{bmatrix}^T$. For this initial condition and $\zeta = -p/2 = -0.2828$, we require

$$z_0 \in [0.1358, 0.5822].$$

Choosing arbitrarily $z_0 = 0.2$, we obtain the simulation shown in Figures 11.18 and 11.19.

Looking at the evolving parameter estimates (Figure 11.20), we notice a peculiar phenomenon: the estimates converge on values other than the true values. A plot of the estimation error, however (Figure 11.21), shows it converging exponentially to zero. It is suspected that the problem rests with an insufficiently wide variety of data points once the voltage stabilises. If the actual load characteristic is

$$p_L = g(1 + v)^2 + i_0(1 + v) + p$$

but $v$ stabilises and remains in a sufficiently small neighbourhood of zero before the estimator has stabilised, then the estimates can conceivably converge to erroneous values $g + \Delta g$, $i_0 + \Delta i_0$, $p + \Delta p$ provided only that

$$\Delta g + \Delta i_0 + \Delta p = 0.$$
Figure 11.18: Voltage response with 100A constant-power load, \( z_0 = 0.2 \)

Figure 11.19: Current response with 100A constant-power load, \( z_0 = 0.2 \)
Figure 11.20: Load estimation with 100A constant-power load

Figure 11.21: Estimation error for Figure 11.20
Figure 11.22: Voltage response with ohmic and switched constant-power load

In practice it can be argued that this is a non-problem, because it only arises if \( \mathbf{x} \) attains an equilibrium, and the integrator ensures that the only equilibrium is at the origin. Nonetheless, this phenomenon bears future investigation.

As a further test, we examine the performance of the controller under variable load conditions. Figures 11.22 and 11.23 depict a scenario where the system starts from shutdown with a 50A ohmic load, and an additional 50A constant-power load is switched on at 20ms (or after five control periods). At the sixth period (24ms) the controller detects the deterioration in the system state (Condition 2 for integrator reset) and the integrator is reset to the value 0.5 based on the UBLE solution with the current values of \( \dot{\zeta} \) and \( \mathbf{v} \). After this point, the system stabilises again relatively quickly. Importantly, had the integrator not been reset the voltage oscillations would have persisted longer and with a much larger amplitude.

Figure 11.24 shows the evolving parameter estimates, and Figure 11.25 the estimation error. Again, the convergence after load switching is toward erroneous values, but it will be noted that the estimator is still very much in a transient state at the end of this simulation. In any event, the erroneous estimates do not affect the performance of the controller.
Figure 11.23: Current response with ohmic and switched constant-power load

Figure 11.24: Load estimation with ohmic and switched constant-power load
Figure 11.25: Estimation error for Figure 11.24
Part IV

Conclusion
Chapter 12

Conclusions and Recommendations

12.1 Concluding Remarks

This thesis has covered a range of topics arising from the concept of ‘managed’ DC power reticulation systems. Chapter 1 presented the argument for a ‘paradigm shift’ toward the use of direct current for electric power distribution in commercial buildings, industrial sites and residential subdivisions. A shift towards DC is a radical suggestion in what is now a conservative industry, but in fact this industry has had extensive historical experience with DC—whether this be in town supply schemes through the first half of the twentieth century, in railways, in telephone exchanges or in automotive electrical systems. A shift in thinking is timely now that there is impetus to replace ‘plain old electricity supply’ with smarter technologies.

The ‘management’ of electricity encompasses a broad spectrum of issues including demand management, ‘intelligent’ operation of facilities, power quality, condition monitoring and process automation. We have focussed here on the issues relating to the performance of the system in its electrical aspects, dividing these into ‘static’ and ‘dynamic’ control problems. The static problems, as formulated in Part II, have to do with the allocation of demand and discretionary currents in the steady state. Dynamic problems, the subject of Part III, relate to the control of voltages and currents, and in particular to the switching control of the AC–DC converters which would provide the main supply source to our envisaged DC reticulation system.

Discussion of static problems focussed on performance improvement through minimisation of network losses and prevention of distribution cable overloads. Losses can be minimised through an optimal allocation of discretionary currents in a managed
network (discussed in Chapter 4 and Appendix B), or through reconfiguration of the network topology (discussed for radially-operated networks in Chapter 5). Within our management framework, overload conditions can be anticipated and prevented by assessing the effect of requested demand on cable currents throughout the network (Chapter 6).

Our approach to the Loss Minimisation Problem in Chapter 4 was deliberately quite ‘solution-focussed’. Space constraints permitted only a brief digression into the Kuhn-Tucker conditions for the problem and the ‘active constraints’ approach before proceeding to a straightforward solution based on established algorithms. When discretionary loss minimisation is combined with the overload prevention strategy of Chapter 6, it is not sufficient simply to know the minimal-loss solution for an isolated discretionary profile; one also needs information on the global structure of the discretionary map. Such information is provided by the more abstract theory developed in Appendix B; this will however require further elaboration before it can be applied to a practical algorithm.

The material in Chapter 5 was to some extent a digression from the main thrust of the thesis. It was motivated primarily by the perceived applicability of the sensitivity analysis of Chapter 3 to solving in a novel manner an established problem for traditional power system operators. The reconfiguration algorithm can of course also be applied to managed DC networks, but the radial operation constraint is of dubious relevance to such networks. It has been suggested in personal communications to the author that the Managed DC Bus is best operated as a single-ring rather than a tree network. A worthwhile project in future would therefore be to extend the algorithm to handle networks of small positive nullity, if such an extension is indeed feasible.

The study of dynamic problems starts with the reduction of the converter-fed reticulation system to a tractable model. Chapter 7 provided several such models, varying with the desired degree of complexity and the method of converter control. Of these, the Generalised Asynchronous Switching (GAS) model is specially suited to the detailed study of generalised switching schemes for switch-mode (PWM) converters, the subject of Chapter 8.

Of the various switching schemes for current-driven converters proposed in Chapter 8, the sixfold scheme shows most promise on grounds of simplicity and input current distortion. Its chief drawback is the 4 : 3 ratio of peak to average switching
frequency. The twelvefold switching scheme eliminates this disadvantage, but at the expense of low-frequency current harmonics. The latter occur at only four discrete frequencies and can be filtered, but this will add to the cost of the conversion system. The consequences of twelvefold switching for higher-level control design are also unclear, and require further investigation.

In addition to a low-level switching strategy, switch-mode converters require a 'synchronising controller' that synchronises the AC input current with the supply voltage while mimicking a programmable current or voltage source on the DC side. After examining the limitations of unity-power-factor operation in Chapter 9, we presented synchronising control designs in Chapter 10 for both voltage-driven and current-driven converters. Synchronising control for VDCs has been well-studied in the literature, and this part of the work largely involved reviewing existing techniques. We have, however, provided a novel justification for the popular PCFF or 'synchronous regulator' scheme, which emerges as one of the better-performing alternatives. For the CDC synchronising control we have presented a 'quasilinear' design of our own, which has good response-time and robustness properties.

With these designs in place, we were amply equipped to turn in Chapter 11 to the problem of voltage control and stability for DC systems with ideal controllable sources. A number of results were obtained here for the simplest case of a single-node network with a current-driven source and voltage-driven load. Bifurcation theorems were presented and illustrated with the practically relevant example of a constant-power load, demonstrating the existence of subcritical Hopf bifurcations in systems containing such loads, and the consequent need for closed-loop regulatory control.

The regulatory control problem was analysed as a specific instance of the more general LTIBU regulator problem, characterised by (asymmetrically) bounded control inputs and underdetermined initial conditions. A design was proposed based on saturated deadbeat control, with one-step LQ control as a backup strategy. As the robustness of this controller (and indeed of any linear controller) is severely limited under the load variations to be expected, the controller must be augmented with an identification algorithm that provides estimates of ohmic, constant-current and constant-power components of the load.

This controller, designed for a single-node power system, is adequate for more general systems with a single point of supply provided the voltage drops within the net-
work are not excessive. Where there is more than one point of supply, a multiple-input
design must be undertaken, and where distribution cables run over long distances,
a ‘large’ system model must be used to take into account cable network dynamics.
Detailed controller designs for the general case will need to proceed on a case-by-case
basis.

The work in this thesis is highly theoretical in nature, and the reader will have
noted a lack of laboratory experiments. All empirical results in the thesis have been
obtained using detailed, albeit somewhat idealised, simulation studies. While the
author has every confidence that experimental tests will confirm the theoretical con-
cclusions, it is understood that the final verification of this work awaits practical
laboratory demonstrations, which will need to be undertaken in the future.

12.2 Directions for Further Work

Geometry of the Loss Minimisation Problem

As seen in Chapter 4, the general solution to the Loss Minimisation Problem for
a managed DC network is a piecewise linear discretionary map, partitioned into a
finite number of linear ‘domains’. Within any one of these domains the demand
sensitivity matrix $T$ is constant, and so the effect of a change in demand on the cable
currents can be easily calculated. However, $T$ changes whenever the prospective
demand profile crosses a domain boundary. Thus, for overload prevention purposes
it is useful to know where these boundaries are located, and the demand sensitivity
matrix pertaining to each.

In Appendix B, it was shown that these piecewise linear domains correspond to
the outward projections of faces of a convex polytope in an $(N-1)$-dimensional space.
Various results have been obtained concerning the location of these faces and their
outward projections, including the outline of a recursive algorithm for determining
the outward projections. Further work is needed, however, before these results can
be used as the basis of a full-fledged algorithm. In general, the problem of finding
the outward projections is NP-complete, but it is conjectured that for networks of
sufficiently low complexity, the orthogonality of ‘most’ of the constraint directions
permits a more computationally tractable approach.
Reconfiguration Problems for Non-Radial Networks

In Chapter 5, we presented an algorithm for finding minimal-loss radial configurations for general distribution networks. This algorithm was based on an enumeration of the spanning trees in the network graph, for which a variety of standard algorithms are available.

One can readily formulate a generalised version of this problem, in which it is required to find a minimal-loss configuration of (maximum) nullity $\nu_0$ in a network of nullity $\nu$. (Without any such restriction on the nullity, it is conjectured that the original network itself is a minimal-loss configuration. Similarly, it is conjectured on the basis of intuition that the losses in a network of nullity $\nu_0$ are \textit{(ceteris paribus)} no greater than in any of its subnetworks, hence that there is a minimal-loss network of nullity $\nu_0$. To the author’s knowledge no proof exists of this statement.)

This generalised problem raises two subproblems: firstly, the efficient enumeration of subnetworks of nullity $\nu_0 > 0$; and secondly, the efficient transformation of the sensitivity matrix from one such subnetwork to another. That these problems are of a higher order of complexity than the equivalent problems for trees casts doubt on the prospects for a feasible extension of the algorithm of Chapter 5 to the generalised problem. Nonetheless, there has been no investigation into these questions as part of this thesis.

Improved Understanding of CDC Switching Behaviour

The development of the GAS model permitted us to investigate ‘nonstandard’ switching schemes for PWM converters, such as the twelvefold scheme for CDCs in Chapter 8. Naturally, the GAS model has serious limitations when it comes to studying non-ideal aspects of switching such as reverse recovery and blanking time. The aim in our model development was to produce simple approximations to the real behaviour in order to develop a control theory for switch-mode converters. In this respect we follow the paradigm established by Wu, Dewan and Slemon [78] among others.

It is nonetheless essential that a more detailed investigation be carried out to study the non-ideal effects. Attention also needs to be given to the effect of variable space vectors on the time-averaged dynamics of the converter; a study of this kind could be expected to pin down the precise origin of the empirically observed low-
frequency harmonics. Even if the twelvefold CDC switching scheme is ultimately deemed impractical, further investigation of its behaviour is warranted on theoretical grounds.

**CDC Synchronising Control**

In Chapter 10 we presented a synchronising control scheme for the CDC (with sixfold switching) based on linear methods. As with our solution to the Loss Minimisation Problem in Chapter 4 by least-squares methods, this design was guided by simplicity. First a quasilinear approximation was made to allow the design to be carried out by linear methods; then the available degrees of freedom were restricted to the minimum necessary to achieve sufficient performance.

The quasilinear design methodology is vindicated by our review of synchronising control techniques for VDCs earlier in that chapter, in which it was found that quasilinear controllers generally perform better than those employing overt nonlinearities. Nonetheless, the two-gain pole placement design suggested in Chapter 10 is not the only design possible, and may be far from ‘optimal’ according to some reasonable criterion. There is much scope for further work, both in optimising the pole placement design and in seeking superior alternatives.

**The General Regulatory Control Problem**

As noted in Chapter 11, much more work is required in extending the regulatory control design to DC systems with multiple discretionary sources and extensive cable networks. Even in the single-supply, single node case, there is scope for further investigations into the robustness of the saturated deadbeat control and the conditions for SD-controllability given the load parameters and initial state. The proposed parameter estimator is the simplest conceivable, and superior designs are no doubt possible here also.
Part V

Appendices
Appendix A

Graph Theory and Electrical Networks

In this appendix we review some important concepts and results of network theory, which are used throughout the main text. Standard sources for the material in this appendix include [69, 6, 5, 13].

A.1 Graph Definitions

A graph $\mathcal{G}$ is a nonempty set of elements (the vertices or nodes of $\mathcal{G}$) together with a set of unordered pairs of elements (the edges of $\mathcal{G}$). Graphs may be pictured using dots to represent vertices and line segments connecting dots to represent edges. If a vertex is contained within an edge, the two are said to be incident.

A directed graph $\mathcal{G}$ is a nonempty set of elements together with a set of ordered pairs of elements (the arcs of $\mathcal{G}$). Any graph may be viewed as a directed graph by assigning arbitrary orientations to its edges; conversely, any directed graph has an underlying graph obtained by ignoring arc orientations. The definitions given below for graphs have obvious extensions to directed graphs.

We are mostly concerned with simple graphs and directed graphs, in which there are no edges (arcs) connecting a vertex to itself, and at most one edge (arc) connecting any two given vertices. (We adopt the convention here that a directed graph is simple if and only if its underlying graph is simple.) A subgraph of a (directed) graph is a subset $V'$ of the set of vertices together with a subset $E'$ of the set of edges (arcs), such that each edge (arc) in $E'$ contains only vertices in $V'$.
A path in a graph is an ordered set of distinct edges \( \{e_1, e_2, \ldots, e_n\} \) such that \( e_i \) and \( e_{i+1} \) have a vertex in common for \( 1 \leq i < n \) and all such vertices are distinct. The vertex in \( e_1 \) not shared with \( e_2 \), and the vertex in \( e_n \) not shared with \( e_{n-1} \), are together referred to as the endpoints of the path, and the path is said to connect these vertices. If the endpoints of a path are identical, the path is a closed path or loop. Note that a loop in a simple graph must contain at least three vertices. For our purposes, we define a path in a directed graph as an ordered set of arcs that correspond to a path in the underlying graph; if all arcs are oriented in the direction of the path, we call this a directed path.

A graph \( \mathcal{G} \) is connected if there exists a path connecting any two distinct vertices in \( \mathcal{G} \). Any graph \( \mathcal{G} \) can be partitioned into connected components, such that each component is connected and no edge of \( \mathcal{G} \) joins vertices in different components.

A connected graph with no loops is a tree. A tree has the minimum number of edges of any graph with a specified number \( N \) of vertices, namely \( N - 1 \) edges. A union of trees with distinct vertices is a forest. Sometimes one vertex of a tree is distinguished and referred to as the root, giving a rooted tree. Similarly, a rooted forest has one root vertex for each tree. A rooted tree is uniform if there is a directed path from the root vertex to every other vertex; a uniform rooted forest is a union of uniform rooted trees.

A spanning subgraph of a graph \( \mathcal{G} \) is a subgraph \( \mathcal{G}' \) containing all the vertices of \( \mathcal{G} \) and having the same number of connected components as \( \mathcal{G} \). A minimal spanning subgraph is a spanning forest or, if \( \mathcal{G} \) is connected, a spanning tree.

A cut set is a maximal set of edges (arcs) which separate the vertices of \( \mathcal{G} \), in the sense that the vertices may be partitioned into sets \( V_1 \) and \( V_2 \) such that every edge in the cut set contains one vertex in \( V_1 \) and one in \( V_2 \). A cut set in a directed graph is uniform if its arcs are directed uniformly from one vertex set to the other.

### A.2 Graph Algebra

Algebraically, a simple (directed) graph is commonly represented by its incidence matrix. Suppose the graph has \( N \) vertices and \( E \) edges (arcs); the incidence matrix is then a \( N \times E \) matrix \( \mathbf{D} \) with entries from \( \{1, 0, -1\} \). For a directed graph, a general element \( d_{ij} \) is equal to 1 if arc \( j \) is directed towards node \( i \), to \(-1\) if arc \( j \) is directed
away from node \(i\), and to 0 if arc \(j\) and node \(i\) are not incident. For a graph, \(d_{ij}\) is 1 if arc \(j\) and node \(i\) are incident and 0 otherwise.

The rank of a graph \(G\) is defined as the rank of its incidence matrix. A graph with \(N\) nodes and \(C\) connected components has rank \(N - C\), which is also the number of edges in any spanning forest.

If \(G\) has \(E\) edges, then the incidence matrix \(D\) can be regarded as a linear mapping \(D\) from an \(E\)-dimensional vector space (the edge space) to an \(N\)-dimensional vector space (the vertex space). Since the incidence matrix has rank \(N - C\), the image of the edge space under the mapping \(D\) is a subspace of the vertex space of dimension \(N - C\). The mapping \(D\) accordingly has a kernel of dimension

\[
\nu = E - N + C
\]

referred to as the nullity of \(G\). The nullity is always nonnegative, and is zero if and only if \(G\) is a forest.

We may represent an element of the edge space as follows. Let \(S\) be a set of edges, and form a vector \(v \in \mathbb{R}^E\) by setting \(v_i = 1\) if \(e_i \in S\) and \(v_i = 0\) if \(e_i \notin S\). This gives a generic element of the edge space for an (undirected) graph. For directed graphs, we associate an arbitrary direction with each arc in \(S\), and set \(v_i = 1\) if the direction of arc \(e_i\) in \(G\) coincides with its direction in \(S\), \(v_i = -1\) if the directions are opposed, and \(v_i = 0\) if \(e_i \notin S\).

In either case, we have that \(v\) belongs to the kernel of \(D\) if it is associated with a (uniformly directed) loop in \(G\). Furthermore, \(v\) belongs to the orthogonal complement of \(D\) if it is associated with a (uniform) cut set in \(G\). (The uniformity here relates to the orientations assigned in \(S\), not necessarily to those in the graph itself.) The kernel of \(D\) is accordingly referred to as the loop subspace, and its orthogonal complement as the cut subspace. These subspaces consist of all actual loops and cuts, as well as their ‘linear combinations’.

A set of \(\nu\) loops in \(G\) whose vectors \(v\) are linearly independent in \(\mathbb{R}^E\) provides a basis for the loop subspace. Such a basis may be constructed for a connected graph as follows. Form a spanning tree (forest) \(T \subset G\); this leaves \(\nu\) edges, referred to as the cotree of \(T\). With each of these edges \(e\), associate the unique loop contained within the subgraph \(T \cup \{e\}\). This gives a set of \(\nu\) loops, which provide a basis for the loop subspace; these are called the fundamental loops with respect to \(T\).
In a similar manner we may find a collection of $N - C$ cut sets which provide a basis for the cut subspace. Namely, with each edge $e$ in the spanning tree (forest) $T$, associate the unique cut set formed from $e$ and edges in the cotree $G \setminus T$. These $N - C$ sets are called the fundamental cut sets with respect to $T$.

We assume henceforth that all graphs are connected. Since any connected graph with $N$ nodes has rank $N - 1$, removing any one row from the incidence matrix of a connected graph leaves a full-rank matrix with $N - 1$ rows, which we call the reduced incidence matrix and denote by $A$. The row removed to form $A$ corresponds to a particular vertex of $G$ which we shall call the datum node.

A selection of $k$ edges from $G$ corresponds to a selection of $k$ corresponding columns from $A$. If we select $N - 1$ edges, we obtain a square matrix; this matrix is nonsingular if and only if the $N - 1$ edges form a spanning tree $T$. Given $T$, we may construct a matrix $C$ whose rows represent the fundamental loops with respect to $T$ as follows: let $A_T$ denote the submatrix of $A$ corresponding to $T$, and let $A_{T'}$ denote the remaining columns of $A$. Then

$$ C = \begin{bmatrix} -(A_{T'}^{-1} A_T)^T & I \end{bmatrix} $$

where $I$ is a $N \times N$ identity matrix. Similarly, we may construct a matrix $K$ whose rows represent the fundamental cut sets with respect to $T$ as

$$ K = \begin{bmatrix} I_{N-1} & (A_{T'}^{-1} A_T) \end{bmatrix}. $$

Note that $K$ is the row-reduced echelon form of $A$, after a permutation of columns.

### A.3 Electrical Networks

Let $R$ be a ring. A flow in a directed graph is an assignment of currents $i_j \in R$ to each arc $e_j$ such that Kirchhoff’s current law (KCL) is satisfied: the sum of currents entering a vertex equals the sum of currents leaving the vertex. Algebraically, KCL may be written $Di = 0$, where $D$ is the incidence matrix and $i$ the vector of arc currents. It may also be written $Ai = 0$ or $Ki = 0$ without any loss of information (here $K$ is the fundamental cut set matrix defined above).

Since $K$ provides a basis for the cut subspace, KCL is equivalent to the statement that the sum of (directed) currents across any cut set is zero. Equivalently one may
say that \( i \) is an element of the loop subspace (where the edge space is interpreted as \( \mathbb{R}^E \)). A directed graph with an associated flow is usually called a network.

A potential field in a directed graph is an assignment of voltage drops \( v_j \in \mathbb{R} \) to each arc \( e_j \) such that Kirchoff's voltage law (KVL) is satisfied: the sum of (directed) voltage drops around any loop is zero. KVL may be written in the form \( Cv = 0 \), where \( C \) is the fundamental loop matrix. Equivalently, one may say that \( v \) is an element of the cut subspace.

Any assignment of potentials \( u_k \in \mathbb{R} \) to the vertices of the graph induces a potential field in which the voltage drop on any arc is the difference in potential between its 'start' and 'end' vertices. Two assignments of potentials are equivalent, or induce the same potential field, if and only if they differ by a constant. In this sense there is no 'absolute zero' of potential.

An electrical network is a directed graph endowed with a flow and a potential field. The orientations chosen for the arcs are purely arbitrary, serving only to impart a sign convention to voltages and currents. The symbol \( \mathcal{N} \) will be used henceforth both for an electrical network and (with some abuse of notation) for its underlying directed graph.

Physically, the vertices of \( \mathcal{N} \) represent the nodes or buses of the network, and the arcs represent connections between nodes: cables, sources or other circuit elements. As well as associating with each arc \( e_j \) a current \( i_j \) and voltage drop \( v_j \), we assign a potential \( u_k \) to each vertex \( k \), such that \( u_i - u_k = v_j \) for each arc \( e_j =< k, l > \). It follows that the vector \( \mathbf{u}' \) of vertex potentials satisfies \( D^T \mathbf{u}' = \mathbf{v} \).

To fix \( \mathbf{u}' \), one vertex is usually identified with the network ground and assigned a zero potential. It is convenient to identify this vertex with the datum node when forming the reduced incidence matrix. In this case, the vector \( \mathbf{u} \) of potentials at vertices 1 through \( N - 1 \) satisfies \( A^T \mathbf{u} = \mathbf{v} \).

For analysis purposes, the description of a network \( \mathcal{N} \) is completed by the specification of branch characteristics relating the current and voltage drop in each arc. The combined branch characteristics can in principle be expressed as a functional relation \( F(i, v, t) = 0 \), where \( t \) is a time variable. If \( F \) contains derivatives with respect to \( t \), the network is a dynamic network; otherwise it is a static or resistive network. The network is linear, nonlinear, time-invariant or time-varying according as \( F \) is.

When analysing dynamic networks, it is common to introduce flux variables \( \phi \)
and charge variables \( \mathbf{q} \), satisfying \( \dot{\phi}_j = v_j \) and \( \dot{q}_j = i_j \). Dynamic elements are then classified into two broad types: \textit{inductors}, having a static relationship between flux and current, and \textit{capacitors}, having a static relationship between charge and voltage. A dynamic network has an equivalent static network obtained by replacing each inductor with an independent current source \( i_j - I_j(t) = 0 \), and each capacitor with an independent voltage source \( v_j - V_j(t) = 0 \).

A linear static network consists only of voltage sources, current sources and elements satisfying \textit{Ohm’s law} \( v_j = z_j i_j \). \( z_j \) is referred to as the \textit{impedance} of arc \( e_j \), and its inverse \( y_j = 1/z_j \) as the \textit{admittance}. The sources may be independent, or may depend on voltages or currents in other parts of the network, as in the case of ideal transformers. All networks considered in the body of this thesis contain only independent sources.

A \textit{source assignment} for an electrical network \( \mathcal{N} \) is a function associating a value with each independent voltage and current source in \( \mathcal{N} \). Given a source assignment for \( \mathcal{N} \), a \textit{solution} of \( \mathcal{N} \) is a complete assignment of node potentials and branch currents in \( \mathcal{N} \) compatible with Kirchoff’s laws, the branch characteristics and the source assignment. \( \mathcal{N} \) is \textit{well-posed} if there exists a unique solution of \( \mathcal{N} \) for any source assignment.

For proofs of the following theorems, refer to [69].

\textbf{Theorem A.1} \textit{A linear static network} \( \mathcal{N} \) \textit{is well-posed if and only if there exists a partition of the branches of} \( \mathcal{N} \) \textit{into a tree} \( \mathcal{T} \) \textit{and cotree} \( \mathcal{T}' \), \textit{such that} \( \mathcal{T} \) \textit{contains all the voltage sources of} \( \mathcal{N} \) \textit{and} \( \mathcal{T}' \) \textit{contains all the current sources of} \( \mathcal{N} \).

\textbf{Theorem A.2} \textit{A linear dynamic network} \( \mathcal{N} \) \textit{is well-posed if and only if its equivalent static network is well-posed}.

Two other fundamental theorems concerning linear static networks are the \textit{superposition theorem} and \textit{reciprocity theorem}. The superposition theorem states that in a network with multiple sources, the voltage or current in any arc is the sum of contributions from each source when all others are suppressed (their current or voltage set to zero). The reciprocity theorem states that in a network with a single source, if the current and voltage in some arc \( e_j \) are \( I \) and \( V \) respectively, then moving the source to that arc produces the same current and voltage in the arc originally bearing the source.
Appendix B

A Geometric Approach to Loss Minimisation and Other Minimum-Distance Problems

B.1 Preliminaries

It was seen in Chapter 4 that the problem of minimising losses by discretionary control of a managed distribution network is equivalent to that of finding the minimum distance to a convex polytope in a (finite-dimensional) Hilbert space. In that chapter we suggested a practical algorithm for solving the problem. In this appendix, we outline an alternative approach based on a deeper understanding of the underlying geometry. This approach, while not developed here into a precise algorithm, nonetheless promises a more direct and exact solution to the problem in many cases. Due to space considerations, and because this appendix represents work essentially still in progress, a number of proofs have been omitted. As is seen in Chapter 6, the geometric theory we develop is of importance for overload prevention schemes based on a minimal-loss discretionary map.

The following theorem (see [44] for a proof) will be of key importance in what follows.

Orthogonal Projection Theorem Let $S$ be a convex set in a Hilbert space $H$ with inner product $(a, b)$ and norm $\|a\|$. Let $x$ be any element of $H$. Then there is a unique element $x_\perp \in S$ such that $\|x - x_\perp\| < \|x - a\|$ for all $a \in S$; in plain language, $x_\perp$ minimises the distance from $x$ to the set $S$. We refer to $x_\perp$ as the orthogonal
projection of $\mathbf{x}$ onto $S$. A necessary and sufficient condition that $\mathbf{x}_\perp$ be the correct orthogonal projection is that

$$(\mathbf{x} - \mathbf{x}_\perp, \mathbf{a} - \mathbf{x}_\perp) \leq 0$$

(B.1)

for all $\mathbf{a} \in S$.

We require a few more definitions.

**Definition B.1** Given the Hilbert space $\Lambda_{\Sigma,r}$, the subspace $\Phi_k$ is defined, for $1 \leq k \leq N - 1$, as the set $\{\mathbf{x} \in \Lambda_{\Sigma,r} : x_k = 0\}$. In addition, we define the subspace $\Phi_N$ as the set $\{\mathbf{x} \in \Lambda_{\Sigma,r} : \operatorname{bal} \mathbf{x} = 0\}$. For $1 \leq k \leq N$, the orthogonal complement of $\Phi_k$ is denoted $\Phi_k^\perp$.

Note that $\Phi_N$ could have been defined, by analogy with the definitions of $\Phi_k$ for $k < N$, as the set

$$\Phi'_N = \{\mathbf{x} : x_N = 0\} = \{\mathbf{x} : \operatorname{bal} \mathbf{x} = \operatorname{bal} \mathbf{r}\}.$$ 

This set, however, is in general not a subspace of $\Lambda_{\Sigma,r}$, so instead we choose $\Phi_N$ as the unique translate of $\Phi'_N$ containing the origin, giving the definition stated above.

For each $k$, $\Phi_k$ is an $(N - 2)$-dimensional subspace of $\Lambda_{\Sigma,r}$, and hence $\Phi_k^\perp$ is a 1-dimensional subspace, spanned by any single element other than zero. This suggests the following:

**Definition B.2** The $k$th constraint direction is the unique vector $\phi^{(k)} \in \Lambda_{\Sigma,r}$ such that $\phi^{(k)} \in \Phi_k^\perp$ and

$$\phi^{(k)}_k = 1, \quad k < N$$

$$\operatorname{bal} \phi^{(k)} = -1, \quad k = N.$$ 

(B.2)

**Lemma B.1** Given $\mathbf{x} \in \Lambda_{\Sigma,r}$, $\beta \in \mathbb{R}$ and $1 \leq k \leq N$, the unique orthogonal projection of $\mathbf{x}$ onto the set $\{\mathbf{a} : a_k = \beta\}$ is the vector

$$\mathbf{x}_\perp = \mathbf{x} + (\beta - x_k)\phi^{(k)}.$$ 

(B.3)
Proof. It follows immediately from the definition of $\phi^{(k)}$ that $x_\perp$ as defined in (B.3) has $\beta$ for its $k$th component. The uniqueness of $x_\perp$ follows from the fact that $A_{\Sigma_r}$ is the direct sum of $\Phi_k$ and $\Phi_k^\perp = \text{span} \phi^{(k)}$, and the set $\{a : a_k = \beta\}$ is a translate of $\Phi_k$. However, we may also verify the lemma using the criterion (B.1) whereupon we find that $x - x_\perp$ is a vector proportional to $\phi^{(k)}$ (hence in $\Phi_k^\perp$), while if $a$ is any member of the set in question, then $a - x_\perp$ has zero for its $k$th component and therefore belongs to $\Phi_k$. It follows that $(x - x_\perp, a - x_\perp) = 0$ for any choice of $a$, satisfying (B.1). £

Lemma B.1 provides a simple way to determine immediately whether zero, one, or more than one constraint is active, and to find $\hat{d}$ in the former two cases.

**Theorem B.1** Assume $N \geq 2$. For $r_b \in A_{\Sigma_r}$ and $1 \leq k \leq N$, set

$$
c_k = \begin{cases} 
    r_b - r_k \phi^{(k)} & \text{if } r_k < 0 \\
    r_b - (r_k - D_k) \phi^{(k)} & \text{if } r_k > D_k \\
    r_b & \text{if } 0 \leq r_k \leq D_k
\end{cases}
$$

Then

1. Either all $c_k = r_b$, or there is at most one $c_k$ which is feasible.

2. If all $c_k = r_b$, then $r_b$ is feasible and $\hat{d}_b = r_b$.

3. If precisely one $c_k$ is feasible, then the constraint applied in the derivation of $c_k$ from $r_b$ is the one and only constraint active at $\hat{d}$, and $\hat{d}_b = c_k$.

4. If none of the $c_k$ are feasible, then there are two or more constraints active at $\hat{d}$.

*Proof.* The definition of $c_k$ implies that $c_k \neq r_b$ unless $0 \leq r_k \leq D_k$. Thus, if $c_k = r_b$, for all $k$ then $r_b$ satisfies all the constraints and so is feasible, proving (2). Conversely, if $r_b$ is feasible then $c_k = r_b$ for all $k$, and are themselves feasible.

Suppose now that among the vectors $c_k$, some are feasible and some infeasible. $r_b$ is then infeasible by the above reasoning. Let $k$ be such that $c_k$ is feasible; then, by Lemma B.1 and the defining equation (B.4), $c_k$ is the orthogonal projection of $r_b$ onto the feasible set. Since this set is convex, we infer by the Orthogonal Projection Theorem that $c_k = \hat{d}_b$, and, moreover, that $c_k$ is unique. It follows that there is only one constraint active at $\hat{d}$, namely that applied in the derivation of $c_k$ from $r_b$, proving (3). (The assumption $N \geq 2$ ensures that (2) and (3) are distinct cases.)
The uniqueness of the optimal solution vector given by the Orthogonal Projection Theorem implies that there can be only one orthogonal projection from an infeasible point onto the feasible set, hence that only one $c_k$ can be feasible when $r_i$ is not, proving (1).

The converse to (3) is also true: if only one constraint is active at $\hat{d}$, say the constraint on $\hat{d}_k$, then $c_k$ will be feasible. If on the other hand none of the $c_k$ are feasible, we infer from this that at least two constraints are active, proving (4). √

To aid in visualising the geometry of the problem, it may help to picture the problem as it appears when transformed to a standard Euclidean space of $N - 1$ dimensions. More explicitly, let $R$ denote the square root of $\Sigma$. Then for all $a, b \in \Lambda_{\Sigma R}$ we have

$$(a, b)_\Sigma = b^* \Sigma a = b^* R R a = a' \cdot b'$$

where $a' = Ra$, $b' = R^* b = Rb$, and $a' \cdot b'$ denotes the standard Euclidean inner product. Thus, the matrix $R$ expresses a linear transformation from $\Lambda_{\Sigma R}$ to the Euclidean space $\mathbb{R}^{N-1}$. Convexity of sets is preserved by this transformation. Note, however, that if we visualise the problem in this way, the discretionary orthotope $\Gamma_D$ ceases to have orthogonal faces.

### B.2 Outward Projections for an Arbitrary Convex Polytope

We now investigate the problem of locating the orthogonal (minimum-distance) projection from an arbitrary point onto an arbitrary convex polytope in an arbitrary number of dimensions. Several useful results can be obtained for this more general problem, which we proceed to apply to the Loss Minimisation Problem in the following sections.

Consider a $D$-dimensional closed convex polytope $\mathcal{P}$, embedded in a Hilbert space $H^D$ of $D$ dimensions. We denote the inner product of two vectors $a, b \in H^D$ by $(a, b)$ and the norm induced on $a$ by $\|a\|$.

Throughout this section, a closed convex set of topological dimension $n$ is referred to as an $n$-cell. Thus $\mathcal{P}$ itself is a $D$-cell, and is bounded by a finite set of $(D - 1)$-cells
termed the facets of $\mathcal{P}$. We let $N_F$ be the number of facets, and denote an individual facet by $F_i$, with $1 \leq i \leq N_F$.

With each facet we associate several objects, viz:

- The extension of $F_i$, defined as the unique hyperplane $\overline{F}_i$ in $H^D$ containing $F_i$;
- The outward normal of $F_i$, defined as the unique vector $\mathbf{n}_i$ such that $\mathbf{n}_i$ is orthogonal to $F_i$, $\|\mathbf{n}_i\| = 1$, and $(\mathbf{x} - \mathbf{a}, \mathbf{n}_i) \leq 0$ for all $\mathbf{x} \in \mathcal{P}$ and $\mathbf{a} \in \overline{F}_i$; and
- The associated half-space of $F_i$, defined as the set

\[ \hat{F}_i = \{ \mathbf{x} : (\mathbf{x} - \mathbf{a}_i, \mathbf{n}_i) > 0 \} \quad (B.5) \]

where $\mathbf{a}_i$ is an arbitrary vector in $\overline{F}_i$.

We denote by $[F_i F_j \ldots F_k]$ an intersection of facets $F_i \cap F_j \cap \ldots \cap F_k$, called the join of the facets. For the sake of brevity, if $I$ is an index set on the set of facets (that is, a set of integers $i_j$ such that $1 \leq i_j \leq N_F$ for all $j$ and $i_j \neq i_k$ when $k \neq j$), we denote the join of the corresponding facets by $[F_I] = [F_{i_1} F_{i_2} \ldots F_{i_{|I|}}]$. The number $|I|$ is referred to as the order of the join. A set of facets $\{F_i\}$ is termed (mutually) adjacent if their join $[F_I]$ has topological dimension $D - |I|$ or greater, and nondegenerate if their join has topological dimension precisely $D - |I|$. (Thus in Euclidean 3-space, two facets are adjacent if they meet at an edge, but not if they only meet at a corner.)

If two facets $F_i$ and $F_j$ are adjacent, their join $[F_i F_j]$ is a $(D - 2)$-cell. In fact any join of dimension $D - 2$ or greater is nondegenerate (since we assume $\mathcal{P}$ contains an interior point), but this is not necessarily true for dimension $D - 3$ or for lower dimensions. In general, the dimension of a nonempty join $[F_I]$ between facets $F_I$ is equal to $D - d$, where $d$ is the dimension of the subspace spanned by the normals $\mathbf{n}_j$. We refer to $d$ as the codimension of the join; a nondegenerate set, then, is simply one whose join has codimension equal to its order. It follows that a set of facets is nondegenerate if and only if they are mutually adjacent and their normals are linearly independent. In particular, a nondegenerate set contains at most $D$ facets, any mutually adjacent set of facets contains a nondegenerate set having the same join, and any set of facets having nonempty join but not mutually adjacent is contained in a nondegenerate set having the same join.
The above comments allow us to restrict our attention to joins of order at most $D$ between nondegenerate facets. In fact, we shall restrict this further and consider only sets of facets that form nondegenerate chains. By a \textit{chain} we mean a set of facets \( \{ F_I \} \) which can be ordered so that for each $k$ ($1 < k \leq |I|$) the facet $F_{i_k}$ is adjacent to at least one of the facets $F_{i_j}$ with $j < k$. Thus, a chain is built up by successively joining adjacent facets. (Any facet on its own is also considered to be a chain, as is the null set.) The nondegenerate chains are a subclass of the nondegenerate facet sets, but since any $k$-cell of $\mathcal{P}$ may be identified with the join of a nondegenerate chain, we lose nothing by restricting attention to this subclass.

The nondegenerate joins of order $D$ consist of single points (0-cells), known as the \textit{vertices} of $\mathcal{P}$. It can be shown that if precisely $D$ facets meet at each vertex (the polytope is \textit{simple}), then any mutually adjacent set of facets forms a nondegenerate chain. In this case we can say that two joins (or two chains) are distinct if they have distinct index sets, which simplifies matters a great deal. Otherwise, there will exist distinct sets of facets having the same join; we may say however that the joins corresponding to index sets $I$ and $I'$ are identical if and only if $\{ F_{I \cup I'} \}$ is a set of mutually adjacent facets. We denote by $\text{adj}(I)$ the index set $I'$ such that $F_{I'}$ is the maximal mutually adjacent set containing $[F_I]$. (Note that $\text{adj}(I) = I$ for all $I$ if and only if $\mathcal{P}$ is simple.) It can be shown that for any $I$ (provided $[F_I]$ is nonempty), the set $\text{adj}(I)$ is a (possibly degenerate) chain, and that $\{ F_{\text{adj}(I)} \}$ is identical with the set of all facets that contain $[F_I]$.

In what follows, $a_i$ denotes an arbitrary vector in $\overline{F_i}$, while $a_I$ denotes an arbitrary vector in $\bigcap_{i \in I} \overline{F_i}$. These vectors are used for the purpose of deciding on which side of $\overline{F_i}$ a given point $x$ lies. For any index set $I$ corresponding to a set of mutually adjacent facets, the set

$$
\Phi_I = \bigcap_{i \in I} \overline{F_i} - a_I
$$

is a subspace of $H^D$, such that

$$
\text{span}\{ n_I \} = \Phi_I ^{\perp}
$$

where $\Phi_I ^{\perp}$ denotes the orthogonal complement of $\Phi_I$. Equation (B.7) restates in more precise terms the above assertion concerning the codimension of the join $[F_I]$.

The following is the key definition of this section.
Definition B.3  Let $I$ be an index set such that the join $[F_I]$ is nondegenerate and of order $n \leq D$, and set

$$[F_I] = \bigcup_{i \in \text{adj}(I)} \hat{F}_i.$$  

The outward projection $[F_I]^+$ of $[F_I]$ is defined as the set of all $x \in [F_I]$ such that:

1. the orthogonal projection of $x$ onto $H^D \setminus [F_I]$ lies in $[F_I]$; and

2. $x$ does not belong to any outward projection of order less than $n$.

The outward projection of order zero is defined as $[\emptyset]^+ = \mathcal{P}$.

Condition (2) in the above definition has been added merely to ensure that the set of all outward projections defines a proper partition on the space $H^D$. In fact, the addition of this condition means that the restriction to $x \in [F_I]$ is, strictly speaking, unnecessary. For if $x$ belongs to the set $H^D \setminus [F_I]$, then its orthogonal projection onto this set is identical with $x$, thus is in $[F_I]$ only if $x$ itself is in $[F_I]$ and hence in $\mathcal{P} = [\emptyset]^+$. It follows that $x$ cannot belong to $[F_I]^+$ if $I$ is nonempty.

We use $\text{adj}(I)$ rather than simply $I$ in the definition of $[F_I]$ to ensure that $[F_I]^+ = [F_I]_r^+$ whenever $[F_I] = [F_I]_r$, thus giving a true one-to-one correspondence between the outward projections and the various $k$-cells of $\mathcal{P}$.

The importance of outward projections to our problem is suggested by the following theorem.

**Theorem B.2** If $I \neq \emptyset$ and $a \in [F_I]^+$, then there exists a vector $x_0 \in [F_I]$ which uniquely minimises $\|a - x\|$ over all $x \in \mathcal{P}$.

**Proof.** Theorem B.2 is a simple consequence of the Orthogonal Projection Theorem. By definition of $[F_I]^+$, the orthogonal projection of $a$ onto $H^D \setminus [F_I]$ lies in $[F_I] \subset \mathcal{P}$. This projection therefore satisfies the conditions for an orthogonal projection onto $\mathcal{P} \subset H^D \setminus [F_I]$, and this projection is unique by the Orthogonal Projection Theorem. $\square$

Before stating the conditions under which a vector belongs to a particular outward projection, some further definitions are necessary. To simplify the statements below, we say that an index $j$ is compatible with an index set $I$ if $\{F_I, F_j\}$ is a nondegenerate chain. It follows by definition that if $j$ is compatible with $I$, then $\{F_I\}$ is itself a
nondegenerate chain, and \( j \not\in I \). Two indices \( j \) and \( k \) are said to be *jointly* compatible with \( I \) if both are compatible with \( I \) and \( \{F_I, F_j, F_k\} \) is a nondegenerate chain.

Let \( I \) be the index set of a nondegenerate chain of order \( n \), and let the elements \( i_k \) of \( I \) be ordered to give a ‘chain ordering’ of the facets \( \{F_I\} \). (In general there will be many possible chain orderings.) Now define \( I_m \) as the subset containing the first \( m \) elements of \( I \), where \( 1 \leq m \leq n \). It follows that each \( I_m \) is itself the index set of a nondegenerate chain of order \( m \). Also define \( I_0 = \emptyset \), with the convention \( F_\emptyset = \emptyset \).

Let \( \text{norm}(a) \) denote the normalisation of a vector \( a \), that is, the vector \( a/\|a\| \). (It follows that \( a \) and \( \text{norm}(a) \) always differ by a positive real factor.) For all nondegenerate chain index sets \( I \) of order \( n < D \), suitably ordered, and all \( j \) compatible with \( I \), define vectors \( [F_I]_j \) recursively as follows:

\[
[\emptyset]_j = n_j
\]
\[
[F_I]_j = \text{norm} \left( n_j - \sum_{m=1}^{n} ([F_{I_{m-1}}]_j, n_j) [F_{I_{m-1}}]_j \right). \tag{B.8}
\]

What is in fact being described here is simply the Gram-Schmidt procedure for obtaining an orthonormal set from a linearly independent set of vectors. In other words, the set

\[
\{[F_{I_0}]_1, [F_{I_1}]_2, \ldots, [F_{I_n}]_j = [F_I]_j \}
\]

is obtained by the Gram-Schmidt procedure from the set \( \{n_I, n_j\} \). It is a well-known fact that the value of the last vector obtained in this set (in this case \( [F_I]_j \)) is independent of the order in which the first \( n \) initial vectors (the \( n_I \)) are taken. Thus \( [F_I]_j \) is well-defined by (B.9) even if we allow the chain ordering on \( I \) to be purely arbitrary. Note also that the set (B.10) forms an orthonormal basis for the subspace \( \Phi^+_{I \cup j} \). (For the sake of brevity we shall use the notation \( I \cup j \) in place of \( I \cup \{j\} \) to denote an index set augmented by one element.)

We are now ready to give the necessary and sufficient conditions for a vector to belong to \( [F_I]^+ \), given \( I \).

**Theorem B.3** Let \( \{F_I\} \) be a nondegenerate chain of order \( n \leq D \). A vector \( x \in H^D \) is in \( [F_I]^+ \) if and only if

\[
(x - a_{F_I}, [F_I]_j) > 0 \quad \tag{B.11}
\]

\[338\]
for all nondegenerate chain subsets $I' \subset \text{adj}(I)$ of order $n - 1$ and all $j \in \text{adj}(I)$ compatible with $I'$, and

$$ (x - a_I, [F_I]_{i_j}) \leq 0 $$  \hspace{1cm} (B.12)

for all $j$ compatible with $I$.

If $n = D$, then only conditions of the form (B.11) apply. Similarly, if $n = 0$ then only conditions of the form (B.12) apply.

If $\text{adj}(I) = I$, then (B.11) reduces to

$$ (x - a_I, [F_I \setminus \{i_m\}]_{i_m}) > 0 $$  \hspace{1cm} (B.13)

for $1 \leq m \leq n$.

We note the following special cases.

- When $n = 0$, we have $[\emptyset]^+ = \mathcal{P}$ and $[\emptyset]_j = n_j$ for $1 \leq j \leq N_F$. A vector $x$ is in $\mathcal{P}$ if and only if $(x - a_I, n_j) \leq 0$ for all $j$, a fact consistent with (B.12).

- When $n = 1$, $[F_i]^+$ is a semi-infinite prism in the half-space $\hat{F}_i$ with end face $F_i$ (not included in $[F_i]^+$), and a side face corresponding to each facet $F_j$ adjacent to $F_i$ (since two facets form a nondegenerate chain if and only if they are adjacent). The normals to the side faces are given by (B.9) as

$$ [F_i]_{i_j} = \text{norm}(n_j - (n_i, n_j)n_i). $$  \hspace{1cm} (B.14)

The set $[F_i]^+$ is convex, though not closed, and infinite in extent. A vector $x \in [F_i]^+$ if and only if $(x - a_i, n_i) > 0$ and $(x - a_{ij}, [F_i]_{i_j}) \leq 0$ for all $j$ such that $F_j$ is adjacent to $F_i$.

- When $n = D - 1$, all the vectors $[F_I]_{i_j}$ are identical up to a sign change, as they span a 1-dimensional set. The applicable $j$ correspond to the facets $F_j$ having a vertex in common with the facets $\{F_I\}$ and being adjacent to at least one of them.

The conditions (B.11) and (B.12) for membership of $[F_I]^+$ are somewhat unwieldy to work with directly. However, the process of assigning an arbitrary point to a particular outward projection can be simplified with the help of some ‘background knowledge’ concerning the signs of the inner products $(n_j, n_k)$ and, by extension, the inner products $([F_I]_{i_j}, [F_I]_{i_k})$ for various index sets $I$. We begin by proving the following result, which is a generalisation of (B.14) to arbitrary orders.
Lemma B.2 Let \( \{F_i\} \) be a nondegenerate chain (possibly empty), and let \( j \) and \( k \) be jointly compatible with \( I \). Then

\[
[F_{I,j}\vert_{[F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}}] = \text{norm}(\langle [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}, [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}] \rangle) \\
= \frac{[F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}}{\sqrt{1 - (\langle [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}], [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}] \rangle)^2}}. 
\]

(B.15)

Proof. Let \( n = |I| \), choose a chain ordering on \( I \) and let \( I_m \), for \( 0 \leq m \leq n \), be the subset sequence induced by this ordering. Now, for some positive real normalisation constants \( \alpha_j, \alpha_k \) we have

\[
\langle [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}, [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}] \rangle \\
= \alpha_j \alpha_k \left( n_j - \sum_{m=1}^{n} \langle [F_{I_{m-1}}]_{i_m}, n_j \rangle [F_{I_{m-1}}]_{i_m} n_k - \sum_{m=1}^{n} \langle [F_{I_{m-1}}]_{i_m}, n_k \rangle [F_{I_{m-1}}]_{i_m} n_j \right) \\
= \alpha_j \alpha_k \left( n_j n_k - 2 \sum_{m=1}^{n} \langle [F_{I_{m-1}}]_{i_m}, n_j \rangle \langle [F_{I_{m-1}}]_{i_m}, n_k \rangle \right) \\
+ \sum_{m=1}^{n} \sum_{m'=1}^{n} \langle [F_{I_{m-1}}]_{i_m}, n_j \rangle \langle [F_{I_{m'-1}}]_{i_{m'}}, n_k \rangle \langle [F_{I_{m'-1}}]_{i_{m'}}, [F_{I_{m-1}}]_{i_m} \rangle \\
= \alpha_j \alpha_k \left( n_j n_k - \sum_{m=1}^{n} \langle [F_{I_{m-1}}]_{i_m}, n_j \rangle \langle [F_{I_{m-1}}]_{i_m}, n_k \rangle \right).
\]

The last line follows from the fact that, since the recursive definition of the \([F_{I_{m-1}}]_{i_m}\) follows the Gram-Schmidt process, we have

\[
\langle [F_{I_{m-1}}]_{i_m}, [F_{I_{m'-1}}]_{i_{m'}} \rangle = \delta(m, m')
\]

where \( \delta(m, m') \) is the Kronecker delta function, equal to 1 when \( m = m' \) and zero otherwise.

By virtue of the above identity, we have

\[
\alpha_k ([F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}, n_k) = \alpha_j \alpha_k \left( n_j - \sum_{m=1}^{n} \langle [F_{I_{m-1}}]_{i_m}, n_j \rangle [F_{I_{m-1}}]_{i_m} n_k \right) \\
= \langle [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}, [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}] \rangle
\]

and similarly

\[
\alpha_j ([F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}, n_j) = \langle [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}}, [F_I]_{[F_I]_{[F_I]_{[F_I]_j}}] \rangle.
\]

340
So, augmenting the index set \( I \) by the index \( j \) gives
\[
[F_{I \cup j}]_k = \text{norm} \left( n_k - \sum_{m=1}^{n} ([F_{I_{m-1}}]_{i_m}, n_k) [F_{I_{m-1}}]_{i_m} - ([F_I]_j, n_k) [F_I]_j \right)
\]
\[
= \text{norm}([F_I]_k - \alpha([F_I]_j, n_k) [F_I]_j)
\]
\[
= \text{norm}([F_I]_k - ([F_I]_j, [F_I]_k) [F_I]_j).
\]

To determine the normalisation factor for \([F_{I \cup j}]_k\) we calculate the (inner product induced) norm of the vector in parentheses above:
\[
\|[F_I]_k - ([F_I]_j, [F_I]_k) [F_I]_j\|^2
\]
\[
= \|[F_I]_k\|^2 - 2([F_I]_j, [F_I]_k)^2 + ([F_I]_j, [F_I]_k)^2\|[F_I]_j\|^2
\]
\[
= 1 - ([F_I]_j, [F_I]_k)^2.
\]

To obtain \([F_{I \cup j}]_k\) we divide by the norm to obtain
\[
[F_{I \cup j}]_k = \frac{[F_I]_k - ([F_I]_j, [F_I]_k) [F_I]_j}{\sqrt{1 - ([F_I]_j, [F_I]_k)^2}}
\]

which is the result stated above. √

The following lemmas establish some additional compatibility conditions for non-degenerate chains which will be necessary later.

**Lemma B.3** Let \( \{F_I\} \) be a nondegenerate chain of order 2 or greater. Then the index set \( I \) has a representation
\[ I = I_0 \cup \{j, k\} \quad (B.16) \]

where \( I_0 \) is the index set of a nondegenerate chain (possibly empty) and \( j \) and \( k \) are jointly compatible with \( I_0 \).

**Proof.** By identifying the vertices of a graph with the facets of a chain, and connecting two vertices by an edge whenever the corresponding facets are adjacent, one can deduce a correspondence between nondegenerate chains of order \( n \) and connected graphs on \( n \) nodes. Given such a graph representation of a nondegenerate chain, it follows that every connected subgraph obtained by deleting vertices also corresponds to a nondegenerate chain. The result we require then proceeds immediately from the observation that a spanning tree in a connected graph always has at least two vertices of degree 1; deletion of either or both of these vertices leaves the graph connected. These vertices correspond to two facets \( F_j \) and \( F_k \) whose indices, by definition, are jointly compatible with \( I_0 \). √
Corollary B.1 Let \( \{F_I\} \) be a nonempty nondegenerate chain and let \( j \) be compatible with \( I \). Then there exists \( k \in I \) such that \( I_0 = I \setminus k \) is the index set of a nondegenerate chain, and \( j \) and \( k \) are jointly compatible with \( I_0 \).

Proof. Let \( I'' = I \cup j \) and apply Lemma B.3 to \( \{F_{I''}\} \) (which is a nondegenerate chain by assumption). In our graph-theoretic analogy, the graph of \( \{F_{I''}\} \) has a spanning tree in which \( F_j \) is a vertex of degree 1. We know there is at least one other such vertex, which we identify with \( F_k \). The result then follows. √

Lemma B.4 Let \( \{F_I\} \) be a nonempty chain of codimension \( d \) and let \( \{F_{I'}\} \subset \{F_I\} \) be a nondegenerate chain of order \( d - 1 \). Then there exists \( j \in I \) compatible with \( I' \). Furthermore, the indices compatible with \( I' \) correspond to those facets in \( \{F_{I\setminus \text{adj}(I')}\} \) which are adjacent to facets in \( \{F_{I'}\} \).

Proof. Since \( \{F_I\} \) is of codimension \( d \), there exist facets in \( \{F_I\} \) whose intersection with \( [F_{I'}] \) is not equal to \( [F_{I'}] \). It is easily seen that these facets are precisely the set \( \{F_{I\setminus \text{adj}(I')}\} \), and any one of these facets forms a nondegenerate set with \( \{F_{I'}\} \), by virtue of an increase in codimension. Since \( \{F_I\} \) is a chain, it follows that at least one of these facets is adjacent to some facet in \( \{F_{I'}\} \), hence forms a (nondegenerate) chain with \( \{F_{I'}\} \). √

Lemma B.5 Let \( \{F_I\} \) be a chain of codimension \( d \geq 2 \), and let \( \{F_{I_0}\} \) be a nondegenerate chain of order \( d - 2 \) with \( I_0 \subset \text{adj}(I) \). Given any \( j \in \text{adj}(I) \) compatible with \( I_0 \), there exists \( k \in \text{adj}(I) \) such that \( j \) and \( k \) are jointly compatible with \( I_0 \).

Proof. Consider a facet \( F_i \in \{F_{I_0}\} \). Since \( [F_i] \) is of lower dimension than \( [F_{I_0\cup j}] \), there is a facet \( F_k \) adjacent to \( F_i \) such that the intersection \( F_i \cap F_k \) contains \( [F_i] \) but does not contain \( [F_{I_0\cup j}] \). It follows that \( k \) belongs to \( \text{adj}(I) \) but not to \( \text{adj}(I_0 \cup j) \), whence \( k \) is compatible with both \( I_0 \) and \( I_0 \cup j \) by Lemma B.4. Thus \( j \) and \( k \) are jointly compatible with \( I_0 \) by definition. √

These results allow us to make some general observations about the relationship between various sets related to the outward projections. To simplify the statement of these results we make one further definition:

Definition B.4 Given vectors \( a \) and \( r \) in \( H^D \), we denote by \( H(r, a) \) the half-space \( \{x : (x - a, r) > 0\} \).
We observe in passing that \( H(r, a) = H(r, b) \) if and only if \((a - b, r) = 0\), and that 
\( H(n_i, a_i) = \hat{F}_i \) for any \( a_i \in \hat{F}_i \).

**Theorem B.4** Let \( \{F_I\} \) be a nondegenerate chain of order 2 or greater, and let 
\( I = I_0 \cup \{j, k\} \) where \( j \) and \( k \) are jointly compatible with \( I_0 \). Define

\[
[F_I]_{j,k} = H([F_{I_0,j}]_k, a_I) \cap H([F_{I_0,k}]_j, a_I) = [F_I]_{k,j}.
\]  

(B.17)

Then:

1. \( [F_I]^+ \subseteq [F_I]_{j,k} \subseteq H([F_{I_0,j}], a_I) \cup H([F_{I_0,k}], a_I) \).

2. If \(([F_{I_0,j}], [F_{I_0,k}] \geq 0\), then \([F_I]_{j,k} \subseteq H([F_{I_0,j}], a_I) \cap H([F_{I_0,k}], a_I)\).

3. If \(([F_{I_0,j}], [F_{I_0,k}] \leq 0\), then \([F_I]_{j,k} \supseteq H([F_{I_0,j}], a_I) \cap H([F_{I_0,k}], a_I)\).

**Proof.** Suppose \( x \in [F_I]_{j,k} \), and set \( c = \sqrt{1 - ([F_{I_0,j}], [F_{I_0,k}]^2)\}. \) Then, by Lemma B.2,

\[
c(x - a_I, [F_{I_0,j}]_k) = (x - a_I, [F_{I_0,k}] - ([F_{I_0,j}], [F_{I_0,k}]) (x - a_I, [F_{I_0,j}] > 0 \) \quad \text{(B.18)}
\]
\[
c(x - a_I, [F_{I_0,k}]_j) = (x - a_I, [F_{I_0,k}] - ([F_{I_0,j}], [F_{I_0,k}]) (x - a_I, [F_{I_0,k}] > 0 \) \quad \text{(B.19)}
\]

Now suppose that \( x \not\in H([F_{I_0,j}], a_I) \cup H([F_{I_0,k}], a_I) \); we show that this leads to a contradiction. This assumption gives \((x - a_I, [F_{I_0,j}] \leq 0 \) and \((x - a_I, [F_{I_0,k}] \leq 0 \). If

in addition \(([F_{I_0,j}], [F_{I_0,k}] \leq 0 \), we have an immediate contradiction, since then neither

of the conditions (B.18) and (B.19) are satisfied. Assume then that \(([F_{I_0,j}], [F_{I_0,k}] > 0 \),

and consider the case \((x - a_I, [F_{I_0,j}] \leq (x - a_I, [F_{I_0,k}] \leq 0 \). For (B.19) to be satisfied

we require

\[
([F_{I_0,j}], [F_{I_0,k}] \frac{(x - a_I, [F_{I_0,k}])}{(x - a_I, [F_{I_0,j}])} > 1.
\]

But by assumption this cannot hold since \(([F_{I_0,j}], [F_{I_0,k}] \leq 1 \) by the Cauchy-Schwarz inequality. Symmetrical reasoning shows that the remaining case \((x - a_I, [F_{I_0,k}] \leq (x - a_I, [F_{I_0,j}] \leq 0 \) leads to a similar contradiction. Therefore \([F_I]_{j,k} \subseteq H([F_{I_0,j}], a_I) \cup \)

\( H([F_{I_0,k}], a_I) \). But \( [F_I]_{j,k} \) is by definition a convex set (being the intersection of two

half-spaces), whereas the union of two half-spaces cannot be convex unless the normals

are linearly dependent, and linear dependence of \([F_{I_0,j}], [F_{I_0,k}] \) would contradict the

nondegeneracy of \( \{F_I\} \). The outward projection \( [F_I]^+ \) is contained in the set \( [F_I]_{j,k} \)

by condition (B.11) of Theorem B.3, establishing part 1 of the theorem.

343
If \( ([F_0]_j, [F_0]_k) \geq 0 \), then neither \( (x-a_I, [F_0]_j) \) nor \( (x-a_I, [F_0]_k) \) can be zero; nor can they have opposite signs, as this would falsify one of the conditions (B.18) or (B.19). Together with part 1, this implies part 2 of the theorem.

Finally, if \( ([F_0]_j, [F_0]_k) \leq 0 \), then both conditions (B.18) and (B.19) are satisfied whenever \( (x-a_I, [F_0]_j) \) and \( (x-a_I, [F_0]_k) \) are both positive, establishing part 3. \( \square \)

According to Theorem B.4, much can be learned about the geometry of the outward projections from a knowledge of the signs of the inner products \( ([F_I]_j, [F_I]_k) \) for various index sets \( I \) and compatible indices \( j \) and \( k \). In particular, it can be seen that whenever one of these inner products is equal to zero, the corresponding set \( \widetilde{[F_{I\cup\{j,k\}}]_{j,k}} \), which contains the outward projection \( [F_{I\cup\{j,k\}}]^{+} \), is precisely equal to the intersection of the two half-spaces defined by \( [F_I]_j \) and \( [F_I]_k \). Here \( I \) can be the null set, in which case we obtain information about the outward projection \( [F_{jF_k}]^{+} \) in terms of the half-spaces \( \hat{F}_j \) and \( \hat{F}_k \).

If \( I \) is not empty, we may be able to carry the process a step further. Let \( [F_I]^* \) be the set of all points satisfying just the conditions (B.11) of Theorem B.3. In other words,

\[
[F_I]^* = \bigcap H([F_I]_j, a_I) \tag{B.20}
\]

where the intersection is taken over all nondegenerate chain subsets \( I' \subset \text{adj}(I) \) of order \( n-1 \) and all \( j \in \text{adj}(I) \) compatible with \( I' \). Consider one such ‘admissible pair’ \( I', j \). By Corollary B.1 there exists \( k \in I' \) such that the pair \( I'', k \) where \( I'' = (I' \setminus k) \cup j \) is also an admissible pair. Accordingly, the set \( [F_I]^* \) is contained in the intersection

\[
H([F_I]_j, a_I) \cap H([F_{I''}]_k, a_I) = \widetilde{[F_{I''\cup j}]_{j,k}}.
\]

Applying this argument over all admissible pairs gives the identity

\[
[F_I]^* = \bigcap \widetilde{[F_{I']}_{j,k}} \tag{B.21}
\]

where the intersection is now taken over all nondegenerate chain subsets \( I' \subset \text{adj}(I) \) of order \( n \) and all \( j, k \in I' \) such that \( I' \setminus j \) and \( I' \setminus k \) are index sets of nondegenerate chains. (Corollary B.1 then guarantees that \( I' \setminus \{j, k\} \) is also a nondegenerate chain index set, with which \( j \) and \( k \) are jointly compatible.) Note that if \( I = \text{adj}(I) \), then there is only one choice for \( I' \), namely \( I \) itself.
With the aid of Corollary B.1, we may restate the identity (B.21) as

\[ [F_I]^* = \bigcap [F_{I^{n,j,k}}]_{j,k} \]  

(B.22)

with the intersection taken over all nondegenerate chain subsets \( I'' \subset \text{adj}(I) \) of order \( n - 2 \), and all \( j \) and \( k \) jointly compatible with \( I'' \).

Suppose now (for example) that \( ([F_{I''}]_j, [F_{I''}]_k) = 0 \) for all such \( I'' \), \( j \) and \( k \). By Theorem B.4 we then have

\[ [F_I]^* = \bigcap H([F_{I''}]_j, a_I) \]  

(B.23)

with the intersection taken over all nondegenerate chain subsets \( I'' \subset \text{adj}(I) \) of order \( n - 2 \) and all \( j \in \text{adj}(I) \) compatible with \( I'' \). (Thanks to Lemma B.5, we are able to drop the reference to joint compatibility at this point.) Equation (B.23) is directly analogous to Equation (B.20) for \( [F_I]^* \), except that we have now succeeded (thanks to some favourable orthogonality conditions) in reducing the order from \( n - 1 \) to \( n - 2 \).

If the inner products \( ([F_{I''}]_j, [F_{I''}]_k) \) are merely uniform in sign, rather than strictly zero, we can still derive weaker results based on Theorem B.4 and the observation that, if \( \{A_1, A_2, \ldots, A_N\} \) and \( \{B_1, B_2, \ldots, B_N\} \) are two collections of sets such that \( A_i \subseteq B_i \) for each \( i \), then the intersection of all the \( A_i \) is contained in the intersection of all the \( B_i \). Before giving these more general results, we provide recursive formulae for the inner products concerned.

**Lemma B.6** Let \( \{F_I\} \) be a nondegenerate chain and let \( j, k \) be jointly compatible with \( I \). Then

\[ ([F_{I^{n,j}}]_k, [F_{I^{n,j}}]_j) = -\frac{([F_I]_j, [F_I]_k)}{1 + ([F_I]_j, [F_I]_k)}. \]  

(B.24)

**Lemma B.7** Let \( \{F_I\} \) be a nondegenerate chain, and let \( i, j, k \) be such that \( \{i, j\} \) and \( \{i, k\} \) are each jointly compatible with \( I \). Then

\[ ([F_{I^{n,i}}]_j, [F_{I^{n,i}}]_k) = \frac{([F_I]_j, [F_I]_k) - ([F_I]_i, [F_I]_j)([F_I]_i, [F_I]_k)}{\sqrt{(1 - ([F_I]_i, [F_I]_j)^2)(1 - ([F_I]_i, [F_I]_k)^2)}}. \]  

(B.25)

It is worth noting that this last result has a fairly simple geometric interpretation. Consider a triangular cone \( C \) in Euclidean 3-space with apex angles \( \alpha, \beta \) and \( \gamma \). Let

345
\( \theta \) denote the dihedral angle opposite \( \alpha \); that is, the angle in a cross-section of \( C \) taken perpendicular to the edge opposite \( \alpha \). Then

\[
\cos \theta = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma}.
\]

The inner products in (B.25) may be identified with the cosines of the angles in \( C \) in an obvious manner, whereby it transpires that \( C \) is just the cone formed by the vectors \([F_i]_j, [F_i]_j\text{ and } [F_i]_k\), and the inner product \(([F_{i,i}]_j, [F_{i,i}]_k)\) is the cosine of the dihedral angle at the edge defined by \([F_i]_i\). This is a geometrical consequence of the fact that \([F_{i,i}]_j\) is defined to be orthogonal to \([F_i]_i\) and to lie in the span of \([F_i]_i\) and \([F_i]_j\), and similarly for \([F_{i,i}]_k\).

Using Lemma B.6 and (particularly) Lemma B.7, it is possible to generate all inner products of the form \(([F_i]_j, [F_i]_k)\) or \(([F_{i,i}]_j, [F_{i,i}]_k)\) recursively from the inner products of the normals \( n_i \), without explicitly calculating the vectors \([F_i]_j\). The following Theorem (the proof of which was sketched above) shows how these inner products may be used in certain special cases to derive reduction formulae for the sets \([F_i]^*\).

**Theorem B.5** Let \( \{F_i\} \) be a nondegenerate chain of order \( n \geq 2 \), and define \([F_i]^{(m)}\) for \( 0 \leq m < n \) by

\[
[F_i]^{(m)} = \bigcap H([F_i]_j, a_I)
\]

where the intersection is taken over all nondegenerate chain subsets \( I' \subset \text{adj}(I) \) of order \( m \) and all \( j \in \text{adj}(I) \) compatible with \( I' \). Then

1. \([F_i]^{(n-1)} = [F_i]^*\), the set of all points satisfying just the conditions (B.11) of Theorem B.3. It immediately follows that \([F_i]^+ \subseteq [F_i]^{(n-1)}\), with equality if \( n = D \).

2. \([F_i]^{(0)}\) is an intersection of fundamental half-spaces:

\[
[F_i]^{(0)} = \bigcap_{i \in \text{adj}(I)} \hat{F}_i.
\]

3. If \(([F_i]_j, [F_i]_k) \geq 0 \text{ for all nondegenerate chain subsets } I' \subset \text{adj}(I) \text{ of order } m - 1 \text{ and all } j, k \in \text{adj}(I) \text{ jointly compatible with } I'\), then

\[
[F_i]^{(m)} \subseteq [F_i]^{(m-1)}.
\]
4. If \([F_I], [F_I]' \leq 0\) for all nondegenerate chain subsets \(I' \subset \text{adj}(I)\) of order \(m - 1\) and all \(j, k \in \text{adj}(I)\) jointly compatible with \(I'\), then
\[
[F_I]^{(m)} \geq [F_I]^{(m-1)}.
\] (B.29)

### B.3 Computing the constraint directions \(\phi^{(k)}\)

We now proceed to apply the above theory to the loss minimisation problem, by calculating the vectors \(\phi^{(k)}\). We show that the \(\phi^{(k)}\) can be obtained in a simple manner directly from the cable network topology.

Recall that for \(1 \leq k \leq N - 1\), \(\phi^{(k)}\) is orthogonal to the subspace \(\Phi_k\) of vectors \(x \in \Lambda_{\Sigma, r}\) such that \(x_k = 0\). A basis for this subspace is the set
\[
\{e_1, e_2, \ldots, e_{k-1}, e_{k+1}, \ldots, e_{N-1}\}
\]
and \(\phi^{(k)}\) is accordingly \(\Sigma\)-orthogonal to each of these vectors. We therefore have
\[
(\phi^{(k)}, e_i)\Sigma = e_i^\top \Sigma \phi^{(k)} = \sigma_i^\top \phi^{(k)} = 0, \quad i \neq k
\] (B.30)
where \(\sigma_i\) denotes the \(i\)th row of \(\Sigma\). These \(N - 2\) equations, together with the defining equation \(\phi^{(k)}_k = 1\), completely determine \(\phi^{(k)}\).

Now let \(\gamma_k\) denote the \(k\)th column of the matrix \(\Sigma^{-1} = A\Sigma^T\). From the definition of matrix inverses it follows that \(\sigma_i^\top \gamma_k = 0\) whenever \(i \neq k\), thus \(\gamma_k\) satisfies the \(N - 2\) equations (B.30). Since any scalar multiple of \(\gamma_k\) will satisfy the same equations, we may immediately write
\[
\phi^{(k)} = Y_{kk}^{-1} \gamma_k
\] (B.31)
where \(Y_{kk}\) is the \(k\)th component of \(\gamma_k\), or equivalently the \(k\)th element on the main diagonal of \(A\Sigma^T\). A simple calculation will show that this is just the sum of the admittances \(y_j\) of all cable segments connected to the node \(\bar{k}\), and is therefore always positive in a connected network of positive resistances.

Equation (B.31) for \(\phi^{(k)}\) yields the following result:

**Theorem B.6** Let \(\mathcal{N} = \mathcal{C} \cup \mathcal{E}\) be a managed DC distribution network with strictly positive cable resistances and \(N\) external circuits. Let \(\phi^{(k)}\) be the \(k\)th constraint direction associated with \(\mathcal{N}\), where \(1 \leq k \leq N - 1\). If \(i \neq k\), then \(\phi^{(k)}_i = -y_{ki}/y_{kk}\) where \(y_{kk}\) is the sum of cable admittances meeting at \(\bar{k}\), and \(y_{ki}\) is the admittance of the cable segment(s) connecting \(\bar{k}\) and \(\bar{i}\).
Proof. In (B.31), the \( i \)th component of \( \gamma_k \) is equal to the sum of \( a_{kj}a_{ij}y_j \) over all cable segments \( j \). If \( i = k \) (giving \( Y_{kk} \)), then the product \( a_{kj}a_{ij} = a_{kj}^2 \) is 1 when segment \( j \) is incident with \( \overline{k} \) and zero otherwise. If on the other hand \( i \neq k \), the product \( a_{kj}a_{ij} \) is zero except when segment \( j \) connects \( \overline{k} \) and \( \overline{7} \), in which case it is \(-1\). The result follows immediately. \( \sqrt{} \)

**Corollary B.2** If \( 1 \leq k \leq N - 1 \), then bal \( \phi^{(k)} = Y_{kN}/Y_{kk} \) where \( Y_{kN} \) is the admittance of the cable segment(s) connecting \( \overline{k} \) to the datum node \( \overline{N} \). In particular, if \( \overline{k} \) has no connections to \( \overline{N} \), then \( \phi^{(k)} \) is balanced.

The remaining constraint direction \( \phi^{(N)} \) is calculated as follows. The vector we seek must be \( \Sigma \)-orthogonal to the balanced subspace \( \Phi_N \) in \( \Lambda_{\Sigma,r} \), a basis for which is

\[
\{ e_1 - e_2, e_2 - e_3, \ldots, e_{N-2} - e_{N-1} \}.
\]

We therefore require

\[
(e_i^* - e_{i+1}^*) \Sigma \phi^{(N)} = (\sigma_i^* - \sigma_{i+1}^*) \phi^{(N)} = 0, \quad 1 \leq i \leq N - 2.
\]

(B.32)

These equations, together with the condition bal \( \phi^{(N)} = -1 \), completely determine \( \phi^{(N)} \).

Now set \( b = \sum_{k=1}^{N-1} \gamma_k \), that is, the sum across all columns of \( AYA^T \). Then for all \( i \) we have \( \sigma_i^*b = 1 \), again by the definition of matrix inverses, and \( (\sigma_i^* - \sigma_{i+1}^*)b = 1 - 1 = 0 \). In other words, \( b \) or any scalar multiple of it will satisfy the equations (B.32), so we may write

\[
\phi^{(N)} = -B^{-1}b = -B^{-1} \sum_{k=1}^{N-1} \gamma_k
\]

(B.33)

where \( B = \text{bal } b \). This leads to the following result:

**Theorem B.7** Let \( \mathcal{N} = \mathcal{C} \cup \mathcal{E} \) be a managed DC distribution network with strictly positive cable resistances and \( N \) external circuits. Let \( \phi^{(N)} \) be the \( N \)th constraint direction associated with \( \mathcal{N} \). Then \( \phi^{(N)}_i = -Y_{iN}/Y_{NN} \) where \( Y_{NN} \) is the sum of cable admittances meeting at \( \overline{N} \), and \( Y_{IN} \) (defined in Corollary B.2) is the admittance of the cable segment(s) connecting \( \overline{i} \) and \( \overline{N} \).

**Corollary B.3** The \( N \) constraint directions are a linearly dependent set of vectors which nonetheless span \( \Lambda_{\Sigma,r} \). The directions are related by the dependency equation

\[
\sum_{k=1}^{N} Y_{kk} \phi^{(k)} = 0.
\]

(B.34)
Proof. For 1 ≤ k ≤ N − 1 we have Y_{kk} φ^{(k)} = γ_k, while Y_{NN} φ^{(N)} = −∑_{k=1}^{N-1} γ_k. Equation (B.34) follows immediately.

Since AYAT is nonsingular, the vectors γ_k and thus φ^{(k)} for 1 ≤ k ≤ N − 1 together span ΛΣ_r. (B.34) then guarantees that if any one vector is removed from the set φ^{(k)}, 1 ≤ k ≤ N, the remaining vectors span ΛΣ_r. √

Two immediate but very useful results concerning the φ^{(k)} are contained in the following lemma.

**Lemma B.8** Let 1 ≤ i ≤ N, 1 ≤ j ≤ N with j ≠ i. Then

\[ \|φ^{(i)}\|_Σ = \frac{1}{\sqrt{Y_{ii}}} \quad \text{and} \quad (φ^{(i)}, φ^{(j)})_Σ = -\frac{Y_{ij}}{Y_{ii}Y_{jj}}. \]  

**Proof.** Set γ_N = −∑_{i=1}^{N-1} γ_i. Then the kth component of γ_N is −Y_{kN}, its balance is bal γ_N = −Y_{NN}, and φ^{(N)} = Y_{NN}^{-1} γ_N.

The Σ-norm of φ^{(i)} for all i is

\[ \|φ^{(i)}\|_Σ = \sqrt{(φ^{(i)}, φ^{(i)})_Σ} = \frac{1}{\sqrt{Y_{ii}}} \sqrt{γ_i^2}. \]

If i < N, then Σγ_i = e_i by definition of Σ, so

\[ \|φ^{(i)}\|_Σ = \frac{1}{\sqrt{Y_{ii}}} \sqrt{γ_i^2 e_i} = \frac{1}{\sqrt{Y_{ii}}}, \]

as required. If on the other hand i = N, then Σγ_N = −1_{N−1} and so

\[ \|φ^{(N)}\|_Σ = \frac{1}{Y_{NN}} \sqrt{-bal γ_N} = \frac{1}{\sqrt{Y_{NN}}}. \]

By a similar argument, when i < N and j < N, j ≠ i we have

\[ (φ^{(i)}, φ^{(j)})_Σ = \frac{1}{Y_{ii}Y_{jj}} γ_j^* Σγ_i = \frac{1}{Y_{ii}Y_{jj}} γ_j^* e_i = \frac{-Y_{ij}}{Y_{ii}Y_{jj}}. \]

When i = N and j < N we have

\[ (φ^{(N)}, φ^{(j)})_Σ = \frac{1}{Y_{NN}Y_{jj}} γ_j^*(-1_{N−1}) = \frac{-bal γ_j}{Y_{NN}Y_{jj}} = \frac{-Y_{jN}}{Y_{NN}Y_{jj}}. \]

again as required. The remaining case i < N, j = N follows by symmetry. √

In the light of the results of the preceding section, the following corollary to Lemma B.8 is of great significance.

349
Corollary B.4 Let $\mathcal{N}$ be a managed DC distribution network with positive cable resistances and $N$ external circuits, and let $\phi^{(i)}$ denote the $i$th constraint direction, where $1 \leq i \leq N$. Then the following statements hold for any $j \neq i$, $1 \leq j \leq N$:

1. If nodes $i$ and $j$ are connected by a cable segment, then $(\phi^{(i)}, \phi^{(j)})_{\Sigma} < 0$.
2. If nodes $i$ and $j$ are not connected by a cable segment, then $(\phi^{(i)}, \phi^{(j)})_{\Sigma} = 0$.

B.4 Elements of an Algorithm

In order to apply the results of Section B.2, we need to identify the facets of the polytope $\mathcal{P}$. For each $i$, denote by $L_i$ the constraint $d_i \geq 0$, and by $U_i$ the constraint $d_i \leq D_i$. Each of these constraints has an associated half-space, the intersection of which is $\mathcal{P}$. Assuming for the time being that all $D_i > 0$ (see Section 4.2.3 for the case where they are not), $\mathcal{P}$ has topological dimension $D = N - 1$. Since there are just $2N$ constraints, it follows that $N \leq N_F \leq 2N$.

We say a constraint is potentially active if it generates a facet of $\mathcal{P}$, equivalently if there exists $r_i \in A_{\Sigma,r}$ such that the constraint is active when the Loss Minimisation Problem is solved at $r_i$. We have the following simple criterion for determining whether a constraint is potentially active:

Lemma B.9 Let $\Sigma$, $r$, $D$ be the parameters for the Loss Minimisation Problem.

1. The constraint $U_i$ is potentially active if and only if $\text{bal } r > D_i$.
2. The constraint $L_i$ is potentially active if and only if $\text{bal } r < \text{bal } D - D_i$.

If the constraint $L_i$ is potentially active, we denote by $F_i$ the corresponding facet of $\mathcal{P}$; if $U_i$ is potentially active, we denote the corresponding facet by $F_{N+i}$. (Note that the facets do not necessarily have consecutive indices under this scheme.) We then have $n_i = -\phi^{(i)}$ and $n_{N+i} = \phi^{(i)}$ for $1 \leq i \leq N$.

Knowing the normals $n_i$, many of which are mutually orthogonal, the results of the previous section provide much information about the location of the outward projections $[F_i]^+$ of $\mathcal{P}$. Through recursive applications of Lemmas B.6 and B.7, we may easily determine the sign of an arbitrary inner product $([F_i]_j, [F_i]_k)$ from the network topology. Theorem B.5 then immediately provides a large part of the information we require to pin down the outward projections.
As a consequence of Theorem B.2, once a given demand profile \( \mathbf{r} \), has been assigned to a particular outward projection \([F_i]^+\), we may state which constraints are active in the Kuhn-Tucker solution simply by reading off the elements of the index set \( I \). Elements \( i \) between 1 and \( N \) correspond to constraints \( L_i \), and elements greater than \( N \) to constraints \( U_{i-N} \). As explained in Section 4.2.2, the knowledge of which constraints are active is sufficient to determine the discretionary profile \( \mathbf{d} \).

We have thus outlined a geometrical method for solving the Loss Minimisation Problem by explicitly identifying the active constraints in the Kuhn-Tucker solution, and thereby identifying a piecewise-linear solution domain in which the demand profile \( \mathbf{r} \) belongs. At present the method and supporting results are very sketchy; much further work is required to make it rigorous and serviceable.
Appendix C

The Number of Spanning Trees in a Graph

In Chapter 5 we presented the outlines of an efficient brute-force algorithm for the solution of the network reconfiguration problem, and illustrated its use with a typical distribution network of moderate complexity. Regardless of the algorithm used to generate the spanning trees, the question whether it is feasible in practice to apply this algorithm to a particular distribution system, rather than one of the many suboptimal but faster alternatives, depends on the tree-number $\tau(C)$ not being excessively large. It is therefore of great practical importance to have a method for estimating the tree-number based on quantities that are easily read off the system diagram, such as the number of tie lines, or the number of branches in the ring mains formed by closing particular tie lines.

Tree-number calculations to date have depended almost exclusively on the familiar formula based on a matrix determinant [6]:

$$\tau(C) = \det(AA^T). \quad (C.1)$$

The matrix $AA^T$ is not difficult to describe: its rows and columns are indexed by node number; its diagonal elements are the degrees of each node (the number of incident cable segments); and the off-diagonal elements are $-1$ if the two nodes in question are connected by a cable segment and 0 otherwise. Nonetheless, this formula is difficult to apply in a practical setting, particularly when the number of nodes is large.

Our guarded optimism concerning the applicability of the brute-force algorithm to real-world reconfiguration problems stems from the observation that in many practical
networks, the number of tie lines is small by comparison with both \( N \), the number of nodes, and \( C \), the number of cable segments. Intuitively one might expect that all else being equal, a graph with low nullity will have far fewer spanning trees than one with high nullity. However, this is not apparent from the standard formula (C.1) for the tree-number. This is another factor motivating the search for a topological formula, relying on topological features such as nullity and fundamental loops.

Consider a graph \( C_1 \) with \( \nu = 1 \); that is, a graph having a single loop. Let \( n_1 \) be the number of branches in this loop. A moment’s reflection leads to the conclusion that

\[
\tau(C_1) = n_1
\]

since removing any one of the \( n_1 \) branches of the loop leaves a spanning tree.

Now consider a graph \( C_2 \) having \( \nu = 2 \); such a graph has two fundamental loops. (Recall that the fundamental loops with respect to a reference tree \( T \) are the \( \nu \) unique loops each of which includes precisely one branch of the cotree \( C \setminus T \).) Let \( n_1 \) be the number of branches in the first loop, and \( n_2 \) the number of branches in the second. There are two topologically distinct cases; either the loops have branches in common, or they do not. However, if we define \( n_{12} \) to be the number of branches common to both loops (with \( n_{12} = 0 \) in the case of no overlap), we find that in all cases

\[
\tau(C_2) = n_1 n_2 - n_{12}^2.
\]

This is established by a simple counting argument; it is the number of ways of choosing one branch from each of the fundamental loops so as to leave behind a connected graph. The nontrivial case with the overlapping loops is viewed as three ‘chains’; one of \( n_1 - n_{12} \) branches, one of \( n_2 - n_{12} \) branches and one of \( n_{12} \) branches, as shown in Figure C.1. This gives a three-term expression, each term corresponding to a choice of branch from a particular chain. Since there are two branches to be chosen and the final spanning tree has no regard to the order in which they are chosen, this strategy actually counts each spanning tree twice, giving:

\[
2\tau(C_2) = (n_1 - n_{12})n_2 + (n_2 - n_{12})n_1 + n_{12}(n_1 + n_2 - 2n_{12})
\]

which reduces to (C.3) above.

This counting strategy may be extended to graphs with higher nullity, although the expressions become rapidly more complicated. When \( \nu = 3 \), there are six topologically
Figure C.1: Graph of nullity $\nu = 2$ with two overlapping loops

distinct cases as shown in Figure C.2. Depending how the loops are defined, there may be branches common to all three fundamental loops, their number denoted by $n_{123}$ in keeping with the general pattern. In principle we may treat each case in turn, identify the ‘chains’ and the number of branches in each in terms of $n_1, n_2, n_3$ and so on, and count the number of distinct triples that may be removed while leaving the graph connected. The general formula resulting from this is

$$\tau(C_3) = n_1n_2n_3 - n_1n_{23}^2 - n_2n_{13}^2 - n_3n_{12}^2 + 4n_{123}^3 - 2n_{12}n_{13}n_{23}. \quad (C.5)$$

The assurance that there are no more than six distinct cases with $\nu = 3$ stems from the observation that, by contracting as many edges as possible while preserving the nullity, and by moving self-loops from one node to another (which leaves the essential topological characteristics unchanged), any connected graph may be reduced to a graph of the same nullity in which

1. if there is more than one node, then every node is of degree 3 or greater, excluding self-loops;

2. at most one node bears self-loops, with two graphs deemed equivalent if they differ only in the location of the self-loops; and

3. there are no bridges (a bridge being a branch whose removal disconnects the graph).

Let us use the term loop graph to refer to a connected graph satisfying these three properties. Then there are in principle just two items of information we require in order to calculate the tree-number of a graph $C$: the corresponding loop graph, and the number of actual branches in $C$ corresponding to each branch in the loop graph (the ‘chains’ we referred to earlier).
Figure C.2: Topologically distinct graphs with nullity $\nu = 3$
The enumeration of loop graphs of a given nullity is assisted by use of the pigeon-hole principle. In a loop graph on \( n > 1 \) nodes, the sum of the node degrees is at least \( 3n \). This must be equal to twice the number of branches, or \( 2(n - 1 + \nu) \). It follows that a nontrivial loop graph of nullity \( \nu \) has at most \( 2(\nu - 1) \) nodes, and that a loop graph with precisely this number of nodes is 3-regular. Notwithstanding the above, there is always a trivial loop graph with one node and \( \nu \) self-loops. (For \( \nu = 1 \) this is the only possible loop graph.)

With \( \nu = 2 \) for example, there are just two possibilities for a loop graph: one node and two branches, or two nodes and three branches. In the first category there is only one possibility, the trivial graph with two self-loops, corresponding to the case of two nonoverlapping ring mains. In the second category there is also only one possible graph that omits bridges; in this graph the three edges are parallel, corresponding to the case of two loops having branches in common.

Returning now to the case \( \nu = 3 \), we find that there are in fact only six loop graphs (see Figure C.3): the trivial loop graph, two graphs on two nodes, one on three nodes, and two on four nodes. Figure C.3 may be compared with Figure C.2. Each loop graph gives an alternative derivation of (C.5) or one of its special cases, with each edge of the loop graph contributing a term to an expression of the form (C.4).

The specific form of these terms depends on the choice of the loops defining \( n_1, n_2 \) and \( n_3 \). For a graph of nullity 3, there are 28 ways of choosing 3 linearly independent loops, though not all are fundamental loops with respect to a tree. In general, the number of independent systems of \( \nu \) loops in a graph of nullity \( \nu \) is

\[
\frac{1}{\nu!} \prod_{\mu=0}^{\nu-1} (2^{\nu} - 2^\mu). \tag{C.6}
\]

We can define an algebra of loop-systems, under the operation of addition of branches modulo 2. Such an algebra is isomorphic to the Galois field \( GL(2^\nu) \), and its elements may be represented as sequences of \( \nu \) binary digits. \( \nu \) such elements constitute a
complete system of loops, which is linearly independent if and only if there is no subset whose sum in $GL(2^\nu)$ is zero. Equation (C.6) was derived using this formalism. Consider the process of choosing $\nu$ linearly independent elements of $GL(2^\nu)$: after $\mu$ elements have been chosen, there are $2^\mu$ possible linear combinations of these elements (including zero), and hence $2^\nu - 2^\mu$ choices remaining at the next step. As the order of choice is irrelevant, we divide the final result by $\nu!$.

Unfortunately, it does not appear as easy to characterise the conditions under which a system of $\nu$ loops is a fundamental system with respect to some reference tree. In addition, when loops are topologically separate, our algebra admits as elements disconnected sets of two or more loops, which though formally correct goes against the spirit of a topological formula.

A theory of loop graphs and algebras will need to be developed to a much greater extent before it can be usefully applied to more complex graphs. Even when $\nu = 4$, calculations analogous to those above are excessively laborious. Nonetheless, detailed investigations of this kind will be necessary in order to generalise from expressions such as (C.3) and (C.5), even in approximate form. Looking again at the 33-bus example of Table 5.1 with nullity 5, a quick calculation shows that the crudest approximation, the product $n_1 n_2 n_3 n_4 n_5$, severely overestimates the number of trees, while the ‘second-order’ approximation obtained by subtracting terms of the form $n_i^2 n_k n_l n_m$ severely underestimates this number. While the necessary theoretical elaboration is beyond the scope of this thesis, we hope we have provided sufficient practical motivation to explore a challenging open problem in graph theory.
Appendix D

The DHO Family

In the analysis of circuit models, such as those based on the Small Model of Section 7.3, we frequently need to solve damped harmonic oscillator (DHO) problems of the form

$$x'' + 2\zeta x' + x = 0.$$  \hfill (D.1)

In standard textbook treatments, the solution of (D.1) falls into three separate classes, depending whether $|\zeta|$ is greater than, less than, or equal to 1. To avoid explicitly treating these three cases separately, we define the DHO family of functions $C_\zeta(\tau)$ and $S_\zeta(\tau)$, parametrised by $\zeta$. This permits a unified treatment, at least formally.

The formal definitions are via Laplace transforms. Recall that the (one-sided) Laplace transform of an exponentially-bounded function $f(t)$ is a function of $s$ defined as

$$\mathcal{L} \{ f(t) \} = F(s) = \int_0^\infty e^{-st} f(t) \, dt.$$  

For $\zeta, \tau \in \mathbb{R}$, the DHO family of functions has the defining identities

$$\mathcal{L} \{ e^{-\zeta \tau} C_\zeta(\tau) \} = \frac{s + \zeta}{s^2 + 2\zeta s + 1} \quad \mathcal{L} \{ e^{-\zeta \tau} S_\zeta(\tau) \} = \frac{1}{s^2 + 2\zeta s + 1} \quad (D.2)$$

for $\tau > 0$, with the values for $\tau \leq 0$ found by analytic continuation. The explicit formulae for these functions are given in Table D.1. As special cases we obtain the standard circular functions $\cos \tau$, $\sin \tau$ (when $\zeta = 0$) and hyperbolic functions $\cosh \tau$, $\sinh \tau$ (when $\zeta = \sqrt{2}$). The DHO family satisfies a number of identities reminiscent of those for circular and hyperbolic functions.

- Values at zero: $C_\zeta(0) = 1$ and $S_\zeta(0) = 0$.  

\[
\begin{array}{ll}
C_\zeta(\tau) & S_\zeta(\tau) \\
|\zeta| < 1 & \cos \sqrt{1 - \zeta^2 \tau} \quad \frac{1}{\sqrt{1 - \zeta^2}} \sin \sqrt{1 - \zeta^2 \tau} \\
|\zeta| = 1 & 1 \quad \tau \\
|\zeta| > 1 & \cosh \sqrt{\zeta^2 - 1} \tau \quad \frac{1}{\sqrt{\zeta^2 - 1}} \sinh \sqrt{\zeta^2 - 1} \tau
\end{array}
\]

Table D.1: The DHO family of functions

- **Even-odd symmetry in \( \tau \):** \( C_\zeta(-\tau) = C_\zeta(\tau) \) and \( S_\zeta(-\tau) = -S_\zeta(\tau) \).

- **Even symmetry in \( \zeta \):** \( C_{-\zeta}(\tau) = C_\zeta(\tau) \) and \( S_{-\zeta}(\tau) = S_\zeta(\tau) \).

- **Addition formulae:**
  \[
  S_\zeta(\tau_1 \pm \tau_2) = S_\zeta(\tau_1)C_\zeta(\tau_2) \pm C_\zeta(\tau_1)S_\zeta(\tau_2) \\
  C_\zeta(\tau_1 \pm \tau_2) = C_\zeta(\tau_1)C_\zeta(\tau_2) \mp (1 - \zeta^2)S_\zeta(\tau_1)S_\zeta(\tau_2)
  \]

- **The quasi-Pythagorean identity:**
  \[
  C^2_\zeta(\tau) + (1 - \zeta^2)S^2_\zeta(\tau) = 1.
  \]

For all \( \zeta \in \mathbb{R} \) the functions \( C_\zeta(\tau) \) and \( S_\zeta(\tau) \) are analytic, by definition. Their derivatives are

\[
\frac{d}{d\tau} S_\zeta(\tau) = C_\zeta(\tau) \quad \frac{d}{d\tau} C_\zeta(\tau) = (\zeta^2 - 1)S_\zeta(\tau).
\]

The DHO functions are also smooth when considered as functions of \( \zeta \) for fixed \( \tau \), a fact not readily apparent from the definition. (Analyticity with respect to \( \zeta \) is an open question.) The first derivatives with respect to \( \zeta \) are

\[
\frac{d}{d\zeta} C_\zeta(\tau) = \zeta \tau S_\zeta(\tau) \\
\frac{d}{d\zeta} S_\zeta(\tau) = \begin{cases} 
\frac{\zeta}{1 - \zeta^2} (S_\zeta(\tau) - \tau C_\zeta(\tau)) & |\zeta| \neq 1 \\
\frac{1}{3\zeta^2 \tau^3} & |\zeta| = 1.
\end{cases}
\]

Evaluation of these derivatives at \( |\zeta| = 1 \) is delicate, but may be accomplished with the aid of the asymptotic series for \( \zeta^2 = 1 - \epsilon \):

\[
C_{\pm \sqrt{1-\epsilon}}(\tau) = \cos(\epsilon^{1/2}\tau) = 1 - \frac{1}{2}\tau^2 \epsilon + \frac{1}{4!}\tau^4 \epsilon^2 - \ldots
\]

\[
S_{\pm \sqrt{1-\epsilon}}(\tau) = \epsilon^{-1/2} \sin(\epsilon^{1/2}\tau) = \tau - \frac{1}{3!}\tau^3 \epsilon + \frac{1}{5!}\tau^5 \epsilon^2 - \ldots
\]

360
The series for $\zeta^2 = 1 + \epsilon$ are identical to the above, with $-\epsilon$ in place of $\epsilon$.

We also have the following integral formulae:

$$
\int_0^T e^{-\zeta \tau} c_\zeta (\tau) \, d\tau = \zeta (1 - e^{-\zeta T} c_\zeta (T)) - (\zeta^2 - 1) e^{-\zeta T} s_\zeta (T) \quad (D.9)
$$

$$
\int_0^T e^{-\zeta \tau} s_\zeta (\tau) \, d\tau = 1 - e^{-\zeta T} (c_\zeta (T) + \zeta s_\zeta (T)). \quad (D.10)
$$

The following theorem illustrates the use of the DHO family in solving DHO problems of the form (D.1).

**Theorem D.1** Let $x : [0, \infty) \to \mathbb{R}$, with derivative $x'$, be the unique solution of the homogeneous DHO (D.1) with $\zeta \in \mathbb{R}$ and prescribed initial conditions $x(0) = x_0$, $x'(0) = x'_0$. Then

$$
\begin{bmatrix}
  x(\tau) \\
  x'(\tau)
\end{bmatrix} = e^{-\zeta \tau} \begin{bmatrix}
  c_\zeta (\tau) + \zeta s_\zeta (\tau) & s_\zeta (\tau) \\
  -s_\zeta (\tau) & c_\zeta (\tau) - \zeta s_\zeta (\tau)
\end{bmatrix} \begin{bmatrix}
  x_0 \\
  x'_0
\end{bmatrix}. \quad (D.11)
$$

**Proof.** The original problem can be written $x' = Ax$, where $x = [ x \ x' ]^T$ and

$$
A = \begin{bmatrix}
  0 & 1 \\
  -1 & -2\zeta
\end{bmatrix}.
$$

The solution has the form $x(\tau) = e^{A\tau} x(0)$ where

$$
e^{A\tau} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2\zeta s + 1} \begin{bmatrix}
  s + 2\zeta & 1 \\
  -1 & s
\end{bmatrix} \right\}.
$$

The result now follows simply by applying the definitions (D.2). √

As a corollary to Theorem D.1 we note that the unit step response of (D.1), given by the initial conditions $x_0 = -1$ and $x'_0 = 0$, is

$$
x(t) = -e^{-\zeta \tau} (c_\zeta (\tau) + \zeta s_\zeta (\tau)) \quad x'(t) = e^{-\zeta \tau} s_\zeta (\tau). \quad (D.12)
$$
Appendix E

Polyphase Signals: Definition and Properties

In this appendix we introduce a formalism for the description of polyphase AC quantities, and recall some basic properties of polyphase signals.

E.1 Definitions

A \textit{p-phase signal} is a vector $\mathbf{X}$ of $p$ real signals, $X_0$ through $X_{p-1}$, all of which are periodic with common period $T$ and fundamental frequency $\Omega = 2\pi/T$. The \textit{phase constant} of this signal is the complex number $\alpha = e^{j\theta_0}$. Two such signals are \textit{synchronous} if they have the same number of phases and the same fundamental frequency.

We shall mostly be dealing with \textit{sinusoidal} or \textit{linear} signals, in which all the $X_k$ are sinusoidal quantities (or zero). In such signals we may identify for each component a magnitude $|X_k| \geq 0$ and a phase $\phi_k^X$ (up to a multiple of $2\pi$), such that $X_k = \sqrt{2}|X_k|\cos(\Omega t + \phi_k^X)$. If we fix a time origin and require $\phi_k^X \in (-\pi, \pi]$, these magnitudes and phases are unique. The \textit{phasor representation} $\tilde{\mathbf{X}}$ of $\mathbf{X}$ is the vector of complex numbers $\tilde{X}_k = |X_k|e^{j\phi_k^X}$.

A polyphase signal $\mathbf{X}$ is \textit{balanced} if it is sinusoidal and $\tilde{X}_k = \alpha^{-k}\tilde{X}_0$ for all $k$. This implies in particular that all the $|X_k|$ are identical; this quantity we label $|\mathbf{X}|$, the magnitude of the signal. The \textit{relative phase shift} of two synchronous balanced signals $\mathbf{X}$ and $\mathbf{Y}$ is the quantity $\phi_Y^X - \phi_0^X$; we say that $\mathbf{Y}$ \textit{leads} $\mathbf{X}$ if this is positive and \textit{lags} $\mathbf{X}$ if it is negative.
E.2 Elementary Properties

As is well known, balanced polyphase signals have a number of ‘nice’ properties. These may be derived as consequences of the following simple lemma.

Lemma E.1 Let \( \theta(t) \) be any real signal, and let \( p \) be a positive integer. Then for all \( m \in \mathbb{Z} \) we have

\[
\sum_{k=0}^{p-1} \cos \left( \theta(t) - \frac{2\pi mk}{p} \right) = \begin{cases} 
\frac{p}{2} \cos \theta(t) & \text{if } m \text{ is a multiple of } p \\
0 & \text{otherwise.} 
\end{cases} 
\]

Proof. The sum on the left is equal to

\[
\text{Re} \left( \sum_{k=0}^{p-1} e^{j(\theta(t) - \frac{2\pi mk}{p})} \right) = \text{Re} \left( e^{j\theta(t)} \sum_{k=0}^{p-1} \alpha^{-mk} \right). 
\]

If \( m = \mu p \) for some \( \mu \in \mathbb{Z} \), then \( \alpha^{-mk} = (\alpha^p)^{-\mu k} = 1 \) and the result follows immediately. Otherwise, \( \alpha^{-m} \neq 1 \) and

\[
\sum_{k=0}^{p-1} (\alpha^{-m})^k = \frac{1 - \alpha^{-mp}}{1 - \alpha^{-m}} = \frac{1 - 1^{-m}}{1 - \alpha^{-m}} = 0. 
\]

\[\checkmark\]

For the following, we recall the bal \( \mathbf{x} \) notation from Chapter 3.

Theorem E.1 (Vanishing-Mean (VM) Property) If \( \mathbf{X} \) is a balanced polyphase signal, then \( \text{bal } \mathbf{X} = 0 \).

Proof. Apply Lemma E.1 with \( \theta(t) = \Omega t \) and \( m = 1 \). \(\checkmark\)

Theorem E.2 (Constant-Power-Flow (CPF) Property) Let \( \mathbf{X} \) and \( \mathbf{Y} \) be synchronous \( p \)-phase signals with \( p \geq 3 \), of the form \( \mathbf{X} = \mathbf{X}' + X_d \mathbf{1} \), \( \mathbf{Y} = \mathbf{Y}' + Y_d \mathbf{1} \) where \( \mathbf{X}' \), \( \mathbf{Y}' \) are balanced signals with relative phase shift \( \phi \) (leading or lagging) and \( X_d \), \( Y_d \) are arbitrary constants (possibly zero). Then

\[
\mathbf{X}'^T \mathbf{Y} = p(|\mathbf{X}'| |\mathbf{Y}'| \cos \phi + X_d Y_d). 
\]

(E.2)
Proof. Without loss of generality assume that $\phi_0^\bar{X} = 0$, and suppose $\mathbf{Y}$ leads $\mathbf{X}$. Then

$$X_k Y_k = 2 |\mathbf{X}'| |\mathbf{Y}'| \cos \left( \Omega t - \frac{2\pi k}{p} \right) \cos \left( \Omega t + \phi - \frac{2\pi k}{p} \right) + X_d Y'_k + Y_d X'_k + X_d Y_d$$

$$= |\mathbf{X}'| |\mathbf{Y}'| \left( \cos \phi + \cos \left( 2\Omega t + \phi - \frac{4\pi k}{p} \right) \right) + X_d Y'_k + Y_d X'_k + X_d Y_d.$$

Now, by the VM property $\text{bal} \mathbf{X}' = \text{bal} \mathbf{Y}' = 0$. The result then follows on applying Lemma E.1 with $\theta(t) = 2\Omega t + \phi$ and $m = 2$. Substituting $-\phi$ for $\phi$ yields the same result when $\mathbf{Y}$ lags $\mathbf{X}$. $\sqrt{\text{V}}$

The CPF property is usually stated in the case where $\mathbf{X}$ is the voltage drop in a polyphase circuit element, $\mathbf{Y}$ is the current, and $\mathbf{X}^T \mathbf{Y}$ is the associated (constant) real power. Nonetheless, this property holds for any two synchronous signals of the requisite form and, as stated here, also generalises to the case where a common DC offset is added to $\mathbf{X}$ and/or $\mathbf{Y}$.

### E.3 The Park Transform for Three-Phase Signals

In certain contexts, the analysis of models involving three-phase quantities can be greatly simplified by transforming the three-phase variables in a particular way. For the switch-mode converter circuits of Chapters 7 through 10, extensive use is made of one such transformation, the Park transform.

The Park transform was developed originally by R. H. Park as part of his general theory of AC machines [59, 60]. Today it is applied in two forms: a time-invariant ‘stationary’ form, and a time-varying ‘synchronous’ form.

#### E.3.1 Stationary Park Transform

Let $\mathbf{X}$ be a three-phase signal. The stationary Park transform $\tilde{\mathbf{X}}$ of $\mathbf{X}$ is defined as

$$\tilde{\mathbf{X}} = \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix} = \frac{\sqrt{2}}{3} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} \equiv \mathbf{P}_0 \mathbf{X}. \quad (E.3)$$

The form of this transformation is similar to that of the symmetrical-component transformation used in the analysis of asymmetrical faults on power systems [33]. Indeed, the component $X_z$ is precisely the zero-sequence component of $\mathbf{X}$.
If zero-sequence components are of no concern (or nonexistent, as with balanced signals), the Park transform may be represented as a complex number:

\[ \ddot{X} = X_a + jX_b = \frac{\sqrt{2}}{3}(X_0 + \alpha X_1 + \alpha^2 X_2) \]  

(E.4)

where \( \alpha = e^{2\pi/3} \) is the phase constant for three-phase signals. A slight abuse of notation has been committed in using \( \ddot{X} \) to refer to both representations, but the intended representation will usually be clear from the context.

Applications of the Park transform in the literature differ in minor respects from each other and from (E.3), chiefly in the choice of scaling factor. Park himself, and many authors subsequently, used \( 2/3 \) rather than \( \sqrt{2}/3 \); others use \( \sqrt{2}/3 \). The choice is arbitrary; the present author uses \( \sqrt{2}/3 \) in preference to \( 2/3 \) as the former yields RMS rather than peak amplitudes.

In a machine-theoretic context, the components \( X_a \) and \( X_b \) are referred to as the direct-axis and quadrature-axis components and denoted \( X_d \) and \( X_q \) (or sometimes \( X_q \) and \( X_d \)). We eschew these labels in the present work because the context of interest is switch-mode circuits rather than machines, and because a fair degree of confusion surrounds the identification of the hypothetical axes. The terms ‘direct’ and ‘quadrature’ suggest that in a complex-number representation, the former would lie on the real axis and therefore correspond to \( X_a \); this is consistent with Park’s original presentation. However, because in a synchronous machine the EMF is developed on the quadrature axis, some authors [7, 12] prefer to identify the quadrature axis with \( X_a \). To avoid confusion, we avoid the use of the terms ‘direct’ and ‘quadrature’ entirely.

A key property of the stationary Park transform is the following:

**Lemma E.2** If \( X \) is a balanced signal then \( \ddot{X} \) has the complex-number representation

\[ \ddot{X} = |X|e^{j(\phi_0 + \Omega t)}. \]

(E.5)

In words, \( \ddot{X} \) rotates anticlockwise through the complex plane, and its value at \( t = 0 \) is identical to the phasor \( \ddot{X}_0 \).

A note on the proof will be found in the next section.

From the matrix \( P_0 \) we get \( P_0 P_0^T = P_0^T P_0 = (1/3)I \), from which \( P_0^{-1} = 3P_0^T \). This gives a useful identity for the product of three-phase signals, similar to that for symmetrical components:
Lemma E.3 Let $X$ and $Y$ be three-phase signals. Then
\[ X^T Y = 3X^T \tilde{Y}. \] (E.6)

Dynamical systems involving three-phase quantities are easily converted, by means of a linear change of variables, to equivalent systems involving their Park transforms. In particular, the time derivative of the (stationary) Park transform of a three-phase signal $X$ is equal to the (stationary) Park transform of its derivative $\dot{X}$.

### E.3.2 Synchronous Park Transform

Again, let $X$ be a three-phase signal. The *synchronous Park transform* $\tilde{X}$ of $X$ is defined as
\[
\tilde{X} = \frac{1}{3} \begin{bmatrix}
X_\alpha \\
X_\beta \\
X_\gamma
\end{bmatrix} = \begin{bmatrix}
\cos(\Omega t) & \cos(\Omega t - \frac{2\pi}{3}) & \cos(\Omega t + \frac{2\pi}{3}) \\
-\sin(\Omega t) & -\sin(\Omega t - \frac{2\pi}{3}) & -\sin(\Omega t + \frac{2\pi}{3}) \\
1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}
\end{bmatrix}
\begin{bmatrix}
X_0 \\
X_1 \\
X_2
\end{bmatrix}
\] (E.7)
or in brief
\[
\tilde{X} = P_\Omega(t)X.
\]

This is a time-varying transformation with an additional parameter, the synchronous frequency $\Omega$. With $t = 0$ or $\Omega = 0$ we recover the stationary Park transform:
\[
P_\Omega(0) = P_0 = P_0(t).
\]

As our synchronous frequency will not vary, we shall usually write $P(t)$ in place of $P_\Omega(t)$.

Neglecting zero-sequence components, $\tilde{X}$ has a complex-number representation, obtained from that of $\dot{X}$ by a time-dependent rotation:
\[
\tilde{X} = X_\alpha + jX_\beta = e^{-j\Omega t} \dot{X}.
\] (E.8)

The most convenient property of the synchronous Park transform is that it transforms balanced three-phase signals into constant quantities.

**Lemma E.4** If $X$ is a balanced signal at the synchronous frequency $\Omega$, then $\tilde{X}$ has the complex-number representation
\[
\tilde{X} = |X|e^{j\Phi X}.
\] (E.9)

Thus $\tilde{X}$, as a complex number, is equal to the phasor $\tilde{X}_0$. 

367
Lemma E.4 may be proved with the aid of the CPF property; it implies Lemma E.2 as a consequence of (E.8). The CPF and VM properties also entail that $P^{-1}(t) = 3P^T(t)$, and so Lemma E.3 carries over unchanged to the synchronous case:

**Lemma E.5** Let $X$ and $Y$ be three-phase signals. Then

$$X^T Y = 3\hat{X}^T\hat{Y}.$$ \hfill (E.10)

The fact that constants in the synchronous Park domain equate to balanced three-phase signals in the time domain (assuming zero-sequence components are avoided) is extremely convenient when applied to control problems involving three-phase signals. Such a control problem can often be converted to a regulation problem (having a desired fixed-point equilibrium at the origin) in the synchronous Park variables. To carry out the transformation, we must know how derivatives are affected by its time-varying nature. We have

$$\frac{d\hat{X}}{dt} = \frac{d}{dt}P(t)X = P(t)\dot{X} + \dot{P}(t)X.$$ 

One easily verifies that

$$\dot{P}(t) = \Omega J P(t), \quad J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ \hfill (E.11)

yielding the following formula for the synchronous Park transform of $\dot{X}$:

$$P(t)\dot{X} = \frac{d}{dt}\hat{X} - \Omega J \hat{X}.$$ \hfill (E.12)
Appendix F

The Asynchronous Thyristor Switching Model

Where a DC reticulation system is fed by a thyristor converter, the Large Model or Small Model of Chapter 7 will be adequate for analysis and control design on time scales significantly larger than the thyristor switching period \((F/2p, \text{where } F \text{ is the AC frequency and } 2p \text{ the converter pulse number})\). However, a controller fast enough to operate on the level of individual device switchings will need a more detailed model of the converter itself. The Asynchronous Thyristor Switching (ATS) model provides a more accurate representation of the converter dynamics at these time scales.

In a thyristor converter, the control action expresses itself not as an extraneous input variable, but rather in the timing of switching events. Thus, the control model developed will be an asynchronous one, tracking the system state from one switching event to the next, with the control law establishing the spacing of events.

The development of the ATS model proceeds as follows. In Section F.1 we derive equivalent (linear) circuits for the various switching modes of the converter. We consider only the most common case, in which current flow is uninterrupted and commutation periods are separate from one another. There are then just two essentially different switching modes: single conduction and commutation, analysed in Sections F.2 and F.3 respectively. These are drawn together in Section F.4 to yield an asynchronous discrete-time model for the thyristor converter.
Figure F.1: ‘Simple’ bridge configuration

F.1 Equivalent Circuits

The details of the model will depend on the specific converter topology used. The basic building block for thyristor converters is the simple bridge configuration, which takes an odd number of input phases $p$ and connects each phase to a common cathode (CK) and common anode (CA) through a pair of switching devices (Figure F.1).

Two or more simple bridges may be connected in series or in parallel to form more complicated circuits. One commonly used topology is formed by connecting two bridges in series and using two $p$-phase supplies, where $p$ is again odd, and the second supply is shifted in phase by $\pi/2p$ relative to the first. This achieves a doubling in the output pulse number from $2p$ to $4p$.

Let the input on the $k$th phase be

$$V_k = \sqrt{2}V_{AC} \cos \left( \Omega t - \frac{2\pi k}{p} \right)$$

and define $V_L$ according to (7.18). We allow for AC source inductance $L_S$ per phase, but ignore DC-side resistance as it complicates the analysis unduly for only a small improvement in accuracy. Assuming a simple bridge converter, and leaving aside complications such as simultaneous commutation and discontinuous conduction, we have two conduction modes to consider:

1. Single conduction, in which two devices conduct in series (Figure F.2); and

2. Commutation, in which current passes from one device to another, with the two devices conducting in parallel, and in series with a third device (Figure F.3).

We assume a ‘standard’ switching sequence, meaning that a maximal phase displacement is maintained between the input phases switched into the circuit. (In the terminology of Section 7.4, the switching sequence is restricted to maximally
driven c-states.) This sequence determines a unique standard order for gating the devices in the converter, and the converter output is invariant with respect to cyclic permutations of this ordering. Accordingly, we may speak of start-of-commutation events (SCE) and end-of-commutation events (ECE)—the time instants at which the equivalent circuit representations swap over—without reference to the exact devices involved.

Consider first a diode converter with no source reactance, and choose the origin of time at an arbitrary SCE or ECE (the two being simultaneous in this case). Then, in the single conduction mode, the net input voltage is

\[ V_S = \sqrt{2} V_L \cos \left( \Omega t - \frac{\pi}{2p} \right). \] (F.1)

In a thyristor converter, the SCE occurs at a delay of \( \alpha \) electrical degrees relative to that in a diode converter, and the effect of source reactance introduces an additional delay \( \gamma \) between SCE and ECE. The converter enters single conduction mode at the ECE, which occurs \( t_0 \) seconds later than in the diode converter, where \( \Omega t_0 = \alpha + \gamma \). Equation (F.1) still holds, provided we understand the ECE to occur at \( t = t_0 \) instead of \( t = 0 \).

The transition to the commutation mode occurs at the SCE, which is delayed by
$t_1$ seconds relative to that in the ideal diode converter, with $\Omega t_1 = \alpha$. Once again choosing the time origin as the SCE in the diode converter, we have for commutation from phase $a$ to phase $b$

$$V_a - V_c = \sqrt{2}V_L \cos \left( \Omega t + \frac{\pi}{2p} \right)$$
$$V_b - V_c = \sqrt{2}V_L \cos \left( \Omega t - \frac{\pi}{2p} \right).$$

The actual supply voltage $V_S = (V_a + V_b)/2 - V_c$ (found by Thevenin equivalence) is the arithmetic mean of these two expressions, giving

$$V_S = \sqrt{2}V_L \cos \frac{\pi}{2p} \cos(\Omega t). \quad (F.2)$$

The equivalent circuit inductance is easily found as $L + L_S + L_S/2 = L + 3L_S/2$.

In the compound bridge configuration, four devices conduct in the single conduction mode, and five in the commutation mode. The equivalent circuits nonetheless have the same form, with the addition of two more series AC sources (and inductance $2L_S$). The net voltage of these two sources, in either case, is equivalent to that of a simple bridge in single conduction mode, with a $\pi/2p$ phase shift. The equivalent line-to-line voltage $V'_L$ for the compound converter is given by

$$V'_L = 2V_L \cos \frac{\pi}{4p} = 4V_{AC} \cos \frac{\pi}{4p} \cos \frac{\pi}{2p}. \quad (F.3)$$

For single conduction mode, we have

$$V_S = \sqrt{2}V'_L \cos \left( \Omega t - \frac{\pi}{4p} \right)$$

with source inductance $L + 4L_S$. In commutation mode, reasoning as above gives

$$V_S = \sqrt{2}V'_L \cos \frac{\pi}{4p} \cos(\Omega t)$$

with source inductance $L + 3L_S + L_S/2 = L + 7L_S/2$. In either case the source voltage is equivalent to that in the simple bridge upon setting $p' = 2p$ and $V'_{AC} = V_L$.

To maintain full generality, we refer to the circuit inductance in the single conduction and commutation modes simply as $L_1$ and $L_2$, respectively.

### F.2 Dynamics in the Single Conduction Mode

For simplicity, we consider only the case of a lumped, voltage-driven load, as in the Small Model, and develop explicit model equations only for the special case of a linear
load. Note, however, that owing to the very short time scales involved (particularly when \( p \) is increased beyond 3), a nonlinear load may often be approximated in this model by a time-varying linear load.

We work in LC units, with normalisation factors \( \omega_1 = 1/\sqrt{L_1C} \) and \( Z_1 = \sqrt{L_1/C} \). Define the dimensionless quantities \( \sigma_1 = \Omega/\omega_1 \) and \( v_L = \sqrt{2}V_L/V_R \). Denoting the bus voltage by \( v \), and the load function by \( f(v, \tau) \), we have

\[
v'' + D_1 f(v, \tau) v' + v = v_L \cos \left( \sigma_1 \tau - \frac{\pi}{2p} \right) - 1 - D_2 f(v, \tau).
\]  (F.4)

If \( f \) is linear in \( v \), write

\[
f = i_0(\tau) + 2\zeta(\tau)v.
\]

Then

\[
v'' + 2\zeta v' + (1 - 2\zeta' )v = v_L \cos \left( \sigma_1 \tau - \frac{\pi}{2p} \right) - 1 - i_0.
\]  (F.5)

If \( i_0 \) and \( \zeta \) are slowly varying with respect to \( \tau = \omega_1 t \), then we may approximate \( i_0(\tau) \) and \( \zeta(\tau) \) by their values at the most recent ECE, and treat (F.5) as time-invariant. In the time-invariant case the contribution of \( i_0 \) vanishes and the homogeneous part of (F.5) is a simple damped harmonic oscillator (DHO). The response (see Appendix D) takes the form

\[
v(\tau) = A \cos(\sigma_1 \tau - \phi) - 1 + e^{-\zeta(\tau - \tau_0)}(v_0 C \zeta(\tau - \tau_0) + (\zeta v_0 + v_0') S \zeta(\tau - \tau_0)).
\]  (F.6)

The initial conditions are \( v(\tau_0) = v(0) \) and \( v'(\tau_0) = v'(0) \) where

\[
\tau_0 = \sigma_1^{-1}(\alpha_n + \gamma_n).
\]  (F.7)

Here \( \alpha_n, n = 0,1,2\ldots \) is the phase delay at the \( n \)th SCE, and \( \gamma_n \) represents the commutation delay between the \( n \)th SCE and the \( n \)th ECE. The sequence \( \{\alpha_n\} \) represents the control input.

The numbers \( A, \phi \) in (F.6) are independent of the initial conditions, and are found upon calculation to be

\[
A = \frac{v_L}{\sqrt{(1 - \sigma_1^2)^2 + 4\zeta^2\sigma_1^2}}
\]  (F.8)

\[
\phi = \frac{\pi}{2p} + \text{argtan}(2\zeta \sigma_1, 1 - \sigma_1^2)
\]  (F.9)

where \text{argtan} denotes the four-quadrant arctangent, \( \text{argtan}(y, x) = \text{arg}(x + jy) \). Given \( A \) and \( \phi \), we may evaluate the constants \( v_0 \) and \( v_0' \) in (F.6), which are the initial
conditions for the residual ‘natural part’ of the response. Setting \( \tau = \tau_0 \) in (F.6) we find

\[
\begin{align*}
v_0 &= v_{(0)} + 1 - A \cos(\alpha_n + \gamma_n - \phi) \\
v_0' &= v_{(0)}' + \sigma_1 A \sin(\alpha_n + \gamma_n - \phi).
\end{align*}
\] (F.10)

Relative to the time origin chosen, the following SCE will occur at

\[
\tau = \tau_f = \sigma_1^{-1} \left( \frac{\pi}{p} + \alpha_{n+1} \right).
\] (F.11)

Let \( v_{(1)} \), \( v_{(1)}' \) denote the state variables at the transition to the commutation mode, and define

\[
\begin{align*}
\phi^* &= \frac{\pi}{p} - \phi = \frac{\pi}{2p} - \text{arctan}(2\xi \sigma_1, 1 - \sigma_1^2) \\
\hat{\tau}_n &= \tau_f - \tau_0 = \sigma_1^{-1} \left( \frac{\pi}{p} + (\Delta \alpha)_n - \gamma_n \right)
\end{align*}
\] (F.12)

where \((\Delta \alpha)_n = \alpha_{n+1} - \alpha_n\). Then

\[
\begin{align*}
v_{(1)} &= A \cos(\alpha_{n+1} + \phi^*) - 1 + e^{-\xi \hat{\tau}_n}(v_0 c_\xi(\hat{\tau}_n) + (\zeta v_0 + v_0') s_\xi(\hat{\tau}_n)) \\
v_{(1)}' &= -A \sigma_1 \sin(\alpha_{n+1} + \phi^*) + e^{-\xi \hat{\tau}_n}(v_0' c_\xi(\hat{\tau}_n) - (v_0 + \zeta v_0') s_\xi(\hat{\tau}_n))
\end{align*}
\] (F.13)

with \( v_0, v_0' \) as in (F.10). The expressions for \( v_{(1)} \) and \( v_{(1)}' \) fall into two parts: a ‘forced response’ component which depends on \( \alpha_{n+1} \) but not on \( \alpha_n \), and a ‘natural response’ component which depends on both. The constants \( A, \phi, \phi^* \) depend on the parameters \( v_L, p, \zeta \) and \( \sigma_1 \) but not on \( \alpha_n \) or \( \gamma_n \). The initial state \((v_{(0)}, v_{(0)}')\) enters via the numbers \( v_0 \) and \( v_0' \).

**F.3 Dynamics in the Commutation Mode**

In the commutation mode, the LC units must be normalised with respect to the constants \( \omega_2 = 1/\sqrt{L_2 C} \) and \( Z_2 = \sqrt{L_2/C} \). We define \( \sigma_2 = \Omega/\omega_2 \) and \( \eta = Z_2/Z_1 = \sqrt{L_2/L_1} \). Assuming that \( \eta \zeta' \ll 1 \) and \( \eta \zeta'_0 \ll 1 \) we have

\[
v'' + 2\eta \zeta v' + v = v_L \cos \frac{\pi}{2p} \cos(\sigma_2 \tau) - 1.
\] (F.16)

The response has the same general form as (F.6):

\[
v(\tau) = B \cos(\sigma_2 \tau - \psi) - 1 + e^{-\eta \zeta (\tau - \tau_*)}(v_* c_{\eta \zeta}(\tau - \tau_*) + (\eta \zeta v_* + v_*') s_{\eta \zeta}(\tau - \tau_*)).
\] (F.17)

374
The initial conditions are \( v(\tau_*) = v(1) \) and \( v'(\tau_*) = v'(1) \) where \( v(1), v'(1) \) are as above and

\[
\tau_* = \sigma_2^{-1} \alpha_{n+1}. \tag{F.18}
\]

Note that \( \sigma_2 \tau_* = \sigma_1 \tau_f - \pi/p \), reflecting the change in time scale and origin. Reasoning as above we have

\[
B = \frac{v_L \cos(\pi/2p)}{\sqrt{(1 - \sigma_2^2)^2 + 4\eta^2 \zeta^2 \sigma_2^2}} \tag{F.19}
\]

\[
\psi = \arg \tan(2\eta \zeta \sigma_2, 1 - \sigma_2^2) \tag{F.20}
\]

\[
v_* = v(1) + 1 - B \cos(\alpha_{n+1} - \psi) \tag{F.21}
\]

\[
v'_* = v'(1) + \sigma_2 B \sin(\alpha_{n+1} - \psi). \tag{F.22}
\]

We seek to know the state at the next ECE, which occurs at

\[
\tau = \tau_F = \sigma_2^{-1}(\alpha_{n+1} + \gamma_{n+1}) \tag{F.23}
\]

where \( \sigma_2^{-1} \gamma_{n+1} \) is the time taken for the current in the previous thyristor to decay to zero after switching. It is the calculation of \( \gamma_{n+1} \) that presents us with our greatest technical difficulty.

Without loss of generality, suppose commutation occurs between phases \( a \) and \( b \) on the common cathode, with device currents \( i_a \) and \( i_b \) respectively, directed from AC side to DC side. Let \( v_c \) represent the series AC sources not taking part in commutation, and let \( i_S = i_a + i_b \) be the total source current. Set \( l_S = \omega_2 L_S/Z_2 = L_S/L_2 \); this is the value of the source inductance in LC units.

We may now calculate \( v \) using KVL in either of two ways. Taking a circuit that includes \( v_a \) but not \( v_b \), we obtain

\[
v = (v_a - v_c) - 1 - l_S i'_a - \left( 1 - \frac{l_S}{2} \right) i'_S = v_L \cos \left( \sigma_2 \tau + \frac{\pi}{2p} \right) - 1 - i'_S - l_S i'_a + \frac{l_S}{2} i'_S.
\]

On the other hand, combining \( v_a \) and \( v_b \) in parallel gives

\[
v = \left( \frac{v_a + v_b}{2} - v_c \right) - 1 - i'_S = v_L \cos \frac{\pi}{2p} \cos(\sigma_2 \tau) - 1 - i'_S.
\]

375
Equating these two expressions and integrating from \( \tau_s \) to \( \tau_F \) we obtain

\[
\frac{l_s}{2} (i_s(\tau_F) - i_s(\tau_s)) = l_s (i_a(\tau_F) - i_a(\tau_s)) + \int_{\tau_s}^{\tau_F} v_L \sin \frac{\pi}{2p} \sin(\sigma_2 \tau) \, d\tau
\]

\[
= -l_s i_s(\tau_s) + \frac{v_L}{\sigma_2} \int_{\alpha_{n+1} + \gamma_{n+1}}^{\alpha_{n+1} + \gamma_{n+1}} \sin(\sigma_2 \tau) \, d(\sigma_2 \tau)
\]

\[
= -l_s i_s(\tau_s) + \frac{v_L}{\sigma_2} \left( \cos(\alpha_{n+1}) - \cos(\alpha_{n+1} + \gamma_{n+1}) \right).
\]

Noting that \( \sigma_2 l_s = (\Omega L_S)/Z_2 = x_S \) is the AC source reactance in LC units, we finally obtain

\[
\cos(\alpha_{n+1}) - \cos(\alpha_{n+1} + \gamma_{n+1}) = \frac{x_S}{v_L \sin(\pi/2p)} \left( \frac{i_s(\tau_s) + i_s(\tau_F)}{2} \right).
\]  

(24)  

Note that if \( i_s \) is constant and \( p = 3 \), then (24) reduces to the usual textbook formula. (See for example [50, p.146].) On the other hand, if \( i_s \) varies then (24) normally leads to a transcendental equation for \( \gamma_{n+1} \), as the latter determines \( \tau_F \) and hence \( i_s(\tau_F) \).

In our case we have \( i_s = v' + i_0 + 2\eta \zeta v \). Denote by \( v_{(2)}, v'_{(2)} \) the final state at \( \tau_F \). Assuming that \( i_0 \approx (i_0)_{n+1} \) is approximately constant over the commutation period, we obtain

\[
\frac{i_s(\tau_s) + i_s(\tau_F)}{2} = v_{(1)} + v'_{(2)} + \frac{(i_0)_{n+1} + 2\eta \zeta v_{(1)} + v_{(2)}}{2}.
\]  

(25)  

\( v_{(2)} \) and \( v'_{(2)} \) are in turn related to \( \gamma_{n+1} \) by equations analogous to (14) and (15):  

\[
v_{(2)} = B \cos(\alpha_{n+1} + \gamma_{n+1} - \psi) - 1
\]

\[
+e^{-\frac{\gamma_{n+1}}{\sigma_2}} \left( v_s \mathcal{C}_{\eta \zeta} \left( \frac{\gamma_{n+1}}{\sigma_2} \right) + (\eta \zeta v_s + v'_s) \mathcal{S}_{\eta \zeta} \left( \frac{\gamma_{n+1}}{\sigma_2} \right) \right)
\]

(26)  

\[
v'_{(2)} = -B \sigma_2 \sin(\alpha_{n+1} + \gamma_{n+1} - \psi)
\]

\[
+e^{-\frac{\gamma_{n+1}}{\sigma_2}} \left( v'_s \mathcal{C}_{\eta \zeta} \left( \frac{\gamma_{n+1}}{\sigma_2} \right) - (v_s + \eta \zeta v'_s) \mathcal{S}_{\eta \zeta} \left( \frac{\gamma_{n+1}}{\sigma_2} \right) \right)  
\]

(27)  

with \( B, \psi, v_s, v'_s \) given by (19) through (22).

Substituting for \( v_{(2)} \) and \( v'_{(2)} \) in (25), and inserting the result in (24), we obtain a transcendental equation to be solved for \( \gamma_{n+1} \). This equation is expressible in the form

\[
K_0 + K_1 \cos(\gamma_{n+1} + \theta) + e^{-\frac{\gamma_{n+1}}{\sigma_2}} \left( K_2 \mathcal{C}_{\eta \zeta} \left( \frac{\gamma_{n+1}}{\sigma_2} \right) + K_3 \mathcal{S}_{\eta \zeta} \left( \frac{\gamma_{n+1}}{\sigma_2} \right) \right) = 0
\]  

(28)  

for some \( K_0, K_1, K_2, K_3, \theta \), of which \( K_1 \) and \( \theta \) are insensitive to the intermediate state \( v_{(1)}, v'_{(1)} \), though all are sensitive to \( \alpha_{n+1} \).  

376
F.4 An Asynchronous Discrete-Time Model

Combining the results of Sections F.2 and F.3, we obtain a mapping from \((v_0, v'_0)\) to \((v_2, v'_2)\) determined by

- the constant parameters \(v_L, p, x_S, \sigma_1, \sigma_2\) and \(\eta\), and
- the variable parameters \(\alpha, \zeta\) and \(i_0\).

We may view this as a system which maps the state at the \(n\)th ECE to the state at the \((n + 1)\)th ECE, the equations of which are obtained from (F.14), (F.15), (F.28), (F.26) and (F.27). This is essentially a discrete-time system, but since ECEs do not generally occur at regular intervals, it will be an asynchronous one, in which the sample times are a function of the control input.

Let \(\tau_n, n = 0, 1, 2 \ldots\) denote the time at which the \(n\)th ECE occurs, normalised as for the single conduction mode, with the convention \(\tau_0 = 0\). Also denote by \(\tau^*_n\), \(n = 1, 2, 3 \ldots\) the time of the \(n\)th SCE, so that \(\tau_{n-1} < \tau^*_n \leq \tau_n\). Define

\[
\mathbf{x}_n = \begin{bmatrix} v(\tau_n) \\ v'(\tau_n) \end{bmatrix}, \quad \mathbf{x}^*_n = \begin{bmatrix} v(\tau^*_n) \\ v'(\tau^*_n) \end{bmatrix}. \tag{F.29}
\]

Assume that at \(\tau = 0\) the converter is in single conduction mode, in one of the valid switching states, and let \(v_S\) denote the converter output voltage. There will then exist some \(\alpha_0 \in [0, \pi)\) such that

\[
v_S = v_L \cos \left(\sigma_1 \tau + \alpha_0 - \frac{\pi}{2p}\right) - 1 \tag{F.30}
\]

in some interval \(\tau \in [0, \epsilon)\) with \(\epsilon > 0\). One easily verifies that, if \(\gamma_0\) is taken arbitrarily to be zero, this choice of \(\alpha_0\) is consistent with the analysis of Section F.2 for \(n = 0\). (If \(\tau'\) is the time variable in Section F.2, then \(\tau' = \tau + \sigma_1^{-1} \alpha_0\).)

We assume a linear load with slowly-varying parameters \(i_0(\tau)\) and \(\zeta(\tau)\), so that \((i_0)_n = i_0(\tau_n) \approx i_0(\tau^*_n)\) and \(\zeta_n = \zeta(\tau_n) \approx \zeta(\tau^*_n)\). For \(n > 0\) we have a discrete-time system \(\mathcal{S}: \mathbf{x}_{n-1} \mapsto \mathbf{x}_n\) with the explicit form

\[
\mathbf{x}^*_n = F_1(\mathbf{x}_{n-1}, \alpha_n, \gamma_{n-1}, \zeta_{n-1})
\]

\[
0 = G(\mathbf{x}^*_n, \alpha_n, \gamma_n, (i_0)_n, \zeta_n) \tag{F.31}
\]

\[
\mathbf{x}_n = F_2(\mathbf{x}^*_n, \alpha_n, \gamma_n, \zeta_n)
\]

377
and initial conditions \( x_0, \alpha_0 \) and \( \gamma_0 = 0 \). In principle (F.31) could be rewritten to eliminate \( \gamma_n \), but in practice it is more convenient to regard \( \gamma_n \) as an auxiliary state variable.

The equations relating the times \( \tau_n, \tau_n^* \) to the control inputs \( \alpha_n, n \geq 0 \), are

\[
\begin{align*}
\tau_0 & = 0 \\
\tau_{n+1}^* & = \tau_n + \tau_n + \sigma_1 \left( \frac{\pi}{p} + (\Delta \alpha)_n - \gamma_n \right) \\
\tau_{n+1} & = \tau_{n+1} + \sigma_1^{-1} \gamma_{n+1} + \sigma_1^{-1} \left( \frac{\pi}{p} + (\Delta \alpha)_n + (\Delta \gamma)_n \right).
\end{align*}
\] (F.32)

Taking partial sums we obtain an explicit expression for \( \tau_n, n > 0 \):

\[
\begin{align*}
\tau_n & = \tau_0 + \sum_{m=0}^{n-1} (\tau_{m+1} - \tau_m) \\
& = \sigma_1^{-1} \sum_{m=0}^{n-1} \left( \frac{\pi}{p} + (\Delta \alpha)_m + (\Delta \gamma)_m \right) \\
& = \frac{1}{\sigma_1} \left( \frac{n\pi}{p} + \alpha_n + \gamma_n - \alpha_0 \right). \\
\end{align*}
\] (F.33)

The control input is constrained such that \( 0 \leq \alpha_n < \pi - \gamma_n \), which implies in turn that

\[
\frac{n\pi}{p} - \alpha_0 \leq \sigma_1 \tau_n < \frac{(n+p)\pi}{p} - \alpha_0. \\
\] (F.34)

We now have an accurate model for the dynamics of the switched circuit, provided simultaneous commutation does not occur. Of course, as either \( p \) or \( L_S \) increases, there is a greater likelihood that the ECE on one leg of the converter will be delayed beyond the SCE on another leg, so that there is no longer a neat alternation of SCEs with ECEs. The extension of the ATS model to include simultaneous commutation, as well as the possibility of discontinuous conduction at very low loadings, is left for future investigations.

If we can justify ignoring the effects of finite commutation time altogether, the model can be greatly simplified, as we then have for \( n \geq 0 \)

\[
\begin{align*}
x_{n+1} & = F_1(x_n, \alpha_{n+1}, \alpha_n, \zeta_n) \\
\tau_n & = \frac{1}{\sigma_1} \left( \frac{n\pi}{p} + \alpha_n - \alpha_0 \right)
\end{align*}
\] (F.35)

with \( F_1 \) given by Equations (F.14) and (F.15).
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Author/s:
Morton, Anthony Bruce

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