Hilbert and Hardy type inequalities

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Abstract


In 1967 Richard Rado said ‘there are ... most ingenious applications of Hölder’s inequality which reveal the tremendous power of this elementary formula’ [64]. Elliott [33] used Hölder’s inequality to obtain a simple proof of the well-known Hardy’s inequality in 1926.

I use novel splittings of conjugate exponents in Hölder’s inequality and other techniques to obtain new inequalities of Hilbert, Hilbert–Pachpatte and Hardy type for series and integrals.


Declaration

This is to certify that

(i) the thesis comprises only my original work except where indicated in the preface

(ii) due acknowledgement has been made in the text to all other material used

(iii) the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

George Donald Handley  Date
I would like to thank Associate Professor Jerry Koliha of the Department of Mathematics and Statistics at The University of Melbourne for courageously agreeing to supervise a mature–age PhD student; for his encouragement and being freely available for consultation; his lectures on Linear and Functional Analysis, Measure Theory, Operator Theory, Spectral Theory and C*-algebras provide excellent vistas of contemporary functional analysis. My interest in those areas started as a civil engineering undergraduate buying a copy of *An Introduction to Hilbert Space and the Theory of Spectral Multiplicity* by Paul Halmos. I am indebted to Professor Josip Pečarić, my strategic supervisor, of The University of Zagreb, Croatia for guidance in the vineyard of inequalities. Nevertheless any shortcomings are mine. I appreciate the support of the Department of Mathematics and Statistics at The University of Melbourne in providing financial assistance and an excellent working environment and enabling a visit to Melbourne by Professor Pečarić; the Department’s Information Technology Group; Alégra, Belinda and Loretta of Room 162; and the Glenferrie Courtyard Café.

In the Department of Civil Engineering at The University of Melbourne Professor A. J. Francis and Professor L. K. Stevens were inspirational lecturers. They approved a six–year course in Civil Engineering and the Honour School of Mathematics which was unheard of at the time.

This research commenced while I was a Lecturer in the School of Mathematical Sciences at Swinburne University which allocated time for research under the informal supervision of Professor Russell Love.

I was fortunate to have the opportunity of carrying out undergraduate re-
search projects supervised by Professor Sir Thomas Cherry and by his student Professor George Batchelor when he visited Melbourne. G. K. Batchelor\textsuperscript{1} was instrumental in setting up the Department of Applied Mathematics at Cambridge University and like Cherry was inspired by the work of Poincaré. Some years later I carried out research towards a Master’s degree on Legendre functions under the supervision of Professor Love. The legendary lectures and research of these men retained my interest in mathematics during decades working as a civil engineer.

Russell Love was acknowledged by G. H. Hardy in the preface to \cite{62} as follows: I have inserted a large number of new examples from the Mathematical Tripos during the last twenty years . . . These were collected for me by Mr. E. R. Love who has also read all the proofs and corrected many errors. It is necessary to work through a set of examples in \cite{62} to appreciate the challenge of the task.

Tom Cherry was Isaac Newton Scholar at Trinity College, Cambridge and his dissertation for the then rare degree of PhD in the early 1920’s was supervised in part by J. E. Littlewood, while H. F. Baker was his principal supervisor.\textsuperscript{2}

Last but not least I would like to thank my son Tim for his encouragement and support; and Micaela for her inspiration.

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\textsuperscript{1} Batchelor, “The don who understood turbulence”, in 1947 wrote up his own masterly interpretation of the great Russian mathematician A.N. Kolmogorov’s 1941 seminal theory for the structure of small scale turbulence. Until then, as Kolmogorov’s own students used to say, the Russians themselves did not really understand this great break through.

\textsuperscript{2} Appendix E contains a brief note on T. M. Cherry.
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Preface

In 1967 Richard Rado said ‘there are . . . most ingenious applications of Hölder’s inequality which reveal the tremendous power of this elementary formula’ [64]. Elliott [33] used Hölder’s inequality to obtain a simple proof of the well–known Hardy’s inequality in 1926.


A recurring thread is the use of novel splittings of conjugate exponents for Hölder’s inequality to obtain new inequalities of Hilbert, Hilbert-Pachpatte and Hardy type for series and integrals.

The specialised notation for Chapters 2, 3, 4 and 8 is presented at the beginning of each chapter. Outside those chapters symbols such as $\beta_i$ revert to the usual meaning of a simple variable with a subscript.

Towards the end of my work on this Thesis Alois Kufner and Lars–Erik Persson’s book Weighted inequalities of Hardy type [77] appeared. There is a certain amount of overlap but on the whole the book approaches the subject from a somewhat different perspective and using a different kit of tools. It could be said that the approaches in the book and in the Thesis are to a certain extent complementary.
Chapter 1

Introduction

1.1 Summary

This summary is also a catalogue of new results.

The well-known Hilbert’s inequality (1.1) has been generalised in many directions by a number of mathematicians [18, 41, 42, 63, 91]. A recent paper by Handley, Koliha and Pečarić [46] derived a new inequality of Hilbert type, which subsumes, as a special case, a recent result of Pachpatte [109, Theorem 1].

In Chapter 2 we obtain a new class of multivariable integral inequalities of Hilbert type. By specializing the upper estimate functions in the hypothesis and the parameters we obtain many special cases which include, in particular, the integral inequalities derived recently by Pachpatte.

In Chapter 3 our investigation is continued of multivariable integral inequalities of the type considered by Hilbert and recently by Pachpatte by focusing on fractional derivatives. Our results apply to integrable not necessarily continuous functions, and we are able to relax the original conditions to admit negative exponents in the weight functions.

In Chapter 4 we use a new approach to obtain a class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte and inequalities obtained by Handley, Koliha and Pečarić.

In Chapter 5 we derive integral inequalities of Hardy type which are gen-
eralisations of the inequalities of Mohapatra and Russell [94], who in turn extend the elegant inequalities of Davies and Petersen [28]. A new concept of $\alpha$–submultiplicative and $\alpha$–supermultiplicative functions is used. The exponent is split (or partitioned) into three components using a method of Hanjš, Love and Pečarić [52] (the HLP method) to form conjugate Hölder exponents. We introduce an $\alpha$–submultiplicative function $\varphi$ into the integrand and solve a differential equation for $\varphi$. In the main theorem there is a product of fractional powers of integrals on the right hand side.

In Chapter 6 we further generalise Hardy–type inequalities of Pečarić and Love [119]. Chapter 6 was supervised by the late Professor Love at the beginning of my candidature.

In Chapter 7 we consider discrete Hardy–type inequalities. The convergence of the series is examined, filling gaps in [69] and [70]; this fact is non–trivial and depends on the relation between two series. We find an ‘infimum principle’ leading to ‘best possible’ constants for the inequalities for the class of suitably convergent sequences $(a_n)$ for separable weight functions $\beta_n \lambda_n$. Our ‘infimum principle’ does not presuppose that the best possible constant, if it exists, will be expressible in terms of known functions. Existing 2–level inequalities are generalised to partitioned or multiple exponents by the HLP method, further extending Elliott’s simplest method of proof of discrete Hardy’s Theorem (1.2.5). Elliott’s method has been previously extended by Copson (1928), Hwang–Yang (1990), Pachpatte (1994), Hwang (1996) and others. Inequalities of three or higher levels\footnote{See Definition 7.2.1 of a $k$–level inequality.} are relatively easily constructed if the exponent is not partitioned, but if the exponent is partitioned it is not possible to extend beyond two levels using Elliott’s method.

In Chapter 8 we re–design the notation used by previous authors. This enables rigorous proofs and extensions of the discrete Hardy–type inequalities of Hwang (1996) which have a separable weight function $\beta_i(x_j) \lambda_i(x_j)$ and multiple summations. Existing 2–level, 1–exponent inequalities are generalised to partitioned (i.e. split) exponents by the HLP method. They are extended to 3–level inequalities using generalised Hölder’s inequality with three\footnote{The HLP method uses two conjugate Hölder exponents.} conjugate expo-
1.2. PROLEGOMENA

onents, in order to utilise the tremendous power of Hölder’s inequality described by Rado (loc. cit.).

The infimum principle is used to determine ‘best possible’ constants for the inequalities for the class of suitably convergent sequences \((a_n)\) with an arbitrary separable weight function \(\beta_i(x_j)\lambda_i(x_j)\). The convergence of the series is examined using the technique of Chapter 7, filling a gap in [70].

Worked examples exhibit the structure, orders of magnitude and sharpness of the inequalities in Chapter 8, and the effects of monotonic increasing and oscillating weight functions \(\beta_i(x_j)\). The theoretical reasons for these effects are a topic for further research along with the effects of convexity, \(\alpha\)-submultiplicativity and other properties of \(\beta_i(x_j)\).

1.2 Prolegomena

We start with a brief survey of the early history of Hilbert’s and Hardy’s inequalities and the body of literature generated by them.\(^3\) Much of this section may be dug out of [61] and [63].

1.2.1 Hilbert’s theorem and extensions

At the close of the 19\(^{th}\) century a theorem of great elegance and simplicity was discovered by serendipity by the world’s leading mathematician, D. Hilbert. It is

\textbf{Theorem 1.2.1.} Hilbert’s double series theorem (weak discrete version): the

\(^3\)n. pl. prolegomena a preliminary discussion, especially a formal essay introducing a work of considerable length or complexity.’ The term was used by Hardy in his Prolegomena to a chapter on inequalities [61], and seems appropriate to a discussion of inequalities. Hardy’s Prolegomena was a Presidential Address at the annual meeting of the London Mathematical Society of 8 November, 1928. It was a masterly discussion of 20 years of research into inequalities by the most distinguished mathematicians and a precursor to the classic book Inequalities by Hardy, Littlewood and Pólya [63]. The aims of this Prolegomena are much more modest. Hardy, Littlewood and Pólya was the only book produced by the Hardy–Littlewood collaboration. The joint author ‘Hardy–Littlewood’produced 97 papers of the highest quality and was recognised as the best mathematician in the world for a decade or so. Hardy produced 279 papers of comparable quality in his own right. Littlewood produced 90 individual papers and 116 joint papers with various authors, including the 97 papers of which Hardy was a co–author.
series
\[ \sum_{m,n=1}^{\infty} \frac{a_m a_n}{m+n} \]  
\[ (1.1) \]

is convergent whenever \( \sum a_m^2 \) is convergent.

The search for a simple and elementary proof of Theorem 1.2.1 attracted the attention of distinguished mathematicians for thirty years. It is unclear when the theorem was discovered since it was not published – Hilbert’s relatively complicated proof was contained in his lectures on integral equations. The proof was outlined in 1908 by H. Weyl [137] in his doctoral Dissertation and is shown in Appendix A as a matter of interest.

A. M. Fink [36, p.129] refers to Hardy, Littlewood and Pólya’s philosophy that ‘generally an inequality that is elementary should be given an elementary proof, the proof should be inside the theory it belongs to, and the proof should try to settle the cases of equality’ [63, p. 7]. In 1925 a stronger version of Theorem 1.2.1 was published by G. H. Hardy [58]:

**Theorem 1.2.2.** Hilbert’s inequality (strong discrete version): If \( p > 1, \ p' = \frac{p}{p-1}, \) and \( \sum a_m^p \leq A, \ \sum b_n^{p'} \leq B, \) the summations running from 1 to \( \infty, \) then
\[ \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \frac{A^{1/p}}{B^{1/p'}} \]  
\[ (1.2) \]

unless (a) or (b) is null.

The determination of the constant \( \pi/\sin(\pi/p) \) in Theorem 1.2.2 and the following integral analogue of Hilbert’s inequality are both due to I. Schur [129].

**Theorem 1.2.3.** Hilbert’s inequality (strong integral analogue): If \( p > 1, \ p' = \frac{p}{p-1}, \) and
\[ \int_0^\infty f^p(x) \, dx \leq F, \ \int_0^\infty g^{p'}(y) \, dy \leq G, \]  
\[ (1.3) \]
then
\[ \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \frac{F^{1/p}}{G^{1/p'}}, \]  
\[ (1.4) \]

unless \( f \equiv 0 \) or \( g \equiv 0. \)
1.2. PROLEGOMENA

1.2.2 Hardy’s theorem and extensions

Hardy noted that ‘it was a considerable time before any really simple proof of Hilbert’s double series theorem was found’ [63, p. 239]. Hilbert appears to have been too busy to publish his discovery or search for a simple proof, and as mentioned above his original proof was contained in unrelated material and not amenable to publication. In 1915 Hardy proved Theorem 1.2.4 [54] which he later described as ‘a by–product of a prolonged attempt to find a really simple and elementary proof of Hilbert’s double–series theorem’:

**Theorem 1.2.4.** Hardy’s inequality (weak discrete version): If \( \sum a_n^2 \) is convergent then

\[
\sum_{n=1}^{\infty} \left( \frac{A_n}{n} \right)^2 < \infty
\]

(1.5)

where \( A_n = a_1 + a_2 + \cdots + a_n \).

Curiously, [54] is not mentioned in the bibliography of Hardy, Littlewood and Pólya’s Inequalities [63]. The journal in which it appeared, Messenger of Mathematics, seems not to be available in libraries, probably due to its incorporation in the Quarterly Journal of Mathematics in 1929. Nevertheless [54] may be found in Hardy’s Collected Papers [64, Volume V], where Hardy’s earlier papers are organised chronologically rather than by topic. His other papers on inequalities are collected by topic in [64, Volume II]. In 1919 Hardy showed that Hilbert’s Theorem 1.2.1 could be deduced from Theorem 1.2.4 [55] and his proof is given in Appendix B. In the same paper he also announced a strong version of Theorem 1.2.4:

**Theorem 1.2.5.** Hardy’s inequality (strong discrete version): If \( p > 1, a_n \geq 0 \), and \( A_n = a_1 + a_2 + \cdots + a_n \) then

\[
\sum_{n=1}^{\infty} \left( \frac{A_n}{n} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p,
\]

(1.6)

unless all the \( a_n \) are zero. The constant is the best possible.

1.2.3 The search for best–possible constants

Considerable effort was expended by distinguished mathematicians in finding ‘best possible constants’ for the new inequalities. In 1926 E. Landau in a beau-
A somewhat literal English translation is provided as Appendix C, showing Landau’s analysis of asymptotic behaviour of the right hand side as a function of an introduced parameter $\epsilon$.

We observe that equality occurs in (1.6) when the $a_n$ are equal. The ‘best possible’ constant applies to a class of inequalities$^4$ and is not necessarily sharp for a specific sequence $(a_n)$.

### 1.2.4 Integral analogues

In 1920 Hardy [56] stated but initially did not prove an integral version of Theorem 1.2.4, saying he was ‘occupied primarily with the corresponding theorem for infinite series’. It is

**Theorem 1.2.6.** Hardy’s inequality (strong integral analogue): If $p > 1$, $f(x) \geq 0$, and $F(x) = \int_0^x f(t) \, dt$, then

$$\int_0^\infty \left( \frac{F}{x} \right)^p \, dx < \left( \frac{p}{p - 1} \right)^p \int_0^\infty f^p \, dx$$

(1.7)

where $\int_0^\infty f^p \, dx$ exists.

In 1925, after prompting by his friend Landau, Hardy published a proof of Theorem 1.2.6 [59]. The two inequalities, together known as ‘Hardy–Hilbert’s inequality’, have many applications and are the subject of a considerable body of literature, a small sample of which is in the bibliography. Hardy was half of the author ‘Hardy–Littlewood’, which succeeded Hilbert as the world’s pre-eminent mathematician. In 1920 Hardy observed that ‘Hilbert’s beautiful double-series theorem is the subject of at least five essentially different proofs’ [55]. By 1928 Hardy was able to say that ‘There is a still greater variety of proofs of Hilbert’s theorem: I have given nine in a lecture at Oxford, and more have been found since then’ [61]. The proofs described by Hardy in 1920 are:

(i) ‘Hilbert’s original proof which was sketched in 1908 in the *Inaugural-Dissertation* of his student H. Weyl [137]. The proof was based on considerations drawn from the theory of Fourier series in the course of Hilbert’s investigations in the theory of integral equations.

$^4$See e.g. [72].
(ii) A proof in 1910 by F. W. Wiener is genuinely elementary, but decidedly artificial [141].

(iii) Two proofs in 1912 by I. Schur [129]; the first depends on the general theory of quadratic and bilinear forms in an infinity of variables. The second is the simplest and most elegant given at that time; but inasmuch as it depends on a change of variables in a double integral, it cannot be regarded as elementary."

In 1919 Hardy demonstrated that his Theorem 1.2.4 was equivalent to Hilbert’s double series theorem, showing that ‘each may be deduced from the other, by arguments of an entirely simple and elementary kind’ [55]. In 1925 Hardy also provided a new and simpler proof of his theorem 1.2.4 [59]. This proof he describes as ‘not lacking in simplicity’ and he acknowledges a contribution by M. Riesz. He notes that the best possible constants for the strong versions of Hilbert’s and Hardy’s inequality (Theorems 1.2.2 and 1.2.5) may not be deduced from one another, not surprisingly since they differ.

In 1926 E. B. Elliott produced an even simpler proof than Hardy’s for Theorem 1.2.4. Elliott’s method has been extended by later authors and provides the basis for proving the more complicated Hardy–type inequalities which appear in this dissertation [33].

We note that alternative proofs of Hardy’s theorems were subsequently given by the following writers: in 1927 Kaluza and Szegö [74]; in 1928 Grandjot [45] and Knopp [75]; and in 1930 A. E. Ingham [71].

In 1928 E. T. Copson published two generalised Hardy’s discrete inequalities [26] which are reproduced as our Theorems 7.1.1 and 7.1.2. In 1975 after a resurgence of interest in such inequalities Copson produced an extension of Hardy’s integral inequalities [27]. Other distinguished mathematicians who worked in those early years on the preceding theorems and their generalisations in different directions, were in 1921 L. Fejér and F. Riesz [35], in 1928 E. C. Francis and J. E. Littlewood [40] and G. Pólya and G. Szegö [121].

1.2.5 Rado’s assessment of Hardy’s work

R. Rado was the member responsible for the papers on inequalities of the Editorial Committee appointed by the London Mathematical Society to produce the
Collected Papers of G. H. Hardy [64]. He observed that ‘much of Hardy’s work on inequalities was suggested in the first instance by specific applications in real or complex analysis, but soon most of the problems were pursued for their own sake’ [64, p.379].

Rado’s editorial comments provide an excellent overview of the correspondence concerning the equivalence of various versions and theorems between Hardy, Landau, F. Riesz, Schur, Pólya and others. Perchance my impression of Rado coincides with that of

(a) P. Erdős: ‘I was good at discovering perhaps difficult and interesting special cases and Richard (Rado) was good at generalising them and putting them in their proper perspective’ [34], and

(b) C. A. Rogers in Biographical Memoirs of Fellows of the Royal Society of London: ‘Richard Rado was fascinated by mathematical beauty and sought after it. He always tried to formulate his results at their natural level of generality, so that their full power was exhibited, without their content being obscured by over-elaboration’ [125].

So it is without apology that Rado is further quoted.

‘Various stages of perfection are recognisable in Hardy’s work on inequalities.

(a) The determination of all systems of values of the parameters \( p_i \) of the problem for which the inequality holds with some suitable multiplicative constant \( K \) on the right-hand side,

(b) the determination for each such system \( p_i \) of the best possible value \( K^* \) of \( K \),

(c) the characterisation, for each system \( p_i \) and the corresponding \( K^* \), of all cases of equality.

Many of the results obtained are of type (b) and some of type (c).

Of the methods employed there are

(i) ‘Elementary’ methods depending only on tools such as Hölder’s and Minkowski’s
inequalities and the estimation of sums by means of integrals,

(ii) ‘Advanced’ methods relying on the calculus of variations,

(iii) Highly original methods involving rearrangements of sequences of numbers and a comparison between quantities derived from these sequences.

This classification is very inadequate at times. Thus there are under (i) most ingenious applications of Hölder’s inequality which reveal the tremendous power of this elementary formula’ (*ibid*).

Hilbert lived until 1943 and his student E. Schmidt was a mentor of Rado. Thus Rado was in the intersection of Hilbert’s circle and the locus of English and European mathematicians who collaborated with Hardy and Littlewood.

The parameters listed above provide a framework for considering the work presented herein. But simple propositions do not always have simple proofs and if the statement of a theorem is complicated, then its proof will be also.
Chapter 2

New Hilbert–Pachpatte type integral inequalities

In Chapter 2 we obtain a new class of multivariable integral inequalities of Hilbert type. By specialising the upper estimate functions in the hypothesis and the parameters, we obtain many special cases, which include, in particular, the integral inequalities derived recently by Pachpatte.

2.1 Introduction

Hilbert’s double series inequality and its integral version [63, Theorem 316] have been generalised in several directions [18, 41, 42, 63, 91, 109, 110, 144]. Recently, Pachpatte [111, 114, 118] considered inequalities similar to those of Hilbert. A representative sample is the following.

Theorem 2.1.1. (Pachpatte [111, Theorem 1].) Let $n \geq 1$ and $0 \leq k \leq n - 1$ be integers. Let $u \in C^n([0, x])$ and $v \in C^n([0, y])$, where $x > 0$, $y > 0$, and let $u^{(j)}(0) = v^{(j)}(0) = 0$ for $j \in \{0, \ldots, n - 1\}$. Then

$$\int_0^x \int_0^y \frac{|u^{(k)}(s)||v^{(k)}(t)|}{s^{2n-2k-1} + t^{2n-2k-1}} \, ds \, dt \leq M(n, k, x, y) \left( \int_0^x (x - s)|u^{(n)}(s)|^2 \, ds \right)^{1/2} \left( \int_0^y (y - t)|v^{(n)}(t)|^2 \, dt \right)^{1/2} \tag{2.1}$$

where

$$M(n, k, x, y) = \frac{1}{2 \left((n - k - 1)!\right)^2 (2n - 2k - 1)} \sqrt{xy}. \tag{2.2}$$
The purpose of Chapter 2 is to derive a new class of related integral inequalities from which results of Pachpatte in [111, 114, 118] are obtained by specialising the parameters and the functions $\Phi_i$ in (2.3) below.

### 2.2 Notation and preliminaries for Chapter 2

The following notation and hypotheses are used throughout Chapter 2. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ have their usual meaning, $\mathbb{R}^+$ denotes the interval $[0, \infty)$.

$I = \{1, \ldots, n\}$  \quad n \in \mathbb{N}$

$m_i, i \in I$  \quad m_i \in \mathbb{N}$

$k_i, i \in I$  \quad k_i \in \{0, \ldots, m_i - 1\}$

$x_i, i \in I$  \quad x_i \in \mathbb{R}, x_i > 0$

$p_i, q_i, i \in I$  \quad p_i, q_i \in \mathbb{R}^+, 1/p_i + 1/q_i = 1$

$p, q$  \quad 1/p = \sum_{i=1}^{n} (1/p_i), 1/q = \sum_{i=1}^{n} (1/q_i)$

$a_i, b_i, i \in I$  \quad a_i, b_i \in \mathbb{R}^+, a_i + b_i = 1$

$w_i, i \in I$  \quad w_i \in \mathbb{R}, w_i > 0, \sum_{i=1}^{n} w_i = 1$

$\alpha_i, i \in I$  \quad \alpha_i = (a_i + b_i q_i) (m_i - k_i - 1)$

$\beta_i, i \in I$  \quad \beta_i = a_i (m_i - k_i - 1)$

$u_i, i \in I$  \quad u_i \in C^{m_i'}([0, x_i])$ for some $m_i' \geq m_i$

$\Phi_i, i \in I$  \quad \Phi_i \in C^1([0, x_i]), \Phi_i \geq 0$

Here $u_i$ are given functions of sufficient smoothness, and $\Phi_i$ are subject to choice. The coefficients $p_i, q_i$ are conjugate Hölder exponents to be used in applications of Hölder’s inequality, and the coefficients $a_i, b_i$ will be used in exponents to factorise integrands. The coefficients $w_i$ will act as weights in applications of the geometric–arithmetic mean inequality; this will enable us to pass from products to sums of terms. The coefficients $\alpha_i$ and $\beta_i$ arise naturally in the derivation of the inequalities.

The key to the results derived in Chapter 2 are the inequalities (2.3). Such inequalities are always available with some continuous nonnegative functions $\Phi_i$ provided the $u_i$ are sufficiently smooth and their derivatives at 0 satisfy certain conditions (vanish).

### 2.3 The main result

The theorem of this section forms an abstract basis for obtaining a class of concrete inequalities by selecting suitable functions $\Phi_i$ in (2.3); as noted above,
2.3. THE MAIN RESULT

such functions $\Phi_i$ always exist under suitable hypotheses on the $u_i$.

**Theorem 2.3.1.** Let $u_i \in C^{m_i}([0, x_i])$ for $i \in I$. If

$$\left| u_i^{(k_i)}(s_i) \right| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i) \, d\tau_i, \ s_i \in [0, x_i], \ i \in I,$$

then

$$\int_0^{x_1} \ldots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i + 1)/(q_i w_i)}} \, ds_1 \ldots ds_n \leq U \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i} \, ds_i \right)^{1/p_i}$$

where

$$U = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]}.$$

**Proof.** Factorize the integrand on the right side of (2.3) as

$$(s_i - \tau_i)^{(\alpha_i/q_i + b_i)(m_i - k_i - 1)} \times (s_i - \tau_i)^{(\alpha_i/p_i)(m_i - k_i - 1)} \Phi_i(\tau_i)$$

and apply Hölder’s inequality [93, p. 106]. Then

$$|u_i^{(k_i)}(s_i)| \leq \left( \int_0^{s_i} (s_i - \tau_i)^{\alpha_i + b_i q_i} (m_i - k_i - 1) \Phi_i(\tau_i) \, d\tau_i \right)^{1/q_i} \times$$

$$\times \left( \int_0^{s_i} (s_i - \tau_i)^{\alpha_i(m_i - k_i - 1)} \Phi_i(\tau_i) p_i \, d\tau_i \right)^{1/p_i}$$

$$= \frac{s_i^{(\alpha_i + 1)/q_i}}{(\alpha_i + 1)^{1/q_i}} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i) p_i \, d\tau_i \right)^{1/p_i}.$$

Using the inequality of means [93, p. 15]

$$\prod_{i=1}^n s_i^{(\alpha_i + 1)/q_i} \leq \sum_{i=1}^n w_i s_i^{(\alpha_i + 1)/(q_i w_i)},$$

we get

$$\prod_{i=1}^n |u_i^{(k_i)}(s_i)| \leq W \sum_{i=1}^n w_i s_i^{(\alpha_i + 1)/(q_i w_i)} \prod_{i=1}^n \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{1/p_i}.$$
where \( W = \frac{1}{\prod_{i=1}^{n} (\alpha_i+1)/\alpha_i} \). In the following estimate we apply Hölder’s inequality and, at the end, change the order of integration:

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i \omega_i)}} ds_1 \cdots ds_n
\]

\[
\leq W \prod_{i=1}^{n} \left[ \int_0^{s_i} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} d\tau_i \right]
\]

\[
\leq W \prod_{i=1}^{n} x_i^{1/q_i} \left( \int_0^{x_i} \left( \int_0^{x_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right) d\tau_i \right)^{1/p_i}
\]

\[
= \frac{W}{\prod_{i=1}^{n} (\beta_i + 1)/p_i} \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - \tau_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}
\]

This proves the theorem.

**Corollary 2.3.2.** Under the assumptions of Theorem 2.3.1,

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i \omega_i)}} ds_1 \cdots ds_n
\]

\[
\leq p^{1/p} U \prod_{i=1}^{n} x_i^{1/q_i} \left( \sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p}, \quad (2.6)
\]

where \( U \) is given by (2.5).

**Proof.** By the inequality of means, for any \( A_i \geq 0 \),

\[
\prod_{i=1}^{n} A_i^{1/p_i} \leq p^{1/p} \left( \sum_{i=1}^{n} \frac{1}{p_i} A_i \right)^{1/p}
\]

The corollary then follows from the preceding theorem.

In the following sections we discuss various choices of the functions \( \Phi_i \).

### 2.4 The first inequality

**Theorem 2.4.1.** Let \( u_i \in C^{m_i}([0, x_i]) \) be such that \( u_i^{(j)}(0) = 0 \) for \( j \in \{0, \ldots, m_i-1\}, \ i \in I \). Then

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i \omega_i)}} ds_1 \cdots ds_n
\]
2.4. THE FIRST INEQUALITY

\[
\leq U_1 \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} |u_i^{(m_i)}(s_i)|^{p_i} \, ds_i \right)^{1/p_i}
\]

(2.7)

where

\[
U_1 = \frac{1}{\prod_{i=1}^{n} [(m_i - k_i - 1)! (\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]}.
\]

(2.8)

Proof. By [111, Equation (7)],

\[
u_i^{(k_i)}(s) = \frac{1}{(m_i - k_i - 1)!} \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} u_i^{(m_i)}(\tau_i) \, d\tau_i.
\]

Inequality (2.7) is proved when we set

\[
\Phi_i(s_i) = \frac{|u_i^{(m_i)}(s_i)|}{(m_i - k_i - 1)!}
\]

(2.9)
in Theorem 2.3.1.

Corollary 2.4.2. Under the hypotheses of Theorem 2.4.1,

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n
\]

\[
\leq p^{1/p} U_1 \prod_{i=1}^{n} x_i^{1/q_i} \left( \sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} |u_i^{(m_i)}(s_i)|^{p_i} \, ds_i \right)^{1/p}
\]

(2.10)

where \( U_1 \) is given by (2.8).

We discuss a number of special cases of Theorem 2.4.1. Similar examples apply also to Corollary 2.4.2.

Example 2.4.3. If \( a_i = 0 \) and \( b_i = 1 \) for \( i \in I \), then (2.7) becomes

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(q_i m_i - q_i k_i - q_i + 1)/(q_i w_i)}} \, ds_1 \cdots ds_n
\]

\[
\leq U_1 \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} |u_i^{(m_i)}(s_i)|^{p_i} \, ds_i \right)^{1/p_i}
\]

(2.11)

where

\[
U_1 = \frac{1}{\prod_{i=1}^{n} [(m_i - k_i - 1)! (q_i m_i - q_i k_i - q_i + 1)^{1/q_i}]}.
\]

(2.12)
Example 2.4.4. If \( a_i = 0, b_i = 1, q_i = n, w_i = 1/n, p_i = n/(n-1), m_i = m \) and \( k_i = k \) for \( i \in I \), then

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{m-nk-n+1}} ds_1 \cdots ds_n \\
\leq \frac{1}{n} \left( \frac{1}{(m-k)!} \right)^n (nm-nk-n+1) \times \\
\prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m_i-k_i} |u_i^{(m_i)}(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}.
\]

(2.13)

For \( q = p = n = 2 \) this is \([111, \text{Theorem 1}]\). Setting \( q = p = 2, k = 0 \) and \( n = 1 \), we recover the result of \([118]\).

Example 2.4.5. Let \( a_i = 1 \) and \( b_i = 0 \) for \( i \in I \). Then (2.7) becomes

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n u_is_i^{(m_i-k_i)/(q_i w_i)}} ds_1 \cdots ds_n \\
\leq \tilde{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m_i-k_i} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right)^{1/p_i},
\]

(2.14)

where

\[
\tilde{U}_1 = \frac{1}{\prod_{i=1}^n (m_i - k_i)!}.
\]

(2.15)

Example 2.4.6. Set \( a_i = 1, b_i = 0, q_i = n, w_i = 1/n, p_i = n/(n-1), m_i = m \) and \( k_i = k \) for \( i \in I \). Then (2.7) becomes

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k)}(s_i)|}{\sum_{i=1}^n s_i^{m-nk}} ds_1 \cdots ds_n \\
\leq \frac{1}{n} \left( \frac{1}{(m-k)!} \right)^n \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m_i-k_i} |u_i^{(m_i)}(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}.
\]

(2.16)

Example 2.4.7. Let \( p_1, p_2 \in \mathbb{R}_+, 1/p_1 + 1/p_2 = 1 \). If we set \( n = 2, w_1 = 1/p_1, w_2 = 1/p_2, k_i = 0, m_i = 1 \) for \( i = 1, 2 \) in Theorem 2.4.1, then by our assumptions \( q_1 = p_2, q_2 = p_1, \) and we obtain

\[
\int_0^{x_1} \int_0^{x_2} \frac{|u_1(s_1)||u_2(s_2)|}{p_2 s_1^{p_1-1} + p_1 s_2^{p_2-1}} ds_1 ds_2 \\
\leq \frac{x_1^{1/p_2} x_2^{1/p_1}}{p_1 p_2} \left( \int_0^{x_1} (x_1 - s_1)^{|u'_1(s_1)|^{p_1}} ds_1 \right)^{1/p_1} \left( \int_0^{x_2} (x_2 - s_2)^{|u'_2(s_2)|^{p_2}} ds_2 \right)^{1/p_2},
\]

(2.17)

which is \([114, \text{Theorem 2}]\). (The values of \( a_i \) and \( b_i \) are irrelevant.)
2.5  The second inequality

**Theorem 2.5.1.** Let \( u_i \in C^{m_i+1}([0,x_i]) \) be such that \( u^{(j)}(0) = 0 \) for \( j \in \{0,\ldots,m_i\} \), and let \( \rho \in C^4([0,\infty)) \). Then

\[
\int_{0}^{x_1} \ldots \int_{0}^{x_n} \frac{\prod_{i=1}^{n} |u^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \ldots ds_n
\leq U_1 \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left[ \int_{0}^{x_i} (x_i - s_i)^{\beta_i+1} \frac{s_i^{p_i-1}}{\rho(s_i)^{p_i}} \left( \int_{0}^{s_i} |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i} d\sigma_i \right) ds_i \right]^{1/p_i}
\]  

(2.18)

where \( U_1 \) is given by (2.8).

**Proof.** According to [111, Equation (14)],

\[
u_i^{(k_i)}(s_i) = \frac{1}{(m_i - k_i - 1)!} \int_{0}^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \left( \frac{1}{\rho(\tau_i)} \int_{0}^{\tau_i} (\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))' d\sigma_i \right) d\tau_i.
\]

By Hölder’s inequality,

\[
\int_{0}^{\tau_i} |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'| d\sigma_i \leq \tau_i^{1/q_i} \left( \int_{0}^{\tau_i} |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i} d\sigma_i \right)^{1/p_i},
\]

and inequalities (2.3) hold with

\[
\Phi_i(\tau_i) = \frac{1}{(m_i - k_i - 1)!} \frac{\tau_i^{1/q_i}}{\rho(\tau_i)} \left( \int_{0}^{\tau_i} |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i} d\sigma_i \right)^{1/p_i}.
\]

The theorem is then proved by an application of Theorem 2.3.1.

**Corollary 2.5.2.** Under the assumptions of Theorem 2.5.1,

\[
\int_{0}^{x_1} \ldots \int_{0}^{x_n} \frac{\prod_{i=1}^{n} |u^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)/(q_i w_i)}} ds_1 \ldots ds_n
\leq p^{1/p} U_1 \prod_{i=1}^{n} x_i^{1/q_i} \times
\left[ \sum_{i=1}^{n} \frac{1}{p_i} \int_{0}^{x_i} (x_i - s_i)^{\beta_i+1} \frac{s_i^{p_i-1}}{\rho(s_i)^{p_i}} \left( \int_{0}^{s_i} |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i} d\sigma_i \right) ds_i \right]^{1/p}
\]

(2.19)

where \( U_1 \) is given by (2.7).
Example 2.5.3. Let $a_i = 0$ and $b_i = 1$ for $i \in I$. Then (2.18) becomes

\[
\int_0^x \cdots \int_0^x \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{m_i - q_i k_i + q_i - 1}/(q_i w_i)} \, ds_1 \cdots ds_n \leq \widehat{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{s_i^{-1} \rho(s_i)^{p_i}} \left( \int_0^{s_i} |\rho(s_i) u_i^{(m_i)}(\sigma_i)|^{p_i} \, d\sigma_i \right) \, ds_i \right)^{1/p_i}
\]

(2.20)

where $\widehat{U}_1$ is given by (2.12).

Example 2.5.4. Let $a_i = 0$, $b_i = 1$, $q_i = n$, $p_i = n/(n - 1)$, $w_i = 1/n$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (2.18) becomes

\[
\int_0^x \cdots \int_0^x \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n s_i^{n - nk - n + 1}} \, ds_1 \cdots ds_n \leq \frac{1}{n} \frac{1}{(m - k - 1)!! n^{n - nk - n + 1}} \times \prod_{i=1}^n \left[ \int_0^{x_i} (x_i - s_i)^{s_i^{1/(n-1)} \rho(s_i)^{n/(n-1)}} \left( \int_0^{s_i} |\rho(s_i) u_i^{(m_i)}(\sigma_i)|^{n/(n-1)} \, d\sigma_i \right) \, ds_i \right]^{(n-1)/n}
\]

(2.21)

For $q = p = n = 2$ this is [111, Theorem 2].

Example 2.5.5. Set $a_i = 1$ and $b_i = 0$ for $i \in I$. Inequality (2.18) becomes

\[
\int_0^x \cdots \int_0^x \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{m_i - k_i}/(q_i w_i)} \, ds_1 \cdots ds_n \leq \widehat{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left[ \int_0^{x_i} (x_i - s_i)^{m_i - k_i} s_i^{p_i - 1} \rho(s_i)^{p_i} \left( \int_0^{s_i} |\rho(s_i) u_i^{(m_i)}(\sigma_i)|^{p_i} \, d\sigma_i \right) \, ds_i \right]^{1/p_i}
\]

(2.22)

where $\widehat{U}_1$ is given by (2.15).

Example 2.5.6. Set $a_i = 1$, $b_i = 0$, $q_i = n$, $w_i = 1/n$, $p_i = n/(n - 1)$, $m_i = m$ and $k_i = k$ for $i \in I$. Then (2.18) becomes

\[
\int_0^x \cdots \int_0^x \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n s_i^{m_i - k_i}} \, ds_1 \cdots ds_n
\]


\[ \leq \frac{1}{n} \left( \frac{1}{(m-k)!} \right)^n \times \prod_{i=1}^{n} \left[ \int_0^{x_i} (x_i - s_i)^{m-k} \frac{s_i^{1/(n-1)}}{\rho(s_i)^{n/(n-1)}} \left( \int_0^{s_i} |(\rho(\sigma_i)u_i^{(m)}(\sigma_i))'|^{n/(n-1)} d\sigma_i \right) ds_i \right]^{(n-1)/n}. \] (2.23)

**Example 2.5.7.** Let \( p_1, p_2 \in \mathbb{R}_+, 1/p_1 + 1/p_2 = 1 \). If we set \( n = 2, w_1 = 1/p_1, w_2 = 1/p_2, k_i = 0, m_i = 1 \) for i = 1, 2 in Theorem 2.5.1, then \( q_1 = p_2, q_2 = p_1 \) and we obtain
\[
\int_0^{x_1} \int_0^{x_2} \frac{|u_1(s_1)||u_2(s_2)|}{p_2s_1^{p_1-1} + p_1s_2^{p_2-1}} ds_1 ds_2 \\
\leq \frac{x_1^{1/p_1}x_2^{1/p_2}}{p_1p_2} \left( \int_0^{x_1} (x_1 - s_1) \frac{s_1^{p_1-1}}{\rho(s_1)^{p_1}} \int_0^{s_1} |(\rho(\sigma_1)u'_1(\sigma_1))'|^{p_1} d\sigma_1 ds_1 \right)^{1/p_1} \times \\
\times \left( \int_0^{x_2} (x_2 - s_2) \frac{s_2^{p_2-1}}{\rho(s_2)^{p_2}} \int_0^{s_2} |(\rho(\sigma_2)u'_2(\sigma_2))'|^{p_2} d\sigma_2 ds_2 \right)^{1/p_2}.
\] (2.24)

This result is parallel to [114, Theorem 2].

### 2.6 The third inequality

**Theorem 2.6.1.** Let \( u_i \in C^{2m_i}([0, x_i]), \rho \in C^m([0, \infty)) \) with \( m = \max_i m_i, u_i^{(j)}(0) = 0 \) and \( (\rho(s_i)u_i^{(m_i)}(s_i))^{(j)} = 0 \) at \( s_i = 0 \) for \( j \in \{0, \ldots, m_i - 1\} \), \( i \in I \). Then
\[
\int_0^{x_1} \cdots \int_0^{x_n} \prod_{i=1}^{n} u_i^{(k_i)}(s_i) ds_1 \cdots ds_n \leq U_3 \prod_{i=1}^{n} x_i^{1/q_i} \times \\
\times \prod_{i=1}^{n} \left[ \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \frac{s_i^{q_i(m_i-1)+1}(p_i-1)}{\rho(s_i)^p_i} \left( \int_0^{s_i} |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}|^{p_i} d\sigma_i \right) ds_i \right]^{1/p_i},
\] (2.25)

where
\[
U_3 = \frac{1}{\prod_{i=1}^{n} [(m_i-1)!(m_i-k_i-1)!(q_i(m_i-1)+1)^{1/q_i}(\alpha_i+1)^{1/q_i}(\beta_i+1)^{1/p_i}]}. \] (2.26)
Example 2.6.3. Let $a_i = 0$ and $b_i = 1$ for $i \in I$. Inequality (2.25) becomes

$$
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n
$$

where $U_3$ is given by (2.26).

Corollary 2.6.2. Under the hypotheses of Theorem 2.6.1,

$$
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n \leq p^{1/p} U_3 \prod_{i=1}^n x_i^{1/q_i} \times
$$

$$
\times \left( \sum_{i=1}^n \int_0^{x_i} (x_i - s_i)^{\beta_i+1} \frac{s_i^{(q_i(m_i-1)+1)/(q_i w_i)-1}}{p_i/p_i} \, ds_i \right)^{1/p} \int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n
$$

(2.27)

The result then follows from Theorem 2.3.1.
2.6. THE THIRD INEQUALITY

\[
\leq U_3^{\frac{n}{p_i}} \prod_{i=1}^{n} x_i^{1/p_i} \times \\
\times \prod_{i=1}^{n} \left[ \int_{0}^{x_i} (x_i - s_i)^{s_i (q_i m_i - q_i + 1)(p_i - 1)} \rho(s_i)^{p_i} \left( \int_{0}^{s_i} |(\rho(\sigma_i) u_i^{(m_i)}(\sigma_i))(m_i)|^{p_i} d\sigma_i \right) ds_i \right]^{1/p_i}
\]

where

\[
U_3 = \frac{1}{\prod_{i=1}^{n} (m_i - 1)! (m_i - k_i - 1)! (q_i m_i - q_i k_i - q_i + 1)^{1/q_i} (q_i m_i - q_i + 1)^{1/q_i}}.
\]

Example 2.6.4. Set \(a_i = 0, b_i = 1, q_i = n, w_i = 1/n, p_i = n/(n - 1), m_i = m\) and \(k_i = k\) for \(i \in I\). Then (2.25) becomes

\[
\int_{0}^{x_i} \cdots \int_{0}^{x_n} \left\{ \frac{\prod_{i=1}^{n} u_i^{(k_i)}(s_i)}{\sum_{i=1}^{n} w_i u_i^{(m_i)}(s_i)/(q_i w_i)} \right\} ds_1 \cdots ds_n
\]

\[
\leq \frac{1}{n} \frac{1}{[(m - 1)!]^n [((m - k - 1)!]^n (nm - nk - n + 1)(nm - n + 1)]^{(n-1)/n}} \times \\
\times \prod_{i=1}^{n} \left[ \int_{0}^{x_i} (x_i - s_i)^{s_i (nm - n + 1)/(n - 1)} \rho(s_i)^{n/(n - 1)} \left( \int_{0}^{s_i} |(\rho(\sigma_i) u_i^{(m)}(\sigma_i))(m)|^{n/(n - 1)} d\sigma_i ds_i \right) \right]^{(n-1)/n}
\]

For \(q = p = n = 2\) this becomes [111, Theorem 3].

Example 2.6.5. Set \(a_i = 1\) and \(b_i = 0\) for \(i \in I\). Inequality (2.25) becomes

\[
\int_{0}^{x_i} \cdots \int_{0}^{x_n} \left\{ \frac{\prod_{i=1}^{n} u_i^{(k_i)}(s_i)}{\sum_{i=1}^{n} w_i s_i^{(m_i - k_i)/(q_i w_i)}} \right\} ds_1 \cdots ds_n \leq U_3^{\frac{n}{p_i}} \prod_{i=1}^{n} x_i^{1/p_i} \times \\
\times \prod_{i=1}^{n} \left[ \int_{0}^{x_i} (x_i - s_i)^{m_i - k_i s_i^{(q_i m_i - q_i + 1)(p_i - 1)}} \rho(s_i)^{p_i} \left( \int_{0}^{s_i} |(\rho(\sigma_i) u_i^{(m)}(\sigma_i))(m)|^{p_i} d\sigma_i \right) ds_i \right]^{1/p_i}
\]

where

\[
U_3 = \frac{1}{\prod_{i=1}^{n} (m_i - 1)! (m_i - k_i)!(q_i m_i - q_i + 1)^{1/q_i}}.
\]

Example 2.6.6. Set \(a_i = 1, b_i = 0, q_i = n, w_i = 1/n, p_i = n/(n - 1), m_i = m\) and \(k_i = k\) for \(i \in I\). Then (2.25) becomes

\[
\int_{0}^{x_i} \cdots \int_{0}^{x_n} \left\{ \frac{\prod_{i=1}^{n} u_i^{(k_i)}(s_i)}{\sum_{i=1}^{n} s_i^{m_i - k_i}} \right\} ds_1 \cdots ds_n
\]
\begin{align*}
&\leq \frac{1}{n} \left[ (m-1)! \right]^n \left[ (m-k)! \right]^n (nm-n+1) \times \\
&\times \prod_{i=1}^{n} \left[ \int_{0}^{x_i} (x_i - s_i)^{m-k} \frac{s_i^{(nm-n+1)/(n-1)}}{\rho(s_i)^{n/(n-1)}} \int_{0}^{s_i} \left| (\rho(\sigma_i) u_i^{(m)}(\sigma_i))^{(m)} \right|^{n/(n-1)} d\sigma_i ds_i \right]^{(n-1)/n}.
\end{align*}
(2.33)
Chapter 3

Hilbert–Pachpatte inequalities – fractional derivatives

In Chapter 3 we continue our investigation of multivariable integral inequalities of the type considered by Hilbert and recently by Pachpatte by considering fractional derivatives. Our results apply to integrable not necessarily continuous functions, and we are able to relax the original conditions to admit negative exponents in the weight functions.

3.1 Introduction and preliminaries

The purpose of Chapter 3 is to derive new integral inequalities for fractional derivatives related to those obtained originally by Hilbert, and their recent analogues and generalisations involving classical derivatives due to Pachpatte and other authors. Recall the original Hilbert’s double integral inequality:

**Theorem 3.1.1.** [63, Theorem 316] If \( p > 1 \), \( q = p/(p - 1) \) and

\[
\int_0^\infty f^p(x) \, dx \leq F, \quad \int_0^\infty g^q(y) \, dy \leq G,
\]

then

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} F^{1/p} G^{1/q}
\]

where \( f, g \) are nonnegative measurable functions not identically zero.

Pachpatte [109, 110, 111, 115, 118] obtained analogues and generalisations of the theorem in several directions. We are interested in the results represented for classical derivatives by the following theorem.
CHAPTER 3. INEQUALITIES FOR FRACTIONAL DERIVATIVES

Theorem 3.1.2. (Pachpatte [111, Theorem 1]) Let \( n \geq 1 \) and \( 0 \leq k \leq n - 1 \) be integers. Let \( u \in C^n([0, x]) \) and \( v \in C^n([0, y]) \), where \( x > 0 \), \( y > 0 \), and let \( u^{(j)}(0) = v^{(j)}(0) = 0 \) for \( j \in \{0, \ldots, n - 1\} \). Then

\[
\int_0^x \int_0^y \frac{|u^{(k)}(s)||v^{(k)}(t)|}{s^{2n-2k-1} + t^{2n-2k-1}} \, ds \, dt \leq M(n, k, x, y) \left( \int_0^x (x-s)|u^{(n)}(s)|^2 \, ds \right)^{1/2} \left( \int_0^y (y-t)|v^{(n)}(t)|^2 \, dt \right)^{1/2}
\]

where

\[
M(n, k, x, y) = \frac{1}{2} \frac{\sqrt{xy}}{((n-k-1)!)^2(2n-2k-1)}. \tag{3.2}
\]

In [46], we obtained extensions and modifications of Pachpatte’s results, again with classical derivatives, recovering many of the theorems in [109, 110, 111, 118] as special cases. Our aim is to derive theorems of this type for fractional derivatives, and treat integrable instead of continuously differentiable functions.

For the purpose of our exposition we survey some facts about fractional derivatives needed in the sequel; for more details see the monograph [128, Chapter 1].

Let \( x > 0 \). By \( C^m([0, x]) \) we denote the space of all functions on \([0, x]\) which have continuous derivatives up to order \( m \), and \( AC([0, x]) \) is the space of all absolutely continuous functions on \([0, x]\). By \( AC^m([0, x]) \) we denote the space of all functions \( g \in C^{m-1}([0, x]) \) with \( g^{(m-1)} \in AC([0, x]) \). For any \( \alpha \in \mathbb{R} \) we denote by \( [\alpha] \) the integral part of \( \alpha \) (the integer \( k \) satisfying \( k \leq \alpha < k + 1 \)).

Let \( \alpha > 0 \). For any \( f \in L(0, x) \) the Riemann–Liouville fractional integral of \( f \) of order \( \alpha \) is defined by

\[
I^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t) \, dt, \quad s \in [0, x], \tag{3.3}
\]

and the Riemann–Liouville fractional derivative of \( f \) of order \( \alpha \) by

\[
D^\alpha f(s) = \left( \frac{d}{ds} \right)^m I^{m-\alpha} f(s) = \frac{1}{\Gamma(m-\alpha)} \left( \frac{d}{ds} \right)^m \int_0^s (t-s)^{m-\alpha-1} f(t) \, dt \tag{3.4}
\]

where \( m = [\alpha] + 1 \). In addition, we stipulate

\[
D^0 f := f =: I^0 f, \quad I^{-\beta} f := D^\beta f \text{ if } \beta > 0, \quad D^{-\alpha} f := I^\alpha f \text{ if } 0 < \alpha \leq 1. \tag{3.5}
\]

If \( \alpha \) is a positive integer, then \( D^\alpha f = (d/ds)^\alpha f \).
3.1. INTRODUCTION AND PRELIMINARIES

Let \( \alpha > 0 \) and \( m = [\alpha] + 1 \). The space \( I^\alpha(L_1) \) consists of all functions \( f \) on \( [0,x] \) of the form \( f = I^\alpha \varphi \) for some \( \varphi \in L(0,x) \) (see [128, Chapter I, Definition 2.3]). According to [128, Theorem 1.2.3], this is equivalent to the condition

\[
I^{m-\alpha} f \in AC^m([0,x]),
\]

(3.6)

\[
\left( \frac{d}{ds} \right)^j I^{m-\alpha} f(0) = 0, \quad j = 0, 1, \ldots, m - 1.
\]

(3.7)

A function \( f \in L(0,x) \) satisfying (3.6) is said to have an integrable fractional derivative \( D^\alpha f \) [128, Chapter I, Definition 2.4]. In particular, \( D^\alpha f \) is an integrable fractional derivative if \( D^\alpha f \) exists at each point \( s \) of \( [0,x] \). We find it convenient to express these conditions in terms of fractional derivatives.

**Lemma 3.1.3.** Let \( \alpha > 0 \) and \( m = [\alpha] + 1 \). A function \( f \in L(0,x) \) has an integrable fractional derivative \( D^\alpha f \) if and only if

\[
D^{\alpha-k} f \in C([0,x]), \quad k = 1, \ldots, m, \quad \text{and} \quad D^{\alpha-1} f \in AC([0,x]).
\]

(3.8)

Further, \( f \in I^\alpha(L_1) \) if and only if \( f \) has an integrable fractional derivative \( D^\alpha f \) and satisfies the conditions

\[
D^{\alpha-k} f(0) = 0 \quad \text{for} \quad k = 1, \ldots, m.
\]

(3.9)

**Proof.** Observe that, in view of the definition of fractional derivative and of the equation \([\alpha - m + k] + 1 = k\),

\[
\left( \frac{d}{ds} \right)^k I^{m-\alpha} f = \left( \frac{d}{ds} \right)^k I^{k-(\alpha-m+k)} f = D^{\alpha-m+k} f.
\]

Then (3.8) is equivalent to (3.6) and (3.9) is equivalent to (3.7). (For \( k = m \) we use the stipulation \( D^{\alpha-m} f = I^{m-\alpha} f \) in (3.8).)

**Definition 3.1.4.** We say that \( f \in L(0,x) \) has an \( L^\infty \) fractional derivative \( D^\alpha f \) in \( [0,x] \) if conditions (3.8) are satisfied and \( D^\alpha f \in L^\infty(0,x) \).

The next result is a version of Taylor’s theorem for fractional derivatives with an integral remainder, which will be needed later in the paper. For a generalisation of this result see [134].

**Lemma 3.1.5.** [128, Chapter I, Theorem 2.2] Let \( \alpha \geq 0 \), \( m = [\alpha] + 1 \) and \( f \in AC^m([0,x]) \). Then the fractional derivative \( D^\alpha f \) exists almost everywhere in \( [0,x] \) and

\[
D^\alpha f(s) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{\Gamma(k - \alpha + 1)} s^{k-\alpha} + I^{m-\alpha} f^{(m)}(s), \quad s \in [0,x].
\]

(3.10)
We will also need the following result on the law of indices for fractional integration and differentiation using the unified notation (3.5) and restricting ourselves to real exponents.

Lemma 3.1.6. [128, Chapter I, Theorem 2.5] The law of indices

\[ I^u I^v f = I^{u+v} f \]  

(3.11)

is valid in the following cases:

(i) \( v > 0, u + v > 0 \) and \( f \in L(0,x) \);

(ii) \( v < 0, u > 0 \) and \( f \in I^{-v}(L_1) \);

(iii) \( u < 0, u + v < 0 \) and \( f \in I^{-u-v}(L_1) \).

3.2 Preparatory inequalities

We start with an inequality needed in the proof of the main theorem.

Proposition 3.2.1. Let \( \Phi \in L^\infty(0,s) \) be nonnegative on \((0,s)\), where \( s > 0 \). Let \( 1/p + 1/q = 1 \) with \( p, q > 1 \), let \( r > -1 \), and let \( a, b \in \mathbb{R} \) satisfy

\[ a \geq 0, \quad b \geq 0, \quad a + b = 1; \quad b > \frac{r + 1}{(1 - q)r} \text{ if } r < 0. \]  

(3.12)

Then

\[ \int_0^s (s-t)^r \Phi(t) \, dt \leq \frac{s^{((a+by)^r+1)/q}}{((a+by)^r+1)^{1/q}} \left( \int_0^s (s-t)^{ar} \Phi(t)^p \, dt \right)^{1/p}. \]  

(3.13)

Proof. First assume that \(-1 < r < 0\). We observe that \( t \mapsto (s-t)^{r} \Phi(t) \in L(0,s) \) as \( |(s-t)^{r} \Phi(t)| \leq \text{const}(s-t)^{r} \) a.e. in \((0,s)\), and \( t \mapsto (s-t)^{r} \in L(0,s) \) if \( r > -1 \). Factorize the integrand on the left in (3.13) as

\[ (s-t)^{r} \Phi(t) = (s-t)^{(a/q+b)r} \left[ (s-t)^{ar/p} \Phi(t) \right] . \]  

(3.14)

From (3.12) we deduce that \((a+by)r > -1\). For \( a = 0 \) or \( b = 0 \) this is obvious, for \( a > 0 \) and \( b > 0 \) we have

\[ (a+by)r = (1 - b + by)r = b(q-1)r + r > -r - 1 + r = -1. \]
Hence the first factor \( t \mapsto (s-t)^{(a/q+b)r} \) in (3.14) is in \( L^q(0,s) \). Further, \( ar > -1 \), and the second factor \( (s-t)^{ar/p} \Phi(t) \) in (3.14) is in \( L^p(0,s) \).

We then apply Hölder’s inequality to obtain

\[
\int_0^s (s-t)^a \Phi(t) \, dt = \int_0^s (s-t)^{(a/q+b)r} \left[(s-t)^{ar/p} \Phi(t)\right]^{1/q} \left(\int_0^s (s-t)^{ar/p} \Phi(t) \, dt\right)^{1/q} \leq \left(\int_0^s (s-t)^{(a+bq)r} \, dt\right)^{1/q} \left(\int_0^s (s-t)^{ar} \Phi(t)^p \, dt\right)^{1/p} \tag{3.14}
\]

from which (3.13) follows.

Let \( r \geq 0 \). As \( \Phi \in L^\infty(0,s) \) and the exponent \( r \) in the weight function \( (s-t)^r \) is nonnegative, we can apply Hölder’s inequality without restriction, and the result follows.

In our main theorem below, \( u_i \) are given functions, the coefficients \( p_i, q_i \) are conjugate Hölder exponents to be used in applications of Hölder’s inequality, and the coefficients \( a_i, b_i \) are used in exponents to factorise integrands. The coefficients \( w_i \) will act as weights in applications of the geometric–arithmetic mean inequality.

**Theorem 3.2.2.** For each \( i \in \{1, \ldots, n\} \) let \( x_i > 0, \, u_i \in L(0,x_i) \) and \( \Phi_i \in L^\infty(0,x_i) \) be nonnegative, \( r_i > -1 \), let \( p_i, q_i > 1 \) satisfy \( 1/p_i + 1/q_i = 1 \), \( w_i > 0 \) satisfy \( \sum_{i=1}^n w_i = 1 \), and \( a_i, b_i \in [0,1] \) satisfy \( a_i + b_i = 1 \); in addition, \( b_i > (r_i + 1)/1 - q_i r_i \) for those \( i \) for which \( r_i < 0 \). If

\[
|u_i(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{r_i} \Phi_i(\tau_i) \, d\tau_i, \quad s_i \in [0,x_i], \quad i = 1, \ldots, n, \tag{3.15}
\]

then

\[
\prod_{i=1}^n |u_i(s_i)| \leq \prod_{i=1}^n w_i s_i^{((a_i+b_i q_i) r_i + 1)/q_i} \int_0^{s_i} (x_i - s_i)^{a_i r_i + 1} \Phi_i(s_i)^{p_i} \, ds_i \tag{3.16}
\]

where

\[
\Omega = \prod_{i=1}^n \left[\left(\frac{(a_i+b_i q_i) r_i + 1}{a_i r_i + 1}\right) q_i \right]. \tag{3.17}
\]

**Proof.** According to Proposition 3.2.1,

\[
|u_i(s_i)| \leq \frac{s_i^{((a_i+b_i q_i) r_i + 1)/q_i}}{((a_i+b_i q_i) r_i + 1)^{1/q_i}} \left(\int_0^{s_i} (s_i - \tau_i)^{a_i r_i + 1} \Phi_i(\tau_i)^{p_i} \, d\tau_i\right)^{1/p_i}.
\]


Using the inequality of means \([93, \text{p. 15}]\)

\[
\prod_{i=1}^{n} s_i^{(a_i+b_i q_i) r_i+1)/(q_i w_i)} \leq \sum_{i=1}^{n} w_i s_i^{(a_i+b_i q_i) r_i+1)/(q_i w_i)},
\]

we get

\[
\prod_{i=1}^{n} |u_i(s_i)| \leq \Theta \sum_{i=1}^{n} w_i s_i^{(a_i+b_i q_i) r_i+1)/(q_i w_i)} \prod_{i=1}^{n} \left( \int_{0}^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}
\]

where

\[
\Theta = \frac{1}{\prod_{i=1}^{n} ((a_i + b_i q_i) r_i + 1)/q_i}.
\]

In the following estimate we apply Hölder’s inequality and, at the end, change the order of integration. The existence and finiteness of integrals is shown as in the proof of Proposition 3.2.1.

\[
\int_{0}^{x_1} \cdots \int_{0}^{x_n} \prod_{i=1}^{n} |u_i(s_i)| \sum_{i=1}^{n} w_i s_i^{(a_i+b_i q_i) r_i+1)/(q_i w_i)} ds_1 \cdots ds_n
\]

\[
\leq \Theta \prod_{i=1}^{n} \left[ \int_{0}^{s_i} \left( \int_{0}^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} ds_i \right]
\]

\[
\leq \Theta \prod_{i=1}^{n} x_i^{1/q_i} \left( \int_{0}^{s_i} \left( \int_{0}^{s_i} (s_i - \tau_i)^{a_i r_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right) ds_i \right)^{1/p_i}
\]

\[
= \Theta \prod_{i=1}^{n} \frac{x_i^{1/q_i}}{(a_i r_i + 1)^{1/p_i}} \prod_{i=1}^{n} \left( \int_{0}^{s_i} (s_i - \tau_i)^{a_i r_i+1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}.
\]

This proves the theorem.

**Remark 3.2.3.** The condition that \( \Phi \in L^\infty(0, x) \) in Proposition 3.2.1 and the preceding theorem can be replaced by the assumption that the function \( t \mapsto (s-t)^{a r} \Phi(t)^{p} \) is integrable on \((0, s)\); this allows unbounded \( \Phi \).

**Corollary 3.2.4.** Under the assumptions of Theorem 3.2.2,

\[
\int_{0}^{x_1} \cdots \int_{0}^{x_n} \prod_{i=1}^{n} |u_i(s_i)| \sum_{i=1}^{n} w_i s_i^{(a_i+b_i q_i) r_i+1)/(q_i w_i)} ds_1 \cdots ds_n
\]

\[
\leq p^{1/p}\Omega \prod_{i=1}^{n} x_i^{1/q_i} \left( \sum_{i=1}^{n} \frac{1}{p_i} \int_{0}^{s_i} (x_i - s_i)^{a_i r_i+1} \Phi_i(\tau_i)^{p_i} ds_i \right)^{1/p},
\]

where \( \Omega \) is given by (3.16) and \( p = p_1 + \cdots + p_n \).
P r o o f. By the inequality of means, for any \( A_i \geq 0, \)
\[
\prod_{i=1}^{n} A_i^{1/p_i} \leq p^{1/p} \left( \sum_{i=1}^{n} \frac{1}{p_i} A_i \right)^{1/p}.
\]
The corollary then follows from the preceding theorem.

### 3.3 Inequalities for fractional derivatives

Our first result is an integral representation of the fractional derivative \( D^\alpha f \) which will enable us to apply Theorem 3.2.2.

**Lemma 3.3.1.** Let \( \alpha \geq 0, \beta > \alpha, \) let \( f \in L(0, x) \) have an \( L^\infty \) fractional derivative \( D^\beta f \) in \([0, x]\), and let \( D^{\beta-k} f(0) = 0 \) for \( k = 1, \ldots, [\beta] + 1 \). Then
\[
D^\alpha f(s) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^s (t - s)^{\beta - \alpha - 1} D^\beta f(t) \, dt, \quad s \in [0, x]. \tag{3.19}
\]

**P r o o f.** Set \( u = \beta - \alpha > 0 \) and \( v = -\beta < 0 \). According to Lemma 3.1.3, \( f \in I^{-v}(L_1) \). Then case (ii) of Lemma 3.1.6 guarantees that the law of indices holds for this choice of \( u, v \), namely
\[
I^{\beta - \alpha} D^\beta f = I^{\alpha + v} f = I^{-\alpha} f = D^\alpha f;
\]
this is (3.19).

The following theorem is the main result of this section, a new inequality for fractional derivatives derived from (3.16).

**Theorem 3.3.2.** Let \( n \in \mathbb{N} \). For each \( i \in \{1, \ldots, n\} \) let \( x_i > 0, \alpha_i \geq 0, \beta_i > \alpha_i \). Let \( p_i, q_i > 1 \) satisfy \( 1/p_i + 1/q_i = 1 \), \( w_i > 0 \) satisfy \( \sum_{i=1}^{n} w_i = 1 \), and \( a_i, b_i \in [0, 1] \) satisfy \( a_i + b_i = 1 \); if \( \beta_i < \alpha_i + 1 \), let in addition \( b_i > (\beta_i - \alpha_i)/(q_i - 1)(1 - \beta_i + \alpha_i) \). Write \( r_i = \beta_i - \alpha_i - 1 \). If, for each \( i \in \{1, \ldots, n\} \), \( f_i \in L(0, x_i) \) has an \( L^\infty \) fractional derivative \( D^{\beta_i} f_i \) and \( D^{\beta_i - 1} f_i(0) = 0 \) for \( j = 1, \ldots, [\beta_i] + 1 \), then
\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |D^{\alpha_i} f_i(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(a_i + b_i)(r_i + 1)/(q_i w_i)}} \, ds_1 \cdots ds_n 
\leq \Omega_1 \prod_{i=1}^{n} x_i^{1/p_i} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i)^{a_i r_i + 1} |D^{\beta_i} f_i(s_i)|^{p_i} \, ds_i \right)^{1/p_i}, \tag{3.20}
\]
where
\[
\Omega_1 = \frac{1}{\prod_{i=1}^{n} \Gamma(r_i + 1)((a_i + b_i q_i)(r_i + 1))^{1/p_i}}. \tag{3.21}
\]
Theorem 3.3.6. Let \( a \) and \( \alpha \) as before, we first obtain an integral representation of \( D^m \) if \( \alpha \in \Omega \) where \( \Omega \) is given by
\[
\Omega = \{0, 1, \ldots, m\} - \{0, 1, \ldots, \alpha - 1\}.
\]
Set
\[
\Phi_i(t_i) = \frac{|D^\beta f_i(t_i)|}{\Gamma(r_i + 1)}.
\]
Then Theorem 3.2.2 applies with \( r_i = \beta_i - \alpha_i - 1 > -1 \).

Remark 3.3.3. Instead of the hypothesis \( D^\beta f_i \in L^\infty(0, x_i) \) in the preceding theorem we may assume that for each \( s_i \in [0, x_i] \) the functions defined by \( t_i \mapsto (s_i - t_i)^{\alpha_i r_i} |D^\beta f_i(t_i)|^{p_i} \) are integrable on \((0, x_i)\). This allows unbounded fractional derivatives \( D^\beta f_i \).

Corollary 3.3.4. Under the assumptions of Theorem 3.3.2,
\[
\int_0^{x_1} \cdots \int_0^{x_n} \prod_{i=1}^n |D^{\alpha_i} f_i(s_i)| \frac{1}{\sum_{i=1}^n w_i s_i^{((a_i+b_i) r_i+1)/(q_i w_i)}} \, ds_1 \cdots ds_n
\]
\[
\leq p^{1/p} \Omega_1 \prod_{i=1}^n x_i^{1/q_i} \left( \sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{a_i r_i+1} |D^\beta f_i(s_i)|^{p_i} \, ds_i \right)^{1/p}, \tag{3.22}
\]
where \( \Omega_1 \) is given by (3.21) and \( p = p_1 + \cdots + p_n \).

A useful specialisation of Theorem 3.3.2 is to functions possessing classical derivatives. From the definition of \( AC^m([0, x]) \) it follows that if \( f \in AC^m([0, x]) \), the derivative \( f^{(m)} \) exists almost everywhere in \([0, x]\) and is integrable there.

As before, we first obtain an integral representation of \( D^\alpha f \), a consequence of Lemma 3.1.5.

Lemma 3.3.5. Let \( \alpha \geq 0, f \in AC^m([0, x]) \) and let \( f^{(k)}(0) = 0 \) for \( k = 0, \ldots, m-1 \), where \( m = [\alpha] + 1 \). Then the derivative \( D^\alpha f \) exists in \([0, x]\), and
\[
D^\alpha f(s) = \frac{1}{\Gamma(m-\alpha)} \int_0^s (s-t)^{m-\alpha-1} f^{(m)}(t) \, dt, \quad s \in [0, x]. \tag{3.23}
\]

Theorem 3.3.6. Let \( n \in \mathbb{N} \). For each \( i \in \{1, \ldots, n\} \) let \( x_i > 0, \alpha_i \geq 0, m_i = [\alpha_i] + 1 \), let \( p_i, q_i > 1 \) satisfy \( 1/p_i + 1/q_i = 1 \), \( w_i > 0 \) satisfy \( \sum_{i=1}^n w_i = 1 \), and \( a_i, b_i \in [0, 1] \) satisfy \( a_i + b_i = 1 \); let also \( b_i > (m_i - \alpha_i)/(q_i - 1)(1 - m_i + \alpha_i) \) if \( \alpha_i \) is not an integer. Write \( r_i = m_i - \alpha_i - 1 \). If, for each \( i \in \{1, \ldots, n\} \), \( f_i \in AC^{m_i}([0, x_i]), \Phi_i^{(m_i)} \in L^\infty(0, x_i) \) and \( f_i^{(j)}(0) = 0 \) for \( j = 0, \ldots, m_i - 1 \), then the inequality (3.20) holds with \( \beta_i = m_i \) and with \( \Omega_1 \) defined by (3.21).
3.3. INEQUALITIES FOR FRACTIONAL DERIVATIVES

If \( \alpha_i = k_i \in \mathbb{N} \) and \( \beta_i = m_i \in \mathbb{N} \) for \( i = 1, \ldots, n \), Theorem 3.3.2 specializes to the following.

**Theorem 3.3.7.** For each \( i \in \{1, \ldots, n\} \) let \( m_i \in \mathbb{N} \), \( u_i \in \mathcal{C}^{m_i}(0, x_i) \) be such that \( u_i^{(j)}(0) = 0 \) for \( j \in \{0, \ldots, m_i - 1\} \), and let \( k_i \in \{0, \ldots, m_i - 1\} \). Further, let \( p_i, q_i \in (1, \infty) \) satisfy \( 1/p_i + 1/q_i = 1 \), and let \( a_i, b_i \in [0, 1] \) satisfy \( a_i + b_i = 1 \). Write \( r_i = m_i - k_i - 1 \). Then

\[
\begin{align*}
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(u_i+b_i q_i) r_i+1}/q_i w_i} ds_1 \cdots ds_n \\
\leq K \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i)^{a_i r_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right)^{1/p_i}
\end{align*}
\]

where

\[
K = \frac{1}{\prod_{i=1}^{n} |r_i!((a_i + b_i q_i) r_i + 1)^{1/q_i (a_i r_i + 1)^{1/p_i}}|}.
\]

**Example 3.3.8.** Suppose that \( \beta_i \geq \alpha_i + 1 \) for \( i \in \{1, \ldots, n\} \). Then we can choose \( a_i = 1 \) and \( b_i = 0 \) \( (i = 1, \ldots, n) \) in Theorem 3.3.2. The inequality (3.20) becomes

\[
\begin{align*}
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |D^{a_i f_i(s_i)}|}{\sum_{i=1}^{n} w_i s_i^{(r_i+1)/(q_i w_i)}} ds_1 \cdots ds_n \\
\leq \frac{1}{\prod_{i=1}^{n} \Gamma(r_i + 2)} \prod_{i=1}^{n} x_i^{1/q_i} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i)^{r_i+1} |D^{\beta_i f_i(s_i)}|^{p_i} ds_i \right)^{1/p_i}. \tag{3.24}
\end{align*}
\]

[49, Example 4.5].

**Example 3.3.9.** In Theorem 3.3.2 set \( \alpha_i = \alpha, \beta_i = \beta \), where \( \beta \geq \alpha + 1 \). Then we can choose \( a_i = 1 \) and \( b_i = 0, q_i = n, w_i = 1/n, p_i = n/(n-1) \) for \( i = 1, \ldots, n \). The inequality (3.20) becomes

\[
\begin{align*}
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |D^{a_i f_i(s_i)}|}{\sum_{i=1}^{n} s_i^{\beta - \alpha}} ds_1 \cdots ds_n \\
\leq \frac{(x_1 \cdots x_n)^{1/n}}{n^{\Gamma(n (\beta - \alpha + 1))}} \prod_{i=1}^{n} \left( \int_0^{x_i} (x_i - s_i)^{\beta - \alpha} |D^{\beta f_i(s_i)}|^{n/(n-1)} ds_i \right)^{(n-1)/n}. \tag{3.25}
\end{align*}
\]

[49, Example 4.6].
Example 3.3.10. In Theorem 3.3.2 set $a_i = 0$ and $b_i = 1$ for $i = 1, \ldots, n$. Then
\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha} f_i(s_i)|}{\sum_{i=1}^n w_i s_i^{(q_i r_i + 1)/(q_i w_i)}} ds_1 \cdots ds_n \\
\leq \frac{1}{\prod_{i=1}^n \Gamma(r_i + 1)} \left( \prod_{i=1}^n x_i^{1/q_i} \right) \left( \sum_{i=1}^n \frac{1}{w_i} \left( \frac{1}{s_i^{(\beta - \alpha - 1) + 1}} \right) \int_0^{x_i} (x_i - s_i) |D^{\beta} f_i(s_i)|^{p_i} ds_i \right)^{1/p_i}.
\]
(3.26)

[49, Example 4.3].

Example 3.3.11. In Theorem 3.3.2 set $\alpha_i = \alpha$, $\beta_i = \beta$, $a_i = 0$ and $b_i = 1$ for $i = 1, \ldots, n$. Set further $q_i = n$, $w_i = 1/n$, and $p_i = n/(n - 1)$, for $i = 1, \ldots, n$. Then
\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |D^{\alpha} f_i(s_i)|}{\sum_{i=1}^n s_i^{n(\beta - \alpha - 1) + 1}} ds_1 \cdots ds_n \\
\leq \frac{(x_1 \cdots x_n)^{1/n}}{n \Gamma^n(\beta - \alpha)(n(\beta - \alpha - 1) + 1)} \left( \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) |D^{\beta} f_i(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n} \right).
\]
(3.27)

[49, Example 4.4]. Setting $q = p = n = 2$, we obtain [111, Theorem 1] with $k = 0$. If $q = p = 2$ and $n = 1$, we recover the result of [118].

We observe that in the special case when $\alpha_i = 0$ for $i = 1, \ldots, n$, the inequalities obtained in Theorems 3.3.2 and 3.3.6, Corollary 3.3.4 and the preceding examples reduce to new inequalities that estimate functions in terms of their fractional derivatives.
Chapter 4

Hilbert type multidimensional inequalities

In Chapter 4 we use a new approach to obtain a class of multivariable integral inequalities of Hilbert type from which we can recover as special cases integral inequalities obtained recently by Pachpatte and Handley, Koliha and Pečarić.

4.1 Introduction

The integral version of Hilbert’s inequality [63, Theorem 316] has been generalised in several directions (see [18, 41, 42, 63, 65, 91, 144, 150, 149]). Recently, inequalities similar to those of Hilbert were considered by Pachpatte in [109, 110, 111, 114, 113, 117]. In [46] and Chapter 2 we established a new class of related inequalities, which were further extended by Dragomir and Kim [30]. Two and higher dimensional variants were treated by Pachpatte in [115, 116]. In Chapter 4 we use a new systematic approach to these inequalities based on Theorem 4.3.1 which serves as an abstract springboard to classes of concrete inequalities.

To motivate our investigation, we give a typical result of [115]. In this theorem, $H(I \times J)$ denotes the class of functions $u \in C^{(n-1,m-1)}(I \times J)$ such that $D^i u(0,t) = 0, 0 \leq i \leq n-1, t \in J, D^j u(s,0) = 0, 0 \leq j \leq m-1, s \in I$, and $D^i D^j u(s,t)$ and $D^n D^m u(s,t)$ are absolutely continuous on $I \times J$. Here $I, J$ are intervals of the type $I_\xi = [0, \xi)$ for some real $\xi > 0$.

**Theorem 4.1.1.** (Pachpatte [115, Theorem 1].) Let $u(s,t) \in H(I_x \times I_y)$ and $v(k,r) \in H(I_z \times I_w)$. Then, for $0 \leq i \leq n-1, 0 \leq j \leq m-1$, the following
inequality holds:

\[
\int_0^x \int_0^y \left( \int_0^z \int_0^w \frac{|D_1^i D_2^j u(s, t) D_1^i D_2^j v(k, r)|}{s^{2n-2i-1} t^{2m-2j-1} + k^{2n-2i-1} r^{2m-2j-1}} \, dk \, dr \right) \, ds \, dt \\
\leq \frac{1}{2} [A_{i,j} B_{i,j}]^2 \sqrt{xyzw} \left( \int_0^x \int_0^y (x-s)(y-t)|D_1^n D_2^m u(s, t)|^2 \, ds \, dt \right)^{1/2} \\
\cdot \left( \int_0^z \int_0^w (z-k)(w-r)|D_1^n D_2^m v(k, r)|^2 \, dk \, dr \right)^{1/2},
\]

where

\[
A_{i,j} = \frac{1}{(n-i-1)!(m-j-1)!}, \quad B_{i,j} = \frac{1}{(2n-2i-1)(2m-2j-1)}.
\]

The purpose of Chapter 4 is to obtain a simultaneous generalisation of Pachpatte’s multivariable results [115], and of our results in [46] and Chapter 2. The single variable results [111, 114, 112, 117] follow as special cases of our theorems. Our treatment is based on Theorem 4.3.1, in particular on the abstract inequality \((4.1)\), which yields a variety of special cases when the functions \(\Phi_i\) are specified.

### 4.2 Notation and preliminaries for Chapter 4

The following notation and hypotheses will be used throughout Chapter 4. By \(\mathbb{Z} (\mathbb{Z}_+)\) and \(\mathbb{R} (\mathbb{R}_+)\) we denote the sets of all (nonnegative) integers and all (nonnegative) real numbers. We will be working with functions of \(d\) variables, where \(d\) is a fixed positive integer, writing the variable as a vector \(s = (s_1, \ldots, s_d) \in \mathbb{R}^d\).

A multiindex \(m\) is an element \(m = (m_1, \ldots, m_d)\) of \(\mathbb{Z}_+^d\). As usual, the factorial of a multiindex \(m\) is defined by \(m! = m_1! \cdots m_d!\). An integer \(j\) may be regarded as the multiindex \((j, \ldots, j)\) depending on the context. For vectors in \(\mathbb{R}^d\) and multiindices we use the usual operations of vector addition and multiplication of vectors by scalars. We write \(s \leq \tau (s < \tau)\) if \(s_j \leq \tau_j (s_j < \tau_j)\) for \(1 \leq j \leq d\). The same convention will apply to multiindices. In particular, \(s \geq 0 (s > 0)\) will mean \(s_j \geq 0 (s_j > 0)\) for \(1 \leq j \leq d\).

If \(s = (s_1, \ldots, s_d) \in \mathbb{R}^d\) and \(s > 0\), we define the cell

\[
Q(s) = [0, s^1] \times \cdots \times [0, s^j] \times \cdots \times [0, s^d];
\]

replacing the factor \([0, s^j]\) by \(\{0\}\) in this product, we get the face \(\partial_j Q(s)\) of \(Q(s)\).
Let $s = (s^1, \ldots, s^d)$, $\tau = (\tau^1, \ldots, \tau^d) \in \mathbb{R}^d$, $s, \tau > 0$, let $k = (k^1, \ldots, k^d)$ be a multiindex and let and $u : Q(s) \to \mathbb{R}$. Write $D_j = (\partial/\partial s^j)$. We use the following notation:

$$s^\tau = (s^1)^{\tau^1} \cdots (s^d)^{\tau^d},$$
$$D^k u(s) = D_1^{k_1} \cdots D_d^{k_d} u(s),$$
$$\int_0^s u(\tau) \, d\tau = \int_0^{s^1} \cdots \int_0^{s^d} u(\tau) \, d\tau^1 \cdots d\tau^d.$$ 

An exponent $\alpha \in \mathbb{R}$ in the expression $s^\alpha$, where $s \in \mathbb{R}^d$, will be regarded as a multiexponent, that is, $s^\alpha = (s^\alpha, \ldots, s^\alpha)$.

Another positive integer $n$ will be fixed throughout.

$I = \{1, \ldots, n\}$ \hspace{1cm} $n \in \mathbb{N}$

$m_i, \ i \in I$ \hspace{1cm} $m_i = (m^1_i, \ldots, m^d_i) \in \mathbb{Z}^d_+$

$x_i, \ i \in I$ \hspace{1cm} $x_i = (x^1_i, \ldots, x^d_i) \in \mathbb{R}^d$, $x_i > 0$

$p_i, q_i, \ i \in I$ \hspace{1cm} $p_i, q_i \in \mathbb{R}_+$, $1/p_i + 1/q_i = 1$

$p, q$ \hspace{1cm} $1/p = \sum_{i=1}^n (1/p_i)$, $1/q = \sum_{i=1}^n (1/q_i)$

$a_i, b_i, \ i \in I$ \hspace{1cm} $a_i, b_i \in \mathbb{R}_+$, $a_i + b_i = 1$

$w_i, \ i \in I$ \hspace{1cm} $w_i \in \mathbb{R}$, $w_i > 0$, $\sum_{i=1}^n w_i = 1$

Throughout Chapter 4, $u_i, v_i, \Phi$ will denote functions from $[0, x_i]$ to $\mathbb{R}$ of sufficient smoothness. If $m$ is a multiindex and $x \in \mathbb{R}^d$, $x > 0$, then $C^m[0, x]$ will denote the set of all functions $u : [0, x] \to \mathbb{R}$ which possess continuous derivatives $D^k u$, where $0 \leq k \leq m$.

The coefficients $p_i, q_i$ are conjugate Hölder exponents used in applications of Hölder’s inequality, and the coefficients $a_i, b_i$ are used in exponents to factorise integrands. The coefficients $w_i$ act as weights in applications of the geometric-arithmetic mean inequality; this enables us to pass from products to sums of terms.

### 4.3 The main result

First we present a theorem that can be regarded as a template for concrete inequalities obtained by selecting suitable functions $\Phi_i$ in (4.1). A special case of this theorem is Theorem 2.3.1.
Theorem 4.3.1. Let $v_i, \Phi_i \in C(Q(x_i))$ and let $c_i$ be multiindices for $i \in I$. If
\[ |v_i(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{c_i} \Phi_i(\tau_i) \, d\tau_i, \quad s_i \in Q(x_i), \quad i \in I, \quad (4.1) \]
then
\[
\int_0^{x_i} \cdots \int_0^{x_n} \prod_{i=1}^n |v_i(s_i)| \frac{ds_1 \cdots ds_n}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)/(q_i w_i)}} \leq U \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{s_i} (s_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i} \, ds_i \right)^{1/p_i}, \quad (4.2)
\]
where $\alpha_i = (a_i + b_i q_i) c_i$, $\beta_i = a_i c_i$, and
\[ U = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{1/q_i} (\beta_i + 1)^{1/p_i}]} . \quad (4.3) \]

Remark 4.3.2. Remembering our conventions, we observe that, for example,
\[ x_i^{1/q_i} = (x_i^{1/q_i})^{1/q_i} \cdots (x_i^{1/q_i})^{1/q_i}, \quad \prod_{i=1}^n (\alpha_i + 1)^{1/q_i} = \prod_{i=1}^d (\alpha_i + 1)^{1/q_i}. \]

Proof. Factorise the integrand on the right side of (2.3) as
\[ (s_i - \tau_i)^{(\alpha_i/q_i + b_i) c_i} \cdot (s_i - \tau_i)^{(\alpha_i/p_i) c_i} \Phi_i(\tau_i) \]
and apply Hölder’s inequality [93, p. 106] and Fubini’s theorem. Then
\[
|v_i(s_i)| \leq \left( \int_0^{s_i} (s_i - \tau_i)^{(a_i + b_i q_i) c_i} \, d\tau_i \right)^{1/q_i} \\
\cdot \left( \int_0^{s_i} (s_i - \tau_i)^{a_i c_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{1/p_i} \\
= \frac{s_i^{(\alpha_i + 1)/q_i}}{(\alpha_i + 1)^{1/q_i}} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{1/p_i} .
\]
Using the inequality of means [93, p. 15]
\[ \prod_{i=1}^n s_i^{(\alpha_i + 1)/q_i} \leq \sum_{i=1}^n w_i s_i^{(\alpha_i + 1)/(q_i w_i)} , \]
we get
\[
\prod_{i=1}^n |v_i(s_i)| \leq W \sum_{i=1}^n w_i s_i^{(\alpha_i + 1)/(q_i w_i)} \prod_{i=1}^n \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} \, d\tau_i \right)^{1/p_i}.
\]
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where
\[ W = \frac{1}{\prod_{i=1}^{n}(\alpha_i + 1)^{1/q_i}}. \]

In the following estimate we apply Hölder’s inequality and Fubini’s theorem, and, at the end, change the order of integration:

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n}\left|v_i(s_i)\right|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i + 1)/(q_i w_i)}} \ ds_1 \cdots ds_n \\
\leq W \prod_{i=1}^{n} x_i^{1/q_i} \left( \int_0^{x_i} \left( \int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} ds_i \right)^{1/p_i} \\
= \frac{W}{\prod_{i=1}^{n} (\beta_i + 1)^{1/p_i}} \prod_{i=1}^{n} x_i^{1/q_i} \left( \sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi(\tau_i)^{p_i} d\tau_i \right)^{1/p_i},
\]

This proves the theorem.

If \( d = 1 \) and \( v_i \) are replaced by the derivatives \( u_i^{(k)} \), the preceding theorem reduces to Theorem 2.3.1.

**Corollary 4.3.3.** Under the assumptions of Theorem 4.3.1,

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n}\left|v_i(s_i)\right|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i + 1)/(q_i w_i)}} \ ds_1 \cdots ds_n \\
\leq p^{1/p} U \prod_{i=1}^{n} x_i^{1/q_i} \left( \sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi(\tau_i)^{p_i} d\tau_i \right)^{1/p}, \quad (4.4)
\]

where \( U \) is given by (4.2).

**Proof.** By the inequality of means, for any \( A_i \geq 0 \),

\[
\prod_{i=1}^{n} A_i^{1/p_i} \leq p^{1/p} \left( \sum_{i=1}^{n} \frac{1}{p_i} A_i \right)^{1/p}.
\]

The corollary then follows from the preceding theorem.

The preceding corollary reduces to Corollary 2.3.2 in the special case when \( d = 1 \) and \( v_i \) are replaced by \( u_i^{(k)} \).
4.4 Applications to derivatives

In this section we shall assume that \( m_i, k_i \) are multiindices satisfying \( 0 \leq k_i \leq m_i - 1 \), and write

\[ \alpha_i = (a_i + b_i q_i)(m_i - k_i - 1), \quad \beta_i = a_i(m_i - k_i - 1). \] (4.5)

Recall that according to our conventions, \( m_i - k_i - 1 = (m_1^i - k_1^i - 1, \ldots, m_d^i - k_d^i - 1) \).

**Theorem 4.4.1.** Let \( u_i \in C^{m_i}(Q(x_i)) \) be such that \( D_j^r u_i(s_i) = 0 \) for \( s_i \in \partial_j Q(x_i), \) \( 0 \leq r \leq m_j^i - 1 \), \( 1 \leq j \leq d \), \( i \in I \). Then

\[
\int_{x_1^i}^{x_n^i} \cdots \int_{0}^{x_n^i} \prod_{i=1}^{n} \frac{|u_i^{(k_i)}(s_i)|}{w_i^{s_i^{(\alpha_i+1)/(\beta_i+1)}}} \, ds_1 \cdots ds_n
\leq U_1 \prod_{i=1}^{n} x_i^{1/p_i} \prod_{i=1}^{n} \left( \int_{0}^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} \, ds_i \right)^{1/p_i} \) (4.6)

where

\[ U_1 = \prod_{i=1}^{n} \frac{1}{[(m_i - k_i - 1)!(\alpha_i + 1)^{1/q_i}(\beta_i + 1)^{1/p_i}]} \] (4.7)

**Proof.** Under the hypotheses of the theorem we have the following multivariable identities established in [100],

\[ u_i^{(k_i)}(s) = \frac{1}{(m_i - k_i - 1)!} \int_{s_i}^{x_i} (s_i - \tau_i)^{m_i - k_i - 1} u_i^{(m_i)}(\tau_i) \, d\tau_i, \quad i \in I. \]

Inequality (4.4.1) is proved when we set \( v_i(s_i) = u_i^{(k_i)}(s_i) \), \( c_i = m_i - k_i - 1 \), and

\[ \Phi_i(s_i) = \frac{|u_i^{(m_i)}(s_i)|}{(m_i - k_i - 1)!} \] (4.8)

in Theorem 4.3.1.

**Corollary 4.4.2.** Under the hypotheses of Theorem 4.4.1,

\[
\int_{0}^{x_1^i} \cdots \int_{0}^{x_n^i} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i^{s_i^{(\alpha_i+1)/(\beta_i+1)}}} \, ds_1 \cdots ds_n
\leq p^{1/p} U_1 \prod_{i=1}^{n} x_i^{1/q_i} \left( \sum_{i=1}^{n} \frac{1}{p_i} \int_{0}^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} \, ds_i \right)^{1/p_i} \) (4.9)

where \( U_1 \) is given by (4.7).
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Proof. The result follows by applying the inequality of means to the preceding theorem.

Single variable analogues of the preceding two results were obtained in Theorem 2.4.1 and Corollary 2.4.2.

We discuss a number of special cases of Theorem 4.4.1 with similar examples applying also to Corollary 4.4.2.

Example 4.4.3. If $a_i = 0$ and $b_i = 1$ for $i \in I$, then (4.6) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{m_i-k_i-1}/(q_i w_i)} ds_1 \cdots ds_n$$

$$\leq \mathcal{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) \left| u_i^{(m_i)}(s_i) \right|^p_i ds_i \right)^{1/p_i}.$$  \hspace{1cm} (4.10)

where

$$\mathcal{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!/(q_i m_i - q_i k_i - q_i + 1)^{1/q_i}].}$$  \hspace{1cm} (4.11)

Example 4.4.4. If $a_i = 0$, $b_i = 1$, $q_i = m$, $w_i = 1/n$, $p_i = n/(n-1)$, $m_i = m$ and $k_i = k$ for $i \in I$, then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n s_i^{m_i-k_i-n+1}} ds_1 \cdots ds_n$$

$$\leq \frac{1}{n [(m - k - 1)!/(m(m - k - 1) + 1)^{1/m}]} \times$$

$$\times \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i) \left| u_i^{(m)}(s_i) \right|^{n/(n-1)} ds_i \right)^{(n-1)/n}.$$  \hspace{1cm} (4.12)

For $d = 2$ and $q = p = n = 2$ this is Pachpatte’s theorem [115, Theorem 1] cited in the Introduction; if $d = 1$ and $q = p = n = 2$, we obtain [111, Theorem 1].

Example 4.4.5. Let $a_i = 1$ and $b_i = 0$ for $i \in I$. Then (4.6) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(m_i-k_i)/((q_i w_i)}} ds_1 \cdots ds_n$$

$$\leq \tilde{U}_1 \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m_i-k_i} \left| u_i^{(m_i)}(s_i) \right|^{p_i} ds_i \right)^{1/p_i}.$$  \hspace{1cm} (4.13)

where

$$\tilde{U}_1 = \frac{1}{\prod_{i=1}^n [(m_i - k_i - 1)!/(m_i - k_i)].}$$  \hspace{1cm} (4.14)
Example 4.4.6. Set \(a_i = 0, b_i = 1, q_i = n, w_i = 1/n, p_i = n/(n - 1), m_i = m\) and \(k_i = k\) for \(i \in I\). Then (4.6) becomes

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(m)}(s_i)|}{\sum_{i=1}^n s_i^{m-k}} ds_1 \cdots ds_n \\
\leq \frac{\sqrt[n]{x_1 \cdots x_n}}{n-1} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m-k} |u_i^{(m)}(s_i)|^{n/(n-1)} ds_i \right)^{(n-1)/n}.
\]

(4.15)

In the following theorem we establish another inequality similar to the integral analogue of Hilbert’s inequality.

Theorem 4.4.7. Let \(u_i \in C^{m_i+1}(Q(x_i))\) be such that \(u_i^{(m_i)}(s_i) = 0\) for \(s_i \in \partial Q(s_i), 1 \leq j \leq d, i \in I\). Then

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(m_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} ds_1 \cdots ds_n \\
\leq \prod_{i=1}^n x_i^{1/q_i} \prod_{i=1}^n \left( \int_0^{x_i} (x_i - s_i)^{m_i+1} |u_i^{(m_i+1)}(s_i)|^{p_i} ds_i \right)^{1/p_i}.
\]

(4.16)

Proof. Under the hypotheses of the theorem we have the following multivariable identities established in [100] for \(m_i = (0, \ldots, 0)\):

\[
u_i^{(m_i)}(s_i) = \int_0^{s_i} u_i^{(m_i+1)}(\tau_i) d\tau_i, \quad i \in I.
\]

(4.17)

In Theorem 4.3.1 set \(v_i(s_i) = u_i^{(m_i)}(s_i), c_i = 0, \Phi_i(s_i) = |u_i^{(m_i+1)}(s_i)|\), and the result follows.

In the special case that \(d = 2, m_i = (0, 0), p = q = n = 2,\) and \(w_i = 1/2\), the preceding theorem reduces to [115, Theorem 2].

When we apply the inequality of means to the preceding theorem, we get the following corollary which generalises the inequality obtained in [115, Remark 3].

Corollary 4.4.8. Under the hypotheses of Theorem 4.4.7,

\[
\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |v_i^{(m_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{1/(q_i w_i)}} ds_1 \cdots ds_n \\
\leq p^{1/p} \prod_{i=1}^n x_i^{1/q_i} \left( \sum_{i=1}^n \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{m_i+1} |u_i^{(m_i+1)}(s_i)|^{p_i} ds_i \right)^{1/p}.
\]

(4.18)
Chapter 5

Generalisations of Mohapatra and Russell’s extension of Hardy’s integral inequality

This chapter generalises the results of Davies and Petersen [28], Mohapatra and Russell [94], as well as those of Copson [26], Beesack [6] and others.

5.1 Davies and Petersen’s theorem

In 1985 Davies and Petersen [28] obtained the following elegant theorem:

**Theorem 5.1.1.** Let $1 \leq p < \infty$ and $z(\cdot)$ be non-negative and integrable over $(0, x)$. Then

$$
\left( \int_0^x z(t) \, dt \right)^p = p \int_0^x z(t) \left( \int_0^t z(u) \, du \right)^{p-1} \, dt; \quad (5.1)
$$

the result holds for $0 < p < 1$ provided that $\int_0^t z(u) \, du > 0$ for $0 < t < x$.

5.2 Mohapatra and Russell’s theorems

Mohapatra and Russell point out that Davies and Petersen’s proof of Theorem 5.1.1, stated for $p > 1$, holds for $0 < p < 1$ under the given positivity hypothesis. They then proved Theorem 5.2.1 by writing $F(t) = \int_t^\infty z(u) \, du$ so that $F'(t) = -z(t)$ a.e. on $(x, \infty)$; and Theorem 5.2.1 follows from $d/dx \{|F(x)|^p\} = -p \int_x^\infty |F(t)|^{p-1} F'(t) \, dt$. 

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CHAPTER 5. MOHAPATRA–RUSSELL’S INTEGRAL INEQUALITY

Theorem 5.2.1. Let \( 1 \leq p < \infty \) and \( z(\cdot) \) be non-negative and integrable over \((x, \infty)\). Then

\[
\left( \int_x^\infty z(t) \, dt \right)^p = p \int_x^\infty z(t) \left( \int_t^\infty z(u) \, du \right)^{p-1} \, dt; \tag{5.2}
\]

the result holds for \( 0 < p < 1 \) provided that \( \int_t^\infty z(u) \, du > 0 \) for \( x < t < \infty \).

Starting from Theorems 5.1.1 and 5.2.1 Mohapatra and Russell prove the following two theorems.

Theorem 5.2.2. Assume that \( a(\cdot, \cdot) \) is defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \), with \( a(x, t) \geq 0 \) for \( 0 < t < x \), \( a(x, t) = 0 \) for \( t > x \), and suppose that, for some constant \( k_1 \geq 1 \),

\[
a(x, t) \leq k_1 a(y, t) \text{ for } x > y > t. \tag{5.3}
\]

Let \( g(x) \geq 0 \) (\( x \in \mathbb{R}_+ \)) and \( g(\cdot)a(\cdot, t) \in L(0, \infty) \) for each \( t > 0 \), and write

\[
G_2(t) := \int_t^\infty g(x) a(x, t) \, dx \quad (t > 0). \tag{5.4}
\]

Let \( f(t) \geq 0 \) (\( t \in \mathbb{R}_+ \)) and \( a(x, \cdot)f(\cdot) \in L(0, x) \) for each \( x > 0 \), and write

\[
F_1(x) := \int_0^x a(x, t)f(t) \, dt \quad (x > 0). \tag{5.5}
\]

(a) If \( 1 < p < \infty \), \( 0 < m \leq \infty \), \( g(x) > 0 \) on \((0, m)\), then

\[
\int_0^m gF_1^p \, dx \leq \left( pk_1^{p-1} \right)^p \int_0^m g^{1-p}(G_2f)^p \, dx. \tag{5.6}
\]

(b) If \( 0 < p < 1 \), \( 0 \leq r < \infty \), \( F_1(x) > 0 \) on \( \mathbb{R}_+ \), then

\[
\int_r^\infty gF_1^p \, dx \geq \left( pk_1^{p-1} \right)^p \int_r^\infty g^{1-p}(G_2f)^p \, dx. \tag{5.7}
\]

(c) If \( p = 1 \) then hypothesis (5.3) is not required: (5.6) \((0 < m < \infty)\) and (5.7) \((0 < r < \infty)\) hold, with equality in (5.6) \((m = \infty)\) and (5.7) \((r = 0)\).

Theorem 5.2.3. Assume that \( a(\cdot, \cdot) \) is defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \), with \( a(x, t) \geq 0 \) for \( 0 < x < t \), \( a(x, t) = 0 \) for \( x > t \), and suppose that, for some constant \( k_2 \geq 1 \),

\[
a(x, t) \leq k_2 a(y, t) \text{ for } x < y < t. \tag{5.8}
\]
Let \( g(x) \geq 0 \) (\( x \in \mathbb{R}_+ \)) and \( g(\cdot)a(\cdot,t) \in L(0,t) \) for each \( t > 0 \), and write
\[
G_1(t) := \int_0^t g(x)a(x,t) \, dx \quad (t > 0).
\]
(5.9)

Let \( f(t) \geq 0 \) (\( t \in \mathbb{R}_+ \)) and \( a(x,\cdot)f(\cdot) \in L(\cdot,\infty) \) for each \( x > 0 \), and write
\[
F_2(x) := \int_x^\infty a(x,t)f(t) \, dt \quad (x > 0).
\]
(5.10)

(a) If \( 1 < p < \infty \), \( 0 \leq r < \infty \), \( g(x) \geq 0 \) on \((r,\infty)\), then
\[
\int_r^\infty gF_2^p \, dx \leq \left( pk_2^{p-1} \right)^p \int_r^\infty g^{1-p} (G_1f)^p \, dx.
\]
(5.11)

(b) If \( 0 < p < 1 \), \( 0 < m \leq \infty \), \( F_2(x) > 0 \) on \( \mathbb{R}_+ \), then
\[
\int_0^m gF_2^p \, dx \geq \left( pk_2^{p-1} \right)^p \int_0^m g^{1-p} (G_1f)^p \, dx.
\]
(5.12)

(c) If \( p = 1 \) then hypothesis (5.8) is not required: (5.11) \( (0 < r < \infty) \) and (5.12) \( (0 < m < \infty) \) hold, with equality in (5.11) \( (r = 0) \) and (5.12) \( (m = \infty) \).

5.3 Two main theorems

**Hypotheses.** Throughout it is assumed that all the functions considered in the paper are nonnegative and measurable on their domains of definition, which is usually \( \mathbb{R}_+ := [0,\infty) \). In particular, \( a(\cdot,\cdot) \) is a nonnegative measurable function defined on \( \mathbb{R}_+ \times \mathbb{R}_+ \).

Let \( f, g \) be nonnegative measurable functions defined on \( \mathbb{R}_+ \). For any \( x, t \in \mathbb{R}_+ \), we set
\[
F_1(x) = \int_0^x a(x,u)f(u) \, du, \quad F_2(x) = \int_x^\infty a(x,u)f(u) \, du, \quad (5.13)
\]
\[
G_1(t) = \int_0^t g(u)a(u,t) \, du, \quad G_2(t) = \int_t^\infty g(u)a(u,t) \, du \quad (5.14)
\]
assuming that the integrals are finite. The weight function \( w \) is measurable and positive. The function \( \varphi \) is always assumed nonnegative and continuously differentiable in \( \mathbb{R}_+ \). In appropriate places in the paper it is further assumed that \( \varphi' \) is nonnegative, increasing and \( \alpha \)-submultiplicative, or nonnegative, decreasing and \( \alpha \)-supermultiplicative. In general, \( \alpha \)-submultiplicativity and \( \alpha \)-supermultiplicativity cannot be replaced by mere submultiplicativity and supermultiplicativity.

**Definition 5.3.1.** Let \( \alpha > 0 \). A function \( f: J \subset \mathbb{R}_+ \to \mathbb{R}_+ \) is \( \alpha \)-submultiplicative if the function \( \alpha f \) is submultiplicative; equivalently,

\[
f(xy) \leq \alpha f(x)f(y) \quad \text{for all } x, y \in J;
\]

\( f \) is \( \alpha \)-supermultiplicative if the reverse inequality holds, and \( \alpha \)-multiplicative if it is both \( \alpha \)-submultiplicative and \( \alpha \)-supermultiplicative.

Typical functions in these classes are as follows:

**Example 5.3.2.** Let \( \alpha > 0, c \geq 0, p > 0, \) and define

\[
f(x) = \alpha^{-1}(c + x)^{p-1}, \quad x \geq 0.
\]

The following properties of \( f \) may be verified:

(i) If \( c = 0 \), then \( f \) is \( \alpha \)-multiplicative.

(ii) If \( c \geq 1 \) and \( p > 1 \), then \( f \) is \( \alpha \)-submultiplicative.

(iii) If \( c \geq 1 \) and \( 0 < p < 1 \), then \( f \) is \( \alpha \)-supermultiplicative.

**Theorem 5.3.3.** Assume that the function \( a(\cdot, \cdot) \) satisfies

\[
a(x,t) = \begin{cases} 
0 & \text{if } t > x, \\
\leq k_1 a(y,t) & \text{if } x > y > t > 0 \quad (\text{for some constant } k_1 \geq 1) 
\end{cases}
\]  

(5.15)

and \( F_1, G_2 \) are as in Theorem 5.2.2.

(i) Suppose that \( 1 < p < \infty, p' = p/(p-1), 0 < m \leq \infty, g(x) > 0 \text{ on } (0,m), \) and that \( \varphi' \) is nonnegative, increasing and \( \alpha \)-submultiplicative for some \( \alpha > 0 \). Then

\[
\int_0^m g(x)\varphi(F_1(x)) \, dx
\]
5.3. TWO MAIN THEOREMS

\[ \leq \alpha \varphi'(k_1) \left\{ \int_0^m [w(x)f(x)G_2(x)]^p dx \right\}^{1/p} \left\{ \int_0^m \frac{\varphi'(F_1(x))p'}{w(x)p'} dx \right\}^{1/p'}. \]  

(5.16)

(ii) Suppose that \( 0 < p < 1, \) \( p' = p/(p-1), \) \( 0 \leq r < \infty, \) \( F_1(x) > 0 \) on \( \mathbb{R}_+, \) and that \( \varphi' \) is nonnegative, decreasing and \( \alpha \)-supermultiplicative for some \( \alpha > 0. \) Then

\[ \int_r^\infty g(x)\varphi(F_1(x)) \, dx \geq \alpha \varphi'(k_1) \left\{ \int_r^\infty [w(x)f(x)G_2(x)]^p dx \right\}^{1/p} \left\{ \int_r^\infty \frac{\varphi'(F_1(x))p'}{w(x)p'} dx \right\}^{1/p'}. \]  

(5.17)

Proof. (i) Let \( \varphi' \) be nonnegative, increasing and \( \alpha \)-submultiplicative.

In order to apply the hypotheses and Hölder’s inequality, it is necessary to express \( \varphi(F_1(x)) \) in a suitable form. In the following argument the integral \( F_1(x) = \int_0^x a(x,s) f(s) \, ds \) is differentiated as a function of the upper terminal. To this end it is stipulated that the variable \( x \) in \( a(x,u) \) is not coupled with the variable \( x \) in the upper terminal of the integral. Consider

\[ \varphi(F_1(x)) = \int_0^x \left( \frac{d}{dt} \varphi(F_1(t)) \right) \, dt = \int_0^x \varphi'(F_1(t)) \frac{d}{dt} F_1(t) \, dt \]
\[ = \int_0^x \varphi'(\int_0^t a(x,u)f(u) \, du) \left( \frac{d}{dt} \int_0^t a(x,u)f(u) \, du \right) \, dt \]
\[ = \int_0^x \varphi'(\int_0^t a(x,u)f(u) \, du) a(x,t)f(t) \, dt. \]

Hence

\[ \varphi(F_1(x)) = \int_0^x \varphi'(\int_0^t a(x,u)f(u) \, du) a(x,t)f(t) \, dt. \]  

(5.18)

Equation (5.18) is a generalisation of [28, Lemma 2] which is recovered by letting \( a(x,t) = 1 \) for \( x > t > 0 \) and \( \varphi(x) = x^p, \ p > 1. \)

Since \( \varphi'(x) \) is increasing and \( a(x,u) \leq k_1 a(t,u) \) in (5.18), it follows that

\[ \varphi(F_1(x)) \leq \int_0^x \varphi'(k_1 \int_0^t a(t,u)f(u) \, du) a(x,t)f(t) \, dt. \]  

(5.19)

Since \( \varphi'(x) \) is \( \alpha \)-submultiplicative,

\[ \varphi(F_1(x)) \leq \alpha \varphi'(k_1) \int_0^x \varphi'(F_1(t)) a(x,t)f(t) \, dt \]  

(5.20)
and 
\[ \int_0^m g(x)\varphi(F_1(x)) \, dx \leq \alpha \varphi'(k_1) \int_0^m g(x) \left\{ \int_0^x a(x,t)f(t)\varphi'(F_1(t)) \, dt \right\} \, dx \]
\[ = \alpha \varphi'(k_1) \int_0^m f(t)\varphi'(F_1(t)) \left\{ \int_t^m g(x)a(x,t) \, dx \right\} \, dt \]
\[ \leq \alpha \varphi'(k_1) \int_0^m f(t)\varphi'(F_1(t))G_2(t) \, dt \]

by changing the order of integration. Hence

\[ \int_0^m g(x)\varphi(F_1(x)) \, dx \leq \alpha \varphi'(k_1) \int_0^m \{w(t)f(t)G_2(t)\} \left\{ \frac{\varphi'(F_1(t))}{w(t)} \right\} \, dt. \quad (5.21) \]

Applying Hölder’s inequality with the conjugate indices \( p \) and \( p' \) yields (5.16).

(Recall that all functions are measurable and nonnegative.)

(ii) Proceeding analogously as in (i), we obtain (5.19) and (5.20) with the inequalities reversed since \( \varphi' \) is decreasing and \( \alpha \)–supermultiplicative. If \( g = 0 \) a.e. on \( \mathbb{R}_+ \) then (5.17) holds trivially. Hence suppose that \( g > 0 \) on a set \( A \) of positive measure and \( g = 0 \) on \( \mathbb{R}_+ \setminus A \). Then multiplying the reverse inequality (5.20) by \( g(x) \) and integrating over \( E = A \cap (r, \infty) \), we have

\[ \int_E g(x)\varphi(F_1(x)) \, dx \geq \alpha \varphi'(k_1) \int_E g(x) \left\{ \int_0^x a(x,t)f(t)\varphi'(F_1(t)) \, dt \right\} \, dx \]
\[ = \alpha \varphi'(k_1) \int_0^\infty f(t)\varphi'(F_1(t)) \left\{ \int_{E \cap (t, \infty)} g(x)a(x,t) \, dx \right\} \, dt \]
\[ \geq \alpha \varphi'(k_1) \int_E f(t)\varphi'(F_1(t))G_2(t) \, dt. \]

Since \( E \subset (r, \infty) \) and \( g = 0 \) on \( (r, \infty) \setminus E \), we have

\[ \int_r^\infty g(x)\varphi(F_1(x)) \, dx \geq \alpha \varphi'(k_1) \int_r^\infty \{w(t)f(t)G_2(t)\} \left\{ \frac{\varphi'(F_1(t))}{w(t)} \right\} \, dt. \quad (5.22) \]

Applying Hölder’s reverse inequality for \( 0 < p < 1 \) (see [63, 6.9.3]) to the right hand side of this inequality with the conjugate indices \( p \) and \( p' \), we obtain (5.17).

The following theorem is proved in an analogous fashion.

**Theorem 5.3.4.** Assume that the function \( a(\cdot, \cdot) \) satisfies

\[ a(x,t) \begin{cases} = 0 & \text{if } t < x, \\ \leq k_2a(y,t) & \text{if } 0 < x < y < t \quad \text{(for some constant } k_2 \geq 1). \end{cases} \quad (5.23) \]

and \( F_2, G_1 \) are as in Theorem 5.2.3.
5.4. FURTHER INEQUALITIES

(i) Suppose that $1 < p < \infty$, $p' = p/(p-1)$, $0 < r < \infty$, and that $\varphi'$ is nonnegative, increasing and $\alpha$–submultiplicative for some $\alpha > 0$. Then
\[
\int_r^\infty g(x)\varphi(F_2(x))\,dx \\
\leq \alpha \varphi'(k_2) \left\{ \int_r^\infty \left[ w(x)f(x)G_1(x)\right]^p \,dx \right\}^{1/p} \left\{ \int_r^\infty \frac{\varphi'(F_2(x))^p}{w(x)^p} \,dx \right\}^{1/p'}.
\]
(5.24)

(ii) Suppose that $0 < p < 1$, $p' = p/(p-1)$, $0 < m \leq \infty$, and that $\varphi'$ is nonnegative, decreasing and $\alpha$–supermultiplicative for some $\alpha > 0$. Then
\[
\int_0^m g(x)\varphi(F_2(x))\,dx \\
\geq \alpha \varphi'(k_2) \left\{ \int_0^m \left[ w(x)f(x)G_1(x)\right]^p \,dx \right\}^{1/p} \left\{ \int_0^m \frac{\varphi'(F_2(x))^p}{w(x)^p} \,dx \right\}^{1/p'}.
\]
(5.25)

5.4 Further inequalities

In this section the weight function $w$ is chosen in such a way that (5.26) is satisfied, and the factor $\int_0^m (w^{-1}\varphi'(F_1))^{1/p'} \,dx$ can be eliminated from the right hand side of (5.16) and (5.17). (Similarly for (5.24) and (5.25).)

**Theorem 5.4.1.** Assume that the function $a(\cdot, \cdot)$ satisfies (5.15) and $F_1, G_2$ are as in Theorem 5.2.2.

(i) Suppose that $1 < p < \infty$, $0 < m \leq \infty$, and $g(x) > 0$ on $(0, m)$, and that the solution $\varphi$ to the differential equation
\[
\left( \frac{\varphi'(F_1(x))}{w(x)} \right)^{p/(p-1)} = g(x)\varphi(F_1(x)), \quad x > 0,
\]
is such that $\varphi'$ is increasing and $\alpha$–submultiplicative for some $\alpha > 0$. Then
\[
\int_0^m g(x)\varphi(F_1(x))\,dx \leq (\alpha \varphi'(k_1))^p \int_0^m [w(x)f(x)G_2(x)]^p \,dx.
\]
(5.27)

(ii) Suppose that $0 < p < 1$, $0 \leq r < \infty$, and $F_1(x) > 0$ on $\mathbb{R}_+$, and that the solution $\varphi$ to the differential equation (5.26) is such that $\varphi'$ is decreasing and $\alpha$–supermultiplicative for some $\alpha > 0$. Then
\[
\int_r^\infty g(x)\varphi(F_1(x))\,dx \geq (\alpha \varphi'(k_1))^p \int_r^\infty [w(x)f(x)G_2(x)]^p \,dx.
\]
(5.28)
Proof. (i) Let \( p' = p/(p-1) \). When differentiating the integral \( \int_0^x a(x,t) f(t) \, dt \) as a function of the upper terminal, it is again assumed that the variable \( x \) in \( a(x,t) \) is not coupled with the terminal \( x \).

Let \( \varphi \) be a solution to (5.26). Then \( (\varphi'(F_1)/w)^{p'} = g\varphi(F_1) \) implies

\[
\frac{d}{dx} \varphi(F_1)^{1/p} = \frac{1}{p} \frac{1}{w} g^{1/p'} F_1',
\]

and

\[
\varphi(F_1(x)) = \left\{ \frac{1}{p} \int_0^x w(t) g(t)^{1/p'} a(x,t) f(t) \, dt \right\}^p. \tag{5.29}
\]

From \( \varphi'(F_1) = w g^{1/p'} \varphi(F_1)^{1/p'} \) we get

\[
\varphi'(F_1(x)) = w(x) g(x)^{1/p'} \left\{ \frac{1}{p} \int_0^x w(t) g(t)^{1/p'} a(x,t) f(t) \, dt \right\}^{p-1}. \tag{5.30}
\]

We may check that the differential equation (5.26) has a solution \( \varphi \) given by (5.29) whose derivative is given by (5.30). From these equations it follows that the composite functions \( \varphi \circ F_1 \) and \( \varphi' \circ F_1 \) are nonnegative. If, in addition, \( \varphi' \) is increasing and \( \alpha \)-submultiplicative, we can examine the proof of Theorem 5.3.3 (i) to conclude that the theorem applies also in this situation. Then

\[
\int_0^m g\varphi(F_1) \, dx \leq \alpha \varphi'(k_1) \left( \int_0^m (w f G_2)^p \, dx \right)^{1/p} \left( \int_0^m g\varphi(F_1) \, dx \right)^{1/p'}. \tag{5.27}
\]

Dividing by \( (\int_0^m g\varphi(F_1) \, dx)^{1/p'} \) and raising both sides to the power \( p \) we obtain

\[
\int_0^m g\varphi(F_1) \, dx \leq \alpha \varphi'(k_1) \left( \int_0^m (w f G_2)^p \, dx \right)^{1/p} \left( \int_0^m g\varphi(F_1) \, dx \right)^{1/p'}. \tag{5.27}
\]

Part (ii) is proved similarly.

We observe that we can eliminate \( \varphi \) from the inequalities (5.27) and (5.28) on substituting (5.29):

**Corollary 5.4.2.** Let the function \( a(\cdot, \cdot) \) satisfy condition (5.15).

(i) Under the hypotheses of Theorem 5.4.1 (i),

\[
\int_0^m g(x) \left\{ \int_0^x w(t) g(t)^{(p-1)/p} a(x,t) f(t) \, dt \right\}^p \, dx \leq (\alpha \varphi'(k_1))^p \int_0^m |w(x) f(x) G_2(x)|^p \, dx. \tag{5.31}
\]
5.4. FURTHER INEQUALITIES

(ii) Under the hypotheses of Theorem 5.4.1 (ii),

\[
\int_0^m g(x) \left\{ \int_0^x w(t)g(t)^{(p-1)/p} a(x,t) f(t) \, dt \right\}^p \, dx \\
\geq \left( \alpha \varphi'(k_1) \right)^p \int_0^m [w(x)f(x)G_2(x)]^p \, dx. \tag{5.32}
\]

**Remark 5.4.3.** In the proof of Theorem 5.4.1 an explicit formula was derived for the composite function \( \varphi' \circ F_1 \) without assuming anything special about the nature of \( \varphi' \). In the case that \( \lim_{x \to \infty} F_1(x) = \infty \), we can obtain an explicit formula for \( \varphi' \), and test whether it has the properties required by Theorem 5.4.1. If \( \lim_{x \to \infty} F_1(x) = \infty \), then \( F_1 \) is surjective as a function on \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \), and there exists a right inverse \( H : \mathbb{R}_+ \to \mathbb{R}_+ \) for \( F_1 \). According to (5.30),

\[
\varphi'(u) = w(H(u))g(H(u))^{1/p'} \left\{ \frac{1}{p} \int_0^{H(u)} w(t)g(t)^{1/p'} a(H(u),t) f(t) \, dt \right\}^{p-1}.
\tag{5.33}
\]

We can then decide whether the right hand side of this equation considered as a function of \( u \) is increasing and \( \alpha \)–submultiplicative (or decreasing and \( \alpha \)–supermultiplicative).

If the functions involved in Theorem 5.4.1 are continuous and \( F_1(x_1) = k_1 \) for some \( x_1 \in \mathbb{R}_+ \), then \( \varphi \) and \( \varphi' \) may be eliminated from (5.27) and (5.28) altogether:

**Corollary 5.4.4.** Suppose that \( a \) satisfies condition (5.15), that the functions \( a, f, g \) and \( w \) are continuous, and that there exists \( x_1 \in \mathbb{R}_+ \) such that \( F_1(x_1) = k_1 \).

(i) If the hypotheses of Theorem 5.4.1 (i) hold, then

\[
\int_0^m g(x) \left\{ \int_0^x w(t)g(t)^{(p-1)/p} a(x,t) f(t) \, dt \right\}^p \, dx \\
\leq \alpha^p w(x_1)^p g(x_1)^{p-1} \left\{ \int_0^{x_1} w(t)g(t)^{(p-1)/p} a(x_1,t) f(t) \, dt \right\}^{p(p-1)} \\
\cdot \int_0^m [w(x)f(x)G_2(x)]^p \, dx.
\]
(ii) If the hypotheses of Theorem 5.4.1 (ii) hold, then
\[
\int_0^m g(x) \left\{ \int_0^x w(t)g(t)^{(p-1)/p}a(x,t)f(t)\,dt \right\}^p \,dx \\
\geq \alpha^p w(x_1)^p w(x_1)^{p-1} \left\{ \int_0^{x_1} w(t)g(t)^{(p-1)/p}a(x_1,t)f(t)\,dt \right\}^{p(p-1)} \\
\cdot \int_0^m \left[ w(x)f(x)G_2(x) \right]^p \,dx.
\]

**Proof.** It is enough to observe that, under the assumptions of continuity, equation (5.30) holds pointwise, rather than almost everywhere.

For certain special choices of the weight function \(w\), the differential equation (5.26) has a required type of solution independently of \(a, f\) and \(g\). Such an example is illustrated in the following corollary of Theorem 5.4.1 which yields the inequalities obtained by Mohapatra and Russell in [94, Theorem 1].

**Corollary 5.4.5.** Let the function \(a(\cdot, \cdot)\) satisfy (5.15), let \(g(x) > 0\) on \((0, m)\) and let \(w = g^{(1-p)/p}\).

(i) Suppose that \(1 < p < \infty\) and \(0 < m \leq \infty\). Then the differential equation (5.26) admits a solution \(\varphi(x) = p^{-1}x^{p-1}\) with \(\varphi'(x) = p^{1-p}x^{p-1}\), which is nonnegative, increasing and \(\alpha\)-submultiplicative with \(\alpha = p^{p-1}\), and
\[
\int_0^m g(x) F_1(x)^p \,dx \leq (ph_1^{p-1})^p \int_0^m g(x)^{1-p}[f(x)G_2(x)]^p \,dx. 
\]  
(5.34)

(ii) Suppose that \(0 < p < 1\), \(0 \leq r < \infty\), and \(F_1(x) > 0\) on \(\mathbb{R}_+\). Then the differential equation (5.26) admits a solution \(\varphi(x) = p^{-1}x^{p-1}\) with \(\varphi'(x) = p^{1-p}x^{p-1}\), which is nonnegative, decreasing and \(\alpha\)-submultiplicative with \(\alpha = p^{p-1}\), and
\[
\int_r^\infty g(x) F_1(x)^p \,dx \geq (ph_1^{p-1})^p \int_r^\infty g(x)^{1-p}[f(x)G_2(x)]^p \,dx. 
\]  
(5.35)

**Proof.** (i) The differential equation (5.26) becomes \((d/dx)\varphi(F_1)^{1/p} = p^{-1}F_1'\) with the solution \(\varphi(F_1) = (p^{-1}F_1)^p\). Hence \(\varphi(x) = p^{-1}x^{p-1}\) and \(\varphi'(x) = p^{1-p}x^{p-1}\). In view of Example 5.3.2, \(\varphi'\) is \(p^{p-1}\)-(sub)multiplicative. Substituting into (5.27), we get
5.5. SPECIAL CASES

\[
\int_0^m g \left( \frac{F_1}{p} \right)^p \, dx \leq (p^{p-1})^p \left( \frac{k_1}{p} \right)^{p-1} \int_0^m g^{1-p} f^p G_2^p \, dx,
\]

from which (5.34) follows.

The proof of (ii) is analogous.

There is also the following counterpart of Theorem 5.4.1.

**Theorem 5.4.6.** Suppose that the function \( a(\cdot, \cdot) \) satisfies condition (5.23) and \( F_2, G_1 \) are as in Theorem 5.2.3.

(i) If \( 1 < p < \infty \), \( 0 < m \leq \infty \), \( g(x) > 0 \) on \((0, m)\), and the solution \( \varphi \) to the differential equation (5.26) is such that \( \varphi' \) is increasing and \( \alpha \)-submultiplicative, then

\[
\int_r^\infty g(x) \varphi(F_2(x)) \, dx \leq (\alpha \varphi'(k_2))^p \int_r^\infty [w(x) f(x) G_1(x)]^p \, dx. \tag{5.36}
\]

(ii) If \( 0 < p < 1 \), \( 0 \leq r < \infty \), \( F_1(x) > 0 \) on \( \mathbb{R}_+ \), and the solution \( \varphi \) to the differential equation (5.26) is such that \( \varphi' \) is decreasing and \( \alpha \)-supermultiplicative, then

\[
\int_r^\infty g(x) \varphi(F_2(x)) \, dx \geq (\alpha \varphi'(k_2))^p \int_r^\infty [w(x) f(x) G_1(x)]^p \, dx. \tag{5.37}
\]

The preceding theorem admits corollaries analogous to Corollaries 5.4.2–5.4.5. We leave the formulation to the reader. Specialising \( w \) in the preceding theorem to \( w = g^{(1-p)/p} \) as in Corollary 5.4.5, we recover [94, Theorem 2].

5.5 Special cases

The first result is a convolution inequality generalising [94, Theorem 3].

**Corollary 5.5.1.** Assume that \( s(t) \geq 0 \) if \( t > 0 \) and \( s(t) = 0 \) if \( t < 0 \), that for some constant \( k \geq 1 \), \( s(x) \leq ks(y) \) for \( x > y > 0 \), and \( 1 < p < \infty \). Let \( S(x) = \int_0^x s(t) \, dt \) and \( F(x) = \int_0^x s(x-t) f(t) \, dt \) for \( x > 0 \). Suppose that the solution \( \varphi \) to the differential equation

\[
\left( \frac{\varphi'(F(x))}{w(x)} \right)^{p/(p-1)} = S(x)^{-p} \varphi(F(x)), \quad x > 0,
\]

is such that \( \varphi' \) is increasing and \( \alpha \)-submultiplicative for some \( \alpha > 0 \). Then

\[
\int_0^\infty S(x)^{-p} \varphi \left( \int_0^x s(x-t) f(t) \, dt \right) \, dx \leq \left( \frac{p \alpha \varphi'(k)}{p-1} \right)^p \int_0^\infty [w(x) S(x)^{1-p} f(x)]^p \, dx. \tag{5.39}
\]
CHAPTER 5. MOHAPATRA–RUSSELL’S INTEGRAL INEQUALITY

Proof. Write \( G(t) = \int_t^\infty S(x)^{-p} s(x - t) \, dx \) for \( t > 0 \), and take \( a(x, t) = s(x - t) \), \( g(x) = S(x)^{-p} \), \( r = 0 \), \( m = \infty \) in Theorem 5.4.1 (i). As in [94, Theorem 3],

\[
G(x) \leq p(p - 1)^{-1} S(x)^{1-p}.
\]

The result then follows when we substitute in (5.27).

An analogous result is obtained for \( 0 < p < 1 \), in which case \( \varphi' \) is assumed to be \( \alpha–\text{supermultiplicative} \), and the inequality in (5.39) is reversed.

Remark 5.5.2. If we set \( w(x) = S(x)^{p-1} \) in the preceding theorem, the differential equation (5.38) has the solution \( \varphi(x) = p^{-p} x^p \) for which \( \varphi'(x) = p^{1-p} x^{-p} \) is nonnegative, increasing and \( \alpha-\text{multiplicative} \) with \( \alpha = p^{p-1} \). We thus recover Theorem 3 of Mohapatra and Russell [94],

\[
\int_0^\infty s(x)^{-p} \left( \int_0^x s(x - t)t(f) \, dt \right) \, dx \leq \left( \frac{p^2 k^{p-1}}{p - 1} \right)^p \int_0^\infty \left[ f(x) \right]^p \, dx.
\]

(correcting a misprint in [94], where the constant is given as \( \left[ p^2 k/(p - 1) \right]^p \)).

Theorems 1 and 2 of Mohapatra and Russell [94] were seen as special cases of our Theorem 5.4.1 and Theorem 5.4.6. In that case we made a special choice of the weight function \( w \), and obtained \( \varphi \) as a solution of the differential equation (5.26)—there was no freedom of choice for \( \varphi \). In the following example we apply a reverse choice in Theorem 5.3.3: We first select \( \varphi \), and still have freedom of choice for the weight function \( w \).

The following is a new inequality of Hardy type from which we recover Theorem 1 of [94] by setting \( c = q = 0 \).

Corollary 5.5.3. Let \( a(\cdot, \cdot) \) satisfy (5.15) and \( F_1 \) be as in Theorem 5.2.2, let \( 0 < p < \infty \), \( p \neq 1 \), \( q > 0 \), \( c \in \{0\} \cup [1, \infty) \), \( F_1(x) > 0 \) and \( g(x) > 0 \).

(i) If \( 1 < p < \infty \) and \( 0 < m \leq \infty \), then

\[
\int_0^m g(c + F_1)^{p+q} \, dx \leq \left[ (p+q)(c+k_1)^{p+q-1} \right]^p \int_0^m g^{1-p}(c + F_1)^q(fG_2)^p \, dx. \quad (5.40)
\]

(ii) If \( 0 < p < 1 \), \( q < 1 - p \) and \( 0 \leq r < \infty \), then

\[
\int_r^\infty g(c + F_1)^{p+q} \, dx \geq \left[ (p+q)(c+k_1)^{p+q-1} \right]^p \int_r^\infty g^{1-p}(c + F_1)^q(fG_2)^p \, dx. \quad (5.41)
\]
5.5. SPECIAL CASES

Proof. (i) Write \( p' = p/(p - 1) \). Set \( \varphi(x) = (p + q)^{-1}(c + x)^{p+q} \) in Theorem 5.3.3 (i). Then \( \varphi'(x) = (c+x)^{p+q-1} \) is submultiplicative (multiplicative for \( c = 0 \)), and (5.16) specialises to

\[
\int_0^m g(c + F_1)^{p+q} \, dx \leq (p + q)(c + k_1)^{p+q-1} \left( \int_0^m [wG_2]^p \, dx \right)^{1/p} \cdot \left( \int_0^m (w^{-1}(c + F_1)^{p+q-1})^p \, dx \right)^{1/p'}.
\]

Choosing the weight function \( w = g^{-1/p'}(c + F_1)^{q/p} \), we obtain

\[
\int_0^m g(c + F_1)^{p+q} \, dx \leq (p + q)(c + k_1)^{p+q-1} \left( \int_0^m [w^{-1}(c + F_1)^{q/p}fG_1]^p \, dx \right)^{1/p} \cdot \left( \int_0^m g(c + F_1)^{p+q} \, dx \right)^{1/p'}.
\]

Dividing by \( \left( \int_0^m g(c + F_1)^{p+q} \, dx \right)^{1/p'} \) and raising both sides to the power \( p \) yields (5.40).

For the proof of part (ii) it is noted that under the hypotheses, \( p + q - 1 < 0 \), and \((c + x)^{p+q-1}\) is supermultiplicative. The rest is proved similarly as in part (i).

There is also the following counterpart to the preceding corollary. For \( q = c = 0 \) this reduces to [94, Theorem 2].

Corollary 5.5.4. Let \( a(x) \) satisfy (5.23) and \( F_2 \) be as in Theorem 5.2.3, let \( 0 < p < \infty, \, p \neq 1, \, p' = p/(p - 1), \, q > 0, \, c \in \{0\} \cup [1, \infty), \, F_2(x) > 0 \) and \( g(x) > 0 \).

(i) If \( 1 < p < \infty \) and \( 0 \leq r < \infty \), then

\[
\int_r^\infty g(c + F_2)^{p+q} \, dx \leq [(p+q)(c+k_2)^{p+q-1}]^p \int_r^\infty g^{1-p}(c + F_2)^q(fG_1)^p \, dx. \quad (5.42)
\]
(ii) If $0 < p < 1$, $q < 1 - p$ and $0 < m \leq \infty$, then
\[ \int_0^m g(c + F_2)^{p+q} dx \geq \left[ (p+q)(c+k_2)^{p+q-1} \right]^p \int_0^m g^{1-p}(c + F_2)^q (f G_1)^p dx. \tag{5.43} \]

Several new inequalities of Hardy type are now derived that will yield results of Hardy et al. [63], Beesack [6] and Copson [26] by making a special choice of parameters.

**Example 5.5.5.** In Corollary 5.5.3 set $g(x) = x^{-\beta}$ for some $\beta > 1$, $a(x,t) = 1$ if $0 < t < x$, $a(x,t) = 0$ if $x < t$, $k_1 = 1$, $c \in \{0\} \cup [1, \infty)$ and $m = \infty$. We note that $G_2(x) = (\beta - 1)^{-1} x^{1-\beta}$. If $1 < p < \infty$, then
\[ \int_0^\infty x^{-\beta}(c + F_1(x))^{p+q} dx \leq \left( \frac{(p+q)(c+1)^{p+q-1}}{\beta - 1} \right)^p \int_0^\infty x^{-\beta}(c + F_1(x))^q (x f(x))^p dx. \tag{5.44} \]
If $0 < p < 1$, $\beta > 1$ and $q < 1 - p$, the inequality is reversed. This generalises [63, Theorem 330] which is obtained when $c = 0$ and $q = 0$. When in addition $\beta = p$, we obtain Theorem 1.2.6, Hardy’s original strong integral analogue.

**Example 5.5.6.** In Corollary 5.5.4 set $g(x) = x^{-\beta p}$ for some $\beta > 0$, $a(x,t) = t^{\beta - 1}$ for $0 < x < t$, $k_2 = 1$, $c \in \{0\} \cup [1, \infty)$ and $r = 0$. If $1 < p < \infty$ and $\beta < 1/p$, then
\[ \int_0^\infty x^{-\beta p} \left( c + \int_x^\infty t^{\beta - 1} f(t) dt \right)^{p+q} dx \leq \left( \frac{(p+q)(c+1)^{p+q-1}}{1 - \beta p} \right)^p \int_0^\infty \left( c + \int_x^\infty t^{\beta - 1} f(t) dt \right)^q (f(x))^p dx. \tag{5.45} \]
When we set $c = q = 0$, we obtain [63, Equation (9.9.9)].

In the last three examples we assume that $\psi$, $f$ are positive and measurable on $\mathbb{R}_+$, and that
\[ \Psi(x) = \int_0^x \psi(t) dt, \quad F(x) = \int_0^x f(t) \psi(t) dt, \quad H(x) = \int_x^\infty f(t) \psi(t) dt. \]

**Example 5.5.7.** Suppose that $1 < p < \infty$, $q > 0$, $c \in \{0\} \cup [1, \infty)$ and $0 < m < \infty$. Then
\[ \int_0^m (c + F)^{p+q} \psi \Psi^{-1} dx \]
Example 5.5.8. Suppose that $1 < p < \infty$, $q > 0$, $c \in \{0\} \cup [1, \infty)$ and $0 < r < \infty$. Then

$$\int_r^\infty (c + H)^{p+q} \psi \Psi^{-1} \, dx \leq [(p+q)(c + 1)^{p+q-1}]^p \int_0^m (c + F)^q f^p \Psi^{p-1} \left( \log \frac{\Psi(x)}{\Psi(r)} \right)^p \, dx. \quad (5.46)$$

If $0 < p < 1$ and $q < 1 - p$, this inequality is reversed.

Inequality (5.46) is obtained from Corollary 5.5.3 by setting $a(x,t) = \psi(t)$ if $0 < t < x$, $k_1 = 1$, $g(x) = \psi(x)\Psi(x)^{-1}$ if $0 < x < m$ and $g(x) = 0$ if $x > m$.

The choice $q = c = 0$ leads to [6, (28), (32)] and [26, Theorem 5].

Example 5.5.9. (i) Suppose that $1 < p < \infty$, $q > 0$, $c \in \{0\} \cup [1, \infty)$, $\beta > 1$ and $0 < m \leq \infty$. Then

$$\int_0^m (c + F)^{p+q} \psi \Psi^{-\beta} \, dx \leq \left( \frac{(p+q)(c + 1)^{p+q-1}}{\beta - 1} \right)^p \int_0^m (c + F)^q \psi \Psi^{-\beta} f^p \, dx. \quad (5.48)$$

(ii) Suppose that $0 < p < 1$, $q > 0$, $q < 1 - p$, $c \in \{0\} \cup [1, \infty)$, $\beta > 1$, $0 \leq r < \infty$ and $\lim_{s \to \infty} \Psi(s) = \infty$. Then

$$\int_r^\infty (c + F)^{p+q} \psi \Psi^{-\beta} \, dx \geq \left( \frac{(p+q)(c + 1)^{p+q-1}}{\beta - 1} \right)^p \int_r^\infty (c + F)^q \psi \Psi^{-\beta} f^p \, dx. \quad (5.49)$$

The inequalities are obtained from Corollary 5.5.3 on setting $a(x,t) = \psi(t)$ if $0 < t < x$, $k_1 = 1$, $g(x) = \psi(x)\Psi(x)^{-\beta}$. We have

$$G_2(t) = (\beta - 1)^{-1} \psi(t) \Psi(t)^{1-\beta} - A, \quad A = \lim_{s \to \infty} \Psi(s)^{1-\beta}.$$

For part (i) we use the inequality $G_2(t) \leq (\beta - 1)^{-1} \psi(t) \Psi(t)^{1-\beta}$, in part (ii) we need $A = 0$. When we set $q = c = 0$, we obtain Theorems 1 and 2 of [26].
The preceding example has a companion result for $\beta < 1$ obtained by an analogous procedure from Corollary 5.5.4. Setting $q = c = 0$, we recover Theorem 3 and 4 of [26].
Chapter 6

Extensions of Hardy–Copson–Hanjš–Love–Pečarić integral inequalities

Some further generalisations of Hardy–type inequalities derived by Love and Pečarić [119] are given. There are some similarities to Hanjš, Love and Pečarić [52] but different weight functions and criteria for integrability are used. The motivation for this chapter was suggested by the late Professor Love.

6.1 Introduction

One of the many fundamental discoveries of G. H. Hardy is the integral inequality [60] and [63, Theorem 330]:

\[
\text{Theorem 6.1.1. If } p > 1, m \neq 1, \text{if } f(t) \text{ is non–negative measurable, and } F \text{ is defined on } (0, \infty) \text{ by}
\]

\[
F(x) = \begin{cases} 
\int_0^x f(t) \, dt & \text{for } m < 1, \\
\int_x^\infty f(t) \, dt & \text{for } m > 1 
\end{cases}
\]

(6.1)

then

\[
\int_x^\infty x^{-m} F^p(x) \, dx < \left( \frac{p}{|m-1|} \right)^p \int_x^\infty x^{-m+p} f^p(x) \, dx
\]

(6.2)

unless \( f(t) \equiv 0 \). The constant on the right is the best possible. When \( p = 1 \), the two sides of (6.1) are equal.

In [119] there is
CHAPTER 6. COPSON–HANJŠ–LOVE–PECARIĆ INEQUALITIES

Theorem 6.1.2. Let \( 0 \leq a < b < \infty, c > 0, p > 0 \) and \( q > 0 \) be constants. Let \( r(x) \) be positive and locally absolutely continuous in \([a, b)\), and \( f(x) \) be almost everywhere non-negative and measurable on \((a, b)\). Let

\[
F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)f(t)}{t \log(b/t)} \, dt
\]

(6.3)

for all \( x \in [a, b) \), and

\[
F(x) = o \left( (b - x)^{-q/p} \right)
\]

(6.4)

as \( x \to b- \). If \( p > 1 \) and

\[
1 + \frac{pxr'(x)}{q} \frac{\log b}{x} \geq \frac{1}{c}
\]

(6.5)

for almost all \( x \in (a, b) \), then

\[
\left( \int_a^b F^p(x) \left( \frac{\log(b/x)}{x} \right)^{q-1} \, dx \right)^{1/p} \leq \frac{cp}{q} \left( \int_a^b f^p(x) \left( \frac{\log(b/x)}{x} \right)^{q-1} \, dx \right)^{1/p}.
\]

(6.6)

If \( 0 < p < 1 \) and the reverse inequality (6.5) holds, then the reverse inequality (6.6) holds also.

In [52] there are six theorems which may be characterised by the presence of single or iterated integrals and the interval of integration. Four such theorems are as follows.

Theorem 6.1.3. (Theorem 1 in [52]). Let \( m > 1, p \geq 1, q \geq 0 \) and \( X > 0 \). Let \( s(x), w(x) \) and \( z(x) \) be absolutely continuous and positive on \([0, X]\), with \( z'(x) \) essentially bounded and positive. If \( f(x) \) is non-negative and \( L^p \) on \((0, X)\),

\[
F(x) = \frac{1}{s(x)} \int_0^x \frac{s(t)z'(t)}{z(t)} f(t) \, dt \quad \text{for} \quad 0 \leq x \leq X,
\]

(6.7)

and

\[
0 < \frac{1}{\alpha} \leq \text{ess inf} \left\{ 1 + \frac{1}{m-1} z(x) \left( p + q \frac{z'(x)}{s(x)} - \frac{w'(x)}{w(x)} \right) \right\}.
\]

(6.8)

Then

\[
\int_0^X \frac{z'(x)}{z^m(x)} w(x) F^{p+q}(x) \, dx
\]

\[
\leq \left( \frac{(p + q)\alpha}{m-1} \right)^p \int_0^X \frac{z'(x)}{z^m(x)} w(x) F^{q}(x) f^p(x) \, dx.
\]

(6.9)

This conclusion also holds with \( X \) replaced by \( \infty \) if the hypotheses on \( s, w, z \) and \( z' \) hold locally in \([0, \infty)\) and \( X \) is replaced by \( \infty \) in the bound on \( 1/\alpha \); but the integrals may not then be convergent.
6.1. INTRODUCTION

Theorem 6.1.4. (Theorem 2 in [52]). Let $m > 1$, $p + r \geq 1$, $p \geq 0$, $q \geq 0$, $r \geq 0$ and $X > 0$. Let $s(x)$, $w(x)$, $z(x)$, $z'(x)$, $X$ and $F$ be defined as in Theorem 6.1.3. If $f(x)$ is non–negative and $L^{p+r}$ on $(0,X)$, and

$$0 < \frac{1}{\alpha} \leq \text{ess inf}_{0<x<X} \left\{ 1 + \frac{1}{m-1} \frac{z'(x)}{z(x)} \left( (p+q+r) \frac{s'}{s(x)} - \frac{w'}{w(x)} \right) \right\}$$

(6.10)

then

$$\int_{0}^{X} \frac{z'(x)}{z^m(x)} w(x) F^{p+q}(x) f^r(x) \, dx$$

$$\leq \left( \frac{(p+q+r)\alpha}{m-1} \right)^p \int_{0}^{X} \frac{z'(x)}{z^m(x)} w(x) F^q(x) f^{p+r}(x) \, dx.$$  (6.11)

This conclusion also holds with $X$ replaced by $\infty$ if the hypotheses on $s$, $w$, $z$ and $z'$ hold locally in $[0,\infty)$ and $X$ is replaced by $\infty$ in the other hypotheses.

Theorem 6.1.5. (Theorem 6 in [52]). Let $m < 1$, $p + r \geq 1$, $q \geq 0$, $r > 0$ and $X > 0$. Let $s(x)$, $w(x)$ and $z(x)$ be locally absolutely continuous on $[X,\infty]$, and have positive lower and upper bounds thereon. Let $z'(x)$ be essentially bounded and essentially positive on $[X,\infty)$ and

$$0 < \frac{1}{\alpha} \leq \text{ess inf}_{0<x<X} \left\{ 1 + \frac{1}{m-1} \frac{z'(x)}{z(x)} \left( (p+q+r) \frac{s'}{s(x)} - \frac{w'}{w(x)} \right) \right\}$$

(6.12)

If $f(x)$ is non–negative and in $L^1 \cap L^p$ on $(0,\infty)$ and

$$F(x) = \frac{1}{s(x)} \int_{x}^{\infty} \frac{s(t)z'(t)}{z(t)} f(t) \, dt$$

(6.13)

then

$$\int_{X}^{\infty} \frac{z'(x)}{z^m(x)} w(x) F^{p+q}(x) f^r(x) \, dx$$

$$\leq \left( \frac{(p+q+r)\alpha}{m-1} \right)^p \int_{X}^{\infty} \frac{z'(x)}{z^m(x)} w(x) F^q(x) f^{p+r}(x) \, dx.$$  (6.14)

and these integrals are convergent. This conclusion also holds with $X$ replaced by $0$ if the hypotheses on $s$, $w$, $z$ and $z'$ hold locally in $[0,\infty)$ and $X$ is replaced by $0$ in the bound on $1/\alpha$.

Theorem 6.1.6. (Theorem 3 in [52]). Let $m > 1$, $p \geq 1$, $X > 0$ and all hypotheses involving $k$ hold for $k = 1,2,\ldots,n$. Let $s_k(x)$, $w(x)$ and $z(x)$ be positive and
absolutely continuous on the finite interval $(0, X)$, with $z'(x)$ essentially bounded and essentially positive. For any $\phi$ non-negative and $L^p$ on $(0, X)$ let

$$\tilde{I}_k \phi(x) = \frac{1}{s_k(x)} \int_0^x \frac{s_k(t)z'(t)}{z(t)} \phi(t) \, dt, \quad \text{for } 0 \leq x \leq X. \quad (6.15)$$

If $f(x)$ is non-negative and $L^p$ on $(0, X)$,

$$\tilde{F}_0(x) = f(x), \quad \tilde{F}_k(x) = \tilde{I}_k \tilde{I}_{k-1} \ldots \tilde{I}_1 f(x), \quad (6.16)$$

and

$$0 < \frac{1}{\alpha_k} \leq \text{ess inf}_{0 < x < X} \left\{ 1 + \frac{1}{m-1} \frac{z(x)}{z'(x)} \left( \frac{p}{s_k(x)} - \frac{w'(x)}{w(x)} \right) \right\} \quad (6.17)$$

then

$$\int_0^X \frac{z'(x)}{z^{m(x)}} w(x) \tilde{F}_n^p(x) \, dx \leq \left( \frac{p}{m-1} \right)^{np} \left( \prod_{i=1}^n \alpha_i \right)^p \int_0^X \frac{z'(x)}{z^{m(x)}} w^{p(x)} \, dx. \quad (6.18)$$

This conclusion also holds with $X$ replaced by $\infty$ if the hypotheses on $s$, $w$, $z$ and $z'$ hold locally in $[0, \infty)$ and $X$ is replaced by $\infty$ in the other hypotheses; but then the integrals may not be convergent.

### 6.2 Main Results

**Theorem 6.2.1.** (extension of Theorem 6.1.3) Let $0 \leq a < b < \infty$, $p_1 \geq 0$, $p_2 \geq 1$, $q > 1$ be constants, with $p_1 + p_2 > 1$. Let $r(x)$ and $w(x)$ be positive and locally absolutely continuous in $[a, b)$, and $f(x)$ be almost everywhere non-negative and in $L^{p_2}$ on $[a, b)$, with $|h'(t)|$ essentially bounded and essentially positive,

$$F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)|h'(t)|}{h(t)} f(t) \, dt < \infty, \quad (6.19)$$

and

$$0 < \frac{1}{c} \leq \text{ess inf}_{0 < x < X} \left\{ 1 + \frac{1}{q} \frac{h(x)}{|h'(x)|} \left( \frac{p_1 + p_2}{r(x)} - \frac{w'(x)}{w(x)} \right) \right\} \quad (6.20)$$

then

$$\int_a^b F^{p_1+p_2}(x) h^{q-1}(x)|h'(x)| w(x) \, dx$$
6.2. MAIN RESULTS

\[ \leq \left( \frac{(p_1 + p_2)c}{q} \right)^{p_2} \int_a^b F^{p_1}(x) f^{p_2}(x) h^{q-1}(x) |h'(x)| w(x) \, dx. \] (6.21)

If \(0 < p_1 + p_2 < 1\) and the reverse inequality (6.20) holds, then the reverse inequality (6.21) holds also.

Proof.

(a) Since \(h(t) \geq h(b) > 0\), \(1/h(t)\) is bounded above on \([a, b]\). Further, \(|h'(t)|\) is bounded so \(h(t)\) is bounded on \([a, b]\). Since \(r(t)\) is bounded, the integrand in

\[ F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)|h'(t)|}{h(t)} f(t) \, dt \]  (6.22)

is integrable and the integral exists and is absolutely continuous on \([a, b]\). Since \(r(x)\) is continuous and positive on \([a, b]\) it has a positive lower bound thereon; consequently \(1/r(x)\) is absolutely continuous on \([a, b]\). So \(F(x)\) is absolutely continuous on \([a, b]\), being the product of two such functions. From the boundedness above of \(h(x)\), proved above, we have boundedness of \(h^{q-1}(x)\). Also \(|h'(x)|\) and \(w(x)\) are bounded. So the whole integrand on the left side of (6.21) is bounded; thus the integral on the left in (6.21) is convergent as \(b \to \infty\).

(b) Since \(h(x)\) is absolutely continuous and positive on \([a, b]\), so also is \(h^{q-1}(x)\). Also \(|h'(x)|\) is essentially bounded, and \(|h'(x)| = -h(x)\). Now \(w(x)F^{p_1+p_2}(x)\) is also absolutely continuous on \([a, b]\), so the following integration by parts holds.

\[ \int_a^b F^{p_1+p_2}(x) q h^{q-1}(x) |h'(x)| w(x) \, dx \]
\[ = - [h^{q}(x) F^{p_1+p_2}(x) w(x)]_a^b \]
\[ + \int_a^b h^{q}(x) \left\{ w'(x) F^{p_1+p_2}(x) + w(x)(p_1 + p_2) F^{p_1+p_2-1}(x) F'(x) \right\} \, dx \]
\[ = -w(b) F^{p_1+p_2}(b) h^{q}(b) \]
\[ + \int_a^b h^{q}(x) \left\{ w'(x) F^{p_1+p_2}(x) + w(x)(p_1 + p_2) F^{p_1+p_2-1}(x) \right\} \left( \frac{|h'(x)|}{h(x)} f(x) - \frac{r'(x)}{r(x)} \right) \, dx. \]  (6.23)

(c) It follows that

\[ q \int_a^b F^{p_1+p_2}(x) h^{q-1}(x) |h'(x)| w(x) \, dx + w(b) F^{p_1+p_2}(b) h^{q}(b) \]
\[ \int_a^b h^q(x) |h'(x)| w(x) \left\{ \frac{h(x)}{|h'(x)|} \right\} \frac{w'(x)}{w(x)} - (p_1 + p_2) \frac{r'(x)}{r(x)} \] 
\[ F^{p_1 + p_2}(x) + (p_1 + p_2) F^{p_1 + p_2 - 1}(x) f(x) \right\} \, dx 
\]
\[ = \int_a^b h^q(x) |h'(x)| w(x) \left( q - q \left\{ 1 + \frac{h(x)}{|h'(x)|} \right\} \right) \left( p_1 + p_2 \right) \frac{r'(x)}{r(x)} \] 
\[ F^{p_1 + p_2}(x) \right\} + (p_1 + p_2) F^{p_1 + p_2 - 1}(x) f(x) \right\} \, dx 
\]
\[ = \int_a^b h^q(x) |h'(x)| w(x) \left( \frac{w'(x)}{w(x)} - (p_1 + p_2) \frac{r'(x)}{r(x)} \right) F^{p_1 + p_2}(x) \right\} \, dx 
\]
\[ + \int_a^b h^q(x) |h'(x)| w(x) (p_1 + p_2) F^{p_1 + p_2 - 1}(x) f(x) \right\} \, dx. \quad (6.24) \]

Now (6.23) is finite, as shown in (a), so (6.24) is finite. And
\[ h^q(x) |h'(x)| w(x) (p_1 + p_2) F^{p_1 + p_2 - 1}(x) f(x) \right\} \]

is integrable since \( f(x) \) is in \( L^{p_2} \) and \( p_2 \geq 1, |h'(x)| \) is bounded, and the other factors in this expression are absolutely continuous. Writing
\[ W(x) = h^q(x) |h'(x)| w(x) \]
\[ and \quad S(x) = 1 + \frac{1}{q} \frac{h(x)}{|h'(x)|} \left( p_1 + p_2 \right) \frac{r'(x)}{r(x)} - \frac{w'(x)}{w(x)} \]

the right side of (6.24) is equal to
\[ \int_a^b W(x) \left( q - q S(x) \right) F^{p_1 + p_2 - 1}(x) f(x) \right\} \, dx + (p_1 + p_2) \int_a^b W(x) F^{p_1 + p_2}(x) f(x) \right\} \, dx \quad (6.26) \]

this separation into the sum of two integrals being correct because the integral (6.24) and the integral (6.26) exist as (finite) Lebesgue integrals, so that, by additivity, the first integral in (6.26) exists, and (6.24) and (6.26) are equal. We obtain, on dividing by the positive number \( q \),
\[ \int_a^b W(x) F^{p_1 + p_2}(x) \right\} \, dx \leq \int_a^b W(x) (1 - S(x)) F^{p_1 + p_2}(x) \right\} \, dx 
\]
\[ + \left( \frac{p_1 + p_2}{q} \right) \int_a^b W(x) F^{p_1 + p_2 - 1}(x) f(x) \right\} \, dx 
\]
so that
\[ \int_a^b W(x) S(x) F^{p_1 + p_2}(x) \right\} \, dx \leq \frac{p_1 + p_2}{q} \int_a^b W(x) F^{p_1 + p_2 - 1}(x) f(x) \right\} \, dx \]
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Since \( S(x) \geq 1/c \) almost everywhere

\[
\frac{q}{(p_1 + p_2)c} \int_a^b W(x) F^{p_1 + p_2}(x) \, dx 
\leq \int_a^b W(x) F^{(p_1 + p_2 - 1/p_2/p_1)}(x) F^{p_2/p_1}(x) f(x) \, dx 
\leq \left\{ \int_a^b W(x) F^{p_1 + p_2}(x) \, dx \right\}^{1-1/p_2} \left\{ \int_a^b W(x) F^{p_1}(x) f^{p_2}(x) \, dx \right\}^{1/p_2} \tag{6.27}
\]

by Hölder’s inequality with indices \( p_2/p_1 \) and \((p_1 + p_2 - 1 - p_2/p_1)\).

(d) Since \(|h'(x)| > 0\) almost everywhere in \([a, b]\) and \( h^{q-1}(x)w(x) > 0\) everywhere, \( W(x) > 0\) almost everywhere. So if the first factor on the right in (6.27) were to vanish, we should have \( F(x) = 0 \) almost everywhere, and therefore everywhere since \( F \) is continuous. This would make \( f(t) = 0 \) almost everywhere, so that both sides in (6.27) would vanish and (6.27) would hold trivially. There being nothing to prove, we can exclude this situation. Thus the first factor on the right in (6.27) may be supposed positive, (and it is finite as shown in (a)); so we can divide through (6.27) by it, obtaining

\[
\frac{q}{(p_1 + p_2)c} \left\{ \int_a^b W(x) F^{p_1 + p_2}(x) \, dx \right\}^{1/p_2} \leq \left\{ \int_a^b W(x) F^{p_1}(x) f^{p_2}(x) \, dx \right\}^{1/p_2} \tag{6.28}
\]

Theorem 6.2.1 follows immediately from this.

6.2.1 A second proof of Theorem 6.2.1

We transform Theorem 6.1.3. Let \( m > 1, \sigma > 2 \) and replace the parameters \( p, q, \alpha \) respectively in Theorem 6.1.3 by \( p_1 > 1, p_2 \geq 0 \) and \( c \). Let

\[
h^{\sigma-1}(x) := z^{-m}(x) \quad \text{for} \quad 0 \leq x \leq X. \tag{6.29}
\]

Since \( z(x) > 0 \), \( z^{-m}(x) \) is positive and decreasing because \( m > 0 \) and \( z(x) \) is increasing. Thus \( h^{\sigma-1}(x) \) is decreasing if \( \sigma > 1 \), and

\[
\frac{d}{dx} \left\{ h^{\sigma-1}(x) \right\} = (\sigma - 1)h^{\sigma-2}(x)h'(x) = -(\sigma - 1)h^{\sigma-2}(x)|h'(x)| \\
= \frac{d}{dx} \left\{ z^{-m}(x) \right\} = -mz^{-m-1}(x)z'(x) = -m \frac{z'(x)}{z(x)z^m(x)}
\]
so
\[
\frac{m}{z(x)} \frac{z'(x)}{z^m(x)} = 1 - \frac{\sigma}{m} h^{\sigma - 2}(x) h'(x) z(x) = 1 - \frac{\sigma}{m} h^{\sigma - 2+1/(1-\sigma)/m}(x) h'(x)
\]
and
\[
\frac{z'(x)}{z^m(x)} = \frac{\sigma - 1}{m} h^{[(m-1)\sigma+1-2m]/m}(x) |h'(x)|.
\]
Also
\[
\frac{z'(x)}{z^m(x)} = \frac{\sigma - 1}{m} |h'(x)|.
\]
Let
\[
F(x) = \frac{1}{s(x)} \int_0^x \frac{s(t)z'(t)}{z(t)} = \frac{\sigma - 1}{ms(x)} \int_0^x \frac{s(t)|h'(t)|}{h(t)} f(t) dt := \frac{\sigma - 1}{m} G(x).
\]
Then
\[
F^{p_1+p_2}(x) = \left(\frac{\sigma - 1}{m}\right)^{p_1+p_2} G^{p_1+p_2}(x). \tag{6.30}
\]

The left side of (6.9) is
\[
\int_0^x \frac{z'(x)}{z^m(x)} w(x) F^{p_1+p_2}(x) dx
\]
\[
= \left(\frac{\sigma - 1}{m}\right)^{p_1+p_2+1} \int_0^x G^{p_1+p_2}(x) h^{[(m-1)\sigma+1-2m]/m}(x) |h'(x)| w(x) dx.
\]

The right side of (6.9) is
\[
\left(\frac{(p_1 + p_2)c}{m-1}\right)^{p_1} \int_0^x \frac{z'(x)}{z^m(x)} w(x) F^{p_2}(x) f^{p_1}(x) dx
\]
\[
= \left(\frac{(p_1 + p_2)c}{m-1}\right)^{p_1} \int_0^x \frac{1 - \sigma}{m} h^{[(m-1)\sigma+1-2m]/m}(x) |h'(x)| w(x)
\]
\[
\cdot \left(\frac{1 - \sigma}{m}\right)^{p_2} G^{p_2}(x) f^{p_1}(x) dx
\]
\[
= - \left(\frac{(p_1 + p_2)c}{m-1}\right)^{p_1} \left(\frac{1 - \sigma}{m}\right)^{p_2+1}
\]
\[
\cdot \int_0^x h^{[(m-1)\sigma+1-2m]/m}(x) |h'(x)| w(x) G^{p_2}(x) f^{p_1}(x) dx.
\]

Theorem 6.1.3 transforms to
\[
\left(\frac{\sigma - 1}{m}\right)^{p_2+1} \int_0^x G^{p_1+p_2}(x) h^{[(m-1)\sigma+1-2m]/m}(x) |h'(x)| w(x) dx
\]
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\[ \leq - \left( \frac{(p_1 + p_2)c}{m-1} \right)^{p_1} \left( \frac{1 - \sigma}{m} \right)^{p_2+1} \]

\[ \cdot \int_0^X h^{(m-1)\sigma + 1 - 2m/m(x)}h'(x)|w(x)G^{p_2}(x)f^{p_1}(x)\,dx \]

\[ = - \left( \frac{m}{m-1} \frac{(p_1 + p_2)c}{m-1} \right)^{p_1} \]

\[ \cdot \int_0^X h^{(m-1)\sigma + 1 - 2m/m(x)}h'(x)|w(x)G^{p_2}(x)f^{p_1}(x)\,dx. \]

Writing

\[ \ell = \frac{(m-1)\sigma + 1 - 2m}{m} + 1 = \frac{(m-1)(\sigma-1)}{m} \]

we see that \( \ell > 1 \) if and only if

\[ \sigma > 2 + \frac{1}{m-1} + 1 \quad \text{or} \quad m > 1 + \frac{1}{\sigma-2}. \]

Then

\[ \int_0^X h^{\ell-1}(x)|h'(x)|w(x)G^{p_1+p_2}(x)\,dx \]

\[ \leq \left( \frac{(p_1 + p_2)c}{\ell} \right)^{p_1} \int_0^X h^{\ell-1}(x)|h'(x)|w(x)G^{p_2}(x)f^{p_1}(x)\,dx. \]

We obtain Theorem 6.2.1 by replacing the interval of integration \([0, X]\) by the sub-interval \([a, b]\), \(\ell\) by \(q\) and letting \(q > 1\) and

\[ G(x) := \frac{1}{s(x)} \int_0^x \frac{s(t)|h'(t)|}{h(t)} f(t) \,dt. \] (6.31)

Extension of the proof to the interval \(1 < \sigma \leq 2\) is not necessary or feasible because we already have \(q > 1\); \(\sigma = 1\) gives the trivial case \(h(x) \equiv 1\).

**Corollary 6.2.2.** If we let \(h(x) = \log(b/x)\) in Theorem 6.2.1, then

\[ F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)f(t)}{t\log(b/t)} \,dt < \infty \] (6.32)

and

\[ 0 < \frac{1}{c} \leq \operatorname{ess inf}_{a<x<b} \left\{ 1 + \frac{x\log(b/x)}{q} \left( (p_1 + p_2) \frac{r'(x)}{r(x)} - \frac{w'(x)}{w(x)} \right) \right\} \] (6.33)

then

\[ \int_a^b F^{p_1+p_2}(x) \frac{(\log(b/x))^{q-1}}{x} w(x) \,dx \]
Let $0 \leq a < b < \infty$, $c > 0$, $p_1 > 0$, $p_2 \geq 0$, $p_3 \geq 0$, $q > 1$ be constants with $p_1 + p_3 > 1$. Let $r(x)$, $w(x)$, $h(x)$ and $s(x)$ be defined as in Theorem 6.2.1. Let

$$F(x) = \frac{1}{r(x)} \int_a^x \frac{r(t)|h'(t)|}{h(t)} f(t) \, dt \leq \infty$$

and

$$0 < \frac{1}{c} \leq \inf_{a < x < b} \left\{ 1 + \frac{1}{q} \frac{h(x)}{h'(x)} \left( (p_1 + p_2 + p_3) \frac{r'(x)}{r(x)} - \frac{w'(x)}{w(x)} \right) \right\}.$$  

Then

$$\int_a^b F^{p_1+p_2}(x)f^{p_3}(x)h^{q-1}(x)|h'(x)|w(x) \, dx \leq \left( \frac{(p_1 + p_2 + p_3)c}{q} \right)^{p_1} \int_a^b F^{p_2}(x)f^{p_1+p_3}(x)h^{q-1}(x)|h'(x)|w(x) \, dx.$$  

If $0 < p_1 + p_2 < 1$, and the reverse inequality (6.36) holds, then the reverse inequality (6.37) holds also.

**Proof.** By Theorem 6.2.1 with $p_1$ replaced by $p_1 + p_3$ and Hölder’s inequality

$$\int_a^b W(x)f^{p_1+p_2}f^{p_3}(x) \, dx = \int_a^b W(x) \left\{ F^{p_2/p_1+p_3}(x)f^{p_3}(x) \right\} \left\{ F^{p_1+p_3-p_2p_3/(p_1+p_3)}(x) \right\} \, dx$$

$$\leq \left\{ \int_a^b W(x)f^{p_2}(x)f^{p_1+p_3}(x) \, dx \right\}^{p_1/(p_1+p_3)} \cdot \left\{ \int_a^b W(x)f^{p_1+p_2+p_3}(x) \, dx \right\}^{p_3/(p_1+p_3)}$$

$$\leq \left( \frac{(p_1 + p_2 + p_3)c}{q} \right)^{p_1} \int_a^b W(x)f^{p_2}(x)f^{p_1+p_3}(x) \, dx \cdot \left\{ \int_a^b W(x)f^{p_1+p_2+p_3}(x) \, dx \right\}^{p_3/(p_1+p_3)}.$$
6.2. MAIN RESULTS

Theorem 6.2.4. Let $0 \leq a < b < \infty$, $c > 0$, $p > 1$, $q > 1$ and all hypotheses involving $k$ hold for $k \in \mathbb{N}$. Let $s_k(x)$ and $w(x)$ be positive and absolutely continuous on the finite interval $(a, b)$ and $h(t)$ be decreasing and positive and absolutely continuous on $[a, b]$, and $|h'(t)|$ be essentially bounded and essentially positive. For any $\phi$ non–negative and integrable on $(a, b)$ let

$$
\tilde{I}_k \phi(x) = \frac{1}{s_k(x)} \int_a^x \frac{s_k(t)||h'(t)|}{h(t)} \phi(t) \, dt \quad \text{for } a \leq x \leq b. \tag{6.39}
$$

If $f(x)$ is non–negative and integrable on $(a, b)$,

$$
\tilde{F}_0(x) = f(x), \quad \tilde{F}_k(x) = \tilde{I}_k \tilde{I}_{k-1} \ldots \tilde{I}_1 f(x) \tag{6.40}
$$

and

$$
0 < \frac{1}{\alpha_k} \leq \text{ess inf}_{a < x < b} \left\{ 1 + \frac{1}{q} \frac{h(x)}{|h'(x)|} \left( \frac{p s_k(x)}{s_k(x)} - \frac{w'(x)}{w(x)} \right) \right\} \tag{6.41}
$$

then

$$
\int_a^b h^{q-1}(x)|h'(x)|w(x)\tilde{F}_n^p(x) \, dx
\leq (\frac{p}{q})^{np} \left( \prod_{i=1}^n \alpha_i \right)^p \int_a^b h^{q-1}(x)|h'(x)|w(x)\tilde{F}_{n-1}^p(x) \, dx. \tag{6.42}
$$

If $0 < p < 1$ and the reverse inequality (6.41) holds, then the reverse inequality (6.41) holds.

Proof. Since $h(x)$ is continuous and positive on $[a, b]$ it has positive lower and upper bounds thereon; and the same holds for $s_k(x)$ and $w(x)$. Since $s_k(x)|h'(x)|$ is essentially bounded and $\phi(t)$ is integrable on $[a, b]$, $\tilde{I}_k \phi(x)$ exists and is continuous on $[a, b]$. In particular, $\tilde{I}_k \phi(x)$ is integrable and clearly non–negative. It follows that $\tilde{F}_k(x)$ exists and is continuous and non–negative. In Theorem 6.2.1 replace $s(x)$ by $s_k(x)$ and $f(x)$ by $\tilde{F}_{k-1}(x)$. Then $\alpha$ is replaced by $\alpha_k$ and $F(x)$ by $\tilde{I}_k \tilde{F}_{k-1}(x) = \tilde{F}_k(x)$. These give

$$
\int_a^b h^{q-1}(x)|h'(x)|w(x)\tilde{F}_k^p(x) \, dx
\leq (\frac{p\alpha_k}{q})^{np} \int_a^b h^{q-1}(x)|h'(x)|w(x)\tilde{F}_{k-1}^p(x) \, dx. \tag{6.43}
$$

Iteration of this inequality gives the stated conclusion.
CHAPTER 6. COPSON–HANJŠ–LOVE–PEČARIĆ INEQUALITIES

An alternate proof of Theorem 6.2.4 follows.

By (6.30) \( G(x) = \frac{m}{\sigma - 1} F(x). \) (6.44)

Let
\[
g(x) = \frac{m}{\sigma - 1} f(x), \quad \tilde{G}_k(x) = \tilde{I}_k \tilde{I}_{k-1} \cdots \tilde{I}_1 g(x) = \frac{m}{\sigma - 1} f(x),
\]

\[
\tilde{I}_k \phi(x) = \frac{1}{s_k(x)} \int_0^x \frac{s_k(t) z'(t)}{z(t)} dt, \quad 0 \leq x \leq X, \quad \tilde{F}_k(x) = \tilde{I}_k \tilde{I}_{k-1} \cdots \tilde{I}_1 f(x).
\]

Then
\[
\int_0^X \frac{z'(x)}{z(x)} w(x) \tilde{F}^p(x) dx \leq \left( \frac{p}{m - 1} \right)^{np} \left( \prod_{i=1}^n \alpha_i \right)^p \int_0^X \frac{z'(x)}{z(x)} w(x) f^p(x) dx.
\]

Using (6.29), the left side of (6.18) is
\[
\int_0^X \frac{z'(x)}{z(x)} w(x) \tilde{F}^p(x) dx
\]
\[
= \frac{\sigma - 1}{m} \int_0^X h^{[(m-1)\sigma+1-2m]/m}(x)|h'(x)|w(x) \left( \frac{\sigma - 1}{m} \right)^{np} \tilde{G}_n^p(x) dx
\]
\[
= \left( \frac{\sigma - 1}{m} \right)^{np+1} \int_0^X h^{\ell-1}(x)|h'(x)|w(x) \tilde{G}_n^p(x) dx.
\]

The left side of (6.18) is
\[
\left( \frac{p}{m - 1} \right)^{np} \left( \prod_{i=1}^n \alpha_i \right)^p \int_0^X \frac{\sigma - 1}{m} h^{\ell-1}(x)|h'(x)|w(x) f^p(x) dx
\]
so that
\[
\int_0^X h^{\ell-1}(x)|h'(x)|w(x) \tilde{G}_n^p(x) dx
\]
\[
= \left( \frac{m}{\sigma - 1} \right)^{np-1} \left( \frac{p}{m - 1} \right)^{np} \left( \prod_{i=1}^n \alpha_i \right)^p \int_0^X h^{\ell-1}(x)|h'(x)|w(x) f^p(x) dx.
\]

**Theorem 6.2.5.** Let \( n, p, k, s_k(x), w(x) \) and \( h(x) \) be as in Theorem 6.2.1. For any \( \phi(x) \) non-negative and integrable on \([a,b]\) let

\[
\overline{I}_k \phi(x) = \frac{1}{s_k(x)} \int_a^x s_k(t)|h'(t)|\phi(t) dt, \quad \text{for } a \leq x \leq b.
\]
If \( f(x) \) is non-negative and in \( L^p \) on \( (a,b) \) let

\[
\mathcal{F}_0(x) = f(x), \quad \mathcal{F}_k(x) = \mathcal{T}_k \mathcal{T}_{k-1} \ldots \mathcal{T}_1 f(x), \quad q > (n-1)p + 1,
\]

and

\[
0 < \frac{1}{\alpha_k} \leq \text{ess inf}_{a < x < b} \left\{ 1 + \frac{1}{q - (n-k)p} \frac{h(x)}{|h'(x)|} \left( \frac{p s_k(x)}{s_k(x)} \frac{w'(x)}{w(x)} \right) \right\} \quad (6.45)
\]

then

\[
\int_a^b h^{q-1}(x)|h'(x)|w(x)\mathcal{F}_n^p(x) \, dx \\
\leq \left( \prod_{i=1}^n \frac{p \alpha_i}{q - (n-i)} \right)^p \int_a^b h^{q-1}(x)|h'(x)|w(x)f^p(x) \, dx. \quad (6.47)
\]

**Proof.** Let \( g(x) = h(x)f(x) \) be non-negative and integrable on \([a,b]\), so that

\[
\mathcal{T}_k g(x) = \frac{1}{s_k(x)} \int_a^x \frac{s_k(t)|h'(t)|}{h(t)} h(t)g(t) \, dt, = \tilde{I}_k (h(x)g(x)).
\]

As in Theorem 6.2.4 this is non-negative and integrable. Similarly

\[
\mathcal{F}_k(x) = \mathcal{T}_k (\mathcal{F}_{k-1}(x)) \\
= \frac{1}{s_k(x)} \int_a^b \frac{s_k(t)|h'(t)|}{h(t)} h(t)\mathcal{F}_{k-1}(t) \, dt \\
= \tilde{I}_k (h(x)\mathcal{F}_{k-1}(x)).
\]

This, and therefore \( \mathcal{F}_{k-1}, \) are non-negative and integrable. So Theorem 6.2.1 gives

\[
\int_a^b h^{q-1}(x)|h'(x)|w(x)\mathcal{F}_n^p(x) \, dx \\
\leq \left( \frac{p \alpha_k}{q} \right)^p \int_a^b h^{q-1}(x)|h'(x)|w(x) (h(x)\mathcal{F}_{k-1}(x))^p \, dx.
\]

Replacing \( q \) by \( q - (n-k)p \geq q - (n-1)p > 1, \)

\[
\int_a^b h^{q-(n-k)p-1}(x)|h'(x)|w(x)\mathcal{F}_k^p(x) \, dx \\
\leq \left( \frac{p \alpha_k}{q - (n-k)p} \right)^p \int_a^b h^{q-(n-k)p-1}(x)|h'(x)|w(x) (h(x)\mathcal{F}_{k-1}(x))^p \, dx.
\]

Iteration of this inequality gives the stated conclusion of Theorem 6.2.5.
Chapter 7

Hardy’s discrete inequality – I

7.1 Introduction

In Chapter 7 we extend results obtained by Hwang–Yang [69] in 1990 and by Pachpatte [107] in 1994. The method used is an extension of the simplest proof of Hardy’s Theorem 1.2.5 developed in 1926 by Elliott [33]. The method has been used in various forms by subsequent authors including Copson, Pachpatte and Hwang–Yang. We devise a new method of creating conjugate Hölder exponents. The new material in this chapter may be summarised as follows.

(i) Hwang–Yang’s exponent $p$ is partitioned into components $p + q$ then to $p + q + r$ and finally to $p_1 + p_2 + \cdots + p_n$ and the components are then used to create new Hölder exponents.

(ii) A multi–level version of Hwang–Yang’s inequality (loc. cit.) is proved.

(iii) The convergence of the series is examined, filling a gap in [69]; this fact is non–trivial and depends on the relation between two series. It is based on the requirement that when Hölder’s inequality is used to split a convergent series into a product of two series, both of the resulting series should converge. This leads us to an infimum principle that determines ‘best possible’ constants for inequalities for the class of all suitably convergent sequences $(a_n)$ for separable weight functions $\beta_{ij}\lambda_j$. When the weight functions are specialised to yield (1.6), our constants coincide with Hardy–Landau’s ‘best possible’ constant (see Appendix C).

\footnote{$p'$ differs from the exponent '$p$' customarily used in Hölder’s inequality.}
(iv) A multi-level version of Pachpatte’s inequality [69] is proved. His technique is extended to handle the case of $n$ components of the exponent.

### 7.1.1 Copson’s inequalities

In 1928 Copson [26] established the following Hardy-type inequalities involving series of positive terms.

**Theorem 7.1.1.** If $p > 1$, $\lambda_n > 0$, $a_n > 0$, $A_n = \sum_{i=1}^{n} \lambda_i$, $A_n = \sum_{i=1}^{n} \lambda_i a_i$ and $\sum_{n=1}^{\infty} \lambda_n a_n^p$ converges, then

$$\sum_{n=1}^{\infty} \lambda_n \left( \frac{A_n}{A_n} \right)^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{7.1}$$

The constant is the best possible.

**Theorem 7.1.2.** Let $p_n$, $\lambda_n$, $a_n$, $\Lambda_n$ and $A_n$ be as in Theorem 7.1.1 and let $H(u)$ be a real-valued positive convex function defined for $u > 0$. If $\sum_{n=1}^{\infty} \lambda_n H^n(a_n)$ converges, then

$$\sum_{n=1}^{\infty} \lambda_n H^n \left( \frac{A_n}{A_n} \right) \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n H^n(a_n). \tag{7.2}$$

The constant is the best possible.

### 7.1.2 Hwang–Yang’s inequalities

In 1990 Hwang–Yang [69] established the following two theorems.

**Theorem 7.1.3.** Let $p > 1$, $\lambda_n > 0$, $\alpha_n > 0$, $\beta_i > 0$, $\sum_{n=1}^{\infty} \lambda_n a_n^p$ converge and further let

$$\Lambda_n = \sum_{i=1}^{n} \beta_i \lambda_i, \ A_n = \sum_{i=1}^{n} \beta_i \lambda_i a_i, \ \alpha_n = \frac{A_n}{\Lambda_n}. \tag{7.3}$$

If there exists $\kappa > 0$ such that

$$p - 1 + \frac{(\beta_{n+1} - \beta_n) \Lambda_n}{\beta_{n+1} \beta_n \lambda_n} \geq \frac{p}{\kappa}, \ n \in \mathbb{N} \tag{7.4}$$

then

$$\sum_{n=1}^{\infty} \lambda_n \alpha_n^p \leq \kappa^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{7.5}$$

The case $\beta_n = 1, n \in \mathbb{N}$ and $\kappa = p/(p-1)$ shows the constant in (7.5) to be the best possible.
Theorem 7.1.4. Let $H$ be a real-valued positive convex function defined on $\mathbb{R}_+$, and let $p, \beta_n, \lambda_n, a_n, A_n, \Lambda_n, \kappa$ be as in Theorem 7.1.3. If $\sum_{n=1}^{\infty} \lambda_n H^p(a_n)$ converges, then
\[
\sum_{n=1}^{\infty} \lambda_n H^p (\alpha_n) \leq \kappa \sum_{n=1}^{\infty} \lambda_n H^p (a_n).
\] (7.6)

7.1.3 Pachpatte’s inequalities

In 1994 Pachpatte [107] proved the following theorem.

Theorem 7.1.5. If $p > 1$ is a constant, $a(n) \geq 0$ for $n \in \mathbb{N}$ and
\[
\begin{align*}
A(n) & = \frac{1}{n} \sum_{m_1=1}^{n} \sum_{m_2=1}^{m_1} \ldots \sum_{m_r=1}^{m_{r-1}} a(m_r) \quad \text{(7.7)}
\end{align*}
\]
with $m_0 = n$, then
\[
\sum_{n=1}^{\infty} A^p(n) \leq \left( \frac{p}{p-1} \right)^{rp} \sum_{n=1}^{\infty} a^p(n). \quad \text{(7.8)}
\]
The equality holds in (7.8) if $a(n) = 0$ for $n \in \mathbb{N}$.

7.2 Copson–Pachpatte–Hwang–Yang inequalities

Definition 7.2.1. An inequality in $\mathbb{R}^n$ is $k$–level if the kernel on the left hand side contains a product of $r$–fold summations by $(r-1)$–fold summations and the kernel on the right hand side contains a product of terms containing $r$–, $(r-1)$–, $\ldots$, $(r-k+1)$–fold summations.

7.2.1 $2$–component, $2$–level

The first result extends Copson’s inequality stated in our Theorem 7.1.1, and the inequality of Hwang and Yang [69, Theorem 1].

Theorem 7.2.2. Let $p > 1$, $q \geq 0$, $\beta_n > 0$, $\lambda_n > 0$, $a_n > 0$ for all $n \in \mathbb{N}$, and define
\[
\begin{align*}
A_n &= \sum_{i=1}^{n} \beta_i \lambda_i a_i, \quad \Lambda_n = \sum_{i=1}^{n} \beta_i \lambda_i, \quad \rho_n = \frac{\beta_{n+1} - \beta_n}{(\beta_{n+1} \beta_n) \lambda_n}, \quad n \in \mathbb{N}. \quad \text{(7.9)}
\end{align*}
\]
Suppose that $\sum_{n=1}^{\infty} \lambda_n a_n^p (A_n/\Lambda_n)^q$ converges and that
\[
\sigma = \inf \frac{\rho_n}{n} > 1 - p - q. \quad \text{(7.10)}
\]
Then
\[ \sum_{n=1}^{\infty} \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^{p+q} \leq \left( \frac{p+q}{p+q-1+\sigma} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p \left( \frac{A_n}{\Lambda_n} \right)^q. \] (7.11)

**Proof.** The first step is to show that the series on the left in (7.11) converges. This will fill in the gap in the proof of [69, Theorem 1]. We observe that if \( \beta_n \) is not a monotonic function of \( n \) then \( \rho_n \) and \( \sigma \) may be negative. Set
\[ \omega_n = \frac{A_n}{\Lambda_n}. \] (7.12)
Following the first part of the proof of [69, Theorem 1] with \( p \) replaced by \( p+q \), we obtain
\[ \sum_{n=1}^{N} (p+q-1)\lambda_n \omega_n^{p+q} + \sum_{n=1}^{N-1} \lambda_n \rho_n \omega_n^{p+q} \leq (p+q) \sum_{n=1}^{N} \lambda_n a_n \omega_n^{p+q-1}. \] (7.13)
Let
\[ \kappa = \frac{p+q}{p+q+\sigma-1}, \quad \mu = \max\{\kappa, \frac{p+q}{p+q-1}\}. \]
We observe that \( \mu = \kappa \) if \( \sigma < 0 \) and \( \kappa \to \infty \) as \( \sigma \to 1 - p - q \). We find a lower estimate of the left hand side of this inequality:
\[ \sum_{n=1}^{N} (p+q-1)\lambda_n \omega_n^{p+q} + \sum_{n=1}^{N-1} \lambda_n \rho_n \omega_n^{p+q} \geq \frac{p+q}{\mu} \lambda_N \omega_N^{p+q} + \frac{p+q}{\mu} \sum_{n=1}^{N-1} \lambda_n \omega_n^{p+q} \]
\[ \geq \frac{p+q}{\mu} \sum_{n=1}^{N} \lambda_n \omega_n^{p+q}. \] (7.14)
Combining this with (7.13), we have
\[ \sum_{n=1}^{N} \lambda_n \omega_n^{p+q} \leq \mu \sum_{n=1}^{N} \lambda_n a_n \omega_n^{p+q-1}. \]
Applying Hölder’s inequality with indices \( p \) and \( p/(p-1) \) we get\(^2\)
\[ \sum_{n=1}^{N} \lambda_n \omega_n^{p+q} \leq \mu \sum_{n=1}^{N} \lambda_n a_n \omega_n^{p+q-1} \]
\(^2\) \( q \) differs from the \( q \) used in Hölder’s inequality to denote \( p/(p-1) \).
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\[ \mu \sum_{n=1}^{N} \left( \frac{\lambda_n^{1/p} a_n \omega_n^{q/p}}{p} \right) \left( \frac{\lambda_n^{(p-1)/p} \omega_n^{(q-1)/p}}{p} \right) \]

\[ \leq \mu \left\{ \sum_{n=1}^{N} \left( \frac{\lambda_n^{1/p} a_n \omega_n^{q/p}}{p} \right)^{1/p} \left( \sum_{n=1}^{N} \left( \frac{\lambda_n^{(p-1)/p} \omega_n^{(q-1)/p}}{p} \right) \right)^{(p-1)/p} \right\}^{1/p} \]

\[ = \mu \left\{ \sum_{n=1}^{N} \frac{\lambda_n^{p} a_n^{p} \omega_n^{q}}{p} \right\}^{1/p} \left\{ \sum_{n=1}^{N} \lambda_n^{(p+q-1)/p} \right\} \]  

\( (7.15) \)

Dividing by the last factor on the right and raising to the power of \( p \), we get

\[ \sum_{n=1}^{N} \lambda_n^{\omega_n^{p+q}} \leq \mu p \sum_{n=1}^{N} \lambda_n^{a_n^{p} \omega_n^{q}}. \]

This proves that the series \( \sum_{n=1}^{\infty} \lambda_n^{\omega_n^{p+q}} \) converges. In view of (7.15), the series \( \sum_{n=1}^{\infty} \lambda_n^{a_n^{p+q}-1} \) also converges. This is our infimum principle.

Returning to (7.14) and observing that the term \( (p+q-1) \lambda_n^{\omega_n^{p+q}} \) is non-negative, we see that

\[ \sum_{n=1}^{N-1} (p+q-1+\rho_n) \lambda_n^{\omega_n^{p+q}} \leq (p+q) \sum_{n=1}^{N} \lambda_n^{a_n^{p} \omega_n^{q}+q-1}, \]

that is,

\[ \sum_{n=1}^{N-1} \lambda_n^{\omega_n^{p+q}} \leq \kappa \sum_{n=1}^{N} \lambda_n^{a_n^{p} \omega_n^{q}+q-1}. \]

Since the series on both sides converge, we have

\[ \sum_{n=1}^{\infty} \lambda_n^{\omega_n^{p+q}} \leq \kappa \sum_{n=1}^{\infty} \lambda_n^{a_n^{p} \omega_n^{q}+q-1}. \]

Applying Hölder’s inequality to the right hand side as above for the finite sums, we obtain (7.11).

Setting \( q = 0 \) in the preceding theorem, we get a version of [69, Theorem 1] with the constant \( \kappa \) given explicitly as \( \kappa = p/(p-1+\sigma) \).

**Corollary 7.2.3.** Let \( p > 1, \beta_n > 0, \lambda_n > 0, a_n > 0 \) for all \( n \in \mathbb{N} \), and define

\[ A_n = \sum_{i=1}^{n} \beta_i \lambda_i a_i, \quad \Lambda_n = \sum_{i=1}^{n} \beta_i \lambda_i, \quad \rho_n = \frac{(\beta_{n+1} - \beta_n) A_n}{(\beta_{n+1} \beta_n) \lambda_n}, \quad n \in \mathbb{N}. \]  

\( (7.16) \)
Suppose that \( \sum_{n=1}^{\infty} \lambda_n a_n^p \) converges and that \( \sigma = \inf_n \rho_n > 1 - p \) exists. Then
\[
\sum_{n=1}^{\infty} \lambda_n \left( \frac{A_n}{\Lambda_n} \right)^p \leq \left( \frac{p}{p-1+\sigma} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p. \tag{7.17}
\]

**Corollary 7.2.4.** Setting \( \beta_n \equiv 1 \) in (7.17) we obtain Copson’s inequality in Theorem 7.1.1.

**Corollary 7.2.5.** Setting \( \beta_n \equiv \lambda_n \equiv 1 \) in (7.17) we obtain Hardy’s inequality in Theorem 1.2.5.

**Remark 7.2.6.** In the proof of [69, Theorem 1], Hwang and Yang use the inequality
\[
\sum_{n=1}^{N-1} \lambda_n \omega_n^p \leq \kappa \sum_{n=1}^{N} \lambda_n a_n \omega_n^{p-1}, \quad N \in \mathbb{N}, \tag{7.18}
\]
(for a positive \( \kappa \)) to prove the desired inequality by taking the limit as \( N \to \infty \). However, the convergence of the two series is not established in [69]. This fact is nontrivial and depends on the relation between the two series. The rôle of the infimum in (7.10) is illustrated in Example 8.7.3.

### 7.2.2 3-component, 2-level

From Theorem 7.2.2 we can deduce the following more general result.

**Theorem 7.2.7.** Let \( p > 1 \), \( q > 0 \), \( r \geq 0 \), \( a_n > 0 \), \( \lambda_n > 0 \), \( \beta_n > 0 \) for all \( n \in \mathbb{N} \), and let \( A_n, \Lambda_n \) and \( \rho_n \) be as in (7.9). Suppose that \( \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} (A_n/\Lambda_n)^r \) converges and that \( \sigma = \inf_n \rho_n > 1 - p - q - r \). Then
\[
\sum_{n=1}^{\infty} \lambda_n a_n^p \left( \frac{A_n}{\Lambda_n} \right)^{q+r} \leq \left( \frac{p+q+r}{p+q+r-1+\sigma} \right)^q \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \left( \frac{A_n}{\Lambda_n} \right)^r. \tag{7.19}
\]

**Proof.** Set \( \omega_n = A_n/\Lambda_n \) and \( \kappa = (p+q+r)/(p+q+r-1+\sigma) \). We apply Hölder’s inequality with indices \( (p+q)/p \) and \( (p+q)/q \) (for this reason we need to assume \( q > 0 \) rather than \( q \geq 0 \) as we did in Theorem 7.2.2), and then we apply Theorem 7.2.2 with \( q \) replaced by \( q + r \):
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\[ \sum_{n=1}^{\infty} \left( \lambda_n a_n^{p+q} \omega_n^{r} \right)^{p/(p+q)} \left( \lambda_n \omega_n^{p+q+r} \right)^{q/(p+q)} \]

\[ \leq \left( \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^{r} \right)^{p/(p+q)} \left( \sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q+r} \right)^{q/(p+q)} \]

\[ \leq \left( \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^{r} \right)^{p/(p+q)} \left( \sum_{n=1}^{\infty} \lambda_n \omega_n^{p+q+r} \right)^{q/(p+q)} , \]

by Theorem 7.2.2, that is,

\[ \left( \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^{r} \right)^{p/(p+q)} \leq \kappa^{p/(p+q)} \left( \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \omega_n^{r} \right)^{p/(p+q)} \]

Raising both sides of (7.20) to the power \((p+q)/p\) yields (7.19).

7.2.3 2–component, 2–level

Setting \(r = 0\) in the preceding theorem, we get

**Corollary 7.2.8.** Let \(p > 1, q > 0, a_n > 0, \lambda_n > 0, \beta_n > 0 \) for all \(n \in \mathbb{N}\), and let \(A_n, \Lambda_n\) and \(\rho_n\) be as in (7.9). Suppose that \(\sum_{n=1}^{\infty} \lambda_n a_n^{p+q}\) converges and that \(\sigma = \inf_n \rho_n > 1 - p - q\). Then

\[ \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \left( \begin{array}{c} p+q \\ \rho_n \end{array} \right) \left( \begin{array}{c} p \\ q \end{array} \right) \sum_{n=1}^{\infty} \lambda_n a_n^{p+q}. \]

**Proof.** Inequality (7.21) follows from (7.19) by setting \(r = 0\).

7.2.4 \(n\)–component, 2–level

From Theorem 7.2.7 we can deduce the following more general result.

**Theorem 7.2.9.** Let \(p = p_1 + \cdots + p_i, p_1 > 1, p_i > 0, q = p_{i+1} + \cdots + p_{i+j}, \)
\(r = p_{i+j+1} + \cdots + p_{i+j+k} \geq 0, p_i > 0, i \in \mathbb{N}\setminus\{1\}, a_n > 0, \lambda_n > 0, \) for \(n \in \mathbb{N}\).

Let \(A_n, \Lambda_n\) and \(\rho_n\) be as in (7.9). Suppose that

\[ \sum_{n=1}^{\infty} \lambda_n a_n^{p_1+\cdots+p_i+p_{i+j}} \left( \begin{array}{c} p_{i+j+1}+\cdots+p_{i+j+k} \\ \Lambda_n \end{array} \right) \]

converges. Let

\[ \rho_m = \left[ \frac{\beta_{m+1} - \beta_m} {\beta_{m+1} \beta_m} \right] \Lambda_m, \quad n \in \mathbb{N} \]

(7.23)
and \( \sigma = \inf_{m} \rho_{m} > 1 - (p_{i+1} + \cdots + p_{i+j}) \). Then
\[
\sum_{n=1}^{\infty} \lambda_{n} a_{n}^{p_{i} + \cdots + p_{i+j}} \left( \frac{A_{n}}{\Lambda_{n}} \right)^{p_{i+1} + \cdots + p_{i+j} + \cdots + p_{i+j+k}} \\
\leq \left( \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} - 1} \right)^{p_{i+1} + \cdots + p_{i+j}} \\
\cdot \sum_{n=1}^{\infty} \lambda_{n} a_{n}^{p_{i} + \cdots + p_{i+j}} \left( \frac{A_{n}}{\Lambda_{n}} \right)^{p_{i+j+1} + \cdots + p_{i+j+k}}.
\] (7.24)

The equality holds in (7.24) if \( a(x) = 0 \) for all \( n \in \mathbb{N} \).

**Proof.**

In Theorem 7.2.7 let \( p = p_{1} + \cdots + p_{i}, \ p_{1} > 1, \ q = p_{i+1} + \cdots + p_{i+j} > 0, \)
\( r = p_{i+j+1} + \cdots + p_{i+j+k} \geq 0 \) and
\[ \kappa = \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} + \sigma_{m} - 1}, \ \mu = \max \left\{ \kappa, \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} - 1} \right\}. \]

We now present a multilevel version of Corollary 7.2.3.

### 7.2.5 1–exponent, \((n + 1)\)–level

**Theorem 7.2.10.** Let \( p > 1, \ m \in \mathbb{N}, \ a_{j} > 0, \ \beta_{ij} > 0, \ \lambda_{j} > 0 \) for \( i \in \{1, \ldots, m\} \) and all \( j \in \mathbb{N} \). For \( i \in \{1, \ldots, m\} \) and \( n \in \mathbb{N} \). Write \( \beta_{(i+j),k} = \beta_{i+j,k}, \ \beta_{i,(j+k)} = \beta_{i,k}, \ A_{(i-j),k} = A_{i-j,k} \) and define
\[
A_{0n} = a_{n}, \ A_{in} = \sum_{j=1}^{n} \beta_{ij} \lambda_{j} A_{i-1,j}, \ \Lambda_{in} = \sum_{j=1}^{n} \beta_{ij} \lambda_{j}, \ \rho_{in} = \frac{(\beta_{i,n+1} - \beta_{in}) \Lambda_{in}}{(\beta_{i,n+1} \beta_{in}) \lambda_{n}}.
\]

Suppose that \( \sum_{n=1}^{\infty} \lambda_{n} a_{n}^{p} \) converges and \( \sigma_{i} = \inf_{n} \rho_{in} > 1 - p, \ i \in \{1, \ldots, m\} \).

Then
\[
\sum_{n=1}^{\infty} \lambda_{n} \left( \frac{A_{mn}}{\Lambda_{mn}} \right)^{p} \leq \left( \prod_{i=1}^{m} \frac{p}{p - 1 + \sigma_{i}} \right)^{p} \sum_{n=1}^{\infty} \lambda_{n} a_{n}^{p}.
\] (7.25)

**Proof.** Let \( \omega_{in} = A_{in}/\Lambda_{in}, \ \kappa_{i} = p/(p - 1 + \sigma_{i}), \ i \in \{0, 1, \ldots, m\}, \ n \in \mathbb{N} \).

Applying Corollary 7.2.3, we conclude that \( \sum_{n=1}^{\infty} \lambda_{n} \omega_{in}^{p} \) converges, and
\[
\sum_{n=1}^{\infty} \lambda_{n} \omega_{in}^{p} \leq \kappa_{1}^{p} \sum_{n=1}^{\infty} \lambda_{n} \omega_{0n}^{p}.
\]
Continuing in this way we show that
\[ \sum_{n=1}^{\infty} \lambda_n \omega_n^p \leq \kappa_p \sum_{n=1}^{\infty} \lambda_n \omega_{i-1,n}^p, \quad i \in \{1, \ldots, m\}. \]
Applying this inequality \( m \) times, starting with \( i = m \), we obtain (7.25).

Remark 7.2.11. Setting \( \beta_{ij} = 1, \lambda_j = 1 \) for \( i \in \{1, \ldots, m\}, j \in \mathbb{N} \), we have \( \sigma_i = 0 \) for all \( i \) in the preceding theorem, and recover Pachpatte’s result [107, Theorem 1].

7.2.6 Generalisations to convex functions

Theorem 7.2.12. Let \( H \) be a real-valued positive convex function defined on \((0, \infty)\) and let \( p > 1, q \geq 0, \beta_i > 0, \lambda_i > 0, a_i > 0 \) for all \( i \in \mathbb{N} \). Let \( A_n, \Lambda_n \) and \( \rho_n \) be as in (7.9), and let
\[ F_n = \sum_{i=1}^{n} \beta_i \lambda_i H(a_i), \quad n \in \mathbb{N}. \] (7.26)
Suppose that \( \sum_{n=1}^{\infty} \lambda_n H^p(a_n)(F_n/\Lambda_n)^q \) converges and that \( \sigma = \inf \rho_n > 1 - p - q \). Then
\[ \sum_{n=1}^{\infty} \lambda_n H^{p+q}(A_n/\Lambda_n) \leq \left( \frac{p+q}{p+q-1+\sigma} \right)^p \sum_{n=1}^{\infty} \lambda_n H^p(a_n)(F_n/\Lambda_n)^q. \] (7.27)

Proof. Write \( \Phi_n = F_n/\Lambda_n, n \in \mathbb{N}, \kappa = (p+q)/(p+q-1+\sigma) \). We apply Theorem 7.2.2 with \( a_i \) replaced by \( H(a_i) \). Then \( A_n \) is replaced by \( F_n \), and \( \omega_n = A_n/\Lambda_n \) by \( \Phi_n \). Inequality (7.11) then becomes
\[ \sum_{n=1}^{\infty} \lambda_n \Phi_n^{p+q} \leq \kappa_p \sum_{n=1}^{\infty} \lambda_n H^p(a_n)\Phi_n^q. \] (7.28)
Since \( H \) is convex, we can apply Jensen’s inequality to obtain
\[ H(\omega_n) = H\left( \frac{A_n}{\Lambda_n} \right) = H\left( \sum_{i=1}^{n} \frac{\beta_i \lambda_i}{\Lambda_n} a_i \right) \leq \sum_{i=1}^{n} \frac{\beta_i \lambda_i}{\Lambda_n} H(a_i) = \Phi_n. \]
Substituting this in (7.28), we get (7.27).

The choice \( H(u) = u \) in the preceding theorem yields Theorem 7.2.2, noting that \( H \) is not strictly convex. Setting \( q = 0 \), we recover [69, Theorem A]. If
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$q = 0$, $\beta_i = 1$ and $\sigma = 0$, the preceding theorem reduces to Copson’s result in [26] (see our Theorem 7.1.2).

The next result generalises Theorem 7.2.12 in the same way that Theorem 7.2.7 generalises Theorem 7.2.2. The proof is left to the reader.

**Theorem 7.2.13.** Let $H$ be a real-valued positive convex function defined on $(0, \infty)$ and let $p > 1$, $q > 0$, $r \geq 0$, $\beta_i > 0$, $\lambda_i > 0$, $a_i > 0$ for all $i \in \mathbb{N}$. Let $A_n$, $\Lambda_n$ and $\rho_n$ be as in (7.9), and $F_n$ as in (7.26). Suppose that $\sum_{n=1}^{\infty} \lambda_n H^r(a_n)(F_n/\Lambda_n)^{p+q}$ converges and that $\sigma = \inf_n \rho_n > 1 - p - q - r$. Then

$$\sum_{n=1}^{\infty} \lambda_n H^{q+r} \left( \frac{A_n}{\Lambda_n} \right) \left( \frac{F_n}{\Lambda_n} \right)^p \leq \left( \frac{p + q + r}{p + q + r - 1 + \sigma} \right) \sum_{n=1}^{\infty} \lambda_n H^r(a_n) \left( \frac{F_n}{\Lambda_n} \right)^{p+q}.$$  

(7.29)

### 7.2.7 2–component, 4–level

In this section we extend Pachpatte’s results in [107]. In order to deal with the two components we extend Pachpatte’s algebraic technique (see (7.38)). In 1994 Pachpatte proved the following [107, Theorem 2]:

**Theorem 7.2.14.** If $p > 1$ is a constant, $b_{mn} \geq 0$ for $m, n \in \mathbb{N}$ and

$$B_{mn} = \frac{1}{mn} \sum_{s=1}^{m} \sum_{t=1}^{n} \frac{1}{st} \sum_{x=1}^{s} \sum_{y=1}^{t} b_{xy} \quad \text{for} \quad m, n \in \mathbb{N},$$  

(7.30)

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}^p \leq \left( \frac{p}{p - 1} \right)^{4p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^p.$$  

(7.31)

The equality holds in (7.31) if $b_{mn} = 0$ for $m, n \in \mathbb{N}$.

We now prove

**Theorem 7.2.15.** Let $p > 1$, $q \geq 0$, $b_{mn} > 0$ for $m, n \in \mathbb{N}$ and let

$$B_{mn} = \frac{1}{mn} \sum_{s=1}^{m} \sum_{t=1}^{n} \frac{1}{st} \sum_{i=1}^{s} \sum_{j=1}^{t} b_{ij} \quad \text{for} \quad m, n, i, j \in \mathbb{N}.$$  

(7.32)

If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^p B_{mn}^q$ converges, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}^{p+q} \leq \left( \frac{p}{p - 1} \right)^{3p} \left( \frac{p + q}{p + q - 1} \right)^p \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^p B_{mn}^q.$$  

(7.33)

The equality holds in (7.33) if $b_{mn} = 0$ for $m, n \in \mathbb{N}$.
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Proof. The proof is an adaptation of the proof of Theorem 7.2.14, which in turn extends an idea first used by Elliott in 1926 [33].

Let \( L, M \in \mathbb{N} \) and define

\[
\omega_{mn} := \frac{1}{n} \sum_{t=1}^{n} \frac{1}{t} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{t} b_{ij},
\]

(7.34)

\[
n\omega_{mn} - (n-1)\omega_{m,n-1} = \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{x=1}^{s} \sum_{y=1}^{t} b_{xy}
\]

(7.35)

and

\[
S_{ML} := \sum_{m=1}^{M} \sum_{n=1}^{L} B_{mn}^{p+q} = \sum_{m=1}^{M} \sum_{n=1}^{L} \omega_{mn}^{p+q}.
\]

(7.36)

Then the left hand side of (7.33) may be written as

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn}^{p+q}.
\]

(7.37)

Thus

\[
\omega_{mn}^{p+q} - \left( \frac{p + q}{p + q - 1} \right) \left\{ \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{x=1}^{s} \sum_{y=1}^{t} b(x,y) \right\} \omega_{mn}^{p+q-1}
\]

\[
= \omega_{mn}^{p+q} - \left( \frac{p + q}{p + q - 1} \right) \{n\omega_{mn} - (n-1)\omega_{m,n-1}\} \omega_{mn}^{p+q-1}
\]

\[
= \left\{ 1 - \left( \frac{p + q}{p + q - 1} \right) n \right\} \omega_{mn}^{p+q}
\]

\[
+ \left( \frac{p + q}{p + q - 1} \right) (n-1)\omega_{m,n-1}\omega_{mn}^{p+q-1}
\]

\[
= \left\{ 1 - n - \left( \frac{1}{p + q - 1} \right) n \right\} \omega_{mn}^{p+q}
\]

\[
+ \left( \frac{p + q}{p + q - 1} \right) (n-1)\omega_{m,n-1}\omega_{mn}^{p+q-1}
\]

\[
\leq \left\{ 1 - n - \left( \frac{1}{p + q - 1} \right) n \right\} \omega_{mn}^{p+q}
\]

\[
+ \left( \frac{p + q}{p + q - 1} \right) (n-1) \frac{1}{p + q} \left\{ \omega_{m,n-1}^{p+q} + (p + q - 1)\omega_{mn}^{p+q} \right\}
\]

\[
= \left( \frac{1}{p + q - 1} \right) \left\{ (n-1)\omega_{m,n-1}^{p+q} - n\omega_{mn}^{p+q} \right\}.
\]

(7.38)
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Keeping $m$ fixed in (7.38) and letting $n = 1, 2, \ldots, L$ and adding the inequalities we have

$$
\sum_{n=1}^{L} \omega_{mn}^{p+q} = \left( \frac{p+q}{p+q-1} \right) \sum_{n=1}^{L} \left\{ \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{n} b_{ij} \right\} \omega_{mn}^{p+q-1}
$$

$$
= \left( \frac{1}{p+q-1} \right) \sum_{n=1}^{L} \left\{ (n-1)\omega_{m,n-1}^{p+q} - n\omega_{mn}^{p+q} \right\}
$$

$$
= - \left( \frac{1}{p+q-1} \right) L\omega_{m,L}^{p+q}
$$

$$
\leq 0.
$$

(7.39)

This is [107, Equation (24)] with $p$ replaced by $p+q$. From (7.39), using Hölder’s inequality with indices $p$ and $p/(p-1)$, we obtain

$$
\sum_{n=1}^{L} \omega_{mn}^{p+q} \leq \left( \frac{p+q}{p+q-1} \right) \sum_{n=1}^{L} \left\{ \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{n} b_{ij} \right\} \omega_{mn}^{p+q-1}
$$

$$
\leq \left( \frac{p+q}{p+q-1} \right) \left\{ \sum_{n=1}^{L} \left\{ \omega_{mn}^{q/p} \left\{ \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{n} b_{ij} \right\} \right\} \right\}^{1/p}
$$

$$
\cdot \left\{ \sum_{n=1}^{L} \left\{ \omega_{mn}^{(p-1)(1+q/p)} \right\} \right\}^{(p-1)/p}
$$

(7.40)

Dividing both sides of (7.40) by the last term on the right and raising to the $p$th power, we have

$$
\sum_{n=1}^{L} \omega_{mn}^{p+q} \leq \left( \frac{p+q}{p+q-1} \right)^{p} \sum_{n=1}^{L} \omega_{mn}^{q} \left\{ \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{n} b_{ij} \right\}^{p}
$$

$$
= \left( \frac{p+q}{p+q-1} \right)^{p} \sum_{n=1}^{L} \omega_{mn}^{q} m^{p} n^{-p/\beta_{mn}}.
$$

(7.41)

If we define

$$
\beta_{mn} = \frac{1}{m} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{n} b_{ij}
$$

(7.42)

so that

$$
\frac{m}{n} \beta_{mn} = \frac{1}{n} \sum_{s=1}^{m} \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{n} b_{ij}
$$

(7.43)
then from (7.36) and (7.41),

$$S_{ML} \leq \left( \frac{p + q}{p + q - 1} \right)^p \sum_{n=1}^{L} n^{-p} \sum_{m=1}^{M} m^{-q} \omega_{mn}^{q} \beta_{mn}^{p}$$

$$= \left( \frac{p + q}{p + q - 1} \right)^p \sum_{n=1}^{M} m^{-q} \sum_{m=1}^{L} n^{-p} \omega_{mn}^{q} \beta_{mn}^{p}, \quad (7.44)$$

and

$$m \beta_{mn} - (m - 1) \beta_{m-1,n} = \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \geq 0, \quad (7.45)$$

and $m \beta_{mn}$ is an increasing function of $m$. We note that

$$\omega_{mn} - \omega_{m-1,n} = \frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{t} b_{ij} \geq 0, \quad (7.46)$$

$$\beta_{mn} - \beta_{m,n-1} = \frac{1}{m} \sum_{i=1}^{m} b_{in} \geq 0. \quad (7.47)$$

From (7.42) and using the inequality (B.1) we deduce that

$$m^{-q} \omega_{mn}^{q} \beta_{mn}^{p} - \left( \frac{p}{p - 1} \right) m^{-q} \omega_{mn}^{q} \left\{ \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \right\} \beta_{mn}^{p-1}$$

$$= m^{-q} \omega_{mn}^{q} \beta_{mn}^{p} - \left( \frac{p}{p - 1} \right) m^{-q} \omega_{mn}^{q} (m \beta_{mn} - (m - 1) \beta_{m-1,n}) \beta_{mn}^{p-1}$$

$$= \left\{ 1 - \left( \frac{p}{p - 1} \right) m \right\} m^{-q} \omega_{mn}^{q} \beta_{mn}^{p-1}$$

$$+ \left( \frac{p}{p - 1} \right) (m - 1) m^{-q} \omega_{mn}^{q} \beta_{m-1,n} \beta_{mn}^{p-1}$$

$$\leq \left\{ 1 - m - \left( \frac{1}{p - 1} \right) m \right\} m^{-q} \omega_{mn}^{q} \beta_{mn}^{p-1}$$

$$+ \left( \frac{p}{p - 1} \right) (m - 1) \left( 1 - m \right) m^{-q} \omega_{mn}^{q} (\beta_{m-1,n}^{p} + (p - 1) \beta_{mn}^{p})$$

$$= \left( \frac{1}{p - 1} \right) m^{-q} \omega_{mn}^{q} ((m - 1) \beta_{m-1,n}^{p} - m \beta_{mn}^{p})$$

$$\leq 0 \quad (7.48)$$

by (7.43). Keeping $n$ fixed in (7.48) and letting $m = 1, 2, \ldots, M$ and adding the
inequalities we have
\[
\sum_{m=1}^{M} m^{-q} \omega_{mn}^q \beta_{mn}^p \leq \left( \frac{p}{p-1} \right) \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \left\{ \frac{1}{m} \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \right\} \beta_{mn}^{p-1}
\]
\[
\leq \left( \frac{p}{p-1} \right) \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \left( (m-1) \beta_{m-1,n}^p - m \beta_{mn}^p \right)
\]
\[
\leq 0 \quad (7.49)
\]
(the sum of negative terms, by (7.43)). Let
\[
\gamma_{mn} := \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{m} b_{ij},
\]
so that
\[
n \gamma_{mn} - (n-1) \gamma_{m,n-1} = \sum_{i=1}^{m} b_{in} \geq 0. \quad (7.50)
\]
From (7.49), following the same procedure as used from (7.39) to (7.41)
\[
\sum_{m=1}^{M} m^{-q} \omega_{mn}^q \beta_{mn}^p \leq \left( \frac{p}{p-1} \right) \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \left\{ \frac{1}{m} \sum_{x=1}^{m} \sum_{y=1}^{n} b(x,y) \right\} \beta_{mn}^{p-1}
\]
\[
= \left( \frac{p}{p-1} \right) \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \left( \frac{n}{m} \right) \gamma_{mn} \beta_{mn}^{p-1}
\]
\[
\leq \left( \frac{p}{p-1} \right) \left\{ \sum_{m=1}^{M} \left\{ m^{-q/p} \omega_{mn}^{q/p} \left( \frac{1}{m} \sum_{x=1}^{m} \sum_{y=1}^{n} b(x,y) \right) \right\} \right\}^{1/p}
\]
\[
\cdot \left\{ \sum_{m=1}^{M} \left\{ m^{-q(p-1)/p} \omega_{mn}^{q(p-1)/p} \beta_{mn}^{(p-1)/p} \right\} \right\}^{(p-1)/p}
\]
\[
= \left( \frac{p}{p-1} \right)^p \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \beta_{mn}^p \gamma_{mn} \quad (7.51)
\]
so that
\[
\sum_{m=1}^{M} m^{-q} \omega_{mn}^q \beta_{mn}^p \leq \left( \frac{p}{p-1} \right)^p \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \left\{ \frac{1}{m} \sum_{x=1}^{m} \sum_{y=1}^{n} b(x,y) \right\}^p
\]
\[
= \left( \frac{p}{p-1} \right)^p n^p \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \beta_{mn}^p \gamma_{mn} \quad (7.52)
\]
From (7.44) we observe that

\[ S_{ML} \leq \left( \frac{p}{p-1} \right)^p \left( \frac{p+q}{p+q-1} \right)^p \sum_{m=1}^{M} m^{-p-q} \sum_{n=1}^{L} \omega_{mn}^q \gamma_{mn}^p. \]  

(7.53)

From (7.50) and using the inequality (B.1) we observe that

\[ m^{-p-q} \omega_{mn}^q \gamma_{mn}^p \left\{ \frac{p}{p-1} \right\} m^{-p-q} \omega_{mn}^q \left\{ \sum_{i=1}^{m} b_{in} \right\} \gamma_{mn}^p \]

\[ = m^{-p-q} \omega_{mn}^q \gamma_{mn}^p \left\{ 1 - \left( \frac{p}{p-1} \right) n \right\} \]

\[ + \left( \frac{p}{p-1} \right) (n-1) m^{-p-q} \omega_{mn}^q \gamma_{mn,n-1}^p \gamma_{mn}^{p-1} \]

\[ \leq m^{-p-q} \omega_{mn}^q \gamma_{mn}^p \left\{ 1 - \left( \frac{p}{p-1} \right) n \right\} \]

\[ + \left( \frac{1}{p-1} \right) (n-1) m^{-p-q} \omega_{mn}^q \left\{ \gamma_{mn,n-1}^p + (p-1) \gamma_{mn}^p \right\} \]

\[ = \left( \frac{1}{p-1} \right) m^{-p-q} \omega_{mn}^q \left\{ (n-1) \gamma_{mn,n-1}^p - n \gamma_{mn}^p \right\} \]

\[ \leq 0 \]  

(7.54)

by (7.43). Keeping \( m \) fixed in (7.54), letting \( n = 1, 2, \ldots, L \) and adding the inequalities we have

\[ m^{-p-q} \sum_{n=1}^{L} \omega_{mn}^q \gamma_{mn}^p \]

\[ \leq \left( \frac{p}{p-1} \right) m^{-p-q} \sum_{n=1}^{L} \omega_{mn}^q \left\{ \sum_{i=1}^{m} b_{in} \right\} \gamma_{mn}^{p-1} \]

\[ \leq \left( \frac{p}{p-1} \right) m^{-p-q} \left\{ \sum_{n=1}^{L} \left\{ \omega_{mn}^q \left\{ \sum_{i=1}^{m} b_{in} \right\} \right\} \right\}^{1/p} \]

\[ \cdot \left\{ \sum_{n=1}^{L} \left\{ \omega_{mn}^q \gamma_{mn}^p \right\} \right\}^{(p-1)/p} \]

\[ = \left( \frac{p}{p-1} \right) m^{-p-q} \left\{ \sum_{n=1}^{L} \omega_{mn}^q \left\{ \sum_{i=1}^{m} b_{in} \right\} \right\}^{1/p} \]  

\[ \cdot \left\{ \sum_{n=1}^{L} \omega_{mn}^q \gamma_{mn}^p \right\}^{(p-1)/p}. \]
Following similar procedures as used to derive (7.49) from (7.39) with a further application of Hölder’s inequality we arrive at

\[
S_{ML} \leq \left( \frac{p}{p-1} \right)^p \left( \frac{p+q}{p+q-1} \right)^p \sum_{m=1}^M m^{-p-q} \sum_{n=1}^L \omega_m^q \gamma_{mn}^p \tag{7.55}
\]

and

\[
\sum_{n=1}^L \omega_{mn}^q \gamma_{mn}^p \leq \left( \frac{p}{p-1} \right)^p \sum_{n=1}^L \omega_m^q \left\{ \sum_{i=1}^m b_{in} \right\}^p \tag{7.56}
\]

From and (7.53) and (7.56) we observe that

\[
S_{ML} \leq \left( \frac{p}{p-1} \right)^{2p} \left( \frac{p+q}{p+q-1} \right)^{2p} \sum_{n=1}^L \sum_{m=1}^M m^{-p-q} m^p \omega_{mn}^q \delta_{mn}^p \tag{7.57}
\]

where

\[
\delta_{mn} := \frac{1}{m} \sum_{i=1}^m b_{in}, \tag{7.58}
\]

and

\[
m \delta_{mn} - (m-1) \delta_{m-1,n} = b_{mn} \geq 0. \tag{7.59}
\]

From (7.59) and using the inequality (B.1) we observe that

\[
\omega_{mn}^q \delta_{mn}^p - \left( \frac{p}{p-1} \right)^q \omega_m^q b_{mn} \delta_{mn}^{p-1}
\]

\[
= \omega_{mn}^q \delta_{mn}^p - \left( \frac{p}{p-1} \right)^q \omega_m^q \{ m \delta_{mn} - (m-1) \delta_{m-1,n} \} \delta_{mn}^{p-1}
\]

\[
= \omega_{mn}^q \delta_{mn}^p \left\{ 1 - \left( \frac{p}{p-1} \right)^q m \right\} + \left( \frac{p}{p-1} \right)^q (m-1) \omega_m^q \delta_{m-1,n} \delta_{mn}^{p-1}
\]

\[
\leq \omega_{mn}^q \delta_{mn}^p \left\{ 1 - \left( \frac{p}{p-1} \right)^q m \right\}
\]

\[
+ \left( \frac{1}{p-1} \right)^q (m-1) \left\{ \delta_{m-1,n}^p + (p-1) + \omega_m^q \right\} \omega_{mn}^q
\]

\[
= \omega_{mn}^q \delta_{mn}^p \left\{ 1 - m - \left( \frac{1}{p-1} \right)^q m \right\} + (m-1) \omega_m^q \delta_{mn}^p
\]

\[
+ \left( \frac{1}{p-1} \right)^q (m-1) \omega_{mn}^q \delta_{m-1,n}^p
\]
= \left(\frac{p}{p-1}\right)^q \omega_{mn}^q \left\{ (m-1)\delta_{m-1,n}^p - m\delta_{mn}^p \right\}
\leq 0. \quad (7.60)

Keeping \( n \) fixed in (7.60), letting \( m = 1, 2, \ldots, M \) and adding the inequalities we have

\[ \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \delta_{mn}^p \leq \left(\frac{p}{p-1}\right)^q \sum_{m=1}^{M} m^{-q} \omega_{mn}^q b_{mn}^p \delta_{mn}^{p-1} \]

\leq \left\{ \sum_{m=1}^{M} \left( m^{-q/p} \omega_{mn}^{q/p} b_{mn}^p \right)^{p/(p-1)} \right\} \left( \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \delta_{mn}^p \right)^{1-1/p} \quad (7.61)

so that

\[ \sum_{m=1}^{M} m^{-q} \omega_{mn}^q \delta_{mn}^p \leq \left(\frac{p}{p-1}\right)^q \sum_{m=1}^{M} m^{-q} \omega_{mn}^q b_{mn}^p. \quad (7.62) \]

From (7.57) and (7.62) we observe that

\[ S_{\text{ML}} \leq \left(\frac{p}{p-1}\right)^{3p} \left( \frac{p+q}{p+q-1} \right)^p \sum_{m=1}^{M} \sum_{n=1}^{L} b_{mn}^p \omega_{mn}^q \]

\[ = \left(\frac{p}{p-1}\right)^{3p} \left( \frac{p+q}{p+q-1} \right)^p \sum_{m=1}^{M} \sum_{n=1}^{L} b_{mn}^p B_{mn}^q. \quad (7.63) \]

By letting \( L, M \) tend to infinity in (7.63) we get the desired inequality (7.33).

The proof is complete.

If we set \( q = 0 \) in Theorem 7.2.15, we recover [107, Theorem 2].

7.2.8 3–component, 2–level

The following result extends the preceding theorem in a similar way that Theorem 7.2.7 extends Theorem 7.2.2.
Theorem 7.2.16. Let \( p > 1, q \geq 0, r \geq 0, b_{mn} > 0 \) for \( m, n \in \mathbb{N} \) and let \( B_{mn} \) be given by (7.32). If \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q} B_{mn}^{r} \) converges, then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p} B_{mn}^{q+r} \leq \left( \frac{p}{p-1} \right)^{3(p+q)} \left( \frac{p + q + r}{p + q + r - 1} \right)^{p+q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q} B_{mn}^{r}. \tag{7.64}
\]

Proof. Using Hölder’s inequality with indices \((p + q)/p\) and \((p + q)/q\), we get

\[
\sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p} B_{mn}^{q+r} = \sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p+q} B_{mn}^{r} \left( \frac{p}{p+q+r} \right)^{p+q} \sum_{m=1}^{M} \sum_{n=1}^{L} b_{mn}^{p+q} B_{mn}^{r} \left( \frac{q}{p+q+r} \right)^{q+q}.
\]

Then

\[
\sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p+q} B_{mn}^{r} \left( \frac{p}{p-1} \right)^{3p+q} \left( \frac{p + q + r}{p + q + r - 1} \right)^{p} \sum_{n=1}^{L} \sum_{m=1}^{M} b_{mn}^{p+q} B_{mn}^{r} \left( \frac{q}{p+q+r} \right)^{q+q}.
\]

Raising both sides to the power \((p + q)/p\) yields (7.64).

Corollary 7.2.17. If we set \( r = 0 \) in Theorem 7.2.16 we obtain

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p} B_{mn}^{q} \leq \left( \frac{p}{p-1} \right)^{3(p+q)} \left( \frac{p + q}{p + q - 1} \right)^{p+q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q}. \tag{7.66}
\]
7.3 Extension to higher dimensions

In [107] Pachpatte notes that Theorem 7.2.14 may be easily extended to higher dimensions. Our extension is

**Theorem 7.3.1.** If \( p > 1 \) is a constant, \( b_{mn} \geq 0 \) for \( m, n \in \mathbb{N} \) and

\[
B_{mn} = \frac{1}{mn} \sum_{s=1}^{m} \sum_{t=1}^{n} \frac{1}{st} \sum_{x=1}^{s} \sum_{y=1}^{t} b_{xy} \quad \text{for} \quad m, n, s, t, x, y \in \mathbb{N},
\]

\[
D_{uv} = \frac{1}{uv} \sum_{q=1}^{u} \sum_{r=1}^{v} \frac{1}{qr} \sum_{m=1}^{q} \sum_{n=1}^{r} B_{mn} \quad \text{for} \quad q, r, u, v \in \mathbb{N},
\]

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn}^{p} \leq \left( \frac{p}{p-1} \right)^{8p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p}.
\]  \hspace{1cm} (7.67)

The equality holds in (7.67) if \( b_{mn} = 0 \) for \( m, n \in \mathbb{N} \).

**Proof.** Applying Theorem 7.2.14 twice we have

\[
\sum_{u=1}^{\infty} \sum_{v=1}^{\infty} D_{uv}^{p} \leq \left( \frac{p}{p-1} \right)^{4p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}^{p} \leq \left( \frac{p}{p-1} \right)^{8p} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p}.
\]

An extension of our Theorem 7.2.16 with partitioned exponent is

**Theorem 7.3.2.** Let \( p > 1, q \geq 0, r \geq 0, s \geq 0, b_{mn} > 0 \) for \( m, n \in \mathbb{N} \) and let \( B_{mn}, D_{mn} \) be as in Theorem 7.3.1. If \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q+r} B_{mn}^{s} \) converges, then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p} D_{mn}^{p+q+r+s} \leq \left( \frac{p}{p-1} \right)^{3(p+q)} \left( \frac{p+q}{p+q-1} \right)^{3(p+q+r)} \cdot \left( \frac{p+q+r+s}{p+q+r+s-1} \right)^{2(p+q)+r} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q+r} B_{mn}^{s}.
\]  \hspace{1cm} (7.68)

**Proof.** Applying (7.64) with \( r \) replaced by \( r + s \)

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p} D_{mn}^{p+q+r+s} \leq \left( \frac{p}{p-1} \right)^{3(p+q)} \left( \frac{p+q+r+s}{p+q+r+s-1} \right)^{p+q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q+r+s} B_{mn}^{s}.
\]  \hspace{1cm} (7.69)

Applying (7.64) to the right hand side of (7.69) with \( p, q, \) and \( r \) replaced by \( p + q, r \) and \( s \) respectively we get (7.68).

We also have
Corollary 7.3.3. Let $p > 1$, $q \geq 0$, $r \geq 0$, $b_{mn} > 0$ for $m, n \in \mathbb{N}$ and let $B_{mn}$, $D_{mn}$ be as in Theorem 7.3.1. If $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q+r}$ converges, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p} D_{mn}^{q+r} \leq \left( \frac{p}{p-1} \right)^{3(p+q)} \left( \frac{p+q}{p+q-1} \right)^{3(p+q+r)} \cdot \left( \frac{p+q+r}{p+q+r-1} \right)^{p+q} \left( \frac{p+q+r}{p+q+r-1} \right)^{p+q+r} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn}^{p+q+r}. \quad (7.70)$$

Proof. Let $s = 0$ in Theorem 7.3.2.
Chapter 8

Hardy’s discrete inequality – II

8.1 Summary

In this chapter Hwang’s many–variable discrete Hardy’s inequality [70, Theorem 1] and [53, Theorem 3] are generalised as follows.

(i) Hwang’s notation is re–designed to eliminate ambiguities and describe summation processes more precisely. Hwang’s theorems are proved more rigorously and we then proceed to extensions. The bound $\sigma_j = \inf_{x_j} \rho_i(x_j)$ arising from the separable weight function $\beta_i(x_j) \lambda_i(x_j)$ plays an essential rôle in the proof.

(ii) We take the new Hwang and Hanjiˇs–Pearce–Peˇcari´c 2–level, 1–exponent inequalities and partition the exponent $p$ into components $p + q$, $p + q + r$ and $p_1 + p_2 + \cdots + p_n$ in order to utilise the tremendous power of Hölder’s inequality described by Rado (loc. cit.). The components are combined to create novel Hölder exponents and new 2–level inequalities with multiple components.

(iii) The 2–level inequality for partitioned exponents is extended to a 3–level inequality by using a new set of conjugate exponents with generalised Hölder’s inequality.

(iv) The convergence of the series is examined, filling a gap in [69] using the techniques of Chapter 7.

(v) Worked examples are given of 2–level, 2–component type inequalities to exhibit the summation processes, orders of magnitude and determination of the

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1 $p$ may differ from the exponent $p$ customarily used in Hölder’s inequality.

2 See Definition 7.2.1 of a $k$–level inequality.
constants \( \mu_i \). We explore the effects of the following behaviours of \( \beta_i(x_j) \) as \( x_j \to \infty \) on the convergence of the infinite series on the right hand side of the inequality: monotonic increasing convergence, monotonic increasing divergence, finite and infinite oscillation and oscillating convergence. The theoretical reasons for the above effects on \( \sigma_j = \inf_{x_j} \rho_i(x_j) \) are proposed for further research, and the effects, if any, of convexity, \( \alpha \)–submultiplicativity and other properties of \( \beta_i(x_j) \).\(^3\)

8.2 Introduction

Much of the literature is concerned with finding sharper constants in terms of known functions for inequalities with specific weight functions. Our infimum principle does not presuppose that the best possible constant, if it exists, will be expressible in terms of known functions.

Hwang’s algebraically somewhat complicated proofs are an extension of Elliott’s simplest proof \cite{33} of Hardy’s Theorem 1.2.5. His weight functions are more general than Pachpatte’s and the theorems and proofs are accordingly more complex than Elliott’s original proof.

One of our aims was to generalise \cite{70} simultaneously to an exponent split into multiple components and to levels greater than 2.

Our new method generates the geometric mean of two series on the right hand side whereas the previous inequalities have only one series on the right hand side. Thus although we succeed in our aim, the new inequalities are of a different character.

The resulting series exhibit some parallels with the integral inequalities of Chapter 5 which contain products of integrals on the right hand side such as Theorem 5.3.3.

8.3 Notation and Preliminaries for Chapter 8

The notation in \cite{70} suffers from ambiguities. It is desirable to clarify the meaning of expressions such as \( \sum_{x_i=1}^{n} \lambda_i(x_i), \sum_{x_i=1}^{n} \sum_{y_j=1}^{z_i} \lambda_i(y_j) \) and \( \sum_{i=1}^{n} \sum_{y_j=1}^{z_i} \lambda_i(y_j) \).

\(^3\)A. Kufner–L. E. Persson discuss monotonicity conditions for kernels and other characterisations of more general weight functions \cite[pp. 77 et seq.]{77}. 
Hwang’s proof depends on a telescoping series

\[
\frac{\Lambda_n(x_n)}{\beta_n(x_n)} \omega_p^n(x_n) - \frac{\Lambda_n(x_n + 1)}{\beta_n(x_n + 1)} \omega_p^n(x_n + 1)
\]  

(8.1)

which generalises Elliott’s telescoping series [33]. \( \Lambda_n(x_n + 1) \) is undefined in Hwang’s notation and we are not assured that his series actually telescopes. In order to prove Hwang’s theorem more rigorously and proceed to extensions, we re-define \( \Lambda_n(x_n + 1) \), \( \beta_n(x_n + 1) \) and \( \omega_n(x_n + 1) \) precisely and set out proofs in detail. For instance it was found that our inability to construct a telescoping series

\[
\frac{\Lambda_n(x_n)}{\beta_n(x_n)} \omega_q^n(x_n) - \frac{\Lambda_n(x_n + 1)}{\beta_n(x_n + 1)} \omega_q^n(x_n + 1)
\]

(8.2)

or show that each term in (8.2) is negative precludes a proof by Elliott’s method of a 3-level inequality with two components \( p, q \).

8.3.1 Existing Notation

First we survey the notation used by Hwang [70] and Hanjš, Pearce, and Pečarić [53].

Hwang’s notation

Hwang’s notation [70, p. 126] is as follows. ‘Let \( B \) be a subset of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) defined by \( B = x \in \mathbb{R}^n : e \leq x < \infty \) where \( e = (1, 1, \ldots, 1) \in \mathbb{R}^n \). For a function \( u : B \to \mathbb{R} \), we use the following notations

\[
\sum_{B} u(y) = \sum_{y_1=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} u(y_1, y_2, \ldots, y_n).
\]  

(8.3)

and

\[
\sum_{B(e, x)} u(y) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} u(y_1, y_2, \ldots, y_n).
\]  

(8.4)

where \( e = (1, \ldots, 1) \in B \), \( x = (x_1, x_2, \ldots, x_n) \in B \) such that \( e \leq x \), i.e. \( 1 \leq x_i \).

Hwang’s result

Hwang’s result\(^4\) [70, Theorem 1], in his notation, is

\(^4\)Theorem 8.3.1 will be presented again, slightly modified and with revised notation, as Theorem 8.4.4.
Theorem 8.3.1. If \( p > 1, a(x) > 0, \beta_i(x_i) > 0, \lambda_i(x_i) > 0, i \in \{1, 2, \ldots, n\} \) for \( x_i \geq 1, \) and
\[
\sum_{\mathcal{B}} \prod_{i=1}^{n} \lambda_i(x_i) a^p(x), \ x \in \mathcal{B}
\]
converge, and further let
\[
\Lambda_i(x_i) = \sum_{y_i=1}^{x_i} \beta_i(y_i) \lambda_i(y_i), i \in \{1, 2, \ldots, n\}
\] (8.5)
and
\[
A(x) = \sum_{\mathcal{B}(e,x)} \left( \prod_{i=1}^{n} \beta_i(y_i) \lambda_i(y_i) \right) a(y), \ x \in \mathcal{B}.
\] (8.6)
If there exist \( \kappa_i > 0 \) such that
\[
p - 1 + \frac{[\beta_i(x_i + 1) - \beta_i(x_i)] \Lambda_i(x_i)}{\beta_i(x_i + 1) \beta_i(x_i) \lambda_i(x_i)} \geq \frac{p}{\kappa_i}
\] (8.7)
for \( x_i \geq 1, i = 1, 2, \ldots, n, \) then
\[
\sum_{\mathcal{B}} \prod_{i=1}^{n} \lambda_i(x_i) \left( \frac{A(x)}{\prod_{i=1}^{n} \Lambda_i(x_i)} \right)^p \leq \left( \prod_{i=1}^{n} \kappa_i \right)^p \sum_{\mathcal{B}} \prod_{i=1}^{n} \lambda_i(x_i) a^p(x).
\] (8.8)
The equality holds in (8.8) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\}. \)

We give generalisations of these results. Hwang’s variables \( \beta_i(x_i) \lambda_i(x_i) \) and \( a(x) \) replace the variables \( \lambda_n \) and \( a_n \) respectively of Chapter 7. Hwang’s inequality appears to be more general than Pachpatte’s [106] but we show that Pachpatte’s inequality cannot be expressed as a special case of Hwang’s. Hwang’s \( \mathcal{B}(e,x) \) defined in [70] is the same as \( \mathcal{C}_n(x) \) used in [53].

8.3.2 Hanjš, Pearce and Pečarić’s notation

The notation of Hanjš, Pearce, and Pečarić [53] goes a considerable way towards rectifying Hwang’s and is sufficient for their induction proof of Theorem 8.3.1. Their notation is:

Let
\[
\mathcal{B}_n := \{ x = (x_1, \ldots, x_n) \in \mathbb{J}_+: e \leq x < \infty \}
\] (8.9)
where \( e = (1, 1, \ldots, 1) \in \mathbb{J}_+, \) (i.e. \( 1 \leq x_i \)) and \( u : \mathcal{B}_n \rightarrow \mathbb{R}, \) and
\[
\sum_{\mathcal{B}_n} u(y) := \sum_{y_1=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} u(y_1, \ldots, y_n)
\]
8.3. NOTATION AND PRELIMINARIES FOR CHAPTER 8

\[ C_n(x) := \{ y = (y_1, \ldots, y_n) : y \in \mathbb{B}_n, y_i \leq x_i \ (i = 1, \ldots, n) \}, \quad (8.10) \]

\[ \sum_{C_n(x)} u(y) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \sum_{y_n=1}^{x_n} u(y_1, \ldots, y_n). \quad (8.11) \]

In order to discover new theorems and rectify proofs of existing ones a redesigned notation is required.

8.3.3 New notation

The specific aims of our new notation are to

(i) Eliminate ambiguities through precision.

(ii) Show overall relationships by compactness.

(iii) Describe the summation processes carefully for the purpose of constructing examples and modelling.

Some special forms are set out for ease of reference. As an example of our new notation we have two simple forms for (8.8):

\[ \sum_{\mathbb{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) \left( \frac{\omega_n(x_1, \ldots, x_n)}{\prod_{i=1}^{n-1} \lambda_i(x_i)} \right)^p \leq \left( \prod_{i=1}^{n} \kappa_i \right)^p \sum_{\mathbb{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) a^p(x_1, \ldots, x_n) \]

\[ = \left( \prod_{i=1}^{n} \kappa_i \right)^p \sum_{\mathbb{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) \Omega_p(x_1, \ldots, x_n). \quad (8.12) \]

\[ \sum_{\mathbb{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) \Omega_p^n(x_1, \ldots, x_n) \leq \left( \prod_{i=1}^{n} \kappa_i \right)^p \sum_{\mathbb{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) \Omega_0^n(x_1, \ldots, x_n). \quad (8.13) \]

Our new definitions are as follows.

**Definition 8.3.2.** \( C_n(x) \) defined in (8.10) is replaced by

\[ C_{n-j}(x_1, \ldots, x_{n-j} + s, x_{n-j+1}, \ldots, x_n) \]

\[ := \{ y = (y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) : y \in \mathbb{B}_n, y_i \leq x_i \ (i = 1, \ldots, n-j) \}. \quad (8.14) \]

If \( j = 0, \ n = 1 \) then \( C_1(x_1, \ldots, x_n) = \{ y = (y_1) = y_1 : y_1 \leq x \}. \)
Definition 8.3.3. Let \( u : \mathbb{B}_n \to \mathbb{R} \). We replace \( \sum_B \) in (8.3) and \( \sum_{B(x,n)} \) in (8.4) by

\[
\sum_{\mathbb{B}_n} u(y) := \sum_{y_1=1}^{\infty} \cdots \sum_{y_n-1=1}^{\infty} \sum_{y_n=1}^{\infty} u(y_1, \ldots, y_n), \tag{8.15}
\]

and

\[
\sum_{\mathbb{C}_{n-j}(x_1, \ldots, x_{n-j+s}, x_{n-j+1}, \ldots, x_n)} u(y) := \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=1}^{x_{n-j+s}} u(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) = \sum_{\mathbb{C}_{n-j}(x_1, \ldots, x_{j}, \ldots, x_n)} u(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) + \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=x_{n-j}+1}^{x_{n-j+s}} u(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n), \tag{8.16}
\]

\( j \in \{0, 1, 2, \ldots, n\}, s \in \mathbb{N}. \)

We note the distinction between \( x_{n-j+s} \) and \( x_{n-j} + s \).

(i) If \( j = 0, s = 1 \),

\[
\sum_{\mathbb{C}_{n}(x_1, \ldots, x_{n+1})} u(y) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \sum_{y_n=1}^{x_n+1} u(y_1, \ldots, y_{n-1}, y_n) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \sum_{y_n=1}^{x_n} u(y_1, \ldots, y_{n-1}, y_n) + \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} u(y_1, \ldots, y_{n-1}, x_n + 1) \tag{8.17}
\]

(ii) If \( j = n-1, s = 0 \),

\[
\sum_{\mathbb{C}_{j}(x_1, \ldots, x_n)} u(y) = \sum_{y_1=1}^{x_1} u(y_1, x_2, \ldots, x_n). \tag{8.18}
\]

Definition 8.3.4. Hwang’s \( \beta_i(x_i) \) and \( \lambda_i(x_i) \) may be regarded as functions of the two variables \( i \) and \( x_i \) if \( i \) is uncoupled from \( x_i \) and as separable components of the weight function \( \lambda_i(x_j) \beta_i(x_j) \).
Hwang’s kernel \( \prod_{k=1}^{n} \beta_k(y_k)\lambda_k(y_k) \) is replaced by
\[ \prod_{k=1}^{n} \beta_k(y_k)\lambda_k(y_k), \quad 0 \leq j \leq n - 1. \] We define
\[ \lambda_i(0) = 1, \quad \beta_i(0) = 1, \quad i \in \{1, 2, \ldots, n\}. \] (8.19)

**Definition 8.3.5.** \( \Lambda_i(x_i) \) defined in (8.5) is replaced by
\[ \Lambda_n(x_n + s) := \sum_{y_n = 1}^{x_{n-j+s}} \beta_m(y_{n-j})\lambda_m(y_{n-j}), \quad m, s \in \mathbb{N}, i, j, k \in \{0, 1, 2, \ldots, n\}, \]
\[ = \sum_{y_{n-j}=1}^{x_{n-j}} \beta_m(y_{n-j})\lambda_m(y_{n-j}) + \sum_{y_{n-j}=x_{n-j}+1}^{x_{n-j+s}} \beta_m(y_{n-j})\lambda_m(y_{n-j}). \] (8.20)

(i) If \( s = j = 0 \) in (8.20)
\[ \Lambda_m(x_n) = \sum_{y_n=1}^{x_n} \beta_m(y_n)\lambda_m(y_n). \] (8.21)

(ii) If \( m = 1, n = 1 \) then \( \Lambda_1(x_1) = \sum_{y_1=1}^{x_1} \beta_1(y_1)\lambda_1(y_1) \). We recover the \( \Lambda_n \) defined in (7.3) by replacing \( y_1 \) by \( i \), \( x_1 \) by \( n \), \( \beta_1(y_1) \) by \( \beta_i \) and \( \lambda_1(y_1) \) by \( \lambda_i \).

(iii) If \( m = n + 1 \) in (8.21)
\[ \Lambda_n(x_n + 1) = \sum_{y_n = 1}^{x_{n+1}} \beta_n(y_n)\lambda_n(y_n) = \sum_{y_n = 1}^{x_n} \beta_n(y_n)\lambda_n(y_n) + \beta_n(y_n + 1)\lambda_n(y_n + 1) \]
\[ = \Lambda_n(x_n) + \beta_n(y_n + 1)\lambda_n(y_n + 1). \] (8.22)

(iv) If \( j = n - 1 \) and \( s = 0 \) in (8.20), \( \Lambda_1(x_1) = \sum_{y_1=1}^{x_1} \beta_1(y_1)\lambda_1(y_1) \).

(v) We define
\[ \Lambda_i(0) = 0 \quad \text{for} \quad i \in \{1, 2, \ldots, n\}. \] (8.23)

**Definition 8.3.6.** Hwang’s \( A(x) \) is a function of \( (x_1, \ldots, x_n) \) and of the number of summations, which may be less than \( n \). (8.6) may be written
\[ A(x) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} \left\{ \prod_{i=1}^{n} \beta_i(y_i)\lambda_i(y_i) \right\} a(y_1, \ldots, y_n), \quad x \in \mathbb{B}. \] (8.24)

We replace \( A(x) \) by the more general
\[ A_{n-j}(x_1, \ldots, x_{n-j} + s, x_{n-j+1}, \ldots, x_n) \]
\[ \sum_{C_{n-j}(x_1, \ldots, x_{n-j+s}, x_{n-j+1}, \ldots, x_n)} \{ \prod_{k=1}^{n-j} \beta_k(y_k) \lambda_k(y_k) \} a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) \]

\[ = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=1}^{x_{n-j}} \{ \prod_{k=1}^{n-j} \beta_k(y_k) \lambda_k(y_k) \} a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) \tag{8.25} \]

\( j \in \{0, 1, 2, \ldots, n-1\} \), \( s \in \mathbb{N} \), where \( n-j \) denotes the number of summations.

This also removes the ambiguity whereby Hwang’s \( A(x) \) can be interpreted as either \( A_n(x_1, \ldots, x_n) \) or \( A_n(x_1, \ldots, x_n + 1) \).

(i) If \( s = 0 \),

\[ A_{n-j}(x_1, \ldots, x_{n-j}, x_{n-j+1}, \ldots, x_n) \]

\[ = \sum_{C_{n-j}(x_1, \ldots, x_{n-j}, x_{n-j+1}, \ldots, x_n)} \{ \prod_{k=1}^{n-j} \beta_k(y_k) \lambda_k(y_k) \} a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) \]

\[ = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=1}^{x_{n-j}} \{ \prod_{k=1}^{n-j} \beta_k(y_k) \lambda_k(y_k) \} a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n). \tag{8.26} \]

(ii) If \( j = 0 = s \),

\[ A_n(x_1, \ldots, x_n) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} \{ \prod_{k=1}^{n} \beta_k(y_k) \lambda_k(y_k) \} a(y_1 \ldots, y_n). \tag{8.27} \]

If also \( n = 1 \),

\[ A_1(x_1) = \sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1) a(y_1). \tag{8.28} \]

(iii) If \( j = 1, s = 0 \),

\[ A_{n-1}(x_1, \ldots, x_n) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \{ \prod_{k=1}^{n-1} \beta_k(y_k) \lambda_k(y_k) \} a(y_1 \ldots, y_{n-1}, x_n). \tag{8.29} \]

(iv) If \( j = 0, s = 1 \),

\[ A_n(x_1, \ldots, x_n + 1) = \sum_{C_{n}(x_1, \ldots, x_n+1)} \{ \prod_{i=1}^{n} \beta_i(y_i) \lambda_i(y_i) \} a(y_1, \ldots, y_n) \]

\[ = A_n(x_1, \ldots, x_n) \]
\[ + \beta_n(x_n + 1)\lambda_n(x_n + 1) \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n + 1). \]

(v) If \( j = n - 1, s = 0, \)
\[
A_1(x_1, \ldots, x_n) = \sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1) a(y_1, x_2, \ldots, x_n). \quad (8.31)
\]

(vi) We define \( A_0(x_1, \ldots, x_n) = a(x_1, \ldots, x_n). \)

Definition 8.3.7. We define
\[
\Omega_{n-j}(x_1, \ldots, x_n) := \frac{A_{n-j}(x_1, \ldots, x_n) + s, x_{n-j+1}, \ldots, x_n)}{\prod_{k=1}^{n-j} \Lambda_k(x_k)}
\]
\[
= \sum_{c_{n-j}(x_1, \ldots, x_{n-j} + s, x_{n-j+1}, \ldots, x_n)} \frac{\left\{ \prod_{k=1}^{n-j} \beta_k(y_k) \lambda_k(y_k) \right\} a(y_1, y_2, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n)}{\prod_{k=1}^{n-j} \Lambda_k(x_k)}
\]
\[
= \sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1) \cdots \sum_{y_{n-j}=1}^{x_{n-j}} \beta_{n-j}(y_{n-j}) \lambda_{n-j}(y_{n-j})
\cdot \left\{ \sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1) \cdots \sum_{y_{n-j}=1}^{x_{n-j}} \beta_{n-j}(y_{n-j}) \lambda_{n-j}(y_{n-j}) \right\}
\cdot \left\{ \prod_{k=1}^{n-j} \beta_k(y_k) \lambda_k(y_k) \right\} a(y_1, y_2, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n). \quad (8.32)
\]

(i) If \( j = s = 0, \)
\[
\Omega_n(x_1, \ldots, x_n) = \frac{A_n(x_1, \ldots, x_n)}{\prod_{k=1}^{n} \Lambda_k(x_k)}. \quad (8.33)
\]

(ii) If \( j = 0, n = 1, s = 0, \)
\[
\Omega_1(x_1) = \frac{A_1(x_1)}{\Lambda_1(x_1)} = \frac{a(x_1)}{\sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1)}. \quad (8.34)
\]

(iii) We define \( \Omega_0(x_1, \ldots, x_n) = A_0(x_1, \ldots, x_n) = a(x_1, \ldots, x_n)|_{n=0} = a. \)
**Definition 8.3.8.** Hwang [70, p. 127] defines \( a_{n-j}(x_{n-j}) \) (it should be \( \alpha_{n-j}(x_{n-j}) \)) by:

\[
a_{n-j}(x_{n-j}) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=1}^{x_{n-j}} \left\{ \prod_{i=1}^{n-j} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) \Lambda_{n-j}^{-1}(x_{n-j}).
\]

(8.35)

Authors since Hardy, Littlewood and Pólya have used \( \alpha \) to denote expressions of this type. It is a source of errors due to the visual similarity between \( \alpha \) and \( a \). We replace \( \alpha_{n-j}(x_{n-j}) \) defined in (8.35) by

\[
\omega_{n-j}(x_1, \ldots, x_{n-j} + s, x_{n-j+1}, \ldots, x_n) := \frac{A_{n-j}(x_1, \ldots, x_{n-j} + s, x_{n-j+1}, \ldots, x_n)}{\Lambda_{n-j}(x_{n-j} + s)}
\]

\[
= \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-j}=1}^{x_{n-j}+s} \left\{ \prod_{i=1}^{n-j} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n)
\]

\[
= \frac{\sum_{y_{n-j}=1}^{x_{n-j}+s} \beta_m(y_{n-j}) \lambda_m(y_{n-j})}{\sum_{y_{n-j}=1}^{x_{n-j}+s} \beta_m(y_{n-j}) \lambda_m(y_{n-j})}
\]

(8.36)

for \( j \in \{0, 1, 2, \ldots, n-1\} \), \( s \in \mathbb{N} \) and \( x_{n-j} \geq 1 \).

(i) If \( s = 0 \),

\[
\omega_{n-j}(x_1, \ldots, x_{n-j}, x_{n-j+1}, \ldots, x_n) = \frac{A_{n-j}(x_1, \ldots, x_{n-j}, x_{n-j+1}, \ldots, x_n)}{\Lambda_{n-j}(x_{n-j})}
\]

\[
= \left\{ \sum_{y_{n-j}=1}^{x_{n-j}} \left( \prod_{i=1}^{n-j} \beta_i(y_i) \lambda_i(y_i) \right) a(y_1, \ldots, y_{n-j}, x_{n-j+1}, \ldots, x_n) \right\}
\]

\[
\cdot \Lambda_{n-j}^{-1}(x_{n-j}).
\]

(8.37)

for \( j \in \{0, 1, 2, \ldots, n-1\} \), \( s \in \mathbb{N} \) and \( x_{n-j} \geq 1 \).

(ii) If \( j = s = 0 \),

\[
\omega_{n}(x_1, \ldots, x_n) = \frac{A_{n}(x_1, \ldots, x_n)}{\Lambda_{n}(x_{n})}
\]

\[
= \frac{\sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \sum_{y_n=1}^{x_n} \{ \prod_{i=1}^{n} \beta_i(y_i) \lambda_i(y_i) \} a(y_1, \ldots, y_n)}{\sum_{y_{n-1}=1}^{x_{n-1}} \beta_m(y_{n-1}) \lambda_m(y_{n-1})}
\]

(8.38)

(iii) If \( j = 1, s = 0 \),

\[
\omega_{n-1}(x_1, \ldots, x_n) = \frac{A_{n-1}(x_1, \ldots, x_n)}{\Lambda_{n-1}(x_{n-1})}
\]
8.4. EXTENDED HARDY–HWANG INEQUALITIES

\[ \frac{\sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n)}{\sum_{y_{n-1}=1}^{x_{n-1}} \beta_{n-1}(y_{n-1}) \lambda_{n-1}(y_{n-1})} \]  
(8.39)

(iv) If \( j = n - 1, s = 0 \),

\[ \omega_1(x_1, \ldots, x_n) = \frac{A_1(x_1, \ldots, x_n)}{A_1(x_1)} = \frac{\sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1) a(y_1, x_2, \ldots, x_n)}{\sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1)}. \]  
(8.40)

(v) If \( n = 1, j = s = 0 \),

\[ \omega_1(x_1) = \frac{A_1(x_1)}{A_1(x_1)} = \frac{\sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1) a(y_1, x_2, \ldots, x_n)}{\sum_{y_1=1}^{x_1} \beta_1(y_1) \lambda_1(y_1)}. \]  
(8.41)

\( \omega_1(x_1) \) may be identified with \( \omega_n \) defined in (7.12) by replacing \( x_1 \) by \( n \) and dropping the subscript 1.

(vi) We have

\[ \Omega_{n-j}(x_1, \ldots, x_{n-j}+s, x_{n-j+1}, \ldots, x_n) = \frac{\omega_{n-j}(x_1, \ldots, x_{n-j}+s, x_{n-j+1}, \ldots, x_n)}{\prod_{i=1}^{n-j-1} \Lambda_i(x_i)}. \]  
(8.42)

(vii) We define

\[ \omega_n(x_1, \ldots, x_j, \ldots, x_n) = 0 \quad \text{if one of the } x_j = 0. \]  
(8.43)

We now prove some theorems.

8.4 Extended Hardy–Hwang inequalities

8.4.1 2–component, 2–level

The first theorem generalises [70, equation (12)] to two components, providing a more complete statement and a basis for subsequent theorems.

**Theorem 8.4.1.** Let \( p > 1, q \geq 0, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0 \), for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathbb{B}_n, x = (x_1, \ldots, x_n) \in \mathbb{J}_+ \) and \( e \leq x, i.e. 1 \leq x_i \). Suppose that

\[ \sum_{x_n=1}^{N} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^p(x_1, \ldots, x_n), \quad x \in \mathbb{B}_n \]
converges. Let
\[ \rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)] \Lambda_m(x_n)}{\beta_m(x_n + 1) \beta_m(x_n) \lambda_m(x_n)} \]  
(8.44)

for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \), and
\[ \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p - q, \]  
(8.45)

\[ \kappa_m = \frac{p + q}{p + q + \sigma_m - 1}, \]

\[ \mu_m = \max \left\{ \kappa_m, \frac{p + q}{p + q - 1} \right\}. \]  
(8.46)

Then
\[ \sum_{x_n=1}^{\infty} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \leq \mu_n^{p+q} \sum_{x_n=1}^{\infty} \lambda_n(x_n) \Lambda_{n-1}^p(x_1, \ldots, x_n) \omega_n^q(x_1, \ldots, x_n) \]
\[ = \mu_n^{p+q} \sum_{x_n=1}^{\infty} \lambda_n(x_n) \Lambda_{n-1}^p(x_1, \ldots, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n). \]  
(8.47)

The equality holds in (8.47) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\} \).

Proof. First we sum Elliott’s telescoping series. We use (8.38) and (B.1); and define \( \Lambda_i(0) = 0 \) for \( i \in 1, 2, \ldots, n \). We have for \( x_n = 0, 1, 2, \ldots \)
\[
- (p + q) \lambda_n(x_n + 1) \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n + 1) \omega_n^{p+q-1}(x_1, \ldots, x_n + 1)
= -(p + q) \beta_n(x_n + 1) \lambda_n(x_n + 1) \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n + 1) \omega_n^{p+q-1}(x_1, \ldots, x_n + 1)
\]
\[
= -(p + q) \left\{ \omega_n(x_1, \ldots, x_n + 1) \Lambda_n(x_n + 1) - \omega_n(x_1, \ldots, x_n) \Lambda_n(x_n) \right\} \frac{\omega_n^{p+q-1}(x_1, \ldots, x_n + 1)}{\beta_n(x_n + 1)}
\]
\[
= \frac{1}{\beta_n(x_n + 1)} \left\{ -(p + q) \Lambda_n(x_n + 1) \omega_n^{p+q}(x_1, \ldots, x_n + 1)
+ (p + q) \Lambda_n(x_n) \omega_n(x_1, \ldots, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n + 1) \right\}
\]
\[ \leq \frac{1}{\beta_n(x_n + 1)} \left\{ -(p + q)\Lambda_n(x_n + 1)\omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
+ \Lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) + (p + q - 1)\Lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n + 1) \right\}. \]

Then
\[
\frac{[\beta_n(x_n + 1) - \beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n + 1)\beta_n(x_n)\lambda_n(x_n)} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) \\
+ (p + q - 1)\lambda_n(x_n + 1)\omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
- (p + q)\lambda_n(x_n + 1) \cdot \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \\
\cdot \left\{ \prod_{i=1}^{n-1} \beta_i(y_i)\lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n + 1)\omega_n^{p+q-1}(x_1, \ldots, x_n + 1) \\
\leq \frac{[\beta_n(x_n + 1) - \beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n + 1)\beta_n(x_n)\lambda_n(x_n)} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) \\
+ (p + q - 1)\lambda_n(x_n + 1)\omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
- (p + q)\frac{\Lambda_n(x_n + 1)}{\beta_n(x_n + 1)} \omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
+ (p + q - 1)\frac{\Lambda_n(x_n)}{\beta_n(x_n + 1)} \omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
= (p + q - 1)\frac{\Lambda_n(x_n + 1)}{\beta_n(x_n + 1)} \omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
+ \frac{\Lambda_n(x_n)}{\beta_n(x_n)} \omega_n^{p+q}(x_1, \ldots, x_n) - (p + q)\frac{\Lambda_n(x_n + 1)}{\beta_n(x_n + 1)} \omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
= \frac{\Lambda_n(x_n)}{\beta_n(x_n)} \omega_n^{p+q}(x_1, \ldots, x_n) - \frac{\Lambda_n(x_n + 1)}{\beta_n(x_n + 1)} \omega_n^{p+q}(x_1, \ldots, x_n + 1) \quad (8.48) \]

using (8.22) and (8.30). Summing (8.48) from \( x_n = 0 \) to \( x_n = N - 1 \),
\[
(p + q - 1) \sum_{x_n=0}^{N-1} \lambda_n(x_n + 1)\omega_n^{p+q}(x_1, \ldots, x_n + 1) \\
+ \sum_{x_n=0}^{N-1} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) \frac{[\beta_n(x_n + 1) - \beta_n(x_n)]\Lambda_n(x_n)}{\beta_n(x_n + 1)\beta_n(x_n)\lambda_n(x_n)} \\
- (p + q) \sum_{x_n=0}^{N-1} \lambda_n(x_n + 1) \cdot \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \\
\cdot \left\{ \prod_{i=1}^{n-1} \beta_i(y_i)\lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n + 1)\omega_n^{p+q-1}(x_1, \ldots, x_n + 1) \]
\[
\sum_{x_n=0}^{N-1} \left\{ \frac{\Lambda_n(x_n)}{\beta_n(x_n)} \omega_n^{p+q}(x_1, \ldots, x_n) - \frac{\Lambda_n(x_n+1)}{\beta_n(x_n+1)} \omega_n^{p+q}(x_1, \ldots, x_n+1) \right\}. \tag{8.49}
\]

Using (8.19), (8.23) and (8.43)

\[
(p + q - 1) \sum_{x_n+1=1}^{N} \lambda_n(x_n+1) \omega_n^{p+q}(x_1, \ldots, x_n+1) \\
+ \sum_{x_n=0}^{N-1} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \left[ \frac{\beta_n(x_n+1) - \beta_n(x_n)}{\beta_n(x_n+1) \beta_n(x_n)} \Lambda_n(x_n) \right] \\
- (p + q) \sum_{x_n+1=1}^{N} \lambda_n(x_n+1) \sum_{y_1}^{x_n+1} \cdots \sum_{y_{n-1}}^{x_n-1} \\
\cdot \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_{n-1}, x_n+1) \omega_n^{p+q-1}(x_1, \ldots, x_n+1) \\
\leq \frac{\Lambda_n(0)}{\beta_n(0)} \omega_n^{p+q}(x_1, \ldots, x_{n-1}, 0) - \frac{\Lambda_n(N)}{\beta_n(N)} \omega_n^{p+q}(x_1, \ldots, N) \\
\leq 0 \tag{8.50}
\]

since the first term in the second–last line is zero. Replacing \(x_n + 1\) by \(x_n\) as the variable of summation in the first and third terms

\[
(p + q - 1) \sum_{x_n=1}^{N} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \\
+ \sum_{x_n=0}^{N-1} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \left[ \frac{\beta_n(x_n+1) - \beta_n(x_n)}{\beta_n(x_n+1) \beta_n(x_n)} \Lambda_n(x_n) \right] \\
\leq (p + q) \sum_{x_n=1}^{N} \lambda_n(x_n) \sum_{y_1}^{x_n} \cdots \sum_{y_{n-1}}^{x_n-1} \\
\cdot \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} \\
\cdot a(y_1, \ldots, y_{n-1}, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n) \\
= (p + q) \sum_{x_n=1}^{N} \lambda_n(x_n) \Lambda_n-1(x_1, \ldots, x_n) a(y_1, \ldots, y_{n-1}, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n). \tag{8.51}
\]

We may verify that (8.48) is valid for \(x_n = 0\) by using (8.38). Various authors use different assumptions here. Elliott says ‘it being understood that any number with suffix 0 is zero’ while Copson says ‘write \(\Lambda_0 = 0\’). Hwang–Yang use (8.38)
and stipulate that \( x_n \geq 1 \). Using (8.44)

\[
(p + q - 1) \sum_{x_n=1}^{N} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) + \sum_{x_n=1}^{N-1} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n)\rho_n(x_n)
\]

\[
\leq (p + q) \sum_{x_n=1}^{N} \lambda_n(x_n)A_n-1(x_1, \ldots, x_n)\omega_n^{p+q-1}(x_1, \ldots, x_n).
\]  

(8.52)

We find a lower estimate of the left–hand side of (8.52). Using (8.46)

\[
(p + q - 1) \sum_{x_n=1}^{N} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) + \sum_{x_n=1}^{N-1} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n)\rho_n(x_n)
\]

\[
= (p + q - 1)\lambda_n(N)\omega_n^{p+q}(x_1, \ldots, N)
\]

\[
+ \sum_{x_n=1}^{N-1} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n)\{p + q - 1 + \rho_n(x_n)\}
\]

\[
\geq \frac{p + q}{\mu_n} \sum_{x_n=1}^{N} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n).
\]  

(8.53)

Combining this with (8.52) and following a similar procedure to that in Theorem 7.2.2 we have

\[
\sum_{x_n=1}^{N} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n)
\]

\[
\leq \mu_n \sum_{x_n=1}^{N} \lambda_n(x_n)A_n-1(x_1, \ldots, x_n)\omega_n^{p+q-1}(x_1, \ldots, x_n).
\]  

(8.54)

Applying Hölder’s inequality with indices \( p \) and \( p/(p - 1)^5 \)

\[
\sum_{x_n=1}^{N} \lambda_n(x_n)\omega_n^{p+q}(x_1, \ldots, x_n) \leq \mu_n \sum_{x_n=1}^{N} \left( \lambda_n^{(p-1)/p}(x_n)\omega_n^{(p-1)(1+q/p)}(x_1, \ldots, x_n) \right)
\]

\[
\cdot \left( \lambda_n^{1/p}(x_n)A_n-1(x_1, \ldots, x_n)\omega_n^{q/p}(x_1, \ldots, x_n) \right)
\]

\[
\leq \mu_n \left\{ \sum_{x_n=1}^{N} \left( \lambda_n^{(p-1)/p}(x_n)A_n-1(x_1, \ldots, x_n)\omega_n^{q/p}(x_1, \ldots, x_n) \right)^{1/p} \right\}
\]

\[
\cdot \left\{ \sum_{x_n=1}^{N} \left( \lambda_n^{(p-1)/p}(x_n)\omega_n^{(p-1)(1+q/p)(x_1, \ldots, x_n)} \right)^{(p-1)/p} \right\}^{1/(p-1)/p}
\]

\(^5q\) differs from the \( q \) customarily used in \( 1/p + 1/q = 1 \) for Hölder’s inequality.
\[ \mu_n \left\{ \sum_{x_n=1}^{N} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^q(x_1, \ldots, x_n) \right\}^{1/p} \]

\[ \cdot \left\{ \sum_{x_n=1}^{N} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \right\}^{(p-1)/p} \quad . \quad (8.55) \]

Dividing by the last factor on the right

\[ \left\{ \sum_{x_n=1}^{N} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \right\}^{1/p} \]

\[ \leq \mu_n \left\{ \sum_{x_n=1}^{N} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^q(x_1, \ldots, x_n) \right\}^{1/p} \quad . \quad (8.56) \]

Raising both sides to the power \( p \), we get

\[ \sum_{x_n=1}^{N} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \]

\[ \leq \mu_n^p \sum_{x_n=1}^{N} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^q(x_1, \ldots, x_n) . \quad (8.57) \]

This proves that the series

\[ \sum_{x_n=1}^{N} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \]

converges. In view of (8.55) and (8.56) the series

\[ \sum_{x_n=1}^{N} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n) \]

also converges. Returning to (8.53) and observing that the term

\[ (p + q - 1) \lambda_n(N) \omega_N^{p+q}(x_1, \ldots, x_n) \]

is nonnegative and using (8.52), we see that

\[ \frac{p + q}{\mu_n} \sum_{x_n=1}^{N-1} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \]
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\[ \leq \sum_{x_n=1}^{N-1} \{p + q - 1 + \rho_n(x_n)\} \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \]

\[ \leq (p + q) \sum_{x_n=1}^N \lambda_n(x_n) A_{n-1}(x_1, \ldots, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n). \quad (8.58) \]

Since the series on both sides converge, we have

\[ p + q \mu_n \sum_{x_n=1}^\infty \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \]

\[ \leq \mu_n \sum_{x_n=1}^\infty \lambda_n(x_n) A_{n-1}(x_1, \ldots, x_n) \omega_n^{p+q-1}(x_1, \ldots, x_n). \quad (8.59) \]

Applying Hölder’s inequality to the right hand side in a similar way to the above for finite sums, we obtain

\[ \sum_{x_n=1}^\infty \lambda_n(x_n) \omega_n^{p+q}(x_1, \ldots, x_n) \leq \mu_n^p \sum_{x_n=1}^\infty \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^q(x_1, \ldots, x_n). \quad (8.60) \]

8.4.2 1–exponent, 2–level

Setting \( q = 0 \) in (8.47) we have a more complete version of [70, equation (12)] with a proof that is now more soundly based.

**Theorem 8.4.2.** Let \( p > 1, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0 \), for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathbb{B}_n, x = (x_1, \ldots, x_n) \in J_+ \) and \( e \leq x \), i.e. \( 1 \leq x_i \). Suppose that \( \sum_{x_n=1}^\infty \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \) converges. Let

\[ \rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)] \Lambda_m(x_n)}{\beta_m(x_n + 1) \beta_m(x_n) \lambda_m(x_n)} \]

for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \), \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p \) and

\[ \kappa_m = \frac{p}{p + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{p}{p - 1} \right\}. \]

Then

\[ \sum_{x_n=1}^\infty \lambda_n(x_n) \omega_n^p(x_1, \ldots, x_n) \leq \mu_m^p \sum_{x_n=1}^\infty \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \]
= \mu_p^{n} \sum_{x_n=1}^{\infty} \lambda_n(x_n) \Lambda_{n-1}^p (x_{n-1}) \omega_{n-1}^p (x_1, \ldots, x_n). \quad (8.61)

The equality holds in (8.61) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\} \).

**Remark 8.4.3.** In the proof of [70, Theorem 1], Hwang uses the inequality

\[
\sum_{x_n=1}^{N-1} \lambda_n(x_n) \omega_n^p (x_1, \ldots, x_n) \leq \kappa \sum_{x_n=1}^{N} \lambda_n(x_n) \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \left\{ \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right\} \cdot a(y_1, \ldots, y_{n-1}, x_n) \omega_{n-1}^p (x_1, \ldots, x_n), \quad N \in \mathbb{N}, \quad (8.62)
\]

(for a positive \( \kappa \)) to prove the desired inequality by taking the limit as \( N \to \infty \).

The convergence of the two series is not established in [70]. This fact is nontrivial and depends on the relation between the two series.

**8.4.3 \( 1 \)-exponent, \( (n+1) \)-level**

The following is a more complete version of Hwang's Theorem 8.3.1 with a proof that is now more soundly based.

**Theorem 8.4.4.** Let \( p > 1 \), \( a(x_1, \ldots, x_n) > 0 \), \( \beta_i(x_i) > 0 \), \( \lambda_i(x_i) > 0 \), \( i \in \{1, 2, \ldots, n\} \) for \( x_i \geq 1 \), and

\[
\sum \prod_{i=1}^{n} \lambda_i(x_i) a^p (x_1, \ldots, x_n), \quad x \in \mathbb{B}_n
\]

converge. If there exist \( \kappa_i > 0 \) such that

\[
p - 1 + \frac{[\beta_i(x_i+1) - \beta_i(x_i)] \Lambda_i(x_i)}{\beta_i(x_i+1) \beta_i(x_i) \lambda_i(x_i)} \geq \frac{p}{\kappa_i}
\]

for \( x_i \geq 1 \), \( i = 1, 2, \ldots, n \), and \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p \),

\[
\kappa_m = \frac{p}{p + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{p}{p - 1} \right\}.
\]

Then

\[
\sum \prod_{i=1}^{n} \lambda_i(x_i) \left( \frac{A_n(x_1, \ldots, x_n)}{\prod_{k=1}^{n} \Lambda_k(x_k)} \right)^p \leq \left( \prod_{i=1}^{n} \mu_i \right)^p \sum \prod_{i=1}^{n} \lambda_i(x_i) a^p (x_1, \ldots, x_n)
\]

(8.63)
or more compactly
\[
\sum_{\mathcal{B}_n} \prod_{i=1}^{n} \lambda_i(x_i)^{\Omega_n^p(x_1, \ldots, x_n)} \leq \left( \prod_{i=1}^{n} \mu_i \right)^p \sum_{\mathcal{B}_n} \prod_{i=1}^{n} \lambda_i(x_i)^{\Omega_n^p(x_1, \ldots, x_n)}. \tag{8.64}
\]

The equality holds in (8.63) if \(a(x_1, \ldots, x_n) = 0\) for all \(x_i, i \in \{1, 2, \ldots, n\}\).

**Proof.** Using (8.15), (8.30), (8.40), (8.61) and Fubini’s theorem we have
\[
\sum_{\mathcal{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) \left\{ \frac{A_n(x_1, \ldots, x_n)}{\prod_{k=1}^{n} \Lambda_k(x_k)} \right\}^p
= \left\{ \prod_{k=1}^{n-1} \Lambda_k(x_k) \right\}^{-p} \sum_{\mathcal{B}_n} \prod_{i=1}^{n} \lambda_i(x_i)^{\omega_n^p(x_1, \ldots, x_n)}
\leq \sum_{x_1=1}^{\infty} \cdots \sum_{x_{n-1}=1}^{\infty} \prod_{i=1}^{n-1} \lambda_i(x_i) \left\{ \prod_{k=1}^{n-1} \Lambda_k(x_k) \right\}^{-p} \mu_n^n \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n)
= \mu_n^n \sum_{x_1=1}^{\infty} \cdots \sum_{x_{n-2}=1}^{\infty} \prod_{i=1}^{n-2} \lambda_i(x_i) \left\{ \prod_{k=1}^{n-2} \Lambda_k(x_k) \right\}^{-p}
\cdot \sum_{x_{n-1}=1}^{\infty} \lambda_{n-1}(x_{n-1}) A_{n-1}^{-p}(x_{n-1}) A_{n-2}^p(x_1, \ldots, x_n). \tag{8.65}
\]

Now by following exactly the same arguments as in the proof of Theorem 8.4.1 we have, using (8.20) and (8.65) and Fubini’s theorem,
\[
\sum_{\mathcal{B}_n} \prod_{i=1}^{n} \lambda_i(x_i) \left\{ \frac{A_n(x_1, \ldots, x_n)}{\prod_{k=1}^{n} \Lambda_k(x_k)} \right\}^p
\leq (\mu_{n-1} \mu_n)^p \sum_{x_{n-1}=1}^{\infty} \sum_{x_n=1}^{\infty} \lambda_{n-1}(x_{n-1}) A_n(x_n)
\cdot \sum_{x_1=1}^{\infty} \cdots \sum_{x_{n-2}=1}^{\infty} \prod_{k=1}^{n-2} \lambda_k(x_k) \left\{ \prod_{j=1}^{n-2} \Lambda_j(x_j) \right\}^{-p} A_{n-2}(x_1, \ldots, x_n). \tag{8.66}
\]

Continuing in this way, we finally obtain (8.63). The proof of the theorem is complete.

**8.4.4 3–component, 2–level**

We now prove three–component versions of [70, equation (12)] and Theorems 8.4.1 and 8.4.2.
The equality holds in (8.69) Theorem 8.4.5. Let \( p > 1, q > 0, r \geq 0, a(x_1,\ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0, \) for all \( i, j \in \{1,2,\ldots,n\} \) where \( e = (1,1,\ldots,1) \in \mathbb{B}_e, x = (x_1,\ldots, x_n) \in \mathbb{J}_+, \) and \( e \leq x, i.e. 1 \leq x_i. \)

Suppose that

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n), \quad x \in B_n
\]  

(8.67)

converges. Let

\[
\rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)] \Lambda_m(x_n)}{\beta_m(x_n + 1) \beta_m(x_n) \lambda_m(x_n)}
\]

(8.68)

for \( x_i \geq 1, i \in \{1,2,\ldots,n\} \) and \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p - q - r, \)

\[
\kappa_m = \frac{p + q + r}{p + q + r + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{p + q + r}{p + q + r - 1} \right\}.
\]

Then

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n)
\]

\[
\leq \left( \frac{p + q + r}{p + q + r - 1 + \sigma_n} \right)^q \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n).
\]  

(8.69)

The equality holds in (8.69) if \( a(x_1,\ldots, x_n) = 0 \) for all \( x_i, i \in \{1,2,\ldots,n\}. \)

Proof. We apply Hölder’s inequality with indices \((p+q)/p\) and \((p+q)/q\) (for this reason we need to assume \( q > 0 \) rather than \( q \geq 0 \) as we did in Theorem 8.4.1) and then apply Theorem 8.4.1 with \( q \) replaced by \( q + r. \)

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n)
\]

\[
= \sum_{x_n=1}^{\infty} \left( \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n) \right)^{p/(p+q)} \left( \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n) \right)^{q/(p+q)}
\]

\[
\leq \left( \sum_{n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n) \right)^{p/(p+q)}
\]

\[
\cdot \left( \sum_{n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1,\ldots, x_n) \omega_n^r(x_1,\ldots, x_n) \right)^{q/(p+q)}
\]
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\[
\leq \left( \sum_{n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_n(x_1, \ldots, x_n) \right)^{p/(p+q)}
\cdot \left( \sum_{n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_n^{q+r}(x_1, \ldots, x_n) \right)^{q/(p+q)}
\]

by Theorem 7.2.2, that is,

\[
\leq \kappa_n \left( \sum_{n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_n(x_1, \ldots, x_n) \right)^{p/(p+q)}
\]

Raising both sides of (8.70) to the power \((p+q)/p\) yields (8.69). The case \(\beta_n \in N, q = 0, \kappa = (p+q+r)/(p+q+r-1)\) shows the constant in (8.70) to be the best possible.

Theorem 7.2.7 is a special case of the preceding theorem:

Corollary 8.4.6.

Proof. In Theorem 8.4.5 letting \(n = 1\) we obtain

\[
\sum_{x_1=1}^{\infty} \lambda_1(x_1) a^y \omega_1(x_1) \leq \left( \frac{p + q + r}{p + q + r - 1 + \sigma_1} \right)^q \sum_{x_1=1}^{\infty} \lambda_1(x_1) a^{p+q}(y_1) \omega_1(x_1).
\]

(8.71)

Write \(x_1 = n, y_1 = i\), and then replace \(a(y_1)\) by \(a_i\), \(\lambda_1(y_1)\) by \(\lambda_n\), \(A_1(n)\) by \(A_n\), \(\Lambda_1(n)\) by \(\Lambda_n\), \(\sigma_1\) by \(\sigma\) and let \(\beta_1(y_1) \equiv 1\). Then

\[
\omega_n = \frac{A_n}{\Lambda_n} = \frac{\sum_{i=1}^{n} \lambda_i a_i}{\sum_{i=1}^{n} \lambda_i}
\]

and (8.71) becomes

\[
\sum_{n=1}^{\infty} \lambda_n a_n^p \left( \frac{\sum_{i=1}^{n} \lambda_i a_i}{\sum_{i=1}^{n} \lambda_i} \right)^{q+r} \leq \left( \frac{p + q + r}{p + q + r - 1 + \sigma} \right)^q \sum_{n=1}^{\infty} \lambda_n a_n^{p+q} \left( \frac{\sum_{i=1}^{n} \lambda_i a_i}{\sum_{i=1}^{n} \lambda_i} \right)^r
\]

which is (7.19).
8.4.5 $n$–component, 2–level

Theorem 8.4.7. Let $p = p_1 + \cdots + p_i$, $p_1 > 1$, $q = p_{i+1} + \cdots + p_{i+j} > 0$, $r = p_{i+j+1} + \cdots + p_{i+j+k} \geq 0$, $p_i > 0$, $i \in \mathbb{N}\setminus\{1\}$, $a(x_1, \ldots, x_n) > 0$, $\beta_i(x_j) > 0$, $\lambda_i(x_j) > 0$, for all $i, j \in \{1, 2, \ldots, n\}$ where $e = (1, 1, \ldots, 1) \in \mathbb{B}_n$, $x = (x_1, \ldots, x_n) \in \mathbb{J}_+$ and $e \leq x$, i.e. $1 \leq x_i$.

Suppose that

$$
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p_{i+1} + \cdots + p_i + p_{i+j} + \cdots + p_{i+j+k}} (x_1, \ldots, x_n) \omega_n^{p_{i+j+1} + \cdots + p_{i+j+k}} (x_1, \ldots, x_n) \quad (8.73)
$$

converges. Let

$$
\rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)] \Lambda_m(x_n)}{\beta_m(x_n + 1) \beta_m(x_n) \lambda_m(x_n)} \quad (8.74)
$$

for $x_i \geq 1$, $i \in \{1, 2, \ldots, n\}$ and $\sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - (p_{i+1} + \cdots + p_{i+j})$,

$$
\kappa_m = \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} - 1} \right\}.
$$

Then

$$
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p_{i+1} + \cdots + p_i} (x_1, \ldots, x_n) \omega_n^{p_{i+1} + \cdots + p_{i+j} + \cdots + p_{i+j+k}} (x_1, \ldots, x_n)
$$

\begin{align*}
&\leq \left( \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} - 1} \right)^{p_{i+1} + \cdots + p_{i+j}} \\
&\cdot \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p_{i+1} + \cdots + p_i} (x_1, \ldots, x_n) \omega_n^{p_{i+j+1} + \cdots + p_{i+j+k}} (x_1, \ldots, x_n)
\end{align*}

\begin{align*}
&= \left( \frac{p_{i+1} + \cdots + p_{i+j}}{p_{i+1} + \cdots + p_{i+j} - 1} \right)^{p_{i+1} + \cdots + p_{i+j}} \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1} (x_n) \\
&\cdot \omega_{n-1}^{p_{i+1} + \cdots + p_{i+j}} (x_1, \ldots, x_n) \omega_n^{p_{i+j+1} + \cdots + p_{i+j+k}} (x_1, \ldots, x_n).
\end{align*}

(8.75)

The equality holds in (8.75) if $a(x_1, \ldots, x_n) = 0$ for all $x_i$, $i \in \{1, 2, \ldots, n\}$.

Proof.

Let $p = p_1 + \cdots + p_i$, $p_1 > 1$, $q = p_{i+1} + \cdots + p_{i+j} > 0$, $r = p_{i+j+1} + \cdots + p_{i+j+k} \geq 0$ in Theorem 8.4.5.
8.4.6 Another 2–component, 2–level theorem

Setting \( r = 0 \) in Theorem 8.4.5, we get

**Theorem 8.4.8.** Let \( p > 1, q > 0, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0, \) for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathbb{B}_n, x = (x_1, \ldots, x_n) \in \mathbb{J}_+ \) and \( e \leq x, i.e. \ 1 \leq x_i. \) Suppose that

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{+q}(x_1, \ldots, x_n), \quad x \in \mathbb{B}_n
\] (8.76)

converges. Let

\[
\rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)] \Lambda_m(x_n)}{\beta_m(x_n + 1) \beta_m(x_n) \lambda_m(x_n)}
\]

for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \) and \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p - q, \)

\[
\kappa_m = \frac{p + q}{p + q + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{p + q}{p + q - 1} \right\}.
\]

Then

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^p(x_1, \ldots, x_n) \omega_n^q(x_1, \ldots, x_n)
\]

\[
\leq \left( \frac{p + q}{p + q - 1 + \sigma_n} \right)^q \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n).
\] (8.77)

**Proof.** Set \( r = 0 \) in (8.69).

8.5 Multi–component, multi–level inequalities

8.5.1 2–component, 3–level

We apply generalised Hölder’s inequality to obtain three levels of summation. A new combination of Hölder exponents results in the geometric mean of series on the right hand side whereas the inequalities previously considered have a single series on the right hand side.

**Theorem 8.5.1.** Let \( p > 1, q > 1, r \geq 0, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0, \) for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathbb{B}_n, x = (x_1, \ldots, x_n) \in \mathbb{J}_+ \) and \( e \leq x, i.e. \ 1 \leq x_i. \) Suppose that

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_n^r(x_1, \ldots, x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n),
\] (8.78)
CHAPTER 8. HARDY’S DISCRETE INEQUALITY – II

\begin{align*}
\sum_{x_n=1}^\infty \lambda_n(x_n)\omega_n^{q+r}(x_1, \ldots, x_n)A_{n-1}^p(x_1, \ldots, x_n) & \quad (8.79) \\
\sum_{x_n=1}^\infty \lambda_n(x_n)\omega_n^q(x_1, \ldots, x_n)A_{n-2}^p(x_{n-2})\omega_{n-2}^p(x_1, \ldots, x_n) & \quad (8.80)
\end{align*}

and converge. Let

\[ \rho_m(x_n) = \left[ \frac{\beta_m(x_n + 1) - \beta_m(x_n)}{\beta_m(x_n + 1)\beta_m(x_n)\lambda_m(x_n)} \right] \Lambda_m(x_n) \]  

\[ (8.81) \]

for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \),

\[ \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - q, \quad \kappa_m = \frac{q}{q + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{q}{q - 1} \right\}. \]

Then

\begin{align*}
\sum_{x_n=1}^\infty & \lambda_n(x_n)\omega_n^{q+r}(x_1, \ldots, x_n)A_{n-1}^p(x_1, \ldots, x_n) \\
\leq & \left( \frac{p}{p-1} \right)^{p/2} \left( \frac{q}{q-1} \right)^q \\
& \cdot \left\{ \sum_{x_n=1}^\infty \lambda_n(x_n)A_{n-1}^{p+q}(x_{n-1})\omega_n^{q+r}(x_1, \ldots, x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n) \right\}^{1/2} \\
& \cdot \left\{ \sum_{x_n=1}^\infty \lambda_n(x_n)\omega_n^q(x_1, \ldots, x_n)A_{n-2}^p(x_{n-2})\omega_{n-2}^p(x_1, \ldots, x_n) \right\}^{1/2}. \quad (8.82)
\end{align*}

The equality holds in (8.82) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\} \).

**Proof.** In Theorem 8.4.7 let \( i = 1, j = 1 \) and \( k = 2 \) and (8.75) becomes

\begin{align*}
\sum_{x_n=1}^\infty & \lambda_n(x_n)\omega_n^{q+r}(x_1, \ldots, x_n)A_{n-1}^p(x_1, \ldots, x_n) \\
\leq & \left( \frac{q}{q-1} \right)^q \sum_{x_n=1}^\infty \lambda_n(x_n)\omega_n^r(x_1, \ldots, x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n). \quad (8.83)
\end{align*}

Using (8.83) the series on the right hand side is equal to

\begin{align*}
\sum_{x_n=1}^\infty & \lambda_n(x_n)\omega_n^r(x_1, \ldots, x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n) \\
& \cdot A_{n-1}^{-(p+q)/2q}(x_1, \ldots, x_n)A_{n-1}^{(p+q)/2q}(x_1, \ldots, x_n).
\end{align*}
\begin{align*}
&= \sum_{x_n=1}^{\infty} \lambda_n(x_n)\omega^r_{n}(x_1, \ldots, x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n) \\
&\quad \cdot A_n^{-\frac{p+q}{2q}}(x_1, \ldots, x_n)A_{n-1}^{\frac{(p+q)/2q}{(x_n-1)}\omega_{n-1}^{(p+q)/2q}(x_1, \ldots, x_n)} \\
&= \sum_{x_n=1}^{\infty} \left( \lambda_{n-1}^{q-1}/q(x_n)A_{n-1}^{p+q}(q-1)/q(x_1, \ldots, x_n) \right) \\
&\quad \cdot \omega_n^{(q-1)/q}(x_1, \ldots, x_n) \left( \lambda_n^{1/2q}(x_n)A_{n-1}^{(p+q)/2q}(x_n-1) \right) \\
&\quad \cdot A_n^{(p+q)/2q}(x_1, \ldots, x_n)\omega_n^{r/q}(x_1, \ldots, x_n) \left( \lambda_n^{1/2q}(x_n)\omega_{n-1}^{(p+q)/2q}(x_1, \ldots, x_n) \right).
\end{align*}

We apply generalised Hölder’s inequality (B.18) with indices \(q/(q-1), 2q, 2q\).

\[\sum_{x_n=1}^{\infty} \lambda_n(x_n)\omega^r_{n}(x_1, \ldots, x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n) < \left( \frac{q}{q-1} \right)^q\]

\[\cdot \left\{ \sum_{x_n=1}^{\infty} \left( \lambda_{n-1}^{q-1}/q(x_n)A_{n-1}^{p+q}(q-1)/q(x_1, \ldots, x_n) \right)^{(q-1)/q} \right\}^{1/2q}\]

\[\cdot \left\{ \sum_{x_n=1}^{\infty} \left( \lambda_n^{1/2q}(x_n)A_{n-1}^{(p+q)/2q}(x_n-1) \right) \right\}^{1/2q}\]

\[= \left( \frac{q}{q-1} \right)^q \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n)\omega_n^{r}(x_1, \ldots, x_n) \right\}^{(q-1)/q} \]

\[\cdot \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_{n-1}^{2r}(x_1, \ldots, x_n) \right\}^{1/2q}\]

\[\cdot \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)\omega_{n-1}^{p+q}(x_1, \ldots, x_n) \right\}^{1/2q}. \quad (8.84)\]

Dividing both sides of (8.84) by the first sum on the right–hand side

\[\left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)\omega_{n-1}^{p+q}(x_1, \ldots, x_n) \right\}^{1/q}\]

\[< \left( \frac{q}{q-1} \right)^q \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)\omega_{n-1}^{p+q}(x_1, \ldots, x_n) \right\}^{1/2q}.\]
Raising both sides to the power $2q$

$$
\left( \sum_{x_n=1}^{\infty} \lambda_n(x_n) \omega_{n-1}^r(x_1, \ldots, x_n) \right)^2 < \left( \frac{q}{q-1} \right)^2 \left( \sum_{x_n=1}^{\infty} \lambda_n(x_n) \omega_{n-1}^p(x_1, \ldots, x_n) \right)
\cdot \left( \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_{n-1}^{2r}(x_1, \ldots, x_n) \right). \quad (8.86)
$$

Applying Theorem 8.4.2 to the first series on the right hand side

$$
\left( \sum_{x_n=1}^{\infty} \lambda_n(x_n) \omega_{n-1}^r(x_1, \ldots, x_n) \right)^2 < \left( \frac{p}{p-1} \right)^p \left( \frac{q}{q-1} \right)^{2q^2}
\cdot \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_{n-1}^{2r}(x_1, \ldots, x_n)
\cdot \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-2}^{p}(x_2, \ldots, x_n) \omega_{n-2}^p(x_1, \ldots, x_n)
\cdot \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{q}(x_1, \ldots, x_n) \omega_{n-1}^q(x_1, \ldots, x_n). \quad (8.87)
$$

from which (8.82) follows.

### 8.6 Special cases

Interchanging the rôles of $p$ and $q$ in the final application of Theorem 8.4.2 yields a parallel inequality.

**Corollary 8.6.1.**

$$
\sum_{x_n=1}^{\infty} \lambda_n(x_n) \omega_{n-1}^r(x_1, \ldots, x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) < \left( \frac{q}{q-1} \right)^{q(q+1/2)}
\cdot \left( \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-1}^{p+q}(x_1, \ldots, x_n) \omega_{n-1}^{2r}(x_1, \ldots, x_n) \right)^{1/2}
\cdot \left( \sum_{x_n=1}^{\infty} \lambda_n(x_n) A_{n-2}^{q}(x_2, \ldots, x_n) \omega_{n-2}^q(x_1, \ldots, x_n) \right)^{1/2}. \quad (8.88)
$$
8.6. SPECIAL CASES

Since \( p > 1, q > 1 \) in (8.87) we are unable to recover earlier theorems by letting \( p = 0 \) or \( q = 0 \). The structure of this inequality is different due to the different Hölder exponents.

**Corollary 8.6.2.** Let \( p > 1, q > 1, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0 \), for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathcal{B}_n, x = (x_1, \ldots, x_n) \in \mathcal{J}_+ \) and \( e \leq x \), i.e. \( 1 \leq x_i \). Suppose that

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{p+q}(x_{n-1})A_{n-1}^{p+q}(x_1, \ldots, x_n) \tag{8.89}
\]

and

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-2}^{p}(x_{n-2})\omega_{n-2}^{p}(x_1, \ldots, x_n)\omega_{n-1}^{q}(x_1, \ldots, x_n) \tag{8.90}
\]

converge. Let

\[
\rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)]A_m(x_n)}{\beta_m(x_n + 1)\beta_m(x_n)\lambda_m(x_n)} \tag{8.91}
\]

for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \), \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - q \),

\[
\kappa_m = \frac{q}{q + \sigma_m - 1}, \quad \mu_m = \max\left\{ \kappa_m, \frac{q}{q - 1} \right\}.
\]

Then

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{p+q}(x_1, \ldots, x_n) < \left( \frac{p}{p - 1} \right)^{p/2} \left( \frac{q}{q - 1} \right)^{q^2} \tag{8.92}
\]

The equality holds in (8.92) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\} \).

**Proof.** Let \( r = 0 \) in (8.87).

**Corollary 8.6.3.** Let \( p > 1, r \geq 0, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0 \), for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathcal{B}_n, x = (x_1, \ldots, x_n) \in \mathcal{J}_+ \) and \( e \leq x \), i.e. \( 1 \leq x_i \). Suppose that

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{2p}(x_{n-1})A_{n-1}^{2p}(x_1, \ldots, x_n) \tag{8.93}
\]
and
\[ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-2}^p(x_{n-2})\omega_{n-2}^p(x_1, \ldots, x_n)\omega_{n-1}^p(x_1, \ldots, x_n) \] (8.94)

converge. Let
\[ \rho_m(x_n) = \left[ \beta_m(x_n + 1) - \beta_m(x_n) \right] \Lambda_m(x_n) \]
(8.95)
for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \), \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p \),
\[ \kappa_m = \frac{p}{p + \sigma_m - 1}, \quad \mu_m = \max \left\{ \kappa_m, \frac{p}{p - 1} \right\}. \]

Then
\[ \sum_{x_n=1}^{\infty} \lambda_n(x_n)\omega_n^{2p}(x_1, \ldots, x_n)A_{n-1}^{2p}(x_1, \ldots, x_n) < \left( \frac{p}{p - 1} \right)^{p(p+1/2)} \]
\[ \cdot \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{2p}(x_{n-1})A_{n-1}^{2p}(x_1, \ldots, x_n)\omega_n^{2p}(x_1, \ldots, x_n) \right\}^{1/2} \]
\[ \cdot \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-2}^p(x_{n-2})\omega_{n-2}^p(x_1, \ldots, x_n)\omega_{n-1}^p(x_1, \ldots, x_n) \right\}^{1/2}. \] (8.96)

The equality holds in (8.96) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\} \).

**Proof.** Let \( q = p \) in (8.87) or (8.88).

**Corollary 8.6.4.** Let \( p > 1, a(x_1, \ldots, x_n) > 0, \beta_i(x_j) > 0, \lambda_i(x_j) > 0, \) for all \( i, j \in \{1, 2, \ldots, n\} \) where \( e = (1, 1, \ldots, 1) \in \mathbb{B}_n, x = (x_1, \ldots, x_n) \in \mathbb{J}_+ \) and \( e \leq x \), i.e. \( 1 \leq x_i \). Suppose that
\[ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-1}^{2p}(x_{n-1})A_{n-1}^{2p}(x_1, \ldots, x_n) \] (8.97)
and
\[ \sum_{x_n=1}^{\infty} \lambda_n(x_n)A_{n-2}^p(x_{n-2})\omega_{n-2}^p(x_1, \ldots, x_n)\omega_{n-1}^p(x_1, \ldots, x_n) \] (8.98)

converge. Let
\[ \rho_m(x_n) = \frac{[\beta_m(x_n + 1) - \beta_m(x_n)]\Lambda_m(x_n)}{\beta_m(x_n + 1)\beta_m(x_n)\lambda_m(x_n)} \] (8.99)
for \( x_i \geq 1, i \in \{1, 2, \ldots, n\} \), \( \sigma_m = \inf_{x_n} \rho_m(x_n) > 1 - p \),

\[
\kappa_m = \frac{p}{p + \sigma_m - 1}; \quad \mu_m = \max \left\{ \kappa_m, \frac{p}{p - 1} \right\}.
\]

Then

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) A^{2p}_{n-1}(x_1, \ldots, x_n)
\]

\[
< \left( \frac{p}{p-1} \right)^{p(p+1/2)} \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n) A^{2p}_{n-1}(x_n) A^{2p}_{n-1}(x_1, \ldots, x_n) \right\}^{1/2}
\]

\[
\cdot \left\{ \sum_{x_n=1}^{\infty} \lambda_n(x_n) A^{p}_{n-2}(x_n-2) \omega^{p}_{n-2}(x_1, \ldots, x_n) \omega^{p}_{n-1}(x_1, \ldots, x_n) \right\}^{1/2}.
\]

(8.100)

The equality holds in (8.100) if \( a(x_1, \ldots, x_n) = 0 \) for all \( x_i, i \in \{1, 2, \ldots, n\} \).

**Proof.** Let \( r = 0 \) in (8.96).

### 8.7 Examples

The examples test and illustrate Theorem 8.4.1 for 2–component, 2–level inequalities. The effects of the following behaviours of \( \beta_i(x_j) \) as \( x_j \to \infty \) on \( \sigma_i = \inf_{x_j} \rho_i(x_j) \) are examined: monotonic increasing convergence, monotonic increasing divergence, finite and infinite oscillation and oscillating convergence. In the examples considered \( \sigma_i \) exists where \( \beta_i(x_j) \) is a monotonic increasing\(^6\) function of \( x_j \). In the examples where \( \beta_i(x_j) \) oscillates \( \sigma_i \) fails to exist whether the oscillation be finite, infinite or damped. In that case (8.45) is not satisfied and the inequality is not in the class of suitably convergent series on the right hand side of the inequality.

First we display all summations in Theorem 8.4.1:

\[
\sum_{x_n=1}^{\infty} \lambda_n(x_n) \left\{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} \left( \prod_{i=1}^{n} \beta_i(y_i) \lambda_i(y_i) \right) a(y_1, \ldots, y_n) \right\}^{p+q} \sum_{y_n=1}^{\infty} \beta_n(y_n) \lambda_n(y_n)
\]

\[
\leq \mu_n^{p} \sum_{x_n=1}^{\infty} \lambda_n(x_n) \left\{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} \left( \prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) \right) a(y_1, \ldots, y_{n-1}, x_n) \right\}^{p}
\]

\(^6\) c.f. the monotonicity conditions in (5.15) and (B.13).
\[
\left( \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} \left\{ \prod_{i=1}^{n} \beta_i(y_i) \lambda_i(y_i) \right\} a(y_1, \ldots, y_n) \right)^q \left( \sum_{y_n=1}^{x_n} \beta_n(y_n) \lambda_n(y_n) \right)^p.
\]

(8.101)

**Example 8.7.1.** Here \( \beta_i(x_j) \) is an unbounded monotonic increasing function of \( x_j \). In (8.101) let \( n = 2, \ x_1 = 2, \ p = 2, \ q = 2, \ a(x_1, x_2) = x_1^{-1} x_2^{-1}, \ \lambda_i(x_j) = x_j. \) Let \( \beta_i(x_j) = x_j / \kappa. \) Then we have

\[
\sum_{x_2=1}^{\infty} x_2 \left( \sum_{y_1=1}^{2} \sum_{y_2=1}^{x_2} \left\{ \prod_{i=1}^{2} y_i^2 \right\} y_1^{-1} y_2^{-1} \right)^4 \leq \mu^2 \sum_{x_2=1}^{\infty} \left( \sum_{y_1=1}^{2} \sum_{y_2=1}^{x_2} \left\{ \prod_{i=1}^{2} y_i^2 \right\} y_1^{-1} y_2^{-1} \right)^2 \sum_{y_2=1}^{x_2} \frac{1}{y_2^2}.
\]

(8.102)

which reduces to

\[
\sum_{x_2=1}^{\infty} \frac{x_2}{(2x_2 + 1)^4} \leq \left( \frac{\mu^2}{3} \right)^2 \sum_{x_2=1}^{\infty} \frac{1}{x_2(2x_2 + 1)^2}.
\]

(8.103)

Using (B.5)

\[
\Lambda_i(x_i) = \sum_{y_i=1}^{x_i} \beta_i(y_i) \lambda_i(y_i) = 1^2 + 2^2 + \cdots + x_i^2 = \frac{1}{6} x_i(x_i + 1)(2x_i + 1)
\]

\[
\Lambda_1(x_1) = 1^2 + 2^2 = 5, \ \Lambda_2(x_2) = 1^2 + 2^2 + \cdots + x_2^2 = \frac{1}{6} x_2(x_2 + 1)(2x_2 + 1),
\]

\[
\prod_{i=1}^{2} \Lambda_i(x_i) = \frac{5}{6} x_2(x_2 + 1)(2x_2 + 1) \sim \frac{5x_2^3}{3} \text{ as } x_i \to \infty.
\]

Let us check the equivalent of (8.7) which is

\[
p + q + \frac{\{ \beta_i(x_i + 1) - \beta_i(x_i) \} \Lambda_i(x_i)}{\beta_i(x_i + 1) \beta_i(x_i) \lambda_i(x_i)} \geq \frac{p + q}{\kappa_i}.
\]

(8.104)

In general \( \beta_i(x_i) \) and \( \lambda_i(x_i) \) are functions of both \( i \) and \( x_i \), which are uncoupled, and consequently so is \( \kappa_i \). Here

\[
\beta_i(x_j + 1) - \beta_i(x_j) = 1 \quad \text{and} \quad \beta_i(x_j + 1) \beta_i(x_j) \lambda_i(x_j) = (x_j + 1)x_j^2
\]

Letting \( i = j = 2 \) and solving for \( \kappa_2 \) where this is a function of \( x_2 \) only

\[
\kappa_2 \geq \frac{24x_2}{26x_2 + 1} = \frac{12}{13 + (1/2x_2)} \sim \frac{12}{13}
\]
as \( x_2 \to \infty \). From (8.44) with \( i = j = 2 \) we have

\[
\rho_2(x_2) = \frac{1}{3} + \frac{1}{6x_2}, \quad \sigma_2 = \inf_{x_2} \rho_2(x_2) = \frac{1}{3} > 1 - p - q = -3,
\]

\[
\kappa_2 = \frac{p + q}{p + q + \sigma_2} = \frac{6}{5}, \quad \mu_2 = \max \left\{ \kappa_2, \frac{p + q}{p + q - 1} \right\} = \max \left\{ \frac{6}{5}, \frac{4}{3} \right\} = \frac{4}{3}.
\]

Substituting for \( \mu_2 \) in (8.103)

\[
\sum_{x_2=1}^{\infty} \frac{x_2}{(2x_2 + 1)^4} \leq \left( \frac{2}{3} \right)^4 \sum_{x_2=1}^{\infty} \frac{1}{x_2(2x_2 + 1)^2}.
\]

Numerical evaluation gives the left hand side of (8.105) as approximately 0.0186 and the right hand side as approximately 0.0289. The error is approximately 0.36 of the right hand side or 0.56 times the left hand side. The terms on the left hand side and on the right hand side of (8.105) are both \( O\left(\frac{1}{x_2^3}\right) \) as \( x_2 \to \infty \).

**Example 8.7.2.** Here \( \beta_i(x_j) \) is an unbounded monotonic increasing function of both \( i \) and \( x_j \). As in Example 8.7.1 in (8.101) let \( n = 3, x_1 = 2, x_2 = 3, p = 3, q = 2, j = 0, a(y_1, y_2, y_3) = y_1^{-2}y_2^{-2}y_3^{-3}, \lambda_i(y_j) = y_j \). Let \( \beta_i(y_j) = iy_j \not\to \infty \) as both \( i \) and \( y_j \to \infty \). Then

\[
\Lambda_i(x_1) = \frac{i}{6} x_i(x_i + 1)(2x_i + 1), \quad \Lambda_2(x_2) = \frac{1}{3} x_2(x_2 + 1)(2x_2 + 1) = 28,
\]

\[
\Lambda_3(x_3) = \frac{1}{2} x_3(x_3 + 1)(2x_3 + 1), \quad \prod_{i=1}^{n-1} \Lambda_i(x_i) = 14x_3(x_3 + 1)(2x_3 + 1),
\]

\[
\prod_{i=1}^{n-1} \beta_i(y_i) \lambda_i(y_i) = \prod_{i=1}^{2} iy_i^2 = 2y_1^2y_2^2, \quad \prod_{i=1}^{n} \beta_i(y_i) \lambda_i(y_i) = \prod_{i=1}^{3} iy_i^2 = 6y_1^2y_2^2y_3^2.
\]

and

\[
\sum_{x_3=1}^{\infty} x_3 \left\{ \sum_{y_2=1}^{2} \sum_{y_1=1}^{3} \sum_{y_3=1}^{x_3} 2y_3^{-1} \right\} \left\{ \frac{1}{3(1^2 + 2^2 + 3^2 + \cdots + x_3^2)} \right\}^5 \left( \sum_{y_2=1}^{3} \sum_{y_1=1}^{3} \sum_{y_3=1}^{x_3} 2y_3^{-1} \right) \left\{ \frac{1}{3(1^2 + 2^2 + 3^2 + \cdots + x_3^2)} \right\}^2, \quad (8.106)
\]
which reduces to
\[ \sum_{x_3=1}^{\infty} x_3 \left\{ \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_3}}{x_3(x_3 + 1)(2x_3 + 1)} \right\}^5 \leq \left( \frac{\mu_3}{6} \right)^3 \sum_{x_3=1}^{\infty} x_3^{-8} \left\{ \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_3}}{x_3(x_3 + 1)(2x_3 + 1)} \right\}^2. \]

Let us check the equivalent of (8.7) which is
\[ 5 + \frac{2x_3 + 1}{6x_3} \geq \frac{5}{\kappa_3}, \quad \kappa_3 \geq \frac{3}{4 + 1/2x_3} \not\to \frac{3}{4} \quad \text{as } x \to \infty \]
and
\[ \sum_{x_3=1}^{\infty} x_3 \left\{ \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_3}}{x_3(x_3 + 1)(2x_3 + 1)} \right\}^5 \leq \left( \frac{5}{25.3} \right)^3 \sum_{x_3=1}^{\infty} x_3^{-8} \left\{ \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x_3}}{x_3(x_3 + 1)(2x_3 + 1)} \right\}^2. \]

Numerical evaluation gives the left hand side of (8.108) as approximately 0.000129 and the right hand side as approximately 0.000251. The error is approximately 0.48 of the right hand side or 0.94 times the left hand side. The terms on the left hand and right hand sides are \( O \left( (\log x_3)^5/x_3^{14} \right) \) and \( O \left( (\log x_3)^2/x_3^{14} \right) \) respectively as \( x_3 \to \infty \). The right hand side converges faster than the left hand side.

**Example 8.7.3.** Here \( \beta_i(x_j) \) oscillates infinitely between \( x_j/2 \) and \( 3x_j/2 \) as \( x_j \to \infty \) and \( \rho_i(x_j) \) oscillates infinitely between \( \pm 8x_j/9 \). Following Example 8.7.1 we let \( n = 2, \ x_1 = 2, \ p = 2, \ q = 2, \ a(y_1,y_2) = y_1^{-1}y_2^{-1}, \ \lambda_i(y_j) = y_j \). Let \( i, \ x_j, \ k \in \mathbb{N} \) and

\[ \beta_i(x_j) = \left\{ 1 + \frac{1}{2}(-1)^{x_j} \right\} x_j. \]  

(8.109)

Then \( \beta_i(x_j) \) and \( \lambda_i(x_j) \) are independent of \( i \) and

\[ \rho_m(x_n) \begin{cases} < 0, & x_n = 2k; \\ > 0, & x_n = 2k + 1 \end{cases} \]  

(8.110)
in (8.47). We have

\[
\beta_i(x_i + 1) - \beta_i(x_i) = \left\{1 - \frac{1}{2}(-1)^i\right\} (x_i + 1) - \left\{1 + \frac{1}{2}(-1)^i\right\} x_i,
\]

\[
= 1 - \frac{2x_i + 1}{2} (-1)^x_i, \quad \beta_i(2k + 1) = \frac{2k + 1}{2},
\]

\[
\lambda_i(2k + 1) = 2k + 1, \quad \beta_i(2k + 1)\lambda_i(2k + 1) = \frac{(2k + 1)^2}{2},
\]

\[
\beta_i(2k) = 3k, \quad \lambda_i(2k) = 2k, \quad \beta_i(2k)\lambda_i(2k) = 6k^2,
\]

\[
\beta_i(2k - 1) = \frac{2k - 1}{2}, \quad \lambda_i(2k - 1) = 2k - 1,
\]

\[
\beta_i(2k - 1)\lambda_i(2k - 1) = \frac{(2k - 1)^2}{2}, \quad \beta_i(2k + 1) - \beta_i(2k) = \frac{1 - 4k}{2} < 0,
\]

\[
\beta_i(2k) - \beta_i(2k - 1) = \frac{1 + 4k}{2} > 0,
\]

\[
\beta_i(2k + 1)\beta_i(2k)\lambda_i(2k) = 3k^2(2k + 1),
\]

\[
\beta_i(2k)\beta_i(2k - 1)\lambda_i(2k - 1) = \frac{3k(2k - 1)^2}{2},
\]

\[
\Lambda_i(x_i) = \sum_{y_i=1}^{x_i} \beta_i(y_i)\lambda_i(y_i) = \sum_{y_i=1}^{x_i} \left\{1 + \frac{1}{2}(-1)^y_i\right\} y_i^2
\]

\[
\Lambda_i(2k) = \sum_{j=1}^{k} \left\{\beta_i(2j - 1)\lambda_i(2j - 1) + \beta_i(2j)\lambda_i(2j)\right\}
\]

\[
= \sum_{j=1}^{k} \left\{\frac{(2j - 1)^2}{2} + 6j^2\right\} = 6 \sum_{j=1}^{k} j^2 + \frac{1}{2} \sum_{j=1}^{k} (2j - 1)^2
\]

\[
= \frac{k(2k + 1)(8k + 5)}{6} \sim \frac{8k^3}{3} \text{ as } x_i \to \infty,
\]

\[
\Lambda_i\{2(k - 1)\} = \frac{(k - 1)(2k - 1)(8k - 3)}{6} \sim \frac{8k^3}{3} \text{ as } k \to \infty,
\]

\[
\Lambda_i(2k - 1) = \Lambda_i\{2(k - 1)\} + \beta_i(2k - 1)\lambda_i(2k - 1)
\]

\[
= \frac{k(2k - 1)(8k - 5)}{6} \sim \frac{8k^3}{3} \text{ as } k \to \infty
\]

\[
\rho_i(2k) = \frac{5}{36k} - \frac{1}{3} - \frac{8}{9}k \sim -\frac{8}{9}k \text{ as } k \to \infty.
\]

\[
\rho_i(2k - 1) = \frac{(1 + 4k)k(2k - 1)(8k - 5)}{2} \frac{2}{6} \frac{3k(2k - 1)^2}{18(2k - 1)}
\]

\[
= \frac{(4k + 1)(8k - 5)}{18(2k - 1)} = \frac{32k^2 - 12k - 5}{18(2k - 1)}
\]
CHAPTER 8. HARDY’S DISCRETE INEQUALITY – II

\[ \sim \frac{8k}{9} \text{ as } k \to \infty. \]

Thus \( \rho_i(x_j) \) oscillates infinitely as \( x_j \to \infty \) and \( \sigma_i \) does not exist. The inequality is not in the class of suitably convergent series.

**Example 8.7.4.** Here \( \beta_i(x_j) \) oscillates finitely between 1/2 and 3/2 as \( x_j \to \infty \) and \( \rho_i(x_j) \) again oscillates infinitely. Following Example 8.7.1 in (8.101) we let \( n = 2, x_1 = 2, p = 2, q = 2, a(y_1, y_2) = y_1^{-1}y_2^{-1} \) and \( \lambda_i(y_j) = y_j \). Let

\[
\beta_i(x_j) = \left\{ 1 + \frac{1}{2}(-1)^{x_j} \right\} = \begin{cases} \frac{3}{2}, & x_j = 2k, \\ \frac{1}{2}, & x_j = 2k + 1, \end{cases} \quad i, x_j, k \in \mathbb{N}. \quad (8.111)
\]

\( \beta_i(x_j) \) and \( \lambda_i(x_j) \) are independent of \( i \) and

\[
\rho_i(x_j) \begin{cases} < 0, & x_j = 2k, \\ > 0, & x_j = 2k + 1, \end{cases} \quad k \in \mathbb{N}, \quad (8.112)
\]
in (8.44) and (8.46). We have

\[
\beta_i(2k + 1) - \beta_i(2k) = -1, \quad \beta_i(2k + 1) - \beta_i(2k - 1) = 1,
\]

\[
\beta_i(2k + 1)\lambda_i(2k + 1) = \frac{2k + 1}{2}, \quad \beta_i(2k)\lambda_i(2k) = 3k,
\]

\[
\beta_i(2k - 1)\lambda_i(2k - 1) = \frac{2k - 1}{2},
\]

\[
\beta_i(2k + 1)\beta_i(2k)\lambda_i(2k) = \frac{3k}{2}, \quad \beta_i(2k)\beta_i(2k - 1)\lambda_i(2k - 1) = \frac{3(2k - 1)}{4},
\]

\[
\Lambda_i(x_i) = \sum_{y_i=1}^{x_i} \beta_i(y_i)\lambda_i(y_i) = \sum_{y_i=1}^{x_i} \left\{ 1 + \frac{1}{2}(-1)^{y_i} \right\} y_i,
\]

\[
\Lambda_i(2k) = \sum_{j=1}^{k} \left\{ \beta_i(2j - 1)\lambda_i(2j - 1) + \beta_i(2j)\lambda_i(2j) \right\}
\]

\[
= \sum_{j=1}^{k} \left\{ \frac{2j - 1}{2} + 3j \right\} = \sum_{j=1}^{k} 4j - \sum_{j=1}^{k} \frac{1}{2}
\]

\[
= \frac{k(4k + 5)}{2} \sim 2k^2 \text{ as } k \to \infty,
\]

\[
\Lambda_i(2k - 1) = \Lambda_i(2k) - \beta_i(2k)\lambda_i(2k)
\]

\[
= \frac{k(4k + 5)}{2} - 6k = \frac{k(4k - 1)}{2} \sim 2k^2 \text{ as } k \to \infty
\]

\[
\rho_i(2k) = -\frac{4k}{3} - \frac{5}{3}.
\]
\[ \rho_i(2k - 1) = \frac{2k(4k - 1)}{3(2k - 1)} = \frac{4k}{3} + \frac{2}{3(2k - 1)} \sim \frac{4k}{3} \quad \text{as } k \to \infty. \]

Thus \( \rho_i(x_j) \) oscillates infinitely as \( x_j \to \infty \) and \( \sigma_i \) does not exist. The inequality is not in the class of suitably convergent series.

**Example 8.7.5.** Here \( \beta_i(x_j) \) shows damped oscillation as \( x_j \to \infty \) and \( \rho_i(x_j) \) oscillates infinitely. Following Example 8.7.1 in (8.101) we let \( n = 2, x_1 = 2, p = 2, q = 2, a(y_1, y_2) = y_1^{-1} y_2^{-1} \) and \( \lambda_i(y_j) = y_j \). Let

\[ \beta_i(x_j) = \frac{1 + \frac{1}{2}(-1)^x_j}{x_j} = \begin{cases} \frac{3}{4k}, & x_j = 2k, \\ \frac{1}{2(2k+1)}, & x_j = 2k + 1, \end{cases} \quad i, x_j, k \in \mathbb{N}. \quad (8.113) \]

\( \beta_i(x_j) \) and \( \lambda_i(x_j) \) are independent of \( i \) and

\[ \rho_i(x_j) \begin{cases} < 0, & x_j = 2k, \\ > 0, & x_j = 2k + 1. \end{cases} \quad (8.114) \]

We have

\[ \beta_i(2k + 1) - \beta_i(2k) = -\frac{(4k + 3)}{4k(2k + 1)}, \quad \beta_i(2k) - \beta_i(2k - 1) = \frac{4k - 3}{4k(2k - 1)}, \]
\[ \beta_i(2k) \lambda_i(2k) = \frac{3}{2}, \quad \beta_i(2k + 1) \lambda_i(2k + 1) = \frac{1}{2}, \]
\[ \beta_i(2k - 1) \lambda_i(2k - 1) = \frac{1}{2}, \quad \beta_i(2k + 1) \beta_i(2k) \lambda_i(2k) = \frac{3}{4(2k + 1)}, \]
\[ \beta_i(2k) \beta_i(2k - 1) \lambda_i(2k - 1) = \frac{3}{8k}, \]
\[ \Lambda_i(x_i) = \sum_{y_i=1}^{x_i} \beta_i(y_i) \lambda_i(y_i) = \sum_{y_i=1}^{x_i} \left\{ 1 + \frac{1}{2}(-1)^{y_i} \right\} \]
\[ \Lambda_i(2k) = \sum_{j=1}^{k} \left\{ \beta_i(2j - 1) \lambda_i(2j - 1) + \beta_i(2j) \lambda_i(2j) \right\} = 2k \]
\[ \Lambda_i(2k - 1) = \Lambda_i(2k) - \beta_i(2k) \lambda_i(2k) = \frac{4k - 3}{2} \]
\[ \rho_i(2k) = -\frac{2(4k + 3)}{3} \sim \frac{8k}{3} \quad \text{as } k \to \infty, \]
\[ \rho_i(2k - 1) = \frac{(4k - 3)^2}{3(2k - 1)} \sim \frac{8k}{3} \quad \text{as } k \to \infty. \]

Thus \( \rho_i(x_j) \) oscillates infinitely as \( x_j \to \infty \) and \( \sigma_i \) does not exist. The inequality is not in the class of suitably convergent series.
Appendix A

Hilbert’s and Hardy’s theorems

A.1 Hilbert’s double–series theorem

The following outline of Hilbert’s original proof of his double–series theorem appeared in his lectures on integral equations and is reproduced in [137].

‘The proof depends on the identity

\[
\int_{-\pi}^{\pi} t \left\{ \sum_{r=1}^{n} (-1)^r (a_r \cos r t - b_r \sin r t) \right\}^2 dt = 2\pi (S - T),
\]

where

\[
S = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{a_r b_s}{r + s}, \quad T = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{\prime a_r b_s}{r - s}
\]

(the dash implying that pairs \( r, s \) for which \( r = s \) are omitted). From this it follows that

\[
2\pi |S - T| \leq \pi \int_{-\pi}^{\pi} \left\{ \sum_{r=1}^{n} (-1)^r (a_r \cos r t - b_r \sin r t) \right\}^2 dt = \pi^2 \sum_{r=1}^{n} (a_r^2 + b_r^2). \quad (A.1)
\]

If \( a_r = b_r, T \) disappears, and we obtain

\[
S = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{a_r b_s}{r + s} \leq \pi \sum_{r=1}^{n} a_r^2 \quad \text{ (A.2)}
\]

and from (A.1) and the theory of bounded bilinear forms, since a symmetric bilinear form has a bound equal to that of the corresponding quadratic form, we
deduce

\[ S = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{a_r b_s}{r+s} \leq \pi \left( \sum_{r=1}^{n} a_r^2 \right)^{1/2} \left( \sum_{r=1}^{n} b_r^2 \right)^{1/2} \leq \frac{\pi}{2} \left( \sum_{r=1}^{n} a_r^2 + \sum_{r=1}^{n} b_r^2 \right). \tag{A.3} \]

From (A.1) and (A.3) it follows that

\[ |T| = \left| \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{a_r b_s}{r-s} \right| \leq \pi \left( \sum_{r=1}^{n} a_r^2 + \sum_{r=1}^{n} b_r^2 \right); \]

and hence, on the grounds of homogeneity, that

\[ \left| \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{a_r b_s}{r-s} \right| \leq 2 \pi \left( \sum_{r=1}^{n} a_r^2 \right)^{1/2} \left( \sum_{r=1}^{n} b_r^2 \right)^{1/2}. \]

### A.2 Deduction of Hilbert’s theorem from Hardy’s

In 1919 Hardy [55] deduced Hilbert’s weak discrete double series Theorem 1.2.1 from Hardy’s weak discrete Theorem 1.2.4, as follows.

‘... we have only to observe that the convergence of the two series \( \sum a_n^2 \) and \( \sum (A_n/n)^2 \) involves that of

\[ \sum \frac{a_n A_n}{n} \]

and \textit{a fortiori} of

\[ \sum a_n \sum_{m=1}^{n} \frac{a_m}{m+n} \]

or of

\[ \sum \frac{a_m a_n}{m+n}, \]

In 1929 Hardy [61] presented his proof in more detail. We note his use of the Cauchy–Schwarz theorem).

‘Hilbert’s series does not exceed

\[ 2 \sum_{n} a_n \sum_{m \leq n} \frac{a_m}{m+n} \leq 2 \sum_{n} \frac{a_n A_n}{n} \leq 2 \left( \sum a_n^2 \right)^{1/2} \left\{ \sum \left( \frac{A_n}{n} \right)^2 \right\}^{1/2}. \]

He observed that, as the constants \( \pi \) and 4 \((p = 2)\) suggest, there is no such direct connection between Theorems 1.2.2 and 1.2.5, the ‘strong discrete’ versions.
Appendix B

Algebraic preliminaries

B.1 The inequality \( s^p + (p-1)t^p \geq pst^{p-1} \)

Professor Pečarić observes that the elementary inequality

\[
    s^p + (p-1)t^p \geq pst^{p-1}
\]  

where \( s \) and \( t \) are non–negative numbers and \( p > 1 \), is a simple consequence of the weighted geometric–arithmetic mean inequality for two numbers \( s \) and \( t \) on \( p \) and 1 with weights 1 and \( p - 1 \) respectively. The inequality is a starting point for many theorems in the Thesis and an indication of its genesis is appropriate. Hardy [61, p.67] observes that it is easily proved by using the processes of the calculus, but a proof using more elementary methods would be more satisfactory. He provides such a proof in [62, pp.143–4] for \( p \in \mathbb{N} \), easily extends it to rational \( p \). He notes that it is equally true for irrational \( p \), ‘‘for which case it is not proved in either book ([62] or [19]); and it is obvious that it cannot be proved as a limiting case of the inequality for rational \( p \), unless we have determined lower bounds for the excesses independent of the \( n \) of the approximating \( p_n \).

We address similar difficulties in taking limits in Theorem 7.2.2.

We obtain a calculus proof dividing both sides of (B.1) by \( t^p \) and letting \( x = s/t > 0 \).

\[
    f(x) := x^p - px + (p - 1) \geq 0, \quad f(1) = 1 \tag{B.2}
\]

\[
    f'(x) = p(x^{p-1} - 1), \quad f'(1) = 0 \tag{B.3}
\]

\[
    f''(x) = p(p-1)x^{p-2} \geq 0 \quad \text{if} \quad p > 1.
\]

so \( f \) is convex, with a minimum at \( x = 1 \), and \( f(x) \geq 0 \).


APPENDIX B. ALGEBRAIC PRELIMINARIES

B.2 Sums of powers of $n$ integers

The following formulae for sums of powers of $n$ integers are used in Examples 8.7.1, 8.7.2 and 8.7.3.

\[ \sum_{r=1}^{n} r = \frac{n(n+1)}{2} = s_1, \quad \text{(B.4)} \]

\[ \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6} = s_2, \quad \text{(B.5)} \]

\[ \sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = s_3, \quad \text{(B.6)} \]

\[ \sum_{r=1}^{n} (2r)^2 = 4s_2 = \frac{2n(n+1)(2n+1)}{3} \quad \text{(B.7)} \]

\[ \sum_{r=1}^{n} (2r-1)^2 = \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} (2r)^2 = \frac{n(4n^2-1)}{3}. \quad \text{(B.8)} \]

B.3 Generalised Hölder’s inequality

Bollobás’ generalised Hölder’s inequality for $k$–series [8, p. 14, ex. 14] is as follows.

**Lemma B.3.1.** If

\[ M_s(abc) = \left( \sum_{i=1}^{n} p_i a_i^r \right)^{1/r} \quad \text{(B.9)} \]

where $p_1, \ldots, p_n > 0$, $\sum_{i=1}^{n} p_i = 1$ and $p, q, r, s > 0$ are such that

\[ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{s} \quad \text{(B.10)} \]

then

\[ M_s(abc) \leq M_p(a)M_q(b)M_r(c) \quad \text{(B.11)} \]

for all positive sequences $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ and $c = (c_1, \ldots, c_n)$ with equality iff all $a_k$ are 0 or $|a_k|^p = s|b_k|^q = t|c_k|^r$ and $a_kb_kc_k = e^{i\theta}|a_kb_kc_k|$ for all $k$ and some $s$, $t$ and $\theta$. The analogous inequality holds for $k$–series.

Hardy in [63, p. 24, Theorem 12] uses the strict inequality

\[ M_s(abc) < M_p(a)M_q(b)M_r(c) \quad \text{(B.12)} \]
unless \( a^p_k, b^p_k, c^p_k \) are proportional or one of the factors on the right is zero.

We adopt (B.12).

Following [8, p.5] let us fix \( p_1, \ldots, p_n > 0 \) with \( \sum_{i=1}^{n} p_i = 1 \). Given a continuous and strictly monotonic function \( \varphi : (0, \infty) \rightarrow \mathbb{R} \), the \( \varphi \)-mean of a sequence \( a = (a_1, \ldots, a_n) \) \( a_i > 0 \) is defined as

\[
M_\varphi(a) = \varphi^{-1} \left( \sum_{i=1}^{n} p_i \varphi(a_i) \right) \quad (B.13)
\]

If \( \varphi(t) = t^r \) \( (-\infty < r < \infty, r \neq 0) \) then we write \( M_r \) for \( M_\varphi \).

Re-writing (B.12)

\[
\left( \sum_{i=1}^{n} p_i (a_i b_i c_i)^s \right)^{1/s} < \left( \sum_{i=1}^{n} p_i a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} p_i b_i^q \right)^{1/q} \left( \sum_{i=1}^{n} p_i c_i^r \right)^{1/r} \quad (B.14)
\]

\( M_r(a) \) is a continuous monotone increasing function of \( r \) [8, p. 6].

Let \( p_i = 1, \ i \in \mathbb{N} \) and raise (B.14) to the \( s^{th} \) power

\[
\sum_{i=1}^{n} (a_i b_i c_i)^s < \left( \sum_{i=1}^{n} a_i^p \right)^{s/p} \left( \sum_{i=1}^{n} b_i^q \right)^{s/q} \left( \sum_{i=1}^{n} c_i^r \right)^{s/r} \quad (B.15)
\]

Letting \( s = 1 \) we recover a 3–dimensional version of generalised Hölder’s inequality.

\[
\sum_{i=1}^{n} a_i b_i c_i < \left( \sum_{i=1}^{n} a_i^p \right)^{1/p} \left( \sum_{i=1}^{n} b_i^q \right)^{1/q} \left( \sum_{i=1}^{n} c_i^r \right)^{1/r}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1. \quad (B.16)
\]

Replacing \( p, q, r \) by \( p/(p-1), 2p, 2p \)

\[
\sum_{i=1}^{n} a_i b_i c_i < \left( \sum_{i=1}^{n} a_i^{p/(p-1)} \right)^{(p-1)/p} \left( \sum_{i=1}^{n} b_i^{2p} \right)^{1/2p} \left( \sum_{i=1}^{n} c_i^{2p} \right)^{1/2p} \quad (B.17)
\]

According to [63, p. 24, Theorem 167] (B.17) remains valid when the series concerned are infinite, so that

\[
\sum_{i=1}^{\infty} a_i b_i c_i < \left( \sum_{i=1}^{\infty} a_i^{p/(p-1)} \right)^{(p-1)/p} \left( \sum_{i=1}^{\infty} b_i^{2p} \right)^{1/2p} \left( \sum_{i=1}^{\infty} c_i^{2p} \right)^{1/2p} \quad (B.18)
\]

unless \( a^p_k, b^p_k, c^p_k \) are proportional or one of the factors on the right is zero.

The following theorems are converses of Hölder’s inequality.
Theorem B.3.2. Converse of Hölder’s inequality – weak version [63, Theorem 15]. If $k > 1$ and $\sum ab$ is convergent for all $b$ for which $\sum b^{k'}$ is convergent, then $\sum a^k$ is convergent.

Theorem B.3.3. Converse of Hölder’s inequality – strong version [63, Theorem 161] Suppose that $k > 1$, that $k'$ is conjugate to $k$, and that $B > 0$. Then a necessary and sufficient condition that $\sum a^k \leq A$ is that $\sum ab \leq A^{1/k}B^{1/k'}$ for all $b$ for which $\sum b^{k'} \leq B$.

Hardy Littlewood and Pólya observe that Theorem B.3.2 may be deduced directly from a theorem of Abel. The extension of the converses of Hölder’s inequality to generalised Hölder’s inequality for $k$–series is a topic for further investigation.
Appendix C

Note of Professor E. Landau

The following rather literal translation is provided of a beautiful paper [78] by E. Landau in German. Its interest in particular concerns the proof that constants are the ‘best possible’ using an analysis of asymptotic behaviour. We note that (C.2) is just (B.3).

‘A NOTE ON A THEOREM CONCERNING SERIES OF POSITIVE TERMS: EXTRACT FROM A LETTER OF PROF. E. LANDAU TO PROF. I. SCHUR (communicated by G. H. Hardy)

...I am now in a position to prove exactly after Hardy’s model that

Theorem C.0.4. the relation

(i)

\[
\left( \frac{\kappa}{\kappa - 1} \right)^\kappa \sum_{i=1}^{N} a_i^\kappa \geq \sum_{i=1}^{N} \left( \frac{A_n}{n} \right)^\kappa \quad \left( a_n \geq 0, \, \kappa > 1, \, A_n = \sum_{i=1}^{n} a_{\nu} \right); \quad \text{(C.1)}
\]

and (C.1) is also true for \( N = \infty \), if the series on the left hand side converges.

(ii) further to show that for each finite \( N \) or for \( N = \infty \), equality applies when and only when all \( a_n = 0 \);

(iii) finally, that \( [\kappa/(\kappa - 1)]^\kappa \) is the ‘real’ (i.e best possible) constant.

Proof.
APPENDIX C. NOTE OF PROFESSOR E. LANDAU

(i) For \( y \geq 0 \) we have

\[
y^\kappa - \kappa y + \kappa - 1 \geq 0
\]

(C.2)
as the derivative \( \kappa(y^{\kappa-1} - 1) \) shows. Therefore for \( y_1 \geq 0, \ y_2 \geq 0 \)

\[
y_1^\kappa - \kappa y_1 y_2^{\kappa-1} + (\kappa - 1)y_2^\kappa \geq 0,
\]

(C.3)
so if we let

\[
y_1 = b_n \geq 0, \ y_2 = \frac{\kappa - 1}{\kappa} B_n, \ B_n = \sum_{1}^{n} b_n
\]

we have

\[
b_n^\kappa - \kappa b_n \left( \frac{\kappa - 1}{\kappa} \frac{B_n}{n} \right)^{\kappa-1} + (\kappa - 1) \left( \frac{\kappa - 1}{\kappa} \frac{B_n}{n} \right)^{\kappa} \geq 0,
\]

(C.4)

\[
\sum_{1}^{N} b_n^\kappa - \left( \frac{\kappa - 1}{\kappa} \right)^{\kappa-1} \sum_{1}^{N} \kappa b_n \left( \frac{B_n}{n} \right)^{\kappa-1} + (\kappa - 1) \left( \frac{\kappa - 1}{\kappa} \right)^{\kappa} \sum_{1}^{N} \left( \frac{B_n}{n} \right)^{\kappa} \geq 0.
\]

(C.5)

Using (C.3)

\[
\kappa b_n B_n^{\kappa-1} = \kappa b_n B_n^{\kappa-1}(B_n - B_{n-1}) \geq B_n^\kappa - B_{n-1}^\kappa,
\]

and summing from 1 to \( N \)

\[
\sum_{1}^{N} \kappa b_n \left( \frac{B_n}{n} \right)^{\kappa-1} \geq \sum_{1}^{N} B_n^\kappa \left( \frac{1}{n^{\kappa-1}} - \frac{1}{(n+1)^{\kappa-1}} \right) \geq (\kappa - 1) \sum_{1}^{N} \frac{B_n^\kappa}{(n + 1)^{\kappa}}.
\]

Therefore

\[
\sum_{1}^{N} b_n^\kappa \geq \left( \frac{\kappa - 1}{\kappa} \right)^{\kappa} \sum_{1}^{N} B_n^\kappa \left( \frac{\kappa}{(n+1)^{\kappa}} - \frac{\kappa - 1}{n^{\kappa}} \right) = \left( \frac{\kappa - 1}{\kappa} \right)^{\kappa} \sum_{1}^{N} \frac{c_n}{(n + 1)^{\kappa}},
\]

(C.6)

where \( c_n \to 1 \), so \( c_n \geq 0 \) for \( n > \Lambda(\kappa) \), where \( \Lambda > 0 \) and it is complete. For later I observe that according to Principle (C.4) on the right hand side of (C.4) it will be possible to add \( \sum_{1}^{N} \{ \text{left hand side of (C.4)} \} \) for each \( m \leq N \). For the proof of (C.1) there is only the limitation \( a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_N \). I set, like you in your paper, \( b_1 = \ldots = b_m = a_1, \ b_{m+1} = \ldots = b_{2m} = a_2, \ldots \),
\[ b_{(N-1)m+1} = \ldots = b_{Nm} = a_N, \text{ and take } m > \Lambda(\kappa). \] Then from (C.4) with \( Nm \) in place of \( N \),
\[
\left( \frac{\kappa - 1}{\kappa} \right) m \sum_{1}^{N} a_{n}^{\kappa} \geq (-|c_1| - \ldots - |c_{\Lambda}|) A_{1}^{\kappa} + (c_{\Lambda+1} + \ldots + c_{m}) \left( \frac{A_1}{1} \right)^{\kappa} \\
+ (c_{m+1} + \ldots + c_{2m}) \left( \frac{A_2}{2} \right)^{\kappa} + \ldots \\
\ldots + (c_{(N-1)m+1} + \ldots + c_{Nm}) \left( \frac{A_{N}}{N} \right)^{\kappa},
\]
where the abovementioned part may be added on the right. Therefore
\[
+ \left\{ \left( \frac{\kappa - 1}{\kappa} \right)^{\kappa} - \kappa \frac{\kappa}{\kappa - 1} + \kappa - 1 \right\} ma_{1}^{\kappa} = p_{\kappa} ma_{1}^{\kappa} \left( p_{\kappa} > 0 \text{ since } \frac{\kappa}{\kappa - 1} > 1 \right).
\]
If we divide by \( m \), and let \( m \to \infty \) (C.1) follows, even with the auxiliary member \( p_{\kappa} a_{1}^{\kappa} \) on the right.

(ii) If there were equality in (C.1), without \( a_{n} \equiv 0 \) (here \( N \) can be finite or \( \infty \)), then the sequence \( (a_{n}) \) is not monotonic increasing, and \( a_{1} > 0 \); the auxiliary member thus results in a contradiction.

(iii) After the Paradigm of Hardy, Note on a theorem of Hilbert, *Math. Zeitschr.* 6 (1920), [56, p. 317] we have
\[
a_{n} = \frac{1}{n^{\lambda+\epsilon}} \left( \lambda = \frac{1}{\kappa}, 0 < \epsilon < 1 - \lambda \right) \\
\sum_{1}^{\infty} a_{n}^{\kappa} = \sum_{1}^{\infty} \frac{1}{n^{1+\kappa}} \to \infty \text{ as } \epsilon \to 0,
\]
then
\[
\frac{A_{n}}{n} = \frac{1}{n} \sum_{1}^{n} \frac{1}{\nu^{\lambda+\epsilon}} > \frac{1}{n} \int_{1}^{n} \frac{d\nu}{\nu^{\lambda+\epsilon}} \\
= \frac{1}{1 - \lambda - \epsilon} \frac{n^{1-\lambda-\epsilon} - 1}{n} \geq \frac{1}{1 - \lambda} \frac{1}{n^{\lambda+\epsilon}} \left( 1 - \frac{1}{n^{1-\lambda-\epsilon}} \right) \\
\left( \frac{A_{n}}{n} \right)^{\kappa} > \frac{1}{(1 - \lambda)^{\kappa}} \frac{1}{n^{1+\kappa \epsilon}} \left( 1 - \frac{\kappa}{n^{1-\lambda-\epsilon}} \right) \text{ by (C.3)},
\]
\[
\sum_{1}^{\infty} \left( \frac{A_{n}}{n} \right)^{\kappa} > \left( \frac{\kappa}{\kappa - 1} \right)^{\kappa} \left( \sum_{1}^{\infty} a_{n}^{\kappa} - \kappa \sum_{1}^{\infty} \frac{1}{n^{1-\lambda}} \right) \sim \left( \frac{\kappa}{\kappa - 1} \right)^{\kappa} \sum_{1}^{\infty} a_{n}^{\kappa}. \quad (C.8)
\]

*Göttingen, 22 June, 1921.*
Appendix D

Possible further research

Possible avenues for further research are as follows.

Chapter 6

(i) Integrals on \((X, \infty)\) not already included and discrete analogues.

(ii) There is an expression for best possible constant in the Mohapatra and Russell extension Theorem 5.3.3 which may or may not lead to an analogous algorithm for best possible constant in the discrete case.

Chapter 7

(i) Examples of Theorems 7.2.2 and 7.3.1.

(ii) Theorems for the case \(0 < p < 1\).

(iii) Arbitrary convex functions and integral analogues.

Chapter 8

(i) The infimum principle by which the constants \(\mu_i\) are determined and possible extension to integral analogues; the rôle of the infimum in (7.10) and (8.46).

(ii) Proofs by induction similar to those in [53].

(iii) Integral analogues; the more complicated series at the end of Chapter 8 may be analogous to the integrals in Mohapatra and Russell extension theorems of Chapter 5.
APPENDIX D. POSSIBLE FURTHER RESEARCH

(iv) Formalise the determination in Section 8.5 of new combinations of components for use as conjugate Hölder exponents by setting up a system of equations.

(v) Explore further the effects of the following behaviours of $\beta_i(x_j)$ as $x_j \to \infty$: monotonic decreasing convergence, monotonic divergence, finite and infinite oscillation and oscillating convergence. The theoretical reasons for the widely varying effects on $\sigma_j = \inf_{x_j} \rho_i(x_j)$ may be further considered. Consider the effects of convexity, $\alpha$–submultiplicativity and other properties of $\beta_i(x_j)$ on $\sigma_j$.

(vi) The infimum principle for analogous integral inequalities.

(vii) Specialising the infimum principle to see whether known ‘best possible’ constants can be generated.

(viii) Find an algorithm to determine sharp constants for the class of suitably convergent series inequalities with weight function $\beta_i(x_j)\lambda_i(x_j)$. It is expected that the function

$$\Psi(x_j) = \beta_i(x_j)\lambda_i(x_j)a(x_1, \ldots, x_j, \ldots, x_n)$$

would be arbitrary within constraints as $\varphi$ was constrained to be submultiplicative in Chapter 7, Theorem 5.3.3.

(ix) The extension of the converses of Hölder’s inequality to generalised Hölder’s inequality for $k$–series.

(x) The original title of the Thesis included Applications to Operator Theory which is now a possible topic for further research.
Sir Thomas (T. M.) Cherry was a successor of Poincaré in dynamical systems [4, pp. 155, 221]. He stands between J. Hadamard–G. D. Birkhoff and A. N. Kolmogorov–V. I. Arnol’d–J. Moser. Cherry was influenced by his PhD supervisors H. F. Baker [3], J. E. Littlewood [83, 84] and by Sir Edmund Whittaker [140] who all worked in the English tradition of differential equations and dynamical systems inherited from Sir Isaac Newton [97]. In the 1937 edition of Whittaker’s definitive *Treatise on the analytical dynamics of particles and rigid bodies, with an introduction to the problem of three bodies* [140] he cites only the following 10 out of 235 authors more frequently than Cherry: Bertrand, Euler, Hamilton, Jacobi, Lagrange, Levi–Civita, Newton, Painlevé, Poincaré and Whittaker [140, pp. 385, 396, 412, 434, 449].

With a series of papers in the 1920’s young Cherry enjoyed a meteoric rise in the mathematical firmament. The papers are hard analysis in the vein of Hardy–Littlewood showing considerable originality and virtuosity with hints of Ramanujan’s influence. Cambridge at that time was in awe of the fabulous virtuosity of the mystic untutored Indian mathematician Ramanujan who strongly influenced Hardy, Littlewood and Whittaker. On a scale of 100 Hardy rated himself 20, Littlewood 30 and Ramanujan 100. Cherry created dynamical and Hamiltonian systems which concretised very general and at times obscure theorems of Poincaré and Birkhoff, adding considerably to the meagre stock of known

---

1 The KAM method was named after Kolmogorov–Arnol’d–Moser; C. L. Siegel co-authored an important book with Moser [131].

2 See e.g. [140, pp. 434–435].
integrable and non–integrable Hamiltonian systems. His early research led him to the topology of dynamical systems and he moved towards a formal approach to topological dynamics in [14, 15]. In [14] he explores all possible relations between a trajectory and its $\alpha$– and $\omega$–limit sets.

Cherry returned to Melbourne at the age of 29 to take up the post of Professor of Mathematics, Pure and Mixed (i.e. Applied), immersed himself in education and was involved in the Schools curriculum in mathematics down to Year 9, with positive results for mathematical education in Victoria [12]. His military service in 1918 no doubt helped him in dealing with the world at large.

Cherry knew much more than he ever wrote. His lectures contained highly original unpublished work in several areas of pure and applied mathematics. He lectured on van der Pol’s equation which two years later was the subject of a paper by his PhD supervisor Littlewood [83, 84]. As the first Soviet Sputnik orbited the earth Cherry’s lectures predicted the perturbations of its orbit. I was fortunate to attend several of Cherry’s lecture courses in Particle and Rigid Body Dynamics, Differential Equations and Orbits, Complex Analysis, the Calculus of Variations, Special Functions and Fluid Dynamics. Although he referred to books his treatment was always original with fascinating insights. He introduced an element of originality, a breadth of approach, and a depth of thought. A full account of Cherry’s mathematical work is in the Bulletin of the London Mathematical Society [88, pp. 230–233].
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