TEACHING AND LEARNING INTRODUCTORY DIFFERENTIAL CALCULUS WITH A COMPUTER ALGEBRA SYSTEM

Margaret Kendal

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Department of Science and Mathematics Education
The University of Melbourne
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DECLARATION

This thesis contains no material that has been accepted for any other degree in any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where reference is given in the text. The thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies, and appendices.

Signature:

Date:
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ABSTRACT

Computer Algebra Systems (CAS), a powerful mathematical software currently available on handheld calculators, is becoming increasingly available to assist secondary students learn school mathematics. This study investigates how two teachers taught introductory differential calculus to their Year 11 classes using multiple representations in a CAS-supported curriculum. This thesis aims to explore the impact of the teaching on students’ understanding of the concept of derivative.

Understanding of the concept of derivative was gauged using an innovative Differentiation Competency Framework that was developed to describe understanding of the concept of derivative. It consists of eighteen competencies for formulation and interpretation of derivatives with, and without, translation between different representations. It clarified the objectives of the curriculum, purpose for using particular CAS activities, and also guided the construction of individual test items on the Differentiation Competency Test that enabled individual and class learning about the concept of derivative to be identified.

The Framework also helped identify each teacher’s privileging characteristics and facilitated analysis of the learning in relation to the teaching.

This study found that using multiple representations was important in developing understanding of the concept of derivative but that the graphical and the symbolic representations were the most useful and important to emphasize and link.

Analysis of the teaching actions showed that the teachers used CAS in ways that were consistent with their teaching approach and preferred use of representations and that a conceptual teaching method and student-centred style supported understanding of the concept of derivative.

Teaching is directly linked to learning and each class developed a different understanding of the concept of derivative that related to the combined effect of their teacher’s privileging characteristics: calculus content, teaching approach, and use of CAS.

This study also shows that if a CAS-supported curriculum is to be successfully implemented, it needs to acquire institutional status including a corresponding change in assessment to legitimize new teaching practices.
Computer Algebra Systems [CAS] technology, readily available on large frame computers since the early 1970s and on microcomputers since the early 1980s, is now accessible on hand held CAS calculators that are becoming increasingly affordable for secondary school students. CAS capabilities include performance of algebraic procedures including calculus, drawing graphs, and the execution of numerical, vector, matrix, and statistical calculations. CAS calculators have the capabilities of graphics calculators together with capabilities for symbolic algebra while graphics calculators have the capabilities of numeric calculators together with capabilities for graphics.

Powerful mathematical software such as CAS changes the way in which mathematics is done beyond school, and also offers many opportunities for new learning experiences. In consequence, it is imperative and urgent that research is conducted into teaching and learning secondary school mathematics with this new, important and increasingly available technology.

This thesis reports about teaching in a time of transition, from a classroom without CAS support, to teaching in a classroom where CAS has the potential to be an integral part of students’ doing and learning mathematics. It examines in detail two teachers’ pioneering use of CAS calculator technology while teaching their secondary school students introductory differential calculus. It explores a wide range of factors related to introducing new technology into the classroom, how these factors influenced the teachers’ pedagogy, and in consequence what the students learned about the concept of derivative. This chapter provides background information about teaching calculus with technology (Section 1.1), briefly describes how this study originated (Section 1.2), and outlines the organization of the thesis (Section 1.3).

### 1.1 CAS in Secondary School Calculus

Calculus is an important area of study in the secondary school curriculum and is very suitable to teach with CAS. The advent of personal computing with CAS in the 1970s coincided with the Calculus Reform Movement in the 1980s with its principal aim of making calculus in USA universities more understandable for more students. During the last twenty years, a body of literature has emerged, including from the Reform
Movement, espousing the potential of CAS to improve the teaching and learning of functions and calculus. General claims have been made that CAS would be a useful tool to assist students in understanding calculus concepts, that it can reduce the necessity to remember rules and procedures for differentiation and integration, and that it can assist in solving more realistic problems (Arnold, 1991; Fey, 1989; Tall & West, 1986; White, 1990). In particular, claims were made that using CAS for multiple representations (symbolic, numerical and graphical) of functions and linking them would facilitate understanding of the concepts (Heid, 1988; Keller & Russell, 1997; Porzio, 1994; Repo, 1994). However, other research suggested that using multiple representations was not always advantageous (Ferrini-Mundy & Graham, 1993; Slavit, 1998). This difference of opinion needs to be resolved. In addition, most of this work has been done with tertiary students so there is less known about using CAS with younger students at secondary school.

Teachers bring to their classrooms different knowledge of mathematics and different underlying beliefs about mathematics and how it should be taught. The literature review presented in Chapter 2 suggests that these factors influence teacher decisions about what mathematical content they will emphasize and the teaching practices they will adopt. There is also evidence to suggest that this privileging by teachers influences the ways they implement technology in their teaching. New technologies offer alternative approaches to solving problems giving teachers even more options for teaching. In consequence it is likely that, with more use of technology, there will be greater variety in what is taught, how it is taught, and in what students learn. In this study, privileging by teachers includes decisions about what is taught and how it is taught: what is emphasized in the content (what is stressed and what is not stressed), what representations are preferred and ignored, the attention paid to procedures and concepts, the attention paid to rules and meaning, and how much is explained or left to the students to work out for themselves.

This study is one of a growing number of studies exploring the possibilities of using CAS with students at school. It examines the teaching and learning that occurred when CAS was used to teach introductory differential calculus in Year 11. The study provides a broad range of information about how a standard curriculum could be modified to incorporate technology, how assessment emphasis could be changed from symbolic to
include numerical and graphical representations, about the teachers’ decisions regarding emphasis and teaching methods, and about what the students learned. It also provides a unique look at differences in teaching, when two teachers introduce the written calculus curriculum they helped to plan to two similar classes at the same school. In this study, the term CAS will be used in the general sense to refer to all the capabilities of the CAS calculator, whereas the term symbolic algebra will be used to describe the facility of the CAS for manipulation of algebraic variables including factorising and expanding expressions, solving literal equations, differentiation, and integration. The term algebra refers to the same symbolic manipulations without CAS.

1.2 Development of the Main Study from the Preliminary Study

This section explains how the main study, conducted in 1999, evolved from the 1998 preliminary study. The preliminary study is described in Section 1.2.1 and the main study in Section 1.2.2. In this section some specific references that relate to the preliminary study are given. Otherwise, literature overviewed in this chapter is more fully reported in the following chapters.

1.2.1 The preliminary study

This section describes how the present study was designed to answer questions that arose from a preliminary trial of teaching with CAS in Australian secondary schools. In 1998, a CAS-active introductory calculus teaching trial was conducted in two secondary schools. Three teachers (from two schools) were invited by university staff to be involved in the preliminary study after their participation in professional development related to teaching with graphics calculators. The same teachers were subsequently invited to participate in the follow-up main study because it was considered important that the project be carried through well. The university staff believed that the three teachers were competent, interested in teaching with graphics calculators, and willing to trial teaching with CAS calculators. The three teachers taught their Year 11 classes (of 16-17 year olds) a carefully designed introductory calculus course with a strong focus on developing understanding of the concept of derivative. As mentioned above, a feature of the new technology that has received much attention in the literature is that it provides for use of numerical, graphical, and symbolic representations of functions and derivatives and it enables easy translations among them. As some authors see
translations between representations as a strong indicator of understanding, the calculus course strongly emphasized links among the representations. The teachers used the CAS calculator as a teaching tool to enhance understanding and as a mathematical assistant to carry out algebraic procedures including symbolic differentiation. The primary aim of the preliminary study was to explore the possibility of introducing CAS to senior school mathematics, and the consequent likely changes in curriculum and assessment. The trial was very successful. The Year 11 students (with CAS) were able to perform Year 12 differentiation problems as successfully as a group of Year 12 students with an additional year’s experience of calculus (without CAS). In addition, the students’ understanding of concepts was generally better than a comparable group of Year 11 students taught without CAS. In addition, their by-hand algebraic manipulation skills seemed not to be adversely affected through use of CAS. These results are reported by McCrae, Asp, and Kendal (1999, 2000).

Although the overall achievement on the tests was similar for all three classes, each made different use of CAS and their different success rates on items seemed to relate directly to the way they had been taught. The teachers gave different priority to particular ways of doing mathematics and using technology. Following Wertsch (1990), this overall set of biases was termed the teachers’ privileging. The teachers’ privileging seemed to explain the differences in methods selected by the students. The students in the first class preferred to use (and even overuse from the authors’s perspective) symbolic algebra, rarely used graphs, and were successful on symbolic items but not on conceptual items. Their teacher privileged using technology, particularly symbolic algebra and procedures for standard tasks. The students in the second class, whose teacher privileged understanding and by-hand algebra (i.e., symbolic manipulation), preferred by-hand algebra, used graphs to enhance understanding, and were more successful on conceptual items. The students in the third class (with relatively weak algebraic skills both at the beginning and end of the study) used the graphical features of CAS to great advantage. To compensate for their weak symbolic manipulation skills, they substituted graphical procedures whenever possible, and were the most successful class on conceptual items. Their teacher privileged understanding and graphical methods. In total, three significant privileging characteristics were identified: calculus
content, teaching approach, and use of CAS. These results have been reported in Kendal and Stacey (1999a, 1999b).

The word privileging was adopted to describe the (mostly unconscious) biases of teachers. Privileging is a term used by Wertsch (1990) who explained how different forms of mental functioning dominate in different socio-cultural contexts. For teaching and learning in a technological context, Berger (1998) wrote “the social setting and values . . . may elevate one form of mental functioning over another and in this way privilege a particular form of mental operation such as algebraic or graphical reasoning” (p.19). Kendal and Stacey (2001a, 2001b) have further explored the notion of privileging. They believe that it reflects the teacher’s underlying beliefs about the nature of mathematics and how it should be taught and is derived from the interplay of teachers’ beliefs and interrelated knowledge sources (content, content pedagogical, and pedagogical), and is moderated by institutional knowledge about students and school constraints.

The preliminary study had established a number of important parameters for teaching introductory calculus within the constraints of the local schools and the state’s official mathematics curriculum, the Victorian Certificate of Education [VCE] Mathematics Methods 1 & 2 Course (Board of Studies, 1996a):

- A set of lessons that focussed on using CAS to develop understanding of calculus concepts had been trialled successfully.
- Students could learn to use CAS in the context of learning a new topic area, calculus, especially with a background of using graphics calculators.
- Understanding of differentiation could be achieved through teaching numerical, graphical, and symbolic representations; and translations between the representations.
- The teachers’ role in implementing curriculum with technology was seen to be crucial.

The preliminary study had also shown that the testing program (for the VCE Mathematics Methods course) was essentially symbolic and understanding of numerical and graphical representations and translations among all three representations were only incidentally monitored. The preliminary study had also highlighted the differences in
teachers’ privileging, the way that the new technology apparently gave more opportunities for differences, and provided initial support for strong links between teacher privileging and students’ learning. It was therefore decided to continue the investigation of some of these issues by conducting a further, more specific study with the same teachers and same calculus curriculum (and same provision for use of CAS) but with an assessment tool that was more inclusive of multiple representations of derivative and translations between different representations. The second study is reported on in this thesis and is referred to as the main study.

In summary, the preliminary study showed that teaching calculus in Year 11 with CAS calculators could enhance learning and demonstrated that further work towards long-term introduction of CAS into senior mathematics subjects in the VCE was worthwhile. The main study takes up this challenge, especially examining students’ learning of the multiple representations of differentiation and links between them and how the varieties of teachers’ privileging and use of CAS impacted on this learning.

1.2.2 The main study

This section describes the conduct and purpose of the main study that was conducted in 1999 after two of the three teachers who had participated in the preliminary study were willing to teach the calculus course for the second time (as discussed above). The third teacher was on leave in 1999. The teachers (from the same private Victorian girls’ school) and their new cohort of Year 11 students would once again have full access to CAS with its numerical, graphical, and symbolic capabilities and the teaching and student learning about differentiation would be monitored. The results of this second study are reported in this thesis.

Informed by the experience of the preliminary study, and motivated by the possible introduction of CAS into secondary schools and uncertainty about the ways in which CAS can profitably be used in teaching, this thesis aims to explore the impact of teaching on students’ understanding of the concept of derivative in an introductory calculus course using multiple representations and the functional and pedagogical CAS activities provided in the curriculum. Of particular interest is the effect that the teacher privileging characteristics (calculus content, teaching approach, and use of CAS) has on what the students learned. Understanding of the concept of derivative is gauged by
acquisition of differentiation competencies based on an innovative conceptual framework (inclusive of different representations and translations between them). Effects of student-related factors on learning are also observed including school attainment level, attitude to calculus and CAS, and use of technology. The principal questions to be addressed are listed below and further expanded in Section 4.1.1.

Research Questions
Research Question 1. What was the role of CAS in assisting students to develop understanding of the concept of derivative?
Research Question 2. How did teacher privileging (with respect to calculus content, teaching approach, and use of CAS) influence the learning that occurred in each class?
Research Question 3. Did the teachers’ privileging stay the same when they taught with CAS for a second time?

Use of a test, based on the framework, helped establish what each class understood about the concept of derivative and enabled similarities and differences in understanding between the classes to be identified.

This study shows that the learning that occurred in each class is related to their teacher’s particular variety of privileging.

1.3 Outline of the Thesis
The thesis will be developed as follows:

Chapter 2 presents a review of recent literature about teaching calculus with CAS technology and the influence of teachers on student learning outcomes.

Chapter 3 outlines the development of the theoretical framework and the test instrument for measuring learning about different representations of derivative and translations between them.

Chapter 4 describes the conduct of the main study, including the design of further instruments for measuring learning and the observation of the teaching.

Chapter 5 describes the teachers’ lessons and gives, as examples, a detailed description of three calculus lessons. It identifies each teacher’s privileging characteristics (together with some of the reasons for these). Changes in privileging
characteristics from the preliminary study to the main study are also described, providing an answer to Research Question 3.

Chapter 6 reports the learning of the concept of derivative that was achieved by each class. Learning was gauged by class results on a written test that was designed to measure learning in each of the representations of differentiation (symbolic, numerical and graphical) and translations between them.

Chapter 7 reports the results of a second written test, a questionnaire, an evaluation of use of the calculator (completed by all students), and student interviews. These instruments provide further insights into what each class learnt about the concept of derivative, how they used symbolic algebra, attitudes towards using CAS, and ways they were taught. Finally, limitations of the methodology are reported.

In Chapter 8, teaching is linked to learning. The ways each teacher taught, including with CAS, (reported in Chapter 5) are linked to what the students in each class learned (reported in Chapter 6 and Chapter 7) enabling Research Questions 1 and 2 to be answered and the answers to the three Research Questions are summarized before analysis in Chapter 9.

Finally, in Chapter 9, an overall assessment of the impact of teaching a CAS-supported CAS curriculum and its impact on learning is presented.
2. THE LITERATURE REVIEW

This chapter reviews the literature that guided the development of this study. It describes the influence of the Calculus Reform Movement on new approaches to teaching introductory calculus (Section 2.1) and reports studies where understanding calculus concepts is associated with use of technology (Section 2.2) and with use of multiple representations (Section 2.3). The necessity and rationale for developing a conceptual framework for the concept of derivative are also discussed (Section 2.4). Next, teacher-related studies about factors that influence the ways teachers teach with technology, curriculum design (Section 2.5), and teacher characteristics (Section 2.6) are reported. Finally, studies that link understanding of calculus concepts with particular teaching practices are reviewed (Section 2.7) and the issues arising from this literature review are summarized (Section 2.8).

2.1 Calculus Reform

The Calculus Reform Movement’s influence on modern approaches to teaching introductory calculus is discussed (Section 2.1.1) and the capabilities of hand-held technology with a CAS facility (Section 2.1.2).

2.1.1 The Calculus Reform Movement

Historically, the study of calculus is considered to be an extremely important component of mathematics. However, by the mid 1980s USA educators believed it was in crisis due to the difficulties that students were experiencing in understanding the concepts involved and an over reliance on algebraic techniques. Hughes-Hallett et al. (1994) wrote:

> Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences. Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures —thereby losing sight of both the mathematics and of its practical value. (p.vii)

At the same time, personal computer technology was becoming increasingly available in college classrooms offering new approaches for teaching calculus and a better opportunity for students to understand underlying calculus concepts. Ferrini-Mundy and
Graham (1991) stated “technology is proposed as a means of freeing students from algebraic manipulation, reducing the drudgery of calculation, supporting the learning of fundamental ideas, and allowing the exploration of concepts” (p.628). These sentiments were central to the Calculus Reform Movement although Tall (1996) believes that a wider range of factors motivated the reforms including:

- altruistic desires to make calculus more understandable for a wider range of students,
- commercial desires to produce saleable products, practical considerations of what actually needed to be taught, reflection on the type of mathematics that is suitable in a technological age, and a growing aspiration to research the learning process to understand how individuals conceptualize calculus concepts. (p.291)

Thus, the Calculus Reform Movement inspired changes in the ways calculus was taught and learned. Tucker and Leitzel (1994) reported:

The hallmarks of calculus reform - changes in modes of instruction and use of technology along with increased focus on conceptual understanding and decreased attention to symbolic manipulation - are finding their way into both pre-calculus and post-calculus mathematics courses. (p.1)

The study reported in this thesis continues the spirit of the Calculus Reform Movement by introducing school students to differential calculus using technology (a hand held calculator with CAS) with the objective of assisting them to develop understanding of the concept of derivative.

### 2.1.2 Hand-held calculators with CAS

By the early 1970s the first computer algebra systems (such as MACSYMA of MIT) were available on main frame computers. Since then the technology has rapidly evolved. The first CAS to run on a personal computer, muMATH, was in use by the early 1980s but was quickly superseded by programs like Mathematica, Maple, and Derive. A similar evolution has occurred with hand-held pocket calculators. Simple programmable calculators with BASIC programming were developed in the early 1980s. By the late 1980s sophisticated graphics calculators were available and by 1995, graphics calculators with computer algebra systems (CAS) were available (Wilf, 1982; see also Arnold, 1996). Currently, the unit price of CAS calculators is reducing so they are becoming increasingly affordable for students in secondary school classrooms, making research into teaching and learning with them increasingly urgent.

CAS (including calculators such as the TI-92) can perform all the routine procedures of mathematics normally covered in secondary school. It executes exact calculations,
draws graphs, calculates with vectors and matrices, and does statistics. It also does algebra (e.g., simplify expressions) and calculus (e.g., compute routine symbolic derivatives and integrals). Additional facilities include animation of graphs and 3-dimensional plotting. In short, it automates most of the calculation skills taught in school mathematics. The TI-92 (equipped with a version of the computer software Derive) is an ideal tool for use in upper secondary school level mathematics (and lower tertiary level) since it allows students to develop understanding of mathematical ideas, through investigation and the modelling of real-life situations, without the difficulties imposed by complex algebra (Barling, 1996; Brown, 1996b). The TI-92 CAS calculator was selected for use in this secondary school study of teaching and learning calculus supported by CAS.

2.2 Use of Technology for Understanding Calculus Concepts

Claims that traditional difficulties associated with learning calculus may be overcome with use of technology are reported (Section 2.2.1) and the potential of technology to assist students in developing understanding of calculus concepts is discussed including the involvement of multiple representations (Section 2.2.2). Next, the types of representations involved in this study are clarified (Section 2.2.3).

2.2.1 Technology and traditional difficulties with calculus

This section reports the traditional difficulties experienced by many students while learning introductory calculus without technology and how technology might assist in overcoming some of them.

A considerable body of literature has identified common student difficulties including:

- Algebraic manipulation (Orton, 1983).
- Understanding limits (Barnes, 1991; Cornu, 1992; Ferrini-Mundy & Graham, 1991; Orton, 1986; White, 1993).
- Use of notations (Frid, 1992; Orton, 1986; Tall, 1985).
- Understanding the concept of function (Markovits, Eylon, & Bruckheimer, 1988; Tall & Bakar, 1992; White, 1993).
There is a commonly held belief that problems associated with algebraic manipulations can easily be overcome by using the symbolic algebra facility of CAS (Bennett, 1995; Day, 1993; Heid, 1988; Tall, 1996). Bennet (1995) and his teaching colleagues monitored a one-term general education calculus course with several classes of undergraduate students of varying abilities. The college teachers informally evaluated the course and they found that their students were able to tackle problems in a variety of different ways. Bennet found that he spent less class time dealing with algebraic problems and, in consequence, he was able to spend more time discussing calculus concepts with the students. He strongly recommends use of CAS (Derive) to minimize the detrimental effects of students’ poor algebraic backgrounds.

Researching understanding limits with CAS, Monaghan, Sun, and Tall (1994) investigated students’ understanding of the limit concept by comparing a group of nine college students who had access to a CAS (Derive) with a comparable group of nineteen senior secondary school students who had studied an identical curriculum without CAS. The researchers matched the groups very carefully so that the differences between the school and college students would be minimized. The results indicate that the limit concept is not “easy” to understand and that simple approaches, with and without CAS, do not allow students to appreciate the depth of the limit concept. However, there were indications that a combination of alternative approaches including use of CAS might assist students to develop a more flexible proceptual view of limit.


The graphical approach allows the limit notion to be handled implicitly, for instance, by magnifying the graph to see it looking ‘locally straight’ so that the gradient required is that of the visibly straight graph. This helps the move into elementary calculus without stumbling over the limit concept but the deficiency may need to be addressed at a later stage when further cognitive reconstruction may prove necessary to cope with formal concepts. (p.295)

Frid (1992), in a wider study of comparative methods of instruction for college calculus, interviewed students and observed that those from the class that had used Tall’s approach were better able to explain the underlying notion of local linearity using the notion of local straightness.
Thus, for this present study of elementary introductory calculus it was decided to adopt Tall’s alternative approach using the graphical facility of the CAS calculator to help students understand local linearity of a curve at a point without formal reference to limits. Formal treatment of differentiation using “first principles” would be deferred until later in the course or until the following year. Using the CAS calculator should also help students overcome algebraic difficulties. Other traditional problems associated with calculus are not addressed in this thesis.

2.2.2 Using technology assists understanding of calculus concepts
This section reports claims that improved understanding of calculus concepts (in general) is associated with use of technology and suggests reasons why this may be the case.

Improved understanding of calculus concepts
Computers can perform time-consuming tedious and laborious activities (e.g., drawing graphs, solving complex equations, and finding derivatives) theoretically making it possible for students to solve more realistic problems and develop understanding of calculus concepts. Since the early 1980s numerous general claims have been made about the likely benefits of using computer tools to improve understanding of calculus concepts (Arnold, 1991; Fey, 1989; Tall & West, 1986; White, 1990). For example, Heid (1988, p.4), commenting on a body research conducted during the previous ten years, states "Computing devices are natural tools for the refocusing of the mathematics curriculum on concepts." Subsequent research also supports this claim (Cooley, 1996; Ellison, 1994; Hillel, 1993). Ellison’s study indicated that using TI-81 graphing calculators and computer software assisted her 10 college students to mentally construct an appropriate concept image of the concept of derivative. However, not all the students developed a mature concept image. Reporting informally on a remedial teaching program (for 22 college students) that integrated a CAS (Maple) into a course of calculus Hillel (1993, p.46) observed benefits to student learning: "students coming out of it had acquired different types of insights and knowhows than the traditionally-prepared students - insights and knowhows which we felt were closer to the essence of calculus.” Cooley (1996) also reports the positive effect on achievement and understanding the concepts (limit, derivative, instantaneous rates of change, integral,
maximum and minimum, and curve sketching) after integrating a CAS (Mathematica) into an introductory college calculus course. The students in the technology group scored significantly higher in the conceptual area of derivative (plus limit and curve sketching) than the non-technology group. The technology group also scored higher on the traditional computational calculus questions indicating that these students did not suffer any loss of computational skills and that the conceptual bias of the teaching may also have helped them perform algorithmic processes.

Other research also suggests that skill acquisition is not a prerequisite for understanding of concepts or applications and that, when conceptual understanding of calculus concepts is emphasized first, computational skills are not lost (see Heid, 1984, 1988; Judson, 1988; Palmiter, 1991; Repo, 1994; Schrock, 1990).

For example, Heid (1988) conducted a study in which two classes of college students were taught calculus for fifteen weeks. Computers were used as a tool to facilitate the investigation of the concepts that underpinned the calculus being studied. The experimental class focused on concept development for twelve weeks and they were then taught skills during their final three weeks. The control group studied the skills for the entire fifteen weeks. Heid found students in the experimental group showed better understanding of the concepts of the calculus (such as the meaning of the derivative) than the control group and there was little difference on a final exam of routine skills. Their learning of concepts greatly improved and the students performed almost as well in routine skills as the control group. Specifically, they were able to express ideas in their own words and their conceptualizations were broader, clearer, more flexible and more detailed than students in the control group. Heid interpreted these results as evidence that students can understand calculus concepts, without prior mastery of basic calculus skills, showing that it was possible to reorganize the order in which calculus is taught to students, to focus on concepts prior to teaching procedures. The students reported feeling that the computer relieved them of some of the manipulative aspects of calculus work, that it gave them confidence on which they based their reasoning, and it helped them focus on more global aspects of problem solving. During the instruction the students were involved in discussing ideas and were required to make sense of calculus related language, including terminology and symbols. Frid (1992) suggested
that the nature of instruction including ways the computers were used, may have also influenced the nature and role of the conceptualizations the students made.

Palmiter (1991) reports a study in which 40 college students were taught calculus using a CAS (MACSYMA) for five weeks, and they used the software for techniques. The 40 control students were taught by traditional methods for ten weeks using paper and pencil computations. At the end, both groups took a conceptual examination under identical conditions and a computational examination during which the MACYSMA group was allowed to use the technology for one hour and the control group was allowed two hours. The MACYSMA group achieved higher scores on the test of conceptual knowledge of integration (definition, purpose, and application of the integral). They also scored significantly better than the control group on the computational exam, in half the time.

These studies (and others) attributed improved understanding of calculus concepts to use of CAS and since the concepts before skills structure was suitable for the concept of derivative it was adopted for the present study with its focus on understanding.

**Reasons for improved conceptual understanding**

Visualization and use of multiple representations have been proposed to explain why technology assists students to be successful on conceptual items. Tall and West (1986) attribute improved understanding to the power of the visual image to building up concepts:

> The human brain is powerfully equipped to process visual information. By using computer graphics it is possible to tap this power to help students gain a greater understanding of many mathematical concepts. Furthermore, dynamic representations of mathematical processes furnish a degree of psychological reality that enables the mind to manipulate them in a far more fruitful way than could ever be achieved starting from static text and pictures in a book. (p.107)

Dubinsky and Tall (1991) believe that computers can give meaning to "abstract" ideas by representing them as "concrete" objects (symbols, numbers, and figures) able to be manipulated through the interactive potential of the software and reflected upon and, in consequence, more likely to be understood. They state that "Indeed, it is in general true that whenever a person constructs something on a computer, a corresponding construction is made in the person’s mind (p. 235)." However, use of
multiple representations; easier to access with technology, is reported as the most important aid to conceptual development (Fey, 1989; Norman, 1993, and others).

Students’ understanding of and facility with the multiple representations of calculus concepts and processes - symbolic, graphical, and numerical - are essential for thorough understanding of the calculus and problems that require calculus. (Norman 1993, p.70)


The ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allow one to see rich relationships, develop a better conceptual understanding, broaden and deepen one’s understanding, and strengthen one’s abilities to solve problems. Connectedness between different representations develops insights into understandings of the essence as well as well as the many facets of a concept. (p.105)

Thus, use of multiple representations appears to be very important and studies that report its association with understanding calculus concepts is delayed until Section 2.3. In Section 2.3, the large body of literature that analyses the “general” notion of representations and translations between them is acknowledged. The representations involved in this study are clarified next in Section 2.2.3.

2.2.3 Representations in this study

The collections of articles in Problems of Representations in the Teaching and Learning of Mathematics (Janvier, 1987a) and in two consecutive Special Issues of the Journal of Mathematical Behavior (Numbers 1 & 2 of Volume 17, 1998) address fundamental theoretical and practical issues related to representation. These articles show that a full treatment of the role of representations in mathematics is complex. Goldin (1998) describes theoretical interpretations of the term "representation" and "system of representation" in connection with mathematical learning, teaching, and development. He also explains in detail the notion of mathematical cognition taking place in several internal representational systems that are not directly observable.

In this study the term representation is used to refer to three external representations used for functions and their derivatives. These are the numerical, graphical, and symbolic representations that are used in traditional mathematics and described by Goldin and Kaput (1996, p.415) as part of the "formal" and "mathematical system of representation". Goldin and Kaput (1996) regard external configurations as those "accessible to direct observation (speech, written words, formulas, concrete manipulatives, computer microworlds as they appear on the screen . . (p.402)". 
In response to the literature review, the CAS-active teaching program devised for this study gave relatively equal emphasis to these three external representations of derivative, and to linking them. These external representations were also available on the CAS (TI-92) through the three distinct software modules: numerical, graphical, and symbolic, modules that were easy to link. The author monitored the three external representations that the teachers used in their teaching and that the students used while they were being assessed. She also monitored translations among the different representations of derivative.

Janvier (1987, p.29) states "A translation involves two modes of representation. For the modes of equation and graph, for instance, we have the translations: “graph \rightarrow equation” and “equation \rightarrow graph”. To achieve directly (and correctly) a given translation, one has to transform the source “target-wise” or, in other words, to look at it from a target point of view and derive the results. Kaput (1992, p.524) has included “translations between notation systems” as one of the four classes of school mathematics and explained the value of technology in enabling manipulation of mathematical concepts both within and between these different representations. Goldin and Kaput (1996) believe that teaching representations with CAS has changed the nature of graphical and numerical representations (and linking among the three representations) from static to action that is perceived as beneficial to understanding representations. This study provides an opportunity to explore representations and translations among representations using a CAS-active curriculum.

Goldin and Kaput (1996) describe internal configurations as "those characteristics of the reasoning individual that are encoded in the human brain and nervous system and are to be inferred from observation (p.402)". In this study, students’ understanding of formulation and interpretation of derivatives (defined below) will be monitored from their behaviour as they use the numerical, graphical, and symbolic representations, and translate among them.
2.3 Use of Multiple Representations for Understanding Calculus Concepts

Studies are now reported in which improved understanding of calculus concepts is associated with use of multiple representations with CAS (Section 2.3.1) and others in which use of multiple representations was problematic (Section 2.3.2).

2.3.1 Using multiple representations assists understanding of calculus concepts

A considerable body of research has been conducted over the last twenty years that supports the contention that conceptual understanding of calculus concepts is enhanced through use of multiple representations (numerical, graphical, and symbolic) and linking the representations using computer or calculator technology (Heid, 1988; Keller & Russell, 1997; Porzio, 1994; Repo, 1994).

For example, in Heid’s (1988) study discussed above (Section 2.2.2) college students demonstrated better understanding of the concepts of calculus associated with their use of multiple representations: computer generated graphs, tables of values, and algebraic techniques to solve real world problems.

Computers lent flexibility to the analysis of problem situations. Their easy display in a large range of representations made feasible consideration of more difficult problems, opened avenues for exploring several methods for solution for a single problem, and created an environment amenable to convenient exploration of the effects of changing parameters. (p.10)

Keller and Russell (1997) report a large study involving hundreds of students studying college calculus with CAS (TI-92, a hand-held calculator) or without CAS. It explores students’ ability to solve problems and found, almost without exception, that the students who used CAS were more successful at symbolically solving problems than the comparison students. With appropriate instructional emphasis they were empowered to make sense of problems and they frequently used graphical and numerical representations to help them reason about the symbols involved.

As a further example, Repo (1994) reports an empirical investigation dealing with the teaching of the concept of derivative, with focus on developing conceptual understanding, to a class of 17 upper secondary school students (age 16-19) and control classes. In the investigation the experimental students constructed the concept of derivative using a CAS (Derive) involving numerical, graphical, and symbolic
representations of derivative, in student-centred interaction with the teacher and other pupils. The control group was given standard mathematics teaching in differential calculus. Both groups of students were tested at the conclusion of the study and a written retest was administered six months later. The results indicated that the pupils in the CAS group had constructed higher conceptual understanding than in the control group and had retained their algorithmic skills better.

Porzio (1994) reports a study in which different classes of college students were taught calculus by alternative methods: with CAS (Mathematica), or with a graphics calculator, or traditionally without technology. The group of students who used CAS was the most proficient in using different forms of representations (symbolic, numerical, and graphical) and combinations of different representations. They were also superior in making connections between the representations. The study concluded that student understanding of calculus concepts is likely to improve more when the curriculum includes suitable activities to emphasize connections between different representations through appropriate use of technology than if technology is simply added to an existing curriculum.

These studies present convincing evidence that use of multiple representations is likely to improve students’ conceptual understanding of calculus concepts and so a CAS supported curriculum that focused on multiple representations of derivative was adopted for this study. However, some specific difficulties associated with use of multiple representations have also been reported and are discussed below.

2.3.2 Difficulties associated with using multiple representations

Some studies have reported students’ difficulties linking representations and moving flexibly between different representations. For example, Ferrini-Mundy and Graham (1993), in a synthesis of some of the research related to calculus learning, commented that college students were often comfortable with different results in different representations and did not always realize that the results were inconsistent.

[Students’] concepts, methods of reasoning, and repertoire of heuristics are radically different in each of these representations, and if students are comfortable with the inconsistencies, contradictions, and competing meanings that emerge as a result, then the challenge of helping them reach a workable means of connecting these representations is very complex. An additional variable of course, is the part to be played by technology. (p.44)
Other studies report difficulties related to translating between representations and report that students may have a surface ability to link representations without an understanding of the deeper conceptual links between them (see Greer & Harel, 1998; also Hong, Thomas, & Kwon, 2000)

Slavit (1998) reports that three secondary school students (using graphics calculators and with predominantly symbolic notions of algebra) experienced difficulties, both procedurally and conceptually, linking symbolic and graphical representations. They preferred to use previously learnt strategies rather than more recently taught new approaches.

Thus, it is important to explore the role of technology and the learning mechanisms in moving between representations, particularly since there is a paucity of studies about how this occurs. Even (1998) acknowledges that more research is necessary.

However, even though there is an agreement among mathematics educators today on the importance of different representations in the learning of mathematics, not much is known about the nature of the processes involved in working with different representations. (p.105)

Although there is a significant body of literature that claims that using CAS will assist students (mostly college students) to develop understanding of calculus concepts, there are very few studies reporting about secondary school students using multiple representations with CAS to develop understanding of the concept of derivative, particularly when the course of study is their very first introduction to calculus. Research involving secondary school students is important because it addresses this gap in the research. It also contributes to understanding how using multiple representations with powerful hand-held personal technology should be adopted in classrooms prior to its inevitable use (in the near future) by secondary school students. In addition to the benefits to understanding the concept of derivative, this study will also be able to explore difficulties associated with using multiple representations.

2.4 The Concept of Derivative
This section reports theoretical approaches to understanding calculus concepts and establishes the requirements of a framework for the concept of derivative (Section 2.4.1) and the need to assess multiple representations of derivative (Section 2.4.2).
2.4.1 Towards a conceptual framework for the concept of derivative

This section reports different approaches to understanding calculus concepts and identifies cognitions that should be included in a framework for the concept of derivative.

Tall (1996) outlines three possible teaching approaches for understanding the calculus. The first is related to understanding real-world calculus, and the second, suitable in elementary calculus, is a more theoretical approach that involves numerical, symbolic, graphical representations. The third approach is the formal definition-theorem-proof approach of analysis. Tall proposes a model that links these approaches through “doing” and “undoing actions” but it includes calculus content outside the requirements of the curriculum for this introductory study of the concept of derivative.

Based on his own personal experiences of teaching calculus with graphics calculators with CAS, Dick (1996) focuses attention on teaching the concept of derivative involving numerical, graphical, and symbolic representations.

The numeric and graphic capabilities of the new technology are bringing truly new tools to the calculus classroom. In particular, we now have the means to take a multi-representational approach to the concept of derivative: numerically using the calculator to approximate difference quotients quickly and easily, graphically to exploit the local linearity of differentiability in a dynamic visual way, and symbolically using perhaps both traditional paper-and-pencil methods as well as some of the symbolic algebra capabilities found even on hand-held machines. The opportunity is there for us to help students build a rich fabric of understanding of the derivative, with many connections between the representations. (pp.44-45)

These recommendations for teaching differentiation (possible with CAS) in each representation are very important and were used in this study (see Section 2.5.3 for further discussion) and the cognitions associated with differentiating in each representation will need to be accommodated in a framework suitable for the concept of derivative, together with translations between representations as proposed by Janvier (1987b) (see Section 2.2.3).

Vergnaud (1998) proposes that a full definition of the concept of derivative includes procedural knowledge of differentiation and links between representations.

. . . the concept of derivative involves knowledge of its numerical, graphical, symbolic verbal (and probably kinaesthetic) forms (representations) and an awareness of the links between them. Each representation of differentiation is associated with specific procedural knowledge. (p.177)
O’Callaghan (1998) identifies four components required in problem solving involving different representations:

1. Modelling: the transition from problem situation to a mathematical representation of the situation.
2. Interpreting: understanding different representations in terms of real-life applications.
3. Translating: the ability to move from one representation of function to another.
4. Reifying: creation of a mental object from what was initially perceived as process or procedure.

For this study of introductory differential calculus, three of O'Callaghan’s components of knowledge for the concept of function proved to be useful as a suitable basis for the construction of the Differentiation Competency Framework for understanding the concept of derivative in a multiple representational environment. These are modelling (relabelled as formulation), interpretation, and translation. The proposed framework accommodates Dick’s approach to teaching differentiation in each representation, Janvier’s (1987b) definition of translation, and Vergnaud’s (1998) definition for the concept of derivative and is also consistent with Tall’s approach (excluding the formal definition-theorem-proof approach of analysis).

This section has identified the component cognitions, associated with a full understanding of the concept of derivative, that have guided the development of a suitable framework for differentiation. The resulting Differentiation Competency Framework will enable the objectives of the curriculum to be defined and the purpose for particular CAS activities to be clarified. In addition, it will enable appropriate test items to be constructed to monitor students’ understanding of each competency, involving multiple representations and translation between different representations (see Section 2.4.2 below).

2.4.2 Assessment of multiple representations of derivative

This section emphasizes the need for a suitable assessment tool that reflects the objectives of the teaching by testing understanding of the concept of derivative involving multiple representations.
As teachers learn to teach with technology and give more emphasis to multiple representations of calculus concepts, traditional conceptual and skill items may no longer be the most appropriate to test the learning that has occurred (Dugdale et al., 1995). Penglase and Arnold (1996) observe that a significant aspect of learning to use the new tools involves learning to ask new questions to match the new emphases in instruction.

It seems that differences in instruction which are associated with graphics calculator use would call for test items, and possibly test emphases, which may vary considerably from those of traditional tests. (p.83)

Thus, in a study of derivative an assessment tool is required that monitors students’ learning about numerical, graphical, and symbolic representations of the concept of derivative, and how they link together to form a holistic understanding of the concept of derivative. The assessment instrument should test the cognitive processes in each representation and translations between representations. Individual test items would test individual competencies that together comprise the concept of derivative. The Differentiation Competency Framework (described above, Section 2.4.1) would allow this to be achieved.

2.5 Calculus Curriculum with Technology
It has been established that with technology, students have an opportunity to develop understanding of the concept of derivative. However, the construction of an appropriate calculus curriculum and its implementation by teachers are also important. This section reports studies that have adopted particular curriculum designs (Section 2.5.1) and that have recommended ways to help teachers prepare to implement a CAS-supported curriculum (Section 2.5.2). Finally, types of CAS activities that should be incorporated in the curriculum are discussed (Section 2.5.3).

2.5.1 Varieties of curricula with technology
Since the mid-1980s a variety of technology-supported reform curricula have been designed and implemented with the goal of helping more students understand calculus concepts better (Hillel, 1993; Murphy, 2000). Some of these designs are outlined below, grouped into two main categories with the same overall focus: organization of the curriculum, and teaching approach to implement the curriculum.
Variation in curriculum organization

Research studies have adopted different curriculum designs that have had a primary focus on curriculum structure.

- **Computer-integrated approach:** Suitable computer activities for students to use are integrated into an existing curriculum (Connors, 1996; Dunham, 1991; Martinez-Cruz, 1993; Rochowicz, 1996).

- **Resequence calculus courses:** Calculus concepts and applications are emphasized and taught before manipulative skills in the belief that students could learn the necessary skills once they had a good understanding of the underlying concepts (Dubinsky & Schwingendorf, 1991; Heid, 1984, 1988; Judson, 1988; and others).

- **Applications/problem solving approach:** Emphasis on applications and real life modelling is viewed as essential for the development of conceptual understanding. O’Callaghan (1998) supports this notion in the context of functions and Kissane (1996) in the context of calculus problems.

In addition to the designs above, two other approaches have particularly focused on conceptual understanding.

- **Multiple representations approach.** In the introduction to their textbook, *Calculus*, Hughes-Hallett et al. (1994) write:

  "One of the guiding principles is the 'Rule of Three,' which says that wherever possible topics should be taught graphically and numerically, as well as analytically [algebraically]. The aim is to produce a course where the three points of view are balanced, and where students see a major idea from several angles. (p.121)"

  With this approach, emphasis is given to understanding numerical and graphical representations in addition to the traditional symbolic representation (Ellison, 1994; Hart, 1992, Heid, 1984, 1988; Klein, 1994; Park, 1993; Porzio, 1994; Repo, 1994).

- **Alternative representations approach:** There is a body of literature that argues for broadening the number of representations. For example, Kennedy (2000) recommends that the verbal representation be included as a fourth essential representation for a "Rule of Four". Kaput (1998) also describes the inadequacy of the "Big Three" to link representations and urges that students experience real
situations and phenomena and imbed their use of function in real data. Tall (1996) recommends that for a mature understanding of calculus concepts the range of representations be widened beyond the numerical, graphical, and symbolic representations to include enactive (visuo-spatial) and formal representation. Bowers (1999) and Leinbach (1996) claim that animation (“moving pictures”) will help students to visualize (“see”) the mathematical properties under consideration such as the “limiting case” in differential calculus and properties associated with standard graphical transformations. Lin (1993) also recommends using interactive dynamic programs to help students observe the effects of changes in graphical and symbolic representations and to meaningfully link the representations.

As discussed earlier in Section 2.2.2, the present study was designed with a curriculum focus and calculus concepts were emphasized prior to emphasis on skills (re-sequence calculus course approach) involving numerical, graphical, and symbolic representations (multiple representations approach). In addition, CAS activities consistent with the alternative representations approach were incorporated into the curriculum to help students to develop deeper understanding the concept of derivative. For example, dynamic interactive CAS activities were incorporated into the teaching program to allow the students to experience changes in representations and to visualize the mathematical properties in their mind’s eye.

Variation in teaching approach

Other curriculum designs focused on teaching approaches. The first approach outlined below is a teacher-centred method where the teacher guides students’ learning whereas in the other two approaches are student-centred and the students have more input and control over their own learning.

- **Traditional topped up with technology approach:** Computer-based teacher demonstrations are added to the traditional curriculum and lecture style teaching methods and students are given assignments that involve use of technology (Campbell, 1996; Klein, 1994).

- **A laboratory approach:** Students are encouraged to observe, identify, explore, analyze, and explain so that emphasis was moved from calculation to

- **Cooperative learning approach and student construction of meaning**: Students are encouraged to construct meaning for themselves through explorations with computers (Bosche, 1998; Crocker, 1991; Repo, 1994). For example, Keller and colleagues report that student centred instruction (cooperative learning, laboratory experiments, variety of discussion formats) had a positive effect on students' overall success in calculus (Keller & Russell, 1997; Keller, Russell, & Thompson, 1999a, 1999b).

Classroom organization and its impact on learning was also important in this study since the preliminary teaching trial showed that one teacher adopted student-centred teaching practices and the other teacher-centred practices. Classroom organization was not prescribed for the present study because one important aim of the study was to see if the teachers used the same teaching approaches during their second experience of teaching the curriculum (main study), or if they had changed them from the preliminary study.

Thus, features from a variety of curriculum designs that have been successfully used for teaching the concept of derivative supported by use of technology were incorporated into the CAS-supported curriculum for this study.

Recent literature, not available at the beginning of the study, has started to assess the value of different classroom organization and the role of CAS itself in helping students to learn. For example, Guin and Trouche (1999) and Trouche (2000) report an innovative, teaching approach that was found to be successful over three years with ten classes of secondary school students who used graphics calculators for three months, then CAS calculators for six months. The traditional organization of the classrooms was changed and the teachers used new management styles that involved guiding the work of a "sherpa student" who, using an overhead projection of the calculator screen, plays a central role as a guide, assistant and mediator. This arrangement promotes various levels of discussion for meaning within the classroom and enables the teacher to be more aware of and responsive to student thinking.
Based on a two-year classroom experiment with CAS in secondary school classrooms by the DIDIREM team (University of Paris), Lagrange (1999a, 1999b) found that when the students used CAS for technical work associated with problem solving at various levels of difficulty, they developed mathematical meaning for the tasks when discussion and reflection accompanied the performance of the task. He argues that the learning potential of CAS is to be found in the wide range of varied and richer techniques that CAS has to offer. When the technical work is non-trivial, it plays an important role in facilitating conceptual understanding.

2.5.2 Preparing teachers to implement a CAS curriculum

As suggested above, the success of a curriculum depends on its effective implementation by teachers. Nowakowski's (1992) study involved him in showing secondary school teachers how to integrate CAS into their teaching practices using computers. He reports that the teachers who participated in the project were subsequently able to integrate the use of CAS into their traditional mathematics instruction if they were provided with hands on experience with CAS, given sessions that focused on pedagogical issues, and if they were involved in developing and presenting CAS based curriculum topics. Thomas, Tyrrell, and Bullock (1996) observe that the twelve teachers in their one-year study required a long period of familiarization before they felt confident in teaching with computers in their lower level secondary school classrooms. For the present study, the teachers who volunteered to participate in the study were assisted in learning how to use the CAS calculator, over a six-month period, prior to the commencement of the study. They were also given the opportunity to help refocus the school’s existing curriculum for Mathematics Methods (Board of Studies, 1996a, 1996b) towards conceptual understanding of the concept of derivative (based on the Differentiation Competency Framework). This included the decision to teach concepts prior to teaching procedures (based on Heid’s (1998) and Repo’s (1994) research). Appropriate CAS activities were devised that focused principally on multiple representations (numerical, graphical, symbolic) of derivative and links between them (see next Section 2.5.3). The curriculum did not prescribe the pedagogy since part of the research was to explore how each teacher implemented the curriculum that they had helped to modify.
2.5.3 CAS activities for the curriculum

This section addresses the types of CAS activities that should be incorporated in the curriculum.

Etlinger (1974), writing at the beginning of the hand-held calculator era, envisaged that electronic four-function calculators could be used functionally or pedagogically. His definitions of the extremes of these types of use are still applicable to the more recent CAS calculator technology. “Pure-functional” use of the calculator includes the easy performance of calculations in order to get an answer to a problem. In contrast, “pure-pedagogical” use of the calculator is designed to facilitate learning. Ruthven (1995) describes three possible roles for the calculator in the curriculum: functional use for routine computational tasks, neutral use for checking work, and pedagogical use for testing conjectures, exploration and analysis.

On some occasions it is intended to accelerate the execution of routine computational tasks, releasing more time for conjectural thinking; and to provide valuable support for pupils who could otherwise be defeated or overwhelmed by the computational demand of the task. On other occasions, the calculator is seen as providing a neutral authority over which pupils can check their findings or test out mathematical conjectures. Finally, the calculator can be taken as embodying mathematical ideas, making them available for exploration and analysis through interactions with the machine. (p.249)

In this study, the type of use of the calculator is related to its purpose. The object of functional use of CAS is for getting an answer and pedagogical use of CAS is for understanding. As described above (see Section 2.5.1) the curriculum in this present study is focused on developing conceptual understanding using multiple representations of derivative and translating between them. As briefly reported above (see Section 2.4.1), Dick (1996) suggested finding the derivative of a function “at a point” by any of the following methods (CAS can be used):

- Symbolic differentiation followed by substitution (symbolic representation of differentiation).
- Drawing a graph and the tangent at a point (an automated process) and deducing the gradient of the tangent from its equation (graphical representation of differentiation).
- Calculating difference quotients using nearby values of the function and inferring the limit (numerical representation of differentiation).
- Finding the value of a limit expression using symbolic manipulation.
• Zooming in on the curve at a point and using nearby co-ordinate values of the curve (which appears as a straight line) and inferring the limit.

• Drawing a graph and some nearby secants from which the limit of their gradients can be determined.

Using the CAS calculator, the TI-92, these methods of differentiation, involving numerical, graphical, and symbolic representations of derivative were also suggested by other mathematics educators such as Berry (1997), Brown (1996a), and Kissane (1996) and will be used in this study. Additional pedagogical CAS activities such as investigations are required to promote understanding of the links between representations (Berry, 1997; Cappuccio, 1996; D'Ambrosia, Spitznagel, & Carroll, 1996). In addition, Kaput (1998), Leinbach (1996), Lin (1993), and Tall (1996) suggest that appropriate CAS activities need to involve real experiences and dynamic programs (see Section 2.5.1).

The CAS activities included in this study’s curriculum took up some of these recommendations. They included activities intended for finding answers, designated functional CAS activities, together with a range of activities intended to help students understand the links between representations, designated pedagogical CAS activities. Where possible, physical representations associated with the collection and interpretation of real data using data loggers and dynamic programs were also included. These gave students the opportunity to visualize the mathematical properties under consideration as previously discussed.

This section has reported on studies related to curriculum design, teacher preparation for implementation of the curriculum, and types of CAS activities included in the curriculum.

2.6 Teaching the Curriculum

It has been established that with an appropriate curriculum and technology, students will have the opportunity to develop understanding of the concept of derivative. The extent to which this occurs is largely dependent on the teachers. This section reports different teacher beliefs and conceptions (Section 2.6.1) and the influence of different types of teacher knowledge on teaching practices (Section 2.6.2). Finally, the relationship
between teaching practices and different teacher characteristics, including teacher content knowledge (Section 2.6.3), is explored (Section 2.6.4).

2.6.1 Teachers’ beliefs and conceptions of mathematics

This section describes teacher characteristics and beliefs and practices that impact on teaching practices.

In a comprehensive literature review of teachers’ beliefs and conceptions of mathematics Thompson (1992) discusses the importance of teachers’ conceptions of mathematics to their instructional practices. Several studies are reported where teachers’ professed beliefs about the nature of mathematics are consistent with their teaching practices, including her personal study (Thompson, 1984) that concludes:

Although the complexity of the relationship between conceptions [of the nature of mathematics] and practice defies simplicity of cause and effect, much of the contrast in teachers’ instructional emphases may be explained by differences in their prevailing views of mathematics. (p. 119)

From a range of possible classifications of teachers’ conceptions of mathematics and styles of teaching, Thompson (1992) identifies Kuhn and Ball's (1986) model of mathematics teaching “as constituting a consensual knowledge base regarding models of teaching” (p.137).

Kuhn and Ball’s preferred model identifies four different teaching approaches:

- **Learner focused**: mathematics teaching that focuses on the learner’s personal construction of mathematics knowledge;
- **Content-focused with an emphasis on conceptual understanding**: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;
- **Content-focused with an emphasis on performance**: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures; and
- **Classroom focused**: mathematics teaching based on knowledge about effective classrooms. (Thompson, 1992, p.136)

Thompson draws attention to the distinguishing feature of the **content focused view with an emphasis on conceptual understanding** as the only approach with “dual influence of content and learner. On one hand, content is focal, but on the other, understanding is viewed as constructed by the individual” (Kuhn & Ball, 1986, p.15).
Kuhn and Ball’s (1986) model was found to be very appropriate and useful in this thesis and proved helpful in distinguishing between the teaching characteristics of both teachers.

2.6.2 Teachers’ knowledge

This section identifies the wide variety of knowledge employed by teachers while they teach and identified as having an impact on teaching practices, including:

• Knowledge of mathematics (Fennema & Franke, 1992; Shulman, 1986).
• Knowledge of technology (Heid, 1995).
• Knowledge of school-related institutional and cultural factors (Gutstein & Mack, 1998; Llinares, 2000; Lumb, Monaghan & Mulligan, 2000; Moreira & Noss, 1995).
• Underlying beliefs and mathematical conceptions (Davis, 1992; Gutstein & Mack, 1998; Thompson, 1992).

Fennema and Franke (1992) divide knowledge for teaching into teacher content knowledge of mathematics, content pedagogical knowledge, and pedagogical knowledge (of teaching).

First, content knowledge is described:

[Content knowledge] includes teacher knowledge of the concepts, procedures, and problem solving processes within the domain in which they teach, as well as in related domains. It includes knowledge of the concepts underlying the procedures, the interrelatedness of these concepts, and how these concepts and procedures are used in various types of problem solving. Crucial also to teacher knowledge of content is the manner in which the knowledge is organised, indicating teacher knowledge of the relationships between mathematical ideas. (p.162)

Second, pedagogical knowledge is described:

Pedagogical knowledge includes teachers’ knowledge of teaching procedures such as effective strategies for planning classroom routines, behaviour management techniques, classroom organizational procedures, and motivational techniques. (p.162)

Third, content pedagogical knowledge is described:

[Content pedagogical knowledge] includes knowledge of pedagogy, as well as understanding the underlying processes of mathematical concepts, knowing the relationship between different aspects of mathematical knowledge, being able to interpret that knowledge for teaching, knowing and understanding students’ thinking, and being able to assess student knowledge to make instructional decisions. (p.161). This also involves taking complex subject matter and translating into representations that can be understood by students. (p.153)
These distinctions about teacher knowledge are useful in this thesis. Each teacher’s content knowledge of the concept of derivative is assessed in order to ascertain its influence on how the teachers taught and what the students learned. In the preliminary study both teacher’s pedagogy impacted on student learning and an aim of the main study was to determine how the teachers’ knowledge of calculus impacted on their teaching approaches, teaching styles, and on student learning.

This section has described multiple strands of teacher knowledge that blend to form each teacher’s personal conceptions of mathematics and teaching mathematics and for this study, important teacher knowledge includes: content, content pedagogical, pedagogical, how students learn, technology, and school related institutional and cultural influences. Each teacher’s personal beliefs influence their personal conceptions of mathematics and are likely to influence their teaching practices and specific examples to illustrate this are discussed next.

2.6.3 Teachers’ pedagogy relates to beliefs about mathematics and how it should be taught

This section reports studies that show how teachers’ conceptions and beliefs influenced their teaching practices including teaching with graphics calculator technology. The research reported in this section is primarily focused on graphics calculators since CAS calculators have only recently been introduced into secondary school classrooms. It is appropriate to rely on them to inform this study because the technologies are similar and all the studies involve pioneering teachers (and students) attempting to adapt to teaching (and learning) in ways that are different from their usual practices.

Underlying belief systems and philosophies appear to guide teaching actions. Davis (1992) states:

Both tutors and teachers also enter the situation with a well developed (though possibly tacit) philosophy. Looking carefully at actions may give us clues about what these philosophies are, and that, in turn, may give us a far deeper ability to communicate with one another. (p. 359)

Several studies report that teaching practices adopted by teachers reflect their mathematical conceptions and beliefs about mathematics. Hoyles (1992) claims that a Microworlds project which explores the “interactions of the teachers with the computer activities . . . and the ways they incorporated them into their practice [provided] a window on their views and beliefs about mathematics teaching” (p. 39).
Moreira and Noss (1995, p.173) concur with this opinion and with Thompson (1992) but they believe that the teaching context also plays a role in how the teachers taught. "In essence we find much in tune with Thompson's notion of a 'dialectic between teachers' beliefs and practice'; moreover the beliefs and attitudes held by the two participants were interrelated with the situations in which they were engaged." Penglase and Arnold (1996) also urge researchers not to treat the graphic calculator in isolation from the context in which it is being used.

In this present study of two teachers pioneering teaching with CAS, the context is important. While teaching with CAS they will be constrained by the social and institutional parameters that influence their normal teaching practices (e.g., future examinations will not allow use of CAS, the school expectation that successful teaching is measured by the pass rate on examinations, and many others). In practice they will not be entirely free to take full advantage of the symbolic algebra capabilities of the CAS calculator.

Some studies report teachers adopting teaching practices with technology that are consistent with their professed views of teaching and learning. For example, Jost (1992) explores teachers’ beliefs about teaching (based on in-depth interviews) and compares them with their perception of the role of programmable graphics calculators. Jost categorized the teachers’ perceptions of the role of the calculator along a continuum from a computational tool (for doing procedures) to that of an instructional tool (for promoting understanding). Jost found that the teachers who valued the computer as a computational tool stressed content-orientated goals and viewed learning as listening. The teachers who came to appreciate the strength of the calculator as an instructional tool had student-centred and discipline level goals for their students: interactive, inquiry-driven teaching styles; and student-centred views on learning including the view that students can learn through interactions.

Fine and Fleener (1994) report that for three preservice teachers their personal beliefs and attitudes towards graphics calculator use had a significant influence on their pedagogical beliefs and attitudes towards calculator use in their classrooms. "Just having experiences with calculators does not seem to be enough to unlock the minds of these preservice teachers to the possibilities of calculators as instructional tools. (p.98)"
Fleener (1995) surveyed 94 middle and secondary school mathematics teachers and concluded that the type and degree of graphics calculator use in their mathematics lessons was influenced by their personal belief about whether or not students should have conceptual mastery before calculators are used and their personal experience with using calculators.

Other studies report teaching practices with technology that were consistent with current teaching practices. For example, Simmt (1997) observed six teachers and concluded that they used graphics calculators as an extension of their normal teaching practices without technology. They used similar activities, but their differing conceptions of mathematics affected how they followed up those activities with questions and summary notes.

Although all of the teachers were provided with the same curricular constraints, each of the teachers brought forth the mathematics curriculum within the context of his or her personal philosophies of mathematics and mathematics education. (p.269)

Bottino and Furinghetti (1996) report case studies from their study of curriculum reform involving 120 teachers. They comment about the larger study that the teachers made choices while introducing technology to their mathematics classrooms that were consistent with their prior pedagogy teaching technology.

From these studies a significant fact has emerged: even if the proposals by the institutions (new programs) and the teacher training are quite homogeneous, classroom implementation shows a variety of different trends regarding both curriculum (the nature and form of content) and instructional decisions (how content relates to learners in the instructional setting). (p.113)

These studies indicate that the ways individual teachers take up using technology in their mathematics classrooms is usually consistent with their normal teaching practices. This is important in this study of two very different teachers pioneering new technology in their classrooms.

Some studies observed that the teachers changed their use of technology over time. These are important because the teachers involved in this present study will be monitored over a two-year period. For example, Tharp, Fitzsimmons, and Brown Ayers (1997) studied 261 mathematics and science teachers while they participated in a four-month support course for introducing graphics calculators. They noted that teachers generally came to see the graphics calculators as enhancing understanding and promoting exploration. However, they found that teachers they classified as having a
A rule-based view of mathematics quickly abandoned inquiry approaches, tended to control the amount and type of calculator use by students, and were more likely to believe that the calculators may hinder learning.

Other studies report on individual classroom teachers who changed their pedagogy in response to knowledge of their students. For example, Lumb, Monaghan, and Mulligan (2000) report that one teacher’s initial focus on CAS (Derive) technology subsided over time (Steve) and Zbiek (1995) reports one teacher’s technology focus and guidance increased (LeAnne). After reflection about the behaviour of these two teachers, Zbiek (2001) indicates that she believes Steve and LeAnne made changes to their use of CAS in response to their personal perceptions of their students’ needs. Zbiek (2001, p.5) also indicates that LeAnne believed her first-year high students (grade 9, aged 15-17 years) were less mathematically capable than other students and required more practice (including specific directions about how to use the technology) whereas Steve’s different response was precipitated by his fear that his secondary school students (A level) were bored. These studies offer plausible explanations why some teachers changed their use of technology over time.

Some teaching with technology has not been successful and this is also attributed to the teaching practices of the teachers. For example, Manoucherhri’s (1999) research with 181 secondary school mathematics teachers indicates that teachers only used computers for drill and practice. Other reasons for failure to use the computers in ways that enhance mathematics instruction have been proposed. Manoucherhri (1999) and Rochowicz (1996) both suggest that the teachers did not successfully integrate use of computers into their classroom practices because they were not given an opportunity to participate in prior curriculum development, to help them feel confident about using the computers and to understand how to connect the computer activities to the curriculum. Thomas, Tyrrell, and Bullock (1996) examine the issues involved in teachers implementing use of computers into their mathematics lessons and suggest that teachers are more likely to be successful if they adopt classroom practices that support student-centred teaching practices, such as guided discovery learning, and that focus on the mathematics and its implications rather than the computer. Student-centred learning occurs when the student has the opportunity to actively participate in his own learning.
This section has reported studies that show how teacher beliefs about mathematics and how it should be taught influenced their teaching practices (including teaching with graphics calculators). The present study is important because it explores this aspect of teaching in depth. It also starts to fill a gap in research about teachers changing from teaching with graphics calculators to teaching with CAS calculators. Most teachers and students taking up CAS calculators in the future will have had prior experience with graphics calculators.

The next section describes literature that links learning outcomes to teacher knowledge and particular pedagogy. Before this, the influence of teacher content knowledge on their pedagogy is discussed.

2.6.4 Teachers’ content knowledge

Teacher content knowledge influences teacher pedagogy. For example, Lloyd and Wilson’s (1998) report in depth how one secondary school teacher’s deep and integrated knowledge of functions impacted on his first implementation of reform curriculum involving use of a graphics calculator. Over a six-week period his well-articulated ideas about features of a variety of relationships in numerical, graphical, and symbolic representations of functions supported his meaningful classroom discussions with students.

Gutstein and Mack’s (1998) fine-grained analysis of Mack’s “expert” teaching actions showed that her instructional decisions mirrored her content knowledge, pedagogical content knowledge, and knowledge of students and helped to teach fractions with understanding. Although this study is not related to teaching calculus or with CAS, it gives an example of the importance to student learning of teachers having deep understanding of mathematical ideas and the ability to communicate these ideas to their students.

Even and Tirosh (1995) conducted a study involving 162 prospective secondary mathematics teachers during which they conducted in-depth interviews with ten students. Even and Tirosh attributed ineffective teaching to lack of content knowledge by the teachers. They reported that teachers with insufficient knowledge of the subject-matter (functions in this case) found it difficult to understand the sources of students’ responses (i.e., students’ cognitions).
Thus, teacher content knowledge is highly influential in how teachers teach and will be explored in this study.

2.7 Teaching Linked to Learning

This section reports three studies in which instructional choices of teachers are linked to learning outcomes. Thompson (1992) notes that there is little known about this important aspect of learning.

Documented research shows that teachers’ conceptions, for the most part, are reflected in their instructional practices. But we know little about how instructional practices, in turn communicate those conceptions or others to students, if they do so at all. . . . Furthermore since teachers are the primary mediators between the subject matter of mathematics and the students, it is also natural to infer that the teachers’ conceptions are indeed communicated to students through practices in the classroom. . . There is a great deal that can be learned from insightful analyses of the nature of that interaction. (p.141)

First, Hart (1992) investigated how 324 college students’ (from 12 institutions) understanding of the concepts of differentiation and integration were affected by using handheld graphics calculators while studying an experimental curriculum where the teaching focused on conceptual understanding of calculus concepts through use of multiple representations. The experimental students showed greater facility with graphical and numerical representations and exhibited better ties among the three representations than parallel classes studying calculus by traditional methods without technology. Individual students showed definite preferences for representations but different factors influence their choices. Hart reports that the role of the teacher appeared to be particularly important in terms of management of representations and of the calculator.

Second, Roddick (1995) investigated students’ conceptual and procedural understanding of calculus within the context of an engineering mechanics course. Four students who were taught a traditional calculus course were compared with three students from a reform calculus course using Mathematica. Over ten weeks, each student participated in task-based interviews and the results show a distinct difference in approach to solving engineering mechanics problems involving calculus. The reform calculus students using Mathematica learned calculus with a conceptual emphasis and were found to be more likely to solve problems from a conceptual point of view than the traditional students who were more likely to focus on procedures.
Third, Frid (1992, 1994) reports a study in which different instructional methods were employed and different learning outcomes were observed for each instructional method. It involves three classes in different colleges learning calculus using multiple representations. While in the first class learning techniques were emphasized, the other two classes adopted approaches that focused on conceptual understanding. In the second class, a concepts-first approach was followed by a traditional emphasis on formal definitions and proofs that preceded skill development. In the third class, concepts were explored intuitively involving discussions of ideas using numerical infinitesimal methods related to non-standard analysis. The three methods of calculus instruction were compared and one of the conclusions was that “the nature and role of students’ conceptualizations can be influenced by the nature of the instruction. . . Instruction which emphasises use of visual and physical calculus [as occurred in Classes 2 and 3], is likely to enhance students’ sense of personal understanding of calculus” (Frid, 1992, p.91). Frid also reported that the instruction style in the third class required the students to use infinitesimal language (words and symbols) as tools by which to describe, explain, and justify particular mathematical situations and that language use proved to be essential for students to conceptualize calculus.

The present author distinguishes two aspects of the instructional approaches reported by Frid (1992) above: teaching method, and teaching style. In the first class, the teacher used an approach that was technique orientated using an essentially teacher-centred style of teaching. The second class used an approach that focused on understanding using an essentially teacher-centred style of teaching while the third class used an approach that focused on understanding together with an essentially student-centred style of teaching. Thus, this study showed that teaching method and teaching style, and combinations of them, appear to have influenced the learning that occurred.

In this present study it is necessary to distinguish between teaching method and style. The term teaching method describes the teacher’s general learning focus as either procedural method (to teach procedures) or conceptual method (to communicate understanding). Teaching style describes the general organization of the class as either teacher-centred (where the teacher controls learning) or student-centred (where individual students have significant input into their own learning e.g., through
interactions with other students or the teacher). *Teaching approach* is a general term that includes both teaching method and teaching style.

In some of the successful studies reported above in Section 2.3.1, (Heid, 1988; Keller & Russell, 1997; Porzio, 1994; Repo, 1994) where understanding of the concept of derivative was associated with use of CAS technology, it is interesting to observe that the instruction also used a conceptual method and a student-centred style of teaching. For example, Repo (1994) reported that students in the experimental group were assisted to reflect about the mathematics they were learning by interacting with the teachers and the other students.

In this program, the computer was considered a cognitive and didactic tool that helps the pupils (and the teacher) to externalise abstract thinking procedures by using different representation modes which inspire students to reflection. An important result is that the average level of conceptual understanding can be improved by involving pupils in the sequence of critical activities and by changes in the quality of social interaction. (p.11)

Heid's (1988) study also reported a conceptual method and a student-centred style of teaching and that the experimental group showed better understanding of calculus concepts (see Section 2.2.2).

More recently, Keller, Russell, & Thompson (1999a, 1999b) report on a large scale (N = 687) study that extended and replicated Keller and Russell's (1997) study. It examined the effects of the TI-92 graphics calculator on students' success during calculus instruction that emphasized a sense-making approach (i.e., conceptual method). It explored a range of instructional methods and showed that cooperative learning, lab experiments, and variety of discussion formats (i.e., student-centred style) had a positive effect on students' calculus exam scores. They attributed the positive effect of the student-centred style of teaching to the greater diversity of student approaches it supported, enabling more students to make sense of the mathematics under consideration.

Thus, in studies where success has been associated with use of CAS, maybe it is due to the combined effect of conceptual method and student-centred style, making the exact role of CAS uncertain. It is likely that effective teaching requires more than merely including CAS into the curriculum. The studies above indicate that teaching instruction is critical and both teaching method and style are highly influential in student learning. This was also observed by Kendal and Stacey (1999a, 1999b) in the
preliminary study to this present study. As reported in Chapter 1, they reported links between specific learning outcomes and particular pedagogy. In the preliminary study, teacher privileging of the mathematics taught and how it was taught (method and style of teaching and including use of CAS) were found to be highly influential on student learning. This result, previously reported in Section 1.2.1, was an impetus for the main study that aimed to see if this result could be replicated, by the same teachers, using the same curriculum and technology, but with new classes of students and a more appropriate assessment tool, based on a conceptual framework.

This section has reported studies in which the instructional choices of teachers (including use of technology) were directly linked to particular learning outcomes and also successful studies where teachers have used a combination of conceptual method and student-centred teaching style. When the teaching emphasis was on understanding the students demonstrated evidence of understanding calculus concepts and in most of the studies reported the method of instruction gave the students the opportunity to make personal sense of the mathematics. This section opens up new questions about how teachers’ beliefs and knowledge and consequent privileging of calculus content, instructional choices, and use of technology impact on the learning outcomes of their students. This thesis will address these questions.

2.8 Summary of Issues Arising from Literature Review

This section summarizes the conclusions and issues that have arisen from this literature review.

- There is a need to research how secondary school students learn with CAS calculator technology since there is very little research at this level of school mathematics.
- Differential calculus is an appropriate and important part of the school curriculum in which to conduct research.
- There is a need to research the learning process to understand how individuals conceptualize the calculus concept of differentiation involving multiple representations.
• There is a need for a suitable conceptual framework to structure the individual components of cognition (competencies) that contribute to understanding the concept of derivative.

• There is a need to understand how using multiple representations contributes to understanding of the concept of differentiation (made possible with use of a CAS calculator).

• There is a need to clarify the role of a CAS calculator.

• There is a need for a suitable assessment tool to monitor the learning about differentiation that occurs where multiple representations are emphasized.

• There is a need to understand the role that teachers play in learning and the effect of teaching method and teaching style.

• There is a need to understand how teachers adapt to the change from a graphics to a CAS calculator.

Thus, the purpose of this thesis has been clarified. It explores the impact of teacher privileging characteristics on the understanding of the concept of derivative achieved by students. It will explore if understanding of the concept of derivative is achievable through teaching introductory differential calculus using multiple representations of derivative and linking the representations using hand-held technology, a CAS calculator. Different categories of competencies will be identified through the development of a conceptual framework and the construction of test items to monitor demonstration of each competency. Each teacher’s privileging characteristics will be monitored and each student’s understanding of the concept of derivative will be established enabling the impact of each teacher’s privileging characteristics on learning to be assessed.
3. THE DIFFERENTIATION COMPETENCY FRAMEWORK AND DIFFERENTIATION COMPETENCY TEST

The thesis explores how teacher-related factors, including use of CAS, influenced students’ learning about the concept of derivative. In consequence, it is essential to first establish the criteria for students’ understanding of differentiation and then devise a suitable test instrument.

The purpose of this chapter is to establish these criteria within the Differentiation Competency Framework. As indicated in Section 2.4.1, the Framework identifies a set of fundamental component competencies associated with a holistic understanding of the concept of derivative. The Differentiation Competency Test was developed as the primary test instrument for the competencies in the Framework and its design is included in this chapter, prior to the description of methodology in Chapter 4. The results of the Differentiation Competency Test are reported in Chapter 6.

In this chapter, multiple representations of differentiation are more precisely defined than in Chapter 2 (see Section 3.1) and the development and structure of the Differentiation Competency Framework is explained (Section 3.2). The creation of the Differentiation Competency Test from the Framework and related sample test items are also explained (Section 3.3). Finally, the efficacy of the Differentiation Competency Framework and the Differentiation Competency Test is discussed (Section 3.4).

3.1 Multiple Representations of Differentiation

The primary purpose of the calculus curriculum is to assist students to develop an understanding of the concept of derivative through learning about numerical, graphical, and symbolic differentiation and to make strong, cognitive, and reversible links among the three representations of derivative. The conceptual framework incorporates these features.

This section defines numerical, graphical, and symbolic differentiation techniques and illustrates them using a concept map (Section 3.1.1), describes how CAS activities are incorporated into the curriculum to link different representations (Section 3.1.2), and reports complexities related to differentiation in different representations (Section 3.1.3).
3.1.1 Numerical, graphical, and symbolic representations of derivative

This section defines numerical, graphical, and symbolic differentiation techniques and then discusses the author’s concept map for differentiation in relation to the study.

CAS calculators, such as the TI-92, have excellent graphical, and numerical computational capabilities as well as a symbolic algebra capability that allows manipulation of symbols according to certain algorithmic procedures. This enables a wide range of algebraic tasks including differentiation to be performed quickly and accurately, as briefly outlined in Chapter 1 and Section 2.1.2. As mentioned in the literature review (see Section 2.4.1), Dick (1996) strongly advocates involving multiple representations in teaching the concept of derivative: “for teaching calculus with a multiple representation approach, the numerical and graphical avenues opened up by these machines [calculators with CAS] are almost essential” (p.45). He explains that zooming in on the graph of a differentiable function makes the notion of a unique linear approximation self-evident and that with a graphics calculator “the difference quotient need to be defined just once and the convergence of a sequence of its values can be investigated easily” (p.37). Dick’s numerical differentiation procedure is described below together with the straightforward graphical and symbolic differentiation procedures that have been widely adopted for use with CAS calculators to determine derivatives in the corresponding representations. These three differentiation procedures and links between the different representations of derivative are illustrated in the author’s concept map of differentiation (see Figure 3.1).

Definitions of numerical, graphical, and symbolic differentiation

For the purpose of this thesis, three aspects of the concept of differentiation are defined: numerical, graphical, and symbolic. These reflect the meaning of differentiation in each representation and draw on Dick’s (1996) vision for teaching calculus from a “multi-representational” point of view (based on his experiences of the Calculus Connections Project involving over 400 secondary school teachers in the USA) involving both graphics and CAS calculators.
**Numerical differentiation**

From a numerical (tabular) representation of function, differentiation is most readily understood as finding the rate of change near a value (see Figure 3.1). Rate of change may be found (by-hand or using the CAS calculator) through a difference quotient calculation of two nearby values:

Average rate of change at \( x \) over the interval \([x, x+h]\) = \[
\frac{f(x+h) - f(x)}{h}
\]

The instantaneous rate of change is the limit as \( h \) approaches zero. In practice, in standard problems, the limit can often be inferred from the average rate of change. For example, for the function, \( y = x^5 + 2x \), taking \( x = 1 \) and \( h = 0.001 \),

Average rate of change over the interval \([1, 1.0001]\) = \[
\frac{(1.0001)^5 + 2(1.0001) - [(1)^5 + 2(1)]}{0.001} = 7.01001
\]

Students familiar with the conventions of school mathematics have no trouble identifying the correct answer (difference quotient = 7) for the instantaneous rate of change at \( x = 1 \). (A more sophisticated approach would use a sequence of values of \( h \) and to deduce the limit). Thus, the numerical representation of differentiation involves determination of an arbitrarily fine difference quotient, the numerical derivative.

**Graphical differentiation**

From a graphical representation of a (differentiable) function, differentiation is most readily understood as finding the gradient of the curve (see Figure 3.1). The gradient of the curve can be interpreted intuitively as the slope of the line that approximates the curve at high resolution (i.e., zooming in to see how the curve is locally straight) or as the gradient of the tangent to the curve. The graphical representation of derivative involves finding the gradient of the tangent to the curve "at a point", either by constructing the tangent on a drawn graph (then calculating its gradient) or using the CAS calculator. At the push of a button, the tangent is drawn on the displayed graph, and its equation in slope and intercept form appears on the screen. For the example above, the equation of the tangent drawn at \( x = 1 \) is \( y = 7x - 4 \), and hence the gradient of the tangent to the curve at \( x = 1 \) is 7. In fact the derivative (as gradient) obtained by the CAS calculator is derived by approximation rather than as a limit, but this is not detectable in standard introductory problems.
Symbolic differentiation

From a symbolic representation of function, differentiation can be seen as a symbolic process, so a symbolic derivative is found by following symbolic rules (see Figure 3.1). For example if \( y = x^5 + 2x \), the symbolic derivative is found by standard rules (by-hand or using CAS) giving \( \frac{dy}{dx} = 5x^4 + 2 \). For a particular point, the \( x \) value is substituted into the symbolic derivative (by-hand or using CAS). Consequently, at \( x = 1 \),

\[
\frac{dy}{dx} = 5(1)^4 + 2 = 7
\]

The three representations Numerical (N), Graphical (G), and Symbolic (S), have therefore been associated with three parts of the concept map for differentiation.

   Numerical – rate of change, Graphical – gradient, Symbolic – symbolic derivative.

In this section, the numerical, graphical, and symbolic differentiation procedures used in the main study’s curriculum have been described. The term derivative is generally used for the result of symbolic differentiation and the phrases numerical derivative and graphical derivative are used when it is necessary to distinguish those representations. The term derivative is used to describe both the derivative function and its value at a specified point (unless otherwise specified).

Concept map of differentiation

A concept map of differentiation (see Figure 3.1) illustrates the three differentiation procedures described above and the links between the three different representations of derivative. The author developed the concept map to clarify how the numerical, graphical, and symbolic representations applied to teaching the concept of derivative and how to assess what the students had learned. It accommodates:

- Average rates of change and the gradients of secants (without limits being taken) that arise when teaching differentiation to students for the first time.
- Exact derivatives (with limits being taken).
- Physical representations (considered as numerical since the physical situations present as rates of change).
- Differentiation from first principles that integrates numerical, graphical, and symbolic representations.
In Figure 3.1, the numerical, graphical, and symbolic representations of derivative are each represented by one circle, and the solid arrows that link the circles indicate translations between representations. Within each circle, the heavy dotted line separates situations where the limit has been taken from those where it has not been taken. Thus, in the graphical circle, the "gradient of secant of curve" (where the limit has not been taken) is separated from the gradient of the curve and from the gradient of the tangent to curve at a point (where limits have been taken). Similarly, in the numerical circle, the "average rate of change" (where the limit has not been taken) is separated from "instantaneous rates of change" (where limits have been taken). The physical representations of differentiation such as speed and rate of inflation (rates of change) had little prominence in the teaching program (unfortunately) and hence the ideas involved are not analysed or depicted in any detail in this study. However, real world rates of change were included in the curriculum (for problem solving) and the dashed line links the physical circle to the numerical circle to indicate that the physical representation (involving rates of change) was regarded as belonging to the numerical representation. Symbolic differentiation incorporates the limiting process when the standard rules of differentiation are employed and the resultant derivative is a symbolic function or its value at a specified point. Differentiation from "first principles" is defined as the limit of a sequence of gradients of secants and is calculated by difference quotient (numerical derivative). It is often visually presented on a coordinate grid as the gradient of the tangent to the curve (graphical derivative) and the derivative obtained is a symbolic function or its value at a specified point. However, since differentiation from first principles was not actually taught in this study, its links to the numerical, graphical and symbolic representations are not included in the concept map.
A CONCEPT MAP OF REPRESENTATIONS OF DIFFERENTIATION

**Figure 3.1.** Concept map of differentiation in numerical, graphical, and symbolic representations for the main study.
3.1.2 Linking representations of derivative

The curriculum includes CAS activities to help students understand the concept of derivative (as discussed in Section 2.5.1) some of which link the pairs of representations and are incorporated into Figure 3.1. The full set of CAS activities that are incorporated into the calculus curriculum to link the representations is to be presented and discussed in Section 4.2.2. This section describes in detail how one CAS activity is incorporated into the curriculum to link different representations.

One CAS activity links the numerical and graphical representations. Using a CBR interactive program, individual students walk at different rates (numerical representation), and in different directions, simultaneously experiencing the effect of their speed (physical representation) on the slope (graphical representation) of the straight lines and curves they generate with their movement. While using this activity, the teachers have the opportunity to use personal teaching actions to help the students develop understanding of the link between the two representations, such as employing enactive representations (e.g., modelling tangent lines by outstretched arms) or encouraging the students to visualize the relationship between representations.

Thus, the conceptual Framework needs to accommodate differentiation in each representation and translations between pairs of representations of derivative, and the test instrument needs to test them.

3.1.3 Complexities related to differentiation in different representations

This section describes two of the many complexities associated with differentiation in different representations.

The first complexity relates to differentiation “at a point” and “at all points”. Derivatives “at a point” for the numerical, graphical, and symbolic representations are instantaneous rate of change, gradient of curve or tangent to curve at a point, and derivative at a specified \( x \) value. The three examples of differentiation given to illustrate the definitions in Section 3.1.1 involve differentiation “at a point”. Differentiation in the symbolic representation usually provides a derivative function “at all points” whereas the graphical and numerical representations usually provide a derivative “at a point”.

Thus, the teaching program of introductory elementary differential calculus, developed
for both the preliminary and main studies, emphasized differentiation “at a point”, particularly for numerical and graphical differentiation.

The second complexity is confusion between exact and approximate values associated with the limiting process. As indicated in Section 2.2.1, the concept of limit causes students difficulties (Barnes, 1991; Ferrini-Mundy & Graham, 1991; Orton, 1977; White, 1993). By using the standard rules of differentiation the symbolic representation incorporates the limiting process within the derivative. In contrast, differentiation involving the numerical representation is inherently approximate since it involves two numbers close together. Differentiation involving the graphical representation both by-hand and with CAS is also, in practice, approximate because a (graphical) tangent cannot usually be constructed precisely. The procedure for graphical differentiation, available on the CAS calculator (described in Section 3.1.1), makes it appear that the tangent of the curve has been identified so that the limiting process (giving an exact slope of the tangent) has been carried out, rather than merely an approximation. This misconception is perhaps more likely to be held by students who find it easy to conceptualise the gradient of the tangent to the curve (and the gradient of curve itself) as an exact derivative quantity already encompassing the idea of a limit.

In summary, two complexities associated with multiple representations of differentiation have been reported in this section: the differentiation procedures in the numerical and graphical representations are primarily “at a point”, rather than for functions “at all points”, and the graphical differentiation procedures with CAS may mislead students into thinking the limiting procedure has been performed and the derivatives are exact.

### 3.2 The Differentiation Competency Framework

In this section, the motivation for the development of a theoretical Framework is discussed (Section 3.2.1) and the theoretical basis for the development Differentiation Competency Framework is described in detail. Its main features and its limitations are discussed (Section 3.2.2) and the characteristics of the Framework competencies are also described (Section 3.2.3).
3.2.1 Motivation for the Differentiation Competency Framework

The motivation for the development of the Differentiation Competency Framework is discussed in this section. The primary impetus for the main study (described in Section 2.4.1) was the desire to appropriately monitor the learning about differentiation that occurred during a course of study that concentrated on multiple representations of differentiation and that was supported by use of a technological tool. In addition, although the calculus teaching program and conduct of research in the preliminary study proved satisfactory, its testing program did not comprehensively reflect the teaching focus of understanding differentiation using numerical, graphical, and symbolic (i.e., multiple) representations and links between them. One of the specific objectives of the preliminary study reported by McCrae, Asp, and Kendal (1999, 2000) was to ascertain if Year 11 students using a CAS calculator could perform differentiation with the same level of success as Year 12 students taught without CAS but with an additional year of experience of calculus. A set of traditional Victorian Certificate of Education (VCE) test questions was used to assess achievement on the preliminary study tests. Some of the questions were selected from previous statewide Year 12 Mathematical Methods exam papers while others were at Year 11 standard. All the test questions were essentially symbolic or symbolic with graphical data. There were three types of questions. First, concept items tested understanding of the concept of derivative and did not require use of CAS. Second, symbolic items tested algebraic techniques and could be solved using the symbolic algebra capabilities of CAS or by-hand (in most cases). Third, options items were problems that could be solved either by using symbolic differentiation (with symbolic algebra or by-hand) or alternatively by using graphical solutions (with CAS). Student use of the numerical, graphical, and symbolic algebra facilities of the CAS was monitored on the test but understanding of the multiple representations of differentiation and the translations between them was not monitored. The main study research aimed to address this deficiency by matching the assessment to the goals of the curriculum. The Differentiation Competency Framework was constructed to:

- Identify the set of fundamental competencies associated with multiple representations of differentiation and translations between them that are associated with a holistic understanding of the concept of derivative.
• Clarify the objectives of the calculus curriculum and help identify the purpose of the teaching activities (including those involving CAS).

• Guide the development of a more appropriate assessment tool that matched the goals of the curriculum, the Differentiation Competency Test, a new type of test instrument that monitored learning of differentiation in each representation and differentiation involving translations between representations.

• Facilitate the monitoring of students’ learning about the concept of derivative and the teaching practices of the teachers.

Thus, the Differentiation Competency Framework was primarily developed to identify the competencies, related to a full understanding of the concept of derivative. The Framework delineates the competencies that need to be taught, supported by CAS activities, and then assessed.

3.2.2 The Differentiation Competency Framework

As described in Section 2.4.1, the construction of the Differentiation Competency Framework was broadly based on three of O'Callaghan’s (1998) components of knowledge for functions and it accommodated Janvier’s (1987b) notion of translation between different representations, Vergnaud’s (1998) definition of the concept of derivative, and Tall’s (1996) approach for introductory calculus. This section describes the Differentiation Competency Framework, and how it was developed, in detail.

The construction of the Differentiation Competency Framework was informed by consideration of the cognitive demands required in introductory calculus and was specially designed to show how aspects of differentiation interacted with the different representations of function. The intention was to focus on understanding rather than procedures of accurately calculating a derivative. During the preliminary study, the author became aware that some students had used their CAS calculators to find equivalent expressions rather than perform differentiation, indicating that they did not really understand the cognitive processes involved with performing symbolic differentiation. She systematically examined in detail tasks commonly given to students of introductory calculus using about 150 questions from textbooks, school assignments, state examination questions. This process showed her that the students needed to understand when and how to use the differentiation process, be able to give meaning to
derivatives, move among representations, and recognize equivalent derivatives. Thus, the characteristics of the cognitive processes associated with derivative were identified and they guided the structure of the competencies in a comprehensive Framework (involving 3 representations, translations among them, formulating and understanding each derivative). A few additional competencies were designed to fill the gaps that typical tasks ignored (particularly with respect to the numerical representation, and to a lesser extent the graphical representation). Figure 3.3 displays the 18 fundamental competencies (see definition in Section 3.2.3) that were considered to be comprehensive in the Framework for differentiation. An earlier version of the Framework included three compound competencies (combinations of fundamental competencies) to give a total of 21 competencies (see Kendal & Stacey, 2000). These compound competencies have now been excluded from the Framework of fundamental competencies. Thus, solving problems involving differentiation requires being able to work with its common representations, numerical, graphical, symbolic and translating between pairs of representations. It requires competence in formulating, calculating, interpreting, and translating between the three types of derivative: instantaneous rates of change, the gradients of curves and tangents, and symbolic derivatives (described above in Sections 3.1.1 & 3.1.2 and displayed in Figure 3.1). Each of these derivatives can be “at a point” or “at all points”. However, in this study of introductory calculus, differentiation “at all points” was more accessible with the symbolic representation than in the other two representations. Although the physical representations of differentiation such as speed and inflation did not figure prominently in the introductory school curriculum from which the Framework was derived they are clearly identifiable as rates of change and were regarded as numerical representations (as Figure 3.1 shows).

Other representations most likely to be used by the teachers, such as visualization and enactive representations (mentioned above, Section 3.1.2), were not directly included in the Differentiation Competency Framework. Finally, “first principles” calculation of symbolic derivatives (not emphasized in either the preliminary or main study curriculum) was also not included in the Differentiation Competency Framework due to complexities related to its dependency on limits in different representations (see Figure 3.1 above). As reported in the literature review (Section 2.4.1), authors such as Tall (1996) and Dick (1996) believe that the limit approach, associated with “first
principles” methods, is no longer necessary in an introductory course of calculus study if it is replaced by zooming in on the graph of a differentiable function (see Section 2.2.1). This CAS activity was also incorporated into the curriculum (see Section 4.2.2) and the teaching of “first principles” was postponed until later in the school year.

This section has described the development Differentiation Competency Framework. It was initially created, in a way that was consistent with other definitions and approaches, by examination of the characteristics of tasks commonly given to students of introductory calculus. It focuses on the traditional representations associated with differentiation, namely, the numerical, graphical, and symbolic representations of derivative and on linking them.

3.2.3 Characteristics of Differentiation Competency Framework competencies

The structure of the Differentiation Competency Framework, associated with a full understanding of the concept of derivative (across three representations of differentiation), is explained in this section.

The Differentiation Competency Framework consists of the minimum comprehensive set of fundamental competencies (18 in all) used in solving problems associated with the concept of derivative. Each competency involves a cognitive process and a single representation or pair of representations and the word competency refers to a specific cognitive component part of the concept of derivative. In this present study, the word competency is used differently from Goldin's (1998) use of the word. He claims that learning can be defined operationally as the “acquisition of competencies” (p. 147) and that “Human competence refers to the ability to perform a task some of the time, under conditions which are partially but incompletely specified” (p.147). He distinguishes between directly observable mathematical behaviour (i.e., "performance") and competence, an operational construct. Goldin stresses that competence is always inferred, never observed directly “. . . we cannot define competence unreservedly as successful performance at a particular level.” In consequence, some readers may prefer to use the term sub-competency rather than competency to indicate its more minor status.
Each of the 18 competencies in the Framework\textsuperscript{1} is identified by three characteristics:

1. The representation of the \textit{input} derivative that is dependent on the representation of the data in the question.

2. The representation of the \textit{output} derivative that is required by the representation of the goal of the question.

3. The cognitive \textit{process} (either formulation of a derivative or interpretation of the meaning of the derivative) that is needed to achieve the \textit{output} derivative. If the \textit{output} representation is different from the \textit{input} derivative, a translation accompanies the formulation or interpretation.

The representation of a derivative is identified in two different ways. First, it is identified by its natural language descriptor. The phrase “rate of change” denotes a numerical representation of derivative. Similarly, the word “gradient” (of curve or of the tangent to curve) denotes the graphical representation of derivative and the word “derivative” in a question normally denotes the symbolic representation. Second, the representation of a derivative is dictated by the method of differentiation carried out in its determination. For example, calculation of a difference quotient (using ordered pair data) results in a numerical representation of derivative. Similarly, finding the gradient of the tangent to the curve results in a graphical representation of derivative while symbolic manipulation of a function results in a symbolic representation of derivative. Figure 3.2 displays how the three representations of derivative may be identified by the differentiation procedure performed or by the natural language phrase associated with the given or resultant derivative.

<table>
<thead>
<tr>
<th>Identification</th>
<th>Representation of Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>Differentiation procedure</td>
<td>Numerical</td>
</tr>
<tr>
<td>Calculation of a difference quotient</td>
<td></td>
</tr>
<tr>
<td>Determination of the gradient of a tangent to the curve</td>
<td></td>
</tr>
<tr>
<td>Use of symbolic rules</td>
<td></td>
</tr>
<tr>
<td>Natural language terminology (word or phrase to describe the given or resultant derivative)</td>
<td>Rate of change (instantaneous or average)</td>
</tr>
</tbody>
</table>

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Identification & Numerical & Graphical & Symbolic \\
\hline
Differentiation procedure & Calculation of a difference quotient & Determination of the gradient of a tangent to the curve & Use of symbolic rules \\
\hline
Natural language terminology (word or phrase to describe the given or resultant derivative) & Rate of change (instantaneous or average) & Gradient & Derivative \\
\hline
\end{tabular}
\caption{Identification of the representation of a derivative by differentiation procedure and natural language terminology.}
\end{figure}

\textsuperscript{1} An earlier version of the Framework identified 21 competencies (see Kendal & Stacey, 2000).
The rules for deciding on the representations of the input and output derivatives for an item are now described.

**Characteristic 1. The representation of the input derivative**

The representation of the input derivative is dependent on the data and is classified as:

- Numerical (N), if the data is a numerical derivative (instantaneous or average rate of change) or enables a difference quotient to be calculated (and possibly its limit) using ordered pair data or a table of values.

- Graphical (G), if the data is a graphical derivative (gradient of curve or tangent to curve) or enables the gradient of the tangent to the curve "at a point" to be determined.

- Symbolic (S), if the data is a symbolic derivative or enables a symbolic derivative to be determined using the rules for symbolic differentiation.

**Characteristic 2. Representation of the output derivative**

The representation of the output derivative is dependent on the goal of the question asked and is classified:

- Numerical (n), if the question requires finding or explaining a "rate of change".

- Graphical (g), if the question requires finding or explaining a "gradient" of curve or tangent.

- Symbolic (s), if the question requires finding or explaining a "derivative", given as a formula.

Note that the upper case letters are used for Characteristic 1 while lower case letters are used for Characteristic 2.

The third characteristic of differentiation competencies is related to whether the goal of the question requires a derivative to be constructed or interpreted.

**Characteristic 3. Cognitive process linking the input and output representations**

The cognitive process is needed to achieve the output derivative. It is either formulation or an interpretation.
• Formulation (F) is the ability to recognize the representation of input derivative from the data supplied and to know how to perform the corresponding differentiation, numerically, graphically, or symbolically.

• Interpretation (I) is the ability to reason about an input derivative supplied in the data, (namely, to explain it in natural language, or to give it meaning including its equivalence to a derivative in a different representation).

For example, consider Questions A and B.

**Question A.** A curve has the equation \( g(x) = 5x^3 - 6x^2 + 3x - 6 \). Find the gradient of the curve at the point \( P \), where \( x = -1 \).

The data is a symbolic function so the input representation is symbolic (S). If a symbolic differentiation is performed using symbolic rules on the function data, the process is formulation of a symbolic derivative. The phrase “Find the gradient” indicates that a graphical output derivative (g) is required. The competency is coded [process, input representation, output representation], \([F,S,g]\).

**Question B.** The derivative function of \( f(x) \) is given by \( f'(x) = x^3 - 5x + 3 \). What is the gradient of the tangent to the curve \( y = f(x) \), when \( x = 1 \)?

The data is a symbolic derivative so the input representation is symbolic (S). The symbolic derivative needs to be reinterpreted as a gradient. The process is interpretation (I) of the given symbolic derivative as a gradient. The phrase “Find the gradient” indicates that a graphical output derivative (g) is required. The competency is coded \([I,S,g]\).

In total, the three characteristics give rise to the Differentiation Competency Framework. Figure 3.3 below, displays the 18 individual competencies, set out in a 3 by 3 grid of nine blocks (with two competencies per block). The first letter (in italic capitals with (F) for formulation and (I) for interpretation) designates the cognitive process (Characteristic 3 above) The second letter (upper case N, G, S) designates the input derivative representation and is governed by the data as noted above (Characteristic 1). The third letter (in lower case: n, g, s) designates the output representation (Characteristic 2) that is required by the goal of question (expressed in natural language). In each of the nine blocks, one competency involves formulation and the other interpretation.


<table>
<thead>
<tr>
<th>Output</th>
<th>Numerical (N)</th>
<th>Graphical (G)</th>
<th>Symbolic (S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical (n)</td>
<td>$F_{Nn}$</td>
<td>$F_{Gn}$</td>
<td>$F_{Sn}$</td>
</tr>
<tr>
<td>Graphical (g)</td>
<td>$I_{Ng}$</td>
<td>$I_{Gg}$</td>
<td>$I_{Sg}$</td>
</tr>
<tr>
<td>Symbolic (s)</td>
<td>$F_{Ns}$</td>
<td>$F_{Gs}$</td>
<td>$F_{Ss}$</td>
</tr>
</tbody>
</table>

*Figure 3.3. The eighteen differentiation competencies in the Differentiation Competency Framework.*

Thus, $[FSg]$ for example, represents formulating ($F$) a symbolic derivative ($S$) from the data to find a gradient (g) of tangent (or curve) "at a point". This is the competency involved in Example A above. Similarly, the competency involved in Example B above, $[ISg]$, represents reinterpretation ($I$) of a given symbolic derivative ($S$) as a gradient (g). Further examples of competencies are discussed fully (in relation to the Differentiation Competency Test questions) in Section 3.3.2 below.

A further distinction in the competencies can also be observed in Figure 3.3. *Translation* of representation occurs if the representation of the output derivative is different from the representation of the input derivative. The representations of the input and output derivatives are identical in the three diagonal blocks, and different in the remaining six blocks (shaded), indicating translations are involved in 12 competencies.

The roles of the input and output representation are somewhat different. To start the problem, students need to identify (and sometimes choose) the input derivative while the output derivative (expressed in natural language) is predetermined. We therefore designate the six competencies in the first column as the numerical competencies (input derivative is numerical (N)). Three of these competencies, $[F_{Nn}]$, $[F_{Ng}]$ and $[F_{Ns}]$, involve formulation ($F$) of a numerical derivative whereas $[I_{Nn}]$, $[I_{Ng}]$ and $[I_{Ns}]$, involve interpretation ($I$) of a numerical derivative. Four of the six competencies involve translation of representation. $[F_{Ng}]$ and $[I_{Ng}]$ have graphical (g) output representations while $[F_{Ns}]$ and $[I_{Ns}]$ have symbolic (s) output representations. The two remaining competencies $[F_{Nn}]$ and $[I_{Nn}]$ do not involve translation since their output representation is also numerical (n). Similarly, the second and third columns display the corresponding six graphical and six symbolic competencies. In total, nine
competencies are associated with formulation of derivatives, (six with translation) and nine with interpretation of derivatives (six with translation). Thus, the three characteristics of each competency are reflected by its code. For example \([FGg]\), requires *formulation-without-translation* of a graphical derivative, using the data supplied in the question to give a graphical derivative response while \([FSg]\) requires *formulation-with-translation* of a symbolic derivative formulated using the symbolic data supplied in the question, translated to give a graphical derivative response. Incidentally, monitoring pairs of competencies such as \([FGs]\) and \([FSg]\) will be used to indicate *flexibility* between representations: the ability to move between competencies involving the same pair of representations. It is also possible to increase the range of competencies beyond the Framework by combining two or more fundamental competencies to construct *compound competencies*.

In summary, the Differentiation Competency Framework consists of 18 fundamental competencies associated with one of the numerical, graphical, or symbolic representations of derivative. Each competency involves either a formulation or interpretation process, with or without translation between different representations of derivative. The structure of the framework facilitates construction of suitable test questions to assess student and class understanding across all of the 18 fundamental competencies, and of individual competencies and groups of competencies associated with each input representation: formulation overall, formulation-with-translation, formulation-without-translation, interpretation overall, interpretation-with-translation, and interpretation-without-translation. The structure of the Framework also facilitates the monitoring of the teaching practices of the teachers and their use of the CAS activities provided in the curriculum.

### 3.3 The Differentiation Competency Test

The creation of the Differentiation Competency Test is explained and sample test questions are discussed to show their relationship to particular Framework competencies.

The Differentiation Competency Framework guided the construction of a comprehensive assessment instrument, the Differentiation Competency Test [DCT]. It consists of 18 questions and each tests one of the Framework competencies set out in
Figure 3.3 above. These questions were chosen to be representative of the 150 questions that were collected during examination of the textbooks, school assignments, test questions, and state examination questions (discussed previously). Some of the questions were not appropriate for the DCT because they involved compound competencies, and the intention was to use items that tested only the 18 fundamental competencies. From the set of 150 questions that were analysed there were approximately 8 items per competency (for most competencies) to examine in detail. The characteristics of suitable questions were identified and analyzed. For example, on the 16/5/99, the author identified the following behaviours associated with formulating graphical derivatives while planning DCT items. These are listed in Figure 4.4.

<table>
<thead>
<tr>
<th>Differentiate graphically by-hand:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Graph function</td>
</tr>
<tr>
<td>(1.1) construct graph by-hand or</td>
</tr>
<tr>
<td>(1.2) draw graph with calculator (*)</td>
</tr>
<tr>
<td>2. Construct tangent at point</td>
</tr>
<tr>
<td>(2.1) by-hand using graph of function (need gridlines)</td>
</tr>
<tr>
<td>3. Find gradient of tangent</td>
</tr>
<tr>
<td>(3.1) Identify coordinates of points on the tangent line</td>
</tr>
<tr>
<td>(3.2) Calculate rise/run</td>
</tr>
</tbody>
</table>

Note (*): The nature of the function is likely to determine whether or not the students are likely to use CAS to construct the graph at this stage.

<table>
<thead>
<tr>
<th>Differentiate graphically using the CAS calculator:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Graph function</td>
</tr>
<tr>
<td>(1.2) draw graph with calculator</td>
</tr>
<tr>
<td>2. Construct tangent at point</td>
</tr>
<tr>
<td>(2.2) use CAS calculator</td>
</tr>
<tr>
<td>3. Find gradient of tangent</td>
</tr>
<tr>
<td>(3.3) read gradient of tangent line from its equation</td>
</tr>
</tbody>
</table>

**Figure 3.4.** Author’s notes related to graphical differentiation while planning DCT items.

While being mindful of the competencies involved, the author included as many of these options as possible in the DCT questions (see Appendix 1.1). For example, while designing suitable questions that involved graphical differentiation (FG), the author included DCT Question 5 (competency [FGg]) on which the students were likely to draw the graph of the quadratic function using CAS, step 1.2 and then use steps 2.2 and 3.3 to find the gradient (or the by-hand alternative). For Question 6 (competency [FGs]) and Question 19b (competency [FGn]) the students had to use the by-hand option. On Question 6, the students needed to use steps 3.1 and 3.2 whereas on Question (19b) they
would also need the preliminary step 2.1. In addition, symbolic functions that were accessible to the students using by-hand techniques were selected so that students’ achievement of competencies was not obscured by their inability to do symbolic manipulations either by-hand or with CAS. The same types of considerations were applied to the other representations, ensuring that the DCT items were representative of the questions that had been collected. The 18 DCT items were sufficient because they tested not only competencies that are normally tested but in addition, several unusual competencies that are not normally tested (see Section 3.2.2). Several university colleagues performed reliability checks on the classification of the competencies on the 18 DCT items.

In the following discussion of four sample DCT questions, the way the Framework competencies are embedded in the test items is explained. The Framework competencies were tested on Test A (together with a small number of other questions). Test A is found in Appendix 1.1, and a complete guide to the location of the 18 DCT questions is found in Appendix 1.2. Four sample questions are considered and their location on Test A is shown in brackets. For example, Sample 1 is Question 1 on Test A.

**Sample 1. (A, Q.1)**

Find the derivative of \( y = x^5 + 4x^3 - x + 10 \)

Using the function data, a differentiation (using algebraic rules) is required. The process is formulation \( F \) of a symbolic derivative (symbolic input representation, S) and the phrase “Find the derivative” indicates that the output derivative is also symbolic (s). The competency is \([FSs]\) and translation of representation is not involved.

**Sample 2. (A, Q.11)**

The values of a function close to \( x = 5 \) are displayed in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>4.997</th>
<th>4.998</th>
<th>5.000</th>
<th>5.001</th>
<th>5.002</th>
<th>5.003</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>15.470</td>
<td>15.482</td>
<td>15.500</td>
<td>15.508</td>
<td>15.515</td>
<td>15.520</td>
</tr>
</tbody>
</table>

Find the best estimate of the derivative \( f'(5) \) at \( x = 5 \).

Using the table values closest to \( x = 5 \), a difference quotient is calculated. The process is formulation \( F \) of a numerical derivative (numerical input representation, N) and the phrase “Find the best estimate of the derivative” indicates that the output derivative is symbolic (s). In this case, the symbolic derivative is just one number. A translation from
the numerical to symbolic representation is involved and the competency is \( [F\text{Ns}] \).

[Note: a symbolic derivative can just be a number.]

**Sample 3. (A, Q.9)**

At 1.00p.m., the rate of change of temperature in your house was +3\(^\circ\text{C}\) degrees Celsius per hour. Immediately after 1.00p.m., is the temperature in the house most likely to decrease, stay the same or increase? Give a reason for your answer.

This question requires interpretation of rate of change data (numerical derivative, *input representation*, \( N \)). The *process* is interpretation (\( I \)), because the comparison task relies on students understanding the meaning of rate of change. The *output representation* is determined from the words of the question “Is the temperature likely to increase?” The focus on numerical values going up and down means the *output representation* is also numerical (\( n \)). There is no change of representation and the competency is \( [I\text{Nn}] \).

**Sample 4. (A, Q.14)**

The derivative function of \( f(x) \) is given by \( f'(x) = x^3 - 5x + 3 \). What is the gradient of the tangent to the curve \( y = f(x) \) when \( x = 1 \)?

The data requires that an algebraic formula (a symbolic derivative, *input representation*, \( S \)) be reinterpreted (\( I \)) as its equivalent graphical derivative (output representation, \( g \)). The competency is a translation of a symbolic derivative to its graphical equivalent \( [I\text{Sg}] \).

This section has shown how four sample DCT questions tested particular Framework competencies.

### 3.4. Summary and Discussion

This chapter has shown that Differentiation Competency Framework is a classification of the fundamental cognitive processes involved with understanding of the concept of differentiation involving numerical, graphical, and symbolic representations of derivative and translations between them. It provides a structure of competencies associated with the concept differentiation across three representations.

The Framework enables the goals of the calculus curriculum to be specified. It facilitates analysis of student learning of competencies by their representation and process. Strong cognitive reversible links between the three representations of derivative are identified by means of processes involving translation between
representations. In addition, the Framework enabled the author to monitor teachers’ use (and non-use) of particular representations in their personal teaching practices. Also, in an interview, the author was able to monitor the teachers’ understanding of differentiation competencies by providing them with the opportunity to demonstrate many of the Framework competencies.

Student behaviour on these four categories of DCT items enables student learning to be monitored. Success (i.e. the competency is demonstrated) on an item that tests the formulation-without-translation competency indicates that the student knows when and how to use the data in the question to carry out a differentiation procedure in the particular representation. Success indicates that she had made an appropriate decision to differentiate in the particular representation even if she made procedural errors (see Section 4.6.1). Similarly, success on a formulation-with-translation competency indicates that the student knows when and how to use the data in the question to carry out a differentiation procedure in one representation to achieve a derivative in a different representation. Success on an interpretation-without-translation item indicates that the student can give meaning to the derivative in the particular representation while success on interpretation-with-translation indicates that she can recognize pairs of equivalent derivatives in different representations without actually carrying out a differentiation procedure. This latter category of competency is not often specifically used or tested in textbooks, tests and other forms of assessment.

The DCT identifies class achievements on individual and groups of competencies and facilitates easy comparison between individual students and between all of the students (in the same class) taught by a particular teacher. The DCT is used to pinpoint differences between the classes and the results of this analysis are fully discussed in Chapter 6. In turn, these differences in learning between the classes will be explored in relation to the provision for use of CAS in the curriculum and the privileging of the two teachers. This information will enable Research Questions 1, 2, and 3 to be answered.
4. CONDUCT OF MAIN STUDY

This chapter describes the methodology that was adopted to establish what the students learned about the concept of derivative and how the teachers taught a calculus course that emphasized multiple representations of differentiation using a CAS supported curriculum.

An overview of the conduct of the main study includes the strategy to explore the three Research Questions (Section 4.1). The scope of the calculus curriculum is described (Section 4.2). Background information about the teachers is provided (Section 4.3) and the ways the teaching was monitored are described (Section 4.4). Background information about the students is also presented (Section 4.5) and the ways student learning was assessed are described (Section 4.6). Finally, the chapter is summarized and the conduct of the study is evaluated (Section 4.7).

4.1 Overview

This section provides an overview of the conduct of the main study involving a CAS supported calculus curriculum. The three Research Questions, intended to help establish the impact of each teacher’s teaching on the learning of the students in their class, are listed in Section 4.1.1. In addition, the research strategy adopted to answer the Research Questions is outlined (Section 4.1.2).

4.1.1 Research Questions

The three Research Questions below are expanded from Section 1.2.2 and include sub-questions for Research Question 1 and suggestions for data required to answer Research Questions 2 and 3.

Research Question 1

What was the role of CAS in assisting students to develop understanding of the concept of derivative?

1(a) How did the teachers incorporate functional and pedagogical CAS activities (outlined in Section 2.5.3) into their personal teaching programs?

1(b) What types of CAS activities promote understanding of the concept of derivative?
Research Question 2
How did teacher privileging influence the learning that occurred in each class? (See Section 1.1 for the working definition of privileging.)

To answer this question, the following information is required.

- What were the particular strengths and weaknesses of each class?
- What were the characteristics of each teacher’s privileging?
- Did the members of each class develop competencies related to the particular privileging characteristics of their teacher?
- Did students in the same school-based attainment group (see Section 4.5), but in different classes, demonstrate any differences in understanding?

Research Question 3
Did the teachers’ privileging stay the same when they taught with CAS for a second time?

- For each teacher, which privileging characteristics were changed during the main study from the preliminary study?

Exploring these three Research Questions will assist in the determination of the impact of teaching calculus in a CAS supported curriculum (with focus on multiple representations) on what the students learned about the concept of derivative.

4.1.2 Research strategy
This section describes how the themes for the main study were developed and the research strategy that was adopted to examine the three Research Questions. General background information about the setting of the main study was previously outlined in Section 1.2.1. It explained briefly how the main study developed out of the preliminary study that had assessment as its primary focus. This is now described in more detail.

While the author (researcher) observed approximately half of the lessons in the preliminary study, she noticed that the three teachers made very different use of the CAS calculator while using the curriculum materials that they had helped to prepare. Using the detailed data base she had constructed, the author undertook further exploratory analysis looking for alternative themes. She investigated several hypotheses
including the notion that greater use of symbolic algebra and CAS activities would lead to better student understanding of calculus concepts, accompanied by fewer conceptual errors (and fewer procedural errors) on the essentially symbolic test items. However, careful analysis of the teachers’ and the students’ use of CAS, students’ achievement on different types of test items, and patterns of students’ errors showed that greater use of CAS by the teachers and students did not automatically lead to better understanding of calculus concepts. The analysis also showed that a range of factors related to teacher behaviour appeared to influence what the students learned. She discovered that, in addition to using technology differently, each teacher had a very different approach to teaching and gave different emphasis to the symbolic and graphical representations of derivative. These factors impacted on student learning. She linked different types of achievement by the students in each class with each teacher’s different ‘privileging’ (of representations used, teaching approach, and use of technology). These results are briefly reported in Section 1.2.1 and more fully by Kendal and Stacey (1999a, 1999b). Her university colleagues agreed that the themes she had identified were legitimate and also reported them (see McCrae, Asp, & Kendal, 1999). To explore these themes longer term, research questions for the main study were proposed (and accepted by a university committee). This gave the author opportunities to examine the same teachers’ privileging in more depth, to explore its the stability over two years, and to monitor its impact on students’ learning using a new testing regime that focused on the numerical, graphical representations as well as the symbolic representation. This new testing regime also better matched the teaching planned in the curriculum.

To answer these research questions a variety of data is required for analysis. First, information about the potential opportunities for learning from the range of functional and pedagogical CAS activities provided in the curriculum is required. These data were obtained by analysis of the representations involved in the CAS activities included in the curriculum (see Section 3.1.2) in relation to the Framework competencies.

Second, data about each teacher’s privileging are needed. The author obtained information about both teachers’ classroom teaching practices through the observation and audiotaping of all the lessons. Comprehensive field notes were written by the author about the pedagogy of each teacher: the aspects of calculus they emphasized (including representations of differentiation), how they related to their students and involved them
in the learning process, and the ways they used the CAS calculator activities they chose to use from those provided in the curriculum. In addition, each teacher was interviewed in-depth twice: prior to, and after the main study.

Third, data about student learning are needed. This was achieved principally by means of the DCT described in the previous chapter (Section 3.3) that was given to the students in both classes at the end of the teaching program. Student learning was measured by achievement on the DCT. A second test instrument, the Preference for Representation Test [PRT] (to be discussed in Section 4.6.2) was used to confirm the class achievement on a range of competencies that were previously tested on the DCT. The PRT also identifies how the students used symbolic algebra and gives other insights into students’ understanding of the concept of derivative including their preferences for different representations. Further insights about student learning were obtained from the student interviews, questionnaire, and evaluation of the TI-92 (all to be discussed in Section 4.6). Since the students had no prior exposure to calculus, a calculus pre-test was not given, but the students’ prior attainment on algebra, graphs, and other mathematics was identified (see Section 4.5). This data proved useful in monitoring student learning that had occurred during the main study (see Section 6.4.3).

Achievement on different categories of competencies indicated the students’ understanding of different aspects of differentiation such as knowledge of differentiation in each representation, ability to formulate, ability to interpret, and ability to translate between representations. Class achievement (defined in Section 4.6.1) on different categories of competencies also provides a basis for comparing what each class (on the average) learned, and what (and how) each teacher taught and the opportunities for learning provided by the CAS activities in the curriculum.

Thus, exploring the Research Questions will involve identifying and assessing a range of factors that impacted on student learning. In particular, Research Question 1 involves identification of the opportunities provided in the curriculum for learning about different categories of differentiation competencies using functional and pedagogical CAS activities while Research Question 2 explores how teacher privileging (manifest through instructional differences with the CAS activities available in the curriculum and without CAS) related to student understanding of different categories of differentiation competencies. Research Question 3 establishes the stability of each teacher’s privileging
by comparing it prior to the main study (based on the first teacher interview and privileging observed during the preliminary study) with that observed during the main study and confirmed by the second teacher interview.

This section has outlined the directions that were taken to address the three Research Questions posed in this thesis. The data collected will allow assessment of the impact of the teaching in a CAS supported curriculum on students’ learning of the concept of derivative.

4.2 The Main Study Calculus Curriculum

In this section, the principal features of calculus curriculum used for the main study are reported. As mentioned in Section 2.5.1, the curriculum was essentially the school’s standard calculus curriculum based on the official VCE curriculum for Year 11 (VCE Mathematics Methods Course 1 & 2, 1998). However, it was modified to accommodate innovations for teaching calculus with technology. Its scope is described (Section 4.2.1), the CAS activities incorporated into the curriculum are described in detail in (Section 4.2.2), and the implementation of the curriculum is reported (Section 4.2.3).

4.2.1 Scope of curriculum

For the main study, the following official VCE curriculum aims were adopted.

- To develop intuitive understanding of average and instantaneous rates of change through familiar situations.
- To develop practical understanding by measuring average and instantaneous rates of change using numerical and graphical approaches.
- To develop understanding and proper use of symbolic derivative notation \( f'(x) \) and \( \frac{dy}{dx} \).
- To apply differentiation to problems such as finding rates of change, solving maximum and minimum problems, and using turning points to assist in sketching graphs of simple polynomials.
- To treat limits informally.
Additionally, compared with the VCE curriculum, the main study:

- Gave more emphasis to understanding the concept of differentiation through numerical, graphical, and symbolic representations and links between them involving CAS calculator technology.

- Gave less emphasis to differentiation from “first principles”. (The reasons for this decision are given in the literature review (Section 2.2.1)).

- Excluded antidifferentiation due to school time constraints.

Since the CAS calculator made symbolic algebra available, the following additional content from the Year 12 course of study (VCE Mathematics Methods 3 & 4, 1998) was included in the main study curriculum.

- Derivatives of rational functions.
- The chain and product rule.
- Rates of change problems.
- Deducing the graph of a gradient function from the graph of a function.

Thus, the main study’s curriculum (as for the preliminary study) was essentially the state’s standard curriculum that was modified slightly for use of CAS by both the teachers and the students. These modifications are discussed in Section 4.2.3.

4.2.2 Incorporation of technology into the curriculum

This section describes in more detail than in Sections 2.5.3 and 3.1.2 the CAS activities that were incorporated into the calculus curriculum for the main study.

CAS use was embedded into the teaching program. The teachers had the opportunity to incorporate CAS activities, intended for both functional and pedagogical use, into their teaching practices. Etlinger (1974) coined these terms when scientific calculators were first introduced (see Section 2.5.3).

CAS activities intended for functional use, involving numerical, graphical, and symbolical differentiation, were previously described in Section 3.1.1. They are displayed as Activities 1 – 3 respectively in Figure 4.1 below. In all, four CAS activities related to differentiation procedures are displayed. In addition, six CAS activities, intended for pedagogical use, were included in the curriculum to explain and link derivatives and are shown as Activities 5 – 10 in Figure 4.2 below.
## Representation of Differentiation

### Functional CAS Activities for Routine Differentiation Procedures

<table>
<thead>
<tr>
<th>Representation of Differentiation</th>
<th>Functional CAS Activities for Routine Differentiation Procedures</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Numerical (N)</strong></td>
<td><strong>Activity 1</strong></td>
</tr>
<tr>
<td>Numerical (N)</td>
<td>Numerical difference quotient calculations using nearby values of the function and inferring the limit</td>
</tr>
<tr>
<td><strong>Graphical (G)</strong></td>
<td><strong>Activity 2</strong></td>
</tr>
<tr>
<td>Graphical (G)</td>
<td>Determination of the gradients of tangents to curves at points by drawing a graph and the tangent at a point (an automated process) and deducing the gradient of the tangent from its equation</td>
</tr>
<tr>
<td><strong>Symbolic (S)</strong></td>
<td><strong>Activity 3</strong></td>
</tr>
<tr>
<td>Symbolic (S)</td>
<td>Differentiation using symbolic algebra symbolic followed by substitution or directly in one step</td>
</tr>
<tr>
<td><strong>Links N → S</strong></td>
<td><strong>Activity 4</strong></td>
</tr>
<tr>
<td>Links N → S</td>
<td>Determination of exact limits from a first principles approach using symbolic algebra</td>
</tr>
</tbody>
</table>

### Miscellaneous Procedures

- Numerical calculations
- Graph sketching
- Symbolic algebra such as substitution, solving equations, factorising etc.

### Differentiation Procedures

**Activity 1**

\[
\frac{\Delta f}{\Delta x} = \frac{f(x + h) - f(x)}{h}
\]

<table>
<thead>
<tr>
<th>(h)</th>
<th>(f(x))</th>
<th>(\Delta f)</th>
<th>(\frac{\Delta f}{\Delta x})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.998</td>
<td></td>
<td></td>
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<tr>
<td>1.999</td>
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<td>2.002</td>
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</tr>
</tbody>
</table>

**Activity 2**

1. Draw a graph and the tangent at a point (an automated process).
2. Deduce the gradient of the tangent from its equation.

**Activity 3**

- \(d(f(x)) = f'(x)\)
- Or \(f'(a)\) directly

**Activity 4**

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

*Figure 4.1.* CAS activities intended for functional use associated with differentiation in the main study curriculum.
<table>
<thead>
<tr>
<th>Representations of Differentiation Linked</th>
<th>Pedagogical CAS Activities to Explain and Link Derivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Activity 5</strong></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>Zooming-in on curves to show local linearity and the apparent coincidence of the zoomed-in curve and the tangent</td>
</tr>
<tr>
<td><strong>Activity 6</strong></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td>Dynamic program showing the relationship between the gradients of secants, curve and tangent viz., the limit of the secant line is the tangent to the curve at the point</td>
</tr>
<tr>
<td>N &amp; G</td>
<td></td>
</tr>
<tr>
<td><strong>Activity 7</strong></td>
<td></td>
</tr>
<tr>
<td>N &amp; G</td>
<td>Dynamic program linking rates of walking with slopes of curves using data logging devices</td>
</tr>
<tr>
<td><strong>Activity 8</strong></td>
<td></td>
</tr>
<tr>
<td>S or G &amp; S</td>
<td>Building up patterns amongst calculated derivatives and/or gradients to guess symbolic rules for derivative functions</td>
</tr>
<tr>
<td><strong>Activity 9</strong></td>
<td></td>
</tr>
<tr>
<td>S or G &amp; S</td>
<td>Plotting sets of gradients and/or derivatives as a function to predict or check derivative rules</td>
</tr>
<tr>
<td><strong>Activity 10</strong></td>
<td></td>
</tr>
<tr>
<td>N &amp; G &amp; S</td>
<td>Using the split screen to emphasize the links between different representations of derivative.</td>
</tr>
</tbody>
</table>

Figure 4.2. CAS activities intended for pedagogical use: To explain and link derivatives in different representations included in the main study curriculum.
Several of the pedagogical CAS activities also allow students to “experience” links between representations as suggested in the literature (Kaput, 1998; Leinbach, 1996; & Tall, 1996) (see Section 2.5.1) and discussed in Section 3.1.2.

Activity 5. The zoom-in capability of the calculator allows the notion of local linearity to be explored and the informal concept of derivative as the gradient of the curve itself to be developed (see Tall, 1996).

Activity 6. The programming capability permits dynamic programs to be incorporated into the teaching. First, a graph of a function say \( f(x) = 3x^2 \) is drawn and then a secant is drawn through a nominated point (say \( x = 1 \)) and a nearby point (say \( x = 2 \)). A difference quotient is calculated that represents the gradient of the secant. The process is repeated using a set of nearby points that get progressively closer to \( x = 1 \) (say \( x = 1.8, x = 1.5, x = 1.1, x = 1.01, x = 1.001 \) etc.). For each secant, a difference quotient is automatically calculated until the limiting difference quotient value at \( x = 1 \) is observed. At this stage, the secant appears to be coincident the tangent. This activity provides an opportunity to relate the limiting difference quotient to the limit of the gradient of the secants, the gradient of the curve \( x = 1 \), and the gradient of the tangent to the curve at \( x = 1 \), thereby developing intuitive links between difference quotients and slopes of lines and curves (see Bowers, 1999; Leinbach, 1996).

Activity 7. When connected to a Computer-Based Laboratory data logger, [CBL], the TI-92 has the capacity to collect data from a real-world physical situation (e.g., the distance and time of a moving object from an initial point) and then represent it in both tabular and graphical forms, thereby developing links between rates of change (e.g., speed) and slopes of lines and curves (see Kaput, 1998).

Activity 8. Patterns amongst calculated gradients and/or gradients may be quickly built up to guess symbolic rules for derivative functions.

Activity 9. For a given function, a set of gradients and/or derivative values at points are plotted to predict, or check, the derivative rule for the given function.

Activity 10. The split screen facility enables different representations of differentiation to be linked by simultaneous viewing.

This section has described ten CAS activities (four intended for functional use and six for pedagogical use) that were incorporated into the standard calculus curriculum for
the main study (and used previously for the preliminary study) to support teaching for understanding the concept of derivative with CAS.

4.2.3 Implementation of the curriculum

In preparation for the main study the two teachers initiated minor modifications to the calculus curriculum they had previously taught with the CAS calculator (TI-92) during the preliminary study. They wished to avoid some of the pitfalls previously experienced so they recommended changes for the main study. This section describes the way the main study’s curriculum was implemented including some minor modifications suggested by the teachers. First, they decided to reduce the overall number of (45 minute) lessons from 26 to 22. In turn, the number of lessons devoted to initial concept development was reduced from 12 to 9 while the number of application lessons was maintained. In addition, 90 minutes was available for each test. The decision to reduce teaching time was motivated by the teachers’ confidence that they could teach the course more efficiently the second time round and a school requirement that they teach additional mathematical topics during the semester. Second, they decided to encourage by-hand algebra more than in the preliminary study and in consequence reduce the emphasis on using the symbolic facility of the CAS calculator. Both teachers were well aware that their students would have to sit future exams (including external state examinations in 2000 and beyond) without access to a CAS calculator (this was an institutional school pressure). In Victoria, during examinations, students are permitted to use graphics calculators but not CAS calculators. Third, both teachers strongly recommended simpler procedures for numerical differentiation. The students in the preliminary study had experienced difficulties setting up tables and had wasted quite a deal of time for little perceived benefit since the preliminary study assessment regime was essentially symbolic (with graphical options) and did not assess numerical procedures (see McCrae, Asp, & Kendal, 1999). In consequence, the procedures for numerical differentiation were simplified for the main study. The teachers made these recommendations with full knowledge of the capabilities of the students in their classes (to be discussed in Section 4.5).

Since the textbook used by the school provided examples that focused mainly on the symbolic representation, the teachers also helped develop a set of worksheets, one for
each lesson, that were available for student use during the 22 calculus lessons. As for the preliminary study, the teachers taught the students trigonometry to help them learn about the capabilities of the CAS calculator.

This section has reported details of the main study’s curriculum for teaching an introductory calculus course with a focus on understanding differentiation in numerical, graphical, and symbolic representations and linking the representations. The teaching program offered a smorgasbord of CAS activities. The functional CAS activities were intended to be used for differentiation procedures while the pedagogical CAS activities were intended to enhance students’ understanding of the concept of derivative.

4.3 Background Information About Teachers
Teachers A and B, colleagues at the same secondary school, were highly experienced teachers of mathematics who had used graphics calculators in their classrooms for several years prior to teaching the experimental calculus course for both the main study and the previous preliminary study. They both had prior experience teaching calculus to Year 11 and 12 students (using the state-wide VCE curriculum, described above, in Section 4.2.1). Teacher B was the mathematics coordinator at his school and, over several years, had regularly attended professional development at the University of Melbourne to help him introduce graphics calculators into his classroom and to assist his staff to use them in their classes. Subsequently, he was invited by university staff to participate in the preliminary trial conducted in 1998. Teacher C, from a different school and another participant in the graphical calculator professional development, was also invited to participate. Teacher A, a colleague of Teacher B, volunteered to participate in the preliminary trial. He was also well known to the university staff. In the six months prior to the preliminary study these three teachers participated in on-going professional development about how to use the CAS calculator and how to use it for teaching introductory calculus. They shared ideas and helped establish the content to be taught, and how it was to be taught. Teachers A and B were able to participate in the main study and they came to it with prior (albeit limited) experience of teaching with CAS calculators.

In summary, although the teachers had comparable prior experience of teaching mathematics and teaching mathematics with technology, they taught differently in the
preliminary study and their students achieved differently but in ways that were consistent with their teachers’ privileging. The main study provided an opportunity to explore these themes, initially identified in preliminary study, again in more depth.

4.4 Teaching Monitored
This section reports how the teaching practices and attitudes of each teacher were monitored during the main study. These observations will be used to determine the privileging characteristics of each teacher. Of particular interest are the calculus they emphasized, their teaching approach, and their attitudes towards and manner of use of the CAS activities they incorporated into their personal teaching programs. Each teacher was interviewed twice (Section 4.4.1). All lessons by both teachers were observed and detailed observations of classroom teaching were made. How the classroom data was gathered, processed, and treated is also discussed (Section 4.4.2).

4.4.1 Teacher interviews
The conduct of the two interviews with the teachers is described in this section. On both occasions, the teachers were interviewed separately to gain an insight into their mathematical conceptions, content knowledge of the concept of differentiation, and personal use of technology. The first interview was conducted nine months after the end of the preliminary study and ten weeks prior to the main study and the second interview was conducted ten weeks after the end of the main study. Both interviews were tape recorded and transcribed.

First Teacher Interview
The purpose of the first teacher interview (see Appendix 3.1) was fourfold:

- To investigate the depth of each teacher’s understanding of differentiation using items that tested their knowledge of multiple representations of derivative.
- To identify if, and how, each teacher privileged particular representations of differentiation and derivative.
- To gauge each teacher’s attitude toward the CAS calculator.
- To gain an understanding of each teacher’s teaching methods and style of teaching.

The first teacher interview required the teachers to discuss a selection of problems. After their initial response, each teacher was asked to describe other methods they
thought their students might adopt (with CAS available). The teachers were told that they did not have to actually solve the problem and were encouraged to discuss their ideas about the advantages and disadvantages of using CAS.

There were nine teacher interview items, six of which were later given to the students, one on the DCT (that tested the Framework competencies, see Section 4.6.1) and five on the PRT (the second test and its purpose is discussed in Section 4.6.2). However, three of the teacher interview items were not given to the students including a more complex question involving reasoning about symbolic derivatives. A guide to the location of each interview question on Tests A and B is given in Appendix 3.2. Using different representations of differentiation some items could be solved in alternative ways. The author encouraged this by stating:

I’ve got a set of calculus questions here to talk about. Could you just tell me about how you would solve each of these problems. You don’t actually have to solve the problems, just describe your method. Secondly, I’m very interested in the different ways the girls might solve the problems having had access to the TI-92 during the calculus lessons. Can you tell me about the methods you think the girls (different methods from yourself or the same methods as yourself)? You don’t have to solve the problems, but just tell me about how you think they would solve them.

Second Teacher Interview
Ten weeks after the teaching program both teachers participated in a second interview (see Appendix 3.3). It consisted of six open-ended questions that were designed to encourage the teachers to reflect about their teaching practices in the main study, particularly the way they used the CAS calculator to support their student’s understanding of the concept of derivative and for routine procedures.

4.4.2 Classroom teaching
The author monitored the classroom teaching of both teachers and observed that they had very different methods and styles of teaching and different ways of teaching with technology. As mentioned above (Section 4.1.2), this was also observed during the preliminary study. To monitor changes in teacher privileging (if any) that occurred during the main study, the author collected classroom data while observing every lesson by both teachers. This section describes how the data about the teaching was gathered, processed, and treated. The aim was to monitor the following aspects of the teachers’ pedagogy:
• The personal teaching method of each teacher (e.g., purpose and cognitive focus of lesson).
• The personal teaching style of each teacher (e.g., teaching strategies, questioning techniques, and relationship with students).
• Issues related to how differentiation was taught using multiple representations (e.g., frequency of use of particular representations and teacher attitude towards different representations, time spent on different types of activities).
• Issues related to how the CAS calculator was used (e.g., technical use and teacher attitude towards using technology, and ways the students were encouraged or discouraged from using the technology).

To monitor each teacher’s pedagogy, the author collected classroom data that is incorporated into six Appendices (4.1 - 4.6). Appendix 4.1 includes the author’s notes written down during 8 minutes of one of Teacher A’s lessons while Appendices 4.2 and 4.3 show how the notes for this lesson were analysed. Appendices 4.4 and 4.5 summarize similar observations for both teachers during the entire teaching program. Finally, Appendix 4.6 summarizes each teacher’s use of each CAS activity throughout the study. The author audiotaped every lesson and made detailed notes about how each lesson was conducted. Videotaping lessons was tried and abandoned. Using an audiotape and making careful notes during the lesson proved to be more effective because several classroom events could be monitored simultaneously. The author regularly noted the times at which activities changed and when significant classroom events occurred. She kept track of the calculus content taught by making an accurate copy of the board notes and the TI-92 displays on the projector screen. She observed the ways the teachers used the TI-92 and noted their explicit attitudes toward its use, and occasions when it was not used. She wrote down the teachers’ questions and students’ answers (as far as possible) and noted the teachers’ interactions with individual students and the whole class. She recorded the students who were absent from class and how the teachers compensated for this in future lessons, the students who failed to bring the TI-92 to the class, and the teachers’ and students’ use of the TI-92. However, she did not monitor student-student interactions in any depth.
As an example, consider the notes taken by the author during 8 minutes of Teacher A’s lesson on the 16/8/99. The original set of notes for this part of the lesson is located in Appendix 4.1. Figure 4.3 gives a typed and slightly expanded version that is easier to follow. It shows that the author observed that Teacher A’s emphasis was symbolic differentiation (involving the S representation) using two different CAS techniques (F3 menu & 2nd 8 button). A third technique, finding multiple derivatives simultaneously, was also demonstrated during the lesson. Teacher A devoted 2 minutes to verifying the symbolic result graphically, thereby linking the symbolic and graphical representations (S to G). Teacher A’s teaching method was to emphasize three different differentiation procedures for symbolic differentiation with CAS. Teacher A’s teaching style was also observed. First, he directed questions to particular students and then demonstrated their responses to the class using CAS and overhead projection. Second, in response to Hannah’s question, he immediately directed her attention back to her copy of his notes. The author was also able to observe that, while Teacher A used the CAS calculator (for symbolic differentiation), he missed an opportunity to differentiate graphically using the CAS calculator. Teacher A’s “doing” teaching actions were to emphasize the symbolic differentiation procedures using symbolic algebra and his “understanding” actions were to verify the gradient as -12 (approximately) thus linking the G and S representations.

On the same day, as soon as possible after each lesson, the author reflected on the lesson and, after checking the audiotape, wrote up a comprehensive set of observations about the lesson, describing up to 52 characteristics of the lesson. Appendix 4.2 shows Lesson Observation Sheet (1) for the entire lesson described above. The lesson plan number, 7, refers to written guidelines for teaching the calculus curriculum. It indicates the specific mathematical content and relevant CAS activities that were planned by the author and both teachers. Characteristics that were monitored include teachers’ questions, students’ questions, teaching strategies, techniques used to check students’ understanding, technical issues, use of representations, and misconceptions. Thus, the teaching approach of both teachers was carefully monitored through examination of their questioning techniques, methods of communicating with students, and teaching strategies. The teachers’ and students’ use (and non-use) of technology were also observed. Finally, the teachers’ use of individual representations and simultaneous use of pairs of representations (or all three representations) were also recorded.
10.44 Lesson started.
Teacher A asked the class the question “How do you find a derivative?
Students in the class responded together “Find gradient”.
Teacher A wrote the following revision question on the black board.
Find the derivative at $x = -2$ on the graph $y = 2x^2 - 4x + 1$

10.46 The audiotape was started
The girls worked quietly
Teacher A: “What answer did you get Jessica?”
Jessica: “-12”.
Teacher A worked Jessica’s CAS procedure for symbolic differentiation on the TI-92 (using the F3 menu, differentiate) using overhead projection for the other students in the class to see.
$$d(2x^2 - 4x + 1, x) = -12$$

10.48 Teacher A repeated the symbolic differentiation showing the girls how to use the 2nd 8 button as an alternative to the F3 menu.

10.50 Teacher A used the TI-92 to draw the graph of $y = 2x^2 - 4x + 1$ and adjusted the window to see the graph properly.
Teacher A added in the tangent to the graph at $x = -2$, by-hand.

$$y$$

-2

Teacher A estimated the validity of the answer ‘-12’ from the slope of the graph.
Teacher A did not use the graphical method available with the CAS calculator to find the gradient of the tangent line from its equation, $y = -12x - 7$.

10.52 Teacher A gave the students a second example.
Find the derivatives at the points $x = -2, -1, 0, 1, 2$ on the graph $f(x) = 3x^2$.
Hannah commented that she did not know what to do.
Teacher A queried whether Hannah had checked her notes.

Figure 4.3. Author’s notes for 8 minutes of Teacher A’s lesson (slightly expanded version).

Next, Lesson Observation Sheet (2) was compiled (for each teacher and every lesson) from the author’s original lesson observations together with the reflections and observations recorded on the Lesson Observation Sheet (1). For example, the Lesson Observation Sheet (2) for Teacher A’s lesson above (see Appendix 4.3) shows the time (shaded in blocks) that Teacher A spent on the symbolic and graphical representations. It also shows the time he spent using the TI-92 and by-hand procedures, and the time he devoted to different teaching approaches. For example, the column headed ‘Symbol’ shows that Teacher A used the symbolic representation from 10.45am to 11.15am (30
minutes). The column headed ‘Graph’ shows that he spent from 10.50am to 10.52am (two minutes) simultaneously using the graphical representation. Thus, Teacher A spent a total of 30 minutes on the S representation, 2 minutes linking the S to G representation, and 28 minutes on the individual S representation. The columns headed ‘By-hand’ and ‘With TI-92’ show that he spent 28 minutes using the TI-92 and 2 minutes using by-hand procedures. Similarly, he spent 2 minutes conducting lectures, 12 minutes conducting class discussions, 11 minutes helping individual students, 0 minutes allowing the students to work without teacher intervention, and 10 minutes doing administrative tasks at the beginning of the lesson.

Appendix 4.4 provides an overview of the time each teacher spent on the three representations during the entire teaching program. It shows, lesson by lesson, the time that each teacher devoted to each representation individually, linked to other representations, and in total. For example, Appendix 4.4 shows that, on the 16/8, Teacher A spent 28 minutes on the S representation (individually, i.e., not linked to other representations) and 2 minutes linking S and G so the total time spent on S was 30 minutes. Appendix 4.4 shows, over the entire teaching trial, the amount of time that each teacher allocated to particular representations (individually, linking the specified pairs of representations, and in total). For example, Teacher A spent 286 minutes on the S (individually), 40 minutes on linking G and S, and 341 minutes (in total) on the S representation. Similarly, the times for Teacher B are also shown. The time each teacher allocated to the representations broadly indicates their relative importance to the teacher (relevant in Section 5.2.2). During these times, the teachers used relevant CAS activities, lectured students, conducted class discussions, worked with individual students, and allowed the students time to work on the worksheets without direct help.

Similarly, Appendix 4.5 provides an overview of the time each teacher devoted to a range of other specified teaching activities during the entire teaching program. It shows, lesson by lesson and over the entire teaching trial, the time each teacher devoted to each teaching activity. For example, it shows that overall Teacher A spent 185 minutes using the TI-92, 260 minutes lecturing Class A, and 92 minutes in engaging the class in discussion. Similarly, the times for Teacher B are also shown. Once again, the total time each teacher devoted to each activity broadly indicates its relative importance to the teacher (relevant in Section 5.2.3).
Appendix 4.6 is also derived from all of the original lesson notes. It collates all instances, by each teacher, of teaching related to 8 of the 10 CAS activities given in the curriculum (see Figures 4.1 and 4.2). This was done so that specific teaching could later be able to be linked to student learning outcomes that will be discussed in Chapter 8. For example, Appendix 4.6 shows that Teacher A taught material related to CAS activities on five days, using several methods. In contrast, Teacher B did this on two days, using fewer methods. This data is used as a basis for further analysis about the teaching that is reported in Section 5.2.2 and Section 8.1.1

Ensuring quality of initial observations

The successful conduct of the main study depended on high quality of lesson observations. Quality was ensured by the prior experience of the author and confirmation of her observations by colleagues. The author was a highly experienced classroom teacher with 25 years experience of teaching calculus. As the mathematics coordinator in her school, she was often required to observe other teachers and trainee teachers. In addition, her prior research involved observation of teachers. During the preliminary study, three of her University colleagues, who were also involved in the research project, observed several lessons taught by Teacher A, Teacher B, and Teacher C. After each of these lessons, her colleagues extensively discussed the author’s lesson records and verified her written observations about the mathematical content (representations) the teachers were teaching and the ways the teachers were teaching it. The main study involved the same author and the same teachers teaching the same curriculum and the author refined her method of data collection by audiotaping every lesson. This was useful to verify her observations, for Lesson Observation Sheets (1) and (2), about the content taught by the teachers and their interactions with the students.

This section has described the conduct of both the teacher interviews and classroom observations. It has also described how the classroom data was gathered, processed, and treated. The first interview, together with the lesson observations from the preliminary study, gave insights into each teacher’s beliefs, knowledge of differentiation, the calculus emphasized, teaching method, teaching style, and attitudes towards technology prior to the main study. The second interview, together with the main study lesson observations, enabled changes in privileging that occurred during the main study to be monitored and Research Question 3 to be addressed.
4.5 Background Information About Students

This section provides background information about the students who participated in the main study.

Thirty-three Year 11 students (16-17 year olds) from an independent Melbourne girls’ school, participated in the main study calculus course. There were fourteen students in Class A (taught by Teacher A) and nineteen in Class B (taught by Teacher B). The students in both classes were studying only one Year 11 mathematics subject, Mathematics Method 1 & 2, and most intended to continue with Mathematics Method 3 & 4 (with its curriculum, assessment and examinations determined by Victoria’s mathematics curriculum authority, the Board of Studies) during the following year. These mathematics subjects counted towards entry to tertiary study.

The classes were mixed ability with students of diverse school attainment levels and mathematical knowledge. \textit{School attainment} was determined by averaging four recent school test assessments: in Year 10 (1998), Semester 2 exam; and in Year 11 (1999), Semester 1 exam, Semester 1 algebra test, and Semester 1 graphing test. Class A attained a mean score of 70\% (SD 19) and Class B 58\% (SD 16) and the number of students in each class who attained in each attainment level from A to G is displayed in Table 4.1 below.

<table>
<thead>
<tr>
<th>Attainment (%)</th>
<th>99-90</th>
<th>89-80</th>
<th>79-70</th>
<th>69-60</th>
<th>59-50</th>
<th>49-40</th>
<th>&lt;40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade (Class A (N=14))</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>G</td>
</tr>
<tr>
<td>Class A (N=14)</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Class B (N=19)</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

The distribution of attainment groups was different in Classes A and B and neither distribution was a normal distribution. Clearly Class A had a much higher proportion of higher achieving students and Class B had a much higher proportion of lower achieving students. Approximately one-third of Class A achieved grades of A-B, one-third achieved grades C-D, and one-third achieved grades E-G. In contrast, about one-tenth of Class B achieved grades of A-B, approximately one-third achieved grades of C-D, and more than half achieved grades of E-G.

These results matched the students’ perceptions of their overall mathematical ability prior to commencing the main study (see Appendix 5.1). On a Likert scale, Class A
students rated their success higher than did Class B students, (good cf. moderate) for mathematics, and (very good-good cf. good-moderate) for symbolic algebra. This was consistent with their class achievements on the symbolic algebra–based tests. Class A achieved an average of 70% and Class B 59% showing that Class B students were weaker with symbolic algebra than Class A students. Prior to the calculus unit, all the students from both classes had routinely used graphics calculators (TI-83) and felt comfortable using them. On a Likert scale, both classes rated their success as (good-moderate) for graphing. These results matched their success on class tests related to graphing.

Immediately prior to the commencement of the main study the teachers conducted a ten-lesson study of circular functions to familiarize the students with CAS (using the TI-92 calculator). The students were given the opportunity to experience (and hopefully acquire) a set of pre-determined pre-skills (see Appendix 6.1) including graphical skills and knowledge about performing basic algebraic procedures using CAS.

This section has established that the classes had different distributions of attainment groups and neither was normal. Class A had a higher proportion of students in the highest attainment groups and who felt very confident with algebra. Both groups of students had chosen to study Mathematical Methods 1 & 2 and although they were reasonably able at mathematics did not see themselves as maths specialists.

**4.6 Student Learning Monitored**

The purpose of this section is to report the instruments (and how they were administered) that were used to establish what the students in each class learned.

Student understanding was assessed, and use of technology gauged, using five instruments: two tests, one interview, a questionnaire, and a written evaluation of the TI-92. The first test, the DCT, is the most important assessment tool. It was specially designed to test individual Framework competencies and its creation was previously discussed (Section 3.3). Its implementation is to be discussed next (Section 4.6.1). The design and implementation of the second test, the PRT, is presented (Section 4.6.2). In addition, the following are also discussed: student interviews (Section 4.6.3), the "Calculus and CAS” questionnaire involving a Likert scale (Section 4.6.4), and students’ personal evaluation of the TI-92 (Section 4.6.5).
4.6.1 Differentiation Competency Test

This section describes the administration of the 18 questions on the DCT, outlines the criteria used to judge the achievement of the Framework competencies embedded in the questions, and defines class achievement of competencies.

Administration of the DCT

At the conclusion of the teaching program Class A (14 students) and Class B (19 students) sat Test A (see Appendix 1.1) with a time allocation of 90 minutes (sufficient time for all students to complete it). Test A tested the 18 Framework competencies together with some additional items. These additional items were simple like Q.19(c) or more challenging like Q.19(e) that involved more than one competency. Appendix 1.2 lists each of the 18 Framework competencies and the Test A question that tested it. For example, the competency $[FSs]$ was tested in Question 1 (coded A Q.1).

Since the DCT is essentially a conceptual test it did not require complicated algebraic manipulations and because the DCT items required only symbolic algebra achievable by-hand the students’ abilities to demonstrate Framework competencies were not masked by their inability to use the symbolic algebra facility of CAS. Since some of the DCT items were unusual (in that they were easy in the sense that they did not involve a lot of work) the students were alerted to the fact that several items did not require any calculation. At the beginning of the test the students were given these verbal instructions by the author (9th September 1999).

The test is not as long as it first appears because some of the questions require very little calculation, in some cases, no calculation. It is important to read each question carefully. Think about what the question is really asking you to find out before you do it. Don’t forget to tick the boxes to indicate how you did the question: with the TI-92 or by-hand. Did you use algebra, graphs or number calculations?

As indicated by this statement, student use of CAS (TI-92 calculator) was closely monitored including the context of its use (algebra, graphs, or number calculations). Students indicated when and how they used CAS and representations by ticking the appropriate check boxes at the end of each Test A question (see Appendix 1.1).

Judging if a competency has been demonstrated

Each item of the differentiation competency test was designed to test one competency that was assessed as either demonstrated or not demonstrated by the student. The competency was demonstrated if the written solution on the test item indicated
awareness of all the necessary cognitive steps. The competency was not demonstrated if conceptual errors were made. The first type of conceptual error was incorrect formulation through an inappropriate selection of representation, the second, a failure to demonstrate critical understanding (such as choosing two points on the curve instead of on the tangent to the curve when differentiating graphically), and the third type was interpretation mistakes. Competency was also considered demonstrated if only procedural errors were made (including algebraic, graphical, calculation, or careless mistakes e.g., transcription).

Although it is highly desirable to replicate the testing of each competency to increase test reliability, school time constraints prevented this from occurring. A test with 36 or more questions was not useable for this study. However, most of the formulation competencies were retested on a second test, the PRT (discussed in the next Section 4.6.2).

Class achievement of competencies
Class achievement of a competency on the DCT (and on PRT, discussed below) was defined to be the percentage of students in each class who successfully demonstrated the competency. It provided a measure for the level of understanding of each differentiation competency for each class. Class achievement on groups of competencies (such as on the six competencies in the numerical, graphical, and symbolic representations and on the formulation and interpretation categories of competencies) was determined by averaging the relevant individual competencies.

This section has described how the DCT was administered, outlined the criteria used to judge the achievement of the Framework competencies and defined class achievement of competencies.

4.6.2 Preference for Representation Test
This section reports the development, focus, and administration of the second test, the PRT (Test B, see Appendix 2.1) that was given to the students at the conclusion of the teaching program. Whereas each DCT question (on Test A) required the students to use a particular representation to solve a problem, some of the PRT questions (on Test B) allowed the students to choose their own method (using a recognized or preferred representation) to solve the problem. The PRT also provided the students with an
opportunity to use the symbolic algebra facility of the CAS (not particularly useful on the DCT). The focus of each PRT question, its location on Test B, and the Framework competency it tests is listed in Appendix 2.2. For example, Questions 1(a)-(f) focus on use of symbolic algebra and test the [FSs] competency.

Development of PRT questions
Many of the PRT items were initially developed by the author to explore student knowledge of, and preference for numerical, graphical, and the symbolic representations that were being taught in the preliminary study’s experimental curriculum but not necessarily tested. The test questions used in the preliminary study were mainly standard VCE test items that were essentially symbolic (as reported briefly in Section 1.2.1). Thus, to explore how the students were managing the numerical and graphical representations (in addition to the symbolic representation) and to identify the students’ preference for representations, potential PRT items were piloted as regular weekly homework items during the preliminary study. The author noticed that the students from different classes tended to answer the questions differently, in ways that related to their teacher’s different emphasis on different representations. For comparison, some of the homework items were also given to a group of secondary school students that regularly used graphics calculators but who were not involved in the preliminary trial, and other differences were noticed. Thus, for the main study, many of the trialled items were incorporated into the PRT. In addition, some revised items and new items were also included. These items were trialled by university colleagues who also checked their classification.

Focus of PRT questions
The PRT consisted of nine questions (sixteen individual items) each of which has one focus from the following: to monitor student use of symbolic algebra, to identify the influence of alternative data on choice of representation for differentiation, to identify students’ abilities to solve the same problem using two different representations, to explore students’ abilities to use different numerical data, and to explore students’ abilities to achieve a compound competency (formulation of a graphical derivative and then its interpretation, numerically, graphically or symbolically). Table 4.2 shows the questions on the PRT. Items with a similar focus are grouped together. For example, the
focus of Questions, 1(a)-(f), and Questions 8(a) - (b) is ‘use of symbolic algebra’. Sample questions for each type focus on the PRT are discussed below.

Table 4.2  Focus of Each Question on the Preference for Representation Test

<table>
<thead>
<tr>
<th>Type</th>
<th>Focus of PRT Question</th>
<th>Questions on Test B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Use of symbolic algebra</td>
<td>To monitor students’ use of CAS or non-use (i.e. by-hand) symbolic differentiation.</td>
</tr>
<tr>
<td>2</td>
<td>Influence of alternative data</td>
<td>To identify the influence of different data on choice of representation for differentiation.</td>
</tr>
<tr>
<td>3</td>
<td>Knowledge of two representations</td>
<td>To identify students’ abilities to solve the same problem using two different representations.</td>
</tr>
<tr>
<td>4</td>
<td>Knowledge of different data</td>
<td>To explore students’ abilities to use numerical data</td>
</tr>
<tr>
<td>5</td>
<td>Knowledge of a compound competency</td>
<td>To explore students’ abilities to use graphical data and understanding of local linearity.</td>
</tr>
</tbody>
</table>

**Use of symbolic algebra**

Questions with this focus give the students an opportunity to use symbolic algebra (not required by DCT questions) and the author an opportunity to monitor CAS use. For example, consider Sample 5. Students have the option to perform the differentiation using by-hand techniques while Sample 6 can only be differentiated with symbolic algebra.

**Sample 5. (B Q.1(b))**  (competency [FSs])

Differentiate \( x^3 (2x + 1)^2 \)

Using the function data, a differentiation (using algebraic rules) is required. The *process* is formulation (\( F \)) of a symbolic derivative (symbolic *input* representation, \( S \)) and the required *output* derivative is also symbolic (\( s \)). Hence, the competency is \([FSs]\). Students may elect to use symbolic algebra or perform the differentiation using by-hand techniques.
Sample 6. (B Q.8(b)) (competency [FSn])

Find the rate of change of \( y \) with respect to \( x \) if \( y = (1.8)^x \) when \( x = 3 \).

The students in this study need to use symbolic algebra to perform this differentiation, because they do not yet know the derivative of \( a^x \).

Influence of alternative data

Questions with this focus are matched so that one of the questions has additional alternative data (or use of data restricted). The presence of particular data may act as a stimulus for students to use the representation associated with it enabling class preference for representation to be ascertained and the influence of different data to be monitored. For example, the influence of adding a graph to symbolic data is explored by comparing the ways the students respond to Sample 7 (with a graph) and Sample 8 (without a graph).

Sample 7. (B Q.4(a))
The data supplied is a function rule for \( y \) accompanied by its graph and involves the competencies \([FSg]\) or \([FNg]\) or \([FGg]\).

Consider the curve \( x^3 - x \). Find the gradient of the curve at \( x = 1 \), stating whether it is accurate or approximate.

Sample 8. (A Q.3)
The data supplied is the function rule for \( g(x) \) only (without its graph) and also involves the competencies \([FSg]\) or \([FNg]\) or \([FGg]\).

A curve has the equation \( 6x^3 - 6x^2 + 3x - 6 \). Find the gradient of the curve at the point \( P \), where \( x = -1 \).

Both Samples 7 and 8 require a graphical output derivative (g) since both questions ask "Find the gradient of the curve" at particular points, \( x = 1 \) and \( x = -1 \). Both questions supply function data that facilitates three viable solutions, each involving a different input derivative representation (numerical, graphical or symbolic). (Note that for Example 8, the competency \([FSg]\) is demonstrated when a student determines the gradient of the curve by differentiating \( g(x) \) using symbolic rules and substituting \( x = -1 \). The competency \([FNg]\) is demonstrated when a student determines the gradient of the curve by calculating a difference quotient using two close \( x \) values, \(-1\) and \((-1 + h)\) and their corresponding function values \( g(-1) \) and \( g(-1 + h) \), obtained by substitution. The competency \([FGg]\) is demonstrated when a student determines the gradient of the curve
directly by constructing the tangent line and determining its gradient at \( x = -1 \). The student’s choice of method of differentiation indicates her preference for representation and leads to achievement of either \([FSg]\), or \([FNg]\), or \([F\!Gg]\) enabling the effect of the additional graphical data supplied in Sample 6 to be easily observed.

**Knowledge of two representations**

Questions with this focus enable identification of students who have the ability to solve the same problem using two different representations. For example, in (B, Q 4(b) the students were asked to repeat Sample 7 (B, Q 4(a) using an alternative method.

**Knowledge of numerical data**

Questions with this focus show the influence of different types of numerical data. Sample 9 provides the students with the opportunity to demonstrate their understanding of rate of change data.

**Sample 9. (B Q.6)**

(competency \([INn]\))

Suppose you travelled 100km in your car in two hours. Must there have be a time when your instantaneous speed was exactly 50 km per hour? Explain your reasoning carefully.

**Knowledge of a compound competency**

Questions with this focus explore how competencies can be combined. Sample 10 involves formulating a graphical derivative and then explaining it as the slope of the curve gradient or reinterpreting it as a rate of change or derivative. It combines the competencies \([FGg]\) with [Interpretation of G] as G, or as its equivalent derivatives N or S.

**Sample 10. (B Q.7(b))**

(compound competency \([FGg + IGn\text{ or } IGg\text{ or } IGs]\))

Suppose you used the Zoom-In function on the calculator 5 times on the graph of \( y = x^2 + 3 \) around the point \( P \) where \( x = 2 \). If you knew the coordinates of two points on the Zoomed-In graph, what could you find out about the graph at the point \( P \)?

**Administration of the PRT**

At the conclusion of the teaching program Class A (14 students) and Class B (18 students) sat Test B (Appendix 2.1) with a time allocation of 90 minutes (sufficient time for all students to complete the test). (Note that while 19 Class B students participated in the main study, due to the illness of one student, only 18 undertook the PRT.) Once
again, the students indicated when and how they used CAS and representations by ticking the appropriate boxes. Student responses were judged to be successful if they demonstrated the competency as for the DCT (see Section 4.6.1) using the particular representation chosen. When the student’s response did not involve differentiation (possible on Questions 2 and 9) the response was judged successful if a correct solution was obtained. Overall class success on the PRT was calculated as the average percentage of successful responses achieved by each class on the 16 items.

In summary, this section reported the administration of the PRT and described the focus of the questions to monitor different aspects of students’ learning and use of symbolic algebra. This focus includes the achievement of particular competencies related to use of different data, knowledge of two representations to solve the same problem, and knowledge of numerical data.

4.6.3 Student interviews

This section reports the purpose and administration of the student interviews (Appendix 7.1).

Focus of student interviews

The purpose of the interviews was to observe how the students used CAS on a range of essentially symbolic items that allowed the students to show their capabilities with the symbolic algebra facility of CAS and that enabled them to indicate their preference for different representations. It also provided an opportunity to confirm the findings of the written tests.

The eight interview questions (see Appendix 7.1) were similar in format to either DCT or PRT items (see Appendix 7.2) and gave students the opportunity to demonstrate up to seven different Framework competencies (the majority symbolic). For example, the interview question (IQ) Q.1 is very similar in format to the DCT question (A Q.1) and tests the competency [FSs]. The eighth interview question (IQ) Q.8, is similar to the PRT question (B Q.4(a-b)) and provides students the opportunity to demonstrate two of three competencies from [FNs], [FGs], and [FSs]. Every question gave students the opportunity to use the symbolic representation for formulation (either formulation-without-translation or formulation-with-translation) and several items gave students the opportunity to use CAS for symbolic algebra. Student interview items, (IQ) Q.7 and
(IQ) Q.8 gave students a choice between representations and (IQ) Q.5 and (IQ) Q.9 gave students the opportunity to use non-algebraic methods to solve maximum or minimum problems.

**Administration of the student interviews**

Almost half of the students in each class were interviewed for 45 minutes. The interviews were held concurrently with the testing program. The intention was to interview students of comparable ability in each class but some of the selected students were unable to participate due to unforeseen end-of-term school commitments. By chance, the school attainment levels of the 15 students who participated in the interviews were quite disparate. The six Class A students achieved an average of 81% (Grade B) and the nine Class B students achieved 57% (Grade E). Table 4.3 below, displays the number of students in each school attainment level and shows that most of the Class A students interviewed were high achievers while the Class B students were middle and low achievers. Thus, the students interviewed were not representative of their classes. Comparison with Table 4.1 above shows that most of the high attainment Class A students were interviewed while no high attainment Class B student was interviewed.

**Table 4.3  Number of Students Interviewed by Class and Attainment (%) on Four Recent School Tests**

<table>
<thead>
<tr>
<th>Attainment (%)</th>
<th>99-90</th>
<th>89-80</th>
<th>79-70</th>
<th>69-60</th>
<th>59-50</th>
<th>49-40</th>
<th>&lt;40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grade</td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td>F</td>
<td>G</td>
</tr>
<tr>
<td>Class A (N=14)</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Class B (N=19)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The students were asked to solve the set of interview problems while thinking out-loud and explaining their use of the TI-92 (if they had opted to use it). The interview questions were designed to observe the student’s use of the CAS calculator, competence with CAS and differentiation, and ability to translate between representations. During the interview, each student was asked to say how she was using the CAS and the author commented about her CAS use (onto the tape) if the student failed to do so. Immediately after each interview, the author audiotaped and then wrote up summary notes about her impressions of what the student had understood (and not understood) about differentiation, noted links made between different representations, specific use of
CAS, and she noted interesting observations including student attitudes toward using CAS.

In summary, student interviews were conducted to observe how the students solved differentiation problems (with CAS available) on a range of problems similar to items that were on either the DCT or the PRT.

4.6.4 Calculus and CAS Questionnaire
The student Calculus and CAS questionnaire (see Appendix 8.1) was given to all students towards the conclusion of the teaching program. Its purpose was to gain an insight into the attitudes of Class A and Class B students towards calculus, the importance to them of computer algebra and by-hand algebra, and how each class used symbolic algebra for differentiation. Students were expected to indicate their response to each questionnaire statement by circling a digit on a Likert scale of responses: 4 (strongly agree), or 3 (agree), or 2 (disagree), or 1 strongly (disagree). A 4-point scale was adopted so that the students did not give neutral responses.

4.6.5 Evaluation of the TI-92
At the conclusion of the final calculus lesson, each student evaluated the TI-92 (see Appendix 9.1). In response to the open-ended prompts, students wrote down descriptive comments about learning with the TI-92. They commented on the mathematics they had learnt, benefits and disadvantages of using the TI-92, their beliefs about the capabilities of the TI-92, and their preference for algebra with symbolic algebra or by-hand. They also made suggestions about how to make the TI-92 more useful in their mathematics classes.

4.7 Summary and Discussion
This chapter has provided an overview of the design for the conduct of the main study. The calculus curriculum supported a range of functional and pedagogical activities that were expected to assist students in developing understanding of the concept of derivative. Background information about the teachers and students was provided and the ways the teaching was to be monitored and student learning was to be assessed were described.
The strategy devised should enable the three Research Questions to be answered. The second teacher interview and lesson observations of the main study will establish each teacher’s privileging during the main study that can be compared with their privileging during the preliminary study (based on the first teacher interview and lesson observations of the preliminary study). The DCT will establish student and class learning of individual Framework competencies, on all the competencies, and on groups of competencies including the four categories of competencies: formulation-with-translation, formulation-without-translation, interpretation-with-translation, and interpretation-without-translation (see Section 3.2.3). The PRT should verify class achievements on both categories of formulation competencies and other Framework competencies tested on the DCT. In addition, the PRT will give additional insights into student and class use of symbolic algebra, preference for particular representations, knowledge of two representations, class proficiency with compound competencies, and the influence of data on students’ preference for representations (Section 4.6.2). The student interviews, questionnaire, and written evaluation of the TI-92 will also give new insights into what the students had learned about differentiation and how they had been taught.

The strengths and weaknesses of each class will be identified (rated according to success on categories of differentiation competencies and compared on a variety of other characteristics, eg., use of symbolic algebra) and provide a basis for the judgement about the role of CAS, use of multiple representations, and the influence of the teaching on learning.
5. TEACHER PRIVILEGING

The preliminary study indicated that the privileging of Teachers A and B impacted on the learning of their students during a calculus course with CAS calculator technology (reported in Section 1.2.1 and more fully by Kendal and Stacey (1999a, 1999b)). As discussed in Section 4.1.2, an important objective of the main study is to confirm this effect: with the same teachers, same curriculum, but with different students. To achieve this, each teacher’s privileging needs to be identified for the main study and then examined in relation to student learning outcomes (Research Questions 1 and 2). Another objective of the study is to see if each teacher’s privileging characteristics stayed the same the second time. This is achieved by comparing their privileging characteristics during the main study and second teacher interview with the preliminary study and first teacher interview (Research Question 3).

This chapter establishes the initial privileging characteristics and privileging profile for each teacher prior to the main study (Section 5.1) and then for the main study (Section 5.2). Research Question 3 is answered by considering the changes in teacher privileging that occurred during the main study (Section 5.3). Finally, a summary and discussion are presented (Section 5.4).

5.1 Teacher Privileging Prior to Main Study

During the preliminary study Teacher A and Teacher B’s privileging characteristics were established by observation of a majority of their lessons (see Kendal & Stacey, 1999a, 1999b). Nine months later and ten weeks prior to the commencement of the main study, each teacher participated in the first teacher interview (see Appendix 3.1). This section shows that during the first teacher interview each teacher’s privileging characteristics were consistent with those observed during the earlier preliminary study.

Teacher A and B’s knowledge of the concept of derivative, demonstrated during the first teacher interview, is identified and compared (Section 5.1.1). Their privileging characteristics are identified and compared (Section 5.1.2) and shown to be consistent with those demonstrated during the earlier preliminary study, enabling privileging profiles for Teachers A and B, prior to the main study, to be developed (Section 5.1.3).
5.1.1 Comparison of Teacher A and B’s knowledge of differentiation

This section establishes Teacher A and Teacher B’s understanding of the concept of derivative in the numerical, graphical, and symbolic representations. First, it identifies each teacher’s personal knowledge of a range of formulation and interpretation Differentiation Competency Framework competencies and second, it compares their personal knowledge of the concept of derivative.

Knowledge of formulation competencies

Six interview questions (Questions 1, 2, 3, 4, 6, 7, Appendix 3.1) gave the teachers the opportunity to demonstrate eight of the nine Framework formulation competencies ([FNN] was not tested). Each of these questions, except Question 7, could be answered in more than one way involving different representations (and different competencies). Figure 5.1 displays each teacher’s responses to the six interview questions, the associated formulation competencies, and their use (and non-use) of CAS. Six months later, the students were tested on these same questions (on the PRT) but with slightly different data for Questions 3 and 7.

<table>
<thead>
<tr>
<th>Interview Question</th>
<th>Competency Representation and Code</th>
<th>Proposed Solution With CAS</th>
<th>Without CAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Numerical</td>
<td>FNN (not tested)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(N)</td>
<td>FNg</td>
<td>B</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>FNs</td>
<td>B</td>
</tr>
<tr>
<td>6</td>
<td>Graphical</td>
<td>FGn</td>
<td>B</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>FGn</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(G)</td>
<td>FGg</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>FGs</td>
<td>B</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>FGs</td>
<td>A</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>FSN</td>
<td>A</td>
</tr>
<tr>
<td>1</td>
<td>Symbolic</td>
<td>FSG</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(S)</td>
<td>FSS</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>FSS</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.1. The formulation competencies involved in the solutions suggested by Teachers A and B on six first teacher interview questions.

Figure 5.1 shows that Teacher B nominated fifteen possible solutions (2 numerical, 7 graphical, 6 symbolic) on the six questions compared with nine by Teacher A (5 graphical, 4 symbolic). On occasions, the teachers proceeded to solve the problems using the methods they had suggested, particularly Teacher B. Figure 5.1 also shows that both teachers nominated the three graphical and three symbolic formulation
competencies: \([FGn], [FGg], [FGs]; \) and \([FSn], [FSg], [FSs].\) Only Teacher B nominated the two numerical formulation competencies \([FNg]\) and \([FNs]\) and he more frequently nominated use of the CAS calculator for symbolic and graphical differentiation.

**Knowledge of interpretation competencies**

Both teachers attempted to solve the remaining three interview questions (Questions 5, 8, 9) that tested different interpretation of derivative competencies. The corresponding competencies were \([INn], \) the compound competency \([FGg + INn], \) and \([ISs].\) Later, the students were also tested on Question 5 on the PRT (previously discussed as Sample 9 B Q.6 in Section 4.6.2). Figure 5.2 displays the success of each teacher on these interpretation competencies. It shows that Teacher B successfully interpreted the numerical derivative (reasoning about simple rates of change) and both the more challenging graphical and symbolic derivatives while Teacher A successfully interpreted only the numerical derivative, showing that he was more capable at calculating symbolic and graphical derivatives than interpreting them.

<table>
<thead>
<tr>
<th>Teacher</th>
<th>Numerical (N) Question 5 Reason numerically about rates of change ([INn]) Interpretation-without-translation</th>
<th>Graphical (G) Question 8 Reason graphically to find a rate of change ([FGg + INn]) Interpretation-with-translation</th>
<th>Symbolic (S) Question 9 Reason about a symbolic derivative ([ISs]) Interpretation-without-translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Successful</td>
<td>Partially successful</td>
<td>Unsuccessful</td>
</tr>
<tr>
<td>B</td>
<td>Successful</td>
<td>Successful</td>
<td>Successful</td>
</tr>
</tbody>
</table>

*Figure 5.2.* Teacher A and B’s success on interpretation of numerical, graphical, and symbolic derivatives during the first teacher interview.

Teacher A and B’s knowledge of differentiation during the first teacher interview is summarized:

- Teacher B differentiated using symbolic, graphical, and numerical representations of differentiation (demonstrating formulation competencies) while Teacher A differentiated using symbolic and graphical techniques but failed to differentiate numerically.
Teacher B translated between the symbolic, graphical, and numerical representations demonstrating formulation-with-translation competencies while Teacher A translated only between the symbolic and graphical representations.

Teacher B reasoned successfully about numerical, graphical, and symbolic derivatives while Teacher A successfully reasoned numerically, had difficulty reasoning graphically, and did not successfully interpret symbolically.

Teacher B nominated a greater number of differentiation strategies in different representations (up to four per question) while Teacher A nominated fewer (up to two per question).

Thus, during the first interview Teacher B demonstrated a comprehensive, deep, and integrated knowledge about the concept of derivative including knowledge of formulation-without-translation and formulation-with-translation competencies in the numerical, graphical, and symbolic representations, and interpretation-without-translation of numerical, graphical, and symbolic competencies. Interpretation-with-translation was not tested. Teacher A displayed less depth and less integrated knowledge about the concept of differentiation. He demonstrated knowledge of formulation-without-translation and formulation-with-translation competencies in the graphical and symbolic representations but not in the numerical representation and was able to interpret-without-translation only the numerical representation.

5.1.2 Privileging characteristics of Teachers A and B during the first teacher interview

This section describes Teacher A and B’s privileging with respect to three characteristics: calculus content (gauged by preference for representations), teaching approach (gauged by teaching methods used or suggested), and use of CAS (gauged by use of CAS or attitude expressed towards using it) during the first teacher interviews. These privileging characteristics were first identified during the preliminary study’s teaching trial (see Kendal & Stacey, 1999a, 1999b).
Characteristic 1. Calculus content (preference for representations)
This section describes each teacher’s preference for representations of derivative (numerical, graphical, or symbolic) during the first teacher interview.

Teacher A believed a symbolic approach was superior and that solving problems graphically was a second rate option. For example, while solving the first teacher interview Question 2 (discussed in detail below as Sample 11, a maximum/minimum problem that can be solved either using differentiation, or graphical procedures), he observed, “I think when they are doing it graphically, it’s because they have missed out that you can do it algebraically”. During the interview, he reluctantly used by-hand graphical techniques only when it was absolutely essential, and did not suggest the numerical approach. In contrast, Teacher B preferred both algebraic and graphical approaches and often used them both to explain problems (see discussion below) but was also well aware of the usefulness of the numerical representation.

The teachers’ responses to Sample 7 illustrate differences between them. Sample 7 was given to the teachers by the author (researcher) during the first teacher interview (see Question 1, Appendix 3.1) and later it was also given to the students on the PRT.

Sample 7 (B Q.4(a)-(b)).

Consider the curve \( y = x^3 - x \). A diagram of the graph is supplied.

(a) Find the gradient of the curve at \( x = 1 \), stating whether it is accurate or approximate.

(b) Confirm your result by finding the gradient using another method, stating whether it is accurate or approximate.

Teacher A: “For me, they’ve got the equation of the curve, I would differentiate and put in \( x = 1 \).”

Researcher: “Now can you answer the second part and then go back and think about the girls.”

Teacher A: “Confirm your result by finding the gradient using another method, stating whether it is accurate or approximate. Oh, well mine is accurate, and the approximate method which would be really approximate would be actually drawing a tangent at \( x = 1 \) and then working out the gradient by locating two points and doing rise over run.
But I don’t think the girls would ever attempt that because they hate anything where they have to guess or where the answer might be really different.

They’re not inclined to put it on the graphical calculator. That’s not their first instinct.”

This discussion reveals Teacher A’s strong preference for symbolic differentiation and his wariness of the “approximate” graphical method.

In response to the same question, Teacher B also indicated a strong preference for the symbolic solution but was cognizant of both graphical and numerical solutions.

Teacher B: “O.K. Find the gradient of the curve at $x = 1$, stating whether it is accurate or approximate. Well I would just find $\frac{dy}{dx}$ and substitute $x = 1$ and I would say it had an accurate value.

Let me think about it. I’ve got to make sure I am right here. Yes, find the derivative, sub in $x = 1$ and that will give me an accurate gradient at that point.

The second part. Confirm your result by finding the gradient using another method, stating whether it is accurate or approximate. Well I’d probably um, get the gradient which is approximate and I’d take two close points. So I’d take $f$ at say 1.001 minus $f$ at 1 all over 0.001, So I’d work that out, and that would be an approximate one. Third method. Do you want a third method?”

Researcher: “If you know a third one.”

Teacher B: “Well, third, um, estimate from the graph, by drawing a tangent.”

Researcher: “Now if the girls are using a CAS, do you think they would do it any other way?”

Teacher B: “Well, I’d say on a CAS you’d just go $d/dx$ of $x^3-x$ and evaluate that.”

Researcher: “Right, so they’d still use a symbolic approach?”

Teacher B: “Oh well, that’s my feeling and I think it’s the best method of all of those and it’s an exact approach.”
During the first teacher interview, across all of the questions, Teacher A proposed use of the symbolic representation while Teacher B suggested use of the symbolic representation supported by use of the graphical representation to understand the symbolic representation. In addition, he discussed numerical solutions, for example, 

\[
\frac{f(1.001) - f(1)}{0.001}
\]

Characteristic 2. Teaching approach (methods suggested)

This section describes the ways each teacher approached solving problems during the first teacher interview including the methods they suggested (and sometimes used).

Teachers A and B responded to the interview questions differently, giving the following different emphasis to routines and explanations:

- Teacher A read each question silently and selected one method while Teacher B read the question out loud and selected a first method.
- Teacher A discussed procedures out loud and on some occasions symbolized some steps while Teacher B verbalized the method and logically symbolized each step.
- Teacher B checked the proposed method by re-reading the question and re-checking the steps.
- Teacher A suggested a second method only after prompting while Teacher B suggested a second method (sometimes a third and fourth) without prompting.

Thus, Teachers A and B talked through their solutions differently. Teacher A spoke in terms of procedures that the students and that he (the author believes) needed to remember and cues for procedures. In contrast, Teacher B approached the problem from several points of view, explained what he was doing and why, made sense of his answer, re-checked his work, and re-checked the question. For example, see the discussion below that relates to Sample 11. It was Question 2 on the first teacher interview and was later given to the students on the PRT (B Q.2(a)-(b)).
Sample 11 (B Q.2(a)-(b))

After a few days of heavy rain the river burst its banks and the surrounding area was flooded. Suppose the area of land under water is given by $A = 16t - t^2$ where $A$ is the area in hectares and $t$ is the number of days since the flood began.

(a) When (after how many days) was the most land under water?

(b) Can you suggest an alternative way to find when the most land was under water?

Teacher A: “See, most of them will avoid, and in fact the way I try to teach them is to pull away from the real situation because sometimes it just runs interference to the process, and some of them can’t go back and assess what their answer is in relation to the question. They can do the maths but can’t look at the maths and relate it to the theme. That’s why calculus is so difficult.”

Researcher: “Yes.”

Teacher A: Read Question 2 silently then commented, “So, most, so it’s a maximum. So differentiate. How many days was the land under water? Hang on. So its maximum area and you want time, so equate it to zero, $\frac{dA}{dt}$ equals zero and find the time. “I’d hope they’d focus in on the word *most* because when they do these sorts of questions I try and lead them into picking out key words like *most*, and then saying O.K. that means a *maximum*, and they will look at the equation and say “I need to find the *maximum* of that. They would have to look for clues, they have to look for key words to solve it.”

Researcher: “Now, part (b), can you suggest an alternative way?”

Teacher A: Considered the question for about 20 seconds.

Researcher: “Think of the girls, bearing in mind they’ve got a calculator.”

Teacher A: “Now, yes, they’d probably graph it. Um, some of them will change it to $t^2$ dealing with a quadratic is so much easier than a cubic and they will just type in $t^2$ and they’ll get a maximum which is what they want. And there’s no cues in there to show that they have put in the wrong equation.”
Researcher: “What would they draw the graph of?”

Teacher A: “They’ll be plotting this (points to $A = 16t - t^3$), they’ll get confused. I can’t see the graph because I haven’t got [it] in my head so what is it?”

Teacher A: Drew a sketch of the function (by-hand) and said “And it’s a negative cubic so it starts here (quadrant 3) so it goes like that. They’ll get that hopefully. They’ll think that it should only have positive $t$ but I don’t think they will.

I think when they do it graphically, it’s because they have missed out you can do it algebraically. That’s in a sense the teaching method too, because they have such trouble with this concept that we try to treat it in a very mathematical way so that they can at least get the maths right.”

In contrast to Teacher A’s emphasis on procedures, Teacher B solved the same maximum/minimum problem (Sample 11) with a focus on understanding by drawing a graph to help him interpret and solve the problem. Only when he had a “picture” of the problem did he nominate algebraic procedures.

Teacher B: Quietly read the question out loud to himself and stated the question

“When (after how many days) was the most land under water?”

He drew the graph while he spoke and thought about the question “O.K. Um, I’d look at the graph, on the TI-92, which is going to look like, well it’s –4, 0 and 4 isn’t it?”

He crossed out the section of the graph with negative domain, stating “It’s a negative cubic, so it goes that way. You don’t want that bit of the graph, so you want $t$ greater than and equal to nought.”

He marked a cross on the highest point on the graph and said “So you actually want to locate that point there, locate maximum, um, and therefore $dA/dt$ wants to be zero, solve for $t$, that’s all you want. You actually want the value of $t$: you don’t want the value of $A$. Let me just re-read that. It’s the **most** land under water, so that’s $A$, so you want the largest area, that’ll be right, that’s what we want. O.K. Let me think if I would do it differently to that?”
He wrote down a short sequence of steps to solve the problem using differentiation. “I would do it algebraically, perhaps now I’m thinking about it, I might do this as well as my first thing, simply finding that point, find the maximum using the calculator.”

Thus, during the first interview Teacher A privileged teaching routines and differentiation procedures while Teacher B privileged understanding of the concepts. In class, during the preliminary study they also used these teaching approaches (see Kendal & Stacey, 1999a, 1999b) and during the main study (see the description of three lessons by both teachers in Section 5.2.1 below).

Characteristic 3. Use of CAS
This section describes Teacher A and Teacher B’s different attitudes towards the use of CAS during the first teacher interview.

While responding to interview Question 6 Teacher A enthusiastically endorsed using CAS for a range of procedures including symbolic differentiation, as his comments below indicate. This question was later given to the students on the PRT and is discussed as Sample 6 (B Q.8(b) in Section 4.6.2).

Teacher A: Question 6. Ah, exponentials! [Find the rate of change of \( y \) with respect to \( y = (1.8)^x \)] Ahhh. Well they’d use their calculators wouldn’t they? But, remember we did that last year and it was actually an area where they were pretty confident remember? Because they knew the calculator would do it, so if it was outrageously different then they knew they could use their calculators, Yes, so I think they probably thought “Oh yeh, but I know the calculator can do this”. I think they’d do it on one step or two steps in their calculator, you just don’t see it on paper.”

Researcher: “After one year’s experience how do you feel overall about the CAS, its advantages, disadvantages?”

Teacher A: “I loved it. I thought it was great. I pined for it when I went back to the TI-83 [at the end of the preliminary study]. It [TI-83] was very limited and inaccurate. I really liked the exact and approximate, I really liked the spreadsheets [tables of values]. I loved graphing from the spreadsheet
used to introduce the students to using CAS in a study of trigonometry]. I loved the display on the TI-92 but as well I found the menus were more accessible. It’s a bit like a computer, you know what’s in there somewhere. Um, yes, I thought they were fantastic. And the girls did too. When they went back to their normal graphical calculators in fourth term, we all missed it.”

Researcher: “So what about learning? Do you think it would help them with their learning?”

Teacher A: “I don’t think they did a lot of learning.” [checking of their algebraic solutions in the context of the problem].

Teacher B had a different appreciation of what the CAS technology should do. He believed that once the students had grasped the fundamental concept (by-hand) the CAS calculator could assist students to investigate (and develop better understanding) by performing the more tedious and complicated procedures.

Researcher: “Now with the experience of hindsight, how have you found the CAS? What effect do you think that using a TI-92 for the computer algebra part of the work has affected their learning?”

Teacher B: “So if you have a conceptual understanding of a derivative, then so long as that’s O.K. and you can do it [by-hand] for simple functions, then why do it by-hand for complicated functions when you’ve got your calculator? Potentially, I suppose it enables you to do a bit more investigation, in terms of looking at more complex functions and you haven’t got the problem of finding a derivative. So you look at other application work where you don’t just have a look at an algebraic function, you look at everything else, so you’re able to use their concept, their understanding of a derivative to flow over to complicated functions but they don’t necessarily know how to differentiate themselves.”

During the interview, Teacher A viewed CAS as a tool to perform routines and differentiation procedures while Teacher B appeared confident with technology and appreciated its potential to go beyond procedures to assist students in understanding
concepts. This will also be demonstrated through the subsequent teaching during the main study (see the description of three lessons by both teachers in Section 5.2.1).

**Comparison of Teacher A and B during the first teacher interview**

Teacher A and B’s use of representations, teaching method and attitudes to technology, observed during the first teacher interview and discussed in this section, are summarized in Figure 5.3.

<table>
<thead>
<tr>
<th>Teacher A</th>
<th>Teacher B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preference for representations</strong></td>
<td></td>
</tr>
<tr>
<td>• Preferred the symbolic representation</td>
<td>• Preferred the symbolic and graphical representations and linking them</td>
</tr>
<tr>
<td><strong>Teaching method</strong></td>
<td></td>
</tr>
<tr>
<td>• Talked about routines to solve problems</td>
<td>• Explained solutions, used different approaches, and checked solutions</td>
</tr>
<tr>
<td><strong>Use of technology</strong></td>
<td></td>
</tr>
<tr>
<td>• Suggested CAS use for routine procedures</td>
<td>• Suggested CAS use for investigations and for performing more complex algebraic procedures for understanding</td>
</tr>
</tbody>
</table>

*Figure 5.3.* Teacher A and B’s use of representations, teaching approaches, and use technology during the first teacher interview.

### 5.1.3 Privileging profiles of Teachers A and B prior to the main study

This section compares Teacher A and B’s privileging characteristics during the first teacher interviews (just described) and during the earlier preliminary study (described by Kendal & Stacey, 1999a, 1999b).

In the preliminary study, Teacher A privileged the symbolic representation, teaching procedures, and frequent use of CAS for symbolic algebra, while Teacher B privileged the symbolic and graphical representations and teaching for understanding. He used the calculator for graphical procedures and CAS for symbolic algebra associated with investigations to aid understanding.

During the first teacher interview, both teachers’ privileging was entirely consistent with the preliminary study, nine months earlier. Teacher A showed good understanding
of the symbolic representation and used it whenever it was possible. He showed partial appreciation of the graphical representation and used it only when necessary and ignored the numerical representation. He showed enthusiasm for using CAS (particularly symbolic algebra) but had a limited vision for the role of the CAS in promoting understanding. In contrast, Teacher B showed comprehensive understanding of differentiation across numerical, graphical, and symbolic representations and used the graphical representation to assist with understanding the symbolic representation. Although he did not endorse use of symbolic algebra for simple algebra, he understood the role of CAS in promoting understanding.

Thus, the first teacher interviews confirmed the teacher privileging that occurred in the preliminary study and together they provide a sound basis on which to monitor the changes in privileging that occurred in the main study during the teachers’ second teaching experience. The observations from both the preliminary study and first teacher interviews are collated in Figure 5.4 that displays a privileging profile for each teacher prior to the main study.

<table>
<thead>
<tr>
<th>Privileging Characteristics of Teachers A and B Prior to the Main Study</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher A</td>
</tr>
<tr>
<td><strong>Calculus content</strong></td>
</tr>
<tr>
<td>Preference for representations</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Teaching approach</strong></td>
</tr>
<tr>
<td>Teaching style</td>
</tr>
<tr>
<td>Teaching method</td>
</tr>
<tr>
<td><strong>Use of CAS</strong></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

*Figure 5.4. Personal privileging profile for Teacher A and Teacher B prior to the main study.*
5.2 Teacher Privileging During Main Study

This section establishes each teacher’s privileging characteristics during the main study through observation of every lesson (see Section 4.4.2) and verification by the second teacher interview. The three privileging characteristics, described above, are identified for each teacher: the attention they paid to the numerical, graphical, and symbolic representations of derivative and links between them; their teaching methods and styles; and the use they made of the CAS activities provided in the curriculum. This information is essential to determine the impact of the teaching on learning and assess changes in privileging from the first teacher interviews and preliminary study (and to answer the three Research Questions). It has also been fully reported by Kendal and Stacey (2001a). To illustrate differences in the privileging of the two teachers, three 45-minute lessons from the main study (the first two lessons were similarly delivered in the preliminary study) are described in detail in Section 5.2.1. These lessons were considered to be typical of each teacher because of the representations they used (both teachers preferred use of the S representation and Teacher B simultaneously involved the graphical representation), the teaching approaches they used, and the ways they used CAS. Next, the three teacher privileging characteristics, observed over the entire teaching program (Section 4.4.2 describes the guidelines for classroom observations) and confirmed by the second teacher interview, are reported with respect to: calculus content (see Section 5.2.2), teaching approach (see Section 5.2.3), and how the teachers incorporated CAS use into their teaching programs (Section 5.2.4). These characteristics are combined into a privileging profile for each teacher for the main study (Section 5.2.5).

5.2.1 Description of three lessons

Sample Lesson 1

The aim of the first lesson is to teach the general rule for differentiating a power function (i.e. that \( d(ax^n)/dx = n.ax^{(n-1)} \)) and to apply it to positive and negative powers. In an earlier lesson, in both classes, the gradients of tangent lines to the curve \( f(x) = 3x^3 \) had been found at \( x = \{ -2, -1, 0, 1, 2 \} \) using CAS. The resultant gradients had been compared with the derivatives that resulted from the one-line CAS symbolic
generation of derivatives and substitution (given a set of x values), \( d(3x^2, x)/x = \{-2, -1, 0, 1, 2\} \), thereby linking gradients of functions with symbolic derivatives.

Outline of Teacher A’s Method

Teacher A presented carefully planned mathematical content with a lecture and demonstration style. Students observed the teacher carry out the CAS procedures, which they imitated with help from the TI-92 flow chart written on the white board (see Figure 5.5).

<table>
<thead>
<tr>
<th>( f(x) = 3x^2 )</th>
<th>( f(x) = x^3 )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>f ’(x)</td>
<td>x</td>
<td>f ’(x)</td>
</tr>
<tr>
<td>-2</td>
<td>-12</td>
<td>-2</td>
<td>12</td>
</tr>
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<td>3</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td><strong>Multiply by 6</strong></td>
<td><strong>Pattern 3x^2</strong></td>
<td><strong>ax^0</strong></td>
<td><strong>n.ax^{n-1}</strong></td>
</tr>
</tbody>
</table>

Flow chart for TI-92 procedure

Example: Finding \( f'(2) \) for \( f(x) = 3x^3 \) was worked with and without CAS. Firstly, the formula \( f'(x) = n.ax^{n-1} \) was applied and \( x = 2 \) substituted. Then
the differentiation and substitution in a one-step CAS routine was performed, which was also recorded on the board using a flow chart (see Figure 5.5). Most students imitated the CAS procedure then copied down the detailed board notes.

- Practice problems were set including the differentiation of $\frac{1}{x^2}$, the first negative power encountered, and the students began to work on them using the formula or CAS. Unfinished problems were set for homework.

- When a student asked for help, the teacher worked another example on the board. This was a common practice.

Outline of Teacher B’s Method

Teacher B explored carefully planned mathematical content through a teacher-led class discussion, drawing on individual students’ contributions. His blackboard notes are given in Figure 5.6.

<table>
<thead>
<tr>
<th>$f(x) = x^2$</th>
<th>$f(x) = x^3$</th>
<th>$f(x) = x^4$</th>
<th>$f(x) = ax^n$</th>
<th>Patterns</th>
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<tbody>
<tr>
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<td>$f,'(x)$</td>
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<td>4</td>
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<td>32</td>
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<tr>
<td>$f(x) = 2x^2$</td>
<td>$f(x) = 2x^3$</td>
<td>$f(x) = 2x^4$</td>
<td>Special cases</td>
<td>with diagrams</td>
</tr>
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<td>$f,'(x) = 6x^2$</td>
<td>$f,'(x) = 8x^3$</td>
<td>$f(x) = ax = ax^1$</td>
<td>$dy/dx = 35x^6$</td>
</tr>
<tr>
<td>$f(x) = 3x^2$</td>
<td>$f(x) = 3x^3$</td>
<td>$f(x) = 3x^4$</td>
<td>$f(x) = ax = ax^1$</td>
<td>$f(x) = 1/x^4$</td>
</tr>
<tr>
<td>$f,'(x) = 6x$</td>
<td>$f,'(x) = 9x^2$</td>
<td>$f,'(x) = 12x^3$</td>
<td>$f(x) = a$</td>
<td>$f(x) = x^4$</td>
</tr>
<tr>
<td>$f(x) = ax^2$</td>
<td>$f(x) = ax^3$</td>
<td>$f(x) = ax^4$</td>
<td>$f(x) = a = ax^0$</td>
<td>$f,'(x) = -4x^5$</td>
</tr>
<tr>
<td>$f,'(x) = 2ax$</td>
<td>$f,'(x) = 3ax^2$</td>
<td>$f,'(x) = 4ax^3$</td>
<td>$f,'(x) = 0$</td>
<td>$f(x) = x'$</td>
</tr>
</tbody>
</table>

Figure 5.6. Teacher B’s board notes (diagrams, notation section, with some examples omitted).

- Students generated the values of the derivative of $f(x) = x^2$ at five points using the CAS command to differentiate and then substitute the given set of $x$ values in one line, $d(x^2, x)/x = \{-2, -1, 0, 1, 2\}$. The teacher moved around the classroom checking individuals’ screens and giving students help with the calculator.
• Student generated values were written on the board (see Figure 5.6) and individual students were challenged to find and explain how the pattern was developed, and to verify the resultant symbolic pattern, derivative = 2x.

• The procedure was repeated for $f(x) = 2x^2$ and $3x^2$. For $f(x) = ax^2$ the derivative rule $f'(x) = 2ax$ was spontaneously suggested by one student and confirmed by others.

• A student observed that quadratic function gave a linear derivative. This was observed from a sketch of $f(x) = 2x^2$ drawn on the board, and awareness that the derivative, $4x$, represented a linear function and straight line.

• Another student posed the question "What about $x^3$?" Students generated a table of derivative values using the TI-92 and guessed the derivative rule for $f(x) = x^3$.

• Teacher B asked individuals to predict the derivative rules for $f(x) = 2x^3$, $f(x) = 3x^3$, and eventually for $f(x) = ax^3$.

• A student observed that a cubic function gave a quadratic derivative and the teacher helped all the students understand why. (Diagrams were drawn on the board and links with gradients made). The algebraic patterns were revised again.

• Students generated the derivative rule for $f(x) = x^4$ assisted by tables of derivative values which they generated using the TI-92 (see board notes in Figure 5.6).

• Individual students predicted the derivative rule for $f(x) = 2x^4$, $f(x) = 3x^4$, and eventually for $f(x) = ax^4$.

• Using the summary table of $f(x)$ and $f'(x)$ for the general expressions, for quadratic, cubic and quartic polynomials, after class discussion a student volunteered the derivative rule for the polynomial $ax^n$.

• Teacher B carefully explained the $f'(x)$ notation and checked students’ understanding by nominating individuals to explain to the class how to use it. He again revised the general rule for differentiation of a polynomial function.

• Students progressively copied the notes and sketches from the board including a section devoted to alternative notations, $f'(x)$ and $dy/dx$. 

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• The special functions \( f(x) = ax \) and \( f(x) = c \) were sketched on the board, the gradients determined, and the outcomes linked to the formula \( nx^{n-1} \).

• Examples using the rule were worked by-hand, including \( f(x) = 5x^7 \) and \( f(x) = \frac{1}{x^4} \).

• Practice problems were set for homework and included the differentiation of \( \frac{1}{x^5} \).

Students were instructed to do the items by-hand and to check their answers by repeating the items with the TI-92.

Sample Lesson 2
This lesson gave students the opportunity to solve a maximum and minimum problem. The students had previously learned that the derivative = 0 at a maximum or minimum and the following standard textbook problem was used:

From the corners of a rectangular piece of cardboard, 32 cm by 12 cm, square sides of length \( x \) cm, are cut out and the edges turned up to form a box. Find the value of \( x \) if the volume of the box is a maximum.

Outline of Teacher A’s Method
• Firstly, Teacher A drew a diagram of a rectangle on the blackboard (representing the cardboard) and with one student’s help wrote down the rule for the volume of the box, \( V = x(32-2x)(12-2x) \).

• Next, the differentiation and how to solve for zero derivatives was demonstrated, in a two-step, one line CAS procedure \([\text{solve } d(x(32-2x)(12-2x),x) = 0,x] \).\)

• Finally, Teacher A demonstrated how to set up a table of slope values to decide the nature of the stationary points, using CAS to calculate the derivative and substitute nearby values between the zeros into it. A rough graph was constructed from the slope values with focus on the zero gradients.

• The appropriate \( x \) value was noted, and the second value discarded.

• The students copied the notes from the board.
Outline of Teacher B’s Method

- Teacher B constructed a cardboard box and discussed the problem with the students using a diagram on the blackboard.

- Next, Teacher B led a class discussion which generated the dimensions of the box (in terms of x) and its volume: \[ V = x(32 - 2x)(12 - 2x) \]

- During further class discussion, the students differentiated the function and found the zeros of the derivative by-hand. This resulted in two values of x at which the derivative is zero.

- Students graphed the function on their calculators, and Teacher B illustrated the algebraic result graphically from a white board sketch of the graph.

- Finally, the necessity to ignore one of the algebraic solutions (for the real box) was discussed.

Sample Lesson 3

Independently, prior to the tests, the teachers organized revision for their classes.

Outline of Teacher A’s Revision Program

- The students were reminded of the initial dynamic program CAS Activity 7 that relates rates of walking to the gradient of the resultant lines and curves (see Figure 4.2).

- The dynamic program (showing the connection between the gradient of secants, curve and tangent to the curve) was demonstrated several times (CAS Activity 6).

- Using the overhead projector screen and the CAS calculator (TI-92), numerical and graphical and differentiation procedures were performed, accompanied by notes on the board which students were expected to copy.

- The corresponding CAS calculator instructions were hand written on the whiteboard.

- The students were reminded of the types of problems they could expect on the tests including maximum/minimum problems and rates of change problems.
• The students completed a differentiation worksheet, with instructions to perform the differentiation by-hand and check the result on the TI-92. (e.g., Differentiate $x\sqrt{x^4 - x + 9}$)

• Meanwhile, Teacher A solved the Worksheet 10 items on the TI-92 using the overhead display.

Outline of Teacher B’s Revision Program

• The students were given two previous school test papers to solve, Year 11 Mathematics Methods B (Calculus) 1995 and 1996. Each was accompanied by a set of solutions.

• The students were told to solve the problems by-hand and to use CAS to check their answers.

• If they requested help, the students were given individual help as they worked on the essentially symbolic items.

• Each student’s progress was checked during the lesson.

These three sample lessons highlight differences in the teaching methods and styles of the two teachers. Both teachers worked through the planned teaching program, which they had helped develop, but during its implementation they taught differently. They communicated with their students in different ways and made different pedagogical choices about the calculus content that was important to teach, what was important to emphasize, and how to use CAS into their lessons. Considering the entire teaching program, and supported by comments during the second teacher interview (see comments below) each teacher’s privileging characteristics are now clarified (Sections 5.2.2 - 5.2.4) and Teacher A and B’s privileging profiles during the main study are established (Section 5.2.5).
5.2.2 Characteristic 1: Privileging of calculus content

This section identifies each teacher’s preference for representations, the indicator of privileging of calculus content in this study.

Teacher A taught all three representations: numerical, graphical, and symbolic, but had a strong personal preference for the symbolic representation. This is illustrated by Sample Lessons 1 and 2 above, which are both entirely in the symbolic representation. Teacher A explained (in a discussion with the author after a classroom observation) that, after he had participated in the first teacher interview during which he discussed items that the students could be tested on later in the course, he became more interested in and aware of the multiple representations of derivative in relation to the capabilities of the CAS. Classroom observations showed that he confidently began to use CAS in new ways and emphasized the graphical and numerical representations for the first time. Processing all of the classroom observations showed that Teacher A allocated more of his teaching time than Teacher B to linking the graphical and numerical representations (86 minutes cf. 5 minutes, reported in Appendix 4.4) and that he devoted less time than Teacher B to linking the symbolic and graphical representations (40 minutes cf. 120 minutes, see Appendix 4.4). The trends discussed so far were also observed in relation to both teachers’ use of the CAS activities provided in the curriculum. Figure 5.7 shows that Teacher A used symbolic algebra and by-hand procedures for the symbolic CAS Activities 3 and 8. It also shows that he incorporated numerical and graphical representations into his main study classroom practices using Activities 1, 2, and 5 and linked them in Activities 6 and 7. Figure 5.7, based on Appendix 4.6, was developed to identify the particular emphasis that each teacher gave to each CAS activity during the entire teaching program. This was achieved by examining the original data from the lessons again in relation to teacher use of the CAS activities (see Section 4.4.2). Processing of lesson observations also showed that Teacher B frequently used the symbolic representation (see Appendix 4.4) mostly using by-hand procedures (see Appendix 4.5). Figure 5.7 shows that he used rarely used CAS for Activity 1 except to support CAS Activity 8. Teacher B also used CAS Activity 2 for the graphical differentiation. Teacher B stressed the symbolic representation believing it was the most important and useful representation of derivative. In addition, as both sample Lessons 1
and 2 show, he often used a graphical representation to interpret the symbolic derivative visually. At the second interview, he said a highlight was

Teacher B: “. . . getting the tangent idea through to them, what the gradient actually represents, what the derivative represents and the relationship between them - I think we’ve done very nicely with the calculator.”

Teacher B continuously used teaching actions that emphasized understanding the links between representations, particularly between the symbolic and graphical, and on occasions, the numerical rate of change “at all points” to its symbolic representation. Teacher B frequently linked differentiation ideas to the real world, using physical representations of slope and speed and taught the students to visualize gradients. He consistently and deliberately stressed the importance of the symbolic derivative and gave it meaning by linking it to the slopes of tangents to the curve using an enactive representation (arm movements to represent the tangent line) and to rates of change (i.e., to graphical and numerical representations). He devoted 120 minutes to linking the symbolic and graphical representations (compared with Teacher A’s 40 minutes, see Appendix 4.4) and a negligible 5 minutes linking the numerical and graphical representations (see Appendix 4.4). In an after-class discussion with the author he explained that he had rejected numerical differentiation (particularly involving difference quotients) because he believed it was not as important or useful as symbolic and graphical differentiation and that his less able cohort of students would not be able to cope with a third representation.

In summary, during the main study, Teacher A privileged use of the symbolic, graphical, and numerical representations and the link between the graphical and numerical representations while Teacher B privileged use of the symbolic representation and the graphical representation that he linked to the symbolic representation.

5.2.3 Characteristic 2: Privileging of teaching approach

This section identifies each teacher’s method and style of teaching; the indicators of their teaching approach in this study.

From a range of possible classifications of teachers’ conceptions of mathematics and styles of teaching, described in Chapter 2, Teacher A’s teaching approach is best
characterized as *content-focused with an emphasis on performance* (as defined in Section 2.6.1). He demonstrated and emphasized mathematical rules and procedures and generally taught using a lecture style with relatively little interaction with individual students or interactive class discussion. His teaching method was to “automatize” computational procedures and taught Class A to respond to a range of specific data cues, words, and context clues in the question. In the second teacher interview after the teaching program he explained:

Teacher A  
“I’d say, when you see these words it means between two points, and when you see this word that means at a point, and a maximum means that you let the derivative equal zero . . . [I am] giving them strategies.”

Teacher B’s teaching approach is best characterized as *content-focused with an emphasis on conceptual understanding* where understanding is viewed as constructed by the individual (as defined in Section 2.6.2). Teacher B’s teaching style was to deliberately orchestrate student-centred interactive class discussions during which the mathematical content was explored through individual students’ contributions. He questioned each student every lesson and encouraged them all to share their ideas with him and the other students. Teacher B’s teaching method was to focus on understanding. For example, he encouraged his students to develop their intuitive ideas about rate of change, slope and the limit concept. He encouraged conjecture, negotiation of meaning with other students in the class, analysis of information, decision-making, drawing conclusions, and proving ideas. He also made regular checks that all the students understood the concept under discussion. For example, in Sample Lesson 1 above, he checked that the students understood how the derivative rules fitted the patterns of numbers. Appendix 4.5 shows that over the entire teaching program, Teacher A spent 260 minutes lecturing Class A (Teacher B spent 4 minutes) while Teacher B spent 354 minutes conducting class discussions with Class B (Teacher A spent 92 minutes) and Teacher B spent 266 minutes helping individual students (Teacher A spent 78 minutes).

In summary, Teacher A privileged knowledge of routine procedures and rules using a teacher-centred teaching style. Teacher B privileged conceptual understanding of mathematical ideas using a student-centred, guided discovery teaching style that assisted students to construct meaning for themselves. Teacher A and Teacher B’s
pioneering teaching with a CAS calculator is further discussed in Kendal, Stacey, and Pierce (in press).

5.2.4 Characteristic 3: Privileging of use of CAS

This section identifies each teacher’s use of CAS in this study. As just mentioned in Section 5.2.2, Figure 5.7 shows the particular CAS activities the teachers elected to use from the curriculum’s four functional activities and six pedagogical activities (see Figures 4.1, 4.2). It also shows how frequently they used CAS or if they used by-hand procedures on Activities 1 - 4. Together, they indicate the emphasis that each teacher gave to particular representations and to linking pairs of representations, implicit in each activity. Figure 5.7 highlights differences between the teachers. Broadly, it shows that Teacher A used CAS more frequently on more functional and pedagogical CAS activities than Teacher B who used by-hand procedures more frequently for symbolic, graphical, and numerical differentiation. These results are consistent with the classroom observations of the teachers’ use of the TI-92 (see Appendix 4.5). Teacher A used the CAS calculator nearly twice as often as Teacher B (185 minutes cf. 109 minutes).

In a classroom discussion with the author, Teacher A indicated that, after the first teacher interview, he realized that each representation of derivative would be tested and that although he had always strongly preferred the symbolic representation he had decided to use the graphical and numerical representation of derivative for the main study (as discussed above, Section 5.2.2). He also indicated his realization that he could use symbolic algebra to generate ordered pairs for numerical differentiation through the calculation of a difference quotient (Activity 1, see Figures 4.1, 5.7), and that CAS would give an “exact” (note that this is not strictly correct) gradient for the tangent to a curve by graphing a function, drawing a tangent, and reading the gradient from the equation of the tangent (Activity 2, see Figures 4.1, 5.7). He also briefly explored the local linearity of curves (Activity 5, see Figures 4.2, 5.7). He linked the numerical and graphical representations by generating ordered pairs, for use in difference quotient calculations, for an “excellent” approximation to the gradient of the curve and gradient of the tangent and used CAS Activity 6 in this way, spending a significant amount of teaching and revision time emphasizing this particular aspect of differentiation. Teacher A demonstrated use of the technology to the students using an overhead projector.
display (see Sample Lesson 1, Section 5.2.1) and then allowed students free use of the calculator. Most of his personal teaching actions were associated with differentiation techniques even when using pedagogical CAS Activities 6 and 7. Sample Lesson 1 demonstrates this. At the second teacher interview, Teacher A commented:

I used it . . . because it was so easy. I hooked it up at the beginning of the lesson and used it much more than I would use a graphics calculator in the classroom. We just used it (the TI-92) all the time, . . . routine procedures like the product rule and the chain rule, umm, yes, . . . They actually hadn’t made the distinction that there were functions that they could only find the derivative of on their calculator.

Thus, Teacher A privileged use of CAS and used it in ways that were consistent with his privileging of calculus content and teaching approach. He used it principally to emphasize numerical, graphical, and symbolic differentiation techniques while using both functional CAS Activities 1, 2, 3 and pedagogical CAS Activities 5, 6, 7, and 8. In contrast, Teacher B rarely used CAS at all in the main study. He allowed the students to use CAS for graphical differentiation, Activity 2, but restricted use of symbolic algebra to procedures that supported understanding of the underlying concepts (see Sample Lesson 1, Section 5.2.1) and he ignored the numerical representation for calculating difference quotients but did teach average rates of change (and merely demonstrated CAS Activities 6 and 7). His focus was on promoting understanding. For example, he encouraged his students to quickly and accurately gather symbolic data (using CAS Activity 1) from which the rules of differentiation were later inferred (see CAS Activity 8). Otherwise, the students practised exercises by hand, without CAS. During the second teacher interview Teacher B explained his position thus:

I liked the routine procedures. For example, when you’re trying to induce symbolic rules, you haven’t got all that time wasting. You can very nicely do a lot of the algebra so simply on the calculator and you’re avoiding wasting time doing a lot of repetitive calculations.

It’s [the CAS] good for discovery because it takes a lot of the hack work out of teaching for understanding but you still need to teach pen and paper skill. I think there’s certain skills that the kids have to have, even if you can use the technology to do it. I think the kids have to have the (algebraic) skills as well, without the technology, I think that’s essential for their understanding. It’s not sufficient to just use the calculator, they have to have the understanding, what’s behind it.
Curriculum Activity
F = functional, P = pedagogical

<table>
<thead>
<tr>
<th>Identification of Activity</th>
<th>Degree of CAS Use</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency of use</td>
</tr>
<tr>
<td></td>
<td>Reps used</td>
</tr>
</tbody>
</table>

**Activity 1**
Numerical differentiation
F N High A
Moderate B

**Activity 2**
Graphical differentiation
F G High A
Moderate B

**Activity 3**
Symbolic differentiation
F S High A
Moderate B

**Activity 4**
First principles differentiation
F N, S High A
Moderate B

**Activity 5**
Zoom-in on graph
P G High A
Moderate B

**Activity 6**
Dynamic program to link gradients of secants, curve, and tangent at point
P G (intuitive G, N) High A
Moderate B

**Activity 7**
Dynamic program to link average rates of change to slopes of generated straight lines and curves
P G, N High A
Moderate B

**Activity 8**
Patterns amongst calculated derivatives to induce symbolic rules
P only for S High A
Moderate B

**Activity 9**
Gradient points plotted to predict symbolic rules
P G, S High A
Moderate B

**Activity 10**
Use of split screen to emphasize links between pairs of representations
P N, G, S High A
Moderate B

**Figure 5.7.** Frequency and degree of CAS use (or by-hand) by Teachers A and B on functional and pedagogical activities during the main study.

Teacher B believed that symbolic routines carried out in a “black box” did not assist understanding and insisted that (apart from the use described above) students perform algebraic procedures by-hand. He also warned his students not to become dependent on
CAS since they would not be permitted to use it for exams after the main study experimental unit. In contrast, he strongly supported use of graphics calculators (permitted in the state examination system) and had downloaded a set of programs onto his students’ graphics calculators including some symbolic programs such as one to solve a quadratic equation by the formula. Teacher B did not privilege use of CAS apart from using it (in an investigative way) for supporting understanding of the concept of derivative, particularly related to the symbolic and graphical representations, and for performing graphical differentiation.

Teacher A included the additional representations and related CAS activities because he became aware his students were to be tested on these representations (mentioned above). In contrast, Teacher B restricted student use of symbolic algebra because he knew his students were very weak, particularly in algebra, and that they would have to sit future examinations without the assistance of CAS (see Section 4.5). This is reported more fully by Kendal and Stacey (2001b).

Figure 5.8 provides a summary of each teacher’s high - moderate use of CAS activities during the main study and it shows (based on Figure 5.7) that Teacher A used five of the ten activities provided in the curriculum (both functional and pedagogical) whereas Teacher B used only two. In summary, Teacher A privileged use of CAS for procedures (and to a lesser extent for understanding) while Teacher B rarely used it except for essential graphical procedures and for developing student understanding.

<table>
<thead>
<tr>
<th>High - moderate use of CAS Activities by Teachers A and B in main study</th>
<th>Class A</th>
<th>Class B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type</strong></td>
<td><strong>No</strong></td>
<td><strong>CAS activities</strong></td>
</tr>
<tr>
<td>Functional</td>
<td>1</td>
<td>N differentiation</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>G differentiation</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>S differentiation</td>
</tr>
<tr>
<td>Pedagogical</td>
<td>8</td>
<td>To understand S</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>To understand G</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>Stressed N, G differentiation while linking N and G</td>
</tr>
<tr>
<td>Neutral</td>
<td></td>
<td>Checked by-hand solutions with CAS</td>
</tr>
</tbody>
</table>

* CAS Activity 3 was used in conjunction with CAS Activity 8.

*Figure 5.8. Teacher A’s and Teacher B’s high - moderate use of CAS activities during the main study.*
5.2.5 Privileging profiles of Teachers A and B during the main study

This section provides a privileging profile for Teachers A and B. Information from Sections 5.2.2, 5.2.3, and 5.2.4 above is collated into Figure 5.9 below which summarizes the observations made about each teacher’s privileging throughout the main study teaching program and verified by the teacher’s own opinions expressed in the second teacher interview.

<table>
<thead>
<tr>
<th>Calculus content</th>
<th>Teacher A</th>
<th>Teacher B</th>
</tr>
</thead>
</table>
| Preference for representations | • Preferred S  
• Used G  
• Used N  
• Linked G and N | • Preferred S  
• Used G  
• Linked S and G |

<table>
<thead>
<tr>
<th>Teaching approach</th>
<th>Teacher A</th>
<th>Teacher B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teaching style</td>
<td>• Teacher-centred lectures</td>
<td>• Student-centred, guided discovery discussions</td>
</tr>
<tr>
<td>Teaching method</td>
<td>• Used rules for routine procedures</td>
<td>• Promoted understanding of routine rules and procedures using enactive representations and visualization</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Use of CAS</th>
<th>Teacher A</th>
<th>Teacher B</th>
</tr>
</thead>
</table>
| • Used frequently  
• Emphasis on procedures | • Rarely used  
• Emphasis on understanding |

Figure 5.9. Personal privileging profile for Teacher A and Teacher B during the main study.

5.3 Changes in Teacher Privileging During the Main Study

This section addresses research Question 3. Did the teachers’ privileging stay the same when they taught with CAS for a second time?

Figure 5.4 shows Teacher A and B’s privileging characteristics prior to the main study and Figure 5.9 shows them during the main study, enabling the changes that occurred during the main study to be easily identified. Figure 5.1 shows these changes from the preliminary study (nine months earlier) and the first teacher interview (10 weeks earlier) to the main study. It shows that both teachers changed only one of the
three privileging characteristics after teaching introductory calculus using CAS for the second time and the other two privileging characteristics did not change. Neither teacher’s method or style of teaching changed, nor their approach to using CAS. Teacher A continued to lecture his students about how to use rules and procedures and he also continued to use CAS procedurally for differentiation procedures and Teacher B continued to promote understanding of routine rules and procedures through student-centred discussion and he continued to use CAS for symbolic algebra only when it would be beneficial for understanding.

However, both teachers changed the calculus content characteristic of their privileging by altering the representations of derivative they stressed. Teacher A included the numerical and graphical representations for the first time while Teacher B reduced his emphasis on the numerical representation. Both teachers changed the CAS activities they used for the main study (see Figure 7.8) and the changes corresponded with the changes in representations. Teacher A used CAS more for numerical and graphical differentiation procedures while Teacher B failed to use it for numerical differentiation.

### Changes in Privileging Characteristics of Teachers A and B During the Main Study

<table>
<thead>
<tr>
<th></th>
<th>Teacher A</th>
<th>Teacher B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Calculus content</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Preference for representations</td>
<td>• Used G</td>
<td>• Reduced use of N</td>
</tr>
<tr>
<td></td>
<td>• Used N</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Linked N and G</td>
<td></td>
</tr>
<tr>
<td><strong>Teaching approach</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teaching style</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Teaching method</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Use of CAS</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Figure 5.10. Changes in privileging characteristics for Teacher A and Teacher B during the main study.*

Reasons for these changes have been proposed. Teacher A included additional representations, in response to different assessment demands, and broadened his procedural use of the technology to include these additional representations. Teacher A decided to emphasize the numerical and graphical representations (and linking them) after the first teacher interview when he realized what the new assessment procedures involved and his increased confidence in using the CAS calculator (with experience)
enabled him to incorporate numerical and graphical procedures with CAS. During his second experience of teaching with CAS, Teacher A’s pleasure and interest in using symbolic algebra increased his confidence in teaching with the CAS calculators (unlike with graphics calculators) and he began to teach the additional representations using them. However, he did this in a procedural way that matched his teaching style.

Although Teacher B’s privileging characteristics were essentially stable, he reduced his use of the numerical representation in response to new knowledge about his students in the main study. He believed his students were not highly mathematically competent and would not cope with the numerical representation, in addition to the symbolic and graphical representations. Unlike Teacher A, Teacher B promoted use of graphics calculators in his teaching and was very aware of how to use CAS for calculus (evident from his first teacher interview). However, during his second teaching experience, he did not make changes in his use of CAS apart from reducing use of the numerical representation (in response to his perception of his students’ abilities) because the activities offered in the curriculum did not provide him with the opportunity to use CAS in ways that matched his teaching style and method. These issues have been fully discussed by Kendal and Stacey (2001b).

5.4 Summary and Discussion
This chapter has addressed Research Question 3 by identifying each teacher’s privileging prior to, and during, the main study. It has shown that, over two years, two of the privileging characteristics were stable: teaching approach, and use of CAS. The third characteristic, calculus content, changed. Both teachers made changes to the representations of derivative they emphasized in response to “institutional” school-related influences. One teacher, with increased awareness of new assessment procedures, increased the number of representations he stressed. He maintained his positive attitude towards use of CAS and, with increased confidence with teaching with CAS calculators, used them with the new representations (calculus content) by stressing procedures in a way that was consistent with his teaching method and style of teaching. The second teacher responded to his knowledge of his students (that the majority were low achievers) by reducing the representations he stressed. In spite of his own competency with the CAS calculator, he minimized its use in his classroom (even more
than in the preliminary study) and was enthusiastic about its use only when he felt it promoted understanding.

The teachers also used graphics and CAS calculators differently. Teacher A did not like using graphics calculators (in his normal teaching practice) whereas he was enthusiastic about the CAS calculator during the teaching trials. In contrast, Teacher B promoted use of the graphics calculators (in his normal teaching practice) but restricted use the CAS calculator. This occurred because Teacher A was unable to teach procedurally with the graphics calculator while Teacher B generally felt unable to teach for understanding with the CAS calculator.

This study shows that the teachers only changed their pedagogy in ways that were compatible with their individual personal privileging characteristics, namely, teaching approach and preferred way of using the CAS technology. They both changed the mathematics they emphasized in response to external “institutional” factors, different for each teacher. One teacher expanded the mathematical content he taught and, with increasing confidence using technology, was able to adapt his procedural manner of teaching to using technology procedurally with the new content. The second teacher reduced the mathematical content he taught, and although technologically proficient himself, did not develop his teaching with technology beyond using activities in the curriculum that would assist his students to understand concepts, the motivation for his teaching approach.

Thus, this study highlights some of the complexities associated with teachers using technology in their classrooms. The teachers’ personal content knowledge and privileging characteristics impinged on their personal pedagogy and how they taught with technology. Both teachers’ teaching approaches and use of CAS were stable. The only significant change that they made to their pedagogy was related to the mathematical content they taught and the motivation for this change was in response to new “institutional” knowledge.

What underpins the teachers’ privileging? Although this issue is outside the formal scope of this thesis, there were hints from the teachers about why they taught as they did. While Teacher A solved problems (in class and in interviews) he talked about finding clues and hints to identify the required procedure. In consequence, he influenced his students to adopt these strategies that reflected his own view about learning. On the
other hand, he indicated that the purpose of his teaching was to help his students achieve success on examination and tests, and some of his students acknowledged this characteristic to the author (they mentioned it casually in class). This caused a conflict for Teacher A. To maximize his students’ chances of success on the main study trial test, he promoted use of symbolic algebra and used graphical and numerical CAS procedures (using representations he had not used in the preliminary study). He also recommended that the students not rely on symbolic algebra so that they could develop the by-hand skills that he considered necessary for future school examinations. Thus, Teacher A’s view of the purpose of his teaching also influenced his pedagogy. Teacher B believed that it was his responsibility to help his students understand mathematical ideas. He believed that they learned best using the symbolic representation (by-hand) because it was effective and efficient, particularly if the symbols were made meaningful (usually through linking the symbol to a graphical representation). The only time he promoted use of symbolic algebra was when he believed it would contribute to the students’ understanding. Because he also believed students learned best while they constructed meaning for themselves he habitually orchestrated classroom discussions that involved every student (on an individual basis) in the class. His personal mathematical content knowledge seemed to help him manage these classroom discussions well. Thus, Teacher B’s view of learning was a powerful influence on the pedagogy he adopted. In addition, he was also acutely aware of the capabilities of his students (the majority were low achievers) and he modified the curriculum by reducing (compared with the preliminary study) his emphasis on numerical differentiation procedures. In addition, he was highly committed to helping his students achieve on school examinations and tests and, in consequence, he promoted by-hand skills in preference to using symbolic algebra. Thus, Teacher B’s view of the purpose of teaching also influenced his pedagogy. Overall, for both teachers, their views about the purpose of teaching and their views of learning influenced the privileging they adopted.

Using the information from this chapter about the ways each teacher taught calculus content to their classes with (and without) use of CAS, together with information about what their students learned (identified in Chapters 6 and 7), enables Research Questions 1 and 2 to be explored and the impact of each teacher’s privileging on the learning of the students in their class to be established in Chapter 8 and assessed in Chapter 9.
6. STUDENT LEARNING IN MAIN STUDY: THE DIFFERENTIATION COMPETENCY TEST

This chapter reports on the learning about differentiation that occurred in Classes A and B during the main study teaching program and identifies differences between the two classes.

The learning was established primarily by use of the DCT (see Appendices 1.1 & 1.2), the conduct of which was described in Section 4.6.1. It tested each student’s understanding of the eighteen competencies of the Differentiation Competency Framework, and the rationale for the development of both the Framework and the DCT was described in Chapter 3. Each competency is associated with an input representation of derivative (numerical, graphical, or symbolic) and a differentiation process: either a formulation or interpretation and, dependent on the output representation, with or without translation between two representations.

In this chapter, an overview of class achievement on the DCT is provided for both Classes A and B together with a broad comparison of the classes (Section 6.1). In addition, class achievement on the test overall is reported (Section 6.2) and on various competencies grouped in different ways including: the four categories (the two processes each with, and without, translation, in Section 6.3), the three input representations (Section 6.4), the twelve groups of category by input representation (Section 6.5), and the six pairs of inverse translations (Section 6.6). Finally, details of class achievement on the eighteen individual competencies and differences (and similarities) between Classes A and B are reported (Section 6.7) and a summary of the DCT results is given (Section 6.8).

6.1 Overview of Class Achievement on Competencies on Differentiation Competency Test

This section provides a broad overview of Class A’s and of Class B’s class achievement on each of the eighteen competencies on the DCT and a visual comparison of the classes that is intended to illustrate differences between the classes. As defined in Section 4.6.1, class achievement of a competency is defined as the percentage of students in each class who demonstrated the competency.
Table 6.1 shows the class achievements of Class A and Class B for each individual competency, listed in categories that are determined by process and translation status. The input representation for each competency is also displayed. Although Table 6.1 provides class achievements on individual competencies, discussion about them is delayed until Section 6.7.

Figure 6.1 displays these class achievements diagrammatically and the symbols N, G, and S indicate the numerical, graphical, and symbolic representations respectively. The solid arrows of varying width indicate class achievements greater than 40% while the dotted arrows indicate competencies with class achievement between 25% and 40%. Class achievements less than 25% are left blank.

For easy comparison, the translation competencies are shown in the triangular diagrams. Consider the following examples:

- Class B’s achievement on the "formulation-with-translation" competency \([FSg]\) was over 70%, indicated by the thick arrow from S to G. (Note: A symbolic derivative was calculated (input representation S) to find a graphical derivative (output representation g)).

- Class B’s achievement on the formulation-with-translation competency \([FGs]\) was between 40% and 49%, indicated by the thin arrow from G to S. (Note: A graphical derivative was calculated (input representation G) to find a symbolic derivative (output representation s)).

- Class A’s class achievement on the formulation-with-translation competency \([FGn]\) was less than 25% so there is no arrow from N to G.

Thus, the broad picture is that both classes achieved very well on the formulation-without-translation of symbolic derivative competency and also on the interpretation-without-translation of graphical derivative competency. For both classes, the strongest pair of translation competencies (overall) is between graphical and symbolic representations. However, each class had other different strengths (and deficiencies) that will be discussed in the following sections.
Table 6.1  *Class Achievement (%) on Individual Competencies on the Differentiation Competency Test in each Category and Input Representation by Class*

<table>
<thead>
<tr>
<th>Category</th>
<th>Input Representation</th>
<th>Competency</th>
<th>Class A (N=14)</th>
<th>Class B (N=19)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Formulation-</td>
<td></td>
</tr>
<tr>
<td>without-translation</td>
<td>Numerical</td>
<td>FNn</td>
<td>50</td>
<td>58</td>
</tr>
<tr>
<td></td>
<td>Graphical</td>
<td>FGg</td>
<td>50</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>Symbolic</td>
<td>FSs</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Formulation-with-translation</td>
<td>FNg</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGs</td>
<td>64</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FSn</td>
<td>36</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FGs</td>
<td>64</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FSg</td>
<td>50</td>
<td>73</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Interpretation-without-translation</td>
<td>/Nn</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IGS</td>
<td>21</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IGS</td>
<td>64</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sn</td>
<td>36</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Interpretation-with-translation</td>
<td>/Ng</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IGs</td>
<td>21</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Igs</td>
<td>64</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sn</td>
<td>64</td>
<td>74</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 6.1.** Diagrammatic depiction of Class A and Class B’s achievement on the eighteen competencies in the Differentiation Competency Test.
6.2 Overall Class Achievement on the Differentiation Competency Test

Class A’s and Class B’s overall class achievement on the DCT is reported in this section. On average, both classes demonstrated about half of the competencies. Over all eighteen items of the DCT, the average class achievement for Class A (N = 14) was 46% (SD 20%) and for Class B (N=19) 47% (SD 18%). The fairly low class achievement of each class was consistent with previous low school attainment. However, the almost identical result was unexpected since Class A had a higher proportion of students achieving higher grades than Class B in all school assessments during the previous nine months (see Section 4.5).

This section has shown that neither Class A nor Class B developed a complete understanding of the concept of derivative and that, unexpectedly, Class B achieved as well overall as Class A.

6.3 Class Achievement on Each Differentiation Category

This section reports Class A’s and Class B’s class achievement on each of the four differentiation categories. As described in Section 3.2.3, each DCT question is associated with one category of differentiation and Table 6.1 above, displays the competencies in each category: 3 formulation-without-translation, 6 formulation-with-translation, 3 interpretation-without-translation, and 6 interpretation-with-translation.

Table 6.2 Class Achievement (%) on the Differentiation Competency Test on Each Category of Differentiation by Process, Translation Status, and Class

<table>
<thead>
<tr>
<th>Translation status</th>
<th>Class</th>
<th>Process</th>
<th>Formulation</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A (N=14)</td>
<td>FNN, FGg, FSs</td>
<td>I Nn, I Gg, I Ss</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B (N=19)</td>
<td>67</td>
<td>55</td>
<td></td>
</tr>
<tr>
<td>Without-translation</td>
<td>A</td>
<td>67</td>
<td>55</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>70</td>
<td>67</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3*2 = 6 items)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>With-translation</td>
<td>FNg, F Gn, FSn</td>
<td>I Ng, I Gn, I Sn</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>FNs, FGs, FSg</td>
<td>INs, IGs, ISg</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>43</td>
<td>35</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>35</td>
<td>38</td>
<td></td>
</tr>
<tr>
<td>(6*2 = 12 items)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2 shows the average class achievement on each category (tested by DCT questions that tested the relevant Framework competencies). Both classes achieved well on the formulation-without-translation and interpretation-without-translation categories. These proved to be easier than formulation and interpretation with-translation and
overall, the formulation process was generally easier than interpretation. Comparing classes, 67% of Class B students (almost 13 of 19 students) were successful on the three interpretation-without-translation items compared with 55% of Class A students (almost 8 of 14 students). For this study, superior achievement is indicated by greater class achievement of at least 10%. This is considered to be a reasonable lower difference between classes to indicate differences in learning. Thus, Class B had superior achievement of 12% (shown shaded) across the three interpretation-without-translation competencies and Class A was more successful (to a lesser extent) across the six formulation-with-translation competencies.

Although none of the results reported in this study is statistically significant (due in part to the small class sizes), they do indicate interesting differences between the classes, particularly in light of the different compositions of the classes. The trends observed were consistent with the teaching that had been observed and increased confidence in the results (to be discussed further in Chapter 8). A series of independent t-tests were conducted to compare classes and, at the 5% level, and none was found to be significant.

This section has shown that although both classes achieved comparably on each category of competencies, some differences were apparent. Class B was superior on the interpretation-without-translation category (across three competencies) and Class A tended towards better achievement on the formulation-with-translation category (across six competencies).

### 6.4 Class Achievement by Input Representation

This section reports Class A and Class B’s class achievement on each of the three input representations of derivative, numerical, graphical, and symbolic. As described in Section 3.2.3, each input representation is associated with six competencies. For example, the symbolic competencies are $F_{Sn}$, $F_{Sg}$, $F_{Ss}$, $I_{Ss}$, $I_{Sn}$, and $I_{Sg}$ (see Table 6.1). The class achievement of Classes A and B in each input representation was determined as the average of the six items testing the competencies (discussed in Section 4.6.1). In this section, class achievement on each input representation (Section 6.4.1) and student proficiency on more than one input representation (Section 6.4.2) are reported. In addition, how these are different in each class is discussed (Section 6.4.3).
6.4.1 Class achievement on each input representation

Table 6.3 shows class achievement on each input representation of derivative. For each class, the class achievement was highest for the symbolic representation, second highest for the graphical representation and lowest for the numerical representation. Comparing classes, although the majority of students in both classes were successful on the symbolic representation competencies Class B achieved a higher average class achievement (see Section 6.3) notable in light of the weaker algebraic skills of Class B students (see Section 4.5).

Table 6.3 Achievement (%) in Each Input Representation of Derivative by Class on the Differentiation Competency Test

<table>
<thead>
<tr>
<th>Competencies</th>
<th>Numerical</th>
<th>Graphical</th>
<th>Symbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>FNn, FN, FNs,</td>
<td>FN, FN, FN</td>
<td>FG, FG, FG</td>
<td>FSn, FS, FS</td>
</tr>
<tr>
<td>In, In, In</td>
<td>In, In, In</td>
<td>In, In, In</td>
<td>In, In, In</td>
</tr>
<tr>
<td>Class A (N=14)</td>
<td>36</td>
<td>48</td>
<td>55</td>
</tr>
<tr>
<td>Class B (N=19)</td>
<td>34</td>
<td>43</td>
<td>64</td>
</tr>
</tbody>
</table>

6.4.2 Mastery of more than one input representation

The purpose of the calculus course was to assist students to develop conceptual understanding across three representations of derivative. This section reports the proportion of students in both classes who demonstrated proficiency on three, two, one, and zero representations of derivative.

If a student was successful on at least three of the six competencies in a particular representation, proficiency (at a minimum level) was deemed to have occurred. Table 6.4 displays the proficiency of each class on the specified combinations of input representations of derivative. It shows that the proportion of students in both classes who demonstrated proficiency on three, two, one, and zero representations of derivative was very similar. Most students in both classes demonstrated proficiency on at least one representation. By combining representations, it can be seen that almost half of the students in each class were proficient with more than one representation while nearly one quarter of the students in each class were proficient in all three representations.

Table 6.4 Percentage of Proficient Students by Class and Types of Input Representations of Derivatives on the Differentiation Competency Test

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A (N=14)</td>
<td>21</td>
<td>7</td>
<td>0</td>
<td>14</td>
<td>0</td>
<td>14</td>
<td>29</td>
<td>14</td>
</tr>
<tr>
<td>Class B (N=19)</td>
<td>21</td>
<td>5</td>
<td>5</td>
<td>16</td>
<td>0</td>
<td>21</td>
<td>21</td>
<td>11</td>
</tr>
</tbody>
</table>
6.4.3 Effect of class on proficiency in more than one input representation

This section compares Class A and Class B students at each school grade with respect to their proficiency on three, two, one, and zero representations of derivative.

Each student was allocated a school grade that indicated a ranked attainment level within the two classes. Four school-based test scores (conducted in the previous nine months) were averaged to give the school grade. Grade A corresponded to an average score in the range (90-100), B (80-89), C (70-79), D (69-60), E (50-59), F (40-49) and G (<40). For each school grade (in both classes), the number of students who attained proficiency in three, two, one, or zero representations of derivative is displayed in Table 6.5.

A comparison of students at the same school grade shows that, for most grades levels, Class B students were proficient in more representations than Class A students. For example, two of Class B’s “C” grade students were proficient in two representations and one was proficient in three representations. In contrast, the two of Class A’s “C” grade students were proficient in only one representation. Similar results apply for the grades “D”, “E”, and “G”. The only grade in which Class A students performed better was “B”.

In summary, this section has shown that Class B students were likely to be proficient in more representations than Class A students with identical school grades.

Table 6.5 Number of Students in Each School Grade and Class who Demonstrated Proficiency on Three, Two, One, or Zero Input Representations of Derivative

<table>
<thead>
<tr>
<th>School Grade</th>
<th>Number of Input Representations in which Proficient</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
</tr>
<tr>
<td>Class A</td>
<td>3</td>
</tr>
<tr>
<td>(N=14)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Class B</td>
<td>3</td>
</tr>
<tr>
<td>(N=19)</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>
6.5 Class Achievement on Each Differentiation Category by Input Representation

This section reports Class A and Class B’s class achievement on each differentiation category by input representation and highlights differences between the classes.

Each input representation (numerical, graphical, and symbolic) is associated with a number of competencies in each category: 1 formulation-without-translation, 2 formulation-with-translation, 1 interpretation-without-translation, and 2 interpretation-with-translation. Table 6.6 displays the class achievement for each category of competency by input representation (averaged on the pairs of translation competencies) and the shaded blocks indicate noteworthy superior class achievement, although not significantly different (as noted above).

The majority of Class A and Class B students were successful and achieved similarly (difference in class achievement less than 10%) on the formulation-without-translation competencies in the numerical, graphical, and symbolic representations. On the formulation-with-translation competencies, Class A was superior (shown shaded) in the numerical representation and graphical representations (with average class achievement < 50%). In contrast, Class B was superior (shown shaded) on the interpretation of the graphical and numerical derivatives, the interpretation-with-translation of symbolic derivatives, and on formulation-with-translation of symbolic derivatives.

Table 6.6 Average Class Achievement (%) in Each Category by Input Representation on the Differentiation Competency Test

<table>
<thead>
<tr>
<th>Category of Competency</th>
<th>Class</th>
<th>Input Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>N</td>
</tr>
<tr>
<td>Formulation-without-translation</td>
<td>(1*3 = 3 items)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
</tr>
<tr>
<td>Formulation-with-translation</td>
<td>(2*3 = 6 items)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
</tr>
<tr>
<td>Interpretation-without-translation</td>
<td>(1*3 = 3 items)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
</tr>
<tr>
<td>Interpretation-with-translation</td>
<td>(2*3 = 6 items)</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B</td>
</tr>
</tbody>
</table>
6.6 Achievement on Inverse Differentiation Translation Competencies

This section compares Class A’s and Class B’s class achievement on inverse differentiation translation competencies.

Inverse competencies associated with formulation-with-translation competencies such as $[FGs]$ and $[FSG]$ and interpretation-with-translation competencies such as $[ISg]$ and $[IGs]$ were grouped together to give a measure for the strength of flexible links between representations. Greater flexibility between representations indicates greater ability to link the pair of representations for both of the possible input (and output) representations of derivative. For example, consider the pair of inverse competencies $[FGs]$ (A Q.6) and $[FSG]$ (A Q.3). Question 6 provides graphical data (input representation is graphical) and asks the students to find the derivative of a function (output representation is symbolic) whereas Question 3 provides symbolic data (input representation is symbolic) and asks the students to find a gradient (output representation is graphical). When grouped together the average achievement on the pair of competencies indicates a student’s ability to link together the numerical and graphical representations.

Class achievements on the pairs of inverses of competencies were averaged and are displayed in Table 6.7. For both classes, the competencies marked with $\otimes$ indicate a translation that was poorly achieved while its inverse was successfully achieved (difference in class achievement ranged from 37% to 58%). The shaded blocks indicate the class with superior class achievement (at least 10% higher).

Both Class A and Class B showed some flexibility in moving between the symbolic and graphical representations (S $\leftrightarrow$ G) and partial flexibility between the numerical and symbolic representations (N $\leftrightarrow$ S). However, neither class developed flexibility between the numerical and graphical representations (N $\leftrightarrow$ G) but Class A’s flexibility was better developed since Class A students achieved better on formulation (with translation) competencies in the N $\leftrightarrow$ G and N $\leftrightarrow$ S pairs while Class B was marginally better on the N $\leftrightarrow$ S interpretation (with translation) competencies.
Table 6.7  *Class Achievement (%) on Inverse Translation Competencies for Formulation, Interpretation, and Overall by Class*

<table>
<thead>
<tr>
<th>Class</th>
<th>Formulation-with-translation (2 items)</th>
<th>Class achievement</th>
<th>Interpretation-with-translation (2 items)</th>
<th>Class achievement</th>
<th>Overall Inverse Translations (4 items)</th>
<th>Class achievement</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (N=14)</td>
<td>⊗ FNgn</td>
<td>32</td>
<td>⊗ INg</td>
<td>18</td>
<td>N ↔ G</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>FNg</td>
<td>18</td>
<td>I Gn</td>
<td>8</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>B (N=19)</td>
<td>FGs</td>
<td>57</td>
<td>I Sg</td>
<td>50</td>
<td>S ↔ G</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>F Sg</td>
<td>58</td>
<td>I Gs</td>
<td>53</td>
<td></td>
<td>55</td>
</tr>
<tr>
<td>A</td>
<td>FSn</td>
<td>39</td>
<td>⊗ INs</td>
<td>36</td>
<td>N ↔ S</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>FNs</td>
<td>29</td>
<td>I Sn</td>
<td>42</td>
<td></td>
<td>38</td>
</tr>
</tbody>
</table>

⊕ For both classes, class achievement on this competency was much less than for its inverse competency.

This section shows that both classes developed flexibility between the symbolic and graphical representations and inflexibility between the symbolic and numerical representations. Class A also showed evidence of some development between the graphical and numerical representations but this was not so for Class B.

### 6.7 Achievement on Specific Differentiation Competencies

In this section, specific competencies *understood* (and *not understood*) by each class and differences between Class A and Class B are identified.

**Specific competencies *understood* (and *not understood*) by Classes A and B**

The class achievement for each individual competency in each category is listed in Table 6.1. Competencies *understood* (class achievement of at least 50%) by each class are located in Table 6.8 and those *not understood* (class achievement of less than 50%) in Table 6.9. In both Tables 6.8 and 6.9, superior achievement is denoted by a * accompanied by the percentage superiority. In Table 6.9, class achievement of less than 25% is indicated by a smaller font.

Table 6.8 shows that both classes *understood* seven identical competencies, the formulation-without-translation competencies, ([FNn], [FGg], [FSs]) and those involving S and G representations ([FSg], [IGg], [IGs], [ISn]). Class A *understood* 2
additional formulation-with-translation competencies ([FNg], [FGs]) and Class B, 2 additional, different interpretation-without-translation competencies ([INn], [ISg]).

Table 6.8 Specific Competencies Understood by Class, Category, and Input Representation

<table>
<thead>
<tr>
<th>Category of Competency</th>
<th>Numerical (N)</th>
<th>Graphical (G)</th>
<th>Symbolic (S)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Class A</td>
<td>Class B</td>
<td>Class A</td>
</tr>
<tr>
<td>Formulation-without-translation (3 competencies in total)</td>
<td>FNn</td>
<td>FNn</td>
<td>FGg</td>
</tr>
<tr>
<td>Formulation-with-translation (6 competencies in total)</td>
<td>*FNg (20%)</td>
<td>*FGs (22%)</td>
<td>FSg</td>
</tr>
<tr>
<td>Interpretation-without-translation (3 competencies in total)</td>
<td>*INn (20%)</td>
<td>IGg</td>
<td>*IGg (11%)</td>
</tr>
<tr>
<td>Interpretation-with-translation (6 competencies in total)</td>
<td>IGs</td>
<td>IGs</td>
<td>ISn</td>
</tr>
</tbody>
</table>

* denotes class was superior by the percentage given in brackets

The behaviours demonstrated by achievement on categories of DCT items (initially examined in Section 3.4) are now discussed in relation to Class A’s and Class B’s achievements as displayed in Table 6.8 above

Class A students (on average):

- Knew when and how to use the data in the question to carry out a differentiation procedure in the numerical, graphical, and symbolic representations (i.e., formulate-without-translation a derivative in each representation, 3 items).
- Knew when and how to use the data in the question to carry out a differentiation procedure in the numerical, graphical, and symbolic representations to achieve a derivative in a different representation (i.e., formulate-with-translation a derivative in each representation, 3 items).
- Reason about a gradient (i.e., interpret a graphical derivative-without translation, 1 item).
• Recognized the equivalence of a derivative to a given gradient and the equivalence of an instantaneous rate to a given derivative (i.e. interpretation-with-translation competencies, 2 items).

Class B students (on average):
• Knew when and how to use the data in the question to carry out a differentiation procedure in the numerical, graphical, and symbolic representations (i.e., formulate-without-translation a derivative in each representation, 3 items, the same as Class A).
• Knew when and how to use the data in the question to carry out a differentiation procedure in the symbolic representations to achieve a derivative in the graphical representation (i.e., formulate-with-translation a derivative in the symbolic, 1 item).
• Reason about a gradient and rate of change (i.e., interpret a graphical and numerical derivative-without-translation, 2 items).
• Recognized the equivalence of a derivative to a given gradient and the equivalence of an instantaneous rate and gradient to a given derivative (i.e. interpretation-with-translation competencies, 3 items).

Table 6.9 shows seven competencies, shown in the small font, ([FNs], [INs], [FSn], [INg], [FGn], [IGn], [ISs]) were not understood by at least one class. Six of these competencies involved the numerical representation either as input or output derivative and mostly involved the more unusual translations of the numerical derivative to find the symbolic derivative (N to S), or of the graphical derivative to find the numerical derivative (G to N).
Table 6.9  Specific Competencies Not Understood by Class, Category, and Input Representation

<table>
<thead>
<tr>
<th>Category of Competency</th>
<th>Input Representation</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Numerical (N)</td>
<td>Graphical (G)</td>
<td>Symbolic (S)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Class A Class B Class A Class B Class A Class B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Formulation-without-translation (3 competencies in total)</td>
<td>*FNs (22%)</td>
<td>FN</td>
<td>FNs</td>
<td>FNs</td>
<td>FNs</td>
<td>FNs</td>
</tr>
<tr>
<td>Formulation-with-translation (6 competencies in total)</td>
<td>I/Nn</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
</tr>
<tr>
<td>Interpretation-without-translation (3 competencies in total)</td>
<td>I/Nn</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
</tr>
<tr>
<td>Interpretation-With-translation (6 competencies in total)</td>
<td>*I/Ng (16%)</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Ng</td>
<td>I/Sg</td>
</tr>
</tbody>
</table>

* denotes class was superior by the percentage given in brackets
The very small font denotes class achievements of < 25%

Differences between Class A and Class B

There were nine competencies on which Class A and Class B achieved differently (with class achievements different by at least 10%, shown bolded). Tables 6.7 displays these competencies by the type of competency and by representation. Where a class achieved superiority on a particular competency, it is starred.

Comparing classes (using Tables 6.8 and 6.9), Class A achieved higher on 4 competencies, Class B, on 5 competencies (indicated by the *'s). Class A achieved better on formulation-with-translation competencies involving numerical and graphical input representations. In contrast, Class B achieved better on both types of interpretation competencies and on competencies involving the symbolic input representation. These differences explain the previous observations about the performance of each class with respect to input representation (Section 6.4.1), category by input representation (Section 6.5), and ability to move flexibly between representations (Section 6.6).

This section shows that while both classes understood seven of the same competencies, both did not understand another seven, and they achieved differently on nine. Class A was superior on four competencies, mostly formulation-with-translation,
while Class B was superior on five competencies, mostly involving the interpretation process and involving the symbolic representation.

6.8 Summary and Discussion
This chapter has shown that the average class achievement for Class A and Class B was almost 50%. At least half the students in each class showed that they each understood 9 competencies (7 identical and 2 different in each class) of the 18 competencies. Few students in both classes understood the remaining 7 competencies, 6 of which involved the numerical representation and less traditional ways of asking questions about differentiation, arising from use of multiple representations. The seventh competency involved interpretation-without-translation of a symbolic derivative. The failure to achieve at least 50% on these competencies suggests that the numerical representation was not adequately taught and that the students were not challenged to link representations in less conventional ways.

Both classes achieved well on the formulation-without-translation of derivatives, interpretation within the graphical representation and graphical-symbolic translations, the representations more traditionally used for differentiation. In each class the proportion of students showing proficiency in three, two, one or zero representations of derivative was almost identical although different attainment groups achieved differently.

Both classes could also move flexibly between the graphical and symbolic representations, less flexibly between the graphical and numerical representations and inflexibly between the symbolic and numerical representations.

A hierarchy of processes was apparent. Both classes found the formulation-without-translation competencies easiest, then interpretation-without-translation, formulation-with-translation, and the most difficult was the interpretation-with-translation competencies.

Class A’s particular strengths are mostly associated with the numerical and graphical representations and translation between representations. Specific competencies include formulation in the numerical and graphical representations with-translation. Class A students (not successful overall) were more flexible than Class B between the numerical and graphical representations.
Class B’s particular strengths are mostly associated with the symbolic and graphical representations and interpretation competencies. Specific competencies include interpretation-without-translation of graphical and numerical derivatives, interpretation of symbolic derivatives with-translation to the graphical and numerical representations, and formulation of a symbolic derivative with-translation to the graphical representation. Significantly, lower achieving Class B students were able to match the performances of the higher achieving Class A students on proficiency with more than one representation of differentiation.

These class differences clearly indicate that some different teacher related factors are operating in each class to bring about these differences in class achievement on the DCT. The class achievements of Class A and Class B on each competency is to be matched with Teacher A and B’s teaching actions using (and not using) the CAS activities provided in the curriculum in Chapter 8 (in Table 8.1). Together with additional insights from the PRT and other student questionnaires (to be discussed in Chapter 7) the full impact of Teacher A and B’s privileging (from Chapter 5) on the learning of Class A and Class B students will be analyzed in Chapter 8 and assessed in Chapter 9.
7. FURTHER RESULTS OF MAIN STUDY

This chapter reports on further learning outcomes, based on the Preference for Representation Test, CAS and Calculus questionnaire, written evaluation of the TI-92, and student interviews. The first objective of using these additional assessment tools was to gain further insights about what the students in each class understood about the concept of differentiation and to identify factors that influenced their understanding. The second objective was to confirm class achievement on the Framework competencies tested and to monitor other notable differences between the classes. This chapter discusses the PRT results (Section 7.1), the Calculus and CAS questionnaire (Section 7.2), students’ evaluation of the TI-92 (Section 7.3), and aspects of the student interviews (Section 7.4). Limitations of the methodology are discussed (Section 7.5) and finally, the summary and discussion of the chapter is presented (Section 7.6).

7.1 Preference for Representation Test

This section reports the results of the second test, the PRT (see Appendices 2.1 & 2.2), the construction and conduct of which were described in Section 4.6.2. The PRT provided an opportunity to verify some of the DCT results and to explore student use of symbolic algebra that was not tested on the DCT.

Overall class achievement on the nine PRT questions (16 items) is reported (Section 7.1.1). Results related to each different focus of the PRT are also reported: class use (and non-use) of the symbolic algebra (Section 7.1.2), the influence of different types of data on class preference for representation (Section 7.1.3), knowledge of two representations (Section 7.1.4), and class responses to different types of numerical data (Section 7.1.5). Class achievement of a compound competency is also discussed (Section 7.1.6) and finally a summary and discussion are presented (Section 7.1.7).

7.1.1 Overall class achievement on the Preference for Representation Test

Class A and Class B’s class achievement on the PRT are now reported and the nature of the results explained. Class A’s overall class achievement on the PRT was 58% (SD 27%) while Class B achieved 49% (SD 26%). Class B’s success rate was very similar to the overall class achievement of competencies demonstrated on the DCT, while Class
A’s success rate was marginally higher (see Section 6.1). These results are consistent with their school-based attainment (Section 4.5) and the difference between the classes was not unexpected since the PRT included several questions involving formulation-with-translation in different representations and Class A was superior on more of these competencies on the DCT (see Table 6.6).

7.1.2 Use of CAS

Class A and Class B made different use of CAS for symbolic algebra on PRT questions involving optional and essential use of symbolic algebra.

Optional use of CAS

Six items on the PRT (B Q.1(a)-(e), see Appendix 2.1) required students to differentiate polynomial functions including items with negative powers, items with fractional powers, and items involving the chain or product rules. Students could choose to differentiate these by-hand or with CAS. For each class, use of CAS and the percentage of successful attempts (and percentage of attempts with errors) on the six symbolic differentiation items are displayed in Table 7.1. Cells are shaded where the percentage of one class is more than 10% greater than the other class.

Table 7.1 Percentage of Attempts on Symbolic Differentiation on Preference for Representation Test Q.1(a)-(e) by Technology Use, Success, Error Type, and Class (6 items)

<table>
<thead>
<tr>
<th>Type of Attempt</th>
<th>Class A (N=14)</th>
<th>Class B (N=18)*</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>With CAS</td>
<td>Without CAS (By-hand)</td>
<td></td>
</tr>
<tr>
<td>Successful</td>
<td>A 30</td>
<td>41</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td>B 8</td>
<td>49</td>
<td>57</td>
</tr>
<tr>
<td>Conceptual errors</td>
<td>A 1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>B 0</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Procedural errors</td>
<td>A 5</td>
<td>21</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>B 2</td>
<td>33</td>
<td>35</td>
</tr>
<tr>
<td>Total</td>
<td>A 36</td>
<td>63</td>
<td>99</td>
</tr>
<tr>
<td></td>
<td>B 10</td>
<td>86</td>
<td>96</td>
</tr>
</tbody>
</table>

*The number of Class B students who took the PRT was reduced from 19 to 18 due to the illness of one student.

Although both classes performed differentiation more by-hand than with CAS, Class A used CAS more than Class B and, in consequence, was more successful overall in obtaining correct answers. The success rates on items attempted by-hand were not markedly different in the two classes but Class B students failed to use CAS to compensate for their poorer algebraic skills (described in Section 4.5) and they made
more procedural errors. Using classroom observation data, counts of the students who used the calculator for extended periods of time in class showed that more Class A students habitually used the CAS calculator than Class B students.

However, all of the students in both classes demonstrated the competency [FSs] on at least one item (and most students on the six items), consistent with the DCT class achievement of 100%. The most frequent type of error was symbolic manipulation (procedural) and the most common type of conceptual error was when the students found an equivalent expression instead of differentiating (e.g. \( f'(2\sqrt[t]{t}) = \frac{1}{2t^\frac{1}{2}} \)).

**Essential use of CAS**

The students were familiar with polynomial and trigonometric functions but not with exponential functions. For this reason, symbolic algebra was needed to successfully differentiate the PRT item (B Q.8(b), described earlier as Sample 6 in Section 4.6.2):

Find the rate of change of \( y \) with respect to \( x \) if \( y = (1.8)^x \) when \( x = 3 \).

After differentiating the function, a substitution of \( x = 3 \) into the derivative is required. The competency involved is \([FSn]\) (formulation-with-translation of a symbolic derivative to the numerical representation). Table 7.2 displays, for each class, the percentage of successful attempts and use of CAS on Sample 6 (B Q.8(b)) and the percentage of attempts with errors.

**Table 7.2 Percentage of Attempts on Symbolic Differentiation on Sample 6 by CAS Use, Success, and Error Type**

<table>
<thead>
<tr>
<th>Type of Attempt</th>
<th>Class A (N=14)</th>
<th>Class B (N=18)</th>
<th>With CAS</th>
<th>Without CAS By-hand</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Successful</td>
<td>A</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Conceptual Errors</td>
<td>A</td>
<td>50</td>
<td>0</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>22</td>
<td>0</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>Procedural Errors</td>
<td>A</td>
<td>0</td>
<td>7</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>0</td>
<td>50</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Total attempts</td>
<td>A</td>
<td>57</td>
<td>7</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>22</td>
<td>50</td>
<td>72</td>
<td></td>
</tr>
</tbody>
</table>

Class A’s errors were mainly conceptual, a failure to formulate the derivative. Instead, \( x = 3 \) was substituted directly into the function, using CAS, \((1.8)^3 = 5.83\). They were distracted by the performance of the CAS procedure and failed to formulate the
derivative. In contrast, Class B knew to differentiate and their errors were principally procedural, related to incorrect differentiation techniques. Many Class B students attempted to differentiate using techniques suitable for polynomials including $f'(1.8^x) = x(1.8)^{x-1}$ and $f'(1.8)^3 = 3(1.8)^2$. An explanation for these incorrect procedures is that the students attempted Sample 6 (B Q. 8(b)) immediately after applying the techniques used for successful differentiation of polynomials for the function $N = t^{1.8}$ (B Q. 8(a)) and on most of the Class B attempts, the students did not (and could not) distinguish between the polynomial and exponential functions. One-quarter of Class B (and nearly one-third of Class A) balked at the unfamiliar function and did not attempt the differentiation at all. Thus, the students from both classes did not have enough experience with differentiating different types of functions to know that they could not differentiate $y = (1.8)^x$ unless they used CAS or did not know how to use the CAS on an unfamiliar function.

This section shows that Class A made more use of CAS, for symbolic differentiation and for substitution, than Class B who rarely used it, preferring by-hand methods. It also shows that Class A and Class B made different types of errors. Class A was more likely to make conceptual errors than Class B. Examination of the test scripts (and confirmed later in interviews) showed that this occurred because the students focused on the CAS procedure rather than on the formulation of the derivative and subsequent differentiation procedure. Class B’s errors were principally procedural errors related to lack of algebraic proficiency. To make more discriminating use of CAS for symbolic differentiation both classes needed greater exposure to a larger variety of functions to know when using CAS would be useful. This would have assisted both classes, particularly Class B, with its higher proportion of algebraically weak students.

7.1.3 Influence of data on use of representations
Choice of representation is influenced by an individual’s personal preference for representation and the data that accompanies the question (including data restricted from use by the question). This section illustrates the influence of data on each class’s use of representations. Further examples are discussed: Sample 12 (B Q.3), and class results on Samples 7 and 8 (described earlier in Section 4.6.2) are compared.
Sample 12. \((B \ Q.3)\)

Find the derivative of \( y = 2x^3 + 1 \) at \( x = 2 \), without using the rule for differentiating or using the derivative function on the TI-92.

With function data the derivative can be formulated numerically, graphically, or symbolically to achieve the competencies \([F\text{Sn}]\) or \([F\text{Sg}]\) or \([F\text{Ss}]\). In this question, symbolic differentiation is banned. The percentages of Class A and Class B students who chose, numerical, graphical, and the banned symbolic representation (shown starred) are shown in Table 7.3. Class differences of at least 10% have been highlighted.

Table 7.3  Percentage of Students Attempting to Find a Derivative on Sample 14 by Class, Data, and Representation

<table>
<thead>
<tr>
<th>Class</th>
<th>Representation and competency</th>
<th>Numerical (F\text{Ns})</th>
<th>Graphical (F\text{Gs})</th>
<th>Symbolic (F\text{Ss})</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (N=14)</td>
<td></td>
<td>50</td>
<td>0</td>
<td><strong>43</strong></td>
</tr>
<tr>
<td>B (N=19)</td>
<td></td>
<td>17</td>
<td>22</td>
<td><strong>57</strong></td>
</tr>
</tbody>
</table>

* Inappropriate choice of representation

Nearly half of Class A and more than half of Class B responded to the function data by differentiating symbolically even though it was banned. However, nearly half of Class A used function data to generate ordered pairs necessary to calculate a numerical difference quotient while nearly one quarter of Class B used the function data to find the gradient of the tangent.

The next example involves a comparison of student responses to Samples 7 and 8. Sample 7 has symbolic and graphical data whereas Sample 8 has only symbolic data.
Sample 7. (B Q.4(a))

Consider the curve \( y = x^3 - x \). Find the gradient of the curve at \( x = 1 \), stating whether it is accurate or approximate.

![Graph of the curve \( y = x^3 - x \).](image)

Sample 8. (A Q.3)

A curve has the equation \( g(x) = 5x^3 - 6x^2 + 3x - 6 \). Find the gradient of the curve at the point P, where \( x = -1 \). (No graph is given.)

Both Samples 7 and 8 have function data that provides students with the opportunity to formulate a derivative in the representation of their choice, numerical, graphical or symbolic. The symbolic competency \([FSg]\) is demonstrated if the function is differentiated symbolically (followed by a substitution at the required \( x \) value). The graphical competency \([FGg]\) is demonstrated if the gradient of the tangent to the curve at the required \( x \) value is determined using CAS or by drawing the tangent by-hand. The numerical competency \([FNg]\) is demonstrated by using the function to generate coordinate values on the curve together with the estimation of the limit of the difference quotient. Sample 7 is accompanied by the presence of a graph enabling its influence to be determined.

Table 7.4 shows the percentage of students in each class who chose numerical, graphical, and symbolic representations on Samples 7 and 8. Class differences of at
least 10% have been highlighted. With only symbolic function data available (Sample 8), Class A showed a strong first preference for the symbolic representation and some preference for numerical representation. In the presence of both graphical and symbolic data, Class A increased its use of the graphical representation at the expense of the symbolic representation. In the presence of only symbolic data, Class B showed a high preference for the symbolic representation but increased its use of the graphical representation (more than Class A) in the presence of the graphical data at the expense of the symbolic representation. Class B showed little preference for the numerical representation.

Table 7.4 Percentage of Students Attempting to Find a Gradient of Curve on Samples 7 and 8 by Class, Data, and Representation

<table>
<thead>
<tr>
<th>Sample</th>
<th>Data Type of Function</th>
<th>Class</th>
<th>Representation and competency</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>Symbolic</td>
<td>A (N=14)</td>
<td>Numerical: FNg 29, Graphical: FGg 7, Symbolic: FSg 64</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B (N=19)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Symbolic</td>
<td>A (N=14)</td>
<td>Numerical: FNg 29, Graphical: FGg 36, Symbolic: FSg 29</td>
</tr>
<tr>
<td></td>
<td>Graphical</td>
<td>B (N=18)</td>
<td>Numerical: FNg 6, Graphical: FGg 50, Symbolic: FSg 44</td>
</tr>
</tbody>
</table>

While both classes were responsive to function data for symbolic differentiation, Class A also used it for numerical differentiation and with the visual cue of the graph both classes used it for graphical differentiation, especially Class B. This result is supported by Even (1998) who acknowledges the importance of context (problem presentation) and underlying notions in students’ abilities to translate from one representation to another.

These results support the results from the DCT on which Class A had superiority on formulation-with-translation competencies [FNs] and [FNg] and Class B on [FSg] (see Tables 6.8 & 6.9).

7.1.4 Knowledge of two representations

As discussed in Section 4.6.2, the students were given the opportunity to demonstrate a different competency when repeating Sample 7 (B Q.4(a) for Q.4 (b)). This section reports students’ knowledge of two representations.

Table 7.5 shows the representations adopted by each class over two attempts at the same question. A majority of Class A solved it using two different representations,
particularly the numerical and graphical representations while the majority of Class B students used a single representation, either the symbolic representation or the graphical representation (favoured in the presence of such a powerful graphic cue of graph and grid). These results are consistent with the DCT result that Class A was developing overall flexibility between the N and G representation (see Table 6.7).

Table 7.5  Percentage of Students by Class and Representation/s Used on Sample 7 Repeated to find a Gradient

<table>
<thead>
<tr>
<th></th>
<th>Single Representation</th>
<th></th>
<th>Pairs of Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>G</td>
<td>S</td>
</tr>
<tr>
<td>Class A (N=14)</td>
<td>14</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Class B (N=18)</td>
<td>0</td>
<td>39</td>
<td>28</td>
</tr>
</tbody>
</table>

7.1.5  Data associated with the numerical representation

This section compares Class A’s and Class B’s achievement on competencies involving the numerical representation and a variety of different data. Table 7.6 shows class achievement (superior class achievement is highlighted) on the six Framework competencies on the DCT together with some of the same competencies tested on the PRT.

Table 7.6  Class Achievement (%) of Class A (N=14) and Class B (N=18) on Numerical Competencies on the DCT (Test A) and PRT (Test B) by Associated Data

<table>
<thead>
<tr>
<th>Question Identity</th>
<th>Question Output</th>
<th>Competency</th>
<th>Associated Data</th>
<th>Class</th>
<th>Class Achievement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A Q.19(a)</td>
<td>Average rate of change</td>
<td>FNn</td>
<td>Graphical data to deduce coordinates</td>
<td>A</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>58</td>
</tr>
<tr>
<td>A Q.2</td>
<td>Gradient</td>
<td>FNg</td>
<td>List of ordered pairs</td>
<td>A</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>37</td>
</tr>
<tr>
<td>B Q.4(a, b)</td>
<td>Sample 7</td>
<td>FNg</td>
<td>Symbolic function and graph</td>
<td>A</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>28</td>
</tr>
<tr>
<td>A Q.11</td>
<td>Sample 2</td>
<td>FNs</td>
<td>Table of ordered pairs</td>
<td>A</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>21</td>
</tr>
<tr>
<td>B Q.3</td>
<td>Derivative</td>
<td>FNs</td>
<td>Symbolic function</td>
<td>A</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>17</td>
</tr>
<tr>
<td>B Q.6</td>
<td>Sample 9</td>
<td>FNn</td>
<td>Discrete numerical</td>
<td>A</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>56</td>
</tr>
<tr>
<td>A Q.9</td>
<td>Sample 3</td>
<td>FNn</td>
<td>Rate of change as symbolic function</td>
<td>A</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>63</td>
</tr>
<tr>
<td>A Q.15</td>
<td>Interpret rate of change as gradient</td>
<td>FNg</td>
<td>Rate of change symbolic function</td>
<td>A</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>16</td>
</tr>
<tr>
<td>A Q.16</td>
<td>Interpret rate of change as derivative</td>
<td>FNs</td>
<td>Rate of change symbolic function</td>
<td>A</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td>11</td>
</tr>
</tbody>
</table>
Table 7.6 shows that Class A was able to work with a bigger variety of numerical data, including ordered pairs and functions to generate ordered pairs. Their use led to superior achievement on the numerical formulation-with-translation competencies \([FNg]\) and \([FNs]\) (shown to be almost identical on the DCT and PRT as above). Table 7.6 also shows that both Class A and Class B were able to work with graphs to deduce coordinates to calculate an average rate of change and achieve the formulation of numerical derivative competency \([FNn]\). Class A was more capable of reasoning about the numerical derivative using discrete data (Sample 9) while Class B was more capable of interpreting rate of change function data (Sample 3). Thus, dependent on the data, each class was superior on \([INn]\) the interpretation of numerical derivative competency.

In summary, Class A was able to use a greater variety of numerical data than Class B, except for rate of change functions “at all points”. This led to greater superiority on the numerical formulation-with-translation competencies \([FNg]\) and \([FNs]\) and supported the results from the DCT (see Table 6.1).

### 7.1.6 A compound competency

As described in Section 4.6.2 Sample 10 tested a compound competency (see below). It involves a formulation of a graphical derivative followed by an interpretation. The class achievement for each class is shown in Table 7.7 below.

**Sample 10. (B Q.7(b))**  
(competency \([FGg + IGn, or IGg, or IGs]\))

Suppose you used the Zoom-In function on the calculator 5 times on the graph of \(y = x^2 + 3\) around the point P where \(x = 2\). If you knew the coordinates of two points on the Zoomed-In graph, what could you find out about the graph at the point P?

Most Class A students but only half of Class B students answered this question, first mentally formulating a gradient. A majority of Class A students interpreted the situation as the gradient of the curve or the gradient of the tangent to the curve at the point (graphical) and a minority as a rate of change (numerical) whereas more Class B students perceived the situation as a derivative (symbolic). These interpretations reflected the way that the classes were taught including teacher’s use of CAS Activity 5 to Zoom-in to show its local linearity.

The students from both classes found it more difficult to achieve the two-step compound competency of formulation of a graphical derivative and its interpretation.
than the single interpretation competencies tested on the DCT. For comparison, Table 7.8 displays class achievement on these single competencies (from Table 6.1). It clearly shows that both Class A and B students found the \([IGs]\) competency easier than the \([FGg + IGs]\) competency and the \([IGg]\) competency easier than the \([FGg + IGg]\) competency. Both classes found the \([IGn]\) and \([FGg + IGn]\) difficult especially Class B.

Table 7.7 Class Achievement (%) of Particular Compound Competencies on Sample 10

<table>
<thead>
<tr>
<th>Competency Interpretation</th>
<th>([FGg + IGn]) Rate of change</th>
<th>([FGg + IGg]) Gradient of curve at P</th>
<th>([FGg + IGs]) Derivative at (x = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>17</td>
<td>64</td>
<td>6</td>
</tr>
<tr>
<td>Class B</td>
<td>0</td>
<td>17 + 17 (gradient of tangent at P)</td>
<td>22</td>
</tr>
</tbody>
</table>

Table 7.8 Class Achievement (%) of Interpretation of Graphical Derivative on the Differentiation Competency Test

<table>
<thead>
<tr>
<th>Competency Interpretation</th>
<th>([IGn]) Rate of change</th>
<th>([IGg]) Gradient of curve (or tangent) at P</th>
<th>([IGs]) Derivative at (x = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class A</td>
<td>21</td>
<td>79</td>
<td>64</td>
</tr>
<tr>
<td>Class B</td>
<td>0</td>
<td>90</td>
<td>68</td>
</tr>
</tbody>
</table>

7.1.7 Summary and discussion of Preference for Representation Test

The PRT has shown:

- The PRT results supported the DCT results, particularly class achievement on formulation-with-translation competencies and Class A’s developing flexibility between the numerical and graphical representations.
- Class A used symbolic algebra more than Class B but neither class used it to best advantage.
- Over several questions, both classes responded consistently to different data. Both classes had a preference for using symbolic function data (particularly Class B) and responded to it by differentiating symbolically (even when it was banned) and to a lesser extent graphically. Both classes were responsive to the presence of graphical cues by differentiating graphically (particularly Class B). However, only Class A was able to use symbolic data to differentiate numerically.
- Questions involving more than one competency were harder than questions involving a single competency.
7.2 Student Questionnaire – Calculus and CAS

This section reports results and the conclusions drawn from the Student Questionnaire – Calculus and CAS.

As outlined in Section 4.6.4, at the conclusion of the teaching program the questionnaire Calculus and CAS (Appendix 8.1) was given to all the students in both classes. Its purpose was to gain an insight into the attitudes of Class A and B students towards calculus, the relative importance of computer algebra and by-hand algebra, and how each class used the CAS calculator for differentiation. Students were requested to indicate their response to each questionnaire statement by circling a digit, 4 (strongly agree), or 3 (agree), or 2 (disagree), or 1 (strongly disagree). For analysis some items were grouped and the responses averaged. Individual and group responses are summarized and displayed in Appendix 8.2. There are some similarities between the classes and while the differences between the classes are not huge some trends are evident.

The majority of students in Classes A and B believed that they understood new ideas better when doing algebra by-hand and felt it was necessary to remember the algebraic rules for differentiation. Both classes indicated that they preferred to use the graphical rather than the symbolic algebra facilities of the calculator. They found the TI-92 fairly easy to use and reported minor problems learning to use it.

Although Class A students claimed to prefer by-hand to CAS, they demonstrated a much greater awareness of the benefits of using CAS and reported using it more frequently than Class B for algebra (and differentiation). This behaviour was also observed in the interviews (see below). Class B students, who reported that they understood calculus words and symbols fairly easily and new ideas better with graphs, used graphs more frequently for differentiation. Class A did not particularly enjoy learning calculus overall, whereas Class B was neutral towards calculus.

In summary, both classes tolerated the study of calculus, believed that they understood algebra and differentiation more when they used by-hand techniques and claimed that they preferred using the CAS calculators for graphs. Comparing classes, Class A understood considerably more about the capabilities of the CAS calculator, used it for a greater variety of algebraic procedures and differentiation, better
understood the advantages of differentiating with CAS, and preferred to use it for symbolic differentiation.

On the self-evaluation of their use of the TI-92, the attitudes of the students to algebra were consistent with the emphasis they gave algebra in their normal mathematics lessons and their usual practices with the graphics calculators. Their reported use of CAS was consistent with their actual use of it. Class A made more use of symbolic algebra for symbolic differentiation on the PRT than Class B who rarely used it (see Section 7.1.2).

7.3 Student Evaluation of TI-92

At the conclusion of the final calculus lesson, each student wrote down comments about the ways she had used the TI-92. These student evaluations of the TI-92 (Appendix 9.1) are reported in this section. Key statements prompted individual student responses some of which are reported below and are collated in Table 7.9. It shows the percentage of students in each class who made the particular type of comment and differences greater than 10% are highlighted.

On average, Class A students made eight written comments per student and Class B six. These comments show that Class A students had a much greater experience with using the CAS calculator, were more realistic about its capabilities, and were far more aware of the advantages and disadvantages associated with using it. For example, Susan and Julie wrote about things they felt were great about the TI-92.

Susan: “In ways it is easier and harder to use than the TI-83 [the graphics calculator normally used]. The TI-83 does not have as much buttons and things on it, therefore it is not as confusing as the TI-92 but the TI-92 has many things that you can’t do on the TI-83.”

Julie: “Does more things than other calculators. It can do everything →
but is really only useful if you can understand it in the first place.”

In contrast, comments by Anna and Louise indicate Class B’s inexperience with using the calculators and unrealistic beliefs about its capabilities.

Anna: “It does hard problems for you.”

Louise: “It has so many programs to do nearly anything.”
Commenting on the mathematics that the TI-92 helped them to learn Class A students, mentioned learning about differentiation mostly through functional use of the calculator (see Caitlin and Angela’s comments). In addition, several students such as Susan and Nanette showed that the TI-92 also helped them with understanding.

Caitlin “Differentiation → cheating answers, learning new techniques such as d(  

Angela: “Derivatives – the equation and function d(\(x^2, x\)) / x = *”

“Using the TI-92 to find tangents and gradients of tangents.”

Susan: “Finding the derivative i.e., gradient/tangent at a point or between two points”

Nanette: It helped really to understand questions to a deeper extent, to visualize graphs and points and also gradient points on the graph.

When commenting about the mathematics the TI-92 helped them to learn, most Class B students, including Maureen and Elizabeth, appreciated the graphical capabilities of the TI-92 and several students such as Steph nominated its ability to find derivatives. However, a few students such as Maggie recognized other ways that the TI-92 could help her learn calculus.

Maureen: “The whole topic of calculus, especially working out curve sketching and of drawing a graph.”

Elizabeth: “Calculus and understanding of graphs and how they all relate to each other.”

Steph: “Calculus – how to find the derivatives of equations.”

Maggie: “Derivatives and tangent – related to gradient at a point.”

Most Class A students had a realistic understanding of the difficulties involved in learning to use the TI-92 effectively. Julie and Caitlin’s suggestions to make the TI-92 more useful in mathematics classes reflect this.

Julie: “Maybe, using it all the time, and not just for certain topics (or not at all) because it takes a lot of practise to get used to.”

Caitlin: “If it was taught throughout the years of your schooling.”
More Class B students than from Class A made superficial suggestions to make the TI-92 more useful in class. Comments by Hannah and Sam illustrate this. Emma, however, made a more thoughtful comment that indicated that TI-92 was not fully used in Class B’s lessons.

Hannah: “Make it more compact. The letter keyboard is not needed. The screen is a good size though.”

Sam: “It would probably be better if it was smaller, more portable.”

Emma: “More extensive lessons on what they can be used for. A clearer explanation of its capacity and the ways in which this can be maximized.”

The comments by all of the students from both classes were categorized and collated into Table 7.9.

Table 7.9 shows that a higher percentage of Class A students:

- Indicated that the TI-92 had helped them learn mathematics related to differentiation procedures including numerical, graphical, and symbolic differentiation, understanding calculus concepts, and solving calculus problems, in eight different categories.
- Stated that the TI-92 was useful for checking answers, making maths easier, constructing tables and for other unspecified purposes.
- Had a realistic appreciation of the capabilities of the TI-92.
- Experienced problems while learning how to use the TI-92 and while using it (indicative of use).
- Suggested that the TI-92 would be more useful in mathematics classes if they learnt how to use it properly.

Table 7.9 also shows that a higher percentage of Class B students:

- Indicated that the TI-92 had helped them learn “calculus” (in general).
- Stated that the TI-92 was beneficial for graphing.
- Made unrealistic claims about the capabilities of the TI-92.
- Found fault with the calculator.
- Suggested that the TI-92 would be more useful in mathematics classes if the design of the calculator were improved.
Table 7.9 Percentage of Students in each Class Who Made Each Type of Comment on the Evaluation of the TI-92 CAS Calculator

<table>
<thead>
<tr>
<th>Type of comment</th>
<th>Class A (N=14)</th>
<th>Class B (N=19)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mathematics learnt while using the TI-92</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Calculus (in general)</td>
<td>7</td>
<td>37</td>
</tr>
<tr>
<td>• Symbolic differentiation</td>
<td>71</td>
<td>42</td>
</tr>
<tr>
<td>• Graphical differentiation</td>
<td>57</td>
<td>16</td>
</tr>
<tr>
<td>• Numerical differentiation</td>
<td>29</td>
<td>5</td>
</tr>
<tr>
<td>• Linked pairs of derivatives</td>
<td>29</td>
<td>11</td>
</tr>
<tr>
<td>• Understand calculus</td>
<td>21</td>
<td>0</td>
</tr>
<tr>
<td>• Understand calculus concepts</td>
<td>21</td>
<td>0</td>
</tr>
<tr>
<td>• Visualization</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>• Solve calculus problems</td>
<td>21</td>
<td>11</td>
</tr>
<tr>
<td><strong>Benefits of using the TI-92</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Saves time</td>
<td>7</td>
<td>16</td>
</tr>
<tr>
<td>• Checks answers</td>
<td>36</td>
<td>5</td>
</tr>
<tr>
<td>• Makes maths easier than by-hand, in head, or on TI-92</td>
<td>43</td>
<td>32</td>
</tr>
<tr>
<td>• Useful for algebra</td>
<td>21</td>
<td>26</td>
</tr>
<tr>
<td>• Useful for graphs</td>
<td>43</td>
<td>53</td>
</tr>
<tr>
<td>• Useful for tables</td>
<td>21</td>
<td>0</td>
</tr>
<tr>
<td>• Has additional useful capabilities</td>
<td>79</td>
<td>47</td>
</tr>
<tr>
<td><strong>Beliefs about the capabilities of the TI-92</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Realistic beliefs</td>
<td>14</td>
<td>5</td>
</tr>
<tr>
<td>• Unrealistic beliefs</td>
<td>0</td>
<td>37</td>
</tr>
<tr>
<td><strong>Disadvantages of using the TI-92</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Problems with the calculator</td>
<td>57</td>
<td>84</td>
</tr>
<tr>
<td>• Problems encountered learning how to use the calculator</td>
<td>64</td>
<td>53</td>
</tr>
<tr>
<td>• Practical problems encountered while using the calculator</td>
<td>64</td>
<td>37</td>
</tr>
<tr>
<td>• Criticism of learning with TI-92</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>• Preference for TI-92 or by-hand</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Preference for symbolic algebra or by-hand</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Preferred CAS to by-hand</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>• Preferred by-hand to CAS</td>
<td>21</td>
<td>16</td>
</tr>
<tr>
<td><strong>Suggestions to make the TI-92 more useful in mathematics classes</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Need to learn how to use it properly</td>
<td>57</td>
<td>21</td>
</tr>
<tr>
<td>• Need to use it all the time</td>
<td>14</td>
<td>11</td>
</tr>
<tr>
<td>• Needs to be smaller and easier to use</td>
<td>0</td>
<td>37</td>
</tr>
</tbody>
</table>

This section shows that the students in both classes had a similar attitude towards the importance of by-hand algebra for learning about differentiation and the importance of graphing. It also shows that Class A had a wider experience of using CAS for algebra including differentiation.
7.4 Additional Information From Student Interviews

Although a detailed overall analysis of the fifteen individual student interviews is not included in this report, additional information from the student interviews is reported in this section.

The purpose of the student interviews (Appendix 7.1) was to observe how the students used the CAS calculator on a range of essentially symbolic items and to determine their preference for, and knowledge of, different representations. The results of the student interviews are reported in Appendix 7.3 where class behaviour on each interview question is described. The student interviews confirmed class achievement of seven of the DCT competencies and confirmed the results of the PRT with respect to student use of symbolic algebra, influence of data on choice of representation, and knowledge of two representations. However, as reported in Section 4.6.3, the students from each class who participated in the interviews were very different in mathematical ability. Most of the Class A students had been high achievers on prior school tests while most of the students from Class B were middle and low achievers. This made meaningful overall comparison of the class achievement on the interview items difficult and, as it did not add new information to the data already reported, will not be reported in detail in this thesis.

The student interviews did however confirm the observations about teaching calculus with CAS. For example, they confirmed that the students solved the maximum/minimum problem (Question 5, Appendix 7.1) similarly to ways they had been taught (see Lesson 2, reported in Section 5.2.1). All the students from both classes approached the problem symbolically differentiating \( P = 300n - n^3 \) by hand. Next, two of the six Class A students used CAS to factorize the resultant derivative while five of the nine Class B students used a graph (on the calculator, or sketched, or modelled in the air). One Class B student drew the graph of \( P \) on the TI-92 and explained “I have to find that point, turning point. . . The gradient of the tangent is zero because it’s a straight line.” Four of the Class B students attempted to justify their algebra by using graphs to show that the gradient of the tangent to the curve was zero at the turning point. In contrast, one Class A student used a sign diagram to justify her method. These results mirror the ways that Teacher A taught Class A and Teacher B taught Class B to solve the maximum/minimum problem during lessons. Thus, the student interviews
were useful to confirm student learning outcomes and the observations about teaching calculus with CAS.

The sections above have reported data from a wide variety of sources that gives a very full picture of learning and teaching in both classes. The varieties of data from the students supported the results of the tests and observations about teaching. The results are summarized and discussed in Section 7.6 after a discussion, in the next section, that addresses the adequacy of the methodology for studying student learning.

### 7.5 Limitations of the Study

This section addresses two aspects of the methodology: appropriateness of the Differentiation Competency Framework (Section 7.5.1) and limitations of the DCT (Section 7.5.2).

#### 7.5.1 Appropriateness of Differentiation Competency Framework

The Differentiation Competency Framework was designed to complement a calculus curriculum that focused on developing understanding of the concept of derivative using multiple representations. The Framework involved the essential representations in a course of introductory calculus for students beginning to learn calculus: numerical, graphical, symbolic representations of derivative. As discussed in Section 3.2.2, the Framework did not directly include the physical representation and it was considered as numerical. Nor did it include enactive representations or visualization, used by the teachers while teaching. Limited emphasis was given to the notion of limits and teaching “first principles” was deferred in this school until the students were more experienced in calculus.

The Differentiation Competency Framework proved to be an effective, hierarchical, structure that monitored students’ knowledge of how to differentiate using numerical, graphical, and symbolic representations of derivative, understanding of each representation of derivative, knowledge of how to differentiate using numerical, graphical, and symbolic representations of derivative to find a derivative in a different representation, and understanding how to reinterpret derivatives in one representation into a different representation.
Adopting the Framework enabled a comprehensive approach to learning to use all three representations. It assisted with identification of competencies that are rarely part of teaching and rarely assessed, especially if the testing is mainly directed at determining symbolic derivatives. The Framework also provides a working definition of some of the goals of teaching calculus. For example, students’ ability to move flexibly between representations was assessed by comparing competence on pairs of inverse translations.

Overall though, the Differentiation Competency Framework proved to be rather robust, led to interesting results, and captured the main features of the current school curriculum well. It also facilitated critical observation of the teachers’ teaching emphasis on developing each aspect of the Differentiation Competency Framework and showed the potential to be expanded by combining competencies to yield more difficult challenging items.

### 7.5.2 Suitability of Differentiation Competency Test

In this study, the DCT was the main indicator of learning. It was designed to give the students the opportunity to demonstrate each competency in the Framework (based on key aspects of the calculus curriculum). The DCT was essentially a conceptual test designed to test formulation and interpretation of numerical, graphical, and symbolic representations of derivative and the translations between them in a comprehensive and balanced manner, giving the three representations equal weighting. In general, the DCT did not test procedures for differentiation that could easily be performed using CAS (see Section 4.6.1). It assessed student behaviours related to differentiation that are fully described in Section 3.4.

The DCT is a tool for early calculus learning since it distinguishes different meanings that are soon integrated with experience of calculus. (For example, while solving problems, a rate of change is very quickly thought of as symbolic if an algebraic differentiation is used. The underlying translation between the numerical and symbolic representations has become “invisible”.)

A range of difficulties was encountered in the construction of DCT questions. First, the question designers who were calculus “experts” had a thoroughly integrated knowledge of differentiation and an almost unconscious awareness of the interwoven
nature of the representations of derivative and had to pay particular attention to using the most appropriate language for each individual representation. (For example, it was very easy to perceive a rate of change function as symbolic and the coordinates of points on a curve as numerical).

Second, it was intended that each DCT question would target only one competency. However, on three of the eighteen questions with symbolic function data (A Q.3, A Q.5, A Q.8, see Appendix 1.2) demonstration of more than one competency was theoretically possible. For example, A Q.3 has three viable solutions (see Sample 8, Section 4.6.2) including the targeted competency [FSG]. Appendix 1.2 shows that 6% of responses were [FNG] and 5% were [FGG]. Students were awarded these competencies (using the established criteria listed in Section 4.6.1) if they demonstrated them even on questions that did not specifically target these competencies. In consequence, the competency [FNG] was awarded, unplanned, to four (29%) Class A students. This problem in the design of DCT questions did not unduly affect the effectiveness of the DCT. In fact, it revealed that Class B made maximum use of symbolic data for symbolic differentiation whereas Class A students knew how to use symbolic function data to differentiate numerically.

The DCT questions were primarily focused on finding the derivative “at a point”. Having now used the DCT once, an improved set of items with sharper focus has been constructed (see Appendix 10.1) including both “at a point” and “at all points” function-based questions where possible. Even’s (1998) study into students’ knowledge of representations suggests that the combination of the two methods is likely to be most powerful for student learning. Designing even more challenging questions that combine more than one DCT competency is possible. The new questions could involve two or more processes and representations. One of these was used in the PRT and it proved to be more difficult than individual competencies for all of the students.

Third, DCT questions did not test (in general) students’ capabilities to actually use the symbolic algebra facility of CAS. However, the second test, the PRT provided the opportunity to see how the students used symbolic algebra under test conditions, and to see if it related to class use.

Fourth, each Framework competency was tested only once on the DCT. As mentioned in Section 4.6.1, more school time was not available for repeat testing.
However, the following competencies were retested on the PRT: two of the three formulation-without-translation competencies, \([FSs]\) and \([FGg]\); five of the six of formulation-with-translation competencies, \([FNg]\), \([FNs]\), \([FGs]\), \([FSn]\), and \([FNg]\); and one of the three interpretation-without-translation competencies, \([INn]\). The class achievement on each these competencies was consistent with the achievement shown on the DCT except for \([FSn]\) and \([INn]\). On the competency \([FSn]\) Class B achieved 50% on the PRT but only 37% on the DCT. The question involved is (A, 8(d)). The PRT showed that Class B was highly susceptible to the context of questions and the prior parts of the question 8(a), (b), (c) served to distract the students. Class A was not similarly distracted and achieved 36% on both items. As discussed above, for the \([INn]\) competency Class A achieved 71% on the PRT with discrete data compared 43% on the DCT with rate of change function data reflecting Teacher A’s particular emphasis on “at a point”, while Class B achieved similarly on both items (56%, 63%) reflecting Teacher B’s emphasis on understanding.

In conclusion, overall, the DCT successfully enabled identification of the types of competencies acquired by each class with respect to each category (process by representation) and it facilitated comparison between the two classes.

7.6 Summary and Discussion
This chapter reported on further learning outcomes based on the PRT, the CAS and Calculus questionnaire, the evaluation of the TI-92, and student interviews. The PRT showed that Class A used symbolic algebra more than Class B and the questionnaire, evaluation of the TI-92, and student interviews verified this by showing that Class A students were much more knowledgeable about using symbolic algebra and CAS and about its limitations. The PRT showed that neither class used CAS for symbolic algebra particularly effectively. This occurred because the teachers did not expose the students to a sufficiently wide variety of functions and the students were not able to identify functions for which CAS use would be highly appropriate and beneficial to successfully performing the symbolic procedure. The teachers were apprehensive that their students would come to rely on CAS when it would not be available for future official school examinations.
The PRT confirmed class achievement on the formulation competencies on the DCT and also differences. It confirmed the differences between classes that were observed in the DCT such as Class A’s developing flexibility between the N and G representations.

In addition, the PRT gave additional insights into the influence of data on the competencies each class achieved were obtained. Class B was overly responsive to function data for symbolic differentiation although it could be used for graphical differentiation particularly in the presence of a graph. Class A also preferred symbolic data for symbolic differentiation, could also differentiate graphically, and most significantly numerically. Overall, Class A was more discriminating in its choice of representation to suit the limitations of particular problems.

These further results indicate the complexity of working with functions and derivatives and support Even’s (1998) conclusion that “knowledge about different representations is not independent, but is interconnected with knowledge about different approaches to functions, knowledge about the context of the presentation, and knowledge of underlying notions” (p. 120). In this study some of this complexity is associated with the different ways the teachers taught the students and this chapter has given explicit insights into differences between the classes. The PRT, student interviews, and the questionnaire confirmed the classroom observations of the teaching with CAS. In Chapter 8 what the students learnt about the concept of derivative will be related to the teaching actions of their teachers using a CAS supported curriculum and the impact of teacher privileging on learning will be determined and then evaluated in Chapter 9.
8. THE IMPACT OF TEACHING ON LEARNING

In this chapter, the links between teaching and learning are established. What is known about teaching is reported in Chapter 5 and what is known about student understanding of the concept of differentiation is reported in Chapter 6 (summarized in Section 6.8) and supported by the results in Chapter 7 (summarized in Section 7.6).

Teacher actions, with and without use of CAS, are identified in relation to the calculus being taught and are shown to correspond directly with class achievement of individual Framework competencies on the DCT (Section 8.1). This enables class achievement on different categories of Framework competencies to be related to teaching actions and the links between teaching and class learning about the concept of derivative to be determined. Research Question 1 clarifies the effect of including CAS in the curriculum (Section 8.2) and Research Question 2 explores the impact of teacher privileging on students’ learning (Section 8.3). Finally, the implications of exploring the Research Questions (including Research Question 3) are summarized (Section 8.4). The overall impact of teaching on learning is assessed in Chapter 9.

8.1 Teaching Actions’ Impact on Class Achievement of Individual Competencies

This section illustrates how teaching actions relate to class achievement of individual competencies. First, teaching actions are classified with respect to their purpose and the representations involved (Section 8.1.1). Second, examples show how class achievement on competencies involving particular representations correspond to particular teaching actions (Section 8.1.2).

8.1.1 Classification of teaching actions

This section classifies the teachers’ teaching actions with respect to how they taught the calculus content in the curriculum. Specifically, their teaching actions are identified in relation to each pair of representations in the Framework and the CAS activities they elected to use.

Organization of calculus content

Understanding of the concept of derivative, involves achievement of up to eighteen Framework competencies tested on the DCT. To facilitate analysis of the teaching
actions, the competencies with a similar teaching focus were paired (making nine pairs). The paired competencies have identical input and output representations and one of the pair involves a formulation process \((F)\) and the other an interpretation \((I)\) processes. For example, the pair \([FNs], [INs]\) have an input representation of \(N\) and an output representation of \(S\). Three pairs involve a single representation only: \([FNN], [INn]; [FGg], [IGg]; \) and \([FSs], [ISs]\). The other six pairs involve a translation between representations. For example, the pairs \([FNg], [INg]\) and \([FGn], [IGn]\) involve the \(G\) and \(N\) representations. Similarly, \([FNs], [INs]\) and \([FSn], [ISn]\) involve the \(N\) and \(S\) representations while \([FGs], [IGs]\) and \([FSg], [ISg]\) involve the \(S\) and \(G\) representations.

**Classification of teaching actions**

Two types of teaching actions were observed during the main study. They reflect each teacher’s personal manner and style of teaching introductory differentiation:

- A “doing” action emphasizes differentiation procedures in a single representation or when linking a pair of representations.
- An “understanding” action focuses on understanding a derivative or the relationship between pairs of derivatives in different representations.

The teachers used these teaching actions in conjunction with the functional CAS activities and some of the pedagogical CAS activities from the curriculum, and on other occasions when CAS was not involved in the teaching.

**Classification of teaching actions associated with particular calculus content**

As the teachers taught the calculus content they paid attention to pairs of representations and, on occasions, they elected to use a particular CAS activity to support their particular emphasis. Sometimes they preferred by-hand procedures. Thus, the teachers’ “doing” and “understanding” teaching actions were associated with particular pairs of representations that related to particular CAS activities (or to by-hand techniques). The CAS activities that were planned to be functional are numbered 1 to 4 (as in Figure 4.1) and those planned to be pedagogical from 5 to 10 (as in Figure 4.2), in *italics* to distinguish them from the functional CAS activities.

Figure 8.1 displays Teacher A and Teacher B’s teaching actions while they taught the calculus curriculum. While the author reviewed the original observations of individual lessons to establish the teachers’ precise use of CAS activities (see Appendix
4.6) she identified the two different types of teaching actions. She realized that the teachers did not always use the CAS activities in the ways that were planned in the curriculum guidelines. In particular, she realized that the teachers did not always use "understanding" actions while using the pedagogical activities. Subsequently, the teachers’ teaching actions while using CAS activities (and otherwise) were identified from the original observations of lessons. Figure 8.1 was gradually developed to report the teachers’ teaching actions in relation to particular pairs of representations (see above) and the related CAS activities where applicable. Initially, the representations involved in particular CAS activities were identified. For example, CAS Activity 1 involves the N representation while CAS Activities 6 and 7 involve the N and G representations. Next, the teachers’ teaching actions in relation to each pair of representations were identified and included in Figure 8.1. Finally, Class A’s and Class B’s class achievement of competencies (from Table 6.1) were incorporated into Figure 8.1. Their inclusion enabled the impact of teaching actions on class achievement of competencies to be examined.

For example, in Figure 8.1, consider the first pair of Framework competencies \([F_{Nn}]\) and \([I_{Nn}]\) that involve only the N representation. Figure 8.1 shows that Teacher A’s "doing" actions were to emphasize numerical differentiation procedures. Firstly, he showed the students how to calculate difference quotients using functional CAS Activity 1 and pedagogical CAS Activity 6. Secondly, he emphasized the calculation of average rates of change in association with use of pedagogical CAS Activity 7. He showed the students how to use three different types of data to calculate difference quotients: (i) coordinates from curves or graphs, (ii) function values generated from functions, and (iii) function values. Teacher B’s "doing" actions included calculating difference quotients (N differentiation using CAS Activity 1) and emphasizing average rates of change, associated with CAS Activity 7. His "understanding" action was to conduct a class discussion about rates of change in realistic contexts. Figure 8.1 also shows that Question 19(a) and Question 9 from the DCT tested these competencies. Class A achieved 50% on \([F_{Nn}]\) and Class B achieved 43%. On \([I_{Nn}]\), Class A achieved 58% and Class B achieved 63%.

As a second example, consider the third pair of competencies, \([F_{Ss}]\) and \([I_{Ss}]\). Both teachers used "doing" actions for S differentiation. Teacher A encouraged use of CAS
for CAS Activity 3 while Teacher B preferred by-hand techniques except when using pedagogical CAS Activity 8 when he expected the students to use CAS Activity 3. Both teachers used "understanding" actions in conjunction with pedagogical CAS Activity 8. Teacher A quickly explored the patterns of derivatives while Teacher B conducted a class discussion that enabled the students to "discover" the rules to differentiate polynomial functions and then relate them to realistic contexts (see Sample Lesson 1, Section 5.2.1). Both classes achieved 100% on the [FSs] competency. On [ISs], Class A achieved 43% and Class B 47%.

Figure 8.1 shows generally, that where teaching was provided, class achievement was higher. One obvious example is that both Teachers A and B devoted time to teaching [FGg] and [JGg] and these competencies were understood by Class A and Class B. Where teaching was not provided the classes failed to succeed (e.g., on [FGn], [JGn] and [FNs] and [INs]). Figure 8.1 also shows that the teachers used "doing" actions in association with functional CAS Activities 1, 2, and 3 and also in association with pedagogical CAS Activities 1, 2, and 3 and also in association with pedagogical CAS Activities 6, 7, and 8. Figure 8.1 also shows that sometimes the teachers used "understanding" actions (as anticipated) in association with pedagogical CAS Activities 5, 6, and 8, and that sometimes the teachers’ teaching actions did not involve any use of CAS.

This section has classified teaching actions with respect to the ways the calculus content was taught and identified their use with respect to functional CAS activities, pedagogical CAS activities, and non-use of CAS activities. This organization enables particular teaching actions (with and without CAS activities) to be matched to class achievement of Framework competencies involving the same pair of representations and allows the correspondence of teaching actions and class achievement of particular competencies to be judged.
<table>
<thead>
<tr>
<th>Class</th>
<th>Class Achievement (%)</th>
<th>Location on DCT (Test A)</th>
<th>&quot;Doing&quot;</th>
<th>&quot;Understanding&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Doing actions give emphasis to differentiation procedures, in a single representation, or when linking two representations</td>
<td>Understanding actions give emphasis to understanding derivatives, singly, or to the relationship between derivatives in different representations</td>
</tr>
<tr>
<td>F/Nn</td>
<td>Q.19(a)</td>
<td>50</td>
<td>43</td>
<td>Emphasised N differentiation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Calculated difference quotients</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Calculated average rates of change</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Emphasised data types (i), (ii), &amp; (iii)</td>
</tr>
<tr>
<td>B</td>
<td>58</td>
<td>63</td>
<td></td>
<td>Conducted class discussion about rates of change in realistic contexts</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(to understand N derivative)</td>
</tr>
<tr>
<td>F/Gg</td>
<td>Q.5</td>
<td>50</td>
<td>79</td>
<td>Emphasised G differentiation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Demonstrated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Emphasised and explained G differentiation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 2 and used</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• ‘by-hand’ procedures</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Demonstrated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Conducted class discussion relating slope to gradient of curve using enactive representation for tangent</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(to understand G derivative)</td>
</tr>
<tr>
<td>F/Ss</td>
<td>Q.1</td>
<td>100</td>
<td>43</td>
<td>Encouraged S differentiation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Quickly examined patterns of derivatives to develop algebraic rules for differentiation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(to understand S derivative)</td>
</tr>
<tr>
<td>B</td>
<td>100</td>
<td>47</td>
<td></td>
<td>Demonstrated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 3 but preferred</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• ‘by-hand’ procedures</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Expected use of</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 3 during</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Thorough discussion of patterns of derivatives to develop algebraic rules for differentiation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Related rules to realistic contexts</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(to understand S derivative)</td>
</tr>
<tr>
<td>F/Ng</td>
<td>Q.2</td>
<td>57</td>
<td>14</td>
<td>Repeatedly demonstrated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>to generate data type 1(ii) for N differentiation (difference quotient)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 1 to find the gradient of the tangent (linking N-G)</td>
</tr>
<tr>
<td>B</td>
<td>37</td>
<td>16</td>
<td></td>
<td>Demonstrated</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 7</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>to generate data type 1(i) for</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>• CAS Activity 1 (av r of change)</td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>Q.19(b)</th>
<th>Q.13</th>
<th>Repeated</th>
<th>CAS Activity 6 (indirectly linking G-N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 7</td>
<td>21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B 0</td>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FGs</th>
<th>KGs</th>
<th>Emphasized G differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.6</td>
<td>Q.17(a)</td>
<td>CAS Activity 2 to find a derivative (linking G-S)</td>
</tr>
<tr>
<td>A 64</td>
<td>64</td>
<td></td>
</tr>
<tr>
<td>B 42</td>
<td>68</td>
<td>Conducted class discussion linking gradients of tangents to derivatives using visualization and enactive representations (to understand the G-S link)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FSg</th>
<th>Sg</th>
<th>Emphasized S differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.3</td>
<td>Q.14</td>
<td>CAS Activity 3 and by-hand to find a gradient (linking S-G)</td>
</tr>
<tr>
<td>A 50</td>
<td>36</td>
<td>Class discussion where students were encouraged to visualize the derivative as the gradient of tangent at a point, using an enactive representation (to understand S-G link)</td>
</tr>
<tr>
<td>B 73</td>
<td>53</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FNs</th>
<th>Ns</th>
<th>Demonstrated N differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.11</td>
<td>Q.16</td>
<td>CAS Activity 1 to find a derivative (linking N-S)</td>
</tr>
<tr>
<td>A 43</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>B 21</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>FSn</th>
<th>Sn</th>
<th>Demonstrated S differentiation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.8(d)</td>
<td>Q.10</td>
<td>CAS Activity 3 and by-hand to find rate of change (linking S-N)</td>
</tr>
<tr>
<td>A 36</td>
<td>64</td>
<td>Conducted class discussion of derivatives as real world rates of change (to understand S-N link)</td>
</tr>
<tr>
<td>B 37</td>
<td>74</td>
<td></td>
</tr>
</tbody>
</table>

Class achievement is the percentage of students in each class who successfully demonstrated the competency (see Section 4.6.1). CAS Activity 1 Data type (i) using coordinates from curves or graphs Data type (ii) using function values generated from a function Data type (iii) using given function values "By-hand" indicates that the functional procedures were performed without CAS. CAS activities used by the teachers and intended as pedagogical are indicated by an italicised font whereas CAS activities intended as functional are not italicised.

Figure 8.1. Teaching actions matched to class achievement (%) on competencies related to pairs of representations.
8.1.2 Teaching actions’ impact on class achievement of individual competencies

This section illustrates how the teachers’ teaching actions correspond to class achievement of individual competencies. Using the data about teaching and learning collated in Figure 8.1, two examples show how class achievement on particular competencies correspond to a particular teaching action or actions associated with the same representations.

First, consider the graphical competencies $[FGg]$ and $[IGg]$ involving only the G representation. Figure 8.1 shows that both teachers used functional CAS Activity 2 to show their students how to differentiate graphically, a “doing” action, and each class understood (achieved at least 50%) the formulation-without-translation competency $[FGg]$ with similar class achievement. Figure 8.1 also shows that both classes were more successful on the interpretation-without-translation competency $[IGg]$. Both teachers demonstrated pedagogical CAS Activity 6 and, in addition, Teacher A zoomed-in on the curve (pedagogical CAS Activity 5) drawing attention to the local linearity of curves, an “understanding” action. However, Teacher B’s “understanding” action, a classroom discussion that emphasized the equivalence of the slope of the curve to the gradient of tangent to the curve including use of an enactive representation (arms spread out to represent the tangent line) seems to have contributed to Class B’s superior achievement on $[IGg]$.

Second, consider the competencies $[FNg]$ and $[INg]$ involving the N and G representations. Figure 8.1 shows that only Class A understood the formulation-without-translation competency $[FNg]$ and that Teacher A’s teaching actions seem to have contributed to Class A’s superior achievement of this competency. While repeating CAS Activity 6, he calculated difference quotients, a “doing” action using CAS Activity 1 using data type (ii) to find gradients, thereby emphasizing the intuitive link between the N and G representations. He also demonstrated the walking program, pedagogical CAS Activity 7, using it to link the N and G representations. In contrast, Teacher B demonstrated CAS Activities 6 and 7 once and then focused on calculating average rates of change associated with Activity 7 using Data type (i). Figure 8.1 also shows that both classes achieved poorly on the interpretation-without-translation competency $[INg]$. 

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seemingly because neither teacher employed teaching actions that helped students succeed on this rarely needed competency.

In this section, the likely impact of particular teaching actions on class achievement of four competencies has been demonstrated. "Doing" teaching actions were more likely to be associated with class achievement of formulation competencies and "understanding" teaching actions were more likely to result in class achievement of interpretation competencies. However, the actual competencies achieved were dependent on the representations involved. Class success on the other Framework competencies, and on groups of competencies, is similarly related to the teaching actions involving particular representations. Thus, teacher privileging, manifest through personal teaching actions on representations, in association with either functional or pedagogical CAS activities or without CAS involvement, appeared to impact on each class’s understanding of the concept of derivative.

8.2 Exploration of the Role of CAS in Assisting Students Develop Understanding of the Concept of Derivative

This section addresses Research Question 1. It monitors the provision of CAS activities in the curriculum, their use by the teachers, and their likely impact on learning. This is achieved by combining what has been learned while exploring Research Questions 1(a), 1(b), and 1(c) in Section 8.2.4. Research Question 1(a) investigates how the teachers used the functional and pedagogical CAS activities (Section 8.2.1). Research Question 1(b) links teacher use of CAS activities with class understanding of different categories of competencies (Section 8.2.2) and Research Question 1(c) explores the effect on learning of greater use of symbolic algebra (Section 8.2.3).

8.2.1 Purpose for teacher use of CAS activities

This section addresses Research Question 1(a). How did the teachers incorporate functional and pedagogical CAS activities into their personal teaching programs? It reports how the teachers used the four functional and six pedagogical CAS activities provided in the curriculum (see Figures 4.1 & 4.2).

Figure 8.1 shows that between them the teachers used seven of the ten CAS activities provided in the curriculum. Teacher A used three of the four functional CAS activities (Activities 1, 2, 3) and four pedagogical CAS activities (Activities 5, 6, 7, & 8) in
association with the nine pairs of representations (N-N, G-G, S-S, N-G, G-N, G-S, S-G, N-S, S-N) while Teacher B used functional CAS Activities 1, 2, and 3 (for investigative purposes) and three pedagogical CAS activities (Activities 6, 7, & 8) on four pairs of representations (N-N, G-G, S-S, N-G). Neither teacher used the variations of CAS activities possible with Activities 8, 9, and 10 (see Figure 4.2) to assist students understand other pairs of representations, particularly (N & S) or (G & S).

In summary, functional CAS activities were used for “doing” differentiation in each representation, sometimes in conjunction with linking representations while the pedagogical CAS activities were used to emphasize “understanding” of graphical and symbolic derivatives or as a stimulus for “doing” numerical differentiation and linking it with gradient (N-G). The teachers did not use pedagogical activities to link other representations.

Exploration of Research Question 1 indicates that, overall, Teachers A and B made limited use of the CAS activities provided in the curriculum and that neither initiated more creative use of CAS. Teacher A enjoyed using the CAS calculator and used more CAS activities than Teacher B. He used functional CAS activities to emphasize procedures and pedagogical CAS activities to emphasize procedures while linking different representations, and to understand derivatives. In contrast, Teacher B made limited use of both functional and pedagogical CAS activities, preferring his students to carry out procedures by-hand. He orchestrated class discussions to help students understand derivatives and the link between derivatives.

### 8.2.2 CAS activities associated with understanding the concept of derivative

This section addresses Research Question 1(b). What types of CAS activities promote understanding of the concept of derivative? It indicates how teacher usage of functional and pedagogical CAS activities in the curriculum relates to student learning of each different category of Framework competencies and also shows learning that relates to teaching without CAS.

**Functional CAS activities**

Figure 8.1 shows the teachers used functional CAS activities for two categories of competencies. First, both teachers used differentiation procedures to teach formulation-
without-translation competencies. Accompanied by “doing” teaching actions, Teacher A and Teacher B used functional CAS Activities 1, 2, and 3 (and they also used by-hand techniques), and Class A and Class B students demonstrated that they understood the related formulation-without-translation competencies \([FNn]\), \([FGg]\), and \([FSs]\). In addition, the same differentiation procedures were used in association with formulation-with-translation competencies. Using the context of pedagogical CAS Activity 6 Teacher A showed the students how to calculate a numerical derivative (using functional CAS Activity 1) in order to find a gradient and only Class A understood the competency \([FNg]\). Teacher A also stressed finding a graphical derivative (using functional CAS Activity 2) in order to find a derivative and only Class A understood the competency \([FGs]\). Teacher A and Teacher B both used a symbolic derivative in order to find a gradient and both classes understood the competency \([FSg]\). However, Class B’s performance was superior, possibly due to Teacher B’s additional “understanding” teaching action of conducting a class discussion (see Table 8.1). It is worth noting that some neither class understood \([FGn]\), \([FNs]\), \([FSn]\).

Thus, use of functional CAS activities by both teachers using “doing” teaching actions was associated with both classes learning three formulation-without-translation competencies. In addition, some of the formulation-with-translation competencies were achieved in association with use of functional CAS activities. However, both teachers also taught the differentiation procedures using by-hand techniques, particularly Teacher B.

**Pedagogical CAS Activities**

Figure 8.1 shows that the teachers used pedagogical activities in two ways. They used them with “doing” teaching actions or with “understanding” teaching actions. For example, Teacher A used Activity 5 with an “understanding” teaching action and Class A understood the interpretation-without-translation competency \([IGg]\) and when Teacher A used Activity 6 with a “doing” teaching action, Class A understood the interpretation-with-translation competency \([FNg]\). When both teachers used Activity 8 with both “doing” and “understanding” actions both classes understood the formulation-without-translation competency \([FSs]\). However the students in both classes found the DCT item that tested the \([ISs]\) competency too difficult.
This section has shown that the teachers did not make best use of the pedagogical CAS activities provided in the curriculum. It has also shown that when the teachers did use a CAS activity what the students learned was likely to have been influenced by the type of teaching actions their teachers used.

Non-CAS Activities
Class B understood interpretation-with-translation competencies \([IGs], [ISg], \) and \([ISn]\) and interpretation-without-translation competencies \([INn]\) and \([IGg]\) that seems to relate to Teacher B’s strong emphasis on the S representation and on linking symbolic derivatives with gradients (G representation) and rate of change functions (N representation) during in-depth class discussions that related the ideas to realistic contexts when feasible.

Research Question 1(b) indicates that, in general, teacher use of functional CAS activities with emphasis on “doing” may have assisted some students learn both formulation-without-translation and formulation-with-translation competencies. Teacher use of pedagogical activities with emphasis on “doing” for linking derivatives may have assisted some students to learn formulation-with-translation competencies while teacher emphasis on “understanding” may have assisted some students to learn one interpretation-without-translation competency. However, some competencies were learned (particularly interpretation competencies) supported by teacher actions alone, without use of any CAS activities. Interestingly, CAS activities planned for pedagogical use were sometimes used for functional purposes (Activities 6, 7, & 8) while CAS activities planned for functional use (Activities 1, 2, & 3) appeared to indirectly assist with understanding links between representations.

8.2.3 Effect on learning of use of symbolic algebra
The purpose of this section is to address Research Question 1(c). Was greater use of symbolic algebra advantageous for student learning? It reports provision for use of symbolic algebra within the curriculum, class use of symbolic algebra during lessons, class use of symbolic algebra on PRT items, and class understanding of symbolic derivative.
Provision for use of symbolic algebra within the curriculum

The curriculum was designed so that concepts would be taught prior to problem solving so symbolic algebra was used mainly for early exploratory activities (see CAS Activity 8 & Figure 4.2). As discussed above, Teacher B discouraged use of CAS Activity 3, for symbolic differentiation and both teachers elected not to use CAS Activities 9 and 10 that would involve use of symbolic algebra (see Figure 4.2).

Teacher and class use of symbolic algebra during lessons

Despite the limited need to use it, Teacher A and Class A used symbolic algebra more than Teacher B and Class B. Class A students were observed to use CAS more frequently in lessons than Class B students and both the student questionnaire (Section 7.2) and student evaluation of the TI-92 (Section 7.3) showed that Class A students had a much greater degree of familiarity with symbolic algebra and were more aware of its capabilities and limitations.

The teachers made different uses of symbolic algebra. Teacher A permitted his students to use symbolic algebra at will while Teacher B insisted his students perform symbolic differentiation by-hand except when needed for investigative purposes such as discovering and making sense of the rules for symbolic differentiation (see Sample Lesson 1 in Section 5.2.1 & Figure 5.7).

Class use of symbolic algebra on the PRT

The PRT specifically tested the proficiency of each class on questions involving symbolic differentiation (Section 7.1.2) and the results show that Class A used CAS for symbolic algebra more frequently and made fewer procedural errors than Class B students whose attempts were mostly by-hand with fewer conceptual errors (see Table 7.1). However, Class A’s inappropriate use of CAS was accompanied by more conceptual errors (see Table 7.2). The main error was mistaking finding an equivalent expression for formulating a derivative (see Section 7.1.2). The PRT results also show that neither class made very discriminating use of symbolic algebra possibly due to lack of exposure to problems requiring symbolic algebra during lessons.

Research Question 1(c) suggests that, in this study, using symbolic algebra was not entirely beneficial to student learning. Neither teacher took full advantage of the symbolic capabilities of the CAS calculators so that the students did not gain enough experience to use it strategically. The class that made greatest use of symbolic algebra
was also more prone to making conceptual errors that prevented them from formulating the derivative correctly. On the other hand, its use afforded most students the opportunity to explore and analyse data that had been quickly and accurately generated.

8.2.4 The role of CAS in assisting students develop understanding of the concept of derivative

This section combines and summarizes the conclusions above (Sections 8.2.1, 8.2.2, & 8.2.3) to answer Research Question 1. It outlines the role of CAS in the curriculum in assisting students to develop understanding of the concept of derivative.

While the curriculum provided an adequate range of functional and pedagogical CAS activities to assist students to differentiate in each representation and to understand the links between different representations the teachers failed to make full use of some activities and failed to use others at all. The CAS activities that were used appeared to support learning in the four categories of competencies associated with the concept of derivative but not every competency. Functional CAS activities principally assisted students to formulate derivatives, and the actual competencies understood by each class depended on their teacher’s teaching actions. Overall, the teachers did not take full advantage of the pedagogical CAS activities to help students learn about the concept of derivative and the competencies understood once again depended on teaching actions including some that did not involve use of CAS activities.

8.3 Exploration of the Influence of Teacher-Related Privileging Characteristics on the Learning that Occurred in Each Class

This section addresses Research Question 2. It explains how each teacher’s particular privileging characteristics impacted on the learning of the students in their class. Each teacher’s preference for and use of particular representations of derivative (calculus content), teaching approach, and use of CAS together seemed to influence the categories of competencies learned by each class.

During lessons Teacher A privileged the symbolic representation and he used and linked the graphical and numerical representations. Teacher B showed a strong preference for the symbolic representation, used the graphical representation, and limited his use of the numerical representation. He also linked the symbolic and graphical representations (see Figure 5.9). Class A and Class B understood the symbolic
representation and Class B was more successful. However, Class A and Class B achieved similarly overall on the graphical representation and surprisingly they both achieved poorly on the numerical representation (it was expected Class A would be superior, see Table 6.3). Clearly, class achievements were not entirely dependent on teacher preference for representations and both classes had extremely variable results on different categories of competencies in both the numerical and the graphical representations (see Table 6.5).

Teacher A’s and Teacher B’s teaching approach appeared to influence class achievement of different categories of competencies. Teacher A emphasized rules and procedures while Teacher B promoted student understanding. Both classes understood the formulation-without-translation competencies in the symbolic, numerical, and graphical representations. In addition, Class A understood the formulation-with-translation competencies in the numerical representation and was superior (but with low achievement) in the graphical representation while Class B understood and was superior on the interpretation-without-translation competencies in both the numerical and graphical representations (see Table 6.6).

The combined effects of teacher privileging preference for representations, teaching approach and method, and use of CAS is incorporated into a flow chart (Figure 8.2) that links the cumulative effects of these privileging factors to the learning that occurred in each class on each different category of differentiation competencies. Figure 8.2 shows that Teacher A’s three privileging characteristics influenced what Class A learned about the concept of derivative. He taught the symbolic, graphical, and numerical representations and linked the numerical and graphical representations as discussed above. He used a teacher-centred style emphasizing rules for routine procedures and he focused on the process of formulation in the three representations. As has just been shown (see Section 8.2.2) the greater proportion of Teacher A’s teaching actions were "doing" using both functional and pedagogical CAS activities (see Figure 8.1) that seemed to helped students learn the formulation competencies. In fact, Class A understood formulation competencies across the three representations: three formulation-without-translation competencies, *[FNg], *[FGs], and *[FSg]. The starred competencies indicate superior achievement. On occasions, Teacher A taught for
understanding and Class A achieved one interpretation-without-translation competency $[JGg]$ and two interpretation-with-translation competencies, $[JGs]$ and $[JSn]$. The competencies Class A did not understand were unusual inverse competencies that also related to Teacher A’s lesser emphasis on understanding differentiation processes.

Figure 8.2 also shows Class B’s achievement was influenced by Teacher B’s three privileging characteristics. He taught the symbolic and graphical representations and emphasized the links between them. Using a student-centred style he emphasized understanding of rules for routine procedures and understanding the connections between representations. He largely directed his teaching actions, with minimal CAS involvement, towards ”understanding” that helped students learn the interpretation competencies. Class B achieved mostly on competencies involving the symbolic and graphical representations: two interpretation-without-translation competencies $*[FNN]$ and $*[FGG]$ and three interpretation-with-translation involving the symbolic and graphical representations, $[FGs]$, $*[FSS]$ and $*[FGS]$. In addition, Teacher B taught differentiation procedures by-hand, emphasizing symbolic differentiation and interpreting it as a gradient (graphical representation of derivative). Class B understood the three formulation-without-translation competencies $[FNN]$, $[FGG]$, and $[FSS]$ and the formulation-with-translation $*[FSG]$ that linked the symbolic and graphical representations. The competencies not understood by Class B (including the unusual inverse competencies described above) relate mostly to Teacher B’s failure to emphasize the numerical representation and its link to the graphical and symbolic representations. It is also noteworthy that Teacher B’s approach enabled students in lower attainment groups to become as proficient on as many representations of derivative as students in higher attainment groups in Class A (see Table 6.6).

Thus, the interplay of each teacher’s privileging characteristics (and decision to ignore competencies that were not useful for problem solving) seems to explain the class achievement of each class on particular categories of competencies reported above. For example, although Teachers A and B both privileged the symbolic representation and used the graphical representation (see Figure 5.9) their privileging of teaching approach and use of CAS was different and appears to explain Class A’s superior achievement on formulation-with-translation competencies and Class B’s superior achievement on the interpretation-without-translation mentioned above. These
patterns of success help explain how Class A and Class B achieved similarly overall on the graphical and numerical representation (see Table 6.3). The same pattern of superiority on formulation-with-translation and interpretation-without-translation competencies also occurred for the numerical representation even although Teacher B did not emphasize the representation (see Table 6.6).

Both classes developed some flexibility between the S and G representations seemingly as a consequence of teacher emphasis on the pairs of inverse competencies and additionally Class A developed some flexibility between the N and G representations (see Table 6.7) due to his emphasis on linking the numerical and graphical representations Since Teachers A and B ignored the less useful inverse competencies to link S to N and G to N both classes failed to develop flexibility between the N and G representations.

This section has addressed Research Question 2. It has shown how teacher privileging characteristics appeared to influence what the students learned about differentiation. Their interplay led to "doing" or "understanding" teaching actions (with, and without, linking and with, and without, CAS) and mostly involved preferred representations from the nine pairs accommodated in the calculus curriculum. They resulted in teacher emphasis on particular processes and categories of competencies that were essentially those understood by the students in their classes. The competencies not understood by the students were those not emphasized by their teacher.
Figure 8.2. Effect of teacher privileging on Class A and Class B’s understanding of competencies associated with each category of differentiation.
8.4 Summary

This chapter has provided considerable evidence that the teaching corresponded to the learning of the students. Research Question 1 shows that while CAS had the potential to assist students develop understanding of the concept of derivative its influence seemed to be determined by the ways the teachers elected to use it. Functional CAS activities had the potential to help students learn differentiation procedures in each representation and differentiation that linked representations. Pedagogical CAS activities could assist students to understand derivatives in each representation and to understand the link between different representations of derivative.

The analysis related to Research Question 2 suggests that what the students learned was largely determined by the interplay of the teacher’s particular privileging characteristics. Using their preferred representations (calculus content), the teachers used teaching actions (with and without CAS) that were consistent with their teaching approaches (method and style). In turn, the students in each class generally understood the competencies involving the representations their teachers preferred and the processes and categories of competencies that their teacher stressed.

The analysis related to Research Question 3 (previously discussed in Section 5.4) suggests that two privileging characteristics, teaching approach and use of CAS were stable over two years and the results in this chapter have shown that these two characteristics are indeed very closely linked. Both teachers used teaching actions (with CAS) that were consistent with their teaching approaches. The third characteristic, the calculus content taught, was shown to be less stable during the two years and the teachers were more susceptible to changing this privileging characteristic. In turn, this may have been responsible for changing what the students learned about the concept of derivative.

Thus, the three Research Questions have been addressed and issues arising from them and the effectiveness of teaching introductory calculus with CAS using multiple representations are discussed in Chapter 9.
9. EFFECTIVENESS OF TEACHING CALCULUS WITH CAS

This thesis contributes to knowledge about introducing CAS into secondary school classrooms. It was planned as a study of teaching and learning calculus but it has developed into a broader study of teaching. Results about teaching introductory calculus with CAS and learning about the concept of derivative were reported in Chapters 5, 6, and 7. The dependency of the learning on the teaching was established in Chapter 8 and is discussed in this chapter. What has been discovered during this study about teaching and learning introductory differential calculus with multiple representations involving use of CAS and their wider implications are discussed (Section 9.1) and the conclusions are presented (Section 9.2).

9.1 Discussion

This section reports on the effectiveness of: the curriculum design, the Differentiation Competency Framework, teaching with multiple representations, and teaching with CAS. It also considers the combined impact of teacher privileging characteristics on learning, discusses an effective teaching approach, compares teaching with graphics and CAS calculators, and finally makes recommendations for future teaching with CAS.

Effectiveness of the curriculum

This study has shown that a concepts-first approach curriculum, based on studies by Heid (1988) and Repo (1994), supported the teaching of introductory differential calculus and assisted secondary school students to learn about the concept of derivative. During the lessons, the students used CAS for numerical, graphical, and symbolic differentiation as recommended by Dick (1996) and for a range of pedagogical CAS activities (see Figure 4.2) intended to help students understand the links between different representations of derivative (incorporating visualization and use of dynamic programs as suggested by Tall, 1996; Leinbach, 1996; Kaput, 1998). The curriculum also paid attention to derivatives “at a point” (derivative at a specified x value, instantaneous rate of change, and gradient of curve or tangent to curve at a point) and “at all points” (particularly symbolic derivative functions, and to a lesser extent rates of change and gradient functions).
Effectiveness of the Differentiation Competency Framework and the DCT

The Differentiation Competency Framework developed for this study was compatible with curriculum expectations of learning about the concept of derivative and involved competencies associated with formulating and interpreting derivatives with, and without, translation between external representations of numerical, graphical, and symbolic representations of derivative.

In this study, the teachers’ and students’ use of the three external representations of derivative and translations among them were observed using the Framework that proved to be comprehensive for this early stage in learning about calculus. The Framework facilitated observations of the teachers’ and students’ use of the external representations. It proved to be effective in monitoring the teaching that occurred and it was also extremely useful for inferring what the students understood about the concept of derivative, including formulation and interpretation of derivatives in each representation with-translation and without-translation between representations of derivative.

Behaviour on the DCT (based on the Framework) questions showed that Class A students (on average) were successful on items involving the three representations and on other items that involved translation between different representations (see Table 8.6). In particular, they were adept at knowing when and how to use the data in questions to find numerical, graphical, and symbolic derivatives by carrying out differentiation procedures in these representations. They also knew how to use the resultant derivatives to find derivatives in other representations. In addition, Class A students were also able to reason about a gradient, recognize the equivalence of a symbolic derivative to a gradient, and recognize the equivalence of a rate of change to a symbolic derivative (see Section 6.8). The DCT also showed that Class B’s strengths were mostly associated with the symbolic and graphical representations and the majority involved the interpretation process (see Table 8.6). In particular, Class B students (on average) were adept at knowing when and how to use the data in questions to carry out differentiation procedures in the numerical, graphical, and symbolic representations. They also knew when to carry out symbolic differentiation to find a gradient. In addition, Class B students were able to reason about a gradient. They also recognized the equivalence of a symbolic derivative to a gradient (and vice versa) and the equivalence of a rate of change to a symbolic derivative (see Section 6.8).
The DCT showed that the majority of students in both classes were successful overall on the symbolic representation and nearly half of the students in both classes developed proficiency in at least one other representation. In addition, some of the most capable students from both classes were proficient in all three representations while the least capable were proficient in none. These observations are in line with Tall’s (1996) suggestion that using technology, gifted students move more flexibly between representations than average students who may find it difficult to switch from one representation to another. Krutetskii (1976) first developed this notion of flexibility of reasoning in the gifted.

The Differentiation Competency Framework also allowed the content taught by the teachers to be identified including the emphasis they gave to particular representations. While most competencies in the Framework are often used in problem solving and are normally assessed, others link representations that are rarely used, not required in problem solving, and not normally assessed (such as N to S representations, also G to N). The teachers mostly ignored these particular competencies in their teaching and in consequence the students did not develop flexibility across all representations. However, they did develop some flexibility across those representations emphasized by their teacher. Thus, the goal of teaching all of the eighteen competencies in the Differentiation Competency Framework proved to be unrealistic.

Finally, since the DCT proved to be very effective in identifying individual students’ (and classes’) particular strengths and weaknesses in the early stages of learning about differentiation its future use is recommended and new test items are provided in Appendix 10.1.

Effectiveness of teaching with multiple representations
The benefits of teaching calculus with multiple representations were discussed in Section 2.3.1 and the difficulties encountered in other studies were reported in Section 2.3.2. This study confirms previous research (see Section 2.3.2) that shows that understanding the links between representations is not "easy".

At the beginning of the study it was assumed, from the literature (Section 2.3.1), that teaching students to differentiate using numerical, graphical, and symbolic representations, and to link them by teaching about different representations at the same
time, would be relatively straightforward with the support of a CAS calculator. In consequence, the curriculum was planned assuming that a strong and reasonably equal emphasis on the three external representations (see Section 2.2.3) would improve calculus teaching and student learning. However, this was not entirely true in the present study. Although Teacher A used the numerical, graphical, and symbolic representations he favoured the symbolic representation (see Section 5.2.2). Teacher B also gave very strong emphasis to the symbolic representation and supported its use with the graphical representation. However, he gave less priority (and less time) to the numerical representation (see Section 5.2.2), and his students were not disadvantaged. The students in Class B achieved as well overall on the DCT as the students in Class A (see Section 6.2). They also demonstrated better understanding overall of the symbolic representation (see Table 6.3) that appears to be related to Teacher B’s teaching approach and the greater emphasis he gave to the symbolic representation. In addition, Class B students were proficient on more representations when compared with students of similar ability from Class A (see Table 6.5) and they also understood more interpretation-with-translation competencies, the most difficult category of competency in the hierarchy (see Sections 6.7, 6.8). As mentioned above, only the most capable students in both classes developed proficiency in all three representations (see Table 6.5). Finally, Class B students developed similar flexibility (as Class A students) between the S and G representations and similar (partial) flexibility between the S and N representations (see Table 6.7). These learning outcome for Class B students are impressive because their symbolic algebra and mathematical skills were generally weaker than the students in Class A (see Table 4.1). Thus, Teacher B did not disadvantage his low achieving students when he streamlined the calculus content he taught achieved by reducing his emphasis on the N representation and giving greater emphasis to the more important S and G representations, and to derivatives “at all points”. Interestingly, while both teachers recommended simpler procedures for numerical differentiation after their first experience of teaching calculus with CAS (Section 4.2.3), only Teacher B achieved it.

On the basis of the discussion above, it is recommended that strong emphasis be given to the important and useful symbolic and graphical representations and that less
emphasis be given to the numerical representation because it is time consuming to teach for no apparent additional conceptual benefit.

**Effectiveness of using CAS**
Since this study was concerned with understanding the role CAS played in helping students learn about the concept of derivative, the role of CAS activities and use of symbolic algebra in relation student learning are now discussed.

**CAS activities**
Analysis of the results has shown that the CAS activities provided in the curriculum supported student learning about the concept of derivative. Functional CAS activities seemed to assist some students to learn formulation competencies associated with the symbolic, graphical, and numerical representations of differentiation while pedagogical CAS activities seemed to assist some students to learn some of the interpretation competencies and formulation competencies to link representations (see Section 8.2).

What the students in each class learned, however, was related to the emphasis of their teacher. For example, one pedagogical CAS activity gave students the opportunity to understand the link between two representations of derivative and demonstrate the competency interpretation-with-translation. While one teacher stressed procedures to link the representations and his class was successful on the unexpected formulation-with-translation competency, the second teacher did not develop the ideas further and his students were not successful on either competency. Thus, the teachers used the CAS activities in ways that matched their teaching objectives but not always in ways that were anticipated by the author.

The second teacher did not rely on CAS activities to any extent. During carefully orchestrated class discussions, he emphasized understanding of derivatives by relating them to the real world (physical representations) and using enactive representations (e.g., arms outstretched to represent slope) and his class was more successful on a range of other interpretation competencies that were learned without reference to any CAS activity.

**Use of symbolic algebra**
In this study, the student’s use of symbolic algebra was more limited than intended. Because both classes had a majority of lower achieving students, the time devoted to
concepts took longer than anticipated and, in consequence, the students mostly solved relatively simple problems and fewer more challenging problems. Thus, during the six-week research project, use of CAS for symbolic algebra was minimized and most of the algebra used in the teaching was achievable by-hand. In addition, the algebraic demands and on the DCT were deliberately simple so as not to confuse understanding Framework competencies with ability to use symbolic algebra.

The potential of substantial use of symbolic algebra to help students learn has been highlighted by recent research (see Guin & Trouche, 1999; Lagrange, 1999; Tiwari, 1999). They indicate that using CAS (including symbolic algebra) as an instructional tool will assist students to learn. They suggest that purposeful use of CAS’s symbolic algebra and graphing capabilities will help students to develop conceptual understanding alongside procedural skills if they are given the opportunity to engage cognitively with a variety of challenging mathematics problems requiring use of different representations and translations between them.

Finding the place for challenging and more complex questions that needed CAS was difficult in this study. Although CAS was available to the students on calculus tests associated with the research, it would never be available for these students to use on significant school examinations (to obtain their VCE) in the future. These constraints did not allow a prominent role for CAS since the teachers felt obliged to not fully exploit its symbolic capabilities. Hence there was never a genuine need to consider more complex questions that could not be solved without it. Artificially placing CAS into a curriculum for a short period of time is not an entirely satisfactory way to introduce CAS. It needs to be legitimately available for all class activities and for assessment, into the future as well as currently, since using CAS reduces the necessity for by-hand skills and impacts significantly on how mathematical content is taught and tested. Both teachers in this present study, like the teachers reported in other studies (see for example Artigue, 2001; Schneider, 2000), found it difficult to give adequate status to using CAS and they both still valued by-hand symbolic manipulations to meet external examination requirements.
Combined impact of teacher privileging characteristics on learning

This thesis suggests that what the students learned about the concept of derivative was related to the combined effect of three teacher privileging characteristics: calculus content, teaching approach, and use of CAS. For each teacher, these characteristics were initially identified during the preliminary study and explored again during the main study. The mechanism of their impact was understood only after the teachers’ teaching actions (with and without CAS) were linked to the calculus content the teachers taught and to the learning of the students in their classes.

The first characteristic, calculus content, determined the mathematics the teachers taught and emphasized during their lessons and it was influenced by their personal content knowledge of mathematics. During the repeat study, one teacher broadened his content knowledge of calculus (motivated by the realization of new assessment procedures at the conclusion of the main study and new awareness of the capabilities of CAS) and began to teach the numerical and graphical representations of derivative in addition to the symbolic representation. The second teacher, with deeper knowledge of the concept of derivative, was empowered to support meaningful classroom discussions, with, and between, students and he gave greater emphasis to symbolic functions (“at all points”) and its links with the graphical representation while deliberately deciding to reduce the use of the numerical representation. Lloyd and Wilson (1998) report a similar outcome for a teacher with deep understanding of function. He made informed decisions about what content should be emphasized and what should be abandoned and he confidently and successfully supported student learning during classroom discussion.

The second teacher’s content knowledge of calculus also assisted him to identify sources of difficulties for students as suggested by Even and Tirosh (1995) and in response to his perception that his students were not very mathematically capable he reduced his emphasis on the numerical representation (and related use of CAS). This is consistent with the behaviour of the teachers reported by Zbiek (1995) and Lumb, Monaghan, and Mulligan (2000) who changed their use of CAS in response to their personal perceptions of their students’ needs.

For both teachers in the present study, several aspects of teacher knowledge (such as personal mathematical knowledge and knowledge of students) contributed to the decisions they made regarding the calculus content they would teach. In turn the
representations they emphasized impacted on their students’ learning and in general each class was more likely to be successful on those representations preferred by their teacher.

The second characteristic, teaching approach (i.e., method and style of teaching), also significantly influenced what students learnt about the concept of derivative. Over two years, neither teacher changed these highly influential aspects of their privileging. Other research has also shown that teaching style tended to be unchanged over the long term by the addition of technology (see Tharp, Fitzsimmons, and Brown Ayers, 1997). The teacher-centred teacher continued to lecture his students about rules and procedures and his students were generally more successful on formulation competencies involving differentiation procedures across the numerical, graphical, and symbolic representations. The student-centred teacher continued to foster understanding through classroom discussions that challenged the students to communicate, explain their ideas, and construct their own knowledge through discussion using real-world contexts. He did this mostly in the context of the symbolic representation and explained symbolic derivatives graphically and his students were generally more successful on competencies involving interpretation or the symbolic representation.

The third privileging characteristic, teacher use of CAS, was also stable over two years (see Section 5.3). Both teachers used CAS in ways that were consistent with their preference for representations and teaching method and style. After the first experience of teaching with CAS, the teacher who promoted the use of CAS increased his confidence in teaching with technology and as he expanded his use of representations he simultaneously widened his use of the CAS calculator to carry out “exact” differentiation procedures in the three representations. He gave the students keystroke procedures for the CAS so that they could replicate the solutions he had demonstrated. In contrast, the second teacher continued to discourage use of CAS unless he perceived that the activity would help his students develop understanding of the symbolic and graphical derivatives.

As Figure 8.2 illustrates, both classes understood competencies that related to the combined effect of the three privileging characteristics of their teachers evidenced by their teaching actions that clearly showed what calculus content was taught (representations emphasized), how it was taught (the differentiation process
emphasized), and how CAS activities were used (or not used) to emphasize the same process in the same representations.

Each teacher’s privileging characteristics, in combination, appear to have had a significant impact on what the teachers taught and how they taught it. These privileging characteristics seem to be underpinned by a combination of the teacher’s view of learning and his view of purpose for teaching, as fully discussed in Section 5.4. In particular, the privileging characteristics that were stable, teaching approach and use of CAS, seem to be underpinned by the teachers’ views of learning while the variable privileging characteristic, calculus content, seems to be underpinned by their views of the purpose for teaching.

**Effectiveness of teaching approach**

The teacher with deeper content knowledge was more effective in his teaching even although he did not make best use of CAS for student learning. As mentioned above, his privileging appeared to be influenced both by his underlying beliefs about the purpose of teaching and his view of learning. In response to these influences, his teaching approach was characterized by teaching for understanding using a student-centred style. His less capable students achieved as well overall on the DCT, were superior on a range of interpretation competencies and on one formulation competency, and were proficient in more representations than students from the other class with identical school grades.

As reported in the literature review, some studies report that teaching for understanding (i.e., conceptual method) together with a style that promotes students negotiating meaning (i.e, student-centred style) empowers students to learn mathematics with greater understanding (see Frid, 1992 whose study involved use of multiple representations; also Heid, 1988; Keller, 1997; Porzio, 1994; Repo, 1994 whose studies involved use of CAS). However, there is also a substantial body of literature that attributes improved understanding of concepts to use of technology but does not report on the teaching approaches that were adopted in the studies (see Section 2.2.2).

Currently, successful studies are being reported that combine a conceptual method and student-centred style for teaching. For example, research by Guin and Trouche (1999) report the success of an innovative teaching approach involving both graphics and CAS calculators. It enables the teacher to guide the work of a “sherpa student” who
plays a central role as a guide, assistant and mediator. This arrangement promotes various levels of discussion for meaning within the classroom, between the teacher and students and between students and enables the teacher to be more aware of student thinking.

In the main study however, it was also observed that the teacher-centred teacher who emphasized procedures expanded the calculus content he taught as he gained confidence in teaching with CAS and, in consequence, his students learned more. Thomas, Tyrrell, and Bullock (1996) also found that teacher experience with technology, over a period of time, contributed to more effective teaching. Although they found that effective teaching with technology was not restricted to particular teaching styles, they proposed a range of student-centred teaching strategies that they believed teachers would need to adopt in the long run to teach effectively with computers. Thomas, Tyrrell, and Bullock (1996) also acknowledged that it would be difficult for some teachers to change their teaching approach and use of technology (identified as stable privileging characteristics in this study).

Since 1993, in many schools and classrooms, Artigue (2001) and her colleagues have observed that teacher use of CAS evolves: after an initial explosion of methods, a reduction of methods occurs. This occurs as CAS use becomes more streamlined in schools and gradually acquires greater institutional status. In this study, this phenomenon was also observed. Over a period of time, both teachers expanded their methods of teaching but at different times. For one teacher it occurred during the main study when he recognized new ways to teach procedures using CAS and the expansion of methods was accompanied by an increase in his confidence in teaching with CAS. The second teacher expanded his methods of teaching with a graphics calculator just prior to his volunteering to participate in the study. During the preliminary study he recognized and adopted new ways to teach for understanding using CAS but during the main study he limited his use of CAS to quite specific uses. The next step is to institutionalise preferred ways of using CAS in the classroom.

Comparing teaching with graphics and CAS calculators

In this study, both teachers used CAS calculators differently from the ways they used graphics calculators (approved for official examinations) in their normal classes but in ways that were consistent with their observed teaching method and style. This result is
consistent with the studies reported in the literature review, Section 2.6.3 (Fine & Fleener, 1994; Fleener, 1995; Simmt, 1997; and others). The first teacher avoided using graphics calculators because he believed using graphs did not generally lead to "exact" solutions. However, with the CAS calculator he used procedures for numerical and graphical differentiation believing they were "exact". In contrast, the second teacher enthusiastically embraced the use of graphics calculators since he viewed them as a powerful partner for helping students to develop understanding but he did not feel free to use CAS in the same way due to constraints associated with future examination.

Recommendations for teaching with CAS

This study has shown that CAS is potentially a powerful partner for assisting secondary students to do and understand mathematics. It was beneficial for quick and accurate performance of a wide range of numerical, graphical, and symbolic procedures and to simultaneously work in two of these representations using its split screen facility. It was used routinely, for exploration of patterns, investigation of new ideas, and to test conjectures. To be able to use CAS effectively, the students need to have the confidence to explore the capabilities of the tool and personalize its use, identify the purpose for its use, and have the technical facility to use it easily. In this study, school constraints prevented full utilization of its symbolic algebra capabilities.

Based on two years of observations of two teachers teaching introductory calculus with CAS, the following recommendations are made for how CAS should be used in future teaching. The CAS needs to be available at all times, including for all examinations, as an easy and natural addition to the mathematics classroom, including projection of a calculator screen for use by the teacher and students. (This recommendation is made because the present study encountered problems because CAS was not available at all times). As occurred in this study, CAS should be introduced to students using suitable tasks that would be difficult, boring or tedious without it. This enables the students to appreciate the capabilities of the CAS calculator and understand the benefits of using it. Students should experience a wide range of tasks that challenge them to deepen their knowledge and explore new ideas and they should be given the opportunity to negotiate meaning for their ideas with their teacher and other students. Students should be encouraged to explore different ways of using the calculator and to develop their own ways of using it. For calculus, they should be given engaging tasks
that necessarily and naturally involve different representations and linking them and that
require use of CAS. This will empower them actively to construct knowledge, to
develop deep conceptual understanding, to acquire insightful problem solving skills, to
develop higher levels of thinking and to gain an understanding of how to interpret and
validate solutions (see Kendal & Pierce, 2000).

Further implications
This section describes how the findings of this thesis could be generalized to other areas
of mathematics and identifies further research that follows on from the research reported
in this thesis.

The outcomes of this research about teaching introductory calculus with CAS could
be generalized to other areas of mathematics that involve multiple representations and
use of technology (e.g., functions, trigonometry, mensuration, investigations, problem
solving, and others). This study suggests that teachers play an enormously important
role in students’ learning of mathematics. The teachers’ privileging (underpinned by
their beliefs in the purpose of teaching and view of learning) influenced the
mathematical content they taught and how they taught it, and in consequence what
cognitive processes were emphasized or ignored, and how the teachers used the
technology. In turn, these privileging characteristics appeared to influence what the
students learned. The same types of teacher influences on student learning that occurred
in this study are likely to occur when other areas of mathematics are studied.

The following research questions are proposed following on from the research
reported in this study.

- What is the relative importance of numerical, graphical, and symbolic
  representations in other content areas (e.g., functions, trigonometry, and
  mensuration, investigations, problem solving, etc.)? What is the role of other
  representations (e.g., internal / external representations)?
- How do teacher preferences for representations influence student preferences for
  representations?
- What is the effect of using a curriculum that requires students to make
  significant use of the symbolic algebra capabilities of the CAS calculator to
  learn introductory calculus?
• What is the role of teacher knowledge for effective teaching with CAS?
• What teaching approaches are most effective for teaching with CAS?

This study has important implications for teacher educators. Teaching introductory calculus with CAS seems to have most chance of success if the teachers have: deep mathematical knowledge, sound pedagogical knowledge, the confidence and willingness to use CAS, use student-centred teaching styles, use teaching methods that promote understanding of the concept of derivative and knowledge of differentiation procedures, and use a wide range of CAS activities (including those that require that use of symbolic algebra for learning). In turn, students’ learning should be greater.

9.2 Conclusion

In this study, the Differentiation Competency Framework was developed to identify the fundamental competencies associated with multiple representations of the concept of derivative, made possible with a CAS supported curriculum with a range of functional and pedagogical CAS activities to support students’ understanding of the concept of derivative. The Differentiation Competency Framework guided construction of the test instrument, monitoring of the teaching, and analysis of the learning in relation to the teaching.

This thesis generally supports the notion that teaching introductory calculus with multiple representations gives students the opportunity to develop understanding of the concept of derivative. Multiple representations are important as claimed in the literature, but it does not follow that all representations are equally important. Some streamlining is necessary for efficient teaching and the graphical and the symbolic representations proved to be the most useful and important representations of differentiation to emphasize and link.

This thesis also reveals the importance of the three privileging characteristics that reflected each teacher’s beliefs and conceptions about mathematics and how it should be taught. Each teacher’s approach to using CAS was consistent with his privileging of teaching method and both these characteristics were stable over two years (and most likely underpinned by their views of learning). However, the actual mathematics taught with CAS was influenced by each teacher’s content knowledge and was susceptible to change (and most likely underpinned by their view of the purpose for teaching).
time of transition from teaching with a graphics calculator to a CAS calculator, each teacher’s approach to using CAS was constrained by its low institutional status, and successful implementation of a CAS supported curriculum needs a corresponding change in assessment to legitimise new teaching practices.

The three privileging characteristics were initially perceived as independent but after very detailed analysis of teaching actions their interdependence became evident. Both teachers used teaching actions that accommodated their personal teaching approach (method and style of teaching) and they used the same teaching actions in association with both functional and pedagogical CAS activities. These teaching actions were used in conjunction with their preferred representations of derivative (even when they varied over time.)

Analysis of the teaching actions provides evidence that a teaching approach that focuses on understanding using a student-centred style of teaching is likely to enhance students’ understanding of the concept of derivative as has often been claimed in the literature. However, in research involving technology, their impact on learning is sometimes overlooked or underestimated.

This thesis directly links learning outcomes to teaching practices. It shows the powerful influence of teacher privileging on learning. The classes developed different understanding of the concept of derivative that seem to relate directly to the combined effect of their teacher’s preference for representations, method and style of teaching, and use of CAS technology.

Thus, prior to the introduction of CAS into mainstream secondary school mathematics, this thesis has provided valuable and useful insights into the experiences of two pioneering teachers as they made the transition from teaching with graphics calculator technology to this powerful hand-held personal technology and integrated it into their everyday classroom practices.
REFERENCES


Kuhn, T., & Ball, D. (1986). *Approaches to teaching mathematics: Mapping the domains of knowledge, skills and dispositions.* East Lansing: Michigan State University, Center on Teacher Education.


[On line]. Available: [http://ltsn.mathstore. ac.uk /came/events/freudenthal](http://ltsn.mathstore.ac.uk/came/events/freudenthal)
Appendix 1.1

NAME: ___________________________
DATE: ___________________________
TEACHER: __________________________

Year 11, 1999
CALCULUS - TEST A

This test contains 19 questions to check your understanding of differentiation.

Write all your answers in the spaces provided. Do not use scrap paper.

You may use your TI-92 as much as you like in answering these questions however there will be some questions where it will be of no use in helping you reach the correct answer.

At the end of each question, you will find a grid as shown below. Please tick the relevant box to show whether or not you used the TI-92 to:

- calculate with numbers
- do algebra (e.g. expand, factorize, solve equations and differentiate etc)
- use a graph

If you did the problem without any calculations, algebra or graphs, leave the boxes blank.

Sample only

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
**Question 1**

Find the derivative of $y$ given $y = x^5 + 4x^3 - x + 10$.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

**Question 2**

The CAS calculator was used to find values of the function $y = f(x)$ near $x = 3$.

(3.000, 0.000)  (3.103, -0.701)  (3.051, -0.353)  
(3.011, -0.079)  (2.990, 0.071)  (2.999, 0.007)

Find the best estimate of the gradient of the graph of $y = f(x)$ at $x = 3$.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

**Question 3**

A curve has the equation $g(x) = 5x^3 - 6x^2 + 3x - 6$.

Find the gradient of the curve at the point P, where $x = -1$.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
Question 4

The graph of \( y = f(x) \) is sketched below. A series of points A to J are marked along the curve. Consider the statements below and decide if they are true or false.

True or false?

(a) The gradient of the curve at F is greater than the gradient of the curve at B

(b) The gradient of the curve at A is greater than the gradient of the curve at H

(c) The gradient of the curve at I is less than the gradient of the curve at F

(d) The gradients of the curve at E and O are approximately equal

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

Question 5

Use a graph of \( y = x^2 + x - 10 \) to find the gradient of the tangent to the curve at \( x = 3 \).

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
**Question 6**

The graph of the function $h(x)$ is sketched below. The tangent at point P on the curve of $h(x)$ has also been drawn. Find the value of the derivative of $h(x)$ at P.

**Question 7**

(a) If $d = t^3 + 10t + 20$, find value(s) of $t$ where the derivative has the value of 58.

(b) If $d$ is the distance (in metres) that Jill has run in a race after $t$ (seconds), explain your answer to 7(a) above.
Question 8

The height of a plant can be determined by the formula

\[ H(t) = 7t^3 - 3t^2 \]

where \( H \) is the height of the tree in metres, and \( t \), is the number of years since the tree was first planted.

(a) What was the height of the tree when it was first planted? (i.e. \( t = 0 \))

(b) What was the height of the tree 2 years after it was planted? (i.e. \( t = 2 \))

(c) What was the average rate of change of the height of the tree during the first two years?

(d) Find the rate of increase of the plant’s height 2 years after it was planted. (i.e. \( t = 2 \))
**Question 9**

At 1.00pm, the rate of change of temperature in your house was +3 degrees Celsius (°C) per hour. Immediately after 1.00pm, the temperature in the house is most likely to:

- Decrease
- Stay the same
- Increase

Circle your choice and give the reason for your answer.

Reason:

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
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</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
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<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

**Question 10**

The derivative of the function $g(t)$ is given by the rule

$$g'(t) = t^3 - 5t$$

To find the rate of change at $t = 4$, you should:

A. Differentiate $g'(t)$ and then substitute $t = 4$

B. Substitute $t = 4$ into $g'(t)$

C. Find where $g'(t) = 0$

D. Find the value of $g'(0)$

E. None of the above

Circle the letter corresponding to your answer.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
**Question 11**

The values of a function close to \( x = 5 \) are shown in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>4.997</th>
<th>4.998</th>
<th>5.000</th>
<th>5.001</th>
<th>5.002</th>
<th>5.003</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>15.470</td>
<td>15.482</td>
<td>15.500</td>
<td>15.508</td>
<td>15.515</td>
<td>15.520</td>
</tr>
</tbody>
</table>

Find the best possible estimate of the derivative, \( f'(x) \) at \( x = 5 \).

**Tick a box for each method you used.**

- **I USED the TI-92 for:**
  - Number calculations
  - Algebra, Differentiation
  - Graphs

- **NOT USING the TI-92 I did:**
  - Number calculations
  - Algebra, Differentiation
  - Graphs

**Question 12**

If \( y \) is a function of \( x \), explain in words, the meaning of the equation; \( \frac{dy}{dx} = 5 \) when \( x = 10 \).

**Tick a box for each method you used.**

- **I USED the TI-92 for:**
  - Number calculations
  - Algebra, Differentiation
  - Graphs

- **NOT USING the TI-92 I did:**
  - Number calculations
  - Algebra, Differentiation
  - Graphs

**Question 13**

\( P \) (2, 7) is a point on the curve \( y = f(x) \), and at \( P \), the gradient of the curve is 3.

\( Q \) (2.011, 7.351) is a second point on the curve \( P \).

What is the instantaneous rate of change of \( y \) with respect to \( x \) at the point \( P \), when \( x = 2 \)?

(Note: An exact answer is required.)

**Tick a box for each method you used.**

- **I USED the TI-92 for:**
  - Number calculations
  - Algebra, Differentiation
  - Graphs

- **NOT USING the TI-92 I did:**
  - Number calculations
  - Algebra, Differentiation
  - Graphs
Question 14

The derivative function of \( f(x) \) is given by \( f'(x) = x^3 - 5x + 3 \).
What is the gradient of the tangent to the curve \( y = f(x) \), when \( x = 1 \)?

Tick a box for each method you used.

I USED the TI-92 for: Number calculations Algebra, Differentiation Graphs
NOT USING the TI-92 I did: Number calculations Algebra, Differentiation Graphs

Question 15

A curve has the function rule \( y = f(x) \).
If the rate of change of \( y \) with respect to \( x \) is given by the rule \( 5x + 7 \), what is the gradient function of the curve?

Tick a box for each method you used.

I USED the TI-92 for: Number calculations Algebra, Differentiation Graphs
NOT USING the TI-92 I did: Number calculations Algebra, Differentiation Graphs

Question 16

An eagle follows a flight path where its height depends on the time since it flew out of its nest.
The rule for finding the height of the bird above its nest, is a function of time, \( H(t) \) where \( H \) is the height of the bird (in metres) above the nest and \( t \) the flight time (in seconds).

5 seconds after take-off, the 4 kg eagle was observed to be 100m above its nest and climbing at the rate of 3 metres/second. What is the value of \( f'(5) \)?

Tick a box for each method you used.

I USED the TI-92 for: Number calculations Algebra, Differentiation Graphs
NOT USING the TI-92 I did: Number calculations Algebra, Differentiation Graphs
**Question 17**

The gradient function of f(x) is sketched below.

![Graph of f'(x)](image)

(a) From the list of gradient function rules listed below, select the rule that best represents the derivative of f(x). Circle the letter corresponding to your answer.

A. \( f'(x) = x^3 + x^2 + 2 \)
B. \( f'(x) = -x^3 + x^2 + 2 \)
C. \( f'(x) = -3x^2 - 2x + 2 \)
D. \( f'(x) = 3x^2 + 2x + 2 \)
E. \( f'(x) = 3x + 2 \)

**Tick a box for each method you used.**

<table>
<thead>
<tr>
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<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

(b) If \( g'(x) = 3x^2 - 4x + 2 \) What is the rule for \( g(x) \)?

**Tick a box for each method you used.**

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
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<td>NOT USING the TI-92 I did:</td>
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<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
**Question 18**

The graph of the function $y = f(x)$ is shown below.

Which one of the following graphs shown below could be the graph of the gradient function of $f(x)$? Circle the letter that corresponds to your answer. Give reasons for your choice.

A.  

B.  

C.  

D.  

E.  

Tick to indicate how you answered this question.

I used the TI-92 to:
- Do a number calculation
- Draw a graph
- Do algebra / differentiation

I did not use the TI-92 to:
- Do a number calculation
- Draw a graph
- Do algebra / differentiation
**Question 19**

One day during the school holidays, a family went on a bush walk. The graph below shows their distance from the start of the walk (in kilometres) as a function of the number of hours walked.

![Graph showing distance from start of walk vs. time]

(a) What was the family’s average rate of walking between 9am and 12 noon?

(b) What was the family’s speed (rate of walking) at 11.00am?

---

**Tick a box for each method you used.**

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
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<th>Graphs</th>
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<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
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<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
(c) When did the family have lunch?

*Tick a box for each method you used.*

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
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<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

(d) Mark S on the graph to show when the family was walking at their slowest speed, after lunch.

*Tick a box for each method you used.*

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

(e) Do you think they were going uphill or downhill during the first five hours? Circle your choice and give the reason for your answer.

Uphill  Downhill  Can’t tell

Reason:

*Tick a box for each method you used.*

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
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<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

(e) Find a time on the return journey when the walking speed was the same as at 11.00am on the forward journey.

*Tick a box for each method you used.*

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
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</tbody>
</table>
Appendix 1.2

Competency Characteristics and Location on Test A of Differentiation Competency Test Questions

<table>
<thead>
<tr>
<th>DCT Framework Competency</th>
<th>Representation Process</th>
<th>Competency Characteristics</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Input Derivative</td>
<td>Output Derivative</td>
</tr>
<tr>
<td>FNn</td>
<td></td>
<td>N</td>
<td>n</td>
</tr>
<tr>
<td>FNn</td>
<td></td>
<td>N</td>
<td>n</td>
</tr>
<tr>
<td>FNg</td>
<td></td>
<td>N</td>
<td>g</td>
</tr>
<tr>
<td>FNn</td>
<td></td>
<td>N</td>
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</tr>
<tr>
<td>FNn</td>
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<td>N</td>
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</tr>
<tr>
<td>FNn</td>
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<tr>
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</tr>
<tr>
<td>FSs</td>
<td></td>
<td>S</td>
<td>s</td>
</tr>
</tbody>
</table>

* t indicates a translation of representation is involved

8(d) ♥ theoretically valid alternative competency $[FNn]$ adopted by 2 Class A and 1 Class B students
3* theoretically valid alternative competency $[FNg]$ adopted by 4 Class A and 0 Class B students
5♣ theoretically valid alternative competency $[FNg]$ adopted by 0 Class A and 1 Class B students
8(d) ♦ theoretically valid alternative competency $[FGn]$ adopted by 0 Class A and 1 Class B students
3** theoretically valid alternative competency $[FGg]$ adopted by 1 Class A and 2 Class B students
Appendix 2.1

NAME: _________________________________
DATE: _________________________________
TEACHER: _________________________________

Year 11, 1999

CALCULUS - TEST B

This test contains 9 questions.

Write all your answers in the spaces provided. Do not use scrap paper.

You may use your TI-92 as much as you like in answering these questions however there will be some questions where it will be of no use in helping you reach the correct answer.

At the end of each question, you will find a grid as shown below. Please tick the relevant box to show whether or not you used the TI-92 to:

- calculate with numbers
- do algebra (e.g. expand, factorize, solve equations and differentiate etc)
- use a graph

If you did the problem without any calculations, algebra or graphs, leave the boxes blank.

Sample only

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

220
**Question 1**

Differentiate the following functions

(a) \(y = x^3 - 2x^2 - 7\)

*Tick a box for each method you used.*

**I USED the TI-92 for:**

- Number calculations
- Algebra, Differentiation
- Graphs

**NOT USING the TI-92 I did:**

- Number calculations
- Algebra, Differentiation
- Graphs

(b) \(g(x) = x^3(2x + 1)^2\)

*Tick a box for each method you used.*

**I USED the TI-92 for:**

- Number calculations
- Algebra, Differentiation
- Graphs

**NOT USING the TI-92 I did:**

- Number calculations
- Algebra, Differentiation
- Graphs

(c) \(h(x) = (4x^2 + 9)^7\)

*Tick a box for each method you used.*

**I USED the TI-92 for:**

- Number calculations
- Algebra, Differentiation
- Graphs

**NOT USING the TI-92 I did:**

- Number calculations
- Algebra, Differentiation
- Graphs

(d) \(f(x) = x^{-3}\)

*Tick a box for each method you used.*

**I USED the TI-92 for:**

- Number calculations
- Algebra, Differentiation
- Graphs

**NOT USING the TI-92 I did:**

- Number calculations
- Algebra, Differentiation
- Graphs
(e) \( f(t) = 2\sqrt{t} \)

\[ f(t) = 2\sqrt{t} \]

\[ (e) \quad f(t) = 2\sqrt{t} \]

\[ (f) \quad f(x) = \frac{1}{x^3} \]

\[ (f) \quad f(x) = \frac{1}{x^3} \]

**Question 2**

After a few days of heavy rain the river burst its banks and the surrounding area was flooded. Suppose the area of land under water is given by

\[ A = 16t - t^3 \]

where \( A \) is the area in hectares and \( t \) is the number of days since the flood began.

(a) When (after how many days) was the most land under water?
(b) What was the maximum area under flood?

---

**Tick a box for each method you used.**

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

---

**Question 3**

(a) Find the derivative of \( y = 2x^3 + 1 \) at \( x = 2 \), without using the rule for differentiating or using the derivative function on the TI-92.

---

(b) Is your answer exact or approximate? ____________________
Question 4

Consider the curve \( y = x^3 - x \)

(a) Find the gradient of the curve at \( x = 1 \), stating whether it is accurate or approximate.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

(b) Check your result by finding the gradient using another method, stating whether it is accurate or approximate.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
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<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
**Question 5**

(a) Find the derivative of \( y = f(x) \), sketched below, at \( x = -5 \).

![Graph of a function](image)

(b) Is your answer exact or approximate? _______________

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>

**Question 6**

Suppose you travelled 100 km in your car in two hours. Must there have been a time when your instantaneous speed was exactly 50 km per hour? Explain your reasoning carefully.

Tick a box for each method you used.

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
**Question 7**

Suppose you used the Zoom-In function on the calculator 5 times on the graph of \( y = x^2 + 3 \) around the point P where \( x = 2 \).

(a) Draw a graph of what you would see.

(b) If you knew the co-ordinates of two points on the new graph, what could you find out about the graph of \( y = x^2 + 3 \) at the point P where \( x = 2 \).

**Question 8**

(a) The number of bacteria in a culture, \( N \), was found to depend on the time in seconds, \( t \) secs, since it was first cultured. If \( N = t^{(1.8)} \), find the rate of increase in the number of bacteria after 3 seconds.

(b) Find the rate of change of \( y \) with respect to \( x \) if \( y = (1.8)^x \) when \( x = 3 \).
**Question 9**

On the axes below, sketch the graph of \( y = 4 + 3x^2 - x^3 \)

Draw the graph of \( y \) versus \( x \) for \( 0 < x < 5 \)

- Number the vertical axis
- Show its overall shape
- Find the maximum value of \( y \) and the value of \( x \) for which this occurs.

**Tick a box for each method you used.**

<table>
<thead>
<tr>
<th>I USED the TI-92 for:</th>
<th>Number calculations</th>
<th>Algebra, Differentiation</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT USING the TI-92 I did:</td>
<td>Number calculations</td>
<td>Algebra, Differentiation</td>
<td>Graphs</td>
</tr>
</tbody>
</table>
# Appendix 2.2

## Preference for Representation Test Questions: Focus, Location on Test B, and Associated Framework Competency

<table>
<thead>
<tr>
<th>Focus of PRT Question</th>
<th>No of items</th>
<th>Location of Question</th>
<th>Framework Competency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Use of symbolic algebra or by-hand</td>
<td>6</td>
<td>B 1(a)-(f)</td>
<td>FSs</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>B 8(a)-(b)</td>
<td>FSn / FGn</td>
</tr>
<tr>
<td>2. Influence of alternative data on choice of representation of differentiation</td>
<td>1</td>
<td>B 3(a)</td>
<td>FNs / FGs</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>B 2(a)-(b)</td>
<td>FSs Alternative non-calculus graphical method</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Alternative non-calculus numerical method</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>B 9</td>
<td>FSs Alternative non-calculus graphical method</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Alternative non-calculus numerical method</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Knowledge of two representations to solve a problem in two different ways</td>
<td>1</td>
<td>B 5(a)</td>
<td>FGs</td>
</tr>
<tr>
<td>4. Knowledge of numerical data</td>
<td>1</td>
<td>B 6</td>
<td>INn</td>
</tr>
<tr>
<td>5. Knowledge of a compound competency</td>
<td>1</td>
<td>B 7(b)</td>
<td>Compound competency FGg + In or IGg or IGs</td>
</tr>
</tbody>
</table>

**By-hand graphical differentiation**

<table>
<thead>
<tr>
<th></th>
<th>No of items</th>
<th>Location of Question</th>
<th>Framework Competency</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Knowledge of two representations to solve a problem in two different ways</td>
<td>1</td>
<td>B 4(a)</td>
<td>FNg / FGg / FSg</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Knowledge of a compound competency</td>
<td>1</td>
<td>B 7(b)</td>
<td>Compound competency FGg + In or IGg or IGs</td>
</tr>
</tbody>
</table>

**Total number of items** 16
Appendix 3.1

First Teacher Interview

| Interview date: | Friday 21st May 1999 |
| Teaching commenced | Monday 2nd August 1999 |

The aim of this first teacher interview is to investigate:

- the depth of each teacher’s conceptual understanding of differentiation;
- which particular representations each teacher privileges in his own solutions;
- each teacher’s personal approach to solving problems.

The following methodology was adopted for each first teacher interview.

1. Each teacher was given the set of eight problems.

2. Each teacher was asked to describe, while thinking-aloud:
   - his preferred method of solving each problem;
   - the variety of methods he believed his students might employ to solve each problem (with CAS available);

3. Each teacher given the opportunity to discuss his ideas about the benefits and problems of teaching introductory calculus using a CAS.

Protocol for each Teacher Interview

“I’ve got a set of calculus questions here to talk about.”

For each of the eight questions say to the teacher

First, “Could you tell me about how you would solve this problem. You don’t have to solve the problem, just tell me your method.”

Second, “I am very interested in the different ways the girls might solve this problem having had access to the TI-92 during all the calculus lessons. Can you tell me about the methods you think that the girls are likely to use to solve the problem? You don’t have to solve the problem, just tell me about the methods.”

After asking the set of questions, ask

Third “Can you tell me your opinion about using CAS for Year 11 calculus, some of the benefits, some of the problems.”

Each interview will be audiotaped and transcribed.
First Teacher Interview Questions

**Question 1**

Consider the curve $y = x^3 - x$

(a) Find the gradient of the curve at $x = 1$, stating whether it is accurate or approximate.

(b) Confirm your result by finding the gradient using another method, stating whether it is accurate or approximate.

**Question 2**

After a few days of heavy rain the river burst its banks and the surrounding area was flooded. Suppose the area of land under water is given by

$$A = 16t - t^3$$

where $A$ is the area in hectares and $t$ is the number of days since the flood began.

(a) When (after how many days) was the most land under water?

(b) Can you suggest an alternative way to find when the most land was under water?
**Question 3**

Line $L$ is a **tangent** to the curve $f(x) = 4x^3 - x^4$ at the point $(1,3)$ as indicated.

(a) Find $f'(1)$, stating whether it is accurate or approximate.

(c) Confirm your result by finding the derivative using another method, stating whether it is accurate or approximate.

**Question 4**

Find an approximate value of the derivative of $y = 2x^3 + 1$ at $x = 2$, without differentiating.

**Question 5**

Suppose you travelled 100 km in your car in two hours. Must there have been a time when your instantaneous speed was exactly 50 km per hour?
Question 6

Find the rate of change of $y$ with respect to $x$ if $y = 1.8^x$ when $x = 2$.

Question 7

Using the graph of $f(x)$ shown below, estimate the instantaneous rate of change of $y = f(x)$ at $x = 1$. 

![Graph of f(x)](image-url)
Question 8

The graph below shows, \( h \), the height of a moving object above ground in metres as a function of the time, \( t \), in seconds.

![Graph of Height of a Moving Object with Time](image)

Draw a graph showing the rate of change of height as a function of time.

Question 9

Suppose A’s salary is $x and B’s salary is $y, and \( y \) is a function of \( x \).
If A’s salary increases, how will B’s salary fare in comparison to A’s, if \( dy/dx \) equals the following?

- a. 4
- b. \( \frac{1}{2} \)
- c. -2
- d. 0
## Appendix 3.2

### First Teacher Interview Questions: Location and Relationship to Student DCT and PRT Questions

<table>
<thead>
<tr>
<th>First Teacher Interview Question</th>
<th>Relationship to Corresponding Student Question</th>
<th>Focus of PRT Question (see Appendix 2.2)</th>
<th>Location of DCT Question (see Appendix 2.2)</th>
<th>Competency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Identical</td>
<td>PRT 3</td>
<td>B 4(a)-(b)</td>
<td>FNg</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>FG&lt;sub&gt;g&lt;/sub&gt;</td>
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<td></td>
<td></td>
<td>FG&lt;sub&gt;s&lt;/sub&gt;</td>
</tr>
<tr>
<td>2</td>
<td>Identical</td>
<td>PRT 2</td>
<td>B 2(a)-(b)</td>
<td>FS&lt;sub&gt;s&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>3</td>
<td>Similar data</td>
<td>PRT 3</td>
<td>B 4(a)-(b)</td>
<td>FS&lt;sub&gt;s&lt;/sub&gt;</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>FG&lt;sub&gt;s&lt;/sub&gt;</td>
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<td>4</td>
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<td>PRT 2</td>
<td>B 3(a)</td>
<td>FG&lt;sub&gt;s&lt;/sub&gt;</td>
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<tr>
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<td>B 6</td>
<td>FN&lt;sub&gt;n&lt;/sub&gt;</td>
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<td>B 8(b)</td>
<td>FS&lt;sub&gt;n&lt;/sub&gt;</td>
</tr>
<tr>
<td>7</td>
<td>Similar data</td>
<td>DCT</td>
<td>A 19(b)</td>
<td>FG&lt;sub&gt;n&lt;/sub&gt;</td>
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<tr>
<td></td>
<td>Similar question</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Different</td>
<td></td>
<td></td>
<td>[FG&lt;sub&gt;g&lt;/sub&gt; + I&lt;sub&gt;Gn&lt;/sub&gt;]</td>
</tr>
<tr>
<td>9</td>
<td>Different</td>
<td></td>
<td></td>
<td>IS&lt;sub&gt;s&lt;/sub&gt;</td>
</tr>
</tbody>
</table>
Appendix 3.3

Second Teacher Interview

Monday 15th November 1999

The 1999 calculus course was designed to enhance students’ understanding of differentiation with a TI-92 available for use.

1. How did you feel about teaching introductory calculus this way?

2. Describe how your teaching practices changed while teaching this calculus course and with a TI-92 always available.

3. Describe the ways you used the TI-92 to teach:
   - Conceptual understanding
   - Routine procedures

4. Describe any difficulties/problems associated with the 1999 calculus course?

5. How do you think the students coped with learning about the concept of derivative in such a conceptual way?
   - Benefits
   - Difficulties

6. Any other comments?
Appendix 4.1

Author’s original notes
for 8 minutes of Teacher A’s lesson on 16/8

Original

Teacher A 16/5/99 (Lesson 3) 59

Lesson Plan

Lesson Plan 7

14/15 present

Na2 away

10:35 Administration

10:44 Lesson started after 4 girls split up.

TA’s Q How do you find a derivative?

Class Ans Find gradient

TA’s BB notes - a revision question

Q1 Find the derivative of y = -2 on the

graph y = 2x^2 - 4x + 1

10:46 Tab started

Girls worked quietly

10:47

TA’s Q What answer did you get Jess?

Jess -12

TA used TI-92 to show Jess’s solution

(on O.H. projector using F8 menu)
to the class d(2x^2 - 4x + 1)/dx = -2x^2 - 12

10:48 T.A. repeated the solution to the

same problem showing the girls

how to use the 2nd 8 button (d)

10:50 T.A. used the TI-92 to draw the

graph of y = 2x^2 - 4x + 1

She showed the girls how to adjust

the window to see the entire

graph.

TA drew in the tangent at x = -2

by hand on the projector screen
\[
y = mx + c
\]
\[
y = -12x + c
\]

Was TA's check that the derivative = -12.

T.A. used S and G representations

One girl's TI-82 got jammed (busy mode) -- she came to me for help.

TA did not use the TI-82 to differentiate graphically -- to find the gradient of the tangent from its equation \( y = -12x - 7 \).

10:52 TA gave students a second revision example.

Q. 2. Find the derivative at the points
\[
x = 2, x = -3
\]
\[
x = \{-2, -1, 0, 1, 2\} \text{ on the graph } y = 3x^2
\]

Hannah: I don't know what to do.

T.A.: Have you checked your notes?

Hannah: I don't know what \( f'(x) \) is.

10:53 TA: Told Hannah to look back at her text book and then revised functions.

10:55 \[
\text{Relation } f(x) \text{ with an } g(x) \text{ inappropriate responses to student's question}
\]

11:00 TA asked Grace how to solve Q.2.

TA solved the problem on the OH screen:

\[
\frac{d(3x^2)}{dx} = \{ -2, 1, 0, 1, 2 \}
\]
\[
\text{Ex} = \{-12, -6, 0, 6, 12\}
\]
Appendix 4.2

Observation of Lessons (1) for Teacher A’s lesson on 16/8

Date: 16/8/99  
Teacher: A  
Student attendance

Lesson Plan 1

Teacher questions
1. Style/techniques of teacher questioning
   - Class was questioned as various points in the lesson.

2. Nature of student responses which were acceptable [class response or individual responses]
   - Several individuals were called upon to answer a question.
   - Group response was accepted as derivative rule for f(x) = 3x^3

3. Proportion of students involved in class discussion about a mathematical idea
   - Most

4. Student involvement with commitment to teacher’s questions
   - Very few students thought about the questions as they knew one student only would be required to answer or that the teacher would answer the question herself.

Student questions
5. Value the teacher placed on student questions
   - 3 individuals with Ti-92 problems. Also helped one student who said she didn’t understand what this meant.

6. How the teacher handled student questions
   - Directed it to her notes on functions then spent 5 min. summarizing functions and relations (inappropriately).

7. Were student questions used as a basis for further explanations?
   - 10:08 - showed students how to undo an incorrectly entered line (Ti-92).

8. How were other students involved in answering classmates questions?
   - Quietly helped each other

Teaching strategies
9. Role of BB notes
   - Revision of new work provides a mechanism for the teacher to answer questions to direct students back to their notes, e.g.

10. Techniques used to verify validity of solutions
    - After using a button to find deriv = 12, teacher directed function.

Techniques used to check students’ understanding
11. Communication with students
    - Class questions
    - Homework
    - Assignment
    - Tests

12. Other
    - 15. Other
    - 16. Other

17. Teacher strategy to “catch up” students who have been absent
    - Revision at the beginning of the lesson.

18. Teacher’s private work was done in class
    - No

19. Level of preparation for class
    - Adequate

20. Discipline
    - 1 minute at beginning of lesson.
21. Teacher’s level of personal contact with each student in the class
   helped about 5/14 students

Technical issues
22. How did the teacher explain to the students how to use the TI-92?
   Used O.H. projection and performed the steps on TI-92.

23. Proportion/number of students involved in class discussion about ways to use the TI-92
   All

24. Proportion of students involved in watching the TI-92 demonstrations on the O.H. projector screen
   Most

25. How did the teacher explain/connect/link the mathematical purpose to the use of the TI-92
   Interpreted the derivative as the gradient of the curve at the point.

26. Proportion/number of students involved in the class discussion about the purpose of using the TI-92
   Most

27. Where else could the TI-92 have been used during the lesson?
   Teacher could easily have shown the exact graphical method of differentiation or zoomed-in or approx differentiation

28. Did the teacher not use the technology when appropriate e.g. fail to demonstrate TI-92 using the O.H.
   projector. Was this acknowledged by the teacher? Was the teacher unaware of the opportunity?

29. Was the lesson structured around the use of the TI-92?
   Yes

30. Was TI-92 use encouraged more than by-hand?
   Yes (except HW)

31. Was by-hand use encouraged more than TI-92 use?
   For homework

32. Was approximately equal emphasis on by-hand and TI-92 use? — for HW

33. Problems encountered using the TI-92
   Students could not clear screen or turn off TI-92.
   Problem with variables
   Problem with syntax: d(3x^3 x)^ forgot ten

34. Number of students who came to class without the TI-92
   0

35. Level of co-operation/mathematical/TI-92 discussion between students or did they work alone
   High

36. Teacher attitude to CAS was apparent
   Yes — it was positive

37. Students attitude to CAS was apparent

38. Explicit message was given to students about the value of using the TI-92
   The students were told it was easier to find sets of derivatives using the TI-92, not possible on TI-83.

39. Explicit message was given to the students about problems associated with using the TI-92

———
Representations
40. Teacher used numerical explanations where appropriate

41. Where else could a numerical representation have been used in this lesson?

42. Teacher used algebraic explanations where appropriate
   Yes

43. Where else could a symbolic representation have been used in this lesson?

44. Teacher used graphical explanations where appropriate
   Yes - on graph

45. Where else could a graphical representation have been used in this lesson?
   Could have followed through with the TI-82 to differentiate graphically

46. Teacher used both algebraic and graphical explanations
   Yes

47. Teacher used more than one explanation (numerical, graphical, algebraic, physical etc)
   Twice

Misconceptions
48. Teacher’s mathematical error/misconception/confusion
   No

49. Student’s mathematical error/misconception/confusion
   Query about function notation

50. Were student misconceptions corrected in discussions with the teacher?

51. Outstanding feature of the lesson, special insights by the students, teachers etc

52. Other matters
   The technology forces the teacher to communicate with the students, as there is a need for technology problems to be addressed.
   The teacher continued to tell the girls the answers to questions rather than allow them to make their own discoveries.
Appendix 4.3

Observation of Lessons (2) for Teacher A’s lesson on 16/8

Calculus Project 1999

| Time | Mins | Symbol | Graph | Number | By-hand | CAS | With TI-92 | Class Lecture | Class Discuss | Help Individ | Class worked alone | Investigate | Revise | Discuss HW |
|------|------|--------|-------|--------|---------|-----|------------|----------------|---------------|--------------|---------------|-------------------|------------|--------|------------|
| 10.00| 1    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 36   | 2    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 37   | 3    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 38   | 4    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 39   | 5    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 40   | 6    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 41   | 7    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 42   | 8    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 43   | 9    |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 44   | 10   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 45   | 11   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 46   | 12   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 47   | 13   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 48   | 14   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 49   | 15   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 50   | 16   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 51   | 17   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 52   | 18   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 53   | 19   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 54   | 20   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 55   | 21   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 56   | 22   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 57   | 23   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 58   | 24   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 59   | 25   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 60   | 26   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 61   | 27   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 62   | 28   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 63   | 29   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |
| 64   | 30   |        |       |        |         |     |            |                |               |              |                |                  |            |        |            |

Comments

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<th>Class Discuss</th>
<th>Help Individual</th>
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\[ N = 50 \]

\[ N(\text{total}) = 50 \]
## Appendix 4.4

**Time spent by Teacher A and Teacher B on each representation during the teaching program**

<table>
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<tr>
<th>Lesson Date</th>
<th>Representations (individually)</th>
<th>Representations (linked)</th>
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<td>Total</td>
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*Revision lessons

Total time (minutes) on the N, G, and S representations (total) by Teachers A and B.

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<th>G (Total)</th>
<th>S (Total)</th>
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<tr>
<td>Teacher B</td>
<td>110</td>
<td>181</td>
<td>551</td>
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</table>

(E.g, On the N representation (in total), Teacher A spent 75 (individual) + 86 (N to G) + 15 (N & S) = 176 minutes)

### Preference for representations

The calculus content was taught to both classes during the experimental period. However, the teachers did not necessarily cover the same lesson content on the same date.

_Appearance 4.4 shows the total time spent on each teacher devoted to the representation including performing related CAS Activities, lecturing students, class discussions, working with individuals, and allowing the students to work without direct help. It also shows the time each teacher spent individually on each representation and on linking pairs of representations._

Both teachers spent most total time on the symbolic representation and less on the numerical and graphical representations. In general, less time was devoted to linking representations. An exception is the time Teacher B devoted to the S to G link. In addition, Teacher A devoted substantially more time to the N to G link than Teacher B.

In addition to the representations shown above, Teacher B, on occasions, spontaneously used an enactive representation to link derivative to gradient (G & S). The teachers placed different emphasis on the importance of each representation that is broadly reflected by the total time they allocated to it.
Appendix 4.5

Time spent by Teacher A and Teacher B on different teaching approaches during the teaching program

<table>
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<tr>
<th>Lesson Date</th>
<th>Teacher Used TI-92</th>
<th>Lectured Class</th>
<th>Conducted Class Discussion</th>
<th>Helped Individual Students</th>
<th>Class Worked Without Teacher Support</th>
<th>Time Devoted to Experiment or Investigat’n</th>
<th>Time Revision or Test</th>
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Method and style of teaching

Appendix 4.5 shows the overall time spent by each teacher on a range of teaching activities during each class lesson. It indicates the teachers’ different teaching approaches.

Teacher A’s teaching style was to lecture his students while Teacher B preferred to conduct classroom discussions with the students.

Teacher A’s teaching method was to lecture rules and procedures (260 minutes). Teacher A only helped the students who requested help (78 minutes) and he often directed them back to their notes and left the students to work alone (101 minutes). Teacher B’s teaching method was to use classroom discussion to help students understand what the mathematical ideas under consideration (354 minutes). He followed this up by devoting the remaining class time to helping individual students engaging in meaningful discussion about their mathematical ideas (at least once every lesson 266 minutes). Teacher B frequently initiated this contact. In addition, Teacher A spent more time on classroom administration then Teacher B (117 minutes cf. 73 minutes).

Use of CAS

Teacher A actively used the CAS calculator in class nearly twice as often as Teacher B. In Class A there was an atmosphere of acceptance about using the CAS calculator at any time for any purpose whereas in Class B there was a resistance to using it except for checking work quickly or generating sets of data for subsequent analysis.
### Appendix 4.6

**Teacher A’s and Teacher B’s use of specific CAS activities in the curriculum**

<table>
<thead>
<tr>
<th>CAS Activity</th>
<th>Teachers’ use of CAS Activities</th>
</tr>
</thead>
</table>
| 1 N differentiation       | Teacher A taught average rates of change using CAS Activity 7 (2/8, 3/8) and stressed difference quotients in several lessons and revised this procedure during the revision lessons (1/9). He taught the students how to generate function values to calculate difference quotients linking G to N while performing CAS Activity 6 (5/8, 2/9). He expected the students to use the CAS calculator.  
Teacher B taught average rates of change (2/8, 3/8) using CAS Activity 7 but rarely mentioned difference quotients. He allowed the students to use the CAS calculator but also encouraged them to calculate by hand where possible. |
| 2 G differentiation       | Teacher A taught the students to use the CAS calculator procedures and by hand procedures (9/8, 10/8, 30/8, 1/9).  
Teacher B taught the students to use the CAS calculator procedures and by hand procedures (6/8, 10/8). |
| 3 S differentiation       | Teacher A taught the students to use the CAS calculator procedures and by hand procedures (most lessons from 10/8).  
Teacher B showed the students to use the CAS calculator procedures but strongly emphasized by hand procedures (most lessons after 10/8). |
| 4 First principles        | Not taught by Teacher A.                                                                                                                                                                                                              |
| differentiation           | Not taught by Teacher B.                                                                                                                                                                                                              |
| 5 Understanding G         | Teacher A briefly demonstrated local linearity with Activity 5 (1/9).  
Teacher B did not demonstrate Activity 5.                                                                                                                                                                                                 |
| differentiation           |                                                                                                                                                                                                                                          |
| 6 Linking G/N derivatives | Teacher A used Activity 6 to show students how to find gradients of curves using difference quotients. He taught the students how to generate function values to calculate difference quotients (5/8) and revised the procedure during one revision lesson (2/9).  
He expected the students to use the CAS calculator.  
Teacher B demonstrated Activity 6.                                                                                                                                                                                                 |
| 7 Linking G/N derivatives | Teacher A demonstrated Activity 7 and allowed the students to participate (2/8). He focused on the link between G & N.  
Teacher B demonstrated Activity 7 and allowed the students to participate (2/8). He focused on calculating average rates of change.                                                                                                                          |
| 8 Understanding S         | Teacher A permitted CAS calculator procedures in order to quickly and easily collect data for investigation A (16/8).                                                                                                                                 |
| derivatives               | Teacher B positively encouraged CAS calculator procedures in order to quickly and easily collect data for investigation (16/8).                                                                                                               |
Appendix 5.1

Student Self-Assessment

Name  _________________________________________
Teacher __________________________________________
Date  ___________________________________________________

Thank you for agreeing to help me with my research about how students learn calculus.

During the project you will be asked to complete questionnaires, log sheets, questions and test items.

Throughout the research project your answers are confidential, so what you say will not be seen by your teacher or anyone else at your school. Under no circumstances will any student be able to be identified in any reports of this research.

Please complete the questions below. Please try to answer ALL questions as honestly as you can.

Rate your success in Mathematics and the particular topics listed:

<table>
<thead>
<tr>
<th>Poor</th>
<th>Moderate</th>
<th>Good</th>
<th>Very Good</th>
<th>Excellent</th>
<th>Didn’t do any</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics overall</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Tables and mental arithmetic</td>
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<td></td>
</tr>
<tr>
<td>Numerical calculations with calculator</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Algebra</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solving equations</td>
<td></td>
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</tr>
<tr>
<td>Factorising</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Expanding brackets</td>
<td></td>
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<tr>
<td>Substituting</td>
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<tr>
<td>Graphing</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Trigonometry</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using graphing calculator</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using computers</td>
<td></td>
<td></td>
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</tbody>
</table>
Rate how you generally feel about the following aspects of mathematics.

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<thead>
<tr>
<th></th>
<th>Hate</th>
<th>Dislike</th>
<th>Neutral</th>
<th>Like</th>
<th>Enjoy</th>
<th>Didn’t do any</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics overall</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Tables and mental arithmetic</td>
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<td></td>
</tr>
<tr>
<td>Numerical calculations with calculator</td>
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<tr>
<td>Algebra</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Solving equations</td>
<td></td>
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<tr>
<td>Factorising</td>
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<td></td>
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<tr>
<td>Expanding brackets</td>
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<td></td>
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<tr>
<td>Substituting</td>
<td></td>
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<td></td>
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<tr>
<td>Graphing</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using TI-83</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Trigonometry</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Using computers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. What is your opinion of the graphical calculators?
   - TI-83
   - TI-92

2. Do you have a computer at home?

3. Do you use it for mathematics?

4. Name any computer programs that you have used for maths at home.

5. Name any computer programs that you have used for maths at school.

6. What mathematics do you hope to study in Year 12?
Appendix 6.1

Pre-requisite Skills for CAS

Pre-requisite CAS skills for the main study Year 11 Calculus Course (1999)

Students need to be able to:
- sketch graphs, set windows, zoom functions using the TI-83;
- substitute into functions to get ordered pairs, by-hand and with the TI-92;
- recognize gradient of straight line given its formula;
- appreciate rates of change of graphs – see worksheet provided;
- perform basic algebra e.g. expansion, factorization, solve equations, by-hand and using the TI-92;
- know how to set the TI-92 mode to auto, exact or approximate values (via circular functions);
- perform numerical calculations.
Appendix 7.1

Student Interviews

After Year 11 Calculus Program, 1999

The aims of this interview are to investigate:
• the depth of the student’s conceptual understanding of differentiation
• if and how the student privileges a particular representation
• how the teacher’s conceptions affect the students

Method:
Give each student a selection of problems. Ask each student to solve each problem, while thinking-aloud using his preferred method.

Ask open ended questions which will lead each student to:
• discuss their ideas about the advantages and disadvantages of using a CAS.
• reveal how they see the various aspects of differentiation linking together

Each interview will be audiotaped and transcribed.

Protocol for each Student Interview

I’ve got a set of calculus questions here to talk about.

For each of the questions say to the student

“Could you tell me out-loud how you are solving this problem as you go along. If you decide to use the TI-92, say what keys you are using and why.

At the end of the questions ask:

“Can you tell me your opinion about using CAS for Year 11 calculus, some of the benefits, some of the problems.”
Student Interview Questions

After Year 11 Calculus Program, 1999

Name: ____________________
Date: ____________________

Question 1
Find the derivative of \( y = 7x^9 + 5x^3 - 7 \)

Question 2
Find the rate of change of \( y \) with respect to \( x \) if \( y = 7x^9 + 5x^3 - 7 \)

Question 3
Find the gradient of the curve \( y = (3x + 1)^2(x^2 - 2x - 3) \) at \( x = -2 \)

Question 4
Find the gradient of the tangent to the curve \( y = (3x + 1)^2(x^2 - 2x - 3) \) at \( x = -2 \).

Question 5
A women’s Co-operative manufactured doll’s houses. One of the women employees observed that the amount of profit depended on the number of items manufactured. She found that the profit, $P \), depended on \( n \), the number of doll’s houses made according to the relationship

\[ P = 300n - n^3 \]

(a) Find how many items should the company produce so as to give the maximum profit?

(b) What was the maximum profit?
**Question 6**

Differentiate: \((\sin x)(1 + \ln x)\)

**Question 7**

Find an approximate value of the derivative of \(y = 2x^3 + 1\) at \(x = 2\)

**Question 8**

Line \(L\) is a tangent to the curve \(f(x) = 4x^3 - x^4\) at the point \((1,3)\) as indicated.

(a) Find the derivative of \(f(x)\) at \(x = 1\)

Is your answer accurate or approximate?

(b) Find the derivative using another method.

Is your answer accurate or approximate?

(c) Compare and comment on the two answers obtained.
## Appendix 7.2

### Student Interview Questions: Purpose, Competency, and Link to Similar DCT or PRT Questions

<table>
<thead>
<tr>
<th>Student Interview Question</th>
<th>Alternative Approaches to Problems</th>
<th>Competency</th>
<th>Location DCT</th>
<th>Location PRT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>• CAS or by-hand (polynomial)</td>
<td>FSs</td>
<td>A Q.1</td>
<td>B Q.1(a)-(f)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>B Q.8(a)</td>
</tr>
<tr>
<td>2</td>
<td>• CAS or by-hand (polynomial)</td>
<td>FSn or ISn</td>
<td>A Q.8(d)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Knows immediately rate of change</td>
<td></td>
<td></td>
<td>A Q.10</td>
</tr>
<tr>
<td></td>
<td>= derivative</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>• CAS or by-hand (product of</td>
<td>FSg or FGg</td>
<td>A Q.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>polynomials)</td>
<td></td>
<td></td>
<td>A Q.5</td>
</tr>
<tr>
<td></td>
<td>• Use of G representation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>• CAS or by-hand (product of</td>
<td>FSg or FGg</td>
<td>A Q.3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>polynomials)</td>
<td></td>
<td></td>
<td>A Q.5</td>
</tr>
<tr>
<td></td>
<td>• Use of G representation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Knows immediately that gradient</td>
<td></td>
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<tr>
<td></td>
<td>of curve = gradient of tangent</td>
<td></td>
<td></td>
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<tr>
<td>5</td>
<td>• CAS or by-hand (max/min problem</td>
<td>FSs</td>
<td></td>
<td>B Q.2(a)-(b)</td>
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<tr>
<td></td>
<td>with S data).</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Use of non-calculus graphical</td>
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<tr>
<td></td>
<td>approach</td>
<td></td>
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<td></td>
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<tr>
<td>6</td>
<td>• CAS or by-hand when function</td>
<td>FSs</td>
<td></td>
<td>B Q.8(b)</td>
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<td></td>
<td>requires CAS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>• Use of N representation</td>
<td>FNs or FGs</td>
<td></td>
<td>B Q.3(a)</td>
</tr>
<tr>
<td></td>
<td>(appropriate)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>• Use of G representation</td>
<td></td>
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<tr>
<td></td>
<td>(inappropriate) or</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>• Prefers S representation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(inappropriate)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>8</td>
<td>• Prefers N representation</td>
<td>FNs or FGs</td>
<td></td>
<td>B Q.4(a)-(b)</td>
</tr>
<tr>
<td></td>
<td>(appropriate)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>• Prefers G representation</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(appropriate)</td>
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<td></td>
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</tr>
<tr>
<td></td>
<td>• Prefers S representation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(appropriate)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• CAS or by-hand (polynomial)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Knows two representations</td>
<td></td>
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<td></td>
</tr>
<tr>
<td></td>
<td>• Shows first preference of</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>representation</td>
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</tbody>
</table>
## Appendix 7.3

**Student Interview Results**

Percentage of Attempts with Nominated Behaviours on Student Interview Questions, by Class

<table>
<thead>
<tr>
<th>Q’n</th>
<th>Behaviour During Interview</th>
<th>Code</th>
<th>Class A (N=6)</th>
<th>Class B (N=9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q.1</td>
<td>Found derivative by differentiating the function</td>
<td>FSs</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Q.2</td>
<td>Immediately knew rate of change = derivative</td>
<td>ISn</td>
<td>33</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Found rate of change function by differentiating the identical function again</td>
<td>FSn</td>
<td>67</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>Not attempted</td>
<td></td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>Q.3</td>
<td>Found the gradient of the curve by differentiating the function</td>
<td>FSg</td>
<td>67</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>Used CAS to get correct answer</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Found the gradient of the curve by finding the gradient of the tangent</td>
<td>FGg</td>
<td>17</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Made conceptual errors (failed to differentiate)</td>
<td></td>
<td>0</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>Not attempted</td>
<td></td>
<td>17</td>
<td>11</td>
</tr>
<tr>
<td>Q.4</td>
<td><strong>Question 4 was done separately to Question 3</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Knew gradient of tangent = gradient of curve (answered Q.4 using Q.3 answer)</td>
<td></td>
<td>67</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Found the gradient of tangent to curve by differentiating the identical function again</td>
<td>FSg</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Found the gradient of the tangent to the curve</td>
<td>FGg</td>
<td>33</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>Made conceptual errors (failed to differentiate)</td>
<td></td>
<td>0</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>Not attempted</td>
<td></td>
<td>17</td>
<td>11</td>
</tr>
<tr>
<td>Q.5</td>
<td><strong>Maximum/minimum problem</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Used a symbolic approach</td>
<td>FSs</td>
<td>83</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>In addition, visualized or drew a graph</td>
<td></td>
<td>(0)</td>
<td>(56)</td>
</tr>
<tr>
<td></td>
<td>Verbally justified derivative = 0</td>
<td></td>
<td>(17)</td>
<td>(44)</td>
</tr>
<tr>
<td></td>
<td>Solved the problem entirely graphically, without differentiation in any representation</td>
<td></td>
<td>17</td>
<td>+11</td>
</tr>
<tr>
<td>Q.6</td>
<td>Correctly differentiated function that required CAS</td>
<td>FSs</td>
<td>50</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>Made procedural errors with CAS</td>
<td>FSs</td>
<td>50</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>No idea how to differentiate the function with CAS</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q.7</td>
<td><strong>Symbolic question, symbolic data, symbolic representation excluded</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Inappropriately chose to differentiate symbolically</td>
<td></td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>Appropriately chose a numerical method</td>
<td></td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Appropriately chose a graphical method</td>
<td></td>
<td>0</td>
<td>+11</td>
</tr>
</tbody>
</table>
Q.8  
*Repeat item to show use two representations, symbolic question, symbolic data*

<table>
<thead>
<tr>
<th>1st attempt</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Used numerical representation</td>
<td>FNs</td>
</tr>
<tr>
<td>Used graphical representation</td>
<td>FGs</td>
</tr>
<tr>
<td>Used symbolic representation</td>
<td>FSs</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2nd attempt</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Used numerical representation</td>
<td>FNs</td>
</tr>
<tr>
<td>Used graphical representation</td>
<td>FGs</td>
</tr>
<tr>
<td>Used symbolic representation</td>
<td>FSs</td>
</tr>
<tr>
<td>Not attempted</td>
<td>17 0</td>
</tr>
</tbody>
</table>

*Across the two attempts*

<table>
<thead>
<tr>
<th></th>
<th>FNs</th>
<th>FGs</th>
<th>FSs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numerical representation</td>
<td>8 11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphical representation</td>
<td>33 17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Repeated symbolic representation</td>
<td>50 73</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Not attempted</td>
<td>8 0</td>
<td></td>
<td></td>
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</tbody>
</table>

*Across all attempts on Questions 1-8*

<table>
<thead>
<tr>
<th>Q’s. 1-8</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Checked by-hand differentiation with CAS</td>
<td>0 22</td>
</tr>
<tr>
<td>Used CAS for miscellaneous algebraic procedures, factorizing, solving equations, substitution etc.</td>
<td>1.7 per student 1.1 per student</td>
</tr>
</tbody>
</table>

+ indicates a second, additional response.
Appendix 8.1

Calculus and CAS Questionnaire

Name: __________________________
Date: __________________________

Please think carefully about each statement and circle the letter that best reflects your opinion.

<table>
<thead>
<tr>
<th>SA = Strongly Agree</th>
<th>A = Agree</th>
<th>D = Disagree</th>
<th>SD = Strongly Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Circle your choice</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Attitude to calculus
1.1 Calculus is an enjoyable maths topic to study  

<table>
<thead>
<tr>
<th></th>
<th>SA</th>
<th>A</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Skills associated with using the TI-92
2.1 I learnt how to use the TI-92 very easily
2.2 I find it is easy to remember which keys to use on the TI-92
2.3 I spend more of my time concentrating on how to use the TI-92 than understanding how to do calculus exercises
2.4 There is only one way to use the keys on the TI-92 to get the correct answer for a given procedure
2.5 I make lots of mistakes when entering expressions into the TI-92
2.6 I usually know which MODE the TI-92 should be in

<table>
<thead>
<tr>
<th></th>
<th>SA</th>
<th>A</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>

Attitude to using TI-92 for learning about differentiation
3.1 I find using the TI-92 for algebra really helps me understand about differentiation
3.2 I find using the TI-92 for graphs really helps me understand about differentiation
3.3 Since using the TI-92 I have become more confident with algebra
3.4 Since using the TI-83 & TI-92 I have become more confident with using graphs
3.5 I feel confident that I can get the TI-92 to do the maths that I want it to do

<table>
<thead>
<tr>
<th></th>
<th>SA</th>
<th>A</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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</tbody>
</table>

Use of TI-92 while learning differentiation
4.1 I frequently use graphs on the TI-92 when differentiating
4.2 I frequently use algebra on the TI-92 when differentiating

<table>
<thead>
<tr>
<th></th>
<th>SA</th>
<th>A</th>
<th>D</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Comparing attitude to algebra by-hand with the TI-92 algebra
5.1 I prefer to do algebraic tasks by-hand rather than with the TI-92 SA A D SD
5.2 I prefer to do algebraic tasks with the TI-92 rather than by-hand SA A D SD

Attitude to TI-92 algebra
6.1 Using TI-92 algebra helps me differentiate easily SA A D SD
6.2 There is no advantage to me in using TI-92 algebra SA A D SD
6.3 Knowing how to do algebra by-hand algebra is much more important to me than knowing how to use TI-92 algebra SA A D SD

Use of TI-92 algebra
7.1 I often use the TI-92 to solve equations SA A D SD
7.2 I often use the TI-92 to substitute a value into a formula SA A D SD
7.3 I often use the TI-92 to expand brackets SA A D SD
7.4 I often use the TI-92 to differentiate functions SA A D SD

Use of the TI-92 to differentiate
8.1 I nearly always use the TI-92 to differentiate functions SA A D SD
8.2 I rarely use the TI-92 to differentiate functions SA A D SD

Attitude towards differentiation using the TI-92
9.1 I prefer to solve calculus differentiation problems without the help of the TI-92 SA A D SD

Attitude towards graphing
10.1 It is difficult to read values on graphs using the TI-92 SA A D SD
10.2 A TI-92 graph always shows the important parts of the graph SA A D SD
10.3 It is difficult to change the graph window on the TI-92 SA A D SD
10.4 I find it easy to draw tangents to curves using the TI-92 SA A D SD
10.5 I find it easy to find co-ordinate values of points on a graph SA A D SD

Comparing using the TI-92 for graph and algebra use
11.1 I prefer to use the TI-92 for graphs rather than algebra SA A D SD
11.2 I prefer to find the maximum value of a function using TI-92 algebra rather than a graph SA A D SD
11.3 I prefer to find the maximum value of a function using by-hand algebra rather than a graph SA A D SD
11.4 I prefer to find the maximum value of a function using a graph rather than using TI-92 or by-hand algebra SA A D SD
11.5 I prefer to find the x-intercepts of a function using algebra rather than a graph SA A D SD
Calculations on the calculator compared with by-hand
12.1 I prefer to do most numerical calculations using the TI-92 rather than by-hand

Understanding derivatives with a TI-92 available to use
13.1 It is not necessary to understand about derivatives since the TI-92 can work out the answers
13.2 With a TI-92 it is not necessary to remember the different rules for differentiating functions

Learning style
14.1 I have made at least one maths discovery of my own while using the TI-92
14.2 I understand new ideas best when doing examples by-hand
14.3 I understand new ideas best when doing examples while using the TI-92 algebra
14.4 I understand new ideas best when doing examples while using graphs

Calculus skills
15.1 I have problems understanding and remembering calculus words
15.2 I find calculus symbols such as dy/dx, f '(x) confusing
# Appendix 8.2

## Calculus and CAS Questionnaire: Summary of Class A and Class B Responses

Each number represents the class average of the students’ responses. Based on a 4-point scale, a 0.1 difference between classes represents a difference of opinion of 2.5%. The shaded blocks indicate differences of at least 10%. Class A (N = 14) and Class B (N = 19).

<table>
<thead>
<tr>
<th>No of items</th>
<th>4 Strongly Agree</th>
<th>3 Agree</th>
<th>2 Disagree</th>
<th>1 Strongly Disagree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both classes strongly agreed that knowing how to do algebra by-hand was more important to them than knowing how to use symbolic algebra</td>
<td>1</td>
<td>3.2 (A)</td>
<td>3.3 (B)</td>
<td></td>
</tr>
<tr>
<td>Both classes believed they understood new ideas better when they used by-hand procedures</td>
<td>1</td>
<td>3.1 (A)</td>
<td>3.1 (B)</td>
<td></td>
</tr>
<tr>
<td>Both classes were neutral about understanding new ideas better using CAS</td>
<td>1</td>
<td></td>
<td>2.4 (A)</td>
<td>2.4 (B)</td>
</tr>
<tr>
<td>Both classes found the TI-92 CAS calculator fairly easy to use</td>
<td>5</td>
<td>2.8 (A)</td>
<td>2.9 (B)</td>
<td></td>
</tr>
<tr>
<td>Both classes encountered the same moderate level of technical problems using the TI-92</td>
<td>7</td>
<td>2.6 (A)</td>
<td>2.5 (B)</td>
<td></td>
</tr>
<tr>
<td>Class A demonstrated more understanding about the capabilities of the calculator</td>
<td>1</td>
<td>3.0 (A)</td>
<td></td>
<td>2.4 (B)</td>
</tr>
<tr>
<td>Class A used the symbolic algebra facility on the TI-92 for a greater variety of algebraic procedures</td>
<td>4</td>
<td>2.9 (A)</td>
<td>2.5 (B)</td>
<td></td>
</tr>
<tr>
<td>Class A students more strongly preferred by-hand algebra to symbolic algebra</td>
<td>2</td>
<td>2.9 (A)</td>
<td>2.5 (B)</td>
<td></td>
</tr>
<tr>
<td>Class A better understood the advantages of differentiating with CAS</td>
<td>2</td>
<td>3.1 (A)</td>
<td></td>
<td>2.7 (B)</td>
</tr>
<tr>
<td>More Class A students preferred to use symbolic algebra for differentiation</td>
<td>1</td>
<td></td>
<td>2.6 (A)</td>
<td>2.2 (B)</td>
</tr>
<tr>
<td>Both classes agreed it was necessary to remember differentiation rules even with symbolic algebra to use</td>
<td>2</td>
<td>3.1 (A)</td>
<td></td>
<td>2.8 (B)</td>
</tr>
<tr>
<td>Class A did not agree that they enjoyed calculus whereas Class B was neutral</td>
<td>1</td>
<td></td>
<td>2.4 (B)</td>
<td>1.9 (A)</td>
</tr>
<tr>
<td>Class B found using calculus words and symbols fairly easy</td>
<td>2</td>
<td>2.8 (B)</td>
<td>2.6 (A)</td>
<td></td>
</tr>
<tr>
<td>Both classes preferred to use CAS for graphs rather than algebra</td>
<td>1</td>
<td>2.8 (A)</td>
<td></td>
<td>2.9 (B)</td>
</tr>
<tr>
<td>Both classes were neutral about how helpful graphs were for understanding differentiation</td>
<td>1</td>
<td>2.6 (A)</td>
<td></td>
<td>2.6 (B)</td>
</tr>
<tr>
<td>Class B used graphs more frequently for differentiation</td>
<td>1</td>
<td></td>
<td>2.7 (B)</td>
<td>2.4 (A)</td>
</tr>
<tr>
<td>Both classes were neutral about their preference for finding the maximum value of a function using symbolic rather than graphical procedures</td>
<td>3</td>
<td></td>
<td></td>
<td>2.6 (B)</td>
</tr>
</tbody>
</table>
Appendix 9.1

Evaluation of the TI-92

As you progress through this unit, please describe (in detail if possible) any insights and understandings “magic moments” that have occurred while studying calculus using the TI-92.

Mathematics that the TI-92 has helped me learn

The TI-92 did not help me learn about

Things I think are great about the TI-92!

Things that are frustrating/confusing on the TI-92!

What would make the TI-92 more useful in your mathematics classes?
### Appendix 10.1

**New Test Items for each Framework Competency**

DCT items for each competency “at a point” and “at all points”

<table>
<thead>
<tr>
<th></th>
<th>&quot;At a point&quot;</th>
<th>&quot;At all points&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. FNn</td>
<td>The concentration $C$ units of a drug in a patient’s bloodstream $t$ hours after injection is given by $C = \frac{0.45}{(3 + t)^2}$. What is the average rate of change in concentration of the drug in the patient’s bloodstream over the first two hours after the drug was first administered? [This calculation involves two points.]</td>
<td>⊗</td>
</tr>
<tr>
<td>2. FGg ***</td>
<td>Use the graph of the function $y = f(x)$ below to find the gradient of the curve at $x = 2.5$. [This competency is only shown if the student draws a tangent to the graph. An alternative item is to give the graph context and function rule ♥.]</td>
<td>⊗ Draw a graph of the gradient function of $y = f(x)$ given the graph of the function. <img src="image" alt="Graph" /></td>
</tr>
<tr>
<td>3. FSs ♥</td>
<td>⊗ Find the value of $f'(3)$ when $f(t) = t^3(1-3t)$.[This competency is only shown if the student uses symbolic differentiation.]</td>
<td>Find $g'(x)$ when $g(x) = (x^2 + 2)(x - 2)$.</td>
</tr>
</tbody>
</table>
4. The Zoom-In function on a calculator was used five times on the graph of the function $y = f(x)$ near $P$ where $x = 1$. The values of the function close to $P$ were observed and are listed below. Find the best estimate of the gradient of the curve at $P$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.998</th>
<th>0.999</th>
<th>1.000</th>
<th>1.001</th>
<th>1.002</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>4.593</td>
<td>4.518</td>
<td>4.444</td>
<td>4.370</td>
<td>4.297</td>
</tr>
</tbody>
</table>

5. Values of the function close to $x = -2$ are listed below. Find the best estimate of the derivative at $x = -2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.998</td>
<td>7.975</td>
</tr>
<tr>
<td>-1.999</td>
<td>7.988</td>
</tr>
<tr>
<td>-2.000</td>
<td>8.000</td>
</tr>
<tr>
<td>-2.001</td>
<td>8.012</td>
</tr>
<tr>
<td>-2.002</td>
<td>8.025</td>
</tr>
</tbody>
</table>

6. The number of sugar gliders in a remnant forest was tracked over nine months and the graph shows how the number of gliders ($y$) changed with time where ($x$) is the number of months since the counting commenced. Find the rate of change in the population of the sugar gliders 3 months after counting started.
<table>
<thead>
<tr>
<th>Problem</th>
<th>Equation/Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. FGs</td>
<td>Find $f'(2)$.</td>
</tr>
<tr>
<td>8. FSn</td>
<td>The concentration $C$ units of a drug in a patient’s bloodstream $t$ hours after injection is given by $C = \frac{0.45}{(3+t)^2}$. What is the rate of change in concentration of the drug in the patient’s bloodstream exactly two hours after the drug was first administered? [This competency is only shown if the student uses symbolic differentiation.]</td>
</tr>
<tr>
<td>9. FSg</td>
<td>A curve has function rule $y = x^5 + x^{-3} + 1$. Find the gradient of the curve at $x = -2$. [This competency is only shown if the student uses symbolic differentiation.]</td>
</tr>
<tr>
<td>10. INn</td>
<td>The rate of change of volume of air inside a balloon is $\frac{3}{t}$ where $V$ is the volume in cubic centimetres and $t$ is the time in seconds since the air was first pumped into the balloon. Explain what is happening to the volume of the ball 10 seconds after the air was first pumped into the balloon. [This competency is only shown if the student uses symbolic differentiation.]</td>
</tr>
</tbody>
</table>

The cost of production ($SC$) each week of computers in a factory is related to the number of computers produced ($n$). If $C = 300 - 10n + 0.5n^2$ find the rate of change of cost of production with respect to the number of computers produced.

Find the gradient function of the curve that has function rule $y = x(x-3)(x+1)$.

The rate of change of the volume of a container with respect to the time since air was first pumped into the balloon is determined by the formula $V'(t) = 3t$ where $V$ is the volume in cubic metres and $t$ is the time in seconds since the air was first pumped into the balloon. Explain what is happening during the first five seconds of the pumping.
11. **IGg**

Place a cross on the graph where its gradient has the greatest value and explain how you made your decision.

![Graph]

The gradient function $f'(x) = 5x$ of a function $f(x)$ is sketched below. Explain how the gradient of $f(x)$ changes between $x = -1$ and $x = 1$.

![Gradient Function]

12. **ISs**

For the function $y = f(x)$

$$\frac{dy}{dx} = f'(5) = 3$$

Explain the meaning of this statement.

For the function $G(x)$

$$G'(x) = 6x + 1$$

Explain the meaning of this statement.

13. **INg**

At $P(1, -4)$, a point on the graph of the function $f(x)$, the instantaneous rate of change of $y$ with respect to $x$ is 7.

What is the gradient of the curve at $P$? Explain your answer.

A curve has the function rule $y = f(x)$. The rate of change of $y$ with respect to $x$ is given by the formula $5x^3 + 7x$.

What is the gradient function of the curve? Explain your answer.

14. **INs**

A ball is thrown into the air and its flight path depends on the time since it was thrown. $H$ is the height of the ball above the ground (in metres) and $t$ is the flight time (in seconds) and the height of the ball above the ground is a function of time, $H(t)$.

Five seconds after being thrown in the air, the ball was observed to be 100m above the ground and climbing at the rate of 3 metres/second.

What is the value of $f''(5)$? Why?

A ball is thrown into the air and it height depends on the time since it was thrown. $H$ is the height of the ball above the ground (in metres) and $t$ the flight time (in seconds) and its flight path is modelled by the function $H(t)$.

If the rate of change of height with respect to time, is given by the formula $2t^2 + 7t$, what is the derivative of the function $H(t)$. Why?
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>15.</strong></td>
<td><strong>IGn</strong></td>
</tr>
<tr>
<td></td>
<td><em>P</em> (2,7) and <em>Q</em> (2.001, 7.351) are two points on the curve <em>y</em> = <em>f</em>(<em>x</em>).</td>
</tr>
<tr>
<td></td>
<td>At <em>P</em>, the gradient of the curve is 3.</td>
</tr>
<tr>
<td></td>
<td>What is the instantaneous rate of change of <em>y</em> with respect to <em>x</em> at the point <em>P</em>?</td>
</tr>
<tr>
<td></td>
<td>Explain your answer.</td>
</tr>
<tr>
<td></td>
<td>⊗ The gradient function of <em>H</em>(<em>x</em>) is given by the formula <em>x</em>^2 − <em>x</em>.</td>
</tr>
<tr>
<td></td>
<td>What is the rate of change of <em>H</em>(<em>x</em>) with respect to <em>x</em>?</td>
</tr>
<tr>
<td></td>
<td>Explain your answer.</td>
</tr>
<tr>
<td><strong>16.</strong></td>
<td><strong>IGs</strong></td>
</tr>
<tr>
<td></td>
<td>⊗ At <em>P</em> (-1, 9), the gradient of the function <em>g</em>(<em>x</em>) is −4.</td>
</tr>
<tr>
<td></td>
<td>What is the value of the derivative of the function <em>g</em>(<em>x</em>) at <em>P</em>? Explain your answer.</td>
</tr>
<tr>
<td></td>
<td>The gradient function of <em>K</em>(<em>x</em>) is given by the formula 3<em>x</em>^2 + 10.</td>
</tr>
<tr>
<td></td>
<td>What is the derivative of <em>K</em>(<em>x</em>)? Explain your answer.</td>
</tr>
<tr>
<td><strong>17.</strong></td>
<td><strong>ISn</strong></td>
</tr>
<tr>
<td></td>
<td>The derivative of the function <em>G</em>(<em>t</em>) is the function <em>t</em>^2 − 5<em>t</em>^3.</td>
</tr>
<tr>
<td></td>
<td>To find the rate of change of <em>G</em> with respect to <em>t</em> at <em>t</em> = 4, you should:</td>
</tr>
<tr>
<td></td>
<td>A. Differentiate <em>t</em>^2 − 5<em>t</em>^3 and then substitute <em>t</em> = 4</td>
</tr>
<tr>
<td></td>
<td>B. Find where <em>t</em>^2 − 5<em>t</em>^3 = 0</td>
</tr>
<tr>
<td></td>
<td>C. Find the value of <em>G</em>(<em>4</em>)</td>
</tr>
<tr>
<td></td>
<td>D. Find the value of <em>t</em>^2 − 5<em>t</em>^3 when <em>t</em> = 4</td>
</tr>
<tr>
<td></td>
<td>E. None of these</td>
</tr>
<tr>
<td></td>
<td>Circle the letter corresponding to your answer. Explain your answer.</td>
</tr>
<tr>
<td></td>
<td>⊗ The derivative of the function <em>G</em>(<em>x</em>) is the function 5<em>x</em>^4 + 10<em>x</em> − 8.</td>
</tr>
<tr>
<td></td>
<td>What is the rate of change of <em>G</em> with respect to <em>x</em>? Explain your answer.</td>
</tr>
<tr>
<td><strong>18.</strong></td>
<td><strong>ISg</strong></td>
</tr>
<tr>
<td></td>
<td>The derivative of <em>f</em>(<em>x</em>) is the function <em>f′</em>(<em>x</em>) = <em>x</em>^3 − 5<em>x</em> − 3.</td>
</tr>
<tr>
<td></td>
<td>What is the gradient of <em>f</em>(<em>x</em>) when <em>x</em> = -2?</td>
</tr>
<tr>
<td></td>
<td>Explain your answer.</td>
</tr>
<tr>
<td></td>
<td>⊗ The derivative of <em>H</em>(<em>x</em>) is the function 5<em>x</em>^3 + 10<em>x</em> − 8.</td>
</tr>
<tr>
<td></td>
<td>What is the gradient function of <em>H</em>(<em>x</em>)? Explain your answer.</td>
</tr>
</tbody>
</table>

*** Replacement items for [FGg, FSg, FSn] (all with symbolic function data) that were problematic on the DCT (see Appendix 1.2).

♥ With function data, differentiation “at a point” is theoretically possible in the three representations.

⊗ New items, 3 associated with ‘at a point’, 10 associated with “at all points”.

∅ Not feasible to test.
Author/s: Kendal, Margaret

Title: Teaching and learning introductory differential calculus with a computer algebra system

Date: 2001-09


Publication Status: Unpublished

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