AN EXPLORATION OF ALGEBRAIC INSIGHT AND EFFECTIVE USE OF COMPUTER ALGEBRA SYSTEMS

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ABSTRACT

At a time of transition, when the increasing availability and affordability of Computer Algebra Systems (CAS) presents mathematics educators with new challenges, this thesis explores two facets of students’ abilities and understanding that impact on the use of CAS in teaching and learning mathematics. In this thesis, these are called ‘Algebraic Insight’ and ‘Effective Use of CAS’. A framework is presented and described for each construct and then the frameworks are explored within the context of a course in introductory calculus, taught by the researcher to a class of 21 undergraduate tertiary students.

Algebraic Insight is the subset of Symbol Sense required when using CAS for the mathematical solution phase of problem solving. The framework breaks Algebraic Insight into two aspects: ability to Link Representations (symbolic, numeric, graphical); and Algebraic Expectation, the cognitive skill required to monitor symbolic work (comparable to arithmetic estimation for monitoring numeric work). The framework of Effective Use of CAS is also divided into two aspects: Technical, using syntax and program features; and Personal, the willingness to use CAS in a judicious manner.

Throughout the 15 week course a bank of tests, surveys, observation and interviews assessed students' levels of Algebraic Insight and Effective Use of CAS. The instruments successfully monitored changes and demonstrated class improvement, a finding clarified by 7 detailed case studies. Effective Use of CAS and Algebraic Insight are inter-dependent. First, sufficient Algebraic Insight is needed to begin to use CAS. Second, CAS can be employed successfully as a learning tool for exercises designed to improve Algebraic Insight provided the student demonstrates at least a moderate level of Effective Use of CAS. Third, Algebraic Insight helps students to use CAS in a strategic manner.

The new outcomes of this study that will be of use to teachers and curriculum planners are the frameworks of Algebraic Insight and Effective Use of CAS, the quick Algebraic Insight Quiz and the CAS use survey. Of the terms coined for this study, perhaps the most useful will be Algebraic Expectation. The new terms and new frameworks provide a structure for the new focus that the teaching of algebra must adopt in a CAS environment.
DECLARATION

This is to certify that:

i. the thesis comprises only my original work except where indicated in the preface,

ii. due acknowledgment has been made in the text to all other material used,

iii. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices

Signed

Date
PUBLICATIONS ASSOCIATED WITH THIS THESIS


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CHAPTER 1
INTRODUCTORY OVERVIEW

Historical perspective
This thesis addresses issues related to the teaching and learning of algebra and calculus raised by the increasing availability of Computer Algebra Systems, both in classrooms and for individuals. Over the centuries mathematicians have used many different 'tools' to assist with the speed and accuracy of their calculations. From the earliest times these tools have progressed from stone counters and sand drawings through to the development of books of tables, the abacus, slide rule, difference machines and adding machines through to calculators and computers. In each time, learning mathematics has included learning to ‘effectively use’ those tools available to quickly produce correct solutions to problems and at each stage there has been some reluctance to change. Controversy surrounding change is common; in particular, we are familiar with that ongoing debate which has continued among both professional educators and concerned parents, since hand-held arithmetic calculators became widely and cheaply available in the late 1970's. Recent technological tools have been seen as both an aid and a challenge to mathematics educators because these tools 'remember' mathematical facts and follow algorithms to perform mathematical calculations. The availability of such tools has raised many questions about both what and how mathematics should be taught.

Etlinger, writing in 1974, when four function calculators were first available for classrooms, noted that technology could be used functionally (to produce answers) or pedagogically (to improve learning): a distinction that will be drawn in this thesis. He outlined some of the questions that he felt needed research, in order to inform mathematics teachers’ use of arithmetic calculators. For example:

- Will experimentation with the calculator teach children about numbers and operations or merely about the calculator?
- Will the ideas behind the arithmetic operations be more widely understood because the children have immediate feedback from the calculations? Will the patterns be easier to grasp? Or will the ideas become more mysterious and unintelligible because it is a "black box" which is relating the answer to the calculation? (Etlinger, 1974, p44)
These questions have been behind much of the extensive experimentation that has taken place since the 1970s, seeking to find ways to advantage students through the use of four function calculators. This research has strongly concluded that, in order for such arithmetic calculators to be used as an effective tool, students need to be taught ‘number sense’. This includes the skill of arithmetic estimation. Much has been written about number sense and its importance (McIntosh, Reys and Reys, 1992, provides a thorough summary article). Agreement about the importance of number sense is indicated by the fact that, for some time, it has been promoted as a desirable objective of the mathematics curricula for students in many countries. The Cockcroft report (Cockcroft, 1982) used a different phrase, that of ‘at-homeness with numbers’, but encouraged the same concept, while in the United States the NCTM standards of 1989 list ‘number sense’ as something needing ‘increased attention in content and emphasis’ for K-4 mathematics (p20) and 5-8 mathematics (p7).

**Changes in power and availability of technology**

Just as arithmetic calculators have both challenged and aided primary mathematics teachers, now, at the beginning of the twenty-first century, secondary and tertiary mathematics educators face similar questions about the use of graphical calculators and Computer Algebra Systems (CAS). Amongst other functions a CAS can do ‘exact’ arithmetic, (for example, work with common fractions and surds as well as decimals); graph functions; manipulate symbolic algebraic expressions; and differentiate and integrate functions both symbolically and numerically. Computer Algebra Systems were first created by mathematicians and for mathematicians. They were first written to solve particular problems and were not interactive. Later developments saw this style of program written for different platforms and for interactive use. The use of CAS increased rapidly once Personal Computer versions were released and these programs are now being built into hand-held calculators. These sophisticated programs are both increasingly easy to use and affordable. This technology is very powerful, but it quickly becomes apparent to anyone who tries to use CAS that it is a tool, not a magic box, and as such one needs to learn both the mechanics of using the tool and understand the purpose of its use.
Research informs and challenges

Over the last two decades, teachers and researchers have begun experimenting with the use of CAS in teaching and learning mathematics. Results of research on the use of CAS in teaching mathematics have been published since the late 1980's and some of this literature will be reviewed in detail in Chapter two. In summary, it essentially showed that some mathematics educators were extremely positive about the place of CAS in teaching mathematics. A major early study reported by Kathleen Heid (1988) suggested that those students who used CAS in their mathematics learning gained greater conceptual understandings than those in the control group who were taught in a traditional manner. The experimental group also experienced no significant loss in 'by-hand' skills. Day (1993) enthusiastically wrote that 'The power and flexibility of technology can help change the focus of school algebra from students becoming mediocre manipulators to their becoming accomplished analysts' (p30). Further studies raised questions about the use of CAS in learning mathematics. Allaire and Fabricant (1995), for example, were positive but cautious about the place of CAS in teaching mathematics. They found that the use of CAS showed up gaps in students' knowledge and that students were less successful in their use of CAS with unfamiliar mathematics. They felt that using CAS highlighted their students' weaknesses and so helped to focus their teaching. The issue of algebraic readiness for the use of CAS has been raised by a number of researchers (Hansen, 1993, Mitic & Thomas, 1994, Monaghan, 1993, Stevenson, 1995). These writers ask questions about exactly what algebra it is that students need to know before they can benefit from or make use of CAS. At the extreme, Hansen (1993) was adamant that 'such tools should not be used until the student has mastered the appropriate algorithms for the application being studied' (p414). A more typical position is that expressed by Berry (1996) when he comments that to use CAS effectively, students need a 'good feel for number' and a 'good feel for mathematics'. Berry also suggests taking what has been learnt about teaching with arithmetic calculators and extending this to working with the new CAS technology. These reports asked questions about the approach that should be taken to teaching algebra and the thinking that should be developed, but they only circled the issue in broad vague terms; an indication that it needed exploring in detail.
New technology - familiar questions

Experience and research has shown that to use arithmetic calculators effectively, students need arithmetic estimation skills (McIntosh, 1992). This is part of Berry's (1996) 'good feel for number' concept. The extension of this, for a Computer Algebra System, to 'a good feel for mathematics' will include the ability to scan algebraic expressions and estimate patterns (Fey, 1990), as well as symbol sense (Arcavi, 1994). This thesis explores these proposals in order to suggest how a ‘good feel for number’ concept might be paralleled in algebra.

Scope of this study

This study explores one aspect of 'a good feel for mathematics', which will be described as Algebraic Insight. A review of literature, and reflection on teaching and mathematical experience was used to justify a set of key elements that contribute to a definition of Algebraic Insight. This term, ‘Algebraic Insight’, will be applied to the thinking required to monitor the support and use of CAS. A more precise definition of its scope and purpose is given in Chapter 3.

Successfully using CAS to do mathematics will not only be determined by students’ ability to monitor the mathematics calculations, but also their ability and willingness to use the program. This study will also therefore explore the notion of Effective Use of CAS. Again reflection on the researcher’s own teaching experience and that reported in the literature was used to justify a set of elements that contribute to a definition of Effective Use of CAS, explained in Chapter 3.

As a result, working frameworks were created to define and describe both Algebraic Insight and Effective Use of CAS. These frameworks are seen as a major outcome of this thesis. As a trial of their usefulness, they were applied to a naturalistic, ethnographic style study using a mixture of open and closed data collection techniques. This study maps the progress of a class, of tertiary students, using CAS with respect to each of these abilities – Algebraic Insight and Effective Use of CAS. It then considers possible relationships between the two.
Research questions and data sources

The results of the study will add to our understanding of the thinking required to use CAS in learning mathematics. Specifically, the study addresses five main questions:

- What are the major components of Algebraic Insight?
- What are the major components of Effective Use of CAS?
- Does Algebraic Insight change during a course taught using CAS?
- Does Effectiveness Use of CAS change during a course taught using CAS?
- What are the links between Effective Use of CAS and Algebraic Insight?

The study was conducted with first year tertiary students, at an Australian University, who were taking an introductory calculus course with access to the CAS known as DERIVE, which was taught by the researcher. During the course data was collected by the teacher/researcher using tests, surveys, interviews, students’ work and observation. The study allowed for reflection on the usefulness of the frameworks for planning teaching and for analysing students’ thinking and behaviour with respect to mathematics and the use of CAS.

Outline of this thesis

There are a further eight chapters in this thesis. General literature, which both sets the context and need for this study, is reviewed in Chapter two. Chapter three has two distinct sections: one devoted to Algebraic Insight and the other to Effective Use of CAS. Specific literature on understanding algebra and symbol sense is reviewed, and from this a detailed description of what constitutes Algebraic Insight will be developed and summarised in a framework. Effective Use of CAS will be treated in a similar manner.

These frameworks form the foundation of this thesis and the study, which subsequently follows their definition, provides an opportunity to assess both the use of the frameworks and students progress in these two capabilities.

The setting of this study, undertaken with a small group of first year undergraduate students studying an introductory calculus course, is described in Chapter four. This description includes details of what it meant, in this context, to use CAS in teaching,
learning and assessment. Chapter five outlines the methodology used for data collection to map changes in Algebraic Insight and Effective Use of CAS. The general class results are summarized with some discussion in Chapter six. Chapter seven consists of individual case studies that allow us insights into seven students’ responses. These case studies will both provide more information, and may explain results found in Chapter six. The findings of both the class results and the case studies are discussed along with reflections of the teacher/researcher in Chapter eight. Finally, brief conclusions of the whole study will be presented in Chapter nine.

What contribution will this thesis make to this field of research?

This thesis will add to our knowledge of the place of CAS in mathematics education by providing working definitions of Algebraic Insight and Effective Use of CAS. The study required the development of instruments for assessing the components of these two abilities. This process and the results will serve to illustrate the degree to which the frameworks are of practical use for planning teaching, learning, and assessment of mathematics. The outcomes will add to our understanding of students’ Algebraic Insight and Effective Use of CAS and the relationships that may exist between these abilities.
CHAPTER 2

MOTIVATION FOR STUDYING ALGEBRAIC INSIGHT AND EFFECTIVE USE OF CAS

This study was motivated by two key factors: the researcher's teaching and research experience (which will be outlined in Chapter 4) and a review of the relevant literature. A review of the literature related to this study, showed that the use of CAS for teaching and learning mathematics has been the subject of research publications since the late 1980s. Over this time there has been considerable agreement about what CAS might offer the teaching and learning of mathematics. As these features have been explored in classrooms much has been learnt. From the many reports it may be inferred that students benefit from the use of CAS when learning mathematics because it can produce accurate results quickly using symbolic, graphical and numeric representations. It is also clear that for CAS to be an aid to learning, students need to use CAS effectively and both have and develop Algebraic Insight. These two factors, noted from this literature review, are the main focus of this thesis and are explored (with further literature) and operationalised in Chapter 3 ‘Algebraic Insight and Effective Use of CAS’. First, however, let us consider the literature which precedes this thesis.

What could CAS offer?

In 1982 Herbert S. Wilf (1982) alerted the American Mathematical Monthly to ‘The disk with the college education’. What was new here was the prediction of sudden mass availability of a computer program that could do symbolic manipulation of mathematical expressions. For Wilf, his first evening spent using a CAS was sufficient to make him realise that such technology would inevitably impact on the teaching and learning of mathematics. He wrote his article to alert mathematics educators to be prepared, and to start considering the implications of this next technological wave.

The results of studies considering possible implications of this technology for teaching and learning mathematics took some years to reach publication. The most famous of these must be the work of Kathleen Heid published in 1988. Heid (1988) compared the mathematical working, thinking and test results of two groups of students: one taught
calculus with CAS and one without. The group with access to technology also spent 12 weeks concentrating on concepts and applications without learning the usual by-hand skills, relying instead on the CAS to generate results. Only in the last 3 weeks of the course were they introduced to the associated traditional pen and paper routines. The control group was taught a traditional course in the usual manner with lectures and tutorials. When they were tested at the end of the courses, the experimental group demonstrated both better conceptual understanding than the control group and almost equaled their skill level for pen and paper routines.

As a result of her experience, Heid (1989) reflected on those characteristics of symbolic manipulation systems that could have potential for changing the content and processes of mathematics taught in secondary schools. She summarised these characteristics as follows:

1. Symbolic-manipulation results are exact and free of manipulation errors.
2. Symbolic-manipulation results are quickly generated.
3. A wide range of symbolic capacities is available within a single environment.
4. Symbolic-manipulation systems can handle more complicated problems than most students can do by hand. (p411)

These four characteristics of CAS will, in this thesis, be used to provide a structure for examining studies, concerning the use of CAS in teaching and learning mathematics, reported in the literature. Studies, mainly from the period 1989 to 2000 have informed the work for thesis. As CAS availability is escalating, there is a range of new approaches in very recent literature that is not fully reported here.

Researchers report on CAS use

**CAS results are exact and free of manipulation errors**

Heid (1989) suggested that being able to produce results which are exact and free from manipulation errors aids the teaching of real mathematics, in particular problem solving, pattern finding, theorem production and quantitative analysis of real life situations. The accuracy of results gives students confidence to rely on the answers and go further in exploration. Dugdale et al (1995) felt that the experience of using CAS to find patterns and solve problems then led to a greater understanding of variables and functions. These concepts, she said, were developed by the exposure to many correct examples.
Students felt they had some control over values for variables and their understanding developed as they observed the effect of substituting different values into functions.

Demana and Waits (1990) also commented that CAS encouraged students to use an inductive method to find patterns. These researchers, however, did not find that production of exact solutions was necessarily helpful to their understanding. In a later paper (Waits and Demana, 1992) they suggest that the usefulness of exact CAS solutions may be over emphasized. Who, they ask, apart from mathematics teachers find beauty, much less meaning, in a solution like $\frac{1 + \sqrt{2}}{3}$? Unless students have some sense of what this number means, then the solution will be of no practical significance. These different responses do not represent conflicting views on the usefulness of CAS, but rather different mathematical goals. For the purpose of solving a real world problem, an approximate answer will meet a pragmatic need, while exact answers, which may demonstrate symmetry or other patterns, will be of greater epistemic value.

*Students note patterns and unexpected outcomes*

Demana and Waits are out of step with the majority writers who value these exact solutions and the facility to work with symbols because they help students see patterns. The value of exact or symbolic solutions was reported, for example, by Smith (1996), Gilligan, (1993), Allaire and Fabricant (1995) and Davies and Fitzharris (1995). Using CAS to discover patterns, they suggested, helped students to learn mathematics. Students could use CAS quickly and easily to produce correct solutions to a series of related problems where, for example, the value of some parameter would be changed in each example. Students could make a conjecture based on observed patterns and then use CAS again, to test this conjecture. The outcome may confirm their theory but even obtaining an unexpected outcome can also provide valuable learning for students. Through repeated experience of testing conjectures, based on particular values, students may gain insight into the need for, and concept of, proof. Gilligan (1993) commented that this exploratory approach required students to develop a structural view of mathematics, linking algebra with graphical representations.
More participation of students in the learning process

Bennett (1995) reported that one of the reasons students liked using CAS was because it ‘saved silly mistakes’. This trust that CAS will produce results that are free from manipulation errors increased students’ confidence and hence their willingness to participate.

Mayes (1993) and Yershalmy and Gilead (1997) observed that CAS use encouraged discovery learning. Students’ participation in the learning process, through exploration and discovery, facilitated peer to peer interaction (Geiger, 1998). Their students actually talked about mathematics. Gilligan (1993) and the present author (Pierce, 1999) have also reported that the use of CAS promoted spontaneous classroom discussion of mathematics. Students from that previous study undertaken from 1996 to 1998 (Pierce and Stacey 2001b) said:

In the labs we get together as a group. Something will happen on one machine and everyone will go and look at the graph or equation. In ordinary classes we take down notes but in the labs we discuss what we are learning. (Pierce 2001b, p37)

As a consequence of these discussions, students were better able to use their own words to explain and express their mathematical understanding to each other and their teacher. Heid (1988) commented that the students in her experimental group using CAS were able to do this better in interviews than those students taught using traditional methods. She also observed that when the students who had used CAS made an error or had forgotten a step they were better able to explain from first principles or by using an alternative representation what they understood the problem to be. She was pleased that these students were able to use their own words and explanations, and not just reproduce a learned routine as was common from control-group students.

CAS results are generated quickly

Bennett (1995) also found that students liked using CAS because it ‘saved time’. This increase in speed meant that students could be asked to undertake investigations that would simply have taken too long to do by hand. Atkins, Creegan and Soan (1995) found that such tasks could help students learn to think mathematically. The ability to quickly draw many graphs and link these to algebraic forms helped students gain mathematical insight. Small and Hosack (1991), Mayes (1993) and Dugdale (1995) all
felt that the speed of CAS operations meant that the time saved could be used for exploration. Berry (1997) suggested that with extra time available students should be encouraged to ask ‘What happens if?’ and ‘Why?’ questions. Students could be encouraged to make and test conjectures to increase their understanding of algebra. As well as freeing up some time in the busy secondary mathematics curriculum, the speed of calculation provided by CAS meant that students did not give up or lose sight of the objectives of tasks since they only required concentration for a short time.

**Wide range of symbolic capacities available in a single environment**

Many researchers, such as Tall and Thomas (1991) and Dreyfus (1994) have discussed the value of students seeing different representations of functions. Tall and Thomas outlined the importance of versatile thinking in mathematics. They said that:

> a sequential/logical/analytical way of carrying out a succession of mathematical processes needs to be complemented by a global/holistic overall grasp of the context... [Some brain function is] related to verbal, aural and linguistic abilities...[these] constitute a sequential/verbal mode of thought. Others, relating to visual, tactile and other senses can give only non-verbal output...characterised as global/holistic thought. ...there may be aspects of visual thinking involved in understanding mathematics which are less easily verbalised...(p128-129)

Tall (1992) later wrote that ‘symbols alone cannot provide a total environment for mathematical thinking’ (p64). Dreyfus (1994) wrote that:

> The idea is to use several representations of the same concept in such a way that different aspects of the concept are stressed in different representations, and that students are helped to conceptually link corresponding aspects in different representations (p.204).

Working with different representations of functions enhances students’ understanding as they add different mental images to their schema for these concepts. The ability of CAS to link visual, symbolic, and numeric representations supports and promotes such versatile thinking. CAS facilitates this experience as students move between algebraic, graphical, and numeric representations of functions with the press of a button. Demana and Waits, (1990), Arnold (1992), Day, (1993) and Dugdale et al (1995) all comment on students gaining mathematical meaning by linking and observing the relationships between each of these three representations. Brunner (1998) gives examples of students who, when using CAS, developed new algorithms for solving problems using graphs and tables as well as algebra.
Students use more, different, and unexpected strategies

The wide range of symbolic capacities available through CAS meant that for most problems there was no longer one ‘best’ method of solution (Atkins, Creegan and Soan, 1995). Students might use more than one method (Bennett, 1995), and move between representations (Yerushalmy and Gilead, 1997). It was not uncommon for a student to use strategies other than traditional methods which the teacher may have suggested or expected (Smith, 1996). In summarising the experiences of several French researchers, Artigue (2001) reported that students’ work may not reflect the ‘actual official teaching devoted to various issues’. She suggests that the range of methods students used proceeded through an ‘explosion-reduction’ cycle and that those students gave more importance to the search for coherence between information coming from different applications (symbolic computations, graphical representations and associated approximate values obtained in the graphical application, table of values) than to the search for a decisive argument. (p7)

Not only did CAS provide alternative paths for students, but the range of representations available meant that teachers could take quite different approaches to teaching the same material. Kendal and Stacey (1999a, 1999b) report a study in which three teachers and three classes took part. Each class had access to CAS for all of their mathematics work during the time of the study, and not only were all three teachers supplied with the same teaching guidelines and materials, but they were also involved in the actual planning. Despite this each teacher chose to use the materials and CAS in a different way with emphasis on different methods of solution. CAS facilitates this variety.

Symbolic manipulations systems (CAS) can handle more complicated problems than most students can do by-hand

Kutzler (1994) introduced the image of a multistoried “house of mathematics” and illustrated three levels: arithmetic, elementary algebra and equation solving. He reminded us that at each stage most students’ understanding of topics is incomplete and the gaps may inhibit progression to more interesting mathematics. He suggested that CAS could be viewed as scaffolding to provide support for students with partial mastery of lower level topics. CAS could enable them to work on higher level problems that would be impossible for them without this assistance. This technological support, he claimed, would enable more students to view something of the beauty of mathematics.
Several researchers commented that the use of CAS did indeed enable their students to tackle a broader range of problems (for example Mayes, 1993, Llorens-Fuster 1995). McCrae, Asp and Kendal (1999) found that the students in their study were able to progress more rapidly to harder problems. In their study, conducted with year 11 students using CAS for introductory calculus, they found that students were able to tackle problems usually reserved for year 12 students who were studying calculus for the second time. CAS supported this extension.

*Students take a broader view of mathematical problems*

Using CAS to handle the more complex manipulations and calculations of algebra or calculus frees the student to think more about formulating problems and interpreting solutions. Demana and Waits (1990), Arnold, (1992), and Davies and Fitzharris (1995) all emphasized the importance of students being able to concentrate on interpreting the results, and not just on the potentially long series of steps that lead to the result. In the past, they said, so much effort and time went into techniques, in which students frequently made errors, that the skills of checking and interpreting results were neglected. CAS facilitates that shift in emphasis.

From the teacher and researcher perspectives, we can see that the reports of the last decade provide strong support for Heid’s (1989) contention that these four characteristics of working with CAS do indeed make a positive contribution to students learning of mathematics. It is important also to consider what students’ reaction to CAS had been. Many researchers make some comment on this too.

*Students’ reaction to CAS use*

Students’ reactions to the use of CAS have been mixed. While most students report a positive or neutral response, not all students feel that way. Llorens-Fuster (1995) who studied two groups, one working with and one without CAS, noted that a greater percentage of students in the CAS group completed the course and chose to take the examination. He felt that the CAS had allowed the experimental group to achieve success that had increased their self-confidence in mathematics. Davies and Fitzharris (1995) observed that their students seemed to enjoy using CAS, as did Atkins, Creegan and Soan (1995); however their students reacted against spending extra, out-of-class time in computer laboratories.
Mayes (1998) reported a study of affect in algebra, comparing a treatment (CAS) group and a control group. He found no statistically significant difference between the groups, but from his experience of observing students had doubt about the results. He thought the wording of the questions asked and the early stage of development of the new CAS course might account for this results in this study.

It is a common claim that CAS will relieve students of the drudgery of doing by-hand calculations; however Lagrange (1996) commented that in his experience not all students wanted to be relieved of pen and paper work and many, in fact, enjoyed doing routine calculations. In an earlier study, the present author (Pierce, 1999a) presented a group of students with a series of simple algebraic questions and asked students to specify their preferred tool for speed, accuracy and learning. She found that students commonly chose to use CAS for speed and accuracy but showed a preference for learning by-hand. They felt that working at least simple examples with pen and paper was helpful when they were trying to understand new concepts or solve new types of mathematical problems.

It seems that students’ response to the use of CAS for learning mathematics is mixed. There is evidence of positive responses but also evidence of students having reservations about the place of CAS in their learning. It seems that students certainly do not want to totally replace by-hand mathematics and that some students enjoy successfully carrying out routine calculations by-hand.

**Shared intelligence: CAS does not work alone**

A reading of the published research on doing, teaching, and learning with CAS certainly suggests that there are benefits in CAS for at least some students. However, CAS does not and can not work alone. This was well illustrated when, at the beginning of a course, to my surprise one of my students proceeded to type the entire text of a mathematics problem into DERIVE. They were disappointed to receive a ‘syntax error’ message. (True but unpublished anecdote.)

The results of using CAS are the results of a partnership, an intellectual partnership whose results greatly depend on joint ‘effort’. Salomon, Perkins and Globerson (1991) discussed the role of people and the role of technology in an environment of rapid
development. Salomon et al use this term ‘intellectual partnership’ to describe the relationship. They go on to say that:

In sum, although intelligent computer tools offer a partnership with the potential of extending the user’s intellectual performance, the degree to which this potential is realized greatly depends on the user’s volitional, mindful engagement. It is not only what students are interacting with but also how they do it. (p. 4)

Further, Artigue (2001), discusses the importance of coming to understand the process by which an artifact - piece of technology (CAS, calculator) – becomes an instrument. This process, which she calls ‘instrumental genesis’, could be seen as the process by which a user develops an intellectual partnership with CAS or comes to use it effectively. (When Artigue’s 2001 paper was published, this thesis was almost complete and so the author decided that detailed discussion of the links between the thesis and her paper would be best left to later publications.)

Technology needs to be used effectively

For the range of CAS available in the 1990s it was common for researchers to report that students had minimal difficulty and only took a short time to learn to use the program. French (1998) and Monaghan (1994) both said that in their observation little time was needed for secondary school students to learn how to use CAS for their purposes. Mayes (1995) found that initially laboratory classes were dominated by questions about how to use the program and not about the mathematics, but that his students rapidly became proficient users. Zehavi (1997) reported a similar experience even though her young students had the extra overhead of using an English version of the program when English was their second language. French (1998) did report that it seemed to be teachers rather than students who had difficulty learning to use CAS. Perhaps this was because the students were content with sufficient knowledge to use CAS for ‘today’s’ problems while teachers wanted to know more. Alternatively, it may be that today’s students have learned to use technology throughout their education.

It seems from these reports that students have little difficulty in learning to use CAS as a tool for mathematics but these reports seem to be based on observation and anecdotes. Lagrange (1999) examined results in more detail. Students who had been using a TI-92 for three months were given both a questionnaire and a test. From the questionnaire it
was apparent that students thought that they used the program easily enough. However, the test on basic commands from the ‘Home’ screen, produced an average class mark of only around 50%. Since these students reported few difficulties it seems they were making very limited use of CAS.

*Technical difficulties*

The technical difficulties that must be overcome in learning to use CAS will vary with both the particular hardware and software, but similar difficulties can be found across the board. For example CAS are unforgiving in the syntax required for entering expressions. This may be seen as a good thing since, as Hubbard (1994) reminds us, CAS encourages students to be careful about precision of notation. On the other hand Atkins, Creegan and Soan (1995) felt that entering long command sequences in CAS could detract from mathematics learning.

Wain (1993) noted that students sometimes had difficulties interpreting the program output since it produced exact answers but not always in a familiar form. When using CAS students need to be able to recognise equivalent mathematical expressions.

Graphical output also presented problems. Students had difficulty choosing an appropriate window in which to view the graph and further difficulty understanding the effect of interpolation in graphing programs and how this may obscure points of discontinuity. Demana and Waits (1990) and Dubinsky (1995) warn of the limitations of CAS graph displays. They point out that students need to be able to link algebraic form to key features of a graph since one scaled image may not be enough to see all the features of a graph. In addition, Goldenberg (1988) and Schoenfeld (1992) both remind us that despite the accuracy of CAS graphs, human perception affects what students observe. For example, the vertical translation of a curve may be perceived as a dilation of the curve because human perception tends to focus on the minimum distance between two curves, not the vertical distance. This further reinforces the need to note key features of the symbolic representations of functions.
CAS users needs Algebraic Insight

For students to operate in an intelligent partnership with CAS they need to know not only how to run the program, but also what it is that they expect the technology to do for them.

When Heid (1989) discussed the benefits of ‘exact, free from error’ results she noted that this should not be assumed. The student may easily make an entry error or a syntax error. Students need to be able to check for the reasonableness of the results. This requires the student to have some expectation of what would be a reasonable answer in the circumstances: Reasonable, that is, from the point of view of the mathematics, and reasonable from the point of view of the context, if there is one. Allaire and Fabricant (1995) cited an example of a student experiencing difficulty because they had no expectation of the type of results that they might expect when using a 5th degree polynomial. This difficulty was exacerbated because the student did not understand the difference between rational numbers and rounded approximations.

The results produced by CAS are not always presented in the form that is most commonly used for an equivalent ‘pen and paper’ solution. For example while the solution of the general quadratic equation \( ax^2 + bx + c = 0 \) is commonly written as

\[
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

CAS might present this as

\[
\frac{\sqrt{b^2 - 4ac}}{2a} - \frac{b}{2a}, \quad \frac{-\left(b + \sqrt{b^2 - 4ac}\right)}{2a},
\]

with these two solutions presented either underneath each other, or side by side with one out of the initial window view. To check the reasonableness of results, it is important that students can recognise equivalent algebraic expressions. In this example the student needs firstly to identify the degree of the polynomial and hence expect two possible solutions, then understand the properties of the operations involved in order to recognise the two CAS solutions as equivalent to the ‘pen and paper’ version. This requirement may also be seen as a positive aspect of CAS. Demana and Waits (1990) and Brown (1998) suggested that the development of skills in recognition of equivalent expressions is an important contribution that using CAS brings to the learning of mathematics. Lagrange (1999), on the other hand, saw this as an obstacle to students using CAS as a
tool for checking by-hand results. He found they often lacked the knowledge to recognise the CAS expression as equivalent to their by-hand expression.

As stated above the precision of syntax required to enter expression in CAS may present a technical difficulty, but as well as encouraging students to be careful, CAS forces students to look at the structure of expressions (Zehavi, 1997). Certain features of CAS syntax focus students’ attention on the meaning of algebraic symbols. Heid (1989), mentions two examples. Firstly, considering the ‘=’ sign and noting whether they are dealing with an equation or an expression. Secondly, identifying the difference between \(ht\) which may be used for say, \(\text{height} \times \text{time}\), with the multiplication implicit in ‘pen and paper’ algebra but which to a CAS would represent a single variable, since in CAS a name may be longer than one letter. CAS would require the multiplication of the two variables to be explicit.

The experience of teachers and researchers since 1988 suggests that CAS can be a beneficial tool for many students learning mathematics. It also suggests that in order for students to benefit they need to be able to use this tool effectively and to develop algebraic thinking that allows them to monitor this use. In order to define what thinking is required for the use of CAS it is helpful to consider the questions raised, and some of the solutions proposed, for the use of arithmetic calculators.

**A spiral of progress – learning from experience**

The issues raised by CAS were not new: similar questions were asked about the use of arithmetic calculators. Wilf (1982) recognised this when he alerted the *American Mathematical Monthly* to ‘The disk with the college education’ in the same year that the first ‘pocket computers’ became widely available. He realised that CAS, which was already in limited use, would become cheap and generally accessible, thereby making an impact on the doing, teaching and learning of mathematics. He raised some questions in response to his experience of using CAS:

As teachers of mathematics, our responses might range all the way from a declaration that “no computers are allowed in exams or to help with homework” to the if-you-can’t-lick-em-then-join-em approach (teach the students how to use their clever little computers).

Will we allow students to bring them into exams? Use them to do their homework? How will the content of the calculus course be affected? Will we take the advice that we have
Wilf recognised that the issues raised by CAS for secondary and tertiary mathematics educators closely paralleled those already being addressed by primary teachers as a result of the widespread availability of arithmetic calculators. Etlinger (1974) reported that when calculators were initially used in primary and secondary classes most teachers saw their major use as allowing students to consider more ‘real life’ examples and to check answers. The purpose of allowing calculator use was to improve attitudes and increase student motivation to learn mathematics by increasing its relevance through greater use of applied examples. This led to questions about whether technology provided a ‘learning facility’ or a ‘learning crutch’. Etlinger went on to give specific examples of the pedagogical use of the calculator. He saw that the calculator had ‘impressive potential for making children think, create and learn’. (p 44)  

Arithmetic calculators allowed students to do both more problems and a greater range of problems in a given time, and also offered the possibility of new approaches to teaching arithmetic concepts. This technology made the teaching of some arithmetic processes redundant. For example few teachers, now, would continue to expect students to learn an algorithm to find square roots. CAS provides a natural extension of this facility to other areas of mathematics, most obviously algebra and calculus. If then we ‘should teach more of concepts and less of mechanics’, because CAS allows us to, what concepts should be taught and what concepts will CAS help us to teach? As the spiral of progress continues we could look for more ‘parallels’ in the experience of teaching numeracy. Reflection on teaching experiences in the 1970s and 80s led to the term ‘number sense’ being coined and developed. McIntosh, Reys and Reys (1992) attempted to clarify this term. Briefly they said:

Number sense refers to a person’s general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations. It reflects an inclination and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and mathematics has a certain regularity. (p3)

McIntosh et al presented a framework for number sense. This framework did not provide an all-encompassing list but rather a structure which clarified, organised, and
interrelated the generally agreed upon components of number sense. The framework can be used as a guide both for planning and monitoring the teaching and learning of number sense.

Motivation for this study

It has become apparent from this review of the literature concerning teaching and learning with CAS that if the potential benefits of this technology are to be realised two key areas need to be explored. Development of curriculum and recommended pedagogy for CAS would be assisted by

1. A framework of what it means to use CAS effectively.
2. A framework for the Algebraic Insight needed to work in an intelligent partnership with CAS.

These will be developed and described in Chapter three.
CHAPTER 3

FRAMEWORKS FOR ALGEBRAIC INSIGHT AND EFFECTIVE USE OF CAS

PART A: ALGEBRAIC INSIGHT

This study introduces names for two key clusters of variables which other researchers have reported, in the literature, and which my years of observation as a teacher suggest are associated with students' use of Computer Algebra Systems (CAS) in mathematics classes. These are *Algebraic Insight* and *Effective Use of CAS*. In Part A of this chapter the first of these is explored and a working framework for Algebraic Insight proposed. This has been based on the literature relating to ‘symbol sense’, understanding algebra, reports of studies on students working with CAS and the researcher’s experience of teaching with CAS. Effective Use of CAS is treated in a similar way in Chapter 3 Part B.

Using technology for mathematics requires mathematical sense

It is commonly stated (see Chapter 2) that the computational capabilities of technology should change the emphasis of teaching in mathematics classrooms. A scan of local (Victorian) school mathematics text books up to the 1980s showed a curriculum for arithmetic and algebra dominated by the goal of training students to manipulate numerical and algebraic symbols. Since the introduction of arithmetic calculators, numerical approximations have gained importance along with projects and mathematical modeling. Correct algebraic manipulations and calculations are still essential to make progress to any higher level mathematics. Some students enjoy practicing these routines by-hand but many others find this a difficult, time consuming and frustrating exercise. CAS can perform these tasks accurately and quickly. This does not mean that the machine will do all of the mathematics required in any given problem. As Fey (1990) pointed out:

> Even if machines take over the bulk of computation, it remains important for users of those machines to plan correct operations and to interpret results intelligently. Planning
calculations requires sound understanding of the meaning of operations – of the characteristics of actions that corresponds to various arithmetic operations. Interpretation of results requires judgement about the likelihood that the machine output is correct or that an error may have been made in data entry, choice of operations, or machine performance.

For arithmetic, this understanding and ability both to plan and interpret has been termed Number Sense. The concept of Number Sense required for students to work successfully in partnership with scientific calculators, has been discussed and written about since the late 1980s. (NCTM produced a focus issue of the Mathematics Teacher, on the topic of Number Sense, in February 1989.) It has been referred to as a desirable goal of school mathematics for example, in the United States (NCTM, 1989) and Australia (AEC, 1994). McIntosh, Reys and Reys (1992) defined number sense as:

A propensity for and ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity (makes sense). (p 4)

They go on to present the framework for examining basic Number Sense which is included below as table 3A.1. McIntosh et al say that ‘this framework is an attempt to articulate a structure which clarifies, organises, and interrelates some of the generally agreed upon components of number sense’ (p5). They do not claim that their framework is an exhaustive listing of all possible components of Number Sense but that it identifies key observable components and organises these according to common themes. The list, in their framework, could be used to examine the extent to which a student demonstrates Number Sense and to highlight gaps in students’ basic understandings.

The frameworks presented in this thesis will emulate the style of this framework. Number Sense has been broken down into key components (column 1 of table 3A.1). Then each component is divided into a number of ‘understandings’ that make up that component (column 2, table 3A.1). Finally some important examples of each understanding are listed (column 3, table 3A.1).
### Table 3A.1

**Framework for considering number sense**

<table>
<thead>
<tr>
<th>KEY COMPONENTS</th>
<th>UNDERSTANDINGS</th>
<th>IMPORTANT EXAMPLES</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong> Knowledge of and facility with NUMBERS</td>
<td>Sense of orderliness of numbers</td>
<td>1.1.1 Place Value</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.1.2 Relationship between number types</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.1.3 Ordering numbers within and among number types</td>
</tr>
<tr>
<td></td>
<td>Multiple representations of numbers</td>
<td>1.2.1 Graphical/symbolic</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.2.2 Equivalent numerical forms (including decomposition/recomposition)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.2.3 Comparison of benchmarks</td>
</tr>
<tr>
<td></td>
<td>Sense of relative and absolute magnitude of numbers</td>
<td>1.3.1 Comparing to physical referent</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.3.2 Comparing to mathematical referent</td>
</tr>
<tr>
<td></td>
<td>System of benchmarks</td>
<td>1.4.1 Mathematical</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.4.2 Personal</td>
</tr>
<tr>
<td><strong>2</strong> Knowledge and facility with OPERATIONS</td>
<td>Understanding the effect of operations</td>
<td>2.1.1 Operating on whole numbers</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.1.2 Operation on fractions/decimals</td>
</tr>
<tr>
<td></td>
<td>Understanding mathematical properties</td>
<td>2.2.1 Commutativity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.2 Associativity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.3 Distributivity</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.4 Identities</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.5 Inverses</td>
</tr>
<tr>
<td></td>
<td>Understanding the relationship between operations</td>
<td>2.3.1 Addition/Multiplication</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.3.2 Subtraction/Division</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.3.3 Addition/Subtraction</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.3.4 Multiplication/Division</td>
</tr>
<tr>
<td><strong>3</strong> Applying knowledge of and facility with numbers and operations to COMPUTATIONAL SETTINGS</td>
<td>Understanding the relationship between problem context and the necessary computation</td>
<td>3.1.1 Recognise data as exact or approximate</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.1.2 Awareness that solutions may be exact or approximate</td>
</tr>
<tr>
<td></td>
<td>Awareness that multiple strategies exist</td>
<td>3.2.1 Ability to create and/or invent strategies</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.2.2 Ability to apply different strategies</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.2.3 Ability to select efficient strategies</td>
</tr>
<tr>
<td></td>
<td>Inclination to utilize an efficient representation and/or method</td>
<td>3.3.1 Facility with various methods (mental, calculator, paper/pencil)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.3.2 Facility choosing efficient number(s)</td>
</tr>
<tr>
<td></td>
<td>Inclination to review data and result for sensibility</td>
<td>3.4.1 Recognize reasonableness of data</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3.4.2 Recognize reasonableness of calculation</td>
</tr>
</tbody>
</table>

(McIntosh, Reys and Reys, 1992, p4 with column headings added)
Since CAS will perform algebraic manipulations as well as calculations, students working with CAS will not only need Number Sense, but also an equivalent for working with symbols – *Symbol Sense*. Many writers give examples illustrating the importance of skills comparable of those of Number Sense. These include:

- Heid (1989) who noted the importance of students being able to check for reasonableness of results.
- Allaire and Fabricant (1995) who gave the example of a student in difficulty because they had no expectation of the degree of the polynomial in their problem solution.
- Demana and Waits (1990) who also noted that students need to be able to identify the degree of polynomials and hence form an expectation of the number of solutions.
- Lagrange (1999) who commented that a lack of such expectation hindered students’ use of CAS.
- Mitic and Thomas (1994) who pointed out that a CAS may not always be ‘right’ and certainly may not always present results in a familiar form. At a simple level incomplete information can cause error. This can be seen when an expression like $\frac{x-1}{x^2-1}$ is entered. The CAS may automatically simplify this to $\frac{1}{x+1}$ without noting the restrictions on the domain in the algebra window and without indicating a discontinuity at $x=1$ in the graph.

However, little has been written to try and specifically articulate what it would mean for a student to have Symbol Sense. Fey (1990) and Arcavi (1994) provide the two important, specific, contributions to this task so far.

Fey (1990) suggested basic themes that he thought should be included in a set of goals for teaching Symbol Sense. (Numbering has been added)

- F1 Ability to scan an algebraic expression to make rough estimates of the patterns that would emerge or graphic representation …
- F2 Ability to make informed comparisons of order of magnitude for functions with rules of the form $n_1, n_2, n_3, \ldots$
F3 Ability to scan a table of function values or a graph or to interpret verbally stated conditions, to identify the likely form of an algebraic rule that expresses the appropriate pattern…

F4 Ability to inspect algebraic operations and predict the form of the result, or as in arithmetic estimation, to inspect the result and judge the likelihood that it has been performed correctly…

F5 Ability to determine which of several equivalent forms might be most appropriate for answering particular questions… (pp 80-81 numbering added)

These five abilities are each part of the thinking that enables a mathematician to recognise equivalent expressions or form an expectation of the nature of the result of a problem. For an arithmetic problem, this would equate to the first two components of the McIntosh, Reys and Reys Number Sense framework, (knowledge and facility with numbers and operations -see table 3A.1).

Abraham Arcavi’s 1994 paper on Symbol Sense is the article most commonly referred to by writers who use this term. After considering Fey’s (1990) suggestions, and studying students’ work, Arcavi suggested that Symbol Sense would include the following elements, (again numbering has been added):

A1 An understanding of and aesthetic feel for the power of symbols: understanding how and when symbols can and should be used in order to display relationships, generalisations, and proofs which are otherwise hidden and invisible.

A2 A feeling for when to abandon symbols in favour of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation.

A3 An ability to manipulate and to ‘read’ symbolic expressions as two complimentary aspects of solving algebraic problems. Detached from the meaning or context of the problem and with the symbolic expression viewed globally, symbol handling can be relatively quick and efficient. On the other hand, the reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results.

A4 The awareness that one can successfully engineer symbolic relationships that express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions.

A5 The ability to select a possible symbolic representation of a problem, and, if necessary, to have the courage, first, to recognise and heed one’s dissatisfaction with
that choice, and second, to be resourceful in searching for a better one as replacement.

A6 The realisation of the constant need to check symbol meanings while solving a problem, and to compare and contrast those meanings with one’s own intuitions or with the expected outcome of the problem.

A7 Sensing the different ‘roles’ symbols can play in different contexts.

(p31, numbering added)

Arcavi warns that his ‘catalogue’ is not exhaustive and that Symbol Sense itself could never be completely defined by any catalogue. Even so he has presented us with a list of behaviours to which to aspire in the development of Symbol Sense. It is a broader view than Fey’s (above) and goes beyond the insights which a student need to work in parallel with CAS as will be explained below. If this list were to be compared with the McIntosh, Reys and Reys Number Sense framework it would include all three key components (table 3A.1).

Arcavi presented a comprehensive definition of Symbol Sense that clearly describes both the broad scope and importance of this concept. Many of the Symbol Sense abilities in Arcavi’s list are unaffected by the availability of CAS, since CAS only facilitates manipulations and calculations; it does not decide how best to solve a problem; nor does it interpret results. This thesis addresses the subset of Symbol Sense that can be identified as specifically relating to the use of CAS for teaching and learning mathematics. The development of a manageable framework for what, in this thesis, is called Algebraic Insight, is made possible by focusing on this narrower view of Symbol Sense.

Consider a basic model of the key aspects of problem solving presented below in figure 3A.1. This model lists four main stages in problem solving:

Step 1 FORMULATE Taking a real world problem and formulating algebraic expressions that may aid in the solution

Step 2 SOLVE Carrying out the necessary mathematical calculations and manipulations to solve the mathematical problem created. This task includes checking that the mathematical solution is a reasonable answer to the mathematical problem. For a
quadratic function it may mean checking whether there is a maximum or minimum or that number of solutions accords with key features of the function.

Step 3  INTERPRET  Interpreting the solution in terms of the real world problem, noting appropriate units, etc.

Step 4  CHECK  Checking that the solution does indeed answer the initial real world problem.

In this model it is suggested that most of the process and thinking required to use mathematics to solve a problem remains the same whether or not CAS is involved. CAS can be used to handle the manipulations and calculations of the ‘solve’ procedures. CAS does not formulate, interpret or check the problem. These tasks, for the foreseeable future, remain solely the responsibility of the person solving the problem. The Symbol Sense that is required to do these sections of mathematical thinking will remain the same with or without the presence of CAS. Fey’s list of aspects of Symbol Sense (F1 to F5 above) could all be seen as applying to the ‘solve’ section of the model. Arcavi’s catalogue covers more than the mathematical solution process. His elements A1, A2 and A5 relate to the formulation in algebraic symbols including the decision of whether it is best to use algebra or whether some other method would be more efficient. Elements A3, A4 and A7 are aspects of ‘Symbol Sense’ that will help with finding a mathematical solution to the algebraic, mathematical formulation of a problem. They refer to thinking which is necessary if CAS is to be used appropriately. Finally, on one hand, element A6 is required for interpreting the algebraic result into a ‘real world’ solution. On the other hand, it also points to a need for monitoring of the on-screen progress of an algebraic solution and in this sense can be seen as relating to Algebraic Insight.

This thesis focuses on the use of CAS and is concerned about the Symbol Sense required to use a CAS instead of ‘solving’ by hand. (A preliminary version of this work was presented in Pierce and Stacey (2000) and a revised version, close to that presented here, in Pierce and Stacey (2000c).) Figure 3A.1 illustrates this. It shows that Symbol Sense applies to three of the four processes (formulate, solve and interpret) whereas Algebraic Insight applies only within the mathematical world, between a problem
formulated mathematically and its mathematical solution. It does not apply to the process of moving between the real world (top left figure 3A.1) and the mathematical world (top right figure 3A.1). This subset of Symbol Sense will be referred to as Algebraic Insight and will be outlined below in the light of Symbol Sense and factors other researchers have seen as keys to working with computer algebra systems.

Figure 3A.1 A model of problem solving

**ALGEBRAIC INSIGHT A WORKING FRAMEWORK FOR SYMBOL SENSE WITH CAS**

The following framework, presented in table 3A.2, aims to articulate a structure which clarifies, organises, and interrelates some of the key elements of Algebraic Insight. This has been done in a manner that may assist in examining whether or not students have, or are developing, the skills of Algebraic Insight. This thesis will later use such information to see whether students’ level of Algebraic Insight has any effect on their ability to form an effective intelligent partnership with CAS. It may also form a basis for considering new emphases in teaching and curriculum that should evolve for teaching mathematics in a CAS environment. The framework is presented here, and then it is discussed in overview. This is followed by a detailed description of each component of the framework.
Table 3A.2
*A proposed Framework for Algebraic Insight*

<table>
<thead>
<tr>
<th>Aspects</th>
<th>Elements</th>
<th>Common Instances</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Algebraic Expectation</td>
<td>1.1 Recognition of conventions and basic properties</td>
<td>1.1.1 Know meaning of symbols</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.1.2 Know order of operations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.1.3 Know properties of operations</td>
</tr>
<tr>
<td>1.2 Identification of structure</td>
<td></td>
<td>1.2.1 Identify objects</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.2.2 Identify strategic groups of components</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.2.3 Recognise simple factors</td>
</tr>
<tr>
<td>1.3 Identification of key features</td>
<td></td>
<td>1.3.1 Identify form</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.3.2 Identify dominant term</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.3.3 Link form to solution type</td>
</tr>
<tr>
<td>2. Ability to Link representations</td>
<td>2.1 Linking of symbolic and graphic representations</td>
<td>2.1.1 Link form to shape</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.1.2 Link key features to likely position</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.1.3 Link key features to intercepts and asymptotes</td>
</tr>
<tr>
<td></td>
<td>2.2 Linking of symbolic and numeric representations</td>
<td>2.2.1 Link number patterns or type to form</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.2 Link key features to suitable increment for table</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.2 Link key features to critical intervals of table</td>
</tr>
</tbody>
</table>
An Overview

First of all, it should be emphasised again that this is a working framework, a basis from which to plan teaching, assess, or reflect on that part of students’ Symbol Sense this thesis calls Algebraic Insight. It is not proposed as a catalogue of specific, itemised skills. The divisions within the framework are neither mutually exclusive nor exhaustive. Whilst these features would be desirable the author does not believe they are fully attainable. The elements of each aspect of Algebraic Insight can be used as the springboard for test items or examples, but a particular item may show Common Instances of more than one element of Algebraic Insight. The framework was developed in response to the literature and the author’s experience of teaching with CAS. It is an attempt to analyse what it is that ‘expert’ mathematicians do when they look at a result to an algebraic problem and say ‘there is a mistake here’ or ‘that looks all right’. This is the thinking used in, what the problem solving literature, for example Schoenfeld (1985), calls ‘monitoring’ or ‘control’. Mathematics teaching has, of necessity, focused a great deal of time and attention on algorithmic routines. Since CAS does these effectively, perhaps attention can be directed towards deliberately teaching these skills of Algebraic Insight.

In the framework summarised in table 3A.2, Algebraic Insight is considered from two important aspects. These two aspects, Algebraic Expectation and Ability to Link Representations, form a logical division of Algebraic Insight. This is illustrated by the diagram in figure 3A.2 which reminds us that Algebraic Insight is the part of Symbol Sense that helps in the mathematical solution of algebraically formulated problems. Algebraic Expectation focuses on the application of Algebraic Insight within the symbolic representation of mathematics while ability to Link Representations deals with the students’ ability to move cognitively between symbolic (algebraic) representations and graphical or numeric representations. Such Linking is also concerned with expectations, but expectations across representations. Algebraic Insight will be shown when a student has expectations about graphs and tables that are linked to features of the symbolic representation; for example, when a student asks and answers such questions as:

Given the rule $y=x+3$ what do I expect the graph to look like? Should I expect it to cross the $x$-axis? If I am to construct a table of values for this rule, what might be a suitable increment to use?
Ability to Link Representations is limited to these three representations because these are the representations supported by CAS. In using CAS to do mathematics, being able to link between numeric and graphical representations is also important but, for the purposes of this thesis, this is not considered to be an element of Algebraic Insight. Common Instances of the elements of each aspect, listed in column two of table 3A.1, may be seen when students demonstrate the abilities listed in the third column. These Common Instances do not form a definitive list but have been selected because they are important examples within the context of this study, an introductory calculus course. In general, while the aspects of Algebraic Insight and the elements apply at any level, details of Common Instances will be both content and stage specific. As will be explained later, the phrases ‘poor’ or ‘good’ Algebraic Insight would be judged against Common Instances relevant to the students’ age or stage of learning. With these things in mind, each component of the framework will now be discussed in detail.

![Diagram](image)

*Figure 3A.2* Algebraic Insight: focusing in on the Symbol Sense for solving mathematical problems

**Aspects of Algebraic Insight: Algebraic Expectation**

The term Algebraic Expectation is used here to name the thinking process which takes place when an experienced mathematician considers the nature of the result they expect to obtain as the outcome of some algebraic process. For example, it takes place when a
mathematician looks at two expressions and decides, without doing any explicit calculations or manipulations, whether they are likely to be equivalent. This aspect of Algebraic Insight is part of Symbol Sense. It links loosely to Fey’s (1990) F1, F2, F4 and F5 and Arcavi’s (1994) A3 and A6. The name ‘Algebraic Expectation’ has been chosen because this skill is similar to arithmetic estimation - where a quick assessment of the expected characteristics of an expression is made. For example arithmetic estimation may involve estimating that the answer to $6283 \times 4816$ will be in millions; Algebraic Expectation may involve expecting that $(2-x+x^2)(x^4-x^3+27x-63)$ will be a polynomial of degree six. The word expectation, rather than estimation, has been chosen because estimation involves a sense of approximation and rounding not possible in algebra. In the example above $6283 \times 4816$ could be approximated by rounding to the nearest 1000 giving $6000 \times 5000 = 30000000$. The product of the two polynomials cannot be approximated in this way but with a similar level of understanding, it may be seen that the product will be a polynomial, with the highest powered term of this new polynomial $+x^6$ and the constant term -126. Algebraic Expectation does not involve producing an approximate solution but rather noticing conventions, structure, and key features of an expression that determine features which may be expected in the solution.

After much reflection and analysis of the process of Algebraic Expectation, the author has decided that it can be broken down into three elements: recognition of conventions and basic processes (1.1); identification of structure (1.2); and identification of key features (1.3).

As general concepts these elements do not depend on students’ age or stage of learning, but the Common Instances which demonstrate evidence of these skills will vary with level and content. As well as finding parallels in arithmetic estimation there are also parallels in learning to read. Just as in arithmetic and reading, with algebra the brain needs to learn to look for and identify the clues which give meaning to symbols. With this in mind the three elements of Algebraic Expectation are now described in detail.

**Recognition of conventions and basic properties**

There are established conventions in mathematics just as there are in language, regarding the use of symbols. When writing, for example, we use capital letters to begin the names of people and places. This convention gives the reader important clues. We
expect, for example, that Sunshine, used in the middle of a sentence, may be being used as a name and is not to be given its everyday meaning - because it begins with a capital letter. A similar convention in mathematics would be the use of Sin(x) to indicate that only the principal values of sin(x) should be considered. Further parallels could be drawn with other conventions like word order and parts of speech. These conventions must be followed in order for meaning to be clear. Consider the collection of words ‘basket can in sleep Nellie her’. These words need to be arranged in a conventional order for it to have literary meaning, for example ‘Nellie can sleep in her basket’. Similarly ‘$\int \sin x \, dx$’ needs to be arranged in a conventional manner for this to have algebraic meaning. In this case several alternatives are possible and the meaning is not clear, it could be, for example, $2 \int \sin x \, dx$ or $\int \sin(2x) \, dx$.

Recognising the conventions of mathematics is a skill based on both knowledge and understanding of the meaning of symbols. Much of this knowledge will transfer from experience with numbers and arithmetic processes. Recognition of conventions and basic properties is seen in three Common Instances: when students know the meaning of symbols; the appropriate order of operations; and the basic properties of operations. Each of these will be considered in turn.

**Meaning of symbols**

Users of algebra need to recognise two types of symbols, ‘operators’ and ‘letters’. ‘Operators’ are the symbols which are used to indicate processes like addition (+), subtraction (-), square root ($\sqrt{\cdot}$) and so on. Students become familiar with these symbols in arithmetic but need to be aware that CAS syntax uses some different conventions: for example / for division, * for multiplication, sqrt( ) for square root etc.

The convention in pen and paper algebra of implicit multiplication, where $xy$ means $x \times y$, may also be a source of confusion. Booth (1988) found that many beginning students may read $xy$ as $x+y$. There are further issues here that relate to the use of CAS. The required syntax is CAS specific but since CAS allow variables names to have more than one letter a distinction must be made between a variable $ab$ and $a*b$. However in expressions like $2a$ or $5(q+3)$ multiplication may be implied.
Letters are the symbols most commonly associated with algebra. Mathematics teachers know that the meaning which students assign to letter symbols is fundamental to their use of algebra. Many researchers (for example Küchemann, 1981; Usiskin, 1988; Booth, 1988; Kieran, 1992; MacGregor and Stacey, 1997 and Kinzel, 1999) have shown that there is a great deal of variety in these meanings and a great deal of misunderstanding results from assigning inappropriate meaning to symbols. To show Algebraic Insight students must understand that letters can be used with different meanings in different contexts and be able to recognise which meaning is applicable for a particular expression. The literature referred to above points to issues of which it is important to be aware when considering a student’s thinking as it applies to Algebraic Insight.

This research tells us, for example, that not all students understand ‘letters’ in the same way or at the same level. Küchemann (1981) looked at assessing students' understanding of the symbols or letters used in algebra at five levels. Students classified as performing at his lower levels used letters to represent a 'missing' number or express a relationship, for example:

\[ a + 2 = 5, \text{ read by the student as 'what number plus two gives five?'} \]

or

\[ 120 \text{ m} = 2 \text{ h} \text{ used to abbreviate 120 minutes is the same as 2 hours.} \]

At his higher levels students used letters as generalised numbers or variables for example:

\[ 2x + 3y = 16 \]

or

\[ \text{If } c + d = 10, \text{ what do we know about } d \text{ if } c < 5? \]

Wagner (1983) recounted the story of a student who believed that the next consecutive integer following \( x \) would be \( y \). This fundamental misconception was, more recently, discussed by MacGregor and Stacey (1997) who studied early secondary school students’ understanding of the meaning of letters in algebra. In addition to the meanings discussed by Küchemann (1981), like Wagner, they described students who, based on prior experience, associated letters with their position in the alphabet eg \( a = 1, b = 2, c = 3 \) etc. They also found some students who thought a different letter must be used for each unknown even when a clear relationship between two variables was given. Conversely
some students believed that every different letter must represent a different number. Such misconceptions or limited understanding of the possible meanings of letter symbols hinder students’ Algebraic Expectation.

Students’ understanding (or lack thereof) of the varied use of letters in mathematics is important for algebra with or without the use of CAS. However the use of CAS raises its own specific issues of naming and defining variables and parameters within the program. For example: variable names may be short words not just single letters. Letters, representing parameters, may be assigned specific values that will be used by the program instead of the letter until the assignment is changed or cancelled. Letters that conventionally represent a particular constant in mathematics, for example ‘e’, may either require special syntax or be reserved letters.

A classic quadratic function is commonly expressed as \( y = ax^2 + bx + c \). This requires a student to recognise that \( a, b \) and \( c \) are parameters while \( x \) and \( y \) are variables, two different meaning for letters in one statement. The syntax used to enter such an expression into CAS must make it clear that what is required is \( a \times (x)^2 \) and not \( (ax)^2 \) or \( (a\times x)^2 \) etc. Consider a second example, \( \frac{\sin x}{\cos x} \) where letters are used with two different meanings: to name the processes of sine and cosine, and to identify the variable. A lack of recognition of the meaning of these letter symbols leads students to make errors such as \( \frac{\sin x}{\cos x} = \frac{\sin}{\cos} \) or \( \frac{\sin}{\cos} \). Finally Drijvers (1999) observed that students who do not understand the meaning of letters as they represent variables and parameters, may fail to make use of CAS. Students who equate the notion ‘solve’ with ‘find the number’ may not even try to use the ‘solve’ capabilities of the machine when there are no numbers in the equation. Recognition of the meaning of symbols used in any algebraic expression is foundational to Algebraic Expectation.

**Order of operations**

A second common instance of the recognition of conventions and basic properties is seen when students know the appropriate order of operations. Tall and Thomas (1991) draw attention to the importance of ‘order of operations’. They point out that the expression \( 3x+2 \) is both read and processed from left to right while the equivalent expression \( 2+3x \) is read from left to right as “two plus three \( x \)”, but computed from right
to left. From arithmetic we learn that unless operations are carried out in an agreed order the meaning of an expression like $a + p \div q$ is ambiguous. The acronym BODMAS summarises the agreed convention for order of precedence of operations (Brackets, of, division, multiplication, addition, subtraction) so $a + p \div q$ must be evaluated as $a + \frac{p}{q}$ not working from left to right, giving $\frac{a + p}{q}$ as it might otherwise be read.

The conventions of order of operations also mean that when an expression involves only one type of operation, then it should be read from left to right not right to left. For some processes, which are commutative, this rule has no effect but in cases like $x \div y$ this is very important. These conventions could be learnt as a series of rules, but it is our experience of arithmetic that helps such rules make sense. Booth (1988) noted that, when working by hand, students have tended not to use brackets but rather evaluate everything from left to right.

CAS programs can vary in the way they interpret syntax, but most consider bracketed expressions first, then carry out operations working from left to right. It is therefore necessary to use brackets to make the meaning of expressions clear and ensure that operations are carried out in the order expected. To apply BODMAS to the expression $a + p \div q$ it must be entered as $a + (p/q)$. Similarly to enter an expression like $\frac{\cos(3t)}{3}$ into CAS requires the syntax $f(t) = \frac{\cos(3t)}{3}$ in order to make it clear that the cosine of $3t$ is intended not $\cos(3)$ times $t$. These examples show why ‘knowing order of operations’ is a common instance of the Algebraic Expectation needed to work in an intelligent partnership with CAS.

Properties of operations
‘Recognition of conventions and basic properties’ is also shown when students know the basic properties of operations. This third set of Common Instances extends understanding of the ‘meaning of the symbols and the order of operations’. Their importance for Algebraic Expectation is best shown by examples. The following five examples illustrate how knowing the properties of operations contributes to Algebraic Expectation.
1. For each operation there may be inverse operation. When a letter is simply used to stand for an unknown like $6 + x = 10$ the question becomes '6 plus what equals 10', and $x = 4$. If, however, the problem is $a + x = y$, then it is important to understand that for the addition process there is an inverse, subtraction. In an equation it must be recognised that the ‘=’ symbols tells us that the expression on the left is equal to the expression on the right, not just 'the answer is'. It is this understanding that allows us to rephrase the equation in the problem as $x = y - a$. It also promotes an expectation that, if $a$ is a positive integer then $x$ will always be less than $y$. This gives information for both the graph and table that also represent this family of functions. Such links will be discussed as instances of the second aspect of Algebraic Insight.

2. An understanding of the properties of multiplication and its inverse, division, allows the use of the ‘null factor law’ in finding solutions for polynomials, and alerts the mathematician to watch for division by zero. Awareness of this property leads to an expectation of likely solutions to equations or points of discontinuity.

3. The derivative of a constant is always zero. Awareness of this property leads the mathematician to expect a family of functions as possible results for anti-differentiation. They expect to need further information in order to find a particular solution.

4. Not all operations are commutative. Students should not expect to get the same result for the operation $s/t$ and $t/s$.

5. Some operations are distributive over others; some are not. Over-generalisation of this property (Matz, 1980, calls this ‘laws that make laws’) is a common source of confusion for students, which causes incorrect Algebraic Expectations. For example students learn that $a(b+c) = ab+ac$ and then incorrectly expect that $\ln(a+b) = \ln(a) + \ln(b)$; or that if $f(x)=u(x)g(x)$ then $f'(x)=u'(x)v'(x)$.

These examples show that knowing the basic properties of operations, and being alert to the fact that not all operations have the same properties will inform students’ Algebraic Expectation. Recognition of conventions and basic properties (as shown through the Common Instances of knowing the meaning of symbols, knowing the order of operations and knowing the properties of operations) is, the first foundational element of
Algebraic Expectation. It has special importance for entering expressions into CAS and recognising equivalent expressions.

The second element that contributes to Algebraic Expectation is the ability to identify structure.

**Identify structure**

In written English, structuring texts into paragraphs, sentences and phrases immediately aids our reading. The use of the comma alerts us to there being more information coming, or may signal a list of items. Algebraic expressions also have an identifiable structure. Consider, for example, \( \frac{a(x+1)^5 + b(x+1)^2}{(x+1)} \). The vinculum indicates the first level of structure in this expression. The numerator can be seen as consisting of a strategic group of components with two terms, while the denominator has a single object. Viewed at another level, \((x+1)\) can be identified as an object which is common to each term of the expression. Common Instances of identification of structure occur when students identify objects, strategic groups of components or simple factors. These ‘Common Instances’, with further examples, are outlined below.

**Identify objects**

If written text is only viewed as a series of letters, reading it makes no sense. Reading requires the identification of groups of letters as words. In a similar way identifying ‘objects’ and ‘terms’ helps students to see pattern and structure in mathematical expressions and so form Algebraic Expectations. Tall and Thomas (1991) wrote that versatile thinking in algebra required more than carrying out a succession of mechanistic steps. They said that:

> To be a successful mathematician requires more than the ability to carry out a succession of mechanistic steps, be they steps in carrying out a numerical calculation, solving a linear equation, differentiating a composite function, or writing down a mathematical proof. What is required is an overall picture of the task in hand, so that the appropriate solution path can be selected, and any errors that occur are more likely to be sensed and corrected. Thus the sequential/logical/analytical way of carrying out a succession of mathematical processes needs to be complemented by a global/holistic, overall grasp of the context. (p.128)
The example was given of students who, when faced with a question like 'factorise: \((2x+1)^2 - 3x(2x+1)\)', worked from left to right multiplying out the brackets following a serial strategy rather than taking a global approach and 'seeing' that 2x+1 is a common factor. Algebraic Insight may involve 'seeing' at a glance that this expression contains the repeated object (2x+1) or looking at \((2x+1)^2 - 3x(2x+5)\) and noting that the bracketed objects are different.

Sfard (1991), Tall and Thomas (1991), and Kieran (1992) all agreed that a structural view of algebra is important, and draw attention to the dual role of algebra expressions as both processes and objects. An important example of structure occurs when composite functions are used. In examples like ‘Find \(f(t+1)\) when \(f(x) = 6x-x^2\);’ the process of adding 1 to \(t\) results in an object \((t + 1)\) which must be substituted for the singular object \(x\). CAS supports the use of objects. A function may be named, eg. \(f(x)\), and used to process any object eg. \(t+1, \sin x, \text{or } 2y+5\), by entering \(f(t+1), f(\sin x)\) or \(f(2y+5)\). Tall and Thomas’ global view allows students to make the series of substitutions necessary, for example to:

\[
\text{Find } f(g(x)) \text{ if } g(x) = x + h \text{ and } f(x) = 5x^3 - 2x^2 + x + 7.
\]

To show Algebraic Insight a student would not need to perform the expansion of this new expression, but would realise that every \(x\) must be replaced with the new \((x+h)\). This is a further example of the common instance of identifying objects.

**Identify strategic groups of components**

The second group of Common Instances that demonstrate identification of structure occurs when students identify strategic groups of components in an expression. As part of Algebraic Insight these instances illustrate thinking which will assist students to form an intellectual partnership with CAS. Lagrange (1999) emphasised that, when using CAS, it is important to be able to recognise equivalent expressions. Identifying strategic groups of components helps the recognition of equivalent expressions. For example, in trigonometry \((\sin^2 x + 2\sin x \cos x + \cos^2 x)\) may be grouped as \([2 \sin x \cos x] + [\sin^2 x + \cos^2 x]\) and hence re-expressed as \(\sin(2x)+1\), or viewed as being of the form of a perfect square and so written as \((\sin x + \cos x)^2\).
Algebraic fractions require ‘seeing’ an expression composed of its parts; that is, identifying the structure. For example: \( \frac{a}{c} + \frac{b}{d} \) needs to be viewed as two fractions \( \frac{a}{c} + \frac{b}{d} \) composed of two numerators and two denominators. This must be distinguished from \( \frac{a+b}{c+d} \), which must be seen as one fraction \( \frac{a+b}{c+d} \), composed of one object in the numerator and one in the denominator. Recognising what part of an expression should form a numerator and what a denominator is important for entering expressions into a CAS and copying results from a screen. To detect equivalent expressions it is important to notice the structural detail in expressions that appear similar but are different.

**Recognise simple factors**

A third common instance of the identification of structure is seen when students recognise simple factors. This skill is not necessarily distinct from identifying objects or strategic groups of components since, for example, recognising \(5(x+3) + (x+3)^2\) and \((x+3)(x+8)\) as equivalent expressions requires the student to first identify \(x+3\) as an object then as a factor. Similarly, the identification of a difference of two squares, say \(x^2 - y^2\) and re-expressing it as \((x-y)(x+y)\) may be described as identifying strategic groups of components or as recognition of simple factors.

Identification of ‘common’ factors is often required in order to recognise equivalent expressions produced by CAS. Even when an expression is merely entered into a CAS, and no further process or simplification has been carried out by the user, the output of the CAS may show the original expression in a different, but equivalent, form. Some important examples of recognising ‘common’ factors are shown below. In each of these examples extracting the common factor can make the expression ‘look’ quite different while retaining the same mathematical meaning.

- \(a \sin^2 \theta + a \cos^2 \theta = a \left( \sin^2 \theta + \cos^2 \theta \right) = a\)

- \(xe^x + \frac{e^x}{x} = e^x \left( x + \frac{1}{x} \right)\)
\[ xe^x + e^{2x} = e^x(x + e^x) \]

\[-rt - ps = -(rt + ps)\]

\[ \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{b^2 - 4ac - b}}{2a} \text{ or } \frac{-\left(\sqrt{b^2 - 4ac} + b\right)}{2a} \]

A further example of an apparently changed expression can be seen when an expression such as \(\frac{4a^2 + 20a + 25}{2a + 5}\) is entered. CAS will commonly return this as \(2a + 5\). The user needs to have both recognised the numerator as a perfect square and be aware that the domain for \(a\) is restricted to values other than \(-5/2\). In each of the examples cited here recognition of simple factors contributes to student’s expectations about the form of the expressions their answers may include.

**Identify key features**

‘Identifying key features’ forms the third element of Algebraic Expectation. This could also be seen to have parallels in ‘skim’ reading text for general meaning. Our eyes scan text and identify key words, names, repeated words, and long words. These key features help us make general statements about the text. In a not dissimilar manner, mathematical expressions can be scanned for key features. Features that identify the form of the expression indicating whether it is, for example, trigonometric, exponential, or polynomial. Key features also provide information by which expectations may be formed. For functions, for example, these features may lead to expected number of solutions, solution type, number of maxima and minima, and domain and range. Three Common Instances of this element are now described in more detail.

**Identify form**

‘Identifying form’ for functions can mean noting that \(x^3 + 5x + 1\) is a cubic polynomial, \(2 + e^x\) is exponential and \(4\sin(3x)\) is trigonometric. At a second level it can mean seeing that \(e^{2x} + e^x - 2\) is a quadratic in \(e^x\).

Further, recognition of form promotes expectations about the nature of the graph or table related to the rule: for example, recognising \(2^x\) as an exponential function which will rapidly grow large; \(\sin(3x)\) as a function which will oscillate; or \(x^2 + 4x - 21\) as a quadratic which may be graphed as a parabola. These links to other representations will
be discussed below. Such recognition leads to expectations of properties like number of
derivatives and zeros that are also determined by ‘identifying the dominant term.’ This,
then, forms the second common instance of ‘identifying key features’.

*Identify dominant term*

Identifying the dominant term of an expression informs Algebraic Expectation. For
example, the highest power of a polynomial alerts us to the maximum and minimum
number of solutions to expect for \( f(x) = 0 \). As Fey (1990) says, this may indicate the
family of polynomials to which this function belongs. In a second example, involving
the limit of a function, identifying that a squared term will dominant over a linear term
leads to the expectation that \( \frac{4t^2 - 5}{t+1} \) will tend to infinity as \( t \) approaches infinity. Being
able both to identify the dominant term in an expression, and understand its impact,
enables students to assess the reasonableness of CAS output as the solution to a
mathematical process.

Algebraic Expectation is informed by each of these three elements: recognition of
conventions and basic properties; identification of structure; and identification of key
features. It incorporates some of the abilities that Fey said would be important for
students working with CAS, in particular points F1, F2, F4 and F5; similarly it includes
some of Arcavi’s points A1 and A6 (see earlier this chapter). In this study the Common
Instances of these elements are mostly illustrated by examples from functions and early
calculus because this was the core material of the course used for the study. CAS
allows the representation of functions in three ways: algebraically; graphically; and
numerically, through tables. Algebraic Insight will be both shown and developed by
linking the understanding gained in the symbolic representation with insight gained
from graphs and tables. The ability to Link Representations forms the second aspect of
Algebraic Insight.

*Aspects of Algebraic Insight: Ability to Link Representations*

When using CAS, students can swap between algebraic, numeric and graphical
windows at the press of buttons. This may be used to advantage in examining students’
Algebraic Insight. Fey (1990) (F1 and F3) noted that understanding of algebra would be
demonstrated by students’ ability to move between algebraic, numeric and graphical
representations of expressions. Arcavi (1994) commented (A2) that students need a feeling for when to ‘abandon’ symbols. Sometimes better progress will be made in solving a problem by using an alternative representation. Students would demonstrate Algebraic Insight by being able to make estimates; having a rough idea what a graph of a function may or may not look like; and by having some idea of the sort of range of values that might be expected for a given domain. To show Algebraic Insight a student would, for example, not be expected to complete a least squares linear regression but to recognise that a set of data points might be modelled by a linear function. They would be looking for a function or class of functions to link between the dependent and independent variables. This is not a trivial skill. MacGregor and Stacey (1993) reported that ‘while most students can perceive patterns in tables easily, many do not perceive the functional relationship’ (p.I-181). Similarly, a student would immediately recognise a quadratic function rule as being represented by a parabolic graph and as Arcavi (1994) suggests the student may realise that each representation is equally valid and that changing representations may help them make progress in solving a problem. Common instances of elements of this aspect of Algebraic Insight will now be described in detail.

**Linking of Symbolic and Graphic Representations**

Consideration of the algebraic representation of a function will lead to expectations about the nature of the equivalent graphical representation and vice versa.

*Link form to shape*

A step beyond identifying the *form* of an algebraic expression is the linking of this form to the shape of the graph that will also represent that relationship. For example the graph of a polynomial whose highest degree is an even number will have “both ends up” if the coefficient of the leading term is positive; that is, as \( x \) tends to \( \pm \infty \), \( f(x) \) approaches \( \infty \). Conversely a graph which looks like figure 3A.3 will be an odd powered polynomial at least of order 3.
Linking of shape to form is also shown when a student looks at a function like
\[ p(t) = 5t^2 + 3t + 1054 + \sin(2\pi + 5) \]
and recognises that this is made up of a quadratic plus a sine function so the graph will look like a sine graph ‘bent’ up to follow the curve of the quadratic. In general, identifying form gives enough information about a graph to be able to draw the basic shape ‘in the air’ with a hand wave.

From the other perspective, looking at a graph can give basic information about the form of a function. Consider the three ‘graphs’ in figure 3A.4. Without further information we can say that graph 1 most likely represents a polynomial of even power higher than \( x^2 \) because it has ‘two ends up’ and is ‘flattened’ in the middle. Graph 2 represents a trigonometric function because of the oscillating nature of the curve, and graph 3 most likely represents an exponential function because of the ‘flatness’ to the left and steep gradient to the right.

This first common instance of linking symbols to graphical representation could be thought of as macro level examination. It will be seen that the second and third instances link more specific details.
Link key features to likely position

A second common instance where evidence of Algebraic Insight may be shown through linking symbolic and graphical representations is the linking of key features to the likely position of the graph. At a basic level, linking key features to likely position requires identifying the constant term of a function, and recognising that this will determine the vertical translation of the graph relative to the origin. Looking at the graph, the x and y-intercepts will give information about the algebraic representation of the relationship, since at the y-intercept \( x = 0 \) and at the x-intercepts \( f(x) = 0 \). For example given a function like \( f(x) = x^2 + 5x + 6 \), it can immediately be said that, relative to \( y = x^2 \), the graph will be translated up and to the left.

This feature is very important for students working with CAS. The default settings for graph windows usually centre on the origin and as a consequence students often cannot ‘see’ the graph they are trying to draw. If the constant term of a function is very large, like 1560 for example, identifying this key feature will alert the student to make necessary changes to the standard CAS graph window that is commonly set for a 10 by 10 square.

Identifying key features will alert the student to the domain and range of a function. For example if a function has \( \sqrt{x} \) or \( \frac{1}{x^2} \) and \( x \in R \) then the graph could only occupy the first and fourth quadrants since \( x \) must be non-negative. If the function is an even powered polynomial, then the range will be limited to a maximum or minimum value according to the sign of the leading term. Such limitations on the domain or range give clues as to the likely position of a graph.

As well as the general location of the curve it is important to have some idea of where other ‘interesting’ features of a function occur. These form a third common instance of this ability to link between symbolic and graphical representations.

Link key features to intercepts and asymptotes

Key features of algebraic expressions may be linked to intercepts and asymptotes. Working from the symbolic representation, students should learn to recognise that a rational function like \( g(x) = \frac{x^3 - x + 1}{x - 5} \) will, because of the properties of the operation of
division, be undefined when \( x = 5 \) and that the graph of this function will have a vertical asymptote at \( x = 5 \). Similarly, a function like \( g(x) = \frac{x^2 + x - 2}{x - 1} \) will be undefined when \( x = 1 \) and may be ‘seen’ as the same as a graph of \( g(x) = x - 2 \) except for a ‘hole’ in the curve at \( x = 1 \). Such a hole will not be visible on a current CAS graph but identifying this key feature alerts students to this point of discontinuity.

Looking at the graphical representation of a function gives clues to key features of the symbolic equivalent. For example, consider the graph shown in figure 3A.5. The ‘odd’ asymptote at \( x = -1 \) indicates that there will be division by \((x+1)\) to an odd power and the ‘even’ asymptote at \( x = 2 \) indicates there will be division by \((x-2)\) to an even power. Further, the graph does not look like it crosses the \( x \)-axis, in which case a possible option is that the numerator of the expression will be a constant.

![Figure 3A.5 Graph showing asymptotes](image)

Figure 3A.5 Graph showing asymptotes

Figure 3A.6 provides another example for illustrating that key features of a graph may be linked to key features of the function’s symbolic representation. The shape of the graph tells us it is likely to be an odd powered polynomial. The graph crosses the \( y \)-axis at about \(-2\) so the symbolic form will have a constant of approximately \(-2\) as part of the expression. Next by looking at the \( x \)-intercepts we can identify three distinct points, suggesting that if the symbolic form were factorised there would be three factors, two positive and one negative. The point of inflection at the positive intercept alerts us to coincident zeros, while the shape of the graph tells us that this will be due to an odd powered factor. If, instead of working from the graph, the starting point had been the symbolic representation of this function, \( y = (x-1)^3(x+1)(x+2) \), a reversal of this thinking would have led to expectations about these key features of the graphical representation.
These three Common Instances demonstrate when algebraic insight may be shown in linking symbolic and graphical representations. A similar set of Common Instances will now be used to illustrate the second element of ability to Link Representations; that is the linking of symbolic with numeric representations.

**Linking of Symbolic and Numeric Representations**

CAS provide the facility to look at the numeric representation of functions through tables. Such tables may be created by generating a table from a rule (as was done with DERIVE 2.55) or by a link to the graph (a common method for CAS calculators). In this study the links between graphs and tables will not be discussed because the focus is only on Algebraic Insight. The ability to link symbolic and numeric representations will be shown when students link number patterns to form, and link key features to suitable increments and critical values of tables. These three Common Instances of this element of Algebraic Insight will now be described.

**Link number patterns in tables to form**

The first common instance of linking symbolic with numeric representations may be seen when students link number patterns in tables to algebraic form. Consider first starting with a table of values and linking this to its equivalent symbolic representation. In lower secondary school mathematics classes students practise this in the common ‘what’s my rule’ exercises. MacGregor and Stacey (1993) said of tables of values:

> The ability to perceive a relationship and then formulate it algebraically is fundamental to being able to use algebra. … The results of our study show that whereas most students can perceive patterns in tables easily, many do not perceive the functional relationship (p18).
While students may not perceive a relationship between the independent and dependent variables, the ability to perceive patterns in successive terms may enable students to have some expectation of algebraic form. Patterns in tables such as equal differences or equal ratios alert the mathematician to the likely form of the algebraic representation of these functions. Table 3A.3 below shows two relationships. The relationship between $x$ and $y$ may be identified as quadratic because the second order differences are constant and there is a symmetry in the values for $y$ about $x = 0$. The relationship between $s$ and $t$ may be identified as trigonometric because the values of $t$ cycle periodically. Linking patterns to forms does not find the full rule for the symbolic representation, but informs expectations. If a CAS is used to find the rule, a student would be more likely to notice if they had made an error either in syntax or in entering the data for the table.

Table 3A.3

<table>
<thead>
<tr>
<th>Quadratic</th>
<th>Trigonometric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>-5</td>
<td>28</td>
</tr>
<tr>
<td>-4</td>
<td>19</td>
</tr>
<tr>
<td>-3</td>
<td>12</td>
</tr>
<tr>
<td>-2</td>
<td>7</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>28</td>
</tr>
</tbody>
</table>

Identifying the form of the symbolic representation of a function will lead to expectations about the corresponding tabular representation. For example, if a function is trigonometric then as the independent variable increases there will be a cyclical pattern in the dependent variable. If the function is a positive exponential, and the independent variable is greater than zero, as the independent variable increases the dependent variable will increase very rapidly. Making such links demonstrates an instance of Algebraic Insight. Further insight is shown when students choose a suitable increment for a table so that such features are clear.
**Link key features to a suitable increment for a table**

Students need to be able to choose an appropriate increment for the independent variable; otherwise, important features may be missed. Key features of the algebraic representation of a relation may alert the student to likely features of interest. A scale that draws attention to these features should be chosen for the table.

When students are asked to construct a table-of-values by-hand they commonly choose small integer values for the independent variable, ‘x’. They do this because they expect the calculations to be easier. The process for constructing a table with CAS is of equal ease or difficulty regardless of the starting point or increment. To illustrate the importance of choosing an appropriate increment, consider the effect of using an integer increment in the following four examples.

**Example 1:** If \( f(x) = \frac{x + 4}{3x - 5} \) and the function were represented by a table where \( x \) went from -3 to 3 in steps of 1 or -30 to 30 in steps of 10, the fact that \( f(x) \) is not defined at \( \frac{5}{3} \) would be obscured. The key feature to notice here is that \( f(x) \) is divided by \( 3x - 5 \) so \( x = \frac{5}{3} \) would cause division by zero.

**Example 2:** If \( r(\theta) = \sin(2\pi \theta) \), using an integer increment for a table of values will result in 0 for each \( r(\theta) \), which is not very helpful! The key feature to identify here is the \( 2\pi \) factor since any angle that is a multiple of \( 2\pi \) has a sine of zero.

**Example 3:** If \( g(x) = (2x - 1)(3x + 1)(5x - 1) \), a table with integer increment or larger will miss the main detail of the graph which will be between -1 and 1. The key feature to identify here is that the factors will take zero values at \( \frac{1}{2} \), -\( \frac{1}{3} \) and \( \frac{1}{5} \). Not only will the \( x \)-intercepts occur at these values, but the maximum and minimum values will occur between these \( x \) values.

**Example 4:** Consider \( s(t) = t^2 - 300t + 20000 \). In this case an integer increment will be far too small. This is indicated by the large coefficient of the \( t \) term.

Each of these examples shows that identification of key features of the symbolic expression helps in the selection of an increment for a table that will not only show...
some values of the function but also provide information about the interesting features of the function.

It is similarly important when a numerical set of data is to be used to create a mathematical model that the table of values is sufficiently detailed that all salient points may be approximated from the symbolic representation. This, however, is not a matter of Algebraic Insight since mathematical modeling of data is usually done using the maximum amount of data available, not selected data points.

Creating a table that is a useful representation of the corresponding symbolic form of a function requires not only a good choice of increment but also a good choice of the interval of the table. This provides a third common instance of linking symbolic and numeric representations.

**Link key features to critical intervals of table**

Recognising form and key features alerts the student to features of interest that may be explored using the table. A function may describe a relationship between an infinite number of values, but for practical purposes a table will only list a restricted subset of the possible ordered pairs. Therefore to use a table of values to illustrate a function, the student needs an expectation of the likely location of these key features so that not only can a suitable increment be chosen, but also that an appropriate section of the table needs to be considered. The four examples used in the section above also illustrate this point. In example 1, identifying the \((3x - 5)\) denominator indicates that values around \(5/3\) will be of interest, and similarly the \((x+4)\) indicates that values around \(-4\) may be of interest. These features, along with any practical considerations in the problem, will suggest the interval for the table.

In example 2, recognition that the function will have repeating cycles of values suggests that a detailed representation of one cycle may give useful information about the function. In example 3, as already indicated, the most significant interval will be around \(-1\) to \(1\) and in example 4, identifying that the turning point of the graph will be at \(t = 150\) provides information to help choose a useful interval to illustrate with a table of values.

It should be noted that with CAS, while identifying these key features of the symbolic representation is important, it is often more efficient to look at the graphical...
representation to identify ‘critical’ intervals of the function which could be explored with a table of values. This linking of representations does not require Algebraic Insight but, as Arcavi (1994) noted, Symbol Sense; in this case Algebraic Insight may be shown by recognising when algebra is not the best tool to use and that more efficient progress may be made by using information from other representations.

These examples illustrate the third instance of the Linking of symbolic and numeric Representations. This element of the second aspect of Algebraic Insight completes the framework.

**Algebraic Insight and use of CAS**

A number of researchers, (see for example Allaire and Fabricant, 1995; Heid, 1988; Lagrange, 1999) have stated that students need some knowledge and understanding of algebra in order to work effectively in an intelligent partnership with CAS. These writers do not describe in detail what it is that students need to know. The author believes that Algebraic Insight as outlined in this framework identifies key abilities which students need in order to use CAS effectively both to do and to learn mathematics. Reasons why these key abilities may help students working with CAS are now summarised.

**Algebraic Insight helps students to use CAS**

- Recognising conventions and identifying structure are important for transferring conventional mathematics to the required syntax to enter expression in CAS.

- Identifying structure helps students to recognise equivalent expressions. This is important for students interpreting CAS output and checking the reasonableness of results.

- Identifying key features and understanding their significance gives students guidance as to the nature of results which they should expect. This also helps them check for errors.

- Algebraic Insight helps students check for errors in graphical and tabular representations, as well as the symbolic algebra mode.
• Algebraic Insight helps the student to know whether moving to an alternative representation will assist them in solving the problem.

While Algebraic Insight is needed to make effective use of CAS there can be an iterative process: CAS may be used, in turn, to help students deepen and extend their Algebraic Insight. Experience reported by Heid (1988), Demana (1990), and Palmiter (1991) for example, suggested that the use of CAS in learning mathematics assisted development of conceptual understanding. Reasons why the use of CAS may help students improve their Algebraic Insight are now summarised.

**The use of CAS helps Algebraic Insight**

• Exposure to many examples in a short time helps students notice the effects of key features.

• Ease of swapping between representations means students are more likely to explore other representations and so build their knowledge for Linking Representations.

• Support through un-mastered parts of the mathematical solution process enables students to get to aspects beyond it. For example, In a multi-stage problem, assistance with factorising may help students to obtain solutions for one section of the problem, which are then required in order to begin on a later stage of the problem.

**Learning with CAS requires Effective Use of CAS**

CAS may be used functionally: that is, to carry out manipulations and calculations to find or check answers to problems. CAS may also be used pedagogically, to give students exposure to a range of examples, assist them to see patterns, and develop Algebraic Insight. CAS can assist students both to do and to learn mathematics but this requires Effective Use of CAS, to be unhindered by technical difficulties and to use it for investigation in a strategic manner. These qualities of CAS use, which contribute to such Effective Use of CAS, will be described in Chapter 3 Part B.
FRAMEWORKS FOR ALGEBRAIC INSIGHT AND EFFECTIVE USE OF CAS

PART B: EFFECTIVE USE OF CAS

In Part B of this chapter, the second of the two key clusters of variables associated with students’ use of Computer Algebra Systems (CAS) in mathematics classes will be discussed. This second variable, Effective Use of CAS, will be treated in a way similar to the way the first variable Algebraic Insight was treated in Chapter 3 Part A. Again, a framework will be proposed. This framework is based on a review of the literature and reflection on my experience of teaching mathematics, with CAS available, over the past eight years.

On balance, the review of literature reported in Chapter 2 suggested that students working with CAS show significant gains in their conceptual understanding of mathematics and that time lost in learning to use CAS is more than compensated for by the speed with which CAS can perform routine mathematical tasks. In particular, Harper, Wooff, and Hodgkinson in their “Guide to Computer Algebra Systems” (1991) outlined six reasons why a mathematics student would find using CAS an advantage. As listed below, these provide a reminder of exactly what it is that CAS offers a student of mathematics.

- Using CAS saves time and effort.
- Solutions are more likely to be correct.
- Algebraic, general solutions are often preferable to numerical ones since relationships between variables may not be easily evident from numbers or diagrams.
- Algebraic solutions are exact (no truncation errors). When approximate solutions are required the computer can simplify algebraically first (less rounding errors).
- Fast production of solutions to applied mathematics problems allows more time to be devoted to the properties of the solution.
- CAS allow students to investigate large 'real' problems instead of superficial contrived ones.

These facilities create the opportunities to experiment, referred to by Dubinsky and Leron (1994) that is, to make conjectures and try them out. Arnold (1995) suggested
that computer tools are ideal for investigation and systematically pursuing a line of mathematical thought. As a result CAS can be a valuable tool for both doing mathematics and assisting students’ learning of mathematics. My previous work (Pierce and Stacey, 2001a) reports on a study of teaching with CAS where initially CAS was used to increase the range of problems to be solved and in later years was also used for the teaching and learning of new concepts and procedures.

The mere presence of CAS in a classroom does not increase the probability that more mathematical problems will be solved, nor that more learning will take place. The value of CAS, like any new tool or aid, depends on how effectively it is used. Using CAS to do mathematics requires the student to become familiar with both the hardware and the software associated with this technology. This presents the student with some learning overhead on top of learning mathematics. Students' success in obtaining benefits from the use of CAS will depend on how effectively they learn to use this technology. The efficacy of their use will depend on both technical and personal aspects: whether the student can operate the program with a minimum of difficulty; their attitude towards the use of CAS; and the manner and purpose of their partnership with CAS.

In this chapter these two aspects which effect students use of CAS will be discussed and a working framework for Effective Use of CAS proposed. The purpose of the framework will be briefly outlined before the framework is presented and then the components of the framework will be detailed and justified.

**Effective Use of CAS: A working framework for using CAS for learning mathematics**

The aim of the framework, set out in table 3B.1, is to highlight aspects of CAS use that determine how effectively a student is able to use CAS both to do and to learn mathematics. The framework may provide a guide for systematically examining whether CAS is being used effectively. It may also provide a basis for planning and reflecting on teaching content and practice as they impact on the Effective Use of CAS. The purpose of the framework is not to provide a list of categories such that a test can be created with each item relating to one, and only one category. This may be desirable but does not seem possible.
Table 3B.1  
*Effective Use of CAS Framework*

<table>
<thead>
<tr>
<th>Aspects</th>
<th>Elements</th>
<th>Common Instances</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Technical</td>
<td>1.1 Fluent use of program syntax</td>
<td>1.1.1 Enter syntax correctly</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.1.2 Use a sequence of commands and menus proficiently</td>
</tr>
<tr>
<td></td>
<td>1.2 Ability to systematically change</td>
<td>1.2.1 CAS plot a graph from a rule and vice versa</td>
</tr>
<tr>
<td></td>
<td>representation.</td>
<td>1.2.2 CAS plot a graph from a table and vice versa</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.2.3 Create table from a rule or vice versa</td>
</tr>
<tr>
<td></td>
<td>1.3 Ability to interpret CAS output</td>
<td>1.3.1 Locate required results</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.3.2 Interpret symbolic CAS output as conventional mathematics</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.3.3 Sketch graphs from CAS plots</td>
</tr>
<tr>
<td>2. Personal</td>
<td>2.1 Positive attitude</td>
<td>2.1.1 Value CAS availability for doing mathematics</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.1.2 Value CAS availability for learning mathematics</td>
</tr>
<tr>
<td></td>
<td>2.2 Judicious Use of CAS</td>
<td>2.2.1 Use CAS in a strategic manner</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.2 Discriminate in functional use of CAS</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.2.3 Undertake pedagogical use of CAS</td>
</tr>
</tbody>
</table>

**An overview**

After reflecting on the experience of teaching with CAS over a number of years the author decided that, logically, two key aspects contribute to the degree with which students use CAS effectively. The first aspect depends heavily on the hardware and software used and so is termed *Technical*. The second aspect, which depends on the response of the student to the availability of such technology for doing and learning mathematics - that is, on attitude and the choices made by students, is termed *Personal*. 
These two aspects are both aspects of the user’s knowledge. The Technical aspect involves the intersection of the user’s understanding of mathematics, especially Algebraic Insight, with the facilities of the particular CAS used. It focuses on their ability to use the facility of CAS to do mathematics. The Personal aspect involves the intersection of the user’s thinking about the nature of mathematics and the facilities offered by CAS. It focuses on the choices the user makes in using or not using CAS to do and learn mathematics. Each of the aspects of Effective Use of CAS can be subdivided into elements.

These elements contribute to Effective Use of CAS regardless of the particular hardware or software used and the level of mathematics being studied. Evidence of Effective Use of CAS may be seen when Common Instances of these elements occur. Some important examples of these Common Instances are listed in column three of the framework. These Common Instances listed, are not an exhaustive catalogue of items by which the Effective Use of CAS should be measured, but are examples drawn from my experience of observing students using CAS in my classes. Examples of Common Instances will certainly depend on the teaching context: the particular CAS and the content of the mathematics course for which it is used. In other circumstances with students of a different age or studying different content, the Common Instances could be different. However, the aspects and elements of this framework are designed to be general in their application. A summary of the framework is presented here in table 3B.1; the next sections discuss each aspect of Effective Use of CAS in turn. Each element of these aspects is described and illustrated by examples of important Common Instances.

**The Technical Aspect**

The first aspect of CAS use is *Technical*: in particular, a student needs to have the flow of their thinking uninterrupted by technical difficulties for Effective Use of CAS. Such difficulties potentially occur when a student makes mistakes, or is unsure of how to proceed, either because of their inability to use some part of the CAS program or their inability to interpret the CAS notation or image as conventional mathematics. Arnold (1995) noted that the interface, the method of entry of algebraic forms, manipulation of algebra, and manipulations of graphical and tabular representations are each features of CAS that may be a source of difficulty to CAS users. Most writers report that students learnt to use CAS quite quickly. Technical difficulties vary, and may only impact on
student’s Effective Use of CAS, as they become familiar with the use of different aspects of the program. Many of the technical difficulties will be specific to the particular CAS and version of CAS being used which means no definitive, detailed generalisations can be made. At this stage of the development of both hardware, and software, improvements are being made so that some features of CAS that presented difficulties to the students in this study, for example, may have been altered in more recent updates. Never-the-less, technical difficulties are still a factor to be considered when assessing how effectively a student uses CAS. The fewer technical difficulties a student experiences the more effectively they will be able to use CAS for both doing and learning mathematics.

It is difficult to gain from the literature any overview of technical difficulties with CAS. It is common for writers to make comment on whether or not their students experienced difficulty with the technical aspects of CAS. Yet many writers do not give details, as they are specific to the CAS used and may well have changed by the time of publication. Stevenson (1995) reported on students who found it easy to enter expressions and execute commands but who disliked using CAS while Rosendo and Carvalho e Silva (1994) wrote that their students were at ease and enjoyed using the computer so presumably they experienced few technical problems. Others commented that technical problems caused some confusion. Wain (1993) said that their students needed help with using the CAS. Lagrange (1999) found that that even students who said they found the program easy to use, in fact had not necessarily learnt how to use even the basic home screen commands. Most students will learn just enough about CAS to be able to use it to assist with their current problems. They may therefore report few difficulties because their use is limited. The case studies in Chapter seven will provide examples of this.

Operating CAS can mean that there are more things to think about when doing the mathematics: CAS syntax, CAS menus as well as the mathematics. While concentrating on the process of operating the CAS, students can become distracted from the mathematics, which is the real purpose of the exercise. Hunter, Marshal and Monaghan (1995), for example, observed that the technology could create this cognitive ‘noise’. Their students reported problems, for example, using graphics commands and interpreting the output. Hillel et al (1992) had also reported that students had problems
scaling graphs and so sometimes missed the purpose of an exercise when the salient points were not included in the graph window.

Many writers (see for example Asp, Dowsey and Stacey, (1993); Atkins, Creegan and Soan, (1993) expressed concern that students learning new mathematics with new technology may lose sight of the primary objective: learning mathematics. Lagrange (1999) observed that ‘Technical difficulties in the use of CAS replaced the usual difficulties that students encountered with paper/pencil calculations’ (p144). It is important that the technology should not create another large hurdle for students to overcome in the learning of mathematics.

**Elements of the Technical aspect of Effective Use of CAS**

In this thesis the term *Technical difficulties* is used to refer to the difficulties which students may experience in fluent use of program syntax, ability to systematically change representation and ability to interpret CAS output. The sections below explain these terms and note Common Instances of these elements of the Technical aspect of Effective Use of CAS that the author has observed with first year undergraduate students using DERIVE 2.55.

**Fluent use of program syntax**

Fluent use of program syntax can be summarised as the changing of conventional mathematics to CAS syntax and knowing ‘which buttons to push’ or ‘which commands to execute’ when using the program to perform mathematical calculations. Common Instances of fluent use are when students enter syntax correctly or use a sequence of commands and menus proficiently.

Some writers, including Arnold (1995) express concern that the entry of algebraic expressions should use simple syntax that closely approximates conventional written forms. Most CAS use * for multiplication, / for division and ^ to indicate raising to a power. In my experience these common replacements for conventional symbols do not seem to create difficulties for students. There is little mention in the literature of this adjustment causing problems. In an earlier study (Pierce, 1999a) I observed that most students had little trouble using these common symbols, but many found the rigour of syntax difficult. For example: when to use brackets, which type of brackets or parentheses to use, where to use commas and where to use space. In DERIVE 2.55
these difficulties were heightened by the editing process. It was a DOS-based, not Windows-based, program and editing required students to use Control-D and Control-S to move the cursor within an expression. Since then the user interface of DERIVE and other major CAS have been considerably modified to be more ‘user friendly’.

In order for CAS to be an effective tool students need to be able to enter mathematics both quickly and correctly; and while alternative symbols do not cause difficulty, using brackets appropriately does. For example $\sin 3x$, if entered as $\sin 3x$, was interpreted in DERIVE 2.55 as $\sin 3 \times x$. The expression needed to be entered as $\sin(3x)$. Similarly $\frac{x^2 + 5x + 6}{x - 2}$ needed to be bracketed as $(x^2 + 5x + 6)/(x - 2)$. Students themselves need to identify the structure of expressions and insert brackets to ensure the appropriate order of calculations. Knowledge of the rules applied by the particular CAS is a technical difficulty in that it relates to the required syntax of the CAS. The solution to this difficulty requires not only technical knowledge, but also that the student has sufficient Algebraic Insight to decide where to place the brackets (discussed in Chapter 3 Part A).

For students to use CAS effectively they need to be clear about available ‘built-in’ commands and how to access them. This is not always intuitively obvious even if students understand the mathematical process involved. For example, in the version of DERIVE used in this study, substitution of a value, say, $x = 25$, into the expression $15x^5 + 7x^3 + 2x - 1$ required the sequence shown in the screen dumps in figure 3B.1. First, from the alternatives shown in screen 1, select the Manage menu. Next, screen 2, select the Substitute menu then, screen 3, enter the number of the appropriate expression and finally, screen 4, overwrite the appropriate variable with the value to be substituted. This results in the substitution of the value into the expression $15 \times 25^5 + 7 \times 25^3 + 2 \times 25 - 1$ (this is the way the expression is presented by DERIVE 2.55), which must now be Simplified in order to evaluate the result. Students initially had difficulties with such command sequences but they quickly became familiar with them (Pierce, 1999a). The interface and the menu structure of more recent versions of CAS are less cumbersome but each piece of software has some peculiar features of syntax that present difficulties to the novice user.
Figure 3B.1 DERIVE 2.55 screen dumps showing command sequence for substitution
The difficulties that students experience with fluent use of program syntax relate primarily to unfamiliarity with the symbols and words used within the program. A second source of difficulty, and hence the second element of the technical aspect of Effective Use of CAS, occurs when students try to move between the three representations offered by CAS.

*Ability to systematically change representation*

CAS offer symbolic, graphical and tabular representations of functions. Common Instances of using this facility effectively can be seen when students use CAS to link representations. For example, using CAS to plot a graph from a given rule, plot a graph from a table, create a table from a rule or create a table from a graph as symbolised then illustrated in figures 3B.2 and 3B.3.

*Figure 3B.2* CAS allows simple links between these three representations

*Figure 3B.3* Example of the three representations for $f(x)=2^x$ using DERIVE 2.55
The ability to move quickly between algebraic, graphical and tabular representations of functions is often said to be one of the strengths of CAS. However, initially the student must become familiar with the sequence of commands to use, or buttons to press, to move for example, from the algebra window to the graph window and plot the graph. These details will be CAS specific, and will require some practice to become part of a systematic routine for the student.

Next, as Arnold (1995) reminds us, the CAS must be open to manipulation, permitting adjustment of all parameters in addition to quick and easy facilities for "zooming in" and "zooming out". CAS will not automatically draw the view of a graph that will be most helpful to a student. Setting appropriate graph windows, and adjusting scales can present difficulties. Different CAS automate different features of the graphing window. Some CAS allow the user to specify the maximum and minimum $x$ and $y$ coordinates and automatically adjust the scale. DERIVE 2.55 allowed the user to select $x$ and $y$ scales but not maximum and minimum values. Again to be able systematically to change representation, the student needs to establish a routine for checking and adjusting the graph window.

Similarly when producing a table of values for a function the student must learn a CAS-specific routine to be able to control the section of the table viewed and the increment for the independent variable. Students need to know how to use the CAS buttons, menus or commands to set these appropriate parameters. In order to create a tabular representation of a function in DERIVE 2.55, students to needed to use the correct program syntax. To create a tabular representation of some function $f(t)$ from $t=0$ to $t=8$ using an increment of 0.5, for example, would require the student to select the AUTHOR option and then type in the command, vector([t,f(t)],t,0,8,0.5), taking care with the type of brackets used, position of commas and the order of parameters. $f(t)$ could either be defined before this step (using the DECLARE option) or entered as an expression within the vector command. Such commands are a potential source of technical difficulty that may inhibit students’ ability to change representations systematically and so inhibit their Effective Use of CAS. When students have the ability systematically to change representations, then these Common Instances may be seen but these CAS specific routines will require practice to become automatic and so not interrupt students’ mathematical thinking.
There is a third element in the technical aspect of Effective Use of CAS that can present difficulties. Students must not only enter and use expressions in the algebra window and move between representations but they must also interpret the information on the CAS screen in terms of conventional mathematics.

**Ability to interpret CAS output**

Interpreting CAS output can cause difficulties that will lessen the Effective Use of CAS. In this section, ‘interpret’ refers to the specific understanding needed when the format of results is different from the conventional pen and paper presentation, or is constrained by the limitations of the screen. In a previous study (Pierce, 1999a) reporting on students’ use of CAS, the present author wrote:

In addition they had some problems with DERIVE’s presentation of results. The use of negative indices in expressions that they would normally have written as quotients and dealing with long expressions which do not fit in the width of the window caused confusion. In addition the form of solutions was sometimes not what students expected, for example: exact rather than approximate solutions, and complex as well as real solutions. (p173)

A simple example of results being presented in an unfamiliar form comes with the solutions to a quadratic equation. For example, \( \{x : 2x^2 + 5x - 5 = 0\} \) may be shown separately and configured in a different way from the conventional form \( \frac{-5 \pm \sqrt{65}}{4} \), achieved by using the quadratic formula by hand. The two separate solutions, given by CAS, may be across the screen and require scrolling across to view the second solution, or underneath each other. In either case the student must expect that there will be two solutions and make effort to look for the second solution. The DERIVE 2.55 version of the solution is shown in figure 3B.4.

<table>
<thead>
<tr>
<th>Solution process:</th>
<th>DERIVE 2.55 output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author expression 2x^2+5x-5 press ENTER</td>
<td><img src="image" alt="DERIVE 2.55 output" /></td>
</tr>
<tr>
<td>(Line 1 appears in algebra window)</td>
<td></td>
</tr>
<tr>
<td>TAB to solve command press ENTER or just press L</td>
<td></td>
</tr>
<tr>
<td>(Lines 2 and 3 appear in algebra window)</td>
<td></td>
</tr>
</tbody>
</table>

*Figure 3B.4 Solution of quadratic equation using DERIVE 2.55*
Interpreting graphs from a computer or calculator screen presents special difficulties. These may relate to the method that the software uses to draw graphs, the resolution of the screen and the use of colour. Most CAS produce graphs by evaluating the coordinates of a number of points and interpolating lines between. This may produce

graphs which look like a series of small straight lines, or which obscure points of discontinuity or which lead to vertical lines instead of space between sections of graphs of non continuous hybrid functions. An ‘expert’ mathematician who knows roughly what to expect a graph to look like may look beyond these technical problems. They are, however, confusing to the novice. In a previous study (Pierce, 1999a) the present author wrote:

What ‘experts’ see is not always what ‘novices’ see. This became evident when students were asked to copy sketch graphs of their findings from DERIVE. Some students' sketches of

\[ f(x) = x^n \] for increasing values of \( n \) were increasingly 'horse-shoe' and S-shaped curves.

25% of the students reported that they found DERIVE graphs 'hard to read'. (p174).

![Figure 3B.5 Example of student work showing error in visual perception (from 1998 data published in Pierce and Stacey 2001b, p40)](image)

Some of the problems of interpreting screen images relate to visual perception. Ernst (1986) wrote about the use of spacing of lines, colour and shading to produce illusions. This reminds us that these factors may also accidentally cause confusion for students trying to interpret graphical output. Tuft (1983) systematically looked at factors that affect the visual communication of quantitative information. His observations about peoples' distorted perceptions of lines which are close together are particularly relevant when considering difficulties students may experience using CAS. As mentioned earlier Goldenberg (1988) and Schoenfeld (1992) point out that even in a seemingly simple exercise like asking students to explore the effect of \( a \) in \( f(x)+a \) novice students may not
perceive the vertical translation. Human perception draws the eye to the minimum distance between curves (the horizontal distance in this case) not the equal vertical distances. This issue was discussed by Goldenberg (1988) and can be seen in figure 3B.6 which replicates Schoenfeld’s, (1992, pp 12,13) illustrations. Students can quickly draw several graphs on the one screen using CAS, but this does not mean that all students will be able to correctly perceive the differences in the images.

![Illustration of visual perception difficulty](image)

In summary: from the Common Instances of technical difficulties discussed, it can been seen that the hardware and the software of CAS may each cause the student some difficulties that will make their use of CAS less effective. As CAS are developed and refined so that the interface is simplified and the use more intuitive –that is, more like conventional mathematics, these difficulties are more than likely to diminish. There has been considerable progress in the user friendliness of CAS in the time since this study began and no doubt this will continue to improve. In the interim attention must be given to the technical aspect of Effective Use of CAS.

Even if there were few technical difficulties using CAS the second aspect, the Personal aspect would still impact on how effectively students use CAS.

**The Personal Aspect**

The Personal aspect of Effective Use of CAS is evidenced by two elements: positive attitude towards the use of CAS for doing and learning mathematics, and the ability to make Judicious Use of CAS. These elements will now be discussed separately.
**Positive attitude**

It could be expected that students who have a positive attitude are more likely to use CAS effectively, and students with a negative attitude towards the use of CAS for doing mathematics are more likely to avoid its use and hence miss out on its benefits. Students with a positive attitude towards CAS as a tool to assist in both doing and learning mathematics could be expected to persist with its use and overcome initial technical difficulties due to lack of familiarity with the peculiarities of the program.

**Value CAS availability for doing mathematics**

At the end of 1997 the author (Pierce and Stacey, 2001a) surveyed students who had used CAS throughout their mathematics course in that semester. Most of these students reported a positive attitude towards the use of CAS. They agreed that it had given them confidence and was a helpful tool for producing answers to mathematics problems. When CAS was introduced more widely into the teaching and assessment of this first year course a negative attitude was expressed by some mathematics staff who felt that the students would not really be ‘doing’ mathematics. This proposition was put to students at a group interview session following the release of final results in 1997. Here are responses from three students who had a positive attitude towards using CAS and whose use of CAS for ‘doing’ mathematics was effective because they could see roles for themselves and for the technology.

**Student A:** I reckon that we are actually doing it. The computer only spits out an answer to what you type into it

**Student B:** It’s just like with a calculator…it’s just going a bit further, we’re not just doing multiplication and division quickly, we’re doing differentiation quickly.

**Student C:** Also, you still have to interpret the answer or for that matter interpret the question so you can convert it into what the computer wants…you’re still doing a lot of mathematics.

Students' beliefs about mathematics influence their attitude towards the use of CAS. This view can be sensed in Arnold's (1995) experience. He reported that the beliefs and perceptions of the students in his study impeded their use of algebra software. He found that his students viewed mathematics as ‘answer-based’ and devalued exploration. They valued unaided individual effort and devalued the use of technology. Small and Hosack (1991) reminded us that:
Students tend to measure the importance of activity by the amount of time spent on it and the proportion of the examination allotted to it. Since most of a student's effort on both homework and tests is devoted to algorithmic computation, it is not surprising that students view mathematics as a collection of formulas (to be memorised) and "to do mathematics" is to compute.

What CAS does best is compute. Students who believe that computation equals mathematics may feel that the use of CAS is illegitimate, and students who enjoy doing algorithmic computation, and are good at it, may feel that the availability of CAS devalues their skills. Kaput (1992) wrote that:

Serious use of computers of any kind, let alone the level of computing to be available in the 1990's and beyond, is simply not a part of the culture of schools and schooling. This transitional factor will be in effect for at least a generation. (p 517)

He clearly thought that attitudes and belief about mathematics and technology would impede the effective use of technology in mathematics classrooms. The author has also observed that students’ attitude to the use of CAS is strongly influenced by their school experience. Until recently assessment of school mathematics has been heavily biased towards by-hand skills. When the present author asked her 1998 university students whether they would like to have CAS available for all assessable tasks there was a very positive response: 100% in favour of CAS. However when it came actually to using CAS for assessment not all students demonstrated a positive attitude. For example:

On one of their assessment tasks students were deliberately asked to look at a function they had not met in the course, \( g(x) = \sec(x) \). It was expected that they would use DERIVE to explore this function and the associated family of translated and dilated functions. Some students made no attempt at the question while others were observed to graph the function with DERIVE but not record their answer. (Pierce, 1999b, p )

When the present author questioned these students later they indicated that they felt it would be ‘cheating’ to use CAS because they did not ‘know’ the function. They were also not confident that they would be able to explain the output. Their negative attitude, feeling that its use was illegitimate, impeded their Effective Use of CAS. Students with a positive attitude, who value CAS as a tool for doing mathematics and who see this partnership with CAS as a legitimate tool for solving mathematical problems are more likely to make Effective Use of CAS.
Value CAS availability for learning mathematics

Similarly, students show evidence of a positive attitude when they value the availability of CAS for learning mathematics. Seventy percent of students from the author’s 1997 class (Pierce, 2001a) either agreed or strongly agreed that the use of CAS had helped them to understand the mathematics. Three students who valued the availability of CAS for learning mathematics said:

Student A: …not doing everything by hand you lose a bit of the skills. Then on the other hand you’ve got the computer there…so you can play around and find out what you’ve ‘lost’.

Student B: I think it actually helps me learn new things because when there are new things that I’m learning, while I’m finding them difficult, I can use DERIVE and go through the steps. With more practice and seeing DERIVE go through it, I pick it up myself and then I can feel confident doing it myself without a package.

Student C: I think it helps me. It’s probably psychological as well, just knowing that you’ve got the computer there if you get stuck. It helps me understand the concepts of different problems. Using the package helps me understand the concepts behind the maths as well as doing the maths.

Arnold (1995), however, found that his students, who valued ‘answer-based’ mathematics, devalued exploration and open problem solving (for which CAS may be a great aid). They also devalued the use of any external aids in learning mathematics.

Students from the author’s 1998 class showed a mixed response towards using CAS for pedagogical purposes (Pierce, 1999a). Most students agreed that the use of CAS aided learning but many felt that CAS should only be used after first learning to do the mathematics by-hand. When asked to reflect on which tool (Pen and Paper, Scientific Calculator, Graphics Calculator or DERIVE) they would use most frequently for different purposes, it was clear that most of the time they would prefer pen and paper for learning. When questioned further, these students said they felt that mathematical understanding would be shown if they could do the mathematical manipulation by hand and without help.

This experience suggests that a student’s attitude, their emotional response, to the use of CAS for learning and doing mathematics is affected by their beliefs about learning and understanding mathematics. Those who value the facilities of CAS for exploring multiple examples and different representation are more likely to make the effort to overcome technical difficulties and make Effective Use of CAS.
**Ability to make Judicious Use of CAS**

The second element of the Personal aspect of Effective Use of CAS is seen in students’ cognitive response to CAS, that is, in students’ ability to make Judicious Use of CAS. Drijvers (1999) says that it is important that students develop the ability to decide when and how computer algebra can be useful. In this thesis three Common Instances of this element illustrate what, from the author’s experience, constitutes Judicious Use of CAS.

**Use CAS in a strategic manner**

The term ‘manner of use’, as used in this thesis, refers to the approach students take to using CAS. In his thesis, Arnold (1995) called this their ‘form of mathematical software use’ and pointed out that it may vary from non-use and passive use at one extreme to strategic use at the other.

A student’s ‘manner of use’ can be conceptualised as the degree to which a student engages in the process of using CAS to solve or investigate mathematics problems. A common instance of Judicious Use of CAS is seen when students fully engage in the process by planning to use CAS in a strategic manner. The lowest level of engagement would be non use. However in this thesis non-use is considered as evidence of a negative attitude rather than being a descriptor for a manner of use because participation in the course expected CAS use. The lowest ‘manner of use’ is therefore passive use. Passive use occurs when a student observes another using CAS and merely copies down results without engaging in the process. Sometimes students engage with the machine but with limited mathematical thinking. When some algebraic manipulation is required these students enter the expressions and just ‘try out’ possible commands like solve, factor or expand and ‘see what happens’. Their ‘manner of use’ could be described as random. A student using CAS strategically has an aim in mind and goes about finding the solution to the problem in a systematic manner. They demonstrate a strategic ‘manner of use’. For example, when asked to ‘explore’ a particular family of functions the student would deliberately vary one parameter at a time and record the consequences before systematically examining different combinations of parameters. The student would decide which representations of the function, symbolic, graphical and/or tabular, would yield the most useful information for their purpose, and therefore make efficient use of CAS facilities. In the author’s past experience, when questioned about their
approach to using CAS one student may say “Oh I just did one of everything” while a second may say “I chose these graphs because they show…” The second student shows evidence of the common instance -using CAS in a strategic manner.

*Discriminate in functional use of CAS*

A second common instance of students’ ability to make Judicious Use of CAS is seen when students are discriminating, or selective, in their functional use of CAS. A student’s purpose in using CAS is ‘functional’ when they use it to 'find answers'. These may be answers to problems that they could 'easily' have done with pen and paper; questions they perceive to be hard; or those which are time consuming. Common Instances of a student making Judicious Use of CAS may be seen when a student uses CAS for these two later purposes. Non-discriminating, non-selective use of CAS may be likened to using an arithmetic calculator to add 2+2. Students should not be considered to be making Judicious Use of CAS when they use it to do mathematics that they could have done easily and quickly ‘in their head’. A student solving $x+2=3$ for $x$ ‘in their head’ but using CAS to solve $12x^3 + 9x^2 - 43x = 6$ for $x$ would be discriminating in their functional use of CAS.

*Undertake pedagogical use of CAS*

A third common instance of students’ ability to make Judicious Use of CAS can be seen when students undertake pedagogical use of CAS. The purpose for which students choose to use CAS is pedagogical when this use is to help them learn a new mathematical concept or procedure. The purpose is also pedagogical when its aim is to help the student gain a broader or deeper understanding of familiar mathematics. A typical common instance of a student undertaking pedagogical use of CAS is seen when a student uses CAS to explore a family of functions in order to understand the effect of varying some parameter. This may entail using CAS to look at multiple examples and some non-examples of that family of functions. It can also involve using multiple representations of functions in order to illustrate a concept in a number of ways and hence develop strong schema (Tall, 1987).
A further common instance of a student undertaking pedagogical use of CAS is seen when a student decides to use CAS to explore variations on set problems; that is when they use CAS to help answer the question ‘what happens if…?’

Students’ ability to make Judicious Use of CAS is often seen in a combination of these Common Instances. The way students feel about CAS use and the way they think about using CAS form the second element of the Personal aspect of Effective Use of CAS. To use CAS effectively students need a positive attitude and the ability to make Judicious Use of CAS.

**Algebraic Insight and Effective Use of CAS**

The Algebraic Insight framework outlines the mathematical knowledge and thinking which students require, to initiate and monitor the use of CAS, for both functional and pedagogical purposes. The Effective Use of CAS framework summarises the human-technology interaction factors that will influence the degree to which students choose, or are able to use, CAS for both functional and pedagogical purposes. It has been proposed in Chapters 3, Parts A and B, that to use CAS effectively for learning mathematics students need to develop attributes for both Algebraic Insight and Effective Use of CAS. The meaning of these terms has been operationalised by means of working frameworks that have been described and discussed in detail.

CAS are becoming more widely available, convenient and affordable. It is therefore important that mathematics educators identify, and understand, the thinking and skills necessary for students to gain maximum benefit from the use of this technology. The two frameworks, presented in the chapter, may provide a basis for this process. In order to trial the practical use of these frameworks this thesis goes on to study the progress of a group of tertiary students using CAS. Evidence for change in students’ Algebraic Insight or Effective Use of CAS, and any interaction of factors was collected. For that purpose instruments to measure these constructs were needed. These instruments and the methodology of this study will be detailed in Chapter 5, but first the setting of the study will be described in Chapter 4.
CHAPTER 4

THE CONTEXT FOR THIS STUDY OF ALGEBRAIC INSIGHT AND EFFECTIVE USE OF CAS

Reasons for discussing the context

In this study, two key factors that impact on the use of CAS in teaching and learning mathematics will be examined. These factors, Algebraic Insight and Effective Use of CAS, have been outlined in Chapter 3. Many studies reported in the literature make general reference to the context of their study, but often details of what it meant to ‘use CAS in their teaching’ are hazy. There are commonly phrases like 'taught in a traditional manner’ and 'taught using CAS' without much detail on what this meant in practice. In order to interpret or extrapolate the findings of a study it is important to appreciate the setting (including details of the teaching) in which it was carried out. This chapter includes some basic background information about the context, the students, the course, and the previous research, which provide the setting of this study. Then specific details are given to illustrate what is meant, in this case, by a course in which CAS was used for teaching, learning and assessing mathematics.

History of research

Computer Algebra Systems have been used in the teaching of mathematics courses at the University of Ballarat since 1990 and curriculum changes related to the introduction of CAS have been reported by Yearwood and Glover (1995) then further by Mays, Glover and Yearwood (1996). Initially CAS was simply used as an additional tool to assist students to tackle more difficult problems. In the past students had attended two lectures and two tutorials each week. CAS was first introduced by replacing some tutorials by computer laboratory classes and its use was allowed for projects and in a special, additional, examination. Later, when better quality projection facilities became available, teachers also demonstrated the use of CAS in lectures. As access to technology improved, and it became practical for students to use the computers in most
classes and in their own time, CAS was more fully integrated into the teaching, learning and assessment of a number of mathematics courses at this university.

Initially CAS was used to allow students to do a wider range of problems. This range included problems that required the application of principles, learnt on ‘straight forward’ ‘by-hand’ problems, to ones where students would find the algebra involved to be both time consuming and prone to error. Later, CAS was also used to help students to learn mathematics through explorations involving multiple examples and the use of different representations. This transition from using CAS in the mainstream mathematics course to do mathematics (1995-1996) to using CAS to learn mathematics (1996-1997) is described in Pierce and Stacey (2001a). In addition, in 1997 a new course, ‘Introduction to Calculus and Computer Algebra Systems’ was introduced to enable students with limited secondary school mathematics backgrounds to make the transition to mainstream university mathematics. Gradually accumulated teaching experience with CAS and the experience of others, reported in the literature, informed choices about how CAS might be used in this context. This ‘new’ course provided the setting for this study.

**Reasons for using this setting**

This study took place at the University of Ballarat in 1999 with students studying an introductory calculus course. This course was chosen for several reasons. Firstly, the use of CAS was prescribed in the course description. Secondly, there were some textbooks that provided suggestions for the use of CAS with this course content, (summarised in the section of the course description included as figure 4.1). For example, Berry, Graham, and Watkins, (1993) suggests how CAS may be used for both teaching and exercises; Weimer (1998) incorporates specific instructions on the use of DERIVE and provides sets of exercises which encourage the use of CAS for both functional and pedagogical purposes. There was also information from previous studies (Kneebone and Pierce, 1997; Pierce and Roberts, 1997; Pierce 1999a; Pierce and Kendal 1999; Pierce and Stacey, 2001a, 2001b) to assist in planning the teaching program. Thirdly, the researcher had been assigned to teach and develop this course. In 1998-1999, when this study was planned and conducted, the researcher was also the course teacher.
Typically about 30 students had typically taken this course each year since 1997. In 1999, when the main data collection was planned, 21 students enrolled in the course. Students did not choose this course in preference to other mathematics courses. Rather this was the only mathematics course offered to students at this level and mandatory for education students (primary or secondary) who wished to include mathematics in their program of study. Our experience, from previous years, suggested that such education students would be both willing and interested in participating in this study. This was important because the planned study involved detailed data collection throughout the course, a considerable imposition on these students.

**Opportunities and constraints for research in this setting**

The methods chosen to collect the data that provide information about the key questions addressed by this study were both constrained and facilitated by this setting. The small numbers of students studying this course meant that it was not appropriate to consider an independent-groups experimental research design. It also meant that, while quantitative analysis of data would yield interesting observations and be an effective way to monitor student progress, it would most likely yield few statistically significant results. However in this setting, with the teacher as researcher, it was possible to collect detailed data from observations and interviews and so build up profiles of each student as well as overall group results. The methodology therefore uses detailed case study data, set against a background of full-class quantitative data. Furthermore the attempt to collect quantitative data is helpful in advancing the use of the frameworks by demonstrating how teachers might keep track of students’ progress. The methods used to collect this data are outlined in detail in Chapter 5.

**The students**

The University of Ballarat is a small regional university with a broad cohort of students, eighty percent of whom are, according to government categorisation, socially or economically disadvantaged. Typically the students who undertake this course are from a variety of degree programs including Computing, Humanities, Technology and Education. It is only a prerequisite that these students should have completed at least year 11 mathematics at secondary school. The course is effectively a ‘bridge’ to the ‘standard’ first year, university mathematics course that requires that students have
successfully completed at least year 12 Mathematics Methods, the middle level year 12 mathematics course available in the state of Victoria. There is typically a wide range in the mathematical abilities of students taking this course. It is not assumed that these students have previous experience of using technology for learning mathematics.

**The course**

The teaching in the course was in accordance with the course description (see figure 4.1) that had previously been accredited by the University’s curriculum committee and academic board. Any innovations in teaching were therefore limited, and within the topics and methodologies related to the syllabus and the stated course objectives. The CAS used was DERIVE version 2.55 (because the University had this license) but the program was already outdated. DERIVE has been much improved since version 2.55.

**The Place of CAS in the Teaching, Learning and Assessment**

**Policy on CAS-use in the course**

**Compulsory**

The use of CAS in this course was effectively compulsory. The course description stated that:

> Throughout the unit the concepts and techniques will be introduced and developed by the guided exploratory use of a computer algebra system. (p. 1)

The course objectives (iii), (vii), (x), (xiii) and (xvi) (see figure 4.1) specifically mentioned the use of CAS. A student could, theoretically refuse to use CAS but it was clear that its use was expected and that any student not using CAS could not expect to attain a high result for the course assessment.
Course objectives and content

The course description stated that it was designed to:

i. introduce foundation concepts in algebra, trigonometry and geometry
ii. outline the development and application of calculus
iii. develop skills in the use of computer algebra systems
iv. introduce techniques and concepts from differential calculus
v. introduce techniques and concepts from integral calculus
vi. develop skills in mathematical modelling.

vii. use a computer algebra system to perform tasks including graph plotting and exploration, solving equations and algebraic manipulation
viii. understand and apply the properties of trigonometric functions
ix. understand the idea of limits in an informal sense
x. investigate limits using a computer algebra system
xi. understand geometric ideas underpinning differential calculus
xii. apply the fundamental techniques from differential calculus in problem solving
xiii. use a computer algebra system to investigate derivatives and rates of change of functions
xiv. solve problems involving the application of differential calculus
xv. understand the basic concepts of integral calculus
xvi. use a computer algebra system to investigate problems involving integral calculus
xvii. understand the basic applications of numerical integration
xviii. apply the principles of mathematical modelling to study a simple system using a computer algebra system.

<table>
<thead>
<tr>
<th>Week</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction to Functions, Graphs of functions, Introduction to Derive.</td>
</tr>
<tr>
<td>2</td>
<td>Constructing new functions, linear functions</td>
</tr>
<tr>
<td>3</td>
<td>Polynomial, exponential and logarithmic functions</td>
</tr>
<tr>
<td>4</td>
<td>Slope of a curve and rates of change</td>
</tr>
<tr>
<td>5</td>
<td>Limits, continuity, derivatives. Mid-course examination</td>
</tr>
<tr>
<td>6</td>
<td>Graphs: characteristics and salient points</td>
</tr>
<tr>
<td>7</td>
<td>Product, quotient and chain rules</td>
</tr>
<tr>
<td>8</td>
<td>Antiderivatives and indefinite integrals</td>
</tr>
<tr>
<td>9</td>
<td>Definite integrals and areas under curves</td>
</tr>
<tr>
<td>10</td>
<td>Applications of calculus: optimisation</td>
</tr>
<tr>
<td>11</td>
<td>Trigonometry and Triangles</td>
</tr>
<tr>
<td>12</td>
<td>The Trigonometric Functions - Radian measure, definitions, graphs, equations, identities.</td>
</tr>
<tr>
<td>13</td>
<td>The Calculus of Trigonometric Functions And Revision</td>
</tr>
<tr>
<td>14</td>
<td>No lectures: free study for examinations</td>
</tr>
<tr>
<td>15</td>
<td>Post-course examinations</td>
</tr>
</tbody>
</table>

Figure 4.1 Extract from course description. MA502: Introduction to calculus and computer algebra systems.1997-
Functional and Pedagogical

Both the teacher and the students used CAS for functional and pedagogical purposes. The students were encouraged to use CAS to solve harder problems and as an investigative tool to explore patterns in mathematics.

Exploratory approach encouraged

Exploration was demonstrated and encouraged by the teacher. Weimer (1998) was recommended to the students as a textbook that they should purchase. The exercises, selected from this text for students’ attention, directed their learning through exploration of related examples in multiple representations. Students were expected to use CAS in order to complete their assessable tasks. This is discussed further in the sections on learning and assessment.

Teaching

The course was taught over 52 contact hours, 4 hours per week for 13 weeks, with an expectation that students would spend a further 100 hours over the 15 weeks of the course working on this subject in their own time. Of the 4 hours each week, 2 were held in a classroom where a single computer image could be projected so that all students could view the screen. For the other 2 hours, classes were conducted in a computer laboratory where each student had access to a computer. Maximum access to computers was considered desirable but actual number of hours and sequence of classroom or computer laboratory usage was determined by timetable constraints.

In the classroom sessions the teacher used DERIVE, with its output projected on a screen along-side the whiteboard. Initially the teacher used familiar mathematics to demonstrate how to use the program, commands were introduced and some of the capabilities of DERIVE shown.

During the pre-calculus work, in the first four weeks, DERIVE was used to focus students' attention on functions as both processes and objects (Tall, 1992) and to emphasise the value of looking at symbolic, graphical, and numerical representations as suggested in the earlier CAS studies referred to in Chapter 2. DERIVE was used to explore and illustrate mathematical properties of functions. Romberg (1993) reminds us
that the mind naturally organises repeated experiences into complex networks of concepts, rules, and strategies referred to as schema. For this reason, examples and counter examples were discussed by the teacher and the students and summaries of key features given by the teacher. Most examples were chosen by the teacher but students' suggestions and questions were also explored. During this period the teacher aimed to demonstrate good strategies for using DERIVE to explore mathematics.

In introducing calculus, DERIVE was used to examine the gradient of a curve at a point by graphical, numeric, and then symbolic means. The local 'straightness' of the curve was viewed by repeated 'zooming in' on the graph and the gradient calculated by reading off the coordinates of two points then calculating vertical change over horizontal change. Tables of values were generated to look at the difference quotient for very small increments of x. Lastly, the function was defined and the limit of the difference quotient calculated symbolically. Through this process the rule for finding the derivative of a polynomial function was established by the students. This illustrates how the CAS was used for illustration and discovery.

**Learning**

Students worked with DERIVE available in the computer laboratory sessions, and these facilities were available out of class hours. There were sufficient terminals for students to work alone but they were encouraged to share or work side by side and discuss their findings. This was encouraged because negotiation with peers, the interplay of ideas backwards and forwards, allows each student to refine their thinking. According to writers such as Vygotsky (1978) and Romberg (1993) this process of negotiation is important in encouraging students to assimilate new information into their current schema. This working arrangement in laboratory classes had previously been found to be successful as reported in Pierce (1999b). During these laboratory classes students mostly worked on their weekly worksheets, which consisted of a list of problems from the set textbook, Weimer, (1998), (see example Appendix 1.5) and supplementary exercises written by the teacher (see Appendix 1.4). The list from the week 8 worksheet, figure 4.2, is typical. It can be seen from this list that many more problems were recommended than were required to be submitted for assessment. To cater for the
students’ diverse mathematical backgrounds, the range of problems suggested included both basic foundation exercises and extension problems.

For example, amongst the problems set in week 8, basic skills numbers 1 - 45 required the student to find a range of anti-derivatives of increasing difficulty. Students could use DERIVE as they wished but were encouraged to do simple examples quickly by hand. Some questions asked the student to use the CAS to explore and find patterns by looking at a series of examples. This would be tedious by hand but required only one DERIVE command to produce the whole series of answers. Exercise 6.1, Question 54, for example, asked students to

\[
\text{Discover a formula for } \int x^n e^x \, dx, \quad \text{for } n = 0, 1, 2, \ldots \text{ by evaluating the integral for } n = 0, n = 1, n = 2, n = 3, \text{ and } n = 4. \text{ Examine the results and look for a pattern (Weimer, 1998, p 444).}
\]

Other questions required the students to solve an applied problem:

The growth rate of a population of a city is given by \( 675e^{1.28t} \) where \( t \) is the time in years. If the size of the present population is 89476 estimate the population of the city 11 years from now (Exercise 6.2 number 33, Weimer, 1998, p 451).

Students needed to understand the concept of rate-of-change and the role of anti-differentiation in order to use DERIVE in a meaningful way to solve this problem. (Eleven years is a sufficiently long time that using 11 times the rate-of-change for one year cannot approximate the solution.) For students who understood the mathematics, the CAS gave them confidence to produce a correct solution without worrying about slips in algebra or arithmetic.
This worksheet, including attempted solutions to those questions which are **bold and underlined**, is to be handed in by 5 pm Monday 4th October.

Note: DERIVE Algebra Window output should be copied as necessary and Derive Plots should be turned into sketch graphs. It is not necessary to print from DERIVE but its use should be acknowledged.

### Integration

**Antiderivatives**

Weimer: Exercises 6.1

Basic Skills: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 15, 17, 19, 25, 29, 30, 31, 33, 36, 37, 39, 42, 43, 45

Computer Applications: 53, 54, 57

Making Connections: 1, 2, 3

**Applications of indefinite integrals**

Weimer: Exercises 6.2

Basic Skills: 1, 3

Applications: 7, 9, 10, 11, 13, 15

Computer Applications: 29, 33, 35

---

*Figure 4.2* Typical weekly problem list and directions to students

Students’ common errors and misunderstandings, for example in algebraic fractions, were targeted with sheets like worksheet 8a (figure 4.3 below and Appendix 1.4). One of the aims of this sheet was to encourage students to look at the structure of expressions and notice that fractions are composed of a ratio of two functions.
Take a look at fractions and their sum

**Author**

VECTOR([a/7, 11/a, a/7+11/a, (a+11)/(7+a)],a,2,6)

**Simplify**

Does \( \frac{a}{7} + \frac{11}{a} = \frac{a + 11}{7 + a} \)?

**Author** \( a/7 + 11/a \), then use factorize, and then expand

Repeat this for \( (a+11)/(7+a) \)

Based on this pattern find \( \frac{a}{b} + \frac{c}{a} \)

Based on this pattern find \( \frac{a}{b} + \frac{c}{d} \)

Prove this result is true for all \( a, b, c, d \in \mathbb{R} \)

What happens if \( b = d \)?

Give the general answer and then list some specific examples.

<table>
<thead>
<tr>
<th>( \frac{a}{7} )</th>
<th>( \frac{11}{a} )</th>
<th>( \frac{a + 11}{7 + a} )</th>
</tr>
</thead>
</table>

**DERIVE results**

Check your answer with DERIVE

Check your answer with DERIVE

Check your answers with DERIVE

(Use Manage, Substitute, \( b=d \))

---

**f(x) and g(x)** are both linear functions

Find some examples of such functions so that \( \frac{f(x)}{g(x)} \):

(i) may be simplified by cancelling

(ii) may not be simplified by cancelling

Write your results in a general form illustrating which of these functions may be simplified by cancelling.

**Check your answers with DERIVE**

---

**Figure 4.3 Worksheet targeting on algebraic fractions**

The worksheets which students were required to submit for assessment and feedback included two exploratory projects (based on project questions from Weimer 1998). These exploratory projects were set in the context of real world application. They
required students to look at variations on functions, make conjectures, and describe patterns. Students were encouraged to make use of CAS in doing this work.

**Assessment**

CAS was available for all assessable tasks. Students’ learning in the course was assessed in three ways: worksheets, Mid-course examinations and Post-course examinations. The worksheets were marked and returned to students. These tasks have already been described in the preceding section.

Mid-course and Post-course examinations were held in computer laboratories. Students could make use of CAS as they wished but they were told that full marks would not be given for merely writing down the result that appeared on the screen.

For example, Question 1 from the Post-course examination is shown in figure 4.4. For this question, while students could check their answer with DERIVE, they were expected to state the steps necessary to solve these problems with pen and paper. (Recognition of appropriate by-hand rules was required because the group included pre-service teachers, who would be teaching in secondary schools, where CAS was not available.) In this example, the expression could be keyed into DERIVE and the derivative found using a simple sequence of commands, and/or CAS could also be used to check the algebra at each stage, if demonstration of a differentiation rule was required.

For question 2, shown in figure 4.4, students could use DERIVE to plot the function, copy a sketch graph by hand on paper and then use the integral calculus capability of the CAS to find the required area. The student was expected to explain that they had used integration and clarify how they had dealt with the sections of the curve above and below the x-axis.
1. In writing up your answers to the following questions state any rule used (product, quotient or chain rules) and show all steps involved in reaching the answer.

(a) If \( f(x) = \frac{e^{3x}}{x^2 + 1} \) find \( f'(x) \)

(b) If \( r(\theta) = 3\pi\theta^5 \cos(2\theta) \) find the gradient of this curve when \( \theta = \pi \)

(c) If the distance (metres) travelled by a radio controlled boat in \( t \) minutes may be described by the rule \( P(t) = 2 + 3t - t^2 + 7t^3 - t^4 \) at what speed was the boat travelling after 3 minutes?

2. Sketch the curve \( f(x) = 6x^3 - 5x \) then find the area enclosed by the curve and the \( x \) axis. Explain the reasons for each step.

Figure 4.4 Example of examination questions for which the teacher required more than copying from CAS screen.

One question (figure 4.5) required students to read and interpret the CAS output provided. To answer this question students needed to decide what the letter \( x \) represented and then what calculations had been carried out and why. They needed to explain how each step contributed to the solution of the problem. Finally the students needed to be able to interpret the notation and syntax of the CAS output in the context of the problem. This style of question required students to verbalise and explain the solution process not just say, ‘He used a VECTOR command, he used FIT, he used Simplify etc’. Rather they were expected to say something like:

The promoter considered his data, linking cost and rate of increase in ticket sales, as the coordinates of two points. He then found a rule for a linear function that would fit these two points. [Lines 1, 2 and 3] This gave him a model for estimating the rate of increase of sales given the cost of tickets, so he anti-differentiated this to find the number of tickets sold at each cost. [Lines 4 and 5] He graphed these functions and checked that his model matched what he had observed, i.e. that numbers increased to a certain cost but then decreased. [Shown in graph window.] He soLved to find out when the rate of increase was zero because this linked to the maximum ticket sales (the turning point before the curve started to decline). The promoter found that he could expect to sell the optimum number of tickets if he priced them at $35. [Lines 6 and 7] He then substituted this value into his rule for number of tickets sold and found that at this price he could expect to sell 6125 tickets.’ [Lines 8,9 and 10]
4. An entertainment promoter observed that if concerts were marketed too cheaply then few people attended. It seemed that the ticket price influenced perception about the quality of the concert. He noted however that the rate of increase in patronage tapered off as the tickets became more expensive. For example he noted that when tickets were $10 the rate of increase was 250 people per dollar but at $15 the rate of increase in ticket sales was only 200 people per dollar.

The following screen dumps show the outline of the mathematics done by the promoter in order to estimate an optimum price for tickets. Please explain, by referring to each line of the DERIVE printout in turn, how the promoter worked out this optimum price. How would you interpret the result in line 10?

Figure 4.5 Example of examination question requiring students to interpret CAS output
This study in the context of this course

This course has been monitored since its inception in 1997. Kneebone, Pierce, Roberts and Stacey are all mentioned as co-authors of the papers in which we have reported various findings. In 1997 the course was taught by Kaye Kneebone and since then by the present author, Robyn Pierce who instigated this research. Lyn Roberts, who has taught with technology in other courses, has acted, at times, as an observer for this work. Kaye Stacey, who works at a different institution, has been an advisor for this research. During 1997 data was collected through surveys and from the students’ work. The focus of research that year was the students’ response to the use of this technology for learning and the impact of its availability on assessment. This work was reported in Pierce and Roberts (1997) and Kneebone and Pierce (1997). During 1998 further data was collected and analysed with two foci. First the extent to which CAS provided a ‘scaffolding’, as suggested by Kutzler (1994), from which students could learn calculus was considered and reported in Pierce (1999). Secondly the learning strategies facilitated by the CAS learning environment were analysed and reported in Pierce and Stacey (1999) and further in Pierce and Stacey (2001b). After careful study of students using CAS in this course and previously in another course (Pierce and Stacey, 2001a), it became clear that it was important to pursue the two interrelated issues, of students’ Algebraic Insight and their Effective Use of CAS, in depth. The subsequent study, with data collected in 1999, forms the hub of this thesis. The decision to make the main data collection in 1999 was taken because this study was informed by the work done over the previous two years. In addition, other changes within the university meant that this ‘service’ course might not be required in 2000.

The frameworks outlined in Chapter 3, are ‘working frameworks’. This means that the details of what it means to show Algebraic Insight and what it means to make Effective Use of CAS will depend on the context. In this particular study the examples of Algebraic Insight specifically relate to an overview of functions, introductory calculus and trigonometry. Items assessing Algebraic Insight were developed with both the course content and students’ previous experience in mind. The course was not designed as a radical rethink of the teaching of mathematics but incorporated new approaches and emphases facilitated by the availability of CAS. As Artigue (2001) observes
Professional mathematicians and engineers well know that these sophisticated new tools don’t become immediately efficient mathematical instruments for the user; their complexity does not make it easy to fully benefit from their potential…they know that these new tools have progressively changed their mathematical practices. (p1)

The use of CAS was aimed at enhancing and improving learning in a fairly traditional course. The measures of Algebraic Insight and Effective Use of CAS must be viewed in that context. This chapter has highlighted the key aspects of the context of this study. The details of the 1999 study, the instruments developed, and the methodology employed, follow in Chapter five.
CHAPTER 5

METHODOLOGY

This study focused on three key research questions that emerged both from the literature and the previous experience of the researcher. These key questions related to Algebraic Insight and Effective Use of CAS as described in Chapter three. The aim was to add to what was already known about teaching, doing, and learning mathematics with CAS by investigating these questions, and in the process assess and refine the proposed working frameworks for Algebraic Insight and Effective Use of CAS.

The three key questions were broken down into focus questions for the purposes of collecting data and are detailed below. The methods used to collect data reflect the small size of the group and the close links facilitated by the researcher also being the teacher. This chapter outlines the features of the various instruments used for collecting data and gives examples of test and survey items. It concludes with a report on the data collection, including some problems encountered.

The Research Questions

The importance of Algebraic Insight and Effective Use of CAS in maximising the benefit of CAS for teaching, doing, and learning mathematics has been raised in earlier chapters, especially Chapter three. In order to trial the two frameworks and learn about students’ response and change during a course taught using CAS, three key questions are addressed in this study:

1. Does Algebraic Insight change during a course taught using CAS?
2. Does Effective Use of CAS change during a course taught using CAS?
3. What are the links between Effective Use of CAS and Algebraic Insight?

Evidence for answers to these three key questions was sought by focusing on components of the issues they address and thereby building up an overall picture of the results. In order to achieve this, each key question has been divided into a series of
associated focus questions. The purpose of each focus question and the likely sources of data are outlined below.

**Does Algebraic Insight change during a course taught using CAS?**

This is addressed from two perspectives: first, the overall change as well as the selective change in aspects or elements described in Chapter 3. Second, the characteristics of the students who improved are examined.

**Focus question 1.1: During a course taught using CAS, is there evidence of change in students' overall level of Algebraic Insight?**

Algebraic Insight was assessed at the beginning, middle, and end of the course to see if there had been a 'general' improvement during the course. The study looks both for change in individual student's results and in the class overall. Data related to change in Algebraic Insight was primarily collected using two instruments: an Algebraic Insight Quiz and an Algebraic Insight Interview. These instruments are described later in this chapter. Additional information was obtained by considering students' worksheets, examination scripts and the teacher/researcher's classroom observation notes. This broad-brush judgement could be likened to comparing a student's or class’s overall results in a subject from year to year. It is common knowledge that improvement in knowledge or understanding is seldom consistent across the curriculum so for that reason the second question looks at more detail.

**Focus question 1.2: During a course taught using CAS, is there evidence of differential improvement in some elements of students' Algebraic Insight?**

To answer this question, information was sought about the aspects and elements described in the Algebraic Insight Framework summarised in table 3A.1. The purpose of this breakdown of results was to find out whether students improved more in some areas of Algebraic Insight than in others. The data for this analysis was obtained from the same sources as for focus question 1.1.
Focus question 1.3: What are the characteristics of those students whose Algebraic Insight improved?

Many variables, such as students’ previous mathematical background, attitude towards mathematics, attitude towards the use of technology, skills in using technology, social factors and the quality of instruction, may account for improvements in Algebraic Insight. This study will concentrate, primarily, on students’ initial Algebraic Insight and the Effective Use of CAS as explanatory variables because these are the central concern of this thesis. The data for this investigation comes from the Background Survey, Algebraic Insight Quiz, Algebraic Insight Interview, Post-course Interviews, Post-course Evaluation and the researcher's classroom observation notes.

Does Effective Use of CAS change during a course taught using CAS?

This, too, is addressed from two perspectives: first, overall change and selective change; then the characteristics of the students who improved are examined.

Focus question 2.1: Is there evidence that students' overall Effective Use of CAS improved?

The study examines the effectiveness of students' use of CAS. This first question seeks to gain an overview of both students individually and the class as a whole. Taking into account the descriptors of Effective Use of CAS outlined in Chapter 3 part B, data was collected using primarily the Technical Difficulties and the Judicious Use of CAS Surveys with extra data from the weekly worksheets, examination scripts, and observation.

Focus question 2.2: During a course taught using CAS, is there evidence of differential improvement in some elements of students' Effective Use of CAS?

To answer this question information was sought about the various aspects and elements of the Effective Use of CAS framework summarised in table 3B.1. The purpose of this breakdown of results was to find out whether students improved more in some areas of Effective Use of CAS than others. The data for this analysis was obtained from the same sources as focus question 2.1.
**Focus question 2.3: What are the characteristics of those students whose Effective Use of CAS improved?**

This section focuses on finding the characteristics of those students who have and have not improved in some element of Effective Use of CAS. The data for this investigation comes from the Background survey, the Technical Difficulties survey, the Judicious Use of CAS survey, the Post-course interviews, the Post-course evaluation and the researcher's classroom observation notes.

**What are the links between Effective Use of CAS and Algebraic Insight?**

This examines links between improvements in the aspects of Algebraic Insight (Algebraic Expectation and Linking Representations) and the aspects of Effective Use of CAS (Technical and Personal). Evidence of relationships between these variables was sought by linking the data from the Algebraic Insight instruments with the Background survey, the Technical Difficulties survey, the Judicious Use of CAS survey, the Post-course evaluation and the Post-course interview.

**Instruments**

The researcher, for this study developed ten instruments. A list of these instruments along with a summary of the data collection schedule is included later as table 5.17 and a summary of measures and descriptors may be found in Appendix 4. Two of the instruments focused on measuring Algebraic Insight, three on students' Effective Use of CAS and the other five were intended to examine the links between students’ Algebraic Insight and their Effective Use of CAS. In designing these instruments the researcher aimed to minimise extra demands on students and interference with the set course schedule, yet still obtaining the best possible data.

**Measuring Algebraic Insight**

Algebraic Insight was measured using an Algebraic Insight Quiz and an Algebraic Insight Interview. The quiz was administered to the class as a group but interviews were conducted either singly or in pairs and were audio-recorded for later transcription.
**Algebraic Insight Quiz**

The first instrument used for measuring Algebraic Insight was presented as a 'quick quiz'. This Quiz was to be given three times during the course and so, to minimise loss to teaching time, it was important that it take no longer than 10 minutes to administer. The quiz was therefore restricted to 23 ‘flash-card’ style items. In order to control the students' access time for each item, the quiz was presented as a Microsoft Powerpoint slide-show. Time limits of 5 seconds for comparison items, and 15 seconds for multiple choice items, were established by trialing the quiz with a number of colleagues. These accord with the findings of Geake (1994) for items requiring quick recognition.

The aim of this quiz was to make an assessment of the elements, as defined by the framework, of students' Algebraic Insight. In particular this quiz aimed to assess ability to Link Representations using multiple choice items and Algebraic Expectation in the context of recognising equivalent expressions. Recognition of equivalent expressions is important when doing mathematics with CAS because students need to be able quickly to check for entry syntax errors and therefore need quickly to compare written mathematics with that on the screen. As well, quite often the results produced by CAS are not arranged or simplified in ways familiar from conventional by-hand methods and here again students may need to recognise equivalent expressions. Lagrange (1999) wondered if this need to recognise equivalent expressions meant CAS was of limited use to some students. The Algebraic Insight quiz was not trying to test students' ability to use algorithms and solve problems, but rather to assess their first intuitive responses to the given items. The situation is rather like the ongoing monitoring that a CAS user needs to undertake. This was the rationale for choosing the strict time limit for the Quiz items. However, to make the context familiar to the students, the exercise was likened to ‘checking answers against those in the back of a text book.’ Students very frequently have to identify equivalent expressions when they compare their own answer to textbook answers. A summary of the relationship between the quiz items and the Algebraic Insight framework is shown in table 5.1; each item is shown and discussed in figures 5.1 and 5.2.
Items 1 to 11 are shown in table 5.3 along with the rationale for each item. These first eleven slides presented students with pairs of expressions. The students were asked to compare these expressions as if they were checking answers. They were to think of the expression on the left of each slide in table 5.3 as a student's answer to some problem, and the expression on the right as the text book answer. The task was designed, in this way, to resonate with a common experience that students regularly face. Each slide was shown for approximately five seconds. In this time they were asked to decide the status of that fictitious student's answer using the scale shown in table 5.2. Restricting the time exposure to only 5 seconds meant that students did not have time to carry out calculations but had to make an intuitive response. The 'I have no idea' option was designed to discourage students from merely guessing, as well as representing a legitimate response. Students’ responses were scored from –2 to +2. For example, if the pair of expressions was indeed equivalent, the student would score: -2 for selecting Definitely Wrong, -1 for Probably Wrong, 0 for I have no idea, +1 for Probably Right or +2 for selecting Definitely Right. It was considered that a student who knew that they had ‘no idea’ showed more Algebraic Insight than a student who was confidently wrong.
### Table 5.3.
**Questions from first section of Algebraic Insight Quiz and links to the Algebraic Insight Framework**

<table>
<thead>
<tr>
<th>Algebraic Insight Quiz Item</th>
<th>Rationale for item</th>
<th>Link to Algebraic Insight Framework</th>
</tr>
</thead>
</table>
| **In this series of questions you will see a student’s answers and the textbook answers. For each question ask yourself ‘How likely do you think it is that the student’s answer is right?’** | **Example:** Student: \((x+y)^2\)  
Text: \(x^2+y^2\) | **Example** |
| | **Definitely wrong** | **Probably wrong** | **No idea** | **Probably right** | **Definitely right** | |
| **Example** | | | | | |
| **1. Student Textbook** | **Recognise the form of the expression as quadratic and/or identify the key feature that \(x\) is squared and hence expect two solutions.** | **1.3 Identify Key Features** |
| \(x^2 = 9\)  
\(x = \pm 3\) | | |
| **2. Student Textbook** | **This item was included because CAS allow the options of approximate (decimal) or exact (fraction) presentation of results. Recognition of equivalence in this context is important but involves ‘number insight’ rather than Algebraic Insight. This item was deleted from subsequent analysis.** | |
| \(t = 2.375\) | | |
| \(t = 2\frac{3}{8}\) | | |


<table>
<thead>
<tr>
<th>Algebraic Insight Quiz Item</th>
<th>Rationale for item</th>
<th>Link to Algebraic Insight Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Student Textbook</td>
<td>a + t ≠ a</td>
<td>1.1 Recognise conventions and basic properties of operations</td>
</tr>
<tr>
<td>4. Student Textbook</td>
<td>x ÷ y ≠ y ÷ x</td>
<td>1.1 Recognise conventions and basic properties of operations</td>
</tr>
<tr>
<td>5. Student Textbook</td>
<td>(\sqrt{xy} \neq \sqrt{x+y})</td>
<td>1.1 Recognise conventions and basic properties of operations</td>
</tr>
<tr>
<td>6. Student Textbook</td>
<td>(a + p \neq \frac{a + p}{q})</td>
<td>1.1 Recognise conventions and basic properties of operations</td>
</tr>
<tr>
<td>Algebraic Insight Quiz Item</td>
<td>Rationale for item</td>
<td>Link to Algebraic Insight Framework</td>
</tr>
<tr>
<td>-----------------------------</td>
<td>--------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td><strong>7. Student Textbook</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| \[
\begin{align*}
\frac{a-b}{d-c} &= \frac{a-b}{d-c} \\
\end{align*}
\] | Students who identify the processes \(a-b\) and \(d-c\) as objects will be unlikely to expect these expression to be equivalent. (Tall (1992) wrote: Processes are carried out and represented by symbols which subsequently take on a dual role...5+3 [for example] represents both the process of addition and the concepts of sum...By using the symbolism to evoke a process, it can be used to compute a result, and thinking of it as an object, it can be used as part of higher level manipulation. p.58) | 1.1 Identify Structure |
| **8. Student Textbook**     |                    |                                  |
| \[
\begin{align*}
2+x &= x \\
y+2 &= y
\end{align*}
\] | Recognise \(2+x\) and \(y+2\) as objects and recognise the properties of the operation addition | 1.2 Identify Structure |
| **9. Student Textbook**     |                    |                                  |
| \[
\begin{align*}
\frac{s+p}{t+t} &= \frac{s+p}{t} \\
\end{align*}
\] | Recognise \(\frac{s}{t}\) and \(\frac{p}{t}\) as objects and recognise the properties of the operation of division. | 1.2 Identify Structure |
| **10. Student Textbook**    |                    |                                  |
| \[
\begin{align*}
\frac{x^5 - y^5}{(x^3 + y^3)(x - y)(x + y)} \\
\end{align*}
\] | If students identify the key feature of the highest powers in these expressions they will note that the highest power of \(x\) or \(y\) in the right hand expression is 4 and hence the expressions are not equivalent. | 1.3 Identify Key Features |
These questions tested students' ability to recognise conventions and identify key features of expressions. For example: in question 5, table 5.3 above, the student needed both to notice the '+' symbol and to recognise that this symbol made a significant difference to the expression. (Stacey and MacGregor, 1994, reported that failing to recognise this distinction, making a co-joining error, has been a common problem in conventional classrooms.) Students using CAS make typing errors as well as mathematical errors and must cope with syntax that often, but not always, follows the convention of implicit multiplication. It is therefore important that students recognise this algebraic convention immediately.

In question 10, table 5.3 above, it was hoped that students would quickly check the powers in the expressions and identify that the highest power of $x$ in the 'Textbook' answer would be $x^4$. The quick identification of this key feature allows the student to decide that the 'student's answer’ was definitely incorrect without doing any further expansion of the brackets. CAS commonly present expressions in simplified form, or arranged in a way that is different from the by-hand school convention. Students therefore need to be able to use such Algebraic Insight to check whether the answer is at least of the order they expected. Experts use a range of these simple checks as discussed in Chapter 3.

Items 12 to 23 are shown in table 5.4 where is can be seen that, with the exceptions of numbers 12, 13, and 16, these items assessed students' ability to link between
representations of functions. In this section they were asked to choose the best answer from a set of alternatives. For example question 17 in table 5.4, requires students to make the link between a patterns in the table, a numeric representation of a function, and its symbolic form (ie. quadratic etc), while question 21 requires a link between a graphical shape and key features, and symbolic form.

Fifteen seconds was allowed for each of these items because, again, the aim was to assess whether the student could make an educated guess based on identifying structure or key features. In question 17, table 5.4 the student might scan the relationship between $x$ and $f(x)$ and notice that 2 maps to 4, 3 to 9, $x$ to $x^2$. Alternatively the student might recognise that the $f(x)$ values are all squares. In question 21, table 5.4 the student would be expected either to identify that the graph has 3 intercepts and so will be at least cubic or, similarly, to identify that it has two turning points and so will be at least cubic. This makes alternative (d) the only possible option.

Table 5.4.
Questions from second section of Algebraic Insight Quiz and Links to Algebraic Insight Framework

<table>
<thead>
<tr>
<th>Algebraic Insight Quiz Item</th>
<th>Rationale for item</th>
<th>Link to Algebraic Insight framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>2t = 10</td>
<td>This example was used to discuss the change in answer format.</td>
<td>Example</td>
</tr>
<tr>
<td>$t=$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the following multiple-choice questions circle the letter which applies.

Example:

- (a) 2
- (b) -5
- (c) 5
- (d) -1
- (e) -2
<table>
<thead>
<tr>
<th>Algebraic Insight Quiz Item</th>
<th>Rationale for item</th>
<th>Link to Algebraic Insight Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>12. For the function below when: ( t ) approaches ( \infty ), ( h(t) ) approaches \ldots \ldots</td>
<td>Students may recognise the ( t^2 ) as the dominant term.</td>
<td>1.4 Identify Key Features</td>
</tr>
<tr>
<td>[ h(t) = \frac{5 + 2t^2}{t} ]</td>
<td>Identifying ((t+1)) as an object which replaces ( x ) making the recognition of (b) simple.</td>
<td>1.2 Identify Key Features</td>
</tr>
<tr>
<td>13. Given ( y = 6 - x - x^2 ) If ( x = t + 1 ) then ( y = )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a) ( 6 - t + 1 - t^2 + 1 )</td>
<td>(a) 0</td>
<td>2.1 Symbol &lt;-&gt; Graphical</td>
</tr>
<tr>
<td>(b) ( 6 - (t + 1) - (t + 1)^2 )</td>
<td>(b) 1</td>
<td></td>
</tr>
<tr>
<td>(c) ( 6 - t - 1 - t^2 )</td>
<td>(c) 2</td>
<td></td>
</tr>
<tr>
<td>(d) ( 6 - t - t^2 + 1 )</td>
<td>(d) 3</td>
<td></td>
</tr>
<tr>
<td>(e) ( 6 - t + 1 )</td>
<td>(e) 4</td>
<td></td>
</tr>
<tr>
<td>(f) ( 7 - t - t^2 )</td>
<td>(f) 5</td>
<td></td>
</tr>
<tr>
<td>14. ( h(x) = px^5 + qx^4 + rx^3 + sx^2 + t ) If ( p, q, r, s, ) and ( t ) were real numbers, what is the minimum number of times the graph of this function might cut or touch the ( x )-axis?</td>
<td>The student needs to note the degree of the polynomial and linking this to a function that must cross the ( x )-axis at least once.</td>
<td></td>
</tr>
</tbody>
</table>
### Algebraic Insight Quiz Item

#### 16.

If \( f(x) = (x-1)(x-1)(x-1)(x+5) \)

how many distinct solutions will there be for \( x \) when \( f(x) = 0 \).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0</td>
</tr>
<tr>
<td>(b)</td>
<td>1</td>
</tr>
<tr>
<td>(c)</td>
<td>2</td>
</tr>
<tr>
<td>(d)</td>
<td>3</td>
</tr>
<tr>
<td>(e)</td>
<td>4</td>
</tr>
<tr>
<td>(f)</td>
<td>5</td>
</tr>
</tbody>
</table>

**Rationale for item**

Here the student needs to recognise that they are dealing with two objects \( x-1 \) and \( x+5 \) and that if either equalled zero then \( f(x) \) would equal zero.

**Link to Algebraic Insight Framework**

1.5 Identify Key Features

#### 17.

For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) )</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

- (a) linear
- (b) quadratic
- (c) exponential
- (d) none of the above

**Rationale for item**

Link number patterns in table to form: either identifying the that \( f(x) = x^2 \) or noting that the second differences between \( f(x) \) terms are equal.

**Link to Algebraic Insight Framework**

2.2 Symbolic<->Numeric

#### 18.

For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( f(x) )</td>
</tr>
<tr>
<td>-2</td>
<td>0.04</td>
</tr>
<tr>
<td>-1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
</tr>
<tr>
<td>3</td>
<td>12.5</td>
</tr>
</tbody>
</table>

- (a) linear
- (b) quadratic
- (c) exponential
- (d) none of the above

**Rationale for item**

Link number patterns in table to form, noting the rapid growth in \( f(x) \).

**Link to Algebraic Insight Framework**

2.2 Symbolic<->Numeric
<table>
<thead>
<tr>
<th>Algebraic Insight Quiz Item</th>
<th>Rationale for item</th>
<th>Link to Algebraic Insight Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>20. For the table below the rule for the function which links x to f(x) could be:</td>
<td>Link number patterns in table to form, noting the direct relationship between x and f(x), or the equal intervals between f(x) terms.</td>
<td>2.2 Symbolic &lt;-&gt; Numeric</td>
</tr>
<tr>
<td><strong>x</strong></td>
<td><strong>f(x)</strong></td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>-5</td>
<td>(a) linear</td>
</tr>
<tr>
<td>-1</td>
<td>-4</td>
<td>(b) quadratic</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
<td>(c) exponential</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>(d) none of the above</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>21. The graph of one of the functions is shown. Which function must it be?</td>
<td>Link graphical shape to algebraic form or identifying key features noting for example one local maximum and one local minimum, three x intercepts.</td>
<td>2.1 Symbolic &lt;-&gt; Graphical</td>
</tr>
<tr>
<td>(a) ( f(x) = 4x - 12 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) ( f(x) = 2x^2 + 3x - 2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) ( f(x) = e^{x} - 12 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d) ( f(x) = x^3 + 3x^2 - 4x - 12 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>22. The graph of one of the functions is shown. Which function must it be?</td>
<td>Link graphical shape to algebraic form, noticing one maximum, two x intercepts</td>
<td>2.1 Symbolic &lt;-&gt; Graphical</td>
</tr>
<tr>
<td>(a) ( f(x) = 6 - x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b) ( f(x) = 6 - x^2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) ( f(x) = 6 - 2e^x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d) ( f(x) = x^3 + 6 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
23. The graph of one of the functions is shown. Which function must it be?

<table>
<thead>
<tr>
<th>Algebraic Insight Quiz Item</th>
<th>Rationale for item</th>
<th>Link to Algebraic Insight Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) $f(x) = 3x + 4$</td>
<td>Link graphical shape to algebraic form, noticing no local maxima or minima and one $x$ intercept.</td>
<td>2.1 Symbolic &lt;-&gt; Graphical</td>
</tr>
<tr>
<td>(b) $f(x) = 4 - x^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c) $f(x) = e^{x-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(d) $f(x) = 2x^4 - 3x + 4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Algebraic Insight Interview**

The second instrument, designed to measure further aspects of Algebraic Insight, was a set of 5 problems to be solved in an interview situation with one or two students and the researcher. The Interview items did not target each element of the Algebraic Insight framework. It was felt that identification of structure and key features were tested quite effectively by the Algebraic Insight Quiz, so the Interview focused on Recognition of conventions and basic properties, especially the meaning of symbols and the ability to Link Representations.

Attendance at these interviews was voluntary and so depended on the good will and other time commitments of each student. At the interview students were given the problems on a typed sheet and asked to 'think out aloud' and talk the researcher through their thinking as they worked each problem. They were made aware that the interview was being taped for later transcription and analysis. Since the students had not yet used CAS it was not made available for the Pre-course interview but could be used, if they wanted to, in the Post-course interview.
### Table 5.5

*Algebraic Insight Interview questions and Links to Algebraic Insight Framework*

<table>
<thead>
<tr>
<th>Algebraic Insight Interview Item</th>
<th>Rationale for item</th>
<th>Algebraic Insight Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Think of a number [\text{Multiply two less than the number by two more than the number, then add 8. Take away the square of the number you first thought of. What did you get? Do you always get that? Why?}]</td>
<td>Idea that there would be a pattern. Choosing to use algebraic symbols. Successfully using algebraic symbols. Identifying inverse operations.</td>
<td>1.3.1 1.1.1 1.1.2 1.1.3</td>
</tr>
<tr>
<td>2. Find 4 consecutive, positive, odd, integers whose sum is 40.</td>
<td>Using algebra correctly when this was chosen. Discarding algebra as the most efficient strategy.</td>
<td>1.1.1</td>
</tr>
<tr>
<td>3. Find the area of the rectangle above when [d = \frac{x}{4}] and [d = y]. The lengths (d, x,) and (y) are measured in metres.</td>
<td>Able to express area in symbols. Able to express area using only one unknown. Elegant solution – good choice of variable. Content that answer may be a variable.</td>
<td>1.1.1 1.1.1</td>
</tr>
<tr>
<td>4. If (T(x, x^2, 0, 1)) generates the table</td>
<td>Identify Structure in rule. Link symbols and table (graph). Apply rule in new context (more Linking Representations). using numbers. using symbols. Identifying that (f(y)) may be same as (f(x)), choice of symbol.</td>
<td>1.2 2.2.1 1.1.1 1.2.1 1.1.1</td>
</tr>
</tbody>
</table>
5. $g(x) = x^2 + 4x - 21$

What parts of this rule immediately give you clues to the shape and position of a graph of this function? Explain.

Rewrite this rule so that the x intercepts of its graph can be easily seen.

Rewrite this rule so that the coordinates of the minimum point of its graph can be can be easily seen.

<table>
<thead>
<tr>
<th>Algebraic Insight Interview Item</th>
<th>Rationale for item</th>
<th>Algebraic Insight Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. $g(x) = x^2 + 4x - 21$</td>
<td>Link rule to shape of graph</td>
<td>2.1.1</td>
</tr>
<tr>
<td></td>
<td>Link rule to position of graph</td>
<td>2.1.2</td>
</tr>
<tr>
<td></td>
<td>y-intercept will cross x-axis</td>
<td>2.1.2</td>
</tr>
<tr>
<td></td>
<td>will not be symmetrical about y-axis</td>
<td>2.1.3</td>
</tr>
<tr>
<td></td>
<td>will have + and – intercepts</td>
<td>2.1.3</td>
</tr>
<tr>
<td></td>
<td>Identifying x-intercepts</td>
<td>2.1.3</td>
</tr>
<tr>
<td></td>
<td>Identifying maximum or minimum</td>
<td>2.1.3</td>
</tr>
</tbody>
</table>

**Determining level of Algebraic Insight**

In this study, it will be convenient to compare students on the basis of their Algebraic Insight. Students will be judged to have a poor, good or very good level of Algebraic Insight, as an amalgam of performance on the two aspects (and therefore the five constituent elements) which make-up Algebraic Insight. As will be discussed further in Chapter 6, it is generally the case that, while students may not demonstrate equal ability in all five elements their performance on the two aspects and five elements were generally linked. Thus allocation of an amalgamated level, on a coarse scale (poor to very good) seems justifiable as a general description of students’ knowledge.

A student’s level of Algebraic Insight will be a relative measure. Assessment of Algebraic Insight will be based on examples of the Common Instances of its ‘elements’. For this reason, the measure will always be relative to the Common Instances pertaining to the appropriate stage of learning. The choice of examples of Common Instances used to test Algebraic Insight in this study were representative of the level of Algebraic Insight students would require to be successful in this course which assumed students had previously studied year 11 mathematics.

The coarse scale of ‘poor’, ‘good’ and ‘very good’ is used in an imprecise way based on both formal (the Quiz and Interview) and informal measures (the teacher/researcher’s
experienced judgement). A student could be said to show ‘good Algebraic Insight’ in much the same way as we might describe a student as being ‘good at mathematics’ when we mean that they exhibit competence in most topics relative to their age and stage of learning. While, in this study, evidence of level of Algebraic Insight was collected from several sources the Algebraic Insight Quiz scores give a quick guide. Negative scores are clearly indicative of ‘poor’ Algebraic Insight while scores above and ‘well above’ the class average indicate ‘good’ and ‘very good’ Algebraic Insight in this class. These descriptors are used in the case studies (Chapter 7) where they are supplemented by details of constituent measures.

**Examining Students' Effective Use of CAS**

Four instruments (three surveys and observation) were used to assess students’ Effective Use of CAS. As outlined in Chapter 3, this requires assessment: of the level of technical difficulty experienced; of students’ attitude towards the use of this technology; and of the nature of students’ functional and pedagogical use of CAS. The first two instruments used in assessing these characteristics were combined on one page and administered as the *CAS Use Survey*. This instrument had two sections, a *Technical Difficulties Survey* and a *Judicious Use of CAS Survey*. The third survey, *Post-course Evaluation* was attached to the Post-course CAS Use Survey. Observations of CAS use were recorded during class sessions and during examinations.

**Technical Difficulties Survey**

This survey aimed to assess the level of difficulty that each student was experiencing in the technical aspects of using the CAS. Likely difficulties, identified from previous experience of teaching using DERIVE, were noted by the researcher and were classified in accordance with the elements of the Technical Difficulties aspect of Effective Use of CAS (as set out in table 3B.1). That is:

- **Fluent use of program syntax.** Dealing with authoring syntax and commands sequences
Ability to systematically change representation. Moving systematically between representations, for example, setting the dimensions or scale for an appropriate plot window or setting a starting point and increment for a table.

Ability to interpret CAS output. Interpreting the screen output from CAS as conventional pen and paper mathematics.

In order to assess the level of difficulty students experienced in each of these elements a list of 12 examples of CAS use was constructed. This list, included as table 5.7, was based on the researchers' observation of other students learning to use DERIVE in previous years, but was restricted by the range of examples which were relevant to the curriculum. There is no explicit reference to the use of tables since this representation had only been used occasionally. However, some students would indicate technical difficulties in using the syntax for the Vector command that was used to produce tables of values. Items 1 to 4 assessed program syntax (1.1), items 8 to 10 assessed changing representations (1.2), while items 5 to 7, 11 and 12 assessed interpreting CAS output (1.3).

Students were asked to indicate, on the six-point scale shown in table 5.6, the number of times they had difficulty with that item of CAS use during that day's laboratory class. This was followed by an open response item to allow for student comment. Reflection was restricted to that class because it is well known (see for example: Polit, and Hungler, 1983 or Foddy, 1993) that accuracy of reflective recall data decreases with increase in time-frame considered.

For some analysis purposes the response categories were collapsed (see table 5.6) so that none or one were combined in low level of difficulty, some as mid-level and a lot, and every time as high level. Those students who made little use of CAS indicated ‘Not Applicable’ for many options. These responses were treated as missing values, not zero difficulties.

Table 5.6
Response alternatives and collapsed categories for Technical Difficulties survey

<table>
<thead>
<tr>
<th>Not Applicable</th>
<th>None</th>
<th>One</th>
<th>Some</th>
<th>A lot</th>
<th>Every Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>NA</td>
<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
</tr>
<tr>
<td>NA</td>
<td>Low</td>
<td>Mid</td>
<td>High</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5.7
*Statements from Technical Difficulties Survey with links to Effective Use of CAS Framework*

<table>
<thead>
<tr>
<th>Possible area of difficulty</th>
<th>Link to Effective Use of CAS Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Authoring, using /, ^, *, +, - symbols</td>
<td>1.1</td>
</tr>
<tr>
<td>2. Using brackets to force the structure expressions</td>
<td>1.1</td>
</tr>
<tr>
<td>3. Using syntax for commands eg FIT, VECTOR, F(x)</td>
<td>1.1 1.2</td>
</tr>
<tr>
<td>4. Using sequences of commands eg to substitute</td>
<td>1.1</td>
</tr>
<tr>
<td>5. Interpreting the results of the solve command</td>
<td>1.3</td>
</tr>
<tr>
<td>6. Obtaining exact or approximate solutions</td>
<td>1.3</td>
</tr>
<tr>
<td>7. Working from the screen to ordinary maths symbols</td>
<td>1.3</td>
</tr>
<tr>
<td>8. Moving to a graph window</td>
<td>1.2</td>
</tr>
<tr>
<td>9. Obtaining a window which shows the graph required</td>
<td>1.2</td>
</tr>
<tr>
<td>10. Zooming to show key features of a graph</td>
<td>1.2</td>
</tr>
<tr>
<td>11. Copying from the screen for sketch graphs</td>
<td>1.3</td>
</tr>
<tr>
<td>12. Working out the scale of a graph</td>
<td>1.3</td>
</tr>
</tbody>
</table>

**Judicious Use of CAS survey**

The Judicious Use of CAS Survey (table 5.8) aimed to help assess the second element of the Personal aspect of Effective Use of CAS by establishing the purposes of the students' use. The results of this survey provided information on the three Common Instances associated with Judicious Use of CAS. First it helped to distinguish between those students who only used CAS by following suggestions and those students who took initiative to use CAS for themselves. (This information was used to help validate observations made about students’ manner of use. (See table 3B.1 Common Instance 2.2.1 and table 5.11.) Second, it provided information on how discriminating students were in their functional use of CAS (2.2.2), that is did they use it for questions that they could easily have done by hand, or did they only use it for harder or time consuming problems? Then, thirdly, it also provided information on whether their use was merely functional (just to work out answers) or also pedagogical (2.2.3) (using CAS to explore and so add to their understanding of mathematics).
Students were presented with a series of statements which might have described their use of CAS during that class and asked to indicate those which applied to them. The researcher, based on previous teaching with CAS, constructed the items for this list. Students were given the survey at the end of a laboratory class and, similarly to the Technical difficulty items, asked to reflect on their CAS use during that class.

Students’ Judicious Use of CAS (2.2.2 and 2.2.3) was scored on a scale of 1 to 6 as indicated in table 5.9. Students who used CAS to find answers to questions which they could easily have done by hand were classified as ‘non-discriminating functional’ users. Those who restricted its use to difficult and time consuming problems, were classified as ‘discriminating functional’ users. Students who did not use CAS to explore problems, except when following line by line instructions, were classified as ‘minimum-pedagogical’ users. Those who undertook limited exploration, when exploration was suggested, were classed as ‘limited-pedagogical’ users while those who initiated extended exploration were classed as ‘extended-pedagogical’ users.

Table 5.8  
*Statements from Judicious Use of CAS Survey with links to the Effective Use of CAS framework.*

<table>
<thead>
<tr>
<th>Judicious Use of CAS survey statements</th>
<th>Link to Effective Use of CAS Framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>Please tick as many of the statements below as apply.</td>
<td></td>
</tr>
<tr>
<td>Today I have used DERIVE to:</td>
<td></td>
</tr>
<tr>
<td>Find answers if the computer was suggested.</td>
<td>2.2.2</td>
</tr>
<tr>
<td>Explore problems if the computer was suggested</td>
<td>2.2.3</td>
</tr>
<tr>
<td>Explore variations on the set problems.</td>
<td>2.2.3</td>
</tr>
<tr>
<td>Explore, other than when directed but on the same topic</td>
<td>2.2.3</td>
</tr>
<tr>
<td>Explore other mathematics not on today’s topic</td>
<td>2.2.3</td>
</tr>
<tr>
<td>Find answers I could ‘easily’ have done with pen and paper.</td>
<td>2.2.2</td>
</tr>
<tr>
<td>Find answers to ‘hard’ questions</td>
<td>2.2.2</td>
</tr>
<tr>
<td>Find answers to time consuming questions</td>
<td>2.2.2</td>
</tr>
</tbody>
</table>
Table 5.9  
Scores based on Judicious Use of CAS Common Instances 2.2.2 and 2.2.3

<table>
<thead>
<tr>
<th></th>
<th>Minimum Pedagogical</th>
<th>Limited Pedagogical</th>
<th>Extended Pedagogical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-discriminating</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Functional</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discriminating</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Functional</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Post Course Evaluation

The third instrument used to gain information about students’ Effective Use of CAS was a Post-course evaluation. Students were presented with a number of statements to which they were asked to respond using a 5-point Likert scale from ‘Strongly Disagree’ to ‘Strongly Agree’. Three of the items in particular, shown in table 5.10, were used to help measure the first element of the Personal aspect of Effective Use of CAS, Positive attitude. A negative response to ‘I only use DERIVE if the instructions tell me to’ and positive responses to ‘I try out ideas using Derive’ and ‘I find using Derive helps me to understand Mathematics’ were taken as evidence of 2.1, ‘Positive attitude’.

The other items on this survey were used to provide information for the detailed CAS studies.

Table 5.10  
Statements from Post-course evaluation relating to 2.1 Positive attitude.

<table>
<thead>
<tr>
<th>Statements from Post course Evaluation</th>
<th>Link to Effective Use of CAS framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>I only use DERIVE if the instructions tell me to</td>
<td>Attitude</td>
</tr>
<tr>
<td>I try out ideas using Derive</td>
<td>Attitude</td>
</tr>
<tr>
<td>I find using Derive helps me to understand Mathematics</td>
<td>Attitude</td>
</tr>
</tbody>
</table>

Observation

The fourth method of collecting data about students' use of CAS was observation. The teacher/researcher made observations notes during laboratory classes. These notes recorded the way students were using the CAS and the nature of the questions they
asked. The key focus was noted, for example, when their question related to the mathematics being studied or to the technical problems of using CAS. The aim of this observation was to determine: if the student had technical difficulties in using CAS; the ‘manner’ in which they approached its use; if they demonstrated discriminating functional use; and the extent to which they used CAS for pedagogical purposes. Discussions between the teacher/researcher and students during class sessions also provided information about their attitude to the use of this technology.

In addition to these classroom observations, during the Mid-course and Post-course examinations the teacher/researcher and another experienced colleague made observation notes on the students’ use of CAS. In the laboratory where the examinations were held it was possible to view most screens from the back of the room, or at least from positions which did not disturb the students. Notes were made chronologically during the examinations and later transcribed to extract the details for each student and link this to their examination scripts.

The observation data were used to confirm the classification of students’ functional and pedagogical use and to establish their profile of ‘manner of use’. Students’ manner of use, in particular the degree to which they used CAS in a strategic manner, was determined from their comments and the researcher’s observations. Consideration of the students’ survey responses validated these observations. ‘Manner of use’ was scored 0 to 5 using the criteria in table 5.11. Since a student could operate using different manners during a class session they were given a total, out of 10, based on the sum of their highest and lowest scores.
Table 5.11
Score and criteria for manner of use

<table>
<thead>
<tr>
<th>Score</th>
<th>Manner</th>
<th>Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Passive</td>
<td>User is content for the tools to be operated by another, but takes no personal initiative.</td>
</tr>
<tr>
<td>2</td>
<td>Random</td>
<td>Use is not goal-directed and bears no relation to the context.</td>
</tr>
<tr>
<td>3</td>
<td>Responsive</td>
<td>User makes superficial and automatic use of appropriate tools</td>
</tr>
<tr>
<td>4</td>
<td>Directed Strategic</td>
<td>Use of the tools is deliberate, goal-directed and insightful with minimum oral or written direction</td>
</tr>
<tr>
<td>5</td>
<td>Self initiated Strategic</td>
<td>Use of the tools is initiated by the student and is deliberate, goal-directed and insightful.</td>
</tr>
</tbody>
</table>

(based on Arnold, 1995, pp. 321-322)

Determining a level of Effective Use of CAS

If CAS is to be used to advantage in teaching, doing, and learning mathematics then the students need to be able to use CAS effectively. The framework for Effective Use of CAS provides a guide for collecting information on which to base an assessment of a student’s CAS use. The rearrangement of the words from ‘use CAS effectively’ to Effective Use of CAS draws attention to the fact that the latter is a formal term referring to a measure of students’ performance in each aspect of the framework.

Similarly to the discussion of the level of Algebraic Insight, Effective Use of CAS is a relative measure. In each teaching situation, the detail of Common Instances will depend on the age and stage of the students. Since the Common Instances provide the demonstrable evidence of each element of Effective Use of CAS any measure of this construct will be relative to the tasks for which the CAS could be employed in the particular mathematics course being studied.

Again, in parallel with the discussion of Algebraic Insight, in this study students’ behaviour is often described by allocating a level of Effective Use of CAS on a coarse scale. A student’s level of Effective Use of CAS will be based on an amalgam of their
results for each aspect and summarised using the broad terms ‘low’, ‘moderate’ and ‘high’. Verbal descriptors will also be used to indicate students’ level of performance on the elements of Effective use of CAS. These have been explained in the previous section. A student who exhibits a high level of Effective Use of CAS would score highly on most, but not necessarily all, elements. However it may be assumed that a student’s level of Effective Use of CAS would reflect the degree to which their use of CAS, for both doing and learning mathematics, was effective.

**Linking Algebraic Insight and Use of CAS**

Five instruments were used to collect information about the interaction between students' Algebraic Insight and their Effective Use of CAS. Two of these instruments, the Post–course Evaluation and the method of Observation have been referred to above. Their purpose in relation to this aspect of the research will now be outlined, along with the contributions from a Background survey, Post-course general interview, weekly worksheets and the Mid- and Post-course examinations.

**Background survey**

The first instrument used for gaining ‘base-line’ information, was the Background survey. This is included as table 5.12. The Background survey asked students about their previous mathematics experience (questions 1 and 2) and their school experience of using technology for doing or learning maths (questions 3, 4 and 5). The final question asked students to self-rate their general computing skills. To inform teaching practice, similar surveys had previously been routinely conducted at the beginning of mathematics and statistics units at the University of Ballarat. In this study the information was sought to see if students' learning with CAS would be affected by their previous experience of either mathematics or technology.
Table 5.12
Background survey

1. Highest level maths studied at school: ______________________ Result ______.
2. Previous tertiary mathematics units: ______________________ Result ______.
3. Did you use a scientific calculator at secondary school? Yes [ ] No [ ]
4. Did you use a graphical calculator at secondary school? Yes [ ] No [ ]
   If yes, list the features available on your graphical calculator that you found most helpful.
5. Did you use a computer for mathematics at secondary school? Yes [ ] No [ ]
   If yes name the program(s) and the features which you found most helpful.
6. Please rate your general computing skills by marking an X on the number scale below.

<table>
<thead>
<tr>
<th>Poor</th>
<th>OK</th>
<th>Good</th>
<th>Very good</th>
<th>Excellent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Post–course general interview

The second source of data linking Algebraic Insight and Effective Use of CAS was an interview. Following the Post-course Algebraic Insight interview students were asked a number of questions which required them to reflect on the course and the place of CAS in their learning of mathematics. These interviews were also taped and later transcribed for analysis. Similar questions had been put to students studying other mathematics courses at the University of Ballarat in previous years (Pierce and Stacey, 2001a). On that occasion focus group interviews were conducted, but for this study it was decided to interview students individually or in pairs (their choice) so that it could be done at the same time as the Post-course Algebraic Insight interviews. It was considered that a second post-course appointment was too large an imposition on the students’ good will and that they would have been unlikely to attend two sessions. The questions posed to the students are included here in table 5.13.
Table 5.13

*Post-course General Interview questions*

1. Think about the maths we have been studying. Tell me about what you have found easiest and what you have found hardest and why?

2. Now tell me about your experience with DERIVE. What problems, if any have you had learning to use DERIVE?

3. When have you found DERIVE most helpful? Give examples.

4. Are there some aspects of maths for which you think using DERIVE has enhanced your understanding? Give examples.

5. We can tackle maths problems in different ways using mental calculations, pen and paper, scientific calculators, CAS. What place does each have for you? Give examples of the sort of problems for which you would use each approach.

6. How could CAS be used to help a student develop recognition of patterns

   ◊ in algebraic operations that demonstrate how process becomes object ie \( f(g(x)) \)?

   ◊ for possible numbers of solutions to equations?

   ◊ for critical values?

   ◊ links between symbols, graphs and tables?

*Post-course CAS evaluation*

The third instrument that aimed to collect data linking Algebraic Insight and Effective Use of CAS was a survey. At the end of the course students were asked to complete an expanded CAS Use survey. This consisted of the CAS Use survey shown in tables 5.7 and 5.8 with additional questions to measure students' response to the use of CAS for learning mathematics. The additional items consisted of eleven statements for which students were asked to indicate their response using a given 5-point Likert scale. Three of these items were seen as also giving information about students’ attitude to the use of CAS. These three items, presented in table 5.10, along with the remaining eight items, are shown in table 5.14. These items were designed to assess the students' view of a link between the use of DERIVE and their learning of mathematics. Again, the questions had been used previously in research projects at the University of Ballarat and the results reported in Kneebone and Pierce (1997), Pierce and Roberts (1997) and
Pierce (1999). In these previous studies, students’ responses to these questions were supported by the observations of their teachers over the duration of their courses. This suggested that the survey provided a simple method of obtaining both useful and reliable information that could be used when considering possible links between Algebraic Insight and Effective Use of CAS.

Table 5.14
*Items from Post-course evaluation focusing on students’ response to the use of CAS*

<table>
<thead>
<tr>
<th>Item</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Derive has helped me see patterns in Mathematics</td>
</tr>
<tr>
<td>2. I find using Derive helps me to understand Mathematics</td>
</tr>
<tr>
<td>3. DERIVE can be made to do the working out for maths problems.</td>
</tr>
<tr>
<td>4. I can use DERIVE to check every step of a problem.</td>
</tr>
<tr>
<td>5. A DERIVE plot will tell me everything I need to know about a function.</td>
</tr>
<tr>
<td>6. I only use DERIVE if the instructions tell me to.</td>
</tr>
<tr>
<td>7. I try out ideas using Derive</td>
</tr>
<tr>
<td>8. Compared to my confidence with <em>functions</em> before this unit, my confidence now is much greater.</td>
</tr>
<tr>
<td>9. Compared to my confidence with <em>calculus</em> before this unit, my confidence now is much greater</td>
</tr>
<tr>
<td>10. Compared to my confidence with <em>graphs</em> before this unit, my confidence now is much greater</td>
</tr>
<tr>
<td>11. Compared to my confidence with <em>trigonometry</em> before this unit, my confidence now is much greater</td>
</tr>
</tbody>
</table>

These single response items were followed by a set of questions (table 5.15) that required the student to decide which of the 'tools' at their disposal they would prefer to use when doing mathematics. These questions acknowledge that the students may use pen and paper, a scientific calculator, a graphical calculator, the graphing facilities of DERIVE or the symbolic manipulation, algebra capacity of DERIVE. For a series of simple mathematics questions they were asked which 'tool' they would prefer to use under different circumstances, when their main concern was speed, success or learning. It was expected that students with high Algebraic Insight would prefer not to use technology for the simplest problems.
Table 5.15  
*Students’ preferred ‘tool’ items*

<table>
<thead>
<tr>
<th>For this type of problem</th>
<th>For speed</th>
<th>For success</th>
<th>For learning</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Solve for $x$: $2x + 1 = 5$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Solve for $x$: $(x - 4)(x + 1)=0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. Solve for $x$: $x^5 - x^2 + x=0$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. Find the intercepts of the graph of $y=2x+1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5. Find the $x$ and $y$ intercepts for $y=x^2 + 5x + 6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Find the turning point for $y=x^2 + 5x + 6$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Compare $\log x$ and $\left(\frac{1}{x}\right)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This exercise was then re-administered in a second form. The second form of the items asked students which tool they would prefer to use for repeated solution of similar questions. These questions were expressed in general form using both parameters and variables, for example ‘Find the $x$ and $y$ intercepts for $y=x^2 + 5x + 6$’ became ‘Find the $x$ and $y$ intercepts for $y=ax^2 + bx + c$’. This format was included because previous studies (see for example Pierce and Stacey, 2001a) suggested that students see CAS as valuable for doing problems which 'look harder' or time consuming. Additionally, solving repeated instances of similar items with technology is time efficient because the set up time can be spread across several examples. Their responses therefore indicate both the degree to which they are likely to make discriminating functional use of CAS and their level of Algebraic Insight.

*Weekly worksheets*

The fourth instrument that provided data linking Algebraic Insight and Effective Use of CAS consisted of a set of worksheets. Weekly worksheets were assessable tasks for students taking the course, as discussed above. Similar worksheets had been used as part of the assessment of this course in previous years but the list of problems was reviewed in the light of the key questions of study. In particular, thought was given to selecting problems that might encourage the development of students' Algebraic Insight.
through the exploration of patterns. The work submitted was photocopied, then the originals marked and returned to students the following week.

In order to obtain information about students' Effective Use of CAS and their perceptions of progress in learning, they were asked to place a code beside each solution they submitted for marking. The codes consisted of the first letters of the words chosen to complete each of three statements. These statements and the corresponding response choices are shown below in table 5.16. For example the code DG, H, G would indicate that the student had used DERIVE’s Graphing facility, found the problem Hard to do, but that they felt their understanding of this work was now Good. They were asked to indicate which ‘method’ they had used so the researcher could keep track of whether CAS had been used and link this to the working shown in their solutions to problems. This data provided further indication of students’ Judicious Use of CAS.

### Table 5.16

**System of codes for weekly worksheet questions.**

<table>
<thead>
<tr>
<th>Method(s) Used</th>
<th>PP</th>
<th>DA</th>
<th>DG</th>
<th>SC</th>
<th>GC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pen &amp; Paper</td>
<td>Derive</td>
<td>Derive</td>
<td>Scientific</td>
<td>Graphical</td>
</tr>
<tr>
<td>I found this question</td>
<td>PP</td>
<td>DA</td>
<td>DG</td>
<td>SC</td>
<td>GC</td>
</tr>
<tr>
<td>Very Poor</td>
<td>Very Hard</td>
<td>Hard</td>
<td>OK</td>
<td>E</td>
<td>VE</td>
</tr>
<tr>
<td>Very Poor</td>
<td>Very Good</td>
<td>Good</td>
<td>Neutral</td>
<td>Easy</td>
<td>Very Easy</td>
</tr>
<tr>
<td>My understanding of this work is now</td>
<td>PP</td>
<td>DA</td>
<td>DG</td>
<td>SC</td>
<td>GC</td>
</tr>
<tr>
<td>Very Poor</td>
<td>Very Poor</td>
<td>Poor</td>
<td>Neutral</td>
<td>Good</td>
<td>Very Good</td>
</tr>
</tbody>
</table>

**Observation**

Further information about both students' Algebraic Insight and their Effective Use of CAS was gained from observation during laboratory classes and the examinations, as described above. In her observation notes the teacher/researcher noted the questions students asked in class, their use of DERIVE, and any difficulties they had interpreting its output. During the examinations the teacher/researcher and a second experienced
colleague (Lyn Roberts) acted as observer. As stated previously, they stood to the rear of the computer laboratory in positions where they could observe the computer screens. Each took responsibility for observing half of the students. These observers recorded each student’s use of DERIVE by noting anything which could be seen on their screens and the number of the examination questions they were attempting to answer. This data on students’ use of CAS, when integrated with details from their examination scripts, provided further information linking Algebraic Insight and Effective Use of CAS.

**Examinations**

The final instruments used to collect data were the students' examinations. Students' examination scripts were photocopied before marking. The marked mid-course examination scripts were shown to students, the solutions discussed and then the scripts returned to the teacher for future reference, as is that University's policy. The teacher also retained the Post-course examination scripts and students may request to view these.

The examination scripts were expected to provide evidence of Algebraic Insight. These scripts were supplemented by the detailed information about students' CAS use recorded by observers during the examinations. These notes were used to look for links between Algebraic Insight and Effective Use of CAS.

**Data collection**

Data was collected throughout the fifteen-week course as indicated in table 5.17.

All data collection, apart from the examinations, worksheets and interviews, took place during scheduled classes. One difficulty with being both teacher and researcher was that at all times the teaching had priority. In order to minimise disruption to the teaching program, it was planned that all three Algebraic Insight Quizzes be held at the beginning of classes and other data collected quickly at the end of different classes (attention is drawn to exceptions in table 5.17). It was made clear to students that the results from the various instruments, other than examinations and worksheets, would in no way contribute toward their assessment in this course.
Students were cooperative about the data collection, but unfortunately not all students were present on all occasions. The number of students who contributed data for each instrument is recorded in table 5.17. The progressive results of data collection were not shared with the students. When marked weekly worksheets and examination scripts were returned to students, only the mathematics pertaining to the course was discussed in class.

Interviews were conducted either individually or in pairs. Students were given this choice as some felt nervous and for other, more 'absent minded', students coming with a friend increased compliance with keeping appointments.

The researcher planned to conduct the Post-course Algebraic Insight quiz during the last class of the course (week 13); however, unforeseen problems with the technology meant that this had to be delayed. Details of this are included in the section on constraints, later in this chapter.

**Data storage**

Survey data was coded and entered into text files then read into SPSS for later analysis. A physical folder was also established for each student and photocopies of weekly worksheets and examination scripts were stored in these along with records of observations and transcripts of interviews.

**Data analysis**

The data was analysed at a class level using SPSS to produce summary statistics, explore relationships and test for change over time. The results of this analysis are presented in Chapter 6. A case study approach was used to explore the data at an individual level. A selection of this material is presented in Chapter 7.
Table 5.17
Summary of data collection schedule

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Time</th>
<th>Number of valid responses</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebraic Insight Quiz</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-course</td>
<td>Start of first lecture, week 1</td>
<td>n = 20</td>
</tr>
<tr>
<td>Mid-course</td>
<td>Week 7</td>
<td>n = 12</td>
</tr>
<tr>
<td>Post-course</td>
<td>Prior to examination (week 15)</td>
<td>n = 15</td>
</tr>
<tr>
<td><strong>Algebraic Insight Interview</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-course</td>
<td>Weeks 1 and 2</td>
<td>n = 11</td>
</tr>
<tr>
<td>Post-course</td>
<td>Weeks 13,14,15</td>
<td>n = 14</td>
</tr>
<tr>
<td><strong>CAS use survey</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Part A: CAS Technical</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Difficulties survey</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Part B: Judicious Use of CAS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>survey</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Early-course</td>
<td>End of first laboratory session</td>
<td>n = 15</td>
</tr>
<tr>
<td>Mid-course</td>
<td>Week 7</td>
<td>n = 13</td>
</tr>
<tr>
<td>Late-course</td>
<td>End of last laboratory session</td>
<td>n = 12</td>
</tr>
<tr>
<td><strong>Observation</strong></td>
<td>All laboratory sessions and</td>
<td>Variable</td>
</tr>
<tr>
<td></td>
<td>examinations</td>
<td></td>
</tr>
<tr>
<td><strong>Background survey</strong></td>
<td>First lecture, week 1</td>
<td>n = 20</td>
</tr>
<tr>
<td><strong>Post-course general interview</strong></td>
<td>Weeks 13,14,15</td>
<td>n = 14</td>
</tr>
<tr>
<td><strong>Post-course CAS evaluation</strong></td>
<td>End week 13</td>
<td>n = 12</td>
</tr>
<tr>
<td><strong>Weekly worksheet</strong></td>
<td>Weekly</td>
<td>n = 15</td>
</tr>
<tr>
<td><strong>Examinations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mid-course</td>
<td>Week 6</td>
<td>n = 18</td>
</tr>
<tr>
<td>Post-course</td>
<td>Week 15</td>
<td>n = 15</td>
</tr>
</tbody>
</table>
**Discussion of the methodology**

**Opportunities and constraints of the setting**

The setting of this study was constrained by the researcher's work location. This presented some advantages but also some disadvantages. The class was smaller than expected in 1999. The small numbers meant that it was not appropriate to consider an independent groups experimental research design. While it also meant that quantitative analysis of data would most likely yield few statistically significant results, such analysis was useful for monitoring the progress of both individuals and the group. However, in this setting, being both the teacher and the researcher meant that I knew each student in detail and it was possible to collect data from observations and interviews and therefore build up profiles of each student as well as overall group results. The students’ rapport both within the group and with me meant that they felt happy to give honest opinions and answers to the many questions they were asked regarding CAS and its use during the course.

**Repeated Algebraic Insight quiz and interview**

Table 5.17 indicates that the Algebraic Insight quiz was administered three times and the Algebraic Interview twice. In both cases the items presented to the students were similar, parallel items, but not identical items. While students were not given their results for the quiz and items were not discussed by the teacher/researcher, this would not have prevented students from discussing and comparing answers. It was felt that the time between data-collection episodes was not sufficient for students not merely to remember answers so the items were changed slightly. For each administration of the quiz new items were created. Examples of parallel items are shown in table 5.18. The aim was to write items that were sufficiently different that students could not have merely memorised the previous answers, but sufficiently similar that each item would test essentially the same aspect of Algebraic Insight. It was important to try to maintain the same level of difficulty so that improvement, if any, during the course could be noted. Full copies of each Algebraic Insight quiz and the Algebraic Insight Interview questions are included in Appendix 2.
### Table 5.18
Examples of similar items from Mid- and Post-course Algebraic Insight Quizzes

<table>
<thead>
<tr>
<th>Mid-course Algebraic Insight Quiz</th>
<th>Post-course Algebraic Insight Quiz</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. Student</td>
<td>7. Student</td>
</tr>
<tr>
<td>Textbook</td>
<td>Textbook</td>
</tr>
</tbody>
</table>
| \[
\begin{align*}
    a - b & \quad a - c \\
    d - c & \quad d - b
\end{align*}
\] | \[
\begin{align*}
    p - 3 & \quad p - 3 \\
    q - 4 & \quad q - 4
\end{align*}
\] |

14. \( h(x) = px^4 + qx^4 + rx^3 + sx + t \)

If \( p, q, r, s, \) and \( t \) were real numbers, what is the minimum number of times the graph of this function might cut or touch the \( x \)-axis?

- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) 4
- (f) 5

16. \( h(x) = px^5 + qx^4 + rx^3 + sx + t \)

If \( p, q, r, s, \) and \( t \) were real numbers, what is the maximum number of times the graph of this function might cut or touch the \( x \)-axis?

- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) 4
- (f) 5

18. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.04</td>
</tr>
<tr>
<td>-1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>125</td>
</tr>
</tbody>
</table>

- (a) linear
- (b) quadratic
- (c) exponential
- (d) none of the above

For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.25</td>
</tr>
<tr>
<td>-1</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

- (a) linear
- (b) quadratic
- (c) exponential
- (d) other

### Constraints beyond the control of the researcher

**Students’ aims**

It should also be acknowledged that each member of the class had some personal motivation for doing taking this course. Their motivation will have had an effect on their learning in this course. For three students it was a compulsory unit, but for the others this was taken as an elective. The majority of the students planned to be teachers.
and felt that taking courses in mathematics would enhance their employment prospects. However, this should be balanced against the fact that none of the students planned to major in mathematics and so most were not concerned about achieving high score results.

*Students’ personal organisation*

The Algebraic Insight Interview complemented the Algebraic Insight Quiz for those students who took part. While all students were encouraged to attend an interview and the option of coming in pairs was permitted, their participation was voluntary. All students who finished the course attended a Post-course interview but only eleven attended an Early-course interview. Lack of participation in these Early-course interviews was, according to the students, due to poor time management. These interviews were only conducted in the first two weeks of the course to obtain ‘initial’ results so some students missed this window of opportunity.

*Effect of students’ commitments in other classes*

There was an additional factor that affected many of the students’ ability to gain the most from classes later in the semester. The majority of the students was studying Physical Education and had early morning swimming classes immediately before our mathematics classes. Early in the course this had little impact, as they studied swimming technique, but later in the course they were swimming long distances and came to class both very tired and very hungry. Against usual rules, they were permitted to bring food to class, but their concentration was impaired.

*Power supply failure*

Unfortunately, due to power supply failure, the Post-course Quiz was not administered as planned, in the last lecture of the course. As an alternative, permission was gained and the students agreed to take the Quiz immediately prior to their Post-course Examination, as this was the only occasion on which they would be together on campus again. This was less than ideal as students were focused on the Examination and the style of thinking it required. In addition two further factors impacted on students’ ability to concentrate on the Quiz questions. Some students were preoccupied by ‘test anxiety’ relating to the Examination. Others, not particularly concerned about the
Examination, had participated in what, for them, was a major social event on the previous day and night and they were suffering the consequences. Such factors were beyond the control of the researcher!

**Modified Algebraic Insight Quiz Results**

It seemed that the circumstances were likely to have a negative impact and unreasonably bias some students’ Post-course Algebraic Insight Quiz results. To more fairly represent students’ level of Algebraic Insight a new modified measure was created. For students who took both the Mid-course and Post-course Quizzes the researcher decided to use their maximum result when considering change in Algebraic Insight. This result was given the name Late-course Quiz.

**Use of surveys in assessing Effective Use of CAS**

As outlined above, information from surveys was used to build a measure of students’ engagement with CAS. Part of this was used to make inference about students’ response to the use of CAS. Blackwell (1998) warns that a positive response to a computer program does not always mean an effective use of the program. In his study on software visualisation systems he found that students liked the most graphical program best but also performed worst with this system. Looking at the survey results alone did not provide sufficient information to the researcher. A report published by Lagrange (1999) after this data had been collected confirms that there may be a mismatch between how students feel about using a program and their actual use. As discussed in Chapter 3, he found that many students who said that they were having no problems failed to score more than half marks on a task which tested the use of the basic home screen commands of a CAS calculator. In this present study, students were not tested directly on their use of DERIVE. This was a deliberate decision because the time available for collecting data was limited and it was felt that this information could be gained by observing the students at work in both classes and the examinations. In this study the interpretations of students’ perceptions of their CAS use was mediated by the teacher/researchers’ observations and the analysis of students’ work.

This concludes the description of the data collection instruments, method and timing of data collection.
CHAPTER 6

CLASS RESULTS WITH DISCUSSION

This chapter reports the information obtained from the data collection instruments described in Chapter five that allow us to look at summary measures and patterns in the group as a whole. (Refer to Appendix 4 for table outlining measures and descriptors developed in Chapter 5.) This is complemented in Chapter 7 by a selection of case studies that focus on individual students. First, in this chapter, the class data concerning students' Algebraic Insight is considered, then the measures of Effective Use of CAS and, last, evidence for links between Algebraic Insight and Effective Use of CAS are reported. The analysis provides both information about this class and a general model for using these instruments to monitor students’ progress.

It should be noted that the results obtained from the data collection must all be considered in the light of the characteristics of both the group of students who participated in this study and the course content and teaching. The general background of the students and details of the teaching and assessment program were covered in Chapter 4, but specific information gained from student enrolments and the background surveys is summarised below since this data came from the first of the data collection instruments used in this study.

The students

Twenty-one students, ten males and eleven females, started the course. Seven males and eight females completed all assessable tasks. From the University's class lists it was found that these students were enrolled in a variety of degree programs including Computing, Humanities, Technology and Education. Of those fifteen students who completed the course, seven aim to teach in secondary schools and four in primary schools. These students often asked questions about the relevance of both the mathematics and the technology to their long-term goal of teaching. As a teacher/researcher I also felt constrained by knowing that CAS will not be available in most schools when these students graduate, and so I taught ‘by-hand’ techniques (for
example, the quotient rule for differentiation) which could be considered unnecessary with CAS.

From the background survey responses it was learnt that all except one student had completed a mathematics subject in the final year of secondary education. At school, seven had taken Victorian Certificate of Education Further Mathematics; this course included no calculus and little algebra. The others had all taken Victorian Certificate of Education Mathematical Methods (or equivalent), which includes some algebra, some calculus and has a strong emphasis on functions, graphs and approximate methods. One of these students had also studied Victorian Certificate of Education Specialist Mathematics. This subject covers further calculus and uses exact methods. All students had previously used scientific calculators and forty-one percent had also used graphical calculators for school mathematics. While thirty-eight percent had made some use of computers for school mathematics, only one student had previously used a Computer Algebra System.

Results from the two instruments measuring Algebraic Insight

Algebraic Insight Quiz

As noted in Chapter 5, the Algebraic Insight Quiz was administered to all students present on three occasions throughout the semester. The Quiz was composed of 22 items with 4 or 5 items for each of the five elements of Algebraic Insight. The breakdown was as follows: recognising conventions and basic properties of operations (4 items); identifying structure (5 items); identifying key features (4 items); linking symbolic and graphical representations (5 items); and linking symbolic and numeric representations (4 items). When considering these results it should be remembered that the Quiz was scored positively for correct responses but negatively for incorrect responses (refer back to table 5.2 for response options) so that the range of possible scores was from –44 to 44. The scoring system depended on both the correctness of a student’s response and their level of certainty for this response.

Analysis of the Pre-, Mid- and Post-course results, summarised in table 6.1, shows an overall improvement from Pre-course Quiz to Mid-course Quiz which then seems to have declined towards the end of the course. Fourteen students took both the Pre-course and Post-course Quizzes. Direct comparison of these students’ scores shows that 10
students’ results improved and that the mean score, for this subset of students, improved from 8.3 to 12. A paired t-test (df = 13) yielded t = 1.543 with a p-value of 0.073 (one tailed).

Table 6.1

| Summary statistics for Pre- Mid- and Post-course Algebraic Insight Quiz results |
|-----------------------------|-----------------|-----------------|-----------------|-----------------|
| Number of students | Mean | Std. Deviation | Minimum | Maximum |
| Pre-course | 20 | 6.5500 | 13.3711 | -17.00 | 37.00 |
| Mid-course | 12 | 16.5833 | 10.3876 | 2.00 | 33.00 |
| Post-course | 15 | 10.9333 | 8.5646 | -4.00 | 25.00 |

In the light of classroom observation, interaction with the students and Post-course interviews, the researcher was surprised by the decline shown by some students between their Mid-course and Post-course Quiz scores. As mentioned earlier (See Chapter 5 Constraints beyond the control of the researcher: power supply failure) the timing and circumstances surrounding the administration of the Post-course Quiz would have been likely to adversely affect the students’ performance on this Quiz. In order to more fairly represent the change in students’ Algebraic Insight the researcher decided to construct a Late-course index based on students’ maximum scores for the Mid- or Post-course Quiz. The two aspects of Algebraic Insight have been treated in a similar manner. These have been called the Late-course results. The Pre-course and Late-course results will be discussed here and the results from all administrations of the Algebraic Insight Quiz are included in Appendix 3.

First, the summary statistics for these two tests were considered. Table 6.2 shows the class mean, standard deviation, and the minimum and maximum for the Pre-course and Late-course Algebraic Insight Quizzes. There is a clear improvement in the class mean score from the Pre-course Quiz (mean 6.6) to the Late-course Quiz results (mean 14.6). These statistics, along with the graphs in figure 6.1, show that there was a wide variation in scores for the Pre-course Quiz, and that almost half of the group scored below 0. For the Late-course Quiz scores they show that the class mean was higher, no student scored less than 0 and the class variation was reduced. The overall
improvement observed may be considered to be more than just a chance occurrence since there was a statistically significant difference between the Pre-course and Late-course Quiz results. For those students who did at least two tests, a paired t-test, df =15 gave a t-value of 4.282 with an associated p-value of 0.001. A paired test was used to ensure that the improvement was not just due to students leaving the course because they were having difficulty. In fact, this was not the case. Although only 15 of the initial 20 students completed the course, those who withdrew scored –17, -3, -2, 0, 8, and 29 out of the possible 44 on the first Algebraic Insight Quiz. This study's results are therefore not affected, as they could have been, had only the weakest students dropped out. The student who joined the group after the first Quiz only scored -4 on the Post-course Quiz and so did not contribute positively to the improved class average.

The distribution of individual scores for the Pre-course and Late-course Quiz results is shown by figure 6.1. The information is displayed showing the percentage of students in each category in order to allow comparison, since different numbers of students were involved. These graphs indicate the overall improvement, showing a shift to the right, especially from the –10-0 interval into the 1-22 score range. Those students who initially had the lowest scores showed most improvement in Algebraic Insight. The best students showed little or no improvement on this Quiz because there was little room for them to show improvement on this limited number of items.

Table 6.2

<table>
<thead>
<tr>
<th>Quiz</th>
<th>No. of students</th>
<th>Mean Score</th>
<th>Standard deviation</th>
<th>Minimum (possible -44)</th>
<th>Maximum (possible 44)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-course</td>
<td>20</td>
<td>6.6*</td>
<td>13.37</td>
<td>-17</td>
<td>37</td>
</tr>
<tr>
<td>Late course Max (Mid,Post)</td>
<td>17</td>
<td>14.6*</td>
<td>9.84</td>
<td>-4</td>
<td>32</td>
</tr>
</tbody>
</table>

• Difference significant at the 0.1% level
Aspects of Algebraic Insight

Having established that, as a result of this course taught using CAS, there was a significant improvement in students’ Algebraic Insight, the next step was to explore whether this improvement was the same for both aspects of Algebraic Insight (as described in Chapter 3A). Grouping the Quiz items and checking the summary statistics for each aspect showed that there was more improvement in ability to Link Representations than in Algebraic Expectation but the increase in mean score was of a similar order (see table 6.3). Comparison of mean scores, using paired t-tests, showed that each of these improvements was also statistically significant. (Paired t-test for improvement in mean score for Algebraic Expectation, df = 15, t = 3.16, p = 0.003; paired t-test for improvement in mean score for Linking Representations, df = 15, t = 2.83, p = 0.007.) The variability of the results for the later Quiz was considerably less than for the Pre-course Quiz, with most reduction in variability shown for the Algebraic Expectation scores. The change in the distribution of the results is illustrated by the graphs in figure 6.2. The graphs show a clear shift to the right for both aspects of Algebraic Insight. The change in the distribution of scores for Algebraic Expectation is notable because, while the distribution of Late-course scores is positively skewed all scores were above positive two. This represents an improvement of 10 points (out of the range of 52 points) in the minimum score and all of the weakest students had
improved. In contrast, the Late-Quiz scores for Linking Representations showed some students had not improved but rather still gained negative scores.

Table 6.3
Algebraic Insight Quiz results by Aspect: Algebraic Expectation and Ability to Link Representations

<table>
<thead>
<tr>
<th>Aspect</th>
<th>No. of students</th>
<th>Mean Score</th>
<th>Standard deviation</th>
<th>Minimum (-26)</th>
<th>Maximum (26)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebraic Expectation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-course</td>
<td>20</td>
<td>4.5*</td>
<td>7.80</td>
<td>-8</td>
<td>21</td>
</tr>
<tr>
<td>Late-course</td>
<td>17</td>
<td>9.2*</td>
<td>5.71</td>
<td>2</td>
<td>23</td>
</tr>
<tr>
<td>Ability to Link Representations</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pre-course</td>
<td>20</td>
<td>2.1*</td>
<td>7.33</td>
<td>-9</td>
<td>16</td>
</tr>
<tr>
<td>Late-course</td>
<td>17</td>
<td>5.4*</td>
<td>6.67</td>
<td>-8</td>
<td>16</td>
</tr>
</tbody>
</table>

- Differences significant at the 1% level

Figure 6.2 Comparison of Aspects of Algebraic Insight using mean scores from Algebraic Insight Quizzes

Students’ scores for each aspect of Algebraic Insight in the Pre and Late-course Quizzes were correlated to investigate whether ability in the two aspects was related, and to measure the extent to which weaker students showed more improvement than those students with stronger Algebraic Insight. The correlation coefficients indicated a trend for students either to do well or poorly on the items for both elements of Algebraic Insight (correlations 1 and 2 in table 6.4). While these correlations, 0.56 and 0.49,
indicate significant positive relationships between Algebraic Expectation and ability to Link Representations both Pre-course and Late-course, they also show that these relationships did not hold for all students. The corresponding coefficients of determination, 31% and 24%, suggest that most of the variability in Linking Representations is *not* explained by variability in Algebraic Expectation or vice versa. Amongst this small group there were also students who showed greater strengths in one aspect of Algebraic Insight than the other. Some detail of this will be evident in the Case studies in Chapter 7.

It is perhaps more interesting to note that, while there was a negative relationship between students’ scores on the Pre-course Quiz and Late-course Quiz (correlations number 3 to 5, table 6.4), weaker students made more improvement. This is indicated by the negative correlations (correlations number 6 and 7 table 6.4) in the table. The initial low scores of some students left much scope for change and it is encouraging to see such improvement, by the weaker students, in each aspect of Algebraic Insight.

Table 6.4
*Relationships between Aspects and change during the course*

<table>
<thead>
<tr>
<th>Variable</th>
<th>Correlation coefficient</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Pre-course Algebraic Expectation and</td>
<td>0.560</td>
<td>0.010</td>
</tr>
<tr>
<td>and Pre-course Linking Representations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 Late-course Algebraic Expectation and</td>
<td>0.489</td>
<td>0.024</td>
</tr>
<tr>
<td>and Late-course Linking Representations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Pre-course Algebraic Expectation and</td>
<td>0.424</td>
<td>0.062</td>
</tr>
<tr>
<td>and Late-course Algebraic Expectation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 Pre-course Linking Representations and</td>
<td>0.389</td>
<td>0.090</td>
</tr>
<tr>
<td>and Late-course Algebraic Expectation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 Pre-course Linking Representations and</td>
<td>0.382</td>
<td>0.096</td>
</tr>
<tr>
<td>and Late-course Linking Representations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 Pre-course Algebraic Expectation and</td>
<td>-0.658</td>
<td>0.002</td>
</tr>
<tr>
<td>and Change in Algebraic Expectation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 Pre-course Linking Representations and</td>
<td>-0.680</td>
<td>0.001</td>
</tr>
<tr>
<td>and Change in Linking Representations</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Elements of Algebraic Insight

The results were then broken down further to consider student’s ability with each element of Algebraic Insight. In order to compare students’ performance on each element the mean score for each element was weighted so that the possible maximum score for each element was 100%. (Similarly, the possible minimum was −100%.) The results for both Pre-course and Late-course are presented in figure 6.3. The most striking result is that students had most trouble identifying structure (element 1.2) and did best at recognising conventions (element 1.1). Exploring the change in students’ scores for each element of Algebraic Insight (see table 3A.1) showed that there was improvement in the mean score of every element. Figure 6.3, in which the elements are ordered by framework, gives a visual impression of these improvements. In descending order of increase, the improvement was +26% for element 1.2 (identifying structure), +25% for element 1.3 (identifying key features), 23% for element 2.1 (linking symbolic and graphical representations), +21% for element 2.2 (linking symbolic and numeric representations) and +14% for element 1.1, (recognition of conventions). Thus, while identifying structure was still the group's weakest element, they demonstrated most improvement in this area.

This differential improvement may be related to three factors. Firstly, students did best on element 1.1 on the Pre-course test and this allowed little room for improvement. Secondly, it might reflect the limitations of such a Quiz for assessing some Common Instances of ‘recognising conventions and basic properties of operations’. Students’ responses recorded in the Interviews showed changes in their understanding of the meaning of symbols that could not be detected by this Quiz. Some detail of this is presented in the Case studies (Chapter 7) but briefly, the quick, intuitive responses of the Quiz gave no opportunity to assess what students thought these symbols meant. It focused on recognition of patterns and equivalent expressions, so it may not have picked up some important changes in students’ fundamental understanding of the meaning of letters used.

Thirdly, the prescribed course placed less emphasis on numeric representations than symbolic and graphical representations and it was easier to change to a graph window using DERIVE 2.55 than it was to produce tables. On reflection, the teaching in the course privileged graphical representations over numeric representations so those
students did not have as much exposure to linking symbolic and numeric representations. Kendal and Stacey, (2001), use the term ‘privileging’ to describe the process which occurs when teachers focus on particular aspects of mathematics, or particular methods of solution. They demonstrate that there can be substantial variations in privileging with CAS.

![Figure 6.3 Comparison of mean scores for elements of Algebraic Insight.](image)

**Algebraic Insight Interviews**

Interviews were conducted with volunteers at the beginning and end of the course (see Chapter 5). One of the purposes of these interviews was to assess Common Instances of Algebraic Insight that were not suited to testing with the quick Quiz method. The Algebraic Insight Interviews were therefore heavily weighted towards recognising conventions (1.1, especially meaning of symbols) and linking symbolic and graphical or numeric representations (2.1 and 2.2) of functions (see Chapter 5 for detail). The most valuable contribution of the interviews to this study is the detailed insight that they gave to each student's algebraic thinking.

Interview questions 1 and 3 (see Appendices 2.4 and 2.5) proved successful in revealing students’ understanding of the meaning of symbols. Question 1 required the students to consider the general solution for a given process. At the Early-course interview most students merely suggested trying more examples while at the Post-course interview they suggested the use of a pro-numeral. Question 3 was an area problem with the measurements of side lengths represented by letters. This item gave interesting results because there were several possible correct answers. Most students who understood the meaning of the symbols found the answer in terms of \(x\), but the most ‘elegant’ solution
(which showed more Algebraic Insight) was the solution in terms of \( d \). Responses to this item also revealed some students’ misconceptions about the use of letters in algebra; for example, Catherine carried out the solution process for the area of a rectangle problem without difficulty. However, she then said “I can get that far but I’m sorry I can’t tell you what the answer is” thus indicating that she did not understand the meaning of a letter as a variable.

Question 2 was the least helpful item in terms of providing information about students’ Algebraic Insight. This was because students, in this case, would make progress most easily by abandoning algebra and applying number sense. This question could be useful for assessing Symbol Sense as this understanding corresponds to Arcavi’s A2, (1994), (see Chapter 3) rather than the subset, Algebraic Insight. For those students who chose an algebraic approach this question presented a challenge in terms of recognising basic properties. Since it involved odd numbers (Early-course) and even numbers (Post-course), students needed to recognise that successive numbers in the sequence would be two apart and that to ensure that a number was odd it should be of the form \( 2n+1 \) or even of the form \( 2n \).

Questions 4 and 5 provided information about students’ ability to Link Representation. The fake computer syntax in question 4 required students to identify the structure of this expression. This is an important skill for using CAS and was handled well by these students but since this was an unfamiliar task a number of students required some prompting to begin. Often, re-reading the question to them or re-expressing the task in other words was sufficient for the student to begin. The interactive nature of the interview meant that students were given an opportunity to show what they could do rather than having to leave questions out.

Of the five questions presented in the Interview Question 5 most closely resembled a senior ‘school mathematics’ question. Students showed most confidence in approaching this item often remarking that this was a familiar question. The change between the Early-course interviews and Post-course interviews which was pleasing to observe was a change from students trying to remember a learned routine (again Catherine provided a clear example of this when she said that she wanted to use the ‘flag rule’ but couldn’t remember exactly how. When asked to explain the flag rule, a
term which was genuinely unfamiliar to the researcher, she said she couldn’t explain it, ‘you just do it’) to discussing their thinking in their own words. It seemed that the focus on elements of Algebraic Insight in this course had provided them with both strategies and vocabulary to explain this mathematics.

Further detail of the students’ progression is recorded in the case studies in Chapter seven. However, tables 6.5 and 6.6 provide a broad overview of the results. The students scored only positive points for demonstrating Algebraic Insight according to the marking scheme detailed in the corresponding section of Chapter 5. These results show an improvement in Algebraic Insight (independent t test, df =20.25, t =2.11, p =0.047) with more improvement occurring in Algebraic Expectation than Linking Representations. (The change for each element was not statistically significant: independent t-tests Algebraic Expectation df =16.25, t=1.48, p=0.156; Linking Representations df=22.01, t=1.12, p=0.276 Note: paired t-tests were not used since the effect of reduced sample size outweighed possible gain in power.). Percentage results are included to allow for the different number of items on the parallel but slightly different interview questions (again, see Chapter 5 for detail).

Table 6.5

<table>
<thead>
<tr>
<th></th>
<th>No. of students</th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-Course</td>
<td>11</td>
<td>8.9*</td>
<td>4.44</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>Post-course</td>
<td>14</td>
<td>12.5*</td>
<td>3.94</td>
<td>2</td>
<td>18</td>
</tr>
</tbody>
</table>

Students who did both interviews

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-course</td>
<td>8.5</td>
<td>3.13</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>Post-course</td>
<td>13.1</td>
<td>2.62</td>
<td>8</td>
<td>18</td>
</tr>
</tbody>
</table>

- Difference significant at the 5% level.
Table 6.6
*Algebraic Insight Interview results by Aspects of Algebraic Insight*

<table>
<thead>
<tr>
<th></th>
<th>No. students</th>
<th>Algebraic Expectation</th>
<th>Ability to Link Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Mean</td>
<td>Stdev</td>
</tr>
<tr>
<td>Pre-course</td>
<td>11</td>
<td>4.3</td>
<td>3.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(43%)</td>
<td></td>
</tr>
<tr>
<td>Post-course</td>
<td>14</td>
<td>5.9</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(65%)</td>
<td></td>
</tr>
</tbody>
</table>

Students who did both interviews

<table>
<thead>
<tr>
<th></th>
<th>No. students</th>
<th>Pre-course</th>
<th>Post-course</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Mean</td>
<td>Stdev</td>
</tr>
<tr>
<td>Pre-course</td>
<td>9</td>
<td>3.3</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(33%)</td>
<td></td>
</tr>
<tr>
<td>Post-course</td>
<td>9</td>
<td>5.2</td>
<td>1.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(52%)</td>
<td></td>
</tr>
</tbody>
</table>
Overview of Algebraic Insight instruments and results

This is a summary of key findings based on the class results for Algebraic Insight.

- The group mean score for Algebraic Insight increased
- The majority of students showed some improvement in Algebraic Insight
- All ‘weak’ students improved and these students showed the greatest improvement in Algebraic Insight.
- There was improvement in both aspects of Algebraic Insight but the pattern of improvement was not the same. All the students who initially had very low scores for Algebraic Estimation improved while some students still scored very poorly on Linking Representations.
- There was a positive relationship between students’ scores for Algebraic Expectation and Linking Representations. Students’ Algebraic Expectation explains, in part, their ability to Link Representations (or vice versa) but most of the variation is due to other factors.
- The framework proved useful in analysing students’ work. This approach identified students’ lack of ability to identify structure and strength in recognising conventions.
- The detailed analysis we have achieved indicates that the Algebraic Insight Quiz proved to be a useful, simple and efficient instrument for gathering information about students’ mathematical thinking.
- The Algebraic Insight Interview complemented the Quiz by revealing further details of students’ thinking.
Results from three instruments assessing students' use of CAS

Background
The background survey asked students to rate their general computing skills on a line scale marked 0 to 10 with descriptive words, ‘Poor’ through to ‘Excellent’, indicated as well. (See figure 5.11.) Their responses are summarised in figure 6.4 and showed that most students (64%) rated their general computing skills no higher than 'good'. The student who claimed ‘excellent’ general computing skills was, in fact, in the final semester of his Bachelor of Computing degree.

To take account of students’ varied background both in general technology skills and specifically using technology in mathematics classes some details on the previous use of graphical calculators was also important. As mentioned above, on the background survey 41% of students reported that they had used graphical calculators at school. Of the 15 students who finished the course the percentage with this experience was a little higher 47% but, more importantly, for those who had completed one of the three year 12 mathematics courses offered in Victoria the previous year, 70% owned graphical calculators. They were allowed to continue to use these as they wished, in parallel with CAS, in this course.

![Figure 6.4. Students’ self-assessed level of general computing skills](image)
Technical difficulties survey

Difficulties were categorised, using the Effective Use of CAS Framework elements 1.1, 1.2, and 1.3, as being either difficulty with Program Syntax, Changing Representations, or Interpreting CAS output. Students’ level of difficulty was scored on the basis of their self-reporting on the Technical Difficulties Survey (see table 5.7). The results from these surveys are presented as graphs in figures 6.5 to 6.8 and summarised in tables 6.7 and 6.8. In these tables, those students experiencing only ‘none or one’ difficulty were classified ‘low’, ‘some difficulties’ as ‘medium’ and ‘a lot or everytime’ as ‘high’. Students were given ‘not applicable’ as a legitimate response to any of the suggested possible difficulties. These were treated as missing values, not zero difficulties.

Figure 6.5 illustrates the change in total level of Technical difficulty over the course. These graphs show that the percentage of students experiencing a medium level of difficulty decreased throughout the course, but the percentage experiencing high levels of difficulty decreased by mid-course then increased again late in the course. The Late-course results are split into two groups, one experiencing very few difficulties and the other experiencing a high level of difficulty. While all students found some difficulties adapting to the use of the program no student registered more than 50 % on the total difficulties scale. The grouped results, summarised in table 6.6, emphasise numerically that there was a pattern of improvement followed by further difficulty using the program for about half of the students.

The change, over time, was then more finely analysed to explore differences in the ‘level of difficulty’ experienced with each element of the Technical aspect of Effective Use of CAS. The number of students reporting each level of difficulty (scored 0 to 10) for each of the three elements, on three occasions during the course, is shown in figures 6.6 to 6.8. and is also summarised in table 6.8. The pattern for each element is similar to that reflected in the overall level of difficulty. The change in difficulty experienced using program syntax is typical. The initial distribution was negatively skewed but for each later administration the results were positively skewed. The Late-course results were in two clusters with about half of the students experiencing very few difficulties, but a small group reporting a high number of difficulties. Looking at the trend, in these three elements, during the course, shows that over time the students had fewer
difficulties changing representations and encountered most difficulty interpreting CAS output.

It is important not to look at these results on their own but to see them in the context of the course. Two key factors affected these results: the mathematical topic being studied at the time of each survey and the Personal aspect of each student’s use of CAS. Note again that the survey asked students to reflect on their CAS use on that day. The mid-course mathematics topics involved calculus that led on from the work on functions earlier in the course so the expressions to be entered and interpreted were similar. Students gained familiarity through practice. At the end of the course the topic being studied was trigonometry. This required entering some different expressions, learning more syntax and thinking about the structure of these new expressions in order to Author them correctly. This sequence in the mathematics topic placed varying demands on the students’ use of CAS. The new syntax and command sequences required for these later topics could account for the increase in the number of difficulties students reported on the Late-course Technical Difficulties survey. It seemed that students’ Technical difficulties related to their lack of familiarity with the idiosyncrasies of program syntax, lack of familiarity with the structure and key features of the mathematics, and lack of practice using the program.

Secondly, the Judicious Use of CAS survey, along with the case studies (Chapter 7), suggest that most students progressed in their CAS use, moving beyond just calculating answers to exploring families of functions and variations on set questions. The Technical Difficulties survey (see tables 5.6 and 5.7) offered ‘not applicable’ as an alternative even though on each occasion when data was collected students were expected to work on problems for which these difficulties could have arisen. Across the class the number of times ‘not applicable’ was recorded reduced slightly on the Mid-course survey but halved on the Late-course Survey. This suggests that students were making wider use of CAS at the end of the course. In addition, students were encouraged to initiate CAS use, not just employ DERIVE when it was suggested or when guidance was provided. Later in the course more students did initiate CAS use but they often made errors in the process and so reported more difficulties than those who had only followed instructions. Students initially reported some difficulties using CAS, but with practice this number was reduced and their confidence increased. This
increase in confidence with CAS encouraged students to initiate CAS use themselves, and in the process they encountered more difficulties. Given more time it would seem reasonable to expect that the number of difficulties experienced would again decrease but the increase, decrease cycle may repeat each time a new mathematics topic is introduced.

Table 6.7

*Percentage of students experiencing low, medium or high levels of ‘total’ technical difficulty, at different stages of the course*

<table>
<thead>
<tr>
<th></th>
<th>Low difficulty</th>
<th>Medium difficulty</th>
<th>High difficulty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Early</td>
<td>33</td>
<td>60</td>
<td>7</td>
</tr>
<tr>
<td>Mid</td>
<td>54</td>
<td>46</td>
<td>0</td>
</tr>
<tr>
<td>Late</td>
<td>25</td>
<td>50</td>
<td>25</td>
</tr>
</tbody>
</table>
Early-course

Mid-course

Late-course

Figure 6.5  Number of students reporting each level of total Technical Difficulty

Early-course

Mid-course

Late-course

Figure 6.6  Number of students reporting each level of difficulty using program syntax

Early-course

Mid-course

Late-course

Figure 6.7  Number of students reporting each level of difficulty changing representations

Early-course

Mid-course

Late-course

Figure 6.8  Number of students reporting each level of difficulty interpreting CAS output
Table 6.8
Percentage of students experiencing low, medium or high levels of difficulty, with each technical element, at different stages of the course.

<table>
<thead>
<tr>
<th>Difficulty level</th>
<th>Program syntax</th>
<th>Changing Representations</th>
<th>Interpreting</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Medium</td>
<td>High</td>
</tr>
<tr>
<td>Early</td>
<td>8</td>
<td>83</td>
<td>8</td>
</tr>
<tr>
<td>Mid</td>
<td>44</td>
<td>56</td>
<td>0</td>
</tr>
<tr>
<td>Late</td>
<td>27</td>
<td>27</td>
<td>46</td>
</tr>
</tbody>
</table>

**Judicious Use of CAS survey**

Judicious Use of CAS is evidenced by the three Common Instances: the ‘manner’ in which CAS use is approached; the degree to which discriminating functional use is demonstrated; and the extent to which pedagogical use is undertaken. These terms are described in detail in Chapter 3B. Evidence of these features of CAS use was taken from students’ self reported responses to items on the Judicious Use of CAS survey which accompanied the Technical Difficulties survey on three occasions during the course (see table 5.8 and 5.17). The number of students employing each manner and purpose of use Early, Mid and Late-course is summarised in table 6.9. On the basis of students’ responses to these survey-items and classroom observations, students’ functional and pedagogical use has been categorised. In addition, the ‘manner’ in which students approached the use of CAS has been classified (Table 5.11). While at any one time a student would be working in only one manner, even during one class session their use might vary according to their response to different questions. For this reason some students were classified as using CAS in more than one manner.

It can been seen from the results displayed in table 6.9 that throughout the course all students were able to use CAS for functional purposes, but only about half the students were discriminating in their functional use. This did not improve during the course. However, as the course progressed, more students undertook higher levels of pedagogical use of CAS. While early in the course most students only followed given instructions line by line, ‘exploring problems if the computer was suggested’, by late in the course this had been reversed. At the Late-course stage most students were using CAS to explore ‘other than when directed on the same topic’ and/or ‘other mathematics not on today’s topic’. The students undertook extended pedagogical use of CAS.
Judicious Use of CAS would be demonstrated by students who use CAS in a strategic ‘manner’, but ‘manner of use’ results show us that early in the course at least one third of students used CAS in a passive or random manner. This suggests that they were not confident or did not know how to use the CAS systematically. By Mid-course most students could use it in a strategic manner when this was suggested and when goals were given. This improvement was sustained and by the end of the course more than half of the students would initiate the use of CAS in a strategic manner.

Figure 6.9 represents the percentage of students using CAS in different manners, at different times in the course. Again the same students could be operating at more than one level, but the overall trend is clear. By late in the course most students only operated at the ‘higher levels of use’. The graph (figure 6.9) shows that by late in the semester 90% of students used CAS in a directed strategic manner and 60% would initiate strategic use. Late in the course less than 10% showed evidence of passive or random use.

Table 6.9
Judicious Use of CAS
Number of students demonstrating each manner* or purpose of CAS use

<table>
<thead>
<tr>
<th>Purposes</th>
<th>Early-course</th>
<th>Mid-course</th>
<th>Late-course</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n = 16</td>
<td>n = 14</td>
<td>n = 13</td>
</tr>
<tr>
<td>Functional</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Non discriminating</td>
<td>10</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>Discriminating</td>
<td>6</td>
<td>7</td>
<td>5</td>
</tr>
<tr>
<td>Pedagogical</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Minimum</td>
<td>8</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Limited</td>
<td>6</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Extended</td>
<td>2</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

Manner

<table>
<thead>
<tr>
<th>Score</th>
<th>Manner</th>
<th>Early-course</th>
<th>Mid-course</th>
<th>Late-course</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Passive</td>
<td>6</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>Random</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3.</td>
<td>Responsive</td>
<td>16</td>
<td>14</td>
<td>9</td>
</tr>
<tr>
<td>4.</td>
<td>Directed strategic</td>
<td>10</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>5.</td>
<td>Self initiated Strategic</td>
<td>3</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

* Each student may approach CAS use in more than one manner.
Each student may demonstrate more than one manner and purpose.

*Figure 6.9* Percentage of students demonstrating ‘strategic manner of use’ (2.2.1) and other lower levels of ‘manner’ of use, at different stages of the course.

While the group as a whole progressed in their manner of CAS use not every student showed improvement. Based on the Judicious Use of CAS Survey and the teacher/researcher’s observation students were scored on their ‘manner of use’ by summing the numerical score (tables 5.11 and 6.9) assigned to their highest and lowest use. Figure 6.10 shows that while 50% of the students were operating at a score of 3 early in the course only one person (shown as an outlier on the boxplot) was operating at this level by mid-course. By late in the course 50% were operating above a score of 8 but one student had dropped back. A split in the group between those using CAS in a minimal way and those using it strategically is clearly illustrated in the bar chart depicting Late-course ‘manner of use’ ratings (figure 6.11).
Judicious Use of CAS Common Instances 2.2.2 and 2.2.3 (functional and pedagogical use) were scored using a 6-point scale described in Chapter 5 (table 5.9). Most students improved on this score during the course. This change is illustrated by the graphs presented as figures 6.12 and 6.13. (Note that these graphs are based on data for students who were assessed on all three occasions.) Figure 6.12 shows a shift from 31% of the group scoring more than 3 to 70% scoring more than 3 and 54% scoring 6, the maximum score. The bar charts in figure 6.13 also shows a split in the group. It indicates that a few students made discriminating functional use of CAS but only undertook minimum pedagogical use.
**Figure 6.12** Boxplots of combined functional (2.2.2) and pedagogical (2.2.3) use scores at different stages of the course.

**Figure 6.13** Distribution of combined functional (2.2.2) and pedagogical (2.2.3) of use scores at different stages of the course.

**Reflection on CAS use: Post-course CAS evaluation**

The Post-course Survey was administered at the end of the course. Students were asked to reflect on their use of CAS and indicate their response to a number of statements, using a 5-point Likert scale. The percentage of students giving each response is summarised in table 6.10. Their responses give some insight into the attitude element of the Personal aspect of students’ Effective Use of CAS.

The results show that most students were positive about the use of CAS. Their responses suggest that they saw it as having both functional and pedagogical benefits. No student disagreed that DERIVE could be made to do the working out for mathematics problems, that is all students acknowledged that CAS had functional benefits. Eighty-four percent of students agreed or strongly agreed that DERIVE had...
helped them to see patterns in Mathematics. CAS had been of pedagogical benefit to them. No student disagreed that DERIVE had helped them to see patterns and most students (sixty-seven percent) agreed or strongly agreed that using DERIVE had helped them to understand mathematics. There was, however, a significant minority (seventeen percent) who disagreed that the use of DERIVE had actually helped them to understand mathematics. These results suggest that most, but not all, students had developed a positive attitude towards the place of CAS in this course.

Further evidence of this positive attitude is shown in the fact that most students initiated the use of CAS to explore mathematical ideas. Sixty-seven percent of the group agreed that they tried out ideas using DERIVE and seventy-five percent said they used DERIVE for more than just those times when they were instructed to use it. While most students were prepared to use the facilities of CAS to explore mathematics and were generally positive in their attitude towards CAS, a small group maintained a negative attitude. They used CAS as little as possible within the course and did not, as a result of the structured experiences, decide to initiate use of the program.

**Attitude**

Information from the Judicious Use of CAS survey and the Post-course survey and interview were used to gauge students' Attitude toward the use of CAS for doing and learning mathematics. Students' responses to items on the Judicious Use of CAS survey show us that initially, seven students would have preferred not to use CAS, and two students persisted in this attitude throughout the course. Observation of these students, and discussion with them, revealed that unless they were directed to use it or they were 'stuck' and thought it would be of advantage in solving a problem they preferred not to use CAS. These two students opted to use their graphical calculators or only pen and paper, instead of CAS wherever possible. There was a tendency for the 41% of students who used graphical calculators at school to be less positive about the use of CAS than the 59% who did not have this experience. This trend is illustrated by figure 6.15. This is also discussed further in the case studies in Chapter 7 and general discussions in Chapter 8.

Students' Late-course Attitude was scored on the basis of their responses to four items on the Post-course survey (Table 6.10). These results are illustrated by figure 6.14. It
shows that by late in the course the group was divided by students' attitude toward the use of CAS; three students had a noticeably lower score than the rest of the group. As mentioned above, some students retained a preference for use of graphical calculators. There was also evidence, from an anecdotal student comment, that some of those students with a negative attitude towards CAS, who chose to operate at minimal levels of manner and purpose, in fact had few difficulties using CAS. They preferred not to use CAS as they gained satisfaction from doing the mathematics by hand or by using a graphical calculator.

**Figure 6.14** Distribution of Late course attitude scores

**Figure 6.15** Post-course attitude scores by graphical calculator use
Overview of Effective Use of CAS results
This is a summary of key finding from the class results for Effective Use of CAS.

- Students’ level of Technical difficulty fluctuated. There were new difficulties to be overcome each time new mathematics was introduced that required new commands and syntax.

- Practice, of CAS use, was important for reducing Technical difficulty.

- The Personal aspect of Effective Use of CAS impacted on the Technical aspect. Some students with a negative attitude towards the use of CAS avoided its use.

- Students who had used graphical calculators at school showed a strong preference for this familiar technology over learning to use CAS.

- The majority of students improved in their Judicious Use of CAS. By Late-course most students demonstrated discriminating functional and extended pedagogical use of CAS. They used it to explore a range of examples using multiple representations, and approached this use in a strategic manner.

- Students with a negative attitude seldom made extended pedagogical use of CAS.

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Results from four instruments examining both Algebraic Insight and students' Effective Use of CAS

Post-course CAS evaluation
The Post-course Evaluation also provided information about the interaction of students’ Effective Use of CAS and factors that might affect their Algebraic Insight. In the previous section it was reported that most students considered that using DERIVE had helped them to see patterns and gain understanding of mathematics. ‘Seeing patterns’ is important for identifying structure and key features, this is a fundamental skill for both Algebraic Expectation and ability to Link Representations.

Students were also asked to respond to statements about their use of CAS and their level for confidence in working in the various areas of mathematics dealt with in the course.
These responses are recorded in table 6.10. Statistically significant correlations were found between students' responses to 'I try out ideas using DERIVE' and 'DERIVE has helped me see patterns' ($r=0.76$, $p=0.004$). Those students who strongly agreed that they used DERIVE to try out ideas also strongly agreed that its use had helped them to see patterns. Perhaps these students tried out ideas because they had found that DERIVE helped them. There is no basis here for knowing which is the dependent variable, but this result does suggest that encouraging the students, through guided exercises, to use CAS for pedagogical purposes (to explore families of functions through looking at multiple examples and/or multiple representations) was successful. Most students took the initiative to ‘try out ideas’ themselves and found it assisted them to see patterns in mathematics. There was also a positive correlation between these statements and having 'greater confidence with graphs' ($r=0.59$, $p=0.042$; $r=0.69$, $p=0.012$). Those who ‘tried out ideas’ or found DERIVE helped them to ‘see patterns’ reported an increased confidence with graphs and it was interesting that those who reported greater confidence with graphs reported greater confidence with functions ($r=0.71$, $p=0.010$).

From these results we might have expected to see significant improvement in students’ ability to Link Representations and as noted above there was indeed an improvement in the class mean score for this Aspect of Algebraic Insight. This result is illustrated in figure 6.16 which shows the number of students who achieved various scores on each aspect of Algebraic Insight, both Pre-course and Late-course. These results are coded by students’ response to the ‘I tried out ideas using DERIVE’ statement. For this purpose the students were divided into two groups: ‘Strongly Disagree, Disagree or Neutral’ and ‘Agree or Strongly Agree’. The Algebraic Expectation results for the ‘negative’ group were mixed, -some improved some did not-, while the ‘positive’ group all improved. The Linking Representation results for both groups were mixed but on average the ‘positive’ group achieved higher results on this aspect than the ‘negative’ group.

The students who achieved the greatest improvements in Linking Representations were those who had studied lower level year 12 mathematics and who had not previously used graphical calculators. A general observation by the teacher/researcher was that these students worked with both the symbolic and graphical facilities of DERIVE. Others, however, who had previous experience with graphical calculators, continued to
use these or to use the DERIVE graphs and numeric approximations but made little use of the symbolic facility of CAS. This observation was confirmed in the comments of students like Jocelyn and Rachael who are quoted in the next section.

Table 6.10
Post-course CAS evaluation survey results: percentage of students giving each response

<table>
<thead>
<tr>
<th>Statement</th>
<th>SD</th>
<th>D</th>
<th>N</th>
<th>A</th>
<th>SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Derive has helped me see patterns in Mathematics</td>
<td>0</td>
<td>0</td>
<td>17</td>
<td>67</td>
<td>17</td>
</tr>
<tr>
<td>I find using Derive helps me to understand Mathematics</td>
<td>0</td>
<td>17</td>
<td>17</td>
<td>50</td>
<td>17</td>
</tr>
<tr>
<td>DERIVE can be made to do the working out for maths problems.</td>
<td>0</td>
<td>0</td>
<td>25</td>
<td>58</td>
<td>17</td>
</tr>
<tr>
<td>I can use DERIVE to check every step of a problem.</td>
<td>0</td>
<td>50</td>
<td>33</td>
<td>17</td>
<td>0</td>
</tr>
<tr>
<td>A DERIVE plot will tell me everything I need to know about a function.</td>
<td>8</td>
<td>42</td>
<td>42</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>I only use DERIVE if the instructions tell me to.</td>
<td>0</td>
<td>75</td>
<td>17</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>I try out ideas using Derive</td>
<td>0</td>
<td>17</td>
<td>17</td>
<td>67</td>
<td>0</td>
</tr>
<tr>
<td>Compared to my confidence with functions before this unit, my confidence now is much greater</td>
<td>0</td>
<td>17</td>
<td>42</td>
<td>33</td>
<td>8</td>
</tr>
<tr>
<td>Compared to my confidence with calculus before this unit, my confidence now is much greater</td>
<td>0</td>
<td>0</td>
<td>42</td>
<td>58</td>
<td>0</td>
</tr>
<tr>
<td>Compared to my confidence with graphs before this unit, my confidence now is much greater</td>
<td>0</td>
<td>25</td>
<td>25</td>
<td>50</td>
<td>0</td>
</tr>
<tr>
<td>Compared to my confidence with trigonometry before this unit, my confidence now is much greater</td>
<td>8</td>
<td>42</td>
<td>25</td>
<td>25</td>
<td>0</td>
</tr>
</tbody>
</table>

- SD Strongly disagree  D Disagree  N Neutral  A Agree  SA Strongly agree
- Percentages rounded
Precourse algebraic expectation

80.80 57.70 50.00 42.30 34.60 23.10 7.70 -7.70 -11.50 -26.90

Count 3.0 2.0 1.0 0.0

I try out ideas using DERIVE

SD, D, N
A, SA

Precourse Linking representations

88.90 72.20 33.30 11.10 0.0 -11.10 -16.70 -22.20 -33.30 -38.90

Count 3.0 2.0 1.0 0.0

I try out ideas using DERIVE

SD, D, N
A, SA

Late-course algebraic expectation

88.50 61.50 57.70 50.00 46.20 26.90 19.20 11.50

Count 3.0 2.0 1.0 0.0

I try out ideas using DERIVE

SD, D, N
A, SA

Late-course Linking representations

88.90 77.80 55.60 33.30 11.10 0.0

Count 3.0 2.0 1.0 0.0

I try out ideas using DERIVE

SD, D, N
A, SA

Figure 6.16 Pre and Late course Algebraic Expectation and Linking Representation results coded by response to the statement ‘I try out ideas using DERIVE’.

Post-course general interview

The Post-course interviews were held in the last week of the course and consisted of a series of mathematical questions, reported under the Algebraic Insight section of this chapter, and a series of questions which asked students’ opinions on the use of DERIVE for learning mathematics. Students volunteered to be interviewed either alone or in pairs. Details are described in Chapter 5. Some key quotes from students are recorded here as they relate to particular issues; further quotes are included in the case studies in Chapter 7.

From the Post-course survey it was clear that most but not all students felt that CAS could be used as a tool to help them both to do and learn mathematics. The interview provided an opportunity for students to expand on this, and it was at this stage that the divide between those who used graphical calculators and those who did not became most apparent. Some students were quite open about the fact that they only used
DERIVE when they felt obliged to do so, in laboratory classes for questions which specified that CAS was to be used. The interview also gave students an opportunity to discuss the role that they saw CAS as having in learning mathematics. There was strong agreement that they liked to see and do ‘simple’ examples by hand before using CAS.

All 15 students interviewed agreed that CAS had been of some use to them. The extracts of interviews included below are typical of the students’ comments. They appreciated being able to link the algebraic and graphical representations and the speed and accuracy facilitated by CAS. Students were asked to talk about their experience with DERIVE: about where they had found it most helpful and any difficulties they had experienced. Below are some representative responses. First students appreciated being able to use multiple representations and many examples in a short time.

Yvonne: I’ve found it useful for heaps of stuff… At the start it was hard to figure out what to use, because the language is all new to me, because I haven’t done any of this stuff… I didn’t do [Mathematical] Methods or Specialist [Mathematics] in Year 12 but I’ve found DERIVE helps for the integration and definite integrals… Doing the graphs has helped me to see what the functions look like and from there you can kind of get you’re head around it a little bit better.

Stephen: I like DERIVE for especially proving the theory we’ve learnt in class, I mean it’s ok to say ‘even functions, odd functions’, stuff like that, DERIVE helped, because you could actually see like an even function get squarer between negative 1 and [positive] 1, stuff like that if you can plot, graph it you can see the difference how it changes…..

Jocelyn: I like DERIVE to find the derivative quickly.

Catherine: It gives you, a visual picture while you’re still actually working out the equation and then you can visualise, you can draw a graph of a certain part of the equation to work out whether you’re doing it right or not, and if it doesn’t look like what it should look like, then you’ve got to say well, you must be doing something wrong and change it around, fix it.

All agreed that they liked to use it to find answers quickly for questions they perceived to be difficult or time consuming. Most felt that being able to do numbers of examples quickly and easily, and swap between algebra and graphs was to their advantage.

Louise: Cut down time, draws graphs heaps easily and more accurately.

Annette (interviewed with Louise) : And also things that you’d take a page to work out and it just takes one line. I found that easier.

Louise: It made it easier to see patterns in things, to understand the patterns and figuring out how to find a basic formula.
Emily: It’s helped with really long questions and stuff like that, and when you can’t find exact values and things like that, like when it does it for you.

Some students preferred to do what they perceived to be easy questions in their head or with pen and paper. They were discerning in their use of CAS. Others were anxious about trusting this new technology. Several students preferred to use their more familiar graphics calculator and only use CAS as a last resource.

Stephen: I do the easy stuff, like you would expect, in my head, like simple derivation, but when you’ve got x to a negative power and with square root signs, it seems easier just to plug it straight into DERIVE. …You understand the basic theory but when it comes to applying it to something that looks pretty nasty, I’ve found you just put it straight into DERIVE, save yourself a lot of effort.

Annette: I usually start off with pen and paper. …Sometimes I use Pen and Paper and find you can do it on the computer then I prefer the computer but I don’t start on the computer first or I get confused.

Jocelyn: I don’t like it I find it really daunting I reach for my graphics calculator before I use DERIVE.

Rachael: Same here I use my Graphics Calculator all the time because I know everything on that because I used it all through year 11 and year 12 whereas DERIVE sometimes I just sit there and all class and I don’t use it. I just use my calculator.

Most students admitted that they initially had difficulties using CAS. They reported these difficulties as getting used to the vocabulary of the commands and their symbols. Although the CAS use surveys suggest that these problems persisted, this was not the students’ impression. They each said that it became easy to use.

Annette: I think it’s helpful once you’ve learned.

Louise (interviewed with Annette): Yes you need to know your basics.

Annette: Then it’s helpful.

Interviewer: Do you mean once you know some basic DERIVE or once you know some basic maths?

Annette: DERIVE, once you know how to do it it gets easier and easier. I think it’s a good thing to use but if you don’t know how to use it that well… It just confused me at the start and I’d prefer to do it on pen and paper but now that I know a few basic skills it has been helpful. It’s just a matter of practice.

Stephen: It took me at least the first six weeks to get my head around it, now, I’m not too bad, I can change scales and zoom in easily, in and out. I’ve found integration was made ten times easier by DERIVE.
Jennifer: Well, I found it pretty hard to figure out how to use DERIVE at the start, but once you get used to it it’s good.

**Weekly worksheets**

Students had weekly worksheets, which consisted of a series of problems, to complete. They were asked to code each question according to the ‘tool’ they had used, how difficult they found the problem and their perceived understanding (see table 5.16). The aim of the coding was to help link students’ use of CAS to their Algebraic Insight, but students did not always remember to add a code to each question. Without this information only general observations can be made. On the whole, students used pen and paper only if they were confident about the mathematics and thought that they could do the question easily (according to their comments to the teacher or coding the problem as ‘easy’). All students used technology for questions requiring graphs, even straight-line graphs. Those students familiar with graphical calculators showed a strong preference for their use over DERIVE.

Students seemed to use CAS to do or check all derivatives and integrals except simple polynomials. Students made use of CAS, initially as suggested, but then on their own initiative, they used it to do series of further examples and find patterns. After their first experiences of using CAS to explore, students came to expect to find general patterns and learned to express general relationships using letters as variables and parameters.

**Course Assessment - Examinations**

Students’ course assessment (Total course mark) was based on the two Examinations and their Worksheets. Since this was an additive, linear model there was a high correlation between the Total course mark and the scores for each examination, (correlations 1 and 2, table 6.11). Correlation of Total course mark with a range of ten other variables (Algebraic Insight aspects and elements of Effective Use of CAS) showed that students with strong Algebraic Expectation skills were advantaged. There was evidence of a positive relationship between both Pre- and Late- course Algebraic Expectation and Total course mark (correlations 3 and 4, table 6.11). It was also interesting to note that students’ Late-course ‘attitude’, ‘manner of use’ and ‘functional and pedagogical use’ scores did not provide evidence of a relationship between these Common Instances of the Personal aspect of Effective use of CAS and their course marks (correlations 5-7, table 6.11).
Table 6.11
Correlation of Total course mark with other measures

<table>
<thead>
<tr>
<th>Variables</th>
<th>Correlation coefficient</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Total course mark and Mid-course examination</td>
<td>0.810</td>
<td>p = 0.000</td>
</tr>
<tr>
<td>2 Total course mark and Post-course examination</td>
<td>0.840</td>
<td>p = 0.000</td>
</tr>
<tr>
<td>3 Total course mark and Pre-course Algebraic Expectation</td>
<td>0.684</td>
<td>p = 0.007</td>
</tr>
<tr>
<td>4 Total course mark and Late-course Algebraic Expectation</td>
<td>0.580</td>
<td>p = 0.023</td>
</tr>
<tr>
<td>5 Total course mark and Late-course manner of use</td>
<td>0.040</td>
<td>p = 0.897</td>
</tr>
<tr>
<td>6 Total course mark and Late course functional and pedagogical use</td>
<td>-0.217</td>
<td>p = 0.477</td>
</tr>
<tr>
<td>7 Total course mark and Late course attitude</td>
<td>-0.140</td>
<td>p = 0.649</td>
</tr>
</tbody>
</table>

Examinations indicate links between Algebraic Insight and Effective use of CAS

The examinations offered a good opportunity to observe students’ use of CAS since they were held in a computer laboratory where each screen could be viewed from the back of the room. The teacher/researcher and a second observer recorded students’ use of CAS and this was later matched with the written answers in their examination script books. The following is a brief example of the interaction between Algebraic Insight and Effective use of CAS observed in the Examination. More examples with details for specific students are described in the case studies in Chapter 7. It was interesting to note that the availability of CAS during the examinations did not camouflage students' weaknesses in Algebraic Insight. Nor was there any difficulty in ranking or categorising students' mathematical understandings from examination responses. For example, on the Mid-course examination question 2 (figure 6.17) some students successfully made the substitution by hand in part (a), because they were confident and found it fastest to do the successive multiplication by three in their head while some students used CAS to check their answer. Although some students successfully used CAS syntax to Define
the function \( f(x) \) they could not do parts (b) and (c) because they did not identify the structure of the composite functions. They did not understand that having Defined \( f(x) \) they only need to type in \( f(x+1) \), or \( f(a+b) \) that is replace \( x \) with the new ‘object’ to be subject to the given function rule.

2. \( f(x)=3^x \)
   
   (a) Evaluate \( f(5) \)
   
   (b) Show \( 3f(x) = f(x+1) \)
   
   (c) Show \( f(a+b) = f(a)f(b) \)

*Figure 6.17 Mid-course examination question 2*

**Links between instruments: Algebraic Insight and Effective Use of CAS**

As part of the exploration of links between Algebraic Insight and Effective Use of CAS, Early and Late Algebraic Insight Quiz data was correlated with Early and Late Technical difficulties data. Algebraic Insight was considered at overall and aspect levels while total and elements of Technical difficulties were considered. Looking at these 52 correlation coefficients for statistical evidence of links between Algebraic Insight and Effective Use of CAS produced few statistically significant results. The small sample size also precluded categorizing data and using Chi-squared tests as too many cells had expected frequencies of less than five. In addition interpretation of results is necessarily vague, as it is not possible to determine which was the dependent variable. For these reasons the case studies in the next chapter are especially important for making sense of this data, at least at the level of the individual.

Despite these limitations, correlating the elements of Technical difficulty and the aspects of Algebraic Insight produced some results worth noting. Pearson’s correlation coefficient and associated p-values for selected results are reported in tables 6.12 and 6.13. As indicated in Chapter 5 and earlier in Chapter 6, higher scores on Algebraic Insight measures indicate greater ability or understanding but higher Technical difficulty scores indicate the student was experiencing more problems. Table 6.13 lists the statistically significant correlations while table 6.11 lists those variables that showed
least association. (Since the \( r \) statistic measures linear association scatter plots were checked. These showed no evidence of a non-linear association either.) Correlation 1 in this table suggests that results from the Pre-course Quiz and Early-course Technical Difficulties Survey were not related. There was no trend relating students’ Algebraic Insight and their initial difficulties using DERIVE. Correlations 2 and 3 indicate that early technical difficulties could not be used to predict later achievement in Algebraic Expectation. Early difficulties with the program did not stop it being a useful tool for students’ learning. Perhaps the most interesting results are the last two: these suggest that early technical difficulty in changing representations did not affect the level of improvement students achieved in gaining insight into Linking Representations. Similarly, early difficulty interpreting CAS output did not affect improvement in Algebraic Expectation (as measured by Late-course Quiz results minus Early-course Quiz results). This suggests that there was no general trend in the way students responded to the hurdle of learning to use new technology and software.

Table 6.12

Variables that this evidence suggested were unrelated

(All \( p \)-values >0.8 for test of null hypothesis \( \rho=0 \))

<table>
<thead>
<tr>
<th>Variables</th>
<th>Correlation coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1   Early-course total Technical difficulty And Pre-course Quiz</td>
<td>-0.089</td>
</tr>
<tr>
<td>2   Early-course total Technical difficulty And Late-course Algebraic Expectation</td>
<td>0.002</td>
</tr>
<tr>
<td>3   Early course difficulties using program syntax And Late-course Algebraic Expectation</td>
<td>-0.004</td>
</tr>
<tr>
<td>4   Early course difficulties changing representations And Change in Linking Representations</td>
<td>0.057</td>
</tr>
<tr>
<td>5   Early course difficulties interpreting CAS output And Change in Algebraic Expectation</td>
<td>-0.003</td>
</tr>
</tbody>
</table>

Some relationships were evident. Table 6.13, correlation 1, indicates that one Early-course Technical difficulty was associated with change in Algebraic Expectation. Those students who, in the early part of the course, had fewer difficulties changing representations were likely to make a greater improvement in Algebraic Insight.
Correlations 2 to 4 (table 6.13) indicate that the more technical difficulties (using syntax and changing representations), a student reported the higher their score for Algebraic Expectation. This relationship does not have an obvious explanation. The effect of some students’ preference for using their graphical calculators over DERIVE graphs was considered. Figure 6.18 shows that, on average, these students did experience more technical difficulties late in the course and on average scored higher on Late-course Algebraic Expectation. Perhaps students with high Algebraic Expectation skills were trying out new ideas in CAS. Perhaps students with low Algebraic Expectation skills were trying hard and to learnt to use CAS as an aid. The case studies may suggest some answers.

Table 6.13

<table>
<thead>
<tr>
<th>Variables</th>
<th>Correlation coefficient</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Early-course difficulties changing representations and Change in Algebraic Expectation</td>
<td>-0.674</td>
<td>p = 0.006</td>
</tr>
<tr>
<td>2 Late-course technical difficulties and Late-course Algebraic Expectation</td>
<td>0.624</td>
<td>p = 0.030</td>
</tr>
<tr>
<td>3 Late-course difficulties using program syntax and Late-course Algebraic Expectation</td>
<td>0.660</td>
<td>p = 0.019</td>
</tr>
<tr>
<td>4 Late-course difficulties changing representations and Late-course Algebraic Expectation</td>
<td>0.723</td>
<td>p = 0.008</td>
</tr>
</tbody>
</table>

Figure 6.18 Boxplots illustrating results for Late-course difficulties and Algebraic Expectation on the basis of prior use of graphical calculators.
Overview of links between Algebraic Insight and Effective Use of CAS

This is a summary of key findings that relate Algebraic Insight and students’ Effective Use of CAS

- Students who used CAS to ‘try out ideas’ felt that it helped them to see patterns in mathematics. This led to greater confidence, especially with graphs and functions. This ‘feeling’ is collaborated by results showing their improvement in Linking Representations.

- All students liked being able to use DERIVE to save time and to do what they perceived to be ‘hard’ questions.

- All preferred to do ‘easy’ questions by hand.

- The distinction between ‘easy’ and ‘hard’ questions was individual, depending on skill and confidence.

- Students liked to use familiar technology, in this case graphical calculators, where possible.

- Students’ initial technical difficulties did not appear to affect their long term improvement in Algebraic Insight

Overview of Class Results

From these Algebraic Insight Quiz and Interview class results we can see that, overall, there was a significant improvement in students’ Algebraic Insight. This improvement applied to both aspects, Algebraic Expectation and Linking Representations. However, within the aspect of Linking Representations there was less improvement in Linking symbolic and numeric representations than in other elements. There was a clear indication that weak students made the most progress on the measures used.

There was an overall trend for improvement in Effective use of CAS during the course but different elements showed different trends. The elements of the Technical aspect, as measured by the type and number of difficulties students reported, showed clear improvement to mid-course. The late-course measures, however, showed an increase in
problems for some students. This seemed to be related to students’ changing familiarity with both the mathematics and the CAS. The elements of the Personal aspect of Effective Use of CAS were clearly affected by students’ previous experience of using technology for mathematics, especially graphical calculators, and their level of preference for ‘learning by hand’. Most students’ Judicious Use of CAS improved. They progressed from tentative, non-discriminating functional use to initiating use of CAS for extended pedagogical purposes. While most students responded positively to the use of CAS, a few students with a negative attitude persistently avoided its use. Even these students admitted that they did appreciate its use for some ‘hard’ problems.

The impact of Algebraic Insight on Effective use of CAS or vice versa cannot be described in a neat, clear cut manner. This data showed no relationship between initial difficulties using CAS and Early Algebraic Insight, or overall improvement in Algebraic Insight. The data did suggest that higher Algebraic Expectation was associated with residual Technical difficulties. An explanation for this result will be sought in the detail of the case studies.

The graphs used in this chapter illustrate the fact that the group studied here was not only small but also quite diverse. It is therefore important to look in detail at some of these individuals in order to make some sense of the patterns and apparent associations in the class data. For this reason, Chapter seven consists of a series of case studies. Details about each selected student’s progress through the course will be described and then the outcomes of those investigations will be summarised and linked to the class findings which have been presented and discussed in this chapter.
CHAPTER 7

INDIVIDUAL CASE STUDIES

This chapter explores and enriches the findings made by examining overall class results. The statistical analysis presented and discussed in Chapter 6 indicated improvement in the class average for both Algebraic Insight and Effective Use of CAS, but it was also clear that not all students responded in the same way to the course taught using CAS. Students experienced different levels of Technical difficulty. Investigation of the Personal aspect of Effective Use of CAS indicated important individual differences in students’ attitude toward the use of CAS and, perhaps as a consequence, their Judicious Use of CAS. It is therefore proposed, in this chapter, to review individuals’ results and summarise the rich data obtained from class observations, worksheets, interviews and examinations through case studies of representative students. For this purpose seven case studies have been selected, and will be presented in this chapter. The results from this detailed investigation of individual students will be discussed with the class results in Chapter 8.

Purpose of the Case Studies

The aspects of learning mathematics in a CAS environment, which are examined in this study, are affected by many different factors. While, as stated in Chapter 6, there was a statistically significant improvement in the class mean results, individual results varied and there seemed to be anomalies in the data: for example a decrease, followed by an increase, in Technical difficulties using CAS. The case study material reviews the detail of individual students’ progress. It looks at their results on the various instruments together with specific examples that highlight either progress or problems at each stage. The material also includes the students’ responses to the use of CAS throughout the course.

The results, and examples of responses to particular items and/or work during the course, will be examined for each student, keeping in mind the three broad questions to be discussed in Chapter 8:
1. Did Algebraic Insight change during the course taught using CAS? If so, what was the nature of that change?

2. Did Effective Use of CAS change during the course taught using CAS? If so, what was the nature of the change?

3. Was there evidence of links between Effective Use of CAS and Algebraic Insight?

Choice of Case Studies

Seven case studies were selected for inclusion in this chapter on the basis of three criteria. The first two were the students’ pre-course background in calculus (measured by previous school and tertiary mathematics) and ‘general computing skills’ as reported on the Background Survey. These criteria were used because the focus of the course was an introduction to calculus and it could be expected that those students who already had a background in calculus would be successful in each dimension of the course. In addition, in the researcher’s experience, students who have studied calculus usually have some experience in analysing graphs of functions and could therefore be expected to have better skills in Linking Representations. Similarly those with very good computing skills could be expected to experience few difficulties in making the transition to using a computer program which was new to them. The third criterion, for determining the selection of students for these case studies, was attitude towards the use of CAS for learning mathematics. The researcher expected that students with a positive attitude toward the use of CAS in the course would be more likely to show improvement in Algebraic Insight and demonstrate Judicious Use of CAS. They are perhaps, the best ‘test cases’ for seeing if using CAS promotes Algebraic Insight because they will at least consistently try it. Figure 7.1 gives an impression of the relative position of each student selected with respect to their background in calculus, general computing skills and attitude.

The first four students, Stephen, Louise, Jennifer and Yvonne were selected because each represented a different combination of calculus background and computing skills and exhibited a positive attitude towards the use of CAS. Three other students, Greg, Jocelyn and Samuel, were included because their backgrounds in calculus and computing skills were similar to Stephen and Louise (see figure 7.1) but each expressed a negative attitude towards the use of CAS. These three each said that they would prefer
to work with their graphical calculator or by -hand rather than use CAS. No other students in the group could be ‘paired’ in this way with Jennifer or Yvonne. Please note that all students have been allocated fictitious names for the purpose of this thesis and that some aspects of their backgrounds are reported in broad rather than specific terms in order to preserve their anonymity.

Figure 7.1 Background of students selected for Case studies

Format of case studies
Each of the seven case studies begins with a statement of that student’s place in the selection criteria. This is followed by a graphical figure summarising their results on measures of aspects of Algebraic Insight and the elements of Effective Use of CAS which have been discussed for the class as a whole in Chapter 6. Algebraic Insight and Effective Use of CAS are summarised at different levels because the class results suggested little information was obscured by considering aspects of Algebraic Insight but, at this level, the effect of attitude on Effective Use of CAS would not be evident. There was also evidence that Algebraic Expectation and ability to Link Representations have different effects on each element of the Technical aspect of Effective use of CAS.
For this reason students’ Effective Use of CAS will be analysed at the element level. A list of the abbreviations used in the graphical summaries is provided in table 7.1.

The rest of each study has a chronological approach. One paragraph summarises the students’ background in mathematics and use of technology along with some personal details to help build the reader’s image of the student. Next, students’ results and examples of their Algebraic Insight and Effective Use of CAS are presented and discussed for Early-course, Mid-course and Late-course. Each study finishes with an overview relating the information from that individual to the broad research questions.

Descriptors Used
Some further explanation of the measures and words used to describe features of students’ Algebraic Insight and Effective Use of CAS will now be given. A summary of these measures and descriptors, with links to relevant instruments described in Chapter 5 and results from Chapter 6, is found in Appendix 4.

Levels of Algebraic Expectation and ability to Link Representations
In these Case studies students are referred to as showing ‘poor’, ‘good’ or ‘very good’ Algebraic Expectation or ability to Link Representation. As outlined in Chapter 3, levels of Algebraic Insight are relative and based on the Common Instances that are appropriate for the particular stage of mathematical learning. As detailed in Chapter 5, the items for the Algebraic Insight Quizzes and Algebraic Insight Interviews were based on Common Instances which might be reasonably expected from students who had already studied year 11 mathematics. Since the group had a spread of both mathematical background and ability, a simple way of judging a student’s level of Algebraic Insight was to compare their scores to the class averages. These judgements were validated by observation and analysis of both weekly worksheets and examination scripts.

Technical
Comments on students’ level of Technical difficulty were based on the responses given to the Technical Difficulties survey, students’ comments and the teacher/researchers observations.
Positive Attitude

Since positive attitude is one element of the Personal aspect of Effective Use of CAS, students will be classified as generally having either a positive or negative attitude. Students who persistently made negative comments about using CAS were considered to have a negative attitude. The attitude score, calculated by adding response scores from individual items (as described in Chapter 5) shows these students as scoring less than half the possible total points. Therefore, in this study an attitude score of less than 50% will be considered to be indicative of a negative attitude. Students who made positive comments and were observed to engage willingly in the use of CAS are described as having a positive attitude. These students also scored more than 50% on the survey items that related to attitude.

Judicious Use of CAS

Students’ Judicious Use of CAS is discussed in terms of the relevant Common Instances. Their levels of use are described using the terms set out in Chapter 5 (see tables 5.9 and 5.11). As a summary term some students are referred to a showing a ‘high’ or ‘low’ level of Judicious Use of CAS. These terms are relative and based on the Common Instances examined. The measurement shown in the graphical summaries was constructed by equally weighting each student’s Judicious Use of CAS (2.2.2 and 2.2.3) score against their ‘manner’ of use (2.2.1) score, adding the results, and then transforming these to percentages so that a common scale could be used for the visual representation.

Table 7.1
Abbreviations used in summary figures 7.2, 7.5, 7.8, 7.10, 7.16, 7.17 and 7.22.

<table>
<thead>
<tr>
<th>Abbreviation use in figure</th>
<th>Full name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expectation</td>
<td>Algebraic Expectation</td>
</tr>
<tr>
<td>Link</td>
<td>Ability to Link Representations</td>
</tr>
<tr>
<td>Syntax</td>
<td>Fluent use of program syntax</td>
</tr>
<tr>
<td>Changing</td>
<td>Ability to systematically change representation</td>
</tr>
<tr>
<td>Interpretation</td>
<td>Ability to interpret CAS output</td>
</tr>
<tr>
<td>Attitude</td>
<td>Positive attitude</td>
</tr>
<tr>
<td>Use of CAS</td>
<td>Judicious Use of CAS</td>
</tr>
</tbody>
</table>
CASE STUDY 1  JENNIFER

Jennifer was selected because her prior background in calculus was poor and she judged herself as having limited general computing skills. In addition her attitude towards the use of CAS was positive throughout the course.

![Graphical Summary](image)

*Figure 7.2 Summary of Jennifer’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.*

**Background**

Jennifer was an organised, conscientious student who attended all classes unless ill and handed completed work in on time. Jennifer had studied a low-level year 12 mathematics at school. This included no calculus and only limited algebra. She had used a scientific calculator but neither a graphical calculator, nor a computer, for mathematics. She rated her general computing skills as between OK and Good. So on the criteria for selecting students for these CAS studies (figure 7.1) Jennifer was classified as low in both background in calculus and general computing skills. Throughout the course observations suggested that Jennifer had a positive attitude to trying to use CAS for learning mathematics. This assessment was confirmed by her responses to the survey questions related to attitude.

The graphical summary presented in figure 7.2, above, shows Jennifer’s individual result on measures for aspects of Algebraic Insight and elements of Effective Use of CAS. These measures have been discussed for the class, as a whole, in Chapter 6.
At the beginning of the course

Early-course Algebraic Insight

Jennifer did not attend an Algebraic Insight Interview at the beginning of the course. On the Pre-course Quiz (scored –100% to 100%, class means Algebraic Expectation +17, Linking Representations +12) she scored 23 on the Algebraic Expectation items and –33 on those measuring Linking Representations. This included 6 items for which she had ‘no idea’ (2 items testing element 1.2, 2 items testing element 1.3 and one item each from 2.1 and 2.2). In terms of Algebraic Insight it could be seen as more useful that Jennifer knew that she had ‘no idea’ rather than an entrenched incorrect idea.

Her early worksheets show that she was able to work in general terms with letters (suggesting that she knew basic properties of operations) and made links between graphical and symbolic representations. These early worksheets provided some guided discovery exercises, and some exercises where students were encouraged to use CAS to explore families of curves.

Early-course CAS use

On the Early-course Technical Difficulties Survey, Jennifer indicated that she had no problems Authoring simple expressions or interpreting the results of the soLve command but did have some problems with all the other suggested actions. From the beginning Jennifer embraced the use of CAS for both functional and pedagogical purposes. She demonstrated non-discriminating use of CAS, choosing to use it to find answers to easy, hard and time consuming questions. At this stage she undertook limited pedagogical use of CAS, choosing to use its facilities to explore variations on the set topic (without direction).

The researcher’s observation notes record that Jennifer had trouble with syntax for commands. It often took her several attempts to bracket an expression correctly because she did not identify the structure. Jennifer also had some difficulties linking representations (graphical and symbolic) due to an inability to choose or set an appropriate scale. There was a strong ‘ah-ha’ reaction when an appropriately scaled graph was demonstrated. While Jennifer had trouble setting useful scales for graphs her observations from the screen and translation to sketch graphs were excellent. Her work was carefully and clearly presented.
**Early Course summary**

**Algebraic Insight:**

- Algebraic Expectation: Low
- Linking Representations: Very low

**Effective Use of CAS:**

- Technical: Difficulties: Quite a few with less automated commands and zooming. These difficulties impeded fluent use of syntax and ability to systematically change representation.
- Personal: Attitude: Positive: some uncertainty but prepared to keep trying
- Judicious Use of CAS: Manner of use sometimes passive, but as she gained confidence became strategic: directed and self initiated.
- Functional use non-discriminating
- Pedagogical use limited.

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**In the middle of the course**

**Mid-course Algebraic Insight**

On the Mid-course Algebraic Insight Quiz Jennifer’s Algebraic Expectation score had improved to 50 but her Linking Representations remained very low, down to –44 (class means Algebraic Expectation +25, Linking Representations +23). She still indicated ‘no idea’ for 3 questions (one from each of 1.2, 1.3 and 2.1.).

On the Mid-course examination (included as Appendix 1.2) question 1 required students to set up, solve and graph a simple linear model. Jennifer could not formulate the problem. She recognised that the relationship was linear but that was all. Her attempt at a graph showed a lack of Algebraic Insight. She correctly plotted a starting point but had an incorrect gradient. In question 2 Jennifer correctly substituted a value into the given function but, despite correctly defining the function in DERIVE, could not deal with the composite functions. She did not identify the structure of these expressions and did not replace \( x \) with the new object. In question 5 Jennifer correctly identified the key features of symbolic representations of functions that would cause translation of graphs but she did not correctly identify features indicating dilation.

On her worksheets Jennifer initially had difficulty identifying the key features of rational functions. She had problems identifying their structure, especially identifying factors, however, after seeing lots of examples she showed that she was able to make the
links between the symbolic and graphical forms and began to correctly anticipate patterns.

In the question illustrated in figure 7.3 students were asked to find the area between the two curves. Jennifer only partly linked the graphical and algebraic representations. While she had correctly found the points of intersection and correctly labeled the graph she failed to describe the area to be found by linking this to an appropriate order for the terms in her expression \( \int((1-x^2)-x^2)dx \)

Figure 7.3 Shows Jennifer’s failure to link graphs with symbols

Mid-course use of CAS

On the day of the Mid-course Technical Difficulties Survey Jennifer indicated that she had no problems Authoring simple expressions in DERIVE. She encountered a problem with a more complicated expression that required brackets to identify the structure and had a number of difficulties with syntax and sequences of commands. When linking representations Jennifer found difficulty setting the graph scale and zooming to see key features. She also had a number of difficulties interpreting screen output to conventional mathematics. Jennifer made extensive use of CAS. She used it functionally but was not discriminating in her use. Despite her Technical Difficulties Jennifer undertook extended pedagogical use of CAS, she initiated exploration of mathematics. She worked well at exploring patterns when this was suggested and took initiative to extend this to further variations.
**Mid Course summary**

**Algebraic Insight**
- Algebraic Expectation: OK but still has difficulty with use of letter symbols
- Linking Representations: Improving but still weak

**Effective Use of CAS**
- Technical: Difficulties: More difficulties but initiating more use of CAS. Difficulties with all three technical elements.
- Personal: Attitude: Positive despite technical difficulties
  -Judicious Use of CAS: Improved
  -Manner of use was strategic- both directed, and self initiated
  -Functional use was non-discriminating
  -Undertook extended pedagogical use

**At the end of the course**

*Late-course algebraic insight*

Post-course Algebraic Insight quiz showed improvement from the early course quiz with Algebraic Expectation score of +42 and Linking Representations, +11. (Post-course class means were Algebraic Expectation +19, Linking Representations +16.) Jennifer still gave ‘no idea’ for three items: one element 2.1, two element 2.2s.

Jennifer took part in a late course interview. She showed strong skill in recognising conventions and basic properties and so scored well on Algebraic Estimation but still only scored less than half the points available for Linking Representations. At the Interview, in response to question 1 (questions included as Appendix 2.5), a ‘think of a number and follow given processes’ question, Jennifer suggested just doing a second example to check whether the result would be the same regardless of her initial number. When the question was repeated she suggested finding a general rule and did so using algebra and letters. She seemed comfortable working with letters as variables. Question 4 asked students to identify the structure of the expression by linking to the commonly used symbols for independent and dependent variables x and y. Jennifer identified the form of the relationship as linear (see figure 7.4) and identified the key '+'2' in f(p) indicating the position of the graph on the vertical axis but omitted the letter parameter from the scale choosing specific numbers 1, 2, 3, 4 instead of a+1, a+2, a+3, a+4.
Late-course CAS use

On the Late-course CAS Use Survey Jennifer indicated that she was still having difficulties, especially with syntax and graph windows. However she was using CAS both to find answers and to explore mathematics beyond variations of that day’s problems. Jennifer’s Judicious Use of CAS had improved. She made discriminating use of CAS and undertook extended pedagogical use.

Post-course Examination

Jennifer demonstrated a high level of Effective Use of CAS during the Post-course examination. The observer’s notes suggest that she experienced very few Technical difficulties and had sufficient Algebraic Insight to identify the structure of expressions she wished to Author, and to interpret the screen output.

Post-course reflections on CAS use

On the Post-course Evaluation Survey (included as Appendix 2.9) Jennifer indicated that she agreed that DERIVE had helped her to see patterns and to understand mathematics. She had used it to try out ideas and her confidence with functions, graphs and calculus had improved. She was neutral to the idea that DERIVE could be made to work all problems and that a DERIVE plot would tell all she needed to know about a
function. Jennifer disagreed that she would only use DERIVE when instructed to do so and disagreed that it could be used to check every step of working in a problem.

For speed in doing a set of simple problems (see questions 1 to 7, Post–course Evaluation survey Appendix 2.9) Jennifer would prefer pen and paper for those problems she saw as easy and CAS for those she perceived to be harder. For success she would choose CAS all the time but for learning she would choose pen and paper except for the question she saw as hardest. Given a set of 10 similar questions (questions 8 to 14 Post-course Evaluation survey, Appendix 2.9) expressed using letters, Jennifer indicated that she would prefer to: use pen and paper for learning, and for speed and success in finding intercepts; choose DERIVE for speed and success for all other questions.

In the Post-course interview Jennifer said that she had found DERIVE hard to use at first. She had difficulty remembering the names of the various commands and translating conventional mathematics to CAS, but she especially liked the graphing facility of CAS. She liked to use pen and paper for questions she found easy but preferred CAS for ‘hard’ questions. Overall she said that she thought CAS was ‘good’ for doing and learning mathematics.

### Late Course summary

**Algebraic Insight**
- Algebraic Expectation: improved
- Linking between representations: improved still weak

**Effective use of CAS**
- Technical Difficulties: still similar, a number of problems, but these did not limit her use
- Personal: Attitude: Positive more confident
  - Judicious Use of CAS: Improved
    - Level of manner of use varied: responsive, directed and self initiated; strategic
    - Was becoming discriminating in functional use
    - Undertook extended pedagogical use
**Jennifer – an overview**

Jennifer’s Algebraic Insight improved during the course and she became more certain in her responses. Her Late-course scores for Algebraic Expectation were above the class average but her Linking Representations’ results remained well below the class average, despite the big improvement.

Jennifer experienced some difficulties in using DERIVE 2.55 throughout the course. She had rated her general computing skills as poor and at each stage of the course her level of difficulty with each technical element was higher than the class average. As the mathematics became more familiar in the middle of the course and she gained experience in using CAS in that context her level of difficulty decreased but increased again with new material. Despite these Technical difficulties her attitude towards the use of CAS was positive throughout the course. Jennifer progressed from learning the basics of the CAS program, through non-discriminating functional use, to using CAS to learn mathematics. She undertook extended pedagogical use of CAS. Not only did Jennifer use DERIVE when the set questions said ‘use CAS...’ but she took initiative to increase her own understanding of problems. Sometimes she used CAS in a strategic manner.

Jennifer appreciated and made extensive use of the graphing capabilities of DERIVE. We see the benefit of this experience reflected in her improved ability to Link Representations, notably symbolic form with shape and key features with graph position. It seems that Jennifer’s moderate Algebraic Expectation skills combined with her positive attitude allowed her to benefit from the use of CAS despite the difficulties she had learning to use the actual CAS program.
CASE STUDY 2  YVONNE

Yvonne was selected because her prior background in calculus was poor but she judged herself as having very good general computing skills. In addition her attitude towards the use of CAS was positive throughout the course.

![Figure 7.5 Summary of Yvonne’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.](image)

**Background**

Yvonne viewed herself as a poor mathematics student. It was not uncommon to hear her refer to herself as ‘dumb at maths’. She displayed mathematics anxiety, expecting to have difficulties with each topic, saying ‘I won’t be able to do this’. Yvonne had studied a low-level mathematics at year 12 and had used a scientific calculator for mathematics at school but no other technology. However, Yvonne was confident about her general computing skills, rating them as very good. Later observations confirmed both her confidence and good computing skills.

**At the beginning of the course**

**Early-course Algebraic Insight**

Yvonne’s Pre-course Algebraic Insight Quiz and Interview suggested that she was weak in both aspects of Algebraic Insight. On the Quiz Yvonne scored –8 for Algebraic...
Expectation and –22 for Linking Representations. The possible scores range was from –100% to 100% and Yvonne’s results were both well below the class average (class means Algebraic Expectation +17, Linking Representations +12). There were seven out of the 22 items to which she responded ‘No idea’ (one element 1.2, three element 1.3, two element 2.2 and one element 2.2). Yvonne’s responses in the Algebraic Insight Interview confirmed that she had poor Algebraic Insight.

Yvonne understood that letters may be used to represent numbers but thought a different letter should represent each number. For example when asked to find 4 consecutive, positive, odd numbers whose sum is 40 (Interview question 2, Appendix 2.4) she wrote:

\[40 = x + y + z + a \text{ (all odd numbers).}\]

Yvonne had difficulty both identifying the structure in Interview question 4 (Appendix 2.4) and linking these symbols to the numeric representation. This question required students to write out the elements of tables that would match given ‘rules’. Yvonne had difficulty substituting numbers and when the pattern should have been extended to expressions with parameters (part (ii)) Yvonne did not identify the structure and so squared the \(b\) but not its coefficients. She was confused by the use of \(y\) instead of \(x\), as a variable, in part (iv). The transcript of the interview records that she said ‘But we don’t know what \(y\) is’. This reflects her lack of recognition of the conventions of mathematics and in particular the meaning of symbols (1.1.1).

Yvonne’s response to the last Pre-course Algebraic Insight Interview question, which asked her to link \(g(x)=x^2+4x+21\) to its graphical representation, indicated that she could relate the form of the symbolic expression to the shape but not the position of the graph.

Yvonne’s lack of Algebraic Insight was also reflected in her early worksheets. When asked to prove whether functions were odd or even, according to the given definition, she chose to illustrate with examples using numbers rather than work symbolically.

When using CAS to examine compound functions Yvonne merely copied answers from the screen without considering restrictions to the domain which were necessary in order to perform simplifications. She also had problems linking numerical and symbolic representations of a function. Given two points on a straight line Yvonne had difficulty finding the equation to the line.
Early-course CAS use

On the early-course Technical Difficulties survey, Yvonne indicated that she had few difficulties using DERIVE’s built-in commands and menus, but she wrote a comment saying that she would expect to have problems using syntax for commands that she had to type in. On the early-course Judicious Use survey Yvonne indicated that she used CAS functionally without discriminating between type or difficulty of questions. At this stage she undertook minimal pedagogical use of CAS. She only followed line by line directions.

The researcher’s class observation notes reported that in week 1 Yvonne did not use CAS because she felt that she ‘didn’t know what to do’. She preferred to use the familiar pen and paper. By week 3 she was asking questions in class about the mathematics but not the program. Yvonne still did not know what she wanted the program to do. Classroom observation notes record that as Yvonne started to use DERIVE she had problems moving between conventional mathematics and the CAS. She had problems with bracketing expressions and syntax when entering expressions and then difficulty using the CAS output to answer the set questions.

When she merely copied answers from the computer screen, as described above, Yvonne used CAS unthinkingly. She was unsure about what it was doing for her and how to interpret the output.
Early Course summary

Algebraic Insight:

- Algebraic Expectation: Limited understanding of the meaning of symbols, difficulty identifying structure or key features.
- Linking Representations: sometimes able to link symbolic form to shape but problems with other levels of links.

Effective Use of CAS:

- Technical: Difficulties with all three elements: entering expressions, interpreting output and setting appropriate graph windows.
- Personal: Attitude: Positive, keen but did not know where to begin
  - Judicious Use of CAS: Manner of use varied from Passive and Random to Reflexive (Inhibited by poor Algebraic Insight)
  - Limited non-discriminating functional
  - No pedagogical use unless clearly instructed.

In the middle of the course

Mid-course Algebraic Insight

Unfortunately Yvonne did not take the Midcourse Algebraic Insight Quiz but her Midcourse Examination script indicates both improvement and some residual problems with Algebraic Insight. Her response to question 1 (see figure 7.6) suggests that she formulated the problem numerically, linked this to a graphical representation and then to a symbolic representation. This ability to identify shape and key features is an improvement.
In question 2, however, given \( f(x) = 3^x \) and asked to find \( f(5) \), Yvonne wrote:

\[
 f(5) = 3^x \text{ then } 5 = 3^x .
\]

This suggests that Yvonne did not understand the meaning of the symbols. Her understanding was not even sufficient to allow her to Author the expression in DERIVE and obtain an answer using CAS. She did, in fact, attempt to do this, but did not use the correct syntax to define the function. She Authored \( f(x)=3^x \) instead of \( f(x):=3^x \), then tried \( 5=3^x \) and tried to soLve this obtaining \( 5=e^{1.09861x} \). It is important to note that she had sufficient Algebraic Expectation to know that this was wrong. Yvonne did not record this answer on her examination scripts and after the test she commented that she could not ‘get this to work’.

In her response to question 5 (figure 7.7) Yvonne showed that she was not able to identify the structure of the expressions and link this to the position of the graph.
Figure 7.7 Mid-course examination question 5, showing Yvonne’s inability to link the symbolic expression to the position of the graph.

In question 6 Yvonne was asked to find the equation to a straight line, given two points. While this was similar to question 1 and required linking numbers to a symbolic representation Yvonne tried to follow a rule this time. She found the gradient but did not know how to find the $y$ intercept. This was not a problem in question 1 because $(0,1000)$ was given in the initial information.

Question 7 asked the students to solve $5.7^{10} = 2^n$. Yvonne used CAS functionally, and obtained the right answer, but without understanding the process involved. She commented later “I had no idea about question 7. I just put it in the computer and used solve”. Observations, recorded during the Examination, showed that she did not immediately identify the structure of the expression and so took several attempts to bracket the expression correctly. Her initial solution was in exact form and she was able to convert this to approximate form to obtain a decimal answer.
**Mid-course use of CAS**

On the Mid-course CAS Use Survey Yvonne indicated that she had few problems using DERIVE. She still felt that she would have had trouble using syntax and sequences of commands but did not need to use those in that day’s class. She used CAS to find answers and explore, when this was suggested; exploring variations on set problems and answering hard or time-consuming questions. Her functional use had become discriminating and had expanded to undertaking limited pedagogical use.

Observations made in class suggest that Yvonne usually worked with Stephen (case study 4). She would either watch him use the CAS or do as he instructed. In addition Yvonne was starting to try things out for herself. She explored variations of set problems and called the teacher to ask ‘Is this an example of…?’ When she found patterns or made correct conjectures she was very pleased and expressed confidence.

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**Mid Course summary**

**Algebraic Insight**
- Algebraic Expectation: Improvement in meaning of symbols and identifying key features, problems with identifying structure.
- Linking Representations: Making more links between both graphs and numbers and symbols

**Effective use of CAS**
- Technical: Difficulties: Fewer difficulties but still problems entering expressions and interpreting.
- Personal: Attitude: Positive, trying to use CAS herself
  - Judicious Use of CAS: Manner of use wide ranging, from passive to Self-initiated strategic
  - Functional use becoming discriminating
  - Undertook limited pedagogical use.

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**At the end of the course**

**Late-course algebraic insight**

Yvonne’s responses to the Post-course Algebraic Insight Quiz and Interview showed some improvement in understanding the meaning of symbols, recognising key features and linking between symbolic and graphical representations of functions. Yvonne scored 19 for Algebraic Expectation in the Quiz this time and 56 for Linking
Representations (Post-course class means were Algebraic Expectation +19, Linking Representations +16). Yvonne was more certain of her answers; this was indicated in her increased choice of the ‘definitely’ options. Her results were average for Algebraic Expectation, and above average for Linking Representations. The interview confirmed that Yvonne made significant improvement in both aspects of Algebraic Insight but still had problems with each aspect.

Classroom observations suggest that Yvonne’s problems with mathematical conventions in algebra stemmed from a lack of understanding and skill in arithmetic. There were certainly residual problems with the various uses of symbols in algebra: for example, she asked for help in finding an answer for $\frac{\pi}{2.5}$. She did not recognise $\pi$ as a number.

Late-course CAS use

Responses on the Late-course CAS Use Survey suggested that Yvonne had few problems using DERIVE. On the day of the survey the only problems she encountered were one using brackets correctly and one setting an appropriate graph window. She would use CAS for any question, even simple items that she could have done in her head. This was largely due to her lack of confidence in her ability but also because she liked using CAS. For some problems Yvonne certainly needed the help of CAS but for others she used it to check her mental mathematics.

Post-course reflections on CAS use

Yvonne strongly agreed that using CAS had helped her to see patterns and to understand mathematics. It had helped her to work out answers. She was neutral in her response to statements which suggested that DERIVE could be used to check every step or would tell her all she needed to know about a function. She disagreed with the statement which said that she would only make use DERIVE if so instructed. Yvonne indicated that her confidence had increased in all areas of mathematics studied during the course.

When asked about the tool she would prefer to use for various simple questions (see Appendix 2.9, question 1 to 7) Yvonne was clear that for linear functions she would prefer to use pen and paper for speed, success and learning. For other functions she would prefer to use CAS for speed and success but for learning was still inclined
towards pen and paper. For multiple questions, similar to the first set but expressed in general form (see Appendix 2.9 question 8 to 14), Yvonne would be more likely to choose DERIVE for some learning, too. Yvonne said that ‘DERIVE was handy even though it takes a little getting used to’.

**Late Course summary**

**Algebraic Insight**
- Algebraic Expectation: poor to moderate: improved identifying key features.
- Linking Representations: poor to moderate: improved in linking symbols and graphs

**Effective Use of CAS**
- Technical: Difficulties: Few, still some syntax difficulties related to brackets (structure)
- Personal: Attitude: Positive, gaining confidence
  - Judicial Use of CAS: Improved
    - Manner of use varied from responsive to some self initiated strategic
    - Demonstrated non-discriminating functional use.
    - Undertook extended pedagogical use.
Yvonne – an overview

Yvonne began the course with very poor Algebraic Insight and made significant improvement during the course, especially in Linking Representations. Not only did her results improve but she also showed increased confidence by giving definite answers to all items later in the course. However, despite these improvements, relative to what might be expected at this stage of mathematics study, Yvonne still had limited Algebraic Insight.

Despite starting the course with very good computing skills, Yvonne had difficulty using the CAS program at the beginning of the course. The source of her difficulty, however, was not a problem following commands or the menu structure but in making the links between conventional mathematics and CAS. In the early stages of the course she made little use of CAS because she didn’t know what she wanted the program to do and had difficulty identifying the structure of expressions and hence using brackets appropriately.

Initially Yvonne’s Algebraic Expectation skills were so low that they inhibited her use of CAS. Working as a pair with Stephen, who was quite strong mathematically, assisted her learning and she became confident to work on her own. As the course progressed Yvonne became confident to do mathematics with CAS. She developed positive strategies such as making conjectures, thinking about what she expected the answer or graph to be and learning from the experience of testing these ideas. She used CAS effectively to follow this style of learning in combination with the use of multiple examples and graphical representations.
CASE STUDY 3  LOUISE

Louise was selected because her prior background in calculus was good and she judged herself as having limited general computing skills. In addition her attitude towards the use of CAS was positive throughout the course.

![Figure 7.8 Summary of Louise’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.](Image)

**General background**

Louise missed a few classes and but did all set work. She lacked confidence in her mathematics but was happy to ask questions. In the later part of the semester Louise was distracted, her social life and discussion of parties dominated her conversations. Her questions, for example, ‘How do I find isotopes [asymptotes]?’ reflected her lack of attention to the details of mathematics. Louise had studied a middle level mathematics at year 12 and had used a scientific calculator but never a graphics calculator or computer for mathematics. Louise rated her general computing skills as only OK.

**At the beginning of the course**

*Early-course Algebraic Insight*

On the Early-course Algebraic Insight Quiz Louise scored only 15 for Algebraic Expectation and but 39 for Linking Representations. This result was below the class average for Algebraic Expectation but well above average for Linking Representations. (Class means Algebraic Expectation +17, Linking Representations +12). There was a
lack of confidence in her responses indicated by ‘possibly’ for four items (one 1.1, two 1.2 and one 1.3) and ‘no idea’ for 2 items (one 1.2, one 2.2). In the Algebraic Insight Interview Louise also showed poor Algebraic Expectation especially with the element of recognising conventions but she did make some more correct Links between Representations.

During the Interview, when asked if the result of the processes in a ‘think of a number’ problem (question 1 see Appendix 2.4) would always be true regardless of the number used Louise just asked if she should ‘do another one’, -that is, try a second test integer. Even when prompted she had no notion of using letters as variables. When asked to find 4 consecutive, positive, odd integers that add to 40 Louise said ‘I have no idea. You’d have to sort of just do it’. In response to question 3, an area problem with letters use to designate dimensions, Louise said that she would ‘need to know what \( x \) is’. This statement confirms that she did not grasp the role of letters as variables; however she was able to give logical answers to question 4, linking a rule to a table. Here she identified the structure, made the link to a tabular representation and correctly used letter variables. The final question asked students to link the symbolic form of a quadratic with the associated parabola. Louise was able to link symbolic form to general shape and \( y \) intercept but not key features to position or details of shape.

On her worksheets Louise showed that she was able to work with symbols. She took a general approach but did not identify key features or structure. For example, Louise failed to recognise the basic properties of the operation of division when she cancelled factors like \((x-2)\) without restricting the domain of the function.

Classroom observation records note that Louise had difficulty describing links between representations. Louise had difficulties Linking between Representations because she did not use graphs with appropriate scales that would show key features of shape or position. This seemed to be due to both lack of Algebraic Insight and some difficulty using CAS. She lacked strategies which would inform her expectations of what the graph might look like and tried initially to use CAS without thinking, just trusting that the Plot which ‘appeared’ would be adequate.
Early-course CAS use

On the Early-course CAS Use Survey Louise indicated that she had no difficulties with Authoring expressions, using brackets, or interpreting screen mathematics to conventional mathematics. She did, however, have a problem working out an appropriate graph window, zooming to see key features and making a sketch copy of graphs. In addition she had some problems entering more complex commands which required a sequence of commands or syntax, or interpreting the results of the soLve command.

From the beginning of the course Louise placed great trust in CAS, using it in both functional and pedagogical ways. When it was suggested, she used CAS to find answers and to explore the set mathematics questions. She initiated use of CAS to explore variations on set problems and to find answers to problems she saw as being time-consuming. There was evidence that she made discriminating functional use and undertook limited pedagogical use of CAS.

Louise’s worksheets showed that she made of use of CAS for both graphs and algebra. In these problems she was using graphs to explore patterns and algebra to find answers (eg simplify or solve). Classroom observation showed that she would try things out to see what happened. Her manner was random rather than strategic. She would say ‘I’m not sure what I’m doing but…’

<table>
<thead>
<tr>
<th>Early Course summary</th>
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<tbody>
<tr>
<td><strong>Algebraic Insight</strong></td>
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<tr>
<td>Algebraic Expectation: Poor</td>
</tr>
<tr>
<td>Linking Representations: Moderate</td>
</tr>
<tr>
<td><strong>Effective Use of CAS</strong></td>
</tr>
<tr>
<td>Technical: Difficulties: with more complex command syntax and using graphs (changing representations).</td>
</tr>
<tr>
<td>Personal: Attitude: Positive but uncertain how to use CAS, didn’t expect to ‘think’ about its use.</td>
</tr>
<tr>
<td>Judicious use of CAS: Manner of use varied from random to self initiated strategic</td>
</tr>
<tr>
<td>Demonstrated discriminating functional use.</td>
</tr>
<tr>
<td>Undertook limited pedagogical use.</td>
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</tbody>
</table>
In the middle of the course

Mid-course Algebraic Insight

On the Mid-course Algebraic Insight Quiz Louise scored 15 on the Algebraic Expectation items and 67 on the items relating to Linking Representations (class means Algebraic Expectation +25, Linking Representations +23). Louise still lacked certainty in her responses, answering ‘possibly’ for 4 items (three 1.1 and one 1.2 items.)

On the Mid-course examination Louise showed some Algebraic Insight when linking the symbolic and graphical representation of a function and correctly using a table of values. Other areas of Algebraic Insight showed little sign of improvement. She did not link key features. She showed difficulty in recognising the conventions of mathematical notation when she rewrote the function notation $f(a+b)$ as $f.a+f.b$ ie multiplying by $f$. Despite having DERIVE available Louise graphed $t^n$ and $n'$ by first plotting points. She was not able to describe or compare these functions.

Louise’s solutions to worksheet problems showed that she was starting to see patterns, to look for structure and key features. However there was evidence of a lack of recognition of conventional notation and unthinking use of DERIVE. For example, she would copy answers exactly as they appeared on the screen rather than reverting to conventional notation, in figure 7.9 for example notice the use of the DERIVE symbol $\hat{e}$

\[
\int 675e^{1.78t} \, dt = \frac{33750e^{89/50}}{89} + c
\]

Figure 7.9 Copy of Louise’s solution showing direct transcription from CAS screen.

Louise realised that she had difficulty working from the screen to ordinary mathematics. On the Mid-course CAS use survey Louise indicated that this was the one difficulty that she was experiencing in using DERIVE. She indicated that she now used CAS for any or all questions and made extensive use of CAS to explore variations on problems and to try out things for herself.
Classroom observations suggested that her response to the availability of CAS was ‘to try it and see what happens’. As mentioned earlier, Louise was often distracted from her work. The consequences of this inattention when using CAS was that she seemed to type and copy without thinking. There would be errors on the screen, for example – even obvious mismatches between the mathematical expression on her page and that on the screen – but she would not notice because she was not thinking about the mathematics.

Mid Course summary

Algebraic Insight
Algebraic Expectation: some improvement especially with the use of letters as variables
Linking Representations: improvement when concentrating

Effective use of CAS
Technical: Difficulties: few- interpreting
Personal: Attitude: Positive but still expects too much of CAS, uses without thinking herself
Judicious Use of CAS: Manner of use was sometimes strategic directed but self initiated use was not strategic.
Demonstrated non-discriminating use of CAS
Undertook extended pedagogical use of CAS

At the end of the course

Late-course algebraic insight

On the Post-course Algebraic Insight Quiz Louise still showed poor Algebraic Expectation. She scored –12 on this section while maintaining her previous score of 61 for Linking Representations (Post-course class means Algebraic Expectation +19, Linking Representations +16). The Algebraic Insight Interview confirmed her strength in Linking Representations with a result well above the class average. While her Quiz score was poor, Louise’s interview suggested some improvement in Algebraic Expectation too. The Quiz alone did not show this incremental improvement in her understanding of the meaning of symbols. For example, with the ‘think of a number’ problem Louise still did not suggest using a letter variable to work out a general solution but she did expect that there would be a pattern. She was happy to work with letters to
find the area of a rectangle and simplified this to just one variable. She was content that the answer depended on a variable and could not be expressed as a number. Louise was able to identify structure and link tables to graphs. She was also able to link the quadratic rule given to the shape and position of the graph but not its maximum value.

On the Post-course examination (see Appendix 1.3) Louise identified the structure of expressions when she indicated the use of the quotient and chain rules. When using integration to find the area enclosed by a curve and the \(x\)-axis Louise made effective use of links between representations, linking the symbolic and graphical forms.

Louise did not identify the structure of some trigonometric expressions but clearly had some expectations of the key features of a trigonometric graph. For the example \(\frac{\sin 3x}{3}\)

Louise entered \(\sin 3x/3\) without brackets to identify that the process ‘sine of’ needed to be applied to the expression \(3x\) not just \(3\) and that \((\sin 3x)\) was to be divided by \(3\) not just \(x\). As a result Louise used DERIVE to plot \(\frac{\sin 3x}{3x}\), a linear function. She ‘zoomed’ in repeatedly then out, clearly aware that the graph did not look as she had expected but perhaps she was uncertain and lacked confidence that her expectation would be right. She showed no sign of checking her CAS syntax and so did not identify her error.

Late-course / Post-course CAS use

Louise was absent when the Post-course CAS Use Survey was administered.

At the Post-course interview Louise indicated that she had not found DERIVE difficult to use when she said:

I found using the computer a lot easier than I thought it would be. At secondary school we didn’t really use computers and that…[I had] No difficulties really. Once you get used to what different commands are doing and you put the right amount of brackets in and you got it to stop beeping at you…

Louise also said (at the interview) that “Sometimes you get it (DERIVE) to draw the graph for you and it’s up higher,” which suggests that she continued to have some problems choosing appropriate graph windows. This was because she still had problems with a Common Instance of Algebraic Insight element 2.1, linking key
features to position. A further section of the interview transcript shows that Louise was quite positive about the use of CAS. She preferred to do ‘simple’ problems in her head but harder problems and graphs with DERIVE.

Interviewer: Was DERIVE helpful?
Louise: Yes just cutting down on time...cut down time and it draws graphs heaps easier and more accurately

Interviewer: Did it help your understanding?
Louise: I can’t actually think of anything specific. It made it easier to see patterns in things, to understand the patterns and figure out a basic formula.

Interviewer: How do you prefer to do problems? What tools do you use?
Louise: I use a mix. I do really simple addition in my head but some problems like you couldn’t keep all the information (in your head). I prefer to use computer for graphing and when you want to see all the different equations.

Interviewer: Do you think using DERIVE had been helpful?
Louise: (emphatically) Yes I do actually ...but... you need to know your basics.

Late Course summary

Algebraic Insight:
- Algebraic Expectation: Some improvement but lost concentration.
- Linking Representations: Good for quadratic and linear functions. Some problems with less familiar functions.

Effective Use of CAS
- Technical: Difficulties: Few but bracketing expressions still a problem and changing to a graphical representation.
- Personal: Attitude: Positive, appreciates CAS
  - Judicious Use of CAS: Manner of use can be Self directed strategic but still often unthinking
    - Demonstrated non-discriminating functional use (unthinking).
    - Undertook extended pedagogical use of CAS.
Louise – an overview

Louise showed no improvement on the test score for Algebraic Expectation but further evidence from the Interview and examination suggests there was some improvement at least in her recognition of the meaning of symbols if not in other elements. She did however make clear progress in Linking Representations.

Louise had a positive attitude towards the use of CAS. She tended to use it with blind -trust, in an unthinking way. She had some difficulties using the program but very few which related to features of the program; most were in moving between conventional mathematics and CAS syntax.

Louise had particular difficulty with the syntax of expression because she lacked the Algebraic Expectation skills of identifying structure and key features. Her confidence in doing mathematics with CAS increased during the course but she still lacked certainty without CAS. Louise was content to rely on CAS blindly. She did not make conjectures and think about what the answer to a problem or the shape and position of a graph were likely to be; she had no expectation of results. Louise’ lack of engagement with the process and random manner of use meant that she often made errors. However, through exposure to many related examples she gained an understanding of the use of symbols to represent variables and formed some expectations of links between form of expression and shape of graph or pattern in a table.
CASE STUDY 4 STEPHEN

Stephen was selected because his prior background in calculus was very good and he judged himself as having good general computing skills. In addition his attitude towards the use of CAS was quite positive throughout the course.

![Graph showing Stephen's results](image)

**Figure 7.10** Summary of Stephen’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.

**General background**

Stephen was a conscientious student. His work was always thorough and he asked questions when he felt he needed clarification. Stephen had studied a higher level mathematics at year 12 some years before the other students. Stephen had used a scientific calculator at school but neither used of a graphical calculator nor a computer for learning mathematics. He rated his general computing skills as good.

**At the beginning of the course**

**Early-course Algebraic Insight**

Stephen performed quite well on both the Pre-course Algebraic Insight Quiz and Interview. Despite the negative penalty scoring of the Quiz he scored 81 for the Algebraic Expectation items and 89 for the items requiring Linking of Representations.
(Class means Algebraic Expectation +17, Linking Representations +12) His understanding was such that for four questions on which he lost marks he was in fact uncertain about his response. This indicates that he did not have entrenched incorrect ideas. (He indicated ‘no idea’ for one 1.2 item, possibly for one 1.3 item and two 2.1 items). The Algebraic Insight Interview confirmed his strong Algebraic Insight and demonstrated that he recognised conventions, in particular the use of letters to represent variables. He was the top student in the class on these measures.

In the Algebraic Insight Interview Stephen was asked to do the ‘think of a number’ problem (Appendix 2.4) and consider if the result would always be equal to 4 regardless of the initial number chosen. (Stephen had initially chosen the number 3 and calculated the answer to the given process to be 4). The transcript records that he continued like this:

Stephen: Will I always get that...so try again. Say 6, (here Stephen substitutes 6 into the process)... well based on that I got another 4.

Will I always get that....I could try doing it with x, so x minus 2 times x plus 2, plus 8 minus the square...that would be the algebraic way to work it out [he wrote this down while saying the same words]. That equals 4.

Interviewer: So will that always be true?

Stephen: I’m not sure to tell the truth.

Interviewer: So what did the x represent?

Stephen: x represents the number you first thought of, that would be what basically happens. You’d do your multiplication before your addition and subtraction anyway you have to make sure you add the eight before you take away that square.

Interviewer: So do you think it is always true?

Stephen: I’m not sure.

In this interview Stephen demonstrated that he understood that a letter may be used to stand for a number and recognised the conventions of symbols and operations but he lacked confidence in his result.

In question 4 (Appendix 3) Stephen’s response to part (ii) suggested that he did not link processes to objects, did not always identify structure in algebraic expressions. He noted that \( f(x) = x^2 \) and that \( x \) would take values 0, b, 2b, 3b but expected \( f(x) \) to equal \( b^2, 2b^2, 3b^2. \)

When asked to discuss the function \( g(x) = x^2 + 4x - 21 \) Stephen indicated the shape of the graph and said that it would be a parabola that intersected \( x=0 \) at \(-21\) on the \( y\)-axis and that it would be skinnier (than \( y=x^2 \)) because of the extra \( 4x \) term. He wanted to
factorise the expression in order to find the $x$-intercepts but could not remember how to do this. He knew he wanted ‘($x$ – a number) and ($x$ + a number)’.

On his work sheets Stephen indicated that for those questions which required recognising conventions, identifying key features and linking to tables he had chosen to use pen and paper and found doing them ‘OK’, he felt his understanding was ‘good’. He made use of DERIVE’s graphing facility for questions which linked symbolic to graphical representations. He was careful in his observations and accurately represented the DERIVE graph by a sketch graph. When working with functions that were new to him (exponentials and logarithms) his description of their graphs was detailed but he lacked confidence and indicated that he felt his understanding was only neutral.

Stephen showed many indications of strength in both aspects of Algebraic Insight. His careful approach and habit of looking for what the framework calls ‘key features’ was to his advantage.

_Early-course CAS use_

On the day of the Early-course CAS Use Survey Stephen reported many technical problems using DERIVE. He had problems every time he Authored an expression and sometimes had difficulty with the sequence of commands. Sometimes he had difficulty interpreting conventional mathematics to the required CAS syntax and vice versa. He also ran into some difficulties choosing appropriate graph windows and zooming to view special features of the graph. He commented ‘I still have a bit of trouble navigating the menu’. This may have been because DERIVE 2.55 was a DOS based program which did not have drop down menus with which students had already become familiar. For Stephen the difficulties in using CAS related more to the idiosyncrasies of the program than to the mathematics. As a student who had returned to study after a break he did not like to draw attention to himself. His awareness of his Technical difficulties would have been heightened by the fact the syntax errors were signalled both by an error message and a ‘beep’.

While Stephen reported that he had a lot of Technical difficulties in the early part of the semester the records of classroom observation do not record him asking any questions or requiring any help at all in learning to use the program. Stephen also reported that he
used CAS to find answers and explore mathematics when the teacher or text suggested this and that he also used it to find answers to questions that he classed as ‘hard’.

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<th>Early Course summary</th>
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<tr>
<td>Algebraic Insight</td>
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<tr>
<td>Algebraic Expectation: Good, some weakness in identifying structure</td>
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<tr>
<td>Linking representations: Good</td>
</tr>
<tr>
<td>Effective use of CAS</td>
</tr>
<tr>
<td>Technical: Difficulties: Many but dealt with these himself</td>
</tr>
<tr>
<td>Personal: Attitude: Positive most often content to use pen and paper</td>
</tr>
<tr>
<td>Judicious Use of CAS: Manner of use varied from Responsive to directed strategic</td>
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<tr>
<td>Demonstrated discriminating functional use.</td>
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<td>Undertook limited pedagogical use.</td>
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<th>In the middle of the course</th>
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<td>Mid-course Algebraic Insight</td>
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On the Mid-course Algebraic Insight Quiz Stephen scored 89 on those items measuring Algebraic Expectation and 56 on those items requiring Linking Representations (class means Algebraic Expectation +25, Linking Representations +23). He did make some errors on this second Quiz which he did not make on the Early-course quiz. There was no obvious explanation for this result.

His Mid-course examination script showed that he was able to work with symbols in a formula, for example to create and use a formula for the number of pies left in stock after \( t \) days. Stephen’s response to the question 2, shown in figure 7.11, indicated that he was unable to work structurally using a composite function. His response was to try substituting a value for \( x \) to try to find a pattern empirically. He made no attempt to work with \( a \) and \( b \) instead of \( x \). This item required both an understanding of the meaning and role of symbols (1.1) and the ability to view an expression structurally (1.2).
Figure 7.11 Example of Stephen’s difficulty with identifying structure.

In several questions that were expressed in general form, Stephen chose to substitute a sample set of values for the parameters. He then worked with this particular case to find evidence to enable him to answer the question. This approach is seen in figure 7.12 and was also evident in question 3 which required the students to match a series of algebraic expressions like 

\[ y = -a(x+b)^2 + c \]

Stephen wrote \( a=1, b=2, c=3 \).

He graphed the functions using these particular values for the parameters and based his answers on these specific, numeric results.

Stephen did provide evidence that he could think about an expression structurally when linking the symbolic form of an expression to the position of the graph of this function. Given the graph of \( f(x) \) he was able to correctly sketch \( f(x)+3, f(x-1), \) and \( 2f(x) \).

On his worksheets Stephen indicated that he had done the questions involving aspects of the meaning of symbols (1.1) using pen and paper. He had found these questions Very Easy and felt his understanding was Very Good. Questions that linked symbolic with graphical representations (2.1) were done using DERIVE’s graphing facility in combination with DERIVE algebra or pen and paper. He found these questions ‘OK’ and rated his understanding as ‘Good’. In contrast he indicated that he found other questions that involved identification of structure and key features (1.2, 1.3) Very Hard and felt his understanding was ‘Poor’. He used DERIVE algebra to help him do these
questions. His work shown in figure 7.12 below reflects his difficulties. Given \[ y = \frac{x + 1}{x - 1} \] he was asked to find a pattern for the \( n \)th derivative by examining the pattern of successive derivatives. In fact he identified the structure of the expressions but omitted the alternating sign and had difficulty representing the numeric pattern in the numerator in a general way as \((-1)^n \times 2n!\).

![Figure 7.12 Example of Stephen’s difficulty with identifying structure and key features](image)

Mid-course use of CAS

On the Mid-course DERIVE use survey Stephen reported fewer Technical difficulties. He said that in that day’s class he had had some problems using symbols and moving to an appropriate graph window and one problem interpreting CAS results to conventional mathematics.

At this stage of the course he used CAS for selective functional and limited pedagogical purposes. He used it to find answers and to explore the mathematics if this was suggested and to find answers to questions he considered hard or time consuming.
Observations made during the Mid-course Examination showed that he seemed confident in his DERIVE use but largely relied on pen and paper. He made use of DERIVE for graphs but did not use the algebra facility to help himself in questions like number 2 shown in figure 7.12 above. In a question which asked students to compare exponential and power functions he chose to use DERIVE to explore graphs but used pen and paper to create tables. This may be because in DERIVE 2.55 it was more difficult to create a table than draw a graph. Tables could not be created by using menu selections, instead, it required knowledge of the correct syntax. Stephen’s use of the graphing facility was not efficient. He kept just adding graphs to the graph window and not deleting any. Fortunately he still managed to interpret the information he required from the growing maze of lines.

Observation notes made during class sessions record that Stephen made strategic use of CAS. In week 7 he was observed using CAS to explore a family of functions. Stephen said:

*Guess \((f(x)^2)+2\) will have two solutions. I used DERIVE but I only got complex solutions. That’s why it doesn’t touch or cross the \(x\)-axis.*

He showed Algebraic Insight in expecting two solutions. He showed that he was able to interpret the DERIVE output and make a link between the algebraic and graphical representations. At this stage of the course Stephen was successful in exploring patterns in the symbolic representations of functions and linking them to the graphical representations. He was both pleased and confident.
**Mid Course summary**

**Algebraic Insight**
- Algebraic Expectation: Good but some difficulty identifying structure in expressions
- Linking Representations: Good and symbolic – graph links improving

**Effective Use of CAS**
- Technical: Difficulties: Fewer problems entering expressions and interpreting screen output but still some problem setting appropriate graph windows.
- Personal: Attitude: Positive but mostly content to use pen and paper
  - Judicious use of CAS: Manner of Use: Strategic, sometime self initiated but mostly directed
  - Demonstrated discriminating functional use.
  - Undertook limited pedagogical use.

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**At the end of the course**

**Late-course algebraic insight**

Stephen made some errors on the Quiz and his score dropped to 58 for Algebraic Expectation items and 56 for Linking Representations. However when re-scored and added to the interview results this still showed clear competence in both areas with 86% and 89% (Post-course class means were Algebraic Expectation +19, Linking Representations +16).

Stephen’s approach to answering questions in the Algebraic Insight Interview showed greater speed and confidence. In the ‘think of a number’ problem he looked at one numeric example as the question suggested then generalised correctly and explained his thinking clearly.

The second question asked students to find 4 consecutive, positive, even, integers whose sum was 40. Stephen approached this in a symbolic manner setting up the equation $x+(x+2)+(x+4)+(x+6)=40$ and showing that $x=7$. He demonstrated why it is not possible to find 4, consecutive, positive even numbers whose sum is 40. His ability to link between representations was shown in example 4(iv) (figure 7.13 below) where he linked form to shape (straight line) and identified key features (intercepts) as well as a possible section of the graph required.
Stephen’s Post-course Examination script showed evidence of lack of recognition of conventions and basic properties (1.1) when he included \( \ln \) in his answer. As can be seen in figure 7.14 below he applied the quotient rule and used DERIVE to find the derivative of \( e^{3x} \). He merely copied this answer from the screen without interpreting to conventional mathematics.

In question 2 (figure 7.15) Stephen showed that he could link symbolic and graphical representations of functions when he identified shape and key features.
Late-course CAS use

On the Late-course CAS Use Survey Stephen reported that he still had a lot of difficulties using the program. Essentially he made mistakes entering expressions, using brackets correctly, interpreting algebraic results and linking to the graph window. He had few problems with program commands or making sketch graphs from the screen. Observations made during classes suggested that at this stage of semester the mathematics was unfamiliar to Stephen. His problems with brackets suggest that he had difficulty identifying the structure of these ‘new’ expressions. He was making some errors as he learned to use new symbols and new mathematics but with his attention to detail he was also likely to be aware and report more difficulties than other students were.

He indicated that he was using CAS to find answers to hard or time-consuming problems and using the facility to explore mathematics if that was suggested.

In his Post-course Examination the observer noted that Stephen seemed confident in his use of CAS. He used it strategically and selectively.
Post-course reflections on CAS use

In reflecting on his use of DERIVE Stephen indicated that he Strongly Agreed that using CAS had helped him to see patterns, understand mathematics and workout answers. He was Neutral to the idea that CAS could be used to check every step of a problem and Disagreed both that CAS would tell you everything about a function and that he would only use it if instructed to do so. He Agreed that he used CAS to try out ideas and that its use had increased his confidence in functions, calculus and graphs.

When asked to consider the tool he would prefer to use under different circumstances Stephen chose pen and paper for speed in solving linear equations and finding intercepts and DERIVE for solving harder equations and comparing rational and logarithmic functions. For success he would choose DERIVE algebra for all problems but for learning he preferred pen and paper. Faced with 10 general examples he would prefer to use DERIVE for speed and success but pen and paper for learning except for comparing functions with their derivatives. For this he would prefer to use DERIVE’s graphing facility.

At the Post-course interview Stephen indicated that he found logarithms, limits and oblique asymptotes the most difficult sections of the course. ‘Derivatives were easiest.’

Of using DERIVE Stephen said:

It took me at least the first six weeks to get my head around it, now I’m not too bad. I can change scales and zoom in easily, in and out. I’ve found integration ten times easier by DERIVE. I’m just not happy to do it by hand, just like limits.

I like DERIVE especially for proving the theory we’ve learnt in class… DERIVE helped because… you could plot a graph and see the changes.

I do the easy stuff, like you would expect, in my head, like simple derivation but when you have $x$ to a negative power with square root signs it seems easier to plug it straight into DERIVE. You understand the basic theory but when it comes to applying it to something which looks pretty nasty I’ve found you just put it straight into DERIVE and save yourself a lot of effort.
Late Course summary

Algebraic Insight

  Algebraic Expectation : Good
  Linking Representations: Good

Effective Use of CAS

  Technical: Difficulties: Difficulties seen when using CAS for new mathematics with unfamiliar symbols
  Personal: Attitude: Positive, appreciated facility for exploring and checking
    Judicious Use of CAS: Manner of use was both Directed and self initiated strategic.
      Demonstrated discriminating functional use.
      Undertook limited pedagogical use, rarely extended use.
Stephen – an overview

At the beginning of the course Stephen showed that he had good skills in both Algebraic Expectation and Linking Representations. His initial high scores left little room for improvement on these test instruments and in fact he made some errors on later tests. Although his Algebraic Insight Quiz scores declined Stephen’s approach to questions both in the Algebraic Insight Interview and the Post-course Examination provides evidence suggesting consolidation and improvement in Algebraic Insight.

Initially Stephen made many errors using DERIVE 2.55. Although he said he found it difficult he did not ask for, nor need, help. He knew what he wanted the CAS to do but took some time to become familiar with the commands and syntax. If Stephen could do a problem fairly easily by hand then he did not use DERIVE. This meant that he was often under no pressure to improve his DERIVE skills. By mid-course he was experiencing fewer difficulties, but with unfamiliar mathematics later in the course he again had problems as he learnt both new mathematics and new CAS syntax. The level of difficulty Stephen reported at each stage of the course seemed in excess of that observed by the teacher/researcher. This may be explained by his attention to detail or his self-consciousness when the computer ‘beeped’ to signal an error. However, since Stephen preferred to work by hand, he did not have a lot of practice using CAS.

Stephen showed a high level of Judicious Use of CAS. He did many problems by hand because he found this to be faster and he enjoyed the mental exercise but he had no hesitation in supporting and supplementing this work by using CAS. His attitude towards CAS use was always positive despite his feeling that he had Technical difficulty. He made discriminating use of CAS for functional purposes and any difficulties he encountered only slowed him down a little, rather than causing him significant problems. In this case Stephen’s preference for by hand work rather than any lack of Algebraic Insight impeded his use of CAS.
CASE STUDY 5  GREG

Greg was selected because his prior background in calculus was good and he judged himself as having limited general computing skills. In addition his attitude towards the use of CAS was negative at the beginning of the course. (See comment in introduction about description of attitude and attitude scores.)

![Summary of Greg’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.](image)

**General background**

Greg was a quiet, well mannered, conscientious, student. He worked hard, did all set problems, and frequently sought help outside of class. He always presented neat, detailed work.

Greg had studied a middle level mathematics in year 12 at secondary school. At school Greg had used both a scientific and graphical calculator. He had no previous experience of using a computer for mathematics but rated his general computing skills as OK.

**At the beginning of the course**

*Early-course Algebraic Insight*

On the Pre-course Algebraic Insight Quiz Greg scored 8 on the Algebraic Expectation items and 33 on those testing Linking Representations (scored −100% to 100%, class means were Algebraic Expectation +17, Linking Representations +12). Not only was
his Algebraic Expectation poor but he also lacked certainty in responding to these items, circling ‘no idea’ for three items (two for element 1.2, one element 1.3) and ‘possibly’ for three items (two for element 1.1, one element 1.3). Greg’s responses during the Algebraic Insight Interview made it clear that his Algebraic Expectation skills were poor but he had some ability to Link Representations.

Greg’s weakness in the elements of Algebraic Expectation was fundamental. This fact was illustrated by his responses to the Interview questions (see Appendix 2.4) which indicated that he lacked an understanding of the meaning of symbols in algebra (1.1.1). For example, in response to question 4 (ii) (Appendix 2.4) in the Pre-course Algebraic Insight Interview Greg indicated that he linked letters in algebra to their position in the alphabet when he said:

- It’s got $b$ as the last term. I guess what you start off with is $b$ for $x$.
- It would be going up by – no – actually we’d start off with zero because zero is next to the $b$ and $b$ would be next and then you would have $d$...$b$ could be seen as an even number so the next one would be $d$.

Greg’s inability to conceive of a letter as representing a variable was also evident in his response to question 3, an area problem with dimensions denoted by letters (see Appendix 2.4). In this example he solved the problem by simply replacing the variable with a convenient number. The Interview progressed as follows (note: the Interviewer only asked questions after prolonged pauses):

- **Greg:** I’m thinking about how to do this.
- **Interviewer:** How do you find the area of a rectangle?
- **Greg:** It would be length times width.
- **Interviewer:** So what’s the length?
- **Greg:** I could make the length say 12. $x=12$
- **Interviewer:** Why did you pick that?
- **Greg:** $d=x/4$ so I thought maybe I could make $x$...wait a minute...$d$ is the little bit? OK $x$ would be 12 divided by 4, would be $d$ and then $y$ would be $d$. $d=12/4$ so $y=3$ and the area would be $12*3=36$ and it’s in metres so it would be metres squared.

Greg understood that a letter may stand for a number but had not grasped the concept of a variable. This meant that he was operating at the lowest of Küchemann’s levels (Küchemann, 1981). On the other hand Greg showed quite a good ability to Link Representations. Given tables of values in the Algebraic Insight Quiz he was able to...
identify whether the function represented was linear, quadratic, exponential or none of these. Greg was also able to match graphical and symbolic representations of functions of these types.

In the Algebraic Insight Interview Greg was asked to discuss features of the function \(g(x)=x^2+4x-21\). This time he did not hesitate.

Greg immediately and enthusiastically started: It would be a parabola, a positive parabola it would be down the y-axis 21 spaces, it would cross the y-axis at \(-21\)

Interviewer: Can you tell me anything else from that rule?

Greg: I’d probably have to work it out I’d factorise it.

Interviewer: Why would you factorise it?

Greg: That was the way I was taught to work it out. It would then give me value for x...wait I’ll factorise it and explain. Now I’ll find something that multiplies together to give 21 and adds to give 4 …One of them has to be a negative as it’s negative 21. We have 7 and 3 so 7 will have to be positive it will be negative 3. From this we can work out the values of x so this will tell us where the parabola crosses the x-axis

Interviewer: Why?

Greg: Using the null factor law: the ‘x’ s in the brackets when you add them to the other number the bracket they have to equal zero. So x would equal \(-7\) and 3.

Interviewer: Can you rewrite the rule so that the coordinates of the minimum point of its graph can be easily seen?

Greg: For that you would… I used to know this… I think the equation is something like plus or minus b over 2a.

Interviewer: What does that tell us?

Greg: The b represents the second number in the first equation which is 4 and the a is the first number in front of the x which is one.

Interviewer: How does that help us find the turning point?

Greg: When you substitute those numbers in to the equation it gives you a number and that tells you where it crosses the x axis. And then when you know where it crosses the x axis you would put that value for x. Replace the x values in the equation and work it out to get g(x)

It seemed that in this example Greg was going through a well-practiced drill for describing the graph of a quadratic function. He identified the form of the function as being of the type \(f(x) = ax^2 + bx + c\) and proceeded to describe the graph based on a memorised routine using a, b and c. This routine gave him some skills for Linking Representations and for developing that aspect of Algebraic Insight.

Greg’s lack of Algebraic Insight was also evident in his early worksheets. He described questions, which required the student to prove whether particular functions were odd or even, as ‘confusing’ and his understanding as only ‘OK’. Rather than providing general
solutions he chose to substitute values and thus illustrate the examples. These questions required him to understand the meaning of symbols (1.1.1) and identify some key features of functions (1.3) but he could not do this. Other questions, in which the student was required to specify the domain of given functions, demonstrated that Greg was not totally aware of the properties of division. In simplifying the expression \( \frac{6x^2 - x - 2}{4x - 2} \) Greg cancelled the common factor of 2x-1 without concern for division by zero.

Greg found problems which required Linking Representations (2.1, 2.2) to be ‘easy’ and felt that his understanding was ‘very good’. This was consistent with his responses to the Quiz and Interview, but he did have some learned strategies for tackling this type of problem.

Early-course CAS use

Early in the course Greg reported that he had no difficulties using the program DERIVE. Observation by the researcher and Greg’s coding on his worksheets, suggested that this may be because he made little use of the program at this stage, preferring instead to work by hand. He made limited use of the graphing facility and almost no use of the algebra facility of DERIVE. The researcher’s observation notes record that he didn’t think to move between representations unless prompted, didn’t explore variations unless prompted, and experienced technical difficulties with the program (for example Authoring expressions correctly).

Greg’s made limited, non-discriminating, functional use of CAS. He reported that he used it when he could have used pen and paper, to find answers to hard questions and time-consuming questions. He only used CAS if the teacher or the worksheet instructions suggested its use. Greg preferred not to use CAS, but when he did, his ‘manner’ of use varied between passive and responsive.
Early Course summary

Algebraic Insight

Algebraic Expectation: Poor
Linking Representations: Moderate, follows learned routines

Effective Use of CAS

Technical: Difficulties: few but use of CAS very limited
Personal: Attitude: Negative, he quietly avoided CAS
Judicious Use of CAS: Manner of use varied between ‘non-use’ and responsive
Demonstrated non-discriminating use of CAS.
Undertook only minimal pedagogical use of CAS.

In the middle of the course

Mid-course Algebraic Insight

Greg’s second experience of the Algebraic Insight Quiz showed considerable improvement in his Algebraic Expectation. On these items his score improved to 62 despite a continued lack of certainty. He indicated ‘no idea’ for one 1.2 item and possibly for three items (one 1.1, two 1.2). On the Pre-course Quiz all his responses for identifying key features (1.3) were wrong but this time he responded correctly to all four of these items. For Linking Representations he again scored 33 (class means Algebraic Expectation +25, Linking Representations +23).

Greg’s answers to questions on the Mid-course Examination reflected his limited understanding of the meaning of symbols. The availability of CAS did not obscure this weakness. For example in the question that asked ‘If \(y\) is the number of pies in stock at the end of day \(t\)...’ Greg asked the teacher ‘What does \(t\) stand for?’ He did not understand the meaning of \(t\) as a variable but wanted \(t\) to stand for a particular number.

In a later question where \(f(x)\) was defined to be \(3^x\) the students were required to show that \(f(a+b)\) was equal to \(f(a) + f(b)\). Greg could have used CAS to define the function and evaluate the left and right hand sides of this expression, but he did not have sufficient understanding of the question even to do this. He asked ‘How can you solve \(f(a+b)\) if you don’t have an \(x\)?’ This further illustrates his lack of understanding of the meaning of letters used as variables and parameters. Each time a question was couched in general terms, including linking symbols and graphs, Greg tried to find a specific
example by substituting values for parameters and working with these particular expressions.

At this stage of the course, Greg’s worksheets indicated some improvement in his ability to identify both the structure (1.2) and key features (1.3) of functions, and so link the symbolic with the graphical representations of functions. For example, after examining the vertical asymptotes in a number of examples he wrote:

\[ f(x) = \frac{3(x-1)^2(x+4)^2}{(x+1)^2(x+2)^2(x+3)^2} \]

The vertical asymptotes at \(-3, -2, -1\) are even. All these \(x\) values are negative so the denominator = 0 and the powers are all squared.

Greg has identified the structure \(\frac{f(x)}{g(x)}\) and recognised that there are asymptotes where \(g(x)\) is zero. He noted two key features: \(g(x)\) will be zero if \(x\) is \(-1, -2\) or \(-3\) and each factor is squared.

On another structured worksheet in week 8 Greg showed some improvement in his ability to look at the structure of expressions and to use the result of processes as objects. Given \(p(x) = x^3 + 5x - 4\), \(q(x) = 2x - 1\) and \(h(a) = 3a\) he was able to correctly find composite functions like \(p(h(a))\) or \(q(h(p(x)))\). On the same sheet, in a section focusing on algebraic fractions, after observing that \(\frac{ad + bc}{bd}\) he was asked ‘what happens when \(b=d\)?’. Greg’s response was ‘\(b=(b=d)\’\), illustrating that he still had fundamental problems understanding the meaning of symbols.

**Mid-course use of CAS**

By this stage of the course Greg was starting to make more use of CAS but reported that he was not confident in CAS use and preferred his graphical calculator for calculations. On the day the Mid-course Survey was taken Greg chose not use CAS. The researcher observed, in other classes, that he was now trying to use CAS but had trouble with the required syntax and linking this to conventional mathematics. He often asked the teacher to check: ‘Is this right?’ He wanted to use CAS to explore the mathematics but didn’t know how to go about this.
When coding his work sheets Greg indicated that he made use of both DERIVE algebra and DERIVE graphs for all these questions but classroom observations record that he did not use DERIVE unless it was specifically indicated in a question or suggested by the teacher. His still largely made non-discriminating functional use of the CAS. Although he had started to see the value of CAS as a pedagogical tool, he was still hindered in his use by Technical difficulties. His ‘manner’ of use now varied from ‘passive’ to ‘directed’. He was content to observe the teacher, or watch others showing him how they used the technology, and he chose to use it himself when there were clear directions to follow. He only used CAS in a strategic manner when there were clear guidelines for this use.

Mid Course summary

Algebraic Insight
Algebraic Expectation: Moderate, started to identify structure and key features
Linking Representations: Good, links form to shape and position

Effective Use of CAS
Technical: Difficulties: a lot but making more use of CAS
Personal: Attitude: less negative, values CAS for doing harder problems and for some learning.
Judicious Use of CAS: Manner of use varied between Passive and directed strategic
Non-discriminating functional use.
Undertook limited pedagogical use but interested in trying more.

At the end of the course

Late-course Algebraic Insight
On the post-course Algebraic Insight Quiz Greg scored 23 for Algebraic Expectation and again 33 for Linking Representations (Post-course class means Algebraic Expectation +19, Linking Representations +16). Most of his errors were with items testing element 1.2 and 1.3 (identifying structure and key features). In fact Greg chose the ‘possibly’ categories for nine items and ‘no idea’ for one. This meant that from the first eleven slides he made only one definite answer (which was correct). This level of uncertainty could be explained by his anxiety about the examination to follow. The Post-course Interview confirmed that overall there had been clear improvement in
Greg’s Algebraic Insight. It was especially encouraging to see his improvement in Algebraic Expectation. The element showing most improvement was recognising conventions (1.1). This included some improvement in his understanding of the meaning of symbols (1.1.1). He chose to use symbols to generalise the ‘Think of a number’ problem and was happy to express the area of the rectangle as a variable. The transcript of this second Interview contrasts with that included in the early course section. The excerpt below demonstrates that Greg was able to find the area of a given figure using letter symbols and relationships between these variables. This is evidence of a marked change in his Algebraic Expectation.

*Greg:* (Reads question). Well…the area of this shape here would be this area, [indicates square] plus this area [indicates rectangle]. The area of the square would be $d$ times $y$, so you’d have $dy$ plus… and the area of the rectangle would be $xy$. Now $d$ equals $x$ on 3, so $d$ is a third of $x$ and $y$ is…equal to $d$, so $y$ is $x$ on 3, so you’d have $x$ on 3 by $x$ on 3 plus $x$ times $x$ on 3, which would… well, $x$ on 3 times $x$ on 3 would be equals by fractions, it would be $x$…

*(Interviewer: You can use DERIVE for anything you like if you want to.)*

*Greg:* Ok … So it would be $x$ squared divided by 9, plus $x$ squared divided by 3. Now I have to get a common denominator. Well it would just be simply $x$ squared plus 3 $x$ squared over 9, so you would have 4$x$ squared over 9.

He linked form to shape when moving from the symbolic to graphical representation of functions (2.1.1). At the Post-Course Interview DERIVE was available for students to use as they pleased. Greg made use of DERIVE to simplify algebraic expressions. His ability to Link Representations remained similar but in the Interview it seemed that he was able to talk through and explain his thinking rather than say ‘this is what we were taught to do’.

The codes on Greg’s worksheet questions in the later weeks of the course indicated that he found items requiring recognition of conventions (1.1) and identifying structure (1.2) OK and felt his understanding was Good, while for questions involving linking symbols to graphs his understanding was Very Good. The researcher’s class observation notes suggest that he was making progress with his use of symbols but still had difficulty using symbols to formulate problems.

In the learning tasks for trigonometric functions, a focus had been placed on the use of exact values and the identification of structure and key features when patterns linking
different representations. Greg’s preference for numeric approximations was evident in his response to question 6 on the Post-course Examination (see Appendix 1.3) shown in figure 7.17 below. The observer’s notes show that Greg used DERIVE to produce this graph but he treated CAS like a graphical calculator, obtaining approximate values by moving the cursor to key points rather than using the algebra facilities to obtain exact values.

![Figure 7.17 Example illustrating Greg’s preference for numeric approximations](image)

**Late-course CAS use**

In the later weeks of the course Greg was observed to use DERIVE for almost all questions. His approach often seemed to be functional but unthinking. He would record only the question then the answer with no intermediate steps, explanation of the working or thinking, or check that the answer made sense. In week 11 it was noted that he was having trouble bracketing expressions so that the CAS would execute the expression correctly. This suggests that he still had difficulty identifying the structure of expressions. While Greg made extensive use of CAS he needed to be encouraged to move between representations.

On the Late-course CAS Use Survey Greg reported that he was having lots of problems Authoring expressions, some problems interpreting the results to conventional mathematics and some problems using the graph mode. He indicated that he was using CAS to find answers to problems and to explore the mathematics both when directed and to explore variations on the given topics. The problems he identified were evident in the Post-course examination. He used the CAS in an unthinking manner and merely
wrote down what was on the screen, for example $3e^{3x}\ln(e)$. He failed to choose graph windows that would show all the important features of the function.

Despite the technical difficulties Greg experienced, he was sometimes discriminating in his functional use and, while he undertook limited pedagogical use of CAS, he showed interest in extended use. His manner of use varied from making superficial and automatic use of CAS by following instructions, to self directed strategic use.

*Post-course reflections on CAS use*

At the end of semester Greg agreed that CAS could be used to find answers and plot graphs but was less certain (neutral or disagreed) that CAS helped him to see patterns or understand mathematics. When given a range of question types, Greg said that for speed and success he would choose to use CAS. For learning he felt that pen and paper were better for questions like solving equations, but for intercepts and derivatives DERIVE graphs were more helpful.

At the Post-course Interview Greg said that he had some trouble learning DERIVE. He had found it difficult to remember what to do, but despite this, the use of CAS had aided his understanding of differentiation and integration. He felt that he would choose to use DERIVE for graphs and for what he perceived to be hard algebra, but perhaps would use pen and paper for easy questions. He then revised this statement saying that probably he would use CAS for most questions but still use his graphical calculator for arithmetic.
Late Course summary

Algebraic Insight
Algebraic Expectation: Moderate, most improvement shown in recognising conventions especially the use of letters to represent variables and parameters.
Linking Representations: Good, links form to shape and position

Effective Use of CAS
Technical: Difficulties: a lot but making more use of CAS
Personal: Attitude: Mixed, still fairly negative, preferred graphical calculator, did not think about CAS and mathematics.
Judicious Use of CAS: Manner of use varies between directed and initiated strategic
Still mostly non-discriminating functional and limited pedagogical use.
Greg – an overview

Greg’s overall level of Algebraic Insight improved. His ability to recognise conventions, identify structure and identify key features all showed clear improvement. In terms of items on the Algebraic Insight Quiz and Algebraic Insight Interviews he showed most improvement in recognising conventions (1.1). This was initially his weakest area.

Greg’s ability to link between representations showed no improvement in terms of his score on these items. However, the Post-course Interview and classroom observations suggested that he was following a more reasoned approach, rather than a recipe approach, to explaining what he would expect to find as he moved between representations.

Initially Greg avoided using CAS, but by the end of the course he was using it for functional purposes and in a self-directed strategic manner for pedagogical purposes. While Greg’s manner and purpose in using CAS improved he found that, as he made more use of CAS, he encountered more technical difficulties. He did not reach a high level of competency with the use of CAS during the fifteen weeks of the course.

Greg provides an interesting example because, despite moderately successful secondary school mathematics results, his level of Algebraic Expectation was very low. I had not expected to find students choosing to study an elective unit in introductory calculus who had such a limited understanding of the use of letters in algebra. It was interesting to see that the use of CAS during the course did not obscure his lack of understanding. This fundamental problem meant that Greg had difficulty learning to use alternative symbols and syntax appropriate for DERIVE. Initially he avoided using CAS, but over time he learnt to use the program effectively despite this difficulty. When he had trouble identifying the structure of expressions and therefore bracketing the syntax correctly there was immediate feedback from the screen, this helped him to recognise errors. He could see that the expression on the screen did not match that on the page. Over time his ability to identify structure improved.

Greg was still having Technical difficulties using CAS at the end of the course. The overall class results also show an increase in the number of Technical difficulties
experienced late in the semester. In Greg’s case this result can be explained by his increased and higher level use of CAS. Instead of merely following directions he was initiating use and using CAS to explore variations. This learning process meant that he made more errors but as a result increased his understanding of symbols, structure and key features. The immediate feedback from CAS was well received. He did not give up when he made errors because it was easy to correct them. Being able to do many questions in a short time meant that he also received frequent positive feedback.
CASE STUDY 6  SAMUEL

Samuel was selected because his prior background in calculus was good and he judged himself as having good general computing skills. In addition his attitude towards the use of CAS was negative throughout the course. (See comment in introduction about description of attitude and attitude scores.)

![Graph showing Samuel's results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.]

*Figure 7.18* Summary of Samuel’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.

**General background**

Sam was a quiet and fairly relaxed student who attended all classes and worked steadily during the semester. A somewhat ‘dreamy’ student, Sam’s answers to questions (not just mathematics questions) were slow and often prefaced by ‘What was that again?’ He handed work in on time but showed no evidence of trying to do more than fulfil the minimum requirements.

Sam had studied a middle level mathematics at year 12. At school he had used both a scientific and graphical calculator but not a computer program for mathematics. He classified his general computing skills as between ‘good’ and ‘very good’.
At the beginning of the course

Early-course Algebraic Insight

Early course testing suggested that Sam had weaknesses in both aspects of Algebraic Insight. On the Quiz he scored –8 for Algebraic Expectations items and –11 for the items requiring Linking Representations (scored –100% to 100%, class means were Algebraic Expectation +17, Linking Representations +12). He showed some uncertainty: two ‘possible’ responses (both element 1.3) and one ‘no idea’ (element 1.3). His results on the Algebraic Insight Quiz may have been affected by the quick response required in the time limits, but this did not explain the similar level of understanding shown in the Algebraic Insight Interview for which there were no time constraints.

Sam was asked how he could check whether his result for the ‘think of a number problem’ (Appendix 2.4) would always hold true for any integer used to initiate the process.

Sam: I think it would be to do with less than 2 and more than 2 but I wouldn’t know a rule or anything like that to do it.

Interviewer: Could you work one out?

Sam: Probably not, not in a day or anything anyway.

Sam quite quickly identified the structure in the commands in question 4 of the interview and linked them to the table. He obtained a correct table for part (i) and described his thinking in this way:

I took \( x \) because this is \( x \) here, this is what you do to the number, 0 is where you start and this is what you go up by.

His response to part (ii) suggested that he did not understand the use of letters as variables. His answer, see figure 7.19, showed that he worked ‘alphabetically’. When asked to tell the Interviewer about his answer he replied:

I saw the letter, saw it was using letters instead of numbers.

![Figure 7.19 Example of Sam’s lack of understanding of the use of letters as variables](image)

Figure 7.19 Example of Sam’s lack of understanding of the use of letters as variables
Sam completed a correct table for part (iv) (which was the same as the demonstration example except for the use of $y$ instead of $x$) in terms of $y$ without comment. When asked whether he recognised this example he replied ‘It’s a curve’ thus making a link to its graphical representation but not suggesting he noticed that he was just repeating the given example.

Sam’s worksheets provided evidence of many examples where he had chosen to work with graphical representations of functions on his calculator rather than the algebraic representation either with CAS or by hand. In his answer to one question, he wrote that he had established whether functions were odd or even by graphing them, not by using the algebraic definition given.

His work reflected problems with recognition of the properties of operations. When working with $y = \log_{10} x - \log x$ he wrote that the log $x$ cancels each other. He indicated this by crossing through the two phrases, ‘log’ and ‘$x$’. This is shown in figure 7.20.

![Figure 7.20 Example showing Sam’s lack of recognition of properties of operations](image)

*Figure 7.20 Example showing Sam’s lack of recognition of properties of operations*

**Early-course CAS use**

On the Early Course CAS Use Survey Sam indicated that he had no problems Authoring commands, working between the screen and ordinary mathematics, or using an appropriate graph window. He did however have problems bracketing expressions, using sequences of commands, interpreting solutions, zooming to see key features of graphs and making sketch graphs. He had a lot of problems with commands that required him to write the syntax.
Sam reported that he used DERIVE functionally, but non-selectively, to find answers which he could easily have done with pen and paper, for hard questions and for time-consuming questions. It was interesting to read his responses to this survey as observations made by the researcher, in class, suggested that Sam avoided using DERIVE even when it was suggested. He seemed to make a great deal of use of his graphical calculator. When pushed to use DERIVE, he had problems with syntax and brackets. His worksheets did not include the codes to identify the tools used to do various questions. However, it seems likely that he used his graphical calculator when he produced the answer shown in figure 7.21 for a question asking him to explore functions of the form $y = x^n$ where $n$ is even. The error would have been less likely if Sam had used DERIVE since the graphs would have been in different colours. This example does illustrate the visual perception problems that can hinder correct interpretation of CAS graphical output.

Figure 7.21 Example of Sam’s problem copying from CAS – probably calculator screen
Early Course summary

Algebraic Insight

Algebraic Expectation: Weak, shows problems with meaning of symbols and properties of operations.

Linking Representations: Often chooses to use link to graphs but does not consistently identify key features.

Effective use of CAS

Technical: Difficulties: Lots, especially syntax, in particular use of brackets.

Personal: Attitude: Negative, preferred to use graphical calculator, avoided CAS use

Judicious Use of CAS: Manner of use: Only used CAS if directed by instructions

Demonstrated limited discriminating functional use

Undertook minimal pedagogical use.

In the middle of the course

Mid-course Algebraic Insight

Sam’s results on the Algebraic Insight Quiz improved to a score of 23 on the Algebraic Expectation items and 11 on the Linking Representations items (Mid-course class means Algebraic Expectation +25, Linking Representations +23). The improvement in Algebraic Expectation was on items that required identification of structure or key features. He also improved on linking symbolic form to shape of graphs. His level of certainty was similar, with three answers given as ‘possibly’ – the same three items as on the Early Quiz.

While Sam still favoured the use of graphs wherever possible (Asking “What will I graph for number #?”), by following structured exercises he had begun to work with algebraic notation as well. He observed general patterns and constructed rules.

Mid-course use of CAS

On the Mid-course Technical Difficulties Survey Sam indicated that he had no problems entering expressions and commands but some setting appropriate graph windows and viewing key features. He also had some difficulty interpreting screen output as conventional mathematics. He still did not use CAS to find answers to problems unless he was following explicit instructions.
Observations made during the Mid Course Examination revealed that he still relied almost solely on his graphical calculator. He made use of this in all questions except to find the solution for an exponential function.

<table>
<thead>
<tr>
<th>Mid Course summary</th>
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</thead>
<tbody>
<tr>
<td><strong>Algebraic Insight</strong></td>
</tr>
<tr>
<td>Algebraic Expectation: Moderate</td>
</tr>
<tr>
<td>Linking Representations: Moderate</td>
</tr>
<tr>
<td><strong>Effective Use of CAS</strong></td>
</tr>
<tr>
<td>Technical: Difficulties: Use of syntax improved but some difficulties choosing suitable windows and interpreting screen output</td>
</tr>
<tr>
<td>Personal: Attitude: Negative, no change in attitude</td>
</tr>
<tr>
<td>Judicious use of CAS: Manner of use included passive and directed use.</td>
</tr>
<tr>
<td>Demonstrated limited discriminating functional use</td>
</tr>
<tr>
<td>Undertook minimal pedagogical use (None, unless following explicit instructions)</td>
</tr>
</tbody>
</table>

**At the end of the course**

**Late-course Algebraic Insight**

The Post-course Algebraic Insight Quiz showed Sam continuing to improve in Algebraic Expectation (score 27) but regressing on Linking Representations (score –22) (Post-course class means Algebraic Expectation +19, Linking Representations +16). His level of certainty had not changed. The fuller picture given by the combination of the Quiz and Interview suggested that Sam had made some improvement both in Algebraic Expectation and in Linking Representations.

This time, at the interview (Appendix 2.5) when asked about a general response to the ‘think of a number problem’ Sam initially replied ‘Nah, I wouldn’t be able to tell’. The Interviewer pursued this line of thinking by saying ‘No ideas on how you would go about showing that?’ Sam responded by starting to use a letter variable (x), and carrying out the given process using this letter. This showed that he understood the use of a letter as a variable.
Sam had no difficulty reducing the area problem (question 3) to an expression in terms of one unknown but said that he would like to use DERIVE to simplify this expression correctly. Again he was using a letter as a variable, showing he understood the meaning of symbols but was less sure of the basic properties of operations needed to simplify his expression.

Figure 7.22 shows Sam’s attempt to interpret the ‘command’ given in question 4. He had already completed the other parts of this question, indicating that he understood that what was required was a mapping from \( p \) to \( p+2 \), starting at \( a \) and going to \( a+4 \) in steps of 1. His discomfort with letters as variables is suggested by the lack of ‘\( a \)’s on the vertical axis. Considering Sam’s preference for graphical representations it was also surprising to hear Sam, before he attempted to plot the points, say ‘I’m not sure what the graph looks like at all’ and afterwards ‘I didn’t know what the graph…what I was expecting’. He had either not identified \( f(p)=p+2 \) as a linear function or not identified the structure of the expression.

![Figure 7.22 Example suggesting Sam’s lack of understanding of letters as variables](image)

Sam’s use of the quotient, product and chain rules for differentiation on the Post-course examination (Appendix 1.3) indicated that he could identify the structure of the algebraic expression given in each question. He chose to do some of the calculus by hand but used DERIVE to differentiate \( 3\pi \theta^2 \cos 2\theta \) with respect to theta. Scott was able to enter this expression correctly into the CAS thus indicating that he could identify the structure of the expression correctly. His preference for graphical and approximate methods remained. When asked to find the area enclosed by a curve and the \( x \)-axis he
chose to find the $x$ intercepts approximately from the graph. He linked the idea of area under a curve to integration but did not attempt to find the anti-derivative of the given function. Perhaps he tried to use numeric integration on his calculator. He obtained an inaccurate result. Sam was obviously unsure about identifying key features of graphs and linking this to the symbolic representation. He headed a section where he needed to find the $x$-intercepts of a given graph ‘trial by error’.

Late-course CAS use

Sam’s Post–course DERIVE Use Survey shows that he was still having Technical difficulties using DERIVE. He noted no problems with Authoring and using brackets but indicated some problems with every other option and said that he had problems every time he needed to enter the syntax of a command. He did not complete the Judicious Use of CAS section of the survey that would have indicated his purpose in using CAS.

In class he was still asking quite specific questions about how to use the program (ie ‘how do I enter…?’). He made no use of the program’s help facility. In the Post-course Examination he made little use of DERIVE preferring instead to work by hand or with his graphical calculator. He did not use CAS to check his derivatives or integrals. Post-course reflections on CAS use?

On the Post-course Survey Sam ‘agreed’ that using DERIVE helped him to understand mathematics but also ‘agreed’ that he only used DERIVE if the instructions told him to. He was ‘neutral’ to the ideas that using DERIVE helped him see patterns, DERIVE could be used to do work out problems and that its use had increased his confidence in calculus. Sam ‘disagreed’ with the statements ‘DERIVE can be used to check every step’, ‘I try out ideas using DERIVE’. He did not think his confidence had increased in algebra, graphs or trigonometry. He ‘strongly disagreed’ with the suggestion that a DERIVE plot would tell him all he needed to know about a function.

When presented with a set of questions Sam indicated that for speed and success he would choose pen and paper or graphical calculator except for the hardest question comparing $\log x$ and $1/x$. In all cases he would prefer pen and paper for learning.
Sam felt that the mathematics in the course had been quite easy for him as it revised work he had done at school last year. He said he did not know how to do some things with CAS. He didn’t know how to do things that he saw other people doing but he said ‘When it comes down to it I’d prefer to use my calculator or write it out.’ He did, however, suggest that it was easier to use CAS for things like differentiating and integrating.

Sam:  Pen and paper is fine with me. I like to see it written down, that helps me to do it and makes it easier to understand. I write it down as I do each step then I’d probably go for a calculator for graphing, finding area and finding points. That’s more just to confirm it.

Interviewer: And what about DERIVE?

Sam: It’s probably a bit better when I’m doing bigger sums with more variables in it. Calculators and that can’t really do that. Like draw a number of graphs in different colours.

**Late Course summary**

**Algebraic Insight**

Algebraic Expectation: Improvement in identifying structure and meaning of variables.

Linking Representations: Some improvement in looking for key features but still problems

**Effective use of CAS**

Technical: Difficulties: Only slightly fewer- still some across all three elements

Personal: Attitude: Still negative but gaining some appreciation for the functional use of CAS in some problems.

Judicious Use of CAS: Manner of use responsive: still non-use unless suggested but no longer requires fully specified instructions. Approaching ‘compulsory’ use in a more strategic manner.

Demonstrated limited discriminating functional use

Undertook minimal pedagogical use.
Samuel – an overview

Initially Sam showed poor skills in Algebraic Expectation and Linking Representations. During the course he made clear improvement in Algebraic Expectation and slight improvement in Linking Representations.

From the beginning of the course Sam did not like using CAS and his attitude did not improve. He used it as little as possible, only choosing to use it to help with calculus that he found difficult by hand. His Technical difficulties in using CAS remained similar throughout the course because he did not apply himself to learning to use the program.

Throughout the course Sam showed a preference for using his graphical calculator and so it might have been expected that he would have good skills in linking representation, but he scored poorly on these items. The Quiz and Interview items tested linking symbolic representations to their graphical or numerical equivalents and Samuel preferred to use numerical approximations and particular values rather than general solutions. Some problems and worksheets specifically required the use of CAS to look for patterns in multiple examples. Perhaps these helped Sam build an understanding of variables. He certainly gave better responses, at the Late-course Interview, to questions that required recognition of conventions and basic operations.
CASE STUDY 7  JOCELYN

Jocelyn was selected because her prior background in calculus was good and she judged herself as having good general computing skills. In addition her attitude towards the use of CAS was negative throughout the course. (See comment in introduction about description of attitude and attitude scores.)

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**Figure 7.23** Summary of Jocelyn’s results for aspects of Algebraic Insight and elements of Effective Use of CAS at different stages of the course.

**General background**

Jocelyn attended class regularly, submitted set work and was confident to ask questions in front of the class or individually. She was quite confident about her school mathematics ability but reluctant to learn new methods. Jocelyn had studied a middle level mathematics at year 12. She had previously used both a scientific and graphical calculator but had not used a computer for mathematics. She rated her general computing skills as between good and very good.

**At the beginning of the course**

*Early-course Algebraic Insight*

On the Pre-course Algebraic Insight Quiz Jocelyn scored 35 on the Algebraic Expectation items and 72 on the items testing Linking Representations (scored –100%
to 100%, class means were Algebraic Expectation +17, Linking Representations +12). She was certain about most of her answers, circling possible for just two items (one element 1.2, one 1.3).

In class, despite her preference for her graphical calculator Jocelyn had problems linking representations when finding the slope of a line, given two points and with the concept of local ‘straightness’. She could link symbols to graphs but had difficulty with the reverse process. Jocelyn demonstrated Algebraic Expectation skill in identifying when to apply logarithmic laws. In this case she was identifying form and structure. On her worksheets she confidently used letters as variables and could link symbols with graphs of functions. All of this suggested improving skills of Algebraic Estimation.

*Early-course CAS use*

Jocelyn said that she had a lot of difficulty thinking of, or remembering, the appropriate sequence of DERIVE commands to carry out procedures. She also had some difficulty with entering correct syntax and interpreting exact and approximate solutions. Within the graphical representation Jocelyn could not successfully use the menu commands to zoom and focus on a smaller section of the graph.

Jocelyn’s use of CAS was limited. She demonstrated non-discriminating functional use and undertook a minimum of required pedagogical use. She said that she used DERIVE to find answers to questions only when its use was explicitly suggested. This meant she sometimes used it for questions that she could easily have done in her head.
Early Course summary

Algebraic Insight
Algebraic Expectation: Moderate
Linking Representations: Good

Effective use of CAS
Technical: Difficulties: Lots, most with syntax but some with changing representations and interpreting output.
Personal: Attitude: Negative, she did not want to use DERIVE
Judicious Use of CAS: Manner of use was occasional and random
Demonstrated non-discriminating functional use and undertook minimal pedagogical use. She did not initiate any pedagogical use.

In the middle of the course

Mid-course Algebraic Insight

On the Mid-course Algebraic Insight Quiz Jocelyn scored higher on Algebraic Expectation, 58, but lower on Linking Representations, 56 (Mid-course class means Algebraic Expectation +25, Linking Representations +23). This time Jocelyn showed more uncertainty, choosing ‘possibly’ for 4 items and ‘no idea’ for two. These items were all elements 1.2 and 1.3. She was more confident of her incorrect answers for 2.1 and 2.2.

Jocelyn showed some problems identifying structure in rational functions. She had problems with strategic groupings, on the week 8 worksheet for example. Students were asked to give an example of an expression of the form $\frac{f(x)}{g(x)}$ where the expression may be simplified by canceling. Jocelyn gave the example $\frac{8x + 2}{4x + 2}$ but proceeded to write “cancel the two 2’s, then have $\frac{8x}{4x}$ so cancel”. It was expected that students use CAS to check their conjectures: Jocelyn clearly did not do this. CAS was suggested for this worksheet as there were many examples designed to illustrate structure and pattern. Perhaps Jocelyn did not engage with the activity since she did not like using DERIVE.

On the Mid-course examination Jocelyn showed evidence of being able to link symbolic and graphical representations and being able to identify structure, including using
composite functions. This provided evidence that overall she had made improvement in Algebraic Expectation and did have some ability in Linking Representations.

**Mid-course use of CAS**

Jocelyn still had difficulty entering expressions, using brackets correctly in the syntax, changing representations and setting the appropriate scale for graphs and then interpreting CAS output. She still only used CAS when it was suggested but now discriminated and only used it for ‘hard’ questions. She said “I’m still scared of the computer and try to do most [problems] by pen and paper.”

In class, in week 7, Jocelyn was to explore the family of functions \( \frac{\sqrt{x} - 1}{x^n} \) and their derivatives. Again she said, “I’ve been doing the work by hand because the computer scares me.” The teacher helped her to start to use the computer, prompting and helping as necessary as they set up a table. Later, after she used DERIVE to finish this example herself and then to do other similar ones she declared “Oh that was fine. I didn’t even need to do them all – I could see the pattern”.

In the mid-course examination her use of DERIVE was limited and error-ridden. It was of no advantage to her in answering any question.

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**Mid Course summary**

**Algebraic Insight**

Algebraic Expectation: No improvement

Linking Representations: No improvement

**Effective use of CAS**

Technical: Difficulties: Little improvement

Personal: Attitude: Negative, scared of computer(? CAS?)

Judicious use of CAS: Manner of use: Non-use, random and directed

Some discriminating functional use of CAS.

Minimum pedagogical use.
At the end of the course

Late-course Algebraic Insight

On the Post-course Algebraic Insight Quiz, Jocelyn’s results were only 39 and 11 with a similar lack of certainty as on the Mid-course Quiz (Post-course class means Algebraic Expectation +19, Linking Representations +16). However, her Interview results were more positive, showing ability in both Algebraic Expectation and Linking Representations. These results suggest that Jocelyn performed better without time pressure and that she had made some limited improvement in Algebraic Expectation during the course while maintaining skill in Linking Representations.

The Interview focused on 1.1 and 2.1 (recognition of conventions and basic properties and linking graphs and symbols). Jocelyn showed that she understood the meaning of letters as variables, could identify structure, and link symbolic form and key features with the shape, position and critical points of a graph.

Late-course CAS use

Jocelyn was experiencing fewer Technical difficulties: she reported one difficulty in the class session with three of the examples listed. These all related to fluent use of syntax. She was still not choosing to use CAS to explore mathematics and was not always discriminating in her functional use – availing herself of the CAS facilities to solve easy, hard, and time-consuming problems.

Post-course reflections on CAS use

On the Post-course CAS survey Jocelyn agreed that using DERIVE helped her to see patterns and said that she did use it to try out ideas but she disagreed that it helped her to understand mathematics or could be used to check steps of working. She indicated that her use of DERIVE was no longer restricted to using it when instructed to do so. Jocelyn also admitted that, as a result of the course, her confidence with graphs, functions and trigonometry had improved.

When presented with a series of problems and asked which tool she would prefer, Jocelyn indicated that she would prefer to use a graphical calculator for all problems. If problems were ‘simple’ then she would often choose pen and paper for speed and for learning she liked to use pen and paper as well as the calculator.
She wrote:

I think DERIVE is good but it has sort of let me forget basic steps which I feel are important.

This is an interesting reflection from a student who, by her own admission, made very limited use of CAS!

In the post course general Interview Jocelyn said that she had found maximum and minimum problems the easiest:

We had that stuff drilled into us at high school so that was pretty easy Of DERIVE she said:

I don’t like it. I find it really daunting. I reach for my graphical calculator before it … It is hard to know what to do, all your Manage, Substitute stuff [sequences of commands and syntax].

She could not think of any examples where the use of CAS had enhanced her understanding:

I don’t like it …I don’t understand. … You do it on the computer but still I don’t actually understand what it’s done, but the computer says, so it must be right. I don’t actually know what I am doing… I find using pen and paper or the graphical calculator more satisfying.

## Late Course summary

<table>
<thead>
<tr>
<th>Algebraic Insight</th>
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<tbody>
<tr>
<td>Algebraic Expectation : Good</td>
</tr>
<tr>
<td>Linking Representations: Good</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Effective Use of CAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Technical: Difficulties: Few still with syntax</td>
</tr>
<tr>
<td>Personal: Attitude: Negative but more open to CAS use.</td>
</tr>
<tr>
<td>Judicious Use of CAS: Manner of use Some still random but some directed strategic use</td>
</tr>
<tr>
<td>Demonstrated some discriminating functional use</td>
</tr>
<tr>
<td>Undertook limited pedagogical use.</td>
</tr>
</tbody>
</table>
**Jocelyn – an overview**

Jocelyn showed some improvement but this was not consistent. She showed an understanding of the meaning of symbols but did not understand the structure of algebraic fractions. At the end of semester she still scored poorly on element 1.2 (identifying structure) despite doing worksheets targeted at these problems.

Jocelyn chose to use CAS as little as possible and so found that it was hard to remember procedures that were not intuitively obvious. She had syntax difficulties throughout the course and, while she became more fluent with practice, she remained anxious about using the computer. She persisted as a non-user and random user throughout the course. Despite her dislike of DERIVE and her preference for pen and paper or graphical calculator she was not even consistently discriminating in her use of DERIVE. Jocelyn did not make the necessary effort to become confident in her use of DERIVE and so its use did nothing to improve her Algebraic Insight. When she did use it she was focusing on ‘what to do to run the program’ not on the mathematics.

This concludes the seven case studies. These findings, along with the general class results, will be discussed in Chapter 8.
This chapter will bring together the findings presented as the class results in Chapter 6, and insights from the case studies outlined in Chapter 7. It includes the reflections of the teacher/researcher on both the teaching experience, and the strengths and limitations of the study. This discourse will be structured around the research questions outlined at the beginning of Chapter 5, the frameworks described in Chapter 3, and the measurement strategies employed in the study – as outlined in Chapter 5. Following this discussion the key conclusions to this thesis, along with some implications of these findings, will be presented briefly in Chapter 9.

Discussion of the findings of the study
The first section of this chapter brings together the findings from the class results along with details of individual students recorded in the case studies. This information will be used to address each of the research questions that were the focus of this study. The results described in Chapters 6 and 7 obviously stem from the data collection instruments used, some of which were developed specifically for the purpose of this study. This section accepts the validity of the information gained and further discussion of the quality of these instruments, along with suggestions for strengthening them, is left to later in this chapter.

In keeping with the research questions, set out in Chapter 5, we begin by considering Algebraic Insight and Effective Use of CAS separately, then proceed to exploring the interrelationships between these two abilities.

Does Algebraic Insight change during a course taught using CAS?
The simple answer to this question is yes. During the course the class’s corporate mean Algebraic Insight changed, and so did that of each individual student. The nature of this change will now be explored in some detail.
During a course taught using CAS, was there evidence of change in students' overall level of Algebraic Insight?

The evidence of the class results (taking into account the Algebraic Insight Quizzes, Algebraic Insight Interviews and the teacher/researcher’s observations) presented in Chapter 6 is that overall the class showed significant improvement in Algebraic Insight. Further, these results showed that there was significant improvement in each aspect of Algebraic Insight: both Algebraic Expectation and ability to Link Representations. Someone may well ask what the practical significance of this is. These results could be seen as trivial: the students were taught and they learned something. However before they are dismissed it is important to consider what these results mean in this study.

First, from the perspective of the teacher is was encouraging to see that not only did the class average improve but every student showed improvement in some, if not all, elements of Algebraic Insight. (To be looked at in more detail below). Second, there was a significant trend showing that the weakest students made the greatest improvement. However, the limited improvement shown by the top students (for example, Stephen) may well have been due to an artificial ‘ceiling’ effect resulting from the constraints of the measures used.

Most of the items used for the Algebraic Insight Quizzes and Algebraic Insight Interviews were based on material that students would have been ‘taught’ many times during their secondary school mathematics. Since the students improved during this course it is reasonable to conclude that the teaching and learning facilitated by CAS was instrumental in enabling these weak students to make progress in their understanding of mathematics. Before considering why this might be so the changes seen during the course will be examined in more detail.

During a course taught using CAS, was there evidence of differential improvement in some elements of students' Algebraic Insight?

There was improvement in both aspects of Algebraic Insight but the level of improvement was not uniform across its aspects and elements. For the aspects of Algebraic Insight, there was slightly more improvement in Algebraic Expectation on average, than ability to Link Representations. While all students who initially had low
scores for Algebraic Expectation improved, some students showed little change in ability to Link Representations.

For the elements of Algebraic Insight it was clear, from the Quiz, that initially students were weakest in element 1.2, the ability to identify structure, and best at 1.1, recognising conventions and basic properties. The Quiz results revealed that the group, on average, showed improvement in every element and the improvement was greatest for identifying structure. Despite clear improvement this remained their weakest element of Algebraic Insight. The availability of CAS facilitated the use of multiple structured exercises targeting these elements (worksheets Appendix 1.4) but fifteen weeks is a short time in which to cause change, especially regarding errors which have been engrained for six or more years. For example, \( \frac{a}{c} + \frac{b}{d} = \frac{a+b}{c+d} \), is an error which students may transfer from arithmetic.

The Interviews focused on element 1.1, recognition of conventions and basic properties, and 2.1/2.2, ability to link symbolic with graphical and numeric representations. It was surprising to see, initially, that a number of these students, including those who had succeeded in a middle level mathematics in year 12, had little or no understanding of the meaning of letters in algebra. The improvement shown in this Common Instance of element 1.1 by students like Yvonne, Greg and Samuel was especially gratifying. The learning tasks facilitated by CAS led to an improved understanding of the meaning of symbols. This concurs with the expectations of Dugdale et al (1995), who considered that a focus on functions, so easily supported with three representations by CAS, would enhance students’ understanding of the meaning of letters as symbols in algebra.

**What are the characteristics of those students whose Algebraic Insight improved?**

Several features characterised those students whose Algebraic Insight improved. Firstly, as previously stated, those students who had the lowest scores on the initial algebraic Insight Quiz and Interview showed the greatest improvement. This was, in part, due to the limited test that did not give the ‘top end’ students room to show their improvement. On the results shown the two ‘top’ students did not improve on the Quiz but in fact made slips and showed lower scores at the end of the course. These two ‘top’ students were Stephen and Jocelyn. The details of their Case studies reveal that Stephen’s work
and his interviews showed improvement in Algebraic Insight that was not reflected in his Quiz result. Jocelyn, on the other hand, did not engage in many of the CAS-based exercises in the course and showed very limited improvement in Algebraic Insight. As described in Chapter 4, many of the CAS-based exercises targeted Common Instances of Algebraic Insight. They encouraged students to explore patterns or note features of multiple examples. Jocelyn did not replicate the same range of examples by-hand.

Each of the students described in the case studies (Chapter 7) showed an overall improvement in Algebraic Insight but the detailed profile for each student is different. Consider each in turn:

Jennifer made clear improvement in both aspects but most in Linking Representations. She had not used a graphical calculator at school and we may assume that the experience of exploring families of functions was reasonably new to her. She had a very low Pre-course score, enjoyed this style of learning and made good progress. Jennifer commented that she found the graphs particularly helpful and this is reflected in her improved scores. She moved between the symbolic and graphical representations as she worked, unlike those students who preferred to retain the use of their graphical calculators.

Yvonne improved in both aspects of Algebraic Insight but made most progress in her ability to Link Representations. Her initial results for both aspects were well below the class average. For Yvonne too, the experience of looking at multiple examples linking multiple representations was new, and as the course progressed she took more initiative to use CAS to explore mathematics for herself. While other students had chosen to take this mathematics course, for Yvonne and Stephen it was not optional. Yvonne saw herself as being mathematically poor and did not expect to cope. She learnt successfully to make use of the facilities of CAS to support her learning. Her work, supported by CAS, increased both her level of confidence with algebra and her degree of Insight.

Louise only made progress in Linking Representations. She had used a graphical calculator at school but found the full sized screen with colour made patterns clear. Louise was casual about her work; her lack of attention to detail may explain why she did not make the same progress in Algebraic Expectation. The graphical exercises she completed required her to use multiple examples, linking multiple representations.
Producing families of curves with DERIVE and making sketch copies drew her attention to differences and details which were fairly obvious on the large coloured graphs. However, identifying key features and structure of expressions required much greater concentration. Louise was content to be making some progress in the course and was not aiming to gain a high score for her assessment. She often approached this work casually without monitoring her CAS solutions for errors. This was seen in her failure, for example, to check whether the expression on the screen matched that on her page. Dugdale et al (1995) also comments that:

One important contribution of computers has been to provide environments in which it is easy to learn from mistakes – analyze errors, try again, and experiment within the constraints of a mathematically-accurate environment. (p332)

Students like Louise, who did not bother to check for errors, or Jocelyn and Samuel, who resisted the use of CAS, did not benefit from this facility.

Stephen had the highest level of both Algebraic Expectation and ability to Link Representations on the initial testing. He was a conscientious student who aimed to understand each task he was given. The Quiz scores suggested that his Algebraic Expectation improved but his ability to Link Representations declined. His Interview only showed indication of improvement. Stephen’s anxiety about the Post-course examination, which directly followed the Post-course quiz, may have affected this Quiz score. Stephen took care with the exercises during the course, noted any errors, analysed them and tried again. He made neat sketches of graphs and recorded both the details from CAS and his by-hand working in a careful logical fashion. Stephen also made the effort to write explanations of his answers (as demonstrated in figures 7.14, 7.15).

Greg made great improvement in his Algebraic Expectation but little in Linking Representations. Greg had used a graphical calculator at school. At the Post-course interview he said that he preferred to use this rather than DERIVE and he chose to supplement his CAS use in the examination. As a fairly compliant student, he rarely used his graphical calculator in class; never-the-less it could be assumed that, outside of class, where possible, he used his graphical calculator for graphs, in the same manner that he had at school (see figure 7.17). The mathematics he had studied in year 12 placed emphasis on numeric approximation rather than exact or symbolic solutions. So,
in continuing to use graphical calculator methods rather than CAS, he would have missed the change in focus for this course where the emphasis was not the graphs alone, but on the links between the symbolic and graphical representations of functions. The symbolic facility of CAS was used to find general and exact solutions for zeros and calculus, and these were linked to key features of the graphs. Numeric approximations seldom make these patterns transparent. This would account for Greg’s lack of improvement in ability to Link Representations.

Greg worked slowly, by hand, but was very neat and careful about his work. The use of CAS enabled him to look at many symbolic examples in a short space of time. The initial testing showed that Greg had no understanding of the use of letters to represent variables in algebra. Many problems in this course required students to look at results of some process, for example differentiation (see figure 7.12), for families of functions. Greg’s careful copying of DERIVE output meant that he came to recognise and understand the use of letters to represent variables and became more aware both of key features and the structure of expressions. A shift in Greg’s thinking demonstrated in the Pre- and Post-course Algebraic Insight Interviews was of critical importance. The change was a radical improvement in element 1.1, recognition of conventions and basic properties and in particular the meaning of symbols. Since Greg had been ‘successful’ in a middle-level, year 12, mathematics course, he had clearly been working with algebra for six years without this basic understanding. By the end of this fifteen-week course there was evidence that Greg has changed at least some of his schema. Since the different teaching and learning experiences of this course, described above, provided the main input to his mathematical thinking during this time, it seems reasonable to attribute this improvement to the style of learning facilitated by CAS.

Samuel showed improvement in Algebraic Expectation but little overall in ability to Link Representations. Sam was one of the students who clung to the use of their graphical calculators for graph work outside of class and in examinations. Like others in this category he showed little improvement in his ability to Link Representations. The effect of Sam’s use of a graphical calculator along with his lack of careful observation and copying was shown in the example included as figure 7.20. While Sam did not like using DERIVE even his limited use forced him to examine the structure of
expressions in order to use brackets correctly for the process of entering them into the CAS.

Jocelyn was yet another student who avoided the use of DERIVE but made use of her graphical calculator. Her change in Algebraic Insight followed the same pattern, some improvement in Algebraic Expectation but not in ability to Link Representations.

*Algebraic Insight Improved – but why?*

The use of CAS facilitated a ‘reformed’ approach to teaching functions and introducing calculus. Its availability allowed the teacher to demonstrate, and the students to explore, many parallel examples, use different representations, and observe correct results in a very short space of time. The *purpose* of the exercise was therefore seldom lost in the actual *process* of obtaining these answers. For a majority of the students, this was a fresh approach that clearly had some impact on their basic understanding and knowledge of mathematics. The availability of CAS to act as ‘scaffolding’ (Kutzler, 1994) or to check ‘mental’ algebra also boosted the confidence of most students. The ease of handling mathematics encouraged them to do more examples and sometimes to initiate their own investigations by making conjectures, analysing the results, and trying again until they were satisfied.

CAS facilitated positive learning styles in the classroom. The researcher had observed this in previous studies and reported it in some detail (Pierce and Stacey, 2001b). In this study the students exhibited good camaraderie. This was supported by the dynamics of classes held in computer laboratories where students, who could see each others’ screens, were encouraged but not compelled to work collaboratively. In this environment they both discussed their work with each other and used CAS to test their conjectures.

Many factors contribute to students’ learning: student motivation, quality of teaching, rapport between students and teacher, quality of learning materials, to name a few. It seems reasonable to attribute much of the improvement in these students’ Algebraic Insight to the curriculum, teaching exercises, and style of learning made possible by the availability of CAS.
**Does Effective Use of CAS change during a course taught using CAS?**

Effective Use of CAS, as described by the framework, certainly changed during the course but the change was not uniform across the two aspects, nor across the elements within the Personal aspect. Some students responded positively, while others responded negatively, to the use of CAS for learning mathematics. For this reason, it does not seem sensible to look at change in Effective Use of CAS as a whole but rather to move immediately to examining the changes at the level that the framework calls ‘aspects’.

*Was there evidence that the Technical aspect of students’ Effective Use of CAS improved?*

First it is important to remember that the level of difficulty experienced by any student was never very high. No student scored more than 50% on the Technical difficulty scale at any time during the course. Since the user interfaces for CAS have been improved since the early version used during this study, we could expect students now to experience even fewer Technical difficulties.

The Technical aspect of Effective Use of CAS was measured by considering the level of difficulty experienced by students in entering syntax, changing representations and interpreting CAS output. The pattern of change shown, by the students, during this course, was similar for each of these elements. A quick glance at the summary figures at the beginning of each case study (see for example figure 7.2) shows that students found more difficulty with entering syntax and interpreting CAS output than with changing representations. These figures clearly illustrate the data, showing that between the beginning and middle of the course students improved in the Technical aspect of Effective Use of CAS: that is they experienced fewer difficulties. The figures also show that this trend reversed in the latter part of the course when some students reported more difficulties than at mid-course. As has already been discussed in Chapter 6, it seems that there was an initial overhead in learning to use the CAS. This was followed by some ongoing difficulties such as typing and copying correctly from the written text and then there were new difficulties when dealing with new mathematics and its associated new commands. In this course the use of CAS was introduced with, what was expected to be, familiar mathematics as suggested by Berry (1996). This seemed to work as an initial strategy for most students but inevitably when CAS is being used both to learn and to do mathematics there will be stage where both the mathematics and the CAS
features are new. Then new difficulties come from both two sources: analysing the features of unfamiliar expressions as well as new CAS symbols or commands.

Some of the difficulties experienced by students during this study related specifically to the use of DERIVE 2.55. This is now a very outdated version of this program. (Students taking the same course in 2001 used the version built into the hand-held TI-89.) Newer versions of this CAS are more ‘user friendly’, especially for those who are familiar with a Windows environment. One particular feature of DERIVE 2.55 which students found embarrassing was that syntax errors were drawn to the users’ attention with a ‘beep’ as well as an error message. While some students made light of this, it was not helpful for those few students who were anxious about the use of technology, and it discouraged them from practicing its use in the ‘public’ laboratory situation.

Jocelyn said that using the computer ‘scared her’ but in fact, when she did use CAS she experienced few difficulties. Students like Jennifer and Greg who experienced a range of technical difficulties were not discouraged by this but continued to ask for help and make progress. They overcame their difficulties with the syntax of each new command and improved in their ability to change representations and interpret CAS output.

Was there evidence that the Personal aspect of students' Effective Use of CAS improved?

The Personal aspect of Effective Use of CAS is made up of two elements: 2.1, positive attitude and 2.2, Judicious Use of CAS. It became clear during the study that students’ attitude towards the use of DERIVE 2.55 affected their whole experience in this course, so these two elements will be discussed separately: positive attitude followed by Judicious Use of CAS.

Improvement in positive attitude:

Throughout the course the class could be divided into those with a positive and those with a negative attitude towards the use of DERIVE 2.55 for learning mathematics. For those who began the course with an open mind and an even mildly positive attitude there was improvement during the course. As they gained confidence in using the program, they widened the range of use of the program and, as a consequence, their attitude to the place of CAS in their learning became more positive.
It should be noted that a positive attitude did not preclude Technical difficulties. Three of the four students described in the case studies as having a positive attitude still reported experiencing a wide range of Technical difficulties during the course. To attain a high level of Effective Use of CAS each student must learn how to operate the program correctly. Feeling good about its use is not sufficient.

No student wanted to replace all of his or her by-hand mathematics with the use of CAS. Each student said that they preferred to do ‘easy’ problems by-hand. However, perceptions of what constituted an ‘easy’ problem differed from student to student. There was also observable change during the course. When the students used CAS to develop or support patterns and rules in mathematics, (for example rules for differentiation and anti-differentiation), they would initially use CAS but later, as they gained speed and confidence, many moved to doing this processing mentally or by-hand.

Those students who initially had a negative attitude towards the use of CAS made some progress in this element of Effective Use of CAS during the course. On the Post-course survey and Interview each suggested some appreciation of the place of CAS, such as in doing some problems quickly, but their attitude could only be described as ‘less negative’ than early in the course.

Students with a negative attitude did not necessarily experience greater Technical difficulty but they did make more limited use of CAS.

_Improvement in Judicious Use of CAS:_

All students made some improvement in the level of their Judicious Use of CAS. As might be expected, students initially used CAS only for functional purposes and this use was non-discriminating. In fact they used CAS for simple problems which they might easily have done mentally or by-hand. At this stage most students were learning to use the program rather than learning new mathematics. In order to check their CAS results, students often chose to use CAS for questions that they felt they could confidently do by-hand. They wanted to be sure that they were using the CAS program correctly. All students progressed to undertaking limited pedagogical use of CAS, at least when this was explicitly suggested. Some students took the initiative to use the facility to follow their own variations on problems thus undertaking extended pedagogical use of CAS.
Most students also became discriminating in their functional use of CAS, choosing to use CAS when they thought the problem would be time consuming or difficult. It was interesting to see that the majority of students liked to try to work problems by hand and would often choose to work in parallel with CAS, using its facility to check their by-hand work.

*What were the characteristics of those students whose Effective Use of CAS improved?*

It is the view of the researcher that students may be said to demonstrate a high level of Effective Use of CAS if they are not impeded by Technical difficulties, demonstrate Judicious Use of CAS, and have a positive attitude to the use of CAS for learning mathematics. Their Effective Use of CAS will have improved if they show improvement in one or more of its elements.

Experiencing some level of Technical difficulty is common for most people learning to use new hardware or software. With practice, program syntax becomes familiar and the user develops, to become fluent. Working through this learning stage requires a positive attitude. We see an example of this in the case study of Jennifer. Relative to other students she experienced a high level of Technical difficulty throughout the course but persisted, patiently making corrections and overcoming this hurdle. She progressed to be able to demonstrate a high level of Judicious Use of CAS. It can be seen, by the results across the class, that a low level of Technical difficulty did not impede the use of CAS for pedagogical purposes.

Students improved in their Effective Use of CAS when they practised CAS use. Some students did not practise because they did not like using CAS. Others, like Stephen, could do most of the work easily by hand and so used CAS less. Sam and Jocelyn provide us with examples of students with a negative attitude towards CAS. Both avoided the use of CAS. Sam had considerable Technical difficulty throughout the course and did not progress to using CAS in a strategic manner. Jocelyn was honest and open about her feelings when she said that computers scared her. In fact she experienced a low level of Difficulty when she did use CAS but this did not change her attitude.
The graphical calculator factor

A common feature of those students with a negative attitude, who did not wish to use DERIVE 2.55, was their preference for their familiar graphical calculators. In contrast, those students who had not previously used a graphical calculator commonly commented that they appreciated the graphing facility of DERIVE. Those who had graphical calculators made some comment that the colour helped them to distinguish graphs with DERIVE but the zoom facility of DERIVE version 2.55 was limited and there was no trace facility. In this respect the CAS used was technically inferior to their graphical calculators. Learning to use CAS is more complex than learning to use a graphical calculator so perhaps students felt that the symbolic facilities of CAS did not offer sufficient advantage to warrant the learning overhead.

The courses that many of these students had studied at secondary school, and the facilities of their graphical calculators, encouraged the students to make use of numeric approximations. CAS, on the other hand, facilitates the use of general and exact solutions. While numeric approximations can be helpful for checking the reasonableness of solutions for real-world problems, they do not reveal the patterns within the symbolic representation nor some of the links with graphical representations. The relative value of approximate versus exact answers has been and will be a subject of discussion between mathematics teachers. As mentioned in Chapter 2, for example, Waits and Demana (1992) suggest that the usefulness of exact CAS solutions may be over emphasized. Who, they asked, apart from mathematics teachers, find beauty, much less meaning, in a solution like $\frac{1+\sqrt{2}}{3}$? Numeric approximations for maximum or minimum values and intercepts may be found merely by using the ‘trace’ facility of a graphical calculator. Exact or general solutions require the setting up and solution of equations and the use of calculus. This requires a higher level of mathematical understanding. Some of these students had more than a year’s experience of using their graphical calculators, and had well established routines for finding approximate values, so perhaps their attitude towards CAS may change over time.

The experience described in this study concurs with that of Drijvers (1999). He observed that his students had far more difficulty becoming aware of the facilities of CAS, compared to their graphical calculators. Artigue (2001) wrote about the
importance of the institutional value of various technologies. The students who had used graphical calculators for their ‘high stakes’ Victorian Certificate of Education examinations assumed that these skills should retain their high institutional value. They did not adapt easily to the change. Perhaps this personal factor is the clue to the students’ response. Piez and Voxman (1997) wrote about their experience of introducing their students, who had previously focused on analytical work by-hand, to graphical calculators and the use of multiple representations of functions:

…we model how to use different representations to solve a problem. However, in our work with students, we noticed that students often preferred a certain type of representation and that their preferences were difficult to change. We found that our students often did not use graphs to solve problems or check solutions…(p164)

Their students preferred familiar representations facilitated by familiar by-hand methods, ours preferred familiar graphical calculator representations and methods. This suggests that the graphical calculator factor may be a transient issue, but it had an important impact on the results of this study.

Other factors

Several other factors seem to have an effect on students’ attitude to the use of CAS for learning mathematics. One is their school experience of mathematics: after 13 years of by-hand mathematics, many students felt some unease with the legitimacy of a partnership with CAS. They were unsure whether this was ‘really’ doing mathematics. Some time was spent, in class, discussing what they did and what the machine did. As a result, most students came to see that the user directed the machine’s contribution. In addition, those students who planned to be teachers expected to have to teach mathematics without technology in schools. This was a legitimate concern and for this reason, while CAS was used to enhance and support learning, time was still spent learning by-hand techniques and, even though CAS was available for all assessable tasks, demonstration of a basic knowledge of by-hand techniques was required.

A second factor impacting on students’ attitude to the use of CAS was the enjoyment that some students find in successfully completing routine calculations by-hand. It appeared that some students felt that the simplicity of carrying out such manipulations with CAS devalued competencies they had found hard to achieve and to some extent they resented the CAS. This, together with the consistent response from students that they ‘felt’ that they learnt more effectively by using by-hand methods, underlines the
need for more research on the place of CAS and the place of by-hand methods in learning mathematics.

A third factor was that effective pedagogical use of CAS required thinking use of CAS. Even at a responsive level but certainly for a strategic manner of use students needed to engage in the process. Louise exhibited a lack of this engagement when she ‘just tried things’ without thinking first about what she was trying to achieve or what the results meant. This lack of engagement impeded her progress. This result concurs with the expectation of Salomon, Perkins and Globerson (1991) referred to in Chapter 2.

All students showed some improvement in their level of Effective Use of CAS. At the end of the course all students agreed that CAS could be useful sometimes. Over the fifteen weeks of the course, a low level of Technical difficulty persisted and even increased briefly with the introduction of new commands. However, the level of difficulty experienced did not present a significant obstacle to any of these students’ learning. In this course, those who declined the opportunities afforded by CAS experienced limited progress as the structured learning exercises assumed CAS would be used. Graphical calculator approximations and by-hand work could not replicate the work on patterns in families of functions, variations on problems, testing conjectures and retrying after failure that CAS facilitated. The ease and speed of CAS supported these tasks.

What were the links between Effective Use of CAS and Algebraic Insight?
First consider this briefly in relation to each of the students detailed in the case studies.

Jennifer had moderately poor Algebraic Expectation skills but very poor ability to Link Representations. She experienced Technical difficulties using CAS. Some of these difficulties came from her lack of ability to identify structure and hence bracket expressions appropriately for entering into CAS. However, she persisted with CAS use and increased her level of Judicious Use of CAS. She particularly liked the facility to link symbolic and graphical representations and said that she found this very helpful. Jennifer reported least Technical difficulty changing representations. Exploring families of functions and their derivatives both symbolically and graphically drew her attention to form and key features. The impact of this work is shown in Jennifer’s improvement in Linking Representations.
Louise’ use of CAS was unthinking and not strategic. She showed little improvement in Algebraic Expectation but improved in Linking Representations. She said that she especially liked using the links between symbols and graphs in DERIVE. Although Louise had studied a middle level mathematics at year 12, which required students to look at graphs and families of functions, she had limited experience of using technology for this purpose. The use of CAS gave her a fresh view of this mathematics. The links between form and shape could have been made just as well with a graphical calculator but some key features are made more transparent with CAS exact and general solutions.

Stephen showed the highest level of Algebraic Insight of students in this class and his Algebraic Insight Interview suggested some improvement during the course. Like many other students in the group he preferred to leave the use of CAS for those questions which he saw as difficult or time consuming. This meant that he made little functional use of CAS for his own mathematics although he did help Yvonne to use CAS for hers, pointing out key features and links between representations. He did not have a negative attitude towards CAS for his own use, but rather accepted that it had a place. Perhaps because he did not get much practice using CAS he felt that he had quite a lot of Technical difficulties but the observations made during the Post-course examination showed that he exhibited the highest level of Effective Use of CAS of anyone in the class during this time. His manner of use was strategic; his functional use discriminating; he made use of CAS for exploration (as needed to undertake extended pedagogical use); and his use was successful.

Yvonne initially demonstrated a lack of Symbol Sense, a part of which was her Algebraic Expectation. This was so poor that she could not use CAS. She did not know what it was that she wanted CAS to do for her and she did not know how to begin to enter expressions into CAS because she could not identify structure or use brackets. Fortunately Yvonne was a confident computer user and had little difficulty learning to use the program. She worked with Stephen who helped her through the initial stages. She was used to the rigour of computer syntax and quickly learnt how to enter expressions and correctly use the various commands. Once she gained enough Algebraic Insight to enter expressions, Yvonne used CAS effectively to explore examples. This in turn helped to improve both her Algebraic Expectation and ability to Link Representations.
Greg initially also showed very poor Algebraic Expectation but he also had a negative attitude towards the use of CAS. In the early stages of the course he avoided using CAS, preferring to use his graphical calculator when possible. Greg had problems entering expressions and interpreting CAS output. He was a fairly compliant student and knew that he was expected to learn to use CAS. With help, he started to use CAS for functional purposes. He responded to the immediate feedback from the CAS to syntax errors. Working on entering expressions correctly forced him to think about the structure of expressions and carefully recording CAS output from multiple examples helped him to gain an understanding of the meaning of symbols. In this way, the learning he needed to do in order to use CAS, and his subsequent improved Judicious Use of CAS, both contributed to his improvement in Algebraic Insight.

Samuel had both poor Algebraic Insight skills and a negative attitude towards the use of CAS. He was resolute in his use of his graphical calculator over CAS and failed to show much improvement in either Algebraic Insight or Effective Use of CAS during the course. His attempts to do the course without the use of CAS did not result in the levels of improvement seen in those students who did use the facilities of CAS. As stated previously the numeric approximations offered by graphical calculators do not draw students’ attention to patterns and key features made transparent by CAS’s exact and general solutions.

Jocelyn showed that she had skills in both Algebraic Expectation and Linking Representations at the beginning of the course. She had a negative attitude towards the use of computers and avoided the use of CAS, choosing instead to use her graphical calculator for many questions. In this way she failed to engage the use of CAS for learning mathematics but would still have done exercises by hand. She showed little improvement in Algebraic Insight during the course.

These case studies remind us that learning is always an individual process as each student brings to a course a variety of prior knowledge, skills and attitudes based on their past experiences. While a small-scale study like this highlights this diversity, it can also inform our expectations of the relationship between Algebraic Insight and Effective Use of CAS. It seems that students need some basic level of Algebraic Insight in order to begin using CAS (Yvonne). Then if students are willing to use CAS
carefully (Jennifer, Greg, Stephen, Yvonne and to a lesser extent Louise) the learning style and exercises which CAS facilitate will provide experiences that will be absorbed into students’ schema, leading then to improved Algebraic Insight.

The results of this study support the assertion of Salomon, Perkins and Globerson (1991) that when students use technology the ‘potential realised depends on volitional mindful engagement’ (p4). There are, of course, many potential benefits from the use of technology and this study only focuses on the potential for CAS-use to support improvement in Algebraic Insight. Students who had a positive attitude, and set their mind to using CAS for learning mathematics, showed a clear improvement in their Effective Use of CAS and their Algebraic Insight (Jennifer and Yvonne for example). Students who avoided the use of CAS (Samuel and Jocelyn) did not. Louise’s engagement with CAS was willing but not mindful and as a consequence her learning was limited.

Throughout the course students reported, and were observed to have, fewer difficulties with ability systematically to change representations, than with fluent use of program syntax and ability to interpret CAS output. Difficulties with program syntax were commonly seen as students tried to enter expressions, so many of their difficulties occurred at the interface of by-hand mathematics and CAS use. These seemed to depend, in part, on their Algebraic Expectation, especially recognition of conventions and basic properties and their ability to identify structure. Students like Stephen and Jocelyn, who showed good Algebraic Expectation had little difficulty demonstrating a high level of Effective use of CAS when they wanted to. Levels of Algebraic Expectation and Effective Use of CAS were clearly inter-related but, while moderate to high Effective Use of CAS supported improvement in students’ ability to Link Representations, the converse was not as obvious. However, as students learnt to link the symbolic representation of simple functions to their alternative graphical and numeric representations (by noting key features of the symbolic rule and linking this to key features of the graph or table of values) they built a set of skills and expectations. This, in turn, encouraged them to undertake extended pedagogical use of CAS since it supported their exploration of more complicated functions. As a result of this process they showed improvement in their Effective Use of CAS.
This concludes the discussion of the changes in Algebraic Insight and Effective Use of CAS. The next section will review the two frameworks.

**Discussion of the two frameworks developed for this study**

The decision to explore Algebraic Insight and Effective Use of CAS as two key components of teaching and learning mathematics with CAS required the author to define these terms. This was done in Chapter 3 where a framework for each was proposed and the aspects, elements and some Common Instances of elements described. Each framework was developed and refined, in the light of practice, over some years and it is their most recent form that is presented in this thesis. The next section of this chapter outlines the reflections of the teacher/researcher on the usefulness of each framework in practice.

**Discussion of the framework for Algebraic Insight**

As the teacher/researcher involved in this project I found the Algebraic Insight framework helpful for planning teaching and analysing students’ work. This is hardly surprising since the framework went through a number of revisions in order to capture the key thinking needed when students work and learn in partnership with CAS. The division into aspects emphasised the need to look at students’ thinking as they enter expressions, work within the symbolic representation and interpret this output, and then as they change representation and make links between the symbolic representation and numeric or graphical representations. It became clear that, as expected from the literature, students need a good understanding of symbolic algebra in order to make use of CAS. This understanding is described specifically by Algebraic Expectation. Both good Algebraic Expectation and good ability to Link Representations contributed improvement in students’ level of Effective Use of CAS.

The framework was helpful for providing a direction for teaching strategies when working with students like Yvonne and Greg. For example: the Early-course Algebraic Insight Quiz and Interview highlighted students’ weakness in identifying structure. This was then targeted in a pair of worksheets (worksheets 7a and 8a included in Appendix 1.4). These sheets aimed to focus students’ attention on identifying the structure of fractions and composite functions. Reference to the framework encouraged me, as the teacher, to formally target these weaknesses. These brief exercises did not totally
remedy the problems but their ability to correctly bracket expressions for CAS entry did improve.

One of the Technical difficulties which students commonly acknowledged was the problem of setting an appropriate graph window. The Common Instances of the element ‘linking of symbolic with graphic representations’ drew attention to the knowledge needed to decide on an appropriate graph window –that is, which would show intercepts and critical points. Knowledge of this problem was important for directing teaching toward the links between form and key features of the symbolic representation of functions and their graphical counterparts. This thinking can be paralleled in the linking of symbolic and numeric representations but this was used very little within the teaching of this course. Limited use was made of tabular, numeric representation of functions, because their use in DERIVE 2.55 required the use of a command specifying the starting and end values for the independent variable along with the increment required. These rigid limitations meant that more forethought was required to use tables than graphs. With hindsight I can see that, in my teaching, I did not model the use of tables except for exploring limits and area under curves. As a consequence this section of the framework, Common Instances of this element, has in practice not been tested as much as the other elements and Common Instances outlined in the framework.

When students made errors, the framework provided a structure for classifying the error by pointing to Common Instances of elements that could be the focus of teaching. The framework also helped provide focus for my teaching in terms of the language I used, the questions I asked and the emphasis made in examples. I found that in this way the framework explicitly provides students with a ‘check list’ for functions. For example, students might ask themselves: “What is the form of this function? What is the structure of this function? Is it a composite function and if so which part is the ‘object’? Which is the dominant term? Will this function have zeros? What form are these solutions likely to take?” Mathematicians have always used this thinking to monitor their results, but many students of mathematics have not; rather they have focused on the techniques of algebraic manipulation. This framework outlines thinking which is important whether the ‘calculations’ are done by hand or CAS. The point is that when
CAS is used this thinking is still required in order to monitor working, for detecting errors in entering expressions, or simply pushing the wrong buttons.

After decades of teaching this level of mathematics to secondary and early tertiary students the framework has provided me with a fresh way of looking at students’ thinking, especially their errors in the context of working with CAS. The structure and terms used in the framework are now sufficiently refined that it requires little explanation to colleagues. They too can look at it and suggest examples of such insight, or lack of insight, from their students.

The explicit use of the Algebraic Insight framework became a routine habit so that when introducing a new algebraic expression students were encouraged to think about what they already knew about its symbols, structure and key features before they move further into the question. Considering the various aspects of Algebraic Insight provided students with a basis for deciding what mathematical processes might be appropriate in the solution of a problem and for monitoring that process when using CAS.

**Discussion of the framework for Effective Use of CAS**

The Effective Use of CAS framework outlines elements that can help or hinder a student’s use of CAS. It highlights the fact that the use of CAS brings together the student and the machine. Technical difficulties can occur in this interaction as the student learns to master the program, thereby gaining fluent use of program syntax, ability to systematically change representation and ability to interpret CAS output. However, Effective Use of CAS is equally affected by the student’s attitude towards CAS use and the way in which they choose to use CAS; that is their level of Judicious Use of CAS. When a teacher observes that a student’s use of CAS is limited, each of these factors needs to be considered.

As a teacher I found this framework useful for thinking about the way my students interacted with CAS. For example if we look at two students, Yvonne and Jocelyn, neither experienced many technical difficulties using CAS but Yvonne made extensive use of CAS while Jocelyn did not. The framework alerts us to the necessity of also considering the Personal aspect of CAS use and, in this case, the element of positive attitude. In these cases the difference in attitude accounted for different levels of Effective Use of CAS.
There was one clear difference between the frameworks for Algebraic Insight and Effective Use of CAS. For the Algebraic Insight framework if a student showed a weakness in one element it was likely to reflect in other elements. In particular, if a student had difficulty identifying key features then this would impact on their ability to Link Representations. The elements of Algebraic Insight seem to have a strong positive inter-relationship. In contrast, within Effective Use of CAS, it was possible for a student to experience difficulty with fluent program syntax but still have a positive attitude or demonstrate a moderate to high level of Judicious Use of CAS. Similarly as student with negative attitude did not necessarily experience technical difficulties. The links here were as diverse as the students.

Discussion of the instruments developed for this study

Some information about students’ Algebraic Insight and Effective Use of CAS was taken from observation and tasks that were part of the course’s usual assessment tasks. However, the Algebraic Insight Quiz, Algebraic Insight Interview, Technical Difficulties Survey, Judicious Use of CAS Survey and Post-course Survey and Interview were all developed for this study. The next section of this chapter is a review of the strengths and weaknesses of these instruments.

Review of the instruments developed to assess Algebraic Insight

The Algebraic Insight Quiz

The aim of the Algebraic Insight Quiz was to provide a fast, simple method of gaining data about students’ level of Algebraic Insight. The internal reliability of the Early-course quiz (n=14, 22 items) was assessed yielding a Cronbach’s alpha of 0.7437, which suggested that on the whole the items measured related thinking. It is not surprising that this figure is not higher since the test was comprised of two different styles of items and reflected both aspects of Algebraic Insight.

For external validity the items to be included in the Quiz were viewed by colleagues. There was agreement that those items included were examples that required Algebraic Insight as described in the framework. The results from the Quiz were consistent with other information relating to Algebraic Insight, for example students’ worksheet and examination scripts. It can been seen from the case studies that students’ scores on the
Early-course Quiz were consistent with the difficulties they had in class and the
difficulties they had learning to use CAS.

The administration of the Quiz could have been improved. Some minor features of
layout and wording were improved after the Early-course Quiz but the time allowed for
each item was kept the same for all three quizzes to allow comparison of results. With
hindsight it could be seen that 5 seconds per slide was a little too fast for the first 11
items but 15 seconds was too long for that later items. (See Chapter 5) Allowing
students enough time to read the item and respond but not do calculations was difficult.
Perhaps moving to 7 and 12 seconds would be better. In this style of test even one
second more or less would make a difference. Since then another, similar, version of
this test has been used with a large number of year 11 students. In this study, reported
by Ball, Stacey and Pierce (2001) students were allowed 10 seconds for each item.
This proved too long for the matching items like 1 to 11 but not long enough when there
were a number of alternatives to read like 12 to 23.

The range of response choices for each item provided more detail for analysis than
merely whether the student was right or wrong. The ‘probably’ and ‘definitely’ options
gave the researcher information about the students’ degree of certainty of their response.
In the larger study, Ball et al used this to create a certainty index and explored the
relationship between item facility and certainty. The 'I have no idea’ response
discouraged students from guessing and distinguished missed responses due to lack of
time from lack of insight. For the purpose of planning teaching, it is easier to remedy
the ‘I don’t know’ status of a student’s knowledge than to change the thinking of a
student who gives a definite but incorrect response. These two situations require
different strategies, so being able to use the quiz to distinguish them is of great
advantage.

The Quiz gave some information about students’ strengths and weaknesses at the level
of the elements of the framework, but it was not suitable as a detailed diagnostic tool
because there were too few items. The level of concentration this timed response test
required of the students constrained the number of items that could be used.

The Algebraic Insight Quiz proved to be a useful instrument. It can be used to obtain
information from a large or small number of students, with everyone present. This can
be achieved in a short, tightly controlled time frame. While it is both quick to administer and quick to mark, the students’ responses provide a lot of information beyond the simple right-wrong dichotomy.

The Quiz could easily be used by others and adapted. The items used for the Quiz, in this study, were selected for their applicability to students commencing an introduction to calculus course. For students at other stages of mathematical development different items would be required, but the style of test would still be appropriate. For large numbers the response sheet could easily be changed to allow computer marking. In fact this style of test would be ideally suited for both computer administration and marking. In situations where data projection facilities are not available the use of time controlled screens in Powerpoint could be replaced by the use of very large ‘flash cards’ but these would be more difficult to handle.

Overall the Quiz is a simple and versatile instrument for screening or monitoring students’ Algebraic Insight. It is well known that students value those things that are assessed. The use of such a testing style for assessment would promote the importance of being able to identify structure and key features at a glance. This could encourage a habit of looking for these attributes each time a student sees a new expression: that is developing a habit of practising Algebraic Insight.

*The Algebraic Insight Interview*

For those students who participated, the Interviews provided the researcher with valuable information. Students’ verbal comments revealed very important information that would not have been apparent from multiple choice or constructed response items and was not evident from the Algebraic Insight Quiz. The value of considering individual students’ thinking is shown in studies like that of Schoenfeld, Smith and Arcavi (1993) who recorded and analysed the thinking of one student. Taking time to listen to a student explain their thinking, in detail, as they answer a mathematical problem is revealing, and commonly brings to light interesting and unexpected results which may have generality.
Two of the items were particularly successful in revealing levels of Algebraic Insight. First, question 3 (see appendices 2.4 and 2.5): this appeared to be a simple area problem but the use of three different letters to represent lengths meant that several different correct solutions were possible. Students’ correct and incorrect solutions, along with their verbal commentaries, could be ranked in terms of level of Algebraic Insight. Some students demonstrated their lack of understanding of the meaning of symbols while others not only understood the use of symbols but also could identify structure and key features. This was a very useful item.

Secondly, question 4 (see appendices 2.4 and 2.5): this item made use of ‘fake’ syntax to assess students’ recognition of structure, identification of key features and ability to link representations. Its strengths were in the range of elements that it assessed and, in this context, the simulation of using new CAS syntax. Again this item revealed a lot of information, including students’ understanding of the meaning of symbols.

Administering and analysing the Algebraic Insight Interviews was time consuming but they did reveal information, especially about students’ understanding of the meaning of symbols, which was not obtained in any other way. If, for example, the area problem had been set on a simple written test, students like Catherine would have appeared to have a correct response. Only, in listening to her ‘think aloud’ was it discovered that she was not content because she thought that a ‘letter’ could not be an answer. For these reasons the interviews were worthwhile and the items used were reasonably successful for this purpose.

**Review of the instruments developed to assess Effective Use of CAS**

*The Technical Difficulties Survey*

The Technical Difficulties Survey consisted of statements that reflected difficulties which the teacher/researcher had observed in the past and which reflected the three elements of this aspect of Effective Use of CAS. A reliability analysis of these items for a scale measuring the construct ‘technical difficulty’ yielded a Cronbach’s alpha equal to 0.9094 (n=19, number of items = 12). This means that there was a high degree of correlation between the score on each item and the overall score, suggesting that these items were all basically measuring the same attribute. The students’ responses to these statements matched the difficulties observed in laboratory classes and examinations.
except for Stephen, who reported a higher level of difficulty than was obvious in either classes or examinations.

One weakness in the survey was the lack of statements posed which referred to moving to, or using tabular representations. Some of the students’ problems in this area would have been indicated as syntax difficulties or in their written comments such as ‘I still have difficulty with the VECTOR command’. In the past, the teaching in this unit had placed little emphasis on tables, and so this was not a difficulty that was considered when the list of statements was constructed. As mentioned earlier, the specific difficulty of constructing tables using DERIVE 2.55 does not apply to more recent hand held CAS calculators like the TI-89 which has a button facility labeled ‘TABLE’. This survey instrument proved to be a quick and effective way of gaining information about students’ perceived difficulties, but the actual items are CAS and course specific.

The literature used to inform this study reported on students’ views on the use of CAS. More recent literature suggests that some students’ views about the use of CAS and their perceptions of difficulties they do or do not experience may not be a reflection of their actual skill in using CAS. Lagrange (1999) reported a study in which not only were students surveyed but they were also tested on their facility with home screen commands. In his study students’ facility did not match their perceived skill. Students indicated that they found the CAS easy to use but in fact they did not know how to use many basic commands. Students’ use of CAS was not explicitly tested in this study, but to validate students’ survey responses, and gain further information, the researcher made observations in class (as time permitted) and detailed observations during the two examinations.

The Judicious Use of CAS Survey

The Judicious Use of CAS survey was limited to a series of multiple response items. In the circumstances where the researcher was also the teacher it provided some basic information on how students saw the purpose and manner of their CAS use. It could be suggested that more accurate information would be gained by asking students to make notes or keep a journal that detailed their CAS use. However it is unlikely that this would have been successful since only a few students were compliant with the less arduous task of coding their worksheet questions as instructed.
This survey, in its present form, is not suitable for use with a large group, where it would be difficult to clarify these items and to collect data from other sources. Students’ possible responses to some items may be limited by the exercises done in class that day so the survey could be improved by asking some questions to clarify the nature of the mathematics and the style of questions for which CAS was used.

Since some classroom observation and discussion was possible while teaching and, with the help of a colleague, detailed observations were recorded during the examinations, the limitations of this survey did not adversely impact on this study.

The Post-course Evaluation and Interview

The items on the Post-course evaluation had been selected from surveys used in previous years and were satisfactory for collecting data on students’ response to the use of CAS (in this case DERIVE 2.55). It provided information on whether they felt the use of CAS had contributed to their mathematical learning; whether they felt more confident in the areas of mathematics for which they had been using CAS; and their preferred ‘tool’ for different circumstances.

Again the interviews with students were interesting, but time consuming. The interviews mostly verified details of students’ responses to the main issues addressed by the various quick, closed, surveys. The greatest advantage of the interview was that some students raised issues that had not been addressed in the survey, in particular their preference for graphical calculators over CAS. Recent changes in assessment and lowering prices have resulted in increased graphical calculator use among our local secondary school students, but the rapidity of the impact of this change was not predicted. Since few students in our previous studies had used graphical calculators this was not an issue I had anticipated and therefore it had not been addressed in any of the surveys. The nature of the interviews allowed students to talk openly about the methods and tools they had used and preferred. This contrasted with the closed response items used in the surveys. If the surveys had not been supplemented with at least some interviews, the degree of importance of students’ experience with graphical calculators may not have been recognised. This was certainly an important factor in this study.
**Further use of these instruments**

Further use of variations of each of these instruments would be recommended. The survey and interview instruments used in this study were of the typical styles used in mathematics education research and procured useful data. The closed survey response items were quick for students to complete, an important factor when repeatedly collecting data. In particular the items relating to Technical difficulties and the use of DERIVE could easily be adapted to obtain information about students’ use of other CAS. These self reports of Technical difficulties are of use because a student’s impression of difficulty (or lack of difficulty) can have as significant an impact on their attitude and learning as their actual difficulties. As expected, the interviews enriched the survey/quiz data and allowed for unforeseen factors.

The Algebraic Insight Quiz was a successful innovation because it provided a lot of data in a short time using a method that was not onerous for either the teacher or the students. It adaptability has already been shown in the study reported by Ball et al (2001).

This concludes the discussion of frameworks, instruments and results of this study. The conclusions of this thesis along with implications for future research are left to a final brief chapter.
CHAPTER 9

CONCLUSIONS AND IMPLICATIONS

This chapter presents the conclusions of this thesis, along with suggested implications for both teaching and research.

Conclusions from the results of the study

In a course that promoted the use of CAS in the teaching, learning and assessment of functions and introductory calculus, all students showed improvement in some elements of Algebraic Insight. This improvement was especially evident in students who, at the beginning of the course, demonstrated poor Algebraic Insight. It seems they had not ‘absorbed’ the concepts, nor developed the related abilities of the Common Instances, expected of students at this stage of mathematical study, despite their school mathematics experience.

DERIVE 2.55 is now outdated software but the findings of this study are not. While some items related to technical difficulty were CAS specific, and more recent versions of DERIVE and equivalent CAS have improved user interfaces, all CAS still require the use of syntax that is not part of conventional by-hand mathematics, the ability to navigate menus and the interpretation of CAS output. Given the emphasis of this thesis and the level at which results have been analysed, the use of DERIVE 2.55 does not devalue the relevance of conclusions presented here.

CAS facilitated improvement in Algebraic Insight

In this course, CAS contributed towards the improvement of students Algebraic Insight in the following ways:

- Converting conventional mathematics to the syntax required to enter expressions into CAS, especially the use of brackets to make meaning explicit, forced students to pay attention to the structure of algebraic expressions. Syntax errors received immediate feedback and stopped students from proceeding until the error was corrected. As a result students gained experience in comparing the expression on
the screen with that on paper. This required students to note both the structure and key features of algebraic expressions.

- Students’ careful observation of many correct examples, as facilitated by CAS (for example families of functions), promoted understanding of the use of letters, to represent variables and focused students’ attention on key features of expressions.

- The use of symbolic and graphical representations, as facilitated by CAS, confronted students with links between symbolic form and shape of graphs. After careful observation of graphs and key features of the symbolic form students became aware of further links between key features, shape, position and salient points. Students who persisted with using numeric approximations, facilitated by graphical calculators, did not gain the same level of advantage as those using CAS. They missed seeing the patterns that were only apparent in: CAS-generated; exact value; and symbolic answers. They were less likely to see links between symbolic and graphical representations. The use of tables contributed to students’ understanding of limits, critical points and discontinuity, but they were under-utilised in this course.

**Effective Use of CAS changed**

All students’ showed some improvement in Effective Use of CAS during the course but for any given student this improvement was not necessarily evident in each aspect of Effective Use of CAS.

- Greatest variation between students was shown in the element ‘attitude’. It was clear that a student’s attitude impacted their Judicious Use of CAS. Not surprisingly, students who felt “scared of computers”, or felt that the use of technology is somehow illegitimate avoided its use. Students with a negative attitude made very limited progress in their level of Judicious Use of CAS. Although these students did not necessarily experience a high level of Technical difficulty, when they encountered difficulty they were more likely to ‘give up’. Negative attitude was a strong indicator that a student would not demonstrate a high level on the other elements of Effective Use of CAS. On the other hand, a positive attitude alone was not sufficient to ensure a high level of Effective Use of CAS. Students with a positive attitude were more likely to show improvement in their
Judicious Use of CAS, but some of these students still experienced high levels of Technical difficulty. They were however, prepared to work at overcoming these difficulties.

- Students with very poor Algebraic Insight needed extra help to begin using CAS. Inability to identify structure meant these students had difficulty using brackets and entering expressions into CAS correctly. Even after entering expressions that did not return syntax errors, students who did not identify key features were unaware of typing errors and were unable to recognise patterns without help. These students benefited from one-on-one help from peers or their teacher.

- Students with an initial neutral or positive attitude towards the use of CAS showed most improvement in the other elements of Effective Use of CAS. Their use typically developed from non-discriminating functional use to include strategic, extended pedagogical use. This high level of Judicious Use of CAS was needed for students to gain most benefit from the use of CAS. Students who did not progress to strategic use of CAS showed little or no improvement in Algebraic Insight.

- CAS use is unlikely ever to be totally free of Technical difficulties. After overcoming most initial problems the introduction of new mathematics requires familiarisation with new CAS commands and so often presents new Technical difficulties. Copying errors and typing errors persist, as their equivalents do in working by-hand. It is important therefore that students have sufficient Algebraic Insight to monitor their work and a positive attitude towards the effort required when learning new commands or syntax.

- For CAS to be used as an effective pedagogical tool, students need to ‘engage’ in the learning process just as they do when working by-hand. This means that their manner of use needs to be strategic and they must be prepared to undertake at least limited pedagogical use of CAS. CAS does not provide a magic panacea. Those students who did careful work and paid attention when copying from the screen gained most benefit.

- All students preferred to do some mathematics by-hand. In particular they would rather do the initial ‘simple’ problems in a new section of work by-hand. These students had little experience with the use of technology for mathematics, and expressed uncertainty at the idea of using CAS to explore new mathematics. They
had all gained some pleasure in the past from the challenge of doing mathematics by hand and saw ‘by-hand maths’ as having institutional value. It was also true that all students saw CAS as having a role in doing difficult or time consuming problems. As would be expected, students’ judgements as to what constituted a ‘difficult’ or ‘time consuming’ were individual.

Conclusions about the Frameworks

Algebraic Insight

The Algebraic Insight framework provides a brief, systematic summary of the categories of thinking which students need to develop in order to work in partnership with CAS. The study illustrates the usefulness of this framework as a checklist for planning teaching and analysing students’ errors. It provides direction for the development of new thinking ‘routines’ that should be practised in order to monitor work done with CAS.

Effective use of CAS

The Effective Use of CAS framework similarly provides a brief, systematic summary of the factors that contribute to students’ effective use of CAS. It has been common to report on either students’ technical difficulties or their attitude, but this study suggests that these should both be monitored, along with students’ Judicious Use of CAS, because the Technical and Personal aspects are not always positively related. Within one class, students’ responses, at the level of the elements of Effective Use of CAS, may be combined in many different ways. It was possible for a student to be positive and report few Technical difficulties, but not be making Judicious Use of CAS. It is equally possible for a student with a negative attitude to experience few Technical difficulties, but they are likely to limit their use to functional purposes.

Implications for teaching

The Algebraic Insight Framework provides a global approach to algebra at the secondary and lower tertiary levels. The aspects and elements are common to all these levels of algebra but the Common Instances will be specific to the level of work being studied. The framework provides a guide for curriculum planning, since it focuses on the thinking that is required to monitor CAS working or working by-hand on the ‘solve’ phase. There is no new algebra content suggested by the framework. What the
framework supports is a new focus. A change in focus from the ‘how to’ routines of factorisation, expansion, solution of equations, differentiation and anti-differentiation, to an analysis of expressions and the development of a breadth of experience in order to build schema for algebraic expectation and expected links to the numeric and graphical representations.

If CAS is being used in a classroom for teaching and learning mathematics, the Effective Use of CAS framework provides a reminder of the network of factors that impact on this experience. Such an analytical tool is new and will be helpful to teachers and researchers in monitoring the use of CAS in their class.

The Case studies showed that provided students have enough Algebraic Insight to identify the structure of expressions and recognise typing errors, using CAS in a way that demonstrates a high level of at least some element of Judicious Use of CAS can improve students’ Algebraic Insight. Initially this will require guided exercises but students can learn from these to use CAS to explore parts of mathematics for themselves.

**Implications for research**

This small-scale study has been suitable for developing and trialling the two frameworks and new instruments. More information could now be collected using extended studies: larger numbers of students, different year levels and over a longer period of time. This study also highlighted the need to explore the possible costs and benefits of using CAS instead of just graphical calculators.

**Extended studies**

This in-depth, small-scale study highlights the diversity of students’ Algebraic Insight and Effective Use of CAS. The two frameworks and the instruments (in particular the Algebraic Insight Quiz) used in this study provide a strong basis for large-scale studies which could be expected to distinguish groups of students exhibiting common characteristics. For large-scale studies, at any level, the Algebraic Insight Quiz response sheet could be modified to allow computer marking and therefore data entry.

At the early tertiary level, at which this study was conducted, such information would be helpful for planning both mainstream teaching and remedial support. Given their school
mathematics backgrounds, several of the students in this study showed surprisingly poor Algebraic Insight. It could be of interest to monitor students’ progress in Algebraic Insight, under the current curriculum, from their initial introduction to algebra.

This study was conducted over only a fifteen-week course, with students who had been studying and using algebra for more than six years. It is therefore pleasing that there was improvement in students’ fundamental thinking – that is their Algebraic Insight. If CAS is to be used, Algebraic Insight, appropriate to the age and stage of the students, should be a focus of teaching from the introduction of algebra. It will be important to monitor early attempts at such a change of emphasis. Similarly the Effective Use of CAS framework provides a basis for monitoring the development of students’ use of CAS.

**CAS versus graphical calculators**

In this study, students’ prior experience with graphical calculators proved to have a significant impact on their attitude to and use of CAS. The importance of this factor became apparent during this study, but, because it had not been an issue in the previous years that formed the basis of its planning, limited data was collected. In this current period, when both graphical calculators and CAS are being trialled in schools, there is perhaps a unique opportunity to compare the advantages and disadvantages of each for the teaching and learning of mathematics. Several issues associated with the contribution of the symbolic module from CAS could be explored. For example, in this study the researcher has contended that experience of exact and general solutions is superior for the development of Algebraic Insight than numeric approximations. This deserves further investigation. Also to what extent do students use Algebraic Insight to set up tables or graphing windows and to what extent do they repeatedly ‘Zoom’ in hope of a suitable view? With the availability of the symbolic module in CAS do students use a more strategic approach? Last, why were students who had used graphical calculators reluctant to embrace CAS? Was it an issue of technology – hand-held versus computer? Was it that they attributed more institutional value to graphical calculators than CAS? Did they feel that the gain from using a particular CAS, instead of a graphical calculator was not worth the overhead of learning to use this new technology? Was it evidence of a numerical or graphical approach to problem solving that could not easily switch to algebraic?
A final statement

This thesis began by reflecting on the tools used for mathematics over the centuries. Each of these tools has had an impact on what mathematics is done by the 'mathematician' and what role is allocated to the tool. In this progression Computer Algebra Systems are a catalyst for change in the teaching and learning of algebra in the same manner that arithmetic calculators have recently changed the focus in the teaching of arithmetic, and other calculation devices have changed it in the past. The findings of this study suggest that Effective Use of CAS and Algebraic Insight are interdependent:

◊ First, sufficient Algebraic Insight both to identify the structure of algebraic expressions being studied and to recognise typing errors is needed to begin to use CAS.

◊ Second, CAS can be employed successfully as a learning tool for exercises designed to improve Algebraic Insight provided the student demonstrates at least a moderate level of Effective Use of CAS.

◊ Third, Algebraic Insight helps students to use CAS in a strategic manner.

The study also showed the importance of attitude in determining students’ level of Effective Use of CAS. Students with a positive attitude persisted, despite Technical difficulties, in learning to use CAS strategically for functional (often non-discriminating) and extended pedagogical purposes. Students with a negative attitude made limited use of CAS, typically demonstrating a passive or responsive manner and undertaking only limited pedagogical use, even if they experienced few Technical difficulties.

The new outcomes of this study that will be of use to teachers and curriculum planners are the frameworks for Algebraic Insight and Effective Use of CAS, the quick Algebraic Insight Quiz and the CAS use survey. Of the terms coined for this study, perhaps the most useful will be Algebraic Expectation. The term indicates parallels with but differences from arithmetic estimation, a term that is now well accepted and understood. It sums up the mental skill which users of CAS must develop if they are to be able to monitor the algebraic work they share with CAS. These new terms and new frameworks provide a structure for the new focus that must be adopted for the teaching of algebra in a CAS environment.
REFERENCES


Philadelphia USA : J.B. Lippincott


APPENDIX 1

COURSE MATERIALS

APPENDIX 1.1 COURSE DESCRIPTION
APPENDIX 1.2 MID-COURSE EXAMINATION
APPENDIX 1.3 POST-COURSE EXAMINATION
APPENDIX 1.4 SAMPLE WEEKLY WORKSHEETS
APPENDIX 1.1 COURSE DESCRIPTION

University of Ballarat
SCHOOL OF INFORMATION TECHNOLOGY
AND MATHEMATICAL SCIENCES

Unit Description
Semester 2, 1999

UNIT CODE and TITLE: MA502 - Introduction To Calculus And Computer Algebra

PREREQUISITE: Year 11 Mathematics or MA501

CREDIT POINTS: 15

ORGANISATION:
Lecturer: Robyn Pierce         Room: T151         Ext 9283                     email:r.pierce@ballarat.edu.au

Lecture Times and Locations:
Monday 14:30- 15.30 T237
Tuesday 14.30-16.30 T237/lab
Thursday 13.30-14.30 S114
Friday

Computer Laboratories will be available at appropriate times.

OBJECTIVES:
The modern study of calculus has been significantly influenced over the past few years by the advent of sophisticated computer software that is capable of performing the often complex manipulations required in applying mathematical techniques in problem solving. Modern technology such as Computer Algebra Systems and graphical calculators are capable of carrying out a wide range of algebraic, numeric and graphical operations. The availability of these systems provides a means of removing much of the drudgery from algebraic and calculus-based problem solving and allowing the user to focus on more interesting aspects of mathematics and mathematical modelling. This unit takes advantage of the availability of this new technology to make calculus more accessible. The unit would be particularly valuable to prospective teachers and to students interested in improving their knowledge of mathematics with a view to careers in applied science and engineering.

This unit is designed to:

- introduce foundation concepts in algebra, trigonometry and geometry
- outline the development and application of calculus
- develop skills in the use of computer algebra systems
- introduce techniques and concepts from differential calculus
- introduce techniques and concepts from integral calculus
- develop skills in mathematical modelling.

More specifically, students completing this unit should be able to:

- use a computer algebra system to perform tasks including graph plotting and exploration, solving equations and algebraic manipulation
- understand and apply the properties of trigonometric functions
- understand the idea of limits in an informal sense
- investigate limits using a computer algebra system
- understand geometric ideas underpinning differential calculus
- apply the fundamental techniques from differential calculus in problem solving
- use a computer algebra system to investigate derivatives and rates of change of functions
- solve problems involving the application of differential calculus
- understand the basic concepts of integral calculus
- use a computer algebra system to investigate problems involving integral calculus
- understand the basic applications of numerical integration
- apply the principles of mathematical modelling to study a simple system using a computer algebra system.
CONTENT:

The topics covered in this unit are: an introduction to basic algebra, trigonometry and geometry necessary for the study of calculus, an introduction to the capabilities of computer algebra systems, differential and integral calculus - basic concepts and techniques, applications of calculus and an introduction to mathematical modelling. Throughout the unit the concepts and techniques will be introduced and developed by the guided exploratory use of a computer algebra system.

SEQUENCE:

<table>
<thead>
<tr>
<th>Week</th>
<th>Topic</th>
<th>Required Reading and learning task</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction to Functions, Graphs of functions, Introduction to Derive.</td>
<td>Weimer: 1.1, 1.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Worksheet 1</td>
</tr>
<tr>
<td>2</td>
<td>Constructing new functions, linear functions</td>
<td>Weimer: 1.3, 1.4</td>
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<tr>
<td></td>
<td></td>
<td>Worksheet 2</td>
</tr>
<tr>
<td>3</td>
<td>Polynomial, exponential and logarithmic functions</td>
<td>Weimer: 1.5, 1.6, 1.7</td>
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<tr>
<td></td>
<td></td>
<td>Worksheet 3</td>
</tr>
<tr>
<td>4</td>
<td>Slope of a curve and rates of change</td>
<td>Weimer: 2.1, 2.2</td>
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<td></td>
<td></td>
<td>Worksheet 4</td>
</tr>
<tr>
<td>5</td>
<td>TEST</td>
<td>Weimer: 2.3, 2.4</td>
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<tr>
<td></td>
<td>Limits, continuity, derivatives</td>
<td>Worksheet 5</td>
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<td>6</td>
<td>Graphs: characteristics and salient points</td>
<td>Weimer: 3.1, 3.2, 3.3, 3.4</td>
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<td></td>
<td></td>
<td>Worksheet 6</td>
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<tr>
<td>7</td>
<td>Product, quotient and chain rules</td>
<td>Weimer: 4.1, 4.2, 4.3</td>
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<td>Worksheet 7</td>
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<td>8</td>
<td>Antiderivatives and indefinite integrals</td>
<td>Weimer: 6.1, 6.2</td>
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<td>Worksheet 8</td>
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<td>9</td>
<td>Definite integrals and areas under curves</td>
<td>Weimer: 6.3, 6.4, 6.5</td>
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<td>Worksheet 9</td>
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<td>10</td>
<td>Applications of calculus: optimisation</td>
<td>Weimer: 3.5</td>
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<td>Worksheet 10</td>
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<td>11</td>
<td>Trigonometry and Triangles</td>
<td>Petocz: 11.1</td>
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<td></td>
<td></td>
<td>Worksheet 11</td>
</tr>
<tr>
<td>12</td>
<td>The Trigonometric Functions - Radian measure, definitions, graphs,</td>
<td>Petocz 11.2</td>
</tr>
<tr>
<td></td>
<td>equations, identities.</td>
<td></td>
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<td></td>
<td></td>
<td>Worksheet 12</td>
</tr>
<tr>
<td>13</td>
<td>The Calculus of Trigonometric Functions And Revision</td>
<td>Petocz 11.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Worksheet 13</td>
</tr>
</tbody>
</table>

LEARNING TASKS AND ASSESSMENT:

It is a work requirement of this unit that students maintain a folder containing a complete set of lecture outlines, notes based on the lectures and completed learning tasks and assignments.

The assessment tasks for this unit will comprise two assignments, one test and one examination.

The test will be on material covered in lectures during the first four weeks. Students will be expected to complete simple tasks without the aid of a Computer Algebra System; Derive will be available for use where appropriate. Students may bring a one page (2 sided) summary and a calculator to this test.

The final examination will include material covered in lectures, computer laboratory sessions and material from the prescribed text. DERIVE will be available for students to use where appropriate. Students may bring a two page (4 sides) summary and a calculator to this examination.
Assignment 1 will be ongoing. Each week’s worksheet will have some problems highlighted (bold and underlined). Solutions to these questions form assignment 1. Each week’s questions will be worth 3 marks and the best 9 results will form a mark out of 27 for the total assignment. It is a hurdle requirement of this course that you attempt solutions to these problems. Marks will only be awarded for work handed in by the due date. Worked solutions to worksheet questions will be placed on counter reserve on the Monday following the submission date. Extensions will not be given for instalments of this assignment.


Weimer, page 218 project C or D (not both)
Weimer, page 368 project B,
Weimer, page 499 project B

<table>
<thead>
<tr>
<th>Graded Tasks</th>
<th>Date Issued</th>
<th>Submission Date</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test (45 minutes) based on the learning tasks for Weeks 1-4</td>
<td>August 25</td>
<td>one week after date of issue</td>
<td>10%</td>
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<tr>
<td>Assignment 1</td>
<td>Weekly</td>
<td>one week after date of issue</td>
<td>27%</td>
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<tr>
<td>Assignment 2</td>
<td>Week 1</td>
<td>Friday October 15</td>
<td>18%</td>
</tr>
<tr>
<td>Examination (2 hours)</td>
<td>Final Examination Period</td>
<td>45%</td>
<td></td>
</tr>
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</table>

ASSESSMENT CRITERIA:

Tests and Examinations:
Marks for each question will be shown on the paper. Full marks will be given for complete solutions with detailed explanation/justification. Where a Computer Algebra System is used to perform some of the calculations, the role of the procedures used should be explained. Marks will be deducted for incomplete and/or incorrect answers.

Assignments:
Specific requirements for individual assignments will be provided with each assignment. Assignments should be typed or neatly handwritten.

While group discussion about any assignment is quite permissible, written reports must be entirely the student’s own work. Any collaboration in the exploratory work for the assignment must be clearly acknowledged.

For assignment two late submissions will have 25% of the possible marks deducted if they are up to one week late and 50% of possible marks deducted if they are more than one week late. No work will be accepted after November 26th, 1999.
ACADEMIC REGULATIONS:

Students are advised to acquaint themselves with the academic regulations regarding progress as outlined in the 1998 University handbook, and in particular note the regulations for special consideration in Statute 5.3, The Schedule Part 1, subsection 5.

Note that in cases of absence from a scheduled examination or lateness in submitting an assignment, it is the responsibility of the student (or other party) to notify the lecturer in charge of the unit or the general office of the School of IT & MS (phone 53 279270) on the day or as soon as possible thereafter. Details must be verified in writing within seven (7) days of the scheduled date.

A deferred examination will not be held during lecture times. All such examinations will be held at the end of semester during the normal examination/testing period, and will be included on the official examination timetable.

Supplementary information concerning teaching, learning and assessment may be provided from time to time. Announcement of these matters in classes and placement of a notice on the officially designated School Noticeboard shall be deemed to be official notification.

Regulation 5.3 – Assessment was amended in December 1997. The revised version is published on the official noticeboards.

TEXT BOOK:


REFERENCES:

Finney, R., Thomas, G.B., Calculus, Addison-Wesley
Gillet P., Calculus and Analytic Geometry, Heath.
Suppose the Bean-Inn has 1000 pumpkin and leek pies in stock and sells 20 per day.

(a) If \( y \) is the number of pies in stock at the end of the day, \( t \), find the relationship between \( y \) and \( t \).

(b) How long will it take to exhaust this supply of pies? Illustrate this graphically.

\[1 + 1 \text{ marks}\]

2. \( f(x) = 3^x \)

(a) Evaluate \( f(5) \)

(b) Show \( 3f(x) = f(x+1) \)

(c) Show \( f(a+b) = f(a)f(b) \)
3. Let $a$, $b$, and $c$ all be positive constants. Match the graphs with the rules for the following functions.

(a) $y = -a(x+b)^2 + c$

(b) $y = a(x+b)^2 - c$

(c) $y = -a(x-b)^2 - c$

(d) $y = -a(x-b)^2 + c$

4. Describe and use graphs and tables to demonstrate the differences between $s(t)$ and $r(t)$ when $s(t) = t^n$ and $r(t) = n^t$ if $n$ is a constant, positive integer.
5. If \( f(x) \) may be represented by the following graph draw sketch graphs of
   
   (a) \( f(x) + 3 \)
   
   (b) \( f(x-1) \)
   
   (c) \( 2f(x) \)

6. Find the equation to the straight line which passes through the points \((3, 17)\) and \((10, 66)\).

7. Solve for \( n \) \( \frac{0.6}{n} = 7.5 \) showing your working, step by step.
1. In writing up your answers to the following questions state any rule used (product, quotient or chain rules) and show all steps involved in reaching the answer.

(a) If \( f(x) = \frac{e^{3x}}{x^2 + 1} \) find \( f'(x) \)

(b) If \( r(\theta) = 3\pi \theta^5 \cos(2\theta) \) find the gradient of this curve when \( \theta = \pi \)

(c) If the distance (metres) travelled by a radio controlled boat in \( t \) minutes may be described by the rule \( P(t) = 2 + 3t - t^2 + 7t^3 - t^4 \) at what speed was the boat travelling after 3 minutes?

[2+3+4 = 9 marks]

2. Sketch the curve \( f(x) = 6x^3 - 5x \) then find the area enclosed by the curve and the x axis. Explain the reasons for each step.

[4 marks]
3. Find the following integrals. Show all steps involved in reaching the answer. You may use DERIVE to check your working but an answer copied with no explanation will receive no marks.

(a) \[\int \left(2x^5 + \frac{1}{x} + 5\sqrt{x}\right) dx\]

(b) \[\int \sin(8x) dx\]

[3 + 2 = 5 marks]

4. An entertainment promoter observed that if concerts were marketed too cheaply then few people attended. It seemed that the ticket price influenced perception about the quality of the concert. He noted however that the rate of increase in patronage tapered off as the tickets became more expensive. For example he noted that when tickets were $10 the rate of increase was 250 but at $15 the rate of increase in ticket sales was only 200.

The following screen dumps show the outline of the mathematics done by the promoter in order to estimate an optimum price for tickets. Please explain, by referring to each line of the DERIVE printout in turn, how the promoter worked out this optimum price. How would you interpret the result in line 10?

[7 marks]
5. Sketch the graph of \( p(x) = \frac{x^3 + 4x^2 - 25x - 100}{x - 1} \)

Use the graph to find each of the following and then confirm each result using algebraic methods.

(a) Find the coordinates of the x intercepts.
(b) Locate any asymptotes.
(c) For what values of x is \( p(x) \) increasing? 

[1+2+2=7 marks]

6. (a) Sketch the graph of the function \( g(x) = \frac{\sin 3x}{3} \) for \(-2\pi \leq x \leq 2\pi\)

(b) Write down

(i) the range of \( g(x) \).
(ii) the period of \( g(x) \).

(c) Sketch the graph of \( h(x) = \sin x + \frac{\sin 3x}{3} \) for \(-4\pi \leq x \leq 4\pi\)

(d) What is the period of \( h(x) \)?

[1+2+2+1=6 marks]

7. Demonstrate algebraically that \((\sin y + \cos y)^2 = 1 + \sin 2y\)

[3 marks]
8 The frequency of vibrations of a vibrating violin string is given by

\[ f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} \]

Where \( L \) is the length of the string, \( T \) is its tension and \( \rho \) is its linear density.

(a) Find the rate of change of the frequency with respect to the length if tension and density are constant.

(b) The pitch of a note is determined by the frequency (the higher the frequency, the higher the pitch). Use information about \( f \) and/or the rate of change to show what happens when the effective length of the string is reduced i.e. the player places a finger on the string so that only a shorter section vibrates.

[3 + 1 = 4 marks]
APPENDIX 1.4  SAMPLE WEEKLY WORKSHEETS

MA502   Functions   Week 1

This worksheet, including attempted solutions to those questions which are **bold and underlined**, is to be handed in by 5 pm Friday 7th August.

Note: DERIVE Algebra Window output should be copied as necessary and Derive Plots should be hand copied as sketch graphs. It is not necessary to print from DERIVE but its use should be acknowledged.

### Functions

**Weimer: Exercise 1.1**
- Basic Skills: 1, 2, 5, 6, 7, 13, 15, 16, 20, 23, 27, 28
- Applications: 31, 35, 39, 43
- Going beyond: 46, 49, 51, 52, 53
- Computer Applications: 56, 58, 63
- Making connections: 4

### Graphs of functions: Linear Functions

**Weimer Exercise: 1.2**
- Basic skills: 15, 25, 28
- Computer Applications: 47, 50-55, 56-59
- Making Connections: 1, 3

Each question being handed in should be clearly numbered and have a code added after you complete it. for example **Ex1.1 No 51  DA, E, G**

*The codes are responses to the following questions*

Method(s) Used

<table>
<thead>
<tr>
<th>PP</th>
<th>DA</th>
<th>DG</th>
<th>SC</th>
<th>GC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pen &amp; Paper</td>
<td>Derive Algebra</td>
<td>Derive Graph</td>
<td>Scientific Calculator</td>
<td>Graphical Calculator</td>
</tr>
</tbody>
</table>

I found this question

<table>
<thead>
<tr>
<th>VH</th>
<th>H</th>
<th>OK</th>
<th>E</th>
<th>VE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Hard</td>
<td>Hard</td>
<td>Easy</td>
<td>Very Easy</td>
<td></td>
</tr>
</tbody>
</table>

My understanding of this work is now

<table>
<thead>
<tr>
<th>VP</th>
<th>P</th>
<th>N</th>
<th>G</th>
<th>VG</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Poor</td>
<td>Poor</td>
<td>Neutral</td>
<td>Good</td>
<td>Very Good</td>
</tr>
</tbody>
</table>
DERIVE 1
Exploring DERIVE

Load DERIVE. This program is accessed through the Applications sub-menu. The following screen should appear:

Note:
- At the bottom of the screen you will see a line of commands. These are activated by
  - typing the capitalised letter from the required command. eg type L for soLve
  - highlighting the required command and pressing Enter
- At any stage you can move up one menu level by pressing Esc

TASK 1
- Try out the following instructions.
- Record the results and reminders to yourself about the different commands.
- Explore variations.

<table>
<thead>
<tr>
<th>DERIVE Commands</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author, ( x^2 + 5x + 6 )</td>
<td>Pressing A activates the Author command, type in the required expression then press Enter.</td>
</tr>
<tr>
<td>Factor, Rational</td>
<td>Press F to activate the Factor menu, check the number of the expression to be factorised, press Enter, choose the type of factors required then press Enter. Result: ( (x+2)(x+3) )</td>
</tr>
<tr>
<td>Expand</td>
<td>Expands highlighted expression</td>
</tr>
<tr>
<td>soLve</td>
<td>Solves (highlighted expression)=0</td>
</tr>
</tbody>
</table>
Repeat processes for $x^2 + 5x - 6$
highlight the expression $x^2 + 5x + 6$

Author F3
use Ctrl S to move the cursor left or

Ctrl D to move the cursor right and change +6 to -6 (pressing Insert activates Overwrite, note bottom line of DERIVE screen)

Now factorise $x^2 + 5x - 6$
Solve $x^2 + 5x - 6 = 0$

Manage, Substitute expression #1, overwrite $x$ with 5
Simplify

Find the value of $x^2 + 5x + 6$ if $x = 5$

Find the value of $x^2 + 5x - 6$ if $x = 2$

Author $ax^2 + bx + c = 0$
soLve (for a)
now solve $ax^2 + bx + c = 0$ for $x$

Manage, Substitute, press enter for $x$ and overwrite values $a = 1$, $b = 5$ and $c = 6$
Simplify
Repeat for second solution

Author $x^2 - 1$
Plot, Beside, 40, Plot

Graphs a parabola beside the algebra window starting at column 40.

Explore the effects of changing Scale. or Zoom. Tab between options, overwrite numbers, select alternatives by typing the capitalised letter.

Algebra
Author $x^2 + 5y$
Plot, Beside, 20, Plot

To move to the algebra window
Derive has 3 types of windows algebra, 2D plots and 3D plots

Window, Goto 1
Author $y = 2 - x^2$, Plot, Plot
<table>
<thead>
<tr>
<th>Algebra</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Author $z = xy$</td>
<td></td>
</tr>
<tr>
<td>Plot, Plot</td>
<td></td>
</tr>
<tr>
<td>Algebra</td>
<td></td>
</tr>
<tr>
<td>Author, $y = xy$</td>
<td></td>
</tr>
<tr>
<td>Plot, Plot</td>
<td></td>
</tr>
<tr>
<td>Algebra</td>
<td></td>
</tr>
<tr>
<td>Remove Start 1</td>
<td>End (last expression no.)</td>
</tr>
<tr>
<td>Plot, Delete,</td>
<td>All</td>
</tr>
<tr>
<td>Window, Close window 3</td>
<td></td>
</tr>
</tbody>
</table>
**DERIVE 4**

Differentiation

**TASK 1   Use DERIVE to differentiate polynomials**

Use the following sequence of DERIVE commands to help complete the table below.

- **Author** (type in required polynomial)
- **Calculus, Differentiate, confirm variable x and order 1**
- **Simplify**

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>First derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td></td>
</tr>
<tr>
<td>$x^3$</td>
<td></td>
</tr>
<tr>
<td>$x^5$</td>
<td></td>
</tr>
<tr>
<td>$x^{-1}$</td>
<td></td>
</tr>
<tr>
<td>$x^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$x^{-4}$</td>
<td></td>
</tr>
<tr>
<td>$x^{1/2}$</td>
<td></td>
</tr>
<tr>
<td>$x^{2/3}$</td>
<td></td>
</tr>
<tr>
<td>$x^n$</td>
<td></td>
</tr>
<tr>
<td>$3x^3$</td>
<td></td>
</tr>
<tr>
<td>$7x^9 + x^6 - 2x^4 + 5$</td>
<td></td>
</tr>
</tbody>
</table>
TASK 2   Explore differentiation and tangents

First Load the Utility file DIF_APPS as follows.
Transfer, Load, Utility, file name DIF_APPS

Next graph the function \( y = x^2 \) and the tangent to the curve at the point \( x = 3 \).

Author \( x^2 \)
Plot, Beside, Enter, Plot
Author TANGENT(x^2,x,3)
Simplify  (you may wish to use Expand to express the result in the form \( y=mx+c \))
Plot , Plot

Calculus, Differentiate, the expression \( x^2 \) highlight or correct expression number
Simplify
Manage, Substitute \( x=3 \)
Simplify

The gradient of the tangent to the curve \( x^2 \) at the point where \( x=3 \) is _______.
The value of the derivative of \( x^2 \) evaluated at \( x=3 \) is _______.

Use the above commands as a guide to complete the following table.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Slope of tangent</th>
<th>Value of derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^3 )</td>
<td>( at \ x=5 )</td>
<td>( at \ x=5 )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( at \ x = \frac{\pi}{4} )</td>
<td>( at \ x = \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( x^4-3x^2+x-5 )</td>
<td>( at \ x=2 )</td>
<td>( at \ x=2 )</td>
</tr>
</tbody>
</table>

Summarise your findings so far about tangents and derivatives
TASK 3  Differentiating from first principles

You should recall from lectures that the value of the derivative of $f$ at a point $x_0$ is the gradient of the tangent to the graph of the function at $(x_0, f(x_0))$. This relationship between derivatives and geometry is vital in many applications of differential calculus.

From first principles we have

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

which gives the gradient (slope) of the tangent at $P(x_0, f(x_0))$. The quotient

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

is the gradient of the approximating secant $PQ$. For 'small' values of $h$ it follows that

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

This observation enables us to define an approximate derivative using

$$m(x, h) = \frac{f(x+h) - f(x)}{h}$$

(for some 'small' value of $h$).

To avoid having to retype the difference quotient and to allow us to investigate this idea of an approximate derivative we will **Declare** two functions as follows:

**Declare**, Function
Type $f$ and press Enter *(giving the function a name)*
Enter *(we are not declaring a function value at this stage).*
Type $x$ and press Enter *(the function variable).*
Enter *(in response to the second request for a function variable).*

The expression $F(x):=\;\;\;$ appears on the screen. We now define the difference quotient $m(x,h)$.

**Declare**, Function
Type $m$ and Enter
Type $(f(x + h) - f(x))/h$ and Enter.

The screen should now display

$$M(x,h) := \frac{F(x+h) - F(x)}{h}$$

We can now investigate the relationship between $m(x,h)$ and $f'(x)$ for different functions $f$ and values of the increment $h$ very easily.
Exercise 1: Define the function \( f(x) = x^4 + x^2 + 1 \).

(Declare Function \( f \); then at the request for function value type \( x^4 + x^2 + 1 \)). Set the x-scale to 0.5 and plot this function.

Find and Plot the derivative of \( f(x) \).

Now Author the following:

(i) \( m(x, 1) \)
(ii) \( m(x, 0.5) \)
(iii) \( m(x, 0.1) \)
(iv) \( m(x, 0.0001) \)

Plot each of these on the same graph as the derivative of \( f(x) \) and comment on the results.

Exercise 2: Repeat Exercise 1 with the function \( f(x) = \frac{\sin(x^3)}{x} \).

It is important to remember that not all functions have derivatives at all (or any) points in their domain. The geometry discussed above suggests that if the graph of a function has a tangent at a particular point then it should have a derivative at that point.

Exercise 3: For each of the following functions graph the function, evaluate its derivative and discuss the behaviour of the derivative near \( x = 0 \). Do these functions have a derivative at \( x = 0 \)? Does the graph of these functions have a unique tangent at \( x = 0 \)?

(i) \( |x| \) (Author ABS(x))

(ii) \( \sqrt{|x|} \) (Author SQRT (ABS(x)))

(iii) the 'greatest integer function' (Author FLOOR (x))
Define the following functions

- \( f(t) := t + 1 \)
- \( g(t) := t - 2 \)
- \( h(t) := t + 3 \)
- \( m(t) := (t - 5)(t + 5) \)

Graph \( f(t) \)

Use algebra to solve for \( t \) when \( f(t) = 0 \)

Real solution (s): \( t = \)

Number of distinct real solutions:

Power of polynomial:

Sketch graph

List the points where the graph cuts or touches the \( x \)-axis.

Graph \( m(t) \)

(Hint: zoom out)

Use algebra to solve for \( t \) when \( m(t) = 0 \)

Real solution (s): \( t = \)

Number of distinct real solutions:

Expand \( m(t) \)

Power of polynomial:

Sketch graph

List the points where the graph cuts or touches the \( x \)-axis.

Author \((f(t))^2\)

Graph \((f(t))^2\)

Use algebra to solve for \( t \) when \((f(t))^2 = 0\)

Real solution (s): \( t = \)

Number of distinct real solutions:

Expand \((f(t))^2\)

Power of polynomial:

Sketch graph

List the points where the graph cuts or touches the \( x \)-axis.

Author \((f(t))^2 + 2\)

Graph \((f(t))^2 + 2\)

Use algebra to solve for \( t \) when \((f(t))^2 + 2 = 0\)

Real solution (s): \( t = \)

Number of distinct real solutions:

Expand \((f(t))^2 + 2\)

Power of polynomial:

Sketch graph

List the points where the graph cuts or touches the \( x \)-axis.
How many real solutions do quadratic equations have?  
How many times will a graph of a quadratic function touch or cross the $x$-axis?  
Why is this so?

<table>
<thead>
<tr>
<th>Author</th>
<th>Graph</th>
<th>Use algebra to solve for $t$ when $f(t)g(t)h(t) = 0$</th>
<th>Real solution (s):</th>
<th>Number of distinct real solutions:</th>
<th>Expand $f(t)g(t)h(t)$</th>
<th>Power of polynomial:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)g(t)h(t)$</td>
<td>$f(t)g(t)h(t)$</td>
<td>$f(t)g(t)h(t) = 0$</td>
<td>$t =$</td>
<td></td>
<td>$f(t)g(t)h(t)$</td>
<td></td>
</tr>
<tr>
<td>Sketch graph</td>
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<tr>
<td>Author</td>
<td>Graph</td>
<td>Use algebra to solve for $t$ when $f(t)f(t)h(t) = 0$</td>
<td>Real solution (s):</td>
<td>Number of real solutions:</td>
<td>Expand $f(t)f(t)h(t)$</td>
<td>Power of polynomial:</td>
</tr>
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</tr>
<tr>
<td>$f(t)f(t)h(t)$</td>
<td>$f(t)f(t)h(t)$</td>
<td>$f(t)f(t)h(t) = 0$</td>
<td>$t =$</td>
<td></td>
<td>$f(t)f(t)h(t)$</td>
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</tr>
<tr>
<td>Sketch graph</td>
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</tr>
<tr>
<td>Author</td>
<td>Graph</td>
<td>Use algebra to solve for $t$ when $(f(t))^3 = 0$</td>
<td>Real solution (s):</td>
<td>Number of distinct real solutions:</td>
<td>Expand $(f(t))^3$</td>
<td>Power of polynomial:</td>
</tr>
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</tr>
<tr>
<td>$f(t)f(t)f(t)$</td>
<td>$f(t)f(t)f(t)$</td>
<td>$(f(t))^3 = 0$</td>
<td>$t =$</td>
<td></td>
<td>$(f(t))^3$</td>
<td></td>
</tr>
<tr>
<td>Sketch graph</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Author $f(t)\cdot g(t)\cdot h(t)+9$</td>
<td>Sketch graph</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>----------------------------------</td>
<td>--------------</td>
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</tr>
<tr>
<td>Graph $f(t)\cdot g(t)\cdot h(t) + 9$ Use algebra to solve for $t f(t) \cdot g(t) \cdot h(t) + 9 = 0$</td>
<td>List the points where the graph cuts or touches the $x$-axis.</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Real solution(s): $t =$</td>
<td></td>
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<td></td>
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<tr>
<td>Number of distinct real solutions:</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Expand $f(t) \cdot g(t) \cdot h(t) + 9$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Power of polynomial:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

How many solutions do cubic equations have?

How many times will a graph of a quadratic function touch or cross the $x$-axis?

Why is this so?

<table>
<thead>
<tr>
<th>Author $f(t)\cdot g(t)\cdot h(t)\cdot m(t)$</th>
<th>Sketch graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph $f(t)\cdot g(t)\cdot h(t)\cdot m(t)$ (hint: set scale x:5, y:500) Use algebra to solve for $t f(t) \cdot g(t) \cdot h(t) \cdot m(t) = 0$</td>
<td>List the points where the graph cuts or touches the $x$-axis.</td>
</tr>
<tr>
<td>Real solution(s): $t =$</td>
<td></td>
</tr>
<tr>
<td>Number of distinct real solutions:</td>
<td></td>
</tr>
<tr>
<td>Expand $f(t) \cdot g(t) \cdot h(t) \cdot m(t)$</td>
<td></td>
</tr>
<tr>
<td>Power of polynomial:</td>
<td></td>
</tr>
</tbody>
</table>

If possible create a polynomials to the power 5 which will have the following number of distinct real solutions

| 5 solutions | sketch graphs |
| 4 solutions | |
| 3 solutions | |
| 2 solutions | |
| 1 solution | |
| 0 solutions | |
Suppose \( p(x) \) is a polynomial function whose highest power is \( n \) where \( n \) is an integer.

How many solutions does the equation \( p(x) = 0 \) have?
( don’t forget to consider what happens when \( n \) is odd and when \( n \) is even)

How many times will a graph of \( p(x) \) touch or cross the \( x \)-axis?

Why is this so?
**MA502 Worksheet 8a**  
**Work in groups of 2 or 3**

<table>
<thead>
<tr>
<th>Names</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>

Take a look at fractions and their sum

**Author**

VECTOR([a/7, 11/a, a/7+11/a, (a+11)/(7+a)],a,2,6)

Simplify

Does \( \frac{a}{7} + \frac{11}{a} = \frac{a+11}{7+a} \)?

**Table obtained**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{a}{7} )</td>
<td>11</td>
<td>( \frac{a+11}{7+a} )</td>
</tr>
<tr>
<td>( \frac{a}{7} )</td>
<td>( \frac{11}{a} )</td>
<td>( \frac{a+11}{7+a} )</td>
</tr>
</tbody>
</table>

Author \( \frac{a}{7}+\frac{11}{a} \) then use factorize and then expand

Repeat this for \( \frac{a+11}{7+a} \)

Based on this pattern find \( \frac{a}{b} + \frac{c}{a} \)

Check you answer with **DERIVE**

Based on this pattern find \( \frac{a}{b} + \frac{c}{d} \)

Check you answer with **DERIVE**

Prove this result is true for all \( a, b, c, d \in \mathbb{R} \)

What happens if \( b = d \)?

Give the general answer and then list some specific examples.

Check you answers with **DERIVE**

(Use manage, substitute, \( b = d \))

\( f(x) \) and \( g(x) \) are both linear functions

Find some examples of such functions so that

\[ \frac{f(x)}{g(x)} \]

(i) may be simplified by cancelling

(ii) may not be simplified by cancelling

Note the domain for each function.

Write your results in a general form illustrating which of these functions may be simplified by cancelling.

Check you answers with **DERIVE**

304
Let $f(x)$, $g(x)$, and $h(x)$ all be linear functions. Choose an example for each function then find

\[
\frac{f(x)h(x)}{g(x)} \quad \text{and} \quad \frac{f(x)}{g(x)h(x)}
\]

Note the domain of each function. Graph the two new functions created.

Repeat with a new set of functions.

Compare the two graphs in each pair. Comment particularly on the number and types of asymptotes.

Let $p(x) = x^3 + 5x^2 - 4$
$q(x) = 2x - 1$
$h(a) = 3a$

Find
$p(2)$,
$p(a)$,
p(m),
p(q(x)),
p(q(3))$

$p(h(a))$
$h(p(x))$
$h(q(x))$
$q(h(a))$
$q(h(p(x)))$
$q(h(p(1)))$
$(q(x))^2$ Use DERIVE to check your answers
## Appendix 2

### DATA COLLECTION INSTRUMENTS

<table>
<thead>
<tr>
<th>Appendix 2.x</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
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<tr>
<td>2.2</td>
<td>Mid-course Algebraic Insight Quiz</td>
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<tr>
<td>2.3</td>
<td>Post-course Algebraic Insight Quiz</td>
</tr>
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<td>2.4</td>
<td>Pre-course Algebraic Insight Interview</td>
</tr>
<tr>
<td>2.5</td>
<td>Post-course Algebraic Insight Interview</td>
</tr>
<tr>
<td>2.6</td>
<td>Background Survey</td>
</tr>
<tr>
<td>2.7</td>
<td>Technical Difficulties Survey</td>
</tr>
<tr>
<td>2.8</td>
<td>Judicious Use of CAS Survey</td>
</tr>
<tr>
<td>2.9</td>
<td>Post course Evaluation</td>
</tr>
<tr>
<td>2.10</td>
<td>Post Course General Interview</td>
</tr>
<tr>
<td>2.11</td>
<td>Observation Guidelines</td>
</tr>
</tbody>
</table>
# Appendix 2.1   Pre-course Algebraic Insight Quiz

Could A and B both be correct solutions to some problem?  
Example:  

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x+y)^2$</td>
<td>$x^2+y^2$</td>
</tr>
</tbody>
</table>

No - definitely not  
Yes - certainly  
Possibly

- In all questions $x \neq y \neq a \neq b \neq c \neq d \neq p \neq q$  
- None of the variables = 0 or 1.

For the following multiple-choice questions circle the letter(s) of all the options which apply.  
Example :  

$$2t^2 = 50$$  
$$t =$$

1. A  B  

$x^2 = 9$  
$x = \pm 3$

2. A  B  

$t = 2.43$  
$t = 2 \frac{3}{7}$

3. A  B  

$a + b$  
$a$

4. A  B  

For the function below when: $t$ approaches $\infty$, $h(t)$ approaches . . . . . . . .

$$h(t) = \frac{5+2t^2}{t}$$

5. A  B  

Given $y = 6 - x - x^2$

If $x = t+1$ then $y =$

6. A  B  

How many times might the graph of this function cut or touch the $x$-axis if $p, q, r, s$, and $t$ were real numbers?  

$h(x) = px^4 + qx^3 + rx^2 + sx + t$  

(a) 0  
(b) 1  
(c) 2  
(d) 3  
(e) 4  
(f) 5
4. A   B

\[ \frac{a + p}{q} \quad \frac{q + p}{p} \]

5. A   B

\[ \sqrt{xy} \quad \sqrt{xy} \]

6. A   B

\[ \frac{a + p + q}{q} \quad \frac{a + p}{q} \]

7. A   B

\[ \frac{a - b}{d - c} \quad \frac{a - b}{d - c} \]

15. How many times might the graph of this function cut or touch the x-axis if \( p, q, r, s, \) and \( t \) were real numbers?

\[ g(x) = px^4 + qx^3 + rx^2 + sx \]

(a) 0  
(b) 1  
(c) 2  
(d) 3  
(e) 4  
(f) 5

16. If

\[ f(x) = (x-1)(x-1)(x+2)(x+5) \]

how many solutions will there be for \( x \) when \( f(x) = 0 \)?

(a) 0  
(b) 1  
(c) 2  
(d) 3  
(e) 4  
(f) 5

17. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

(a) linear  
(b) quadratic  
(c) exponential  
(d) none of the above

18. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.04</td>
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<tr>
<td>-1</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>125</td>
</tr>
</tbody>
</table>

(a) linear  
(b) quadratic  
(c) exponential  
(d) none of the above
8. A B

\[
\frac{2+x}{y+2} \quad \frac{x}{y}
\]

9. A B

\[
\frac{s+p}{t} \quad \frac{s+p}{t}
\]

10. A B

\[
x^5 - y^5 \quad (x^2+y^2)(x-y)(x+y)
\]

11. A B

\[
(2x+1)^2 - 3x(2x+1) \quad (2x+1)(1-x)
\]

19. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) linear  
(b) quadratic  
(c) exponential  
(d) none of the above

20. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-5</td>
</tr>
<tr>
<td>-1</td>
<td>-4</td>
</tr>
<tr>
<td>0</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) linear  
(b) quadratic  
(c) exponential  
(d) none of the above

21. The graph of one of the functions is shown. Which function must it be?

<table>
<thead>
<tr>
<th>Function</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( f(x) = 4x-12 )</td>
<td></td>
</tr>
<tr>
<td>(b) ( f(x) = 2x^2+3x-2 )</td>
<td></td>
</tr>
<tr>
<td>(c) ( f(x) = e^{-12} )</td>
<td></td>
</tr>
<tr>
<td>(d) ( f(x) = x^5+3x^3-4x-12 )</td>
<td></td>
</tr>
</tbody>
</table>

22. The graph of one of the functions is shown. Which function must it be?

<table>
<thead>
<tr>
<th>Function</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( f(x) = 6-x )</td>
<td></td>
</tr>
<tr>
<td>(b) ( f(x) = 6-x^2 )</td>
<td></td>
</tr>
<tr>
<td>(c) ( f(x) = 6-2x )</td>
<td></td>
</tr>
<tr>
<td>(d) ( f(x) = x^3-2x^2+3x+6 )</td>
<td></td>
</tr>
</tbody>
</table>
23. The graph of one of the functions is shown. Which function must it be?

(a) \( f(x) = 3x + 4 \)
(b) \( f(x) = 4 - x^2 \)
(c) \( f(x) = e^{-x} \)
(d) \( f(x) = 2x^4 - 3x + 4 \)
### Appendix 2.2 Mid-course Algebraic Insight Quiz

In this series of questions you will see a student’s answers and the textbook answers. For each question ask yourself ‘How likely do you think it is that the student’s answer is right?’

Example:

<table>
<thead>
<tr>
<th>Student</th>
<th>Textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x+y)^2$</td>
<td>$x^2+y^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Definitely</th>
<th>Probably</th>
<th>No idea</th>
<th>Probably</th>
<th>Definitely</th>
</tr>
</thead>
<tbody>
<tr>
<td>wrong</td>
<td>wrong</td>
<td></td>
<td>right</td>
<td>right</td>
</tr>
</tbody>
</table>

---

1. Student | Textbook  
$x^2 = 9$  
$x = \pm 3$

2. Student | Textbook  
$t = 2.375$  
$t = \frac{3}{8}$

3. Student | Textbook  
$a + t$  
$a$

4. Student | Textbook  
$h(x) = px^5 + qx^4 + rx^3 + sx + t$

For the following multiple-choice questions circle the letter which applies.

Example:

- $2t = 10$
- $t = \frac{5}{2}$

1. Student | Textbook  
$x^2 = 9$  
$x = \pm 3$

2. Student | Textbook  
$t = 2.375$  
$t = \frac{3}{8}$

3. Student | Textbook  
$a = t$  
$a$

4. Student | Textbook  
$h(x) = px^5 + qx^4 + rx^3 + sx + t$

If $p, q, r, s,$ and $t$ were real numbers, what is the minimum number of times the graph of this function might cut or touch the $x$-axis?
4. Student Textbook

\[
x + y = y + x
\]

15.

\[g(x) = px^4 + qx^3 + rx + s\]

If \(p, q, r,\) and \(s\) were real numbers, what is the **maximum** number of times the graph of this function might cut or touch the \(x\)-axis?

(a) 0  
(b) 1  
(c) 2  
(d) 3  
(e) 4  
(f) 5

17.

\[
ax + p \div q = \frac{a + p}{q}
\]

18.

For the table below the rule for the function which links \(x\) to \(f(x)\) could be:

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  -2 & 0.04 \\
  -1 & 0.2 \\
  0 & 1 \\
  1 & 5 \\
  2 & 25 \\
  3 & 125
\end{array}
\]

(a) linear  
(b) quadratic  
(c) exponential  
(d) none of the above
<table>
<thead>
<tr>
<th>8. Student</th>
<th>Textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2 + x}{y + 2}$</td>
<td>$\frac{x}{y}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>9. Student</th>
<th>Textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{s + p}{t}$</td>
<td>$\frac{s+p}{t}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>10. Student</th>
<th>Textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^5 - y^5$</td>
<td>$(x^2 + y^5)(x - y)(x + y)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>11. Student</th>
<th>Textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2p+1)^2 - 3(2p+1)$</td>
<td>$(2p+1)(1-p)$</td>
</tr>
</tbody>
</table>

| 19. For the table below the rule for the function which links $x$ to $f(x)$ could be: |
| $\begin{array}{cc}
  x & f(x) \\
  -2 & 3 \\
  -1 & 5 \\
  0 & 8 \\
  1 & 13 \\
  2 & 21 \\
  3 & 34 \\
\end{array}$ |
| (a) linear |
| (b) quadratic |
| (c) exponential |
| (d) none of the above |

| 20. For the table below the rule for the function which links $x$ to $f(x)$ could be: |
| $\begin{array}{cc}
  x & f(x) \\
  -2 & -5 \\
  -1 & -4 \\
  0 & -3 \\
  1 & -2 \\
  2 & -1 \\
  3 & 0 \\
\end{array}$ |
| (a) linear |
| (b) quadratic |
| (c) exponential |
| (d) none of the above |

| 21. The graph of one of the functions is shown. Which function must it be? |
| (a) $f(x) = 4x - 12$ |
| (b) $f(x) = 2x^2 + 3x - 2$ |
| (c) $f(x) = e^{x} - 12$ |
| (d) $f(x) = x^2 + 3x^2 - 4x - 12$ |

| 22. The graph of one of the functions is shown. Which function must it be? |
| (a) $f(x) = 6x$ |
| (b) $f(x) = 6x - x^2$ |
| (c) $f(x) = 6 - 2e^x$ |
| (d) $f(x) = x^2 + 6$ |
23. The graph of one of the functions is shown. Which function must it be?

(a) \( f(x) = 3x + 4 \)
(b) \( f(x) = 4x^2 \)
(c) \( f(x) = e^x - 2 \)
(d) \( f(x) = 2x^4 - 3x + 4 \)
# Appendix 2.3  Post-course Algebraic Insight Quiz

In this series of questions you will see a student’s answers and the textbook answers. For each question ask yourself ‘How likely do you think it is that the student’s answer is right?’

<table>
<thead>
<tr>
<th>Example:</th>
<th>Student</th>
<th>Textbook</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x + y)^2</td>
<td>x^2 + y^2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Definitely Wrong</th>
<th>Probably Wrong</th>
<th>No Idea</th>
<th>Probably Right</th>
<th>Definitely Right</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the following multiple-choice questions circle the letter which applies.

**Example:**

\[2t = 10\]

**t =**

- (a) 2
- (b) -5
- (c) 5
- (d) -1
- (e) -2

### 1. Student Textbook

\[x^2 = 25\]

\[t = \pm 5\]

### 12. For the function below, when \( t \) approaches 1, \( h(t) \) approaches……

\[h(t) = \frac{5 + 2t^2}{t}\]

- (a) 2
- (b) 1
- (c) \(-\)
- (d) 0
- (e) 2t
- (f) 7

### 2. Student Textbook

\[t = \frac{5839}{57}\]

\[t = 102.4386\]

### 13. Given \( y = x^2 + 2x - 1 \)

If \( x = t + 1 \) then \( y = \)

- (a) \( t^2 + 1 + 2t - 1 \)
- (b) \( t^2 + 2t \)
- (c) \((t + 1)^2 + 2(t + 1) - 1 \)
- (d) \( t^2 + 1 + 2(t + 1) - 1 \)
- (e) \( t + 1 + 2t - 1 \)
- (f) \( t^2 + 2(t + 1) - 1 \)
3. \(4m + t = 4m\)

14. \(h(x) = px^5 + qx^4 + rx^3 + sx + t\)

If \(p, q, r, s,\) and \(t\) were real numbers, what is the maximum number of times the graph of this function might cut or touch the \(x\)-axis?

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4  (f) 5

4. \(6b - a = a + 6b\)

15. \(g(x) = px^4 + qx^3 + rx + s\)

If \(p, q, r,\) and \(s\) were real numbers, what is the minimum number of times the graph of this function might cut or touch the \(x\)-axis?

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4  (f) 5

5. \(\sqrt{p + q} = \sqrt{pq}\)

16. \(f(x) = (x-1)(x-1)(x+2)(x+5)\)

If \(f(x) = 0\), how many distinct solutions will there be for \(x\) when \(f(x) = 0\)?

(a) 0  (b) 1  (c) 2  (d) 3  (e) 4  (f) 5
6. Student

\[ x + y \div z \]

Textbook

\[ \frac{x + y}{z} \]

7. Student

\[ \frac{p - 3}{q - 4} \]

Textbook

\[ \frac{p - 3}{q - 4} \]

8. Student

\[ \frac{x + 4}{4 + y} \]

Textbook

\[ \frac{x}{y} \]

9.

17. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
</tr>
</tbody>
</table>

(a) linear
(b) quadratic
(c) exponential
(d) other

18. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.25</td>
</tr>
<tr>
<td>-1</td>
<td>0.5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) linear
(b) quadratic
(c) exponential
(d) other

19. For the table below the rule for the function which links \( x \) to \( f(x) \) could be:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1.33</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.8</td>
</tr>
<tr>
<td>6</td>
<td>0.67</td>
</tr>
</tbody>
</table>

(a) linear
(b) quadratic
(c) exponential
(d) other
9. Student Textbook

\[
a + h \quad \frac{a + h}{t} = \frac{a + h}{t}
\]

10. Student Textbook

\[
(x^4 + y^4)(x-y)(x+y) \quad x^5 - y^5
\]

11. Student Textbook

\[
(2p+q)^2 - 3p(2p+q) \quad (2p + q)(q - p)
\]

20. For the table below the rule for the function which links \(x\) to \(f(x)\) could be:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

(a) linear  
(b) quadratic  
(c) exponential  
(d) other

21. The graph of one of the functions is shown. Which function must it be?

(a) \(y = 6-x\)  
(b) \(y = 6x-x^2\)  
(c) \(y = 6e^x\)  
(d) \(y = x^4+6\)

23. The graph of one of the functions is shown. Which function must it be?

(a) \(y = 4x + 12\)  
(b) \(y = 2x^2+3x-2\)  
(c) \(y = 612\)  
(d) \(y = 3^2+3x^2-4x-12\)
## Appendix 2.4  Pre-course Algebraic Insight Interview

1. Think of a number  
   Multiply two less than the number by two more than the number, then add 8. Take away the square of the number you first thought of.  
   - What did you get?  
   - Do you always get that?  
   - Why?

2. Find 4 consecutive, positive, odd, integers whose sum is 40.

3. Find the area of the rectangle above when  
   \( d = \frac{x}{4} \) and \( d = y \). The lengths \( d, x, \) and \( y \) are measured in metres.
4. If \( T(x, x^2, 0, 1) \) generates the table
\[
\begin{array}{c|c}
    x & f(x) \\
    \hline
    0 & 0 \\
    1 & 1 \\
    2 & 4 \\
    3 & 9 \\
\end{array}
\]
and \( T(x, x^2, 0, 2) \) generates
\[
\begin{array}{c|c}
    x & f(x) \\
    \hline
    0 & 0 \\
    2 & 4 \\
    4 & 16 \\
    6 & 36 \\
\end{array}
\]
What will be generated by
i) \( T(x, 3x^2, 0, 1) \)
ii) \( T(x, x^2, 0, b) \)
iii) \( T(x, x + a, 2, 1) \)
iv) \( T(y, y^2, 0, 2) \)

5. \( g(x) = x^2 + 4x - 21 \)

i) What parts of this rule immediately give you clues to the shape and position of a graph of this function? Explain.

ii) Rewrite this rule so that the \( x \) intercepts of its graph can be easily seen.

iii) Rewrite this rule so that the coordinates of the minimum point of its graph can be easily seen.
### Appendix 2.5  Post-course Algebraic Insight Interview

1. Think of a number. Add 5 and then square the result. Now add 10 times the number you first thought of and take away 20. Finally, take away the square of the number you first thought of.

   What did you get?

   Do you always get that?

   Why?

2. Find 4 consecutive, positive, even, integers whose sum is 40.

3. Find the area of the rectangle above when $d = \frac{x}{3}$ and $d = y$. The lengths $d$, $x$, and $y$ are measured in centimetres.
4. If \( G(x, x + 1, -5, 5, 1) \) generates the graph and \( G(r, r^2, 0, 4, 0.5) \) generates the graph.

What will be generated by the following commands?

(i) \( G(x, x^2, -2, 2, 1) \)

(ii) \( G(x, x^2, -2, 2, 1) \)

(iii) \( G(x, 1/x, -1, 1, 0.5) \)

(iv) \( G(p, p+2, a, a+4, 1) \)
5.

\[ g(x) = 8 - 2x - x^2 \]

iv) What parts of this rule immediately give you clues to the shape and position of a graph of this function? Explain.

v) Rewrite this rule so that the x intercepts of its graph can be easily seen.

vi) Rewrite this rule so that the coordinates of the minimum point of its graph can be easily seen.
Appendix 2.6  Background Survey

Name:_________________________    Student Id No.__________________

Previous Mathematics experience

Highest level maths studied at school:__________________________ Result_______

Previous tertiary mathematics units:____________________________ Result_______

Previous experience of using technology for doing or learning maths

Did you use a scientific calculator at secondary school? Yes [ ] No [ ]

Did you use a graphical calculator at secondary school? Yes [ ] No [ ]

If yes, list the features available on your graphical calculator that you found most helpful.

Did you use a computer for mathematics at secondary school? Yes [ ] No [ ]

If yes name the program(s) and the features which you found most helpful

Program Name ____________________________

Program Name ____________________________

Program Name ____________________________

Program Name ____________________________

Please rate your general computing skills by marking an X on the number scale below.

<table>
<thead>
<tr>
<th>Poor</th>
<th>OK</th>
<th>Good</th>
<th>Very good</th>
<th>Excellent</th>
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<td>4</td>
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### Appendix 2.7 Technical Difficulties Survey

**MA502: Using the program DERIVE**

Please circle the word which best indicates the number of problems you had during today’s lab session with any of the following:

<table>
<thead>
<tr>
<th>Activity</th>
<th>Not Applicable</th>
<th>None</th>
<th>One</th>
<th>Some</th>
<th>A lot</th>
<th>Every time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Authoring , using /, ^, *, +, - symbols</td>
<td>NA</td>
<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
</tr>
<tr>
<td>Using brackets to force the structure expressions</td>
<td>NA</td>
<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
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<tr>
<td>Using syntax for commands eg FIT, VECTOR, F(x)</td>
<td>NA</td>
<td>N</td>
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<td>E</td>
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<tr>
<td>Using sequences of commands eg to substitute</td>
<td>NA</td>
<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
</tr>
<tr>
<td>Interpreting the results of the <code>solve</code> command</td>
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<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
</tr>
<tr>
<td>Obtaining exact or approximate solutions</td>
<td>NA</td>
<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
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<tr>
<td>Working from the screen to ordinary maths symbols</td>
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<td>N</td>
<td>O</td>
<td>S</td>
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<td>E</td>
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<tr>
<td>Moving to a graph window</td>
<td>NA</td>
<td>N</td>
<td>O</td>
<td>S</td>
<td>A</td>
<td>E</td>
</tr>
<tr>
<td>Obtaining a window which shows the graph required</td>
<td>NA</td>
<td>N</td>
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<tr>
<td>Zooming to show key features of a graph</td>
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<td>Copying from the screen for sketch graphs</td>
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<td>Working out the scale of a graph</td>
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Are there any other problems which you have had using DERIVE today?  
***If Yes please give some details:***
### Appendix 2.8  Judicious Use of CAS Survey

Please tick as many of the statements below as apply. Today I have used DERIVE to:

<table>
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<tr>
<th>Statement</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Find answers if the computer was suggested</td>
<td>[ ]</td>
</tr>
<tr>
<td>Explore problems if the computer was suggested</td>
<td>[ ]</td>
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<tr>
<td>Explore variations on the set problems</td>
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<tr>
<td>Explore, other than when directed but on the same topic</td>
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</tr>
<tr>
<td>Explore other mathematics not on today’s topic</td>
<td>[ ]</td>
</tr>
<tr>
<td>Find answers I could ‘easily’ have done with pen and paper.</td>
<td>[ ]</td>
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<tr>
<td>Find answers to ‘hard’ questions</td>
<td>[ ]</td>
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<tr>
<td>Find answers to time consuming questions</td>
<td>[ ]</td>
</tr>
</tbody>
</table>
Appendix 2.9  Post course Evaluation

SD  Strongly disagree  D  Disagree  N  Neutral  A  Agree  SA  Strongly agree

Circle the code that most closely matches your response to the following statements.

Derive has helped me see patterns in Mathematics  SD          D          N          A          SA

I find using Derive helps me to understand Mathematics SD          D          N          A          SA

DERIVE can be made to do the working out for maths problems. SD          D          N          A          SA

I can use DERIVE to check every step of a problem. SD          D          N          A          SA

A DERIVE plot will tell me everything I need to know about a function. SD          D          N          A          SA

I only use DERIVE if the instructions tell me to. SD          D          N          A          SA

I try out ideas using Derive SD          D          N          A          SA

Compared to my confidence with functions before this unit, my confidence now is much greater SD          D          N          A          SA

Compared to my confidence with calculus before this unit, my confidence now is much greater SD          D          N          A          SA

Compared to my confidence with graphs before this unit, my confidence now is much greater SD          D          N          A          SA

Compared to my confidence with trigonometry before this unit, my confidence now is much greater D          N          A          SA
For each of the following mathematical questions please indicate the ‘tool’ you would prefer to use.

PP (pen/paper) DA (Derive-Algebra) DG (Derive Graph) CC (Calculator-computation) CG (calculator-graphs) OT (Other—please specify)

For this type of problem | For speed | For success | For learning
---|---|---|---
1. Solve for $x$: $2x + 1 = 5$
2. Solve for $x$: $(x - 4)(x + 1)=0$
3. Solve for $x$: $x^5 - x^2 + x = 0$
4. Find the intercepts of the graph of $y= 2x + 1$
5. Find the $x$ and $y$ intercepts for $y = x^2 + 5x + 6$
6. Find the turning point for $y = x^2 + 5x + 6$
7. Compare log $x$ and $\left(\frac{1}{x}\right)$

For 10 mathematical questions of the following types please indicate the ‘tool’ you would prefer to use.

PP (pen/paper) DA (Derive-Algebra) DG (Derive Graph) CC (Calculator-computation) CG (calculator-graphs) OT (Other—please specify)

<table>
<thead>
<tr>
<th>For this type of problem</th>
<th>For speed</th>
<th>For success</th>
<th>For learning</th>
</tr>
</thead>
</table>
8. Solve for $x$: $ax + b = c$
9. Solve for $x$: $(x - s)(x + t)=0$
10. Solve for $x$: $ax^5 - bx^2 + cx = 0$
11. Find the intercepts of the graph of $y= mx + c$
12. Find the $x$ and $y$ intercepts for $y= ax^2 + bx + c$
13. Find the turning point for $y= ax^2 + bx + c$
14. Compare $f(x)$ and $f'(x)$

Any other comments: (continue over)
Appendix 2.10  Post Course General Interview

- Think about the maths we have been studying. Tell me about what you have found easiest and what you have found hardest and why?

- Now tell me about your experience with DERIVE. What problems, if any have you had learning to use DERIVE?

- When have you found DERIVE most helpful? Give examples.

- Are there some aspects of maths for which you think using DERIVE has enhanced your understanding? Give examples.

- We can tackle maths problems in different ways using mental calculations, pen and paper, scientific calculators, CAS.
  - What place does each have for you?
  - Give examples of the sort of problems for which you would use each approach.

- How could CAS be used to help a student develop recognition of patterns in algebraic operations:
  - that demonstrate how process becomes object ie \( f(g(x)) \)
  - for possible numbers of solutions to equations
  - for critical values
  - links between symbols, graphs and tables
Appendix 2.11 Observation Guidelines

Stages in answering a question: things to look for:

1. **Deciding what the question requires.**
   What am I supposed to do?

2. **Formulating the problem (if necessary)**
   Interpret words → algebra
   Interpret words → numbers (table)

3. **Decide on mathematical method required to solve problem.**
   Solve equation
   Link information from one equation to another
   substitute
   Simplify expression
   Graph
   Factorise
   Differentiate
   Integrate
   Look at a number of examples and use induction
   Use a general proof – deduction
   May have no strategy: trial and error
   Numerical trial and error
   Random approach to choice of mathematical method

4. **Decide on preferred ‘technology’**
   Pen and paper
   Calculator - Scientific
   Graphical
   CAS Derive algebra
   Derive graphs
   Some combination of the above
   Why?? reasons for choice.

5. **If the student chooses to use CAS?**
   Why?
   Can they use appropriate commands, syntax, authoring to do the maths they want?
   Do they use multiple representations? Graphs, tables, numbers, algebra?
   Do they make links between representations eg notice the link between factors and intercepts
   Do they use CAS to do a number of similar examples?
   Do they change powers, coefficients etc to ‘see what happens’”
   Do they look for patterns? Do they see patterns?
   Do they see links?
   Do they see what we (‘experts’) see? Are the mathematical illustrations of concepts, links, patterns, methods of solution, proofs etc obvious to a ‘novice’?

6. **Interpret results**
   Has the problem been solved?
   What is the solution?
   What does this mean in terms of the initial question? (if applicable)
APPENDIX 3

RESULTS FROM ALL 3 ALGEBRAIC INSIGHT QUIZZES
# Early Course Algebraic Insight Quiz Data – Raw Data

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# Appendix 4

## Measures and Descriptors of Algebraic Insight and Effective Use of CAS

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<tr>
<td><strong>1.</strong> Algebraic Expectation</td>
<td>-26 – 26 -100 – 100% 0 - 10</td>
<td>Algebraic Insight Quiz  Algebraic Insight Interview</td>
<td>T6.2, F6.2 Case Studies T6.5</td>
</tr>
<tr>
<td><strong>1.1.</strong> Recognition of conventions and basic properties Identification of structure Identification of key features</td>
<td>-100 – 100% -100 – 100% -100 – 100%</td>
<td>Algebraic Insight Quiz</td>
<td>F6.3</td>
</tr>
<tr>
<td><strong>2.</strong> Ability to Link Representations</td>
<td>-18 – 18 -100 – 100% 0 – 10</td>
<td>Algebraic Insight Quiz  Algebraic Insight Interview</td>
<td>T6.2, F6.2 Case studies T6.5</td>
</tr>
<tr>
<td><strong>2.1.</strong> Linking of symbolic and graphic representations Linking of symbolic and numeric representations</td>
<td>-100 – 100% -100 – 100%</td>
<td>Algebraic Insight Quiz</td>
<td>F6.3</td>
</tr>
<tr>
<td><strong>2.2.</strong> Fluently Use of program syntax Ability to systematically change representation Ability to interpret CAS output</td>
<td>Low-high Low-high Low-high -100 – 100% 0 – 100% Low-high Low-high Low-high -100 – 100% 0 – 100%</td>
<td>Technical Difficulties survey</td>
<td>F6.6, T6.5 F6.6, T6.7 Case studies F6.7, T6.7 Case studies F6.8, T6.7 Case studies</td>
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<tr>
<td><strong>2.3.</strong> Positive Attitude 2.3.1 Manner of use 2.3.2 Functional use 2.3.3 Pedagogical use</td>
<td>1 - 20 -100 – 100% -100 – 100% 0 – 10 1 - 6</td>
<td>Post-course evaluation  Judicious Use of CAS survey  Judicious Use of CAS survey  Judicious Use of CAS survey</td>
<td>F6.14, 6.15 Case studies Case studies F6.9, 6.10, 6.11 F6.12, F6.13</td>
</tr>
<tr>
<td>Framework</td>
<td>Descriptors</td>
<td>Evidence</td>
<td></td>
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<tr>
<td><strong>Algebraic Insight</strong></td>
<td>Very poor, poor, good, very good</td>
<td>Algebraic Insight Quiz and Interview, observation, examinations, worksheets</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Algebraic Expectation</td>
<td>Very poor, poor, good, very good</td>
<td>Algebraic Insight Quiz and Interview, observation, examinations, worksheets</td>
</tr>
<tr>
<td>1.1</td>
<td>Recognition of conventions and basic properties</td>
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<tr>
<td>1.2</td>
<td>Identification of structure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>Identification of key features</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Ability to Link Representations</td>
<td>Very poor, poor, good, very good</td>
<td>Algebraic Insight Quiz and Interview, observation, examinations, worksheets</td>
</tr>
<tr>
<td>2.1</td>
<td>Linking of symbolic and graphic representations</td>
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<tr>
<td>2.2</td>
<td>Linking of symbolic and numeric representations</td>
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</tr>
<tr>
<td><strong>Effective Use of CAS</strong></td>
<td>Low, moderate, high</td>
<td>Judicious Use of CAS Survey, Post-course survey, observation, examinations, worksheets</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>Technical</td>
<td>Low, moderate, high</td>
<td>Technical difficulties survey, observation</td>
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<tr>
<td>1.1</td>
<td>Fluent Use of program syntax</td>
<td>Few errors</td>
<td>Technical difficulties survey, observation</td>
</tr>
<tr>
<td>1.2</td>
<td>Ability to systematically change representation</td>
<td>Many errors</td>
<td></td>
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<tr>
<td>1.3</td>
<td>Ability to interpret CAS output</td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>Personal</td>
<td>Low, Moderate, High</td>
<td>Surveys, observation, Post-course interview</td>
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<tr>
<td>2.1</td>
<td>Positive Attitude</td>
<td>Negative, Positive</td>
<td>Student comments, Post course evaluation, Post course interview</td>
</tr>
<tr>
<td>2.2</td>
<td>Judyicious Use of CAS</td>
<td>Low, Moderate, High</td>
<td></td>
</tr>
<tr>
<td>2.2.1</td>
<td>Manner of use</td>
<td>Passive, Random, Responsive, Directed Strategic, Self-initiated Strategic</td>
<td></td>
</tr>
<tr>
<td>2.2.2</td>
<td>Functional Use</td>
<td>Non-discriminating, Discriminating</td>
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<tr>
<td>2.2.3</td>
<td>Pedagogical Use</td>
<td>Minimum, Limited, Extended</td>
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